Contribution à l'étude mathématique des plasmas fortement magnétisés

Daniel Han-Kwan

THÈSE DE DOCTORAT

Spécialité :

MATHÉMATIQUES

présentée par

Daniel HAN-KWAN

pour obtenir le grade de

DOCTEUR de l'UNIVERSITÉ PIERRE ET MARIE CURIE

Sujet :

Contribution à l'étude mathématique des plasmas fortement magnétisés

Soutenue publiquement le 8 juillet 2011
devant le jury composé de

Yann BRENIER	Rapporteur
Bruno DESPRÉS	Examinateur
Pierre-Emmanuel JABIN	Examinateur
Nader MASMOUDI	Examinateur
Florian MÉHATS	Examinateur
Clément MOUHOT	Examinateur
Laure SAINT-RAYMOND	Directrice de thèse

Université Pierre et Marie Curie

École Doctorale de Sciences Mathématiques de Paris Centre

Département de Mathématiques et Applications – UMR 8553
École Normale Supérieure, Paris

Remerciements

Mes premiers remerciements vont tout naturellement à Laure Saint-Raymond pour avoir accepté de diriger cette thèse et m'avoir fait découvrir un champ de recherche passionnant. Je la remercie pour le temps passé à lire mes articles et pour les nombreux conseils toujours avisés qui en ont découlé. Je suis également honoré de la confiance qu'elle m'a toujours témoignée, notamment en me laissant choisir librement mes objets de recherche.

Yann Brenier et Walter Strauss m'ont fait l'immense honneur de rapporter sur cette thèse : je les en remercie vivement. J'adresse également mes remerciements les plus sincères à Bruno Després, Pierre-Emmanuel Jabin, Nader Masmoudi, Florian Méhats et Clément Mouhot pour avoir accepté de faire partie de mon jury.

J'ai pris grand plaisir à travailler avec Olivier Glass et le remercie tout particulièrement. Par ailleurs, je n'oublie pas que c'est lui qui a encadré, en 2006, mon premier stage de recherche en m'initiant (en compagnie d'Amic) aux joies du contrôle de l'équation des ondes. Je suis très heureux aujourd'hui d'avoir co-signé avec lui mon premier article en tant que contrôleur et espère que notre collaboration restera longtemps fructueuse.

J'ai eu l'immense chance de travailler au DMA dans des conditions idéales, durant ces trois années de thèse. Je remercie l'ensemble des membres du laboratoire pour leur accueil, ainsi que pour leur bonne humeur et gentillesse au quotidien, en particulier Thomas Alazard, Thibaut Allemand, Diogo Arsénio, François Béguin, Thomas Boulier, Anne-Laure Dalibard, Benoît Desjardins, Vincent Duchêne, Philippe Gravejat, Colin Guillarmou, Cécile Huneau, Cyril Imbert, David Lannes, Jérémie Szeftel, Marie Théret, Amandine Véber, pour les discussions autour d'un repas ou d'un café, mais également pour les bons conseils et les encouragements. Je pense également à Bénédicte, Zaïna et Laurence sans qui la vie au DMA ne serait pas la même. Merci aussi à Guillaume Carlier pour sa sympathie et son magistral cours d'Analyse Fonctionnelle !

Conférences et séminaires me donnèrent l'occasion de rencontrer nombre de personnages hauts en couleur. Certains (peu, rassurez-vous) ont même témoigné de l'intérêt pour mon travail ! Qu'ils en soient ici chaleureusement remerciés.

Pour finir, j'ai une pensée pour mes amis qui ont su me changer les idées au cours de ces trois années. Ma famille m'a toujours soutenu et encouragé : je profite de ces quelques lignes pour leur témoigner ma reconnaissance.

Enfin, merci à Caro pour tout !

Résumé

Cette thèse est consacrée à l'étude mathématique de certains aspects de l'équation de Vlasov-Poisson, qui constitue un modèle cinétique classique en physique des plasmas.

Dans un premier temps, nous nous intéressons à la justification rigoureuse d'approximations de l'équation de Vlasov-Poisson avec un champ magnétique extérieur intense, qui sont couramment utilisées, notamment lors des simulations numériques. Le but est de décrire certains régimes d'intérêt par des modèles asymptotiques, obtenus en faisant tendre un petit paramètre vers 0 (modélisant la physique du problème considéré) dans les équations originelles. Nous étudions pour commencer la limite quasineutre, c'est-à-dire la limite quand la longueur de Debye tend vers 0, pour l'équation de Vlasov-Poisson avec des électrons suivant une loi de Maxwell-Boltzmann. Dans la limite des plasmas froids, à l'aide de la méthode de l'entropie relative et de techniques de filtrage, nous montrons la convergence vers des équations hydrodynamiques compressibles telles que l'équation d'Euler isotherme. Nous nous intéressons ensuite à l'approximation “rayon de Larmor fini” en trois dimensions, qui permet de décrire le comportement turbulent d'un plasma soumis à un champ magnétique intense. Pour cette étude, qui peut en fait être interprétée comme une limite quasineutre anisotrope, nous montrons des résultats très différents selon la dynamique décrite. En effet, dans le cas de la dynamique avec des électrons sans masse, nous exhibons un effet stabilisant qui permet d'obtenir le même résultat que pour le système bidimensionnel, alors que pour la dynamique avec des ions lourds, nous mettons en évidence les conséquences d'instabilités de type multi-fluides.

Dans un second temps, nous nous consacrons à l'étude mathématique du confinement d'un plasma de tokamak. Nous commençons par proposer un modèle hydrodynamique simplifié à deux températures et étudions la stabilité au sens de Lyapunov de deux états stationnaires permettant de modéliser l'équilibre du plasma. Nos résultats sont conformes à l'heuristique physique et mettent de surcroit en évidence qu'un fort gradient de température favorise la stabilité : cela pourrait fournir une explication aux modes de haut confinement (H-modes) dans les tokamaks. Pour finir, nous attaquons ce problème du point de vue de la théorie du contrôle et prouvons des résultats pour l'équation de Vlasov-Poisson en présence de champs extérieurs (typiquement un champ magnétique).

Mots-Clés : théorie cinétique, physique des plasmas, équation de Vlasov-Poisson, limite quasineutre, limite gyrocinétique, régime rayon de Larmor fini, instabilités hydrodynamiques, contrôlabilité de l'équation de Vlasov-Poisson, lemmes de moyenne, entropie relative, méthode du retour.

Abstract

This thesis is concerned with the mathematical study of some aspects of the Vlasov-Poisson equation, which is a classical kinetic model in plasma physics.

First of all, we are interested in the rigorous justification of approximations of the Vlasov-Poisson equation with an external strong magnetic field, which are wildly used, in particular for numerical simulations. The aim is to describe some physically relevant regimes by asymptotic models, obtained by letting a small parameter go to 0. We begin with a study of the quasineutral limit, that is to say the limit when the Debye length vanishes, for the Vlasov-Poisson equation with electrons following a Maxwell-Boltzmann law. In the cold plasma limit, using the relative entropy method and some filtering techniques, we show the convergence towards compressible hydrodynamic equations, such as Isothermal Euler. We are then interested in the finite Larmor radius approximation in three dimensions, which allows to describe the turbulent behaviour of a plasma submitted to a strong magnetic field. For this study, which can actually be interpreted as an anisotropic quasineutral limit, we show different qualitative behaviour, depending on the dynamics we describe. Indeed, in the case of the dynamics with massless electrons, we show a stabilizing effect which allows us to obtain the same result as in the bidimensional case. In the case of the dynamics with heavy ions, we underline the consequences of multi-fluid type instabilities.

In a second time, we are concerned with the mathematical study of the confinement of a tokamak plasma. We start by proposing of a simplified bi-temperature hydrodynamic model and study the stability in the sense of Lyapunov of two stationary states allowing to model the equilibrium of the plasma. Our results comply with the physical heuristics and furthermore underline that a large temperature gradient enhances stability : this could explain the high confinement modes (H-modes) in tokamaks. Finally we tackle this problem from the point of view of control theory and prove some results for the Vlasov-Poisson equation in presence of external force fields (in particular, magnetic fields).

Keywords : kinetic theory, plasma physics, Vlasov-Poisson equation, quasineutral limit, gyrokinetic limit, finite Larmor radius regime, hydrodynamic instabilities, controllability of the Vlasov-Poisson equation, averaging lemmas, relative entropy, return method.

Table des matières

Chapitre 1

Introduction et synthèse

Sommaire

1 Introduction générale

Nous commençons par donner quelques notions élémentaire de physique des plasmas (plus spécifiquement des plasmas pour la fusion magnétique) afin de préciser le contexte dans lequel s'inscrit cette thèse. La principale référence pour cette section est l'ouvrage de Wesson [158]. Nous introduisons également les modèles cinétiques étudiés dans cette thèse.

1.1 Introduction aux modèles cinétiques pour les plasmas

1.1.1 Qu'est-ce qu'un plasma ?

Un *plasma* est un gaz dilué ionisé, formé de particules chargées (ions et électrons) et éventuellement de particules neutres. La terminologie est due au chimiste et physicien Langmuir en 1928, qui a observé des oscillations dans un gaz où survenait une décharge électrique et a choisi le nom par analogie visuelle avec le plasma sanguin. La principale caractéristique d'un plasma est la suivante : contrairement à un gaz, dans lequel les particules sont neutres, les particules sont chargées, ce qui donne lieu à des phénomènes de transport dus aux interactions électromagnétiques entre les différentes particules. Ces interactions se traduisent par la création de champs électriques et magnétiques, suivant les *équations de Maxwell*, qui sont les équations fondamentales de l'électromagnétisme. Nous les rappellerons ultérieurement. Il apparait qu'un gaz chauffé suffisamment passe à l'état de plasma : pour cette raison, l'état plasma est souvent considéré comme le quatrième état de la matière. Les plasmas peuvent être trouvés à l'état naturel dans l'univers, citons les étoiles, l'espace interstellaire, la foudre... En fait, remarquablement, il se trouve que près de 99% de la matière dans l'univers est faite de plasma. Ces dernières décennies, de nombreux plasmas ont été créés en laboratoire, avec souvent des objectifs industriels. On pensera par exemple aux écrans "plasma", ou aux réacteurs à fusion nucléaire. Nous reviendrons par la suite bien plus en détail sur ce dernier exemple.

Introduisons quelques grandeurs caractéristiques, qui vont nous permettre de comprendre les échelles typiques des phénomènes pouvant survenir dans un plasma. L'échelle de temps caractéristique est donnée par l'inverse de la *fréquence "plasma"* ω_p. Celle-ci peut s'interpréter comme la fréquence d'oscillation d'une particule chargée initialement à l'équilibre et que l'on perturbe brutalement. Cet effet correspond à ce qu'a observé Langmuir dans son expérience pionnière. En notant n_α la densité typique des particules chargées de type α, q_α la charge, m_α sa masse et ε_0 la constante (universelle) de permittivité électrique du vide, on a :

$$\omega_p = \sqrt{\frac{n_\alpha q_\alpha^2}{\varepsilon_0 m_\alpha}}. \tag{1.1}$$

L'échelle de temps caractéristique du plasma est donc donnée par :

$$T_p = \frac{1}{\omega_p}. \tag{1.2}$$

Une échelle spatiale typique et d'importance fondamentale en physique des plasmas est la *longueur "de Debye"* λ_D. Elle peut être interprétée physiquement de la manière suivante : supposons que la longueur caractéristique d'observation L_0 soit grande devant λ_D ; alors à cette échelle d'observation, il n'y a pas de séparation de charge, ce qui signifie qu'en tout point la densité des ions semble égale à celle des ions. Autrement dit, le plasma n'est considéré comme non localement neutre que sur des longueurs d'observation inférieures à

λ_D. La longueur de Debye est définie comme suit, pour un type α donné de particules :

$$\lambda_D^{(\alpha)} = \sqrt{\frac{\varepsilon_0 k_B T_\alpha}{n_\alpha q_\alpha^2}}, \tag{1.3}$$

où k_B est la constante (universelle) de Boltzmann, T_α la température moyenne des particules α. Signalons d'ores et déjà que dans les régimes usuels, $\lambda_D \sim 10^{-3} - 10^{-8} m$, ce qui est très souvent petit devant l'échelle d'observation spatiale. Remarquons que cela signifie qu'en pratique, le paramètre ω_p est très grand, ce qui signifie que le plasma a tendance à osciller avec une très grande fréquence.

On peut également calculer la fréquence typique de collision (grâce un calcul élémentaire du libre parcours moyen d'une particule chargée) :

$$\omega_{col} = \frac{e^4 \log \Lambda}{4\pi\varepsilon_0^2 m_\alpha^{1/2}} \frac{n_\alpha}{T_\alpha^{3/2}}, \tag{1.4}$$

avec $\Lambda = 4\pi n_\alpha \left(\frac{\varepsilon_0 T}{n_\alpha e^2}\right)^{3/2}$. Le temps $T_{col} = \frac{1}{\omega_c}$ peut alors s'interprèter comme le temps moyen entre deux collisions entre particules. Ainsi si le plasma est "assez" dilué et si la température est suffisamment élevée (c'est-à-dire si $\Lambda \gg 1$), le plasma peut être considéré comme non collisionnel. En pratique le nombre sans dimension Λ est de l'ordre de $10^{10} - 10^{30}$, ce qui justifie pourquoi cette approximation est très souvent faite. Du fait de leur faible collisionnalité, les plasmas ne sont en général pas à l'équilibre thermodynamique. Par conséquent, pour de nombreuses applications, ils ne sont pas assez précisément décrits par les modèles fluides (de type Magnéto-Hydro-Dynamique, ou MHD)[1]. De ce fait, des *modèles de type cinétique* (plus précisément de champ moyen) sont la plupart du temps nécéssaires pour décrire précisément le comportement d'un plasma.

Nous présentons à présent une brève introduction aux modèles cinétiques, en particulier aux équations de Vlasov, qui constitueront le principal objet d'étude de cette thèse.

1.1.2 Modèles cinétiques pour les plasmas

Principe de la théorie cinétique La *théorie cinétique* est une branche de la physique statistique permettant une description de systèmes formés d'un grand nombre de particules (typiquement un gaz dilué) ; elle se propose de décrire leur comportement statistique plutôt que précisément chaque trajectoire suivie par les particules. Une telle description est donc statistique.

La théorie cinétique a été formalisée grâce aux travaux de Boltzmann [15] et Maxwell [126]. Elle s'appuie sur des travaux précurseurs portant sur la thermodynamique des gaz ; citons notamment les noms de Bernoulli ou Clausius. Originellement elle a permis de modéliser des gaz dilués hors équilibre et d'apporter un nouvel éclairage sur les concepts (à l'époque controversés) de température ou d'entropie. Aujourd'hui, elle permet de modéliser divers autres systèmes, en physique des plasmas, en astrophysique, en biologie...

La théorie cinétique repose fondamentalement sur une description discrète de la nature (les atomes), idée qui remonte à l'Antiquité et qui était toujours très controversée au 19ème siècle. Pour observer un système de particules (par exemple un fluide ou un plasma), plusieurs échelles d'observation sont envisageables. L'échelle *microscopique* consiste à suivre

[1] Signalons toutefois que ces modèles sont moins coûteux à simuler et qu'il est souvent plus facile de les étudier qualitativement que les équations cinétiques. Pour ces raisons, une grande partie de la connaissance physique sur les phénomènes de turbulence nous provient malgré tout de ces modèles.

la position et la vitesse de chaque particule, grâce aux équations de Newton. Cette description est extrêmement précise, mais elle donne beaucoup trop d'informations. Quand N est grand, il est difficile d'étudier ces équations aussi bien du point de vue qualitatif que de celui de l'analyse numérique : en effet pour un système formé de N particules (on peut par exemple penser à un litre d'air dans une pièce à température ambiante, pour lequel $N \sim 10^{25}$), il faudrait résoudre $6N$ équations couplées ! L'échelle *macroscopique* est la plus utilisée pour les applications (comme en météorologie) : cela consiste en une description de l'évolution du fluide par l'évolution de grandeurs macroscopiques, telles que la densité, le courant, la pression, la température. Les modèles typiques sont les équations d'Euler ou de Navier-Stokes. Cette description est bien moins coûteuse en terme de calculs et fournit des informations souvent suffisantes sur le fluide. Cependant elle repose sur l'hypothèse que les particules sont à l'équilibre thermodynamique, ce qui n'est pas toujours vérifié. L'échelle *mésoscopique* (celle de la théorie cinétique) est une échelle intermédiaire entre ces deux échelles. Commençons par considérer Ω_x, Ω_v deux ouverts de $\mathbb{R}^d$ (où $d = 1, 2$ ou 3). L'ouvert Ω_x (borné ou non) constitue le domaine physique où les particules peuvent se déplacer, alors que l'ouvert Ω_v (qui sera souvent $\mathbb{R}^d$ tout entier) sera l'ensemble des valeurs que peut prendre leur vitesse. Le principe est de décrire statistiquement les particules par leur répartition dans l'espace des phases (x, v), plus précisément par une fonction de distribution $f(t, x, v)$ (positive ou nulle), donnant le nombre de particules occupant une position proche de x et d'avoir une vitesse proche de v au temps t. Etant donnés des ensembles mesurables $\omega_x \subset \Omega_x$ et $\omega_v \subset \Omega_v$, la quantité :

$$\int_{\omega_x} \left(\int_{\omega_v} f(t, x, v) dv \right) dx \tag{1.5}$$

donne ainsi le nombre de particules dont la position x est dans ω_x et dont la vitesse est dans ω_v à l'instant t. La plupart du temps, on considérera que le système modélisé comprend un nombre fini de particules, si bien que $f(t, ., .)$ est intégrable. On peut donc la normaliser, de sorte que :

$$\int_{\Omega_x} \left(\int_{\Omega_v} f(t, x, v) dv \right) dx = 1. \tag{1.6}$$

Cette procédure sera systématiquement suivie par la suite. Alors, la quantité (1.5) peut s'interpréter comme la probabilité pour une particule d'être dans ω_x avec une vitesse dans ω_v à un temps t. Une telle description statistique d'un système physique fait sens lorsque celui-ci est constitué d'un grand nombre de particules $N \gg 1$. On doit également supposer que la taille d d'une particule est négligeable, de sorte que dans un volume fixé V, on ait $\frac{Nd^3}{V} \ll 1$: c'est l'hypothèse dite des *gaz dilués*[2].

La fonction de distribution f satisfait alors une équation aux dérivées partielles d'évolution, dite cinétique, posée dans l'espace des phases (x, v). Par rapport à l'échelle microscopique, une telle description permet de réduire de manière conséquente le nombre d'équations à résoudre (on passe de $6N$ équations à une équation à 7 variables). Par rapport à l'échelle macroscopique, elle permet de s'affranchir de l'hypothèse d'équilibre thermodynamique et de décrire plus finement la dynamique des particules, en tenant par exemple compte des effets éventuels des collisions entre particules.

Equations typiques L'équation cinétique la plus "simple" que l'on puisse écrire est l'équation de *transport libre* :

$$\partial_t f + v.\nabla_x f = 0,$$

[2]Cette hypothèse apparait naturellement lorsqu'on essaye de dériver un modèle cinétique à partir des équations de Newton (on se réfère par exemple à [149].

qui exprime que les particules suivent une trajectoire en ligne droite et à vitesse constante[3] lorsqu'elles ne sont soumises à aucune force. L'équation de transport libre est la brique élémentaire de la théorie cinétique.

La modélisation des interactions entre particules et de l'influence du milieu extérieur sur elles dépend des propriétés physiques et chimiques du système considéré. Ainsi, si par exemple on souhaite décrire un gaz ou un plasma, les équations pertinentes peuvent être de nature très différente. Les modèles usuels dépendent fortement du type d'interaction entre particules. Lorsque celles-ci sont de longue portée, elles sont décrites par une *équation de champ moyen.* Lorsqu'elles sont de courte portée, elles sont plutôt décrites par une *équation collisionnelle.*

Une **équation de champ moyen** s'écrit sous la forme :

$$\partial_t f + v.\nabla_x f + F.\nabla_v f = 0, \tag{1.7}$$

où $F(t, x, v)$ est la force effective résultant des interactions particulaires (ainsi qu'éventuellement un forçage extérieur). Elle est en général calculée à partir de la densité f elle-même, le plus souvent à partir de grandeurs macroscopiques, telles que :

- La *densité* $\rho(t, x) := \int f dv$,
- Le *courant* $\rho u(t, x) := \int f v dv$,
- La *température* $T(t, x) := \frac{1}{d} \frac{\int f(t,x,v)|v-u(t,x)|^2 dv dx}{\int f(t,x,v) dv}$, où d est la dimension d'espace.

Une telle équation est parfois appelée équation de Liouville, ou *équation de Vlasov* [156]. Cette dernière dénomination sera la plus souvent employée dans ce manuscrit. On appelle *caractéristiques* les courbes solutions du système d'équations différentielles :

$$\begin{cases} \frac{dX}{dt} = V, \\ \frac{dV}{dt} = F(t, X, V). \end{cases} \tag{1.8}$$

On vérifie formellement que toute solution de (1.7) est constante le long de ces courbes, si bien que l'équation de Vlasov décrit bien l'évolution des particules soumises à la force F.

Dans les plasmas, ce sont les interactions électromagnétiques qui prédominent : ainsi les autres effets (tels que les interactions gravitationnelles ou les collisions) sont souvent négligés.

L'équation la plus fondamentale pour les plasmas est l'équation de *Vlasov-Maxwell* (relativiste), qui consiste comme son nom l'indique en un couplage entre l'équation de Vlasov, que l'on vient d'introduire, et les équations de Maxwell, qui sont les équations de l'électromagnétisme [127]. En notant c la vitesse de la lumière (et en normalisant toutes les autres quantités, pour simplifier), ce système s'écrit[4] :

$$\begin{cases} \partial_t f + \hat{v}.\nabla_x f + \mathrm{div}_v[(E + \frac{1}{c}\hat{v} \wedge B) f] = 0, \\ \partial_t E - c \,\mathrm{rot}\, B = -\int f \hat{v} dv, \quad \partial_t B + c \,\mathrm{rot}\, E = 0, \\ \mathrm{div}\, E = \rho, \quad \mathrm{div}\, B = 0, \end{cases} \tag{1.9}$$

[3]Si l'on veut tenir compte d'effets relativistes, il faudrait considérer l'opérateur $\frac{v}{\sqrt{1+v^2/c^2}}.\nabla_x$, avec c la vitesse de la lumière, à la place de $v.\nabla_x$.

[4]Pour simplifier on a considéré un système constitué d'une seule espèce de particule chargées, si bien qu'il n'y a pas neutralité globale. Ultérieurement, on considérera bien des modèles à plusieurs espèces.

avec $\hat{v} = \frac{v}{\sqrt{1+\frac{v^2}{c^2}}}$.

L'équation de *Vlasov-Poisson* provient de l'approximation électrostatique de Vlasov-Maxwell, obtenue (formellement) quand la vitesse de la lumière c tend vers l'infini.

$$\begin{cases} \partial_t f + v.\nabla_x f + \mathrm{div}_v[Ef] = 0, \\ E = -\nabla_x V, \\ -\Delta_x V = \rho. \end{cases} \tag{1.10}$$

Quand le domaine d'espace est $\mathbb{R}^3$ tout entier, alors la solution élémentaire du Laplacien est connue explicitement et on peut écrire :

$$-\nabla_x V = \frac{x}{|x|^3} *_x \rho(t,x).$$

On parle souvent d'*interaction coulombienne*. Si l'on change le signe du noyau de convolution, auquel cas

$$-\nabla_x V = \frac{-x}{|x|^3} *_x \rho,$$

on parle d'*interaction newtonienne*. L'équation de Vlasov-Poisson avec un tel potentiel attractif est un modèle classique en dynamique des galaxies.

Dans la suite, on en considérera également des versions avec des champs extérieurs additionnels, notamment un champ magnétique extérieur B stationnaire et satisfaisant la condition de divergence nulle :

$$\mathrm{div}_x\, B = 0.$$

La convergence (dans un sens fort) de Vlasov-Maxwell vers Vlasov-Poisson quand la vitesse de la lumière tend vers l'infini a été prouvée rigoureusement et indépendamment par Degond [45], Schaeffer [145], Asano et Ukai [5]. Ce résultat est fondamental, car il justifie rigoureusement que l'équation de Vlasov-Poisson est pertinente dans des régimes où :

$$\frac{L_0}{T_0 c} \ll 1,$$

où L_0 et T_0 sont respectivement la longueur caractéristique et le temps caractéristique d'observation.

Cette approximation sera valable dans la plupart des régimes considérés dans cette thèse, si bien que l'on se concentrera en particulier sur l'équation de Vlasov-Poisson.

Une question fondamentale naturelle est la dérivation rigoureuse des équations de Vlasov du type de (1.11) à partir des équations de la dynamique de Newton. Cela correspond au problème de la justification mathématique du passage du *microscopique* au *mésoscopique* :

$$\begin{cases} \partial_t f + \hat{v}.\nabla_x f + \mathrm{div}_v[Ef] = 0 \\ E = -\nabla G * \rho, \end{cases} \tag{1.11}$$

à partir d'un système de N particules identiques interagissant via un potentiel d'interaction G.

$$\begin{cases} \dot{X}_i = V_i, \\ \dot{V}_i = -\frac{1}{N}\sum_{j\neq i} \nabla G(x_i - x_j). \end{cases} \tag{1.12}$$

Il s'agit d'un problème très délicat, dont on ne connait à ce jour qu'une résolution partielle. Le cas où G est régulier à support compact est bien compris, on se référera par exemple au livre de Spohn [149]. Le résultat optimal à ce jour correspond au cas où $|G|$

(resp. ∇G) se comporte en $\frac{1}{|x|^\alpha}$ (resp. $\frac{1}{|x|^{\alpha+1}}$), avec $\alpha \leq 1$, traité par Hauray et Jabin [97]. Remarquons que le potentiel coulombien tridimensionnel dans l'espace $\mathbb{R}^3$ entier (qui correspond à $\alpha = 2$) n'est pas contenu dans ce résultat.

Une **équation collisionnelle** s'écrit sous la forme :

$$\partial_t f + v.\nabla_x f = Q(f), \tag{1.13}$$

où $Q(f)$ est un opérateur intégro-différentiel modélisant les collisions. Quelques opérateurs classiques (écrits en dimension 3 d'espace) sont

- *L'opérateur de Boltzmann* : il s'agit de l'opérateur le plus communément admis pour la modélisation des collisions dans les gaz dilués neutres ; ces dernières sont supposées binaires et élastiques (elles préservent le moment et l'énergie cinétique).

$$Q_B(f) = \int_{S^2}\int_{\mathbb{R}^3} B(v - v_*, \omega)(f(t,x,v')f(t,x,v'_*) - f(t,x,v)f(t,x,v_*))dv_* d\omega,$$

  où (v, v_*) (respectivement (v', v'_*)) sont les vitesses post-collisionnelles (respectivement pré-collisionnelles) des deux particules en jeu. Elles sont reliées par la relation suivante :

$$\left\{ \begin{array}{l} v' = v - (v - v_*).\omega\,\omega, \\ v'_* = v_* + (v - v_*).\omega\,\omega, \end{array} \right.$$

  pour un certain vecteur ω dans S^2. La fonction B est la noyau de collision, permettant de modéliser la répartition statistique de la déflection..

- *L'opérateur de Landau* : il s'agit de l'opérateur fondamental pour décrire les collisions dans un plasma[5].

$$Q_L(f) = \nabla_v . \left\{ \int_{\mathbb{R}^3} \phi(v - v')[f(t,x,v')\nabla_v f(t,x,v) - f(t,x,v)\nabla_v f(t,x,v')]dv' \right\},$$

  avec $\phi(v) = \Pi(v)|v|^{\gamma+2}$, où $\Pi(v)$ est l'opérateur linéaire de projection orthogonale sur $v^\perp$. Le paramètre γ appartient à l'intervalle $[-3, +\infty[$, le cas le plus pertinent physiquement étant $\gamma = -3$ (interaction coulombienne).

  Citons également *l'opérateur de Balescu-Lenard*, qui est plus complexe que celui de Landau, mais parfois reconnu comme étant plus pertinent .

- *L'opérateur de Fokker-Planck* : cela correspond à une simplification des opérateurs précédents, dans le cas où ce sont les collisions rasantes qui prédominent.

$$Q(f) = \nabla_v(b(v)f) + \Delta_v f,$$

  où b est un terme dû à la friction.

On se réfère à l'article de revue de Villani [153] pour de nombreux détails sur ces modèles, qui ne seront pas étudiés dans cette thèse. Pour les équations collisionnelles, notons simplement qu'il y a un phénomène de relaxation vers l'équilibre thermodynamique, ce qui permet de justifier rigoureusement des modèles de type fluide (autrement dit le passage du *mésoscopique* au *macroscopique*).

Bien entendu, on peut aussi considérer des modèles à la fois de champ moyen et collisionnels : ainsi l'équation de Vlasov-Maxwell-Landau peut raisonnablement être considérée comme la plus fondamentale pour décrire un plasma.

[5] On verra ultérieurement par un argument d'échelle que celles-ci sont souvent négligeables, mais elles existent malgré tout !

1.1.3 Modèles mathématiques étudiés dans cette thèse

Nous allons à présent introduire les modèles qui nous intéresserons le plus dans cette thèse. Comme annoncé précédemment, il s'agit de modèles de type Vlasov-Poisson. De prime abord, il faudrait écrire une équation de transport pour les électrons (de fonction de répartition f_e, de masse m_e, de charge q_e) et pour chaque espèce d'ions (de fonction de répartition f_i, de masse m_i, de charge q_i). Pour simplifier on suppose qu'il n'y a qu'une seule espèce d'ions. Dans le régime électrostatique (Vlasov-Poisson), le système d'équations s'écrit donc :

$$\left\{\begin{array}{l} \partial_t f_i + v.\nabla_x f_i + q_i/m_i \left(E + v\wedge B\right).\nabla_v f_i = 0 \\ \partial_t f_e + v.\nabla_x f_e + q_e/m_e \left(E + v\wedge B\right).\nabla_v f_e = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \frac{1}{\varepsilon_0}\left(q_i \int f_i dv + q_e \int f_e dv\right), \end{array}\right. \tag{1.14}$$

avec B un champ magnétique stationnaire satisfaisant $\mathrm{div}_x\, B = 0$. Pour simplifier la présentation des équations, on suppose par la suite que $B = 0$. L'observation importante est la suivante : étant donné que la masse des ions est très grande devant celle des électrons[6],

$$\frac{m_i}{m_e} \gg 1,$$

les dynamiques se font en réalité sur des échelles de temps différentes, si bien qu'il est possible en première approximation de les découpler. Cela se traduit par l'étude d'une seule équation de Vlasov plutôt que plusieurs. On peut adopter deux points de vue sensiblement différents selon l'échelle de temps choisie :

Approximation de masse nulle (sur les électrons)

$$m_e = 0$$

Dans cette approximation on adopte l'échelle de temps typique des ions. Dans ce régime, les électrons atteignent ainsi instantanément leur équilibre thermodynamique local. Cela peut être obtenu heuristiquement de la façon suivante. Du fait de leur faible masse, le temps moyen entre deux collisions pour les électrons est beaucoup plus court que les ions, de sorte que la dynamique électronique est bien décrite par une équation hydrodynamique.

On fait l'hypothèse d'une loi de pression isotherme, ce qui est raisonnable physiquement. On note n_e la densité et u le courant pour les électrons. Dans un cadre unidimensionnel, pour simplifier, cela s'écrit :

$$\left\{\begin{array}{l} \partial_t n_e + \partial_x(n_e u) = 0, \\ \partial_t u + u\partial_x u = \dfrac{1}{\varepsilon}(\partial_x V - T\dfrac{\partial_x n_e}{n_e}), \end{array}\right.$$

où ε est un petit paramètre traduisant la faible masse des électrons et T la température des électrons En faisant tendre formellement ε vers 0, on obtient que la densité électronique n_e satisfait la loi de Maxwell-Boltzmann :

$$n_e = e^{V/T}.$$

[6] Typiquement dans les cas les moins favorables on a $m_i/m_e \sim 2000$.

Après normalisation de la température, cela signifie que nous avons réduit l'étude de (1.14) à celle de l'équation de Vlasov-Poisson pour les ions, que nous nommerons (VP-I) par la suite :

$$\begin{cases} \partial_t f + v.\nabla_x f + E.\nabla_v f = 0, \\ E = -\nabla_x V, \\ -\Delta_x V = \int f dv - e^V, \\ f_{|t=0} = f_0. \end{cases} \tag{1.15}$$

Remarquons que dans l'état des choses, rien n'assure que le plasma soit globalement neutre : en effet la condition

$$\int (f dv - e^V) dx = 0$$

n'a pas de raison d'être vérifiée. Ce modèle décrit donc des plasmas qui ne sont pas globalement neutres.

Il est possible de considérer que la densité des électrons sans masse suit la loi de Maxwell-Boltzmann normalisée :

$$n_e = \frac{e^V}{\int e^V dx},$$

avec la densité d'ions normalisée :

$$\int f_0 dv dx = 1,$$

et alors la neutralité globale est alors satisfaite. Un tel modèle semble être plus pertinent pour modéliser les plasmas créés en laboratoire, pour lesquels la neutralité globale est toujours vérifiée.

Une approximation très couramment faite en physique des plasmas consiste en la linéarisation de l'exponentielle :

$$n_e = 1 + V,$$

auquel cas l'équation de Poisson sur le potentiel s'écrit plutôt :

$$V - \Delta_x V = \int f dv - 1.$$

Cette approximation est valable dans un régime où l'énergie cinétique des électrons est grande devant leur énergie électrique, c'est-à-dire si :

$$\frac{eV}{kT} \ll 1.$$

Cela constitue une hypothèse simplificatrice très commode quand il s'agit d'approcher numériquement l'équation de Poisson. Du point de vue mathématique, cela permet également de simplifier les équations.

Approximation de masse lourde (sur les ions)

$$m_i = +\infty$$

Dans cette approximation, on se place du point de vue des électrons. L'approximation consiste à dire qu'à cause de leur faible inertie, les ions restent immobiles, de sorte qu'en notant n_i leur densité :

$$\partial_t n_i = 0.$$

On fait en général l'approximation que les ions sont à l'équilibre $n_i = \int f_0 dvdx = 1$. En raison de sa simplicité, c'est cette approximation qui a été la plus étudiée mathématiquement. Nous introduisons ainsi l'équation de Vlasov-Poisson pour les électrons, que nous nommerons (VP-E) par la suite :

$$\begin{cases} \partial_t f + v.\nabla_x f + E.\nabla_v f = 0, \\ E = -\nabla_x V, \\ -\Delta_x V = \int f dv - \int f dv dx, \\ f_{|t=0} = f_0. \end{cases} \tag{1.16}$$

1.2 Physique des plasmas fortement magnétisés

1.2.1 Motivation : la fusion par confinement magnétique

Principe de la fusion nucléaire Il existe deux façons de produire de l'énergie par réaction nucléaire. La première est la *fission nucléaire*, qui est le principe de base de toutes les centrales nucléaires actuelles. Elle consiste à "briser" des atomes d'uranium en deux éléments plus légers grâce à un neutron à grande vitesse. La réaction s'accompagne de l'émission d'autres neutrons qui libèrent de l'énergie et qui vont à leur tour briser d'autres atomes d'uranium : c'est la réaction de fission en chaine.

La *fusion nucléaire*, au contraire, repose sur le principe suivant : il s'agit de faire fusionner deux particules légères, pour former une particule plus lourde. A cause d'un excédent de masse, il y a un excédent d'énergie se traduisant par l'émission d'un neutron à grande vitesse. C'est ce principe qui est à l'oeuvre, par exemple, dans le soleil. Actuellement, pour des raisons logistiques, les éléments considérés pour la fusion sont le deutérium et le tritium (qui sont deux isotopes naturels de l'hyrdogène). Ils forment, lors d'une réaction de fusion, une particule d'hélium 4 (avec un neutron). Pour parvenir aux réactions de fusion, les particules doivent posséder une énergie suffisante pour vaincre les forces répulsives coulombiennes ; cela signifie qu'en pratique, on doit considérer des températures très élevées (de l'ordre de quelques dizaines de keV, ce qui est plus élevé que dans le soleil). Le gaz est alors à l'état de plasma.

Par ailleurs, pour maintenir les réactions de fusion, des critères à la fois nombreux et très restrictifs doivent être vérifiés, de sorte qu'aucun emballement n'est possible (contrairement à la fission). Ainsi, la fusion nucléaire semble beaucoup plus sûre que la fission. Sa maitrise serait une avancée majeure, étant donné qu'elle permettrait en principe une production d'énergie quasi-illimitée. Cela met en jeu un grand nombre de problèmes délicats aussi bien en recherche appliquée (par exemple trouver le bon matériau pour contenir le plasma) qu'en recherche fondamentale (comprendre la turbulence et les instabilités dans les plasmas).

Pour ces raisons, de nombreuses recherches sont actuellement menées pour y parvenir dans un futur proche. Cet effort se manifeste en France particulièrement par le projet du réacteur ITER, dont la construction a commencé récemment à Cadarache.

Confinement d'un plasma de tokamak par un champ magnétique intense Un tokamak est une boite de forme toroïdale, contenant un plasma dans lequel on espère réaliser la fusion. La question du confinement du plasma est absolument primordiale : il faut en effet éloigner le plus possible le plasma des frontières de la boite, afin d'éviter les dégradations du matériau et pour ne pas perturber les réactions de fusion. Le principe du confinement dans un tokamak est celui du *confinement magnétique*[7] : l'idée de base est que le plasma est confiné (loin des bords de la boite) à l'aide d'un champ magnétique *intense*

[7]Un autre principe est celui du *confinement inertiel* ; il ne sera pas étudié dans cette thèse.

bien choisi. Comme nous le verrons bientôt, lorsque l'intensité du champ magnétique est très grande, les particules chargées semblent suivre ses lignes de champ[8].

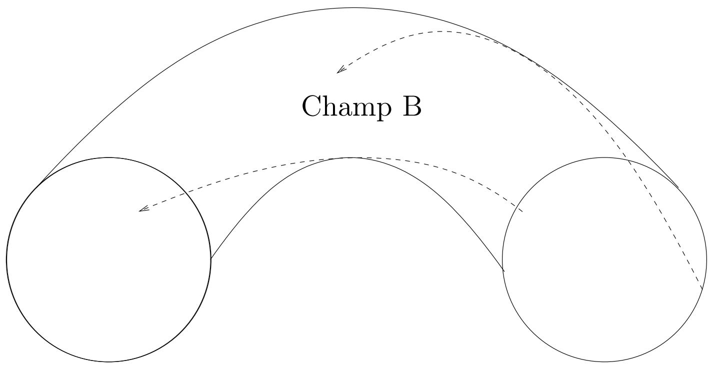


Figure 1.1 – Un tokamak et son champ magnétique "tournant"

On peut donner quelques ordres de grandeur afin de fixer les idées : la concentration de particule est de l'ordre de 10^{20} par m^3. La température est de l'ordre de la dizaine de keV, le champ magnétique va de 1 à 10 Teslas[9].

Il apparaît que dans les tokamaks expérimentaux tels que JET (au Royaume-Uni) ou ToreSupra (à Cadarache en France), les plasmas sont toujours instables, ce qui nuit à la fois à la durée de confinement et au rendement. La physique en oeuvre dans de tels plasmas est très complexe, et malheureusement pour le moment assez mal comprise. Notamment, les expériences indiquent l'apparition des phénomènes "turbulents"[10] engendrés par des micro-instabilités apparaissant à de petites échelles, qui créent du transport (d'énergie et de particules) qui n'est pas prévu par les théories usuelles. Par ailleurs, il existe également de nombreuses sources d'instabilité sur des échelles macroscopiques, qui sont également assez mal comprises [158]. Le projet ITER est précisément un projet de tokamak "avancé", créé afin de démontrer la faisabilité de la fusion thermonucléaire. Son objectif sera de parvenir à un rendement positif pendant un temps "long". Un des aspects fondamentaux du projet consiste à comprendre les instabilités et ultimement à parvenir à les contrôler.

Une caractéristique importante de la physique des plasmas consiste en la coexistence de multiples échelles de temps et d'espace, ce qui peut rendre difficile une simulation numérique efficace des modèles considérés. En effet, plus l'échelle de temps considérée est petite, plus le pas de temps doit être petit dans une simulation. De ce fait, on se rend compte naturellement qu'il y a un besoin de modèles (cinétiques) réduits mais approchant correctement les équations "originelles". Ainsi, en particulier, avoir un champ magnétique intense engendre des petites oscillations en temps pour les particules, qu'il convient de moyenner pour obtenir des modèles homogénéisés.

1.2.2 Approximation centre-guide et dérives particulaires

Notre but est à présent d'expliquer heuristiquement quel est le mouvement typique d'une particule soumise à un champ magnétique intense. Cela nous conduira à introduire deux grandeurs de très grande importance pour les plasmas de tokamaks : la fréquence cyclotron et le rayon de Larmor.

[8] Il semble qu'une telle idée soit due à Alfven et Bohm dans les années 1950.

[9] Pour comparer, le champ magnétique naturellement créé sur Terre est de l'ordre de 50 Micro-Teslas.

[10] c'est-à-dire de transfert d'énergie des petites aux grandes échelles et vice versa

Considérons une particule chargée de masse m et de charge q (pour simplifier, on suppose $q > 0$), soumise à un champ électrique électrique E et un champ magnétique B extérieurs donnés. On suppose que le champ magnétique est intense, de sorte sa norme $|B|$ est de l'ordre de $\frac{1}{\varepsilon}$, où ε est un petit paramètre positif.

Les équations de Newton permettent de décrire la trajectoire de la particule soumise à ce champ :

$$\begin{cases} \dfrac{dX}{dt} = V, \\ m\dfrac{dV}{dt} = qE + qV \wedge B. \end{cases} \tag{1.17}$$

Commençons par nous placer dans le cas le plus simple où B est uniforme et constant. Fixons une base orthonormale (e_1, e_2, e_3) de $\mathbb{R}^3$. On suppose alors par exemple que $B = |B|e_3$.

Mouvement parallèle On remarque qu'en prenant la projection orthogonale sur e_3, on se retrouve avec l'équation :

$$m\frac{dV_3}{dt} = qE_3,$$

ce qui signifie que la particule est accélérée dans cette direction par le champ électrique :

$$V_3 = V_3^0 + \frac{q}{m}\int_0^t E_3 ds, \quad X_3 = X_3^0 + V_3^0 t + \frac{q}{m}\int_0^t \int_0^s E_3 ds' ds.$$

Orbites de Larmor On s'intéresse à présent au mouvement dans le plan (e_1, e_2) (correspondant au plan orthogonal au champ magnétique). On suppose en première approximation que le champ électrique est nul. Cette approximation semble raisonnable puisque l'on étudie particulièrement les régimes où le champ magnétique est intense (et donc on s'attend à ce que ses effets soient prédominants). Les équations du mouvement s'écrivent très simplement :

$$\frac{dV_1}{dt} = \Omega_c V_2, \quad \frac{dV_2}{dt} = -\Omega_c V_1,$$

où l'on a noté :

$$\Omega_c = \frac{q|B|}{m}, \tag{1.18}$$

la *fréquence cyclotron* (ou *fréquence de Larmor*), qui est la fréquence typique des oscillations des particules soumises à un champ magnétique ; le temps caractéristique associé est la période cyclotron :

$$T_c = \frac{1}{\Omega_c}. \tag{1.19}$$

Par un calcul élémentaire, il vient :

$$V_\perp = R(t)V_\perp^0,$$

en notant pour tout vecteur A, $A_\perp$ la restriction de A au plan (e_1, e_2) et $R(t)$ la matrice de rotation définie par :

$$R(t) = \begin{pmatrix} \cos \Omega_c t & \sin \Omega_c t \\ -\sin \Omega_c t & \cos \Omega_c t \end{pmatrix}.$$

Remarquons par ailleurs que $|V_\perp|$ est conservée au cours du mouvement.

En conséquence, on obtient :

$$X_\perp = X_\perp^0 + r_L \begin{pmatrix} \sin\Omega_c t & 1-\cos\Omega_c t \\ \cos\Omega_c t - 1 & \sin\Omega_c t \end{pmatrix} \frac{V_\perp^0}{|V_\perp^0|},$$

en notant r_L le rayon de Larmor défini par :

$$r_L = \frac{V_0}{\Omega_c} = \frac{m|V_\perp^0|}{q|B|}, \tag{1.20}$$

Cela signifie que le mouvement de la particule est un mouvement circulaire de fréquence Ω_c et dont le rayon est r_L. Finalement, en combinant les mouvements parallèle et perpendiculaire, on voit que la particule suit un mouvement hélicoidal.

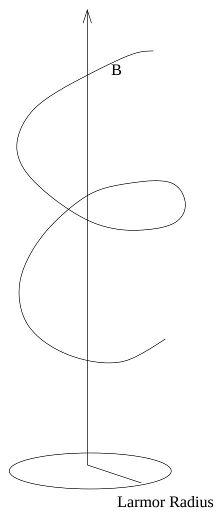


FIGURE 1.2 – Mouvement hélicoïdal à l'ordre dominant

De plus, en rappelant que $|B| \sim \frac{1}{\varepsilon}$, on obtient qualitativement :

$$\Omega_c \sim \frac{1}{\varepsilon}, \quad r_L \sim \varepsilon,$$

ce qui signifie grossièrement que la particule tourne très vite sur un rayon très petit. Le centre instantané de rotation de la particule suit un mouvement rectiligne (dans la direction parallèle), accéléré par le champ électrique parallèle. Ce centre instantané de rotation est parfois appelé centre-guide. L'approximation dite centre-guide consiste à approximer le mouvement de la particule par celui du centre-guide :

$$X_\perp = X_\perp^0, \quad X_3 = X_3^0 + V_3^0 + \frac{q}{m}\int_0^t \int_0^s E_3 ds' ds.$$

Ainsi, le mouvement à l'ordre dominant est simplement rectiligne, dans la direction du champ magnétique. Ce résultat peut être interprèté comme une propriété de confinement, puisque les particules suivent les lignes de champ magnétique. On peut donc penser qu'en choisissant un champ purement toroïdal (faisant le tour du tore), on devrait parvenir à bien confiner les particules.

Cependant, aux ordres suivants, on peut se rendre compte qu'il y a différentes dérives qui ne sont plus négligeables dès qu'on se place sur un temps d'observation long.

Dérive électrique A présent, on ne néglige plus le champ électrique. Supposons dans un premier temps que le champ électrique E est indépendant du temps. Reprenons l'équation sur la vitesse :

$$m\frac{dV}{dt} = qE + qV \wedge B$$

et posons $\tilde{V} = V - \frac{E \wedge B}{|B|^2}$. Alors, la nouvelle vitesse $\tilde{V}$ satisfait l'équation :

$$m\frac{d\tilde{V}}{dt} = q\tilde{V} \wedge B.$$

On a donc comme précédemment :

$$V_\perp = \frac{E \wedge B}{|B|^2} + R(t)\tilde{V}_\perp^0.$$

Le mouvement de la particule est donné par :

$$X_\perp = X_\perp^0 + t\frac{E \wedge B}{|B|^2} + \frac{m}{q|B|}\begin{pmatrix} \sin \Omega_c t & 1 - \cos \Omega_c t \\ \cos \Omega_c t - 1 & \sin \Omega_c t \end{pmatrix} \left(V_\perp^0 - \frac{E \wedge B}{|B|^2} \right).$$

On appelle “dérive électrique” (ou parfois, dérive $E \times B$ ou encore $E^\perp$) la vitesse :

$$v_E = \frac{E \wedge B}{|B|^2}. \tag{1.21}$$

Observons que l'on a $|v_E| \sim \varepsilon$, ce qui justifie a posteriori pourquoi on a négligé le champ électrique à l'ordre dominant. Cependant si on se place sur échelle de temps longue (c'est-à-dire $t \sim \frac{1}{\varepsilon}$) alors on se rend compte que cet effet n'est plus négligeable.

Ainsi, il est naturel que cette dérive joue un rôle important pour l'étude des tokamaks, surtout dans la perspective d'un confinement du plasma sur un temps très long.

De même, si l'échelle d'observation spatiale est du même ordre que le rayon de Larmor[11], alors c'est également un effet d'ordre 1.

Dérives de gradient Il existe également des dérives de nature géométrique, dues à l'inhomogénéité du champ magnétique. Dans ce paragraphe, on ne suppose donc plus que B est uniforme ; on note $B = |B|b$, avec $|b| = 1$. Alors on peut montrer par des arguments heuristiques (voir par exemple [158]) qu'il existe des dérives d'origine géométrique, la dérive de gradient et la dérive de courbure :

$$v_{\text{grad}} = \frac{|V_\perp|^2}{2\Omega_c} \frac{B \wedge \nabla |B|}{|B|^2} \tag{1.22}$$

et

$$v_{\text{curv}} = \frac{|V_\parallel|^2}{2\Omega_c} \frac{B \wedge \nabla |B|}{|B|^2}. \tag{1.23}$$

La dérive de gradient jouera notamment un rôle crucial dans le problème de confinement étudié dans cette thèse (Chapitre 6)

[11] On appellera un tel régime “rayon de Larmor fini” et on expliquera son intérêt physique dans la section 3 de cette introduction.

Dérive de polarisation Supposons à présent que le champ électrique dépend du temps. On obtient alors une dérive supplémentaire, similaire à la dérive électrique. Dans ce cas, l'équation satisfaite par $\tilde{V}$ est la suivante :

$$m\frac{d\tilde{V}}{dt} = q\tilde{V} \wedge B - \frac{m}{|B|^2}\frac{dE}{dt} \wedge B.$$

On peut poser comme précédemment $\bar{V} = \tilde{V} + \frac{m}{q|B|^2}\frac{dE_\perp}{dt}$. L'équation vérifiée par $\bar{V}$ est la suivante :

$$m\frac{d\bar{V}}{dt} = q\bar{V} \wedge B - \frac{m^2}{q|B|^2}\frac{d^2E_\perp}{dt^2}.$$

On néglige le terme $-\frac{m^2}{q|B|^2}\frac{d^2E_\perp}{dt^2}$: on peut en effet penser qu'il s'agit d'un terme d'ordre plus élevé en ε.

Il y a donc une dérive supplémentaire, appelée “dérive de polarisation” :

$$v_p = \frac{1}{\Omega_c|B|}\frac{dE_\perp}{dt}. \tag{1.24}$$

Une analyse d'échelle très sommaire nous apprend que si $|\frac{dE_\perp}{dt}|$ est d'ordre 1, on a $|v_p| \sim \varepsilon^2$, ce qui semble justifier a posteriori que l'on ait négligé cette dérive par rapport à la dérive électrique.

D'autre part, si $|\frac{d^2E_\perp}{dt^2}| \sim 1$, alors le terme $|\frac{m^2}{q|B|^2}\frac{d^2E_\perp}{dt^2}| \sim \varepsilon^2$ et donnera donc naissance à une dérive d'ordre ε^3, donc cette contribution est bien négligeable.

On verra néanmoins dans le Chapitre 5 de cette thèse que ces considérations ne sont en réalité pas toujours valables. En effet le champ électrique est en général fortement oscillant en temps : il s'agit d'un effet non linéaire dû au couplage entre l'équation de transport et l'équation de Poisson.

1.3 Objectifs de la thèse

Un des buts principaux de cette thèse est d'établir rigoureusement certains modèles réduits utilisés par les physiciens et de décrire les régimes pour lesquels ils sont valides. Plus précisément, nous nous intéresserons à l'étude mathématique de la quasineutralité ainsi qu'aux régimes de champ magnétique intense, qui permettent de modéliser en particulier les plamas de tokamaks. L'étude de ces limites pose des problèmes intéressants à la fois physiques et mathématiques. Par exemple, nous étudierons la *limite hydrodynamique* du système (VP-I) vers des systèmes de type Euler compressible, dans le régime où la longueur de Debye et la température tendent vers 0. Cela correspond au passage d'une description cinétique à une description macroscopique d'un plasma quasineutre.

