At Play with Combinatorial Optimization, Integer Programming and Polyhedra

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ou

Excursions en Optimisation Combinatoire, Programmation Entière et Polyhèdres.

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Contents

Glossary

$\alpha(G)$ Stability number of G i.e. size of a maximum cardinality stable set. We sometimes abuse notation and given a subset of vertices U of a graph G, use $\alpha(U)$ for $\alpha(G[V])$.

CIG Circular interval graphs.

Circular interval graphs A *circular interval graph* $G(V,E)$ is defined by the following construction: Take a circle $\mathcal{C}$, a set of vertices V and a mapping $\Phi : V \mapsto \mathcal{C}$ of the vertices on the circle. Take a subset of closed and proper intervals $\mathcal{J}$ of $\mathcal{C}$, with no interval including another, and say that $u, v \in V$ are adjacent (i.e. $(u,v) \in E$) if $\{\Phi(u), \Phi(v)\}$ is a subset of one of the intervals. $(V, \Phi, \mathcal{J})$ is a *interval model* of G.

Circular one matrix Let A be a $m \times n$ matrix with zeros and ones. A is a *circular one matrix* if in each row, the ones appear in consecutive columns (where column 1 is considered consecutive to column n).

Claw A *claw* $(u; v, w, z)$ is a graph with vertex set $\{u, v, w, z\}$ and edge set $\{(u,v), (u,w), (u,z)\}$.

Clique A *clique* is a complete subgraph.

Clique family inequality Let G be a graph and let $\mathcal{F} = \{K_1, \ldots, K_n\}$ be a set of cliques o G, $1 \leq p \leq n$ be integral and $r = n \mod p$. Let $V_{p-1} \subseteq V(G)$ be the set of vertices covered by exactly $(p-1)$ cliques of $\mathcal{F}$ and $V_{\geq p} \subseteq V(G)$ the set of vertices covered by p or more cliques of $\mathcal{F}$. The inequality

$$(p-r-1) \sum_{v \in V_{p-1}} x(v) + (p-r) \sum_{v \in V_{\geq p}} x(v) \leq (p-r) \left\lfloor \frac{n}{p} \right\rfloor$$

is valid for $STAB(G)$ and is called the *clique family inequality* associated with $\mathcal{F}$ and p.

Combinatorial union of polytopes Let $P_i \subseteq \mathbb{R}^n$, $i = 1, ..., k$ be k non empty polyhedra. Let $\lambda^1, ..., \lambda^l \in \{0,1\}^k$. The *combinatorial union* of the polyhedron P_i w.r.t. $\{\lambda_1, ..., \lambda_l\}$ is the set $P = conv.hull(\bigcup_{j=1}^{l} \sum \lambda^j(i) P_i)$ where the sum is the Minkowsky sum.

Composition of strips Let $\mathcal{H} = \{(G^i, \mathcal{A}^i),\ i = 1, ..., k\}$ be a family of vertex disjoint strips. Let $\mathcal{A}(\mathcal{H})$ denote the multi-family of the extremities of those strips, i.e., $\mathcal{A}(\mathcal{H}) = \bigcup_{i=1..k} \mathcal{A}^i$, and let $\mathcal{P} = P_1, P_2, ..., P_q$ be the classes of a partition of $\mathcal{A}(\mathcal{H})$. We associate to the pair $(\mathcal{H}, \mathcal{P})$ the graph G that is made of the disjoint union of the graphs $G^1, \ldots G^k$, with additional edges $E := \{(u,v) : u \neq v$ and u and v belong to different extremities in a same class P_i, for some $1 \leq i \leq q\}$. G is called the *composition of the strips* $\mathcal{H}$ with respect to partition $\mathcal{P}$.

Distance simplicial strip A strip $(G, \mathcal{A})$ is *distance simplicial* if for all $A \in \mathcal{A}$, $\alpha(N_i(A)) \leq 1$ for all $i \geq 0$.

Edmonds' inequality Let G be a graph. Given a set of n cliques and a set of vertices Q covered by at least two of those cliques, one can write an inequality $\sum_{v \in Q} x(v) \leq \lfloor |Q|/2 \rfloor$ which is valid for $STAB(G)$ and such an inequality is called an *Edmonds' inequality*.

FCIG Fuzzy circular interval graphs.

Fuzzy circular interval graphs A graph $G = (V, E)$ is a *fuzzy circular interval graph* if the following conditions hold: (i) There is a map Φ from V to a circle $\mathcal{C}$; (ii) There is a set $\mathcal{J}$ of closed and proper intervals of $\mathcal{C}$, none including another, such that no point of $\mathcal{C}$ is the end of more than one interval and (iia) If two vertices u and v are adjacent, then $\Phi(u)$ and $\Phi(v)$ belong to a common interval ; (iib) If two vertices u and v belong to the same interval, which is not an interval with endpoints $\Phi(u)$ and $\Phi(v)$, then they are adjacent.

$\Gamma(S)$ is the strong neighborhood of S i.e. given a graph $G(V, E)$ and a set $S \subseteq V$, $\Gamma(S) = \{v \in V \setminus S : (u, v) \in E \text{ for all } u \in S\}$.

Intersection property a polyhedra P has the *intersection property* if $P \cap \{x : \sum_{e \in E} x_e = k\}$ is integral for every integer k.

JRP Joint Replenishment Problem, see Section 3.3.

Krausz family Given a graph G, we call a family $\mathcal{F}$ of cliques of G a *Krausz family* (of G) if it satisfies that every edge of G is covered by some clique of $\mathcal{F}$ and every vertex of G is covered by exactly two cliques of $\mathcal{F}$.

Line graph The *line graph* $L(G)$ of a graph G is the intersection graph of the edges of G i.e. there is a vertex for each edge in G and two vertices are adjacent if and only if their corresponding edges in G are incident.

Line strip a strip $(S, \mathcal{A})$ is *line* if it admits a Krausz family $\mathcal{K}$ with $\mathcal{A} \subseteq \mathcal{K}$.

$MATCH(G)$ The matching polytope of a graph G i.e. the convex hull of all incidence vectors of matchings in a graph G.

Matching A *matching* in a (multi-) graph $G(V, E)$ is a subset of pairwise non incident edges.

Minkowsky sum Given two polyhedra P_1, P_2, the *Minkowsky sum* of P_1 and P_2 is denoted by $P_1 + P_2$ and is defined by $P_1 + P_2 = \{x_1 + x_2 : x_1 \in P_1, x_2 \in P_2\}$.

MWSS Maximum Weighted Stable Set.

$N(S)$ the neighborhood of the set S i.e. given an undirected graph $G(V, E)$, and $S \subseteq V$, $N(S) = \{v \in V \setminus S : (u, v) \in E \text{ for some } u \in S\}$.

$N(v)$ the neighborhood of v i.e $N(\{v\})$.

$N[S]$ the closed neighborhood of S i.e. $N(S) \cup S$.

$N[v]$ the closed neighborhood of v i.e $N[\{v\}]$.

$N_i(v)$ the i-th neighborhood of v defined by the following recursion : $N_0(v) = \{v\}$, $N_{i+1}(v) = N(N_i(v)) \setminus \bigcup_{l=1}^{i} N_l(v)$.

Net A *net* $(u_1, u_2, u_3; v_1, v_2, v_3)$ is a graph with vertex set $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ and edge set $\{(u_1, u_2), (u_2, u_3), (u_3, u_1), (u_1, v_1), (u_2, v_2), (u_3, v_3)\}$.

OWMR One Warehouse Multi-Retailer Problem, see Section 3.3.

Partition-cliques Let G be the composition of some strips $\mathcal{H}$ with respect to partition $\mathcal{P}$. By definition the set of vertices in a same class of the partition $\mathcal{P}$ of a composition of strips induces a clique in G. We call such cliques *partition-cliques*.

Quasi-line A graph is *quasi-line* if the neighborhood of any vertex can be covered by two cliques.

Root graph a graph H is a *root graph* of a graph G if G is the line graph of H.

$STAB(G)$ the stable set polytope of a graph G i.e. the convex hull of all incidence vectors of stable sets in a graph G.

Set Packing Polytope Let A be a $m \times n$ matrix with zeros and ones. The set packing polytope is the polytope $conv.hull\{x : Ax \leq \mathbf{1}, x \in \{0,1\}^n\}$.

Set Packing Problem Let A be a $m \times n$ matrix with zeros and ones and $c \in \mathbb{R}^n$, the set packing problem is the problem $\max\{cx : Ax \leq \mathbf{1}, x \in \{0,1\}^n\}$.

Stable set A *stable set* is a set of pairwise non-adjacent vertices in a graph.

Strip A *strip* $(G, \mathcal{A})$ is a graph G (not necessarily connected) with a multi-family $\mathcal{A}$ of either one or two designated non-empty cliques (possibly identical) of G. The cliques in $\mathcal{A}$ are called the *extremities* of the strip.

$U[v]$ the vertices universal to v i.e $U[v] := \{u \in N[v] : (u, w) \in E \text{ for all } w \in N(v)\}$.

Chapter 1

Introduction

Most of the material presented in this manuscript, unless otherwise stated, is based on joint works with some of my co-authors. They are (by alphabetical order) : Fritz Eisenbrand, Yuri Faenza, Jean-Philippe Gayon, Thomas Liebling, Lavanya Marla, Guillaume Massonnet, Luciano Muller Nicoletti, Gianpaolo Oriolo, Ugo Pietropaoli, Sebastian Pokutta, Eleni Pratsini, Christophe Rapine, Alexander Rikun, Bianca Spille, Paolo Ventura and Jean-Philippe Vial.

We are mainly interested in the study of *combinatorial optimization* problems by mean of *integer programming* techniques. *Combinatorial optimization* is a field of applied mathematics that deal with optimization problems over discrete structures. Such problems arise in about all walks of life whether planning the distribution of goods, scheduling tasks over machines or managing inventory levels. *Integer (linear) programming (IP)* is about ways to solve linear optimization problems with discrete or integer decision variables i.e. problems of the form $\min\{cx : Ax \leq b, x \in \mathbb{N}^n\}$ where A, b, c usually have integer entries. IP allows to model many standard combinatorial optimization problems and IP techniques were proven extremely powerful in dealing with combinatorial optimization.

Given the fact that integer programming is $\mathcal{NP}$-hard, we focus mainly on three different types of questions :

(i) understand which particular problems/structures can be optimized efficiently and why ;

(ii) develop generic/problem-specific methods to explore as efficiently as possible the solution space ;

(iii) or develop polynomial time algorithms that find 'approximate' solutions to classic $\mathcal{NP}$-hard problems

Polyhedron are particularly important to address those questions. Indeed integer programming over a polytope $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ is equivalent to linear programming over P_I, the integer hull of P i.e. the polytope $P_I := convex.hull\{x \in \mathbb{N}^n : x \in P\}$. Hence polyhedron play a major role in theory for the design efficient algorithms (exact or approximate). Additionally it also plays an important role in practice as real-world applications are often classic combinatorial optimization problems with additional side constraints and thus a polyhedral characterization of the simpler model allows for stronger formulations and thus faster algorithms.

Our research activities covers different topics in combinatorial optimization, integer programming and polyhedral theory and span both theory and practice. We spent a great part of our time on studying a fundamental polynomial time solvable problem in combinatorial optimization : *the maximum weighted stable set problem in claw-free graphs.* This problem is a fundamental generalization of the matching problem for which the polyhedral structure (i.e. the integer hull of all stable sets in a claw-free graph) is not very well understood. Providing a 'decent' linear description of this polytope is a major open problem in our field. Indeed polyhedral combinatorics has been a powerful unifying theory for combinatorial optimization and often polytime combinatorial optimization algorithms have a nice polyhedral counterpart. We contributed significantly to answer this fundamental question and we also devised new efficient algorithms for both the optimization problem and the separation problem. Our main contributions in this field extend to more general settings and our work in this space is definitely our masterpiece. More recently, we started a stream of research in *inventory control* while working as a research staff member at IBM Zurich Research Lab. There we focussed mainly on the so-called One-Warehouse Multi-retailer problem, a very important inventory control problem arising in the distribution of goods. We devised efficient approximation algorithms for this problem, building on polyhedral techniques. Those results are both important in practice and in theory as they allow to approximate large real-world instances of the problem (we implemented some of those algorithms at IBM) and the techniques we developed appear to offer new possible ways of looking at approximation algorithms for combinatorial problems through extended formulations (i.e. integer programming formulations in higher spaces). Beside those main themes of research, we contributed other fundamental results to combinatorial optimization and integer programming. In particular, we studied the *p-median problem in Y-free graphs* and could unify some results there with standard results from the matching theory. Also we contributed to a better understanding of the classic *Chvátal-Gomory procedure* – an fundamental procedure to derive the integer hull of a polytope – by providing new lower bounds on the CG-rank for general 0/1 polytopes. Finally, we also solved real-world applications such as *cutting stock problems*, *workforce scheduling*, *corporate portfolio management*, pharmaceutical supply chain design and robust aircraft routing.

Those various projects allowed us to learn, master and extend a large spectrum of techniques in combinatorial optimization and integer programming. For instance the claw-free graph project combines (and extends) techniques from dynamic programming, network flows, matching, structural graph theory and polyhedra ; the inventory control project involves the use of convex optimization, dynamic programming and extended formulations ; while our industrial applications build on column generation techniques, cutting plane algorithms and robust optimization. We believe that what remains beside our various problem-specific contributions is precisely this toolkit of standard techniques and extensions that we developed.

Moreover we are convinced that some of those core extensions will find applications in other contexts, e.g. the notion of combinatorial union of polytopes, the reduction of the stable set problem in composition of strips to matching, or the use of extended formulation for the design of approximation algorithm. In some sense our specific research problems also served as catalyzers to generate new ideas and methods that transcend the topics themselves. This bears a resemblance with 'literature under constraints' initiated by OuLiPo in the 60s.

"The more constraints one imposes, the more one frees oneself of the chains that shackle the spirit... the arbitrariness of the constraint only serves to obtain precision of execution"

- Igor Stravinsky

Therefore, in this document, we decided to put the focus on the results that are at the core of all our contributions and that constitute now the toolkit of techniques that we will build on in the future. We try to avoid (as much as possible) the technical details inherent to a particular problem in order to share with the reader the main insights. Of course our main contributions will be discussed along the way as a combination of those different core results. We hope you will enjoy the reading and that you will be able to take something home with you.

In the following, we assume that the reader is familiar with the standard notations and techniques from combinatorial optimization. Nevertheless we sometimes recall some standard but more advanced results that we make use of or at least give pointers. Most of the sections can be read independently if the reader refers to the glossary for terminologies that were defined in other sections.

Chapter 2

Dynamic programming and shortest path

2.1 Inventory control

We start with a very simple inventory control problem. We are interested in managing efficiently the inventory of a single item at a given storing location over a discrete planing horizon of T periods (numbered $1, ..., T$). This is known as a *single product single echelon* inventory control problem. We consider the following model. The storing location, say a warehouse, faces deterministic demand $d_t \geq 0$ over each period $t = 1, ..., T$. Those demands have to be fulfilled on time by ordering those units from an external supplier of infinite capacity. If the warehouse orders in period t from its supplier, it incurs a fixed ordering cost K_t. In addition, a holding cost h_t is paid per unit in stock from period t to $t+1$. We assume that the orders are delivered instantaneously (this is without loss of generality when the lead-time is deterministic). The goal is to place orders (i.e. to decide in which periods to order and which quantities) so that all demands are fulfilled on time and the sum of the fixed ordering costs and holding costs is minimized. Because of the linear holding cost structure and the infinite capacity at the external supplier, one can assume without loss of generality that demand in period t is served from a unique order in period $s \leq t$. The problem can thus be modeled by the following linear integer program, where we denote $h_s^t = \sum_{r=s}^t h_r$.

$$
\begin{array}{lll}
\min & \displaystyle\sum_{s=1}^T y_s K_s + \sum_{t=1}^T \sum_{s:s\leq t} x_s^t h_s^t d_t & \\
\text{s.t.} & \displaystyle\sum_{s:s\leq t} x_s^t = 1, & \text{for all } t = 1, \ldots, T : d_t > 0 \\
& x_s^t \leq y_s, & \text{for all } t = 1, \ldots, T, s = 1, \ldots, t \\
& x_s^t, y_s \in \{0, 1\}, & \text{for all } s = 1, \ldots, T, t = s, \ldots, T
\end{array}
$$

For each $s, t \in \{1, ..., T\}$, $x_s^t \in \{0, 1\}$ indicates if demand d_t is ordered in period s. The variable y_s indicates if order is placed in period s. The first constraint in the formulation ensures that each demand is served by a certain order, while the second ensures that we can serve a demand by an order in period s only if we really place an order in that period. An optimal solution to this integer program can be obtained via dynamic programming. Indeed, let $OPT(t)$ be the minimum cost of a solution when we restrict the problem to the periods $1, ..., t$. If $r \leq t$ is the last period in which an order is placed in this solution (i.e. all demands from period r to t are served by the order in r), then clearly the restriction of the solution must be optimal for periods $1, ..., r-1$ and thus has cost $OPT(r-1)$. This gives the following recursion.

$$OPT(t) = \min_{1 \leq r \leq t} \{OPT(r-1) + K_r + \sum_{s=r}^{t} h_r^s d_s\}$$

with $OPT(0) = 0$.

We can thus compute $OPT(T)$ in time $O(T^2)$ using forward induction. This is known as the Wagner-Within algorithm [98]. Actually, we can also view the problem as a shortest path problem in a directed acyclic graph with nodes $\{1, ..., T+1\}$ and arcs (i,j) for all $i < j$. The cost of arcs (i,j) is then $K_i + \sum_{k=i}^{j-1} h_i^k d_k$ i.e. the cost of ordering in period i and then serving all demand from i to $j-1$ with this order. There is a one to one correspondence between ordering plans and paths from 1 to $T+1$ in this graph and thus a min cost ordering plan can be computed via any shortest path algorithm for acyclic digraphs in time $O(T^2)$. Using the Monge property of the holding cost, Aggarwal and Park [1] proved that the shortest path can actually be computed in time $O(T)$. We will make use of this simple algorithm later to devise a simple and fast 2-approximation algorithm for a more difficult inventory control problem.

2.2 Stable sets in graphs with bounded width stability number

Given a graph $G(V,E)$. A *stable set* is a set of pairwise non-adjacent vertices. Given a weight function $w : V \mapsto \mathbb{R}$, the MWSS problem in G is the problem of finding a stable set S of maximum weight $w(S) := \sum_{s \in S} w(s)$ in G. We denote $\alpha_w(G)$ the optimal value and call it the weighted stability number. If $w = \mathbf{1}$, then we abuse notation and call $\alpha(G) = \alpha_{\mathbf{1}}(G)$ the stability number of G. This problem is known to be NP-hard for general graphs, even for $w = \mathbf{1}$, [37]. However in some particular classes of graphs, this can be solved in polynomial time. This is for instance the case in {claw,net}-free graphs. A *claw* $(u; v, w, z)$ is a graph with vertex set $\{u, v, w, z\}$ and edge set $\{(u,v), (u,w), (u,z)\}$. A *net* $(u_1, u_2, u_3; v_1, v_2, v_3)$ is a graph with vertex set $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ and edge set $\{(u_1,u_2), (u_2,u_3), (u_3,u_1), (u_1,v_1), (u_2,v_2), (u_3,v_3)\}$. Given a set of forbidden subgraphs H, a graph is said to be H-free if it does not contain any graph in H as a induced subgraph. Pulleyblank and Shepherd [83] proved that a {claw,net}-free graph has the property that $\alpha(N_i(v)) \leq 2$ for all $v \in V$ and all $i = 1, ..., |V|$ ($N_i(v)$ is the i-th neighborhood of v)). If we call $max_{v \in V, i=1,...,|V|} \alpha(N_i(v))$ the *width stability number* of a graph G, then {claw,net}-free graphs have thus width stability number at most 2. Pulleyblank and Shepherd have proved that for classes of graphs with width stability number bounded by a fixed parameter B, the maximum weighted stable set problem can be computed in polytime through dynamic programming. This works as follows.

Let z be a vertex in V and $\mathcal{S}_i$ the stable sets in $N_i(z)$ for all $i = 0, ..., \rho$ (including the empty set), where ρ is the last integer such that $N_i(z) \neq \emptyset$. Note that by convention we define $N_0(z) = \{z\}$. Let $s_k \in \mathcal{S}_k$ for some $k \in \{1, ..., \rho\}$. A maximum weighted stable set S of $G[\bigcup_{i=0}^{k} N_i(z)]$ such that $S \cap N_k(z) = s_k$ has clearly the property that $S \cap V(G[\bigcup_{i=0}^{k-1} N_i(z)])$ is a maximum weighted stable set over all stable sets of $G[\bigcup_{i=0}^{k-1} N_i(z)]$ compatible with s_k (compatibility only depends on edges between s_k and $s \cap N_{k-1}(z)$). Hence we can write the following recursion : For all $k = 1, ..., \rho$, for all $s_k \in S_k$

$$OPT(k, s_k) = max\{OPT(k-1, s) + w(s_k) : s \in S_{k-1} \text{ and } s \cup s_k \text{ is a stable set of } G\}$$

with $OPT(0, \emptyset) = 0$ and $OPT(0, \{z\}) = w(z)$.

We can thus compute the maximum weighted stable set in G by computing $OPT(k, s)$ for all $k = 1, ..., \rho$ and $s \in \mathcal{S}_\rho$ using forward induction in time $O(|V|^{2B})$ and then the maximum weighted stable set has value $\max_{s \in S_\rho} OPT(\rho, s)$ and the corresponding solution can be obtained by tracking back the inductive computation of $\max_{s \in S_\rho} OPT(\rho, s)$.

