Some implications of local weak convergence for sparse random graphs

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QUELQUES CONSÉQUENCES DE LA CONVERGENCE LOCALE FAIBLE POUR LES GRAPHES ALÉATOIRES

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À mes frères, à mes parents

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Résumé en français

Dans la limite "diluée" où les nombres d'arêtes et de sommets divergent de manière comparable, on s'attend à ce que le comportement asymptotique de nombreux invariants de graphes ne dépende que de statistiques purement locales. Cette heuristique provient de l'étude thermodynamique de certains systèmes désordonnés en physique statistique, où la contribution microscopique de chaque particule est insensible aux perturbations lointaines. Mathématiquement, une telle absence d'interactions à longue portée se traduit par la continuité de l'invariant vis-à-vis de la topologie de la convergence locale faible. En particulier, l'invariant admettra une limite déterministe le long de la plupart des suites de graphes aléatoires classiques, et pourra être efficacement approximé par des algorithmes locaux et distribués, indépendamment de la taille totale du système.

Dans cette thèse, nous étudions quatre invariants de graphes qui jouent un rôle essentiel en théorie comme dans les applications : la distribution spectrale empirique, la dimension du noyau de la matrice d'adjacence, la taille d'un couplage maximum, et le polynôme énumérant certaines familles de sous-graphes couvrants. Nous montrons qu'il existe une unique manière localement cohérente d'étendre chacune de ces notions aux limites locales faibles de graphes finis, et que ce prolongement est continu. Pour les modèles de graphes aléatoires classiques, les équations de cohérence locale se simplifient en une équation aux distributions que nous résolvons explicitement. Cela conduit à de nouvelles formules asymptotiques, ainsi qu'à la simplification, l'unification et la généralisation de divers résultats jusqu'alors isolés.

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Mots-clés. Graphes aléatoires, convergence locale faible, méthode de la cavité, couplage maximum, distribution spectrale, invariants de graphes

Abstract

In the so-called sparse regime where the numbers of edges and vertices tend to infinity in a comparable way, the asymptotic behavior of many graph invariants is expected to depend only upon local statistics. This heuristic originates from the thermodynamic study of certain disordered systems in statistical physics, where the microscopic contribution of each particle is insensitive to remote perturbations of the system. Mathematically, such a lack of long-range interactions can be formalized into a continuity statement with respect to the topology of local weak convergence of graphs. Among other consequences, continuous invariants are guaranteed to admit a deterministic limit along most of the classical sequences of sparse random graphs, and to be efficiently approximable via local distributed algorithms, regardless of the size of the global structure.

In this thesis, we focus on four graph invariants that play an important role in theory and applications : the empirical spectral distribution, the kernel dimension of the adjacency matrix, the matching number, and the generating polynomial of certain classes of spanning subgraphs. Each of these notions is shown to admit a unique locally self-consistent extension to local weak limits of finite graphs, and this extension is proven to be continuous. When specialized to the classical models of sparse random graphs, the limiting system of local self-consistency equations simplifies into a single distributional equation, which we solve explicitly. This leads to new asymptotic formulae and to the simplification, unification and generalization of various results that were previously relying on model-specific arguments.

2010 Mathematics Subject Classification. Primary 05C80, 05C30, 60K35. Secondary 60B20, 82B23, 82B44.

Key words and phrases. Sparse random graphs, local weak convergence, cavity method, matching number, empirical spectral distribution, subgraph enumeration

Contents

Chapter 1

Introduction

The broad theme of this thesis is the asymptotic study of algebraic, combinatorial or probabilistic parameters associated with graphs as the numbers of edges and vertices tend to infinity in a comparable way. In many situations, it is believed that such asymptotics should depend only upon the limiting local geometry of the graphs, with the following important implications:

1. practically, the parameter of interest can be efficiently approximated via distributed, local message-passing algorithms ;

2. theoretically, a unique self-consistent infinite limit can be defined (and sometimes even explicitly determined) by purely local specifications.

Originating from the study of disordered systems in statistical physics, this general heuristic has given rise to a powerful non-rigorous computational formalism known as the cavity method [79, 77, 78]. Although the exact conditions for its validity are still far from being completely understood, it has already inspired many remarkably efficient approximation algorithms (see e.g. [37, 89, 73]), and some of its most fascinating analytical predictions have recently received a mathematical confirmation. The present thesis aims to contribute to the program of making this ansatz rigorous in a broad setting using the modern framework of local weak convergence. Before describing our results, we start with a brief recall of local weak convergence. Since there are good surveys and books on the subject (among which [8, 7, 47]), we have opted here for a concise presentation.

1.1 The framework of local convergence

1.1.1 Context

In the same way that random walks can be rescaled to converge to the Brownian motion, implying the convergence of many related functionals, classical sequences of random graphs happen to converge in a natural, local sense to a discrete limiting random graph, hence guaranteeing the convergence of any graph parameter that can be shown to be continuous with respect to this local topology. Furthermore, in fortunate circumstances, the structure of the limiting random graph is simple

enough to allow for an explicit determination of the graph parameter. This modern approach was introduced by Benjamini and Schramm [21] in the context of random walks on planar graphs. It was then developed further in *the objective method* by Aldous and Steele [8], and in a more recent survey by Aldous and Lyons [7] ; it now provides a powerful framework for the unified study of sparse random graphs and has already led to several new asymptotic results. Two prototypical examples are the celebrated $\zeta(2)$ limit in the random assignment problem due to Aldous [5], and the asymptotic enumeration of spanning trees in large graphs by Lyons [71]. Since then, other beautiful applications have been found in various contexts, including (but not limited to) planar triangulations [9, 10], planar quadrangulations [39], property testing [92, 22], combinatorial optimization [8, 95, 56, 55], subgraph enumeration [19], spectral graph theory [32], and spin glass models [14, 48, 84].

1.1.2 Graph-theoretical terminology

A (simple unoriented) **graph** is an ordered pair $G = (V, E)$ comprising a countable set V of vertices and a countable set E of edges. Each edge $e = \{i, j\}$ is an unordered pair of distinct vertices i and j, which are declared to be adjacent, or neighbors. The set of neighbors of i is denoted by ∂i, and its cardinality is called the degree of i. When the degree of every vertex is finite, we say that G is **locally finite**. When the set V itself is finite, we say that G is finite. A path of length d from a vertex i to a vertex j is a sequence of $d + 1$ consecutively adjacent vertices, the first one being i and the last one being j. When such a path exists, i is said to be connected to j, and the graph distance from i to j is then defined as the minimum length of a path from i to j. We speak of a simple path when all vertices are pairwise distinct, and of a simple cycle when $d \geq 2$, $i = j$ and all other vertices are pairwise distinct. *Being connected to* is clearly an equivalence relation on V ; the associated equivalence classes are called the connected components of G. When there is only one connected component we say that G is **connected**. A connected graph without simple cycles is called a **tree**. Equivalently, a tree is a graph in which any two vertices are connected by a unique simple path.

1.1.3 Local convergence of rooted graphs

A **rooted graph** is a graph together with the specification of a particular vertex o, called the root. An **isomorphism** from a rooted graph (G, o) to a rooted graph (G', o') is a bijection $\gamma \colon V \to V'$ that preserves

- the root: $\gamma(o) = o'$;
- the edges: $\{i, j\} \in E \iff \{\gamma(i), \gamma(j)\} \in E'$.

When such a γ exists we say that (G, o) is isomorphic to (G', o'), and we write $(G, o) \equiv (G', o')$. We let $\mathcal{G}_\star$ denote the set of locally finite connected rooted graphs, taken up to the equivalence relation $\equiv$. This will constitute our basic workspace. Given $(G, o) \in \mathcal{G}_\star$ and $d \in \mathbb{N}$, we let $[G, o]_d$ denote the (finite) rooted subgraph obtained from (G, o) by keeping only those vertices that lie at graph distance at

most d from o, and all the edges between them. In $\mathcal{G}_\star$, a sequence $\{(G_n, o_n) : n \in \mathbb{N}\}$ **converges locally** to (G, o) is for every radius $d \in \mathbb{N}$, there exists $n_d \in \mathbb{N}$ such that

$$n \geq n_d \Longrightarrow [G_n, o_n]_d \equiv [G, o]_d.$$

It is not hard to see that the function

$$d\colon (G, o), (G', o') \longmapsto 1/\sup\{d \in \mathbb{N} : [G, o]_d \equiv [G', o']_d\}$$

defines a distance which metrizes this notion of convergence and turns $\mathcal{G}_\star$ into a complete separable metric space.

1.1.4 local weak limits of finite graphs

As a consequence (see e.g. [26]), we can endow $\mathcal{G}_\star$ with its Borel $\sigma-$algebra and consider the complete separable metric space of probability measures over $\mathcal{G}_\star$, denoted by $\mathcal{P}(\mathcal{G}_\star)$. In the latter, a sequence $(\mathcal{L}_n)_{n\in\mathbb{N}}$ **converges weakly** to $\mathcal{L}$ if for every bounded continuous function $\varphi\colon \mathcal{G}_\star \to \mathbb{R}$,

$$\mathcal{L}_n[\varphi] \xrightarrow[n\to\infty]{} \mathcal{L}[\varphi],$$

where $\mathcal{L}[\varphi] = \int \varphi(G, o) d\mathcal{L}(G, o)$ denotes the expectation of φ with respect to $\mathcal{L}$. **Uniform rooting** is a natural procedure for turning a finite deterministic graph G into a random element of $\mathcal{G}_\star$: one simply chooses uniformly at random a vertex o to be the root, and restrict G to the connected component containing o. If $(G_n)_{n\in\mathbb{N}}$ is a sequence of (deterministic) finite graphs and if the sequence of their laws under uniform rooting admits a weak limit $\mathcal{L} \in \mathcal{P}(\mathcal{G}_\star)$, we call $\mathcal{L}$ the **local weak limit** of the sequence $(G_n)_{n\in\mathbb{N}}$ and write

$$G_n \xrightarrow[n\to\infty]{LW} \mathcal{L}.$$

In words, $\mathcal{L}$ describes the asymptotic distribution of G_n when viewed locally from a uniformly chosen vertex. This simple and natural notion of convergence happens to have remarkable implications, to the study of which this thesis is devoted. Note that there are various other natural (and complementary) notions of convergence for sparse graphs, which capture non-local rather than local information. For more details, the interested reader is referred to [53, 1, 35] and to the surveys [30, 31].

1.2 Classical examples

Over the past decades, numerous models for random graphs have been introduced and analyzed in great details. In the **sparse regime** where $|V| \to \infty$ and $|E| = \Theta(|V|)$ (sometimes rather called the **extremely sparse** or **diluted** regime), the large majority of these models happen to admit almost surely a natural local weak limit. We list here some of the most important examples. For more details on random graphs models, the interested reader can refer to the classic monographs [29, 52] or to the recent course note [63].

1.2.1 Erdős-Rényi random graphs

Introduced by Gilbert [57] in the late fifties, the **Erdős-Rényi random graph** is undoubtedly the simplest and most popular random graph model. On a set of n vertices, each of the $\binom{n}{2}$ possible edges is randomly and independently drawn with some probability $p \in [0, 1]$. The name is after Paul Erdős and Alfréd Rényi, whose impressive seminal work [54] has spawned an extensive literature. In the sparse regime where the edge probability $p = p(n)$ scales like c/n as $n \to \infty$ for some fixed constant $c > 0$, a sequence of jointly defined Erdős-Rényi random graphs admits almost surely a famous local weak limit $\mathcal{L} \in \mathcal{P}(\mathcal{G}_\star)$: the **Poisson-Galton-Watson** distribution with mean c. This celebrated law is concentrated on rooted trees and was historically used as a model for genealogical family trees. Initially, an isolated root o gives birth to a random number N of children according to the Poisson distribution with parameter c,

$$\mathbb{P}(N = n) = e^{-c}\frac{c^n}{n!}.$$

In turn, each of these children (should there be any) independently gives birth to a random number of children according to the same distribution, and so on. The whole process is called a Galton-Watson branching process. It generates a random locally finite rooted tree whose law is called the Poisson-Galton-Watson distribution with mean c. For a detailed study of branching processes, refer to [12, 61]. A proof of the almost sure local weak convergence of the Erdős-Rényi model can be found in [47, Proposition 2.6].

1.2.2 Random regular graphs

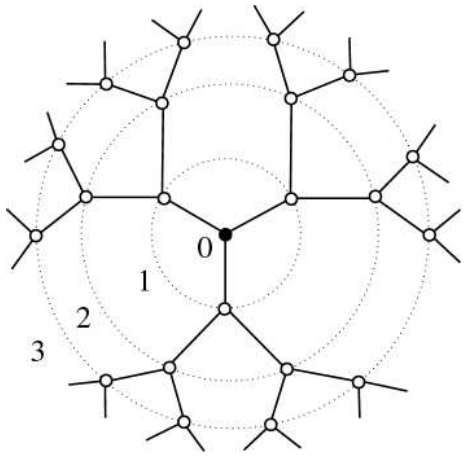


Figure 1.1: A portion of the 3−regular infinite rooted tree

Let $d \geq 3$. A graph in which every vertex has degree d is called $d-$regular. A **random $d-$regular graph** of size n is a graph chosen uniformly at random among all $d-$regular graphs on a fixed set of n vertices. This model has a long history, and we refer the interested reader to the comprehensive survey by Wormald [105].

A sequence of jointly defined random $d-$regular graphs with diverging size admits almost surely a local weak limit $\mathcal{L}$ which is concentrated on a single element of $\mathcal{G}_\star$: the $d-$**regular infinite rooted tree** (also called **Bethe lattice** in statistical mechanics). Here the convergence is a direct consequence of the fact that short cycles are asymptotically rare in random $d-$regular graphs (see e.g. [28]).

1.2.3 Random graphs with prescribed degree profile

In the past few years, the need for more realistic models of various types of complex networks has naturally led to the subtle problem of constructing random graphs in which the degrees approximately have some prescribed distribution $\pi \in \mathcal{P}(\mathbb{N})$ [20, 83, 28, 75]. In the so-called **configuration model**, the degree of each vertex is first generated independently according to the distribution π, and the resulting *half-edges* or *stubs* are then pairwise matched at random to form edges (successively, uniformly and independently). This process may of course result in self-loops and multiple edges, but the former are deleted and the latter merged so as to produce a proper graph. If π has finite mean, the empirical distribution of degrees in the resulting random graph converges to the prescribed degree distribution π as the number of vertices tends to infinity, as established in [38]. A few variants exist, see [63, Chapter 7] or [29, Section 2.4].

If $(G_n)_{n\in\mathbb{N}}$ is a sequence of (jointly defined) random graphs resulting from the above construction, and if their size diverges as $n \to \infty$, then $(G_n)_{n\in\mathbb{N}}$ admits almost surely a particular local weak limit $\mathcal{L}$, namely the **unimodular Galton-Watson distribution** with degree π (see Example 1.1 in [7]). It is the law of the random rooted tree obtained by a Galton-Watson branching process where the root has offspring distribution π and all other genitors have the size-biased offspring distribution $\widehat{\pi} \in \mathcal{P}(\mathbb{N})$ defined by

$$\forall n \in \mathbb{N}, \widehat{\pi}_n = \frac{(n+1)\pi_{n+1}}{\sum_k k\pi_k}. \tag{1.1}$$

Note that both the Poisson-Galton-Watson tree ($\pi = \widehat{\pi} = \text{Poisson(c)}$) and the regular infinite tree ($\pi = \delta_d, \widehat{\pi} = \delta_{d-1}$) are in fact special cases of unimodular Galton-Watson trees. Thus, this local weak limit has some kind of *universality* in the realm of sparse random graphs. A detailed proof of the almost sure local weak convergence can be found in [47, Proposition 2.5]. See also [24] for a closely related result.

1.2.4 Uniform random trees

The most natural way of defining a **random tree** of size n consists in selecting it uniformly at random among the n^{n-2} possible trees that can be built on the vertex set $\{1, \ldots, n\}$. Any sequence of jointly defined such random trees admits almost surely a special local weak limit $\mathcal{L}$ as $n \to \infty$: the (law of the) so-called **Infinite Skeleton Tree**. To describe the latter, consider an infinite sequence of independent random rooted trees (T_1, o_1), $(T_2, o_2), \ldots$ with the Poisson-Galton-Watson distribution with mean 1. Recall that all these trees are finite with probability one. The Infinite

Skeleton Tree may then be constructed by placing an edge between o_k and o_{k+1} for each $k \geq 1$, and declaring o_1 as the overall root. The convergence was first established by Grimmett [60]. Alternative proofs and generalizations can be found in Devroye [50] and Aldous [4].

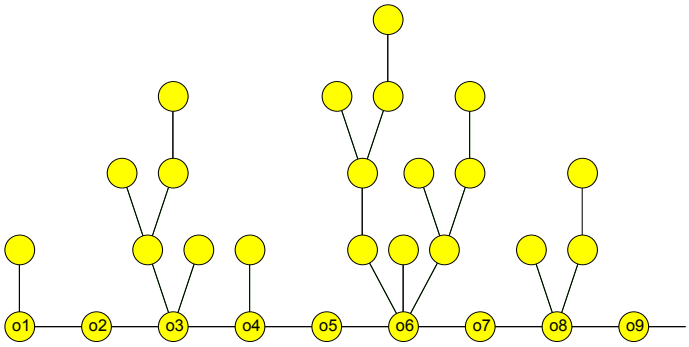


Figure 1.2: A portion of the Infinite Skeleton Tree

1.3 Overview of the results

In this thesis, we use the framework of local weak convergence to investigate the asymptotic behavior of four widely studied graph parameters that play a deep and central role in graph theory:

- the empirical spectral distribution (Chapter 2) ;
- the kernel dimension of the adjacency matrix (Chapter 3) ;
- the matching number (Chapter 4) ;
- the enumerating polynomial for spanning subgraphs subject to local constraints (Chapter 5).

In each case, we show that a suitably normalized version of the parameter ϕ satisfies a continuity theorem of the form

$$\left(G_n \xrightarrow[n\to\infty]{LW} \mathcal{L}\right) \Longrightarrow \left(\phi(G_n) \xrightarrow[n\to\infty]{} \phi(\mathcal{L})\right),$$

where $(G_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of finite graphs. Reasonable conditions are imposed on the local weak limit $\mathcal{L}$ so that a limiting parameter $\phi(\mathcal{L})$ can be properly defined, typically via an infinite set of local equations. We then apply this purely deterministic result to the aforementioned random graph models. Thanks to the particular self-similar nature of their local weak limit $\mathcal{L}$, the set of local equations determining $\phi(\mathcal{L})$ simplifies into a **recursive distributional equation** (see the survey [6] by Aldous and Bandyopadhyay), which we solve explicitly. This leads to new asymptotic formulae or to the simplification, unification and generalization of various known results that were previously relying on model-specific arguments. The reader will find below a slightly shortened version of the introduction of each chapter, together with the main results that it contains.

1.3.1 Empirical spectral distribution (Chapter 2)

A finite graph $G = (V, E)$ on n vertices $V = \{v_1, \ldots, v_n\}$ can be equivalently represented by a $\{0,1\}-$valued $n \times n$ matrix A called its adjacency matrix:

$$A_{ij} = \mathbf{1}_{\{v_i, v_j\} \in E}.$$

Since A is symmetric, it has n real eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$. The empirical spectral distribution of the graph G is the probability distribution

$$\mu_G = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k} \in \mathcal{P}(\mathbb{R}).$$

This graph invariant encodes considerable information about G, and constitutes the very basis of spectral graph theory [42, 41]. It also plays a role in number theory, via the important notion of expander graphs [46]. Many works have been devoted to the typical behavior of μ_G as $|V| \to \infty$. The regime where $|E| >> |V|$ is now well-understood: after a suitable normalization, Wigner's universal semi-circle law typically arises in the limit for many natural models. More details and references will be given in the introduction of Chapter 2. Regarding the (extremely) sparse regime $|E| = \Theta(|V|)$, the earliest result is known as the Kesten-McKay law [74] and concerns random $d-$regular graphs on n vertices:

$$\mu_{G_n}(d\lambda) \xrightarrow[n\to\infty]{\mathcal{P}(\mathbb{R})} \frac{d\sqrt{4(d-1)-\lambda^2}}{2\pi(d^2-\lambda^2)} \mathbf{1}_{(|\lambda|<2\sqrt{d-1})} d\lambda.$$

For the Erdős-Rényi model with $p(n) = c/n$ ($c > 0$ is fixed and $n \to \infty$), a similar convergence was established more recently by Khorunzhy, Shcherbina, and Vengerovsky [67], but in that case the limiting spectral distribution is only implicitly determined. Also, Bhamidi, Evans and Sen [25] established the convergence of the empirical spectral distribution for various models of random trees.

As we have already seen, sparse random regular graphs, sparse Erdős-Rényi graphs and uniform random trees are typical examples of almost surely convergent graph sequences. In light of this observation, the following question naturally arises: can the asymptotic behavior of the empirical spectral distribution be directly read off on the local weak limit of the underlying graph sequence ? Idealistically, we would like to re-interpret the aforementioned results as special cases of a general, unifying statement of the form

$$G_n \xrightarrow[n\to\infty]{LW} \mathcal{L} \qquad \Longrightarrow \qquad \mu_{G_n} \xrightarrow[n\to\infty]{\mathcal{P}(\mathbb{R})} \overline{\mu}_{\mathcal{L}},$$

where $\overline{\mu}_{\mathcal{L}} \in \mathcal{P}(\mathbb{R})$ is some suitably defined spectral distribution associated with the local weak limit $\mathcal{L}$. As we will see, a sense can indeed be given to such an object, provided $\mathcal{L}$ is concentrated on *self-adjoint* graphs. Moreover, in that case, the above implication actually holds.

Theorem (Theorem 2.1)**.** *If $\mathcal{L} \in \mathcal{P}(\mathcal{G}_\star)$ is concentrated on self-adjoint graphs, then the implication*

$$G_n \xrightarrow[n\to\infty]{LW} \mathcal{L} \qquad \Longrightarrow \qquad \mu_{G_n} \xrightarrow[n\to\infty]{\mathcal{P}(\mathbb{R})} \overline{\mu}_{\mathcal{L}}$$

holds true for any sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$.

For this result to be operational, we now need to investigate the question of self-adjointness for graphs that typically arise in the limit of classical models. The notion of self-adjointness of graphs was first considered by Mohar [80], who established that any graph with bounded degree is self-adjoint, and who originally believed that any locally finite graph would be self-adjoint. Counter-examples were soon constructed [81, 85], and to the best of our knowledge, there is no simple graph-theoretical translation of self-adjointness. For more details, we refer the reader to [82, 45]. Our second contribution is an exact characterization in the case of trees (see Theorem 2.2 in Chapter 2 for the precise statement), from which it follows in particular that

Theorem (Corollary 2.3)**.** *A tree with finite branching number is self-adjoint.*

Introduced by Lyons [70] (see also [72]), the branching number is, for many different purposes (including flows, percolation, electrical networks, Hausdorff dimension and random walks), the relevant measurement of *how big an infinite tree is*. It is formally defined as

$$\operatorname{br} T \;=\; \sup\left\{\lambda \geq 1 : \inf_{K\colon \text{cut}} \sum_{e\in K} \lambda^{-|e|} > 0\right\} \in [1,\infty].$$

where $|e|$ denotes the distance from the edge e to the root, and where a cut K is a finite set of edges whose removal makes the connected component of the root finite. (This is easily seen not to depend on the choice of the root.) The branching number of a Galton-Watson tree is simply its average offspring distribution, almost surely on the event of non-extinction ([70, Proposition 6.4]). For a unimodular Galton-Watson tree with degree distribution π, each proper subtree is a Galton-Watson tree whose offspring distribution $\widehat{\pi}$ is given by (1.1). Clearly, the mean of $\widehat{\pi}$ is finite if and only if π has finite variance.

Corollary (Corollary 2.4)**.** *A unimodular Galton-Watson tree whose degree distribution π has finite variance is self-adjoint with probability one.*

In a tree $T = (V, E)$ with an arbitrarily fixed root $o \in V$, an end is an infinite simple path starting from o. If there are finitely many such ends, then clearly $\operatorname{br}(T) = 1$. For example, the Infinite Skeleton Tree has almost surely one end, and is hence almost surely self-adjoint. In fact, a general result due to Aldous and Lyons [7, Proposition 6.3] asserts that the local weak limit of any sequence of finite trees must be concentrated on trees with at most two ends. Thus,

Corollary (Corollary 2.1)**.** *Any random rooted tree arising as the local weak limit of a sequence of finite trees is self-adjoint with probability one.*

In other words, the self-adjointness condition in Theorem 2.1 is automatically fulfilled for trees. We state this separately, since it simplifies and generalizes a recent result due to Bhamidi, Evans and Sen [25, Theorem 4.1].

Corollary (Corollary 2.2)**.** *If a sequence of finite trees admits a local weak limit $\mathcal{L}$, then the sequence of their empirical distributions converges to $\overline{\mu}_{\mathcal{L}}$.*

In addition to the characterization of self-adjointness for trees, Theorem 2.2 also provides a local, recursive description of $\overline{\mu}_{\mathcal{L}}$ via its Stieltjes-Borel transform, which will be crucially used in Chapter 3 to derive new explicit formulae for the multiplicity of the eigenvalue zero.

1.3.2 Rank and nullity (Chapter 3)

The **nullity** $\eta(G)$ of a finite graph $G = (V, E)$ is the multiplicity of the eigenvalue zero in its spectrum:

$$\eta(G) = \dim \ker(\mathrm{A}) = |\mathrm{V}| - \mathrm{Rank}(\mathrm{A}) = |\mathrm{V}|\mu_{\mathrm{G}}(\{0\}).$$

This graph parameter plays an important role in graph theory and computer science, notably as a simple bound for computationally intractable (NP-hard) invariants [101, 3]. It is therefore natural to investigate its typical behavior for classical random graphs. Bauer and Golinelli [15] have computed exactly the expected rank of a uniform random tree of size n, for any $n \in \mathbb{N}^*$. For the Erdős-Rényi model with edge probability $p = p(n)$ on n vertices, the regime where $p(n) = a \log n/n$ ($a > 0$ fixed, $n \to \infty$) has been studied by Costello, Tao and Vu [43] and Costello and Vu [44]. Their results imply that for $a > 1$, the nullity is zero with high probability while for $0 < a < 1$, it scales like n^{1-a}. In the sparse regime where $p(n) = cn$, the answer is only known if $c \leq e$, but a conjecture has been formulated for $c > e$ by Bauer and Golinelli [16].

Conjecture. *In the Erdős-Rényi model with $p(n) = c/n$, for all $c > 0$,*

$$\mu_{G_n}(\{0\}) \xrightarrow[n\to\infty]{a.s.} \lambda_* + e^{-c\lambda_*} + c\lambda_* e^{-c\lambda_*} - 1,$$

where $\lambda_* \in (0, 1)$ denotes the smallest root of $\lambda = e^{-ce^{-c\lambda}}$.