Un autre champ de recherche intéressant et encore largement inexploré mathématiquement est l'étude rigoureuse des phénomènes d'instabilités rencontrés dans de tels plasmas magnétisés. On étudiera dans cette thèse une instabilité, faisant intervenir la dérive électrique ainsi qu'une dérive de gradient due à la géométrie du champ magnétique. Par ailleurs, le projet ITER pose des questions relevant de l'ingéniérie. Parmi ces questions apparaît naturellement le problème du contrôle du plasma, par exemple en cas d'accident ou quand on veut empêcher que le plasma n'endommage un certain dispositif. Ici, on envisagera ce problème du point de vue de la théorie du contrôle des équations aux dérivées partielles. De manière assez surprenante, même de ce point de vue, considérer un champ magnétique additionel peut améliorer les résultats possibles.

2 Outils pour les équations de transport cinétiques et applications au problème de Cauchy pour l'équation de Vlasov-Poisson

Avant de s'intéresser aux propriétés qualitatives, on vérifie que les modèles sont pertinents du point de vue mathématique, c'est-à-dire que le problème de Cauchy est bien posé. Par souci de lisibilité et de simplicité, on se concentrera quasiment exclusivement sur l'équation de Vlasov-Poisson pour les électrons (VP-E) que l'on rappelle :

$$\begin{cases} \partial_t f + v.\nabla_x f + E.\nabla_v f = 0, \\ E = -\nabla_x V, \\ -\Delta_x V = \int f dv - \int f dv dx, \\ f_{|t=0} = f_0. \end{cases}$$

Signalons d'ores et déjà que si l'on considère de surcroit un champ magnétique extérieur régulier, alors tous les résultats qui vont suivre restent valables, les preuves étant seulement très légèrement modifiées.

Nous nous concentrons essentiellement sur les cas $d = 2$ et $d = 3$, ce dernier cas étant bien entendu le plus naturel du point de vue de la physique. En outre, on étudiera essentiellement les équations posées dans des domaines sans bord en espace, plus particulièrement l'espace entier $\mathbb{R}^d$ et le tore périodique $\mathbb{T}^d$.

Commençons par préciser les différentes notions de solutions que nous considérerons.

Définition 1.

(Solution classique) *On dit que $f(t, x, v)$ est une solution classique de (VP-E) sur $[0, T]$ si elle est au moins de régularité $\mathcal{C}^1_{t,x,v}([0, T], \mathcal{C}^1_{x,v})$, et si elle vérifie l'équation au sens fort (avec E au moins lipschitzien).*

(Solution faible) *On dit que $f(t, x, v)$ est solution faible de (VP-E) sur $[0, T]$ si elle appartient à $L^\infty([0, T], L^p_{x,v})$ pour un $p \geq 1$, et si elle vérifie l'équation au sens des distributions, c'est-à-dire si pour toute fonction test $\phi \in \mathcal{C}^\infty_c(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$, on a :*

$$\int\int f(t, x, v) \left(\partial_t \phi + v.\nabla_x \phi + E.\nabla_v \phi\right) dt dx dv = -\int f_0(x, v)\phi(0, x, v) dx dv \tag{2.1}$$

où E vérifie l'équation de Poisson et f est prolongée par 0 pour $t < 0$.

(Solution renormalisée) *On dit que $f(t, x, v)$ est solution renormalisée de (VP-E) si $\beta(f)$ est solution au sens des distributions de l'équation de Vlasov, avec un champ E créé par f, pour toute fonction continue β telle que $\beta(t)$ est sous-linéaire à l'infini (typiquement on peut penser à $\beta(f) = \log(1 + f)$).*

Notre second objectif est de rappeler quelques outils classiques qui nous seront fort utiles par la suite. Plusieurs théorèmes que nous énoncerons s'appuient sur des effets et des propriétés spécifiques aux équations cinétiques. Cela motive l'introduction de ces outils généraux pour les équations de la forme suivante :

$$\begin{cases} \partial_t f + v.\nabla_x f + F.\nabla_v f = g, \\ f_{|t=0} = f_0. \end{cases} \tag{2.2}$$

avec F un champ et g une source donnés. Nous les présenterons dans les paragraphes qui suivent, en gardant comme fil conducteur le problème de Cauchy pour Vlasov-Poisson. Précisons enfin que notre présentation est organisée thématiquement et ne suit pas nécessairement l'ordre chronologique des résultats.

2.1 Estimations a priori

Supposons que l'on ait une solution régulière de (VP-E) : que pouvons nous dire sur les quantités conservées au cours du temps ?

La première remarque que l'on peut faire est la suivante : puisque le champ (v, E) est à divergence nulle, le théorème de Liouville permet d'affirmer que les volumes sont conservés dans l'espace des phases. Cela entraîne la conservation de toutes les entropies $\int \Phi(f)dxdv$, en particulier celle des normes $L^p_{x,v}$, $1 \leq p < \infty$. Par ailleurs, on a un principe du maximum qui permet de montrer la conservation de la positivité et de la norme L^∞.

Lemme 2.1. *Soit f une solution régulière de (VP-E) de condition initiale f_0. Alors on a :*

- *Pour tout $p \in [1, \infty]$, pour tout $t \geq 0$, $\|f(t)\|_{L^p_{x,v}} = \|f(0)\|_{L^p_{x,v}}$.*
- *Si $f_0 \geq 0$ presque partout, alors pour tout $t \geq 0$, $f(t) \geq 0$ presque partout.*

Il est à noter que la conservation de la positivité et de la norme L^1 entraîne la conservation de la masse. L'autre point important provient de la physique et concerne la conservation de l'énergie.

Lemme 2.2. *Soit f une solution régulière de (VP-E). Alors la quantité $\mathcal{E}(t)$ suivante est conservée au cours du temps :*

$$\mathcal{E}(t) = \int f(t,x,v)\frac{|v|^2}{2}dvdx + \frac{1}{2}\int |\nabla_x V|^2 dx. \tag{2.3}$$

Le premier terme correspond à l'énergie cinétique, le second à l'énergie électrique. Cela signifie qu'au cours du temps, il y a un échange entre énergies cinétique et électrique. Pour montrer ce lemme, on calcule explicitement la dérivée en temps de $\int f|v|^2 dvdx$ en intégrant l'équation de Vlasov contre $|v|^2$. On obtient alors l'identité grâce à l'équation de Poisson.

En s'appuyant sur le fait que $f|v|^2$ est dans $L^\infty_t L^1_{x,v}$, on peut montrer que la densité et le courant ρ et ρu (i.e. les moments en vitesse d'ordre 0 et 1) appartiennent à un espace $L^\infty_t L^p_x$ pour un certain $p > 1$. Cela provient du lemme général d'interpolation (réelle) :

Lemme 2.3. *Soit f une fonction mesurable positive.*

- *Pour tout $0 \leq k \leq m$:*

$$\left\| \int f|v|^k dv \right\|_{L^{(m+d)/(d+k)}} \leq C_{d,k}\|f\|_{L^\infty}^{(m-k)/(m+d)} \left(\int f|v|^m dvdx\right)^{(d+k)/(m+d)}, \tag{2.4}$$

où d est la dimension d'espace et $C_{d,k}$ est une constante dépendant de d et k.

- *Pour tout $0 \leq k \leq m$, si $1 \leq p < +\infty$ alors on a l'inégalité suivante :*

$$\left\| \int f|v|^k dv \right\|_{L^{\frac{pm+pd(1-1/p)}{m+pd(1-1/p)+k(p-1)}}} \leq C_{d,k,p}\|f\|_{L^p}^{\frac{m-k}{m+d(1-1/p)}} \left(\int f|v|^m dvdx\right)^{\frac{k+d(1-1/p)}{m+d(1-1/p)}}. \tag{2.5}$$

Pour démontrer ce lemme, on utilise une décomposition de l'espace des vitesses :

$$\int f|v|^k dv = \int_{|v|\leq R} f|v|^k dv + \int_{|v|>R} f|v|^k dv,$$

avec $R > 0$ un paramètre à fixer ultérieurement. Pour la partie "petites vitesses", on s'appuie sur les propriétés d'intégrabilité de f, tandis que pour la partie "grandes vitesses", on s'appuie sur celles de $f|v|^2$. On obtient alors les inégalités voulues en optimisant en R.

2.2 Solutions classiques

Avant d'attaquer le problème complet, il est raisonnable de commencer par résoudre le problème linéaire (2.2). C'est l'objet du paragraphe suivant.

2.2.1 Méthode des caractéristiques

Notons $X(t,x,v)$ et $V(t,x,v)$ les solutions du système d'équations différentielles ordinaires :

$$\begin{cases} \dfrac{dX}{dt} = V, \\ \dfrac{dV}{dt} = F. \end{cases} \tag{2.6}$$

avec $X(0,x,v) = x$ et $V(0,x,v) = v$. Supposons que $F(t,x)$ soit une fonction lipschitzienne et sous-linéaire, c'est-à-dire :

$$\forall t \in \mathbb{R}, \forall R > 0, \exists C(R) > 0, \forall x, y \in B(0,R), \quad |F(t,x) - F(t,y)| \leq C(R)|x-y|,$$

$$\exists C > 0, \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^d, \quad |F(t,x)| \leq C(1+|x|),$$

de sorte que le théorème de Cauchy-Lipschitz s'applique. Alors on a la formule suivante, appelée *formule de Duhamel* :

$$f(t,x,v) = f_0(X(-t,x,v), V(-t,x,v)) + \int_0^t g(s, X(s-t,x,v), V(s-t,x,v))ds. \tag{2.7}$$

Cette formule permet de résoudre le problème de Cauchy pour l'équation de transport (2.2), pour des données régulières, mais également pour des données seulement intégrables.

Proposition 2.1. *Supposons que F soit lipschitzienne et sous-linéaire.*

Si $f_0 \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)$ et $g \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)$ alors il existe une unique solution forte globale f à (2.2) qui soit $\mathcal{C}^1(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$.

Si f_0 est $L^p_{x,v}$ et g est $L^\infty_t L^p_{x,v}$ alors il existe une unique solution globale f à (2.2) dans $L^\infty_t L^p_{x,v}$ au sens des distributions.

2.2.2 Le théorème d'Ukai et Okabe

Nous pouvons maintenant étudier l'équation non-linéaire complète. Nous nous concentrons essentiellement sur le cas de l'espace entier ; le cas du tore périodique sera simplement évoqué ultérieurement. Le résultat fondateur de la théorie est dû à Ukai et Okabe [151]. Par des techniques de point fixe proches de celles utilisées pour l'équation d'Euler incompressible bidimensionnelle par Kato [109], ces auteurs prouvent l'existence globale et l'unicité de solutions régulières en dimension 2 d'espace, sous l'hypothèse (raisonnable) que la condition initiale est régulière et possède des moments en v d'ordre assez élevé. La même méthode permet de montrer l'existence de solutions locales en temps, en dimension 3 d'espace.

Théorème 2.1 (Ukai-Okabe). *Soit $f_0 \in \mathcal{C}^1_{x,v}$ positive telle qu'il existe $\kappa, \kappa' > 0$ et $\gamma > d$:*

$$|f_0(x,v)| \leq \kappa(1+|x|)^{-\gamma-1}(1+|\xi|)^{-\gamma-1} \tag{2.8}$$

$$|\nabla_{x,v} f_0(x,v)| \leq \kappa'(1+|x|)^{-\gamma}(1+|\xi|)^{-\gamma} \tag{2.9}$$

- *Si $d = 2$ l'équation de Vlasov-Poisson admet une unique solution classique globale partant de f_0.*

- *Si $d = 3$, il existe $T_0 > 0$ tel que l'équation de Vlasov-Poisson admet une unique solution classique sur $[0, T_0[$ partant de f_0.*

Concernant l'existence de solutions globales en trois dimensions d'espace, le problème est resté ouvert assez longtemps. Citons pour commencer l'article de Bardos et Degond [7] dans lequel les auteurs montrent l'existence globale en dimension 3 à condition que la donnée initiale soit suffisamment petite. La preuve de ce résultat repose sur les propriétés *dispersives* de l'opérateur de transport libre. C'est l'occasion pour nous de faire une digression sur ce sujet.

2.2.3 Mélange de phase et dispersion

Dans ce paragraphe, on regarde l'équation de transport libre dans l'espace entier $\mathbb{R}^d$:

$$\begin{cases} \partial_t f + v.\nabla_x f = 0, \\ f_{|t=0} = f_0. \end{cases} \tag{2.10}$$

Cette équation très simple possède bien entendu une solution explicite, mais cette prétendue simplicité est trompeuse, car il se produit des phénomènes de dynamique assez complexes.

Le théorème suivant, tiré de [7], décrit un effet dispersif du transport libre.

Théorème 2.2 (Bardos-Degond)**.** *En notant $\rho(t,x) = \int f(t,x,v)dv$, on pour tout $t > 0$:*

$$\|\rho(t)\|_{L^\infty_x} \leq \frac{1}{t^d} \|f_0\|_{L^1_x(L^\infty_v))}. \tag{2.11}$$

Démonstration. On utilise la représentation :

$$\begin{aligned} |\rho(t,x)| &= \int f_0(x - tv, v)dv \\ &= \int f_0(y, \frac{x-y}{t}) \frac{dy}{t^d} \\ &\leq \int \sup_{x \in \mathbb{R}^d} f_0(y, \frac{x-y}{t}) \frac{dy}{t^d} \\ &\leq \int \sup_{z \in \mathbb{R}^d} f_0(y, z) \frac{dy}{t^d}. \end{aligned} \tag{2.12}$$

□

Cette estimation est très intéressante au moins sous deux aspects. Premièrement, elle traduit un phénomène de *dispersion* à l'infini en temps grand ($t \to +\infty$). Cela signifie, conformément à l'intuition, qu'en l'absence de toute interaction, les particules "fuient" vers l'infini. Par ailleurs pour tout $t > 0$ fixé, elle exprime un effet de *mélange* dans l'espace des phases (x, v). Cet effet de mélange est intrinsèquement (et souvent de manière cachée) à la base de nombreux phénomènes typiques aux équations cinétiques, citons par exemple l'hypocoercivité [154], les lemmes de moyenne (voir le paragraphe suivant) ou le controversé amortissement Landau, récemment justifié par Mouhot et Villani [130] (nous y reviendrons brièvement plus tard).

Ultérieurement, dans la note [35], Castella et Perthame ont généralisé ces estimations et les ont utilisées afin de démontrer des estimations de Strichartz pour les équations cinétiques.

Théorème 2.3 (Castella, Perthame)**.** *On a les estimations de type dispersif suivantes :*

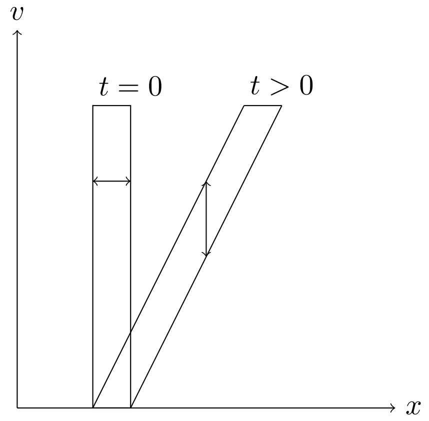


FIGURE 1.3 – Mélange pour le transport libre

- *(Dispersion généralisée) Pour tout* $1 \leq r \leq p \leq +\infty$*, on a pour tout* $t > 0$ *:*

$$\|f(t)\|_{L^p_x(L^r_v))} \leq \frac{1}{t^{d(\frac{1}{r}-\frac{1}{p})}} \|f_0\|_{L^r_x(L^p_v))}. \tag{2.13}$$

- *(Estimations de Strichartz) Pour tous* a, p, q *tels que :*

$$1 \leq p \leq \frac{d}{d-1}, \quad \frac{2}{q} = d\left(1 - \frac{1}{p}\right), \quad 1 \leq a = \frac{2p}{p+1} < \frac{2d}{2d-1}. \tag{2.14}$$

on a l'estimation :

$$\|\rho\|_{L^q_t(L^p_x)} \leq C(d)\|f_0\|_{L^a_{x,v}}. \tag{2.15}$$

Signalons par ailleurs que l'étude de la dispersion pour des équations de Liouville en géométrie courbe non-captive a été menée par Salort [142, 143] : elle montre alors qu'il y a toujours dispersion, dans un sens plus faible. La dispersion dans un sens également faible a été par ailleurs étudiée par le même auteur [144] pour des équations de transport dans un champ de force quelconque.

2.2.4 Le théorème de Pfaffelmoser

Revenons au problème de l'existence globale des solutions fortes. La contribution cruciale est due à Pfaffelmoser [135] : sous l'hypothèse que le support de la condition initiale est borné, alors il existe une solution forte globale en temps, et cela sans hypothèse de petitesse pour la condition initiale. Par ailleurs on est capable d'estimer l'évolution de la taille du support en vitesse de la solution, c'est-à-dire en quelque sorte la dispersion des particules. Pour mesurer cet effet on introduit :

$$Q(t) = 1 + \sup\left\{|v|, \exists (s,x) \in [0,t] \times \mathbb{R}^3 \quad \text{tel que} \quad f(s,x,v) \neq 0\right\}. \tag{2.16}$$

Il y eut par la suite de nombreuses simplifications et améliorations du résultat, citons par exemple [146], qui ont amélioré le taux de croissance de la taille du support en vitesse. Notons cependant que le taux optimal reste à ce jour inconnu. Citons également l'article de Horst [98] dans lequel l'auteur montre comment s'affranchir de l'hypothèse de support compact, en montrant qu'une simple majoration par une fonction suffisamment décroissante à l'infini suffit.

Théorème 2.4 (Pfaffelmoser, Schaeffer). *Soit f_0 une condition initiale $\mathcal{C}^1$ positive à support compact en x et v. Alors le système (VP-E) avec condition initiale f_0 admet une unique solution f de classe $\mathcal{C}^1$, globale en temps. De plus on a :*

$$Q(t) \leq C(p)(1+t)^p, \quad p > 33/17. \tag{2.17}$$

L'approche de Pfaffelmoser est typiquement "lagrangienne", dans la mesure où elle repose sur des estimations fines sur les caractéristiques suivies par les particules.

En ce qui concerne les équations posées sur le tore $\mathbb{T}^2$ ou $\mathbb{T}^3$, il est facile de vérifier que les arguments d'Ukai et Okabe se transposent mutatis mutandis, de sorte que le théorème 2.1 s'applique toujours. Pour le théorème de Pfaffelmoser, c'est moins clair de premier abord, mais Batt et Rein ont vérifié dans [10] que la méthode se transposait également.

2.2.5 Un petit détour par Vlasov-Maxwell

Finissons cette discussion sur l'existence globale de solutions classiques en évoquant brièvement le cas du système de Vlasov-Maxwell, pour lequel il existe des résultats tout à fait remarquables. Citons l'article fondateur de Glassey et Strauss [69] dans lequel les auteurs montrent que seule l'existence de particules ayant une vitesse proche de celle de la lumière peut empêcher l'existence globale d'une solution classique (nous nous référons également aux travaux de Klainerman-Staffilani [110] et Bouchut-Golse-Pallard [20] pour des preuves alternatives). Jusqu'à présent, contrairement au cas de Vlasov-Poisson, on ne sait contrôler l'évolution du support en vitesse (et donc prouver existence globale) qu'en deux dimensions d'espace [67] et en trois dimensions, uniquement pour des cas avec symétrie (voir par exemple [66]) ou pour des données petites (voir par exemple [68]). Le cas général tridimensionnel reste largement ouvert.

2.3 Solutions faibles et solutions renormalisées

2.3.1 Le théorème d'Arsenev

L'existence de solutions faibles pour (VP-E) est due à Arsenev [4], sous l'hypothèse que la condition initiale f_0 est dans $L^1_{x,v} \cap L^\infty_{x,v}$ et d'énergie finie. L'idée sous-jacente de ce résultat est que (VP-E) peut être vue comme le couplage d'une équation hyperbolique (l'équation de Vlasov) avec une équation elliptique (l'équation de Poisson) et on peut s'attendre à ce que le problème elliptique fournisse de la régularité et permette de résoudre le système couplé.

Théorème 2.5 (Arsenev). *Soit $f_0 \in L^1_{x,v} \cap L^\infty_{x,v}$ positive telle que $\mathcal{E}(0) < +\infty$. Alors il existe une solution faible globale f à (VP-E) avec donnée initiale f_0 et d'énergie non croissante.*

La preuve de ce théorème passe par une régularisation de (VP-E), que l'on résout par un théorème de point fixe de type Banach Picard. Il s'agit ensuite de passer à la limite faible, la difficulté étant due au terme non linéaire fE. Il est en effet bien connu que les produits de suites faiblement convergentes n'ont a priori pas de raison de passer aussi à la limite faible. Pour surmonter ce problème, on gagne de la compacité en espace et en temps sur le champ électrique. Dans un premier temps, on remarque que grâce au Lemme 2.3, la densité et le courant appartiennent uniformément à un espace $L^\infty_t L^q_x$ pour un certain $q > 1$. Grâce à l'ellipticité de l'équation de Poisson, on peut en déduire que le champ électrique gagne une dérivée en temps et en espace, ce qui permet de passer à la limite dans le terme non linéaire.

Par ailleurs, signalons qu'il est possible de relaxer la borne L^∞ en la remplaçant par une borne L^p avec p assez grand. Par exemple, Horst et Hünze ont montré dans [99] que $p > p_0 = (12+3\sqrt{5})/11 \sim 1,7$ était suffisant. Dans la suite de cette thèse, nous appellerons *solution faible au sense d'Arsenev* une telle solution. Il s'agit de la notion de solution qui sera la plus souvent utilisée.

Il convient également de préciser qu'il existe des résultats pour des données initiales à énergie infinie ou à masse infinie. Citons à ce sujet les travaux de Jabin [106]. Dans ce cas l'existence de solutions faibles est a priori uniquement locale en temps.

Pour clore ce paragraphe, nous donnons un résultat de stabilité/compacité du à Lions et DiPerna [49] :

Proposition 2.2 (DiPerna-Lions)**.** *Soit $(f_{0,\varepsilon})$ une suite de conditions initiales convergeant faiblement (quand $\varepsilon \to 0$) vers f_0 dans $L^1_{x,v}$ et (f_ε) une suite de solutions globales faibles au sens d'Arsenev à (VP-E) avec données initiales $(f_{0,\varepsilon})$ et convergeant faiblement vers une fonction f dans $L^1_{t,x,v}$.*

Supposons que (f_ε) soit uniformément borné dans $L^\infty_t L^p_x, v$ (pour $p > 2$), alors f est une solution globale faible au sens d'Arsenev à (VP-E) avec donnée initiale f_0 et pour toute fonction test $\Psi(v)$ régulière à support compact on a :

$$\int f_\varepsilon \Psi(v) dv \to \int f \Psi(v) dv,$$

fortement dans $L^q_t L^1_{loc,x}$ pour tout $q \in [1, +\infty[$.

Pour démontrer la compacité sur les moments, l'idée est d'exploiter un effet régularisant en moyenne, spécifique aux équations de transport cinétiques. C'est l'occasion de faire une nouvelle digression et d'introduire cette théorie fondamentale, par ailleurs fort utile dans de nombreuses situations.

2.3.2 Lemmes de moyenne $L^p, 1 < p < +\infty$

Considérons une solution f à l'équation cinétique de transport (2.2). L'équation de transport étant hyperbolique, f n'a aucune raison d'être plus régulière que la source g ou que sa donnée initiale f_0. En effet, par la formule de Duhamel (2.7), on voit bien que les éventuelles singularités de g et f_0 se propagent.

Les lemmes de moyenne (en vitesse) sont des outils fondamentaux pour l'étude des équations cinétiques : il stipulent qu'il y a un effet régularisant (caché) sur les moyennes en vitesse de f, c'est-à-dire sur les quantités macroscopiques :

$$\int f(t,x,v)\Psi(v) dv,$$

où $\Psi(v)$ est régulière et à support compact.

Depuis leur découverte par Golse, Perthame et Sentis [71] (voir aussi [1] pour un résultat voisin indépendant), puis leur formulation "quantitative" par Golse, Lions, Perthame et Sentis [70], ils ont été étudiés et utilisés dans de très nombreuses situations, car ils fournissent souvent de la compacité en espace et éventuellement en temps, ce qui permet de traiter beaucoup de situations non-linéaires.

Dans ce paragraphe, on se restreint à considèrer l'équation de transport cinétique stationnaire :

$$v.\nabla_x f = g, \tag{2.18}$$

sachant que les résultats se généralisent sans peine aux équations dépendant du temps [70]. Le résultat fondamental, tiré de [70], est le suivant :

Théorème 2.6 (Golse, Lions, Perthame et Sentis). *Soit* $1 < p < \infty$. *Soit* $f, g \in L^p(dx \otimes d\mu)$ *vérifiant l'équation de transport :*

$$v.\nabla_x f = g \tag{2.19}$$

Alors pour toute fonction $\psi \in L^\infty_K(d\mu)$, $\rho(x) = \int f(x,v)\psi(v)dv \in W^{s,p}$ *pour tout* s *vérifiant* $0 < s < \inf(1/p, 1/p')$. *De plus,*

$$\|\rho\|_{W^{s,p}_x} \leq C_{\Psi,p}(\|f\|_p + \|g\|_p). \tag{2.20}$$

($C_{\Psi,p}$ est une constante positive dépendant uniquement de Ψ *et* p.*)*

Eléments de preuve. Prouvons le cas $p = 2$. Le cas général s'en déduit par interpolation complexe, en observant que pour les exposants limites $p = 1$ et $p = \infty$, il n'y a a priori pas de gain de régularité. Néanmoins on a les estimations évidentes :

$$\|\rho\|_{L^1_x} \leq \|\Psi\|_{L^\infty_v} \|f\|_{L^1_{x,v}},$$
$$\|\rho\|_{L^\infty_x} \leq \|\Psi\|_{L^\infty_v} |\operatorname{Supp}\Psi| \|f\|_{L^\infty_{x,v}},$$

où $|\operatorname{Supp}\Psi|$ désigne la mesure de Lebesgue du support de Ψ.

Dans le cas $p = 2$, la preuve peut se faire en utilisant la transformée de Fourier en x. Notons ξ la variable duale de x. Les estimations s'appuient sur l'ellipticité du symbole de $v.\nabla_x$ loin de la zone $v.\xi = 0$.

Soit $\hat{f}(\xi, v)$ la transformée de Fourier en x de la fonction $f(x, v)$. L'idée est de découper l'espace des vitesses en deux parties en introduisant un paramètre $\alpha > 0$:

$$\begin{aligned}
\left|\int \hat{f}(\xi,v)\Psi(v)dv\right| &\leq \left|\int \mathbb{1}_{|v.\xi|\leq\alpha}\hat{f}(\xi,v)\Psi(v)dv\right| + \left|\int \mathbb{1}_{|v.\xi|>\alpha}\frac{\hat{f}(\xi,v)(\xi.v)}{\xi.v}\Psi(v)dv\right| \\
&\leq \left(\int |\hat{f}(\xi,v)|^2 dv\right)^{1/2} \left(\int \mathbb{1}_{|v.\xi|\leq\alpha}\Psi^2(v)dv\right) \\
&+ \left(\int |\hat{g}(\xi,v)|^2 dv\right)^{1/2} \left(\int \frac{\mathbb{1}_{|v.\xi|>\alpha}}{(\xi.v)^2}\Psi^2(v)dv\right) \\
&\leq \|\hat{f}\|_{L^2_v} C\frac{\alpha}{|\xi|} + \|\hat{g}\|_{L^2_v} C\frac{1}{\alpha|\xi|}.
\end{aligned}$$

On choisit par exemple $\alpha = 1$ et on conclut par l'inégalité de Plancherel.

□

Par ailleurs, le résultat se généralise si la source est une dérivée de fonctions L^p (avec néanmoins strictement moins d'une dérivée en x). Ce phénomène de régularisation a d'abord été découvert dans un cadre L^2 (avec une dérivée en vitesse) par DiPerna et Lions dans l'article [51] : il s'agit du point crucial pour montrer l'existence de solutions faibles globales au système de Vlasov-Maxwell. Ce principe a ensuite été étudié de façon plus systématique par DiPerna, Lions et Meyer dans [50], puis il a été généralisé par Bézard [14].

Théorème 2.7 (DiPerna-Lions-Meyer, Bézard). *Soit* $1 < p \leq 2$. *Soit* $f, g \in L^p(dx \otimes d\mu)$ *vérifiant l'équation de transport :*

$$v.\nabla_x f = (Id - \Delta_x)^{\tau/2}(Id - \Delta_v)^{m/2} g \tag{2.21}$$

avec $m \in \mathbb{R}^+, \tau \in [0, 1[$. *Alors* $\forall \theta \in \mathcal{C}^\infty_c(\mathbb{R}^d)$, $\rho(x) = \int f(x,v)\theta(v)dv \in W^{s,p}$ *où*

$$s = \frac{1-\tau}{(1+m)p'} \tag{2.22}$$

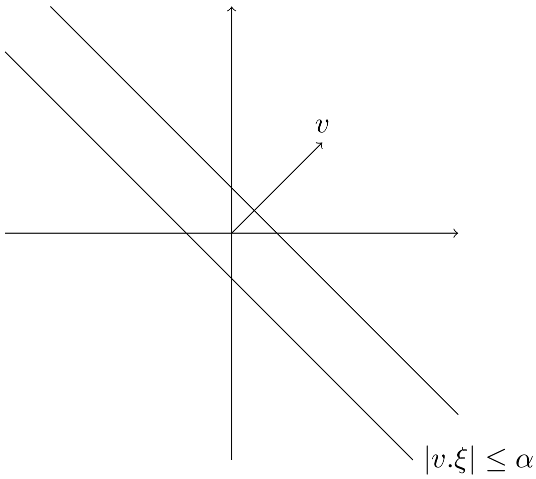


FIGURE 1.4 – Décomposition de l'espace des fréquences

De plus,

$$\|\rho\|_{W_x^{s,p}} \leq C(\|f\|_p + \|g\|_p). \tag{2.23}$$

(C est une constante positive indépendante de f et g)

On notera que l'on peut considérer autant de dérivées en v que l'on veut sur la source, alors qu'on est limité à strictement moins d'une dérivée en x. Cela est dû au fait que l'opérateur de transport est un opérateur différentiel en x d'ordre 1. Pour ce cas limite, il existe néanmoins des résultats intermédiaires, qui sont dus à Perthame et Souganidis [134].

Remarquons que ce théorème est une amélioration du théorème 2.6, même dans le cas sans dérivée sur la source. En effet cela permet d'obtenir l'indice de régularité limite $s = \frac{1}{p'}$ qui échappe au théorème précédent. Par ailleurs, des travaux de Lions [116, 118] ont permis de montrer que cette régularité était optimale (voir aussi [107]).

La preuve de ce théorème repose également sur un argument d'interpolation complexe, mais au lieu d'étudier le cas L^1, pour lequel les choses se passent assez mal, on passe par les espaces de Hardy. Ces espaces ont plusieurs avantages : d'une part, contrairement au cas L^1, les multiplicateurs de Fourier y sont bornés et d'autre part ils se comportent mieux vis à vis de l'interpolation complexe, ce qui permet d'obtenir la régularité limite.

Dans cette thèse, on démontrera dans le Chapitre 3 un lemme de moyenne "anisotrope", qui sera adapté à la structure algébrique du problème considéré. Il s'agit d'une adaptation des lemmes de moyenne précédents, pour des fonctions à valeurs "abstraites" dans un espace de Sobolev d'indice éventuellement négatif.

Lemme 2.4 (Chapitre 3)**.** *Soit $1 < p < +\infty$ et $\lambda \in \mathbb{R}$. Soit $f, g \in L^p_{t,x,v}(W_y^{\lambda,p})$ vérifiant l'équation de transport :*

$$\partial_t f + v.\nabla_x f = (I - \Delta_v)^{m/2} g \tag{2.24}$$

avec $m \in \mathbb{R}^+$. Alors $\forall \Psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\rho_\Psi(t,x) = \int f(t,x,v)\Psi(v)dv \in W_{t,x}^{s,p}(W_y^{\lambda,p})$ pour tout stel que

$$s \leq s_2 = \frac{1}{2(1+m)} \text{ for } p = 2 \tag{2.25}$$

et

$$s < s_p = \frac{1}{(1+m)p'} \text{ for } p \neq 2 \tag{2.26}$$

De plus,

$$\|\rho_\Psi\|_{W^{s,p}_{t,x}(W^{\lambda,p}_y)} \leq C\left(\|f\|_{L^p_{t,x,v}(W^{\lambda,p}_y)} + \|g\|_{L^p_{t,x,v}(W^{\lambda,p}_y)}\right) \tag{2.27}$$

2.3.3 Lemmes de moyenne L^1

Le cas limite $p = 1$ a fait l'objet de développements tout à fait remarquables. Un contre-exemple explicite tiré de [70] montre que sans hypothèse de compacité supplémentaire sur la source ou sur la solution de l'équation de transport, on ne peut pas obtenir de compacité (même faible) dans L^1. Commençons par le rappeler : soit $\Psi \in L^\infty(d\mu)$ à support compact dans $\overline{B}(0,1)$. Soit $(g_n)(x,v)$ une suite bornée de $L^1_{x,v}$ à support compact en v inclus dans $\overline{B}(0,1)$ qui converge faiblement vers un dirac $\delta_0 \otimes \delta_{v_0}$, avec $v_0 \neq 0$. Soit f_n l'unique solution L^1 de l'équation $f_n + v.\nabla_x f_n = g_n$, obtenue grâce à une formule de Duhamel :

$$f_n(x,v) = \int_0^{+\infty} g_n(x - tv, v)e^{-t}dt.$$

Alors on vérifie aisément que (f_n) puis que $(v.\nabla_x f_n)$ sont bornées dans $L^1_{x,v}$. On en déduit que pour toute fonction test χ continue, en utilisant la représentation exacte de (f_n) et la définition de (g_n) :

$$\begin{aligned}
\int \chi(x)\left(\int f_n(x,v)\Psi(v)dv\right)dx &= \int\int\left(\int_0^{+\infty} g_n(x-tv,v)e^{-t}dt\right)\Psi(v)\chi(x)dvdx \\
&= \int_0^{+\infty} e^{-t}\int\int g_n(x-tv,v)\Psi(v)\chi(x)dvdxdt \\
&\to \Psi(v_0)\int_0^{+\infty} e^{-t}\chi(tv_0)dt
\end{aligned}$$

Ceci prouve que $(\int f_n(x,v)\Psi(v)dv)$ converge faiblement vers une mesure portée par la demi-droite $\mathbb{R}_+^* v_0$. En particulier cette famille n'est pas relativement compacte dans L^1.

En revanche, grâce au théorème de Dunford-Pettis, la compacité faible dans L^1 se caractérise localement par l'équiintégrabilité. Ainsi, dès que l'on suppose que la famille de fonctions est équiintégrable (en toutes les variables), alors il est possible de récupèrer de la compacité sur les moments.

Commençons par rappeler la définition de l'équiintégrabilité.

Définition 2. *Soit $(f_\varepsilon(x,v))$ une famille bornée de $L^1_{loc}(dx \otimes dv)$. Elle est dite localement équiintégrable en x et v si et seulement si pour tout paramètre $\eta > 0$ et tout compact $K \subset \mathbb{R}^d \times \mathbb{R}^d$, il existe $\alpha > 0$ tel que pour tout ensemble mesurable $A \subset \mathbb{R}^d \times \mathbb{R}^d$ tel que $|A| < \alpha$, on a pour tout ε :*

$$\int_A \mathbb{1}_K(x,v)|f_\varepsilon(x,v)|dvdx \leq \eta \tag{2.28}$$

On peut alors montrer le résultat suivant, tiré de [70] :

Proposition 2.3 (Golse-Lions-Perthame-Sentis)**.** *Soit (f_ε) une famille de $L^1(dx \otimes dv)$ localement équiintégrable en x et v telle que $v.\nabla_x f_\varepsilon$ soit une famille bornée de $L^1(dx \otimes dv)$. Alors $\forall \psi \in L^\infty_K(dv)$, la famille $\rho_\varepsilon(x) = \int f_\varepsilon(x,v)\psi(v)dv$ est relativement compacte dans $L^1_{loc}(dv)$.*

Une variante de ce résultat dans L^1 constitue un des arguments cruciaux pour la preuve de DiPerna et Lions [52] de l'existence globale de solutions renormalisées à l'équation de Boltzmann.

Dans la note [73], Golse et Saint-Raymond ont montré qu'en fait, seule l'équiintégrabilité en v est nécessaire pour obtenir de la compacité sur les moments. Une telle propriété constitue l'un des points cruciaux de leur preuve de la dérivation de l'équation de Navier-Stokes incompressible à partir de l'équation de Boltzmann ([75]). Avant d'énoncer le théorème, commençons par définir précisément la notion d'équiintégrabilité en v.

Définition 3. *Soit $(f_\varepsilon(x,v))$ une famille bornée de $L^1_{loc}(dx \otimes dv)$. Elle est dite localement équiintégrable en v si et seulement si pour tout paramètre $\eta > 0$ et tout compact $K \subset \mathbb{R}^d \times \mathbb{R}^d$, il existe $\alpha > 0$ tel que pour tout ensemble mesurable $A \subset \mathbb{R}^d$ tel que $|A| < \alpha$, on a pour tout ε :*

$$\int \left(\int_A \mathbb{1}_K(x,v) |f_\varepsilon(x,v)| dv \right) dx \leq \eta \tag{2.29}$$

Théorème 2.8 (Golse-Saint-Raymond). *Soit (f_ε) une famille bornée de $L^1_{loc}(dx \otimes dv)$ localement équiintégrable en v et telle que $v.\nabla_x f_\varepsilon$ est bornée dans $L^1_{loc}(dx \otimes dv)$. Alors :*

- *(f_ε) est localement équiintégrable en x **et** v.*
- *$\forall \psi \in L^\infty_K(dv)$, la famille $\rho_\varepsilon(x) = \int f_\varepsilon(x,v)\psi(v)dv$ est relativement compacte dans $L^1_{loc}(dv)$.*

La preuve du théorème repose de manière cruciale sur le Théorème 2.2, portant sur les propriétés de mélange de l'opérateur de transport libre.

Dans le chapitre 8 de cette thèse, on se propose d'étendre le Théorème 2.8 au cas d'une équation de transport dans un champ de force Lipschitz. Dans le cadre stationnaire, on considère donc l'opérateur de transport :

$$v.\nabla_x + F(x,v).\nabla_v.$$

On se place en dimension quelconque d'espace et on suppose que F est Lipschitz, plus précisément $F \in W^{1,\infty}_{loc,x,v}$ et de divergence nulle en v, c'est-à-dire

$$\mathrm{div}_v F = 0.$$

On démontre le théorème suivant :

Théorème 2.9 (Chapitre 8). *Soit (f_ε) une famille bornée de $L^1_{loc}(dx \otimes dv)$ localement équiintégrable en v et telle que $v.\nabla_x f_\varepsilon + F.\nabla_v f_\varepsilon$ est bornée dans $L^1_{loc}(dx \otimes dv)$. Alors :*

*i. (f_ε) est localement équiintégrable dans les variables x **et** v.*

ii. Pour toute fonction test $\Psi \in \mathcal{C}^1_c(\mathbb{R}^d)$, la famille $\rho_\varepsilon(x) = \int f_\varepsilon(x,v)\Psi(v)dv$ est relativement compacte dans $L^1_{loc}(dv)$.

Dans un premier temps on montre que le premier point implique le second, c'est-à-dire qu'avec une hypothèse d'équiintégrabilité dans toutes les variables, alors on obtient la compacité relative pour les moments en vitesse. Cela correspond à l'analogue du Théorème 2.8 pour l'opérateur de transport dans un champ de force. Pour prouver ce résultat, l'approche est analogue à celle de [70] ; la compacité est obtenue en utilisant un lemme de moyenne L^2, l'idée étant de considérer le terme $F.\nabla_v f$ comme une source avec une dérivée en v.

Rappelons que le lemme de Golse et Saint-Raymond s'appuie en grande partie sur les propriétés de dispersion du transport libre. On peut en fait se rendre compte que le point important consiste plus précisément en les propriétés de mélange du transport libre en temps strictement positif (mais arbitrairement petit). Il s'agit en quelque sorte d'un effet de mélange de phase instantané. On se propose donc de prouver un tel résultat (local en temps) pour les équations de transport avec un champ extérieur. Il est de toutes manières illusoire de montrer la dispersion pour tout temps, étant donné que certains potentiels (tels que l'oscillateur harmonique $V = |x|^2$) sont confinants et ne dispersent pas en temps grand, du moins au sens de l'estimation du théorème 2.2. On prouve donc le résultat suivant :

Proposition 2.4 (Chapitre 8). *Soit $F(x)$ un champ de force localement Lipschitz. Il existe un temps maximal $\tau > 0$ (dépendant seulement de $\|\nabla_x F\|_{L^\infty}$) tel que, si f est solution de l'équation de transport :*

$$\left\{ \begin{array}{l} \partial_t f + v.\nabla_x f + F.\nabla_v f = 0, \\ f_{|t=0} = f^0 \in L^p(dx \otimes dv), \end{array} \right.$$

Alors :

$$\forall |t| \leq \tau, \|f(t)\|_{L^\infty_x(L^1_v)} \leq \frac{2}{|t|^d} \|f^0\|_{L^1_x(L^\infty_v)}. \tag{2.30}$$

Bien entendu, la constante 2 est arbitraire et peut être remplacée par n'importe quel réel r strictement plus grand que 1. La preuve de ce résultat consiste en une étude précise des caractéristiques, ce qui permet de faire un changement de variable en s'inspirant de la preuve de la dispersion pour le transport libre. Le reste de la preuve du lemme de moyenne suit de près l'approche de Golse et Saint-Raymond.

2.3.4 Le théorème de DiPerna et Lions

L'existence de solutions renormalisées à l'équation (VP-E) est montrée par DiPerna et Lions dans [49].

Théorème 2.10 (DiPerna-Lions). *Soit $f_0 \geq 0 \in L^1_{x,v}$, tel que $f_0 \log f_0 \in L^1_{x,v}$ et d'énergie bornée. Alors il existe une solution renormalisée globale à (VP-E) avec donnée initiale f_0.*

Comme on l'a déjà remarqué, une application du Lemme 2.3 ne permet pas de faire gagner de l'intégrabilité sur la densité locale ρ. Comme la théorie elliptique ne fonctionne pas bien dans L^1, il nous faut donc gagner un peu d'intégrabilité. Cela peut être réalisé grâce une inégalité d'interpolation un peu plus fine valable pour toute fonction f mesurable et positive :

$$\int \rho \log^+ \rho \, dx \leq C \int \left(f|v|^2 + f \log^+ f \right) dx.$$

La théorie de Calderon-Zygmund donne alors que $E \in W^{1,1}_{x,loc}$. La preuve du théorème se base finalement sur la théorie dite de DiPerna-Lions pour les équations différentielles ordinaires à coefficients peu réguliers (ici typiquement $W^{1,1}_{x,loc}$). De telles équations ne sont pas couvertes par la théorie classique de Cauchy-Lipschitz.

Nous tenons à souligner que de façon générale, ces théories de solutions faibles sont à comprendre avant tout comme des résultats de stabilité (faible). Les preuves consistent à montrer que si une suite de solutions vérifie l'équation (au sens faible considéré), alors la limite la vérifie également.

2.4 Solutions avec des moments en vitesse d'ordre élevé, à la Lions et Perthame

De manière indépendante à l'approche de Pfaffelmoser, Lions et Perthame ont développé dans [119] une théorie de régularité pour Vlasov-Poisson, s'appuyant sur une propagation des moments en vitesse.

Théorème 2.11 (Lions-Perthame). *Soit $f_0 \in L^1_{x,v} \cap L^\infty_{x,v}$.*

i. Supposons que l'on a initialement :

$$\int f_0(x,v)|v|^m dxdv < +\infty, \tag{2.31}$$

pour tout $m < m_0$ avec $m_0 > 3$. Alors il existe une solution faible de (VP-E) satisfaisant :

$$\sup_{t\in[0,T]} \int f(t,x,v)|v|^m dxdv \leq C(T); \tag{2.32}$$

pour tout $T < +\infty$ et tout $m < m_0$.

ii. La solution f satisfait de plus :

$$\begin{aligned}
&\rho \in \mathcal{C}(\mathbb{R}^+, L^q(\mathbb{R}^3)), \quad && \textit{pour} \quad 1 \leq q < \frac{3+m_0}{3}, \\
&E \in \mathcal{C}(\mathbb{R}^+, L^q(\mathbb{R}^3)), \quad && \textit{pour} \quad \frac{3}{2} < q < \frac{3(3+m_0)}{6-m_0} \quad \textit{si} \quad m_0 \leq 6, \\
&E \in \mathcal{C}(\mathbb{R}^+, \mathcal{C}^{0,\alpha}(\mathbb{R}^3)), \quad && \textit{pour} \quad \alpha < \frac{m_0-6}{3+m_0} \quad \textit{si} \quad m_0 > 6.
\end{aligned}$$

Il s'agit d'une approche "eulérienne" , conceptuellement plus proche des méthodes standards en équations aux dérivées partielles ; elle a le grand mérite de permettre de traiter le cas d'équations de Poisson plus générales. En particulier ces estimations peuvent s'appliquer pour (VP-I). Néanmoins, signalons que la preuve repose sur des effets dispersifs obtenus puisque x vit dans un domaine non borné et qu'en conséquence, on ignore toujours si cette approche fonctionne sur des domaines bornés (par exemple le tore).

Signalons également une amélioration assez récente de cette théorie, due à Gasser, Jabin et Perthame [62]. L'approche de ces auteurs repose sur des estimations de dispersion plus sophistiquées satisfaites par l'équation de transport. Elle permet notamment de montrer que tous les moments en vitesse d'ordre supérieur à 2 sont en fait propagés.

Le problème de l'unicité de la solution faible est une question délicate. Elle a trouvé ses premières réponses dans l'article de Lions et Perthame [119]. Finalement le théorème suivant a été montré par Loeper :

Théorème 2.12 (Loeper). *Etant donnée $f_0 \in \mathcal{M}^+(\mathbb{R}^3 \times \mathbb{R}^3)$, il existe au plus une solution faible à (VP-E), vérifiant :*

$$\|\rho\|_{L^\infty_t(L^\infty_x)} < +\infty. \tag{2.33}$$

Pour construire une solution telle que $\|\rho\|_{L^\infty_t(L^\infty_x)} < +\infty$, l'approche de Lions et Perthame nous apprend qu'il est suffisant d'avoir six moments en vitesse initialement bornés. Citons pour finir le travail récent de Salort [144], qui par des arguments fins de dispersion et d'analyse harmonique a montré que le même résultat s'appliquait avec strictement moins de six moments en vitesse.

2.5 Cas des domaines à bord

Considérons les équations posées dans un domaine Ω dont le bord $\partial\Omega$ est régulier (disons au moins $\mathcal{C}^1$). Il y a plusieurs choix possibles concernant les conditions aux limites pour la fonction de distribution, ce qui modélise la loi de réflexion et/ou d'absorption des particules sur la surface du matériau ainsi que pour le champ électrique, ce qui modélise les caractéristiques électriques du matériau.

Pour la fonction de distribution, on peut choisir en pratique :

- Réflexion spéculaire :
$$f(t,x,v) = f(t,x,v - 2(n_x.v)n),$$
pour tout x sur $\partial\Omega$ et avec n_x la normale sortante au bord.

- Réflexion diffuse :
$$f(t,x,v) = \frac{1}{2\pi}\exp\left(\frac{-v^2}{2}\right)\int_{n_x.w<0} |n_x.w| f(t,x,w)dw,$$
pour tout x sur $\partial\Omega$ et avec n_x la normale sortante au bord au point x. Il est également possible de considérer un autre noyau que la maxwellienne.

- Absorption/Emission :
$$f = g$$
sur $\partial\Omega$, où g est une fonction donnée.

Pour le champ électrique, on choisit en pratique :

- Condition de Dirichlet :
$$V = 0$$
sur $\partial\Omega$. Signalons que cette condition s'interprète physiquement comme la condition de conducteur parfait.

- Condition de Neumann :
$$E.n = 0$$
sur $\partial\Omega$, où n désigne la normale au bord.

On se contente ici de donner quelques références sur ce sujet, sans rentrer dans les détails. En ce qui concerne les conditions au bord pour des équations de transport générales, citons l'article pionnier de Bardos [6]. Concernant l'équation de Vlasov-Poisson, pour le cas de solutions classiques, les travaux fondateurs sont dus à Guo [88], puis étendus par Hwang [100], et très récemment par Hwang et Velazquez [102, 103]. Pour le cas des solutions faibles, citons à ce sujet les travaux de Ben Abdallah [12] et Guo [87]. La stabilité des solutions renormalisées dans un domaine à bord pour des conditions aux limites très générales est étudiée par Mischler [128, 129] (en fait dans le cadre plus général du système de Vlasov-Poisson-Boltzmann).

2.6 Cas de l'équation de Vlasov-Poisson pour les ions (VP-I)

Le seul article traitant explicitement cette équation est celui de Bouchut [19], dans lequel l'auteur montre l'existence globale de solutions faibles pour une version légèrement modifiée de (VP-I) :

$$\begin{cases} \partial_t f + v.\nabla_x f + E.\nabla_v f = 0, \\ E = -\nabla_x V, \\ -\Delta_x V = \int f dv - \frac{de^V}{\int de^V dx}, \\ f_{|t=0} = f_0. \end{cases} \tag{2.34}$$

Dans cette équation d est une fonction dans L^1 telle que :

$$d(x) = e^{-H},$$

où H est un potentiel confinant agissant sur les électrons. Cela assure physiquement que les électrons ne se dispersent pas à l'infini. Mathématiquement, cela permet par ailleurs d'assurer que de^V est bien intégrable.

Le théorème d'existence est l'analogue de celui dû à Arsenev pour (VP-E) :

Théorème 2.13 (Bouchut)**.** *Pour une donnée initiale $f_0 \geq 0$ bornée dans $L^1 \cap L^\infty(\mathbb{R}^6)$ et vérifiant, $\int (1+|x|^2+|v|^2) f_{0,\varepsilon} dxdv < \infty$:*

Soit $\mathcal{G}(t)$ la fonctionnelle d'énergie définie par :

$$\begin{aligned} \mathcal{G}(t) =& \frac{1}{2}\int f|v|^2 dvdx + \int d(x)\left(V - \log\left(\int de^V dx\right)\right)\frac{e^V}{\int de^V dx} dx \\ &+ \frac{1}{2}\int |\nabla_x V|^2 dx. \end{aligned} \tag{2.35}$$

Si $\mathcal{G}(0)$ est finie,il existe $f \in L^\infty_t(L^1_{x,v}) \cap L^\infty_{t,x,v}$ solution globale faible avec $\mathcal{G}(t)$ non croissante.

3 Analyse asymptotique des régimes quasineutres et de champ magnétique intense

Cette section concerne l'étude de limites singulières pour l'équation de Vlasov-Poisson. Cela se traduit mathématiquement par l'introduction d'un petit paramètre $\varepsilon > 0$ (modélisant différents régimes physiques d'intérêt) dans les équations aux dérivées partielles ; on s'intéresse ensuite au comportement des solutions de Vlasov-Poisson quand $\varepsilon \to 0$.