Alternatively we can associate to G an auxiliary directed acyclic graph $D(V(D), A(D))$ (see Fig 2.1). The set $V(D)$ consists of $\{v_S^i : S \in \mathcal{S}_i, i = 0, ..., \rho\}$, together with two special nodes u^*, v^*. $A(D)$ is made of the following arcs: $(u^*, v_{\{z\}}^0)$ and $(u^*, v_\emptyset^0)$; for each $i = 0, ..., \rho - 1$ and S stable set of $G[N_i(z) \cup N_{i+1}(z)]$, the arc $(v_{S \cap N_i(z)}^i, v_{S \cap N_{i+1}(z)}^{i+1})$; for each $S \in \mathcal{S}_\rho$, the arc (v_S^ρ, v^*). We assign weights w' to the arcs of D as follows: for each arc $a = (x, v^*)$, $w'_a = 0$; for each other arc $a = (x, v_S^i)$, $w'_a = \sum_{y \in S} w_y$. The MWSS problem in G is equivalent to the longest directed (u^*, v^*)-path in the acyclic graph D and can thus be solved in time $O(|A(D)|)$, assuming that D is stored via adjacency lists (see e.g. [2]). $|A(D)|$ is easily bounded by $O(|V|^{2B})$ and thus so is the complexity.

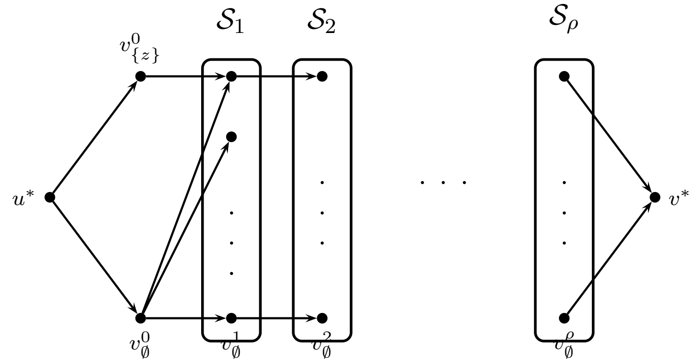

Figure 2.1: A longest path representation

In the special case of {claw,net}-free graphs, both algorithms have complexity $O(|V|^4)$. Building upon results of Brandstadt and Dragan [13], Kloks, Kratsch and Müller [55] and Hempel and Krastch [48] , we could adapt the construction and devise an algorithm with complexity $O(|V||E|)$ [32].

Theorem 1. *[32] The maximum weighted stable set problem on a graph $G(V, E)$ that is {claw,net}-free can be solved in time $O(|V||E|)$.*

Chapter 3

Building upon network flows

3.1 Set packing problem associated with circular one matrices

Let A be a $m \times n$ matrix with zeros and ones (we assume that no row and no column of A is equal to zero). A is a *circular one matrix* if in each row, the ones appear in consecutive columns (where column 1 is considered consecutive to column n). We are interested in the following *set packing problem* : $\max\{cx : Ax \leq \mathbf{1}, x \in \{0,1\}^n\}$. Interestingly this problem can be solved via network flows. Let us consider the problem $\max\{cx : Ax \leq \mathbf{1}, \mathbf{1}x = \beta, x \in \{0,1\}^n\}$ for $\beta \in \{0, ..., n\}$. We have $\max\{cx : Ax \leq \mathbf{1}, x \in \{0,1\}^n\} = \max_{\beta \in \{0,\ldots,n\}} \max\{cx : Ax \leq \mathbf{1}, \mathbf{1}x = \beta, x \in \{0,1\}^n\}$. We are now going to show that for each β, we can solve $\max\{cx : Ax \leq \mathbf{1}, \mathbf{1}x = \beta, x \in \{0,1\}^n\}$ using network flows. Observe first that $\max\{cx : Ax \leq \mathbf{1}, \mathbf{1}x = \beta, x \in \{0,1\}^n\}$ is equivalent to $\max\{cx : Ax \leq \mathbf{1}, \mathbf{1}x = \beta, x \geq 0, x \in \mathbb{Z}^n\}$ as $x \leq \mathbf{1}$ is implicit in the system $Ax \leq \mathbf{1}, x \geq 0$. Following Bartholdi, Orlin and Ratliff [8], we consider the unimodular transformation $x = T\,y$, where T is the unimodular matrix

$$T = \begin{pmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & & & \\ & & & \ddots & & \\ & & & & & 1 \\ & & & & -1 & 1 \end{pmatrix} \quad \text{and where} \quad T^{-1} = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ \vdots & & & \ddots & & \\ & & & & & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix} \tag{3.1}$$

The problem reads now $\max\{cTy : ATy \leq \mathbf{1}, Ty \geq 0, y(n) = \beta, Ty \in \mathbb{Z}\}$, which is itself equivalent to $\max\{cTy : ATy \leq \mathbf{1}, Ty \geq 0, y(n) = \beta, y \in \mathbb{Z}\}$ since T is unimodular (i.e. it preserves integrality). The linear relaxation of this later problem is $\max\{cTy : y \in Q, y(n) = \beta\}$ with Q defined by the system

$$\begin{pmatrix} A \\ -I \end{pmatrix} T\,y \leq \begin{pmatrix} \mathbf{1} \\ 0 \end{pmatrix} \tag{3.2}$$

In the following we denote the inequality system (3.2) by $B\,y \leq d$. Let us rewrite the matrix B as $B = (N|v)$, i.e. v is the n-th column of B. Observe that, by construction, v is also the last column of $\begin{pmatrix} A \\ -I \end{pmatrix}$. Each row of the matrix N has at most one entry which is $+1$ and at most one entry which is -1. All other entries are 0. The matrix N is thus totally unimodular. Actually even better $(N|w)$ is the transpose of a network matrix with $w = -\sum_{i=1}^{n-1} N(i)$. Thus, whenever $y(n)$ is set to an integer $\beta \in \mathbb{Z}$, the possible values for the variables $y(1), \ldots, y(n-1)$ define an integral polytope $Q_\beta = Q \cap \{y \in \mathbb{R}^n \mid y(n) = \beta\}$. We call Q_β the *slice* of Q defined by β. $\max\{cTy : ATy \leq \mathbf{1}, Ty \geq 0, y(n) = \beta, y \in \mathbb{Z}\}$ is thus equivalent to $\max\{cTy : y \in Q_\beta\}$.

In the following, given a vector $x \in \mathbb{R}^n$, we denote by $\overline{x} \in \mathbb{R}^{n-1}$ the vector of the first $n-1$ components of $x \in \mathbb{R}^n$. Defining $c' = cT$ and $d' = d - \beta v$, the problem above is equivalent to $\beta c'(n) + \max\{\bar{c'}\bar{y} : N\bar{y} \leq d'\}$. By duality, we are left to solve the transshipment problem $\beta c'(n) + \min\{d'f : fN = \bar{c'}, f \geq 0\}$ which is a transshipment problem and can be solved by any network flow algorithm. Choosing the best solution over all $\beta = 0, ..., n$ yield the optimal solution to our set packing problem.

3.2 Separating over the set packing polytope associated with circular one matrices

We can use the insight gained from the previous section to dig into the *set packing polytope* associated with circular ones matrices. Let P be the polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq \mathbf{1},\, x \geq 0\}$, where A is still a circular one matrix, as defined in the previous section. Let P_I be the integer hull of P (P_I is the set packing polytope associated with A). We consider the separation problem for the integer hull P_I of P:

> Given $x^* \in \mathbb{R}^n$, determine, whether $x^* \in P_I$ and if not, determine an inequality $c\,x \leq \delta$ which is valid for P_I and satisfies $c\,x^* > \delta$.

Using again transformation T (see (3.1)), the problem then reads, separate $y^* = T^{-1}x^*$ from the integer hull Q_I of the polytope Q.

If $y^*(n)$ is integral, then y^* lies in Q_I if and only if $y^* \in Q_{y^*(n)}$. Therefore we assume in the following that $y^*(n)$ is not integral and let β be an integer such that $\beta < y^*(n) < \beta + 1$ and let $1 > \mu > 0$ be the real number with $y^*(n) = \beta + 1 - \mu$. Furthermore, let Q_L and Q_R be the left slice Q_β and right slice $Q_{\beta+1}$ respectively. A proof of the next lemma follows from basic convexity.

Lemma 2. *The point y^* lies in Q_I if and only if there exist $y_L \in Q_L$ and $y_R \in Q_R$ such that $y^* = \mu\, y_L + (1 - \mu) y_R$*

From the above discussion one has $y^* \in Q_I$ if and only if the following linear constraints have a feasible solution.

$$\begin{array}{rcl} \overline{y_L} + \overline{y_R} & = & \overline{y^*} \\ N\overline{y_L} & \leq & \mu\, d_L \\ N\overline{y_R} & \leq & (1-\mu)\, d_R \end{array}, \tag{3.3}$$

where $d_L = d - \beta v$ and $d_R = d - (\beta + 1)v$.

Using Farkas' Lemma [88], it follows that the system (3.3) is feasible, if and only if $\sum_{i=1}^{n-1} \lambda(i) y^*(i) + \mu f_L d_L + (1 - \mu) f_R d_R$ is nonnegative, whenever λ, f_L and f_R satisfy

$$\begin{array}{rcl} \lambda + f_L N & = & 0 \\ \lambda + f_R N & = & 0 \\ f_L, f_R & \geq & 0. \end{array} \tag{3.4}$$

Now $\lambda + f_L N = 0$ and $\lambda + f_R N = 0$ is equivalent to $\lambda = -f_L N$ and $f_L N = f_R N$. Thus (3.3) defines a feasible system, if and only if the optimum value of the following linear program is nonnegative

$$\min\{-f_L N\, \overline{y^*} + \mu f_L d_L + (1 - \mu) f_R d_R : f_L N = f_R N; f_L, f_R \geq 0\} \tag{3.5}$$

Let w be the negative sum of the columns of N. Then (3.5) is the problem of finding a minimum cost circulation in the directed graph $D = (U, \mathcal{A})$ defined by the edge-node incidence matrix

$$M = \begin{pmatrix} N & w \\ -N & -w \end{pmatrix} \text{ and edge weights } \mu(-N\,\overline{y^*} + d_L), (1 - \mu)(-N\,\overline{y^*} + d_R) \tag{3.6}$$

Thus $y^* \notin Q_I$ if and only if there exists a negative cycle in $D = (U, \mathcal{A})$. The membership problem for Q_I thus reduces to the problem of detecting a negative cycle in D, see [40].

This membership procedure can actually be turned into a separation procedure as a negative cost cycle yields a separating disjunctive cut for y^* i.e. here an inequality which is valid for Q_L and Q_R but not valid for y^*. Indeed, because the inequality $f_L N\overline{y} \leq f_L d_L$ is valid for Q_L and the inequality $f_R N\overline{y} \leq f_R d_R$ is valid for Q_R, the corresponding (valid) disjunctive inequality (see, e.g., [66]) is the inequality

$$f_L N\overline{y} + c(n)y(n) \leq \delta, \text{ where } c(n) = f_L d_L - f_R d_R \text{ and } \delta = (\beta+1) f_L d_L - \beta f_R d_R. \tag{3.7}$$

Recall that the polytopes Q_L and Q_R are defined by the systems

$$\begin{array}{rcl} y(n) & = & \beta \\ N\,\overline{y} + v\,y(n) & \leq & d \end{array} \qquad \text{and} \qquad \begin{array}{rcl} y(n) & = & \beta+1 \\ N\,\overline{y} + v\,y(n) & \leq & d \end{array} \tag{3.8}$$

respectively.

A negative cycle in a graph with m edges and n nodes can be found in time $O(m\,n)$, see, e.g. [2]. We can translate back this inequality in the original space using T^{-1} and give an inequality separating x^* from P_I. This gives the following theorem.

Theorem 3. *The separation problem for P_I can be solved in time $O(m\,n)$.*

3.3 Inventory control in one-warehouse multi-retailer systems as a fixed charge network flow

We now consider two important deterministic inventory control problems: The One-Warehouse Multi-Retailer (OWMR) problem and its special case the Joint Replenishment Problem (JRP). More specifically, we want to optimize the distribution of a single product over a network composed of one warehouse (numbered 0) and N different retailers (numbered $i \in \{1, \ldots, N\}$) over a discrete finite planning horizon of T periods (numbered $1, \ldots, T$). Each retailer i is facing deterministic demands d_{it} for periods $t = 1, \ldots, T$ that have to be fulfilled on time (neither backorders nor lost sales are allowed) by ordering those units from the warehouse (possibly in different periods) which in turn have to be ordered from an external supplier (labeled S) of infinite capacity. Assuming deterministic lead-time, we consider without loss of generality that the orders are delivered instantaneously from one location to another. If the warehouse orders from its supplier in any of the T periods, it incurs a fixed ordering cost K^0 and similarly, a retailer ordering from the warehouse will pay a fixed ordering cost K^i for ordering in any of the T periods, regardless of the number of units. In addition a holding cost is paid to keep units in stock to serve future periods. We consider here a simple linear holding cost $h^i \cdot x$ to keep x units of products in stock from period t to $t+1$ at location i.

The objective of the OWMR problem is to find a planning for the orders at each location (i.e. period and quantity) that minimizes the sum of the fixed ordering costs and holding costs in the system to fulfill the demands at the retailers. The JRP is a special case of the OWMR problem where the warehouse only operates as a cross-docking station (i.e. no inventory can be held). This can be captured in the model via a prohibitive holding cost at the warehouse.

In their discrete time version, both JRP and OWMR problems are known to be NP-hard [3]. We now show how to model this as a (fixed charge) network flow problem. Consider a time expended directed graph $D(V, A)$ with vertex set $V = \{v_t^i$, for each location $i \in \{0, ..., n\} \cup \{S\}$ and each time period $t \in \{1, ..., T\}\}$ and arc set $A = A_1 \cup A_2$ with $A_1 = \{(v_t^i, v_{t+1}^i)$ for each location i and for each $t \leq T-1\}$ and $A_2 = \{(v_t^i, v_t^j)\}$ for each i, j : location i supplies location

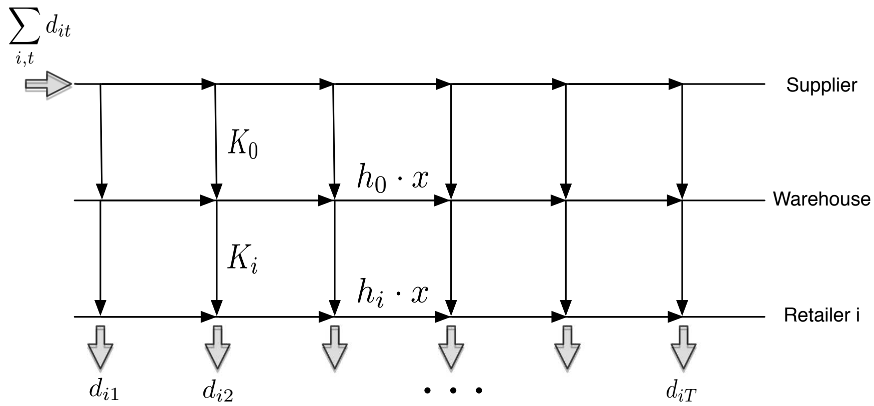

Figure 3.1: An example

$j\}$. Let N be the node-arc incidence matrix of D. Let $b \in \mathbb{R}^{|V|} : b(v) = -\sum_{i=1}^{n} \sum_{t=1}^{T} d_{it}$ if $v = v_1^S$, $b(v) = d_{it}$ if $v = v_t^i$ for $i \in \{1, ..., n\}$, $t \in \{1, ..., T\}$ and $b(v) = 0$ otherwise. Let $c \in \mathbb{R}^{|A|}$: $c(a) = h^i$ if $a = (v_t^i, v_{t+1}^i)$, for $i \in \{0, ..., n\}$ and $c(a) = 0$ otherwise. Let $f \in \mathbb{R}^{|A|} : f(a) = K^j$ if $a = (v_t^i, v_t^j)$, for $j \in \{0, ..., n\}$ and $f(a) = 0$ otherwise. The OWMR problem is the problem of finding a feasible flow in the graph D with demand b (i.e. a vector $x \in \mathbb{R}^{|A|} : Nx = b, x \geq 0$) of minimum cost $\sum_{a \in A} c(a)x(a) + \sum_{a \in A: x(a) > 0} f(a)$. This problem is known as a fixed charge network flow problem. See Fig. 3.1 for an example with a single retailer i. This network flow view on the problem will help later devise a simple approximation algorithm.

Chapter 4

Matching and beyond

4.1 Line graphs and composition of strips

A *matching* in a (multi-) graph $G(V, E)$ is a subset of pairwise non incident edges. The *line graph* $L(G)$ of a graph G is the intersection graph of the edges of G i.e. there is a vertex for each edge in G and two vertices are adjacent if and only if their corresponding edges in G are incident. Interestingly, there is a one-to-one correspondence between matchings in G and stable sets in $L(G)$, which makes the stable set problem polynomial in line graphs as soon as one is given G (or can find such a *root graph* in polynomial time, this can be done in $O(max\{|E(L(G))|, |V(L(G))|\})$-time, see [86] for line graphs of simple graphs and [54, pp 67-68] for line graphs of multi-graphs)

Now given a graph G, each vertex in G can be associated with a clique in the line graph $H = L(G)$ (all edges incident to this vertex are pairwise adjacent in H). Let $\mathcal{F}$ denote the family of cliques of H that are associated with vertices of G. We observe that $\mathcal{F}$ has the following properties: (i) every edge of H is covered by some clique of $\mathcal{F}$; (ii) every vertex of H is covered by exactly two cliques of $\mathcal{F}$.

Suppose now that we are given a (general) graph H. We call a family $\mathcal{F}$ of cliques of H a *Krausz family* (of H) if it satisfies the properties (i) and (ii). Krausz [56] proved the following:

Theorem 4 ([56]). *A graph G is the line graph of a multi-graph if and only if it admits a Krausz family.*

Proof. Necessity follows from the above discussion. Sufficiency goes as follows. Let $\mathcal{K}$ be a Krausz family. We are going to build a root graph H of G. Observe first that we can assume without loss of generality that for any pair of distinct cliques $K_1, K_2 \in \mathcal{K}$, $|K_1 \cap K_2| \leq 1$. Indeed, if $|K_1 \cap K_2| > 1$, the vertices in $K_1 \cap K_2$ are copies (no 3 cliques of $\mathcal{K}$ have a common vertex) and thus we can shrink them to a single vertex and obtain the root graph of G from the root graph of this shrunk graph by duplicating edges. Now an edge of H is uniquely determine by a pair of distinct cliques $K_1, K_2 \in \mathcal{K}$ and because every vertex of G is covered exactly twice, there is a one-to-one correspondence between vertices of G and edges of H. By construction two edges are incident in H if and only if the corresponding vertices in G are in a same clique $K \in \mathcal{K}$. But since every edge of G is covered by a clique of $\mathcal{K}$, it follows that two vertices are adjacent in G if and only if the corresponding edges are incident in H. □

This theorem gives an algorithmic procedure to build line graphs. This procedure requires as input a set of vertices V and a partition $\mathcal{P} = P_1, ..., P_q$ of the multi-set $V \cup V$. It then associates to the pair $(V, \mathcal{P})$ the graph G with vertex set V and edge set $E := \{(u, v) : u \neq v$ and both $u, v \in P_i$, for some $1 \leq i \leq q\}$. Chudnovsky and Seymour generalized the above construction, essentially by replacing *vertices* with *strips*. We borrow (but slightly change) some definitions of theirs.

Definition 5. *A* strip $(G, \mathcal{A})$ *is a graph G (not necessarily connected) with a multi-family $\mathcal{A}$ of either one or two designated non-empty cliques (possibly identical) of G. The cliques in $\mathcal{A}$ are called the* extremities *of the strip.*

Let $\mathcal{H} = \{(G^i, \mathcal{A}^i),\ i = 1, ..., k\}$ be a family of vertex disjoint strips. Let $\mathcal{A}(\mathcal{H})$ denote the multi-family of the extremities of those strips, i.e., $\mathcal{A}(\mathcal{H}) = \bigcup_{i=1..k} \mathcal{A}^i$, and let $\mathcal{P} = P_1, P_2, ..., P_q$ be the classes of a partition of $\mathcal{A}(\mathcal{H})$. We associate to the pair $(\mathcal{H}, \mathcal{P})$ the graph G that is made of the disjoint union of the graphs $G^1, \ldots G^k$, with additional edges $E := \{(u, v) : u \neq v$ and u and v belong to different extremities in a same class P_i, for some $1 \leq i \leq q\}$. G is called the composition of the strips $\mathcal{H}$ with respect to partition $\mathcal{P}$. Observe that by definition the set of vertices in a same class of partition $\mathcal{P}$ induces a clique in G. We call such cliques *partition-cliques*. Note also that, for line graphs, this composition reduces to the above construction, as soon as each graph G^i is made of a single vertex v_i and the corresponding strip is $(\{v_i\}, \{\{v_i\}, \{v_i\}\})$.

Even though the operation of composition of strips builds graphs that are in general non-line, such graphs indeed inherit a "line structure" from its similarity with Krausz composition. Say that a strip $(S, \mathcal{A})$ is *line* if it admits a Krausz family $\mathcal{K}$ with $\mathcal{A} \subseteq \mathcal{K}$. Then, as soon as all *strips* are *line*, the composition is a line graph.

Lemma 6. *Let G be the composition of a family of* line *strips $H^i = (G^i, \mathcal{A}^i), i = 1, ..., k$ with respect to a partition $\mathcal{P}$. Then G is a line graph.*

Proof. Since H^i is a line strip, by definition there exists a Krausz family $\mathcal{K}^i$ of G^i with $\mathcal{A}^i \subseteq \mathcal{K}^i$. Let $\bar{\mathcal{K}}^i := \mathcal{K}^i \setminus \mathcal{A}^i$ and let $\mathcal{K}_\mathcal{P}$ be the set of all partition-cliques of G. The set $\mathcal{K}_\mathcal{P} \cup (\bigcup_{i=1}^k \bar{\mathcal{K}}^i)$ is a Krausz family of G and thus the results follows from Theorem 4. □

We now show that we can solve, in polynomial time, the maximum weighted stable set (MWSS) problem in a graph G that is the composition of strips, as soon as we are able to solve in polytime the same problem on each strip of G. The main tools are Lemma 6 and a simple reduction that replaces each strip with a simple line strip.

More precisely, let $G(V, E)$ be the composition of k strips $H_1 = (G^1, \mathcal{A}^1), \ldots, H_k = (G^k, \mathcal{A}^k)$, with respect to a partition $\mathcal{P}$ and let $w : V(G) \mapsto \mathbb{R}$. We show that we can replace each strip H_i with a simple *line* strip H'_i and reduce the mwss problem on G to the same problem on a graph G^*, that is the composition of the strips $H'_1, H'_2, \ldots, H'_k$ with respect to a suitable partition $\mathcal{P}^*$, where we define a suitable weight function $w^* : V(G^*) \mapsto \mathbb{R}$. As G^* is line from Lemma 6 and we can build a root graph easily (we show this in Corollary 8), we can find a MWSS by solving a matching problem.