We settle this conjecture, thereby also answering one of the open questions in Costello and Vu [44]. More generally, we consider sequences of graphs $(G_n)_{n\geq 1}$ whose local weak limit $\mathcal{L}$ is a unimodular Galton-Watson distribution. If the degree distribution π has a finite second moment, we know from Chapter 2 that the tree is almost surely self-adjoint and that

$$\limsup_{n\to\infty} \mu_{G_n}(\{0\}) \leq \overline{\mu}_{\mathcal{L}}(\{0\}).$$

Our first result is the explicit computation of this natural upper bound. The formula involves a function $M\colon [0, 1] \to \mathbb{R}$ defined in terms of the degree generating function $\phi(\lambda) = \sum_{n=0}^{\infty} \pi_n \lambda^n$ as follows:

$$M(\lambda) = \phi'(1)\lambda\overline{\lambda} + \phi(1 - \lambda) + \phi\left(1 - \overline{\lambda}\right) - 1 \quad \text{with} \quad \overline{\lambda} = \frac{\phi'(1 - \lambda)}{\phi'(1)}.$$

Theorem (Theorem 3.1)**.** *Let $\pi \in \mathcal{P}(\mathbb{N})$ have a finite second moment. Then for the unimodular Galton-Watson distribution $\mathcal{L}$ with degree distribution π,*

$$\overline{\mu}_{\mathcal{L}}(\{0\}) \;=\; \max_{\lambda\in[0,1]} M(\lambda).$$

Moreover, any λ where the above maximum is achieved must satisfy $\lambda = \overline{\overline{\lambda}}$.

To obtain a lower bound, we use the following observation, due to Bauer and Golinelli [16]. A leaf in a graph is simply a vertex of degree 1. The nullity $\eta(G)$ is well-known to be invariant under the action of deleting an arbitrary leaf and its unique neighbor. Iterating this procedure eventually produces a core with minimum degree at least 2, plus a certain number of isolated vertices which is a simple lower bound for $\eta(G)$. In order to determine its asymptotic behavior for large sparse random graphs, we analyze the leaf-removal process directly on the limiting unimodular Galton-Watson tree. We show that the probability that the root eventually becomes isolated is precisely given by the first local extremum of the above-defined function M. Consequently, we obtain a general asymptotic lower bound.

Theorem (Theorem 3.2)**.** *For any sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$ whose local weak limit is a unimodular Galton-Watson distribution,*

$$\liminf_{n\to\infty} \mu_{G_n}(\{0\}) \geq M(\lambda_*),$$

where $\lambda_* \in (0,1)$ is the smallest root of $\lambda = \overline{\overline{\lambda}}$.

Combining the upper and the lower bound, we obtain the following result.

Corollary. *Let $\mathcal{L}$ be the unimodular Galton-Watson distribution with degree distribution π. Assume that π has a finite second moment, and that the first local extremum of M is its global maximum. Then, for any sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$ whose random weak is $\mathcal{L}$,*

$$\mu_{G_n}(\{0\}) \xrightarrow[n\to\infty]{} \overline{\mu}_{\mathcal{L}}(\{0\}) = M(\lambda_*),$$

where $\lambda_* \in (0,1)$ is the smallest root of $\lambda = \overline{\overline{\lambda}}$. Moreover, a simple sufficient condition for the above assumption to hold is that $\log(\phi'')$ is concave on $(0,1)$.

In the Erdős-Rényi case, the local weak limit is almost surely the Poisson-Galton-Watson distribution with mean c, whose degree generating function is $\phi(\lambda) = e^{c\lambda - c}$. ϕ'' is clearly log-concave, and hence the above conjecture is settled. For the $d-$regular infinite tree, the second derivative of $\phi(\lambda) = \lambda^d$ is also trivially log-concave and $M(\lambda_*) = M(0) = 0$, in agreement with the fact that the Kesten-McKay distribution $\overline{\mu}_{\mathcal{L}}$ is absolutely continuous with respect to Lebesgue's measure. However, we will also exhibit simple degree distributions π such that the lower bound $M(\lambda_*)$ does not match the upper bound $\max M$. The asymptotic behavior of the nullity in that case remains an interesting open question.

1.3.3 Matching number (Chapter 4)

A matching on a finite graph $G = (V, E)$ is a subset of mutually non-adjacent edges $M \subseteq E$. The matching number $\nu(G)$ is the largest possible cardinality of a matching on G. This simple invariant plays an important role in graph theory, and we refer the interested reader to the monographs [58, 69] for more details. Karp and Sipser [66] investigated its behavior in the case of the Erdős-Rényi random graph with

average degree $c > 0$ on n vertices. Using a detailed analysis of the leaf-removal process, they showed that

$$\frac{\nu(G_n)}{|V_n|} \xrightarrow[n\to\infty]{} 1 - \frac{\lambda_* + e^{-c\lambda_*} + c\lambda_* e^{-c\lambda_*}}{2}, \tag{1.2}$$

where $\lambda_* \in (0,1)$ is the smallest root of $\lambda = e^{-ce^{-c\lambda}}$. More recently, the same technique has been applied to leafless random graphs with a prescribed log-concave degree distribution (Bohmann and Frieze [27]), resulting in the asymptotic existence of an almost perfect matching in the following sense:

$$\frac{\nu(G_n)}{|V_n|} \xrightarrow[n\to\infty]{} \frac{1}{2}. \tag{1.3}$$

Here we show that the asymptotic behavior of the matching number can in fact be directly deduced from the local weak convergence of the underlying graph sequence, thereby simplifying, unifying and generalizing the two aforementioned results.

Theorem (Theorem 4.1)**.** *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of finite graphs admitting a local weak limit $\mathcal{L}$. Then,*

$$\frac{\nu(G_n)}{|V_n|} \xrightarrow[n\to\infty]{} \overline{\nu}(\mathcal{L}), \tag{1.4}$$

where $\overline{\nu}(\mathcal{L}) \in [0, \frac{1}{2}]$ is described by a local recursion defined directly on the local weak limit $\mathcal{L}$. When the latter is the unimodular Galton-Watson distribution with degree π, we have the explicit formula

$$\overline{\nu}(\mathcal{L}) \;=\; \frac{1}{2} - \frac{1}{2} \max_{\lambda\in[0,1]} M(\lambda).$$

That the atomic spectral mass at zero and the asymptotic matching number be described by the very same function M may surprise at first sight. However, if $G = (V, E)$ is a finite tree, it is well-known that $2\nu(G) = |V| - \eta(G)$. In a sense, the above result thus extends this identity to the case of an infinite unimodular Galton-Watson tree. For Erdős-Rényi random graphs with average degree c, the local weak limit is a.s. the Poisson-Galton-Watson distribution with mean c (i.e. $\phi(\lambda) = \exp(c\lambda - c)$), so we recover precisely Karp and Sipser's formula (1.2). Similarly, for random graphs with a prescribed degree sequence, the log-concave assumption made by Bohmann and Frieze in [27] guarantees that the above maximum is achieved at $\lambda = 0$, hence (1.3) follows automatically.

Nevertheless, Theorem 4.1 is far from being a direct implication of our results for the spectral mass at zero. The main difference, apart from the fact that the convergence (1.4) holds without any self-adjointness assumption, is that the crucial condition $\max M = M(\lambda_*)$ is not imposed anymore. As we will show, this condition is equivalent to a precious property known as *correlation decay*, which usually plays a central role in the objective method. Its failure (called *ergodicity breaking* in [106]) implies non-trivial long-range correlations between the edges in a uniformly chosen largest matching. Our main contribution consists in overcoming this lack of correlation decay. Specifically, we relax the constraint of largest cardinality by means of a variable parameter called the activity, and we show that correlation decay always holds under these relaxed constraints. The size of the largest matching can then be recovered by sending the activity to infinity.

1.3.4 Weighted subgraph enumeration (Chapter 5)

In the last chapter of this thesis, we consider the more general framework of a finite graph $G = (V, E)$ whose spanning subgraphs $(V, F), F \subseteq E$ are weighted according to their local aspect around each vertex as follows:

$$\mu(F) \quad = \quad \prod_{i \in V} \mu_i(F \cap E_i).$$

Here, a spanning subgraph (V, F) is identified with its edge-set $F \subseteq E$, and each μ_i is a given non-negative function over the subsets of $E_i := \{e \in E; e \text{ is incident to } i\}$. We call μ the global measure induced by the local measures $\mu_i, i \in V$. Of particular interest in combinatorial optimization is

$$M(G) \quad = \quad \max\{|F| : F \in \mathrm{supp}(\mu)\},$$

which is the maximum possible size of a spanning subgraph F satisfying the local constraint $\mu_i(F \cap E_i) > 0$ at every vertex $i \in V$. More generally, counting the weighted number of spanning subgraphs of each given size in G, i.e. determining the generating polynomial

$$Z(G; t) \quad = \quad \sum_{F \subseteq E} \mu(F) t^{|F|}$$

is a fundamental task, of which many combinatorial problems are special instances. Intimately related to this is the study of a random spanning subgraph $\mathcal{F}$ sampled from the Gibbs-Boltzmann law:

$$\mathbb{P}_G^t(\mathcal{F} = F) \quad = \quad \frac{\mu(F) t^{|F|}}{Z(G; t)},$$

where $t > 0$ is a variable parameter called the activity. In particular, the expected size of $\mathcal{F}$ is called the energy $U(G; t)$ and is connected to $Z(G; t)$ via the elementary identity

$$U(G; t) \quad = \quad t \frac{d}{dt} \log Z(G; t).$$

Our concern is the behavior of these quantities in the infinite volume limit: $|V| \to \infty$, $|E| = \Theta(|V|)$.

As already explained, the cavity method is a powerful non-rigorous technique for evaluating such asymptotics on graphs that are locally tree-like. The heuristic consists in neglecting cycles in order to obtain an approximate local fixed point equation (the so-called cavity equation) for the marginals of the Gibbs-Boltzmann law. Despite its remarkable practical efficiency and the mathematical confirmation of its analytical predictions for various important models [97, 5, 56, 89, 19, 48, 33], this ansatz is still far from being completely understood, and the exact conditions for its validity remain unknown. More precisely, two crucial questions arise in presence of cycles:

1. **convergence**: is there a unique solution to the cavity equation ?

2. **correctness**: is the latter related to the Gibbs-Boltzmann law ?

Using the theory of negative association for measures initiated by Pemantle [86], we exhibit a new, general condition under which the cavity method is valid for counting spanning subgraphs subject to local constraints. Specifically, we positively answer question 1 for arbitrary finite graphs and their local weak limits (Propositions 5.1 and 5.2), under the only assumption that each local measure enjoys a certain form of negative association which we call the cavity-monotone property. In the exchangeable case, the latter simply boils down to ultra-log-concavity (see Section 5.2 for the precise definitions). Our proof of the uniqueness for any local weak limit relies on a powerful notion of spatial invariance known as unimodularity, to which Aldous and Lyons have dedicated a whole survey [7]. Regarding question 2, we prove asymptotic correctness for any sequence of graphs whose local weak limit is concentrated on trees (Theorem 5.2). In particular, we obtain that the so-called internal energy and thermodynamic pressure,

$$\frac{U(G;t)}{|V|} \qquad \text{and} \qquad \frac{1}{|V|}\log Z(G;t),$$

converge to quantities that are directly described in terms of the limiting cavity equation. Consequently, we state a large deviation principle with speed $|V|$ for the relative size $|\mathcal{F}|/|V|$ of a random spanning subgraph sampled proportionally to μ. Finally, in the important case where the local weak limit is a unimodular Galton-Watson distribution, the cavity equation simplifies into a recursive distributional equation which can be solved explicitly. As an illustration, let us here describe the implications of our work in the special case of b-matchings.

An important combinatorial structure that fits in the above framework is obtained by fixing $b \in \mathbb{N}$ and taking $\mu_i(F) = \mathbf{1}(|F| \leq b)$ for all $i \in V$. The induced global measure μ is then nothing but the counting measure for $b-$matchings in G, i.e. spanning subgraphs with maximum degree at most b. The reader is referred to the monograph [93] for a comprehensive survey on $b-$matchings. The associated quantities $M_b(G)$ and $Z_b(G;t)$ are important graph invariants respectively known as the $b-$matching number and $b-$matching polynomial. The case of matchings ($b = 1$) has been investigated in great detail, notably for the purpose of understanding monomer-dimer systems [62, 23]. Determining $Z_1(G;t)$ is a classical example of a computationally hard problem [100], although efficient approximation algorithms have been designed [18, 14]. The scaled convergence of $Z_1(G;t)$ as $|V| \to \infty$ was established in [62] for the lattice case, and in [19] under a restrictive large girth assumption. Contrastingly, only little is known for $b \geq 2$. To the best of our knowledge, the limit of $\frac{1}{|V_n|}M_b(G_n)$ is only known to exist in the sparse Erdős-Rényi case [55, Theorem 3], and could not be explicitly determined. As a special case of our main result, it will follow that

Theorem (Theorem 5.1)**.** *For any sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$ satisfying $|E_n| = O(|V_n|)$ and whose local weak limit $\mathcal{L}$ is concentrated on trees, the limits*

$$h_b(\mathcal{L};t) := \lim_{n\to\infty} \frac{1}{|V_n|}\log Z_b(G_n;t) \quad \textit{and} \quad m_b(\mathcal{L}) := \lim_{n\to\infty} \frac{M_b(G_n)}{|V_n|}$$

exist and depend only on the local weak limit $\mathcal{L}$. When $\mathcal{L}$ is the unimodular Galton-Watson distribution with degree π, we have the explicit formula

$$m_b(\mathcal{L}) = \frac{b}{2} \min_{\lambda \in [0,1]} \left\{ 2 - g_b(\lambda) - (g_b \circ f_b)(\lambda) + \frac{c}{b} f_b(\lambda)(f_b \circ f_b)(\lambda) \right\},$$

where c, f, g are defined in terms of $\phi(\lambda) = \sum_k \pi_k \lambda^k$ as follows:

$$c = \phi'(1), \;\; f_b(\lambda) = \frac{1}{c} \sum_{k=0}^{b-1} \frac{\lambda^k \phi^{(k+1)}(1-\lambda)}{k!} \;\; \textit{and} \;\; g_b(\lambda) = \sum_{k=0}^{b} \frac{\lambda^k \phi^{(k)}(1-\lambda)}{k!}.$$

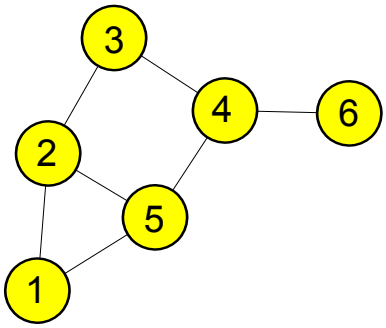
Moreover, any λ achieving this minimum must be a root of $\lambda = (f_b \circ f_b)(\lambda)$.

Chapter 2

Empirical spectral distribution

Joint work with Charles Bordenave and Marc Lelarge

The empirical spectral distribution of a finite graph plays a central role in spectral graph theory, and this chapter is devoted to the study of its asymptotics as the size of the graph tends to infinity. Our first result is that if a sequence of finite graphs admits a local weak limit $\mathcal{L}$, then the corresponding sequence of empirical spectral distributions converges to a certain measure $\overline{\mu}_{\mathcal{L}}$, provided $\mathcal{L}$ is concentrated on self-adjoint graphs. Our second contribution is a characterization of self-adjointness in the important case of trees, as well as a recursive fixed point equation for determining $\overline{\mu}_{\mathcal{L}}$. As a by-product, we obtain that any tree with finite branching number is self-adjoint, which includes in particular all the local weak limits described in the introduction as well as any local weak limit of finite trees.



0	1	0	0	1	0
1	0	1	0	1	0
0	1	0	1	0	0
0	0	1	0	1	1
1	1	0	1	0	0
0	0	0	1	0	0

Figure 2.1: A graph and its adjacency matrix

2.1 Introduction

A finite graph $G = (V, E)$ on n vertices $V = \{v_1, \ldots, v_n\}$ can be equivalently represented by a $\{0,1\}-$valued $n \times n$ matrix A called its adjacency matrix :

$$A_{ij} = \mathbf{1}_{\{v_i, v_j\} \in E}.$$

Since A is symmetric, it has n real eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$. The **empirical spectral distribution** of the graph G is the probability distribution

$$\mu_G = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k} \in \mathcal{P}(\mathbb{R}).$$

This graph invariant encodes considerable information about G, and constitutes the very basis of spectral graph theory [42, 41]. It also plays an important role in number theory, via the notion of expander graphs [46]. Many works have been devoted to the typical behavior of μ_G as the size of G tends to infinity. The word *typical* implicitly assumes the specification of an underlying probability distribution over graphs.

The most natural choice for constructing a random graph G_n on n vertices is undoubtedly the Erdős-Rényi model with edge probability $p = p(n) \in [0,1]$. Up to symmetry, the adjacency matrix of G_n simply consists of i.i.d. Bernouilli random variables. This elementary entry distribution has a finite support, which is much more than what is actually required for the following celebrated principle to hold (see the monograph [13] for a complete introduction to the spectral theory of random matrices): provided $np(n) \to \infty$,

$$\mu_{G_n}\left(\frac{1}{\sqrt{n}\sigma_n} d\lambda\right) \xrightarrow[n\to\infty]{d} \frac{1}{2\pi} \sqrt{4 - \lambda^2} \mathbf{1}_{(|\lambda|<2)} d\lambda, \tag{2.1}$$

where $\sigma_n^2 = p(n)(1 - p(n))$. The limiting distribution is known as the *semicircle law* and was originally discovered by Wigner [104] for Gaussian entries. Its most remarkable feature is its *universality*. To illustrate this, let us replace the Erdős-Rényi graph G_n by a random $d(n)-$regular graphs on n vertices, thereby introducing non-trivial correlations between the entries of the adjacency matrix (since each row/column must now sum to $d(n)$). Remarkably enough, (2.1) still holds with $\sigma_n^2 = \frac{d(n)}{n}(1 - \frac{d(n)}{n})$, provided $d(n) \to \infty$. The proof of this latter fact is very recent [51, 99].

In these two examples, the conditions $np(n) \to \infty$ and $d(n) \to \infty$ have the same meaning : the underlying graphs should be rather dense, in the sense that $|E| >> |V|$. Regarding the (extremely) sparse regime $|E| = \Theta(|V|)$, the earliest result is known as the Kesten-McKay law [74] for random regular graphs : when the degree $d(n) = d$ is kept constant as $n \to \infty$,

$$\mu_{G_n}(d\lambda) \xrightarrow[n\to\infty]{d} \frac{d\sqrt{4(d-1) - \lambda^2}}{2\pi(d^2 - \lambda^2)} \mathbf{1}_{(|\lambda|<2\sqrt{d-1})} d\lambda.$$

In the sparse Erdős-Rényi model ($p(n) = c/n$ where $c > 0$ is fixed), a similar convergence was established more recently [67], but in that case the limiting spectral

distribution is only implicitly determined, and important questions concerning its atomic and absolutely continuous part are still open [17]. Bhamidi, Evans and Sen [25] have established the convergence of the empirical spectral distribution for various models of random trees.

As we have already seen, sparse random regular graphs, sparse Erdős-Rényi random graphs and uniform random trees are typical examples of convergent graph sequences in the local weak sense. In light of this observation, the following question naturally arises : can the asymptotic behavior of the empirical spectral distribution be directly read off on the local weak limit of the underlying graph sequence ? Idealistically, we would like to re-interpret the aforementioned results as special cases of a general, unifying statement of the form

$$G_n \xrightarrow[n\to\infty]{LW} \mathcal{L} \qquad \Longrightarrow \qquad \mu_{G_n} \xrightarrow[n\to\infty]{} \overline{\mu}_{\mathcal{L}},$$

where $\overline{\mu}_{\mathcal{L}} \in \mathcal{P}(\mathbb{R})$ is directly defined on the local weak limit $\mathcal{L}$. As we shall soon see, such an implication holds as soon as $\mathcal{L}$ is concentrated on **self-adjoint graphs** (Theorem 2.1). Our main contribution here is an explicit characterization of self-adjointness of trees, as well as a recursive fixed point equation for determining $\overline{\mu}_{\mathcal{L}}$ (Theorem 2.2). The latter will be crucially used in the next chapter to derive new results concerning the asymptotic multiplicity of the eigenvalue zero for Erdős-Rényi graphs and other popular models of sparse random graphs.

2.2 Rooted spectral measures

For a comprehensive introduction to the theory of linear operators on Hilbert spaces – and in particular the spectral theorem for self-adjoint operators – the reader is referred to the textbooks [87, 88, 98], or to the beautiful expository note [94]. A locally finite graph $G = (V, E)$ can be naturally identified with a certain linear operator A on the Hilbert space

$$\ell^2(V) = \left\{ \psi \colon V \to \mathbb{C} : \sum_{i \in V} |\psi(i)|^2 < \infty \right\},$$

endowed with its canonical inner product $\langle \psi, \varphi \rangle = \sum_{i \in V} \overline{\psi(i)} \varphi(i)$. Specifically, the domain of A is the dense subspace of finitely supported functions $C_c(V) \subseteq \ell^2(V)$, and the action of A on the canonical orthonormal basis $(\delta_i)_{i \in V}$ (where $\delta_i(j) = \mathbf{1}_{\{j=i\}}$) is given by

$$A\delta_i = \sum_{j \in \partial i} \delta_j.$$

A is naturally called the **adjacency operator** of G. It is symmetric, in the sense that $\langle \delta_i | A\delta_j \rangle = \langle A\delta_i | \delta_j \rangle$. In the infinite-dimensional case however, symmetry is not enough to guarantee a suitable spectral decomposition. This translates the fact that the spectrum

$$\sigma(A) = \{ z \in \mathbb{C} : A - z \text{ does not have a bounded inverse} \}$$

is not necessarily real. When it is, A and G are said to be (essentially) **self-adjoint**. Mohar [80] was the first to consider spectra of infinite graphs. His original belief was that any locally finite graph would be self-adjoint, but counter-examples were soon constructed [81, 85]. For more details on spectra of graphs, we refer the reader to [82, 45]. To the best of our knowledge, there is no simple graph-theoretical translation of self-adjointness. This contrasts with many simpler properties of A (for proofs, see e.g. [82]):

1. A is bounded (i.e. continuous) $\iff$ G has bounded degree ;
2. A is compact $\iff$ G has finitely many edges ;
3. $A = \bigoplus_k A_k$, where $\bigoplus$ denotes the orthogonal sum and the A_k are the adjacency operators of the connected components of G.

From these three elementary observations however, it already follows that any graph with bounded degree is self-adjoint, as well as any sub-critical graph (a graph whose connected components are finite). Moreover, since self-adjointness is preserved under addition of a bounded operator (a consequence of Kato-Rellich Theorem [88, Theorem X.12]), we obtain the following simple yet powerful criterion.

Proposition 2.1 (Sufficient condition)**.** *For a graph $G = (V, E)$ to be self-adjoint, it is enough that its edge-set can be written as $E = E_1 \cup E_2$ in such a way that $G_1 = (V, E_1)$ is sub-critical and $G_2 = (V, E_2)$ has bounded degree.*

In particular, the Infinite Skeleton Tree defined in Section 1.2 is almost-surely self-adjoint. In fact, this is true for any local weak limit of trees.

Corollary 2.1. *Any random rooted tree arising as the local weak limit of a sequence of finite trees is self-adjoint with probability one.*

Proof of Corollary 2.1. In a rooted tree $T = (V, E, o)$, an **end** is as an infinite simple path starting from the root o. A result due to Aldous and Lyons asserts that any random rooted tree arising as the local weak limit of a sequence of finite trees has almost surely at most two ends [7, Proposition 6.3]. But any tree with finitely many ends is self-adjoint, since it satisfies the condition of Lemma 2.1 with $E_2 = E \setminus E_1$ and E_1 consisting of those edges that are traversed by an end. □

Similarly, a Galton-Watson tree whose offspring distribution has finite mean can be shown to satisfy the condition of Proposition 2.1 with probability one. We do not go into details since the self-adjointness of Galton-Watson trees will also follow from Corollary 2.3 below. Regarding measurability issues, we simply note that self-adjointness can be expressed as a union and intersection of countably many cylinder events (i.e. events depending only on $[G, o]_d$ for some large enough d). Indeed, by the basic criterium for essential self-adjointness [87, Theorem VIII.3], we have for any countable locally finite graph G with adjacency operator A,

$$\begin{aligned} G \text{ is self-adjoint} &\iff \mathrm{Ran}(\mathrm{A} \pm \mathbf{i})^{-1} \text{ is dense in } \ell^2(\mathrm{V}) \\ &\iff \forall o \in V, \forall n \in \mathbb{N}^*, \exists d \in \mathbb{N}^*, \min_{\mathrm{supp}(\varphi) \subseteq B_{o,d}} \|(A \pm \mathbf{i})\varphi - \delta_o\| < \frac{1}{n}. \end{aligned}$$

with $\mathbf{i} = \sqrt{-1}$ and where $B_{o,d} \subseteq V$ denotes the set of vertices lying at graph distance at most d from o in G. Clearly, the above minimum depends only on $[G, o]_{d+1}$.

We henceforth assume that G is self-adjoint, and we let

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

denote the upper complex half-plane. The spectral theorem for self-adjoint operators then guarantees that for every $\psi \in \ell^2(V)$ with $\|\psi\| = 1$, there exists a unique probability measure $\mu_\psi^* \in \mathcal{P}(\mathbb{R})$ satisfying

$$\langle \psi | (A - z)^{-1} \psi \rangle = \int_{\mathbb{R}} \frac{1}{\lambda - z} \mu_\psi^*(d\lambda), \tag{2.2}$$

for all $z \in \mathbb{H}$. μ_ψ^* is usually called the spectral measure associated with the pair (A, ψ). In particular, for every vertex $o \in V$ of a self-adjoint graph $G = (V, E)$, we may consider the spectral measure associated with the vector δ_o. We denote it by $\mu_{(G,o)}^*$ and call it the **rooted spectral measure** of the rooted graph (G, o). Intuitively, $\mu_{(G,o)}^*$ may be thought of as the local contribution of the vertex o to the spectrum of G. The following Lemma gives a rigorous meaning to this idea in the case where G is finite.