On se concentrera ici sur deux régimes asymptotiques en particulier :

- la limite dite quasineutre, qui correspond au régime où la longueur de Debye est très petite devant les autres longueurs caractéristiques du système.
- la limite de champ magnétique fort, pour une équation de Vlasov-Poisson avec un champ magnétique stationnaire B additionnel, ce qui a pour effet de rajouter la force de Lorentz $F = v \wedge B$ dans l'équation de transport :

$$\begin{cases} \partial_t f + v.\nabla_x f + \left(E + v \wedge \frac{b}{\varepsilon}\right).\nabla_v f = 0, \\ E = -\nabla_x V, \\ -\Delta_x V = \int f dv - 1, \\ f_{|t=0} = f_0. \end{cases} \tag{3.1}$$

et qui correspond grossièrement au cas où l'intensité $|B|$ du champ magnétique tend vers l'infini.

Certains régimes physiquement intéressants constituent une combinaison de ces deux limites, avec éventuellement une interaction entre leurs effets conjugués.

3.1 Sur la limite quasineutre

3.1.1 Analyse formelle

En prenant en compte le fait que la longueur de Debye λ_D (définie en (1.3)) est très petite devant la longueur caractéristique d'observation L_0, on considère le scaling suivant :

$$\frac{\lambda_D}{L_0} \sim \sqrt{\varepsilon},$$

où ε est un petit paramètre, que l'on va faire tendre vers 0. Alors, le modèle classiquement étudié pour la limite quasineutre est l'équation (VP-E) mise à l'échelle suivante :

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + E.\nabla_v f_\varepsilon = 0, \\ E_\varepsilon = -\nabla_x V_\varepsilon, \\ -\varepsilon \Delta_x V_\varepsilon = \int f_\varepsilon dv - 1, \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon,}. \end{cases} \tag{3.2}$$

Pour tout $\varepsilon > 0$ fixé, nous pouvons considérer des solutions globales faibles à ce problème, grâce au théorème d'Arsenev.

L'énergie pour ce système est la suivante :

$$\mathcal{E}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v|^2 dv dx + \frac{\varepsilon}{2}\int |E_\varepsilon|^2 dx. \tag{3.3}$$

Remarquons d'ores et déjà qu'en raison du scaling du système, le champ électrique E_ε n'est pas borné uniformément en ε dans un quelconque espace de Lebesgue. Cela indique qu'on ne pourra passer à la limite "naivement".

On peut faire une analyse formelle de la limite. On fait l'hypothèse de convergence forte suivante : $f_\varepsilon \to f$ et $E_\varepsilon \to E$.

Comme E_ε est un gradient, on peut également écrire $E = \nabla_x p$. La limite formelle est immédiate (seule l'équation de Poisson est affectée par la limite $\varepsilon \to 0$ et dégénère pour devenir $\int f dv = 1$). Le système limite s'écrit :

$$\begin{cases} \partial_t f + v.\nabla_x f + \nabla_x p.\nabla_v f = 0, \\ \int f dv = 1. \end{cases} \tag{3.4}$$

Le champ de vecteurs $\nabla_x p$ peut être vu comme un multiplicateur de Lagrange, ou une pression dans le langage de la mécanique des fluides, associée à la contrainte que la densité locale des électrons est toujours constante égale à 1.

En effet, intégrons l'équation de transport par rapport à v ; étant donné que la densité reste constante, on obtient la contrainte d'incompressibilité :

$$\nabla_x . \int v f dv = 0. \tag{3.5}$$

Ensuite, si on multiplie l'équation par v, qu'on intègre à nouveau par rapport à v et qu'on prend la divergence,on obtient la loi de pression :

$$-\Delta p = -\Delta_x : \int v \otimes v f dv. \tag{3.6}$$

Pour ces raisons, cette limite peut être vue en un certain sens comme une version cinétique de la limite incompressible de la mécanique des fluides. Pour poursuivre l'analogie, remarquons que si on prend f monocinétique, c'est-à-dire de la forme $f(t,x,v) = \rho(t,x)\delta_{v=u(t,x)}$, on récupère exactement l'équation d'Euler incompressible.

$$\begin{cases} \partial_t u + u.\nabla_x u + \nabla_x p = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{3.7}$$

Ainsi, le système (3.4) est une version cinétique de l'équation d'Euler incompressible, comme Brenier l'a remarqué dans [23]. A notre connaissance, ce système est toujours très mal compris du point de vue mathématique. Une de ses caractéristiques intéressante est qu'il fait toujours sens en dimension 1, contrairement à l'équation d'Euler incompressible.

Bien entendu, il est possible de mener la même analyse formelle pour l'équation de Vlasov-Poisson avec des électrons sans masse (VP-I).

- Si l'on considère une loi de Maxwell-Boltzmann linéarisée, auquel cas l'équation de Poisson dans le scaling quasineutre s'écrit :

  $$V_\varepsilon - \varepsilon \Delta_x V_\varepsilon = \int f_\varepsilon dv - 1.$$

  On obtient à la limite le système suivant :

  $$\begin{cases} \partial_t f + v.\nabla_x f + \nabla_x p.\nabla_v f = 0, \\ p = \int f dv. \end{cases} \tag{3.8}$$

  Il s'agit d'une version cinétique de l'équation d'Euler isentropique avec $\gamma = 2$, aussi connue sous le nom d'équation de Saint-Venant (ou Shallow Water). On récupère en effet cette équation en considérant des données monocinétiques :

  $$\begin{cases} \partial_t \rho + \nabla_x(\rho u) = 0, \\ \partial_t u + u.\nabla_x u + \nabla_x \rho = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{3.9}$$

- Si l'on considère une loi de Maxwell-Boltzmann complète, auquel cas l'équation de Poisson dans le scaling quasineutre s'écrit :

  $$-\varepsilon \Delta_x V_\varepsilon = \int f_\varepsilon dv - e^{V_\varepsilon}.$$

  On obtient à la limite le système suivant :

  $$\begin{cases} \partial_t f + v.\nabla_x f + \nabla_x p.\nabla_v f = 0, \\ p = \log \int f dv. \end{cases} \tag{3.10}$$

  De même que précédemment, il s'agit d'une version cinétique de l'équation d'Euler isotherme (Euler isentropique avec $\gamma = 1$), que l'on rappelle ci-dessous :

  $$\begin{cases} \partial_t \rho + \nabla_x(\rho u) = 0, \\ \partial_t u + u.\nabla_x u + \frac{\nabla_x \rho}{\rho} = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{3.11}$$

Dans un premier temps, on se concentre sur la limite quasineutre de l'équation (VP-E).

3.1.2 La limite quasineutre interprétée comme un problème de comportement en temps long

Si on fait subir le changement d'échelles $(t,x) \mapsto (\frac{t}{\sqrt{\varepsilon}}, \frac{x}{\sqrt{\varepsilon}})$ à l'équation de Vlasov-Poisson (VP-E) adimensionnée :

$$\begin{cases} \partial_t f + v.\nabla_x f + E.\nabla_v f = 0, \\ E = -\nabla_x V, \\ -\Delta_x V = \int f dv - 1, \\ f_{|t=0} = f_0, \end{cases} \tag{3.12}$$

on obtient de manière assez remarquable l'équation de Vlasov-Poisson (3.2) dans le scaling quasineutre.

Cette propriété d'échelles indique donc que la limite quasineutre peut s'interpréter comme une limite de "temps grand". A ce titre, son étude est donc forcément liée d'une manière ou d'une autre à l'étude de la stabilité ou de l'instabilité des solutions de l'équation de Vlasov-Poisson, en particulier de ses solutions stationnaires. La notion de stabilité est au sens (de Lyapunov) suivant :

Définition 4. *Soit μ une solution de référence à (3.12). Cette solution est dite stable pour la norme X si pour tout $\eta > 0$, il existe $\delta > 0$ tel que : pour toute solution ρ à (3.12), le contrôle initial $\|f(0) - \mu(0)\|_X \leq \delta$ entraîne que pour tout $t \geq 0$, $\|f(t) - \mu(t)\|_X \leq \eta$.*

Autrement, la solution μ est dite instable pour la norme X.

Depuis les travaux pionniers de Penrose [133] sur la stabilité des solutions stationnaires, on dispose d'un critère pour déterminer si une solution stationnaire homogène est linéairement stable ou non. Sans rentrer dans les détails, et en se restreignant à un cadre $1D$ il permet de discriminer le comportement dynamique d'un équilibre homogène en espace $\mu(v)$:

Si pour tout zéro w de μ',

$$\int_{-\infty}^{+\infty} \frac{\mu'(v)}{v-w} dv \leq 1$$

alors μ vérifie le *critère linéaire de stabilité de Penrose*. Dans le cas contraire il vérifie le *critère linéaire d'instabilité de Penrose*.

En particulier, le critère de Penrose de stabilité stipule que si un équilibre homogène en espace est croissant puis décroissant en vitesse (*condition de monotonie de Penrose*), alors il est linéairement stable. Il y a eu par la suite de nombreux travaux, aussi bien physiques que mathématiques, pour rendre plus rigoureuses les considérations de Penrose et surtout pour justifier ce paradigme pour l'équation non-linéaire. Concernant les travaux mathématiques, citons les travaux sur la stabilité de Rein et al. [11, 9, 137, 22], celui de Guo et Strauss [91] ou celui de Cacéres, Carrillo et Dolbeault [33]. Les travaux les plus récents ont pour trait commun de reposer sur une méthode d'énergie pour démontrer la stabilité de certains équilibres. On peut se rendre compte qu'avec les méthodes actuellement connues, on ne sait pas montrer la stabilité non linéaire, sous la simple condition de monotonie de Penrose : on doit faire l'hypothèse supplémentaire que la donnée initiale minimise une certaine entropie convexe, ce qui implique la condition de monotonie, mais également une propriété supplémentaire de symétrie.

Par ailleurs, cette problématique est également liée à l'*amortissement Landau*, phénomène de relaxation sans dissipation, mis en évidence mathématiquement par Landau sur l'équation de Vlasov-Poisson linéarisée, qui provoqua une grande controverse. Il a néanmoins pu être vérifié expérimentalement par la suite. Ce phénomène a finalement prouvé rigoureusement sur l'équation non-linéaire posée dans le tore, dans le spectaculaire article

[130] de Mouhot et Villani dans le cadre des fonctions à régularité analytique, pour des conditions initiales proches d'équilibres homogènes satisfaisant le condition linéaire de stabilité de Penrose. Ils montrent alors que la solution f à (VP-E) converge exponentiellement vite vers un équilibre homogène, qui est génériquement différent de l'équilibre homogène dont la donnée est proche au départ.

En ce qui concerne le problème d'instabilité, le critère de Penrose stipule que les distributions à "deux bosses" (typiquement une somme de deux Maxwelliennes) sont linéairement instables. Cette assertion a été démontrée pour l'équation non linéaire par Guo et Strauss [89].

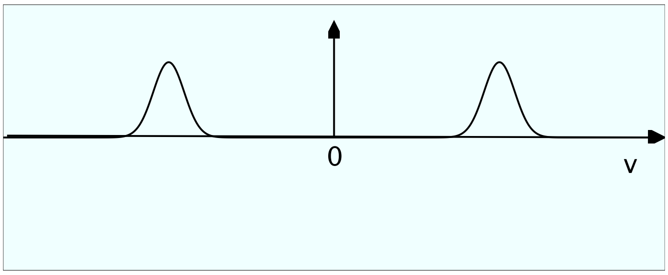


FIGURE 1.5 – Une fonction de distribution $f_0(v)$ à deux bosses, instable au sens de Lyapunov pour Vlasov-Poisson

Ces résultats sont très importants en vue de la limite quasineutre, car ils indiquent que la limite formelle risque d'être fausse pour certaines données initiales, en particulier si le critère d'instabilité de Penrose est vérifié. On s'appuiera en effet sur cette remarque pour exhiber dans le paragraphe suivant un exemple de donnée initiale pour laquelle la limite formelle vers Euler cinétique n'a pas lieu.

Pour finir, signalons le cas des équilibres non homogènes en vitesse (c'est-à-dire dépendant de la variable d'espace) qui semble encore plus délicat, et dont l'étude dépasse largement le cadre de cette thèse. Evoquons ainsi les équilibres BGK (pour Bernstein, Greene et Kruskal) dont l'analyse est un défi majeur, à la fois physique et mathématique. Nous nous référons aux travaux de Guo et Strauss [90] ainsi que Lin et Strauss [115], ainsi qu'à leurs références.

3.1.3 Difficultés et obstructions

Nous essayons maintenant de justifier pourquoi l'analyse rigoureuse du passage à la limite est bien plus délicat que ce que la limite formelle pourrait suggérer.

Oscillations haute fréquence et de grande intensité pour le champ électrique
Nous avions déjà remarqué au début de cette introduction que lorsque la longueur de Debye est petite, le plasma est le siège d'oscillations de fréquence très grande. Ainsi, la première difficulté dans l'analyse mathématique de la limite quasineutre provient de l'existence d'oscillations en temps pour le champ électrique. On peut montrer que ces oscillations sont de fréquence $\mathcal{O}(1/\sqrt{\varepsilon})$ et d'amplitude $\mathcal{O}(1/\sqrt{\varepsilon})$ (ceci est bien entendu compatible avec la conservation de l'énergie $\mathcal{E}_\varepsilon(t)$). Elles sont souvent appelées "ondes plasma" ou "oscillations plasma", et on peut les décrire précisément grâce à l'équation des "ondes" suivante (introduite dans l'article [81] de Grenier), que l'on obtient en combinant l'équation de Poisson et l'équation de Vlasov :

$$\varepsilon\partial_t^2 \operatorname{div} E_\varepsilon + \operatorname{div} E_\varepsilon = \sum_{i,j} \partial^2_{x_i,x_j} \int f_\varepsilon v_i v_j dx_\perp - \varepsilon \sum_i \partial_i (E_{\varepsilon,i} \operatorname{div} E_\varepsilon). \tag{3.13}$$

A cause du second membre qui agit comme une “source”, cela entraine que E_ε se comporte grossièrement comme $1/\sqrt{\varepsilon} e^{\pm it/\sqrt{\varepsilon}}$.

Avec des techniques de filtrage, introduites indépendamment par Schochet [147] et Grenier [82], il est possible de s'en accommoder. On peut alors montrer qu'elles ont un effet non trivial sur le système final. Cependant, en considérant des données “bien préparées”, on peut passer outre ces oscillations.

Instabilités à deux bosses La principale obstruction pour la limite quasineutre est due à des instabilités de type cinétique, en particulier les instabilités de type double-bosse, que l'on a introduit dans le paragraphe précédent.

Le contre-exemple explicite suivant est donné par Grenier dans [84]. Pour simplifier on se restreint au cas unidimensionnel. Considérons l'équilibre homogène $f_0(v)$ défini par :

$$\begin{aligned} f_0(v) &= 1 \text{ pour } -1 \leq v \leq -1/2 \text{ et } 1/2 \leq v \leq 1 \\ &= 0 \text{ ailleurs.} \end{aligned}$$

(Notons que f_0 est une solution stationnaire de (3.2) ainsi que (3.4) avec un champ de force auto-induit réduit à 0.)

Comme il a été montré par Guo et Strauss [89] puis Grenier [84], cette solution est instable par rapport à une certaine perturbation de régularité Sobolev aussi grande que l'on veut.

Théorème 3.1 (Grenier)**.** *Pour tout N et s dans $\mathbb{N}$, et pour tout $\varepsilon < 1$, il existe pour $i = 1,2,3,4$, $v_i^\varepsilon(x) \in H^s(\mathbb{T})$ avec $\|v_1^\varepsilon(x)+1\|_{H^s} \leq \varepsilon^N$, $\|v_2^\varepsilon(x)+1/2\|_{H^s} \leq \varepsilon^N$, $\|v_3^\varepsilon(x) - 1/2\|_{H^s} \leq \varepsilon^N$, $\|v_4^\varepsilon(x)-1\|_{H^s} \leq \varepsilon^N$, telle que la solution $f_\varepsilon(t,x,v)$ associée à la donnée initiale :*

$$\begin{aligned} f_{\varepsilon,0}(x,v) &= 1 \textit{ pour } v_1(x) \leq v \leq v_2(x) \textit{ et } v_3(x) \leq v \leq v_4(x) \\ &= 0 \textit{ ailleurs,} \end{aligned}$$

ne converge pas vers f_0 dans le sens suivant :

$$\liminf_{\varepsilon\to 0} \sup_{t\leq T} \int |f_\varepsilon(t,x,v) - f_0(v)| v^2 dv dx > 0 \tag{3.14}$$

pour tout $T > 0$ et même pour $T = \varepsilon^\alpha$, avec $\alpha < 1/2$.

Remarquons que la convergence considérée est très faible et que, pour $T = 0$, on a bien

$$\liminf_{\varepsilon\to 0} \int |f_{0,\varepsilon}(v) - f_0(v)| v^2 dv dx = 0.$$

En d'autres termes, il existe une petite perturbation (régulière) de f_0 pour laquelle la solution correspondante de (3.2) ne converge pas vers la solution de (3.4) et donc, l'analyse formelle est fausse. A cause de ce type de solutions stationnaires instables, la convergence de (3.2) vers (3.4) n'a en conséquence pas lieu en général.

Donnons à présent quelques idées pour la preuve de ce résultat : il s'agit tout d'abord d'étudier l'instabilité de la solution stationnaire f_0 pour le système avec $\varepsilon = 1$. Cela passe d'abord par une étude de l'instabilité spectrale de l'équation de Vlasov-Poisson linéarisée autour de f_0 ; on observe alors que f_0 vérifie le critère linéaire d'instabilité de Penrose (définie dans un paragraphe précédent), ce qui permet d'exhiber une valeur propre à partie réelle strictement négative au problème linéarisé ([89]). Ensuite, il faut passer à l'instabilité nonlinéaire et pour cela, on peut utiliser les idées de Grenier [85] (introduites dans un autre contexte, pour l'étude de l'instabilité de couches limites pour l'équation d'Euler incompressible). Nous expliquerons ces idées de manière plus détaillée ultérieurement, pour l'étude d'un problème d'instabilité dans un plasma de tokamak. Une fois ce résultat d'instabilité démontré, on revient au système (3.2) par le changement de variable $(t, x) \mapsto (t/\sqrt{\varepsilon}, x/\sqrt{\varepsilon})$ et on traduit le résultat dans les nouvelles variables.

3.1.4 Différentes approches mathématiques

Pour surpasser ces difficultés qui proviennent de la physique du problème, différentes approches ont été développées. Insistons sur le fait qu'elles ne sont pas équivalentes, et qu'en dehors de l'approche de type mesure de défaut, qui s'applique dans toutes les situations, mais dont le résultat ne décrit pas complètement la limite (on obtient une information partielle sur les deux premiers moments uniquement), chaque approche est spécifique à la situation physique décrite. Conformément à l'étude qualitative précédente, le point principal est d'éviter les instabilités double-bosse. A cet effet on peut :

- considérer des données à régularité analytique. Il apparaît en effet que les instabilités de type double bosse n'ont pas d'effet dans ce régime de régularité.
- considérer des profils initiaux stables, ne développant pas d'instabilité cinétique.

L'approche "mesures de défaut" Il semble que la première approche développée dans la littérature mathématique pour l'étude de la limite quasineutre soit celle faisant intervenir des mesures de défaut. Elle a été développée par Brenier et Grenier [28] pour le cas indépendant du temps, puis par Grenier [80] pour le cas dépendant du temps. L'idée est de passer à la limite en faisant apparaître deux mesures de défaut, tenant compte de la non compacité forte en temps (due aux oscillations du champ électrique) et de la non compacité forte en espace (dégénérescence de l'équation de Poisson). Les auteurs se restreignent à la limite pour les deux premiers moments de f_ε, mais il semble que rien n'empêche de considérer des moments d'ordre supérieur, si ce n'est la lourdeur des calculs.

Limite analytique Dans l'article [81], Grenier considère des solutions à (3.2) de la forme :

$$f_\varepsilon(t, x, v) = \int_M \rho^\varepsilon_\Theta(t, x)\delta\left(v - v^\varepsilon_\Theta(t, x)\right)\mu(d\Theta) \tag{3.15}$$

pour $t \geq 0$, $x \in \mathbb{T}^d$, (M, Θ, μ) un espace de probabilité et où Θ est un paramètre permettant de considérer différentes données intéressantes physiquement.

Par exemple $\mu = \delta$ correspond au cas des données dites monocinétiques (cela correspond physiquement au cas où la température est nulle). Plus généralement, cela permet de considérer des données de type "water-bags" (voir par exemple l'article [13] et ses références pour une introduction à ce concept de solutions, qui ne sera pas traité par souci de briéèveté dans cette introduction).

Ensuite, Grenier étudie la limite quasineutre pour le système d'Euler-Poisson sans pression suivant. Ce système, dérivé de (3.2) pour des données ayant la forme que l'on vient d'introduire, peut être interprèté comme un système infini d'équations d'Euler-Poisson sans pression couplées par le parmètre Θ.

$$\begin{cases} \partial_t \rho^\varepsilon_\Theta + \operatorname{div}(\rho^\varepsilon_\Theta v^\varepsilon_\Theta) = 0 \\ \partial_t v^\varepsilon_\Theta + v^\varepsilon_\Theta.\nabla(v^\varepsilon_\Theta) = E^\varepsilon \\ \operatorname{rot} E^\varepsilon = 0 \\ \varepsilon \operatorname{div} E^\varepsilon = \int_M \rho^\varepsilon_\Theta \mu(d\Theta) - 1 \end{cases} \tag{3.16}$$

Pour un tel système hydrodynamique, il s'agit d'éviter les instabilités à deux flux (on se référe à [38] pour leur étude mathématique), qui sont l'analogue des instabilités à deux bosses pour les équations d'Euler "multi-fluides". Il se trouve que ces instabilités ont lieu pour des normes de Sobolev H^s, mais qu'elles n'ont pas d'effet dans le régime de régularité analytique. L'idée est donc de se restreindre à des données analytiques.

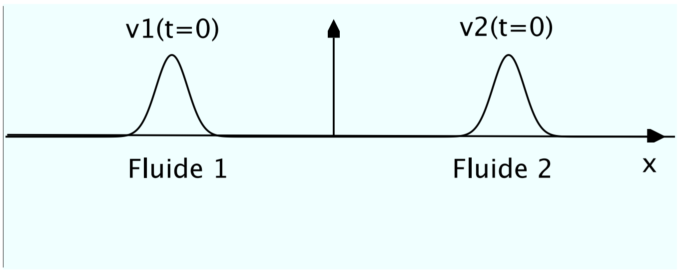


FIGURE 1.6 – Exemple de donnée initiale à deux flux, pour $\Theta \in \{1, 2\}$

En suivant la preuve du théorème de Cauchy-Kovalevskaya donné par Caflisch [34], Grenier prouve l'existence de solutions analytiques pour un temps fini, mais uniforme en ε, ainsi que la convergence après filtrage des ondes plasmas vers le système suivant :

$$\begin{cases} \partial_t \rho_\Theta + \operatorname{div}(\rho_\Theta v_\Theta) = 0 \\ \partial_t v_\Theta + v^\varepsilon_\Theta.\nabla(v_\Theta) = E \\ \operatorname{rot} E = 0 \\ \int \rho_\Theta \mu(d\Theta) = 1, \end{cases} \tag{3.17}$$

qui correspond bien au système (3.4) pour des données de type (3.15).

Méthode d'énergie modulée (ou d'entropie relative) L'équation d'Euler incompressible est bien mieux comprise que sa version cinétique. En effet, en deux dimensions d'espace, on a par exemple l'existence globale pour des données régulières, en trois dimensions d'espace, il y a existence locale de solutions régulières. A contrario, pour la version cinétique, on n'a pour le moment aucun théorème d'existence locale de solutions régulières. Il semble donc raisonnable d'essayer d'obtenir la limite hydrodynamique vers Euler incompressible.

Il s'agit d'imposer qu'asymptotiquement, la solution soit monocinétique (alors qu'à $\varepsilon > 0$ fixé, on souhaite considérer des solutions faibles dans L^p). Pour cela, on utilise la méthode d'entropie relative, qui a été introduite par Brenier [27] pour l'étude de la limite quasineutre. Son approche a ensuite été étendue par Masmoudi [125] et Golse et Saint-Raymond [74]. L'idée fondamentale de cette méthode semble remonter aux travaux de Yau [159] pour l'étude de la limite hydrodynamique d'un modèle de Ginzburg-Landau. Elle est également réminiscente des travaux de Dafermos [43] sur le principe de stabilité fort/faible pour les systèmes hyperboliques. Pour les équations cinétiques, elle a été introduite simultanément et indépendamment par Brenier [27] et Golse [21] dans des contextes différents. La contribution de Golse concerne la limite hydrodynamique de l'équation de Boltzmann vers l'équation d'Euler incompressible.

Etant donnée une certaine fonction régulière $u(t,x)$, l'entropie relative (ou énergie modulée) dans le cas bien préparé ([27]) est la fonctionnelle $\mathcal{H}_\varepsilon(t)$, qui est une version modulée de l'énergie (3.3) :

$$\mathcal{H}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v-u|^2 dvdx + \frac{\varepsilon}{2}\int |E_\varepsilon|^2 dx \tag{3.18}$$

Le but est de montrer que $\mathcal{H}_\varepsilon(t)$ mesure en quelque sorte la distance entre f_ε et sa limite hydrodynamique δ_u.

Le cas bien prépare correspond aux hypothèses suivantes :

$$\begin{aligned} \mathcal{H}_\varepsilon(0) &\to_{\varepsilon\to 0} 0, \\ \operatorname{div} u_{|t=0} &= 0. \end{aligned}$$

Cela signifie en particulier que la donnée initiale converge vers une mesure de Dirac dans le sens faible suivant :

$$\int f_{0,\varepsilon}|v-u_{|t=0}|^2 dvdx \to 0 \tag{3.19}$$

Cela signifie également que $\frac{\varepsilon}{2}\int |E_\varepsilon(t=0)|^2 dx \to 0$. Cela implique qu'il n'y a essentiellement pas d'oscillation pour le champ électrique.

Par un calcul explicite de la dérivée en temps de $\mathcal{H}_\varepsilon(t)$, Brenier montre alors une inégalité de stabilité, qui par un argument de type Gronwall, permet de conclure qu'on a bien $\mathcal{H}_\varepsilon(t) \to 0$, pourvu que u satisfasse la contrainte algébrique d'être une solution dissipative à Euler incompressible. Sans forcément rentrer dans les détails, on rappelle que cette notion de solution est due à Lions [117] et qu'elles coincident avec les solutions fortes si celles-ci existent). Ainsi, si u_0 est suffisamment régulier, un argument de convexité permet finalement de conclure que le courant $J_\varepsilon = \int f_\varepsilon v dv$ converge fortement vers u dans $L^\infty_t L^2_x$ où u est l'unique solution régulière d'Euler incompressible avec donnée initiale u_0.

Finalement, ce résultat est à comprendre avant tout comme un résultat de stabilité : les solutions monocinétiques sont stables dans la limite quasineutre vis à vis des perturbations de l'énergie.

Dans l'article [125], le cas de données initiales générales (pas forcément bien préparées) est considéré, et dans ce cas, une étude détaillée des ondes plasmas devient nécessaire. Citons également les articles de Brenier, Mauser et Puel [30], Puel et Saint-Raymond [136], portant sur la limite quasineutre pour l'équation de Vlasov-Maxwell, utilisant le même type de méthode.

Confrontation entre les différentes méthodes mathématiques

Résumons à présent les avantages et inconvénients des méthodes mathématiques pouvant être envisagées pour étudier la limite quasineutre.

- *Compacité par compensation.* Le principe d'une technique de type compacité par compensation en vue d'étudier la limite quasineutre est d'obtenir des estimations a priori provenant des quantités conservées par le système, puis d'obtenir de la compacité forte en espace ainsi qu'en temps (après un éventuel filtrage) en s'appuyant sur les propriétés régularisantes du système. Pour le système de Vlasov-Poisson, on pense notamment à un gain par régularité elliptique ou par lemme de moyenne. L'objectif est de passer à la limite au sens des distributions dans l'équation afin d'obtenir un système limite homogénéisé. Cette méthode permet en principe de considérer des données physiquement réalistes car les quantités conservées proviennent souvent de la physique du problème (masse, entropie, énergie). Un inconvénient est que l'on s'appuie sur un argument "abstrait" de compacité ; on n'obtient par conséquent pas de taux de convergence explicite et que l'on ne sait donc pas quantifier la validité de l'approximation par l'équation finale.

Hormis pour certains cas très favorables (voir notamment le Chapitre 3), cette méthode ne semble pas adaptée pour l'étude de la limite quasineutre : en effet, à cause de la dégénérescence de l'équation de Poisson, on ne peut pas espérer gagner de la compacité en espace sur le champ électrique (d'où l'introduction de mesures de défaut de [80]). En outre, le scaling de l'énergie fait que l'on n'a même pas d'estimation uniforme pour le champ électrique dans un certain espace L^p. Cela signifie que l'on ne peut pas a priori gagner de la compacité grâce à l'équation de transport. Finalement, avec cette méthode, on ne sait pas s'il est possible de distinguer solutions stables et instables au sens de Penrose, ce qui est problématique.

- *Entropie relative.* Le principe d'une technique de type entropie relative est de moduler l'énergie (ou l'entropie) physique du système considéré, de sorte à mesurer la distance des moments aux solutions de l'équation asymptotique. Elle s'appuie sur une inégalité de stabilité, obtenue par un calcul explicite, qui repose donc sur la structure algébrique des différents systèmes considérés. En général, elle fait intervenir des techniques de filtrage permettant d'obtenir une convergence forte (en espace et en temps). Cette technique a le mérite de permettre (via l'étage de filtrage) de décrire finement les oscillations en temps et leur effet sur le mouvement moyen final. De plus, elle permet d'estimer le taux de convergence (ce qui n'est pas le cas pour la convergence faible). Du point de vue technique, elle nécessite de la régularité (typiquement une borne Lipschitz) sur le système final, mais elle est très peu demandante sur le système initial (typiquement on peut considérer des solutions faibles ou même renormalisées).

  Cette méthode semble bien convenir pour l'étude de la limite quasineutre, car elle est bien adaptée à des données minimisant une certaine entropie, auquel cas c'est cette entropie qu'il faut moduler (voir par exemple [74]) ; cela signifie qu'elle convient plutôt bien à l'étude de certaines solutions stables au sens de Penrose. D'autre part, elle permet de considérer des données à régularité minimale (typiquement la norme d'énergie).

  Cependant, on peut se demander si toutes les solutions stables peuvent être traitées avec une telle technique (notamment les solutions stationnaires vérifiant le critère de stabilité de Penrose sans forcément minimiser une entropie). On peut par exemple penser que la stabilité non linéaire pour ces données linéairement stables est vérifiée dans une norme à plus forte régularité que celle d'énergie (typiquement un espace de Sobolev H^s avec s suffisamment grand).

  Pour finir, un inconvénient du point de vue technique est le suivant : les résultats de convergence ne sont en général que locaux en temps (à cause de l'exigence de régularité sur le système limite).

- *Convergence dans le régime analytique.* Le principe d'une technique se basant sur la régularité analytique est de considérer une norme analytique permettant un bon contrôle de la perte de régularité due aux termes non linéaires. Il s'agit de montrer que l'on sait propager ces normes, uniformément en le petit paramètre, puis de construire une solution via un schéma d'approximation de type Cauchy-Kowalevskaya. La norme analytique contrôle alors suffisamment de normes pour que l'on puisse passer facilement à la limite, après filtrage des éventuelles oscillations en temps.

  Le régime analytique permet de se placer dans des régimes où les instabilités cinétiques de type double bosse n'ont pas d'influence, si bien que l'on peut passer à la limite quasineutre, peu importe le profil. Cependant, il n'est pas clair que l'analyticité soit pertinente physiquement, et elle ne permet de considérer qu'un petit ensemble de données par rapport à l'ensemble des données physiquement admissibles.

3.1.5 Analyse de la limite quasineutre de l'équation de Vlasov-Poisson avec des électrons sans masse (Chapitre 2)

Nous nous intéressons au Chapitre 2 à l'étude de la limite quasineutre pour des ions froids[12], régis par (VP-I), en nous basant sur des techniques d'entropie relative introduites précédemment. Nous considérons le scaling :

$$\frac{\lambda_D}{L_0} = \sqrt{\varepsilon},$$

où λ_D est la longueur de Debye, L_0 désigne la longueur caractéristique d'observation et ε un petit paramètre destiné à tendre vers 0. Pour $x \in \mathbb{R}^3, v \in \mathbb{R}^3, t \in \mathbb{R}^+$, le système étudié s'écrit ainsi :

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + E_\varepsilon.\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\varepsilon \Delta_x V_\varepsilon = \int f_\varepsilon dv - \frac{d e^{V_\varepsilon}}{\int d e^{V_\varepsilon} dx} \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon} \geq 0, \int f_{0,\varepsilon} dv dx = 1. \end{cases} \tag{3.20}$$

Comme dans le paragraphe 2.7, la fonction $d(x) = e^{-H}$ est une fonction intégrable, avec H un potentiel confinant agissant sur le gaz d'électrons. Pour tout ε fixé, nous pouvons appliquer le Théorème 2.13, ce qui nous permet de considérer des solutions globales faibles à (3.20). Avec un tel scaling, l'énergie associée à (3.20) est la fonctionnelle suivante :

$$\begin{aligned} \mathcal{G}_\varepsilon(t) =& \frac{1}{2} \int f_\varepsilon |v|^2 dv dx + \int d(x) \left(V_\varepsilon - \log \left(\int d e^{V_\varepsilon} dx \right) \right) \frac{e^{V_\varepsilon}}{\int d e^{V_\varepsilon} dx} dx \\ &+ \frac{\varepsilon}{2} \int |\nabla_x V_\varepsilon|^2 dx. \end{aligned} \tag{3.21}$$

Il s'agit d'une quantité formellement conservée.

L'analyse formelle de la limite quasineutre, en considérant directement des données monocinétiques (c'est-à-dire des distributions qui sont des Dirac en vitesse), est aisée. On suppose que $f_\varepsilon = \rho_\varepsilon(t,x)\delta_{v=u_\varepsilon(t,x)}$ et qu'on a les convergences formelles suivantes :

$$\begin{aligned} \rho_\varepsilon &\to \rho \\ u_\varepsilon &\to u \\ V_\varepsilon &\to V \end{aligned}$$

[12] Cette terminologie sera justifié par la suite.

On vérifie alors que ρ et u vérifient l'équation d'Euler isotherme avec une force de rappel :

$$\begin{cases} \partial_t \rho + \nabla_x.(\rho u) = 0, \\ \partial_t u + u.\nabla_x u = -\frac{\nabla_x \rho}{\rho} - \nabla_x H, \end{cases} \tag{3.22}$$

que l'on peut reécrire sous la forme symétrique suivante :

$$\begin{cases} \partial_t \log \frac{\rho}{d} + \nabla_x.u + u.\nabla_x \log \frac{\rho}{d} - \nabla_x H.u = 0, \\ \partial_t u + u.\nabla_x u + \nabla_x \log \frac{\rho}{d} = 0. \end{cases}$$

Il s'agit d'un système hyperbolique symétrique (voir par exemple [123]) : on en déduit donc l'existence de solutions fortes locales pour des données initiales $(\log \frac{\rho_0}{d}, u_0)$ dans H^s_x (avec $s > d/2+1$).

Observons que du point de vue physique, considérer des monocinétiques revient à considérer que la température des ions est égale à 0, c'est-à-dire :

$$\int |v - u_\varepsilon(t,x)|^2 f_\varepsilon dv dx = 0,$$

ce qui est pertinent physiquement (cela revient en fait à dire que la température des ions est petite devant celle des électrons, ce qui est toujours le cas dans les tokamaks, par exemple [158]).

Définissons pour simplifier l'écriture la quantité $m_\varepsilon := \frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon} dx}$. En interprétant un des termes de l'énergie physique comme une entropie en $L \log L$, on introduit l'entropie relative, qui est une version modulée de $\mathcal{G}_\varepsilon(t)$:

$$\mathcal{H}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v-u|^2 dv dx + \int (m_\varepsilon \log(m_\varepsilon/\rho) - m_\varepsilon + \rho) dx + \frac{\varepsilon}{2}\int |\nabla_x V_\varepsilon|^2 dx, \tag{3.23}$$

où (ρ, u) sont les quantités vers lesquelles on espère prouver convergence pour les deux premiers moments.

On peut prouver, par des inégalités fonctionnelles élémentaires, que $\mathcal{H}_\varepsilon(t)$ mesure en un certain sens la distance entre les deux premiers moments de f_ε et la solution à Euler isotherme. Le but est d'obtenir une estimation de stabilité sur $\mathcal{H}_\varepsilon(t)$, ce qui nous permettra d'obtenir un contrôle sur cette fonctionnelle.

Cette inégalité de stabilité se base sur un calcul explicite, qui s'appuie sur la structure des équations de Vlasov-Poisson et d'Euler isotherme :

Lemme 3.1 (Chapitre 2)**.** *On a l'identité suivante :*

$$\begin{aligned} \frac{d\mathcal{H}_\varepsilon(t)}{dt} &= \int A(u,\rho).\begin{pmatrix} -de^{V_\varepsilon} + \rho \\ \rho_\varepsilon u - J_\varepsilon \end{pmatrix} dx \\ &+\varepsilon \int \Delta_x V_\varepsilon u.\nabla_x \log \frac{\rho}{d} dx + \varepsilon \int \partial_t \Delta_x V_\varepsilon \log(d/\rho) dx \\ &- \int f_\varepsilon (u-v)\left((u-v).\nabla_x u\right) dv dx + \varepsilon \int D(u) : (\nabla_x V_\varepsilon \otimes \nabla_x V_\varepsilon) dx \\ &-\varepsilon \int \frac{1}{2} |\nabla_x V_\varepsilon|^2 \operatorname{div}_x u dv dx, \end{aligned} \tag{3.24}$$

où A est l'opérateur dit d'accélération :

$$A(u,\rho) = \begin{pmatrix} \partial_t \log \frac{\rho}{d} + \nabla_x.u + u.\nabla_x \log \frac{\rho}{d} - \nabla_x H.u \\ \partial_t u + u.\nabla_x u + \nabla_x \log \frac{\rho}{d} \end{pmatrix}.$$

Evidemment, l'opérateur A s'annule lorsque (ρ, u) est solution d'Euler isotherme. Le théorème est alors le suivant :

Théorème 3.2 (Chapitre 2). *Soit (ρ, u) une solution régulière du système d'Euler isotherme, avec des données initiales (ρ_0, u_0).*

Si $\mathcal{H}_\varepsilon(0) \leq C\varepsilon$, alors il existe $T > 0$, tel que pour tout $t \in [0, T[$, il y a une constante $0 < C_t < +\infty$ avec :

$$\mathcal{H}_\varepsilon(t) \leq C_t \varepsilon.$$

Cela implique les convergences suivantes :

- *m_ε converge fortement vers ρ dans $L_t^\infty L_x^1$.*
- *ρ_ε converge faiblement L^1 vers ρ.*
- *$u_\varepsilon := J_\varepsilon / \rho_\varepsilon$ converge faiblement L^1 vers u et fortement au sens suivant :*

$$\int \rho_\varepsilon |u_\varepsilon - u|^2 dx \to 0.$$

Il s'agit d'une certaine manière d'un prolongement des travaux de Brézis, Golse et Sentis [31] dans lequel seule l'équation de de Poisson (sans couplage avec l'équation de Vlasov) est considérée et de celui de Cordier et Grenier [39], dans lequel les auteurs considèrent une version hydrodynamique de notre système cinétique.

Nous nous intéressons également au cas où l'équation de Maxwell-Boltzmann est linéarisée, c'est-à-dire quand l'équation de Poisson s'écrit :

$$V_\varepsilon - \varepsilon \Delta_x V_\varepsilon = \int f_\varepsilon dv - 1, \tag{3.25}$$

auquel cas on obtient à la limite une équation de Saint-Venant.

Dans ce cas, la méthode est similaire, et la bonne entropie relative est la fonctionnelle suivante :

$$\mathcal{H}_\varepsilon(t) = \frac{1}{2} \int f_\varepsilon |v - u|^2 dv dx + \frac{1}{2} \int (V_\varepsilon - \rho + 1)^2 dx + \frac{\varepsilon}{2} \int |\nabla_x V_\varepsilon|^2 dx, \tag{3.26}$$

où (ρ, u) est une solution forte au système de Saint-Venant correspondant. Les résultats prouvés sont très similaires.

Signalons par ailleurs une différence importante dans l'étude de telles limites quasineutres par rapport à l'étude classique de Brenier pour (VP-E). Dans notre cas, il y a dans l'énergie un terme d'ordre 1 faisant intervenir le potentiel électrique, qui n'a pas d'équivalent dans le cas classique. Cela a un effet stabilisant en temps, dans la mesure où des oscillations en temps n'apparaissent pas, à moins qu'on ait introduit des oscillations en espace au temps initial.

Cela peut être expliqué heuristiquement, dans le cas "linéarisé", en étudiant l'équivalent de l'équation des "ondes" (3.13), qui s'écrit pour notre étude sous la forme :

$$\partial_t^2 (I - \varepsilon \Delta_x) V_\varepsilon - \Delta_x V_\varepsilon = \text{ source}, \tag{3.27}$$

avec une source uniformément bornée en ε dans un espace fonctionnel adéquat. En passant en variables de Fourier, cela nous mène donc à étudier, pour tout $k \in \mathbb{Z}^3$, l'équation :

$$\partial_t^2 \alpha + \frac{|k|^2}{1 + \varepsilon |k|^2} \alpha = \text{ source},$$

dont les solutions se comportent grossièrement en $\sqrt{\frac{|k|^2}{1+\varepsilon|k|^2}}e^{\pm it\sqrt{\frac{|k|^2}{1+\varepsilon|k|^2}}}$. Ainsi pour les modes de Fourier correspondant à $|k| \leq 1/\sqrt{\varepsilon}$, ni la fréquence, ni l'amplitude des oscillations n'explose dans la limite $\varepsilon \to 0$. Pour les modes de Fourier associés à $|k| > 1/\sqrt{\varepsilon}$, on peut penser qu'une régularité suffisante permet de montrer que pour ε très petit, la contribution de ces modes de Fourier est négligeable. En revanche, si on introduit initialement des oscillations en espace, alors cette contribution peut exploser.

Ceci est à comparer avec la limite quasineutre pour (VP-E), pour lequel (3.13) indique que la fréquence et l'amplitude des oscillations plasma sont de l'ordre de $\frac{1}{\sqrt{\varepsilon}}$.

Insistons que cette étude des ondes est avant tout formelle, car elle est difficile à mettre en oeuvre dans notre cadre (en revanche, cela devient rigoureux dans le cadre de la régularité analytique).

3.1.6 Perspectives

Vers une justification de la limite quasineutre "cinétique" ? Par limite quasineutre cinétique, nous entendons la dérivation de l'équation d'Euler cinétique (3.4). A notre connaissance, le seul résultat allant dans cette direction est une conjecture de Grenier [84]. Comme nous venons de le souligner, la limite formelle étudiée est fausse en général, du fait des instabilités de Penrose. On se place dans un cadre unidimensionnel. L'idée de Grenier est de se restreindre à des données initiales $f_0(x, v)$ telles que pour tout x, la fonction $v \mapsto f_0(x, v)$ soit linéairement stable. Plus précisément, il demande à ce que cette fonction soit croissante puis décroissante, et que ce soit son unique changement de monotonie. On s'attend alors, en se restreignant à ce type de données, à ce que les instabilités cinétiques n'apparaissent pas, et ainsi à pouvoir justifier la limite. Plus précisément, considérons l'équation de Vlasov-Poisson posée pour $x \in \mathbb{T}, v \in \mathbb{R}$:

$$\begin{cases} \partial_t f_\varepsilon + v\partial_x f_\varepsilon + E.\partial_v f_\varepsilon = 0, \\ E_\varepsilon = -\partial_x V_\varepsilon, \\ -\varepsilon\partial^2_{xx} V_\varepsilon = \int f_\varepsilon dv - 1, \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon}. \end{cases} \tag{3.28}$$

Il reste beaucoup à faire pour vérifier cette conjecture. Pour commencer, il faut construire des solutions à (3.4), ce qui semble être un problème difficile. De manière assez remarquable, il y a des analogies assez fortes avec le problème de Cauchy pour l'équation d'Euler hydrostatique [26]. Cependant il semble que les méthodes qui fonctionnent pour cette équation ne se transposent pas pour l'équation d'Euler cinétique. La deuxième partie de la conjecture consiste à prouver la convergence vers ces éventuelles solutions à (3.4).

Nous nous intéressons dans un premier temps à la question de la dérivation uniquement. Une idée naturelle consiste à appliquer la méthode de l'entropie relative pour des données initiales qui ne soient pas proches de Dirac en vitesse, mais plutôt proches de profils stables, vérifiant une condition de type Penrose. C'est l'objet d'un travail en cours en collaboration avec Maxime Hauray.

Aussitôt qu'une solution stationnaire homogène $\mu(v)$ est minimiseur d'une certaine entropie convexe Q, alors nous sommes capables de montrer que μ est stable dans la limite quasineutre. La preuve de ce résultat est une nouvelle fois une méthode d'entropie relative, mais contrairement au résultat précédent, ce n'est pas l'énergie physique qui est modulée, mais l'entropie qui est minimisée par la solution finale μ. Le point crucial est ainsi d'introduire l'énergie modulée $\mathcal{L}_\varepsilon(t)$:

$$\mathcal{L}_\varepsilon(t) = \frac{1}{2}\varepsilon \int |E_\varepsilon|^2 dx + \int \big(Q(f_\varepsilon) - Q(\mu) - Q'(f)(f_\varepsilon - \mu)\big)\, dvdx. \tag{3.29}$$

A cet égard, cette méthode peut être vue comme une généralisation de la méthode utilisée par Brenier [27] (dans ce cas, il se trouve que les données monocinétiques minimisent bien l'énergie cinétique, et donc c'est la bonne quantité à moduler). Une telle idée était également présente dans l'article de Golse et Saint-Raymond [74], qui traitait également d'une limite quasineutre, mais couplée à une limite de champ magnétique intense. Le but était également différent, car il s'agissait de montrer que la densité de courant à la limite vérifiait bien la même équation d'Euler que dans le cas proche du monocinétique.

Remarquons à nouveau que nos hypothèses sur μ impliquent une condition de symétrie sur la distribution. Il semble assez raisonnable d'introduire ce que l'on a appelé “condition de Penrose avec symétrie” :

Définition 5. *Une fonction $f(t,x,v)$ satisfait la condition de Penrose avec symétrie s'il existe :*

- *Une profil régulier (au moins $\mathcal{C}^2$) $\varphi(t,x,r) : \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}^+$, strictement croissant par rapport à r,*
- *Une fonction régulière $\bar{v}(t,x) : \mathbb{R}^+ \times \mathbb{T} \to \mathbb{R}$,*

tels que pour tous $t \in [0,T], x \in \mathbb{T}$ et $v \in \mathbb{R}$, on ait :

$$f(t,x,v) = \varphi\left(t, x, -\frac{|v-\bar{v}(t,x)|^2}{2}\right). \tag{3.30}$$

Nous pouvons alors vérifier que de telles fonctions sont très bien adaptées à la méthode d'entropie relative, car on peut considérer des fonctions de la forme $Q(t,x,.)$, convexes par rapport à la dernière variable dans (3.29). On peut alors démontrer des inégalités de stabilité satisfaisantes. Cependant, nous montrons qu'une solution à (3.4) ne peut vérifier cette condition que si elle est de la forme $\mu(v-\overline{v})$, avec $\overline{v} \in \mathbb{R}$ (en particulier elle est stationnaire...).

Analyse de couches limites pour la limite quasineutre Un problème fondamental et important concerne l'analyse des limites quasineutres dans des domaines à bord. Il est en effet bien connu qu'au voisinage des surfaces des dispositifs confinants, la quasineutralité n'est pas vérifiée. A notre connaissance une telle analyse n'a jamais été menée pour l'équation (VP-I). On peut se demander si la méthode d'entropie relative permettrait de démontrer une certaine stabilité.

3.2 Sur les limites de champ magnétique intense

On considère dans cette section l'équation de Vlasov-Poisson (VP-E) décrivant le comportement des électrons avec un champ magnétique extérieur donné (dans un tokamak par exemple, celui-ci est créé par des bobines entourant le dispositif) :

$$\begin{cases} \partial_t f + v.\nabla_x f + (E + v \wedge B).\nabla_v f = 0, \\ E = -\nabla_x V, \\ -\Delta_x V = \int f dv - 1, \\ f_{|t=0} = f_0. \end{cases} \tag{3.31}$$

Le champ B est un champ magnétique extérieur, ayant pour contrainte d'être stationnaire et de vérifier la condition de divergence nulle :

$$\operatorname{div} B = 0. \tag{3.32}$$

On s'intéresse à des régimes dont la principale caractéristique est l'intensité du champ magnétique, c'est-à-dire en notant T_0 le temps caractéristique d'observation et T_c la période cyclotron (définie en (1.19)) :

$$\frac{T_0}{T_c} \sim \frac{1}{\varepsilon}, \tag{3.33}$$

avec ε un petit paramètre, destiné à tendre vers 0.

Au cours des dernières années, il y a eu beaucoup de travaux en physique pour étudier la dérivation (dite "gyrocinétique") de systèmes limites homogénéisés, citons notamment les articles de Littlejohn [120, 121], basés sur une méthode perturbative faisant intervenir des transformation de Lie. Si l'on souhaite une référence plus récente, on pourra consulter l'article de revue de Brizard et Hahm [32] ainsi que ses nombreuses références.

Pour la littérature mathématique, citons à ce sujet les travaux de Frénod et Sonnendrücker [59, 60], Frénod, Raviart et Sonnendrücker [57], Brenier [27], Golse et Saint-Raymond [72, 74, 139, 140].

Signalons également une forte analogie avec les fluides en rotation rapide (la force de Coriolis étant similaire à la force de Lorentz), sujet très porteur ces denières années (on se référera notamment à l'ouvrage référence [36]). La différence fondamentale est que, en ce qui concerne notre problème, les systèmes sont de nature cinétique, tandis que pour les fluides tournants, il s'agit de systèmes hydrodynamiques.

Notre but ici est de présenter différentes idées sur quelques techniques mathématiques (de type convergence faible) qui peuvent être mises en oeuvre dans ce contexte. Pour les résultats mathématiques exposés ultérieurement dans cette introduction, on se concentrera essentiellement sur le cas le plus simple géométriquement où B a une direction et un module fixes. En notant (e_1, e_2, e_3) une base orthonormée fixée de $\mathbb{R}^3$, on peut écrire :

$$B = \frac{1}{\varepsilon} e_3. \tag{3.34}$$

Les articles gyrocinétiques "modernes" de la littérature physique traitent en général de champs magnétiques avec des géométriques générales [32], mais la question du couplage entre transport et champs électromagnétiques n'est pas vraiment traitée. Ce problème a été également récemment étudié dans une série de travaux par Bostan [18, 17], dans un cadre linéaire (sans couplage). L'auteur s'appuie sur des techniques de moyennes issues de la théorie ergodique "élémentaire".

Pour ce qui est des équations auto-consistantes, le cas des géométries plus complexes est évoqué dans l'article [72] Golse et Saint-Raymond.

Nous étudions trois types de régimes dans cette présentation, suivant la hiérarchie choisie entre les différents ordres de grandeur considérés. Ces études ont pour point commun de reposer sur des méthodes de type compacité par compensation.