The rationale in replacing a strip, say H_1, with another strip H'_1 is the following. The only possible obstruction to combine a stable set T of $G \setminus V(G^1)$ and a stable set U of G^1 into a stable set of G are the adjacencies in the partition-cliques involving the extremities of H_1. Because those extremities are cliques, there are four possible configurations describing the interactions between U and the extremities of H_1 (by now assume that H_1 has two extremities): U contains a vertex in both extremities; U contains a vertex in one or the other extremity; U does not contain any vertex in the extremities. When one is interested in a MWSS of G then, given the stable set T for $G \setminus V(G^1)$, one obviously wants the stable set U to be of maximum weight among the stable sets from configurations that are *compatible* with T. Hence, we can replace H_1 with another strip H'_1 as long as they agree, *for each configuration*, on the value of a MWSS.

The strip H'_1 will be fairly trivial: it will be either the strip $H'_1 = (C_1, \{c_1\})$ or the strip $H'_1 = (C_3, \{\{c_1, c_3\}, \{c_2, c_3\}\})$, where we denote by C_k the complete graph on $k \geq 1$ vertices labeled $c_1, ..., c_k$ (hence, C_1 is the graph made of a single vertex, while C_3 is a triangle). However, because the composition, and thus the adjacencies between $G \setminus V(G^1)$ and G^1, are slightly different if: (i) H_1 has only one extremity; (ii) H_1 has two extremities and they are in the same class of the partition $\mathcal{P}$; or (iii) H_1 has two extremities and they are in different classes of the partition $\mathcal{P}$, we need to distinguish those cases. In each case, we define $w'(v) = w(v)$ for $v \notin V(G^1)$.

- In case (i), i.e., when $\mathcal{A}^1 = \{A_1\}$ and there exists $P \in \mathcal{P} : A_1 \in P$, we define: $H'_1 = (C_1, \{c_1\})$; $\delta_1 = \alpha_w(G^1 \setminus A_1)$; $w'(c_1) = \alpha_w(G^1) - \delta_1$; $\mathcal{P}' := (\mathcal{P} \setminus P) \cup (P \cup \{c_1\} \setminus A_1)$.
- In case (ii), i.e., when $\mathcal{A}^1 = \{A_1, A_2\}$ and there exists $P \in \mathcal{P} : A_1, A_2 \subseteq P$, we define: $H'_1 = (C_1, \{c_1\})$; $\delta_1 = \alpha_w(G^1 \setminus (A_1 \cup A_2))$; $w'(c_1) = \max\{\alpha_w(G^1 \setminus A_1), \alpha_w(G^1 \setminus A_2), \alpha_w(G^1 \setminus A_1 \Delta A_2)\} - \delta_1$; $\mathcal{P}' := (\mathcal{P} \setminus P) \cup (P \cup \{c_1\} \setminus \{A_1, A_2\})$.
- In case (iii), i.e., when $\mathcal{A}^1 = \{A_1, A_2\}$ and there exists $P_1 \neq P_2 \in \mathcal{P} : A_i \in P_i \ i = 1, 2)$, we define: $H'_1 = (C_3, \{\{c_1, c_3\}, \{c_2, c_3\}\})$; $\delta_1 = \alpha_w(G^1 \setminus (A_1 \cup A_2))$; $w'(c_1) = \alpha_w(G^1 \setminus A_2) - \delta_1$, $w'(c_2) = \alpha_w(G^1 \setminus A_1) - \delta_1$ and $w'(c_3) = \alpha_w(G^1) - \delta_1$; $\mathcal{P}' := (\mathcal{P} \setminus (P_1 \cup P_2)) \cup ((P_1 \setminus A_1) \cup \{c_1, c_3\}) \cup ((P_2 \setminus A_2) \cup \{c_2, c_3\})$.

The next lemma follows easily from the above discussion.

Lemma 7. *Let G be the composition of k strips $H_1 = (G^1, \mathcal{A}^1), \ldots, H_k = (G^k, \mathcal{A}^k)$, with respect to a partition $\mathcal{P}$ and let $w : V(G) \mapsto \mathbb{R}$. Let G' be the composition of $H'_1, H_2, \ldots, H_k$ with respect to the partition $\mathcal{P}'$ and $w' : V(G') \mapsto \mathbb{R}$, with H'_1, $\mathcal{P}'$ and w' defined above. Then $\alpha_w(G) - \delta_1 = \alpha_{w'}(G')$. Moreover any MWSS of G' (with respect to w') can be converted into a MWSS of G (with respect to w) if the following stable sets are known: a MWSS of G^1; a MWSS of G^1 not intersecting A, for each $A \in \mathcal{A}^1$; a MWSS of G^1 not intersecting $\bigcup_{A \in \mathcal{A}^1} A$; a MWSS of G^1 not intersecting $A_1 \Delta A_2$ (this one is required only if $\mathcal{A}^1 = \{A_1, A_2\}$ and A_1, A_2 are in the same class of $\mathcal{P}$).*

Proof. We only give the proof for case (i) but cases (ii) and (iii) goes along the same line.

(i) We begin with showing that $\alpha_w(G) \leq \delta_1 + \alpha_{w'}(G')$. Let S be a MWSS of G. First suppose that S picks a vertex in A_1. Then $S \cap V(G^1)$ is a mwss in G^1 (otherwise we would swap with a better one in G^1). Also S is not picking any vertex belonging to an extremity in P other than A_1, and therefore $S' = (S \setminus V(G^1)) \cup \{c_1\}$ is a stable set of G'. Therefore, $\alpha_w(G) = w(S) = w'(S') - w'(c_1) + w(S \cap V(G^1)) = w'(S') - w'(c_1) + \alpha_w(G^1) = w'(S') + \delta_1 \leq \alpha_{w'}(G') + \delta_1$. Suppose now that S does not pick any vertex from A_1. Then $S \cap V(G^1)$ is a mwss in $G^1 \setminus A_1$, and $S \setminus V(G^1)$ is a stable set of G'. Therefore, $\alpha_w(G) = w(S) = w(S \cap V(G^1)) + w(S \setminus V(G^1)) \leq \delta_1 + \alpha_{w'}(G')$.

We now show that $\alpha_w(G) \geq \delta_1 + \alpha_{w'}(G')$. Let S' be a MWSS of G'. First suppose that S' picks c_1. In this case, for any stable set S of G^1, $(S' \setminus c_1) \cup S$ is a stable set of G. Therefore, if in particular we choose S as a mwss of G^1, $\alpha_w(G) \geq w((S' \setminus c_1) \cup S) = w'(S') - w'(c_1) + \alpha_w(G^1) = \alpha_{w'}(G') + \delta_1$. Now suppose that S' does not pick c_1. In this case, for any stable set S of $G^1 \setminus A_1$, $S' \cup S$ is a stable set of G. Therefore, if in particular we choose S as a mwss of $G^1 \setminus A_1$, $\alpha_w(G) \geq w(S' \cup S) = w'(S') + \alpha_w(G^1 \setminus A_1) = \alpha_{w'}(G') + \delta_1$.

Therefore, $\alpha_w(G) = \delta_1 + \alpha_{w'}(G')$. Moreover, if S' is a MWSS of G', we may derive from it a MWSS of G, as soon as we are given: a MWSS of G^1; a MWSS of G^1 not intersecting A_1.

□

Trivially, we can apply the above procedure iteratively to each strip H_i. The problem of finding a MWSS on G reduces therefore to the same problem on the graph G^* that is the composition of $H'_1, \ldots, H'_k$ with respect to a suitable partition $\mathcal{P}^*$. The following lemma shows some key properties of G^*.

Corollary 8. *G^* is a line graph and in time $O(k)$ we can build a root graph $\tilde{G}$ with $O(k)$ vertices and edges.*

Proof. It is trivial to see that the strips H_i', $i = 1, ..., k$, are *line strips* (recall that the definition of a line strip differs from that of a line graph), and therefore it follows from Lemma 6 that G^* is a line graph. (Note also that by construction G^* has at most $3k$ vertices). Moreover, the proof of the same lemma, together with Theorem 4 (which is also constructive), suggests how to build a root graph for G^* with $O(k)$ vertices and edges in $O(k)$-time : we skip the details. □ □

Since the number k of strips is bounded by $O(|V(G)|)$, it follows that we have reduced, provided we can efficiently compute the weights w' for the vertices of each strip H_i', the maximum weighted stable set problem on G to a weighted matching problem on the graph $\tilde{G}$, that has $O(|V(G)|)$ vertices and edges. This latter problem can be solved in time $O(|V(G)|^2 \log |V(G)|)$ by [35]. Also note that the computation of the weights w' for the vertices of some strip H_i' requires the solution of some MWSS problems on G^i, where the weight of some vertices is set to 0. Thus, we have proved the following:

Theorem 9. *[70] The maximum weighted stable set problem on a graph G, that is the composition of some set of strips $(G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)$, can be solved in $O(|V(G)|^2 \log |V(G)| + \sum_{i=1,\ldots,k} p_i(|V(G^i)|))$-time, if each G^i belongs to some class of graphs, where the same problem can be solved in time $O(p_i(|V(G^i)|))$.*

4.2 Stable sets in claw-free graphs

Matching is a classic problem in combinatorial optimization and it exhibits some remarkable properties. Many of those properties have been extended and led to very powerful tools and theories like for instance matroid intersection or delta-matroids. In order to extend the matching theory to the stable set setting, it appears that two fundamental properties of matching are crucial: *the augmenting path property* and the *intersection property*.

Petersen observed in 1891 (and Berge proved in 1952) that the symmetric difference of two matchings is made of alternating paths and even cycles. In particular, a matching M is of maximum cardinality in a graph G if and only if there does not exist any augmenting path in G with respect to M. Moreover, as two matchings are adjacent on the matching polytope $MATCH(G)$ – the convex hull of all incidence vectors of matchings in a graph G – if and only if they have a connected symmetric difference, one can easily show that $MATCH(G)$ has the intersection property: $MATCH(G) \cap \{x : \sum_{e \in E} x_e = k\}$ is integral for every integer k.

Interestingly those properties extend to the stable set setting beyond line graphs (alternating paths and cycles being defined in terms of vertices here i.e. a path P is *augmenting* with respect to a stable set S if $(V(P) \setminus S) \cup (S \setminus V(P))$ is a stable set of size $|S|+1$): they are also valid for stable sets in *claw-free graphs*. This was observed by Berge in 1973 for the symmetric difference of stable sets in claw-free graphs and by Calvillo in 1979 for the intersection property. Remarkably, Berge and Calvillo also proved the converse, i.e., a class of graphs exhibits one or the other of those properties for the stable set problem if and only if it is a subclass of claw-free graphs. Hence, with respect to stable sets, claw-free graphs appear to be the right framework to extend the aforementioned properties of matching. We denote by $STAB(G)$ the stable set polytope of a graph G i.e. the convex hull of all incidence vectors of stable sets in a graph G.

Theorem 10 ([10]). *A stable set S is maximum for a claw-free graph G if and only if there are no paths that are augmenting with respect to S.*

Theorem 11 ([14]). *$STAB(G)$ has the intersection property if and only if G is a claw-free graph.*

The problem of finding a maximum weighted stable set in claw-free graphs has been therefore investigated by several people, and its theory has been developing for more than 40 years. Minty [65] was the first to provide a polynomial time algorithm for this problem. We will now propose a strongly polynomial algorithm for solving the MWSS problem in a claw-free graph $G(V, E)$ that runs in $\mathcal{O}(|V|(|E|+|V|\log|V|))-$time, drastically improving the previous best known complexity bound.

Theorem 12. *[32] The maximum weighted stable set problem in a graph $G(V, E)$ can be solved in time $O(|V|(|E| + |V|\log|V|))$.*

Proof. We sketch the proof. Using Theorem 26 presented in the next section, we can determine in time $O(|V||E|)$ if G has $\alpha(G) \leq 3$, if G is $\{claw, net\}$-free or if G is the composition of *distance simplicial strips* and strips with stability number at most three and get the decomposition. A strip $(G, \mathcal{A})$ is distance simplicial if for all $A \in \mathcal{A}$, $\alpha(N_i(A)) \leq 1$ for all $i \geq 0$. If $\alpha(G) \leq 3$, then a MWSS can be found by enumeration and this can be done trivially in time $|V|^3$ and can be improved to $O(|V||E|)$ using simple properties of claw-free graphs (see [32] for details). If G is the composition of strips, then the result follows from Theorem 9, as soon as we observe that we can find a MWSS in a distance simplicial strip with n vertices in time $O(n^2)$ following the dynamic programming approach of Section 2.2. Now if G is {claw,net}-free we follow again Section 2.2 and solve the problem in time $\mathcal{O}(|V||E|)$. The result follows. □

4.3 p-median in Y-free graphs

Let $G = (V, E)$ be a directed graph with vertex set V and arc set E. Let $w_E : E \mapsto \mathbb{R}$ a cost function on the arcs of G and $w_V : V \mapsto \mathbb{R}$ a cost function on the vertices of G. In this section, we use the following notations. Given two sets of vertices $S_1, S_2 \subseteq V$, we denote $E(S_1 : S_2) := \{(u, v) \in E : u \in S_1, v \in S_2\}$ and for short $E(S) := E(S : S)$. For any vertex $v \in V$, we denote $n^+_G(v) := \{u \in V : (v, u) \in E\}$, $n^-_G(v) := \{u \in V : (u, v) \in E\}$, $\delta^+_G(v) := \{(v, u) \in E\}$, $\delta^-_G(v) := \{(u, v) \in E\}$ and $\delta_G(v) := \delta^-_G(v) \cup \delta^+_G(v)$. When no confusion can arise we omit subscripts e.g. we use $\delta^+(v)$ instead of $\delta^+_G(v)$. For any $F \subseteq E$, we denote $w_E(F) = \sum_{e\in F} w_E(e)$.

The *uncapacitated facility location problem* consists in selecting a set of vertices S in V and in defining a mapping $M_S \subseteq E(V \setminus S : S)$ from $V \setminus S$ to S (i.e. to assign the vertices in $V \setminus S$ to the selected nodes S) as to minimize the total cost $w_V(S) + w_E(M_S)$. The *p-median problem* is an uncapacitated facility location problem where the set of vertices S is constrained to have cardinality p.

The following integer linear program is a straightforward formulation of the p-median problem. Relaxing constraint (*) provides a formulation for the uncapacitated facility location problem.

$$
\begin{array}{llcl}
\min & \displaystyle\sum_{(u,v)\in E} w_E(u,v)x(u,v) + \sum_{v\in V} w_V(v)y(v) & & \\
(1) & x(\delta^+(v)) & = & 1 - y(v), \forall v \in V \\
(*) & \sum_{v\in V} y(v) & = & p \\
(2) & x(u,v) & \leq & y(v), \forall v \in V, \forall u \in n^-(v) \\
(3) & y(v), x(u,v) & \in & \{0,1\}, \forall v \in V, (u,v) \in E
\end{array}
$$

The variable $y(v)$ for $v \in V$ is equal to one iff the vertex v is selected in S and the variable $x(u, v)$ is equal to one iff the vertex u is assigned to vertex v. Constraints (1) ensure that a vertex is either selected or assigned to a selected vertex while constraints (2) ensure that a vertex cannot be assigned to a non selected node.

The *uncapacitated facility location polytope* of G, $UFLP(G)$, is the convex hull of the solutions to system (1), (2) and (3). The *p-median polytope* of G, $pM(G)$, is the convex hull of the solutions to system (1),(2),(3) and (*).

The uncapacitated facility location and the p-median problems are NP-hard in general [51] but can be solved in polynomial time in special cases. When p is fixed for instance, the p-median problem can be solved by enumerating all possible choices for the set S and for each of those choices assigning each vertex in $V \setminus S$ to its closest neighbor (with respect to w_E) in S. The set S achieving the minimum cost is the optimal solution. The p-median problem is also known to be polynomially solvable in special classes of graphs, see for instance [51] or [96]. Baiou and Barahona [5] proved recently that it can be solved in polynomial time for a Y-free graph G and they also provide a description of $pM(G)$ in that case. A Y is a graph isomorphic to the graph with vertex set $\{a, b, c, d\}$ and arc set $\{(a, c), (b, c), (c, d)\}$.

Definition 13. *A graph G is said to be Y-*free *if it is a simple graph with no cycle of length two and that does not contain any Y as a (partial) subgraph. Rooted directed trees satisfy those conditions and are therefore Y-free.*

In [4], Avella and Sassano show that there is a strong relation between p-median polytopes and stable set polytopes. We show now that in the special class of Y-free graphs, there is in fact a strong relation between the p-median polytope and the matching polytope. Indeed we show that the uncapacitated facility location and the p-median problems for Y-free graphs can be reduced to matching problems.

We now observe a simple property of the vertices in Y-free graphs and we derive a simplification of the integer programming formulation of the p-median problem in this case.

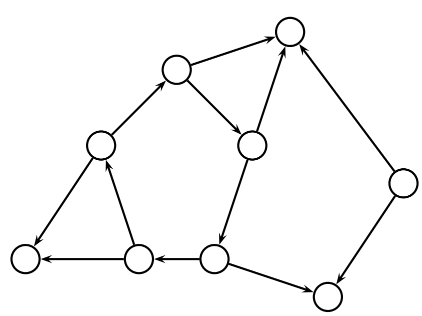

Figure 4.1: An example of Y-free graph

Figure 4.1 gives a simple example of Y-free graph. One can observe that there are only three types of vertices in this graph as proved by Lemma 14

Lemma 14. *Let G be a Y-free graph. The vertices of G can be partitioned into three classes:*

- *sources* $\mathcal{S} := \{v \in V : \delta^-(v) = \emptyset\}$,
- *sinks* $\mathcal{T} := \{v \in V : \delta^+(v) = \emptyset\}$,
- *forks* $\mathcal{F} := \{v \in V : |\delta^-(v)| = 1$ *and* $\delta^+(v) \neq \emptyset\}$.

Proof. By definition of Y-free graphs, if $\delta^+(v) \neq \emptyset$ and $\delta^-(v) \neq \emptyset$, then $|\delta^-(v)| = 1$. □

Now let us reformulate the integer linear program for the p-median problem thanks to a couple of observations.

Using equality (1), we can substitute $y(v)$ in inequalities (2), (*) and in the objective function. The following formulation is thus still valid for the p-median problem (or for the uncapacitated facility location problem when removing (*)).

$$\begin{array}{llcl} \min & w_V(V) + \sum_{(u,v)\in E} (w_E(u,v) - w_V(u))x(u,v) & & \\ (1') & x(u,v) + x(\delta^+(v)) & \leq & 1, \forall v \in V, u \in n^-(v) \\ (*) & \sum_{(u,v)\in E} x(u,v) & = & |V| - p \\ (2') & x(u,v) & \in & \{0,1\}, \forall (u,v) \in E \\ (3') & y(v) & = & 1 - x(\delta^+(v)), \forall v \in V \\ (4') & y(v) & \in & \{0,1\}, \forall v \in V \end{array}$$

Observe now that inequalities (4') are redundant. Also for Y-free graphs, inequality (1') is equivalent to $x(\delta(v)) \leq 1$ for all $v \in \mathcal{S} \cup \mathcal{F}$ and $x(u,v) \leq 1$ for all $v \in \mathcal{T}, u \in \delta^-(v)$. But this last inequality is dominated by $x(\delta(u)) \leq 1$ $(u \notin \mathcal{T})$.

The p-median problem in Y-free graphs can therefore be formulated as the following integer linear program (again removing constraint (*) yield a formulation for the uncapacitated facility location problem).

$$\begin{array}{llcl} \min & w_V(V) + \sum_{(u,v)\in E} (w_E(u,v) - w_V(u))x(u,v) & & \\ (1'') & x(\delta(v)) & \leq & 1, \forall v \in \mathcal{S} \cup \mathcal{F} \\ (*) & \sum_{(u,v)\in E} x(u,v) & = & |V| - p \\ (2'') & x(u,v) & \in & \{0,1\}, \forall (u,v) \in E \\ (3'') & y(v) & = & 1 - x(\delta^+(v)), \forall v \in V \end{array}$$

One can observe that the integral solutions to the system made of inequalities (1") and (2") are matchings of the undirected graph obtained from G by duplicating each sink vertex v with $k = |\delta^-_G(v)|$ copies $v_1, ..., v_k$ and replacing each arc $(u_i, v) \in \delta^-_G(v), i = 1, ..., k$ by an edge (u_i, v_i) ; and then by replacing each remaining arcs $(u,v) \in E$ by a non oriented edge (u,v).

Definition 15. *Given a Y-free graph $G = (V,E)$, the graph obtained from G by the procedure described above is called the* matching-equivalent graph *of G*

Let $G = (V,E)$ be a Y-free graph and $G' = (V', E')$ be its matching-equivalent. In the following, the vertices in G' that correspond to copies of sink vertices in G are also called *sink* even if the graph is undirected. The construction above provides a one-to-one mapping between the arc of G and the edges of G' and between the non sink vertices of G and the non sink vertices of G'. Thus a set of non sink vertices $S \subseteq V$ of G can be regarded as a set $S \subseteq V'$ of non sink vertices in G' and a set of edges $F \subseteq E'$ can be regarded as a set of edges $F \subseteq E$. In particular a vector $x \in \mathbb{R}^E$ can be regarded as a vector in $\mathbb{R}^{E'}$.

Corollary 16. *The uncapacitated facility location problem and the p-median problem in Y-free graphs can be solved in polynomial time (actually in matching time).*

Proof. Let G be a Y-free graph. Let G' be its matching-equivalent. To solve the uncapacitated facility location problem in G, we simply have to solve a matching problem in G' with weight $w_{E'}(u,v) = (w_E(u,v) - w_v(u))$ for all $(u,v) \in E'$. Then we set $y(v) = 1, \forall v \in \mathcal{T}$ and $y(v) = 1 - x(\delta^+_{G'}(v)), \forall v \in \mathcal{S} \cup \mathcal{F}$ i.e. we select as facilities the vertices which are not assigned when selecting the arcs of G corresponding to the edges of the matching in G'. The p-median problem reduces to finding a matching of size $|V| - p$ in G' of minimum weight. □

Chapter 5

Structural graph theory

In this section, we mainly give the logic and intuition behind our results. Indeed, in structural graph theory, the proofs are often based on case analysis and thus can sometimes be long and fastidious.