Lemma 2.1 (Spatial average)**.** *The empirical spectral distribution of a finite graph G is the spatial average of its rooted spectral measures :*

$$\mu_G = \frac{1}{|V|} \sum_{o \in V} \mu_{(G,o)}^*.$$

Proof. In finite dimension, there exists an orthonormal basis of eigenvectors $(\phi_1, \ldots, \phi_n)$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Any vector $\psi \in \mathbb{C}^n$ can be decomposed in this basis, yielding successively

$$\begin{aligned}
(A - z)\psi &= \sum_{k=1}^{n} (\lambda_k - z) \langle \psi | \phi_k \rangle \phi_k, \\
(A - z)^{-1}\psi &= \sum_{k=1}^{n} \frac{1}{\lambda_k - z} \langle \psi | \phi_k \rangle \phi_k, \\
\langle \psi | (A - z)^{-1} \psi \rangle &= \sum_{k=1}^{n} \frac{1}{\lambda_k - z} |\langle \psi | \phi_k \rangle|^2.
\end{aligned}$$

Comparing the last line with (2.2), we see that

$$\mu_\psi^* = \sum_{k=1}^{n} |\langle \psi | \phi_k \rangle|^2 \delta_{\lambda_k}.$$

In words, the spectral measure associated with ψ is a mixture of atoms located at eigenvalues, the mass of each atom being the square norm of the projection of ψ

onto the corresponding eigenspace. Now if $\psi_1, \dots, \psi_n$ is an arbitrary orthonormal basis of $\mathbb{C}^n$, we have :

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} \mu^*_{\psi_i} &= \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{i=1}^{n} |\langle\psi_i|\phi_k\rangle|^2\right)\delta_{\lambda_k} \\ &= \frac{1}{n}\sum_{k=1}^{n} \|\phi_k\|^2 \delta_{\lambda_k} \\ &= \mu_G. \end{aligned}$$

Since $(\delta_i)_{i\in V}$ is an orthonormal basis, the result is proved. □

2.3 Continuity under local convergence

From the orthonormal decomposition of the adjacency operator along connected components, it follows that a locally finite graph G is self-adjoint if and only if all its connected components are self-adjoint. Also, the rooted spectral measure at a vertex o depends only on the connected component containing o. Moreover, these notions are clearly preserved under isomorphism. Consequently, self-adjointness is a well-defined property on $\mathcal{G}_\star$ and $(G, o) \mapsto \mu^*_{(G,o)}$ is a well-defined mapping over the self-adjoint elements of $\mathcal{G}_\star$. This mapping is continuous with respect to the topology of local convergence.

Lemma 2.2 (Continuity of rooted spectral measures)**.** *Let (G, o) and (G_1, o_1), $(G_2, o_2), \dots$ be self-adjoint elements of $\mathcal{G}_\star$. Then,*

$$(G_n, o_n) \xrightarrow[n\to\infty]{\mathcal{G}_\star} (G, o) \qquad \Longrightarrow \qquad \mu^*_{(G_n,o_n)} \xrightarrow[n\to\infty]{\mathcal{P}(\mathbb{R})} \mu^*_{(G,o)}.$$

Proof. By assumption, for any radius $d \in \mathbb{N}$ there exists $n_d \in \mathbb{N}$ such that

$$n \geq n_d \implies [G_n, o_n]_d \equiv [G, o]_d. \tag{2.3}$$

Since the elements of $\mathcal{G}_\star$ have at most countably many vertices and are only considered up to isomorphism, we may without loss of generality embed the graphs $(G, o), (G_1, o_1), (G_2, o_2), \dots$ into the same vertex set V. Their respective adjacency operators $A, A_1, A_2, \dots$ thus act on the same Hilbert space $\ell^2(V)$, the action being defined as zero on the orthogonal complement of the subspace spanned by their proper vertices. A canonical choice for V is the set $\mathbb{N}^* = \bigcup_{d=0}^{\infty} \mathbb{N}^d$ of finite words over integers : the root is represented by the empty-word, and vertices at distance d from the root are represented by words of length d, in such a way that the relation $\equiv$ appearing in (2.3) becomes a true equality. Now fix a word $i \in V$ with length d. By construction, for every $n \geq n_d$, we have that i is a proper vertex of G_n if and only if it is a proper vertex of G. Even better, for $n \geq n_{d+1}$ the neighbors of i are the same in G_n and in G, which precisely mean that $A_n\delta_i = A\delta_i$. By linearity, it follows that any finitely supported vector $\psi \colon V \to \mathbb{C}$ must satisfy

$$n \geq n_\psi \Longrightarrow A_n\psi = A\psi,$$

for some large enough $n_\psi \in \mathbb{N}$. Since finitely supported vectors are dense in $\ell^2(V)$, [87, Theorem VIII.25(a)] guarantees that $A_n \to A$ in the norm resolvent sense, which in particular implies the weak convergence of the rooted spectral measures. Note that this last statement does not depend anymore on the particular way in which the graphs have been embedded. □

We are now in position to prove the following simple but important result, which guarantees that the asymptotic properties of the empirical spectral distribution of sparse graphs may be directly read off on their local weak limit. This idea is not new (see [32]), but the precise statement is.

Theorem 2.1 (Convergence of empirical spectral distributions)**.** *If $\mathcal{L} \in \mathcal{P}(\mathcal{G}_\star)$ is concentrated on self-adjoint graphs, then the implication*

$$G_n \xrightarrow[n\to\infty]{LW} \mathcal{L} \qquad \Longrightarrow \qquad \mu_{G_n} \xrightarrow[n\to\infty]{\mathcal{P}(\mathbb{R})} \overline{\mu}_{\mathcal{L}}$$

holds true for any sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$, with $\overline{\mu}_{\mathcal{L}} = \mathcal{L}\left[\mu^*_{(G,o)}\right]$.

This generalizes various known results that had been proven separately for specific models of sparse random graphs, such as random regular graphs [74] and Erdős-Rényi random graphs [67]. In the special case of trees, the self-adjointness condition is automatically fulfilled by virtue of Corollary 2.1. We state this separately, since it generalizes a recent result [25, Theorem 4.1].

Corollary 2.2 (Trees)**.** *If a sequence of finite trees admits a local weak limit $\mathcal{L}$, then the sequence of their empirical distributions converges to $\overline{\mu}_{\mathcal{L}}$.*

Proof of Theorem 2.1. Fix a bounded continuous function $f\colon \mathbb{R} \to \mathbb{R}$. We want to show

$$\int_{\mathbb{R}} f \, d\mu_{G_n} \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} f \, d\overline{\mu}_{\mathcal{L}},$$

or equivalently, thanks to Lemma 2.1,

$$\mathbb{E}\left[\widehat{f}(X_n)\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[\widehat{f}(X)\right], \tag{2.4}$$

where X_n denotes a random element of $\mathcal{G}_\star$ obtained by uniform rooting of G_n, X denotes a random element of $\mathcal{G}_\star$ with law $\mathcal{L}$, and for every self-adjoint element $(G,o) \in \mathcal{G}_\star$,

$$\widehat{f}(G,o) = \int_{\mathbb{R}} f \, d\mu^*_{(G,o)}.$$

But $\widehat{f}$ is clearly bounded by $\|f\|_\infty$, and continuous by Lemma 2.2, so (2.4) is a direct consequence of the weak convergence $X_n \xrightarrow[n\to\infty]{d} X$, which is precisely what the Theorem asserts. □

2.4 Exact characterization on trees

There are two caveats in Theorem 2.1: first, the class of self-adjoint graphs is not well characterized and second, the limiting spectral distribution $\overline{\mu}_{\mathcal{L}} = \mathcal{L}[\mu^*_{(G,o)}]$ is not very explicit. Here we fix these two problems in the special case of trees. Recall that any probability measure $\mu \in \mathcal{P}(\mathbb{R})$ is characterized by its **Stieltjes-Borel transform** [98], which is the analytic function from $\mathbb{H}$ to $\mathbb{H}$ defined by

$$z \quad \mapsto \quad \int_{\mathbb{R}} \frac{1}{\lambda - z} \mu(d\lambda).$$

In all this section we consider a locally finite tree $T = (V, E)$. Once an arbitrary root $o \in V$ has been specified, we may use the notation $j \to i$ to mean that j is a child of i in the rooted tree (T, o). For $i \in V$ we define T_i as the subtree consisting of i and all its descendants in T, and we let V_i denote its vertex set. Given $z \in \mathbb{H}$, a solution to the **cavity recursion** at activity z on (T, o) is a family $(x_i)_{i \in V} \in \mathbb{H}^V$ satisfying everywhere the local equation

$$x_i \quad = \quad \frac{-1}{z + \sum_{j \to i} x_j}.$$

This recursion is not new (see e.g. Proposition 2.1 in [68]). As it will appear in the proof of Lemma 2.6 below, it can be seen as an operator-theoretic version of Schur's complement formula. Note that any solution to the above equation must immediately satisfy

$$|x_i| \leq \frac{1}{\operatorname{Im} z} \tag{2.5}$$

at every $i \in V$. In particular, the existence of a solution to the cavity recursion is always guaranteed by compactness. The goal of this section is to establish the following recursive characterization.

Theorem 2.2 (Self-adjointness of trees)**.** *The following are equivalent :*

1. *the tree T is essentially self-adjoint;*
2. *for some $z \in \mathbb{H}$, the cavity recursion on (T, o) has a unique solution;*
3. *for every $z \in \mathbb{H}$, the cavity recursion on (T, o) has a unique solution.*

Moreover, in that case, the solution at a given vertex $i \in V$ is exactly the Stieltjes-Borel transform of the rooted spectral measure $\mu^*_{(T_i,i)}$: for all $z \in \mathbb{H}$,

$$x_i(z) \quad = \quad \int_{\mathbb{R}} \frac{1}{\lambda - z} \mu^*_{(T_i,i)}(d\lambda). \tag{2.6}$$

As an application, let us first derive a simple and general sufficient condition for self-adjointness. Introduced by Lyons [70] (see also [72]), the **branching number** of a tree is, for many different purposes (including flows, percolation, electrical

networks, Hausdorff dimension and random walks), the relevant measurement of *how big an infinite tree is*. It is formally defined as

$$\operatorname{br} T \;=\; \sup\left\{\lambda \geq 1 : \inf_{K\,:\,\text{cut}} \sum_{e\in K} \lambda^{-|e|} > 0\right\} \in [1,\infty].$$

where $|e|$ denotes the height of the edge e, and where a cut K is a finite set of edges whose removal makes the connected component of the root finite. Here our interest for $\operatorname{br} T$ lies in the following property.

Lemma 2.3. *Let $T = (V, E)$ be a rooted tree and let $\lambda > \operatorname{br} T$. Then 0 is the only bounded function $f\colon V \to \mathbb{R}_+$ satisfying at every $i \in V$*

$$f(i) \leq \frac{1}{\lambda} \sum_{j\to i} f(j). \tag{2.7}$$

Proof of Lemma 2.3. Let K be a finite cut. Starting from the root o and iterating the inequality (2.7) until the cut K is reached yields

$$f(o) \;\leq\; M \sum_{e\in K} \lambda^{-|e|},$$

where $M = \sup_{i\in V} f(i)$. The fact that $\lambda > \operatorname{br} T$ precisely means that this upper-bound can be made as small as desired by choosing K appropriately, hence $f(o) = 0$. To obtain $f(i) = 0$ for every $i \in V$, simply replace T by the subtree T_i in the above argument (note that $\operatorname{br} T_i \leq \operatorname{br} T$). □

Now if $\mathbf{x} = (x_i)_{i\in V}$ and $\mathbf{x}' = (x'_i)_{i\in V}$ are solutions to the cavity recursion at activity z, then $f\colon i \mapsto |x_i - x'_i|$ is bounded by (2.5) and clearly satisfies (2.7) with $\lambda = (\operatorname{Im} z)^2$. Consequently, we obtain the following result.

Corollary 2.3. *A tree with finite branching number is self-adjoint.*

The branching number of a Galton-Watson tree is its average offspring distribution, almost surely on the event of non-extinction ([70, Proposition 6.4]). In a unimodular Galton-Watson tree with degree distribution π, each subtree has the biased offspring distribution $\widehat{\pi}$ given by (1.1), which has finite mean if and only if π has a finite second moment. Thus,

Corollary 2.4. *A unimodular Galton-Watson tree whose degree distribution π has finite second moment is almost surely self-adjoint.*

Proof of Theorem 2.2

The remaining of the section is devoted to the proof of Theorem 2.2, which we divide into three Lemmas. Since the choice of a root o does not affect the self-adjointness of T, it should not affect either the uniqueness of a solution to the cavity recursion. We first establish this elementary but useful invariance.

Lemma 2.4 (Root invariance)**.** *Fix $z \in \mathbb{H}$ and $o, o' \in V$. Then there is a bijection between the solutions to the cavity recursion at activity z on (T, o) and the solutions to the cavity recursion at activity z on (T, o').*

Proof. By transitivity it is enough to show the result in the case where o' and o are neighbors. By symmetry it is enough to find an injection between the solutions on (T, o) and those on (T, o'). Given a solution $\mathbf{x}$ to the cavity recursion at activity z on (T, o), we construct a solution $\mathbf{x}'$ to the cavity recursion at activity z on (T, o') by setting $x'_i = x_i$ for all $i \notin \{o, o'\}$, then

$$x'_o = \frac{-1}{z + \sum_{i \in \partial o \setminus o'} x_i} \quad \text{and finally} \quad x'_{o'} = \frac{-1}{z + \sum_{i \in \partial o' \setminus o} x_i + x'_o}.$$

Clearly, $\mathbf{x}'$ is a solution to the cavity recursion at activity z on (T, o'), since any $i \in V \setminus \{o, o'\}$ has the same children in (T, o) and in (T, o'), while the set of children of o and o' becomes respectively $\partial o \setminus o'$ and $\partial o'$ in (T, o'). Moreover the map $\mathbf{x} \mapsto \mathbf{x}'$ can be inverted by setting $x_i = x'_i$ for all $i \notin \{o, o'\}$, then

$$x_{o'} = \frac{-1}{z + \sum_{i \in \partial o \setminus o'} x'_i} \quad \text{and finally} \quad x_o = \frac{-1}{z + \sum_{i \in \partial o \setminus o'} x'_i + x_{o'}}.$$

This concludes the proof. Note that the bijection is even constructive. □

For any node $i \in V$, we let $|i|$ denotes its distance to the root and $[o, i]$ denote the set of $|i| + 1$ vertices that compose the unique simple path from i to o. As observed in [80], the adjoint A^* of the adjacency operator A of T is formally defined by

$$(A^*\varphi)(i) = \sum_{j \in \partial i} \varphi(j),$$

and its domain $\mathrm{Dom}(\mathrm{A}^*)$ consists precisely of those $\varphi \in \ell^2(V)$ with $A^*\varphi \in \ell^2(V)$.

Lemma 2.5 (self-adjointness $\Longrightarrow$ uniqueness)**.** *If A is self-adjoint then for each $z \in \mathbb{H}$ there is a unique solution to the cavity recursion on (T, o).*

Proof. To each solution $(x)_{i \in V}$ to the cavity equation at activity $z \in \mathbb{H}$ on (T, o), we associate a function $\varphi \colon V \to \mathbb{C} \setminus \{0\}$ as follows :

$$\varphi(i) \;=\; (-1)^{|i|} \prod_{k \in [o,i]} x_k.$$

Let us first show that for any node $i \in V$ and any finite set $S \subseteq V_i$,

$$\mathrm{Im}(z) \sum_{k \in S} |\varphi(k)|^2 \leq \frac{|\varphi(i)|^2 \mathrm{Im}(x_i)}{|x_i|^2}. \tag{2.8}$$

When S is empty, the thesis is trivial. Now by induction, we may write

$$\begin{aligned}
\mathrm{Im}(z) \sum_{k \in S} |\varphi(k)|^2 &\leq \mathrm{Im}(z)|\varphi(i)|^2 + \mathrm{Im}(z) \sum_{j \to i} \sum_{k \in S \cap T_j} |\varphi(k)|^2 \\
&\leq \mathrm{Im}(z)|\varphi(i)|^2 + \sum_{j \to i} \frac{|\varphi(j)|^2 \mathrm{Im}(x_j)}{|x_j|^2} \\
&\leq |\varphi(i)|^2 \left(\mathrm{Im}(z) + \sum_{j \to i} \mathrm{Im}(x_j) \right) \;=\; \frac{|\varphi(i)|^2 \mathrm{Im}(x_i)}{|x_i|^2},
\end{aligned}$$

where we have successively used the induction hypothesis, the product-form of $\varphi(j)$ and finally the fact that $\mathbf{x}$ is a solution to the cavity equation. Hence (2.8) is proved. In particular, taking $i = o$, we obtain

$$\sum_{k\in S} |\varphi(k)|^2 \leq \frac{\mathrm{Im}\,(x_o)}{\mathrm{Im}(z)},$$

for any finite $S \subseteq V$, which shows that $\varphi \in \ell^2(V)$. Now let $i \in V \setminus o$, and let i' denote its father. Using $\varphi(i) = -x_i\varphi(i')$ and the fact that $\mathbf{x}$ is a solution to the cavity recursion, one finds

$$(A^*\varphi)(i) \;=\; \varphi(i') + \sum_{j\to i}\varphi(j) \;=\; \varphi(i)\left(-\frac{1}{x_i} - \sum_{j\to i} x_j\right) \;=\; z\varphi(i).$$

Similarly, for the case $i = o$, we have

$$(A^*\varphi)(o) \;=\; \sum_{j\to o}\varphi(j) \;=\; -x_o\sum_{j\to o} x_j \;=\; \frac{\sum_{j\to o} x_j}{z + \sum_{j\to o} x_j} \;=\; 1 + zx_o.$$

Summing up, we have shown

$$\varphi \in \mathrm{Dom}(\mathrm{A}^*) \quad \text{and} \quad A^*\varphi = z\varphi + \delta_o. \tag{2.9}$$

To conclude, let $\mathbf{x}, \mathbf{x}'$ both satisfy the cavity recursion at activity z, and let φ, φ' be the vectors resulting from the above construction. Then by (2.9) $\varphi - \varphi'$ must belong to $\mathrm{Ker}(\mathrm{A}^* - \mathrm{z})$. If A is self-adjoint then $\mathrm{Ker}(\mathrm{A}^* - \mathrm{z}) = \{0\}$ ([88, Theorem X.1]), and hence $\varphi = \varphi'$, which is clearly equivalent to the desired identity $\mathbf{x} = \mathbf{x}'$. □

Lemma 2.6 (uniqueness $\Longrightarrow$ self-adjointness)**.** *Fix $z \in \mathbb{H}$. If there is a unique solution $\mathbf{x} = (x_i)_{i\in V}$ to the cavity recursion at activity z on (T, o), then A is self-adjoint and (2.6) holds.*

Proof. For an operator X, we adopt the general notation $X_{ij} = \langle\delta_i|X\delta_j\rangle$. Let A' be any self-adjoint extension of A. Such an extension always exists since A is real and symmetric (this is Von Neumann's Theorem, [88, Theorem X.3]). For every $i \in V$, we let A'_i denote the projection of A' onto $\ell^2(V_i)$, which is also self-adjoint. In particular, $A'_o = A'$. Let S denote the adjacency operator of the *star* formed by the edges that are incident to o. Setting $B = A' - S$, we obtain the following orthogonal decomposition :

$$B = \bigoplus_{i\to o} A'_i,$$

The second resolvent formula [87, Theorem VIII.2] asserts that

$$(B - z)^{-1} - (A' - z)^{-1} = (A' - z)^{-1}S(B - z)^{-1}.$$

Explicitating the definition of S, we obtain for any $j \in V$,

$$(B-z)^{-1}_{oj} - (A'-z)^{-1}_{oj} = \sum_{i\to o}\left((A'-z)^{-1}_{oo}(B-z)^{-1}_{ij} + (A'-z)^{-1}_{oi}(B-z)^{-1}_{oj}\right). \tag{2.10}$$

But since each $\ell^2(V_i), i \to o$ is a stable subspace for B, we know that

$$(B-z)^{-1}_{ij} = \begin{cases} -z^{-1} & \text{if} \quad i = o \text{ and } j = o \\ 0 & \text{if} \quad i = o \text{ and } j \neq o \\ (A'_i - z)^{-1}_{jk} & \text{if} \quad i \to o \text{ and } j \in V_i \\ 0 & \text{if} \quad i \to o \text{ and } j \notin V_i \end{cases}$$

Therefore, taking $j = o$ and $j = i \in \partial o$ in (2.10) yields respectively

$$\begin{aligned} (A'-z)^{-1}_{oo} &= \frac{1}{z}\sum_{i \to o}(A'-z)^{-1}_{oi} - \frac{1}{z} \\ \text{and} \quad (A'-z)^{-1}_{oi} &= -(A'-z)^{-1}_{oo}(A'_i - z)^{-1}_{ii}, \end{aligned}$$

which can be combined into the following recursive identity :

$$(A'_o - z)^{-1}_{oo} = \frac{-1}{z + \sum_{i \to o}(A'_i - z)^{-1}_{ii}}.$$

Iterating this argument, we see that the family $(x_i)_{i \in V}$ defined by

$$x_i = (A'_i - z)^{-1}_{ii}$$

is a solution to the cavity recursion at activity z on (T, o). But there is more. If $j \in V \setminus o$, then j belongs to exactly one of the $V_i, i \to o$, and (2.10) yields

$$(A'-z)^{-1}_{oj} = -x_o(A'_i - z)^{-1}_{ij}.$$

Iterating this argument, we obtain that for every $j \in V$,

$$(A'-z)^{-1}_{oj} = (-1)^{|j|}\prod_{k \in [o,j]} x_k.$$

Consequently, if there is a unique solution $\mathbf{x}$ to the cavity recursion at activity z on (T, o), then it follows that any two self-adjoint extension A', A'' of A must satisfy for all $j \in V$,

$$(A''-z)^{-1}_{oj} = (A'-z)^{-1}_{oj}.$$

By Lemma 2.4, this must in fact hold for any choice of the root $o \in V$. But the resolvent is a bounded operator, and is hence fully determined by its action on the orthonormal basis. Thus, $(A''-z)^{-1} = (A'-z)^{-1}$, i.e. $A'' = A'$: all self-adjoint extensions of A coincide, which means that A is self-adjoint. □

Chapter 3

Rank and nullity

Joint work with Charles Bordenave and Marc Lelarge

In this chapter, we obtain new results concerning the asymptotic multiplicity of the eigenvalue zero in the spectrum of large diluted random graphs. In the special case of Erdős-Rényi graphs with fixed average connectivity, our work answers an open question of Costello and Vu and settles a conjecture due to Bauer and Golinelli. First, we provide a recursive characterization for the atomic mass at zero in the spectrum of a general self-adjoint tree T. Second, we explicitly solve this recursion in the important case where T is a unimodular Galton-Watson tree, leading to a previously unknown formula involving the degree generating function ϕ of T. Third we prove that, under a simple condition on ϕ, the above formula gives indeed the asymptotic spectral mass at zero along any sequence of graphs whose local weak limit is T. Our proofs borrow ideas from analysis of algorithms and random matrix theory. The results presented here have been published in [34].

3.1 Introduction

The **nullity** $\eta(G)$ of a finite graph $G = (V, E)$ is the multiplicity of the eigenvalue zero in its spectrum :

$$\eta(G) = \dim \ker(\mathrm{A}) = |\mathrm{V}| - \mathrm{Rank}(\mathrm{A}) = |\mathrm{V}|\mu_{\mathrm{G}}(\{0\}),$$

where A denotes the adjacency matrix of G and μ_G its empirical spectral distribution, as defined at the beginning of the previous chapter. This graph parameter plays an important role in graph theory and computer science, notably as a simple bound for computationally intractable (NP-hard) invariants [101, 3]. It also have applications in chemistry [36].

It is therefore natural to investigate its typical behavior for classical random graphs. Bauer and Gollineli [15] have computed exactly the expected rank of a uniform random tree of size n, for any $n \in \mathbb{N}$. For the Erdős-Rényi model with edge probability $p = p(n)$ on n vertices, the regime where $p(n) = a \log n/n$ ($a > 0$ fixed, $n \to \infty$) has been studied by Costello, Tao and Vu [43] and Costello and Vu [44]. Their results imply that for $a > 1$, the nullity is zero with high probability while for $0 < a < 1$, it scales like n^{1-a}. In the sparse regime $p(n) = cn$ ($c > 0$), the answer is only known if $c \leq e$, but a conjecture has been formulated for $c > e$ by Bauer and Golinelli [16]:

Conjecture 3.1. *In the Erdős-Rényi model with $p(n) = c/n$, for all $c > 0$,*

$$\mu_{G_n}(\{0\}) \xrightarrow[n\to\infty]{a.s.} \lambda_* + e^{-c\lambda_*} + c\lambda_* e^{-c\lambda_*} - 1, \tag{3.1}$$

where $\lambda_* \in (0, 1)$ denotes the smallest root of $\lambda = e^{-ce^{-c\lambda}}$.

We will settle this conjecture, thereby also answering one of the open questions in Costello and Vu [44]. More generally, we will consider sequences of graphs $(G_n)_{n\geq 1}$ whose local weak limit $\mathcal{L}$ is a unimodular Galton-Watson distribution. Recall from the previous chapter that when the degree distribution π has a finite second moment, the tree is a.s. self-adjoint and

$$\mu_{G_n} \xrightarrow[n\to\infty]{} \overline{\mu}_{\mathcal{L}},$$

In particular, this weak convergence implies that almost surely,

$$\limsup_{n\to\infty} \mu_{G_n}(\{0\}) \leq \overline{\mu}_{\mathcal{L}}(\{0\}). \tag{3.2}$$

Our first result is the explicit computation of this natural upper bound. The formula involves a function $M\colon [0, 1] \to \mathbb{R}$ defined in terms of the degree generating function $\phi(\lambda) = \sum_{n=0}^{\infty} \pi_n \lambda^n$ as follows :

$$M(\lambda) = \phi'(1)\lambda\overline{\lambda} + \phi(1 - \lambda) + \phi\left(1 - \overline{\lambda}\right) - 1 \quad \text{with} \quad \overline{\lambda} = \frac{\phi'(1 - \lambda)}{\phi'(1)}.$$

Theorem 3.1. *Let $\pi \in \mathcal{P}(\mathbb{N})$ have a finite second moment. Then for the unimodular Galton-Watson distribution $\mathcal{L}$ with degree distribution π,*

$$\overline{\mu}_{\mathcal{L}}(\{0\}) \;=\; \max_{\lambda\in[0,1]} M(\lambda).$$

Moreover, any λ where the above maximum is achieved must satisfy $\lambda = \overline{\overline{\lambda}}$.