3.2.1 Scaling centre-guide (ou dérive-cinétique)

Il s'agit du régime le plus "simple" que l'on puisse décrire. En effet sa seule caractéristique est l'intensité du champ magnétique ; on considère en outre que l'échelle d'observation spatiale L_0 est grande devant le rayon de Larmor r_L (défini en (1.20)) et le temps d'observation est de l'ordre de la fréquence plasma T_p, définie en (1.2). On ne décrit donc aucun effet quasineutre.

$$\frac{L_0}{r_L} \sim \frac{1}{\varepsilon}, \quad \frac{T_0}{T_p} \sim 1,$$
$$\frac{L_0}{\lambda_D} \sim 1.$$

On considère un champ magnétique de la forme $B(x) = B(x_1, x_2)e_3$. L'équation de Vlasov-Poisson adimensionnée s'écrit alors :

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + \left(E_\varepsilon + \frac{1}{\varepsilon} v^\perp B(x_1, x_2)\right).\nabla_v f_\varepsilon = 0, \\ E_\varepsilon = -\nabla_x V_\varepsilon, \\ -\Delta_x V_\varepsilon = \int f_\varepsilon dv - 1, \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon}. \end{cases} \tag{3.35}$$

Pour tout vecteur $A = (A_1, A_2, A_3)$, on note $A_\perp = (A_1, A_2, 0)$, $A^\perp = (A_2, -A_1, 0)$ et $A_\parallel = A_3$.

Remarquons que grâce au théorème d'Arsenev, pour tout $\varepsilon > 0$ fixé, nous pouvons effectivement considérer une solution faible globale à ce système d'équations.

L'étude de ce système ($B = 1$) est due à Frénod et Sonnendrücker [59], en utilisant des techniques de convergence à deux échelles, et Golse et Saint-Raymond [72] (pour tout $|B| > 0$) uniquement avec des techniques de convergence faible. L'approche adoptée ici est celle de Golse et Saint-Raymond. On a le résultat suivant :

Théorème 3.3 (Frénod-Sonnendrücker, Golse-Saint-Raymond)**.** *Soit $f_{0,\varepsilon} \in L^\infty_{x,v} \cap L^1_{x,v}$ uniformément en ε et d'énergie uniformément bornée. Alors tout famille (f_ε) de solutions faibles est à extraction près relativement compacte dans $L^\infty_{t,x,v}$-faible* et chacune de ses limites pour $\varepsilon \to 0$ est de la forme :*

$$f \equiv f(t, x, \sqrt{v_1^2 + v_2^2}, v_3),$$

où f est solution de l'équation :

$$\begin{cases} \partial_t f + v_3 \partial_{x_3} f + E_3 \partial_{v_3} f = 0, \\ E = -\nabla_x V, \\ -\Delta_x V = \int f - 1, \\ f(0, x, r, v_3) = \frac{1}{2\pi} \int_{\mathbb{S}^1} f_0(x, r\omega, v_3) d\omega. \end{cases} \tag{3.36}$$

On voit que le système limite est une équation de transport unidirectionnelle, dans la direction de B. Physiquement cela peut être interprété en disant que l'on a négligé les petites rotations des particules et qu'on les a assimilées à leur centre instantanné de rotation.

Eléments de preuve. On montre d'abord facilement que la famille (f_ε) est relativement compacte dans $L^\infty_{t,x,v}$-faible* grâce à la borne uniforme L^∞ obtenue par une application du lemme 2.1.

La deuxième étape consiste à voir qu'une limite faible f est forcément de la forme :

$$f \equiv f(t, x, \sqrt{v_1^2 + v_2^2}, v_3).$$

Pour cela, on multiplie l'équation de Vlasov par ε et on montre que

$$\varepsilon(\partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + E_\varepsilon.\nabla_v f_\varepsilon) \rightharpoonup 0$$

au sens des distributions puisque $(\partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + E_\varepsilon.\nabla_v f_\varepsilon)$ est borné dans des espaces fonctionnels convenables grâce aux estimations a priori uniformes données par les lemmes 2.1 and 2.2. Par conséquent on obtient :

$$v \wedge e_3.\nabla_v f = 0,$$

ce qui montre bien le résultat souhaité puisque l'opérateur $v \wedge e_3.\nabla_v$ engendre le groupe des rotations d'axe e_3 dans l'espace des vitesses.

Pour finir, on intègre l'équation de Vlasov par rapport à l'angle polaire de $(v_1, v_2) = r\omega$ avec $r = |(v_1, v_2)|$ et on passe à la limite. La difficulté concerne les termes non linéaires $f_\varepsilon E_\varepsilon$ et une nouvelle fois on utilise l'équation de Poisson afin de gagner de la compacité en temps et en espace pour le champ électrique.

De manière systématique, remarquons que la méthode mise en oeuvre pour traiter les termes non linéaires dans l'analyse asymptotique est proche des techniques utilisées pour étudier le problème de Cauchy (voir notamment le théorème 2.5). Ce principe général sera mis en oeuvre par la suite.

□

Ce théorème constitue une approximation à l'ordre dominant, il est possible de décrire les ordres suivants de l'approximation, comme l'a fait Saint-Raymond dans [139]. Toujours avec la géométrie la plus simple pour le champ magnétique, on récupère à l'ordre suivant la dérive électrique (introduite en (1.21)). Ce genre d'analyse multi-échelles ne peut pas reposer sur des arguments de compacité faible, mais doit passer par des estimations adéquates de stabilité (pour des normes avec régularité en x et décroissance en v).

3.2.2 Scaling temps long

On s'intéresse au comportement en temps long des particules dans le régime champ magnétique intense, avec une échelle d'observation en espace grande devant le rayon de Larmor ; cela correspond donc au scaling suivant :

$$\frac{L_0}{r_L} \sim \frac{1}{\varepsilon}, \quad \frac{T_0}{T_p} \sim \frac{1}{\varepsilon}.$$

Par ailleurs, on ne décrit aucun effet dû à la quasineutralité :

$$\frac{L_0}{\lambda_D} \sim 1.$$

Le mouvement étant plus rapide dans la direction parallèle, on se doit donc de restreindre l'étude au plan orthogonal. On s'attend alors à observer une dérive électrique macroscopique dans le plan perpendiculaire. En se restreignant effectivement au problème bidimensionnel, on obtient le système de Vlasov-Poisson suivant :

$$\begin{cases} \varepsilon\partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + \left(E_\varepsilon + \frac{1}{\varepsilon}v^\perp\right).\nabla_v f_\varepsilon = 0, \\ E_\varepsilon = -\nabla_x V_\varepsilon, \\ -\Delta_x V_\varepsilon = \int f_\varepsilon dv - 1, \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon}. \end{cases} \tag{3.37}$$

Une nouvelle fois, le théorème d'Arsenev permet de considérer des solutions globales faibles, pour tout $\varepsilon > 0$ fixé.

Si on suit l'étude heuristique menée en section 1.2.2, on s'attend à observer à la limite les effets de la dérive électrique, définie en (1.21). Le théorème prouvé par Golse et Saint-Raymond [72], puis Saint-Raymond [140] va dans ce sens :

Théorème 3.4 (Golse-Saint-Raymond). *Soit $(f_{0,\varepsilon}) \in L^1_{x,v} \cap L^\infty_{x,v}$ d'énergie uniformément bornée en ε et telle que :*

$$\lim_{\varepsilon \to 0} \varepsilon \|f_{0,\varepsilon}\|_{L^\infty_{x,v}} = 0.$$

Soit (f_ε) une famille de solutions faibles au sens d'Arsenev associées. On note $\rho_\varepsilon := \int f_\varepsilon dx$. Alors il existe une sous-suite toujours notée (ρ_ε) et $\rho \in L^\infty_t(\mathcal{M}_+(\mathbb{T}^2))$ tels que :

$$\rho_\varepsilon \rightharpoonup \rho \tag{3.38}$$

dans $L^\infty_t(\mathcal{M}_+(\mathbb{T}^2))$ faible- et ρ est solution de l'équation d'Euler incompressible bi-dimensionnelle, écrite en formulation vorticité :*

$$\begin{cases} \partial_t \rho + \nabla_x.(E^\perp \rho) = 0, \\ E^\perp = \nabla^\perp \Delta^{-1}(\rho - 1). \end{cases} \tag{3.39}$$

Remarque. *Les hypothèses sur $(f_{0,\varepsilon})$ sont très générales : elles autorisent en effet des concentrations en vitesse ou même en position quand ε tend vers 0.*

Eléments de preuve. Commençons par regarder le cas où $f_{0,\varepsilon}$ est uniformément borné dans $L^\infty_{x,v}$. Tout d'abord, il s'agit d'écrire les lois de conservations locales sur la charge et le courant et les combiner pour obtenir l'équation sur ρ_ε :

$$\begin{aligned} &\partial_t \rho_\varepsilon + \nabla_x.(E_\varepsilon^\perp \rho_\varepsilon) = \\ &=(\partial^2_{x_1} - \partial^2_{x_2}) \int v_1 v_2 f_\varepsilon dv + \partial_1\partial_2 \int (v_1^2 - v_2^2) f_\varepsilon dv + \varepsilon \partial_t \nabla_x. \int v^\perp f_\varepsilon dv. \end{aligned} \tag{3.40}$$

Ensuite comme pour le scaling précédent, on vérifie que toute limite faible-* dans $L^\infty_t(\mathcal{M})$ de f_ε est radiale en vitesse.

On peut ensuite essayer de passer à la limite dans (3.40). Pour le membre de gauche, la difficulté vient du terme non-linéaire $\rho_\varepsilon E_\varepsilon^\perp$. Cette fois, l'équation de Poisson permet bien de gagner de la compacité en espace sur le champ électrique, mais pas en temps, à cause du scaling en temps de l'équation de Vlasov. On peut néanmoins gagner de la compacité en temps sur ρ_ε en utilisant un lemme d'Aubin-Lions grâce à (3.40) elle-même.

Le problème majeur vient du fait que les moments d'ordre 2 sont uniquement bornés dans $L^\infty_t(L^1_{x,v})$ par l'inégalité d'énergie, et cela ne suffit pas pour passer brutalement à la limite dans les termes de droite de (3.40) et utiliser les symétries de f. Il y a donc a priori une mesure de défaut (positive) qui apparaît dans le membre de droite. L'article de Golse et Saint-Raymond exhibe même des situations où la mesure de défaut n'est pas nulle. Cependant, un argument dû à Saint-Raymond [140] qui consiste à montrer que l'on a un certain contrôle sur les grandes vitesses, permet de montrer que cette mesure de défaut n'apparaît en fait pas dans l'équation limite.

Pour finir, dans le cas ou l'on a seulement $\lim_{\varepsilon\to 0} \varepsilon \|f_{0,\varepsilon}\|_{L^\infty_{x,v}} = 0$, l'idée est d'utiliser un argument de compacité dû à Delort [46] pour l'équation d'Euler incompressible bi-dimensionnelle (pour prouver l'existence globale de solutions mesures signées dans H^{-1}). Ici, on s'appuie donc vraiment sur la structure de l'équation asymptotique pour justifier rigoureusement la limite.

□

3.2.3 Scaling gyrocinétique (ou rayon de Larmor fini)

Il s'agit ici de dériver une équation limite tridimensionnelle prenant en compte les effets de la dérive électrique. Pour cela, on peut proposer un scaling anisotrope en espace, en considérant que la longueur typique d'observation dans le plan perpendiculaire au champ magnétique est de l'ordre d'un rayon de Larmor (d'où le nom "rayon de Larmor fini"), alors que la longueur caractéristique dans la direction parallèle est très grand devant le rayon de Larmor. Ce scaling est très pertinent physiquement ; en effet certaines expériences indiquent que des phénomènes de type “turbulence” apparaissent sur des échelles de longueur de l'ordre de quelques rayons de Larmor [114], ce qui justifie de se placer à une telle échelle. Observons qu'il est nécessaire de considérer un tel scaling anisotrope, car sinon la dynamique “exploserait” dans la direction parallèle.

On distingue donc une longueur caractéristique $L_\perp$ dans le plan perpendiculaire et une longueur caractéristique $L_\parallel$ dans la direction du champ magnétique

$$\frac{L_\parallel}{r_L} \sim 1/\varepsilon, \quad \frac{L_\perp}{r_L} \sim 1.$$

Par ailleurs des effets quasineutres sont également pris en compte :

$$\frac{\lambda_D}{L_\parallel} \sim \sqrt{\varepsilon}.$$

Rappelons que l' échelle de temps caractéristique est choisie comme suit (hypothèse de champ magnétique intense) :

$$\frac{T_0}{T_c} \sim \frac{1}{\varepsilon}.$$

Ce scaling, couramment utilisé par les physiciens pour la dérivation de modèles réduits, a été introduit dans la littérature mathématique et étudié par Frénod et Sonnendrücker dans [60]. Citons également les travaux récents [16, 64, 55]. Dans le cas où l'on décrit le comportement des électrons avec des ions fixes, le système adimensionné s'écrit :

$$\begin{cases} \partial_t f_\varepsilon + \frac{1}{\varepsilon} v_\perp . \nabla_{x_\perp} f_\varepsilon + v_\parallel \partial_{x_\parallel} f_\varepsilon + \left(E_\varepsilon + \frac{1}{\varepsilon} v^\perp \right) . \nabla_v f_\varepsilon = 0, \\ E_\varepsilon = (-\nabla_{x_\perp} V_\varepsilon, -\varepsilon \partial_{x_\parallel} V_\varepsilon), \\ -\varepsilon^2 \partial^2_{x_\parallel} V_\varepsilon - \Delta_{x_\perp} V_\varepsilon = \int f_\varepsilon dv - 1, \\ f_{\varepsilon, |t=0} = f_{0,\varepsilon}. \end{cases} \tag{3.41}$$

Pour tout $\varepsilon > 0$, nous considérons une solution faible globale au sens d'Arsenev.

Remarquons qu'à l'échelle d'espace considérée, les fluctuations du champ électrique sur une longueur de quelques rayons de Larmor ne sont plus négligeables, si bien que l'approximation centre-guide, qui consistait à approximer une particule son centre-guide n'est plus valable. Néanmoins, chaque particule effectue une rotation rapide par rapport à l'échelle de temps d'observation. Cela justifie l'introduction de la gyromoyenne, qui n'est autre que la moyenne le long d'une rotation rapide pour une particule. La gyromoyenne d'une certaine fonction $A(t, x)$ est définie par la quantité

$$\int_0^{2\pi} A(t, x - |v| e^{i\theta + i\pi/2}) d\theta,$$

en notant pour simplifier $e^{i\theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$.

L'objectif ici, est a priori de justifier rigoureusement, quand $\varepsilon \to 0$, le système utilisé par les physiciens, notamment pour les simulations numériques (citons par exemple le code GYSELA développé au CEA, voir par exemple [78]) :

$$\begin{cases} \partial_t f + v_\parallel \partial_{x_\parallel} f + \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} E^\perp(t, x - |v| e^{i\theta + i\pi/2}) d\theta . \nabla_x f = 0, \\ E = -\nabla_x V, \\ -\Delta_x V = \displaystyle\int f(t, x - v^\perp, v) dv - \int f dv dx. \end{cases} \tag{3.42}$$

Cette équation fait intervenir la dérive électrique dans une version gyromoyennée. Il y a également d'autres termes de dérive, d'origine géométrique (telles que la dérive de courbure ou la dérive de gradient), que nous pouvons occulter ici, puisque nous avons considéré B uniforme et constant.

En se restreignant au cas bidimensionnel (c'est-à-dire en supposant que $f_{0,\varepsilon}$ ne dépend pas la variable parallèle), Frénod et Sonnendrücker [60] ont prouvé un théorème de convergence faible.

Théorème 3.5 (Frénod-Sonnendrücker)**.** *Soit $f_{0,\varepsilon} \in L^1_{x,v} \cap L^\infty_{x,v}$ d'énergie uniformément bornée en ε. Soit (f_ε) une solution faible au sense d'Arsenev, de donnée initiale $f_{0,\varepsilon}$.*

Alors à extraction près, f_ε converge faiblement vers une certaine fonction f. Il existe une fonction G telle que :

$$f = \int_0^{2\pi} G(t, x + \mathcal{R}(\tau)v, R(\tau)v) d\tau, \tag{3.43}$$

et G vérifie :

$$\left\{ \begin{array}{l} \partial_t G + \frac{1}{2\pi} \left(\int_0^{2\pi} \mathcal{R}(\tau)\mathcal{E}(t, \tau, x + \mathcal{R}(-\tau)v) d\tau \right) .\nabla_x G \\ + \frac{1}{2\pi} \left(\int_0^{2\pi} R(\tau)\mathcal{E}(t, \tau, x + \mathcal{R}(-\tau)v) d\tau \right) .\nabla_v G = 0, G_{|t=0} = f_0, \\ \mathcal{E} = -\nabla\Phi, \quad -\Delta\Phi = \int G(t, x + \mathcal{R}(\tau)v, R(\tau)v) dv, \end{array} \right. \tag{3.44}$$

en notant par R et $\mathcal{R}$ les opérateurs linéaires définis par :

$$R(\tau) = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix}, \mathcal{R}(\tau) = (R(\pi/2) - R(\pi/2 + \tau)) . \tag{3.45}$$

Eléments de preuve. La preuve consiste essentiellement à filtrer les oscillations avec le semi-groupe défini par les caractéristiques pénalisées :

$$\left\{ \begin{array}{l} \frac{dX}{dt} = V_\perp, \\ \frac{dV}{dt} = V^\perp. \end{array} \right. \tag{3.46}$$

Une fois le système filtré, on peut passer à la limite en utilisant des outils de convergence à deux échelles (dus à Nguetseng [132] et Allaire [2]). L'équation de Poisson permet de gagner de la compacité en espace sur le champ électrique, tandis que la compacité en temps est obtenue grâce à l'équation de Vlasov, en utilisant un lemme de type Aubin-Lions.

Nous reviendrons plus en détail sur cette preuve, lors que nous démontrerons le résultat analogue dans un cadre tridimensionnel (voir le paragraphe 3.3.2).

□

Il est possible de réécrire l'équation limite (3.44) de manière légèrement plus lisible, grâce au changement de variables :

$$\tilde{G}(t, \tau, x, v) := G(t, \tau, x - v^\perp, v). \tag{3.47}$$

L'équation satisfaite par $\tilde{G}$ est la suivante :

$$\left\{ \begin{array}{l} \partial_t \tilde{G} + \frac{1}{2\pi} \left(\int_0^{2\pi} \mathcal{E}^\perp(t, \tau, x - R(\pi/2 - \tau)v) d\tau \right) .\nabla_x \tilde{G} \\ + \frac{1}{2\pi} \left(\int_0^{2\pi} R(\tau)\mathcal{E}(t, \tau, x - R(\pi/2 - \tau)v) d\tau \right) .\nabla_v \tilde{G} = 0 \\ \mathcal{E} = -\nabla\Phi, \quad -\Delta\Phi = \int \tilde{G}(t, x - R(\pi/2 + \tau)v, R(\tau)v) dv - 1, \\ \tilde{G}_{|t=0} = f_0(x - v^\perp, v), \end{array} \right. \tag{3.48}$$

Nous pouvons alors remarquer le terme

$$\frac{1}{2\pi}\left(\int_0^{2\pi}\mathcal{E}^{\perp}(t,\tau,x-R(\pi/2-\tau)v)d\tau\right).\nabla_x \tag{3.49}$$

correspond à la dérive électrique gyromoyennée, telle qu'elle apparait dans l'équation (3.42). Cependant, dans l'équation de transport, il reste un autre terme, plus précisément :

$$\frac{1}{2\pi}\left(\int_0^{2\pi}R(\tau)\mathcal{E}(t,\tau,x-R(\pi/2-\tau)v)d\tau\right).\nabla_v\tilde{G},$$

dont l'interprétation physique est bien moins immédiate. Nous pouvons en fait même remarquer qu'il est négligé dans (3.42). Dans certains travaux, les auteurs se sont demandés si on pouvait le faire disparaître rigoureusement [16, 55]. Nous reviendrons sur ce problème dans le paragraphe suivant, où nous donnerons une nouvelle interprétation physique de ce terme.

Signalons pour finir que le système gyrocinétique étudié par Brenier dans [27] est en fait équivalent à celui-là. La méthode utilisée est très différente, elle s'appuie sur la méthode d'entropie relative et permet de montrer une convergence vers un système hydrodynamique correspondant, qui se trouve être l'équation d'Euler incompressible bidimensionnelle en formulation "vorticité".

3.2.4 Effets de la dérive de polarisation : l'approximation rayon de Larmor fini bidimensionnelle revisitée (Chapitre 5)

Dans le chapitre 5 de cette thèse, nous expliquons pourquoi l'équation asymptotique (3.44) ne décrit pas seulement la dérive électrique gyro-moyennée mais tient aussi compte d'effets d'ordre supérieur, notamment ceux dus à la dérive de polarisation. Comme nous l'avons vu dans l'étude heuristique des différentes dérives, la dérive de polarisation semble d'ordre plus élevé que la dérive électrique et ne devrait donc pas intervenir à la limite. En fait, ce n'est pas le cas, en raison des oscillations en temps pour le champ électrique (essentiellement dues au couplage entre l'équation de transport et l'équation de Poisson). Ces oscillations ont pour conséquence que $\varepsilon\partial_t E_\varepsilon$ est d'ordre $\mathcal{O}(1)$. Ainsi, la dérive de polarisation a le même ordre que la dérive électrique.

Notre approche donne un point de vue nouveau sur les récents efforts ([16, 64, 55]) pour obtenir l'équation limite (3.42) faisant uniquement intervenir la dérive électrique. L'article [16] repose sur des données initiales "bien préparées", ce qui permet de s'affranchir des oscillations en temps. L'hypothèse principale est de supposer que la suite de conditions initiale $(f_{0,\varepsilon})$ converge vers une fonction f_0 régulière dans le noyau de la perturbation singulière, c'est-à-dire :

$$v.\nabla_x f_0 + v^{\perp}.\nabla_v f_0 = 0. \tag{3.50}$$

Une telle condition est ni imposable, ni vérifiable en pratique. Cela signifie que physiquement, ce résultat n'est pas très satisfaisant.

Les articles [64, 55] concernent en revanche une situation "mal-préparée" (dans la mesure où on ne suppose pas de condition telle que (3.50). Cependant, elles ont pour point commun de supposer qu'il n'y a pas d'oscillation en temps pour le champ électrique pour pouvoir dériver l'équation de dérive, ce qui n'est pas vérifié en général.

Notre approche passe par un changement de variable très simple permettant de faire apparaître explicitement l'influence de la dérive de polarisation. L'idée est de faire le changement de référentiel

$$v \to v + \varepsilon E_\varepsilon^{\perp},$$

puisqu'on s'attend à constater une telle dérive en vitesse. On pose donc la nouvelle fonction de distribution :

$$f'_\varepsilon(t,x,v) = f_\varepsilon(t,x,v+\varepsilon E_\varepsilon^\perp).$$

L'équation satisfaite par f'_ε est alors :

$$\begin{aligned} \partial_t f'_\varepsilon + &\frac{v}{\varepsilon}.\nabla_x f'_\varepsilon + E_\varepsilon^\perp.\nabla_x f'_\varepsilon \\ &+ \underbrace{\Big(-\varepsilon\partial_t E_\varepsilon^\perp - \frac{v+\varepsilon E_\varepsilon^\perp}{\varepsilon}.\nabla_x(\varepsilon E_\varepsilon^\perp) \Big)}_{:=F_\varepsilon}.\nabla_v f'_\varepsilon + \frac{v^\perp}{\varepsilon}.\nabla_v f'_\varepsilon = 0. \end{aligned} \tag{3.51}$$

On appelle F_ε la "force de polarisation", et on montre que F_ε a une contribution à la limite qui se traduit par les termes "mystérieux" de l'équation de Frénod et Sonnendrücker :

$$\frac{1}{2\pi}\left(\int_0^{2\pi} R(\tau)\mathcal{E}(t,\tau,x-R(\tau-\pi/2)v)d\tau\right).\nabla_v \tilde{G},$$

qui ainsi prennent tout leur sens. Nos résultats semblent indiquer qu'il faudrait considérer l'équation (3.44) plutôt que (3.42) pour modéliser et simuler les plasmas magnétisés. Nous nous référons au Chapitre 5 pour les résultats précis.

3.2.5 Perspectives

Vers un scaling plus réaliste ? On peut remarquer que dans le scaling "rayon de Larmor fini" étudié, on a :

$$\frac{r_L}{\lambda_D} = \sqrt{\varepsilon},$$

ce qui signifie que l'on a considéré que le rayon de Larmor est beaucoup plus petit que la longueur de Debye. Il semble qu'il serait physiquement plus pertinent, pour les plasmas de tokamaks actuels ([158]), de considérer le scaling "rayon de Larmor fini" suivant :

$$\frac{L_\parallel}{r_L} \sim 1/\varepsilon, \quad \frac{L_\perp}{r_L} \sim 1,$$

$$\frac{\lambda_D}{L_\parallel} \sim \varepsilon^\alpha,$$

$$\frac{T_0}{T_c} \sim \frac{1}{\varepsilon},$$

avec $\alpha \geq 1$, auquel cas on a :

$$\frac{\lambda_D}{r_L} = \varepsilon^{\alpha-1}.$$

Pour le moment, l'étude d'un tel scaling semble hors de portée. Par exemple, pour le cas limite $\alpha = 0$, le système bidimensionnel mis à l'échelle devient :

$$\left\{ \begin{array}{l} \varepsilon\partial_t f_\varepsilon + v_\perp.\nabla_{x_\perp} f_\varepsilon + \left(E_\varepsilon + v^\perp\right).\nabla_v f_\varepsilon = 0, \\ E_\varepsilon = -\nabla_{x_\perp} V_\varepsilon, \\ -\Delta_{x_\perp} V_\varepsilon = \int f_\varepsilon dv - 1, \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon}. \end{array} \right. \tag{3.52}$$

Cela revient donc à étudier le comportement le temps long de l'équation de Vlasov-Poisson avec un champ magnétique extérieur. Il s'agirait ainsi de comprendre si l'amortissement Landau est possible en considérant un champ magnétique additionel, ce qui constitue une question ouverte.

Approximation rayon de Larmor fini et amortissement Landau "linéaire" Considérons à nouveau l'approximation rayon de Larmor fini en deux dimensions ; on peut se demander comment les résultats sont modifiés si l'on suppose le champ magnétique inhomogène en espace. Avant mise à échelle, cela correspond physiquement à considérer des champs magnétiques à "micro-structures", de la forme $B(x) = b(\frac{x}{\varepsilon})e_3$. Dans un travail en cours avec Clément Mouhot, nous étudions le comportement en temps long d'équations de Vlasov linéaires avec un champ gradient extérieur. Nous montrons, en faisant une hypothèse de complète intégrabilité, que si le champ vérifie une condition de non-dégénérescence (typiquement cela exclut les potentiels harmoniques $|x|^2$), alors il y a convergence faible quand le temps tend vers l'infini vers un équilibre stationnaire à l'équation de Vlasov (ce qui peut être vu comme un amortissement Landau linéaire). Notre objectif serait d'appliquer alors ce résultat à un problème d'homogénéisation et de montrer convergence de la fonction de distribution homogénéisée vers un certain équilibre stationnaire. On peut se demander si l'on peut trouver une certaine géométrie au champ B qui rentrerait dans le cadre de nos hypothèses. Cela permettrait de montrer qu'avec ce choix particulier, la solution à l'équation de transport converge faiblement vers un équilibre stationnaire, au lieu de converger vers une équation de transport par la dérive électrique. Cela pourrait s'interpréter par une propriété de meilleur confinement (car les effets de la "turbulence" électrique disparaitraient).

3.3 Analyse des limites quasineutre et champ magnétique intense combinées (Chapitres 2, 3 et 4)

Nous nous intéressons à présent à l'étude de régimes physiques où l'on combine quasineutralité et champ magnétique intenses.

A notre connaissance, le seul article traitant d'une telle problématique est celui de Golse et Saint-Raymond [74], qui traitait des limites combinées pour Vlasov-Poisson avec des ions lourds fixes (VP-E). Leur méthode repose sur des techniques d'entropie relative, dont nous nous inspirons dans l'étude du paragraphe suivant. Leurs résultats précis ne seront pas discutés ici par souci de briéveté.

3.3.1 Analyse des limites quasineutres et champ magnétique intense combinées pour l'équation de Vlasov-Poisson avec des électrons sans masse (Chapitre 2)

Dans le Chapitre 2, on s'intéresse au régime quasineutre pour (VP-I), en considérant de surcroit un champ magnétique intense, ce qui est particulièrement intéressant pour les plasmas de tokamak. Le champ magnétique est pris de direction et d'intensité fixe pour simplifier. Le système que l'on considère s'écrit :

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + \left(E_\varepsilon + \frac{v \wedge b}{\varepsilon}\right).\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\varepsilon^{2\alpha}\Delta_x V_\varepsilon = \int f_\varepsilon dv - \frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon} dx} \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon} \geq 0, \int f_{0,\varepsilon} dv dx = 1. \end{cases} \tag{3.53}$$

Le scaling étudié ici correspond à :

$$r_L/L_0 = \varepsilon, \quad \lambda_L/L_0 = \varepsilon^\alpha,$$

avec $\alpha \geq 0$ et où r_L est le rayon de Larmor, défini précédemment en (1.20). Ainsi, dans la limite $\varepsilon \to 0$, on ne décrit aucun phénomène de type "rayon de Larmor fini".

Pour simplifier, on considère que $b = e_3$, où (e_1, e_2, e_3) est une base orthonormée choisie d'avance. On peut alors écrire $v \wedge b = v^\perp$.

Physiquement, on note que la longueur de Debye est en général beaucoup plus petite que le rayon de Larmor [158], de sorte que les régimes $\alpha > 1$ semblent les plus pertinents physiquement.

Par un argument de filtrage des oscillations créées par le champ magnétique, on s'attend à converger vers une solution du système suivant :

$$\begin{cases} \partial_t \rho + \partial_{x_\parallel}(\rho w_\parallel) = 0, \\ \partial_t w + w_\parallel \partial_{x_\parallel} w = -\frac{\nabla_{x_\parallel}\rho}{\rho} - \nabla_{x_\parallel} H, \end{cases} \tag{3.54}$$

pour lequel il n'y a plus de dynamique dans la direction orthogonale. Il s'agit d'un système hyperbolique symétrisable et on dispose d'une théorie de Cauchy similaire à celle de (3.22).

En mettant en oeuvre la même méthode d'entropie relative, nous sommes capables de prouver la limite vers un système de type Euler isotherme, sans dynamique dans la direction perpendiculaire, et cela peu importe la valeur de $\alpha > 0$. En effet, en raison de l'absence d'ondes plasmas "violentes", évoquée dans le cas sans champ magnétique au paragraphe 3.1.5, tout se passe comme si les deux limites étaient découplées. Cela constitue une différence importante avec l'étude de Golse et Saint-Raymond portant sur le même système, mais avec l'équation de Poisson :

$$-\varepsilon^{2\alpha}\Delta_x V_\varepsilon = \int f_\varepsilon dv - 1,$$

pour laquelle il est primordial de considérer $\alpha = 1$ (en raison des interactions entre les ondes plasmas et les ondes créées par le champ magnétique).

On obtient pour commencer l'inégalité de stabilité suivante :

Proposition 3.1 (Chapitre 2). *Considérons l'entropie relative définie par :*

$$\mathcal{H}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v - \bar{u}_\varepsilon|^2 dvdx + \int (m_\varepsilon \log(m_\varepsilon/\bar{\rho}_\varepsilon) - m_\varepsilon + \bar{\rho}_\varepsilon)dx + \frac{\varepsilon^{2\alpha}}{2}\int |\nabla_x V_\varepsilon|^2 dx, \tag{3.55}$$

avec $m_\varepsilon = \frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon}dx}$. *Alors on a :*

$$\begin{aligned} \mathcal{H}_\varepsilon(t) \le \mathcal{H}_\varepsilon(0) + G_\varepsilon(t) + C\int_0^t \|\nabla_x \bar{u}_\varepsilon\|_{L^\infty}\mathcal{H}_\varepsilon(s)ds \\ + \int_0^t \int A_\varepsilon(\bar{\rho}_\varepsilon, \bar{u}_\varepsilon). \begin{pmatrix} -m_\varepsilon + \bar{\rho}_\varepsilon \\ J_\varepsilon - \rho_\varepsilon \bar{u}_\varepsilon \end{pmatrix} dxds, \end{aligned} \tag{3.56}$$

avec $A_\varepsilon(t,x)$ *l'opérateur d'accélération défini par :*

$$A_\varepsilon(\bar{\rho}_\varepsilon, \bar{u}_\varepsilon) = \begin{pmatrix} \partial_t \log\left(\frac{\bar{\rho}_\varepsilon}{d}\right) + \nabla_x.\bar{u}_\varepsilon + \bar{u}_\varepsilon.\nabla_x \log\left(\frac{\bar{\rho}_\varepsilon}{d}\right) - \nabla_x H.\bar{u}_\varepsilon \\ \partial_t \bar{u}_\varepsilon + \bar{u}_\varepsilon.\nabla_x \bar{u}_\varepsilon + \nabla_x \log\left(\frac{\bar{\rho}_\varepsilon}{d}\right) - \frac{\bar{u}_\varepsilon^\perp}{\varepsilon} \end{pmatrix}, \tag{3.57}$$

et $G_\varepsilon(t)$ *un terme de reste satisfaisant :*

$$\begin{aligned} G_\varepsilon(t) \le\ & C\varepsilon^\alpha \|\varepsilon^\alpha \nabla_x V_\varepsilon\|_{L^\infty_t(L^2_x)} \times \\ & \Big(\|\nabla_x(\bar{u}_\varepsilon.\nabla_x \log \bar{\rho}_\varepsilon/d)\|_{L^\infty_t(L^2_x)} + \|\log(\bar{\rho}_\varepsilon/d)\|_{W^{1,\infty}_t(H^1_x)}\Big). \end{aligned} \tag{3.58}$$

A cause des oscillations en temps engendrées par le champ magnétique, qui apparaissent dans l'opérateur d'accélération, il est impossible de prouver directement des résultats de convergence forte. On est amené à filtrer ces oscillations, ce qui crée des difficultés techniques supplémentaires. Cela se traduit par ailleurs par l'ajout d'une petite perturbation $\varepsilon y_\varepsilon$ qui disparait à la limite, mais qui permet précisément d'obtenir la convergence forte en temps. Nous ne définirons pas précisément y_ε dans cette introduction (voir le Chapitre 2 pour plus de détails), par souci de simplicité des énoncés. Le résultat prouvé est le suivant :

Théorème 3.6 (Chapitre 2)**.** *Soit* $(\log \rho_0/d, w_0)$ *une condition initiale dans* H^s_x *avec* $s > 5/2$*. Supposons que la suite de données initiales* $(f_{0,\varepsilon})$ *vérifie :*

$$\mathcal{H}_\varepsilon(0) \to 0. \tag{3.59}$$

Soit $(\log \frac{\rho}{d}, w)$ *l'unique solution forte locale de (3.54) avec* $\left(\log \frac{\rho_0}{d}, w_0\right)$ *comme condition initiale. Il existe* y_ε *bornée dans* $L^\infty_t([0,T], H^{s-1})$*, telle que si on définit* $\bar{\rho}_\varepsilon$ *et* $\bar{u}_\varepsilon$ *par la relation :*

$$\begin{pmatrix} \log \frac{\bar{\rho}_\varepsilon}{d} \\ \bar{u}_\varepsilon \end{pmatrix} = \begin{pmatrix} \log \frac{\rho}{d} \\ \mathcal{R}(-t/\varepsilon) w \end{pmatrix} + \varepsilon y_\varepsilon, \tag{3.60}$$

alors, localement uniformément en temps on a :

$$\mathcal{H}_\varepsilon(t) \to 0. \tag{3.61}$$

En particulier, on a que ρ_ε *converge faiblement vers* ρ *et* J_ε *converge faiblement vers* $\rho w_\parallel$ *dans* L^1*.*

La contribution de la “petite” perturbation $\varepsilon y_\varepsilon$ disparaît à la limite, mais permet précisément d'obtenir la convergence forte en temps.

3.3.2 Analyse de l'approximation rayon de Larmor fini tridimensionnelle (Chapitres 3 et 4)

Dans cette partie, nous présentons quelques développements sur l'approximation Rayon de Larmor Fini tridimensionnelle. A notre connaissance, il s'agit des seules contributions existantes dans la littérature mathématique.

Par rapport au système bidimensionnel, la difficulté dans le cadre tridimensionnel provient en particulier de la dégénérescence de la longueur de Debye “effective” dans la direction parallèle au champ magnétique. Il s'agit donc en quelque sorte d'une limite quasineutre anisotrope. Pour l'analyse asymptotique tridimensionnelle, nous distinguons les deux descriptions suivantes :

- ions avec électrons sans masse, ce qui correspond au système (VP-I) (Chapitre 3),
- ou électrons avec ions lourds, ce qui correspond au système (VP-E) (Chapitre 4).

Nous mettrons alors en évidence que des phénomènes très différents peuvent se produire. Mathématiquement, cela se traduit par la mise en oeuvre d'outils adaptés à chaque description.

Remarquons que dans le cadre bidimensionnel, il est en fait inutile de faire une telle distinction pour l'étude de la limite.

L'approximation rayon de Larmor fini tridimensionnelle : cas des ions avec électrons sans masse (Chapitre 3) Rappelons que dans le régime "rayon de Larmor fini", l'équation de Vlasov adimensionnée s'écrit :

$$\begin{cases} \partial_t f_\varepsilon + \frac{1}{\varepsilon} v_\perp . \nabla_{x_\perp} f_\varepsilon + v_\| \partial_{x_\|} f_\varepsilon + \left(E_\varepsilon + \frac{1}{\varepsilon} v^\perp\right) . \nabla_v f_\varepsilon = 0, \\ E_\varepsilon = (-\nabla_{x_\perp} V_\varepsilon, -\varepsilon \partial_{x_\|} V_\varepsilon), \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon}, \end{cases} \tag{3.62}$$

ce système étant couplé avec une équation de Poisson, qui est différente si l'on décrit les ions ou les électrons.

La situation physique où nous sommes en mesure de mener l'analyse de la limite avec des hypothèses faibles sur la donnée initiale correspond à la description cinétique d'un gaz d'ions avec des électrons sans masse, ayant atteint leur équilibre thermodynamique local. Pour simplifier on suppose que leur densité locale suit une distribution de Maxwell Boltzmann linéarisée. Remarquons que cette hypothèse est également faite par les physiciens pour établir leurs modèles (voir notamment [78]). Ainsi, l'équation de Poisson que nous considérons est la suivante :

$$V_\varepsilon - \varepsilon^2 \partial_{x_\|}^2 V_\varepsilon - \Delta_{x_\perp} V_\varepsilon = \int f_\varepsilon dv - \int f_0 dv dx. \tag{3.63}$$

Heuristiquement, le terme V_ε a un certain effet stabilisant sur les ondes plasmas, analogue à ce qui se passe pour la limite quasineutre seule, comme on l'a vu au paragraphe 3.1.5. En revanche, contrairement à la limite quasineutre seule, des ondes sont crées par la partie pénalisée de l'opérateur de transport libre, ce qui entraine que le champ électrique se comporte grossièrement comme $e^{\pm it/\varepsilon}$.

Malgré cela, comme dans le cas bidimensionnel, il est possible de passer à la limite pour des données initiales assez générales, avec une méthode de compacité par compensation. En particulier, nous n'avons donc pas besoin de nous restreindre à des profils en vitesse particuliers, contrairement à la limite quasineutre "classique".

Nous nous donnons ainsi une suite de données initiales $(f_{\varepsilon,0})_{\varepsilon>0}$ satisfaisant les conditions suivantes :

(H1) Pour tout $\varepsilon > 0$, $f_{\varepsilon,0} \geq 0$ et $\int f_{\varepsilon,0} dx dv = 1$.

(H2) L'énergie (rescalée) est uniformément bornée par rapport à ε :

$$\left(\int f_{\varepsilon,0} |v|^2 dv dx + \varepsilon \int V_{\varepsilon,0}^2 dx + \varepsilon \int |\nabla_{x_\perp} V_{\varepsilon,0}|^2 dx + \varepsilon^3 \int |\nabla_{x_\|} V_{\varepsilon,0}|^2 dx \right) \leq C.$$

(H3) $(f_{\varepsilon,0})_{\varepsilon>0}$ est uniformément borné par rapport à ε dans $L^1_{x,v} \cap L^p_{x,v}$ (pour un $p > 3$) et pour tout .

Sous ces hypothèses, le théorème démontré dans le C est l'analogue tridimensionnel de celui de Frénod et Sonnendrücker.

Théorème 3.7 (Chapitre 3). *Pour tout ε, soit $(f_\varepsilon, E_\varepsilon)$ dans $L^\infty_t(L^1_{x,v} \cap L^p_{x,v}) \times L^\infty_t(L^2_x)$ une solution faible au sens d'Arsenev de (3.62) avec l'équation de Poisson (3.63). Alors à extraction près, on a les convergences quand ε tend vers* 0 *:*

$$\begin{array}{rll} f_{\varepsilon,0} & \textit{converge faiblement-* vers} & f_0 \textit{ dans } L^p_{x,v} \\ f_\varepsilon & \textit{converge à 2 échelles vers} & F \textit{ dans } L^\infty_t(L^\infty_{2\pi,\tau}(L^1_{x,v} \cap L^p_{x,v})) \\ E_\varepsilon & \textit{converge à 2 échelles vers} & \mathcal{E} \textit{ dans } L^\infty_t(L^\infty_{2\pi,\tau}(L^{3/2}_{x_\|}(W^{1,\frac{3}{2}}_{x_\perp}))). \end{array}$$

De plus, il existe une fonction $G \in L_t^\infty(L^1_{x,v} \cap L^p_{x,v})$ *telle que :*

$$F(t,\tau,x,v) = G(t, x + \mathcal{R}(\tau)v, R(\tau)v), \tag{3.64}$$

et $(G, \mathcal{E})$ *est solution de :*

$$\begin{aligned}
\partial_t G + v_\parallel \partial_{x_\parallel} G + \frac{1}{2\pi}\left(\int_0^{2\pi} \mathcal{R}(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_x G& \\
+\frac{1}{2\pi}\left(\int_0^{2\pi} R(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_v G = 0&,
\end{aligned}$$

$$G_{|t=0} = f_0,$$

$$\mathcal{E} = (-\nabla_\perp V, 0), \; V - \Delta_\perp V = \int G(t, x+\mathcal{R}(\tau)v, R(\tau)v)dv - \int f_0 dv dx,$$

en notant R *and* $\mathcal{R}$ *les opérateurs de rotation définis par :*

$$R(\tau) = \begin{bmatrix} \cos\tau & -\sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathcal{R}(\tau) = \left(R(-\pi/2) - R(-\pi/2+\tau)\right).$$

Remarque. *i. Observons qu'avec un tel scaling, il y a asymptotiquement une dynamique triviale dans la direction parallèle (il n'y a plus de force parallèle).*

ii. Si f_0 *est régulière (essentiellement* $W^{1,\infty}_{x,v}$ *avec une bonne décroissance en vitesse), alors il est possible de prouver l'unicité pour le système asymptotique. Cela signifie que si toute la suite* $f_{0,\varepsilon}$ *converge faiblement vers* f_0*, alors c'est le cas aussi pour* f_ε*, sans extraction.*

La preuve de ce théorème s'appuie sur les arguments suivants :

Etape 1 : Filtrage de l'équation de transport

La première étape consiste à filtrer les oscillations en temps dues au champ magnétique. Le filtrage est identique à celui de Frénod et Sonnendrücker.

On pose :

$$g_\varepsilon(t,x,v) = f_\varepsilon(t, x+\mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v). \tag{3.65}$$

L'équation satisfaite par g_ε est la suivante :

$$\begin{aligned}
\partial_t g_\varepsilon + v_\parallel.\nabla_x g_\varepsilon + \mathcal{R}(t/\varepsilon)E_\varepsilon(t, x+\mathcal{R}(-t/\varepsilon)v).\nabla_x g_\varepsilon& \\
+R(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v).\nabla_v g_\varepsilon = 0&.
\end{aligned} \tag{3.66}$$

Insistons sur le fait qu'un tel filtrage est nécessaire, ne serait ce que pour obtenir de la compacité en temps. En effet, en raison du scaling de l'équation de transport, on peut seulement montrer que $\varepsilon\partial_t E_\varepsilon \in L^\infty_t L^p_x$ (pour un certain $p > 1$), uniformément en ε. On espère en revanche obtenir de la compacité en temps via l'équation de transport filtrée.

Etape 2 : Gain de régularité perpendiculaire sur le champ électrique

On montre que le champ électrique gagne uniformément un cran de régularité dans la variable perpendiculaire, et aucune dans la variable parallèle ou en temps. Il est à noter que le scaling est utilisé de manière cruciale, notamment le fait que la composante parallèle soit donnée par $-\varepsilon\partial_{x_\parallel}V_\varepsilon$.

Etape 3 : Gain de régularité temporelle et parallèle sur les moments en vitesse

Notre approche finalement par l'utilisation d'un lemme de moyenne, en l'occurence le Lemme 2.4. Il s'agit en quelque sorte d'un outil adapté à la structure anisotrope de l'équation. On peut en effet interpréter l'équation filtrée comme une équation de transport dans la direction parallèle, le point crucial étant qu'elle ne fait pas intervenir d'autres dérivées en $x_\parallel$ (alors qu'elle fait intervenir une dérivée en $x_\perp$).

A l'aide de ce lemme, on peut montrer que l'on peut gagner de la régularité en temps et par rapport à la variable parallèle pour les moments. Pour pouvoir l'appliquer, le point crucial à vérifier est l'absence de dérivée en $x_\parallel$ (si ce n'est l'opérateur de transport). C'est bien le cas, et cela se justifie par le fait que la dérive électrique n'intervient que dans le plan perpendiculaire. Avec ces estimations, qui permettent de gagner de la compacité forte en toutes les variables, il est aisé de passer à la limite double-échelle (en utilisant les outils de [2]).

L'approximation rayon de Larmor fini tridimensionnelle : cas des électrons avec ions fixes (Chapitre 4) Dans le cas où les équations décrivent un gaz d'électrons dans un fond neutralisant d'ions fixes, c'est-à-dire pour un équation de Poisson de la forme :

$$-\varepsilon^2\partial_{x_\parallel}^2 V_\varepsilon - \Delta_{x_\perp} V_\varepsilon = \int f_\varepsilon dv - 1, \tag{3.67}$$

la limite formelle est fausse en général. Une première façon de le voir est de considérer des conditions initiales ne dépendant pas de la variable perpendiculaire. Le champ magnétique disparait alors et on obtient alors exactement le scaling quasineutre en dimension 1, déjà discutée précédemment.

On peut néanmoins essayer de considérer des données vérifiant le critère de monotonie de Penrose, par exemple monocinétiques, comme dans la limite quasineutre telle qu'elle a été traité par Brenier. Cela permet d'éviter les instabilités cinétiques de type double-bosse et il peut sembler raisonnable penser qu'on peut ainsi passer rigoureusement à la limite.

En réalité, de manière assez remarquable, la limite formelle reste malgré cela toujours fausse. Cela est dû à des instabilités causées par le couplage par les variables perpendiculaires. En quelque sorte cela est donc dû à l'anisotropie du système. Dans le Chapitre 4, notre but est d'expliquer et d'apporter un éclairage sur ce phénomène.

Pour cela on considère la plus simple donnée vérifiant la condition de monotonie de Penrose, le Dirac en vitesse. On dérive formellement un système fluide, après changement de variable et en considérant directement des données monocinétiques dans (3.62)-(3.67) :

$$g_\varepsilon(t,x,v) = \rho_\varepsilon(t,x)\delta_{v_\parallel = v_{\parallel,\varepsilon}(t,x)}\delta_{v_\perp=0}.$$

ce[13] qui nous mène au système :

$$\begin{cases} \partial_t\rho_\varepsilon + \nabla_\perp.(E_\varepsilon^\perp \rho_\varepsilon) + \partial_\parallel(v_{\parallel,\varepsilon}\rho_\varepsilon) = 0 \\ \partial_t v_{\parallel,\varepsilon} + \nabla_\perp.(E_\varepsilon^\perp v_{\parallel,\varepsilon}) + v_{\parallel,\varepsilon}\partial_\parallel(v_{\parallel,\varepsilon}) = -\varepsilon\partial_\parallel\phi_\varepsilon(t,x) - \partial_\parallel V_\varepsilon(t,x_\parallel) \\ E_\varepsilon^\perp = -\nabla^\perp\phi_\varepsilon \\ -\varepsilon^2\partial_\parallel^2\phi_\varepsilon - \Delta_\perp\phi_\varepsilon = \rho_\varepsilon - \int \rho_\varepsilon dx_\perp \\ -\varepsilon\partial_\parallel^2 V_\varepsilon = \int\rho_\varepsilon dx_\perp - 1. \end{cases} \tag{3.68}$$

[13]Remarquons que l'on a choisit une vitesse nulle pour la dynamique perpendiculaire : cela correspond à une hypothèse d'isotropie, ce qui est physiquement raisonnable dans les plasmas magnétisés.

et on se propose d'étudier sa limite "quasineutre anisotrope".

Précisons le sens physique des termes de cette équation :

- $\rho_\varepsilon(t, x_\perp, x_\parallel) : \mathbb{R}^+ \times \mathbb{T}^3 \to \mathbb{R}_*^+$ est une densité de charge,
- $v_{\parallel,\varepsilon}(t, x_\perp, x_\parallel) : \mathbb{R}^+ \times \mathbb{T}^3 \to \mathbb{R}$ est un courant "parallèle".
- $\phi_\varepsilon(t, x_\parallel)$ and $V_\varepsilon(t, x)$ sont des potentiels électriques.

La limite formelle est obtenue en prenant $\varepsilon = 0$.

$$\begin{cases} \partial_t \rho + \nabla_\perp.(E^\perp \rho) + \partial_\parallel (v_\parallel \rho) = 0 \\ \partial_t v_\parallel + \nabla_\perp.(E^\perp v_\parallel) + v_\parallel \partial_\parallel (v_\parallel) = -\partial_\parallel p(t, x_\parallel) \\ E^\perp = \nabla^\perp \Delta_\perp^{-1} \left(\rho - \int \rho dx_\perp\right) \\ \int \rho dx_\perp = 1. \end{cases} \tag{3.69}$$

La dynamique dans le plan perpendiculaire est bien donnée par une advection par la dérive électrique $E^\perp$. Celle dans la direction parallèle est non triviale, contrairement au cas du chapitre précédent : elle est dominée par une force de pression provenant d'une contrainte d'incompressibilité en moyenne dans le plan perpendiculaire :

$$\int \rho dx_\perp = 1.$$

Ce système est dans un sens strict compressible, mais également en quelque sorte "incompressible en moyenne", à cause de cette contrainte. Celle-ci permet par ailleurs de calculer la pression p, en prenant la moyenne en $x_\perp$ sur les équations de transport. Un tel système semble nouveau dans la littérature.

Bien qu'on ait considéré des données monocinétiques pour obtenir la dérivation, le système (3.68) apparaît comme un système de type Euler multi-fluides (voir par exemple l'article de [25] pour leur étude). En effet, on observe que ce système peut être interprété comme un système infini de fluides couplés par la variable $x_\perp$, avec par ailleurs une dynamique dans cette variable. Pour cette raison, cette limite n'est pas vraie pour des données générales à régularité Sobolev, principalement à cause des instabilités de type double bosse qui sévissent dans les sytèmes d'Euler multi-fluides (étudiés dans [38]), qui prennent le dessus dans la limite quasi-neutre. Le système limite est d'ailleurs également mal posé dans les espaces de Sobolev.

On montre néanmoins pour des données analytiques l'existence locale (mais uniformément en ε) de (3.68) de solutions analytiques, et leur convergence (toujours localement en temps) vers des solutions du système limite.

Introduisons les espaces analytiques considérés :

Définition 6. *Soit $\delta > 1$. Nous définissons B_δ l'espace des fonctions réelles ϕ définies sur $\mathbb{T}^3$ telles que :*

$$|\phi|_\delta = \sum_{k \in \mathbb{Z}^3} |\mathcal{F}\phi(k)| \delta^{|k|} < +\infty, \tag{3.70}$$

où $\mathcal{F}\phi(k)$ est le k-ième coefficient de Fourier de ϕ défini par :

$$\mathcal{F}\phi(k) = \int_{\mathbb{T}^3} \phi(x) e^{-i2\pi k.x} dx.$$

La preuve se base sur l'approche analytique de Grenier pour la limite quasineutre, que l'on adapte ici pour traiter une dynamique en la variable de couplage, qui ici n'est autre que la variable $x_\perp$.