5.1 Algorithmic decomposition theorem for claw-free graphs

Claw-free graphs are a generalization of line graphs and as such some claw-free graphs can be expressed as a composition of simpler claw-free strips (recall Definition 5). In this section, we would like to understand which claw-free graphs can be decomposed into smaller strips and which are not. Actually we will mainly restrict our attention to *quasi-line graphs*, a subclass of claw-free graphs as the results are easier to expose than for general claw-free graphs. A graph is *quasi-line* if the neighborhood of any vertex can be covered by two cliques.

Partition-cliques are a key feature of the composition of strips : they cover all the adjacencies of the composition of strips beside the edges of the strips themselves. For claw-free graphs, a class of cliques, called *articulation cliques* seems to be closely related and will be our main tool for providing our decomposition result. In order to introduce this notion, we first need a couple of definitions.

Let $G(V, E)$ be a claw-free graph. A vertex $v \in V$ such that $N[v]$ can be covered by two maximal cliques K_1 and K_2 (not necessarily different) is called *regular*, and it is called *strongly regular* when this covering is unique: in this case, we also say that K_1 and K_2 are *crucial* (*for* v). A vertex that is not regular is called *irregular*. Note that each irregular vertex of G is the center of an odd k-anti-wheel with $k \geq 5$, and that G is quasi-line if and only every vertex $v \in V$ is regular.

Definition 17. *A maximal clique K of a claw-free graph G is an* articulation clique *if, for each $v \in K$, K is crucial for v. We denote by $\mathcal{K}(G)$ the family of all articulation cliques of G.*

Note that, by definition, each vertex of an articulation clique is strongly regular. However, the converse does not hold, i.e. it is not true that a maximal clique made of strongly regular vertices is an articulation clique. Indeed, consider a vertex v complete to a path $\{a_1, a_2, a_3, a_4\}$ of length three. The clique induced by $\{v, a_2, a_3\}$ is not an articulation clique, though it is maximal and each vertex is strongly regular. We observe a simple fact that will be extensively used in the following.

Fact 1. *Let G be a claw-free graph and let $K \in \mathcal{K}(G)$ be an articulation clique of G. For each $v \in K$, $N(v) \setminus K$ is a clique.*

Chudnovsky and Seymour [19] recently studied what they call *simplicial cliques* (a clique K is simplicial for a graph $G(V,E)$ if for all $v \in K$, $N(v) \setminus K$ is a clique i.e. each vertex is regular). They give a polytime algorithm to find such cliques in so-called *prime* claw-free graphs and discuss their interest for general graphs. Articulation cliques in claw-free graphs are strongly related and thus we believe that articulation cliques are also of general interest (of course for general graphs, definitions would have to be slightly adapted). The following lemma is easy to prove.

Lemma 18. *Let $G(V,E)$ be a claw-free graph. G has at most $2|V|$ articulation cliques and we can list them all in time $O(|E||V|)$.*

A maximal clique is a *net-clique* if it contains vertices u_1, u_2, u_3 of a net $(u_1, u_2, u_3; v_1, v_2, v_3)$.

We will show that quasi-line graphs that contain at least an articulation clique admit a strip decomposition where each partition-clique is an articulation clique of G. In order to find this decomposition, we therefore need to reverse the operation of composition and define a suitable operation of "ungluing" of articulation cliques. The following lemma says that the other quasi-line graph belong to the class of {claw,net}-free graphs (see [31] for a simple proof).

Lemma 19. *In a quasi-line graph every net-clique is an articulation clique. Therefore a quasi-line graph without articulation clique is net-free.*

Let K be an articulation clique of a *quasi-line* graph G. The ungluing of K requires a partition of the vertices of K into suitable classes. These classes are the equivalence classes defined by an equivalence relation $\mathcal{R}$ on the vertices of K. Call *bound* a vertex of K that belongs to two distinct articulation cliques of G (note that no vertex belongs to *more* than two articulation cliques). Then, for $u, v \in K$, $u\mathcal{R}v$ if and only if:

(i) either $u = v$;

(ii) or $u \neq v$, both u and v are bound and they belong to the same articulation cliques (note that in this case u and v are true twins);

(iii) or $u \neq v$, u and v are neither simplicial nor bound and $(N(v) \setminus K) \cup (N(u) \setminus K)$ is a clique.

We claim that $\mathcal{R}$ define an equivalence relation on the vertices of K (see [31] for a proof). Hence the following definition is consistent. We then skip the straightforward proof of Lemma 21.

Definition 20. *Let G be quasi-line and K an articulation clique of G. We denote by $\mathcal{Q}(K)$ the family of the equivalence classes defined by $\mathcal{R}$ and call each class of $\mathcal{Q}(K)$ a* spike *of K.*

Lemma 21. *Let $G(V,E)$ be a quasi-line graph, K an articulation clique of G and Q a spike of K. Then:*

- *either $Q = \{v\}$, for some simplicial vertex v of G. In this case, $N[Q] = K$ and the spike is called* simplicial*;*
- *or $Q = U[v]$ for some bound vertex v of G. In this case, the vertices in Q are true twins, the unique pair of maximal clique covering $N[Q]$ is $\{K, (N(Q)\setminus K)\cup Q\}$, where $(N(Q)\setminus K)\cup Q$ is also an articulation clique of G, and the spike is called a* bound *spike;*
- *or Q is made of a subset of non-bound and non-simplicial vertices of K. In this case, for each $v \in Q$, the unique pair of maximal clique covering $N[v]$ is $\{K, (N(v) \setminus K) \cup U[v]\}$, and the spike is called* non-trivial.

Before proceeding further, it is convenient to shed some light on the intersections between spikes from different articulation cliques. We will denote by $\mathcal{Q}(\mathcal{K}(G))$ the disjoint union of all spikes of articulation cliques of G: note that $\mathcal{Q}(\mathcal{K}(G))$ is in general a multi-family. The following lemma is straightforward.

Lemma 22. *Let Q_1 and Q_2 be spikes of different articulation cliques. Then either $Q_1 \cap Q_2 = \emptyset$, or Q_1 and Q_2 are bound spikes and $Q_1 = Q_2$.*

We are ready to introduce the operation of ungluing of articulation cliques in quasi-line graphs. We skip the proof of complexity that can be found in [32].

Definition 23. *Let G be a quasi-line graph. The* ungluing *of the cliques in $\mathcal{K}(G)$ consists of removing, for each articulation clique $K \in \mathcal{K}(G)$, the edges between different spikes of $\mathcal{Q}(K)$. We denote the resulting graph by $G|_{\mathcal{K}(G)}$.*

Lemma 24. *Let $G(V, E)$ be a quasi-line graph. We can build the graph $G|_{\mathcal{K}(G)}$ and the family $\mathcal{Q}(\mathcal{K}(G))$ in time $O(|V||E|)$.*

We can now give our algorithmic decomposition theorem for quasi-line graphs. Indeed if G does not have any articulation clique then G is net-free while in case it has, the connected components of $G_{|\mathcal{K}(G)}$ yield the decomposition. Let $\mathcal{C}$ be the connected components of $G_{|\mathcal{K}(G)}$. For each component $C \in \mathcal{C}$, let $\mathcal{A}(C)$ be the family (possibly a multi-family) of spikes in C. The family of strips is defined by $\{(C, \mathcal{A}(C)) : C \in \mathcal{C}\}$ and the partition $\mathcal{P}$ of $\bigcup_{C \in \mathcal{C}} \mathcal{A}(C)$ puts two extremities in the same class if and only if they are spikes from a same articulation clique. G is the composition of the strips $\{(C, \mathcal{A}(C)) : C \in \mathcal{C}\}$ with respect to the partition $\mathcal{P}$. See [31] for proofs and see Figure 5.1 for an example of application.

Theorem 25. *Let $G(V, E)$ be a connected quasi-line graph. In time $O(|V||E|)$ we can:*

(j) either recognizes that G has no articulation cliques and therefore is net-free;

(jj) or provides a decomposition of G into $k \leq |V|$ quasi-line strips $(G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)$, with respect to a partition $\mathcal{P}$ Moreover, for each strip $(G^i, \mathcal{A}^i)$, each extremity $A \in \mathcal{A}^i$ is a spike from some articulation clique of G and the graph G^i is distance simplicial with respect to A.

We now state the result for claw-free graphs that requires some more work and whose proof can be found in [32]

Theorem 26. *Let $G(V, E)$ be a connected claw-free, with $|V| = n$. In time $O(|V||E|)$ we can :*

(i) either recognizes that G has $\alpha(G) \leq 3$;

(ii) or recognize that G is $\{$claw,net$\}$-free

(iii) or provides a decomposition of G into $k+t \leq n$ strips $(F^1, \mathcal{B}^1), \ldots, (F^k, \mathcal{B}^k)$, $(H^1, \mathcal{A}^1), \ldots, (H^t, \mathcal{A}^t)$, with respect to a partition $\mathcal{P}$. Moreover, each graph H^j is a claw-free graph with an induced 5-wheel and stability number at most 3; each graph F^i is distance simplicial with respect to each $B \in \mathcal{B}^i$.

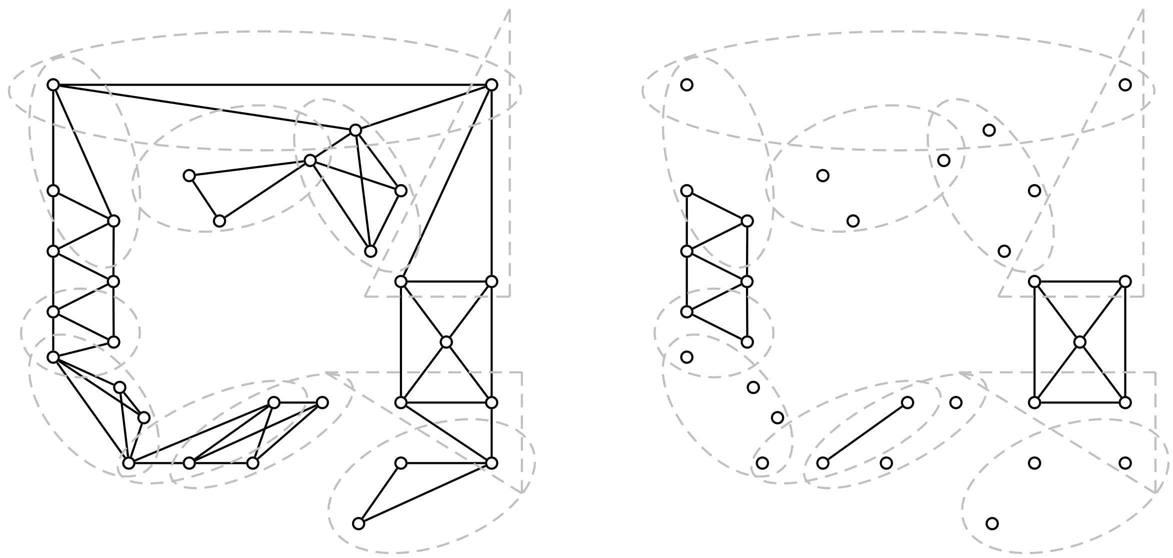

Figure 5.1: On the left: A quasi-line graph G and its articulation cliques (dashed, in gray). On the right: the decomposition of G into distance simplicial strips obtained using our decomposition algorithm: the articulation cliques of G are now the partition-cliques of its decomposition. Recall that, for each strip $(C, \mathcal{A}(C))$, at most two different partition-cliques K_1, K_2 have non-empty intersection with $V(C)$. Thus, $K_1 \cap V(C)$ and $K_2 \cap V(C)$ (if K_2 exists) form the extremities from the set $\mathcal{A}(C)$.

5.2 Recognizing fuzzy circular interval graphs : a simple induction

Fuzzy circular interval graphs are, according to Chudnovsky and Seymour [18], "one of the two principal basic classes of claw-free graphs". They are a generalization of a very well-known class of graphs : circular interval graphs.

Definition 27. [17] *A* circular interval graph (CIG) $G = (V, E)$ *is defined by the following construction: Take a circle* $\mathcal{C}$*, a set of vertices* V *and a mapping* $\Phi : V \mapsto \mathcal{C}$ *of the vertices on the circle. Take a subset of closed and proper intervals* $\mathcal{J}$ *of* $\mathcal{C}$*, with no interval including another, and say that* $u, v \in V$ *are adjacent if* $\{\Phi(u), \Phi(v)\}$ *is a subset of one of the intervals.* $(V, \Phi, \mathcal{J}$ *is a* interval model *of* G*.*

Circular interval graphs (see Figure 5.2) are also known as *proper circular arc graphs*, i.e. they are equivalent to the intersection graphs of arcs of a circle with no containment between arcs [17]. We can thus associate also with G such an *arc model*. Proper circular arc graphs have been studied extensively in the last decades. For instance, given a graph G with n vertices and m edges, there are many polynomial time algorithms that recognize whether G is a proper circular arc graph (i.e. a CIG) and, in case, build an *arc model* (see e.g. [22, 23]). We mainly refer here to the $\mathcal{O}(n+m)$-time algorithm in [23], since it can be trivially adapted to build in $\mathcal{O}(n+m)$-time an *interval model* for G, if any, with n intervals (see Proposition 2.6 in [23]).

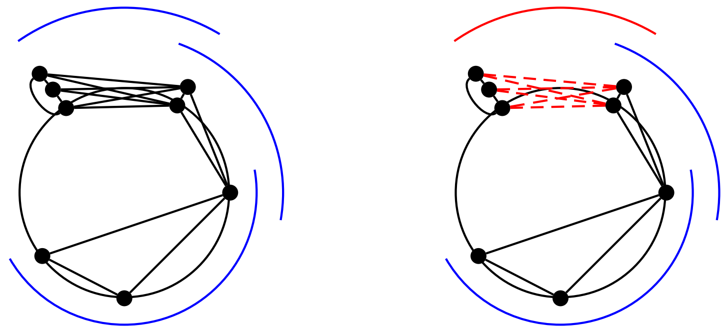

Figure 5.2: A circular interval graph (on the left) and a fuzzy circular interval graph (on the right). Dashed lines represent fuzzy adjacencies.

Definition 28. [17] *A graph $G = (V, E)$ is a* fuzzy circular interval graph (FCIG) *if the following conditions hold: (i) There is a map Φ from V to a circle $\mathcal{C}$; (ii) There is a set $\mathcal{J}$ of closed and proper intervals of $\mathcal{C}$, none including another, such that no point of $\mathcal{C}$ is the end of more than one interval and (iia) If two vertices u and v are adjacent, then $\Phi(u)$ and $\Phi(v)$ belong to a common interval ; (iib) If two vertices u and v belong to the same interval, which is not an interval with endpoints $\Phi(u)$ and $\Phi(v)$, then they are adjacent.*

In other words, in contrast with CIG where adjacencies are defined exactly by the set of intervals, i.e. $u, v \in V$ are adjacent if and only if $\{\Phi(u), \Phi(v)\}$ is a subset of one of the intervals, for a fuzzy circular interval graph additional flexibility is given to adjacencies between vertices at the extremities of an interval: they can be chosen arbitrarily.

For FCIG, we again refer to the triple $(V, \Phi, \mathcal{J})$ as an *interval model* (or simply a *model*) for G; note that while such a triple completely defines a CIG, this is not the case with a FCIG, because of fuzziness. Also, as we discussed above, there are polynomial time algorithms to build an interval model for a given CIG. In contrast, no such algorithm was available for FCIGs, i.e. deciding if a graph is FCIG and if so providing an interval model.

We briefly discuss here a polytime algorithm to recognize whether a graph is a FCIG. The details can be found in [69]. In this algorithm, a crucial role is played by *homogeneous* pairs of cliques: a pair $\{K_1, K_2\}$ of (non necessarily maximal) disjoint cliques of a graph $G = (V, E)$ is is homogeneous if every $v \in V \setminus (K_1 \cup K_2)$ is either complete or anti-complete to each of K_1 and K_2. A subset $S \subset V$, possibly made of a single vertex, is *complete* (respectively, *anti-complete*) to another subset $T \subset V$, with $S \cap T = \emptyset$, if each vertex in S is adjacent (respectively, *anti-adjacent*) to each vertex in T. Moreover a pair of cliques $\{K_1, K_2\}$ is *proper* if K_1, K_2 are neither complete nor anti-complete to each other.

Homogeneous and proper pairs of cliques are fundamental in many structural graph theorems and thus we believe that some of the idea presented here can be useful for other recognition problems.

Our algorithm reduces the problem of recognizing and providing an interval model of a FCIG to the same problem on a suitable CIG and build upon the following simple fact, whose proof is easy.

Lemma 29. *Let G be a fuzzy circular interval graph. If G has no proper and homogeneous pairs of cliques, then G is a circular interval graph.*

The crucial operation of the algorithm is a "reduction" that replaces a proper and homogeneous pair of cliques $\{K_1, K_2\}$ by a pair of cliques, depicted in Fig. 5.3, that is still homogeneous but only *almost-proper*: a pair of non-empty vertex-disjoint cliques $\{K_1', K_2'\}$ is almost-proper if every vertex in K_1' (respectively, K_2') is not complete to K_2' (respectively, K_1') and there exists $u \in K_1'$, $v \in K_2'$ that are adjacent.

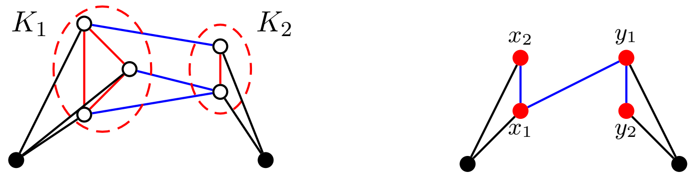

Figure 5.3: A proper and homogeneous pair of cliques $\{K_1, K_2\}$ (on the left) and the reduction of the graph with respect to the pair $\{K_1, K_2\}$ (on the right).Note that $\{\{x_1, x_2\}, \{y_1, y_2\}\}$ defines an almost-proper and homogeneous pair of cliques for the reduced graph
.

Our reduction preserves the property of a graph of being (respectively, being not) a FCIG. The proof is rather long and technical, so we point the reader to the original paper [69].

Theorem 30. *[69] Let G be a connected graph and let $\{K_1, K_2\}$ be a proper and homogeneous pair of cliques. The graph $G|_{\{K_1,K_2\}}$ is connected. Moreover, G is a fuzzy circular interval graph if and only if $G|_{\{K_1,K_2\}}$ is a fuzzy circular interval graph and, from a model for G, one may build in $O(n^2)$-time a model for $G|_{\{K_1,K_2\}}$, and vice versa.*

This result allows for a simple iterative algorithm. Indeed if we start with a graph G with m edges and iterate this reduction, in at most m steps we end up with a graph G' without proper and homogeneous pairs of cliques. Following Theorem 30, G is a FCIG if and only if G' is a FCIG and, by Lemma 29, G' is a FCIG if and only if it is a CIG. The latter fact may be easily tested by applying any existing algorithm for the recognition of CIGs. Even better, Theorem 30 also shows that, if G' is a CIG, we may easily extend an interval model for G' into an interval model for G.

All in all, we can thus show the following.

Theorem 31. *[69] Given a graph G with n vertices and m edges, one can decide in time $O(mn^2)$ if G is a fuzzy circular interval graph and if it is, provide a interval model for G.*

Chapter 6

Polyhedra

6.1 Some properties of the matching polytope

In this section, we recall basic results on the matching polytope. Let $G = (V, E)$ be an undirected graph. The matching polytope of G, $MATCH(G)$, is the convex hull of the characteristic vectors of all matchings in G. Equivalently, $MATCH(G)$ is the integer hull of the polytope $P = \{x \in \mathbb{R}^E_+ : x(\delta(v)) \leq 1, \forall v \in V\}$

It is well-known [24] that the matching polytope can be characterized by adding to the previous polytope the so-called *blossom inequalities*. Blossom inequalities are of the form $x(E(S)) \leq \frac{|S|-1}{2}$ for all odd set of vertices $S \subseteq V$. Hence $MATCH(G) = \{x \in \mathbb{R}^E_+ : x(\delta(v)) \leq 1, \forall v \in V; x(E(S)) \leq \frac{|S|-1}{2}$ for each odd set $S \subseteq V\}$.

Edmonds and Pulleyblank [25] have further studied $MATCH(G)$ and they proved that among blossom inequalities, only a few were facet-defining.

Theorem 32. *[25] The matching polytope $MATCH(G)$ of a graph $G = (V, E)$satisfies $MATCH(G) = \{x \in \mathbb{R}^E_+ : x(\delta(v)) \leq 1, \forall v \in V; x(E(S)) \leq \frac{|S|-1}{2}$, for each odd set $S \subseteq V$ satisfying $|S| \geq 3$ and $G[S]$ is factor-critical and 2-vertex-connected $\}$.*

Factor-critical graphs are graphs H for which $H \setminus v$ has a perfect matching for all $v \in V(H)$. The following theorem gives a characterization of 2-connected factor-critical graphs (see [61]).

Theorem 33. *[61] A graph G is factor-critical and 2-connected if and only if G admits an open odd ear decomposition $G_0, ..., G_k$.*

An *odd ear decomposition* $G_0, ..., G_k$ of a graph G is sequence of graphs with the property that G_0 is an odd cycle, G_i, $i = 1, ..., k$ is obtained from G_{i-1} by *adding an odd ear* to G_{i-1} and $G_k = G$. We say that H' arises from H by adding an odd ear if H' is obtained from H by adding an odd path (i.e. odd number of edges) P from $a \in V(H)$ to $b \in V(H)$ with $V(P) \cap V(H) = \{a, b\}$. An odd ear is said to be *open* when $a \neq b$.

Beside this nice characterization of facets $MATCH(G)$ has the so-called intersection property i.e. intersecting $MATCH(G)$ with an hyperplane of the form $\sum_{e \in E} x_e = k$ for some $k \in \mathbb{Z}$ yields an integral polytope.