To obtain a lower bound, we will analyze the so-called **leaf removal process**, which was originally introduce by Karp and Sipser [66] as a heuristic algorithm for constructing a large matching (subset of pairwise disjoint edges) on a sparse graph G. A **leaf** in a graph is simply a vertex of degree 1. The leaf removal is the action of deleting an arbitrary leaf and its unique neighbor. The nullity $\eta(G)$ is well-known to be invariant under such a transformation (a proof can be found for example in [45]). Iterating this procedure eventually produces a **core** K with minimum degree at least 2, plus a certain number of isolated vertices $\mathrm{LR}(G) \in \mathbb{N}$. Both K and $LR(G)$ are easily checked to be independent of the successive choices that have been made at each removal step. Clearly,

$$\eta(G) = \mathrm{LR}(G) + \eta(K). \tag{3.3}$$

Motivated by the computation of the size of a largest matching, Karp and Sipser [66] analyzed the leaf-removal process on a Erdős-Rényi random graph with connectivity c on n vertices (see [11, 27] for refinements of the method). They approximated the dynamics by a system of differential equations which they explicitly solved in the $n \to \infty$ limit. In particular, they obtained that

$$LR(G_n) \sim n\left(\lambda_* + e^{-c\lambda_*} + c\lambda_* e^{-c\lambda_*} - 1\right),$$

as $n \to \infty$ and that the size of the remaining core K is $o(n)$ if and only if $c \leq e$. Thus, their result implies that the conjecture (3.1) holds when $c \leq e$, as observed by Bauer and Golinelli [16]. However for $c > e$, the size of the core is not negligible and the same argument only leads to the following inequality :

$$\liminf_{n\to\infty} \mu_{G_n}(\{0\}) \geq \lambda_* + e^{-c\lambda_*} + c\lambda_* e^{-c\lambda_*} - 1.$$

To generalize this lower bound, we will define and study the leaf-removal process on an arbitrary unimodular Galton-Watson tree (T, o). As we will see, the probability that the root eventually becomes isolated is exactly given by the first local extremum of the above-defined function M. Consequently,

Theorem 3.2. *For any sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$ whose random weak is a unimodular Galton-Watson distribution,*

$$\liminf_{n\to\infty} \mu_{G_n}(\{0\}) \geq M(\lambda_*), \tag{3.4}$$

where $\lambda_* \in (0, 1)$ is the smallest root of $\lambda = \overline{\overline{\lambda}}$.

Combined with Theorem 3.1, this lower bound yields the following result.

Corollary 3.1. *Let $\mathcal{L}$ be the unimodular Galton-Watson distribution with degree distribution π. Assume that π has a finite second moment, and that the first local extremum of M is its global maximum. Then, for any sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$ whose random weak is $\mathcal{L}$,*

$$\mu_{G_n}(\{0\}) \xrightarrow[n\to\infty]{} \overline{\mu}_{\mathcal{L}}(\{0\}) = M(\lambda_*),$$

where $\lambda_* \in (0,1)$ is the smallest root of $\lambda = \overline{\overline{\lambda}}$. Moreover, a simple sufficient condition for the above assumption to hold is that $\log(\phi'')$ is concave on $(0,1)$.

In the Erdős-Rényi case, the local weak limit is a.s. the Poisson-Galton-Watson distribution with mean c, for which $\phi(\lambda) = e^{c\lambda - c}$. Clearly, ϕ'' is log-concave, and hence Conjecture 3.1 is proved. A plot of M for $c = 4$ is shown in red in Figure 3.1. The yellow and green curves both correspond to a situation where the tree is leafless ($\pi_0 = \pi_1 = 0$), implying $\lambda_* = 0$ and $M(\lambda_*) = 0$. For the yellow one (3−regular random graphs) we see that $\overline{\mu}_{\mathcal{L}}(\{0\}) = 0$, in agreement with the fact that the Kesten-McKay distribution $\overline{\mu}_{\mathcal{L}}$ is absolutely continuous with respect to Lebesgue's measure. Contrastingly, for the green one (75% of vertices have degree 3 and 25% have degree 15), we see that $\overline{\mu}_{\mathcal{L}}(\{0\}) > 0$. Here the lower bound $M(\lambda_*)$ does not match the upper bound $\max M$, and the asymptotic behavior of the nullity in that case remains an interesting open question.

To the best of our knowledge, the above formula was previously unknown. However, this remains only a small achievement for the global understanding of the asymptotic spectral spectral distribution. For example, in the Erdős-Rényi case, the atomic part of the limiting spectral distribution $\overline{\mu}_{\mathcal{L}}$ is dense in $\mathbb{R}$, and nothing is known concerning the atomic masses other than the one at 0. There is also a interesting conjecture concerning the absolutely continuous part of $\overline{\mu}_{\mathcal{L}}$: a measure μ with absolutely continuous part μ_{ac} is said to have *extended states* (resp. no extended state) at $E \in \mathbb{R}$ if $\lambda \mapsto \mu_{ac}(-\infty, \lambda)$ is differentiable at $\lambda = E$ and its derivative is positive (resp. null). This notion was introduced in mathematical physics in the context of spectra of random Schrödinger operators [68] ; a recent treatment can be found in Aizenman, Sims and Warzel [2]. For Erdős-Rényi graphs, Bauer and Golinelli have conjectured that $\overline{\mu}_{\mathcal{L}}$ has no extended state at $E = 0$ when $0 < c \leq e$, and has extended states at $E = 0$ when $c > e$. More generally, one may wonder whether the absolutely continuous part of $\overline{\mu}_{\mathcal{L}}$ is zero when $0 < c \leq e$. Finally, the existence of a singular continuous part in $\overline{\mu}_{\mathcal{L}}$ is apparently unknown.

The remainder of the chapter is organized as follows : in Section 3.2, we provide a recursive characterization of the spectrum at zero for a general self-adjoint tree. In section 3.3 we show that for a unimodular Galton-Watson tree, this recursion simplifies into a recursive distributional equation that can be solved explicitly, leading to Theorem 3.1. Finally, section 3.4 is devoted to the leaf-removal process and to the proof of Theorem 3.2.

3.2 A recursion for the atomic mass at zero

Let $T = (V, E)$ be a self-adjoint tree, and let $o \in V$ be an arbitrarily chosen root. Our goal here is to characterize $\mu^*_{(T,o)}(\{0\})$, the atomic mass at zero of the rooted

spectral measure $\mu^*_{(T,o)}$. More generally, for $i \in V$ we set

$$m_i = \mu^*_{(T_i,i)}(\{0\}) \in [0,1].$$

Proposition 3.1 (Recursive characterization of the spectral mass at zero)**.** *The family $(m_i)_{i\in V}$ is the largest solution in $[0,1]^V$ to the system of equations*

$$m_i = \left(1 + \sum_{j\to i}\left(\sum_{k\to j} m_k\right)^{-1}\right)^{-1}, \tag{3.5}$$

with the conventions $1/0 = \infty$ and $1/\infty = 0$.

Proof. To avoid confusions with our notation for vertices, the canonical complex root of -1 is denoted by $\mathbf{i}$. Taking $z = \mathbf{i}t$ $(t > 0)$ in (2.6) gives

$$\int_{\mathbb{R}} \frac{\lambda + \mathbf{i}t}{\lambda^2 + t^2} \mu^*_{(T_i,i)}(d\lambda) = x_i(\mathbf{i}t).$$

Multiplying this identity by $\mathbf{i}t$ and letting $t \to 0$, we obtain by the dominated convergence theorem

$$-\mathbf{i}t x_i(\mathbf{i}t) \xrightarrow[t\to 0]{} m_i.$$

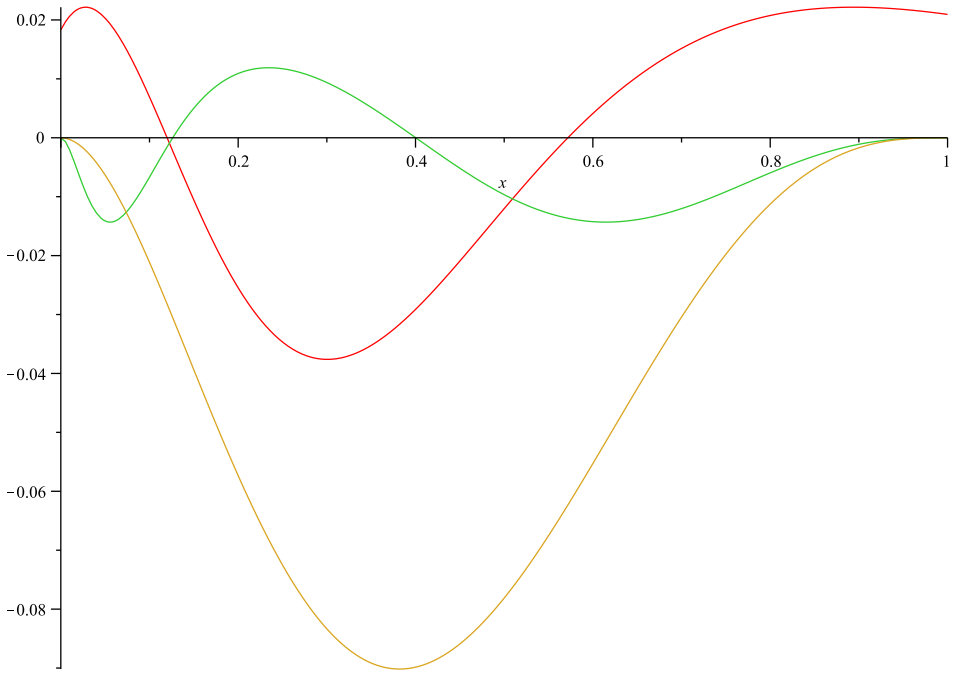


Figure 3.1: Plots of $\lambda \mapsto M(\lambda)$ for $\phi(\lambda) = e^{4\lambda-4}$ (red), $\phi(\lambda) = \lambda^3$ (yellow) and $\phi(\lambda) = \frac{3}{4}\lambda^3 + \frac{1}{4}\lambda^{15}$ (green). The maximum of the curve gives the value of $\overline{\mu}_{\mathcal{L}}(\{0\})$ (upper bound for the asymptotic nullity), and the first local extremum gives the probability that the root ever becomes isolated during the leaf-removal process (lower bound for the asymptotic nullity).

But on the other hands, using twice the fact that $\mathbf{x}$ is the solution to the cavity recursion, we may write

$$\begin{aligned} -\mathbf{i}t x_i(\mathbf{i}t) &= \left(1 + \sum_{j \to i} \frac{x_j(\mathbf{i}t)}{\mathbf{i}t}\right)^{-1} \\ &= \left(1 + \sum_{j \to i} \left(t^2 - \mathbf{i}t \sum_{k \to j} x_k(\mathbf{i}t)\right)^{-1}\right)^{-1}. \end{aligned} \tag{3.6}$$

Letting $t \to 0$ in this last identity gives precisely the recursion (3.5). When the tree T is finite, this recursion characterizes the family $(m_i)_{i \in V}$, since it can be computed iteratively from the leaves up to the root. However, when T is infinite, (3.5) may admit several other solutions. Fortunately, among all of them, $(m_i)_{i \in V}$ is always the largest, as we will now see. For a fixed radius $d \in \mathbb{N}$, we let $x_i^{(d)}(z)$ denote the value at the root of the unique solution to the cavity recursion at activity z on the finite truncated tree $[T_i, i]_d$. Since $[T_i, i]_d$ converges locally to (T_i, i) as $d \to \infty$, Lemma 2.2 guarantees that

$$x_i^{(d)}(z) \xrightarrow[d \to \infty]{} x_i(z).$$

Now consider any solution $(m'_i)_{i \in V} \in [0,1]^V$ to (3.5), and fix $t > 0$. Let us show by induction that for every $d \in \mathbb{N}$,

$$\forall i \in V, \qquad m'_i \leq -\mathbf{i}t x_i^{(2d)}(\mathbf{i}t), \tag{3.7}$$

This will conclude our proof since we may let $d \to \infty$ and then $t \to 0$ to reach the desired $m'_i \leq m_i$. When $d = 0$, our thesis is trivial since the right-hand side is 1. Now, if (3.7) holds for some $d \in \mathbb{N}$, then it must holds for $d+1$ as well since we may write for every $i \in V$,

$$\begin{aligned} m'_i &= \left(1 + \sum_{j \to i} \left(\sum_{k \to j} m'_k\right)^{-1}\right)^{-1} \\ &\leq \left(1 + \sum_{j \to i} \left(t^2 + \sum_{k \to j} m'_k\right)^{-1}\right)^{-1} \\ &\leq \left(1 + \sum_{j \to i} \left(t^2 - \mathbf{i}t \sum_{k \to j} x_k^{(2d)}(\mathbf{i}t)\right)^{-1}\right)^{-1} \\ &\leq -\mathbf{i}t x_i^{(2d+2)}(\mathbf{i}t). \end{aligned}$$

For the last line we have simply applied (3.6) to the tree $[T_i, i]_{2d+2}$. □

3.3 Explicit formula for Galton-Watson trees

In this section, we solve the above recursion in the special case where the rooted tree (T, o) is a unimodular Galton-Watson tree whose degree distribution π has a

finite second moment. Owing to the Markovian nature of the branching process, the set of equations (3.5) characterizing $\mu^*_{(T,o)}$ takes the much simpler form of a **recursive distributional equation** (RDE). The latter is a common ingredient in the objective method (see the survey [6]). Given $\nu, \nu' \in \mathcal{P}(\mathbb{N})$ and $P \in \mathcal{P}([0,1])$, we define $\Theta_{\nu,\nu'}(P) \in \mathcal{P}([0,1])$ to be the distribution of the random variable

$$Y = \left(1 + \sum_{i=1}^{N} \left(\sum_{j=1}^{N_i} X_{ij}\right)^{-1}\right)^{-1}, \tag{3.8}$$

where $N \sim \nu$, $N_i \sim \nu'$, $X_{ij} \sim P$, all of them being independent. With this notation in hands, Proposition 3.1 clearly implies the following.

Corollary 3.2. *The random variable $\mu^*_{(T,o)}(\{0\})$ has distribution $\Theta_{\pi,\widehat{\pi}}(P)$, where $P \in \mathcal{P}([0,1])$ is the largest solution to the RDE*

$$P = \Theta_{\widehat{\pi},\widehat{\pi}}(P). \tag{3.9}$$

In the above sentence the word *largest* refers to the usual stochastic order on $\mathcal{P}([0,1])$: $P_1 \leq P_2$ if for any continuous, increasing function $\varphi\colon [0,1] \to \mathbb{R}$,

$$\int_{[0,1]} \varphi dP_1 \leq \int_{[0,1]} \varphi dP_2.$$

We will now explicitly solve the distributional fixed point equation (3.9). From now on, we assume that $\pi_0 + \pi_1 < 1$, otherwise $\widehat{\pi} = \delta_0$, and the only solution to (3.9) is clearly $P = \delta_1$. We recall that $\phi(\lambda) = \sum_{n\in\mathbb{N}} \pi_n \lambda^n$ denotes the generating function of π, and that for any $\lambda \in [0,1]$,

$$M(\lambda) = \phi'(1)\lambda\overline{\lambda} + \phi(1-\lambda) + \phi(1-\overline{\lambda}) - 1 \text{ where} \qquad \overline{\lambda} = \phi'(1-\lambda)/\phi'(1).$$

First observe that $M'(\lambda) = \phi''(1-\lambda)(\overline{\overline{\lambda}} - \lambda)$. Thus, $M'(0) \geq 0$, $M'(1) \leq 0$, and any $\lambda \in [0,1]$ where M admits a local extremum must satisfy $\lambda = \overline{\overline{\lambda}}$. We will say that M admits an **historical record** at λ if $\lambda = \overline{\overline{\lambda}}$ and $M(\lambda) > M(\lambda')$ for any $0 \leq \lambda' < \lambda$. Since $[0,1]$ is compact and M is analytic, there are only finitely many such records. As we will now see, they are in one-to-one correspondence with the solutions to (3.9).

Proposition 3.2. *Let $\lambda_1 < \ldots < \lambda_r$ denote the locations of the historical records of M. Then, the RDE (3.9) admits exactly r solutions, and they can be stochastically ordered : $P_1 < \ldots < P_r$. Moreover, for any $1 \leq i \leq r$,*

(i) $P_i(\{0\}) = 1 - \lambda_i$;

(ii) $\Theta_{\pi,\widehat{\pi}}(P_i)$ *has mean* $M(\lambda_i)$.

In view of Corollary 3.2, we have in particular

$$\mathbb{E}[\mu^*_{(T,o)}(\{0\})] = \max_{\lambda\in[0,1]} M(\lambda),$$

which is precisely what Theorem 3.1 asserts. The proof of Proposition 3.2 relies on two lemmas, the first one being straightforward.

Lemma 3.1. *For any $\nu, \nu' \in \mathcal{P}(\mathbb{N}) \setminus \{\delta_0\}$, $\Theta_{\nu,\nu'}$ is continuous and strictly increasing on $\mathcal{P}([0,1])$.*

Lemma 3.2. *For any $P \in \mathcal{P}([0,1])$, letting $\lambda = P(\{0\}^c)$, we have*

(i) $\Theta_{\widehat{\pi},\widehat{\pi}}(P)(\{0\}^c) = \overline{\overline{\lambda}}$

(ii) if $\Theta_{\widehat{\pi},\widehat{\pi}}(P) \leq P$, then the mean of $\Theta_{\pi,\widehat{\pi}}(P)$ is at least $M(\lambda)$.

(iii) if $\Theta_{\widehat{\pi},\widehat{\pi}}(P) \geq P$, then the mean of $\Theta_{\pi,\widehat{\pi}}(P)$ is at most $M(\lambda)$;

In particular, if P satisfies (3.9) then $\lambda = \overline{\overline{\lambda}}$ and $\Theta_{\pi,\widehat{\pi}}(P)$ has mean $M(\lambda)$.

Proof of Lemma 3.2. In equation (3.8) it is clear that $Y > 0$ if and only if for each $1 \leq i \leq N$, there exists $1 \leq j \leq N_i'$ such that $X_{ij} > 0$. Denoting by $\widehat{\phi}$ the generating function of $\widehat{\pi}$, this rewrites as follows.

$$\Theta_{\widehat{\pi},\widehat{\pi}}(P)(\{0\}^c) = \widehat{\phi}\left(1 - \widehat{\phi}(1-\lambda)\right).$$

But from (1.1) we see that $\widehat{\phi}(\lambda) = \phi'(\lambda)/\phi'(1)$, i.e. $\widehat{\phi}(1-\lambda) = \overline{\lambda}$, hence the first result. Now let $X \sim P$, $N \sim \pi$, $\widehat{N} \sim \widehat{\pi}$, and let $S, S_1, \ldots$ have the distribution of the sum of $\widehat{N}$ i.i.d. copies of X, all these variables being independent. Then, the mean of $\Theta_{\pi,\widehat{\pi}}(P)$ is

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{1+\sum_{i=1}^N S_i^{-1}}\right] &= \mathbb{E}\left[\left(1 - \frac{\sum_{i=1}^N S_i^{-1}}{1+\sum_{i=1}^N S_i^{-1}}\right)\mathbf{1}_{\{\forall i=1\ldots N, S_i>0\}}\right].\\
&= \mathbb{P}(\forall i = 1\ldots N, S_i > 0) - \sum_{n=1}^{\infty} n\pi_n \mathbb{E}\left[\frac{S^{-1}\mathbf{1}_{\{S>0,\forall i=1\ldots n-1, S_i>0\}}}{1+S^{-1}+\sum_{i=1}^{n-1} S_i^{-1}}\right]\\
&= \phi(1-\widehat{\phi}(1-\lambda)) - \phi'(1)\sum_{n=0}^{\infty}\widehat{\pi}_n \mathbb{E}\left[\frac{S^{-1}\mathbf{1}_{\{S>0,\forall i=1\ldots n, S_i>0\}}}{1+S^{-1}+\sum_{i=1}^{n} S_i^{-1}}\right]\\
&= \phi(1-\overline{\lambda}) - \phi'(1)\mathbb{E}\left[\frac{S^{-1}}{1+S^{-1}+\sum_{i=1}^{\widehat{N}} S_i^{-1}}\mathbf{1}_{\left\{S>0,\forall i=1\ldots\widehat{N}, S_i>0\right\}}\right]\\
&= \phi(1-\overline{\lambda}) - \phi'(1)\mathbb{E}\left[\frac{S^{-1}}{Y^{-1}+S^{-1}}\mathbf{1}_{\{S>0,Y>0\}}\right]\\
&= \phi(1-\overline{\lambda}) - \phi'(1)\mathbb{E}\left[\frac{Y}{S+Y}\mathbf{1}_{\{S>0\}}\right].
\end{aligned}$$

Now, for any fixed $s > 0$, $y \mapsto \frac{y}{s+y}$ is increasing. Thus, depending on whether $\Theta_{\widehat{\pi},\widehat{\pi}}(P) \geq P$ or $\Theta_{\widehat{\pi},\widehat{\pi}}(P) \leq P$, the above quantity is at most/least

$$\phi(1-\overline{\lambda}) - \phi'(1)\mathbb{E}\left[\frac{X}{X+S}\mathbf{1}_{\{S>0\}}\right]. \tag{3.10}$$

To conclude the proof, we simply need to check that this last quantity is precisely $M(\lambda)$. Recall that $\mathbb{P}(X>0) = \lambda$, and set $K = \sum_{i=1}^{\widehat{N}} \mathbf{1}_{\{X_i>0\}}$. Let also X^* have the

law of X conditioned on $\{X > 0\}$, and let $X_1^*, X_2^*, \ldots$ be i.i.d. copies of X^*, all of them being independent of K. Then,

$$\begin{aligned}
\phi'(1)\mathbb{E}\left[\frac{X}{X+S}\mathbf{1}_{\{S>0\}}\right] &= \phi'(1)\lambda\mathbb{E}\left[\frac{X^*}{X^* + \sum_{i=1}^K X_i^*}1_{\{K\geq 1\}}\right] \\
&= \phi'(1)\lambda\mathbb{E}\left[\frac{1}{K+1}1_{\{K\geq 1\}}\right] \\
&= \phi'(1)\lambda\sum_{n=1}^{\infty}\widehat{\pi}_n\sum_{k=1}^{n}\binom{n}{k}\frac{\lambda^k(1-\lambda)^{n-k}}{k+1} \\
&= \sum_{n=1}^{\infty}\pi_{n+1}\sum_{k=1}^{n}\binom{n+1}{k+1}\lambda^{k+1}(1-\lambda)^{n-k} \\
&= \sum_{n=2}^{\infty}\pi_n\left(1-(1-\lambda)^n - n\lambda(1-\lambda)^{n-1}\right) \\
&= 1-\phi(1-\lambda)-\lambda\phi'(1-\lambda).
\end{aligned}$$

Thus, (3.10) is precisely $M(\lambda)$, as desired. □

Proof of Proposition 3.2. Fix $\lambda \in [0,1]$ such that $\overline{\overline{\lambda}} = \lambda$. Set $P_0 = \text{Bernoulli}(\lambda)$ and then iteratively $P_{k+1} = \Theta_{\widehat{\pi},\widehat{\pi}}(P_k)$ for every $k \in \mathbb{N}$. By Lemma 3.2, $P_k(\{0\}^c) = \lambda$ for every $k \in \mathbb{N}$. Since Bernoulli(λ) is the largest element of $\mathcal{P}([0,1])$ putting mass λ on $\{0\}^c$, we have $P_1 \leq P_0$. Consequently, Lemma 3.1 guarantees that $(P_k)_{k\in\mathbb{N}}$ is decreasing and that the limit

$$P_\infty = \lim_{k\to\infty} \downarrow P_k$$

is a solution to (3.9). Note that $\lambda_\infty = P_\infty(\{0\}^c)$ satisfies $\lambda_\infty \leq \lambda$. Moreover, the mean of $\Theta_{\pi,\widehat{\pi}}(P_\infty)$ must be both

- equal to $M(\lambda_\infty)$ by applying lemma 3.2 to P_∞ ;
- at least $M(\lambda)$ since for every $k \in \mathbb{N}$ the mean of $\Theta_{\pi,\widehat{\pi}}(P_k)$ is at least $M(\lambda)$ by applying Lemma 3.2(ii) to P_k.

To sum up, we have just shown both $M(\lambda) \leq M(\lambda_\infty)$ and $\lambda_\infty \leq \lambda$. Consequently, if M admits an historical record at λ, then we must have $\lambda_\infty = \lambda$, so we have constructed a solution to (3.9) satisfying $P_\infty(\{0\}^c) = \lambda$.

Conversely, let P be any solution to (3.9) and set $\lambda = P(\{0\}^c)$. By Lemma 3.2, $\lambda = \overline{\overline{\lambda}}$. We claim that $P = P_\infty$ and that M must admit an historical record at λ. Indeed, the inequality $P \leq \text{Bernoulli}(\lambda)$ implies $P \leq P_\infty$ (since $\Theta_{\pi,\widehat{\pi}}$ is increasing), and in particular $\lambda \leq \lambda_\infty$. But we already had $\lambda_\infty \leq \lambda$, so $\lambda = \lambda_\infty$ and $M(\lambda) = M(\lambda_\infty)$. In other words, the ordered distributions $\Theta_{\widehat{\pi},\pi}(P) \leq \Theta_{\widehat{\pi},\pi}(P_\infty)$ share the same mean, hence are equal. This ensures that $P = P_\infty$. Finally, if $\lambda' < \lambda$ is any historical record location, we may apply the above argument to λ' instead of λ and obtain that

$$P'_\infty = \lim_{k\to\infty} \downarrow \Theta^{(k)}_{\widehat{\pi},\widehat{\pi}}(\text{Bernoulli}(\lambda'))$$

satisfies (3.9) and $P'_\infty(\{0\}^c) = \lambda'$. But Bernoulli$(\lambda') <$ Bernoulli(λ), so $P'_\infty \leq P_\infty$. The latter inequality is in fact strict because $P'_\infty(\{0\}^c) < P_\infty(\{0\}^c)$. Consequently, $\Theta_{\pi,\widehat{\pi}}(P'_\infty) < \Theta_{\pi,\widehat{\pi}}(P_\infty)$ and taking expectations, $M(\lambda') < M(\lambda)$. Thus, M admits an historical record at λ. □

3.4 The leaf-removal process

On a locally finite graph $G = (V, E)$, the **leaf-removal process** can be described as an iterative procedure that produces two non-decreasing sequences of subsets of V, $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$ and $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots$, as follows : initially, $\mathcal{A}_0$ consists of all isolated vertices of G, and $\mathcal{B}_0$ is empty. Then, at each step $t \in \mathbb{N}$, we consider the remaining graph $G_t = G \setminus (\mathcal{A}_t \cup \mathcal{B}_t)$, on which we perform the following action : all leaves of G_t whose unique neighbor is not a leaf in G_t are added to $\mathcal{A}_t$, and all their unique neighbors are added to $\mathcal{B}_t$. Leaves whose unique neighbor is also a leaf (i.e. isolated edges) are ignored. This defines $\mathcal{A}_{t+1}$ and $\mathcal{B}_{t+1}$, and we pursue iteratively. In the limit, the process results in the following two (disjoint) subsets of V:

$$\mathcal{A} = \bigcup_{t=0}^{\infty} \uparrow \mathcal{A}_t \text{ and } \mathcal{B} = \bigcup_{t=0}^{\infty} \uparrow \mathcal{B}_t.$$

Their interest for us lies in the following property (first observed in [16]).