Une étude simple des symboles des équations de Poisson qui sont en jeu donne des bornes adéquates. Le résultat prouvé est résumé dans le théorème suivant :

Théorème 3.8 (Chapitre 4). *i. Soit $\delta_0 > 1$. Soit $\rho_\varepsilon(0)$ et $v_\varepsilon(0)$ deux familles bornées de B_{δ_0} telles que $\int \rho_\varepsilon(0)dx = 1$ et :*

$$\left|\int \rho_\varepsilon(0)dx_\perp - 1\right|_{\delta_0} \leq C\sqrt{\varepsilon}, \tag{3.71}$$

alors il existe $\eta > 0$ tel que pour tout $\delta_1 \in]1, \delta_0[$, pour tout $\varepsilon > 0$, il existe une unique solution forte $(\rho_\varepsilon, v_\varepsilon)$ à (3.68), uniformément bornée dans $\mathcal{C}([0, \eta(\delta_0 - \delta_1)[, B_{\delta_1})$ avec pour conditions initiales $(\rho_\varepsilon(0), v_\varepsilon(0))$. En outre, $\sqrt{\varepsilon}\partial_\| V_\varepsilon$ est uniformément borné $\mathcal{C}([0, \eta(\delta_0 - \delta_1)[, B_{\delta_1})$.

ii. Soit $(\rho_\varepsilon, v_\varepsilon)$ des solutions au système (3.68) pour $0 \leq t \leq T$ et satisfaisant pour un certain indice $s > 7/2$ l'estimation suivante :

$$(H) : \sup_{t \leq T, \varepsilon} \left(\|\rho_\varepsilon\|_{H^s_{x_\perp, x_\|}} + \|v_\varepsilon\|_{H^s_{x_\perp, x_\|}} + \|\sqrt{\varepsilon}\partial_{x_\|} V_\varepsilon\|_{H^s_{x_\|}}\right) < +\infty. \tag{3.72}$$

Alors on a les convergences fortes :

$$\rho_\varepsilon \to \rho,$$

$$v_\varepsilon + \frac{1}{i}(E_+ e^{it/\sqrt{\varepsilon}} - E_- e^{-it/\sqrt{\varepsilon}}) \to v,$$

fortement, respectivement dans $\mathcal{C}([0, T], H^{s'}_{x_\perp, x_\|})$ et $\mathcal{C}([0, T], H^{s'-1}_{x_\perp, x_\|})$ pour tout $s' < s$, et

$$-\sqrt{\varepsilon}\partial_{x_\|} V_\varepsilon - (E_+ e^{it/\sqrt{\varepsilon}} + E_- e^{-it/\sqrt{\varepsilon}}) \to 0,$$

fortement dans $\mathcal{C}([0, T], H^{s'}_{x_\|})$ pour tout $s' < s - 1$, et où (ρ, v) est solution du système asymptotique (3.69) sur $[0, T]$ avec les conditions initiales :

$$\rho(0) = \lim_{\varepsilon \to 0} \rho_\varepsilon(0),$$

$$v(0) = \lim_{\varepsilon \to 0} \left(v_\varepsilon(0) - \int \rho_\varepsilon v_\varepsilon dx_\perp(0)\right)$$

et $E_+(t, x_\|), E_-(t, x_\|)$ sont des correcteurs gradients qui satisfont les équations de transport :

$$\partial_t E_\pm + \left(\int \rho v dx_\perp\right) \partial_{x_\|} E_\pm = 0,$$

avec pour données initiales :

$$E_+(0) = \lim_{\varepsilon \to 0} \frac{1}{2}\left(-\sqrt{\varepsilon}\partial_{x_\|} V_\varepsilon(0) + i\int \rho_\varepsilon v_\varepsilon dx_\perp(0)\right),$$

$$E_-(0) = \lim_{\varepsilon \to 0} \frac{1}{2}\left(-\sqrt{\varepsilon}\partial_{x_\|} V_\varepsilon(0) - i\int \rho_\varepsilon v_\varepsilon dx_\perp(0)\right).$$

Idée de la preuve. Une étape cruciale est l'obtention d'une équation des "ondes", analogue à (3.13), permettant de décrire les oscillations sur la partie "singulière" du champ électrique, qui correspond à $E_{\varepsilon,\|} := \partial_\| V_\varepsilon$.

$$\varepsilon \partial_t^2 \partial_{x_\|} E_{\varepsilon,\|} + \partial_{x_\|} E_{\varepsilon,\|} = \partial_{x_\|}^2 \int \rho_\varepsilon v_\varepsilon^2 dx_\perp - \varepsilon \partial_{x_\|} [E_{\varepsilon,\|} \partial_{x_\|} E_{\varepsilon,\|}] + \partial_{x_\|} \int \rho_\varepsilon (\varepsilon \partial_{x_\|} \phi_\varepsilon) dx_\perp. \quad (3.73)$$

On remarque qu'excepté le dernier terme, on retrouve la même équation des ondes que (3.13). Cela entraine que le champ électrique $E_{\varepsilon,\|}$ se comporte grossièrement en $1/\sqrt{\varepsilon} e^{\pm it/\sqrt{\varepsilon}}$.

Ensuite on met en place un schéma d'approximation de type point fixe dans les espaces analytiques, ce qui permet d'obtenir l'existence et l'unicité de solutions en un temps petit mais uniforme en ε. Cette construction permet par ailleurs d'obtenir des estimations uniformes en ε, permettant de passer à la limite $\varepsilon \to 0$. Il nous faut néanmoins filtrer les oscillations plasmas, ce qui se traduit par l'apparition de correcteurs à la limite. □

Par ailleurs la même méthode permet de démontrer l'existence locale et l'unicité de solutions à (3.69) :

Proposition 3.2 (Chapitre 4)**.** *Soit $\delta_0 > \delta_1 > 1$. Pour toute donnée initiale $\rho(0), v(0) \in B_{\delta_0}$ satisfaisant*

$$\rho(0) \geq 0, \quad (3.74)$$

$$\int \rho(0) dx_\perp = 1 \quad (3.75)$$

et

$$\partial_\| \int \rho(0) v(0) dx_\perp = 0, \quad (3.76)$$

il existe $\eta > 0$ dépendant de δ_0 et des conditions initiales seulement tel qu'il existe une unique solution locale $(\rho, v_\|, p)$ au système (3.69) avec $\rho, v \in \mathcal{C}([0, \eta(\delta_0 - \delta_1)[, B_\delta)$ pour tout $\delta < \delta_1$.

Ce chapitre se finit avec une discussion sur nos résultats. En se basant sur des résultats de Brenier sur les équations d'Euler multi-fluides [25], on montre que nos résultats sur le système asymptotique (3.69) sont essentiellement optimaux : on ne peut pas faire mieux qu'analytique et local en temps pour des solutions fortes. Par ailleurs, on se propose de montrer pourquoi une méthode d'entropie relative ne permet pas de conclure, à cause de l'effet des instabilité à deux bosses.

3.3.3 Perspectives

Il nous semble qu'il reste beaucoup à faire concernant l'approximation rayon de Larmor fini en trois dimensions. Notamment, on peut se demander pour le cas avec des ions lourds s'il est possible de baisser la régularité à Gevrey au lieu de l'analytique. D'autre part, il serait également possible d'étudier directement le système cinétique de départ avec un formalisme analytique, comme cela est mené dans [130].

Finalement, comme précédemment, la question de l'analyse de couches limites se pose toujours.

4 Confinement et contrôle d'un plasma de tokamak

Pour finir, nous nous intéressons au problème du confinement d'un plasma de tokamak, que nous attaquons sous deux angles différents. Dans un premier temps, nous étudions une instabilité due à la dérive électrique et à la "courbure" du champ magnétique, qui permet de comprendre pourquoi considérer un champ magnétique "s'enroulant" autour du tokamak permet en principe d'avoir un meilleur confinement. De manière assez inattendue, notre modèle nous permet également de donner une explication au mode de haut confinement (H-Mode) dans un tokamak. Dans un second temps, on étudie le problème du point de vue de la théorie du contrôle des équations aux dérivées partielles. Le principe ici est que l'on agit sur le plasma via une source placée dans le tokamak et la question que l'on se pose est la suivante : est-il possible de bien choisir la source pour décider de l'état du plasma à un temps donné d'avance ?

4.1 Analyse de la stabilité et de l'instabilité du confinement d'un plasma de tokamak (Chapitre 6)

4.1.1 Le modèle

Dans un premier temps, nous proposons un modèle à 2 fluides miscibles (correspondant aux parties chaudes et froides d'un plasma) afin de modéliser un plasma au bord d'un tokamak. On étudiera la stabilité de deux solutions μ^{good} et μ^{bad} stationnaires modélisant des profils de température dans deux régions du tokamak. Ces régions sont représentées dans la figure ci-dessous :

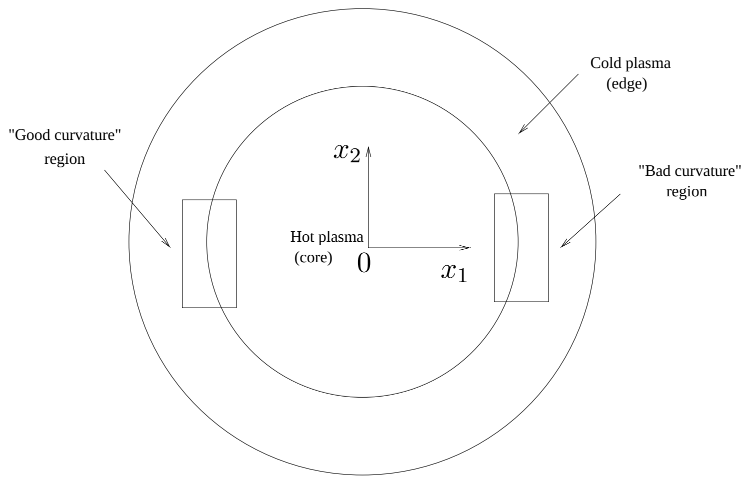


FIGURE 1.7 – Une "tranche" de tokamak

Nous avons ici distingué le plasma de "coeur" et le plasma de "bord", la différence qui nous intéresse étant que le plasma de coeur est bien plus chaud que le plasma de bord.

La solution μ^{good} modélise la région dite à bonne courbure ($x_1 < 0$), tandis que μ^{bad} la région dite à mauvais courbure ($x_1 > 0$). Cette terminologie sera justifiée dans l'heuristique du paragraphe suivant.

Pour ce problème, la dérive électrique $E^{\perp}$ (définie en (1.21)) ainsi que la dérive de gradient (définie en (1.22)) jouent un role crucial dans la dynamique des particules. Le modèle que nous étudions est le suivant :

$$\begin{cases} \partial_t \rho^+ - T^+ \partial_{x_2} \rho^+ + E^\perp . \nabla_x \rho^+ = 0, \\ \partial_t \rho^- - T^- \partial_{x_2} \rho^+ + E^\perp . \nabla_x \rho^- = 0, \\ E = -\nabla_x V, \\ -\Delta_x V = \rho^+ + \rho^- - 1, \\ V = 0 \text{ on } x_1 = 0, L, \\ (\rho^+, \rho^-)_{|t=0} = (\rho_0^+, \rho_0^-) \text{ with } \int \rho_0^+ + \rho_0^- = 1, \end{cases} \tag{4.1}$$

pour $t \geq 0, x \in [0, L] \times \mathbb{R}/L\mathbb{Z}$ où L est la taille de la boite. La quantité ρ^+ (resp. ρ^-) représente la densité de particules "chaudes", de température T^+ (resp. "froides", de température T^-). Par définition, on considère $T^+ > T^-$.

L'opérateur $E^\perp . \nabla_x$ correspond à la dérive électrique, tandis que $-T^\pm \partial_{x_2}$ correspond à la dérive de gradient. Remarquons que le champ de vecteurs $-T^\pm e_2$ a une direction fixe.

Concernant la dérivation de (4.1), nous sommes partis de l'équation de Vlasov-Poisson dans le régime rayon de Larmor fini suivante :

$$\begin{cases} \partial_t f_\varepsilon + \frac{v}{\varepsilon} . \nabla_x f_\varepsilon + (E_\varepsilon + \frac{v^\perp}{\varepsilon} - x_1 v^\perp) . \nabla_v f_\varepsilon = 0, \\ E_\varepsilon = -\nabla_x V_\varepsilon, \\ -\Delta_x V_\varepsilon = \displaystyle\int f_\varepsilon dv - 1. \end{cases} \tag{4.2}$$

La différence avec le système (3.41) est que l'on considère ici un champ magnétique avec une direction fixe mais d'intensité inhomogène et variant lentement (on considère donc un peu plus de géométrie) :

$$B = \frac{1}{1 + \varepsilon x_1} \sim 1 - \varepsilon x_1.$$

Cela correspond au champ créé par une bobine verticale passant par le centre du tokamak (ce qui est utilisé en pratique, on se réfère par exemple à [78]).

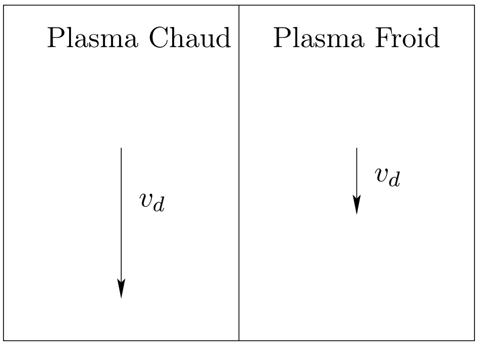
La limite est étudiée en utilisant les mêmes outils que dans [60], la seule différence (3.44) étant de nouveaux termes de transport d'ordre 1, qui modifient la dynamique du système limite (cela correspond à la dérive de gradient). On obtient un système cinétique, dynamiquement proche du système fluide annoncé.

$$\begin{cases} \partial_t G + \left(\frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}(\tau) \mathcal{E}(t, \tau, x + \mathcal{R}(-\tau) v) d\tau + \begin{pmatrix} -v_1(v_2 - x_1) \\ v_2(x_1 - v_2) - \frac{1}{2}(v_1^2 + v_2^2) \end{pmatrix} \right) . \nabla_x G \\ + \left(\frac{1}{2\pi} \int_0^{2\pi} R(\tau) \mathcal{E}(t, \tau, x + \mathcal{R}(-\tau) v) d\tau + \begin{pmatrix} v_2(-x_1 + v_2) \\ -v_1(-x_1 + v_2) \end{pmatrix} \right) . \nabla_v G = 0, \\ G_{|t=0} = f_0, \\ E = -\nabla_x V, \\ -\Delta V = \displaystyle\int G(t, x + \mathcal{R}(\tau) v, R(\tau) v) dv - 1, \end{cases} \tag{4.3}$$

où R et $\mathcal{R}$ sont les opérateurs de rotation déjà définis (3.45). Ce système est à notre sens celui qu'il faudrait considérer. Cependant il paraît trop complexe pour une étude dynamique qualitative. Au prix de plusieurs approximations, (en particulier, on néglige les opérateurs de gyromoyenne et on considère des données "bi-cinétiques", pour faire apparaitre les deux phases), on obtient le système (4.1) (voir le Chapitre 6 pour plus de détails).

4.1.2 Heuristique

Le problème de stabilité/instabilité que l'on étudie peut facilement se comprendre par l'étude heuristique suivante effectuée dans la zone à mauvaise courbure, représentée dans la figure ci-dessous.



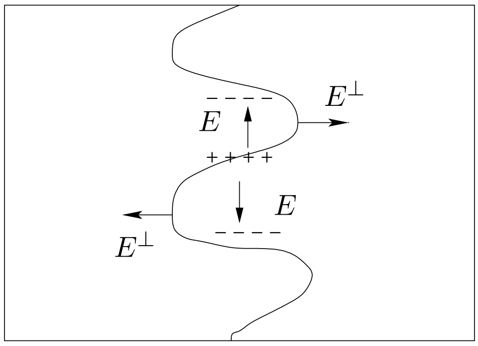


FIGURE 1.8 – Etude heuristique de l'instabilité dans la zone à mauvaise courbure

Les particules du plasma chaud dérivent plus rapidement (figure de gauche), donc s'il y a une petite perturbation (figure de droite), il apparaît spontanément une séparation de charge, ce qui crée un champ électrique E. Ainsi, on voit que la dérive électrique $E^{\perp}$ ne fait qu'accentuer la perturbation, ce qui signifie que l'équilibre est instable. Cette discussion est très classique en physique des plasmas ; cette instabilité est connue comme étant l'une des principales raisons de perte de confinement pour le plasma

On peut réaliser la même étude qualitative dans la région à bonne courbure, et dans ce cas conclure à la stabilité du plasma.

On montrera, conformément à l'intuition physique, la stabilité de μ^{good} en tout régime. Pour μ^{bad}, on montrera effectivement instabilité, mais uniquement si le gradient de température n'est pas trop grand. Au contraire, si ce gradient dépasse un certain seuil, on montre qu'il y a stabilité. Ce phénomène inattendu pourrait fournir une explication aux modes de haut-confinement, qui apparaissent dans les tokamaks. Ces modes, appelés usuellement "H-mode" ont été découverts expérimentalement dans le tokamak ASDEX, voir [157]. Ils constituent aujourd'hui le scénario de base pour les réacteurs à fusion, leur compréhension apparaît ainsi comme primordiale. A notre connaissance, il s'agit de la première justification (qui plus est rigoureuse) du H-mode, avec un modèle très simple mais non ad hoc.

4.1.3 Etude de la stabilité, principes de la preuve

Définissons à présent précisément la notion de stabilité (au sens de Lyapounov) étudiée dans ce chapitre :

Définition 7. *Soit ξ une solution de référence à (4.1). Cette solution est dite stable pour la norme X si pour tout $\eta > 0$, il existe $\delta > 0$ tel que : pour toute solution ρ à (4.1), le contrôle initial $\|\rho(0) - \xi(0)\|_X \leq \delta$ entraîne que pour tout $t \geq 0$, $\|\rho(t) - \xi(t)\|_X \leq \eta$.*

Autrement, la solution ξ est dite instable pour la norme X.

Nous étudierons la stabilité ou l'instabilité autour des solutions stationnaires suivantes :

$$\mu^{bad}(x_1) = \left(\mu^{bad,+} = 1 - \frac{x_1}{L}, \mu^{bad,-} = \frac{x_1}{L}\right), \tag{4.4}$$

$$\mu^{good}(x_1) = \left(\mu^{good,+} = \frac{x_1}{L}, \mu^{good,-} = 1 - \frac{x_1}{L}\right), \tag{4.5}$$

ce qui modélise en fait une transition linéaire entre le plasma chaud et froid. En effet, par exemple pour μ^{bad}, la température du plasma est donnée par la formule :

$$T(t,x) = T^-\frac{x_1}{L} + T^+\left(1 - \frac{x_1}{L}\right).$$

Le théorème prouvé est le suivant :

Théorème 4.1 (Chapitre 6)**.** *Pour le système (4.1) :*

i. *(Instabilité non-linéaire)*
   L'équilibre μ^{good} est non-linéairement stable pour la norme L^2.
   Si le gradient de température $\frac{T^+-T^-}{L}$ satisfait :

$$\frac{T^+ - T^-}{L} > \frac{1}{\pi^2}, \tag{4.6}$$

   alors l'équilibre μ^{bad} est non-linéairement stable pour la norme L^2.

ii. *(Instabilité non-linéaire)*
   Si a contrario le gradient de température satisfait :

$$\frac{T^+ - T^-}{L} < \frac{4}{5\pi^2}, \tag{4.7}$$

   alors il existe des constantes $\delta_0, \eta_0 > 0$ telles que pour tout $0 < \delta < \delta_0$ et tout $s \geq 0$ il existe une solution ρ à (4.1) avec $\|\rho(0) - \mu^{bad}\|_{H^s} \leq \delta$ mais telle que :

$$\|E(t_\delta)\|_{L^2} \geq \eta_0 \tag{4.8}$$

   en notant $E(t_\delta) = \nabla\Delta^{-1}(\rho^+(t_\delta) + \rho^-(t_\delta) - 1)$ le champ électrique au temps $t_\delta = O(|\log\delta|)$.
   En particulier, l'équilibre μ^{bad} est non-linéairement instable pour la norme L^2.

Les ingrédients pour démontrer ce théorème sont les suivants :

i. On utilise le fait que le système (4.1) est structurellement proche de l'équation d'Euler incompressible bi-dimensionnelle pour résoudre facilement le problème de Cauchy. Notamment, pour des données initialement L^∞, on obtient existence et unicité de solutions faibles globales.

   Théorème 4.2 (Chapitre 6)**.** *Soit $\rho_0 = (\rho_0^+, \rho_0^-) \in (L^1((0,L)\times\mathbb{R}/L\mathbb{Z}))^2$ avec ρ_0^+, ρ_0^- positifs et $\int(\rho_0^+ + \rho_0^-)dx = 1$.*

   (a) (Kato, [109]) Si ρ_0 est dans H^s (with $s > 1$) alors il existe une unique solution classique globale ρ à (4.1) dans $\mathcal{C}_t^0([0,\infty[, H^s) \cap \mathcal{C}_t^1([0,\infty[, H^{s-1})$ avec donnée initiale ρ_0.

   (b) (Yudovic, [160]) Si $\rho_0 \in L^\infty$, alors il existe une unique solution faible globale $\rho \in L_t^\infty(L^1 \cap L^\infty)$ à (4.1) avec donnée initiale ρ_0.

ii. Pour démontrer les résultats de stabilité et d'instabilité, les ingrédients sont les suivants. De manière classique, on commence par étudier la structure du linéarisé autour des équilibres stationnaires, ce qui nous permet d'ores et déjà de donner des critères linéaires pour avoir ou non stabilité (notamment, on obtient déjà le fait que si le gradient de température est assez élevé, il y a stabilité linéaire autour de μ^{bad}. L'équation linéarisée au voisinage de μ^{good} ou μ^{bad} s'écrit :

$$\partial_t \begin{pmatrix} \rho^+ \\ \rho^- \end{pmatrix} = L \begin{pmatrix} \rho^+ \\ \rho^- \end{pmatrix},$$

avec L un opérateur linéaire défini par :

$$L \begin{pmatrix} \rho^+ \\ \rho^- \end{pmatrix} = \begin{pmatrix} T^+ \partial_{x_2} \rho^+ \\ T^- \partial_{x_2} \rho^- \end{pmatrix} - \nabla^\perp \Delta^{-1} (\rho^+ + \rho^-) . \nabla_x \mu^\alpha,$$

et $\alpha \in \{good, bad\}$.

A l'aide d'une décomposition en série de Fourier, on peut prouver le lemme suivant :

Lemme 4.1 (Chapitre 6)**.** *Autour de la région à bonne courbure, il n'y a pas de valeur propre à partie réelle strictement positive. Autrement dit, il y a stabilité spectrale linéaire.*

Autour de la région à mauvaise courbure, on a l'alternative suivante :

- *Si* $\frac{T^+ - T^-}{L} < \frac{4\pi}{5}$ *alors il existe une valeur propre à partie réelle strictement positive.*
- *Si* $\frac{T^+ - T^-}{L} \geq \frac{4\pi}{5}$ *alors il n'existe pas une valeur propre à partie réelle strictement positive positive.*

A ce stade de l'étude, une remarque s'impose : on se rend compte que si le gradient de température est assez grand, alors il y a stabilité spectrale linéaire, même dans la zone à mauvaise courbure. Comme nous l'avons dit précédemment, ce phénomène de prime abord inattendu semble en fait correspondre au régime dit du "H-mode". Dans les expériences, ce mode est accompagné d'une hausse du gradient de température, ce qui va dans le sens de nos résultats.

Pour les régimes où on montre l'existence de valeurs propres avec partie réelle positive, une variante du théorème de Weyl nous permet d'obtenir l'existence d'une valeur propre dominante. Cela est dû à la structure du linéarisé :

$$L = A + K,$$

agissant sur L^p, où A est une isométrie et K est un opérateur compact.

iii. Pour prouver la stabilité non linéaire, notre approche se base sur une fonctionnelle explicite, construite sur des normes L^2 adaptées au problème. Le fait remarquable est qu'elles sont exactement conservées (et ne vérifient pas simplement une inégalité de type Gronwall).

Théorème 4.3 (Chapitre 6)**.** *Pour toute donnée initiale* $\rho_0 \in L^\infty$ *avec* $\int (\rho_0^+ + \rho_0^-) dx = 1$, *la solution* ρ *de (4.1) satisfait les conservations suivantes.*

- *Autour de l'état stationnaire à "bonne courbure", la fonctionnelle suivante est conservée :*

$$\mathcal{E}(t) = \|\rho - \mu^{good}\|_{L^2}^2 + \frac{1}{L(T^+ - T^-)} \int |\nabla V|^2 dx \leq \mathcal{E}(0), \tag{4.9}$$

avec $\|\rho - \mu\|_{L^2}^2 = \|\rho^+ - \mu^+\|_{L^2}^2 + \|\rho^- - \mu^-\|_{L^2}^2$ *et* $\nabla V = \nabla \Delta^{-1}(\rho^+ + \rho^- - 1)$.

- *Autour de l'état stationnaire à "mauvaise courbure", la fonctionnelle suivante est conservée :*

$$\mathcal{F}(t) = \|\rho - \mu^{bad}\|_{L^2}^2 - \frac{1}{L(T^+ - T^-)} \int |\nabla V|^2 dx \leq \mathcal{F}(0). \tag{4.10}$$

Pour la région de bonne courbure, "l'énergie" étant la somme de termes positifs, on en déduit immédiatement la stabilité nonlinéaire en norme L^2.

Pour la région de bonne courbure, on utilise une inégalité de type Poincaré pour estimer la norme L^2 du champ électrique en fonction de celle de $\rho - \mu^{bad}$:

$$\int |\nabla V|^2 dx \leq \frac{L^2}{\pi^2} \|\rho - \mu^{bad}\|_{L^2}^2,$$

les constantes étant optimales. Ainsi si :

$$\frac{T^+ - T^-}{L} > \frac{1}{\pi^2},$$

alors il y a stabilité nonlinéaire en norme L^2.

iv. Pour ce qui est de l'instabilité non linéaire, l'idée est de considérer pour donnée initiale un vecteur propre associé à une valeur propre dominante, et montrer grossièrement que la dynamique linéaire domine celle induite par les termes non linéaires, au moins sur un temps assez court. En mettant en oeuvre "naivement" cette idée, il est possible de montrer un résultat d'instabilité non-linéaire, mais dans une norme assez régulière (typiquement $W^{1,p}$ pour p assez grand). Un tel résultat est en fait plus faible que le résultat de stabilité nonlinéaire pour la norme L^2, que l'on a montré précédemment.

Pour montrer l'instabilité pour la norme L^2 il faut procéder avec une méthode plus élaborée. On utilise une technique proposée par Grenier dans [85] pour l'étude d'instabilités hydrodynamiques, basée sur la construction d'une approximation d'ordre élevé de l'équation non-linéaire, permettant un bon contrôle des termes non linéaires. Elle a été imaginée à la base pour étudier l'instabilité de certaines solutions stationnaires de l'équation d'Euler incompressible. Cette approche a par la suite été très fructueuse, citons par exemple Cordier, Grenier et Guo [38], Desjardins et Grenier [47], Guo et Hwang [101], Gérard-Varet [63], Gallaire, Gérard-Varet et Rousset [61], Rousset et Tzvetkov [138] pour l'étude de diverses instabilités en mécanique des fluides ou des plasmas.

Résumons formellement le principe de la technique de Grenier.

Considérons une EDP s'écrivant sous la forme abstraite suivante :

$$\partial_t f = Lf + B(f, f), \tag{4.11}$$

où :

(a) L est un opérateur linéaire anti-autoadjoint engendrant un semi-groupe e^{tL} continu sur les espaces de Sobolev H^s,

(b) B un opérateur bilinéaire avec $B(0,.) = B(.,0) = 0$ satisfaisant l'hypothèse de structure $\int B(f,g)gdx = 0$ et tel qu'il existe une constant $C > 0$ telle que :

$$\int B(f,g)fdx \leq C\|g\|_N \|f\|_{L^2}^2,$$

où N est typiquement un espace de Sobolev à haute régularité (H^s avec s assez grand).

Par exemple pour l'équation étudiée, on a $f = (\rho^+, \rho^-)$,

$$Lf = -\nabla^\perp \Delta^{-1}(\rho^+ + \rho^-).\nabla_x \mu + \begin{pmatrix} T^+ \partial_{x_2} \rho^+ \\ T^- \partial_{x_2} \rho^- \end{pmatrix},$$

(où μ est la solution stationnaire dont on veut étudier la stabilité) et

$$B(f,g) = -\nabla^\perp \Delta^{-1}(f^+ + f^-).\nabla_x g.$$

On suppose une instabilité spectrale linéaire, au sens L admet un certain vecteur propre f_0 associé à une valeur propre λ de partie réelle strictement positive. Pour simplifier, on suppose par la suite que λ est réelle. Il faut également faire des hypothèses supplémentaires sur la structure spectrale de L, nous y reviendrons plus tard.

On souhaite montrer que cette instabilité spectrale se traduit sur l'équation complète par une propriété d'instabilité nonlinéaire. Une première idée consiste à montrer que si l'on part de la perturbation δf_0, où δ est un petit paramètre, alors la solution f va se comporter comme la solution :

$$f_{ap} = \delta f_0 e^{\lambda t}.$$

Pour cela, on souhaite mesurer la quantité $g := f - f_{ap}$ où f est solution de (4.11) avec donnée initiale δf_0 ; elle vérifie l'équation :

$$\partial_t g = (Lg + B(f_{ap}, g) + B(g, f_{ap})) + B(g,g) + B(f_{ap}, f_{ap}), \quad g_{|t=0} = 0.$$

Alors, en multipliant par g et en intégrant, on obtient l'estimation d'énergie L^2 :

$$\frac{d}{dt}\|g\|_{L^2}^2 \leq \gamma \|g\|_{L^2}^2 + C\delta^4 e^{4\lambda t},$$

où γ dépend de B et de f_{ap}. Si jamais $\gamma < 4\lambda$, alors l'inégalité de Gronwall permet de montrer que :

$$\|g\|_{L^2} \leq C\delta^2 e^{2\lambda t}.$$

Il est alors aisé de conclure en choisissant un temps adéquat, de l'ordre de $\log(1/\delta)$.

Pour traiter le cas général ($\gamma > 4\lambda$), l'idée de la construction de Grenier est la suivante : au lieu de montrer que f se comporte comme f_{ap}, on va considérer un développement asymptotique en puissances de δ pour obtenir une approximation d'ordre plus élevé de l'équation (4.11). Plus précisément on construit :

$$f_a p = \delta f_0 e^{\lambda t} + \sum_{i=1}^{N} \delta^{i+1} f_i,$$

Chaque u_i est construit pour tuer les contributions des termes précédents dues au terme bilinéaire :

$$\partial_t f_i = Lf_i + \sum_{j=0}^{i-1} B(f_j, f_{j-i}) + B(f_{i-j}, f_j).$$

Par exemple, pour $i = 1$, cela permet d'annuler le terme $Q(f_0, f_0)$. C'est à cette étape que l'on a besoin d'une hypothèse supplémentaire (non triviale !) sur L ; sa structure doit être telle que l'on puisse obtenir une estimation de la forme :

$$\|\delta^i f_i\|_N \leq C_i \delta^i e^{i\lambda t}.$$

Par exemple cela est vérifié si L admet une valeur propre dominante, ce qui est le cas pour notre étude.

En posant toujours $g := f - f_{ap}$ et en faisant la même estimation d'énergie que précédemment, on a :

$$\frac{d}{dt}\|g\|_{L^2}^2 \leq \mu\|g\|_{L^2}^2 + C\delta^{2(N+2)} e^{2(N+2)\lambda t},$$

On voit donc que la procédure d'approximation d'ordre élevé permet d'obtenir un terme d'ordre plus élevé dans le second membre. En considérant N assez grand, on a bien $\gamma < 2(N+2)\lambda$, ce qui permet de conclure comme avant, avec une inégalité de Gronwall.

4.1.4 Perspectives

Les prolongements possibles à ce travail sont assez naturels. Une première piste consisterait en l'exploration numérique des phénomènes mis en évidence, tout d'abord pour confirmer les prédictions faites dans ce chapitre, puis afin d'initier l'étude de modèles plus complexes.

Notamment, il semble assez crucial de comprendre l'importance de l'hypothèse qui a consisté à négliger l'opérateur de gyromoyenne. Il faudrait donc étendre nos résultats au système suivant :

$$\left\{\begin{array}{l} \partial_t\rho^+ - T^+\partial_{x_2}\rho^+ + \int_0^{2\pi} E^\perp(t, x - \sqrt{T^+}e^{i\theta+i\pi/2})d\theta.\nabla_x\rho^+ = 0, \\ \partial_t\rho^- - T^-\partial_{x_2}\rho^+ + \int_0^{2\pi} E^\perp(t, x - \sqrt{T^+}e^{i\theta+i\pi/2})d\theta.\nabla_x\rho^- = 0, \\ E = -\nabla_x V, \\ -\Delta_x V = \rho^+ + \rho^- - 1, \\ V = 0 \text{ on } x_1 = 0, L, \\ (\rho^+, \rho^-)_{|t=0} = (\rho_0^+, \rho_0^-) \text{ with } \int \rho_0^+ + \rho_0^- = 1. \end{array}\right. \tag{4.12}$$

On pourrait également étudier numériquement les mêmes phénomènes pour des solutions stationnaires plus complexes. Cela pourrait donner une intuition pour étudier rigoureusement ces instabilités du point de vue mathématique.

Il serait également intéressant d'étudier ces instabilités pour des lois de température plus réalistes. Un modèle possible est le système de drift-fluide/Fourier suivant :

$$\left\{\begin{array}{l} \partial_t\rho - T\partial_{x_2}\rho + E^\perp.\nabla_x\rho = 0, \\ \partial_t T + E^\perp.\nabla_x T - \Delta_x T = 0, \\ E = -\nabla_x V, \\ -\Delta_x V = \rho - 1, \\ V = 0 \text{ sur } x_1 = 0, L, \\ \rho_{|t=0} = \rho_0 \text{ avec } \int \rho_0 dx = 1. \end{array}\right. \tag{4.13}$$

pour lequel on étudierait la stabilité de la solution stationnaire $(1, (1-x_1)T_1 + x_1T_2)$.

Une autre piste serait la généralisation de ces résultats pour un modèle cinétique (et donc s'affranchir de l'hypothèse qui nous a amené à considérer des données "bi-cinétiques") :

$$\partial_t + E^{\perp}.\nabla_x f - \frac{1}{2}|v|^2 \partial_{x_2} f = 0 \tag{4.14}$$

(ainsi que sa version gyromoyennée, ou la version plus rigoureuse (4.3).)

Pour de tels modèles, il est plus délicat de choisir une solution stationnaire simple modélisant correctement les zones de bonne ou mauvaise courbure. Encore une fois, une modélisation numérique pourrait aider grandement.

4.2 Contrôle de l'équation de Vlasov-Poisson avec un champ extérieur (Chapitre 7)

Ce chapitre est écrit en collaboration avec Olivier Glass.

4.2.1 Position du problème

Pour ce travail, la motivation est toujours le confinement d'un plasma, mais avec un point de vue différent, celui du contrôle des équations aux dérivées partielles. On considère l'équation de Vlasov-Poisson pour les électrons, posée dans le tore, avec une source g, et un champ de force extérieur $F(t, x, v)$, qui peut être par exemple la force de gravité, la force de Lorentz ou un champ électrique extérieur. On suppose cette force lipschitzienne et au plus sous-linéaire en vitesse à l'infini, ce qui permet de définir les caractéristiques associées. Bien entendu, du point de vue des applications en fusion, le cas du champ magnétique semble le plus intéressant.

$$\begin{cases} \partial_t f + v.\nabla_x f + (E+F).\nabla_v f = \mathbb{1}_\omega G, \\ E = -\nabla_x \Phi, \\ -\Delta_x \Phi = \int f dv - \int f dv dx, \\ f_{|t=0} = f_0. \end{cases} \tag{4.15}$$

Le système est posé pour $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$. La source $\mathbb{1}_\omega G$ est supportée en espace dans un petit ouvert non vide ω du tore, a priori quelconque.

La question de *contrôlabilité* que l'on se pose est la suivante : étant donnés un état $f_1(x, v)$ et un temps $T > 0$, est-il possible de choisir G (le contrôle) pour qu'on atteigne à l'instant T :

$$f(T, x, v) = f_1(x, v). \tag{4.16}$$

Ce problème a été étudié par Glass, pour le cas sans force extérieure $F = 0$ dans [65]. Un fait remarquable de cet article est que l'on n'a pas besoin de supposer que l'ouvert ω vérifie une condition de contrôle géométrique (comme pour le célèbre résultat sur l'équation des ondes par Bardos, Lebeau et Rauch [8]). Cette condition peut dans notre contexte être définie comme suit :

Définition 8. *Un ensemble ω du tore vérifie une condition de contrôle géométrique, si pour tout $x \in \mathbb{T}^d$, pour toute direction $v \in \mathbb{S}^{d-1}$, il existe $t \in \mathbb{R}^+$,*

$$x + tv \in \omega.$$

4.2.2 Résultats et stratégie

On se propose ici d'étendre le résultat de [65] pour les classes de forces suivantes :

i. (Force bornée) $F \in L^\infty_{t,x,v}$. Ce premier cas permet par exemple de traiter les forces dérivant d'un potentiel lipchitzien $-\nabla_x V(x)$.

ii. (Champ magnétique avec direction fixe) En se restreignant à la dimension 2 d'espace, on considère la force de Lorentz $F(t,x,v) = b(x)v^\perp$, pour des intensités b de signe fixe vérifiant la condition géométrique suivante :

   Il existe un compact K du tore sur lequel $\inf b > 0$ *qui vérifie la condition de contrôle géométrique.*

   La condition géométrique permet de s'assurer que les particules sont suffisamment influencées par le champ magnétique.

   Le cas le plus simple auquel on peut penser correspond à celui où l'intensité est constante, mais cette condition permet de considérer des intensités s'annulant sur une grande partie du tore.

Revenir sur le cas sans force extérieure et expliquons brièvement la stratégie suivie dans [65]. Commençons par quelques considérations d'ordre général sur de tels problèmes de contrôle non linéaires. L'idée la plus naturelle consiste à linéariser l'équation, prouver la contrôlabilité du linéarisé, puis prouver que cette propriété est également vérifiée par l'équation non-linéaire, grâce à une technique de type point fixe. Une telle approche conduit à des résultats de contrôlabilité locale, c'est-à-dire pour des données proches d'une certaine solution particulière. Pour passer à des données plus générales (et donc obtenir un résultat de contrôlabilité globale), une possibilité est de s'appuyer sur les propriétés d'échelle de l'équation non-linéaire.

Sans force extérieure, on peut observer que le linéarisé n'est pas toujours contrôlable. En effet, si on linéarise par exemple par rapport à la solution triviale ($f = 0, V = 0$), on obtient l'équation de transport libre suivante :

$$\partial_t f + v.\nabla_x f = \mathbb{1}_\omega G, \tag{4.17}$$

qui n'est pas contrôlable. Rappelons tout d'abord la formule du Duhamel :

$$f(t,x,v) = f_0(x - tv, v) + \int_0^t (\mathbb{1}_\omega G)(s, x + (s-t)v, v))ds. \tag{4.18}$$

On peut alors voir qu'il y a deux types d'obstruction à la contrôlabilité :

- (Petites vitesses) Les particules n'ayant pas une vitesse suffisante à l'instant initial ne sont pas accélérées et ne peuvent donc pas atteindre la zone de contrôle en le temps donné ; ainsi la source g ne peut pas avoir d'influence pour ces particules.

- (Grandes vitesses) Il s'agit d'une obstruction de type contrôle géométrique, dans l'esprit du résultat de Bardos, Lebeau et Rauch pour l'équation des ondes. Si la direction est mauvaise à l'instant initial, alors les caractéristiques peuvent ne jamais passer par la zone de contrôle. Rappelons par ailleurs un résultat bien connu. Pour tout $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, la trajectoire $\{x + tv, t \in \mathbb{R}^+\}$ est soit dense dans le tore (si v est irrationnel), soit périodique (si v est rationnel). Une application du théorème de

Bézout permet alors de conclure qu'il n'y a qu'un nombre fini de mauvaises directions initiales dans $\mathbb{S}^{d-1}$ telles que la trajectoire rectiligne ne rencontre jamais le petit ouvert ω.

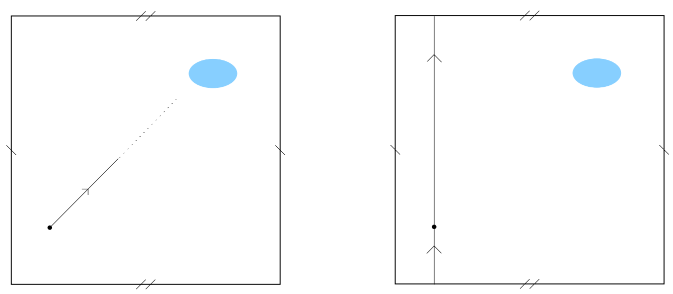


FIGURE 1.9 – Obstructions à la contrôlabilité

On peut néanmoins surmonter ces problèmes en utilisant la méthode du retour, due à Coron, et qui a été utilisée dans de nombreuses équations de la mécanique des fluides, pour lesquelles le linéarisé n'est souvent pas contrôlable [41] (on se réfère à l'ouvrage de Coron pour de nombreuses illustrations et références sur l'utilisation de cette méthode [42]). Cette méthode consiste en la construction d'une solution de référence partant de $(0,0)$ et revenant à $(0,0)$ autour de laquelle le linéarisé est bien contrôlable. Cette construction peut se révéler délicate, et nécessite une compréhension assez poussée de l'équation étudiée.

Tout le problème consiste à trouver un couple de solutions (f,Φ) nulles à l'instant initial et à l'instant final et tel que les caractéristiques associées (X,V) vérifient la propriété :

$$\begin{aligned} \forall x \in \mathbb{T}^d, \quad \forall v \in \mathbb{R}^d, \quad &\exists t \in [T/4, 3T/4] \\ &X(t,0,x,v) \in \omega. \end{aligned} \tag{4.19}$$

Pour $F=0$, ce problème a été résolu dans [65], pour un ouvert ω quelconque (en particulier, sans qu'il y ait une condition de contrôle géométrique). Cela passe par la construction de potentiels harmoniques en dehors de ω, qui permettent de contrôler suffisamment les trajectoires pour les faire passer par ω. Pour cette stratégie, il convient de distinguer les "petites" des "grandes" vitesses pour lesquelles le potentiel construit est différent. Pour les petites vitesses, il s'agit de les accélérer suffisamment ; pour les grandes, il s'agit de leur donner une bonne direction.

Donnons à présent quelques éléments de stratégie pour traiter les cas avec force extérieure non nulle. La propriété (4.19) n'est pas une conséquence directe du cas du transport libre, car les caractéristiques peuvent être très différentes.

i. (Force bornée) L'idée est de s'appuyer sur le comportement *en temps court* de l'équation de Vlasov avec un tel champ de force, et d'utiliser les résultats de contrôlabilité locale dans le cas sans force extérieure pour conclure. Des estimations de stabilité sur les caractéristiques permettent de montrer en effet que les trajectoires restent proches de celles du transport libre. En fixant donc un temps $T' < T$ très court et un petit ouvert $\omega' \subset \omega$ pour lesquels on peut construire une solution de référence pour le transport libre, on montre que cette solution de référence convient pour T et ω.

ii. (Champ magnétique avec direction fixe) Pour ce cas la stratégie précédente ne fonctionne plus, en particulier pour le traitement des grandes vitesses. On peut néanmoins s'appuyer sur les propriétés de la force de Lorentz, à savoir qu'*à grande vitesse, les trajectoires sont proches de celles du transport libre* (car la courbure est très faible), et surtout que *le champ magnétique "mélange" les directions de vitesse*, de sorte que les mauvaises directions ne sont pas conservées longtemps (contrairement au cas du transport libre). L'hypothèse géométrique sur le champ magnétique permet d'assurer que les particules sont suffisamment influencées par la force de Lorentz.

On montre ainsi le fait assez remarquable qu'à grande vitesse, la condition (4.19) est satisfaite pour tout ouvert ω, sans que l'on ait besoin d'agir. Pour les petites vitesses, on peut accélérer les particules comme dans le cas d'une force bornée.

L'idée est ensuite d'utiliser un théorème de point fixe (en l'occurrence un Théorème de Schauder) pour résoudre l'équation pour des données initiales dans un voisinage de $(0,0)$. Cela permet de construire des solutions fortes, similaires à celles construites par Ukai et Okabe [151]. Le voisinage peut être choisi suffisamment petit pour que la propriété (4.19) reste satisfaite. Ce procédé s'accompagne d'un processus d'absorption dans la zone de contrôle. Cette étape s'avère assez technique. On montre ainsi un résultat de *contrôlabilité locale.*

Dans le cas des forces bornées ou pour le champ magnétique avec l'hypothèse géométrique, en dimension 2, on démontre ainsi le même résultat :

Théorème 4.4 (Chapitre 7)**.** *Soit $\gamma > 2$ et $T > 0$. Il existe $\kappa, \kappa' > 0$ assez petits vérifiant les propriétés suivantes. Soit f_0 et f_1 deux fonctions de $C^1(\mathbb{T}^2 \times \mathbb{R}^2) \cap W^{1,\infty}(\mathbb{T}^2 \times \mathbb{R}^2)$, satisfaisant la décroissance en $(x,v) \in \mathbb{T}^2 \times \mathbb{R}^2$ pour $i \in \{0,1\}$,*

$$\begin{cases} |f_i(x,v)| \leq \kappa(1+|v|)^{-\gamma-1}, \\ |\nabla_x f_i| + |\nabla_v f_i| \leq \kappa'(1+|v|)^{-\gamma}, \end{cases} \tag{4.20}$$

et d'autre part,

$$\int_{\mathbb{T}^2\times\mathbb{R}^2} f_0 = \int_{\mathbb{T}^2\times\mathbb{R}^2} f_1. \tag{4.21}$$

Alors il existe un contrôle $G \in C^0([0,T] \times \mathbb{T}^2 \times \mathbb{R}^2)$, telle que (4.15) admette une unique solution égale à f_0 en $t = 0$ et f_1 en $t = T$.

En ce qui concerne la contrôlabilité globale, elle peut être ici obtenue dans le cas des forces bornées par un argument d'échelles sur l'équation considérée. Pour éviter des phénomènes de concentration, on est néanmoins amené à faire certaines hypothèses sur l'ouvert de contrôle ω (mais moins fortes que la condition de contrôle géométrique). Par souci de brièveté, nous renvoyons au chapitre 7 pour les énoncés précis des théorèmes démontrés.

4.2.3 Perspectives

On peut se poser la question naturelle suivante : est-il possible de prouver la contrôlabilité locale, pour toute force F qui soit uniquement globalement Lipschitz et sous-linéaire en vitesse à l'infini ? Cela semble délicat, car à grande vitesse les comportements peuvent être très différents selon les types forces considérées.

Ainsi, par exemple, si on considère la force de type friction $F = \pm v$, alors la dynamique est très différente de celle associée aux champs magnétiques. En effet, les caractéristiques restent des lignes droites. Dans cette géométrie très simple, nous sommes en mesure de prouver la contrôlabilité locale, essentiellement en se basant sur le cas du transport libre. L'étude de ce cas permet d'ouvrir des perspectives nouvelles, en l'occurence l'étude de la contrôlabilité pour de équations fluide-cinétique couplées, telles que Vlasov-Stokes ou Vlasov-Navier-Stokes. Ceci fait d'ores et déjà l'objet de travaux en cours avec Olivier Glass.

Concernant la contrôlabilité de l'équation de Vlasov-Poisson sans force extérieure, un problème naturel serait l'extension des résultats au cas d'un domaine à bord. Ce problème semble délicat en raison des trajectoires rasantes près des bords, qui semblent difficilement contrôlables. Plus généralement, il se pose des questions de type biliards en système dynamique, qui semblent également délicats.

Chapter 2

Quasineutral limit of the Vlasov-Poisson equation with massless electrons

Article à paraitre à Communications in Partial Differential Equations (2011).

Sommaire

Résumé : In this chapter, we study the quasineutral limit (in other words the limit when the Debye length tends to zero) of Vlasov-Poisson like equations describing the behaviour of ions in a plasma. We consider massless electrons, with a charge density following a Maxwell-Boltzmann law. For cold ions, using the relative entropy method, we derive the classical Isothermal Euler or the (inviscid) Shallow Water systems from fluid mechanics. In a second time, we study the combined quasineutral and strong magnetic field regime for such plasmas.

1 Introduction

1.1 Physical motivation

For high enough kinetic temperatures exceeding the atomic ionization energy, atoms tend to decompose into electrons and ions (that is, negatively and positively charged particles): a plasma is a physical or chemical system where such a ionization has occured. Roughly speaking, we simply consider that plasmas are gases made of positive and negative charges. Unlike gases, plasmas are highly conductive. As a consequence, particles interact with each other by creating their own electromagnetic fields which can dramatically affect their behaviour.

The plasma state is considered as the fourth state of matter. Actually it is the most common one in the universe : it is widely recognized that at least 95% of the matter consists of it ! For instance, the suns and other stars are filled with plasma, so is the interstellar medium and so on. Terrestrial plasmas are also quite easy to find: they appear in flames, lightning or in the ionosphere. For the last decades, there has been an increasing interest in creating artificial plasmas, for experimental or industrial purposes. For instance, neon lamps or plasma displays for televisions are now part of our everyday life. An extremely promising application of plasmas consists in the fusion energy research (by magnetic or inertial confinement). This paper specifically aims at rigorously deriving some mathematical models which would help to understand the physics in tokamaks, which are the boxes in which plasmas from magnetic confinement fusion are contained.

1.1.1 Basic kinetic models for plasmas

We adopt a statistical description of the plasma: we describe the behaviour of the charged particles by considering kinetic equations satisfied by their repartition function. That means that we do not follow particles one by one by solving Newton equations but are rather interested in their collective behaviour.

We present the mathematical models we are going to study in the following. In order to establish them, we have to make some standard approximations which we now explain.

- **Assumption 1**: We assume the plasma to be collisionless. Thus, we will consider Vlasov-like equations without collision operators.

- **Assumption 2**: The plasma is non-relativistic and the electric field E is electrostatic. This means that we consider electromagnetic fields that satisfy the electrostatic approximation of the Maxwell equations:

$$\begin{cases} \operatorname{rot} E = 0, \\ \operatorname{div} E = \frac{\rho}{\varepsilon_0}. \end{cases}$$

We can also consider an additional magnetic field B satisfying

$$\operatorname{div} B = 0$$

and it has to be stationary in time in order to be consistent with the electrostatic approximation. The electromagnetic fields act on the charged particles (with charge q) through the Lorentz force:

$$F = q(E + v \wedge B).$$

- **Assumption 3**: The plasma evolves in a domain without boundaries. This means in particular that we may restrict to periodic data in the space variable, which may seem unrealistic, but which is commonly done in plasma physics and mathematics.

We discuss the physical relevancy of these assumptions in the following Remark:

Remark 1.1.
- Discussion on **A1**: Let $\Lambda = 4\pi N_0 \left(\frac{\varepsilon_0 T}{N_0 e^2}\right)^{3/2}$, where N_0 is the average number density of particles, ε_0 is the vacuum permittivity, T the average temperature of the plasma and e is the fundamental electric charge.