Lemma 34. *Let $G = (V, E)$ be a graph and $k \in \mathbb{Z}$. The polytope $P_k = MATCH(G) \cap \{x \in \mathbb{R}^E : \sum_{e \in E} x_e = k\}$ is integral.*

Proof. We sketch the proof, see for instance [61] for a more formal one. Consider two matching M_1, M_2 of G and their characteristic vectors χ^{M_1}, χ^{M_2}. It is easy to prove that χ^{M_1}, χ^{M_2} are adjacent on $MATCH(G)$ if and only if the symmetric difference of M_1 and M_2 is a connected component (if not, we can find several way of writing a point lying on the segment between

χ^{M_1} and χ^{M_2} as a convex combination of matchings). Now it is also well known that connected components of the symmetric difference of two matchings are made of alternating paths and cycles. Hence the cardinality of adjacent matchings on $MATCH(G)$ differs by at most one. So suppose that there exists k such that P_k is fractional. Then there exists a fractional extreme point p in P_k. Since p is an extreme point of P_k it must lie on an edge of $MATCH(G)$. Therefore it is the convex combination of two matchings whose size differs by at most one. But then one the two matchings must have cardinality k and the other $k-1$ or $k+1$. In particular p must coincide with the characteristic vector of the matching of cardinality k, which is a contradiction. □

6.2 p-median polytope of Y-free graphs

We proved in Section 4.3 that the p-median problem in a Y-free graph G reduces naturally to a matching problem of size $|V| - p$ in its matching-equivalent graph G' (see Def. 15), we will use the characterization of the matching polytope of G' to derive the characterization of the p-median polytope of G.

Let us first give a more precise description of 2-connected factor-critical graphs in G'. We recall that given a directed graph $D = (N, A)$, a *circuit* of length k is an ordered (multi-)set of k arcs $(a_1, ..., a_k)$ of D such that for all $i = 1, ..., k$, there exists $v_i \in N$ such that $a_i \in \delta^-(v_i)$ and $a_{i+1} \in \delta^+(v_i)$ with the convention $k + 1 = 1$. When the graph D is simple, a circuit can be represented by the ordered list of vertices v_i. A circuit is said to be *simple* if no vertex is traverse more than once. Observe that in a Y-free graph, all circuit are simple.

Lemma 35. *Let $G = (V, E)$ be a Y-free graph and $G' = (V', E')$ its matching-equivalent graph. Let $U \subseteq V'$ with $|U| \geq 3$ and $|U|$ odd. $G'[U]$ is factor-critical and is 2-vertex-connected if and only if $U \subseteq \mathcal{S} \cup \mathcal{F}$ and $G[U]$ is an simple odd circuit.*

Proof. We only need to prove necessity since sufficiency is immediate. Observe first that U cannot contain any sink vertex since then $G'[U]$ would not be 2-connected.

Claim 36. *No vertex u of U has 2 different arcs in $\delta^-_{G[U]}(u)$.*

Indeed otherwise u is either a sink or $\{u\} \cup N(u)$ contains a Y. □

We know by Theorem 33 that $G'[U]$ can be obtained from an odd cycle $G_0 = (u_0, ..., u_l)$ by adding odd paths. Let $G_0, ..., G_k$ be the sequence of graph provided by Theorem 33. $G[\{u_0, ..., u_l\}]$ is in fact a circuit by Claim 36.

We will prove now that $G_k = G_0$. Suppose to the contrary that $G_k \neq G_0$. Let $P = (v_0, ..., v_t)$, $v_0 \neq v_t$ be the odd path such that G_1 arise from G_0 by adding P. Without loss of generality, we can assume that $v_0 = u_0$. Observe that $u_l \neq v_1 \neq u_1$ (otherwise G has a cycle of length two or G is not simple). $G[\{u_l, u_0, u_1, v_1\}]$ cannot be a Y, thus $v_1 \in \delta^+_G(u_0)$. The same holds for v_{t-1}, $v_{t-1} \in \delta^+(v_t)$. In particular $v_t \neq v_1$. Also the path P is not an oriented path in G and thus there exists a vertex $w \in \{v_1, ..., v_{t-1}\}$ such that $|\delta^-_G(w)| \geq 2$, a contradiction with Claim 36. □

We are now ready to give the polyhedral characterization of $UFLP(G)$ and $pM(G)$ for a Y-free graph G.

Theorem 37. *Let G be a Y-free graph. Then $UFLP(G)$ is characterized by:*

$$\begin{array}{llcl}
(1''') & x(\delta(v)) & \leq & 1, \forall v \in \mathcal{S} \cup \mathcal{F} \\
(2''') & x(C) & \leq & \frac{|C|-1}{2}, \forall \textit{ induced simple odd circuit } C \textit{ of } G \\
(3''') & x(u, v) & \geq & 0, \forall (u, v) \in E \\
(4''') & y(v) & = & 1 - x(\delta^+(v)), \forall v \in V
\end{array}$$

Proof. We know from Theorem 32 and Lemma 35 that inequalities $(1'''), (2'''), (3''')$ define the matching polytope of the matching-equivalent graph of G. Integrality of this polytope implies integrality of the polytope defined by (1"'), (2"'), (3"') and (4"'). (It follows from the fact that if $P := \{x : Ax \leq b\}$ is an integral polytope, then $Q := \{(x, y) : Ax \leq b, y = Bx + d\}$ is also an integral polytope when B, d are integral. Actually it is a simple exercise to prove that if $x_1, ..., x_k$ are the extreme points of P, then $(x_1, Bx_1 + d), ..., (x_k, Bx_k + d)$ are the extreme points of Q). □

Lemma 38. *Let G be a Y-free graph. For any integer k, the polytope defined by the following inequalities is integral.*

$$\begin{array}{lrcl} (i) & x(\delta(v)) & \leq & 1, \forall v \in \mathcal{S} \cup \mathcal{F} \\ (ii) & x(C) & \leq & \frac{|C|-1}{2}, \forall \textit{ induced simple odd circuit } C \textit{ of } G \\ (iii) & \sum_{(u,v)\in E} x(u,v) & = & k \\ (iv) & x(u,v) & \geq & 0, \forall (u,v) \in E \end{array}$$

Proof. We already observe that the set of inequalities (i),(ii),(iv) defines the matching polytope of the matching-equivalent graph of G (from Theorem 32 and Lemma 35). Now from Lemma 34, we know that intersecting this polytope with $\sum_{(u,v)\in E} x(u,v) = k$ results in an integral polytope. □

The following results follows from Lemma 38 by the same argumentation as in the proof of Theorem 37.

Theorem 39. *Let G be a Y-free graph. Then $pM(G)$ is characterized by:*

$$\begin{array}{rcl} x(\delta(v)) & \leq & 1, \forall v \in \mathcal{S} \cup \mathcal{F} \\ x(C) & \leq & \frac{|C|-1}{2}, \forall \textit{ induced odd circuit } C \textit{ of } G \\ x(u,v) & \geq & 0, \forall (u,v) \in E \\ \sum_{v\in V} y(v) & = & p \\ y(v) & = & 1 - x(\delta^+(v)), \forall v \in V \end{array}$$

6.3 Stable set polytope of quasi-line graphs

Recall that the stable set polytope $STAB(G)$ is the convex hull of the characteristic vectors of stable sets of the graph G. We are interested here in the separation problem over $STAB(G)$ when G is claw-free (or quasi-line).

> **Separation problem**: Given a claw-free graph $G(V, E)$ and $x^* \in \mathbb{R}^{|V|}$, determine, whether $x^* \in STAB(G)$ and if not, determine an inequality $cx \leq \delta$ which is valid for $STAB(G)$ and satisfies $cx^* > \delta$.

The polynomial *equivalence between separation and optimization* for rational polyhedra [45, 78, 52] provides a polynomial time algorithm for the separation problem for $STAB(G)$, if G is claw-free (recall from Section 4.2 that the stable set problem in claw-free graphs is polynomial). However, the corresponding algorithms are based on the ellipsoid method [53] and no explicit description of a set of inequalities is known that determines $STAB(G)$ in this case. This apparent asymmetry between the algorithmic and the polyhedral status of the stable set problem in claw-free graphs gives rise to the challenging problem of providing a "...decent linear description of $STAB(G)$" [45], which is still somewhat open today (see [75] for a recent survey).

The matching problem [24] is a well known example of a combinatorial optimization problem in which the optimization problem on the one hand and the facets on the other hand are well understood as already discussed in Section 6.1. Indeed this polytope can be described by a system of inequalities in which the coefficients on the left-hand-side are 0/1 (see Section 6.1). As a consequence so is the stable set polytope of line graphs. More precisely, Let H be a graph and $G = L(H)$ its line graph. Let U be an odd subset of the nodes of H. The blossom inequality defined by U is the inequality

$$\sum_{e \in E(U)} x(e) \leq \lfloor |U|/2 \rfloor \tag{6.1}$$

which is valid for all characteristic vectors of matchings in H. Here, $E(U)$ is the subset of edges of H which have both endpoints in U. This inequality is also valid for $STAB(G)$ and actually since there is a one-to-one correspondence between stable sets in G and matchings in H, there are the only necessary inequalities to describe $STAB(G)$ beside *clique inequalities* (coming from degree constraints) and non negativity constraints. A *clique inequality* is a valid inequality of $STAB(G)$ the form $x(K) \leq 1$ for K a set of nodes of G inducing a *clique*. In the stable set setting inequality (6.1) can be thus seen as follows : each vertex $v \in U$ yields a clique K_v in the line graph G of H consisting of all edges which are incident to v. Let $\mathcal{F}$ be the family consisting of those cliques. Observe that the edges of H in $E(U)$ correspond to vertices of G that are covered by exactly two cliques of $\mathcal{F}$. Summing up all clique inequalities in $\mathcal{F}$ (plus some non-negativity constraints) therefore yield the valid inequality $\sum_{v \in E(U)} 2 \cdot x(v) \leq |U|$. It follows that $\sum_{v \in E(U)} x(v) \leq \lfloor |U|/2 \rfloor$ is a valid inequality for $STAB(G)$ (it is a Chvátal-Gomory cut, see Section 8.1). Generally speaking, given a set of n cliques and a set of vertices Q covered by at least two of those cliques, one can write an inequality $\sum_{v \in Q} x(v) \leq \lfloor |Q|/2 \rfloor$ which is valid for $STAB(G)$ and such an inequality is called an *Edmonds' inequality*.

Chudnovsky and Seymour [17] were able to prove that Edmonds' inequalities are not only useful for line graphs but they allow to characterize completely the stable set polytope of quasi-line graphs that are *not* fuzzy circular interval graphs.

Theorem 40 (Chudnovsky and Seymour [17])**.** *If G is a quasi-line graphs that is not a fuzzy circular interval graph, then all facets of $STAB(G)$ are Edmonds inequalities, clique inequalities or non-negativity constraints.*

For general quasi-line graphs, non 0/1 inequalities are needed [41]. Oriolo [68] introduced a generalization of Edmonds' inequalities : *clique family inequalities*.

Let $\mathcal{F} = \{K_1, \ldots, K_n\}$ be a set of cliques, $1 \leq p \leq n$ be integral and $r = n \mod p$. Let $V_{p-1} \subseteq V(G)$ be the set of vertices covered by exactly $(p-1)$ cliques of $\mathcal{F}$ and $V_{\geq p} \subseteq V(G)$ the set of vertices covered by p or more cliques of $\mathcal{F}$. The inequality

$$(p-r-1) \sum_{v \in V_{p-1}} x(v) + (p-r) \sum_{v \in V_{\geq p}} x(v) \leq (p-r) \left\lfloor \frac{n}{p} \right\rfloor \tag{6.2}$$

is valid [68] for $STAB(G)$ (see Section 8.1 for a short proof) and is called the *clique family inequality* associated with $\mathcal{F}$ and p.

While for claw-free graphs other inequalities are needed [41, 59, 80], clique family inequalities are enough to describe completely $STAB(G)$ when G is quasi-line. Ben Rebea [9] considered first the problem to study $STAB(G)$ for quasi-line graphs while Oriolo [68] formulated a conjecture inspired by his work, which is now a theorem : the Ben Rebea theorem.

Theorem 41. *[27] The stable set polytope of a quasi-line graph $G = (V, E)$ may be described by the following inequalities:*

(i) $x(v) \geq 0$ *for each* $v \in V$

(ii) $\sum_{v \in K} x(v) \leq 1$ *for each maximal clique* K

(iii) *inequalities* (6.2) *for each family* $\mathcal{F}$ *of maximal cliques and each integer* p *with* $|\mathcal{F}| > 2p \geq 4$ *and* $|\mathcal{F}| \mod p \neq 0$.

Because of Theorem 40, in order to prove the Ben Rebea theorem, one only needs to focus on fuzzy circular interval graphs. We can restrict to the special case of circular interval graphs as fuzziness can be dealt with easily. Let G be a circular interval graph and let $\mathcal{K}_{\mathcal{I}}$ the family of cliques stemming from the intervals in the definition of G (see Definition 27). Then $P = \{x \in R^n \mid Ax \leq \mathbf{1}, x \geq 0\}$ where the 0/1 matrix A, corresponding to the cliques $\mathcal{K}_{\mathcal{I}}$, has the circular ones property. But exploiting Theorem 3 further, one can prove the following result, see [27] for details.

Theorem 42. *Let* $P = \{x \in \mathbb{R}^n \mid Ax \leq \mathbf{1}, x \geq 0\}$ *be a polyhedron, where* $A \in \{0,1\}^{mn}$ *is a circular ones matrix and* $\alpha \in \mathbb{N}$ *a positive integer. Any facet of* P_I *is of the form* $a \sum_{v \in T} x(v) + (a-1) \sum_{v \notin T} x(v) \leq a\,\beta$ *where* $T \subseteq \{1, \ldots, n\}$ *and* $a, \beta \in \mathbb{N}$.

Theorem 42 implies that any facet of $STAB(G)$ is of the form

$$a \sum_{v \in T} x(v) + (a-1) \sum_{v \notin T} x(v) \leq a \cdot \beta \tag{6.3}$$

where $T \subseteq V$ and $a, \beta \in \mathbb{N}$.

Now making use of additional properties used in the derivation of Theorem 42, one can easily prove that the inequalities (6.3) are actually clique family inequalities and thus prove Theorem 41 (see [27]). In [93], we could additionally prove that inequalities (6.3) have the property that $\sum_{v \in T} x(v) \leq \beta$ is a rank facet of $STAB(G[T])$.

6.4 Rank-facets for SSP of quasi-line graphs by sequential lifting

In the case of the matching polytope, Pulleyblank and Edmonds [25] could give a description of the blossom inequalities that define facets (see Theorem 32). Such a result is not yet at hand for the stable set polytope of quasi-line graphs. We answer this question for rank facets of fuzzy circular interval graphs in [73] and we briefly discuss this here. The interesting aspect of this work is the combinatorial interpretation of the lifting procedure defined by Padberg in this particular context.

In [77] Padberg proved the following. Let $G = (V, E)$ be a graph and $\sum_{j \in T} a_j x_j \leq b$ be a facet of $STAB(G[T])$, for some $T \subset V$. If $u \in V \setminus T$ and

$$a_u = b - \max_{S \text{ stable set, } S \subseteq T \setminus N(u)} \sum_{j \in S} a_j, \tag{6.4}$$

then the inequality $\sum_{j \in T \cup u} a_j x_j \leq b$ is a facet of $STAB(G[T \cup u])$ and a_u is the *lifting coefficient* of variable x_u. Of course, we can iterate this procedure and, after $\mid V \setminus T \mid$ steps, produce an inequality $\sum_{j \in V} a_j x_j \leq b$ which is a facet of $STAB(G)$ and is a *sequential lifting* of $\sum_{j \in T} a_j x_j \leq b$.

By instantiating a result by Galluccio and Sassano [36] for claw-free graphs to fuzzy circular interval graphs, it is possible to prove the following.

Definition 43. *Let* $n, p \in \mathbb{N}$ *with* $n \geq 2p$. *An* (n,p)-circulant W *is a graph with vertices* $V(W) = \{1, 2, \ldots, n\}$ *and edges* $E(W) = \{(i,j) : 1 \leq i, j \leq n$ *and* $|i-j| \leq p-1\}$.

Lemma 44. *Let* $G = (V, E)$ *be a fuzzy circular interval graph. Every rank facet can be obtained by sequential lifting of an inequality* $\sum_{v \in V(G')} x_v \leq \alpha(G')$ *where* *(i)* G' *is a singleton;*

(ii) G' *is a* $(\alpha\omega + 1, \omega)$*-circulant, for some* $\omega \geq 2$.

This proves that, beside cliques, circulant graphs form the core of the rank facets for the SSP of fuzzy circular interval graphs. Oriolo showed [68] that the rank facets of the stable set polytope of quasi-line graphs have the following property.

Lemma 45. [68] *Let* $G = (V, E)$ *be a quasi-line graph and* Q *a subset of vertices. The inequality* $\sum_{j \in Q} x_j \leq \alpha(Q)$ *is a facet of* $STAB(G)$ *if and only if the following statements hold:*

(i) $\sum_{j \in Q} x_j \leq \alpha(Q)$ *defines a facet of* $STAB(G[Q])$*;*

(ii) $G[Q]$ *is* α*-maximal.*

Given a graph $G(V, E)$ and $W \subseteq V$, $G[W]$ is α*-maximal* if $\alpha(G[W \cup \{v\}]) > \alpha(G[W])$ for every $v \notin W$. We denote by $\Gamma(W)$ the set of vertices in $N(W)$ that are adjacent to *all* vertices of W, that is, $\Gamma(W) = \{u \in V(G) \setminus W : uv \in E, \forall v \in W\}$.

We would like to exploit this result to give a combinatorial characterization of the rank facets of fuzzy circular interval graphs i.e. a description of the set Q when $\sum_{j \in Q} x_j \leq \alpha(Q)$ defines a facet of $STAB(G)$. Circulant graphs, which form the core of the rank facets by Lemma 44, are unfortunately not invariant by sequential lifting in fuzzy circular interval graphs. A superclass of circulants is needed.

Definition 46. *A quasi-line graph* $G(V, E)$ *is an* (n, p)-clique-circulant *if the following statements hold:*

(i) There exists a partition of $V(G)$ *into* n *non-empty cliques* $Q_1, \ldots, Q_n$ *and an integer* p*, with* $n \geq 2p \geq 4$*, such that* $\Gamma(Q_i) \supseteq Q_{i-p+1} \cup \ldots \cup Q_{i-1} \cup Q_{i+1} \cup \ldots \cup Q_{i+p-1}$*, for* $i = 1, ..., n$*.*

(ii) There exists an (n, p)*-circulant* $W = \{v_1, \ldots, v_n\}$ *with* $v_i \in Q_i$ *for* $i = 1, ..., n$*.*

Clique-circulants share an interesting polyhedral properties with circulants. Indeed, given a (n, p)-circulant W, we can associated a valid inequality for $STAB(W) : \sum_{j \in V(W)} x_j \leq \alpha(W)$. Trotter studied when this inequality is facet-defining for $STAB(W)$. [97] [16] [15]

Theorem 47. [97] *If* W *is an* (n, p)*-circulant, then* $\alpha(W) = \lfloor \frac{n}{p} \rfloor$ *and* $\sum_{j \in W} x_j \leq \alpha(W)$ *is a facet of* $STAB(W)$ *if and only if* p *is not a divisor of* n*.*

This results extend to (n, p)-clique-circulant [73].

Theorem 48. *Let* G *be an* (n, p)*-clique-circulant. Then* $\alpha(G) = \lfloor \frac{n}{p} \rfloor$ *and* $\sum_{j \in V(G)} x_j \leq \lfloor \frac{n}{p} \rfloor$ *is a facet of* $STAB(G)$ *if and only if* p *is not a divisor of* n*.*

The following result shows that the sequential lifting of an inequality induced by a non α-maximal clique-circulant is itself a clique-circulant inequality and thus that clique-circulant inequalities are *invariant* by sequential lifting in fuzzy circular interval graphs. The technical proof of this Theorem can be found in [73].

Theorem 49. *[73] Let* $G(V, E)$ *be a fuzzy circular interval graph,* $Q \subset V$ *such that* $G[Q]$ *is an* (n, p)*-clique-circulant and* $v \notin Q$ *such that* $\alpha(Q \cup \{v\}) = \alpha(Q)$*. Then* $G[Q \cup \{v\}]$ *is an* (n', p')*-clique-circulant, with* $\lfloor \frac{n'}{p'} \rfloor = \lfloor \frac{n}{p} \rfloor$*.*

Theorem 49 allows for a combinatorial characterization of *all* the rank facets of the stable set polytope of a fuzzy circular interval graph.

Theorem 50. *Let $G = (V, E)$ be a fuzzy circular interval graph. An inequality $\sum_{v \in Q} x_v \leq \alpha(Q)$, with $\alpha(Q) \geq 2$, is a facet of $STAB(G)$ if and only if $G[Q]$ is an α-maximal (n, p)-clique-circulant, with $n \mod p \neq 0$.*

We can therefore state the following theorem for fuzzy circular interval graphs.

Theorem 51. *Let $G = (V, E)$ be a fuzzy circular interval graph. An inequality $\sum_{v \in V(G')} x_v \leq \alpha(G')$ is a facet of $STAB(G)$ if and only if G' is*

- *a maximal clique or*
- *an α-maximal (n, p)-clique-circulant with $n \mod p \neq 0$.*

Chapter 7

Extended formulations

7.1 Extended formulation for distance claw-free graphs

Let $P \subseteq \mathbb{R}^n$ be a polytope, $Q := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^d : Ax + By \leq d\}$ is an extended formulation of P if for all $x \in \mathbb{R}^n$, $x \in P$ if and only if there exists $y \in \mathbb{R}^d : (x, y) \in Q$. For instance, given k vectors $x_1, ..., x_k$ in $\mathbb{R}^n$, $Q = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^k : x = \sum_{i=1}^k \lambda_i x_i, \lambda \geq 0, \sum_{i=1}^k \lambda_i = 1\}$ is an extended formulation of $P := conv.hull\{x_1, ..., x_k\}$.