Lemma 3.3 (Leaf-removal on finite graphs)**.** *For any finite graph G,*

$$\eta(G) \geq |\mathcal{A}| - |\mathcal{B}|.$$

Proof. Fix $t \in \mathbb{N}$. For each vertex $i \in \mathcal{B}_{t+1} \setminus \mathcal{B}_t$, we may arbitrarily select one of its neighboring leaves i' in G_t, and remove both of them from G_t. It is a well-known and elementary fact that each such leaf removal does not alter the nullity of the underlying graph (a proof can be found for example in [45]). However, the resulting graph $\widetilde{G}_t$ is not yet G_{t+1}, since each $i \in \mathcal{B}_{t+1} \setminus \mathcal{B}_t$ may have other neighboring leaves than i' in G_t. Their total number is clearly $|\mathcal{A}_{t+1} \setminus \mathcal{A}_t| - |\mathcal{B}_{t+1} \setminus \mathcal{B}_t|$, and they are isolated in $\widetilde{G}_t$. An isolated vertex corresponds to an index in the adjacency matrix for which both the line and column are full of zeros, so its removal decreases the nullity by one. Thus,

$$\eta(G_t) = \eta(G_{t+1}) + |\mathcal{A}_{t+1} \setminus \mathcal{A}_t| - |\mathcal{B}_{t+1} \setminus \mathcal{B}_t|.$$

"Integrating" over t, we obtain

$$\eta(G) = \eta(G_t) + |\mathcal{A}_t| - |\mathcal{B}_t| \geq |\mathcal{A}_t| - |\mathcal{B}_t|, \tag{3.11}$$

and the result follows by letting $t \to \infty$. □

When G is infinite the difference $|\mathcal{A}| - |\mathcal{B}|$ does not make any sense. However, we may ask whether a given vertex o belongs to $\mathcal{A}$, or to $\mathcal{B}$ or to none of them. Then, Lemma 3.3 admits the following infinite analog.

Lemma 3.4 (Leaf-removal on local weak limits)**.** *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of finite graphs admitting a local weak limit $\mathcal{L} \in \mathcal{P}(\mathcal{G}_\star)$. Then,*

$$\liminf_{n\to\infty} \mu_{G_n}(\{0\}) \geq \mathcal{L}\left(o \in \mathcal{A}\right) - \mathcal{L}\left(o \in \mathcal{B}\right).$$

Proof. Fix $t \in \mathbb{N}$. By construction, the events $\{o \in \mathcal{A}_t\}, \{o \in \mathcal{B}_t\}$ depend only on $[G, o]_{2t+1}$, so $(G, o) \mapsto \mathbf{1}_{\{o\in\mathcal{A}_t\}}$ and $(G, o) \mapsto \mathbf{1}_{\{o\in\mathcal{B}_t\}}$ are continuous with respect to the topology of local convergence. Now for a finite graph G, (3.11) may be rewritten as

$$\mu_G(\{0\}) \;\geq\; \frac{1}{|V|}\sum_{o\in V} \mathbf{1}_{\{o\in\mathcal{A}_t\}} - \frac{1}{|V|}\sum_{o\in V} \mathbf{1}_{\{o\in\mathcal{B}_t\}},$$

where we recognize expectation with respect to uniform rooting on G. Thus, taking $G = G_n$ and letting $n \to \infty$, $G_n \xrightarrow[n\to\infty]{LW} \mathcal{L}$ implies

$$\liminf_{n\to\infty} \mu_{G_n}(\{0\}) \;\geq\; \mathcal{L}\left(o \in \mathcal{A}_t\right) - \mathcal{L}\left(o \in \mathcal{B}_t\right).$$

Letting finally $t \to \infty$ concludes the proof. □

To prove Theorem 3.2, it now remains to compute $\mathcal{L}\left(o \in \mathcal{A}\right) - \mathcal{L}\left(o \in \mathcal{B}\right)$ when $\mathcal{L}$ is the law of a unimodular Galton-Watson tree. We adopt the notations of the previous section.

Lemma 3.5 (Leaf-removal on a unimodular Galton-Watson tree)**.** *When $\mathcal{L}$ is a unimodular Galton-Watson distribution, we have*

$$\mathcal{L}\left(o \in \mathcal{A}\right) - \mathcal{L}\left(o \in \mathcal{B}\right) \;=\; M(\lambda_*),$$

where λ_* is the smallest root of $\lambda = \overline{\overline{\lambda}}$.

Proof. The argument is close to that appearing in [66, Section 4]. For the leaf-removal process we write $\mathcal{A}_t(G)$ and $\mathcal{B}_t(G)$ when we want to make the dependency upon the underlying graph G explicit. Let $T = (V, E)$ be a tree rooted at o. To obtain a proper recursion, let us first slightly modify our tree so that the leaf removal process evolves only from children to parents. Specifically, we let $\widetilde{T}$ denote the tree obtained from T by attaching an extra infinite simple path to the root o. For every $i \in V$, we let $\widetilde{T}_i$ denote the result of performing this operation on the rooted tree (T_i, i). By construction, a node i satisfies $i \in \mathcal{B}_t(\widetilde{T}_i)$ if and only if at least one of its children $j \to i$ satisfies $j \in \mathcal{A}_t(\widetilde{T}_j)$. Similarly, $i \in \mathcal{A}_{t+1}(\widetilde{T}_i)$ if and only if all its children $j \to i$ satisfy $j \in \mathcal{B}_t(\widetilde{T}_j)$. Therefore, when the initial tree (T, o) is obtained by a homogeneous Galton-Watson branching process with offspring distribution $\widehat{\pi}$, the numbers

$$\alpha_t = \mathbb{P}\left(o \in \mathcal{A}_t(\widetilde{T})\right) \quad \text{and} \quad \beta_t = \mathbb{P}\left(o \in \mathcal{B}_t(\widetilde{T})\right)$$

must satisfy the recursion $\beta_t = 1 - \widehat{\phi}(1 - \alpha_t)$ and $\alpha_{t+1} = \widehat{\phi}(\beta_t)$, where $\widehat{\phi}(\lambda) = \sum_{n=0}^{\infty} \widehat{\pi}_n \lambda^n$ is the generating function of the offspring distribution $\widehat{\pi}$. As already

pointed out, (1.1) implies $\widehat{\phi}(\lambda) = \phi'(\lambda)/\phi'(1)$, and hence $\widehat{\phi}\,(1-\lambda) = \overline{\lambda}$. Thus, we obtain that for every $t \in \mathbb{N}$,

$$\beta_t = 1 - \overline{\alpha_t} \quad \text{and} \quad \alpha_{t+1} = \overline{\overline{\alpha_t}}$$

But $\lambda \mapsto \overline{\overline{\lambda}}$ is non-decreasing and continuous, and $\alpha_0 = 0$ (o is not isolated in $\widetilde{T}$). Thus, the sequence $(\alpha_t)_{t\in\mathbb{N}}$ is non-decreasing and converges to the smallest root of $\lambda \mapsto \overline{\overline{\lambda}}$. We have thus shown

$$\mathbb{P}\left(o \in \mathcal{A}(\widetilde{T})\right) = \lambda_* \quad \text{and} \quad \mathbb{P}\left(o \in \mathcal{B}(\widetilde{T})\right) = 1 - \overline{\lambda_*}.$$

Let us now see how these numbers change when $\widetilde{T}$ is replaced by T. By construction,

1. if all children of o satisfy $i \in \mathcal{B}_t(\widetilde{T}_i)$, then $o \in \mathcal{A}_{t+1}(T)$;
2. if all children of o satisfy $i \in \mathcal{B}_t(\widetilde{T}_i)$, except one which satisfies $i \notin \mathcal{B}_t(\widetilde{T}_i) \cup \mathcal{A}_{t+1}(\widetilde{T}_i)$, then again $o \in \mathcal{A}_{t+1}(T)$;
3. if o has at least two children $i \neq j$ satisfying $i \in \mathcal{A}_{t+1}(\widetilde{T}_i)$ and $j \notin \mathcal{B}_t(\widetilde{T}_j)$, then $o \in \mathcal{B}_{t+1}(T)$;
4. in all other cases, $o \notin \mathcal{A}_{t+1}(T) \cup \mathcal{B}_{t+1}(T)$.

Therefore, if $\mathcal{L}$ is the law of a unimodular Galton-Watson tree with degree distribution π, we obtain

$$\begin{aligned} \mathcal{L}\,(o \in \mathcal{A}_{t+1}) &= \phi(\beta_t) + (1 - \alpha_{t+1} - \beta_t)\phi'(\beta_t) \\ \mathcal{L}\,(o \in \mathcal{B}_{t+1}) &= 1 - \phi(1-\alpha_{t+1}) - \alpha_{t+1}\phi'(\beta_t). \end{aligned}$$

Finally, letting $t \to \infty$ yields

$$\begin{aligned} \mathcal{L}\,(o \in \mathcal{A}) &= \phi(1 - \overline{\lambda_*}) + (\overline{\lambda_*} - \lambda_*)\phi'(1 - \overline{\lambda_*}) \\ \mathcal{L}\,(o \in \mathcal{B}) &= 1 - \phi(1-\lambda_*) - \lambda_*\phi'(1 - \overline{\lambda_*}), \end{aligned}$$

and hence $\mathcal{L}\,(o \in \mathcal{A}) - \mathcal{L}\,(o \in \mathcal{B}) = M(\lambda_*)$, where we have used $\lambda_* = \overline{\overline{\lambda_*}}$. □

To conclude this chapter, we finally show that the concavity of $\log(\phi'')$ is a sufficient condition for $M(\lambda_*) = \max M$ to hold. Setting $h\colon \lambda \mapsto \overline{\overline{\lambda}} - \lambda$, we easily find that for every $\lambda \in (0,1)$,

$$h''(\lambda) = \frac{\phi''\,(1-\lambda)}{\phi'(1)} \frac{\phi''\left(1-\overline{\lambda}\right)}{\phi'(1)} \left[\frac{\phi''(1-\lambda)\phi'''(1-\overline{\lambda})}{\phi'(1)\phi''(1-\overline{\lambda})} - \frac{\phi'''(1-\lambda)}{\phi''(1-\lambda)}\right].$$

Now, if ϕ is log-concave, then $\lambda \mapsto \phi'''(\lambda)/\phi''(\lambda)$ is non-increasing on $(0,1)$, and therefore, the term inside the square brackets is decreasing (as the difference of a decreasing term and a non-decreasing one). Consequently, h'' can vanish at most once on $(0,1)$, hence h' admits at most two zeros on $[0,1]$, and h at most three. The unique root of $\lambda_c = \overline{\lambda_c}$ is always one of them, and if λ is another one, then so is $\overline{\lambda}$. Therefore, only two cases are possible :

- either λ_c is the only zero of h ; then $h(0) > 0$ and $h(1) < 0$, so M is maximum at λ_c.
- or h admits exactly three zeros $\lambda_* < \lambda_c < \overline{\lambda_*}$; in this case the decreasing term inside the brackets has to vanish somewhere in $(0, 1)$, so h'' is positive and then negative on $(0, 1)$. Consequently, h is decreasing, then increasing, and then decreasing again. In other words, M is minimum at λ_c and maximum at $\lambda_*, \overline{\lambda_*}$.

In both cases, the first local extremum of M is its global maximum.

Chapter 4

Matching number

Joint work with Charles Bordenave and Marc Lelarge

In this chapter, we prove that the local weak convergence of a sequence of graphs is enough to guarantee the convergence of their normalized matching numbers. The limiting quantity is described by a local recursion defined directly on the weak limit of the graph sequence. However, unlike most standard applications of the local weak convergence machinery, this recursion may admit several solutions, implying non-trivial long-range dependencies between the edges of a largest matching. We overcome this lack of correlation decay by introducing a perturbative parameter, which we let progressively go to zero. When the local weak limit is a unimodular Galton-Watson tree, the recursion simplifies into a distributional equation, resulting into an explicit formula that considerably extends the well-known one by Karp and Sipser for Erdős-Rényi random graphs. These results can be found in the pre-print [33].

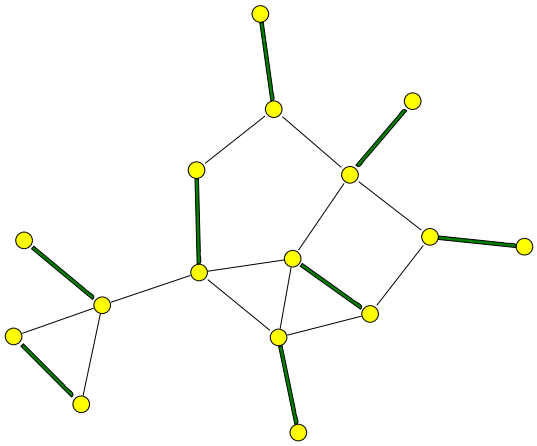


Figure 4.1: A graph and a (perfect) matching on it

4.1 Introduction

A **matching** on a finite graph $G = (V, E)$ is a subset of mutually non-adjacent edges $M \subseteq E$. The **matching number** $\nu(G)$ is the largest possible cardinality of a matching on G. This simple invariant plays an important role in graph theory, and we refer the interested reader to the monographs [58, 69] for more details.

Karp and Sipser [66] investigated its behavior in the case of the Erdős-Rényi random graph with average degree $c > 0$ on n vertices. Using a detailed analysis of the leaf-removal process (introduced in section 3.4), they showed that

$$\frac{\nu(G_n)}{|V_n|} \xrightarrow[n\to\infty]{} 1 - \frac{\lambda_* + e^{-c\lambda_*} + c\lambda_* e^{-c\lambda_*}}{2}, \tag{4.1}$$

where $\lambda_* \in (0, 1)$ is the smallest root of $\lambda = e^{-ce^{-c\lambda}}$. More recently, the same technique has been applied to leafless random graphs with a prescribed log-concave degree distribution (Bohmann and Frieze [27]), resulting in the asymptotic existence of an almost perfect matching in the following sense :

$$\frac{\nu(G_n)}{|V_n|} \xrightarrow[n\to\infty]{} \frac{1}{2}. \tag{4.2}$$

As we have already seen, Erdős-Rényi graphs with fixed average connectivity and random graphs with a prescribed degree distribution are two prototypical examples of convergent graph sequences in the local weak sense. Here we show that the asymptotic behavior of the matching number can in fact be directly deduced from the local weak convergence of the underlying graph sequence, thereby simplifying, unifying and generalizing the aforementioned results.

Theorem 4.1. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of finite graphs admitting a local weak limit $\mathcal{L}$. Then,*

$$\frac{\nu(G_n)}{|V_n|} \xrightarrow[n\to\infty]{} \overline{\nu}(\mathcal{L}),$$

where $\overline{\nu}(\mathcal{L}) \in [0, \frac{1}{2}]$ is described by a local recursion defined directly on the local weak limit $\mathcal{L}$. When the latter is the unimodular Galton-Watson distribution with degree π, we have

$$\overline{\nu}(\mathcal{L}) \;=\; \min_{\lambda\in[0,1]} \left\{ 1 - \frac{1}{2}\lambda\phi'(1-\lambda) - \frac{1}{2}\phi(1-\lambda) - \frac{1}{2}\phi\left(1 - \frac{\phi'(1-\lambda)}{\phi'(1)}\right) \right\},$$

where ϕ is the probability generating function of the degree distribution π. Moreover, any λ achieving the above minimum must satisfy

$$\lambda = \frac{1}{\phi'(1)}\phi'\left(1 - \frac{\phi'(1-\lambda)}{\phi'(1)}\right). \tag{4.3}$$

For Erdős-Rényi random graphs with average degree c, the local weak limit is a.s. the Poisson-Galton-Watson distribution with mean c (i.e. $\phi(\lambda) = \exp(c\lambda - c)$), so that (4.3) becomes $\lambda = e^{-ce^{-c\lambda}}$. We thus recover precisely Karp and Sipser's formula (4.1). Similarly, for random graphs with a prescribed degree sequence, the

log-concave assumption made by Bohmann and Frieze in [27] guarantees that the above minimum is achieved at $\lambda = 0$, hence (4.2) follows automatically. In fact, both these examples fall into the simple situation where a crucial property known as **correlation decay** holds, which allows for a direct application of the objective method (Section 4.2). However, we will exhibit simple examples of graphs on which correlation decay fails to hold. This phenomenon known as *ergodicity breaking* [106] implies non-trivial long-range correlations between the edges in a uniformly chosen largest matching. Our main contribution consists in overcoming this lack of correlation decay. Specifically, we relax the constraint of largest cardinality by means of a variable parameter called the activity, and we show that correlation decay always holds under these relaxed constraints (Section 4.3). The size of the largest matching can then be recovered by sending the activity back to infinity (Section 4.4).

4.2 Correlation decay at infinite activity

In this introductory section, we briefly explain how Karp and Sipser's formula can be recovered by a standard application of the objective method, thanks to correlation decay. Detailed proofs are omitted since the results will anyway be implied by the more sophisticated approach adopted in the subsequent sections. Our starting point is an elementary recursion. On the vertices of a finite tree $T = (V, E)$ rooted at $o \in V$, let us attach $[0, 1]-$valued numbers $x_i, i \in V$ inductively from the leaves up to the root using the following formula.

$$m_i = \left(1 + \sum_{j \to i} \left(\sum_{k \to j} m_k \right)^{-1} \right)^{-1}, \tag{4.4}$$

where $\to$ denotes the child to parent relation, and where an empty sum is zero. What matters to us is the value m_o produced at the root, which we shall rather note $\pi(T, o)$. We then extend this definition to any finite rooted graph (G, o) by applying the recursion (4.4) to a certain rooted tree associated with (G, o), namely the **tree of self-avoiding walks** $T_{(G,o)}$: its vertices are the self-avoiding paths $i = v_0 \dots v_d$ starting at $v_0 = o$ in G, the root is the trivial path v_0, and the parent of a path $i = v_0 \dots v_d$ is the truncated path $i' = v_0 \dots v_{d-1}$. Note that if G is a tree, then there is a unique self-avoiding walk from o to each vertex and therefore $T_{(G,o)} \equiv (G, o)$. It is a well-known observation due to Godsil [59] that the tree of self-avoiding walks $T_{(G,o)}$ captures considerable information about the matchings on G and is easier to work with than G itself. Here, the number $\pi(G, o)$ will play a decisive role for the following reason.

Lemma 4.1. *The number $\pi(G, o)$ is the probability that o remains unmatched in a uniformly chosen largest matching on the finite graph G. In particular,*

$$\frac{\nu(G)}{|V|} = \frac{1}{|V|} \sum_{o \in V} \frac{1 - \pi(G, o)}{2}. \tag{4.5}$$

When the graph G is infinite, the idea of a *uniform largest matching* becomes of course meaningless. However, the tree of self-avoiding walks $T_{(G,o)} \in \mathcal{G}_\star$ is well-defined for any $(G,o) \in \mathcal{G}_\star$, so we may consider the (now infinite) system of equations (4.4) on $T_{(G,o)}$. If the latter admits a unique solution, we say that (G,o) exhibits **correlation decay**, and we define $\pi(G,o)$ to be the resulting value at the root. By compactness and uniqueness, we must then automatically have

$$\pi(G_n, o_n) \xrightarrow[n\to\infty]{} \pi(G,o), \tag{4.6}$$

for any sequence $(G_n, o_n)_{n\in\mathbb{N}}$ of finite rooted graphs converging in the local sense to (G,o). Combining this with Lemma 4.1 (in which we recognize expectation under uniform rooting) immediately yields the following result.

Proposition 4.1. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of finite graphs admitting a local weak limit $\mathcal{L}$, and assume that $\mathcal{L}$ is concentrated on rooted graphs $(G,o) \in \mathcal{G}_\star$ exhibiting correlation decay. Then,*

$$\frac{\nu(G_n)}{|V_n|} \xrightarrow[n\to\infty]{} \overline{\nu}(\mathcal{L}) = \frac{1 - \mathcal{L}\left[\pi(G,o)\right]}{2},$$

This weaker form of Theorem 4.1 is already enough to recover the aforementioned results (4.1)-(4.2). Indeed, on a unimodular Galton-Watson tree, the recursion (4.4) has been solved in the previous chapter (Proposition 3.2) and in particular,

Proposition 4.2. *Let (T,o) be a unimodular Galton-Watson tree with degree generating function ϕ. For $\lambda \in [0,1]$ set*

$$M(\lambda) = \lambda\phi'(1-\lambda) + \phi(1-\lambda) + \phi\left(1 - \frac{\phi'(1-\lambda)}{\phi'(1)}\right) - 1. \tag{4.7}$$

Then (T,o) exhibits a.s. correlation decay if and ony if the maximum of M coincides with the first local extremum and in that case, $\mathcal{L}[\pi(T,o)] = \max M$. Moreover, a sufficient condition for this to hold is that ϕ'' is log-concave.

For Erdős-Rényi random graphs with connectivity c, the limiting degree generating function is $\phi(t) = \exp(c - ct)$, and its second derivative is clearly log-concave, so we recover exactly Karp and Sipser's formula (4.1). Similarly, for random graphs with a prescribed degree sequence, the assumption made by Bohmann and Frieze easily guarantees that $\max M = M(0) = 0$, hence (4.2) follows automatically. However, this is not the end of the story yet : as we have seen in the previous chapter, the degree distribution [75% degree 3, 25% degree 15] provides a simple example of a local weak limit which does not exhibit correlation decay. In such a case, the central compactness/uniqueness argument collapses, and to the best of our knowledge, the asymptotic behavior of the matching number is unknown. Our main result here consists in bypassing correlation decay, i.e. establishing convergence despite the coexistence of several distinct solutions to the limiting recursion.

4.3 The monomer-dimer model

The key idea consists in relaxing the constraint of largest cardinality on matchings by introducing a perturbative parameter $t > 0$, which we call the **activity**. Specifically, given any finite graph $G = (V, E)$, we consider a random matching $\mathcal{M}_G^t$ sampled from the following distribution :

$$\mathbb{P}(\mathcal{M}_G^t = M) = \frac{t^{|M|}}{P_G(t)}, \qquad \text{with} \qquad P_G(t) = \sum_M t^{|M|}.$$

This is known as the **monomer-dimer model** on G [106, 62]. The normalizing factor $P_G(t)$ is a famous graph invariant called the matching polynomial. Its leading term corresponds precisely to the largest matchings on G. Therefore, $\mathcal{M}_G^t$ converges in law to a uniform largest matching as $t \to \infty$, and

$$\mathbb{E}[|\mathcal{M}_G^t|] \xrightarrow[t\to\infty]{} \nu(G). \tag{4.8}$$

Now fix a vertex $o \in V$. A matching on G either leaves o unmatched, or matches it to some neighbor $v \in \partial o$ and hence,

$$P_G(t) = P_{G-o}(t) + t \sum_{v \in \partial o} P_{G-o-v}(t). \tag{4.9}$$

This elementary formula has two important consequences. First, the probability that o remains unmatched in $\mathcal{M}_G^t$ is simply $\pi_t(G, o) = P_{G-o}(t)/P_G(t)$ and second, $\pi_t(G, o)$ can be computed recursively using the following local rule :

$$\pi_t(G, o) = \left(1 + t \sum_{v \in \partial o} \pi_t(G - o, v)\right)^{-1}. \tag{4.10}$$

This may be more conveniently rewritten as a child-to-parent recursion on the tree of self-avoiding walks $T_{(G,o)}$ as follows : any node i in the latter represents a certain simple path $v_0, v_1, \dots v_d$ starting from $v_0 = o$ in G, and we set $x_i = \pi_t\left(G - \{v_0, \dots, v_{d-1}\}, v_d\right)$. Then (4.10) becomes simply

$$x_i = \left(1 + t \sum_{j \to i} x_j\right)^{-1}. \tag{4.11}$$

This recursion is not new : it already appeared in [59, 64, 40, 96, 18, 106]. Here we study (4.11) directly on an infinite tree. A $[0, 1]-$valued solution always exists by compactness, but nothing guarantees anymore its uniqueness. When this is the case, we say that correlation decay holds at activity t on (G, o), and we denote by $\pi_t(G, o)$ the resulting value at the root. Under the change of variable $y_i = i\sqrt{t}x_i$, (4.11) becomes precisely the cavity recursion at activity $z = i/\sqrt{t}$ characterizing the rooted spectral measure of $T_{(G,o)}$. In particular, Theorem 2.2 admits the following corollary.

Corollary 4.1 (self-adjointness $\Longrightarrow$ correlation decay)**.** *A sufficient condition for correlation decay to hold at any activity t on $(G,o) \in \mathcal{G}_\star$ is that $T_{(G,o)}$ is self-adjoint. In particular, it is enough that G has bounded degree.*

Instead of exploiting this correspondence with the previous chapter, we have chosen to present a more direct proof of correlation decay for bounded degree graphs, based on monotony and analyticity with respect to the activity.

Proposition 4.3 (Convergence for the monomer-dimer model)**.** *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of finite graphs with uniformly bounded degree admitting a local weak limit $(G,o) \sim \mathcal{L}$. Then for every activity $t > 0$,*

$$\frac{\mathbb{E}[|\mathcal{M}^t_{G_n}|]}{|V_n|} \xrightarrow[n\to\infty]{} \frac{1 - \mathcal{L}[\pi_t(G,o)]}{2},$$

where $\pi_t(G,o)$ is the value at o of the a.s. unique solution to (4.11) on $T_{(G,o)}$.