  The typical collision frequency is given by:

$$\omega_c = \frac{e^4 \log \Lambda}{4\pi \varepsilon_0^2 m^{1/2}} \frac{N_0}{T^{3/2}}.$$

  So the plasma can be considered as collisionless if it is diffuse and high temperature. Most plasmas can be considered as collisionless to a very good approximation [54].

- Discussion on **A2**: The electrostatic approximation is relevant as soon as

$$\frac{c\tau}{L} \gg 1.$$

  denoting by c the speed of light, τ the characteristic observation time and L the characteristic observation length. Therefore this approximation often appears as reasonable in practical situations for terrestrial plasmas. At least, it is valid for short observation lengths.

- Discussion on **A3**: By ignoring boundary effects, we neglect some important physics, such as the formation of the Debye sheath near walls, which are boundary layers often surrounding plasmas confined in some material. Very few seems to be mathematically known about this phenomenon, starting from a Vlasov-Poisson equation.

Within these approximations, the kinetic system reads:

$$\begin{cases} \partial_t f_i + v.\nabla_x f_i + q_i/m_i \left(E + v \wedge B\right).\nabla_v f_i = 0 \\ \partial_t f_e + v.\nabla_x f_e + q_e/m_e \left(E + v \wedge B\right).\nabla_v f_e = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \frac{1}{\varepsilon_0}\left(q_i \int f_i dv + q_e \int f_e dv\right), \end{cases} \tag{1.1}$$

with $x \in \mathbb{R}^n$ or $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n, v \in \mathbb{R}^n, t \in \mathbb{R}^+$. We can associate to these equations the initial conditions:

$$\begin{cases} f_{i,|t=0} = f_{i,0}, \quad f_{i,0} \geq 0, \quad \int f_{i,0} dv dx = 1, \\ f_{e,|t=0} = f_{e,0}, \quad f_{e,0} \geq 0, \quad \int f_{e,0} dv dx = 1. \end{cases} \tag{1.2}$$

The parameter n is the space dimension, equal to $1, 2$ or 3 in the following. Quantity f_i (resp. f_e) is interpreted as the density distribution of ions (resp. electrons) : $f(t, x, v)dxdv$ is interpreted as the probability of finding particles at time t with position x and velocity v. The parameter m_i (resp. m_e) is the mass of one ion (resp. electron). Likewise, q_i (resp. q_e) is the charge of one ion (resp. electron). For simplicity we will take $q_e = -e$ and $q_i = e$.

We now intend to reduce the two transport equations into only one. To this end, we can observe that the mass ratio between electrons and ions is very small:

$$\frac{m_e}{m_i} \ll 1,$$

so that qualitatively the two types of particles have really different dynamical behaviour. Therefore we make the additional approximation for our idealized model:

- **Assumption 4**: The mass ratio between ions and electrons is infinite: $\frac{m_i}{m_e} = +\infty$.

This remark allows to reduce System (1.1) to only one transport equation. Depending on the interpretation of Assumption 4, we get two classes of models:

Infinite mass ions ($m_i = +\infty$)

One can consider the point of view of electrons, from which ions are very slow, motionless at equilibrium:

$$n_i = \int f_i dv = 1. \tag{1.3}$$

Then, assuming there is no magnetic field, system (1.1) written in dimensionless variables reduces to the classical Vlasov-Poisson system:

$$\begin{cases} \partial_t f + v.\nabla_x f + E.\nabla_v f = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \int f dv - 1 \\ f_{|t=0} = f_0 \geq 0, \int f_0 dv dx = 1. \end{cases} \tag{1.4}$$

This system was intensively studied in the mathematical literature (for $n = 3$ in particular) and the Cauchy-Problem is rather well understood. We refer to the works of Arsenev [4], Horst and Hunze [99] for global weak solutions, DiPerna and Lions [49] for global renormalized solutions, Pfaffelmoser [135], Schaeffer [145] for classical solutions, Lions and Perthame [119] for weak solutions with high order velocity moments and Loeper [122] on the uniqueness problem.

Massless electrons ($m_e = 0$)

Otherwise, one can consider the viewpoint of ions : electrons then move very fast and quasi-instantaneously reach their local thermodynamic equilibrium[1]. Then their density n_e follows the classical Maxwell-Boltzmann law (see [111]) :

$$n_e = \int f_e dv = d(x) \exp\left(\frac{eV}{k_B T_e}\right), \tag{1.5}$$

[1] Actually since $m_e \ll m_i$, the typical collision frequency for the electrons is much larger than for the ions and thus collisions for the electrons may be not negligeable: for this reason they can reach their local thermodynamic equilibrium.

where V denotes the electric potential, k_B is the Boltzmann constant, T_e the average temperature of the electrons, $d \in L^1(\mathbb{R}^n)$ is a term due to an external potential preventing the particles from going to infinity (we also refer to [19] and references therein).

More precisely, we have:

$$d(x) = n_0 e^{\frac{-H(x)}{k_B T_e}}, \tag{1.6}$$

where $n_0 \in \mathbb{R}$ is a normalizing constant and H is the external confining potential.

The Poisson equation then reads:

$$-\Delta_x V = \int f dv - d \exp\left(\frac{eV}{k_B T_e}\right). \tag{1.7}$$

One should notice that in this case, in general

$$\int \left(\int f dv - d \exp\left(\frac{eV}{k_B T_e}\right)\right) dx \neq 0,$$

meaning that global neutrality does not hold, since the total charge of electrons is not a priori fixed[2].

We may also consider the case when the total charge of the electrons is fixed, in which case the Poisson equation reads:

$$-\Delta_x V = \int f dv - \frac{d \exp\left(\frac{eV}{k_B T_e}\right)}{\int_{\mathbb{R}^n} d \exp\left(\frac{eV}{k_B T_e}\right) dx}. \tag{1.8}$$

The existence of global weak solutions to these two systems in dimension three has been investigated by Bouchut [19]. We will recall some of the properties of these solutions in Section 2.

An approximation widely used in plasma physics consists in linearizing the exponential law:

$$n_e = d\left(1 + \frac{eV}{k_B T_e}\right). \tag{1.9}$$

This approximation is valid from the physical point of view as long as:

$$\frac{eV}{k_B T_e} \ll 1,$$

that is as long as the electric energy is small compared to the kinetic energy.

We will consider this law in the case of the torus $\mathbb{T}^n$ (with $n = 1, 2$ or 3), thus we do not need a confining potential (and we take $d = 1$).

In the following we will only focus on models with such Maxwell-Boltzmann laws.

1.1.2 The Debye length

We define now the Debye length $\lambda_D^{(\alpha)}$ as:

$$\lambda_D^{(\alpha)} = \sqrt{\frac{\varepsilon_0 k_B T_\alpha}{N_\alpha e^2}}, \tag{1.10}$$

where k_B is the (universal) Boltzmann constant, T_α and N_α are respectively the average temperature and density of electrons (for $\alpha = e$) or ions (for $\alpha = i$).

[2] This feature will prevent us from studying this system on the torus; instead we will do so on the whole space $\mathbb{R}^3$.

The Debye length is a fundamental parameter which is of tremendous importance in plasmas. It can be interpreted as the typical length below which charge separation occurs.

In plasmas, this length may vary by many orders of magnitude (Typical values go from $10^{-3}m$ to $10^{-8}m$). In practical situations, for terrestrial plasmas, it is always small compared to the other characteristic lengths under consideration, in particular the characteristic observation length, denoted by L. Actually, the condition $\lambda_D \ll L$ is sometimes required in the definition itself of a plasma.

Therefore, if we set:

$$\frac{\lambda_D}{L} = \varepsilon \ll 1,$$

then in many regimes, it is relevant, after considering relevant dimensionless variables, to consider that the Poisson equation formally reads :

$$-\varepsilon^2 \Delta_x V_\varepsilon = \pm (n_i - n_e).$$

The quasineutral limit precisely consists in considering the limit $\varepsilon \to 0$.

1.1.3 Why quasineutral fluid limits ?

From the numerical point of view, kinetic equations are harder to handle than fluid equations. Indeed the main difficulty is that we have to deal with a phase space of dimension 6 (for $x, v \in \mathbb{R}^3$). Actually, another outstanding problem for simulating plasmas is the following : there are characteristic lengths and times of completely different magnitude (think of the Debye length and the observation length) that make numerics really delicate.

In this work, we particularly aim at getting simplified hydrodynamic systems after taking quasineutral limits. Simplified fluid models have some advantages:

- With a fluid description, we deal with a phase space of lesser dimension. Furthermore after taking the limit we now handle only one characteristic time and length. For these reasons, numerical simulations are easier to perform.

  Of course it is well-known that the fluid approximation is not always accurate for simulations of plasmas, but it is nevertheless valid in some regimes that we may describe in the analysis. So it is important to be aware of the physical assumptions we make when we derive the equations.

- Macroscopic quantities, such as charge density or current density are easier to experimentally measure (by opposition, the repartition function is out of reach). So this is a way to check if the initial modeling is accurate or not.

- A simplified fluid description can help us to qualitatively describe the behaviour of the plasma.

The derivation of limit models is deeply linked to issues related to the research project of magnetic confinement fusion. For the last few years, there has been a wider interest in finding simplified systems to model quasineutral plasmas for devices such as tokamaks. Therefore a good mathematical understanding of these becomes important, as it would establish some theoretical basis to compare various models, like gyrokinetic, gyrofluid, MHD or Euler-like equations and understand their range of validity.

In this paper, we will focus on two kinds of quasineutral problems. In both problems, the starting point is the Vlasov-Poisson system with massless electrons (in other words with electrons following a Maxwell-Boltzmann law). First we will investigate the quasineutral limit alone, then we will in addition consider a large magnetic field and study the behaviour of the plasma in this regime.

1.2 Quasineutral limit of the Vlasov-Poisson system with massless electrons

In the first place, we are interested in the quasineutral limit for Vlasov-Poisson systems with Boltzmann-Maxwell laws and without magnetic field. We will focus in particular on the limit $\varepsilon \to 0$ for the following dimentionless system (we refer to the Annex for details on the scaling):

System (S) : Maxwell-Boltzmann law (for $x \in \mathbb{R}^3$,$v \in \mathbb{R}^3, t \in \mathbb{R}^+$)

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + E_\varepsilon.\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\varepsilon \Delta_x V_\varepsilon = \int f_\varepsilon dv - de^{V_\varepsilon} \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon} \geq 0, \int f_{0,\varepsilon} dv dx = 1. \end{cases} \tag{1.11}$$

The method of proof we follow allows also to treat the case of variants of system (S), so we will mention the results we can get, without providing complete proofs, for the systems:

- **System (S')** : Maxwell-Boltzmann law with fixed total charge (for $x \in \mathbb{R}^3, v \in \mathbb{R}^3, t \in \mathbb{R}^+$)

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + E_\varepsilon.\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\varepsilon \Delta_x V_\varepsilon = \int f_\varepsilon dv - \frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon}} \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon} \geq 0, \int f_{0,\varepsilon} dv dx = 1. \end{cases} \tag{1.12}$$

- **System (L)** : Linearized Maxwell-Boltzmann law (for $x \in \mathbb{T}^n, v \in \mathbb{R}^n, t \in \mathbb{R}^+$ and $n = 1, 2, 3$)

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + E_\varepsilon.\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ V_\varepsilon - \varepsilon \Delta_x V_\varepsilon = \int f_\varepsilon dv - 1 \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon} \geq 0, \int f_{0,\varepsilon} dv dx = 1. \end{cases} \tag{1.13}$$

For systems (S) and (S') we will from now on assume the boundedness properties on $d(x) = e^{-H(x)}$:

$$d = e^{-H} \in L^1 \cap L^\infty(\mathbb{R}^3). \tag{1.14}$$

$$\nabla_x H \in W^{s,\infty} \quad \text{for any} \quad s \in \mathbb{N}. \tag{1.15}$$

For instance, this holds for $H(x) = \sqrt{1 + |x|^2}$.

Remark 1.2. From the mathematical viewpoint, we have to add the confining potential H to ensure that the local density of electrons belongs to $L^1(\mathbb{R}^3)$.

1.2.1 Formal derivation of the isothermal Euler system from systems (S) and (S')

We will prove the local in time strong convergence of the charge density and current density:

$$\left(\rho_\varepsilon := \int f_\varepsilon dv, J_\varepsilon := \int f_\varepsilon v dv\right)$$

to the local strong solution $(\rho, \rho u)$ to some Euler-type system, for initial data close (in some sense to be made precise later) to monokinetic data that is,

$$f_\varepsilon(t, x, v) \sim \rho_\varepsilon(t, x) \delta_{v = u_\varepsilon(t,x)},$$

with $u_\varepsilon = \frac{J_\varepsilon}{\rho_\varepsilon}$.

Let us show now how we can guess what is the limit system. First, by integrating the Vlasov equation against 1 and v, we straightforwardly get the local conservation laws satisfied by the first two moments.

$$\partial_t \rho_\varepsilon + \nabla_x . J_\varepsilon = 0, \tag{1.16}$$

$$\partial_t J_\varepsilon + \nabla_x : \left(\int v \otimes v f_\varepsilon dv \right) = \rho_\varepsilon E_\varepsilon. \tag{1.17}$$

Let us directly consider monokinetic data, i.e. $f_\varepsilon(t,x,v) = \rho_\varepsilon(t,x)\delta_{v=u_\varepsilon(t,x)}$. The local conservations laws reduce to:

$$\partial_t \rho_\varepsilon + \nabla_x .(\rho_\varepsilon u_\varepsilon) = 0. \tag{1.18}$$

$$\partial_t (\rho_\varepsilon u_\varepsilon) + \nabla_x : (\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) = -\rho_\varepsilon \nabla_x V_\varepsilon. \tag{1.19}$$

In the case of (S) the Poisson equation reads:

$$-\varepsilon \Delta_x V_\varepsilon = \int f_\varepsilon dv - d e^{V_\varepsilon}.$$

Since ρ_ε and J_ε are uniformly bounded in $L^\infty_t(L^1_x)$, the following convergences hold (up to a subsequence) in the sense of distributions: $\rho_\varepsilon \rightharpoonup \rho$ and $J_\varepsilon \rightharpoonup J$.

If we formally pass to the limit $\varepsilon \to 0$ we get:

$$d e^V = \rho. \tag{1.20}$$

Consequently, we have $V = \log(\rho/d)$ and therefore $-\nabla_x V = -\frac{\nabla_x \rho}{\rho} + \frac{\nabla_x d}{d}$. Notice that $\frac{\nabla_x d}{d} = -\nabla_x H$. Thus, the asymptotic equation we can expect is the following compressible Euler-type model (which can be interpreted as the isothermal Euler equation with an external confining force):

$$\begin{cases} \partial_t \rho + \nabla_x .(\rho u) = 0, \\ \partial_t u + u.\nabla_x u = -\frac{\nabla_x \rho}{\rho} - \nabla_x H. \end{cases} \tag{1.21}$$

In the case of (S') the Poisson equation reads:

$$-\varepsilon \Delta_x V_\varepsilon = \int f_\varepsilon dv - \frac{d e^{V_\varepsilon}}{\int d e^{V_\varepsilon} dx}. \tag{1.22}$$

If we formally pass to the limit $\varepsilon \to 0$ we get:

$$\frac{d e^V}{\int d e^V dx} = \rho. \tag{1.23}$$

Consequently we have $V = \log\left(\frac{\rho}{d} \int d e^V dx\right)$ and so, we get the same Euler equation (1.21).

Remark 1.3. (Physical signification of monokinetic data)

We define:

$$T_{i,\varepsilon} = \frac{1}{3\rho_\varepsilon} \int f_\varepsilon \left| v - \frac{J_\varepsilon}{\rho_\varepsilon} \right|^2 dv dx.$$

The quantity $T_{i,\varepsilon}$ is nothing but the scaled temperature of the ions.

Considering monokinetic data corresponds to the "cold ions" assumption, that is:

$$\int f \left| v - \frac{J}{\rho} \right| dvdx = 0,$$

which means that we consider that the temperature of ions is equal to 0.

More precisely, the cold ions approximation means from the physical point of view that

$$T_i \ll T_e.$$

It turns out that this approximation is highly relevant for terrestrial plasmas and widely used in plasma physics, especially in the study of tokamak plasmas.

There are two main physical reasons why it is relevant to consider that the temperature of electrons is much higher than the temperature of ions : first of all , there exist many plasma sources which can heat the electrons more strongly than the ions. Second notice that energy transfer in a two-body collision is much more efficient if the masses are similar. Thus, since ions and electrons have very different masses, there is almost no transfer of energy from the electrons to the ions. For instance this approximation is used in order to derive the classical Hasegawa-Mima equation ([95]).

Remark 1.4. Nevertheless we observe in the isothermal Euler limit system (1.21) that the ions evolve as if they had the temperature of electrons (of order 1) ! Moreover, ions seem to have better confinement properties than expected, since they feel the confining potential in the limit equation.

Remark 1.5. For (L) the corresponding Euler-type system is the following:

$$\begin{cases} \partial_t \rho + \nabla_x.(\rho u) = 0, \\ \partial_t u + u.\nabla_x u = -\nabla_x \rho. \end{cases} \tag{1.24}$$

Actually System (1.24) can be interpreted in $1D$ or $2D$ as an inviscid Shallow Water system (and ρ is then understood as the depth of the fluid). This quite remarkable fact is one amongst many analogies between geophysics and plasma physics models (see for instance the work of Hasegawa and Mima [95] and the review paper [48]). For instance the concept of "zonal flows" is used in both fields and the mechanism responsible for their generation may be the same. Only the name differs: drift waves for plasma physics, Rossby waves for geophysics, see Cheverry, Gallagher, Paul and Saint-Raymond for a recent mathematical study [37].

1.2.2 Principle of the proof : the relative entropy method

The relative entropy method (also referred to as the modulated energy method) was first introduced in kinetic theory independently by Golse [21] in order to study the convergence of solutions to a scaled Boltzmann equation to solutions of incompressible Euler for well-prepared data and some technical assumptions (see Saint-Raymond [141] for the latest developments on the topic) and by Brenier [27] in order to derive incompressible Euler equations from the quasineutral Vlasov-Poisson equation for electrons in a fixed background of ions.

More precisely Brenier shows the convergence as $\varepsilon \to 0$ of the first two moments $(\rho_\varepsilon, J_\varepsilon) := (\int f_\varepsilon dv, \int v f_\varepsilon dv)$ of the starting system:

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + E_\varepsilon.\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\varepsilon \Delta_x V_\varepsilon = \int f_\varepsilon dv - 1 \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon} \geq 0, \int f_{0,\varepsilon} dvdx = 1. \end{cases} \tag{1.25}$$

to the smooth solution of the limit system which is the classical incompressible Euler system:

$$\begin{cases} \rho = 1 \\ \partial_t u + u.\nabla_x u + \nabla_x p = 0 \\ \operatorname{div} u = 0. \end{cases} \tag{1.26}$$

Brenier treated the case of well-prepared monokinetic data (i.e. cold electrons); the convergence was then generalized by Masmoudi [125] for ill-prepared monokinetic data.

We mention the works [16], [29], [30], [74], [76], [136] which also use the relative entropy method in order to derive fluid equations from Vlasov-like systems.

The principle of the method is the following. For system (S), it can be shown that the following functional is non-increasing:

$$\mathcal{F}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v|^2 dv dx + \frac{\varepsilon}{2}\int |\nabla_x V_\varepsilon|^2 dx + \int d(x)(V_\varepsilon - 1)e^{V_\varepsilon} dx. \tag{1.27}$$

We call this functional the energy of the system.

We then consider the functional $\mathcal{H}_\varepsilon$ which is built as a modulation of this energy:

$$\mathcal{H}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v-u|^2 dv dx + \frac{\varepsilon}{2}\int |\nabla_x V_\varepsilon|^2 dx + \int (de^{V_\varepsilon}\log\left(de^{V_\varepsilon}/\rho\right) - de^{V_\varepsilon} + \rho) dx, \tag{1.28}$$

with (ρ, u) a smooth solution to Isothermal Euler.

Quite surprisingly, it turns out that the last term of $\mathcal{H}_\varepsilon$ is similar to the usual relative entropy for collisional (such as Boltzmann or BGK) equations.

What we want to prove is that this functional is in fact a Lyapunov functional. We will show that indeed, $\mathcal{H}_\varepsilon$ satisfies some stability estimate:

$$\mathcal{H}_\varepsilon(t) \leq \mathcal{H}_\varepsilon(0) + G_\varepsilon(t) + C\int_0^t \|\nabla_x u\|_{L^\infty}\mathcal{H}_\varepsilon(s) ds, \tag{1.29}$$

with $G_\varepsilon(t) \to_{\varepsilon\to 0} 0$ uniformly in time.

Then, assuming that the initial conditions is well-prepared in the sense that

$$\mathcal{H}_\varepsilon(0) \to_{\varepsilon\to 0} 0,$$

this yields that $\mathcal{H}_\varepsilon(t) \to_{\varepsilon\to 0} 0$, thus proving the strong convergence in some sense (which will be made precise later on) to smooth solutions of the isothermal Euler equation, as long as the latter exist. The proof relies on the fine algebraic structure of the nonlinearities in systems (S) and (S'). One major advantage of this method is that it only requires weak regularity on the solutions to the initial system but allows to prove limits in a strong sense. Nevertheless, it requires a good understanding of the Cauchy problem for the limit system (in particular, we must have a notion of stability for the limit). It should be noticed that even if we considered very smooth solutions (say for instance H^s with s large) to the initial system, we would not be able to propagate uniform bounds and thus prove compactness. Indeed, the only uniform controls we have are the energy bound and the conservation of L^p norms of the number density.

Basically this is nothing but a stability result : roughly speaking , this result tells us that monokinetic solutions are stable with respect to perturbations of the energy.

Let us mention that the method used for these two systems can also apply to the quasineutral limit for an isothermal Euler-Poisson version of system (S) studied by Cordier and Grenier [39]. We refer to Section 4.3.

1.3 Quasineutral limit for the Vlasov-Poisson equation with massless equations and with a strong magnetic field

Next we are interested in the behaviour of the plasma if one applies an intense magnetic field. Such a regime is particularly relevant for plasmas encountered in magnetic confinement fusion research. Plasmas are expected to be confined inside tokamaks thanks to this magnetic field. One challenging mathematical problem is to rigorously prove if this strategy is likely to succeed or not.

In this paper, we consider the simplest geometric case of a constant magnetic field with a fixed direction and a fixed (large) intensity.

1.3.1 Scaling of the Vlasov equation

We first introduce some notations:

Notations. *Let $(e_1, e_2, e_{\|})$ be a fixed orthonormal basis of $\mathbb{R}^3$.*

- *The subscript $\perp$ stands for the orthogonal projection on the plane (e_1, e_2), while the subscript $\|$ stands for the projection on $e_{\|}$.*
- *For any vector $X = (X_1, X_2, X_{\|})$, we define $X^{\perp}$ as the vector $(X_y, -X_x, 0) = X \wedge e_{\|}$.*

We consider a strong magnetic field of the form:

$$B = Be_{\|}.$$

Roughly speaking, "strong" means that $|B| \sim 1/\varepsilon$. With this time a quasineutral ordering of the form

$$\frac{\lambda_D}{L} = \varepsilon^{\alpha},$$

where $\alpha > 0$ is an arbitrary parameter, we get in the end the quasineutral system:

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + \left(E_\varepsilon + \frac{v \wedge e_{\|}}{\varepsilon}\right).\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\varepsilon^{2\alpha}\Delta_x V_\varepsilon = \int f_\varepsilon dv - \frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon} dx} \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon}, \quad \int f_{0,\varepsilon} dv dx = 1. \end{cases} \tag{1.30}$$

We refer to the Annex for details on the scaling. The range of parameters $\alpha > 1$ is particularly relevant from the physical point of view.

1.3.2 Comments on the expected result

We will study the limit, once again by using the relative entropy method and show the convergence of the first two moments $(\rho_\varepsilon, u_\varepsilon)$ (defined as before) to smooth solutions to the system:

$$\begin{cases} \partial_t \rho + \partial_{x_{\|}}(\rho w_{\|}) = 0, \\ \partial_t w + w_{\|}\partial_{x_{\|}} w = -\frac{\partial_{x_{\|}}\rho}{\rho} - \partial_{x_{\|}} H. \end{cases} \tag{1.31}$$

We observe that there is no more dynamics in the orthogonal plane (that is, in the $x_\perp$ variable), which can be interpreted as a good confinement result.

For the study of this limit we will have to face more technical difficulties than without magnetic field. Indeed, the strong magnetic field engenders time oscillations of order $\mathcal{O}(1/\varepsilon)$ on the number density. Consequently in order to show strong convergence we will have to:

- filter out the time oscillations.
- add some correction of order $\mathcal{O}(\varepsilon)$ to the limit (ρ, w) in order to get an approximate zero of the so-called acceleration operator (5.13).

The striking point here is that we can study the limit for any value of α. In contrast, for the system describing the electrons with heavy ions:

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + \left(E_\varepsilon + \frac{v\wedge e_\parallel}{\varepsilon}\right).\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\varepsilon^{2\alpha}\Delta_x V_\varepsilon = \int f_\varepsilon dv - 1 \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon}, \quad \int f_{0,\varepsilon} dv dx = 1. \end{cases}$$

it seems primordial to take $\alpha = 1$, so that the Debye length and the Larmor radius vanish at the same rate. This specific scaled system was studied by Golse and Saint-Raymond in [74].

The heuristic underlying reason is that the Poisson equation with a Maxwell-Boltzmann law is in some sense more stable in the quasineutral limit than the "usual" one. Indeed the electric potential is in the limit explicitly a function of ρ, whereas in the "usual" case, it appears as a Lagrange multiplier or equivalently as a pressure.

Remark 1.6. We could as well set $\frac{\lambda_D}{L} = \delta$ and let ε, δ go to 0 independently. One can readily check that we would get the same results.

Remark 1.7. We may also consider the linearized Maxwell-Boltzmann law and perform the same convergence analysis.

1.4 Outline of the paper

The following of this article is organized as follows. First in section 2, we recall some elements on the global weak solutions theory for systems (S), (S') and (L) and recall some useful *a priori* uniform bounds. This theory is due to Arsenev [4] and Bouchut [19]. Then, we will present the local conservation laws satisfied by the two first moments of our solutions.

In section 3, we will focus on the quasineutral limit from the Vlasov-Poisson System (S) to an isothermal Euler system, using the relative entropy method (Theorem 3.1). The crucial step is to show the algebraic identity (3.19) that describes the decay of the relative entropy and from which we will be able to get a stability inequality.

In section 4, we give extensions of the method for systems (S') and (L) (Theorems 4.1 and 4.2), by only sketching the proofs. We will show that this method can also be applied to the quasineutral limit of a system previously studied by Cordier and Grenier [39].

Finally, in section 5, we investigate the combined quasineutral and large magnetic field regime for system (S). The convergence result is stated in Theorem 5.1.

2 Global weak solutions and local conservation laws for the Vlasov-Poisson systems

2.1 Global weak solutions theory

Following Arsenev [4], it is straightforward to build global weak solutions to System (L), for any fixed $\varepsilon > 0$.

Theorem 2.1. *Let $n = 1, 2$ or 3. We consider the functional:*

$$\mathcal{E}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v|^2 dv dx + \frac{1}{2}\int V_\varepsilon^2 dx + \frac{\varepsilon}{2}\int |\nabla_x V_\varepsilon|^2 dx. \tag{2.1}$$

For any $\varepsilon > 0$ and initial data $f_{0,\varepsilon} \geq 0$ bounded in $L^1 \cap L^\infty(\mathbb{R}^{2d})$ such that $\mathcal{E}_\varepsilon(0)$ is finite, there exists a global weak solution to (L) with $f_\varepsilon \in L^\infty_t(L^1_{x,v}) \cap L^\infty_{t,x,v}$ and $\mathcal{E}_\varepsilon(t)$ is non-increasing.

Following Bouchut [19], we obtain the existence of global weak solutions to system (S) and (S'). We recall that d satisfies assumptions (1.14-1.15) (in particular, $d \in L^1(\mathbb{R}^3)$).

Theorem 2.2. *For any $\varepsilon > 0$ and initial data $f_{0,\varepsilon} \geq 0$ bounded in $L^1 \cap L^\infty(\mathbb{R}^6)$ and satisfying $\int(1 + |x|^2 + |v|^2) f_{0,\varepsilon} dx dv < \infty$:*

- ***The case of (S)*** *Let $\mathcal{F}_\varepsilon(t)$ be the functional defined as follows:*

$$\mathcal{F}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v|^2 dv dx + \int d(x)(V_\varepsilon - 1)e^{V_\varepsilon} dx + \frac{\varepsilon}{2}\int |\nabla_x V_\varepsilon|^2 dx. \tag{2.2}$$

If $\mathcal{F}_\varepsilon(0)$ is finite, there exists $f_\varepsilon \in L^\infty_t(L^1_{x,v}) \cap L^\infty_{t,x,v}$ global weak solution to (S) with $\mathcal{F}_\varepsilon(t)$ non-increasing.

- ***The case of (S')****: Let $\mathcal{G}_\varepsilon(t)$ be the functional defined as follows:*

$$\begin{aligned}\mathcal{G}_\varepsilon(t) =& \frac{1}{2}\int f_\varepsilon |v|^2 dv dx + \int d(x)\left(V_\varepsilon - \log\left(\int d e^{V_\varepsilon} dx\right)\right)\frac{e^{V_\varepsilon}}{\int d e^{V_\varepsilon} dx} dx \\ &+ \frac{\varepsilon}{2}\int |\nabla_x V_\varepsilon|^2 dx.\end{aligned} \tag{2.3}$$

If $\mathcal{G}_\varepsilon(0)$ is finite,there exists $f_\varepsilon \in L^\infty_t(L^1_{x,v}) \cap L^\infty_{t,x,v}$ global weak solution to (S') with $\mathcal{G}_\varepsilon(t)$ non-increasing.

In addition, in both cases, we have : $V_\varepsilon \in L^\infty_t(L^6_x)$ and $\operatorname{esssup}_{t,x} V_\varepsilon < \infty$. In particular it means that $e^{V_\varepsilon} \in L^\infty_{t,x}$.

The main difficulties in [19] are to get estimates for the electric potential in the Marcinkiewicz space M^3 to provide some strong compactness, and to use a relevant regularization scheme to preserve the energy inequality.

We assume from now on that the initial data satisfy the uniform estimates :

$$\forall \varepsilon > 0, f_{0,\varepsilon} \geq 0, \tag{2.4}$$

$$\forall \varepsilon > 0, f_{0,\varepsilon} \in L^1 \cap L^\infty(\mathbb{R}^n), \text{ uniformly in } \varepsilon, \tag{2.5}$$

$$\exists C > 0, \forall \varepsilon > 0, \quad \mathcal{E}_\varepsilon(0) \leq C \quad (\text{resp.} \quad \mathcal{F}_\varepsilon, \quad \mathcal{G}_\varepsilon). \tag{2.6}$$

Using a very classical property for Vlasov equations with zero-divergence in v force fields, we get the following unifom in ε estimates.

Lemma 2.1. *For f_ε global weak solution of (L) (resp. (S), resp. (S') we have*

- *(Conservation of L^p norms) For any $p \in [1, +\infty]$, for any $t \geq 0$, $\|f_\varepsilon(t)\|_{L^p_{x,v}} \leq \|f_\varepsilon(0)\|_{L^p_{x,v}}$.*
- *(Maximum principle) If $f_\varepsilon(0) \geq 0$ then for any $t \geq 0, f_\varepsilon(t) \geq 0$.*

- *(Bound on the energy)* $\forall t \geq 0$, $\mathcal{E}_\varepsilon(t) \leq C$ *(resp.* $\mathcal{F}_\varepsilon$, *resp.* $\mathcal{G}_\varepsilon$*).*

Lemma 2.2. *Define* $J_\varepsilon(t,x) = \int f_\varepsilon v dv$. *Then* $J_\varepsilon \in L^\infty_t(L^1_x)$ *uniformly with respect to* ε.

Proof. Actually by the same method, we can also prove that $J_\varepsilon \in L^\infty_t(L^p_x)$ for some $p > 1$ depending on the space dimension, but this result is sufficient for our purpose. The proof is very classical. We can first notice that there exists $C > 0$ independent of ε, such that:

$$\int f_\varepsilon |v|^2 dv dx \leq C.$$

For (L) this is clear by conservation of the energy since all the terms are non-negative.

In the case of (S) we observe

$$\begin{aligned}\int f_\varepsilon |v|^2 dv dx &\leq \mathcal{F}_\varepsilon(t) - 2\int d(V_\varepsilon - 1)e^{V_\varepsilon} dx \\ &\leq \mathcal{F}_\varepsilon(0) + 2\|d\|_{L^1},\end{aligned}$$

since for any $x \in \mathbb{R}, (x-1)e^x \geq -1$. The case of (S') is of course similar. Then we can simply write by positivity of f_ε:

$$\left|\int f_\varepsilon v dv\right| \leq \int f_\varepsilon |v| dv \leq \int_{|v|\leq 1} f_\varepsilon dv + \int_{|v|\geq 1} f_\varepsilon |v|^2 dv,$$

so that: $\|J_\varepsilon\|_{L^\infty_t(L^1_x)} \leq 1 + C$.

□

2.2 Local conservation laws

For system (L), global weak solution in the sense of Arsenev are known to satisfy the following local conservation laws:

Lemma 2.3. *Let* $\varepsilon > 0$. *Let* f_ε *be a global weak solution to (L) with initial data satisfying the assumptions (2.4-2.6). Denote by* $\rho_\varepsilon(t,x) := \int f_\varepsilon(t,x,v)dv$ *and* $J_\varepsilon := \int f_\varepsilon v dv$. *Then the following conservation laws hold in the distributional sense:*

$$\partial_t \rho_\varepsilon + \nabla_x . J_\varepsilon = 0. \tag{2.7}$$

$$\begin{aligned}\partial_t J_\varepsilon + \nabla_x : \left(\int v \otimes v f_\varepsilon dv\right) &= -\frac{1}{2}\nabla_x (V_\varepsilon + 1)^2 \\ &\quad + \varepsilon \operatorname{div}_x(\nabla_x V_\varepsilon \otimes \nabla_x V_\varepsilon) - \frac{\varepsilon}{2}\nabla_x |\nabla_x V_\varepsilon|^2.\end{aligned} \tag{2.8}$$

This is also the case for (S) and (S').

Lemma 2.4. *Let* $\varepsilon > 0$. *Let* f_ε *be a global weak solution to (S) or (S') with initial data satisfying the assumptions (2.4-2.6). We denote the two first moments by* $\rho_\varepsilon(t,x) := \int f_\varepsilon(t,x,v)dv$ *and* $J_\varepsilon := \int f_\varepsilon v dv$ *for any solution* f_ε *to (S) or (S'). The local conservation of charge reads:*

$$\partial_t \rho_\varepsilon + \nabla_x . J_\varepsilon = 0. \tag{2.9}$$

The local conservation of current reads in the case of (S):

$$\partial_t J_\varepsilon + \nabla_x : \left(\int v \otimes v f_\varepsilon dv\right) = -d\nabla_x (e^{V_\varepsilon}) + \varepsilon \operatorname{div}_x(\nabla_x V_\varepsilon \otimes \nabla_x V_\varepsilon) - \frac{\varepsilon}{2}\nabla_x |\nabla_x V_\varepsilon|^2, \tag{2.10}$$

and in the case of (S'):

$$\partial_t J_\varepsilon + \nabla_x : \left(\int v \otimes v f_\varepsilon dv \right) = -\frac{d}{\int de^{V_\varepsilon} dx} \nabla_x (e^{V_\varepsilon}) + \varepsilon \operatorname{div}_x (\nabla_x V_\varepsilon \otimes \nabla_x V_\varepsilon) - \frac{\varepsilon}{2} \nabla_x |\nabla_x V_\varepsilon|^2. \tag{2.11}$$

Proof. Let us briefly explain how we proceed. The main idea is to test the Vlasov equations against test functions of the form $\varphi(t,x)\Psi(\frac{v}{R})$ and $\varphi(t,x)\Psi(\frac{v}{R})v$, with Ψ a compactly supported smooth function, and then to let R go to $+\infty$. The limit is obtained by a dominated convergence argument, using that:

- $f_\varepsilon |v|^2 \in L^\infty_t L^1_{x,v}$, thanks to the energy inequality.
- Using a classical real interpolation argument, we have (in three dimensions, the case of lower dimensions is similar and easier):

  $$\rho_\varepsilon \in L^\infty_t L^{5/3}_x.$$

  By elliptic regularity and Sobolev inequality, we obtain for each Poisson equation (for (L), (S) or (S')):

  $$E_\varepsilon \in L^\infty_t L^{15/4}_x.$$

  For (S) (and also for (S')), this is achieved by considering de^{V_ε} as a source in $L^\infty_t L^\infty_x$.

  We emphasize that this estimate is not uniform in ε, but this does not matter, since we work for any fixed $\varepsilon > 0$.

  This entails that $\rho_\varepsilon E_\varepsilon \in L^\infty_t L^{15/13}_x$.

Finally the local conservation of current in their present forms is obtained through some elementary computations using the Poisson equations.

□

3 From (S) to Isothermal Euler

The isothermal Euler equations (1.21) are hyperbolic symmetrizable. We can perform the change of unknown functions $(\rho, u) \mapsto (\log \frac{\rho}{d}, u)$ that leads to the system:

$$\begin{cases} \partial_t \log(\frac{\rho}{d}) + \nabla_x . u + u . \nabla_x \log(\frac{\rho}{d}) - \nabla_x H . u = 0, \\ \partial_t u + u . \nabla_x u + \nabla_x \log(\frac{\rho}{d}) = 0. \end{cases} \tag{3.1}$$

(we recall that by definition, $d = e^{-H}$).

Therefore, using classical results on hyperbolic symmetrizable systems ([123]), we get the local existence of smooth solutions:

Proposition 3.1. *For any initial data $\rho_0 > 0, u_0$ such that $\rho_0 \in L^1(\mathbb{R}^3)$, $\log(\frac{\rho_0}{d}) \in H^s(\mathbb{R}^3)$ and $u_0 \in H^s(\mathbb{R}^3)$ for $s > \frac{3}{2} + 1$, there is existence and uniqueness of a local smooth solution $\rho > 0$ and u to (3.1) such that :*

$$\log \frac{\rho}{d}, u \in \mathcal{C}^0_t([0, T^*[, H^s(\mathbb{R}^3)) \cap \mathcal{C}^1_t([0, T^*[, H^{s-1}(\mathbb{R}^3)) \tag{3.2}$$

$$\rho \in \mathcal{C}^1([0, T^*[\times \mathbb{R}^3) \tag{3.3}$$

for some $T^* > 0$.

Since shocks may occur for large times, we will have to restrict to local results. We now prove the convergence of the charge and current density to the smooth solution to (3.1), as long as the latter exist.

Theorem 3.1 (The case of (S))**.** *Let $\rho_0 > 0, u_0$ verifying the assumptions of Proposition 3.1 and ρ, u the corresponding strong solutions of system (1.21). We assume that the sequence of initial data $(f_{0,\varepsilon})$ satisfies the assumptions (2.4-2.6) and:*

$$\int f_{0,\varepsilon}|v-u_0|^2 dvdx \to 0, \tag{3.4}$$

$$\|\sqrt{\varepsilon}\nabla_x V_{0,\varepsilon}\|_{L^2} \to 0, \tag{3.5}$$

$$\int \left(de^{V_{0,\varepsilon}}\log(de^{V_{0,\varepsilon}}/\rho_0) - de^{V_{0,\varepsilon}} + \rho_0\right) dx \to 0, \tag{3.6}$$

where $V_{0,\varepsilon}$ is solution of the nonlinear Poisson equation:

$$-\varepsilon\Delta_x V_{0,\varepsilon} = \int f_{0,\varepsilon} dv - de^{V_{0,\varepsilon}}.$$

Then ρ_ε weakly converges to ρ and J_ε weakly converges to ρu in L^1. Furthermore we have the following local in time strong convergences: $u_\varepsilon = J_\varepsilon/\rho_\varepsilon$ strongly converges to u in the following sense:

$$\int |u_\varepsilon - u|^2 \rho_\varepsilon dx \to 0$$

in L^∞_t and

$$\sqrt{de^{V_\varepsilon}} \to \sqrt{\rho}$$

in $L^\infty_t(L^2_x)$.

Remark 3.1 (On the class of admissible initial data satisfying assumptions (2.4-2.6)and (3.4-3.6)). This class is not empty : indeed it includes Maxwellians of the form

$$f_{0,\varepsilon}(x,v) = \frac{\rho_{0,\varepsilon}(x)}{(2\pi T_{i,\varepsilon})^{3/2}} e^{\frac{-|v-u_0(x)|^2}{2T_{i,\varepsilon}}}, \tag{3.7}$$

where $\rho_{0,\varepsilon}$ is computed by the Poisson equation after having previously chosen $V_{0,\varepsilon}$ such that (3.5) and (3.6) hold (for example, we can simply take $V_{0,\varepsilon} = V_0$ with V_0 safisfying $de^{V_0} = \rho_0$ and (3.5) and (3.6) trivially hold) and $T_{i,\varepsilon} \to_{\varepsilon\to 0} 0$ (cold ions approximation).

Proof. Let (ρ, u) verifying the regularity of (3.2-3.3) (for the moment ρ and u do not a priori satisfy the isothermal Euler equations).

We recall that the energy for system (S) is the following functional:

$$\mathcal{F}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v|^2 dvdx + \frac{\varepsilon}{2}\int |\nabla_x V_\varepsilon|^2 dx + \int d(V_\varepsilon - 1)e^{V_\varepsilon} dx.$$

The first two terms correspond to the energy for the quasineutral Vlasov-Poisson limit studied by Brenier in [27]. Therefore we accordingly modulate this quantity by considering:

$$\frac{1}{2}\int f_\varepsilon |v-u|^2 dvdx + \frac{\varepsilon}{2}\int |\nabla_x V_\varepsilon|^2 dx.$$

Let us now look at $m_\varepsilon := de^{V_\varepsilon}$. We can notice that:

$$\int d(V_\varepsilon - 1)e^{V_\varepsilon} dx = \int (m_\varepsilon \log(m_\varepsilon/d) - m_\varepsilon) dx.$$

As mentioned in the introduction, we observe a strong analogy with the relative entropy in collisional kinetic equations (we refer to [141] for a reference on the topic). So by analogy, we modulate this quantity and hence consider the following relative entropy:

$$\mathcal{H}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v-u|^2 dvdx + \int (m_\varepsilon \log(m_\varepsilon/\rho) - m_\varepsilon + \rho) dx + \frac{\varepsilon}{2}\int |\nabla_x V_\varepsilon|^2 dx. \quad (3.8)$$

Later on, the well known inequality (which is a plain consequence of the inequality $x - 1 \geq \log x$, for $x > 0$) will be very useful:

$$\int (\sqrt{a} - \sqrt{b})^2 dx \leq \int (a \log(a/b) - a + b) dx. \quad (3.9)$$

We want to show that $\mathcal{H}_\varepsilon(t)$ satisfies the inequality:

$$\mathcal{H}_\varepsilon(t) \leq \mathcal{H}_\varepsilon(0) + G_\varepsilon(t) + C\int_0^t \|\nabla_x u\|_{L^\infty} \mathcal{H}_\varepsilon(s) ds, \quad (3.10)$$

with $G_\varepsilon(t) \to 0$ uniformly in time.

We show how we can deduce this kind of estimate (the computations can be rigorously justified using the local conservation laws of Lemma 2.4). Since the energy is non-increasing, we have:

$$\frac{d\mathcal{H}_\varepsilon(t)}{dt} \leq I_\varepsilon(t), \quad (3.11)$$

with:

$$\begin{aligned} I_\varepsilon(t) := \int \partial_t f_\varepsilon \left(\frac{1}{2}|u|^2 - v.u\right) dvdx + &\int f_\varepsilon \partial_t \left(\frac{1}{2}|u|^2 - v.u\right) dvdx \\ &+ \int \partial_t \left(m_\varepsilon \log(d/\rho)\right) dx + \int \partial_t \rho dx. \end{aligned} \quad (3.12)$$

Let us first focus on the first two terms of $I_\varepsilon(t)$. Thanks to the Vlasov equation satisfied by f_ε and after integrating by parts, we get:

$$\begin{aligned} \int \partial_t f_\varepsilon \left(\frac{1}{2}|u|^2 - v.u\right) dvdx + &\int f_\varepsilon \partial_t \left(\frac{1}{2}|u|^2 - v.u\right) dvdx \\ &= \int f_\varepsilon \left(\partial_t + v.\nabla_x + E_\varepsilon.\nabla_v\right)\left(\frac{1}{2}|u|^2 - v.u\right) dvdx \\ &= \int f_\varepsilon (u-v).(\partial_t + v.\nabla_x) u dvdx - \int f_\varepsilon E_\varepsilon.u dvdx \\ &= \int f_\varepsilon (u-v).(\partial_t + u.\nabla_x) u dvdx - \int f_\varepsilon (u-v).\left((u-v).\nabla_x u\right) dvdx \\ &- \int \rho_\varepsilon E_\varepsilon.u dx. \end{aligned} \quad (3.13)$$

We can use the Poisson equation to compute the last term of (3.13).

$$\begin{aligned}
-\int \rho_\varepsilon E_\varepsilon.udx &= \int \rho_\varepsilon \nabla_x V_\varepsilon.udx \\
&= \int de^{V_\varepsilon}\nabla_x V_\varepsilon.udx - \varepsilon\int \Delta_x V_\varepsilon \nabla_x V_\varepsilon.udx \\
&= \int d\nabla_x e^{V_\varepsilon}.udx - \varepsilon\int \nabla_x : (\nabla_x V_\varepsilon \otimes \nabla_x V_\varepsilon)udx + \varepsilon\int \frac{1}{2}\nabla_x|\nabla_x V_\varepsilon|^2 udx \\
&= -\int de^{V_\varepsilon}\operatorname{div}_x udx - \int e^{V_\varepsilon}\nabla_x d.udx + \varepsilon\int D(u) : (\nabla_x V_\varepsilon \otimes \nabla_x V_\varepsilon)dx \\
&- \varepsilon\int \frac{1}{2}|\nabla_x V_\varepsilon|^2 \operatorname{div}_x udx,
\end{aligned}$$

where $D(u) = \frac{1}{2}\left(\partial_{x_i}u_j + \partial_{x_j}u_i\right)_{i,j}$ is the symmetric part of $\nabla_x u = (\partial_{x_i}u_j)_{i,j}$.

We now focus on the last two terms of $I_\varepsilon(t)$:

$$\begin{aligned}
\int \partial_t\left(m_\varepsilon \log(d/\rho)\right)dx + \int \partial_t\rho dx &= \int (-de^{V_\varepsilon}/\rho + 1)\partial_t\rho + \int d\partial_t e^{V_\varepsilon}\log(d/\rho)dx \\
&= \int (-de^{V_\varepsilon}/\rho + 1)\partial_t\rho + \varepsilon\int \partial_t\Delta_x V_\varepsilon \log(d/\rho)dx \qquad (3.14)\\
&- \int \operatorname{div}_x J_\varepsilon \log(d/\rho)dx,
\end{aligned}$$

using the Poisson equation and the local conservation of mass:

$$\partial_t\rho_\varepsilon = -\operatorname{div}_x J_\varepsilon.$$

We observe that:

$$\begin{aligned}
&\int f_\varepsilon(u-v).\nabla_x \log\frac{\rho}{d}dvdx \\
&\quad= \int \rho_\varepsilon u.\nabla_x \log\frac{\rho}{d}dx - \int J_\varepsilon.\nabla_x \log\frac{\rho}{d}dx \\
&\quad= \int de^{V_\varepsilon}u.\nabla_x \log\frac{\rho}{d}dx - \varepsilon\int \Delta_x V_\varepsilon u.\nabla_x \log\frac{\rho}{d}dx \\
&\quad+ \int J_\varepsilon.\nabla_x \log\frac{\rho}{d}dx.
\end{aligned} \tag{3.15}$$

Consequently, according to (3.13) and (3.14) we get:

$$\begin{aligned}
I_\varepsilon(t) = &\int -de^{V_\varepsilon}(\partial_t\rho + \nabla_x.u + u.\nabla_x\log\rho)dx + \int \partial_t\rho dx \\
&+ \int f_\varepsilon(u-v)\left(\partial_t u + u.\nabla_x u + \nabla_x\log\frac{\rho}{d}\right)dvdx \\
&+\varepsilon\int \Delta_x V_\varepsilon u.\nabla_x\log\frac{\rho}{d}dx + \varepsilon\int \partial_t\Delta_x V_\varepsilon\log(d/\rho)dx \\
&- \int f_\varepsilon(u-v)\left((u-v).\nabla_x u\right)dvdx + \varepsilon\int D(u) : (\nabla_x V_\varepsilon \otimes \nabla_x V_\varepsilon)dx \\
&-\varepsilon\int \frac{1}{2}|\nabla_x V_\varepsilon|^2\operatorname{div}_x udx.
\end{aligned} \tag{3.16}$$

Let us now introduce the so-called acceleration operator A:

$$A(u,\rho) = \begin{pmatrix} \partial_t \log \frac{\rho}{d} + \nabla_x . u + u.\nabla_x \log \frac{\rho}{d} - \nabla_x H.u \\ \partial_t u + u.\nabla_x u + \nabla_x \log \frac{\rho}{d} \end{pmatrix}. \tag{3.17}$$

We observe that :

$$\begin{aligned} \int \rho(\nabla_x . u + u \nabla_x \log \rho) dx &= \int (\rho \nabla_x u + u.\nabla_x \rho) dx \\ &= \int \nabla_x . (\rho u) dx = 0, \end{aligned} \tag{3.18}$$

and thus $\int A(u,\rho).\begin{pmatrix}\rho \\ 0\end{pmatrix} dx = \int \partial_t \rho dx$.

Gathering the pieces together we have proved:

$$\begin{aligned} I_\varepsilon(t) = &\int A(u,\rho).\begin{pmatrix} -de^{V_\varepsilon} + \rho \\ \rho_\varepsilon u - J_\varepsilon \end{pmatrix} dx \\ &+ \varepsilon \int \Delta_x V_\varepsilon u.\nabla_x \log \frac{\rho}{d} dx + \varepsilon \int \partial_t \Delta_x V_\varepsilon \log(d/\rho) dx \\ &- \int f_\varepsilon (u-v)\left((u-v).\nabla_x u\right) dv dx + \varepsilon \int D(u) : (\nabla_x V_\varepsilon \otimes \nabla_x V_\varepsilon) dx \\ &- \varepsilon \int \frac{1}{2} |\nabla_x V_\varepsilon|^2 \operatorname{div}_x u dv dx. \end{aligned} \tag{3.19}$$

We are now ready to prove that $\mathcal{H}_\varepsilon$ satisfies the expected stability inequality.