Dynamic programming also allows to write simple extended formulations, when the problem can be formulated as a longest path in a directed acyclic graph (Martin, Rardin and Campbell [63] deal with more general cases of dynamic programming). Let us illustrate this on the stable set problem in a graph G with bounded width stability number from Section 2.2. We already noticed that we can associate with $G(V, E)$ an auxiliary directed acyclic graph $D(V(D), A(D))$ (see Fig. 2.1). Recall that for building D, we choose a vertex z in V, and define $\mathcal{S}_i$ to be the stable sets in $N_i(z)$ for all $i = 0, ..., \rho$ (including the empty set). The set $V(D)$ then consists of $\{v^i_S : S \in \mathcal{S}_i, i = 0, ..., \rho\}$, together with two special nodes u^*, v^* and $A(D)$ is made of the following arcs: $(u^*, v^0_{\{z\}})$ and $(u^*, v^0_\emptyset)$; for each $i = 0, ..., \rho - 1$ and S stable set of $G[N_i(z) \cup N_{i+1}(z)]$, the arc $(v^i_{S \cap N_i(z)}, v^{i+1}_{S \cap N_{i+1}(z)})$; and for each $S \in \mathcal{S}_\rho$, the arc (v^ρ_S, v^*). Stable sets in G are in one-to-one correspondence with directed (u^*, v^*)-path in the acyclic graph D. Now, let $Q = \{f \in \mathbb{R}^{|A(D)|} : Nf = d, f \geq 0\}$ with N the node-arc matrix of D and $d(v) = 0$ for all $v \in V(D) \setminus \{u^*, v^*\}$, $d(u^*) = 1$, $d(u^*) = -1$. Q is well known to be the convex hull of all characteristic vectors of (u^*, v^*)-paths. But the characteristic vector x^S of the stable set S of G associated with a (u^*, v^*)-path of G is easily recovered from the characteristic vector of the (u^*, v^*)-path. Indeed for every $u \in N_i(z)$, $x^S(u) = \sum_{S \in \mathcal{S}_i : u \in S} f(\delta^+(v^i_S))$. Hence, the polytope defined by $\{(x, f) \in \mathbb{R}^{|V|} \times \mathbb{R}^{|A(D)|} : Nf = d, f \geq 0$, and for all $i = 1, ..., \rho$, for all $u \in N_i(z)$, $x(u) = \sum_{S \in \mathcal{S}_i : u \in S} f(\delta^+(v^i_S))\}$ is a compact (i.e. polynomial in the input size) extended formulation for $STAB(G)$. This extended formulation for graphs with bounded width stability number is due to Pulleyblank and Shepherd [84].

7.2 Combinatorial union of polytopes

Balas [7] introduced an extended formulation for the union of non empty polyhedra. Let $P_i = \{x \in \mathbb{R}^n : A_i x \leq b_i\}$ for $i = 1, ..., k$ be k non empty polyhedra. Balas showed that $Q = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^k : x = \sum_{i=1}^k x_i, A_i x_i \leq b_i \lambda_i, \lambda \geq 0, \sum_{i=1}^k \lambda_i = 1\}$ is an extended formulation for $conv.hull(\bigcup_i P_i)$. The *Minkowski sum* of two non empty polyhedra P_1 and P_2 in $\mathbb{R}^n$ is the polyhedron $P_1 + P_2 = \{x_1 + x_2 : x_1 \in P_1, x_2 \in P_2\}$. By definition, if $P_i = \{x \in \mathbb{R}^n : A_i x \leq b_i\}$ for $i = 1, 2$, then $Q = \{(x_1, x_2, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : z = x_1 + x_2, A_1 x_1 \leq b_1, A_2 x_2 \leq b_2\}$ is an extended formulation for $P_1 + P_2$. We generalize both concepts as follows. Let $P_i = \{x \in \mathbb{R}^n : A_i x \leq b_i\}$ for $i = 1, ..., k$ be again k non empty polyhedra and let $\lambda^1, ..., \lambda^l \in \{0,1\}^k$. The *combinatorial union* of the polyhedra P_i w.r.t. $\{\lambda^1, ..., \lambda^l\}$ is the set $P = conv.hull(\bigcup_{j=1}^{l} \sum \lambda^j(i) P_i)$. The combinatorial union of polytopes allows to express sophisticated relationship between polyhedra. Now if $P' = conv.hull\{\lambda^1, ..., \lambda^l\}$ can be written $\{\lambda \in \mathbb{R}^k : A'\lambda \leq b'\}$, then it is a simple exercise to show that $Q = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^k : x = \sum_{i=1}^k x_i, A_i x_i \leq b_i \lambda_i, A'\lambda \leq b'\}$ is an extended formulation for the combinatorial union of the polyhedra P_i. We can prove actually even more : the combinatorial union of polytopes 'preserves' integrality i.e. if all the P_i are integral, then Q is integral too. This is the purpose of Lemma 53. This proves particularly useful for the study of integer hulls : if the integer solutions can be describe by mean of a combinatorial union of polytopes for which we have a complete description of the integer hulls, then the corresponding extended formulation is integral.

Before proving Lemma 53, we recall first the following well-known result (see e.g. [89]).

Theorem 52. *A polyhedron Q is integral if and only if $max_{x \in Q} cx$ is an integer for each integral c such that the maximum is attained.*

Lemma 53. *Let $n, q, p \in \mathbb{N}$, and $P = \{y \in \mathbb{R}^n_+ : Ay \leq b\}$ be an integer non-empty polyhedron for some $A \in \mathbb{Z}^{p \times n}$, $b \in \mathbb{Z}^p$. Moreover, for $i \in [q]$, let $n_i, p_i \in \mathbb{N}$, and $P^i = \{x^i \in \mathbb{R}^{n_i} : A^i x^i \leq b^i\}$ be an integer non-empty polyhedron for some $A^i \in \mathbb{Z}^{p_i \times n_i}$, $b^i \in Z^{p_i}$. Last, let $\phi : [q] \to [n]$. Then the polyhedron $Q = \{(x, y) \in \mathbb{R}^{N+n} : y \in P$ and $A^i x^i \leq y_{\phi(i)} b^i$ for each $i \in [q]\}$ is integral, where we set $N = \sum_{i=1}^q n_i$ and $x = \{x^1, \ldots, x^q\}^T$.*

Proof. For each $i \in [q]$ and $\alpha \geq 0$, let $P^i_\alpha := \{x^i \in \mathbb{R}^{n_i} : A^i x^i \leq \alpha b^i\}$. We start with two Claims.

Claim 1. *Let $B \in Z^{n \times q}$ and $c, d \in \mathbb{Z}^q$ for some $n, q \in \mathbb{N}$. For $\alpha \in \mathbb{Q}_+$, let R_α be the polyhedron $\{x \in \mathbb{R}^n : Bx \leq \alpha d\}$ and suppose that $\tilde{x}$ is an optimal solution to $\max_{x \in R_1} cx$. Then for each $\alpha \in \mathbb{Q}_+$, R_α is non-empty and $\alpha\tilde{x}$ is an optimal solution to $\max_{x \in R_\alpha} cx$.*

Let first $\alpha = 0$: the $0-$vector is feasible for R_0. Suppose by contradiction it is not optimal: then there exists $\bar{x} \in \mathbb{R}^n$ such that $B\bar{x} \leq 0$, $c\bar{x} > 0$. But then $(\tilde{x} + \bar{x}) \in R_1$ and $c(\tilde{x} + \bar{x}) > c\tilde{x}$, contradicting the fact that $\tilde{x}$ is optimum for R_1. Now let $\alpha > 0$. Then $\max_{x \in R_\alpha} cx = \max_{x \in \mathbb{R}^n : Bx \leq \alpha d} cx = \alpha \cdot \max_{x \in \mathbb{R}^n : B\frac{x}{\alpha} \leq d} (c\frac{x}{\alpha}) = \alpha \cdot max_{x \in R_1} cx$. Thus, R_α is non-empty and $\alpha \cdot \tilde{x}$ is an optimum solution to $\max_{x \in R_\alpha} cx$. ■

Claim 2. *The projection of Q over the $y-$space coincides with P i.e. P=Proj$(Q)_y := \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^N, (x, y) \in Q\}$*

By definition of Q, Proj$(Q)_y \subseteq P$ so we are left to prove the converse. Let $\bar{y} \in P$, and recall that $\bar{y} \geq 0$. We apply Claim 1 to deduce that for $i \in [q]$ there exists $\bar{x}^i \in P^i_\alpha$. Then the point $\{\bar{y}, \bar{x}^1, \ldots, \bar{x}^q\} \in Q$, concluding the proof. ■

Let $(u, w) \in \mathbb{R}^N \times \mathbb{R}^n$ be an integral cost function such that $max_{(x,y) \in Q} ux + wy$ is attained. This implies that for each $i \in [q]$, $\max u^i x^i$ is obtained in some vertex $\bar{x}^i$ that we can assume integral by hypothesis. Then

$$\max_{(x,y)\in Q} ux + wy = \max_{y\in P}(wy + \max_{x:x^i\in P^i_{y_{\phi(i)}} \text{ for } i\in[q]} \sum_{i=1}^{q} u^i x^i) \tag{7.1}$$

$$= \max_{y\in P}(wy + \sum_{i=1}^{q} \max_{x^i\in P^i_{y_{\phi(i)}}} u^i x^i) \tag{7.2}$$

$$= \max_{y\in P}(wy + \sum_{i=1}^{q} u^i (y_{\phi(i)}\bar{x}^i)) \tag{7.3}$$

$$= \max_{y\in P}(\sum_{j=1}^{n} y_j \cdot (w_j + \sum_{i:\phi(i)=j} u^i \bar{x}^i)) = \max_{y\in P}(\bar{w}y)$$

(7.1) holds by Claim 2, (7.2) by the fact that the polyhedra P^i live in different spaces, and (7.3) by Claim 1. Also, for each $j \in [q]$, $\bar{w}_j = w_j + \sum_{i:\phi(i)=j} u^i x^i$ is a sum of integers and thus an integer itself. Because of the integrality of P and Theorem 52, we conclude that the statement holds. □

Let us now see a simple application of this concept in the context of the stable set polytope of composition of strips. Let G be the composition of a family of strips $H^i = (G^i, \mathcal{A}^i), i = 1, ..., k$ and let G^* be the auxiliary graph as defined in Section 4.1. Each vertex of the auxiliary graph G^* represents one out of the (2 or 4) different possible configurations in a given strip and each stable set in the auxiliary graphs correspond to configurations that are *compatibles* (see Section 4.1 for definitions). Therefore if we associate for each vertex $u \in V(G^*)$ the convex hull of all stable sets of G in the corresponding configuration and denote this by $STAB(G^u)$, the stable set polytope of G can be expressed as the combinatorial union of the polytopes $STAB(G^u)$ with respect to $STAB(G^*)$. But because G^* is a line graph, in order to write an linear extended formulation for $STAB(G)$, it is enough to have a (possibly extended) linear formulation for $STAB(G^u)$.

The combinatorial union of polytopes is a special case of branched polyhedral systems introduced by Kaibel and Loos. [50]. However even though the combinatorial union of polytopes is less general, we believe that it is of interest on its own. For instance it might be possible to exploit this concept to generate new cuts for classic combinatorial optimization problems.

7.3 Extended formulation for the stable set polytope of claw-free graphs

By Theorem 26, we know that in time $O(|V||E|)$ we can distinguish if $\alpha(G) \leq 3$, if G is {claw,net}-free or if G is the composition of distance simplicial strips and strips with stability number at most 3. It is easy to write an extended formulation for a graph G with small stability number. Indeed, let $x_1, ..., x_k$ be all the extreme points of $STAB(G)$ i.e., all stable set of size $0, 1, 2$ or 3. The polytope $Q = \{(x, \lambda) : x = \sum_{i=1}^{k} \lambda_i x_i, \lambda \geq 0, \sum_{i=1}^{k} \lambda_i = 1\}$ is an extended formulation of $STAB(G)$. For {claw-,net}-free graphs, we discussed above how to derive an extended formulation based on a dynamic programming (note that this class also includes graphs that are distance simplicial with respect to some clique). We are left with the case where G is the composition of a family of strips $H^i = (G^i, \mathcal{A}^i), i = 1, .., k$. In this case, building upon Section 7.2, we just need to show that we are able to derive extended formulations for the stable set polytopes associated with the strips. In fact, if either G^i is a distance simplicial graph with respect to some clique, or $\alpha(G^i) \leq 3$, then an extended formulation for $STAB(G^i)$

(or $STAB(G^i[V(G) \setminus \cup_{A \in \mathcal{A}^i} A])$ etc.) follows from the above arguments. We point out that the resulting extended formulation is simple and requires only $O(n)$ extra variables. Moreover, even though it might have exponentially many Edmonds' inequalities (and thus the formulation is non compact), they are separable in polytime [79]. One should also observe that if there would exist a compact extended description of the matching polytope, a well-known open problem, then also our formulation would also be compact. We could provide an alternative extended formulation for the stable set polytope of claw-free graphs that is more intricate but allows for a polynomial time algorithm for the separation problem over $STAB(G)$ when G is claw-free. This separation procedure only requires the solution to a small number of compact linear programs and a oracle for separating over the matching polytope. The reader can find details in [29].

Chapter 8

Cutting-plane and Column generation

8.1 Chvàtal-Gomory cuts, split cuts and the SSP of quasi-line graphs

The Chvàtal-Gomory procedure (see e.g., [43, 44, 20]) is a well-known cutting-plane operator to derive the integral hull of a given polyhedron. More precisely, for $P \subseteq \mathbb{R}^n$ the Chvàtal-Gomory closure is defined as

$$P' := \bigcap_{\substack{(c,\delta)\in\mathbb{Z}^n\times\mathbb{Q} \\ cx\leq\delta \text{ valid for } P}} cx \leq \lfloor \delta \rfloor .$$

It is well-known that P' is a polyhedron again (cf., e.g., [88]) if P is a rational polyhedron. Clearly, $\text{conv}P \cap \mathbb{Z}^n =: P_I \subseteq P'$ and we can iterate the operator by setting $P^{(i+1)} := (P^{(i)})'$ with $P^{(1)} := P'$ and $P^{(0)} := P$ for consistency. The *(Chvàtal-Gomory) rank of a polyhedron* P is then defined to be the smallest $i \in \mathbb{N}$ such that $P^{(i)} = P_I$ holds and we denote it by $\text{rk}(P)$. The rank of a polyhedron P is always finite ([20, 87]) but can be arbitrarily large, even for $n = 2$. If we confine ourselves however to polytopes $P \subseteq [0,1]^n$, the rank of P is bounded by a function of n.

There are many standard combinatorial optimization problem whose natural integer programming formulation has Chvàtal-Gomory rank one. This is the case for matching. Given a graph $G(V,E)$, a natural formulation for the maximum weighted matching in G is $\max\{cx : x(\delta(v)) \leq 1, \text{ for all } v \in V, x_v \in \{0,1\}\}$. Blossom inequalities are easily derived as Chvàtal-Gomory cut from the polytope $P = \{x \in \mathbb{R}^{|E|} : x \geq 0, x(\delta(v)) \leq 1, \text{ for all } v \in V\}$, indeed consider an odd set of vertices S and all inequalities $x(\delta(v)) \leq 1$ for all $v \in S$. Summing up all those inequalities yield $2 \cdot x(S) + x(\delta(S)) \leq |S|$ which implies together with $x(\delta(S)) \geq 0$ the inequality $2 \cdot x(S) \leq |S|$. Hence the blossom inequality $x(S) \leq \lfloor \frac{|S|}{2} \rfloor$ is a Chvàtal-Gomory cut. This results also show that for line graphs, Edmonds' inequalities have Chvàtal-Gomory rank one from the fractional relaxation (i.e. the polytope defined by $\{x \in \mathbb{R}^{|V|} : x \geq 0, x(K) \leq 1, \forall K \in \mathcal{K}\}$) and thus so has the stable set polytope of line graphs. In contrast, the stable set polytope of quasi-line graphs can have rank greater than one [68].

Another well-known cutting plane operator is the split closure. More precisely, for $P \subseteq \mathbb{R}^n$ the split closure is defined in a similar inductive way but with

$$P' := \bigcap_{\substack{(c,\delta)\in\mathbb{Z}^n\times\mathbb{Q} : \exists(\pi,\pi_0)\in\mathbb{Z}^n\times\mathbb{Q}, \\ cx\leq\delta \text{ valid for } P\cap\{x:\pi x\leq\pi_0\} \\ cx\leq\delta \text{ valid for } P\cap\{x:\pi x\geq\pi_0+1\}}} cx \leq \delta.$$

Interestingly, the stable set polytope of quasi-line graphs has split rank one (from the fractional polytope). Indeed clique family inequalities can be derived as a split cut from the fractional stable set polytope as follows. Given a family of n cliques $\mathcal{K}$ and an integer $p \geq 2$, define $V_{\geq p}$ and V_{p-1} the vertices covered by at least p cliques and exactly p-1 cliques respectively. Let $r = n \mod p$. Summing up clique inequalities in $\mathcal{K}$ we get a knapsack constraint :

$$p \cdot \sum_{v \in V_{\geq p}} x_v + (p-1) \cdot \sum_{v \in V_{p-1}} x_v \leq n.$$

Consider the disjunction $\sum_{v \in V_{\geq p} \cup V_{p-1}} x_v \leq \lfloor \frac{n}{p} \rfloor \vee \sum_{v \in V_{\geq p} \cup V_{p-1}} x_v \geq \lfloor \frac{n}{p} \rfloor + 1$. The clique family inequality associated with $\mathcal{K}$ and p is

$$(p-r) \sum_{v \in V_{\geq p}} x_v + (p-r-1) \sum_{v \in V_{p-1}} x_v \leq (p-r) \lfloor \frac{n}{p} \rfloor. \tag{8.1}$$

It is trivially valid for $\{x \in QSTAB(G) : \sum_{v \in V_{\geq p} \cup V_{p-1}} x_v \leq \lfloor \frac{n}{p} \rfloor\}$. For $\{x \in QSTAB(G) : \sum_{v \in V_{\geq p} \cup V_{p-1}} x_v \geq \lfloor \frac{n}{p} \rfloor + 1\}$, we have $(p-r) \sum_{v \in V_{\geq p}} x_v + (p-r-1) \sum_{v \in V_{p-1}} x_v = p \sum_{v \in V_{\geq p}} x_v + (p-1) \sum_{v \in V_{p-1}} x_v - r(\sum_{v \in V_{\geq p}} x_v + \sum_{v \in V_{p-1}} x_v) \leq n - r(\lfloor \frac{n}{p} \rfloor + 1) \leq (p-r) \lfloor \frac{n}{p} \rfloor$ and thus again the inequality holds. This proves that the clique family inequality (8.1) is a split cut and thus by Theorem 41, the stable set polytope of quasi-line graph has split rank one.

It is not known whether the Chvàtal-Gomory rank can be arbitrary large for the stable set polytope of quasi-line graphs. If true, this would be the first 'natural' polytope with split rank one and arbitrary high Chvàtal-Gomory rank and thus makes it an interesting open question.

8.2 A lower bound on the Chvàtal-Gomory rank for polytope in the 0/1 cube

While trying to answer the question raised above, we came up with a nice general lower bound for the Chvàtal-Gomory procedure that yield a new lower bound on the Chvàtal-Gomory rank of polytope in the 0/1 cube. We now describe this result, additional results can be found in [82].

The first known upper bound on the Chvàtal-Gomory rank in the 0/1 cube was exponential in the dimension n and was subsequently reduced to $O(n^3 \log(n))$ (cf. [12]) and later to $O(n^2 \log(n))$ (cf. [28]). On the other hand, the best-known lower bound so far was based on the existence (non-constructive) of a family of polytopes P_n with $\text{rk}(P_n) \geq (1+\epsilon)n$, for $\epsilon \leq 3.12 \cdot 10^{-6}$, leaving a large relative gap of $n \log(n)$. The main tool for proving lower bound is a result of Chvàtal, Cook, and Hartmann.

Lemma 54. *([21, Lemma 2.1]) Let P be a rational polyhedron in $\mathbb{R}^n$. Further let u and v be points in $\mathbb{R}^n$ and $m_1, m_2, \ldots, m_d$ be positive numbers. Write $x^{(j)} = u - \sum_{i=1}^{j} \frac{1}{m_i} v$ for all $j \in [d]_0$. If $u \in P$ and if, for all $j \in [d]$, every inequality $ax \leq b$ valid of P_I with $a \in \mathbb{Z}^n$ and $av < m_j$ satisfies $ax^{(j)} \leq b$, then $x^{(j)} \in P^{(j)}$ for all $j \in [d]_0$.*

While this Lemma is very powerful, it is rather difficult to apply it without, *a priori*, having a precise idea of the sequence of points ones wants to consider. Furthermore, it does not provide an immediate lower bound estimate for the rank. This inconvenience motivated us to introduce a reformulation that is slightly more restricted but has certain advantages: we trade generality for simplicity. In order to apply it, no further knowledge about candidate sequences of points is needed and we readily obtain a lower bound on the rank.

We will now establish a new lemma for proving lower bounds on the Chvàtal-Gomory rank. It is inspired by the techniques established in [21], however we shifted the focus towards the intrinsic geometric progression in order to facilitate its application. Let $P \subseteq [0,1]^n$ be a polytope and $cx \leq \delta$ with $(c,\delta) \in \mathbb{Z}^{n+1}$ be valid for P_I. Then the *depth* of $cx \leq \delta$ (with respect to P) is the minimum number of applications ℓ of the Chvàtal-Gomory procedure so that $cx \leq \delta$ is valid for $P^{(\ell)}$. The maximal depth of all facets of P_I equals the rank of P. We call a polytope $P \subseteq [0,1]^n$ *monotone* (or equivalently: *of anti-blocking type*) if whenever $x \in P$ and $y \in [0,1]^n$ with $y \leq x$ coordinate-wise, then $y \in P$ holds.

Lemma 55. *Let $P \subseteq [0,1]^n$ be a polytope, $Q_I \subseteq P_I$ be monotone and $cx \leq d$ be valid for P_I. Further, let $x^* \in P$ such that $cx^* > d$ and define $\delta := \min_{\{a\in\mathbb{N}^n : ax^* > \max_{x\in Q_I} ax\}} (\max_{x\in Q_I} ax)$. If $\delta > 0$ then the depth of $cx \leq d$ is at least*

$$\kappa = \left\lceil \frac{\ln(\frac{cx^*}{d})}{\ln((\delta+1)/\delta)} \right\rceil \geq \left\lceil \ln\left(\frac{cx^*}{d}\right) \cdot \delta \right\rceil .$$

Moreover if $x^* \leq \frac{1}{k}e$ for some $k \in \mathbb{N}$, then

$$\kappa \geq \left\lceil \ln\left(\frac{cx^*}{d}\right) \cdot \frac{1}{k} \min_{\substack{a\in\{0,1\}^n:\\ a\notin k\cdot Q_I}} (ae-1) \right\rceil .$$

where $k \cdot Q_I$ denotes the Minkowski sum of k copies of Q_I.