Proof. Since the size of a matching is always half the number of matched vertices, we have for any finite graph $G = (V,E)$,

$$\frac{\mathbb{E}\left[|\mathcal{M}^t_G|\right]}{|V|} = \frac{1}{|V|}\sum_{o\in V} \frac{1 - \pi_t(G,o)}{2}.$$

We again recognize expectation under uniform rooting and thus, Proposition 4.3 boils down to showing that $(G,o) \mapsto \pi_t(G,o)$ is well-defined and continuous on graphs with bounded degree. Consider a locally finite tree $T = (V,E)$ with root $o \in V$, and a fixed activity $t > 0$. The recursion (4.11) is nothing but a fixed point equation $\mathbf{x} = \Gamma\mathbf{x}$ in the product space $[0,1]^V$. Specifically, $\Gamma\colon [0,1]^V \to [0,1]^V$ maps $\mathbf{x} = (x_i)_{i\in V}$ to $\mathbf{y} = (y_i)_{i\in V}$, where

$$y_i = \left(1 + t\sum_{j\to i} x_j\right)^{-1}. \tag{4.12}$$

The mapping Γ is clearly decreasing with respect to coordinate-wise order, and it maps the minimal configuration $\mathbf{x}^0 = \mathbf{0}$ to the maximal one, $\mathbf{x}^1 = \Gamma\mathbf{x}^0 = \mathbf{1}$. Consequently, if we define iteratively $\mathbf{x}^{k+1} = \Gamma\mathbf{x}^k$ for every $k \geq 1$, then the limits

$$\mathbf{x}^- = \lim_{k\to\infty} \uparrow \mathbf{x}^{2k} \qquad \text{and} \qquad \mathbf{x}^+ = \lim_{k\to\infty} \downarrow \mathbf{x}^{2k+1} \tag{4.13}$$

always exist in $[0,1]^V$, and any fixed point $\mathbf{x} = \Gamma\mathbf{x}$ must satisfy $\mathbf{x}^- \leq \mathbf{x} \leq \mathbf{x}^+$. Moreover, Γ is obviously continuous with respect to the product topology, so that $\Gamma\mathbf{x}^+ = \mathbf{x}^-$ and $\Gamma\mathbf{x}^- = \mathbf{x}^+$. Thus, correlation decay boils down to the identity $\mathbf{x}^- = \mathbf{x}^+$. The latter clearly holds in the *low activity regime* $t < \deg(T)^{-1/2}$, since in that case the mapping Γ is contracting for the supremum norm on $[0,1]^V$. It then automatically extends to any $t > 0$, thanks to the remarkable fact that $\mathbf{x}^-$ and $\mathbf{x}^+$ depend analytically upon the activity. Indeed, instead of working with each fixed activity separately, we may view Γ as acting at the level of analytic functions of the activity. Specifically, we allow t to take complex values in the right open half-plane

$\mathbb{H}_+ = \{t \in \mathbb{C} : \Re(t) > 0\}$, and we consider the set $\mathcal{K}$ of all analytic functions $x\colon \mathbb{H}_+ \to \mathbb{C}$ that satisfy

$$\Re\left(x(t)\right) \geq 0, \qquad \Re\left(tx(t)\right) \geq 0 \qquad \text{and} \qquad |x(t)| \leq \frac{|t|}{\Re(t)}. \tag{4.14}$$

There are two capital observations to make about $\mathcal{K}$: first, it is easily seen to be stable under the transformation (4.12), so that $\mathbf{x}^k$ remains in $\mathcal{K}^V$ for every $k \in \mathbb{N}$. Second, it is compact within the space of analytic functions on $\mathbb{H}_+$, by Montel's Theorem. This guarantees that the point-wise limits $\mathbf{x}^-$ and $\mathbf{x}^+$ belong to $\mathcal{K}^V$, and in particular, that they are analytic in $t \in \mathbb{H}_+$, as desired. Such an absence of *phase transition* in the monomer-dimer model is well-known [62, 23].

To sum up, for any $t > 0$ and any rooted graph (G, o) with bounded degree, the recursion (4.11) on $T_{(G,o)}$ admits a unique solution. In order to make the dependency upon the activity and the underlying graph explicit, we write $\pi_t^k(G, o)$ for x_o^k and $\pi_t^\pm(G, o)$ for $x_o^\pm$, or simply $\pi_t(G, o)$ when $\pi_t^-(G, o) = \pi_t^+(G, o)$. It remains to show that $(G, o) \mapsto \pi_t(G, o)$ is continuous on the space of rooted connected graphs with bounded degree. Consider any convergent sequence,

$$(G_n, o_n) \xrightarrow[n\to\infty]{} (G, o).$$

By (4.13), we have that for any $k, n \in \mathbb{N}$,

$$\pi_t^{2k}(G_n, o_n) \leq \pi_t^-(G_n, o_n) \leq \pi_t^+(G_n, o_n) \leq \pi_t^{2k+1}(G_n, o_n). \tag{4.15}$$

But by construction, $\pi_t^k(G_n, o_n)$ depends only on $[G_n, o_n]_k$ and must therefore equal $\pi_t^k(G, o)$ as soon as $n \geq n_k$ (local convergence). Consequently, letting $n \to \infty$ and then $k \to \infty$ in (4.15) yields

$$\pi_t^-(G, o) \leq \liminf_{n\to\infty} \pi_t^-(G_n, o_n) \leq \limsup_{n\to\infty} \pi_t^+(G_n, o_n) \leq \pi_t^+(G, o),$$

or simply $\pi_t(G_n, o_n) \xrightarrow[n\to\infty]{} \pi_t(G, o)$ when all graphs have bounded degree. □

4.4 Back to largest matchings

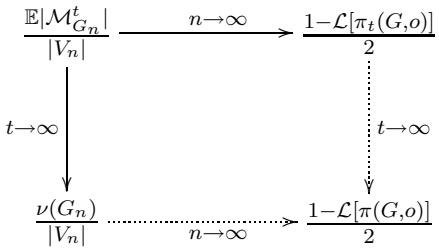
We can finally prove Theorem 4.1. We start with a slightly weaker version.

Proposition 4.4. *Theorem 4.1 holds under the additional assumption that the sequence $(G_n)_{n\in\mathbb{N}}$ has bounded degree, and in that case*

$$\overline{\nu}(\mathcal{L}) = \frac{1 - \mathcal{L}[\pi(G, o)]}{2},$$

where $\pi(G, o)$ is the value at the root of the largest solution to (4.4) on $T_{(G,o)}$.

The proof is divided into two steps : first, we let the activity tend to infinity and establish that the unique solution to (4.11) always converges to the largest solution to (4.4), despite the possible existence of several other solutions (Lemma 4.2). Second, we provide a uniform control which guarantees that the $n \to \infty$ and $t \to \infty$ limits of $\mathbb{E}[|\mathcal{M}_{G_n}^t|]/|V_n|$ may be interchanged (Lemma 4.3), as illustrated by the following diagram.



Lemma 4.2. *Let $T=(V,E,o)$ be a rooted tree with bounded degree, and let $\mathbf{x}(t)\in[0,1]^V$ be the unique solution to the recursion (4.11). Then the limit*

$$\mathbf{m}=\lim_{t\to\infty}\mathbf{x}(t)$$

exists in the product space $[0,1]^V$, and is the largest solution to (4.4).

Proof. With the notations of Section 4.2, we have for every $i\in V$, $n\in\mathbb{N}$,

$$x_i^{n+2}(t)=\left(1+\sum_{j\to i}\left(t^{-1}+\sum_{k\to j}x_k^n(t)\right)^{-1}\right)^{-1}. \tag{4.16}$$

Both $\mathbf{x}^0=\mathbf{0}$ and $\mathbf{x}^1=\mathbf{1}$ are (constant and hence) non-increasing functions of the activity t. By an immediate induction over $n\in\mathbb{N}$ using the above equation, the same must hold for all $\mathbf{x}^n$, $n\in\mathbb{N}$. Letting $n\to\infty$, we finally obtain that the limit $\mathbf{x}$ is non-decreasing with the activity. This proves the existence of the infinite activity limit

$$\mathbf{m}=\lim_{t\to\infty}\downarrow\mathbf{x}(t)\in[0,1]^V.$$

Letting $t\to\infty$ in (4.16), we automatically get that $\mathbf{m}$ must be a solution to the recursion (4.4). Finally, consider an arbitrary solution $\mathbf{m}'\in[0,1]^V$ to (4.4), and let us show by induction over $n\in\mathbb{N}$ that for every $t>0$,

$$\mathbf{m}'\leq\mathbf{x}^{2n+1}(t). \tag{4.17}$$

The statement is trivial when $k=0$ because $\mathbf{x}^1=\mathbf{1}$, and is preserved from n to $n+1$ because for every $i\in V$,

$$\begin{aligned}x_i^{2n+3}(t) &= \left(1+\sum_{j\to i}\left(t^{-1}+\sum_{k\to j}x_k^{2n+1}(t)\right)^{-1}\right)^{-1}\\ &\geq \left(1+\sum_{j\to i}\left(\sum_{k\to j}m_k\right)^{-1}\right)^{-1}=m_i.\end{aligned}$$

Letting $n\to\infty$, then $t\to\infty$ in (4.17) yields $\mathbf{m}'\leq\mathbf{m}$, as desired. □

Lemma 4.3. *For any finite graph $G=(V,E)$, and any activity $t\in(0,1)$,*

$$0\leq\nu(G)-\mathbb{E}\left[|\mathcal{M}^t_G|\right]\leq\frac{|E|\log 2}{\log t}.$$

Proof. $f\colon t \mapsto \mathbb{E}\left[|\mathcal{M}_G^t|\right]$ is non-decreasing with $f(+\infty) = \nu(G)$. Thus, for any $t \in (1,\infty)$,

$$\frac{1}{\log t}\int_1^t \frac{f(s)}{s}ds \leq f(t) \leq \nu(G).$$

Now use $P_G'(s)/P_G(s) = f(s)/s$ to explicitate the left-hand side :

$$\frac{1}{\log t}\int_1^t \frac{f(s)}{s}ds \;=\; \frac{1}{\log t}\log\frac{P_G(t)}{P_G(1)} \;\geq\; \nu(G) - \frac{|E|\log 2}{\log t},$$

where the inequality follows from the fact that the total number of matchings $P_G(1)$ is at most $2^{|E|}$, while $P_G(t)$ is at least $t^{\nu(G)}$. □

Proof of Proposition 4.4 and Theorem 4.1. By the triangular inequality,

$$\begin{aligned}\left|\frac{\nu(G_n)}{|V_n|} - \frac{1-\mathbb{E}[\pi(G,o)]}{2}\right| \;&\leq\; \left|\frac{\nu(G_n)}{|V_n|} - \frac{\mathbb{E}\left[|\mathcal{M}^t_{G_n}|\right]}{|V_n|}\right| \\ &+\; \left|\frac{\mathbb{E}\left[|\mathcal{M}^t_{G_n}|\right]}{|V_n|} - \frac{1-\mathbb{E}[\pi_t(G,o)]}{2}\right| \\ &+\; \left|\frac{\mathbb{E}[\pi_t(G,o)] - \mathbb{E}[\pi(G,o)]}{2}\right|.\end{aligned}$$

By lemmas 4.2 and 4.3, the first and third term can be made arbitrarily small by choosing t large enough (uniformly in n), while the middle term tends to zero as $n \to \infty$ by virtue of Proposition 4.3. This shows Proposition 4.4. We finally need to remove the restriction of bounded degree. To this end, we introduce the *$d-$truncation* G^d ($d \in \mathbb{N}$) of a graph $G = (V,E)$, obtained from G by *isolating* all vertices with degree more than d, i.e. removing any edge incident to them. This transformation is clearly continuous with respect to local convergence. Moreover, its effect on the matching number can be easily controlled :

$$\nu(G^d) \leq \nu(G) \leq \nu(G^d) + \#\{v \in V : \deg_G(v) > d\}. \tag{4.18}$$

Now, consider a sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$ admitting a local weak limit $(G,o) \sim \mathcal{L}$. First, fixing $d \in \mathbb{N}$, we may apply Theorem 4.4 to the bounded-degree sequence $(G_n^d)_{n\in\mathbb{N}}$ to obtain :

$$\frac{\nu(G_n^d)}{|V_n|} \xrightarrow[n\to\infty]{} \frac{1-\mathcal{L}[\pi(G^d,o)]}{2}.$$

Second, we may rewrite (4.18) as

$$\left|\frac{\nu(G_n^d)}{|V_n|} - \frac{\nu(G_n)}{|V_n|}\right| \leq \frac{1}{|V_n|}\sum_{o\in V_n}\mathbf{1}_{(\deg_{G_n}(o)>d)},$$

where we recognize expectation with respect to uniform rooting of G_n. Letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty}\left|\frac{1-\mathcal{L}[\pi(G^d,o)]}{2} - \frac{\nu(G_n)}{|V_n|}\right| \leq \mathcal{L}\left(\deg_G(o) > d\right),$$

This last line is, by an elementary application of Cauchy criterion, enough to guarantee that the limits

$$\lim_{n\to\infty} \frac{\nu(G_n)}{|V_n|} \qquad \text{and} \qquad \lim_{d\to\infty} \frac{1-\mathcal{L}[\pi(G^d,o)]}{2}$$

exist and are equal. Hence the convergence promised by Theorem 4.1. Note that because of the possible absence of correlation decay, the largest solution $\pi(G,o)$ is not a continuous function of $(G,o)\in\mathcal{G}$. In particular, we do not know whether it is always the case that

$$\overline{\nu}(\mathcal{L}) = \frac{1-\mathcal{L}[\pi(G,o)]}{2}, \tag{4.19}$$

as established in Proposition 4.4 for graphs with bounded degree. However, (4.19) holds in the particular case where (G,o) is a unimodular Galton-Watson tree. Indeed, Proposition 3.2 implies that $\mathcal{L}[\pi(G,o)] = \max M$ where $M\colon [0,1]\to\mathbb{R}$ is defined by (4.7). The important point is the fact that $\max M$ depends continuously upon the degree distribution. Now, the $d-$truncation of a unimodular Galton-Watson tree is again a unimodular Galton-Watson tree, whose degree distribution converges to the original one as $d\to\infty$. Hence, $\mathcal{L}[\pi(G^d,o)] \xrightarrow[d\to\infty]{} \mathcal{L}[\pi(G,o)]$, so Theorem 4.1 follows. □

Chapter 5

Weighted subgraph enumeration

Using the theory of negative association for measures initiated by Pemantle, we establish the validity of the cavity method for counting spanning subgraphs subject to local constraints in asymptotically tree-like graphs. Specifically, the thermodynamic pressure is shown to converge along any sequence of graphs whose local weak limit is a tree, and the limit is directly expressed in terms of the unique solution to a limiting cavity equation. On a unimodular Galton-Watson tree, the latter simplifies into a recursive distributional equation which can be solved explicitly. As an illustration, we provide an explicit limit theorem for the $b-$matching number of an Erdős-Rényi random graph with fixed average degree and diverging size, for any $b \in \mathbb{N}$. These results can be found in the pre-print [91].

5.1 Introduction

The general framework we consider is that of a finite graph $G = (V, E)$, in which **spanning subgraphs** $(V, F), F \subseteq E$ are weighted according to their local aspect around each vertex as follows :

$$\mu(F) \;=\; \prod_{i \in V} \mu_i(F \cap E_i). \tag{5.1}$$

Here, a spanning subgraph (V, F) is identified with its edge-set $F \subseteq E$, and each μ_i is a given non-negative function over the subsets of $E_i := \{e \in E; e \text{ is incident to } i\}$. We call μ the global measure induced by the local measures $\mu_i, i \in V$. Of particular interest in combinatorial optimization is

$$M(G) \;=\; \max\left\{|F| : F \in \mathrm{supp}(\mu)\right\}, \tag{5.2}$$

which is the maximum possible size of a spanning subgraph F satisfying the local constraint $\mu_i(F \cap E_i) > 0$ at every vertex $i \in V$. More generally, counting the weighted number of spanning subgraphs of each given size in G, i.e. determining the **generating polynomial**

$$Z(G; t) \;=\; \sum_{F \subseteq E} \mu(F) t^{|F|} \tag{5.3}$$

is a fundamental task, of which many combinatorial problems are special instances. Intimately related to this is the study of a random spanning subgraph $\mathcal{F}$ sampled from the Gibbs-Boltzmann law :

$$\mathbb{P}^t_G(\mathcal{F} = F) \;=\; \frac{\mu(F) t^{|F|}}{Z(G; t)}, \tag{5.4}$$

where $t > 0$ is a variable parameter called the activity. In particular, the expected size of $\mathcal{F}$ is called the **energy** $U(G; t)$ and is connected to $Z(G; t)$ via the elementary identity

$$U(G; t) \;=\; t \frac{d}{dt} \log Z(G; t). \tag{5.5}$$

Our concern is the behavior of these quantities in the infinite volume limit : $|V| \to \infty$, $|E| = \Theta(|V|)$.

As already explained, the cavity method is a powerful non-rigorous technique for evaluating such asymptotics on graphs that are locally tree-like. The heuristic consists in neglecting cycles in order to obtain an approximate local fixed point equation (the so-called cavity equation) for the marginals of the Gibbs-Boltzmann law. Despite its remarkable practical efficiency and the mathematical confirmation of its analytical predictions for various important models [97, 5, 56, 89, 19, 48, 33], this ansatz is still far from being completely understood, and the exact conditions for its validity remain unknown. More precisely, two crucial questions arise in presence of cycles :

1. **convergence**: is there a unique solution to the cavity equation ?

2. **correctness**: is the latter related to the Gibbs-Boltzmann law ?

In this chapter, we exhibit a general condition under which the cavity method is valid for counting spanning subgraphs subject to local constraints. Specifically, we positively answer question 1 for arbitrary finite graphs and local weak limits of such graphs (Propositions 5.1 and 5.2), under the only assumption that each local measure enjoys a certain form of negative association which we call the **cavity-monotone** property (see Section 5.2 for the precise definition). Regarding question 2, we prove asymptotic correctness for any sequence of graphs whose local weak limit is concentrated on trees (Theorem 5.2). In the important case of a unimodular Galton-Watson tree, the cavity equation simplifies into a recursive distributional equation which can be solved explicitly. As a motivation, let us first describe the implications of our work in the special case of **b-matchings**.

An important combinatorial structure that fits in the above framework is obtained by fixing $b \in \mathbb{N}$ and taking $\mu_i(F) = \mathbf{1}(|F| \leq b)$ for all $i \in V$. The induced global measure μ is then nothing but the counting measure for $b-$matchings in G, i.e. spanning subgraphs with maximum degree at most b. The reader is referred to the monograph [93] for a comprehensive survey on $b-$matchings. The associated quantities $M_b(G)$ and $Z_b(G;t)$ are important graph invariants respectively known as the $b-$**matching number** and $b-$**matching polynomial**. Determining $Z_1(G;t)$ is a classical example of a computationally hard problem [100], although efficient approximation algorithms have been designed [18, 14]. The mathematical properties of $Z_b(G;t)$ have been investigated in detail, notably in the case $b = 1$ for the purpose of understanding monomer-dimer systems [62, 23]. The scaled convergence of $Z_1(G;t)$ as $|V| \to \infty$ was established in [62] for the lattice case, and in [19] under a restrictive large girth assumption. Interestingly, the geometry of the complex zeros of $Z_b(G,t)$ has been proven to be quite remarkable (see [62] for $b = 1$, [90] for $b = 2$, and [103] for the general case). The asymptotics of $M_1(G) = \nu(G)$ have been studied in the previous chapter (Theorem 4.1). Contrastingly, only little is known for $b \geq 2$: to the best of our knowledge, the limit of $\frac{1}{|V_n|}M_b(G_n)$ is only known to exist in the Erdős-Rényi case [55], and could not be explicitly determined. As a special case of our main result (Theorem 5.2), it will follow that

Theorem 5.1 ($b-$matchings in locally tree-like graphs)**.** *For any sequence of finite graphs $(G_n)_{n\in\mathbb{N}}$ satisfying $|E_n| = O(|V_n|)$ and whose local weak limit $\mathcal{L}$ is concentrated on trees, the limits*

$$h_b(\mathcal{L}) := \lim_{n\to\infty} \frac{1}{|V_n|} \log Z_b(G_n;t) \quad \textit{and} \quad m_b(\mathcal{L}) := \lim_{n\to\infty} \frac{M_b(G_n)}{|V_n|}$$

exist and depend only on the local weak limit $\mathcal{L}$. In the important case where $\mathcal{L}$ is the unimodular Galton-Watson distribution with degree π, we have the explicit formula

$$m_b(\mathcal{L}) = \frac{b}{2} \min_{\lambda\in[0,1]} \left\{ 2 - g_b(\lambda) - (g_b \circ f_b)(\lambda) + \frac{c}{b} f_b(\lambda)(f_b \circ f_b)(\lambda) \right\},$$

where c, f, g are defined in terms of $\phi(\lambda) = \sum_k \pi_k \lambda^k$ as follows :

$$c = \phi'(1), \;\; f_b(\lambda) = \frac{1}{c} \sum_{k=0}^{b-1} \frac{\lambda^k \phi^{(k+1)}(1-\lambda)}{k!} \;\; \textit{and} \;\; g_b(\lambda) = \sum_{k=0}^{b} \frac{\lambda^k \phi^{(k)}(1-\lambda)}{k!}.$$

Moreover, any λ achieving this minimum must be a root of $\lambda = (f_b \circ f_b)(\lambda)$.

For example, in the case of Erdős-Rényi random graphs with average degree $c > 0$ on n vertices, the local weak limit $\mathcal{L}$ is a.s. the Poisson-Galton-Watson distribution with mean c ($\phi(\lambda) = e^{c\lambda - c}$). Hence,

$$\frac{M_b(G_n)}{n} \xrightarrow[n\to\infty]{a.s.} \frac{b}{2} \min_{\lambda\in[0,1]} \left\{ 2 - f_{b+1}(\lambda) - (f_{b+1} \circ f_b)(\lambda) + \frac{c}{b} f_b(\lambda)(f_b \circ f_b)(\lambda) \right\},$$

and any λ where the minimum is achieved must satisfy $\lambda = (f_b \circ f_b)(\lambda)$, where

$$f_b(\lambda) = e^{-c\lambda} \sum_{k=0}^{b-1} \frac{(c\lambda)^k}{k!}.$$

In the case of matchings ($b = 1$) we recover Karp and Sipser formula [66].

The remainder of the chapter is organized as follows : in section 5.2, we recall the necessary notions and properties pertaining to measures over subsets. In section 5.3, we define and study the cavity equation associated with a finite network. In section 5.4, we extend the results to infinite networks that arise as local weak limits of finite networks. Finally, section 5.5 is devoted to the study of the cavity equation in the limit of infinite activity, and to its explicit resolution in the case of $b-$matchings.

5.2 Preliminaries

In this section, we define the important notions pertaining to (non-negative) measures μ over the subsets of an arbitrary finite **ground set** E. Later on, these will be specialized to the local measures $(\mu_i)_{i\in V}$ attached to the vertices of a graph G. First, μ is characterized by its **multivariate generating polynomial**

$$Z(\mathbf{w}) \;=\; \sum_{F\subseteq E} \mu(F)\mathbf{w}^F,$$

where $\mathbf{w} = (w_e)_{e\in E}$ and $\mathbf{w}^F = \prod_{e\in F} w_e$. Since Z is affine in each $w_e, e \in E$, it can be decomposed as

$$Z(\mathbf{w}) \;=\; w_e Z^{/e}(\mathbf{w}') + Z^{\backslash e}(\mathbf{w}'), \tag{5.6}$$

where $\mathbf{w}' = (w_f)_{f\neq e}$ and $Z^{\backslash e}$, $Z^{/e}$ are the multi-affine polynomials with ground set $E \setminus e$ respectively obtained from Z by setting the variable w_e to 0 (deletion) and differentiating with respect to w_e (contraction). By definition, the **cavity ratio** of the pair (μ, e) is then simply the multi-affine rational function

$$\Gamma^e_\mu(\mathbf{w}') \;=\; \frac{Z^{/e}(\mathbf{w}')}{Z^{\backslash e}(\mathbf{w}')}.$$

When positive values are assigned to the variables (a so-called **external field**), we may consider the probability distribution

$$\mathbb{P}^{\mathbf{w}}_\mu(\mathcal{F} = F) = \frac{\mu(F)\mathbf{w}^F}{Z(\mathbf{w})}.$$

A quantity of interest is the expected size of $\mathcal{F}$ when viewed as a function of the external field. We call this the **energy**:

$$U_\mu(\mathbf{w}) \;=\; \mathbb{E}_\mu^{\mathbf{w}}\left[|\mathcal{F}|\right].$$

From the decomposition (5.6), it follows immediately that

$$\mathbb{P}_\mu^{\mathbf{w}}(e\in\mathcal{F}) \;=\; \frac{w_e\Gamma_\mu^e(\mathbf{w}')}{1+w_e\Gamma_\mu^e(\mathbf{w}')} \tag{5.7}$$

$$U_\mu(\mathbf{w}) \;=\; \sum_{e\in E}\frac{w_e\Gamma_\mu^e(\mathbf{w}')}{1+w_e\Gamma_\mu^e(\mathbf{w}')}. \tag{5.8}$$

The following properties will be of crucial importance throughout the paper.

Definition 5.1 (Cavity-monotone measures)**.** *The measure μ is called*

- ***Rayleigh*** *if every two distinct ground elements $e\neq f$ are negatively correlated in $\mathcal{F}$ under any external field $\mathbf{w}>0$:*

$$\mathbb{P}_\mu^{\mathbf{w}}\left(e\in\mathcal{F}, f\in\mathcal{F}\right)\leq\mathbb{P}_\mu^{\mathbf{w}}\left(e\in\mathcal{F}\right)\mathbb{P}_\mu^{\mathbf{w}}\left(f\in\mathcal{F}\right).$$

- ***Size-increasing*** *if every ground element e has positive influence on the total size $|\mathcal{F}|$ under any external field $\mathbf{w}>0$:*

$$\mathbb{E}_\mu^{\mathbf{w}}\left[|\mathcal{F}|\mathbf{1}_{(e\in\mathcal{F})}\right]>\mathbb{E}_\mu^{\mathbf{w}}\left[|\mathcal{F}|\right]\mathbb{P}_\mu^{\mathbf{w}}(e\in\mathcal{F}).$$

- ***Cavity-monotone*** *if its satisfies $\mu(\emptyset)>0$ and those two properties.*

Rayleigh measures were introduced in the context of matroid theory [102], but soon found their place in the modern theory of negative dependence for probability measures [86, 65]. The size-increasing property can be viewed as a weak form of the so-called normalized matching property, or as a strong form of the so-called Feder-Mihail property (see [65] for definitions). Cavity-monotone measures will play a major role in our study, for the following elementary reason.