It is readily seen that there exists a constant independent of ε such that:

$$\left| \int f_\varepsilon (u-v)\left((u-v).\nabla_x u\right) dv dx \right| \leq C \int f_\varepsilon |u-v|^2 \|\nabla_x u\|_{L^\infty} dv dx,$$

$$\left| \varepsilon \int D(u) : (\nabla_x V_\varepsilon \otimes \nabla_x V_\varepsilon) dx \right| \leq C \int \varepsilon |\nabla_x V_\varepsilon|^2 \|\nabla_x u\|_{L^\infty} dx.$$

We consider now:

$$\begin{aligned} G_\varepsilon(t) \quad &:= \quad \int_0^t \left(\varepsilon \int \Delta_x V_\varepsilon u.\nabla_x \log \frac{\rho}{d} dx + \varepsilon \int \partial_s \Delta_x V_\varepsilon \log(d/\rho) dx \right) ds \\ &= \quad \int_0^t \left(-\varepsilon \int \nabla_x V_\varepsilon . \nabla_x \left(u.\nabla_x \log \frac{\rho}{d} \right) dx - \varepsilon \int \partial_s \nabla_x V_\varepsilon . \nabla_x \log(d/\rho) dx \right) ds \\ &= \quad \int_0^t -\sqrt{\varepsilon} \int \sqrt{\varepsilon} \nabla_x V_\varepsilon . \nabla_x \left(u.\nabla_x \log\left(\frac{\rho}{d}\right)\right) dx ds + \sqrt{\varepsilon} \int_0^t \int \sqrt{\varepsilon} \nabla_x V_\varepsilon . \partial_s \nabla_x \log(d/\rho) dx ds \\ &- \quad \sqrt{\varepsilon} \int \sqrt{\varepsilon} \nabla_x V_\varepsilon(t,x) . \nabla_x \log(d/\rho(t,x)) dx + \sqrt{\varepsilon} \int \sqrt{\varepsilon} \nabla_x V_\varepsilon(0,x) . \nabla_x \log(d/\rho(0,x)) dx. \end{aligned}$$

Thanks to the conservation of the energy, $\sqrt{\varepsilon}\nabla_x V_\varepsilon$ is bounded uniformly with respect to ε in $L^\infty_t(L^2_x)$. Consequently, using Cauchy-Schwarz inequality, we get for any $0 \leq t \leq T$:

$$\begin{aligned} G_\varepsilon(t) \quad \leq \quad & C\sqrt{\varepsilon} \|\sqrt{\varepsilon} \nabla_x V_\varepsilon\|_{L^\infty_t(L^2_x)} \times \\ & \left(\|\nabla_x (u.\nabla_x \log \rho/d)\|_{L^\infty_t(L^2_x)} + \|\log(\rho/d)\|_{W^{1,\infty}_t(H^1_x)} \right), \end{aligned}$$

and so we have $G_\varepsilon(t) \to 0$ when $\varepsilon \to 0$, locally uniformly in time.

Finally since

$$\mathcal{H}_\varepsilon(t) \le \mathcal{H}_\varepsilon(0) + \int_0^t I_\varepsilon(s)ds,$$

we have proved that:

$$\mathcal{H}_\varepsilon(t) \le \mathcal{H}_\varepsilon(0) + \int_0^t \int A(u,\rho).\begin{pmatrix} -de^{V_\varepsilon} + \rho \\ \rho_\varepsilon u - J_\varepsilon \end{pmatrix} dx + G_\varepsilon(t)$$

$$+C\left(\int_0^t \int f_\varepsilon |u-v|^2 \|\nabla_x u\|_{L^\infty} dv dx ds + \varepsilon \int_0^t \int \frac{1}{2}|\nabla_x V_\varepsilon|^2 \|\nabla_x u\|_{L^\infty} dx ds\right). \quad (3.20)$$

Now we can choose ρ and u to be solutions of $A(\rho,u) = 0$, with initial conditions $(\rho,u)_{|(t=0)} = (\rho_0,u_0)$. In other words ρ and u are solutions to the Isothermal Euler system (3.1).

Then we have:

$$\mathcal{H}_\varepsilon(t) \le \mathcal{H}_\varepsilon(0) + G_\varepsilon(t) + C\int_0^t \|\nabla_x u\|_{L^\infty} \mathcal{H}_\varepsilon(s)ds \quad (3.21)$$

and thus, since $\mathcal{H}_\varepsilon(0) \to 0$ and $G_\varepsilon(t) \to 0$, we deduce by Gronwall inequality that

$$\mathcal{H}_\varepsilon(t) \to 0, \quad (3.22)$$

when $\varepsilon \to 0$ (uniformly with respect to time).

By inequality (3.9), this means in particular that

$$\sqrt{de^{V_\varepsilon}} \to \sqrt{\rho}.$$

strongly in $L^\infty_t L^2_x$.

Because of the uniform estimates in $L^\infty_t(L^p_x)$ for some $p > 1$, ρ_ε (resp. J_ε) weakly converges (up to a subsequence) to some $\tilde{\rho}$ (resp. J). In the other hand, in the sense of distributions, thanks to the quasineutral Poisson equation:

$$de^{V_\varepsilon} - \rho_\varepsilon \rightharpoonup 0$$

and thus,

$$de^{V_\varepsilon} \rightharpoonup \tilde{\rho}.$$

Therefore, by uniqueness of the limit we deduce that $\tilde{\rho} = \rho$.

The last step of the proof relies on a by now classical convexity argument. We first get the following Cauchy-Schwarz inequality:

$$\frac{|J_\varepsilon - \rho_\varepsilon u|^2}{\rho_\varepsilon} = \frac{\left(\int f_\varepsilon (v-u)dv\right)^2}{\int f_\varepsilon dv} \le \int f_\varepsilon |v-u|^2 dv. \quad (3.23)$$

The functional $(\rho, J) \to \int \frac{|J-\rho u|^2}{\rho} dx$ is convex and lower semi-continuous with respect to the weak convergence of measures (see [27]). Consequently the weak convergence in the sense of measures $\rho_\varepsilon \rightharpoonup \rho$ and $J_\varepsilon \rightharpoonup J$ leads to:

$$\int \frac{|J-\rho u|^2}{\rho} dx \le \liminf_{\varepsilon \to 0} \int \frac{|J_\varepsilon - \rho_\varepsilon u|^2}{\rho_\varepsilon} dx. \quad (3.24)$$

So $J = \rho u$.

To conclude, the uniqueness of the limit allows us to say that the weak convergences actually hold without any extraction.

□

Remark 3.2 (Rate of convergence). Assume that $\mathcal{H}_\varepsilon(0) \leq C\sqrt{\varepsilon}$. Then the previous estimates show that locally uniformly in time:

$$\mathcal{H}_\varepsilon(t) \leq C\sqrt{\varepsilon}. \tag{3.25}$$

4 Generalization to other quasineutral limits

4.1 From (S') to Isothermal Euler

Similarly, we can prove an analogous theorem for system (S'):

Theorem 4.1 (The case of (S'))**.** *Let $\rho_0 > 0, u_0$ verifying the assumptions of Proposition 3.1 and ρ, u the corresponding strong solutions of system (1.21). We assume that the sequence of initial data $(f_{\varepsilon,0})$ satisfies the assumptions (2.4-2.6) and:*

$$\int f_{0,\varepsilon}|v-u_0|^2 dvdx \to 0, \tag{4.1}$$

$$\sqrt{\varepsilon}\nabla_x V_{0,\varepsilon} \to 0, \tag{4.2}$$

strongly in L^2 and

$$\int \left(\frac{de^{V_{0,\varepsilon}}}{\int de^{V_{0,\varepsilon}}dx} \log\left(\frac{de^{V_{0,\varepsilon}}}{\int de^{V_{0,\varepsilon}}dx}/\rho_0 \right) - \frac{de^{V_{0,\varepsilon}}}{\int de^{V_{0,\varepsilon}}dx} + \rho_0 \right) dx \to 0, \tag{4.3}$$

where $V_{0,\varepsilon}$ is solution of the nonlinear Poisson equation:

$$-\varepsilon\Delta_x V_{0,\varepsilon} = \int f_{0,\varepsilon} dv - \frac{de^{V_{0,\varepsilon}}}{\int de^{V_{0,\varepsilon}}dx}.$$

Then ρ_ε weakly converges to ρ and J_ε weakly converges to ρu in L^1. Furthermore, we have the following local strong convergences: $u_\varepsilon = J_\varepsilon/\rho_\varepsilon$ strongly converges to ρu in the following sense:

$$\int |u_\varepsilon - u|^2 \rho_\varepsilon dx \to 0$$

in L^∞_t and

$$\frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon}dx} \to \rho$$

in $L^\infty_t(L^1_x)$.

Remark 4.1. The convergence result for $\frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon}dx}$ is better than in the limit for (S).

Sketch of proof. According to Theorem 2.2, the functional $\mathcal{G}_\varepsilon(t)$ is non-increasing:

$$\mathcal{G}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v|^2 dvdx + \int d(x)\left(V_\varepsilon - \log\left(\int de^{V_\varepsilon}dx\right)\right)\frac{e^{V_\varepsilon}}{\int de^{V_\varepsilon}dx}dx + \frac{\varepsilon}{2}\int |\nabla_x V_\varepsilon|^2 dx.$$

As in the previous proof, we consider $m_\varepsilon = \frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon}dx}$, and we notice that:

$$\int d\left(V_\varepsilon - \log\left(\int de^{V_\varepsilon}dx\right)\right)\frac{e^{V_\varepsilon}}{\int de^{V_\varepsilon}dx}dx = \int (m_\varepsilon \log(m_\varepsilon/d))dx. \tag{4.4}$$

Since $\int m_\varepsilon dx = 1$ (and thus, $\partial_t \int m_\varepsilon dx = 0$) we can actually add this quantity to the energy so that:

$$\int d\left(V_\varepsilon - \log\left(\int de^{V_\varepsilon}dx\right) - 1\right)\frac{e^{V_\varepsilon}}{\int de^{V_\varepsilon}dx}dx = \int (m_\varepsilon \log(m_\varepsilon/d) - m_\varepsilon)dx, \tag{4.5}$$

and we are in the same case as before. Therefore we can consider the same modulated energy $\mathcal{H}_\varepsilon(t)$. We skip the computations, which are very similar.

The only difference is that since $\int \rho dx = \int \frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon}dx}dx = 1$, we can use the classical Csiszar-Kullback-Pinner inequality, so that the relative entropy controls the L^1 norm of the difference:

$$\frac{1}{4}\left\|\frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon}dx} - \rho_\varepsilon\right\|_{L^1_x}^2 \leq \int\left(\frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon}dx}\log\left(\frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon}dx}/\rho_\varepsilon\right) - \frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon}dx} + \rho_\varepsilon\right)dx. \tag{4.6}$$

□

4.2 From (L) to Shallow-Water

We now treat the case of system (L), following the same methodology as before.

4.2.1 Formal derivation of the Shallow-Water equations

For monokinetic data, i.e. $f_\varepsilon(t,x,v) = \rho_\varepsilon(t,x)\delta_{v=u_\varepsilon(t,x)}$, the conservation laws state:

$$\partial_t \rho_\varepsilon + \nabla_x.(\rho_\varepsilon u_\varepsilon) = 0, \tag{4.7}$$

$$\partial_t(\rho_\varepsilon u_\varepsilon) + \nabla_x : (\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) = -\rho_\varepsilon \nabla_x V_\varepsilon. \tag{4.8}$$

We recall that the Poisson equation is:

$$V_\varepsilon - \varepsilon\Delta_x V_\varepsilon = \int f_\varepsilon dv - 1. \tag{4.9}$$

Since ρ_ε and J_ε are uniformly bounded in $L^\infty_t(L^1_x)$, the following convergences (up to a subsequence) hold in the sense of distributions: $\rho_\varepsilon \rightharpoonup \rho$ and $J_\varepsilon \rightharpoonup J$.

If we formally pass to the limit $\varepsilon \to 0$ we get:

$$V = \rho - 1. \tag{4.10}$$

Consequently the limit system is the following:

$$\left\{\begin{array}{l} \partial_t \rho + \nabla_x.(\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x : (\rho u \otimes u) = -\rho \nabla_x \rho. \end{array}\right. \tag{4.11}$$

or equivalently for smooth data and $\rho > 0$:

$$\left\{\begin{array}{l} \partial_t \rho + \nabla_x.(\rho u) = 0, \\ \partial_t u + u.\nabla_x u = -\nabla_x \rho. \end{array}\right. \tag{4.12}$$

As it has been said before, this system can be interpreted in $1D$ or $2D$ as the Shallow Water equations.

Remark 4.2 (On the kinetic version of the shallow-water limit). In a formal sense, one can also easily perform the kinetic limit $\varepsilon \to 0$ and get the equation:

$$\begin{cases} \partial_t f + v.\nabla_x f - \nabla_x \rho.\nabla_v f = 0, \\ \rho = \int f dv. \end{cases} \tag{4.13}$$

To our knowledge, this equation is very badly mathematically understood. The only existence result we are able to prove is the local existence of analytic solutions. Actually the proof given by Mouhot and Villani [130] (section 9, local in time interaction) in the Vlasov-Poisson case identically holds. Indeed, we notice that in that proof, they do not need the smoothing effect on the force field provided by the Poisson equation. Although it is not explicitly said, the case of the singular force field $F = \pm\nabla_x \rho$ is automatically included in their analysis. Of course this is not the case for the other results of their paper.

Remark 4.3. Let us also mention that the quasineutral together with the gyrokinetic limit of a similar system was performed by the author in [94]. With quite general initial data, we get a limit equation of kinetic nature. In other words we do not need to restrict to particular initial data (or to strong regularity); this rather remarkable fact is due to the anisotropy of the system with the so-called finite Larmor radius scaling ([60]). The Poisson equation then only degenerates in the magnetic field direction but this is overcome thanks to an averaging lemma.

4.2.2 Rigorous derivation for (partially) well-prepared data

As the Shallow Water equations (4.12) are hyperbolic symmetrizable we get the local existence of smooth solutions [123].

Proposition 4.1. *For any initial data ρ_0, u_0 in $H^s(\mathbb{T}^n)$ for $s > \frac{n}{2} + 1$, there is existence and uniqueness of a local smooth solution to (4.12):*

$$\rho, u \in \mathcal{C}^0_t([0, T^*[, H^s(\mathbb{T}^n)) \cap \mathcal{C}^1_t([0, T^*[, H^{s-1}(\mathbb{T}^n))$$

for some $T^* > 0$.

As before, we restrict to finite time intervals.

Theorem 4.2 (The case of (L)). *Let $\rho_0 \geq 0, u_0 \in H^s(\mathbb{T}^n)$ ($s > n/2 + 1$ large enough) and ρ, u the corresponding strong solutions to System 4.12. We assume that the sequence of initial data $(f_{0,\varepsilon})$ satisfies the hypotheses (2.4-2.6) and:*

$$\int f_{0,\varepsilon} |v - u_0|^2 dv dx \to 0, \tag{4.14}$$

$$\int |\sqrt{\varepsilon} \nabla_x V_{0,\varepsilon}|^2 dx \to 0, \tag{4.15}$$

and

$$(Id - \varepsilon\Delta_x)^{-1}(\rho_{0,\varepsilon} - 1) \to (\rho_0 - 1), \tag{4.16}$$

strongly in L^2.

Then ρ_ε weakly- converges to ρ and J_ε weakly converges to ρu in L^1. Furthermore, we have the local strong convergences:*

$$V_\varepsilon \to \rho - 1$$

in $L^\infty_t L^2_x$*,* u_ε *strongly converges to* u *in the following sense:*

$$\int |u_\varepsilon - u|^2 \rho_\varepsilon dx \to 0.$$

Moreover,

$$\int |\sqrt{\varepsilon}\nabla_x V_\varepsilon|^2 dx \to 0.$$

Remark 4.4.

- Assumptions (4.15) and (4.16) are satisfied for some smooth ρ_0 as soon as $\rho_{0,\varepsilon}$ strongly converges to ρ_0 in L^2 (For instance, when $\rho_{0,\varepsilon}$ is uniformly bounded in some H^α with $\alpha > 0$). Indeed, in this case, we notice that:

$$\frac{1}{\sqrt{\varepsilon}}\Big((Id - \varepsilon\Delta_x)^{-1}\rho_{0,\varepsilon} - \rho_{0,\varepsilon}\Big) \tag{4.17}$$

  lies in a compact of H^{-1} endowed with its strong topology.

  This implies, thanks to the Poisson equation that $\sqrt{\varepsilon}\Delta_x V_{0,\varepsilon}$ lies in a compact for the H^{-1} norm. This means that $\sqrt{\varepsilon}\nabla_x V_{0,\varepsilon}$ strongly converges to some $\nabla_x \Psi_0$ in the L^2 norm (up to a sequence). But $V_{\varepsilon,0}$ also strongly converges in L^2 to $\rho_0 - 1$, so by uniqueness of the limit in the sense of distributions, $\nabla_x \Psi_0 = 0$.

- For this reason, an "ill-prepared" case would correspond to some oscillating in space initial data:

$$V_{0,\varepsilon} = \rho_0 - 1 + r_\varepsilon,$$

  where r_ε only weakly converges to 0 and $\sqrt{\varepsilon}\nabla_x r_\varepsilon$ is bounded in L^2 (one can think of $r_\varepsilon(x) = e^{ix/\sqrt{\varepsilon}}$). Then we would have to filter these oscillations in space to prove strong convergences.

  This indicates that the massless electrons have a stabilizing effect on the system, insofar as no time oscillations occur unless some space oscillations are imposed at the initial time.

Remark 4.5. We need some additional regularity on ρ and u in order to handle some non-linear quantities : so we take s large enough (the lower bound $n/2 + 1$ is not sufficient). But we will not dwell on the optimal constant.

Sketch of proof. The functional $\mathcal{E}_\varepsilon(t)$ recalled below is the energy of system (L):

$$\mathcal{E}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v|^2 dv dx + \frac{1}{2}\int V_\varepsilon^2 dx + \frac{\varepsilon}{2}\int |\nabla_x V_\varepsilon|^2 dx.$$

The principle of the proof is to consider the following modulation of $\mathcal{E}_\varepsilon(t)$:

$$\tilde{\mathcal{H}}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v-u|^2 dv dx + \frac{1}{2}\int |V_\varepsilon - (\rho - 1)|^2 dx + \frac{1}{2}\int |\sqrt{\varepsilon}\nabla_x V_\varepsilon|^2 dx. \tag{4.18}$$

Then we can show with similar considerations as previous proofs that $\tilde{\mathcal{H}}_\varepsilon(t)$ satisfies an inequality of the form:

$$\tilde{\mathcal{H}}_\varepsilon(t) \leq \tilde{\mathcal{H}}_\varepsilon(0) + \tilde{G}_\varepsilon(t) + \int_0^t C\|\partial_x u\|_{L^\infty}\tilde{\mathcal{H}}_\varepsilon(s)ds$$

with $\tilde{G}_\varepsilon(t) \to 0$ when $\varepsilon \to 0$, uniformly in time.

□

4.3 Quasineutral limit for the isothermal Euler-Poisson system of Cordier and Grenier

As it was mentioned in the introduction, in [39], Cordier and Grenier study an isothermal Euler-Poisson version of (S) and prove the quasineutral limit to the same kind of Euler equation (1.21). So our result can be seen somehow as a generalization of theirs, since our startpoint is the kinetic equation. Actually the relative entropy method can also apply to their system.

In [39], Cordier and Grenier consider the isothermal Euler-Poisson system (in $1D$):

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \frac{T}{\rho_\varepsilon}\partial_x \rho_\varepsilon = -\partial_x V_\varepsilon \\ -\varepsilon \partial^2_{xx} V_\varepsilon = \rho_\varepsilon - e^{V_\varepsilon}, \end{cases} \tag{4.19}$$

where T is the (scaled) temperature of ions, of order 1.

The authors perform the quasineutral limit $\varepsilon \to 0$ to the so-called quasineutral Euler system:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t u + u \partial_x u + \frac{T+1}{\rho}\partial_x \rho = 0. \end{cases} \tag{4.20}$$

Their proof relies on rather tricky energy estimates obtained by pseudodifferential calculus (using the framework introduced by Grenier [82]).

Relative entropy methods provide an alternative and more direct proof. Indeed, the following functional is an energy for system (4.19):

$$\mathfrak{E}_\varepsilon(t) = \frac{1}{2}\int \rho_\varepsilon u_\varepsilon^2 dx + T\int \rho_\varepsilon(\log \rho_\varepsilon - 1)dx + \int (V_\varepsilon - 1)e^{V_\varepsilon}dx + \frac{\varepsilon}{2}\int |\partial_x V_\varepsilon|^2 dx. \tag{4.21}$$

We can consequently consider the modulated energy:

$$\begin{aligned} \mathfrak{H}_\varepsilon(t) =& \frac{1}{2}\int \rho_\varepsilon |u_\varepsilon - u|^2 dx + T\int \rho_\varepsilon(\log(\rho_\varepsilon/\rho) - 1 + \rho/\rho_\varepsilon)dx \\ &+ \int (m_\varepsilon \log(m_\varepsilon/\rho) - m_\varepsilon + \rho)dx + \frac{\varepsilon}{2}\int |\partial_x V_\varepsilon|^2 dx, \end{aligned} \tag{4.22}$$

with $m_\varepsilon = e^{V_\varepsilon}$. We can show, as in the previous proofs, that H_ε satisfies some stability inequality. In addition to the results obtained for system (S), we get the following strong convergence in $L^\infty_t(L^2_x)$:

$$\sqrt{\rho_\varepsilon} \to \sqrt{\rho}. \tag{4.23}$$

5 Combined quasineutral and large magnetic field limit

We now study the limit $\varepsilon \to 0$ of the following system, which is nothing but system (S') with a strong magnetic field.

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + \left(E_\varepsilon + \frac{v\wedge e_\parallel}{\varepsilon}\right).\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\varepsilon^{2\alpha}\Delta_x V_\varepsilon = \int f_\varepsilon dv - \frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon}dx} \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon}, \int f_{0,\varepsilon} dv dx = 1. \end{cases} \tag{5.1}$$

We shall not dwell on the existence of global weak solutions, since it is very similar to the theory for system (S') that was studied in section 2.

We start with a formal analysis in order to show how we can get the expected limit system.

5.1 Formal analysis

For monokinetic data, i.e. $f_\varepsilon(t,x,v) = \rho_\varepsilon(t,x)\delta(v = u_\varepsilon(t,x))$, the two first conservation laws read:

$$\partial\rho_\varepsilon + \nabla_x.(\rho_\varepsilon u_\varepsilon) = 0,$$

$$\partial_t u_\varepsilon + u_\varepsilon.\nabla_x u_\varepsilon = E_\varepsilon + \frac{u_\varepsilon^\perp}{\varepsilon}.$$

The Poisson equation reads:

$$-\varepsilon^{2\alpha}\Delta_x V_\varepsilon = \rho_\varepsilon - \frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon}dx}.$$

In the limit $\varepsilon \to 0$, assuming that in some sense $\rho_\varepsilon \rightharpoonup \rho, V_\varepsilon \rightharpoonup V$ (as well as $\nabla_x V_\varepsilon \rightharpoonup \nabla_x V$), we get:

$$\rho = \frac{de^V}{\int de^V dx}, \tag{5.2}$$

and this implies that:

$$\nabla_x V = \frac{\nabla_x \rho}{\rho} - \frac{\nabla_x d}{d}.$$

If we multiply the second conservation law by ε we get:

$$u_\varepsilon^\perp = \varepsilon\left(\partial_t u_\varepsilon + u_\varepsilon.\nabla_x u_\varepsilon - E_\varepsilon\right).$$

This implies that $u_\varepsilon^\perp \rightharpoonup 0$. This convergence can not occur in a strong sense because of the oscillations in time of frequency $\mathcal{O}(1/\varepsilon)$, created by the magnetic field, but we can precisely describe the oscillations and consequently the strong convergence.

We denote by $\mathcal{R}(s)$ the rotation of axis $e_\parallel$ and angle s. Explicityly, we have:

$$\mathcal{R}(s) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Following standard methods for singular perturbation problems ([83], [147]), we introduce the filtered momentum w_ε defined by:

$$w_\varepsilon = \mathcal{R}(t/\varepsilon)u_\varepsilon. \tag{5.3}$$

Then it is readily seen that w_ε satisfies the equation:

$$\partial_t w_\varepsilon + \mathcal{R}(-t/\varepsilon)w_\varepsilon.\nabla_x w_\varepsilon = \mathcal{R}(t/\varepsilon)E_\varepsilon. \tag{5.4}$$

We assume then that $w_\varepsilon \to w$ strongly. We take the limit $\varepsilon \to 0$ by time averaging:

$$\mathcal{R}(-t/\varepsilon)w_\varepsilon.\nabla_x w_\varepsilon \to \frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}(-\tau)w_\varepsilon.\nabla_x w_\varepsilon d\tau = w_\parallel \partial_{x_\parallel} w \tag{5.5}$$

and argue similarly for the other terms. We get in the end the following isothermal Euler system (with no dynamics in the $x_\perp$ variable):

$$\begin{cases} \partial_t \rho + \partial_{x_\parallel}(\rho w_\parallel) = 0, \\ \partial_t w + w_\parallel \partial_{x_\parallel} w = -\frac{\partial_{x_\parallel}\rho}{\rho} - \partial_{x_\parallel} H. \end{cases} \tag{5.6}$$

Of course, this system is very similar to the "usual" isothermal Euler system (1.21), so we get the same existence result.

Proposition 5.1. *For any initial data $\rho_0 > 0, w_0$ such that $\rho_0 \in L^1(\mathbb{R}^3)$, $\log(\frac{\rho_0}{d}) \in H^s(\mathbb{R}^3)$ and $w_0 \in H^s(\mathbb{R}^3)$ for $s > \frac{3}{2}+1$, there is existence and uniqueness of a local smooth solution $\rho > 0$ and u to (3.1) such that :*

$$\log \frac{\rho}{d}, w \in \mathcal{C}^0_t([0,T^*[, H^s(\mathbb{R}^3)) \cap \mathcal{C}^1_t([0,T^*[, H^{s-1}(\mathbb{R}^3)) \tag{5.7}$$

$$\rho \in \mathcal{C}^1([0,T^*[\times\mathbb{R}^3) \tag{5.8}$$

for some $T^* > 0$.

From what we have seen, we can thus expect the strong convergence

$$\mathcal{R}(t/\varepsilon)u_\varepsilon \to w, \tag{5.9}$$

that is to say:

$$u_\varepsilon - \mathcal{R}(-t/\varepsilon)w \to 0. \tag{5.10}$$

5.2 Convergence proof

We first give the stability inequality we obtain for system (5.1).

Proposition 5.2. *Let $(f_{0,\varepsilon})$ be a sequence of initial data satisfying assumptions (2.4-2.6) and (f_ε) the corresponding global weak solutions to (5.1).*

Let $s > 3/2+1$. For any sequence $\log \frac{\bar{\rho}_\varepsilon}{d}$, $\bar{u}_\varepsilon$ in $\mathcal{C}^0_t([0,T^*[, H^s(\mathbb{R}^3)) \cap \mathcal{C}^1_t([0,T^*[, H^{s-1}(\mathbb{R}^3))$ we define the modulated energy:

$$\mathcal{H}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon |v-\bar{u}_\varepsilon|^2 dv dx + \int (m_\varepsilon \log(m_\varepsilon/\bar{\rho}_\varepsilon) - m_\varepsilon + \bar{\rho}_\varepsilon) dx + \frac{\varepsilon^{2\alpha}}{2}\int |\nabla_x V_\varepsilon|^2 dx, \tag{5.11}$$

with $m_\varepsilon = \frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon} dx}$. Then the following inequality holds:

$$\begin{aligned}\mathcal{H}_\varepsilon(t) \leq \mathcal{H}_\varepsilon(0) + G_\varepsilon(t) + C\int_0^t \|\nabla_x \bar{u}_\varepsilon\|_{L^\infty} \mathcal{H}_\varepsilon(s) ds \\ + \int_0^t \int A_\varepsilon(\bar{\rho}_\varepsilon, \bar{u}_\varepsilon). \begin{pmatrix} -m_\varepsilon + \bar{\rho}_\varepsilon \\ J_\varepsilon - \rho_\varepsilon \bar{u}_\varepsilon \end{pmatrix} dx ds,\end{aligned} \tag{5.12}$$

with $A_\varepsilon(t,x)$ the so-called acceleration operator defined by:

$$A_\varepsilon(\bar{\rho}_\varepsilon, \bar{u}_\varepsilon) = \begin{pmatrix} \partial_t \log\left(\frac{\bar{\rho}_\varepsilon}{d}\right) + \nabla_x.\bar{u}_\varepsilon + \bar{u}_\varepsilon.\nabla_x \log\left(\frac{\bar{\rho}_\varepsilon}{d}\right) - \nabla_x H.\bar{u}_\varepsilon \\ \partial_t \bar{u}_\varepsilon + \bar{u}_\varepsilon.\nabla_x \bar{u}_\varepsilon + \nabla_x \log\left(\frac{\bar{\rho}_\varepsilon}{d}\right) - \frac{\bar{u}_\varepsilon^\perp}{\varepsilon} \end{pmatrix}, \tag{5.13}$$

and $G_\varepsilon(t)$ satisfying:

$$\begin{aligned} G_\varepsilon(t) \quad \leq \quad & C\varepsilon^\alpha \|\varepsilon^\alpha \nabla_x V_\varepsilon\|_{L^\infty_t(L^2_x)} \times \\ & \Big(\|\nabla_x(\bar{u}_\varepsilon.\nabla_x \log\bar{\rho}_\varepsilon/d)\|_{L^\infty_t(L^2_x)} + \|\log(\bar{\rho}_\varepsilon/d)\|_{W^{1,\infty}_t(H^1_x)} \Big). \end{aligned} \tag{5.14}$$

Proof. The proof is similar to the one given to obtain (3.20) in the proof of Theorem 3.1 and therefore we omit it. □

As we wish to show the strong convergence (after filtering) of ($\rho_\varepsilon := \int f_\varepsilon dv, u_\varepsilon := \frac{1}{\rho_\varepsilon}\int v f_\varepsilon dv$) to solutions to system (5.6), a natural idea would consist in taking ($\bar{\rho}_\varepsilon :=$

$\rho, \bar{u}_\varepsilon = \mathcal{R}(-t/\varepsilon)w)$ where ρ and w are the smooth solution to system (5.6) with initial data ρ_0 and w_0. Unfortunately, we can not prove directly

$$A_\varepsilon(\rho, \mathcal{R}(-t/\varepsilon)w) \to 0$$

in a strong sense. Thus, as in [147] or [74], we add a small correction denoted by $\varepsilon z_\varepsilon$ in order to build a higher order approximation of the equation and make the acceleration operator vanish. The shape of z_ε is precisely chosen in order to "kill" the non-vanishing terms which only weakly converge to 0.

In the following, we will consider the derivative of $\mathcal{R}$:

$$\mathcal{S}(t) := \frac{d\mathcal{R}(t)}{dt}, \tag{5.15}$$

which satisfies:

$$\frac{d\mathcal{S}(t)}{dt} = \mathcal{R}(-t)_\perp := \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem 5.1. *Let ρ_0, w_0 initial data verifying the hypotheses of Proposition 5.1 with $s > 5/2$. We assume that the sequence of initial data $(f_{0,\varepsilon})$ satisfies the assumptions (2.4-2.6) and:*

$$\mathcal{H}_\varepsilon(0) \to 0. \tag{5.16}$$

Let $(\log \frac{\rho}{d}, w)$ the unique strong solution to (5.6) with $\left(\log \frac{\rho_0}{d}, w_0\right)$ as initial conditions. We define $\bar{\rho}_\varepsilon$ and $\bar{u}_\varepsilon$ by the relation:

$$\begin{pmatrix} \log \frac{\bar{\rho}_\varepsilon}{d} \\ \bar{u}_\varepsilon \end{pmatrix} = \begin{pmatrix} \log \frac{\rho}{d} \\ \mathcal{R}(-t/\varepsilon)w \end{pmatrix} + \varepsilon y_\varepsilon, \tag{5.17}$$

with $y_\varepsilon = \begin{pmatrix} z_\varepsilon^\rho \\ \mathcal{R}(-t/\varepsilon)z_\varepsilon^w \end{pmatrix}$ and z_ε^ρ (resp. z_ε^ρ) defined by its Fourier transform $\mathcal{F}z_\varepsilon^\rho$ (resp. $\mathcal{F}z_\varepsilon^w$):

$$\begin{aligned} \mathcal{F}z_\varepsilon^\rho(\xi) = &- \mathbb{1}_{|\xi|\leq\frac{1}{\varepsilon}}\mathcal{F}\left(\nabla_{x_\perp}.(\mathcal{S}(t/\varepsilon)w) - \nabla_x H.\mathcal{S}(t/\varepsilon)w_\perp\right) \\ &- \int_{\mathbb{R}^3} \mathbb{1}_{|\xi-\eta|+|\eta|\leq\frac{1}{\varepsilon}}\mathcal{F}(\mathcal{S}(t/\varepsilon)w_\perp)(\eta).\mathcal{F}\left(\nabla_{x_\perp}\log\left(\frac{\rho}{d}\right)\right)(\xi-\eta)d\eta, \end{aligned} \tag{5.18}$$

$$\begin{aligned} \mathcal{F}z_\varepsilon^w(\xi) = &- \mathbb{1}_{|\xi|\leq\frac{1}{\varepsilon}}\mathcal{F}\left(\mathcal{S}(t/\varepsilon)\nabla_{x_\perp}\log\left(\frac{\rho}{d}\right)\right) \\ &- \int_{\mathbb{R}^3} \mathbb{1}_{|\xi-\eta|+|\eta|\leq\frac{1}{\varepsilon}}\mathcal{F}(\mathcal{S}(t/\varepsilon)w_\perp)(\eta).\mathcal{F}(\nabla_{x_\perp}w)(\xi-\eta)d\eta, \end{aligned} \tag{5.19}$$

where the operator $\mathcal{S}(t)$ is defined in (5.15).

Then, there exists $C > 0$ depending only on w and $\log \frac{\rho}{d}$,

$$\|z_\varepsilon\|_{L_t^\infty([0,T],H^{s-1})} \leq C,$$

and locally uniformly in time we have:

$$\mathcal{H}_\varepsilon(t) \to 0. \tag{5.20}$$

In particular, this means that ρ_ε weakly converges to ρ and J_ε weakly converges to $\rho w_\parallel$. Furthermore, we have the following strong convergences:

$$\int \rho_\varepsilon |u_\varepsilon - \mathcal{R}(-t/\varepsilon)w|^2 dx \to 0$$

and

$$\frac{de^{V_\varepsilon}}{\int de^{V_\varepsilon} dx} \to \rho$$

in $L^\infty_t(L^1_x)$.

Remark 5.1. i. Instead of a cut-off of order $\frac{1}{\varepsilon}$, we could have chosen any function $\xi(\varepsilon)$ such that for some $q \in]3/2, s-1[$:

$$\frac{1}{\xi(\varepsilon)} \to_{\varepsilon \to 0} 0,$$
$$\xi(\varepsilon)^{s-q-2}\varepsilon \to_{\varepsilon\to 0} 0.$$

The choice $\xi(\varepsilon) = \frac{1}{\varepsilon}$ yields a sharp convergence rate.

ii. Concerning the rate of convergence, the proof below actually shows that for any $q \in]3/2, s-1[$, if:
$$\mathcal{H}_\varepsilon(0) \leq C\varepsilon^{\min(s-q-1,1)},$$
then there exists C_q depending on q such that locally uniformly in time:
$$\mathcal{H}_\varepsilon(t) \leq C_q \varepsilon^{\min(s-q-1,1)}.$$

iii. When $s > 7/2$, we can observe in the proof that we actually do not need any cut-off in frequency. In this case the convergence is of order ε.

Proof. We assume that $s \in]5/2, 7/2]$. When $s > 7/2$, the proof is actually much simpler, as we do not need any cut-off in frequency and all the estimates are straightforward.

Step 1 We first show that $\|z_\varepsilon\|_{L^\infty_t([0,T],H^{s-1})} \leq C$. Let us observe that we do not use the cut-off in frequency here. We have:

$$\int_{\mathbb{R}^3} \left(1+|\xi|^2\right)^{s-1} |\mathcal{F}z^\rho_\varepsilon(\xi)|^2 d\xi = \|z^\rho_\varepsilon\|^2_{H^{s-1}}.$$

We then estimate:

$$\begin{aligned}
&\int_{|\xi|\leq \frac{1}{\varepsilon}} \left(1+|\xi|^2\right)^{s-1} \left|\mathcal{F}\left(\nabla_{x_\perp}.(\mathcal{S}(t/\varepsilon)w)\right)\right|^2 d\xi \\
\leq\ & C\int_{|\xi|\leq \frac{1}{\varepsilon}} \left(1+|\xi|^2\right)^{s-1+1} \left|\mathcal{F}\mathcal{S}(t/\varepsilon)w\right|^2 d\xi \\
\leq\ & C\int_{|\xi|\leq \frac{1}{\varepsilon}} \left(1+|\xi|^2\right)^{s} \left|\mathcal{F}\mathcal{S}(t/\varepsilon)w\right|^2 d\xi \\
\leq\ & C\|\mathcal{S}(t/\varepsilon)w\|^2_{H^s} \leq C\|w\|^2_{H^s}.
\end{aligned}$$

Similarly we have:

$$\int_{|\xi|\leq \frac{1}{\varepsilon}} \left(1+|\xi|^2\right)^{s-1} \left|\mathcal{F}\left(\nabla_x H.\mathcal{S}(t/\varepsilon)w_\perp\right)\right|^2 d\xi \leq C\|w_\perp\|^2_{H^{s-1}}. \tag{5.21}$$

Finally we compute:

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(1+|\xi|^2\right)^{s-1}\mathbb{1}_{|\xi-\eta|+|\eta|\le\frac{1}{\varepsilon}}|\mathcal{F}(\mathcal{S}(t/\varepsilon)w_\perp)(\eta).\mathcal{F}\left(\nabla_{x_\perp}\log\left(\frac{\rho}{d}\right)\right)(\xi-\eta)|^2d\eta\\ \le& C\|\mathcal{S}(t/\varepsilon)w\nabla_x\log\frac{\rho}{d}\|^2_{H^{s-1}}\le C\|w\|^2_{H^{s-1}}\|\log\frac{\rho}{d}\|^2_{H^s},\end{aligned} \tag{5.22}$$

since $H^{s-1}(\mathbb{R}^3)$ is an algebra. This proves that there exists a constant depending on w and $\log\frac{\rho}{d}$:

$$\|z_\varepsilon^\rho\|_{L_t^\infty([0,T],H^{s-1})}\le C.$$

Likewise, we prove that:

$$\|z_\varepsilon^w\|_{L_t^\infty([0,T],H^{s-1})}\le C.$$

This yields that $\varepsilon z_\varepsilon\to 0$ in $L_t^\infty([0,T],H^{s-1})$.

Now, let $q\in]3/2,s-1[$ be a fixed parameter. We are interested in the H^{q+1} norm of z_ε. Since $q+2>7/2\ge s$, the H^{q+2} norm of $\log\frac{\rho}{d}$ and w is not necessarily well-defined[3] and we use this time the cut-off in frequency to lower down the regularity to H^s:

$$\begin{aligned}&\int_{|\xi|\le\frac{1}{\varepsilon}}\left(1+|\xi|^2\right)^{q+1}|\mathcal{F}\left(\nabla_{x_\perp}.(\mathcal{S}(t/\varepsilon)w)\right)|^2d\xi\\ \le\ & C\int_{|\xi|\le\frac{1}{\varepsilon}}\left(1+|\xi|^2\right)^s\left(1+|\xi|^2\right)^{q+2-s}|\mathcal{F}\mathcal{S}(t/\varepsilon)w|^2d\xi\\ \le\ & C/\varepsilon^{q+2-s}\int_{|\xi|\le\frac{1}{\varepsilon}}\left(1+|\xi|^2\right)^s|\mathcal{F}\mathcal{S}(t/\varepsilon)w|^2d\xi\\ \le\ & C/\varepsilon^{q+2-s}\|w\|^2_{H^s}.\end{aligned}$$

Treating the other terms similarly, we obtain:

$$\|z_\varepsilon\|_{L_t^\infty([0,T],H^{q+1})}\le C/\varepsilon^{q+2-s}. \tag{5.23}$$

Step 2 We denote $X_\varepsilon=\begin{pmatrix}\rho e^{\varepsilon z_\varepsilon^\rho}\\ w+\varepsilon z_\varepsilon^w\end{pmatrix}$. We introduce the filtered acceleration operator defined by:

$$B_\varepsilon(\bar\rho,\bar u)=\begin{pmatrix}\partial_t\log\left(\frac{\bar\rho}{d}\right)+\nabla_x.\mathcal{R}(-t/\varepsilon)\bar u+\mathcal{R}(-t/\varepsilon)\bar u.\nabla_x\log\left(\frac{\bar\rho}{d}\right)-\nabla_xH.\mathcal{R}(-t/\varepsilon)\bar u\\ \partial_t\bar u+\mathcal{R}(-t/\varepsilon)\bar u.\nabla_x\bar u+\mathcal{R}(-t/\varepsilon)\nabla_x\log\left(\frac{\bar\rho}{d}\right)\end{pmatrix}. \tag{5.24}$$

We show that X_ε is an approximate zero of the filtered acceleration operator B_ε in the sense that (we recall that $q\in]3/2,s-1[$):

$$\|B_\varepsilon(X_\varepsilon)\|_{L_t^\infty([0,T],H^q)}\to 0,$$

when $\varepsilon\to 0$.

By definition of $\mathcal{S}$, we have:

$$\frac{d\mathcal{S}(t/\varepsilon)}{dt}=\frac{1}{\varepsilon}\mathcal{R}(-t/\varepsilon)_\perp.$$

[3]When $s>7/2$, q can be chosen such that $q+2\le s$ and thus we have $\|z_\varepsilon\|_{L_t^\infty([0,T],H^{q+1})}\le C$.

Hence we have:

$$
\begin{aligned}
& \partial_t \mathcal{F}\left(\log(\rho/d) + \varepsilon z_\varepsilon^\rho\right) \\
= & -\mathcal{F}\left(\partial_{x_\parallel} w_\parallel + w_\parallel \partial_{x_\parallel} \log(\rho/d) - \partial_{x_\parallel} H w_\parallel\right) \\
- & \mathbb{1}_{|\xi|\le \frac{1}{\varepsilon}} \mathcal{F}\left(\nabla_{x_\perp}.(\mathcal{R}(-t/\varepsilon)w) - \nabla_x H.\mathcal{R}(-t/\varepsilon)w_\perp\right) \\
- & \int_{\mathbb{R}^3} \mathbb{1}_{|\xi-\eta|+|\eta|\le\frac{1}{\varepsilon}} \mathcal{F}(\mathcal{R}(-t/\varepsilon)w_\perp(\eta))\mathcal{F}(\nabla_{x_\perp}\log\left(\frac{\rho}{d}\right))(\xi-\eta)d\eta \\
+ & D_\varepsilon(t,\xi),
\end{aligned}
$$

where D_ε is defined by:

$$
\begin{aligned}
D_\varepsilon(t,\xi) := & - \varepsilon \mathbb{1}_{|\xi|\le\frac{1}{\varepsilon}} \mathcal{F}\left(\nabla_{x_\perp}.(\mathcal{S}(t/\varepsilon)\partial_t w) - \nabla_x H.\mathcal{S}(t/\varepsilon)\partial_t w_\perp\right) \\
& - \varepsilon \int_{\mathbb{R}^3} \mathbb{1}_{|\xi-\eta|+|\eta|\le\frac{1}{\varepsilon}} \mathcal{F}\mathcal{S}(t/\varepsilon)\partial_t w_\perp(\eta)\mathcal{F}\left(\nabla_{x_\perp}\log\left(\frac{\rho}{d}\right)\right)(\xi-\eta)d\eta \\
& - \varepsilon \int_{\mathbb{R}^3} \mathbb{1}_{|\xi-\eta|+|\eta|\le\frac{1}{\varepsilon}} \mathcal{F}\mathcal{S}(t/\varepsilon)w_\perp(\eta)\mathcal{F}\left(\nabla_{x_\perp}\partial_t\log\left(\frac{\rho}{d}\right)\right)(\xi-\eta)d\eta.
\end{aligned}
$$

Consequently, denoting by $B_{1,\varepsilon}$ the operator in the first line of B_ε (resp. $B_{2,\varepsilon}$ the second operator) we have:

$$
\mathcal{F}B_{1,\varepsilon}(X_\varepsilon) = T_{1,\varepsilon}(t,\xi) + T_{2,\varepsilon}(t,\xi) + D_\varepsilon(t,\xi),
$$

with:

$$
\begin{aligned}
T_{1,\varepsilon}(t,\xi) = & \mathbb{1}_{|\xi|>\frac{1}{\varepsilon}} \mathcal{F}\left(\nabla_{x_\perp}.(\mathcal{R}(-t/\varepsilon)w_\perp) - \nabla_x H.\mathcal{R}(-t/\varepsilon)w_\perp\right) \\
& + \int_{\mathbb{R}^3} \mathbb{1}_{|\xi-\eta|+|\eta|>\frac{1}{\varepsilon}} \mathcal{F}(\mathcal{R}(-t/\varepsilon)w_\perp)(\eta)\mathcal{F}\left(\nabla_{x_\perp}\log\left(\frac{\rho}{d}\right)\right)(\xi-\eta)d\eta,
\end{aligned}
\tag{5.25}
$$

$$
\begin{aligned}
T_{2,\varepsilon}(t,\xi) = & \varepsilon\mathcal{F}\left(\nabla_x.\mathcal{R}(-t/\varepsilon)z_\varepsilon^w + \mathcal{R}(-t/\varepsilon)w.\nabla_x z_\varepsilon^\rho + \mathcal{R}(-t/\varepsilon)z_\varepsilon^w.\nabla_x \log\frac{\rho}{d} - \nabla_x H.\mathcal{R}(-t/\varepsilon)z_\varepsilon^w\right) \\
& + \varepsilon^2 \mathcal{F}(\mathcal{R}(-t/\varepsilon)z_\varepsilon^w.\nabla_x z_\varepsilon^\rho).
\end{aligned}
\tag{5.26}
$$

Remark 5.2. Without corrector ($z_\varepsilon = 0$) we have:

$$
\begin{aligned}
T_{1,\varepsilon}(t,\xi) = & \mathcal{F}\left(\nabla_{x_\perp}.(\mathcal{R}(-t/\varepsilon)w_\perp) + \nabla_x H.\mathcal{R}(-t/\varepsilon)w_\perp\right) \\
+ & \int_{\mathbb{R}^3} \mathcal{F}(\mathcal{R}(-t/\varepsilon)w_\perp)(\eta)\mathcal{F}\left(\nabla_{x_\perp}\log\left(\frac{\rho}{d}\right)\right)(\xi-\eta)d\eta
\end{aligned}
$$

These terms only weakly but not strongly converge to 0 as ε goes to 0 : this is why we have to add the corrector.

When z_ε is defined without without cut-off in frequency, we notice that we exactly have $T_{1,\varepsilon}(t,\xi) = 0$.

Estimating $T_{1,\varepsilon}$

We need the H^s regularity of w and $\log\rho/d$ in order to get some decay in ε for $T_{1,\varepsilon}$, by using, for any $\beta > 0$:

$$
\begin{aligned}
& \mathbb{1}_{|\xi|>\frac{1}{\varepsilon}} \le (1+|\xi|^2)^\beta \varepsilon^{2\beta}, \\
& \mathbb{1}_{|\xi-\eta|+|\eta|>\frac{1}{\varepsilon}} \le 2(|\xi-\eta|^2+|\eta|^2)^\beta \varepsilon^{2\beta}.
\end{aligned}
$$

Therefore we have:

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (1+|\xi|^2)^q \mathbb{1}_{|\xi|>\frac{1}{\varepsilon}} |\mathcal{F}\left(\nabla_{x_\perp}.(\mathcal{R}(-t/\varepsilon)w_\perp)\right)|^2 d\xi \\
\leq & C\varepsilon^{2s-2(q+1)} \int_{\mathbb{R}^3} (1+|\xi|^2)^{q+1+s-(q+1)} |\mathcal{F}\left(\mathcal{R}(-t/\varepsilon)w_\perp\right)|^2 d\xi \\
\leq & C\varepsilon^{2s-2(q+1)} \|w\|_{H^s}^2.
\end{aligned}
$$

We handle the other terms by the same method.There exists a constant C depending only on q and the H^s norm of $\log\frac{\varrho}{d}$ and w such that:

$$
\left(\int_{\mathbb{R}^3} (1+|\xi|^2)^q |T_{1,\varepsilon}|^2 d\xi\right)^{1/2} \leq C\varepsilon^{s-q-1}. \tag{5.27}
$$

Estimating $T_{2,\varepsilon}$

We can use estimate (5.23). As a result, there exists a constant C depending only on q and the H^s norm of $\log\frac{\varrho}{d}$ and w such that:

$$
\left(\int_{\mathbb{R}^3} (1+|\xi|^2)^q |T_{2,\varepsilon}|^2 d\xi\right)^{1/2} \leq C(\varepsilon\times\varepsilon^{s-q-2} + \varepsilon^2\times\varepsilon^{s-q-2}) \leq C\varepsilon^{s-q-1}. \tag{5.28}
$$

Estimating D_ε

We only have $\partial_t w \in H^{s-1}$ and consequently we do not necessarily have $\partial_t w \in H^{q+1}$. Nevertheless, we can use the cut-off in frequency to lower the regularity down to only H^{s-1}:

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (1+|\xi|^2)^q \mathbb{1}_{|\xi|\leq\frac{1}{\varepsilon}} |\mathcal{F}\left(\nabla_{x_\perp}.(\mathcal{S}(t/\varepsilon)\partial_t w)\right)|^2 d\xi \\
\leq & C\int_{\mathbb{R}^3} (1+|\xi|^2)^{q+1} \mathbb{1}_{|\xi|\leq\frac{1}{\varepsilon}} |\mathcal{F}\left((\mathcal{S}(t/\varepsilon)\partial_t w)\right)|^2 d\xi \\
\leq & C\frac{1}{\varepsilon^{2(q+1)-2(s-1)}} \int_{\mathbb{R}^3} \mathbb{1}_{|\xi|\leq\frac{1}{\varepsilon}} (1+|\xi|^2)^{s-1} |\mathcal{F}\left((\mathcal{S}(t/\varepsilon)\partial_t w)\right)|^2 d\xi \\
\leq & C\frac{1}{\varepsilon^{2(q+2-s)}} \|\partial_t w\|_{H^{s-1}}^2.
\end{aligned}
$$

Following the same method for the other terms, we finally obtain:

$$
\left(\int_{\mathbb{R}^3} (1+|\xi|^2)^q |D_\varepsilon|^2 d\xi\right)^{1/2} \leq C\varepsilon\times\varepsilon^{s-2-q} = C\varepsilon^{s-1-q}. \tag{5.29}
$$

Gathering the pieces together, there exists a constant $C>0$ depending on q, the H^s norm of $\log\frac{\varrho}{d}$, w and the H^{s-1} norm of $\partial_t \log\frac{\varrho}{d}$, $\partial_t w$ such that

$$
\|B_{1,\varepsilon}(X_\varepsilon)\|_{H^q} \leq C\varepsilon^{s-q-1}.
$$

As a consequence, we have proved:

$$
\|B_{1,\varepsilon}(X_\varepsilon)\|_{L_t^\infty([0,T],H^q)} \to 0. \tag{5.30}
$$

Arguing similarly for $B_{2,\varepsilon}$, we finally deduce that

$$
\|B_\varepsilon(X_\varepsilon)\|_{L_t^\infty([0,T],H^q)} \to 0.
$$

Step 3 Finally we check that uniformly in time :

$$\int_0^t \int A_\varepsilon(\bar\rho_\varepsilon, \bar u_\varepsilon). \begin{pmatrix} -m_\varepsilon + \bar\rho_\varepsilon \\ J_\varepsilon - \rho_\varepsilon \bar u_\varepsilon \end{pmatrix} dxds \to 0,$$

as ε goes to 0. We recall that $\bar\rho_\varepsilon$ and $\bar u_\varepsilon$ were defined in (5.17).

First we have to check that

$$\|A_\varepsilon(\bar\rho_\varepsilon, \bar u_\varepsilon)\|_{L^\infty_t([0,T],H^q)} \to 0.$$

This is clear in view of Step 2, since we have:

$$A_\varepsilon(\bar\rho_\varepsilon, \bar u_\varepsilon) = \begin{pmatrix} B_{1,\varepsilon}(X_\varepsilon) \\ \mathcal{R}(t/\varepsilon) B_{2,\varepsilon}(X_\varepsilon) \end{pmatrix}.$$

and $\mathcal{R}(t/\varepsilon)$ is an isometry on any $H^s(\mathbb{R}^3)$.