Proof. Let $x_0^* = x^*$ and $x_{l+1}^* = \lambda x_l^*$ for all $l \in \mathbb{N}_+$ with $\lambda = \frac{\delta}{1+\delta}$. We prove first by induction that $x_l^* \in P^{(l)}$ for all $l \geq 0$. Clearly, the hypothesis holds for $l = 0$. Thus let $l \geq 1$ and $ax \leq b$ be a valid inequality for $P^{(l)}$ with $a \in \mathbb{Z}^n$ and let us consider the corresponding inequality $ax \leq \lfloor b \rfloor$, valid for $P^{(l+1)}$. Let a^+ be the restriction of a to its positive coefficients. Observe that since Q_I is monotone it holds $\max_{x\in Q_I} ax = \max_{x\in Q_I} a^+x$. Suppose first that a is such that $a^+x^* \leq \max_{x\in Q_I} ax$. Then $\lfloor b \rfloor \geq \max_{x\in Q_I} ax \geq a^+x^* \geq a^+x_{l+1}^* \geq ax_{l+1}^*$ and thus $x_{l+1}^* \in P^{(l+1)}$. Now suppose that a is such that $a^+x^* > \max_{x\in Q_I} ax = \max_{x\in Q_I} a^+x$. By definition $\max_{x\in Q_I} a^+x \geq \delta$ and thus $\lfloor b \rfloor \geq \max_{x\in Q_I} ax = \max_{x\in Q_I} a^+x \geq \delta$. Then $ax_{l+1}^* = \lambda ax_l^* \leq \lambda b + (1-\lambda)(\lfloor b \rfloor - \delta) \leq \lambda(\lfloor b \rfloor + 1) + (1-\lambda)(\lfloor b \rfloor - \delta) = \lfloor b \rfloor + \lambda - (1-\lambda)\delta = \lfloor b \rfloor$. Again we obtain $x_{l+1}^* \in P^{(l+1)}$.

Next we show that while $l \leq \frac{\ln(\frac{cx^*}{d})}{\ln(1/\lambda)}$ we have $x_l^* \notin P_I$. To this end it suffices to observe that since $cx_l^* = \lambda^l cx^*$ we obtain that $cx_l^* > d$ if and only if $\lambda^l cx^* > d$. We obtain κ as claimed and further we have $\kappa \geq \lceil \ln(\frac{cx^*}{d}) \cdot \delta \rceil$ since $\ln(1/\lambda) \leq \frac{1-\lambda}{\lambda} = 1/\delta$ and the first part of the result follows.

It remains to prove the second statement. Let $k \in \mathbb{N}$ be arbitrary. For $a \in \mathbb{N}^n$ let $\mathrm{supp}(a) \in \{0,1\}^n$ denote the characteristic vector of the support. We claim that $ae/k > \max_{x\in Q_I} ax$ implies that $\mathrm{supp}(a) \notin k \cdot Q_I$. For contradiction suppose that $\mathrm{supp}(a) \in k \cdot Q_I$. Then there exist $x_1, \ldots, x_k \in Q_I$ such that $\mathrm{supp}(a) = \sum_{i\in[k]} x_i$. Thus $ae = \sum_{i\in[k]} ax_i \leq k \cdot \max_{x\in Q_I} ax$ and so $ae/k \leq \max_{x\in Q_I} ax$; a contradiction. Therefore we have $\{a \in \mathbb{N}^n : ae/k > \max_{x\in Q_I} ax\} \subseteq \{a \in \mathbb{N}^n : \mathrm{supp}(a) \notin k \cdot Q_I\}$. If $x^* \leq \frac{1}{k}e$ for some $k \in \mathbb{N}$, then we have

$$\begin{aligned}\delta &\geq \min_{\substack{a\in\mathbb{N}^n:\\ \frac{1}{k}ae > \max_{x\in Q_I} ax}} \left(\max_{x\in Q_I} ax\right) \geq \min_{\substack{a\in\mathbb{N}^n:\\ \mathrm{supp}(a)\notin k\cdot Q_I}} \left(\max_{x\in Q_I} ax\right) \\ &\geq \min_{\substack{a\in\mathbb{N}^n:\\ \mathrm{supp}(a)\notin k\cdot Q_I}} \left(\max_{x\in Q_I} \mathrm{supp}(a)x\right) = \min_{\substack{a\in\{0,1\}^n:\\ a\notin k\cdot Q_I}} \left(\max_{x\in Q_I} ax\right).\end{aligned}$$

Observe that we can assume that $a \notin k \cdot Q_I$ and $a - e_i \in k \cdot Q_I$ for all i with $a_i = 1$; otherwise we could replace a with $a - e_i$. Therefore $\delta \geq \frac{1}{k} \min_{a\in\{0,1\}^n : a\notin k\cdot Q_I} (ae - 1)$. ∎

We now use this result to improve the best known lower bound.

Lemma 56. *Let $P = \text{conv}\{x \in [0,1]^n : ex \leq d\} \cup \{x^*\}$ for $d \in [n]$ and $x^* = \frac{m-1}{m}e$ for $m \in \mathbb{N}_*$. Then $rk(P) \geq \ln\left(\frac{(m-1)\cdot n}{m\cdot d}\right) \cdot d$.*

Proof. It is easy to see that $P_I = \{x \in [0,1]^n : ex \leq d\}$ holds. We apply Lemma 55 with $Q_I = P_I$ to the inequality $ex \leq d$ and choose $k = 1$. As $\min_{a\in\{0,1\}^n : a \notin P_I} \sum_i a_i - 1 \geq d$. The result follows. □

The rank of P in Lemma 56, provided that m tends to ∞, is maximized by choosing d close to $n/\mathfrak{e}$. We obtain the following corollary.

Corollary 57. *For any $\epsilon > 0$ and any $n_0 \in \mathbb{N}_+$, there exists $n \geq n_0 \in \mathbb{N}_+$ and a polytope $P \subseteq [0,1]^n$ with $rk(P) \geq n/\mathfrak{e} - \epsilon$.*

We are now ready to slightly improve the lower bound result of [28].

Theorem 58. *For any $\epsilon > 0$ and any $n_0 \in \mathbb{N}_+$, there exists $n \geq n_0 \in \mathbb{N}$ and a polytope $P \subseteq [0,1]^n$ with $rk(P) \geq (1 + 1/\mathfrak{e})n - 1 - \epsilon$.*

Proof. Let Q be the polytope defined in Corollary 57 with $m = 2$. Define $P := \text{conv} Q \cup A_n$ and note that $P_I = Q_I$ as $(A_n)_I = \emptyset$ (and no 0/1 point in the cube can be expressed as a convex combination of other points from the cube). It is well-known that $\frac{1}{2}e \in A_n^{(n-1)}$ and thus $\frac{1}{2}e \in P^{(n-1)}$. We therefore obtain that $Q \subseteq P^{(n-1)}$ and by Corollary 57 we know that Q has rank of at least $\frac{n}{\mathfrak{e}} - \epsilon$. Together with $\text{rk}(Q) \leq \text{rk}(P^{(n-1)})$ we derive that the rank of P is at least $n - 1 + n/\mathfrak{e} - \epsilon = (1 + 1/\mathfrak{e})n - 1 - \epsilon$. □

8.3 An industrial cutting stock problem by Column Generation

This section is based on a joint work with Luciano Muller Nicoletti and Jean-Philippe Vial. The present real case study was initiated while Luciano was performing a traineeship in “Les Papeteries de Versoix”, a paper mill near Geneva. It appeared that production planning was performed manually every week, without resorting to any computerized tool. What is reported here is a feasibility study to investigate whether an automatized system could improve performance. The idea was to implement a cheap and robust system, based on the standard linear relaxation and some heuristics to reconstruct integer solutions from the relaxation (basically solving a integer program with the column generated for the optimal linear solution). We focus here on the modeling approach and the technique used to solve the linear relaxation of the problem. The reader is referred to [67] for a complete description.

We consider a family of K orders. Order i is characterized by three parameters : l_i, the width of the roll ; D_i, the external diameter of the roll ; and d_i, the number of rolls of that type. The rolls are obtained by cutting parent logs in rolls of appropriate (smaller) dimensions. A cutting pattern is defined by the number a_i of rolls of type i that is contained in the log. It is represented by the integer vector $a = (a_1, \ldots a_K)$. Note that the diameter $\Delta(a)$ associated to a pattern a is given by $\Delta(a) = \max_{i=1,\ldots,K} \{D_i \mid a_i \neq 0\}$.

The cutting stock problem is to find a set of appropriate cutting patterns of the logs and the number of logs to be slit according to those patterns, so as to meet the K orders in the family at minimum cost.

Let us first briefly describe the production process, which transforms pulp into a band of paper in order to understand costs better. Paper flows out of the machine in a continuous band of width $\bar{L}$. The band goes through a winder to form a log of internal diameter δ and external diameter Δ. The size of the log can be adjusted to L, with $\underline{L} \leq L \leq \bar{L}$, by cutting off a band of size $\bar{L} - L$. Since this scrap band does not go into the drying process, it is recycled directly as pulp. The next operation consists of cutting logs according to given cutting patterns and loss in this process are recycled as paper. Recycling paper is $\rho > 1$ times more expensive that recycling pulp.

A production plan may involve rolls in excess of the orders. In principle, these rolls are recycled as paper scrap. However, some of the rolls in excess may be stored to be delivered in a forthcoming week. This alternative is limited by a storage capacity and is subject to a storage cost.

The total cost of recycling $c(a)$ associated with a pattern a can be split into two components $c(a) = c_1(a) + c_2(a)$. The first component corresponds to pulp recycling and is proportional to the volume of the pulp

$$c_1(a) = (\bar{L} - L)(\Delta^2(a) - \delta^2), \quad \text{with } \underline{L} \leq L \leq \bar{L}$$

The other component is the cost of paper recycling. It is proportional to the volume of paper. It involves the cost ratio ρ between paper and pulp. If $l = \sum_{i=1}^{K} a_i l_i$ is the useful section of the log, the cost takes the form:

$$c_2(a) = \rho \sum_{i=1}^{K} a_i l_i \gamma_i(a) + \begin{cases} \rho l_c(\Delta^2(a) - \delta^2), & \underline{L} \leq l + l_c \leq \bar{L} \\ \rho(\underline{L} - l)(\Delta^2(a) - \delta^2), & l + l_c < \underline{L}. \end{cases}$$

In the above expression $\gamma_i(a)$ is the paper waste due to a log diameter $\Delta(a)$ larger than the required diameter D_i of order i. It takes the form

$$\gamma_i(a) = \begin{cases} \Delta^2(a) - D_i^2, & \text{if } \Delta(a) \geq D_i \\ M, & \text{otherwise,} \end{cases}$$

where M is a large enough number to prevent the use of an infeasible diameter.

Finally, we have two user-defined costs g^1 and g^2, associated with storage and excess production, respectively. Storage of rolls of type i is allowed within some range (perhaps null for certain orders). Excess production is lost. (We evaluate it at the scrap paper cost.)

Let I denote the set of all possible patterns. (I is potentially huge.) The cutting stock problem at Versoix factory takes the form:

$$\min\{\sum_{j=1}^{I} c(a^j)\lambda_j + \sum_{i=1}^{K} g_i^1 s_i^1 + \sum_{i=1}^{K} g_i^2 s_i^2 : A\lambda - s^1 - s^2 = d, s^1 \leq \overline{s}, \lambda, s^1, s^2 \in N^K\} \tag{8.2}$$

where A is the matrix of the cutting patterns vectors $a^1, \dots a^I$.

The problem (8.2) is a variant of the classical cutting stock problem [42]. Following [42], we consider instead the linear relaxation of the initial problem (8.2)

$$\min\{\sum_{j=1}^{I} c(a^j)\lambda_j + \sum_{i=1}^{K} g_i^1 s_i^1 + \sum_{i=1}^{K} g_i^2 s_i^2 : A\lambda - s^1 - s^2 = d, s^1 \leq \overline{s}, \lambda, s^1, s^2 \geq 0\} \tag{8.3}$$

which provides a lower bound on the optimal value. Moreover, if the solution of the relaxed problem is integral, this solution is optimal for the initial problem.

The set I is not given explicitly. Therefore, the relaxation problem (8.3) must be solved by a column generation scheme, or, dually, by a cutting plane algorithm. The dual of (8.3) can be formulated as follows:

$$\max\{\sum_{i=1}^{K}(d_i\pi_i + \overline{s}_i\sigma_i) : A^T\pi \le c, -\pi + \sigma \le g^1, \pi \ge -g^2, \, \sigma \le 0\} \tag{8.4}$$

The feasiblity test boils down to solving the knapsack-type problem

$$\max\left\{\sum_{i=1}^{K} a_i\pi_i - c(a) \mid \sum_{i=1}^{K} a_i l_i \le \bar{L} - l_c, \, a \in N^K\right\}, \tag{8.5}$$

where the parameter l_c corresponds to the waste on the extremities. The problem differs from the standard formulation in that the cost $c(a)$ depends on the cutting pattern. This dependence is not trivial, but, nicely enough, it can be broken down into a few simple cases. First, the cost depends on the diameter of the log. Since the number of diameters is finite (and small), it suffices to solve (8.5) for each diameter instance. The second complication stems from the waste cost that accrues when the roll length is smaller than $\underline{L} - l_c$. We are thus lead to consider the two cases:

$$\max\left\{\sum_{i=1}^{K} a_i\pi_i - c(a) \mid \underline{L} - l_c \le \sum_{i=1}^{K} a_i l_i \le \bar{L} - l_c, \, a \in N^K\right\}, \tag{8.6}$$

and

$$\max\left\{\sum_{i=1}^{K} a_i\pi_i - c(a) \mid \sum_{i=1}^{K} a_i l_i < \underline{L} - l_c, \, a \in N^K\right\}. \tag{8.7}$$

Note that problem (8.6) has two constraints and is not any more a simple knapsack problem. We could use however fast dynamic programs (given the instances sizes) to solve those two problems. The cutting plane algorithm was implemented using the analytic center cutting plane method (ACCPM). This implementation lead to a decrease of the operational costs by 4% at the paper mill. Given that the paper industry is capital intensive, this reduction is an important figure.

Chapter 9

Approximation algorithms

9.1 OWMRCD : Performance analysis through convex optimization

We are interested here in a variant of the one warehouse multi-retailer problem presented in Section 3.3 , where each installation faces constant rate demand. More precisely, we consider the inventory control of one single product in a distribution network made of one warehouse and n retailers. The warehouse, referred to as stage 0, supplies the n different retailers, referred to as stage $1, ..., n$. Those retailers face continuous and deterministic demands arriving with constant rate $d_i \geq 0$ for $i = 1, ..., n$. Each retailer i orders from the warehouse and is charged a fixed cost K_i per order while the warehouse, in turn, orders from a supplier (with infinite stock) and is charged a fixed cost K_0 per order. Each location pays a certain holding cost h'_i per unit stored per unit of time where we assume $h'_0 > 0$ and $h'_i \geq h'_0$ for all $i = 1, ..., n$. We denote by h_i the *echelon holding cost* at stage i i.e. $h_0 = h'_0 > 0$ while $h_i = h'_i - h'_0 \geq 0$ for all $i = 1, .., n$. We assume also that all the demand must be met and that the lead times is zero (or deterministic which is equivalent in this setting). We are interested in minimizing the average ordering and inventory costs (over an infinite horizon) subject to the constraint that no shortage is allowed. We call this problem one warehouse multi-retailer with constant demand (OWMRCD).

We would like to evaluate the performance of a solution where each location behaves selfishly and tries to minimize its own cost independently of the others. We first recall how to compute the optimal solution for a single location.

Consider the inventory management of a single facility facing a continuous and deterministic demand arriving at constant rate d. We assume that the facility pays a fixed ordering cost K for each order made and it pays a holding cost h per unit held per unit of time. The lead time for delivery is supposed to be instantaneous (or fixed, this is again equivalent in this setting) and no shortage is allowed. Our goal is to minimize the average inventory cost over an infinite horizon (this cost is made of two conflicting costs: holding costs and ordering costs). It is quite intuitive (see [11] for a rigorous proof) that the optimal strategy for this problem is made of identical ordering cycles i.e. order every T a fixed quantity Q. The profile of the inventory over time will therefore follow the classical sawtooth shape. The optimal strategy consists thus in choosing the cycle length as to minimize the average per unit cost. Let T be the ordering cycle. The average inventory over T is $\frac{T.d}{2}$. Defining $g = \frac{1}{2}hd$, the average cost of ordering and holding inventory is $c(T) = \frac{K}{T} + g \cdot T$. Clearly $c(T)$ is continously differentiable and strictly convex over $T > 0$; moreover $\lim_{T\to 0} c(T) = +\infty$ and $\lim_{T\to +\infty} c(T) = +\infty$ thus it is has a unique minimum σ satisfying $c'(\sigma) = 0$ i.e. for $\sigma = \sqrt{\frac{K}{g}}$. The optimal ordering quantity Q^* corresponding to σ is called the Economical Ordering Quantity (EOQ) and has value $Q^* = d\sqrt{\frac{K}{g}}$.

Let us denote by σ_i, $i = 0, ..., n$ the single-echelon optimal reorder cycle i.e. $\sigma_i = \sqrt{\frac{K_i}{g_i+g^i}}$ for all $i = 1, ..., n$ and $\sigma_0 = \sqrt{\frac{K_0}{\sum_{i=1}^n g^i}}$ A selfish retailer will thus follow its optimal EOQ strategy and will order at time $T \in I_i = \{k\sigma_i, k \in \mathbb{N}\}$ a quantity $Q_i = d_i\sigma_i$. A sellfish warehouse will also follow its optimal reorder period given by the EOQ formula but in order to manage the non constant rate demand it observes, we assume that the warehouse orders only what is needed for each new cycle i.e. at time $T \in I_0 = \{k\sigma_0, k \in \mathbb{N}\}$ it orders a quantity $Q_0(T)$ that matches the demand from the retailers at time T. More precisely if retailer i orders k_i times between T and the next warehouse order (i.e. in interval $[T, T+\sigma_0)$), then the warehouse orders $k_i \cdot Q_i$ for retailer i in period T. Summing over all retailers, $Q_0(T) := \sum_{i\in\{1,...,n\}} Q_i \cdot |I_i \cap [T, T+\sigma_0)|$. We call this policy the *selfish policy.*

Theorem 59. *The selfish policy is $\sqrt{2}$-optimal for OWMRCD.*

Proof. Simple calculation (see [90]) shows that the cost of this policy can be bounded above by

$$2\sqrt{K_0}\sqrt{\sum_{i=1}^{n} g^i} + \sum_{i\geq 1} 2\sqrt{K_i}\sqrt{g_i + g^i} \tag{9.1}$$

In his seminal paper, Roundy [85] gave a lower bound on the optimal solution which reads

$$2\sqrt{K_0 + \sum_{i\in E^*} K_i}\sqrt{\sum_{i\in E^*}(g_i + g^i) + \sum_{i\in L^*} g^i} + 2\sum_{i\in L^*}\sqrt{K_i}\sqrt{g_i} + 2\sum_{i\in G^*}\sqrt{K_i}\sqrt{g_i + g^i} \tag{9.2}$$

for some partition E^*, L^*, G^* of the set $\{1, ..., N\}$. In order to evaluate the quality of the selfish policy, we would like to find a bound (if it exists) on the ratio between (9.1) and (9.2) over all possible choices of K_i, $g^i > 0$, $g_i \geq 0$. We omit the detail here but this ratio can easily proved to be bounded and after some simplifications, it reads

$$\frac{\sqrt{K_0}\sqrt{\sum_{i\geq 1} g^i} + \sum_{i\geq 1}\sqrt{K_i}\sqrt{g^i}}{\sqrt{K_0 + \sum_{i\geq 1} K_i}\sqrt{\sum_{i\geq 1} g^i}}$$

Now fixing all values but K_i and fixing $\sum_{i\geq 1} K_i = \mu > 0$, we can bound the ratio by choosing K_i as to maximize $f(\mathbf{K}) := \sum_{i\geq 1}\sqrt{g^i}\sqrt{K_i}$ over the constraint $h(\mathbf{K}) := \sum_{i\geq 1} K_i - \mu = 0$ (the denominator of the ratio being constant when $\sum_{i\geq 1} K_i$ is constant). This function is strictly concave and continuously differentiable over $(\mathbb{R}_+^*)^n$ and the feasibility region is convex. One can easily check that $K_i^* = g^i \frac{\mu}{\sum_{i\geq 1} g^i}$ for all i and $\lambda = \frac{1}{2}\sqrt{\frac{\sum_{i\geq 1} g^i}{\mu}}$ is a solution to $\lambda \geq 0, \nabla f(\mathbf{K}^*) = \lambda\nabla h(\mathbf{K}^*)$ and $h(\mathbf{K}^*) = 0$ (note that since $g^i > 0$ for all i, $\sum_{i\geq 1} g^i > 0$) . Therefore (see [57]) $\mathbf{K}^*$ is a local maximizer of $f(\mathbf{K})$. But by concavity of the function and convexity of the domain, this local maximizer is also a global maximizer (see again [57]).

It follows that the ratio is bounded by :

$$\frac{\sqrt{K_0}\sqrt{\sum_{i\geq 1} g^i} + \sqrt{\sum_{i\geq 1} K_i}\sqrt{\sum_{i\geq 1} g^i}}{\sqrt{K_0 + \sum_{i\geq 1} K_i}\sqrt{\sum_{i\geq 1} g^i}} = \frac{\sqrt{K_0} + \sqrt{\sum_{i\geq 1} K_i}}{\sqrt{K_0 + \sum_{i\geq 1} K_i}}$$

Maximizing this ratio is equivalent to maximizing $\sqrt{x_1} + \sqrt{x_2}$ over $\{x_1, x_2 > 0 : x_1 + x_2 = 1\}$ and thus is bounded by the maximum of $\sqrt{x_1} + \sqrt{1 - x_1}$ over $x_1 \in [0, 1]$, which has value $\sqrt{2}$. ☐

Using similar techniques, we could prove that a slightly different *selfish* policy obtained in a same way but where we enforce the different location to round their reorder interval to a power-of-two multiple of a common period is 1.27-optimal, see [90] for details.