Lemma 5.1 (Monotony of energy and cavity ratios)**.**

$$\begin{aligned}
\mu(\emptyset)>0 \;&\Leftrightarrow\; \textit{each } \Gamma_\mu^e, e\in E \textit{ is well-defined on } [0,\infty)^{E\setminus e}.\\
\mu \textit{ is Rayleigh} \;&\Leftrightarrow\; \textit{each } \Gamma_\mu^e, e\in E \textit{ is non-increasing in each variable.}\\
\mu \textit{ is size-increasing} \;&\Leftrightarrow\; U_\mu \textit{ is increasing in each variable;}\\
&\Leftrightarrow\; \textit{for each } e\in E, \mathbf{w}'>0, t\mapsto t\Gamma_\mu^e(t\mathbf{w}') \textit{ is increasing.}
\end{aligned}$$

Proof. Differentiating the corresponding quantities and playing with the definition of $\mathbb{P}_\mu^{\mathbf{w}}$ easily yields

$$\begin{aligned}
\frac{\partial\Gamma_\mu^e(\mathbf{w}')}{\partial w_f} \;&=\; \frac{\mathbb{P}_\mu^{\mathbf{w}}\left(e\in\mathcal{F}, f\in\mathcal{F}\right)-\mathbb{P}_\mu^{\mathbf{w}}\left(e\in\mathcal{F}\right)\mathbb{P}_\mu^{\mathbf{w}}\left(f\in\mathcal{F}\right)}{w_ew_f\mathbb{P}_\mu^{\mathbf{w}}\left(e\notin\mathcal{F}\right)^2}.\\
\frac{\partial U_\mu(\mathbf{w})}{\partial w_e} \;&=\; \frac{\mathbb{E}_\mu^{\mathbf{w}}\left[|\mathcal{F}|\mathbf{1}_{(e\in\mathcal{F})}\right]-\mathbb{E}_\mu^{\mathbf{w}}\left[|\mathcal{F}|\right]\mathbb{P}_\mu^{\mathbf{w}}(e\in\mathcal{F})}{w_e}.\\
\frac{\partial\left(t\Gamma_\mu^e(t\mathbf{w}')\right)}{\partial t} \;&=\; \frac{\mathbb{E}_\mu^{t\mathbf{w}}\left[|\mathcal{F}|\mathbf{1}_{(e\in\mathcal{F})}\right]-\mathbb{E}_\mu^{t\mathbf{w}}\left[|\mathcal{F}|\right]\mathbb{P}_\mu^{t\mathbf{w}}(e\in\mathcal{F})}{tw_e\mathbb{P}_\mu^{t\mathbf{w}}\left(e\notin\mathcal{F}\right)^2}.
\end{aligned}$$

□

Remark 5.1 (Matroids)**.** *Interestingly, the support of a cavity-monotone measure admits a remarkable structure : it follows from [102, Theorem 4.6] that for μ Rayleigh with $\mu(\emptyset) > 0$, $\mathcal{I} = \operatorname{supp}\mu$ is a* ***matroid****:*

- *$\mathcal{I}$ is not empty ;*
- *If $B \in \mathcal{I}$ and $A \subseteq B$, then $A \in \mathcal{I}$;*
- *If $A, B \in \mathcal{I}$ and $|A| < |B|$, then $\exists e \in B \setminus A$ such that $A \cup e \in \mathcal{I}$.*

The cavity-monotone property admits a particularly simple characterization in the important case where μ is **exchangeable**, i.e. $\mu(F) = c(|F|)$ for some non-negative coefficients $c(0), \ldots, c(m), m = |E|$:

Lemma 5.2 (The exchangeable case)**.** *An exchangeable measure μ with coefficients $c(0), \ldots, c(m), (m = |E| \geq 1)$ is cavity-monotone if and only if*

1. *c is log-concave, i.e. $c^2(k) \geq c(k-1)c(k+1)$ for all $0 < k < m$, and*
2. *the support $\{0 \leq k \leq m : c(k) > 0\}$ is an interval containing 0 and 1.*

Corollary 5.1. *For any $b \geq 1$, the local measure $\mu_i(F) = \mathbf{1}_{(|F| \leq b)}$ describing the degree constraints in a $b-$matching is cavity-monotone.*

Proof of Lemma 5.2. The result essentially follows from the work of Pemantle [86]. Indeed, Theorem 2.7 therein guarantees that μ is Rayleigh if and only if the sequence c is log-concave and its support is an interval. That the latter must contain 0 is nothing but the last property in the definition of a cavity-monotone measure. That it is not reduced to 0 is imposed by the strict inequality in the size-increasing property. Conversely, let us show that any exchangeable measure μ with $c(0) > 0$ and $c(1) > 0$ is size-increasing. Fix an external field $\mathbf{w} > 0$. By Lemma 2.9 in [86], the law obtained from $\mathbb{P}_\mu^{\mathbf{w}}$ by conditioning on the event $\{|\mathcal{F}| = k\}$ is stochastically increasing in k. By Proposition 1.2 in [86], this implies in particular that for every $e \in E$, the following weak inequality holds :

$$\mathbb{E}_\mu^{\mathbf{w}}\left[|\mathcal{F}| \middle| e \in \mathcal{F}\right] \geq \mathbb{E}_\mu^{\mathbf{w}}\left[|\mathcal{F}|\right].$$

Note that the condition $c(1) > 0$ guarantees that this conditional expectation is well-defined. Since we have not yet used the fact that $c(0) > 0$, the above inequality remains true if one changes the coefficient $c(0)$ to 0. Setting it then back to its initial (positive) value does not affect the left-hand side, but strictly decreases the right-hand side, hence the desired strict inequality. □

5.3 The cavity equation on finite networks

Let $G = (V, E)$ be a finite graph at the vertices of which some local measures $\mu_i, i \in V$ are specified. We call the resulting object a **network**. A **configuration** $\mathbf{x}$ is an assignment of numbers $x_{i\to j} \geq 0$ to every oriented edge $i \to j \in \vec{E}$. Starting from a configuration $\mathbf{x}$, we define a new configuration $\mathbf{y} = \Gamma_G(\mathbf{x})$ by

$$y_{i\to j} = \Gamma_{\mu_i}^{ij}\left(x_{k\to i} : k \in \partial i \setminus j\right), \tag{5.9}$$

where ∂i denotes the set of all neighbors of i. Each $x_{i\to j}$ may be thought of as a message sent by i to j along the edge ij, and Γ_G as a local rule for propagating messages. For $t > 0$, the fixed point equation

$$\mathbf{x} = t\Gamma_G(\mathbf{x}) \tag{5.10}$$

is called the **cavity equation** (also known as Belief Propagation (BP) equation) at activity t on the network G. Its relation to the global measure μ induced by the $(\mu_i)_{i\in V}$ is revealed by the following Lemma.

Lemma 5.3 (Validity on trees)**.** *Assume that G is finite and acyclic. Then, for every activity $t > 0$,*

1. ***convergence**: the cavity equation admits a unique solution $\mathbf{x}(t)$, which can be reached from any initial configuration by iterating $t\Gamma_G$ a number of times equal to the diameter of G ;*

2. ***correctness**: for every $i \in V$, the exact marginal law of $\mathcal{F} \cap E_i$ under the Gibbs-Boltzmann law $\mathbb{P}_G^t$ is given by directly imposing the external field $\{x_{j\to i}(t) : j \in \partial i\}$ onto the local measure μ_i.*

The important consequence is that on trees, the energy $U(G;t)$ can be determined using only local operations :

$$\begin{aligned} U(G;t) &= \frac{1}{2}\sum_{i\in V} U_{\mu_i}(x_{j\to i}(t) : j \in \partial i) && (5.11)\\ &= \sum_{ij\in E} \frac{x_{j\to i}(t)x_{i\to j}(t)}{t + x_{j\to i}(t)x_{i\to j}(t)}, && (5.12)\end{aligned}$$

where the second equality is obtained by applying (5.8) to each $\mu_i, i \in V$.

Proof. The result stated in Lemma 5.3 is well-known (see e.g. Mézard and Montanari [76, Theorem 14.3]). We give a proof for completeness. When i is a leaf, the message $y_{i\to j}$ defined by equation (5.9) does not depend at all on the initial configuration $\mathbf{x}$. Iterating this argument immediately proves the convergence part, and we now focus on correctness. Let $G = (V, E)$ be a finite tree, o a vertex, and i a neighbor of o. We let $G_{i\to o}$ denote the subtree induced by o and all vertices that the edge io separates from o. Now assume that G is equipped with local measures, and let $G_{i\to o}$ inherit from these local measures, except for μ_o which we replace by the trivial local measure with constant value 1. With these notations, any spanning subgraph $F \subseteq E$ can be uniquely decomposed as the disjoint union of a subset $I \subseteq E_o$ and a spanning subgraph F_i on each $G_{i\to o}, i \in \partial o$, with $io \notin F_i$. Thus, writing μ_G for the global measure on the network G, we have

$$\mu_G(F)t^{|F|} = \mu_o(I)\prod_{i\in I} t^{|F_i|+1}\mu_{G_{i\to o}}(F_i \cup io)\prod_{i\notin I} t^{|F_i|}\mu_{G_{i\to o}}(F_i).$$

Fixing I and summing over all possible values for $F_i, i \in \partial o$, we obtain

$$\begin{aligned}\mathbb{P}^t_G(\mathcal{F} \cap E_o = I) &= C\mu_o(I) \prod_{i \in I} \mathbb{P}^t_{G_{i\to o}}(io \in \mathcal{F}) \prod_{i \notin I} \mathbb{P}^t_{G_{i\to o}}(io \notin \mathcal{F}). \\ &= C'\mu_o(I) \prod_{i \in I} \frac{\mathbb{P}^t_{G_{i\to o}}(io \in \mathcal{F})}{\mathbb{P}^t_{G_{i\to o}}(io \notin \mathcal{F})},\end{aligned}$$

where C, C' are normalizing constants that do not depend on I. This already proves that the law of $\mathcal{F} \cap E_o$ can be obtained from the local measure μ_o by imposing on each edge $io \in E_o$ the external field

$$x_{i\to o}(t) := \frac{\mathbb{P}^t_{G_{i\to o}}(io \in \mathcal{F})}{\mathbb{P}^t_{G_{i\to o}}(io \notin \mathcal{F})}. \tag{5.13}$$

In turn, this ratio can now be computed by applying the result to the vertex i in the network $G_{i\to o}$, and using formula (5.7):

$$\frac{\mathbb{P}^t_{G_{i\to o}}(io \in \mathcal{F})}{\mathbb{P}^t_{G_{i\to o}}(io \notin \mathcal{F})} = t\Gamma^{io}_{\mu_i}\left(x_{k\to i}(t) : k \in \partial i \setminus o\right),$$

which shows that the configuration $\mathbf{x}(t)$ defined on G by (5.13) satisfies the cavity equation (5.10). □

There are two distinct parts in Lemma 5.3: convergence and correctness. As we will now show, the convergence part extends to arbitrary graphs under the only assumption that each local measure $\mu_i, i \in V$ is cavity-monotone. Henceforth, such a network will be called a **cavity-monotone network**.

Proposition 5.1 (Convergence on finite cavity-monotone networks)**.** *On a finite cavity-monotone network, the cavity equation admits a unique, globally attractive fixed point* $\mathbf{x}(t)$ *at any activity* $t > 0$.

Proof. Fixing $t > 0$ and starting with the minimal configuration $\mathbf{x}^0 := \mathbf{0}$, we set inductively

$$\mathbf{x}^{k+1}(t) := t\Gamma_G(\mathbf{x}^k(t)),$$

for all $k \in \mathbb{N}$. By Lemma 5.1, the Rayleigh property of the local measures $\mu_i, i \in V$ ensures that Γ_G is coordinate-wise non-increasing on the space of configurations. Therefore, the limiting configuration

$$\mathbf{x}^-(t) := \lim_{k\to\infty} \uparrow \mathbf{x}^{2k}(t) \qquad \text{and} \qquad \mathbf{x}^+(t) := \lim_{k\to\infty} \downarrow \mathbf{x}^{2k+1}(t) \tag{5.14}$$

exist, and any fixed point $\mathbf{x} = t\Gamma_G(\mathbf{x})$ must satisfy $\mathbf{x}^-(t) \le \mathbf{x} \le \mathbf{x}^+(t)$. Moreover, Γ_G is clearly continuous with respect to the product topology on configurations, so that $t\Gamma_G(\mathbf{x}^-(t)) = \mathbf{x}^+(t)$ and $t\Gamma_G(\mathbf{x}^+(t)) = \mathbf{x}^-(t)$. Thus, the existence of unique globally attractive solution to (5.10) boils down to the equality

$$\mathbf{x}^-(t) = \mathbf{x}^+(t). \tag{5.15}$$

Now applying (5.8) to the local measure at a fixed vertex $i \in V$ yields

$$\begin{aligned} U_{\mu_i}\left(x^-_{j\to i}(t) : j \in \partial i\right) &= \sum_{j\in\partial i} \frac{x^-_{j\to i}(t)x^+_{i\to j}(t)}{t + x^-_{j\to i}(t)x^+_{i\to j}(t)}, \\ U_{\mu_i}\left(x^+_{j\to i}(t) : j \in \partial i\right) &= \sum_{j\in\partial i} \frac{x^+_{j\to i}(t)x^-_{i\to j}(t)}{t + x^+_{j\to i}(t)x^-_{i\to j}(t)}. \end{aligned}$$

Summing over all vertices $i \in V$, we therefore obtain

$$\begin{aligned} \sum_{i\in V} U_{\mu_i}\left(x^-_{j\to i}(t) : j \in \partial i\right) &= \sum_{ij\in E} \frac{x^-_{j\to i}(t)x^+_{i\to j}(t)}{t + x^-_{j\to i}(t)x^+_{i\to j}(t)} + \frac{x^+_{j\to i}x^-_{i\to j}(t)}{t + x^+_{j\to i}(t)x^-_{i\to j}(t)} \\ &= \sum_{i\in V} U_{\mu_i}\left(x^+_{j\to i}(t) : j \in \partial i\right). \end{aligned}$$

This implies (5.15), since by Lemma 5.1 each $U_{\mu_i}, i \in V$ is strictly increasing in every coordinate. □

5.4 The infinite volume limit

In the previous section, we have established existence and uniqueness of a cavity solution on any finite cavity-monotone network. Our concern is its asymptotic meaning as the size of the underlying graph tends to infinity. Following the principles of the objective method [8], we replace the asymptotic analysis of finite networks by the direct study of their infinite limits.

5.4.1 Local convergence of rooted networks

The framework of local convergence described here slightly differs from the one given in the introduction or in [7], so as to take into account the local measures that are now attached to the vertices. A **network** is a denumerable graph $G = (V, E)$ whose vertices are equipped with local measures $\mu_i, i \in V$. A **rooted network** (G, o) is a network together with the specification of a particular vertex $o \in V$, called the root. For $\varepsilon \geq 0$, we write $(G', o') \overset{\varepsilon}{\equiv} (G, o)$ if there exists a bijection $\gamma\colon V \to V'$ that preserves

- the root : $\gamma(o) = o'$;
- the adjacency : $\{i, j\} \in E \iff \{\gamma(i), \gamma(j)\} \in E'$;
- the support of the local measures : $\mu_i(F) > 0 \iff \mu'_{\gamma(i)}(\gamma(F)) > 0$, with $\gamma(F) = \{\{\gamma(i), \gamma(j)\} : \{i, j\} \in F\}$.
- the values of the local measures, up to ε: $|\mu'_{\gamma(i)}(\gamma(F)) - \mu_i(F)| \leq \varepsilon$.

We let $\mathcal{G}_\star$ denote the set of all locally finite connected rooted networks considered up to the isomorphism relation $\overset{0}{\equiv}$. In the space $\mathcal{G}_\star$, a sequence $\{(G_n, o_n); n \in \mathbb{N}\}$

converges locally to (G, o) if for every radius $d \in \mathbb{N}$ and every $\varepsilon > 0$, there is $n_{d,\varepsilon} \in \mathbb{N}$ such that

$$n \geq n_d \Longrightarrow [G_n, o_n]_d \overset{\varepsilon}{\equiv} [G, o]_d,$$

where $[G, o]_d$ denotes the finite rooted network obtained by keeping only the vertices lying at graph-distance at most d from o. It is not hard to construct a distance which metrizes this notion of convergence and turns $\mathcal{G}$ into a complete separable metric space. We can thus import the usual machinery of weak convergence of probability measures on Polish spaces.

Uniform rooting is a natural procedure for turning a finite deterministic network G into a random element of $\mathcal{G}_\star$: one simply chooses uniformly at random a vertex o to be the root, and restrains G to the connected component of o. If $(G_n)_{n\in\mathbb{N}}$ is a sequence of finite networks and if the sequence of their laws under uniform rooting admits a weak limit $\mathcal{L} \in \mathcal{P}(\mathcal{G}_\star)$, we call $\mathcal{L}$ the **local weak limit** of the sequence $(G_n)_{n\in\mathbb{N}}$. In [7], it was shown that any such limit enjoys a remarkable invariance property known as **unimodularity**: let $\mathcal{G}_{\star\star}$ denote the space of locally finite connected networks with an ordered pair of distinguished adjacent vertices (G, o, i), taken up to the natural isomorphism relation and endowed with the natural topology. A measure $\mathcal{L} \in \mathcal{P}(\mathcal{G}_\star)$ is called unimodular if it satisfies the **Mass-Transport Principle**: for any Borel function $f \colon \mathcal{G}_{\star\star} \to [0, \infty]$,

$$\mathcal{L}\left[\sum_{i\in\partial o} f(G, o, i)\right] = \mathcal{L}\left[\sum_{i\in\partial o} f(G, i, o)\right], \tag{5.16}$$

where we have written $\mathcal{L}[\cdot]$ for the expectation with respect to $\mathcal{L}$. This is a deep and powerful notion, which we will now use to extend the results of section 5.3 to the infinite setting.

5.4.2 Main result : validity of the cavity method

The definition of Γ_G remains valid for any locally finite network G. When the latter is cavity-monotone, the configurations $\mathbf{x}^-(t), \mathbf{x}^+(t)$ introduced in the proof of Proposition 5.1 remain perfectly well-defined, and the convergence of the cavity method again boils down to the identity $\mathbf{x}^-(t) = \mathbf{x}^+(t)$. However, the proof of the latter involves a summation over all edges, which is no longer valid in the infinite setting. Instead, the desired $\mathbf{x}^-(t) = \mathbf{x}^+(t)$ will be derived from unimodularity, and will thus hold for any local weak limit of finite networks. Indeed, applying the Mass-Transport Principle to the function (which is Borel as the point-wise limit of continuous functions)

$$f(G, o, i) := \frac{x^-_{i\to o}(t) x^+_{o\to i}(t)}{t + x^-_{i\to o}(t) x^+_{o\to i}(t)}$$

yields $\mathcal{L}\left[U_{\mu_o}(x^-_{i\to o}(t) : i \in \partial o)\right] = \mathcal{L}\left[U_{\mu_o}(x^+_{i\to o}(t) : i \in \partial o)\right]$. If this expectation is finite, then the size-increasing property of μ_o implies that $\mathcal{L}-$almost surely, $x^-_{i\to o}(t) = x^+_{i\to o}(t)$ for all $i \in \partial o$. This then automatically extends to every oriented edge since under unimodularity, *everything shows up at the root* (another fruitful application of

the Mass-Transport-Principle, see [7, Lemma 2.3]). We state this as a Proposition below. Note that the energy U_μ of a measure μ is bounded by the **rank** of μ, where

$$\mathrm{rank}(\mu) = \max\{|\mathrm{F}|; \mathrm{F} \in \mathrm{supp}(\mu)\}.$$

In particular, for the measure $\mu_o(F) = \mathbf{1}_{|F|\leq b}$ describing the local constraint at a vertex o for counting $b-$matchings, $\mathrm{rank}(\mu_\mathrm{o}) = \mathrm{b} \wedge \deg(\mathrm{o}) \leq \mathrm{b}$.

Proposition 5.2 (Convergence of the cavity method)**.** *Let $\mathcal{L} \in \mathcal{P}(\mathcal{G}_\star)$ be a unimodular probability measure supported by cavity-monotone networks. If*

$$\mathcal{L}\left[\mathrm{rank}(\mu_\mathrm{o})\right] < \infty,$$

then the cavity equation admits $\mathcal{L}-$a.-s. a unique, globally attractive solution $\mathbf{x}(t)$ at any activity $t > 0$.

By analogy with formula (5.11) in the finite case, the quantity

$$u(\mathcal{L}; t) = \frac{1}{2}\mathcal{L}\left[\sum_{i\in\partial o} \frac{x_{i\to o}(t)x_{o\to i}(t)}{t + x_{i\to o}(t)x_{o\to i}(t)}\right] \tag{5.17}$$

appears as a natural candidate for the limiting energy of any sequence of finite networks whose local weak limit is $\mathcal{L}$. Our main result is precisely the validity of this *cavity ansatz* when $\mathcal{L}$ is concentrated on trees.

Theorem 5.2 (Main result)**.** *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of finite cavity-monotone networks admitting a local weak limit $\mathcal{L}$ which is concentrated on cavity-monotone trees. Assume that the rank of the local measure at a uniformly chosen vertex is uniformly integrable as $n \to \infty$. Then,*

$$\frac{U(G_n; t)}{|V_n|} \xrightarrow[n\to\infty]{} u(\mathcal{L}; t). \tag{5.18}$$

If moreover $|E_n| = O(|V_n|)$ and all the local measures take values in $\{0, 1\}$ or more generally in $\{0\} \cup K$ for a fixed compact $K \subseteq (0, \infty)$, then

$$\frac{1}{|V_n|}\log Z(G_n; t) \xrightarrow[n\to\infty]{} \mathcal{L}[\log \mu_o(\emptyset)] + \int_0^t \frac{u(\mathcal{L}; s)}{s} ds, \tag{5.19}$$

$$\frac{M(G_n)}{|V_n|} \xrightarrow[n\to\infty]{} m(\mathcal{L}) := \lim_{t\to\infty} \uparrow u(\mathcal{L}; t). \tag{5.20}$$

Remark 5.2 (Large deviation principle)**.** *Integrating (5.18) yields*

$$\frac{1}{|V_n|}\log\frac{Z(G_n; t)}{Z(G_n; 1)} \xrightarrow[n\to\infty]{} \int_1^t \frac{u(\mathcal{L}; s)}{s} ds.$$

It will later be checked that $t \mapsto u(\mathcal{L}; t)$ is continuous on $\mathbb{R}_+$. Therefore, for a random spanning subgraph $\mathcal{F}_n$ with the Gibbs-Boltzmann law $\mathbb{P}^1_{G_n}$, Gärtner-Ellis Theorem [49] guarantees that $|\mathcal{F}_n|/|V_n|$ obeys a large deviation principle with speed $|V_n|$ and good rate function $y \mapsto \int_0^\infty (y - u(\mathcal{L}; e^s))^+ ds$.

5.4.3 Proof of the main result

The proof makes use of a classical ingredient known as the **spatial Markov property**, which we first briefly recall. Let $G = (V, E)$ be a finite network and let S be an induced subgraph. We let ∂S denote the boundary of S, i.e. the set of edges having one end-point in S and one in S^c. Any boundary condition $B \subseteq \partial S$ can be used to assign local measures to the vertices of S, namely $\mu_i^B(F) := \mu_i(F \cup (B \cap E_i))$. Note that these local measures differ from the original ones only for vertices that are adjacent to the boundary. The resulting network is denoted by $S|B$. Now, a spanning subgraph $F \subseteq E$ is clearly the disjoint union of a spanning subgraph F_{int} of S, a boundary condition $B \subseteq \partial S$ and a spanning subgraph F_{ext} in S^c. The product form of μ_G immediately yields :

$$\mathbb{P}_G^t(\mathcal{F} = F) = \mathbb{P}_{S|B}^t(\mathcal{F} = F_{\text{int}})\mathbb{P}_G^t(\mathcal{F} \cap \partial S = B)\mathbb{P}_{S^c|B}^t(\mathcal{F} = F_{\text{ext}}). \tag{5.21}$$

In other words, conditionally on the boundary $\mathcal{B} := \mathcal{F} \cap \partial S$, the restrictions of $\mathcal{F}$ to S and S^c are independent with law $\mathbb{P}_{S|\mathcal{B}}^t$ and $\mathbb{P}_{S^c|\mathcal{B}}^t$, respectively.

Lemma 5.4 (Tree approximation)**.** *Let (G, o) be a finite cavity-monotone network, and let $k \in \mathbb{N}$. If $[G, o]_{2k+2}$ is a tree, then for every activity $t > 0$,*

$$U_{\mu_o}\left(x_{i\to o}^{2k}(t) : i \in \partial o\right) \leq \mathbb{E}_G^t\left[|\mathcal{F} \cap E_o|\right] \leq U_{\mu_o}\left(x_{i\to o}^{2k+1}(t) : i \in \partial o\right).$$

Proof. By the spatial Markov property for the tree $S = [G, o]_{2k+2}$, we have

$$\begin{aligned}\mathbb{E}_G^t\left[|\mathcal{F} \cap E_o|\right] &= \sum_{B\subseteq\partial S} \mathbb{P}_G^t(\mathcal{F} \cap \partial S = B)\mathbb{E}_{S|B}^t\left[|\mathcal{F} \cap E_o|\right] \\ &= \sum_{B\subseteq\partial S} \mathbb{P}_G^t(\mathcal{F} \cap \partial S = B)U_{\mu_o}\left(x_{i\to o}^{(B)}(t) : i \in \partial o\right),\end{aligned}$$

where in the second line we have applied Lemma 5.3 to the tree $S|B$ and written $\mathbf{x}^{(B)}(t)$ for the unique solution to the cavity equation at activity t thereon. By monotony of the cavity operator, each $x_{i\to o}^{(B)}(t), i \in \partial o$ must satisfy $x_{i\to o}^{2k}(t) \leq x_{i\to o}^{(B)}(t) \leq x_{i\to o}^{2k+1}(t)$, and since μ_o is size-increasing, we get

$$U_{\mu_o}\left(x_{i\to o}^{2k}(t) : i \in \partial o\right) \leq U_{\mu_o}\left(x_{i\to o}^{(B)}(t) : i \in \partial o\right) \leq U_{\mu_o}\left(x_{i\to o}^{2k+1}(t) : i \in \partial o\right).$$

Re-injecting this into the above equation yields the desired inequalities. □

Proof of Theorem 5.2. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of finite cavity-monotone networks admitting a local weak limit $\mathcal{L}$ which is concentrated on cavity-monotone trees. Denote by $\mathcal{L}_n \in \mathcal{P}(\mathcal{G}_\star)$ the law under uniform rooting of G_n, so that $\mathcal{L}_n \Rightarrow \mathcal{L}$. We will use the short-hand $u_k(G, o) = U_{\mu_o}\left(x_{i\to o}^k(t) : i \in \partial o\right)$, and $\chi_k(G, o)$ for the indicator function that $[G, o]_{2k+2}$ is a tree. Lemma 5.4 guarantees that for any finite cavity-monotone network G and any vertex o,

$$\begin{aligned}\mathbb{E}_G^t\left[|\mathcal{F}| \cap E_o\right] &\geq \chi_k(G, o)u_{2k}(G, o) \\ \mathbb{E}_G^t\left[|\mathcal{F}| \cap E_o\right] &\leq \chi_k(G, o)u_{2k+1}(G, o) + (1 - \chi_k(G, o))\text{rank}(\mu_o).\end{aligned}$$

As functions of (G,o), these lower and upper bounds are continuous on $\mathcal{G}_\star$, since they depend only on $[G,o]_{2k+2}$. Moreover, both are dominated by $(G,o)\mapsto \mathrm{rank}(\mu_{\mathrm{o}})$ which is assumed to be uniformly integrable with respect to the sequence $(\mathcal{L}_n)_{n\in\mathbb{N}}$. Thus, their expectation under $\mathcal{L}_n$ tends to their expectation under $\mathcal{L}$ as $n\to\infty$. But χ_k is zero on the support of $\mathcal{L}$, so we are simply left with

$$\begin{aligned}\liminf_{n\to\infty}\frac{U(G_n;t)}{|V_n|} &\geq \frac{1}{2}\mathcal{L}\left[U_{\mu_o}\left(x^{2k}_{i\to o}(t):i\in\partial o\right)\right]\\ \limsup_{n\to\infty}\frac{U(G_n;t)}{|V_n|} &\leq \frac{1}{2}\mathcal{L}\left[U_{\mu_o}\left(x^{2k+1}_{i\to o}(t):i\in\partial o\right)\right].\end{aligned}$$

Since the local weak limit $\mathcal{L}$ is unimodular, Proposition 5.2 implies that both the lower and upper bounds tend to $\mathcal{L}\left[U_{\mu_o}\left(x_{i\to o}(t):i\in\partial o\right)\right]=u(\mathcal{L};t)$ as $k\to\infty$. Note that the requirement $\mathcal{L}[\mathrm{rank}(\mu_{\mathrm{o}})]<\infty$ in Proposition 5.2 is here automatically fulfilled, thanks to the uniform integrability assumption.