9.2 OWMR : a 2-approximation through extended formulation

We consider the same setting as in Section 3.3. The fixed charge network flow view of the OWMR problem leads to the following simple integer linear program (P) (here we use the fact the network is uncapacitated, hence a demand d_{it} can be assume to be served from a unique pair of orders $\lceil r, s \rfloor$ with $r \leq s \leq t$, i.e. ordered in period r from the supplier and in period s from the warehouse).

$$\begin{array}{lll}
\min & \sum_{r=1}^{T} y_r^0 K_r^0 + \sum_{i=1}^{N}\sum_{s=1}^{T} y_s^i K^i + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{r,s:r\leq s\leq t} x_{rs}^{it} h_{rs}^{it} d_{it} & \\
\text{s.t.} & \sum_{r,s:r\leq s\leq t} x_{rs}^{it} = 1, & i=1,\dots,N, t=1,\dots,T, d_{it}>0 \ (1) \\
& \sum_{r:r\leq s} x_{rs}^{it} \leq y_s^i, & i=1,\dots,N, t=1,\dots,T, s=1,\dots,t \ (2) \\
& \sum_{s:r\leq s\leq t} x_{rs}^{it} \leq y_r^0, & i=1,\dots,N, t=1,\dots,T, r=1,\dots,t \ (3) \\
& x_{rs}^{it}, y_r^0, y_s^i \in \{0,1\}, & i=1,\dots,N, s=1,\dots,T, r=1,\dots,s, t=s,\dots,T
\end{array}$$

For each demand point v_t^i the variable $x_{rs}^{it} \in \{0,1\}$ indicates if demand d_{it} is ordered with respect to the pair $\lceil r, s \rfloor$. The variable y_s^i indicates if retailer i orders in period s and similarly y_r^0 indicates if the warehouse orders in period r. The first constraint in the formulation ensures that each demand is served by a certain pair of orders while the two other constraints ensure that we can serve a demand with a pair of orders $\lceil r, s \rfloor$ only if we order in those periods. h_{rs}^{it} represents the cost of holding a unit of product when ordered through the pair $\lceil r, s \rfloor$ i.e. $h_{rs}^{it} = (s-r)\cdot h^0 + (t-s)\cdot h^i$

We would like to find a 'good' lower bound on the optimal solution that can be computed fast, interpreted easily and converted to a feasible solution without too much effort (the LP relaxation is not satisfactory from this perspective, see [58]). Observe that relaxing constraint (2) or (3) would yield weak lower bounds. We therefore make use of extended formulations to derive a stronger integral lower bound. Let z_r^{it} be the boolean variable indicating if demand d_{it} was ordered from the warehouse at period r i.e. $z_r^{it} = \sum_{s:r\leq s\leq t} x_{rs}^{it}$. In a symetric way, let us also introduce $u_s^{it} = \sum_{r:r\leq s} x_{rs}^{it}$ the boolean variable indicating if demand d_{it} is ordered at the retailer in period s. In addition we define c_r^{it} to be equal to h_{rr}^{it} if $h^i \leq h^0$ and to h_{rt}^{it} if $h^i \geq h^0$. This quantity represents the minimal holding cost to serve demand d_{it} at the retailer with a warehouse order in period r.

Observe that $c_r^{it} z_r^{it} \leq \sum_{s:r\leq s\leq t} x_{rs}^{it} h_{rs}^{it}$ and also $h_{ss}^{it} u_s^{it} \leq \sum_{r:1\leq r\leq s} x_{rs}^{it} h_{rs}^{it}$. Hence we can write for each demand point (i,t) that $\sum_{r,s:r\leq s\leq t} x_{rs}^{it} h_{rs}^{it} \geq \frac{1}{2}\sum_{r\leq t} c_r^{it} z_r^{it} + \frac{1}{2}\sum_{s\leq t} h_{ss}^{it} u_s^{it}$. The optimal solution to the following linear integer program (RP) is thus a lower bound on the optimal solution to the previous one (observe that constraints (1') and (1') are redundant).

$$
\begin{array}{lll}
\min & \sum_{r=1}^{T} y_r^0 K_r^0 + \sum_{i=1}^{N}\sum_{s=1}^{T} y_s^i K^i + \sum_{i=1}^{N}\sum_{t=1}^{T}\left(\frac{1}{2}\sum_{r=1}^{t} c_r^{it} d_{it} z_r^{it} + \frac{1}{2}\sum_{s=1}^{t} h_{ss}^{it} d_{it} u_s^{it}\right) & \\
\text{s.t.} & \sum_{r,s:r\le s\le t} x_{rs}^{it} = 1, & i=1,\ldots,N, t=1,\ldots,T, d_{it}>0 \\
& \sum_{r:r\le t} z_r^{it} = 1, & i=1,\ldots,N, t=1,\ldots,T, d_{it}>0 \ (1) \\
& \sum_{s:s\le t} u_s^{it} = 1, & i=1,\ldots,N, t=1,\ldots,T, d_{it}>0 \ (1') \\
& u_s^{it} \le y_s^i, & i=1,\ldots,N, t=1,\ldots,T, s=1,\ldots,t \\
& z_r^{it} \le y_r^0,) & i=1,\ldots,N, t=1,\ldots,T, r=1,\ldots,t \\
& z_r^{it} = \sum_{s:r\le s\le t} x_{rs}^{it} & i=1,\ldots,N, t=1,\ldots,T, r=1,\ldots,t \ (2) \\
& u_s^{it} = \sum_{r:1\le r\le s} x_{rs}^{it} & i=1,\ldots,N, t=1,\ldots,T, s=1,\ldots,t \ (2') \\
& z_r^{it}, u_s^{it}, x_{rs}^{it}, y_r^0, y_s^i \in \{0,1\}, & i=1,\ldots,N, s=1,\ldots,T, r=1,\ldots,s, t=s,\ldots,T
\end{array}
$$

Now if we relax constraints (2) and (2'), problem (RP) becomes decomposable. Formulation (RP) therefore reduces to solving $N+1$ independent problems : one for the warehouse (RP_0)

$$
\begin{array}{lll}
\min & \sum_{r=1}^{T} y_r^0 K_r^0 + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{r=1}^{t} \frac{1}{2} c_r^{it} d_{it} z_r^{it} & \\
\text{s.t.} & \sum_{r:r\le t} z_r^{it} = 1, & i=1,\ldots,N, t=1,\ldots,T, d_{it}>0 \\
& z_r^{it} \le y_r^0, & i=1,\ldots,N, t=1,\ldots,T, r=1,\ldots,t \\
& z_r^{it}, y_r^0 \in \{0,1\}, & i=1,\ldots,N, r=1,\ldots,t, t=1,\ldots,T
\end{array}
$$

and one for each $i=1,\ldots,N$, (RP_i)

$$
\begin{array}{lll}
\min & \sum_{s=1}^{T} y_s^i K^i + \sum_{t=1}^{T}\sum_{s:s\le t} \frac{1}{2} h_{ss}^{it} d_{it} u_s^{it} & \\
\text{s.t.} & \sum_{s:s\le t} u_s^{it} = 1, & t=1,\ldots,T, d_{it}>0 \\
& u_s^{it} \le y_s^i, & t=1,\ldots,T, s=1,\ldots,t \\
& u_s^{it}, y_s^i \in \{0,1\}, & s=1,\ldots,T, t=s,\ldots,T
\end{array}
$$

Those problems are single echelon inventory control problems (with a modified demand $\overline{d}_{it} = \frac{1}{2}d_{it}$ at each retailer i and a demand $\overline{d}_t = \sum_{i=1}^{N}\frac{1}{2}d_{it}$ at the warehouse).

We now use this lower bound and the optimal solutions of the corresponding single echelon problems (see Section 2.1 to solve those problems) to build a feasible solution to the original problem. We will show then that the cost of the new solution is at most twice the sum of cost of those independent solutions, proving as a consequence a 2-approximation algorithm.

Let $z_r^{it}, y_r^0, u_s^{it}, y_s^i$ be the optimal solutions to the $N+1$ problems above. Let us set $x_{ss}^{it} = u_s^{it}$. If we consider the solution $y_r^0, x_{ss}^{it}, y_s^i$ this is not necessarily feasible in (P).

We define for all demand point d_{it}, the virtual holding cost H_{it} as follows $H_{it} := \frac{1}{2}h_{rt}^{it}d_{it} + \frac{1}{2}h_{ss}^{it}d_{it}$ for $s : x_{ss}^{it} = 1$ and $r : z_r^{it} = 1$ if $h^i \ge h^0$ and $H_{it} := \frac{1}{2}h_{rr}^{it}d_{it} + \frac{1}{2}h_{ss}^{it}d_{it}$ for $s : x_{ss}^{it} = 1$ and $r : z_r^{it} = 1$ if $h^i \le h^0$. We define the total virtual holding cost H as $H := \sum_{i=1}^{N}\sum_{t=1}^{T} H_{it}$. We define now O_i, the virtual ordering cost or retailer i as $O_i := \sum_{s=1}^{T} y_s^i K^i$ and similarly $O_0 := \sum_{r=1}^{T} y_r^0 K_r^0$. The total cost of our current solution in (RP) is $H + O_0 + \sum_i O_i$. We are now going to build a feasible solution for (P) with total cost less or equal to $2.(H + O_0 + \sum_i O_i)$.

In the current solution, we say that an order in period s at retailer i (i.e. $y^i_s = 1$) *crosses* an order at the warehouse in period $r > s$ (i.e. $y^0_r = 1$) if $y^0_{r'} = 0$ for all $r' = s, ..., r-1$ and $y^i_{s'} = 0$ for all $s' = s, ..., r$. In this case we set $y^i_r = 1$ and for all $t \geq r : x^{it}_{ss} = 1$, we set $x^{it}_{ss} = 0$, $x^{it}_{rr} = 1$. The uncrossing operation does not add holding cost (since $h^{it}_{ss} \geq h^{it}_{rr}$ for $r \geq s$) but potentially doubles the number of orders at the retailers. We assume from now on that the solution is non crossing.

For all i and all s such that $y^i_s = 1$, we let $r(s) := \max\{r : 1 \leq r \leq s, y^0_r = 1\}$ ($r(s)$ exists because $y^i_s = 1$ implies that $d_{is} \neq 0$ and thus $z^{is}_r = 1$ for some $1 \leq r \leq s$). Observe that unless $r(s) = s$, there does not exist $t \geq r' \geq s$ such that $y^0_r = 1$ (otherwise we have a crossing order). It follows that unless $r(s) = s$, $H_{it} := \frac{1}{2}h^{it}_{r(s)t}d_{it} + \frac{1}{2}h^{it}_{ss}d_{it}$ for $s : x^{it}_{ss} = 1$ if $h^i \geq h^0$ and $H_{it} := \frac{1}{2}h^{it}_{r(s)r(s)}d_{it} + \frac{1}{2}h^{it}_{ss}d_{it}$ for $s : x^{it}_{ss} = 1$ if $h^i \leq h^0$.

We modify our solution as follows. For all i and s such that $y^i_s = 1$, if $r(s) \neq s$, we do:

- $y^i_s = 0$, $y^i_{r(s)} = 1$, $x^{it}_{ss} = 0$ and $x^{it}_{r(s)r(s)} = 1$ if $h^i \leq h^0$
- $x^{it}_{ss} = 0$ and $x^{it}_{r(s)s} = 1$ if $h^i \geq h^0$

This operation defines a feasible solution to the original problem. It has no effect on the ordering cost but the holding cost is different from the virtual holding cost. The holding cost associated with demand d_{it} is

- $h^{it}_{ss}d_{it}$ if $r(s) = s$ and thus it is less or equal to $2H_{it}$.
- $h^{it}_{r(s)s}d_{it}$ if $r(s) \neq s$ and $h^i \geq h^0$, which is less or equal to $(h^{it}_{r(s)t} + h^{it}_{ss})d_{it} = 2.(\frac{1}{2}h^{it}_{r(s)t}d_{it} + \frac{1}{2}h^{it}_{ss}d_{it}) = 2H_{it}$ (recall $h^{it}_{r(s)s} \leq h^{it}_{r(s)t} + h^{it}_{ss}$).
- $h^{it}_{r(s)r(s)}d_{it}$ if $r(s) \neq s$ and $h^i \leq h^0$, which is less or equal to $2H_{it}$

It follows that the new solution has total holding cost in (P) less or equal to twice the total virtual holding cost. Moreover it has at most twice as much orders as the original solution at each retailer. It has thus a total cost in (P) at most twice the cost of the solution to (RP) which is a lower bound on the optimal cost. We have thus built a feasible solution whose cost is less or equal to twice the cost of the optimal solution. Because each single-echelon problem can be solved in time $O(T)$ (see Section 2.1), we have the following result.

Theorem 60. *The discussion above yields a 2-approximation for the OWMR problem with time varying demand and can be implemented to run in time $O(NT)$.*

This result can be extended to more general cost holding cost structures, see [95]

Chapter 10

Conclusion and perspectives

In this document, we presented the various tools/techniques that were at the core of our major results and explained along the way how we combined them into solutions to complex problems for stable set in claw-free graphs, inventory control and other fundamental and/or practical questions. We would like to conclude now by reviewing in a synthetic way our main problem-specific contributions and by pointing the reader to the relevant papers.

Stable sets in claw-free graphs

Our work on the stable set problem in claw-free graphs constitutes our masterpiece. Our first main contribution to this problem was our proof of the Ben Rebea conjecture, a 25 year old conjecture on the characterization of the stable set polytope of quasi-line graphs [26, 27]. Later, we proposed a new algorithm to find a maximum weighted stable set in a claw-free graph [71] that we subsequently enhanced [32, 31] and whose complexity is now drastically better than the original algorithm by Minty (n^3 versus n^6, where n is the number of vertices). Finally our latter contributions on the topics answered in two different ways the polyhedral question left open since 1980 : we provided a description of the polyhedra in an extended space and a *combinatorial procedure* to separate over the polytope in polynomial-time [29, 30]. Beside those main contributions, we published other important papers on the problem and related questions [71, 72, 69, 73, 60, 70, 94, 93, 91]. We believe that altogether those contributions have laid the fundation for a nice unified theory (in terms of polytope and algorithm) for stable sets in claw-free graphs (see our survey papers [33, 74] for a detailed discussion and the remaining open questions). Many core results developed in this context are promising beyond stable set in claw-free graphs.

- The results from Section 3.1 and 3.2 about set packing was developed in the context of stable sets in circular interval graphs but clearly those results are more general and could even be extended further. They can be generalized to settings where the matrix A has only two consecutive coefficients per row (i.e. not necessarily 0/1) and where the right hand side is of the form $\lambda\mathbf{1}$ for some $\lambda \in \mathbb{N}$. Also it is pretty straightforward to extend those results to the set covering problem.

- The composition of strips is an important graph operation that might play an major role for other classes of graphs (actually Chudnovsky and Seymour discovered the structure of claw-free graphs while studying perfect graphs). In Section 4.1, we showed that the stable set problem in composition of strips (non necessarily claw-free) is polynomiallly solvable if it is for each strip. Section 7.3 shows similar results for the stable set polytope. Both results will certainly find application in other stable set problems.

- The notion of combinatorial union of polytopes introduced in Section 7.2 is a simple yet powerful concept as it generalizes Minkowski sum and union of polytopes. While it has natural applications for stable sets, we are convinced that this concept has much more potential. Actually, we became recently aware that some people in biostatistics [76] study polytope algebra $(\mathcal{P}_d, \oplus, \odot)$ defined by those two operations ($\mathcal{P}_d$ is the set of all polytopes in $\mathbb{R}^d$, $\oplus$ is the convex hull of the union of polytopes and $\odot$ is the Minkowski sum). Thus the combinatorial union of polytopes might have applications far beyond what we can imagine.

Inventory Control Problems

More recently, we studied the OWMR problem and other important inventory control problems with techniques from polyhedral combinatorics and we could show very promising results [90, 39, 38]. Our main contribution is a simple and fast 2-approximation algorithm for the OWMR problem [95]. This result is both important in practice and in theory as it allows to approximate large real-world instances of the problem (we implemented this algorithm at IBM) and the techniques we developed appear to open interesting new research directions. Indeed we used a simple combination of extended formulations and relaxation to derive strong lower bounds on the OWMR problem and we built on top of this to design our approximation result (see Section 9.2). This suggests that extended formulations could find interesting applications in the design of approximation algorithms. This research direction is pretty promising as the field of extended formulation is actively developing those days. Note that to the best of our knowledge, this result is the first of this kind.

Other fundamental contributions

Beside those main themes of research, we have been contributing fundamental results to other areas of discrete mathematics [64, 82, 92]. In particular, we regard our alternative proof of the characterization of the p-median polytope in Y-free graphs [92] as a significant contribution. Here using very standard but powerful polyhedral results, we could give an alternative non technical and simple proof of a result by Baiou and Barahona [5], unifying results for p-median with the classic matching theory. The strong connection that we discovered between matching and p-median in Y-free graphs and the role of the intersection property have application in more general p-median problems as later explored by Baiou, Barahona and Correa [6].

Also our result with Sebastian Pokutta [82] on the Chvátal-Gomory rank of 0/1 polytopes is a notable contribution : it contributes to a better understanding of a well known paradigm for general 0/1 polytopes. We revisited an existing framework by Chvátal, Cook and Hartmann [21] to prove lower bounds on the CG-rank but we made it more accessible. It allowed us to give a very simple construction and improve the lower bound on the rank of general 0/1 polytopes (the previous lower bound relied on an existence theorem by Erdös).

Pratical Applications

Finally, we also worked on real-world applications such as cutting stock problems [67], workforce scheduling [81], corporate portfolio management, pharmaceutical supply chain design and robust aircraft routing [62]. Our 'tutorial' paper with Marla, Pratsini and Rikun [62] reports on our practical experience with the application of various types of robust optimization techniques (including stochastic programming and the framework developed by Bertsimas and Sim) for different optimization problems mentioned above. We believe it will be a valuable document for practitionners.

Perspectives

We would like to close now with a non exhaustive list of problems that we find particularly interesting and that we want to investigate in the future (and that we have sometimes already started to investigate).

Alternative matching algorithms

A long-standing open question in polyhedral combinatorics is that of finding (or prove that it does not exist) a compact extended formulation for the matching problem. We believe that alternative matching algorithms might help to address this question (in the case it is positive). Thus we would like to try to exploit our insight from the stable set problem in claw-free graphs to develop new algorithms for matching. In particular we would like to exploit further Calvillo's result, see Theorem 11. This question is also interesting on its own as matchings still play a very fundamental role in combinatorial optimization.

Extended formulations for non linear systems

Non linear integer programs has attracted a lot of attention lately and software like BONMIN or COUENNE are using polyhedral/convex characterizations of the convex hull associated with some non linear systems to derive efficient algorithms. In this context, we are currently working on extended formulations for so-called multi-linear sets, i.e. sets of the form $\{x \in \mathbb{R}^{n+1} : x_{n+1} = \prod_{i=1}^{n} x_i, l_i \leq x_i \leq u_i, i = 1, ..., n\}$. We already have encouraging results in this direction.

Using extended formulation for approximation algorithms

Primal-dual algorithms have proven very useful in the design of 'good' approximation algorithms. However sometimes, primal-dual algorithms are rather intricate. We would like to see if extended formulations can help simplify/improve some of those algorithms and help design new ones. For instance, Levi et al. [58] gave a sophisticated primal-dual 2-approximation for the OWMR problem. Our recent result [95] shows that using extended formulations, a much simpler 2-approximation can be designed (yielding also similar strong result about the integrality gap of certain relaxations). We would like to exploit this further and develop other approximation algorithms using extended formulations. It would be interesting to start with revisiting standard primal-dual algorithm via extended formulations.

Dominating sets in claw-free graphs

Minimum cardinality dominating set in claw-free graphs was recently proven to be fixed parameter tractable (FPT) [49]. We believe that our new decomposition result for claw-free graphs (see Theorem 26) can be used to give a shorter proof of this result. We would like to see if we can extend this result to the weighted case. Also Fujito, Nagamoshi [34] gave a nice 2-approximation result of the edge-dominating set (or equivalently dominating set in line graphs). We would like to see if we can extend this result to dominating sets in quasi-line graphs or claw-free graphs.

Combinatorial union of polytopes

The combinatorial union of polytopes seems to be an interesting concept for integer programming as it generalizes Minkowski sum and union of polytopes. We would like to explore generic cuts that can be derived from this framework and possibly derive new cuts for important problems like for instance TSP. Those cuts will in particular extend disjunctive cuts. We also would like to explore also the links mentioned earlier with polytope algebra.

Integer decomposition property

Packing polytopes defined by circular one matrices (see Section 3.2) have the so-called *integer decomposition property* [40] ($P \subseteq \mathbb{R}^n$ has the integer decomposition property if for all $k \in \mathbb{N}_*$, $x \in k \cdot P \cap \mathbb{Z}^n \implies \exists x_1, ..., x_k \in P \cap \mathbb{Z}^n : x = \sum_{i=1}^{k} x_i$). Recently Sebó proved that projection of polyhedra defined by TU matrices also have the integer decomposition property. We are working on a possible common generalization of those results.

Generalizations of our approximation algorithms for OWMR

We are working on possible extensions of the technique developed for the deterministic OWMR to more general networks. We could already prove that the natural extension of the OWMR problem to a purely distributive systems with k layers can be approximated within a factor of k. Can we also generalize to other types of network structures ? Also as observed in Section 3.3, the OWMR problem is a special case of fixed charge network flow problem. We would like to extend the technique to more general fixed charge network flow problems. Similarly, the problem can be seen as a special case of facility location. Can we use a similar approach to build efficient approximation algorithm for hierarchical facility location problems ?

Approximation algorithms for stochastic inventory control problems

In practice, inventory control problems are subject to stochastic demands. We would like to develop approximation algorithms for stochastic OWMR problems. Gupta, Krishnaswamy, Molinaro and Ravi [47] recently gave approximation algorithms for stochastic knapsack problems that exploit linear programming relaxations of the problems. We would like to develop similar results for stochastic inventory control problems.

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