It now remains to show (5.19) and (5.20). The identity (5.5) implies that for any activity $t>0$ and any finite network G satisfying $\mu(\emptyset)>0$,

$$\frac{1}{|V|}\log Z(G;t)=\frac{1}{|V|}\sum_{o\in V}\log\mu_o(\emptyset)+\int_0^t\frac{U(G;s)}{s|V|}ds.$$

Now take $G=G_n$ and let $n\to\infty$: the compactness assumption guarantees that $\log\mu_o(\emptyset)$ is bounded uniformly in n, so the first term converges to $\mathcal{L}[\log\mu_o(\emptyset)]$. As per the second one, it tends to $\int_0^t\frac{u(\mathcal{L};s)}{s}ds$ because of (5.18), provided we can show that the uniform domination holds in Lebesgue's dominated convergence Theorem. This is ensured by (5.22) in Lemma 5.5 below, combined with the compactness assumption and the fact that $|E_n|=O(|V_n|)$. The inequality (5.23) easily guarantees (5.20). □

Lemma 5.5 (Uniform controls for the energy)**.** *Let G be a finite cavity-monotone network. As a function of the activity t, the energy $U(G;t)$ increases from 0 to $M(G)$. Furthermore, the rate of convergence to these two extrema can be precisely controlled :*

$$\forall t>0, U(G;t) \leq t\sum_{ij\in E}A(\mu_i)A(\mu_j); \tag{5.22}$$

$$\forall t>1, U(G;t) \geq M(G)-\frac{1}{\log t}\left(|E|\log 2+\sum_{i\in V}\log A(\mu_i)\right). \tag{5.23}$$

where $A(\mu)=\max\mu/\min\mu$, with $\max\mu=\max\{\mu(F):F\in\mathrm{supp}(\mu)\}$ and $\min\mu=\min\{\mu(F):F\in\mathrm{supp}(\mu)\}$.

Proof of Lemma 5.5. That the energy increases with the activity is equivalent by (5.5) to the convexity of $\theta\mapsto\log Z(G;e^\theta)$, a direct consequence of Hölder's inequality. This also implies that for any $t>1$

$$U(G;t)\log t \geq \log\frac{Z(G;t)}{Z(G;1)}.$$

Clearly $Z(G;t) \geq t^{M(G)} \min \mu$ and $Z(G;1) \leq 2^{|E|} \max \mu$ and $A(\mu) \leq \prod_{i \in V} A(\mu_i)$, so (5.23) follows. Regarding (5.22), we have for any $t > 0$

$$\begin{aligned} U(G;t) &= \sum_{ij \in E} \mathbb{P}^t(ij \in \mathcal{F}) \\ &= t \sum_{ij \in E} \frac{\sum_{F \subseteq E \setminus ij} \mu(F \cup ij) t^{|F|}}{\sum_{F \subseteq E} \mu(F) t^{|F|}} \\ &\leq t \sum_{ij \in E} \max_{F \subseteq E \setminus ij, \mu(F) > 0} \frac{\mu(F \cup ij)}{\mu(F)} \\ &\leq t \sum_{ij \in E} A(\mu_i) A(\mu_j), \end{aligned}$$

where the third line uses the standard inequality $\frac{a+b}{c+d} \leq \max(\frac{a}{c}, \frac{b}{d})$ for any $a, b, c, d > 0$, and the fact that $\mu(F \cup ij) > 0 \Longrightarrow \mu(F) > 0$ (Remark 5.1). □

5.5 Explicit resolution for $b-$matchings

The goal of this section is to describe the important quantity $m(\mathcal{L})$ introduced in Theorem 5.2 directly in terms of a certain local equation which we naturally call the **cavity equation at infinite activity**. This will then be used to establish the explicit formulae that have been mentioned in the introduction.

5.5.1 The cavity equation at infinite activity

Let G be a cavity-monotone network. From Lemma 5.1, it follows that $(t, \mathbf{x}) \mapsto t\Gamma_G(t\mathbf{x})$ is increasing in t and decreasing in $\mathbf{x}$. We may thus define a limiting cavity-operator by

$$\overline{\Gamma}_G(\mathbf{x}) := \lim_{t \to \infty} \uparrow t\Gamma_G(t\mathbf{x}).$$

By monotony, $\overline{\Gamma}_G \colon [0,\infty)^{\vec{E}} \to (0,\infty]^{\vec{E}}$ is well-defined without any ambiguity regarding the order in which the limits $t \to \infty$ and $\mathbf{x} \to 0$ are taken. Note also that $\overline{\Gamma}_G$ can be composed with $\Gamma_G \colon (0,\infty]^{\vec{E}} \to [0,\infty)^{\vec{E}}$, yielding a two-step local update rule on $(0,\infty]^{\vec{E}}$ which will now play a crucial role.

Proposition 5.3 (The cavity equation at infinite activity)**.** *Let G be a cavity-monotone network on which the cavity equation at activity t admits a unique globally attractive fixed point $\mathbf{x}(t)$, for every $t > 0$. Then,*

$$\overline{\mathbf{x}} := \lim_{t \to \infty} \uparrow \mathbf{x}(t) \tag{5.24}$$

*exists in $(0,\infty]^{\vec{E}}$, and is the smallest solution to the so-called **cavity equation at infinite activity** on G:*

$$\overline{\mathbf{x}} = \left(\overline{\Gamma}_G \circ \Gamma_G\right)(\overline{\mathbf{x}}). \tag{5.25}$$

In particular, for any unimodular probability measure $\mathcal{L}$ concentrated on cavity-monotone networks, and satisfying $\mathcal{L}[\mathrm{rank}(\mu_\circ)] < \infty$, we have

$$m(\mathcal{L}) = \frac{1}{2}\mathcal{L}\left[U_{\mu_o}\left(\overline{x}_{i\to o} : i \in \partial o\right)\right].$$

Proof of Proposition 5.3. By assumption $\mathbf{x}^k(t) \to \mathbf{x}(t)$ for any $t > 0$, where $\mathbf{x}^0 \equiv \mathbf{0}$ and for all $k \in \mathbb{N}$,

$$\mathbf{x}^{k+1}(t) = t\Gamma_G\left(t\frac{\mathbf{x}^k(t)}{t}\right) \quad \text{i.e.} \quad \frac{\mathbf{x}^{k+1}(t)}{t} = \Gamma_G\left(\mathbf{x}^k(t)\right). \tag{5.26}$$

But $(t, \mathbf{x}) \mapsto t\Gamma_G(t\mathbf{x})$ is increasing in t and decreasing in $\mathbf{x}$, so an immediate induction over k shows that $t \mapsto t^{-1}\mathbf{x}^k(t)$ and $t \mapsto \mathbf{x}^k(t)$ are respectively non-increasing and non-decreasing, for every $k \in \mathbb{N}$. Thus, $t \mapsto \mathbf{x}(t)$ is non-decreasing, hence the existence of (5.24). The identity (5.25) is then obtained by passing to the limits in (5.26). Finally, if $\mathbf{y} \in (0,\infty]^{\vec{E}}$ satisfies $\mathbf{y} = (\overline{\Gamma}_G \circ \Gamma_G)(\mathbf{y})$, then for every $k \in \mathbb{N}$ and $t > 0$,

$$\mathbf{x}^{2k}(t) \le \mathbf{y}. \tag{5.27}$$

Indeed, the property is trivial when $k = 0$, and is preserved from k to $k+1$ because $t\Gamma_G(t\mathbf{w}) \le \overline{\Gamma}_G(\mathbf{w})$ for any $\mathbf{w} \in [0,\infty)^{\vec{E}}$. Letting $k \to \infty$ and then $t \to \infty$ in (5.27) yields $\overline{\mathbf{x}} \le \mathbf{y}$, as desired. □

Remark 5.3 (Continuity with respect to the activity)**.** *Incidentally, we have just obtained that $t \mapsto t^{-1}\mathbf{x}(t)$ and $t \mapsto \mathbf{x}(t)$ are respectively non-increasing and non-decreasing, so that*

$$0 < s \le t \implies \frac{s}{t}\mathbf{x}(t) \le \mathbf{x}(s) \le \mathbf{x}(t).$$

This guarantees the continuity of $t \mapsto \mathbf{x}(t)$, and hence that of $u(\mathcal{L}; t)$, as promised in Remark 5.2.

5.5.2 b−matchings on Galton-Watson trees

As explained in the introduction, many classical sequences of diluted random graphs admit almost surely a particularly simple local weak limit $\mathcal{L}$, namely a unimodular Galton-Watson tree. Recall that this random rooted tree is parametrized by a probability distribution $\pi \in \mathcal{P}(\mathbb{N})$ with finite mean, called its degree distribution, and that it is obtained by a Galton-Watson branching process where the root has offspring distribution π and all other genitors have the size-biased offspring distribution $\widehat{\pi} \in \mathcal{P}(\mathbb{N})$ defined by

$$\forall n \in \mathbb{N}, \widehat{\pi}_n = (n+1)\pi_{n+1} / \sum_k k\pi_k.$$

Thanks to the markovian nature of the branching process, the cavity equation at infinite activity again simplifies into a recursive distributional equation (RDE) (see

[6] for a survey). Let us describe it and solve it in the case of $b-$matchings, where $b \geq 1$ is fixed. The local cavity and energy ratios are

$$\begin{aligned} \Gamma(x_1, \ldots, x_n) &= \frac{\sum_{I \subseteq [n]: |I| \leq b-1} \prod_{i \in I} x_i}{\sum_{I \subseteq [n]: |I| \leq b} \prod_{i \in I} x_i} \\ U(x_1, \ldots, x_n) &= \frac{\sum_{I \subseteq [n]: |I| \leq b} |I| \prod_{i \in I} x_i}{\sum_{I \subseteq [n]: |I| \leq b} \prod_{i \in I} x_i}. \end{aligned}$$

In the infinite activity limit, the local cavity ratio becomes

$$\overline{\Gamma}(x_1, \ldots, x_n) = \frac{\sum_{I \subseteq [n]: |I| = b-1} \prod_{i \in I} x_i}{\sum_{I \subseteq [n]: |I| = b} \prod_{i \in I} x_i},$$

where all conventions regarding degenerate cases are obtained by taking the corresponding limits. Given $Q \in \mathcal{P}((0, \infty])$, we let $\Theta(Q) \in \mathcal{P}([0,1]))$ denote the law of $\Gamma\left(Y_1, \ldots, Y_{\widehat{N}}\right)$, where $\widehat{N}$ has law $\widehat{\pi}$ and $Y_1, Y_2, \ldots$ are i.i.d. with law Q, independent of $\widehat{N}$. Similarly, given $P \in \mathcal{P}([0,1])$, we let $\overline{\Theta}(P) \in \mathcal{P}((0, \infty])$ denote the law of $\overline{\Gamma}\left(X_1, \ldots, X_{\widehat{N}}\right)$, where $\widehat{N}$ has law $\widehat{\pi}$ and $X_1, X_2, \ldots$ are i.i.d. with law P, independent of $\widehat{N}$. With this notation in hands, the law $Q \in \mathcal{P}((0, \infty])$ of a message sent towards the root in the configuration $\overline{\mathbf{x}}$ must satisfy the RDE $Q = (\overline{\Theta} \circ \Theta)(Q)$. Equivalently, $P = \Theta(Q)$ must satisfy $P = (\Theta \circ \overline{\Theta})(P)$. More precisely, letting $H(P)$ denote the expectation of $\frac{1}{2} U\left(Y_1, \ldots, Y_N\right)$, where N has law π and $Y_1, Y_2, \ldots$ are i.i.d. with law $Q = \overline{\Theta}(P)$, independent of N, we may rephrase Proposition 5.3 as follows.

Corollary 5.2. *$m(\mathcal{L}) = H(P)$, where $P \in \mathcal{P}([0,1])$ is the largest solution to the RDE $P = (\Theta \circ \overline{\Theta})(P)$.*

The fixed points of $\Theta \circ \overline{\Theta}$ turn out to be in one-to-one correspondence with the **historical minima** of a certain function $M \colon [0,1] \to \mathbb{R}$ defined in terms of the degree generating function $\phi(\lambda) = \sum_k \pi_k \lambda^k$:

$$M(\lambda) = \frac{b}{2} \left(2 - g(\lambda) - (g \circ f)(\lambda) + \frac{c}{b} f(\lambda)(f \circ f)(\lambda)\right),$$

$$\text{with } c = \phi'(1), f(\lambda) = \frac{1}{c} \sum_{k=0}^{b-1} \frac{\lambda^k \phi^{(k+1)}(1-\lambda)}{k!} \text{ and } g(\lambda) = \sum_{k=0}^{b} \frac{\lambda^k \phi^{(k)}(1-\lambda)}{k!}.$$

A **historical minima** of M is a number $\lambda \in [0,1]$ satisfying $M'(\lambda) = 0$ and $M(\lambda') > M(\lambda)$ for all $\lambda' < \lambda$.

Proposition 5.4 (Resolution of the RDE)**.** *Let $\lambda_1 < \ldots < \lambda_r$ denote the historical minima of the function M. Then, the distributional equation $P = (\Theta \circ \overline{\Theta})(P)$ admits exactly λ solutions, and they are stochastically ordered : $P_1 < \ldots < P_r$. Moreover, for each $1 \leq i \leq r$, we have $H(P_i) = M(\lambda_i)$.*

In particular, $m(\mathcal{L}) = \min M$, which is exactly the formula given in Theorem 5.1. Proposition 5.4 was established in chapter 2 for the special case $b = 1$ (Proposition 3.2), but the proof can easily be adapted to the general case. For the sake of completeness, we have included a general proof.

5.5.3 Resolution of the RDE

First observe that the mappings $\Theta, \overline{\Theta}$ and H are all decreasing with respect to stochastic order, and continuous with respect to weak convergence. Note also that $cf'(\lambda)\lambda = bg'(\lambda)$, so that $M'(\lambda) = cf'(\lambda)\left((f \circ f)(\lambda) - \lambda\right)$. Thus $M'(0) \leq 0$, $M'(1) \geq 0$, and $M'(\lambda) = 0$ if and only if $(f \circ f)(\lambda) = \lambda$.

Lemma 5.6 (Properties of $\Gamma, \overline{\Gamma}$ and U)**.**

1. *Let $(x_1, \ldots, x_n) \in [0,1]^n$ and $y_i = \overline{\Gamma}(x_k : k \neq i)$. Then,*
   (a) $\overline{\Gamma}(x_1, \ldots, x_n) = \infty \iff \sum_{i=1}^n \mathbf{1}_{\{x_i > 0\}} < b$;
   (b) $\displaystyle\sum_{i=1}^n \frac{x_i y_i}{1 + x_i y_i} \mathbf{1}_{\{y_i < \infty\}} = b\mathbf{1}_{\left\{\sum_{i=1}^n \mathbf{1}_{\{x_i > 0\}} > b\right\}}$
2. *Let $(y_1, \ldots, y_n) \in (0, \infty]^n$ and $x'_i = \Gamma(y_k : k \neq i)$. Then,*
   (a) $\Gamma(y_1, \ldots, y_n) > 0 \iff \sum_{i=1}^n \mathbf{1}_{\{y_i = \infty\}} < b$;
   (b) $\displaystyle\sum_{i=1}^n \frac{x'_i y_i}{1 + x'_i y_i} \mathbf{1}_{\{y_i < \infty\}} = U(x'_1, \ldots, x'_n) - b \wedge \sum_{i=1}^n \mathbf{1}_{\{y_i = \infty\}}$.

Proof. Properties 1.*a* and 2.*a* are straightforward from the definition of $\overline{\Gamma}$ and Γ. Regarding property 1.*b*, set $K = \#\{i \in [n] : x_i > 0\}$. If the sum is non-zero then there must be an i satisfying both $y_i < \infty$ and $x_i > 0$. By 1.*a*, this implies $K > b$. Conversely, if $K > b$ then $y_i < \infty$ for every $i \in [n]$. We have just shown

$$\begin{aligned}
\sum_{i=1}^n \frac{x_i y_i}{1 + x_i y_i} \mathbf{1}_{\{y_i < \infty\}} &= \mathbf{1}_{\{K > b\}} \sum_{i=1}^n \frac{x_i y_i}{1 + x_i y_i}. \\
&= \mathbf{1}_{\{K > b\}} \sum_{i=1}^n \frac{\sum_{|I|=b, I \ni i} \prod_{k \in I} x_k}{\sum_{|I|=b} \prod_{k \in I} x_k} \\
&= b\mathbf{1}_{\{K > b\}},
\end{aligned}$$

where the second equality is obtained by replacing $y_i = \overline{\Gamma}(x_k : k \neq i)$ by its definition. For property 2.*b*, set $L = \#\{i \in [n] : y_i = \infty\}$. When $L = 0$, 2.*b* boils down to formula (5.8). The case $1 \leq L \leq b$ can then be obtained by successively setting each of the L variables to ∞, which amounts to condition on the presence of the corresponding ground elements. For $L \geq b$, both sides of the equation are zero. □

Lemma 5.7. *Let $P \in \mathcal{P}([0,1])$ and $P' = (\Theta \circ \overline{\Theta})(P)$. Set also $\lambda = P(\{0\}^c)$ and $\lambda' = P'(\{0\}^c)$. Then,*

1. $\lambda' = (f \circ f)(\lambda)$;
2. $P' \leq P \Longrightarrow H(P) \leq M(\lambda)$;
3. $P' \geq P \Longrightarrow H(P) \geq M(\lambda)$.

In particular, if $P = (\Theta \circ \overline{\Theta})(P)$ then $H(P) = M(\lambda)$ and $M'(\lambda) = 0$.

Proof. In the whole proof, N denotes a generic random integer with law π, $\widehat{N}$ a generic random integer with law $\widehat{\pi}$, $X, X_1, X_2, \ldots$ generic $[0,1]-$valued random variables with law P, $Y, Y_1, Y_2, \ldots$ generic $(0,\infty]-$valued random variables with law $\overline{\Theta}(P)$, and $X', X'_1, X'_2, \ldots$ generic $[0,1]-$valued random variables with law P'. We use the convention that all variables appearing under the same expectation are independent. With these notations, properties $1.a$ and $2.a$ in Lemma 5.6 give

$$\mathbb{P}\left(Y=\infty\right)=\mathbb{P}\left(\sum_{i=1}^{\widehat{N}}\mathbf{1}_{\{X_i>0\}}<b\right)\text{ and }\mathbb{P}\left(X'>0\right)=\mathbb{P}\left(\sum_{i=1}^{\widehat{N}}\mathbf{1}_{\{Y_i=\infty\}}<b\right),$$

which, in view of the definition of f, yields exactly the first claim of the Lemma. Now, using property $1.b$ and $2.b$, we respectively obtain the two following identities:

$$\begin{aligned}
\mathbb{E}\left[\frac{XY\mathbf{1}_{\{Y<\infty\}}}{1+XY}\right] &= \sum_{n\in\mathbb{N}}\widehat{\pi}(n)\mathbb{E}\left[\frac{X\overline{\Gamma}\left(X_1,\ldots,X_n\right)}{1+X\overline{\Gamma}\left(X_1,\ldots,X_n\right)}\mathbf{1}_{\left\{\overline{\Gamma}(X_1,\ldots,X_{n-1})<\infty\right\}}\right]\\
&= \sum_{n\in\mathbb{N}}\pi(n)n\mathbb{E}\left[\frac{X\overline{\Gamma}\left(X_1,\ldots,X_{n-1}\right)}{1+X\overline{\Gamma}\left(X_1,\ldots,X_{n-1}\right)}\mathbf{1}_{\left\{\overline{\Gamma}(X_1,\ldots,X_{n-1})<\infty\right\}}\right]\\
&= \sum_{n\in\mathbb{N}}\pi(n)\mathbb{E}\left[\sum_{i=1}^{n}\frac{X_i\overline{\Gamma}\left(X_k:k\neq i\right)}{1+X_i\overline{\Gamma}\left(X_k:k\neq i\right)}\mathbf{1}_{\left\{\overline{\Gamma}(X_k:k\neq i)<\infty\right\}}\right]\\
&= b\mathbb{P}\left(\sum_{i=1}^{N}\mathbf{1}_{\{X_i>0\}}>b\right)\\
&= b(1-g(\lambda)).
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\frac{X'Y\mathbf{1}_{\{Y<\infty\}}}{1+X'Y}\right] &= \sum_{n\in\mathbb{N}}\widehat{\pi}(n)\mathbb{E}\left[\frac{Y\Gamma\left(Y_1,\ldots,Y_n\right)}{1+Y\Gamma\left(Y_1,\ldots,Y_n\right)}\mathbf{1}_{\{Y<\infty\}}\right]\\
&= \sum_{n\in\mathbb{N}}\pi(n)n\mathbb{E}\left[\frac{Y\Gamma\left(Y_1,\ldots,Y_{n-1}\right)}{1+Y\Gamma\left(Y_1,\ldots,Y_{n-1}\right)}\mathbf{1}_{\{Y<\infty\}}\right]\\
&= \sum_{n\in\mathbb{N}}\pi(n)\mathbb{E}\left[\sum_{i=1}^{n}\frac{Y_i\Gamma\left(Y_k:k\neq i\right)}{1+Y_i\Gamma\left(Y_k:k\neq i\right)}\mathbf{1}_{\{Y_i<\infty\}}\right]\\
&= \mathbb{E}\left[U(Y_1,\ldots,Y_N)\right]-\mathbb{E}\left[b\wedge\sum_{i=1}^{N}\mathbf{1}_{\{Y_i=\infty\}}\right]\\
&= 2H(P)-b(1-g(\lambda))-c\lambda f(\lambda).
\end{aligned}$$

Since the mapping $\frac{xy}{1+xy}$ is increasing in x, claims 2 and 3 follow. □

Proof of Proposition 5.4. Fix $\lambda\in[0,1]$ satisfying $M'(\lambda)=0$, i.e. $(f\circ f)(\lambda)=\lambda$. Define $P_\lambda^{(0)}\in\mathcal{P}([0,1])$ to be the Bernoulli distribution with parameter λ, and set then iteratively

$$P_\lambda^{(k+1)}=(\Theta\circ\overline{\Theta})(P_\lambda^{(k)})$$

for all $k \in \mathbb{N}$. By part 1 in Lemma 5.7, $P_\lambda^{(1)}$ is a distribution on $[0,1]$ satisfying $P_\lambda^{(1)}(\{0\}^c) = \lambda$. Since $P_\lambda^{(0)}$ is the largest such distribution, $P_\lambda^{(1)} \leq P_\lambda^{(0)}$. But both Θ and $\overline{\Theta}$ are decreasing, so by immediate induction, the sequence $(P_\lambda^{(k)})_{k\in\mathbb{N}}$ is non-increasing. Thus, the limit $P_\lambda = \lim_{k\to\infty} \downarrow P_\lambda^{(k)}$ exists in $\mathcal{P}([0,1])$. Setting $\lambda_\infty = P_\lambda(\{0\}^c)$, we claim that

1. P_λ is a fixed point of $\Theta \circ \overline{\Theta}$;
2. $H(P_\lambda) = M(\lambda_\infty)$;
3. $\lambda_\infty \leq \lambda$;
4. $M(\lambda_\infty) \leq M(\lambda)$.

Part 1 follows from the continuity of Θ and $\overline{\Theta}$. Part 2 is guaranteed by Lemma 5.7. Part 3 is a consequence of the fact that $P_\lambda \leq P_\lambda^{(0)}$. Finally, for each $k \in \mathbb{N}$, we have $P_\lambda^{(k)}(\{0\}^c) = \lambda$ and $P_\lambda^{(k)} \geq P_\lambda^{(k+1)}$, so Lemma 5.7 guarantees that $H(P_\lambda^{(k)}) \leq M(\lambda)$. Letting $k \to \infty$ yields exactly part 4.

We are now in position to prove the equivalence between the historical minima of M and the solutions to $(\Theta \circ \overline{\Theta})(P) = P$. If λ is a historical minimum, then parts 3 and 4 force $\lambda_\infty = \lambda$ so $P = P_\lambda$ is a solution satisfying $H(P) = M(\lambda)$, as desired. Conversely, let us now show that any solution P is in fact of the form P_λ for some historical minimum λ. Set $\lambda = P(\{0\}^c)$, which satisfies $M'(\lambda) = 0$ by Lemma 5.7. We claim that $P \leq P_\lambda^{(k)}$ for any $k \in \mathbb{N}$: indeed, this holds for $k = 0$ because $P_\lambda(0)$ is the largest element of $\mathcal{P}([0,1])$ such that $P_\lambda^{(0)}(\{0\}^c) = \lambda$, and it then inductively extends to all $k \in \mathbb{N}$ by monotony of $\Theta \circ \overline{\Theta}$. Letting $k \to \infty$, we obtain $P \leq P_\lambda$; but by Lemma 5.7 we also have $H(P) = M(\lambda)$, and $H(P_\lambda) \leq M(\lambda)$. Thus, $P = P_\lambda$ (H is decreasing). Finally, if $\lambda' < \lambda$ is any historical minimum then $P_{\lambda'}^{(0)} \leq P_\lambda^{(0)}$, which implies $P_{\lambda'} \leq P_\lambda$. In fact the inequality is strict, because $P_{\lambda'}(\{0\}^c) = \lambda' < \lambda = P_\lambda(\{0\}^c)$. Applying the decreasing mapping H yields $M(\lambda') > M(\lambda)$, which shows that λ is a historical minimum. □

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