We denote $A_\varepsilon = \begin{pmatrix} A_{1,\varepsilon} \\ A_{2,\varepsilon} \end{pmatrix}$ and evaluate:

$$\begin{aligned} \left| \int_0^t \int A_{1,\varepsilon}(\bar\rho_\varepsilon, \bar u_\varepsilon). \left(-m_\varepsilon + \bar\rho_\varepsilon\right) dxds \right| &\leq \int_0^t \int |A_{1,\varepsilon}(\bar\rho_\varepsilon, \bar u_\varepsilon) m_\varepsilon| \, dxds + \leq \int_0^t \int |A_{1,\varepsilon}(\bar\rho_\varepsilon, \bar u_\varepsilon)\bar\rho_\varepsilon| \, dxds \\ &\leq C \|A_\varepsilon(\bar\rho_\varepsilon, \bar u_\varepsilon)\|_{L^\infty_t([0,T],L^\infty)} \left(\|m_\varepsilon\|_{L^\infty_t(L^1_x)} + \|\rho\|_{L^\infty_t(L^1_x)} \|e^{\varepsilon z^\rho_\varepsilon}\|_{L^\infty_{t,x}} \right) \\ &\leq C \|A_\varepsilon(\bar\rho_\varepsilon, \bar u_\varepsilon)\|_{L^\infty_t([0,T],H^q)} \left(1 + \|e^{\varepsilon z^\rho_\varepsilon}\|_{L^\infty_{t,x}} \right), \end{aligned}$$

by Sobolev embedding, $H^q(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ $(q > 3/2)$. By the estimates of Step 1, there exists $C > 0$ independent of ε such that:

$$\|e^{\varepsilon z^w_\varepsilon}\|_{L^\infty_{t,x}} \leq C.$$

In the other hand,

$$\begin{aligned} & \left| \int_0^t \int A_{2,\varepsilon}(\bar\rho_\varepsilon, \bar u_\varepsilon). \left(J_\varepsilon - \rho_\varepsilon \bar u_\varepsilon \right) dxds \right| \\ \leq \; & C \|A_\varepsilon(\bar\rho_\varepsilon, \bar u_\varepsilon)\|_{L^\infty_t([0,T],L^\infty)} \left(\|J_\varepsilon\|_{L^\infty_t(L^1_x)} + \|\rho_\varepsilon\|_{L^\infty_t(L^1_x)} \|\mathcal{R}(-t/\varepsilon)(w + \varepsilon z^w_\varepsilon)\|_{L^\infty_{t,x}} \right) \\ \leq \; & \|A_\varepsilon(\bar\rho_\varepsilon, \bar u_\varepsilon)\|_{L^\infty_t([0,T],H^q)} \left(1 + \|w\|_{L^\infty_t(H^s)} + \|\varepsilon z^w_\varepsilon\|_{L^\infty_{t,x}} \right) \end{aligned}$$

and the conclusions follows.

One can also readily check, using (5.14), that $G_\varepsilon(t) \to 0$ uniformly in time.

Finally this proves that $\mathcal{H}_\varepsilon(t) \to 0$ uniformly in time, as soon as $\mathcal{H}_\varepsilon(0) \to 0$.

Using the estimates of Step 1, we check that

$$\left(\rho e^{\varepsilon z^\rho_\varepsilon}, w + \varepsilon z^w_\varepsilon \right) \to (\rho, w)$$

in $L^\infty([0,T], L^\infty)$.

In order to apply Gronwall's inequality to the inequality (5.12), there remains to check that $\|\nabla_x \bar u_\varepsilon\|_{L^\infty}$ is uniformly bounded in ε. It is sufficient to check that $\|\varepsilon \nabla_x z^w_\varepsilon\|_{L^\infty}$ is uniformly bounded. According to (5.23) and by Sobolev embedding $H^q \to L^\infty$, we have:

$$\|\varepsilon \nabla_x z^w_\varepsilon\|_{L^\infty_{t,x}} \leq \|\varepsilon \nabla_x z^w_\varepsilon\|_{L^\infty_t H^q_x} \leq C\varepsilon^{s-q-1} \leq C. \tag{5.31}$$

Then, the other conclusions easily follow as in the end of the proof of Theorem 3.1. □

Appendix

Scaling of the Vlasov-Poisson systems (S), (S') and (L)

Let us introduce the dimensionless variables and unknowns:

$$\tilde{t} = \frac{t}{\tau} \quad \tilde{x} = \frac{x}{L} \quad \tilde{v} = \frac{v}{v_{th}},$$
$$f(t,x,v) = \bar{f}\tilde{f}(\tilde{t},\tilde{x},\tilde{v}) \quad V(t,x) = \bar{V}\tilde{V}(\tilde{t},\tilde{x}) \quad E(t,x) = \bar{E}\tilde{E}(\tilde{t},\tilde{x}).$$

Then the Vlasov equation with Poisson equation (1.5) equation states:

$$\begin{cases} \partial_{\tilde{t}}\tilde{f}_\varepsilon + \frac{v_{th}\tau}{L}\tilde{v}.\nabla_{\tilde{x}}\tilde{f}_\varepsilon + \frac{e\bar{E}\tau}{mv_{th}}\tilde{E}_\varepsilon.\nabla_{\tilde{v}}\tilde{f}_\varepsilon = 0 \\ \frac{\bar{E}L}{\bar{V}}\tilde{E}_\varepsilon = -\nabla_{\tilde{x}}\tilde{V}_\varepsilon \\ -\frac{\varepsilon_0\bar{V}}{L^2}\Delta_{\tilde{x}}\tilde{V}_\varepsilon = e\bar{f}v_{th}^3\int\tilde{f}_\varepsilon d\tilde{v} - e\bar{d}\tilde{d}e^{\frac{e\bar{V}}{k_BT_e}\tilde{V}_\varepsilon} \\ \tilde{f}_{\varepsilon,|\tilde{t}=0} = \tilde{f}_{0,\varepsilon}, \quad \bar{f}L^3v_{th}^3\int\tilde{f}_{0,\varepsilon}d\tilde{v}d\tilde{x} = 1. \end{cases} \tag{5.32}$$

In order to ensure that $\int\tilde{f}d\tilde{x}d\tilde{v} = 1$, it is natural to set:

$$\bar{f}L^3v_{th}^3 = 1.$$

Moreover we consider the normalizations:

$$\frac{v_{th}\tau}{L} = 1, \quad \frac{\bar{E}L}{\bar{V}} = 1,$$
$$\frac{e\bar{V}}{k_BT_e} = 1,$$
$$\bar{f}v_{th}^3 = \bar{d}.$$

This implies that:

$$\frac{e\bar{E}\tau}{mv_{th}} = \frac{v_{th}\tau}{L} = 1.$$

Now we observe that:

$$\frac{\varepsilon_0\bar{V}}{e\bar{f}v_{th}^3} = \frac{\varepsilon_0k_BT_e}{e^2\times 1/L^3} = \lambda_D^2,$$

where λ_D is the Debye length.

The quasineutral scaling consists in considering the ordering:

$$\frac{\lambda_D^2}{L^2} = \varepsilon,$$

with ε a small parameter.

With this scaling we get the following dimensionless system of equations (we forget the $\tilde{}$ for the sake of readability):

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + E_\varepsilon.\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\varepsilon\Delta_x V_\varepsilon = \int f_\varepsilon dv - de^{V_\varepsilon} \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon} \geq 0, \int f_{0,\varepsilon}dvdx = 1. \end{cases} \tag{5.33}$$

This system is nothing but System (S). We get Systems (S') and (L) with the same nondimensionalization.

Remark 5.3. To be rigorous we should also consider the confinement force $-\nabla_x H$ on the ions, but we will not do so for the sake of simplicity and readability. Nevertheless we could handle such an external force with only minor changes.

Scaling of the Vlasov-Poisson equation (1.30)

We once again consider the nondimensionalization analysis of system (S'), this time including the magnetic field:

$$B = \bar{B} e_{\parallel}.$$

This yields:

$$\begin{cases} \partial_{\tilde{t}} \tilde{f}_\varepsilon + \frac{v_{th}\tau}{L} \tilde{v}.\nabla_{\tilde{x}} \tilde{f}_\varepsilon + \left(\frac{e\bar{E}\tau}{m v_{th}} \tilde{E}_\varepsilon + \frac{e\bar{B}}{m} \tau \tilde{v} \wedge e_{\parallel} \right).\nabla_{\tilde{v}} \tilde{f}_\varepsilon = 0 \\ \frac{\bar{E}L}{\bar{V}} \tilde{E}_\varepsilon = -\nabla_{\tilde{x}} \tilde{V}_\varepsilon \\ -\frac{\varepsilon_0 \bar{V}}{L^2} \Delta_{\tilde{x}} \tilde{V}_\varepsilon = e\bar{f} v_{th}^3 \int \tilde{f}_\varepsilon d\tilde{v} - e \frac{\tilde{d} e^{\frac{e\bar{V}}{k_B T_e} \tilde{V}_\varepsilon}}{\int \tilde{d} e^{\frac{e\bar{V}}{k_B T_e} \tilde{V}_\varepsilon} dx} \\ \tilde{f}_{\varepsilon,|\tilde{t}=0} = \tilde{f}_{0,\varepsilon}, \quad \bar{f} L^3 v_{th}^3 \int \tilde{f}_{0,\varepsilon} d\tilde{v} d\tilde{x} = 1. \end{cases} \tag{5.34}$$

We set $\Omega = \frac{e\bar{B}}{m}$: this is the cyclotron frequency (also referred to as the gyrofrequency). We also consider the so-called electron Larmor radius (or electron gyroradius) r_L defined by:

$$r_L = \frac{v_{th}}{\Omega} = \frac{m v_{th}}{e\bar{B}}. \tag{5.35}$$

This quantity can be physically understood as the typical radius of the helix around axis $e_\parallel$ that the particles follow, due to the intense magnetic field.

The Vlasov equation now reads:

$$\partial_{\tilde{t}} \tilde{f}_\varepsilon + \frac{r_L}{L} \Omega\tau \tilde{v}.\nabla_{\tilde{x}} \tilde{f}_\varepsilon + \left(\frac{\bar{E}}{\bar{B} v_{th}} \Omega\tau \tilde{E}_\varepsilon + \Omega\tau \tilde{v} \wedge e_\parallel \right).\nabla_{\tilde{v}} \tilde{f}_\varepsilon = 0.$$

The "strong magnetic field" ordering consists in setting:

$$\Omega\tau = \frac{1}{\varepsilon}, \quad \frac{\bar{E}}{\bar{B} v_{th}} = \varepsilon,$$

$$\frac{r_L}{L} = \varepsilon.$$

The quasineutral scaling we consider is:

$$\frac{\lambda_D^2}{L^2} = \varepsilon^{2\alpha},$$

with $\alpha > 0$. From the physical point of view, it means that we consider that both the Larmor radius and the Debye length vanish. We observe that:

$$\frac{\lambda_D}{r_L} = \varepsilon^{\alpha - 1}$$

In most practical situations, $\lambda_D \ll r_L$, so that the range of parameters $\alpha > 1$ is particularly physically relevant. Finally, by having the same normalizations as before, we get in the end:

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + \left(E_\varepsilon + \frac{v \wedge b}{\varepsilon} \right).\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\varepsilon^{2\alpha} \Delta_x V_\varepsilon = \int f_\varepsilon dv - \frac{d e^{V_\varepsilon}}{\int d e^{V_\varepsilon} dx} \\ f_{\varepsilon,|t=0} = f_{0,\varepsilon}, \quad \int f_{0,\varepsilon} dv dx = 1. \end{cases}$$

Chapter 3

The three-dimensional finite larmor radius approximation: the case of ions with massless electrons

Article paru à Asymptotic Analysis (2010).

Sommaire

Résumé : Following Frénod and Sonnendrücker ([60]), we consider the finite Larmor radius regime for a plasma submitted to a large magnetic field and take into account both the quasineutrality and the local thermodynamic equilibrium of the electrons. We then rigorously establish the asymptotic gyrokinetic limit of the rescaled and modified Vlasov-Poisson system in a three-dimensional setting with the help of an averaging lemma.

1 Introduction and main results

1.1 Physical motivation

We are interested in the behaviour of a plasma (*id est* a gas made of ions with individual charge Ze and mass m_i and electrons with individual charge $-e$ and mass m_e, with $m_i \gg m_e$) which is submitted to a large external magnetic field. It is "well-known" that such a field induces fast small oscillations for the particles and consequently introduces a new small time scale which is very restrictive and inconvenient from the numerical point of view. The simulation of such plasmas appears to be primordial since the model can be applied to tokamak plasmas from magnetic confinement fusion (like for the ITER project).

1.1.1 Heuristic study

Let us give some heuristic formal arguments to investigate the behaviour of the plasma: if we consider the motion of one particle (of charge $q > 0$, mass m, position x and velocity v) submitted to an external constant field B, the fundamental principle of mechanics gives that:

$$\frac{dx}{dt} = v, \ \frac{dv}{dt} = \frac{q}{m}(v \wedge B). \tag{1.1}$$

Straightforward calculations show first of all that the parallel velocity, denoted by $v_{\|}$ (that is to say the component of the velocity in the direction of the magnetic field) is conserved and thanks to the conservation of the kinetic energy, so is the norm of the perpendicular velocity $v_{\perp}$ (the component of the velocity in the perpendicular plane). Actually, we can see that the particle moves on a helix whose axis is the direction of the magnetic field. The rotation period (around the axis) is the inverse of the cyclotron frequency Ω:

$$\Omega = \frac{|q||B|}{m}, \tag{1.2}$$

and the radius is the so-called Larmor radius:

$$r_L = \frac{|v_{\perp}|}{\Omega}. \tag{1.3}$$

In the case where the magnetic field is very strong, Ω tends to infinity whereas r_L tends to zero. More precisely, if we take $|B| \sim \frac{1}{\varepsilon}$ (with $\varepsilon \to 0$) we have:

$$\begin{cases} \Omega \sim \frac{1}{\varepsilon}, \\ r_L \sim \varepsilon. \end{cases}$$

The approximation which consists in considering $r_L = 0$ is the classical guiding center approximation ([77]). This means that each particle is assimilated to its "guiding center" (in other words its "instantaneous rotation center"), which is equivalent to neglect the very fast rotation of the particle around the axis.

If one also applies some external constant electric field E, a similar computation shows that there appears:

i. an acceleration $\frac{E.B}{|B|}$ in the direction of B. If we consider $E \sim 1$, then:

$$\frac{E.B}{|B|} \sim 1 \tag{1.4}$$

ii. a drift $\frac{E \wedge B}{|B|^2}$ in the orthogonal plane. We have:

$$\frac{E \wedge B}{|B|^2} \sim \varepsilon. \tag{1.5}$$

This drift, usually called the electric drift is problematic as regards to the issue of plasma confinement. It is negligible compared to the acceleration in the direction of B, but in the time scale for plasma fusion which is expected to be very long, one can not neglect this small drift, since it creates a displacement of order εt (t represents the time).

At last, note also that if the fields are not constant, various other drifts may appear, whose order in ε is higher than those of the electric drift.

Actually, the fields considered are neither constant, nor external, but self-induced by the plasma itself. The effects we would like to describe are due to the non-linear interaction between the particles and the electromagnetic field.

1.1.2 The mathematical model

In all the sequel, we assume that the magnetic field is external and constant and we suppose that the speed of particles is small compared to the speed of light, so that we can use the electrostatic approximation which consists in reducing the Maxwell equations to the Poisson equation. Finally, we decide to opt for a kinetic description for the ions: in other words, the time and space scales considered here are such that ions are not at a thermodynamic equilibrium and their density is governed by a kinetic equation.

The basic model usually considered for the ions is the following Vlasov-Poisson system:

$$\begin{cases} \partial_t f + v.\nabla_x f + (E + v \wedge B).\nabla_v f = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \int f dv \\ f_{t=0} = f_0. \end{cases}$$

where $f(t,x,v)$ is the density of ions, with $t \in \mathbb{R}^+, x \in \mathbb{R}^d$ or $\mathbb{R}^d/\mathbb{Z}^d, v \in \mathbb{R}^d$ (usually $d = 2$ or 3), meaning that $f(t,x,v)dxdv$ gives the number of ions in the infinitesimal volume $[x, x+dx] \times [v, v+dv]$ at time t (note that in this model, electrons are for the moment neglected).

1.1.3 The gyrokinetic approximation

It is important from a numerical point of view to establish the asymptotic equation when $|B|$ tends to infinity. Indeed, we expect the asymptotic equation to be "easier" to handle: only one time and space scale, perhaps less variables in the phase space to deal with... The derivation of such equations is usually referred to in the mathematic literature as "gyrokinetic approximation".

Rigorous justifications of these derivations with various time and space observation scales have only appeared at the end of the nineties. We refer for instance to the works of Brenier ([27]), Frénod and Sonnendrücker ([59]-[60]), Frénod, Raviart and Sonnendrücker ([57]), Golse and Saint-Raymond ([72]-[74]), Saint-Raymond ([139]-[140]).

The classical "guiding center approximation" corresponds to the following scaling for the Vlasov-Poisson system (from now on and until the end of the chapter, B is a constant vector, say for instance $B = \frac{1}{\varepsilon} e_z$):

$$\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + (E_\varepsilon + \frac{v \wedge e_z}{\varepsilon}).\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\Delta_x V_\varepsilon = \int f_\varepsilon dv \\ f_{\varepsilon,t=0} = f_0. \end{cases} \tag{1.6}$$

The articles [59] and [72] show that when $\varepsilon \to 0$, this leads to a one-dimensional kinetic equation in the direction of B:

$$\begin{cases} \partial_t f + v_\parallel.\nabla_x f + E_\parallel.\nabla_v f = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \int f dv \\ f_{t=0} = f_0. \end{cases} \tag{1.7}$$

Notice that the electric drift does not appear; this was expected since we have seen in the formal analysis that this drift was of higher order in ε than the other effects. This shows in particular that this approximation is not sufficient for the numerical simulation of tokamaks. In order to make this drift appear, there exists to our knowledge two main possibilities:

i. one consists in restricting to a $2D$ problem in the plane orthogonal to B ([72]),

ii. the other consists in rescaling the orthogonal scales in order to get both transport and electric drift at the same order ([60]).

This work directly follows the articles [60] and [57] where the authors considered the "finite Larmor radius approximation". This means that the spatial observation scale in the plane orthogonal to B is chosen smaller than the one in the parallel direction, more precisely with the same order as the Larmor radius r_L, so that one can expect the electric drift to appear in the asymptotic equation.

In some sense, having such a scaling allows the electric field to significantly vary across a Larmor radius, which is not the case for instance in (1.6). Moreover, in this situation, the positions of the particles are no longer assimilated to the position of their "guiding center" and we will have to perform an average over one fast oscillation period (the so-called gyroaverage) in order to get a sort of averaged number density.

1.2 Scaling and existing results

The system we are going to study is based on the "finite Larmor radius scaling" and takes into account the quasineutrality of the plasma.

1.2.1 The (refined) mathematical model

We refer to [60] for a complete discussion on the scaling. Let us recall briefly and quite crudely how it works.

Let $L_\parallel$ be the characteristic length in the direction of the magnetic field and $L_\perp$ be the characteristic length in the perpendicular plane. We consider that $L_\parallel \sim 1$ and $L_\perp \sim \varepsilon$ and define the dimensionless variables $x'_\parallel = \frac{x_\parallel}{L_\parallel}$ and $x'_\perp = \frac{x_\perp}{L_\perp}$. In the same fashion we also define the dimensionless variables t' and v' with characteristic time and velocities with the same order as $L_{\|}$ and introduce the new number density f' defined by $\bar{f}f'(t',x',v') = f(t,x,v)$ (and we define likewise the new electric field and potential $\bar{E}E'(t',x',v') = E(t,x,v)$ and $\bar{V}V'(t',x',v') = V(t,x,v)$). We consider the scaling $\bar{f},\bar{E} \sim 1$ and $\bar{V} \sim \varepsilon$. At last, we introduce the Debye length of the plasma λ_D, which appears in the Poisson equation. In order to take into account the quasineutrality of the plasma, we take from now on $\lambda_D \sim \sqrt{\varepsilon}$.

The Poisson equation states in this scaling:

$$-\varepsilon\Delta_{x'_{\|}}V'_{\varepsilon} - \frac{1}{\varepsilon}\Delta_{x'_{\perp}}V'_{\varepsilon} = \frac{1}{\varepsilon}\left(n^i_{\varepsilon} - n^e_{\varepsilon}\right), \tag{1.8}$$

where $n^i_{\varepsilon} = \int f'_{\varepsilon}dv'$ is the density of ions and n^e_{ε} the density of electrons. The density distribution of ions is normalized so that $\int f'_0 dv'dx' = 1$.

The main difference between Frénod and Sonnendrücker's model and ours lies in the following. Instead of considering a fixed background of ions, and since $\frac{m_e}{m_i} \ll 1$, we make the usual assumption that the (adiabatic) electrons are instantaneously at a local thermodynamic equilibrium, so that their density follows a Boltzmann-Maxwell distribution:

$$n^e_{\varepsilon}(x,t) = \exp\left(\frac{eV'_{\varepsilon}}{k_B T_e}\right), \tag{1.9}$$

where k_B is the Boltzmann constant, $-e$ the charge and T_e the temperature of the electrons. We consider that $\frac{e}{k_B T_e} \sim 1$.

We make the assumption that we are not far from a fixed background of electrons, so that we can linearize this expression:

$$n^e_{\varepsilon}(x,t) = 1 + V'_{\varepsilon}. \tag{1.10}$$

We are obviously aware that this assumption is not really satisfactory from the mathematical point of view; nevertheless it is commonly used in plasma physics. The problem of a fixed background of electrons, i.e. $n^e_{\varepsilon} = 1$, brings actually more interesting formal results; this point will be discussed in the last section.

The Poisson equation can now be written:

$$V'_{\varepsilon} - \varepsilon^2\Delta_{x'_{\|}}V'_{\varepsilon} - \Delta_{x'_{\perp}}V'_{\varepsilon} = \int f_{\varepsilon}dv' - \int f_0 dv'dx'. \tag{1.11}$$

The dimensionless system (1.6) becomes (for the sake of simplicity, we forget the primes):

$$\begin{cases} \partial_t f_{\varepsilon} + \frac{v_{\perp}}{\varepsilon}.\nabla_x f_{\varepsilon} + v_{\|}.\nabla_x f_{\varepsilon} + (E_{\varepsilon} + \frac{v\wedge B}{\varepsilon}).\nabla_v f_{\varepsilon} = 0 \\ E_{\varepsilon} = (-\nabla_{x_{\perp}}V_{\varepsilon}, -\varepsilon\nabla_{x_{\|}}V_{\varepsilon}) \\ V_{\varepsilon} - \varepsilon^2\Delta_{x_{\|}}V_{\varepsilon} - \Delta_{x_{\perp}}V_{\varepsilon} = \int f_{\varepsilon}dv - \int f_0 dvdx \\ f_{\varepsilon,t=0} = f_{\varepsilon,0}, \end{cases} \tag{1.12}$$

with the notation $\Delta_{x_{\|}} = \partial^2_{x_{\|}}$ and $\Delta_{x_{\perp}} = \Delta - \Delta_{x_{\|}}$,

the problem being posed for $(x_{\perp}, x_{\|}, v) \in \mathbb{T}^2 \times \mathbb{T} \times \mathbb{R}^3$ (with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ equipped with the restriction of the Lebesgue measure to $[0,1[$).

1.2.2 State of the art about the Finite Larmor Radius Approximation

Using homogenization arguments, Frénod and Sonnendrücker established the convergence in some weak sense of sequences of solutions $(f_{\varepsilon})_{\varepsilon\geq 0}$ of similar systems, in two cases, namely in some pseudo $2D$ case (assuming that nothing depends on $x_{\|}$ and $v_{\|}$) and in a $3D$ case when the electric field is external. The main tool used to establish the convergence is the "2-scale convergence" introduced by Nguetseng [132] and Allaire [2] that we will recall later on.

i. The 3D case:

Assume that we deal with an external electric field $E_\varepsilon = E \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R}^3)$:

$$\begin{cases} \partial_t f_\varepsilon + v_\parallel.\nabla_x f_\varepsilon + \frac{v_\perp}{\varepsilon}.\nabla_x f_\varepsilon + \left(E + \frac{v\wedge e_z}{\varepsilon}\right).\nabla_v f_\varepsilon = 0 \\ f_{t=0} = f_0. \end{cases}$$

Frénod and Sonnendrücker proved the following theorem:

Theorem 1.1. *For each ε, let f_ε be the unique solution of the scaled Vlasov equation in $L^\infty_t(L^1_{x,v}\cap L^2_{x,v})$. Then the following convergence holds as ε tends to 0:*

$$f_\varepsilon \rightharpoonup f \text{ weak-* } L^\infty_t(L^2_{x,v}) \tag{1.13}$$

where $f \in L^\infty_t(L^2_{x,v})$ is the unique solution to:

$$\partial_t f + v_\parallel.\nabla_x f + \frac{1}{2\pi}\left(\int_0^{2\pi}\mathcal{R}(\tau)E(t,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_x f$$
$$+\frac{1}{2\pi}\left(\int_0^{2\pi}R(\tau)E(t,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_v f = 0,$$

$$f_{|t=0} = \frac{1}{2\pi}\left(\int_0^{2\pi} f_0(x+\mathcal{R}(\tau)v, R(\tau)v)d\tau\right),$$

denoting by R and $\mathcal{R}$ the linear operators defined by:

$$R(\tau) = \begin{bmatrix} \cos\tau & \sin\tau & 0 \\ -\sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathcal{R}(\tau) = \left(R(\pi/2) - R(\pi/2+\tau)\right).$$

ii. The $2D$ case:

The Vlasov-Poisson system considered in this case is the following $2D$ system:

$$\partial_t f_\varepsilon + \frac{v}{\varepsilon}.\nabla_x f_\varepsilon + \left(E_\varepsilon + \frac{v^\perp}{\varepsilon}\right).\nabla_v f_\varepsilon = 0 \tag{1.14}$$
$$f_{\varepsilon|t=0} = f_0 \tag{1.15}$$
$$E_\varepsilon = -\nabla V_\varepsilon, -\Delta_x V_\varepsilon = \rho_\varepsilon, \tag{1.16}$$
$$\rho_\varepsilon = \int f_\varepsilon dv. \tag{1.17}$$

If $v = (v_x, v_y)$, $v^\perp$ is defined by $(v_y, -v_x)$.

We recall that there exist global weak solutions of Vlasov-Poisson systems in the sense of Arsenev ([4]).

Assuming here that $f_0 \geq 0, f_0 \in L^1_{x,v}\cap L^p_{x,v}$ (for some $p > 2$) and that the initial energy is bounded, Frénod and Sonnendrücker proved the following theorem (we voluntarily write an unprecise meta-version of the result)

Theorem 1.2. *For each ε, let $(f_\varepsilon, E_\varepsilon)$ be a solution in the sense of Arsenev to (1.14)-(1.17).*

Then, up to a subsequence, f_ε weakly converges to a function f Moreover, there exists a function G such that :

$$f = \int_0^{2\pi} G(t, x + \mathcal{R}(\tau)v, R(\tau)v)d\tau, \tag{1.18}$$

and G satisfies :

$$\partial_t G + \frac{1}{2\pi}\left(\int_0^{2\pi} \mathcal{R}(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_x G$$
$$+\frac{1}{2\pi}\left(\int_0^{2\pi} R(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_v G = 0,$$
$$G_{|t=0} = f_0,$$
$$\mathcal{E} = -\nabla\Phi, \quad -\Delta\Phi = \int G(t, x + \mathcal{R}(\tau)v, R(\tau)v)dv,$$

denoting by R and $\mathcal{R}$ the linear operators defined by :

$$R(\tau) = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix}, \mathcal{R}(\tau) = (R(\pi/2) - R(\pi/2+\tau)).$$

In this case, we have to introduce an additional variable, the "fast-time" variable τ which comes from the fact that we need to precisely describe the oscillations in order to study the limit in non-linear terms.

Note that the authors actually developped a generic framework that allows them to deal with different scalings and to give a precise approximation at any order. We do not wish to do so in our study.

1.3 A bit of homogenization theory and some useful definitions

Let us now precisely state the "2-scale" convergence tools used in this chapter.

Definition. *Let X be a separable Banach space, X' be its topological dual space and $(.,.)$ the duality bracket between X' and X. For all $\alpha > 0$, denote by $\mathcal{C}_\alpha(\mathbb{R}, X)$ (respectively $L^{q'}_\alpha(\mathbb{R}; X')$) the space of α-periodic continuous (respectively $L^{q'}$) functions on $\mathbb{R}$ with values in X. Let $q \in [1;\infty[$.*

Given a sequence (u_ε) of functions belonging to the space $L^{q'}(0,t;X')$ and a function $U^0(t,\theta) \in L^{q'}(0,T;L^{q'}_\alpha(\mathbb{R};X'))$ we say that

$$u_\varepsilon \text{ 2-scale converges to } U^0$$

if for any function $\Psi \in L^q(0,T;\mathcal{C}_\alpha(\mathbb{R},X))$ we have:

$$\lim_{\varepsilon\to 0}\int_0^T \left(u_\varepsilon(t), \Psi\left(t, \frac{t}{\varepsilon}\right)\right) dt = \frac{1}{\alpha}\int_0^T\int_0^\alpha \left(U^0(t,\tau), \Psi(t,\tau)\right) d\tau dt. \tag{1.19}$$

Theorem 1.3. *Given a sequence (u_ε) bounded in $L^{q'}(0,t;X')$, there exists for all $\alpha > 0$ a function $U^0_\alpha \in L^{q'}(0,T;L^{q'}_\alpha(\mathbb{R};X'))$ such that up to a subsequence,*

$$u_\varepsilon \text{ 2-scale converges to } U^0_\alpha.$$

The profile U^0_α is called the α-periodic two scale limit of u_ε and the link between U^0_α and the weak- limit u of u_ε is given by:*

$$\frac{1}{\alpha}\int_0^\alpha U^0 d\tau = u. \tag{1.20}$$

We also introduce some notations:

Notations. *We define for all* $p \in [1;\infty]$ *the space* $L^p_{x,v}{:=}L^p_x(\mathbb{T}^d, (L^p_v(\mathbb{R}^d)))$.

In the same fashion, we define the spaces $L^p_{t,x}$, $L^p_{t,x,v}\cdots$

Let $L^p_{2\pi,\tau}$ *be the space of* 2π*-periodic functions of* τ *which are in* L^p_τ.

Let $L^p_{x,loc}$ *be the space of functions* f *such that for all infinitely differentiable cut-off functions* $\varphi \in \mathcal{C}^\infty_c$, φf *belongs to* L^p_x. *We will say that a sequence* (f_ε) *is uniformly bounded in* $L^p_{x,loc}$ *if for each compact set* K, *the sequence of the restrictions to* K *is uniformly bounded in* L^p_x *with respect to* ε *(but this bound can depend on* K*).*

We will also use the same notations for Sobolev spaces $W^{s,p}$ $(s \in \mathbb{R})$.

1.4 Statement of the result

In this chapter we prove that the 2-scale convergence established in the previous $2D$ case is also true in our $3D$ framework. The difficulty comes from the fact that there is no uniform elliptic regularity for the electric field because of the factor ε^2 in front of $\Delta_{x_\parallel}$ in the Poisson equation:

$$V_\varepsilon - \varepsilon^2 \Delta_{x_\parallel} V_\varepsilon - \Delta_{x_\perp} V_\varepsilon = \int f_\varepsilon dv - \int f_\varepsilon dv dx$$

In particular there is no a priori regularity on $x_\parallel$ and therefore no strong compactness. Nevertheless, we actually prove that due to the particular form of the asymptotic equation, the moments of the solution with respect to $v_\parallel$ are more regular in $x_\parallel$ than the solution itself. We can then easily pass to the weak limit.

The reason why we have opted for this strange Poisson equation instead of the usual one will appear at the end of the next section and especially in the last one. Roughly speaking it allows us to "kill" the plasma waves which appear in the parallel direction due to the quasineutrality.

Notice that this result is in the same spirit as the proof of the weak stability of the Vlasov-Maxwell system by DiPerna and Lions ([51]), where the authors have regularity on moments, by opposition to the proof of the weak stability of the Vlasov-Poisson system by Arsenev ([4]), where the author has compactness on the electric field. Actually our result is a kind of a hybrid one, since we get on one hand regularity with respect to $x_\perp$ by elliptic regularity and in the other hand regularity with respect to $x_\parallel$ by averaging.

We assume here that the initial data $(f_{\varepsilon,0})_{\varepsilon>0}$ satisfy the following conditions:

- $f_{\varepsilon,0} \geq 0$.
- $(f_{\varepsilon,0})_{\varepsilon>0}$ is uniformly bounded with respect to ε in $L^1_{x,v} \cap L^p_{x,v}$ (for some $p > 3$) and for each ε, $\int f_{\varepsilon,0} dx dv = 1$.
- The initial energy is uniformly bounded with respect to ε:

$$\left(\int f_{\varepsilon,0}|v|^2 dv dx + \varepsilon \int V_{\varepsilon,0}^2 dx + \varepsilon \int |\nabla_{x_\perp} V_{\varepsilon,0}|^2 dx + \varepsilon^3 \int |\nabla_{x_\parallel} V_{\varepsilon,0}|^2 dx\right) \leq C.$$

Theorem 1.4. *For each* ε, *let* $(f_\varepsilon, E_\varepsilon)$ *in* $L^\infty_t(L^1_{x,v} \cap L^p_{x,v}) \times L^\infty_t(L^2_x)$ *be a global weak solution in the sense of Arsenev to (1.12). Then up to a subsequence we have the following convergence as* ε *tends to* 0:

$$f_{\varepsilon,0} \quad \textit{weakly-* converges to} \quad f_0 \in L^p_{x,v} \tag{1.21}$$

$$f_\varepsilon \quad \textit{2-scale converges to} \quad F \in L^\infty_t(L^\infty_{2\pi,\tau}(L^1_{x,v} \cap L^p_{x,v})) \tag{1.22}$$

$$E_\varepsilon \quad \textit{2-scale converges to} \quad \mathcal{E} \in L^\infty_t(L^\infty_{2\pi,\tau}(L^{3/2}_{x_\parallel}(W^{1,\frac{3}{2}}_{x_\perp}))). \tag{1.23}$$

Moreover, there exists a function $G \in L^\infty_t(L^1_{x,v} \cap L^p_{x,v})$ *such that:*

$$F(t,\tau,x,v) = G(t, x + \mathcal{R}(\tau)v, R(\tau)v), \tag{1.24}$$

and $(G,\mathcal{E})$ *is solution to:*

$$\partial_t G + v_\parallel.\nabla_x G + \frac{1}{2\pi}\left(\int_0^{2\pi} \mathcal{R}(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_x G$$
$$+\frac{1}{2\pi}\left(\int_0^{2\pi} R(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_v G = 0,$$
$$G_{|t=0} = f_0,$$
$$\mathcal{E} = (-\nabla_\perp V, 0), \; V - \Delta_\perp V = \int G(t,x+\mathcal{R}(\tau)v, R(\tau)v)dv - \int f_0 dvdx,$$

denoting by R *and* $\mathcal{R}$ *the linear operators defined by:*

$$R(\tau) = \begin{bmatrix} \cos\tau & -\sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathcal{R}(\tau) = \left(R(\pi/2) - R(\pi/2+\tau)\right).$$

As it has been said, for the proof of this theorem, we will first prove a proposition which gives the regularity of moments in $v_\parallel$ of the solution. For this, we use an averaging lemma. The beginning of the proof is very similar to the proof in the $2D$ case, but we will give it again for the sake of completeness.

Remarks 1.1. i. The assumption on the initial energy may, at first sight, look a bit restrictive but in the "usual" Vlasov-Poisson scaling, it only means that the inital electric potential and field are bounded in L^2.

ii. The constant $q = 3$ will come quite naturally from Lemma 2.3 and Proposition 3.1.

iii. This theorem implies that for a given non-negative initial data $G_{|t=0} = G_0$ in $L^1_{x,v} \cap L^p_{x,v}$ (with $p > 3$) and satisfying the energy bound, the asymptotic system admits at least one global weak solution $G \in L^\infty_t(L^1_{x,v} \cap L^p_{x,v})$. With the additional assumptions on the inital data:

$$G_0 \in W^{1,1}_{x,v},$$
$$\|(1+|v|^4)G_0\|_{L^\infty_{x,v}} < \infty,$$
$$\|(1+|v|^4)DG_0\|_{L^\infty_{x,v}} < \infty.$$

we are actually able to prove the uniqueness of the solution, using the same ideas than Degond in [44] (and also used afterwards by Saint-Raymond in a gyrokinetic context ([139])). Hence, it means that if the whole sequence $(f_{\varepsilon,0})$ weak -* converges to some f_0 satisfying these estimates, then by uniqueness of the solution, there is convergence for the whole sequence (f_ε) (without extracting any subsequence).

2 A priori uniform estimates for the scaled Vlasov-Poisson system

2.1 Conservation of L^p norms and energy for the scaled system

In this section we give a priori estimates which are very classical for the Vlasov-Poisson system (used for example in [59], [60], [72]). In order to recall how one can get them, we will give some formal computations. If one wants to have rigorous proofs, one should deal with smooth and compactly supported functions, namely with a sequence $(f_\varepsilon^n)_{n\geq 0}$ of solutions of some regularized Vlasov-Poisson equations then pass to the limit (that is the way one can clasically build a global weak solution in the sense of Arsenev ([4])).

First, as usual for such Vlasov equations, L^p norms are conserved (we work here at a fixed ε):

Lemma 2.1. *For all $1 \leq p \leq \infty$,*

$$\forall t \geq 0, \|f(t)\|_{L^p_{x,v}} \leq \|f(0)\|_{L^p_{x,v}}. \tag{2.1}$$

Moreover, $f_0 \geq 0$ if and only if $\forall t \geq 0, f(t) \geq 0$ (referred to as the maximum principle)

That precisely means that if $f_0 \in L^p_{x,v}$, then $f \in L^\infty_t(L^p_{x,v})$.

Let us now compute the energy for the scaled system:

Lemma 2.2. *We have the estimate:*

$$\mathcal{E}_\varepsilon(t) = \left(\int f_\varepsilon |v|^2 dvdx + \varepsilon \int V_\varepsilon^2 dx + \varepsilon \int |\nabla_{x_\perp} V_\varepsilon|^2 dx + \varepsilon^3 \int |\nabla_{x_\parallel} V_\varepsilon|^2 dx\right) \leq \mathcal{E}_\varepsilon(0). \tag{2.2}$$

In particular if there exists $C > 0$ independent of ε such that $\mathcal{E}_\varepsilon(0) \leq C$, then:

$$\int f_\varepsilon |v|^2 dvdx \leq C. \tag{2.3}$$

Formal proof. We multiply the scaled Vlasov equation by $|v|^2$ and integrate with respect to x and v.

$$\int \partial_t f_\varepsilon |v|^2 dvdx + \int E_\varepsilon . \nabla_v f_\varepsilon |v|^2 dvdx \quad = \quad \frac{d}{dt}\left(\int f_\varepsilon |v|^2 dvdx\right) - 2\int E_\varepsilon(x).v f_\varepsilon dvdx = 0.$$

We then integrate the Vlasov equation with respect to v. We get the so called conservation of charge:

$$\frac{d}{dt}\left(\int f dv\right) + \nabla_{x_\parallel} . \left(\int f v_\parallel dv\right) + \frac{\nabla_{x_\perp}}{\varepsilon} . \left(\int f v_\perp dv\right) = 0. \tag{2.4}$$

Therefore, we have:

$$\begin{aligned}
\int E_\varepsilon(x).v f_\varepsilon dvdx &= -\int (\nabla_{x_\perp} V_\varepsilon, \varepsilon \nabla_{x_\parallel} V_\varepsilon).v f_\varepsilon dvdx \\
&= \int V_\varepsilon \left(\nabla_{x_\perp}.(f_\varepsilon v_\perp) + \varepsilon \nabla_{x_\parallel}.(f_\varepsilon v_\parallel)\right) dvdx \\
&= -\varepsilon \int V_\varepsilon \partial_t f_\varepsilon dvdx.
\end{aligned}$$

Finally, using the Poisson equation, we get:

$$\begin{aligned}
-\varepsilon \int V_\varepsilon \partial_t f_\varepsilon dv dx &= -\varepsilon \int V_\varepsilon \partial_t \left(V_\varepsilon - \varepsilon^2 \Delta_{x_\parallel} V_\varepsilon - \Delta_{x_\perp} V_\varepsilon \right) dx \\
&= -\varepsilon \left(\int V_\varepsilon \partial_t V_\varepsilon dx + \int \nabla_{x_\perp} V_\varepsilon \partial_t \nabla_{x_\perp} V_\varepsilon dx + \varepsilon^2 \int \nabla_{x_\parallel} \partial_t V_\varepsilon \nabla_{x_\parallel} V_\varepsilon dx \right) \\
&= -\varepsilon \frac{1}{2} \frac{d}{dt} \left(\int V_\varepsilon^2 dx + \int |\nabla_{x_\perp} V_\varepsilon|^2 dx + \varepsilon^2 \int |\nabla_{x_\parallel} V_\varepsilon|^2 dx \right).
\end{aligned}$$

Thus it comes:

$$\frac{d}{dt} \left(\int f_\varepsilon |v|^2 dv dx + \varepsilon \int V_\varepsilon^2 dx + \varepsilon \int |\nabla_{x_\perp} V_\varepsilon|^2 dx + \varepsilon^3 \int |\nabla_{x_\parallel} V_\varepsilon|^2 dx \right) = 0. \tag{2.5}$$

□

2.2 Regularity of the electric field

Let us recall a classical lemma obtained by a standard real interpolation argument:

Lemma 2.3. *Let $f(x,v)$ be a mesurable positive function on $\mathbb{R}^3 \times \mathbb{R}^3$. Then:*

$$\int \left(\int f(x,v) dv \right)^{3/2} dx \leq C \|f\|_{L^3_{x,v}}^{3/4} \left(\int |v|^2 f dx dv \right)^{3/4}. \tag{2.6}$$

Proof. For any $R > 0$, we can write the following decomposition:

$$\begin{aligned}
\int f_\varepsilon dv &= \int_{|v| \leq R} f dv + \int_{|v|>R} f dv \\
&\leq CR^2 \|f\|_{L^3_v} + \frac{1}{R^2} \int |v|^2 f dv.
\end{aligned}$$

Then we can take R such that $R^2 \|f\|_{L^3_v} = \frac{1}{R^2} \int |v|^2 f dv$ so that we get:

$$\int f dv \leq C \left(\int f^3 dv \right)^{1/6} \left(\int |v|^2 f dv \right)^{1/2}. \tag{2.7}$$

We then raise the quantities to the power $3/2$, integrate with respect to x and use Hölder's inequality which gives the estimate.

□

By conservation of the L^3 norm and the uniform bound on the intial energy, Lemmas 2.2 and 2.3 entail that:

$$\rho_\varepsilon \in L^\infty_t (L^{3/2}_x), \tag{2.8}$$

and the norm is bounded uniformly with respect to ε.

We now use the Poisson equation to compute the regularity of the electric field. Let us recall that:

$$E_\varepsilon = \left(-\varepsilon \nabla_{x_\parallel} V_\varepsilon, -\nabla_{x_\perp} V_\varepsilon \right)$$

$$V_\varepsilon - \varepsilon^2 \Delta_{x_\parallel} V_\varepsilon - \Delta_{x_\perp} V_\varepsilon = \rho_\varepsilon - \int \rho_0 dx.$$

Lemma 2.4. *With the above notations and assumptions:*
E_ε *is uniformly bounded with respect to* ε *in* $L^\infty_t(L^{3/2}_{x_\parallel}(W^{1,3/2}_{x_\perp}))$.

Proof. Let $\varepsilon > 0$ and $t > 0$ be fixed. For the sake of simplicity we write V instead of V_ε and E instead of E_ε.

For any function $f(x_\parallel, x_\perp)$, define the rescaled function $\tilde{f}(z, x_\perp)$ by

$$\tilde{f}\left(\frac{x_\parallel}{\varepsilon}, x_\perp\right) = \varepsilon^{\frac{2}{3}} f(x_\parallel, x_\perp),$$

so that:

$$\|\tilde{f}\,(z, x_\perp)\,\|_{L^{3/2}_z} = \|f(x_\parallel, x_\perp)\|_{L^{3/2}_{x_\parallel}}. \tag{2.9}$$

The Poisson equation becomes:

$$\tilde{V} - \Delta_z \tilde{V} - \Delta_{x_\perp} \tilde{V} = \tilde{\rho} - \varepsilon^{\frac{2}{3}} \int \rho_0 dx$$

and the scaled electric field is given by:

$$\tilde{E} = \left(-\nabla_z \tilde{V}, -\nabla_{x_\perp} \tilde{V}\right).$$

Since $\rho_\varepsilon(t, ., .)$ and V_ε are uniformly bounded in $L^{3/2}_x$, standard results of elliptic regularity on the torus $\mathbb{T}^2 \times \frac{1}{\varepsilon}\mathbb{T}$ show that there exists $C > 0$ independent of ε such that:

$$\|\tilde{V}\|_{W^{2,3/2}_{z,x_\perp}} \leq C \left\|\tilde{\rho} - \varepsilon^{\frac{2}{3}} \int \rho_0 dx\right\|_{L^{3/2}_{z,x_\perp}}.$$

Remark 2.1. Notice here that due to the dilatation of order $\frac{1}{\varepsilon}$ in the parallel direction, being periodic in this direction does not make things easier.

Thanks to (2.9) we get:

$$\|\tilde{V}\|_{W^{2,3/2}_{z,x_\perp}} \leq C \left\|\rho - \int \rho_0 dx\right\|_{L^\infty_t(L^{3/2}_x)} \leq C_0.$$

with C_0 independent of ε.

Consequently, we have:

$$\|\tilde{E}\|_{L^{3/2}_z(W^{1,3/2}_{x_\perp})} \leq \|\tilde{E}\|_{W^{1,3/2}_{z,x_\perp}} \leq C_0.$$

Finally from (2.9) we get

$$\|E_\varepsilon\|_{L^\infty_t(L^{3/2}_{x_\parallel}(W^{1,3/2}_{x_\perp}))} \leq C_0.$$

$\square$

We can see as expected that the regularity of the electric field with respect to the $x_\parallel$ variable is not sufficient to get some strong compactness.

Remarks 2.1. i. We can write the identity:

$$-\Delta_{x_\perp} V_\varepsilon = -\Delta_{x_\perp}(Id - \varepsilon^2 \Delta_{x_\parallel} - \Delta_{x_\perp})^{-1} \left(\rho_\varepsilon - \int \rho_\varepsilon dx\right), \tag{2.10}$$

so that, thanks to elliptic estimates on the torus $\mathbb{T}^2$, $V_\varepsilon \in L^{3/2}_{x_\parallel}(W^{2,3/2}_{x_\perp})$. Consequently, $\partial_{x_\parallel} V_\varepsilon$ is bounded in $L^{3/2}_{x_\perp}(W^{-1,3/2}_{x_\parallel})$. This implies that $E_{\varepsilon,\parallel} = -\varepsilon \partial_{x_\parallel} V_\varepsilon$ tends to zero in the sense of distributions.

ii. A typical function φ_ε such that φ_ε is bounded in L^p and $\frac{1}{\varepsilon}\varphi_\varepsilon$ is bounded in $W^{-1,p}$ is the oscillating function $\cos(\frac{1}{\varepsilon}x)$. This indicates that $E_{\varepsilon,\parallel}$ oscillates with a frequency of order $\frac{1}{\varepsilon}$ in the parallel direction.

iii. If we work with the usual Poisson equation

$$-\varepsilon^2\Delta_{x_\parallel}V_\varepsilon - \Delta_{x_\perp}V_\varepsilon = \rho_\varepsilon - \int \rho_\varepsilon dx$$

we only get homogeneous estimates for V_ε and we have not been able to deal with such anisotropic estimates in the following of the chapter (namely in the estimates of Proposition 3.1). Roughly speaking, if V is a solution of the Poisson equation $-\Delta V = \rho$ with $\rho \in L^{3/2}(\mathbb{R}^3)$, we can only say that $V \in \dot{W}^{2,3/2}(\mathbb{R}^3)$ (the homogeneous Sobolev space) and not $W^{2,3/2}$.

iv. This difficulty seems to be not only a technical one, but appears to be linked to the existence of plasma waves (with frequence and magnitude of order $\frac{1}{\sqrt{\varepsilon}}$) in the parallel direction which prevents us from passing directly to the limit $\varepsilon \to 0$ (see [81] and last section).

3 Proof of Theorem 1.4

Proof. The first two steps are identical to the one given in [60]. For the sake of completeness we recall here the main arguments and refer to [60] for the details.

Step 1: Deriving the constraint equation

First of all, since (f_ε) is bounded in $L^\infty_t(L^1_{x,v}\cap L^p_{x,v})$, Theorem 1.3 shows that for all $\alpha > 0$:

$$f_\varepsilon \text{ 2-scale converges to } F_\alpha \in L^\infty(0,T;L^\infty_\alpha(\mathbb{R};L^p_{x,v})).$$

Let $\Psi(t,\tau,x,v)$ be an α-periodic oscillating test function in τ and define:

$$\Psi^\varepsilon \equiv \Psi(t,\frac{t}{\varepsilon},x,v)$$

We start by writing the weak formulation of the scaled Vlasov equation against Ψ^ε. Since

$$\nabla_{x_\parallel}.v_\parallel = \nabla_{x_\perp}.v_\perp = \mathrm{div}_v\left(E_\varepsilon + \frac{v\wedge e_z}{\varepsilon}\right) = 0,$$

we get the following equation:

$$\int f_\varepsilon\left((\partial_t\Psi)^\varepsilon + \frac{1}{\varepsilon}(\partial_\tau\Psi)^\varepsilon + v_\parallel.(\nabla_x\Psi)^\varepsilon + \frac{v_\perp}{\varepsilon}.(\nabla_x\Psi)^\varepsilon + \left(E_\varepsilon + \frac{v\wedge e_z}{\varepsilon}\right).(\nabla_v\Psi)^\varepsilon\right) dtdxdv$$
$$= -\int f_0\Psi(0,0,x,v)dxdv.$$

Multiply then by ε and pass up to a subsequence to the (2-scale) limit. We get the so called constraint equation for the α-periodic profile F_α:

$$\partial_\tau F_\alpha + v_\perp.\nabla_x F_\alpha + v\wedge e_z.\nabla_v F_\alpha = 0, \tag{3.1}$$

which means that F_α is constant along the characteristics:

$$\frac{dV}{d\tau} = V \wedge e_z, \tag{3.2}$$
$$\frac{dX}{d\tau} = V_\perp. \tag{3.3}$$

A straightforward calculation therefore shows that there exists $F_\alpha^0 \in L^\infty(0,T,L^p_{x,v})$ such that:

$$F_\alpha(t,\tau,x,v) = F_\alpha^0(t,x+\mathcal{R}(\tau)v, R(\tau)v), \tag{3.4}$$

with:

$$R(\tau) = \begin{bmatrix} \cos\tau & \sin\tau & 0 \\ -\sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathcal{R}(\tau) = \left(R(\pi/2) - R(\pi/2+\tau)\right),$$

i.e. $\mathcal{R}(\tau) = \begin{bmatrix} \sin\tau & \cos\tau - 1 & 0 \\ 1-\cos\tau & \sin\tau & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Since R and $\mathcal{R}$ are 2π-periodic, we will consider the 2π profile: indeed if α and 2π were incommensurable, F_α could not depend on τ and consequently we would have no information on the oscillations.

Step 2: Filtering the essential oscillation

We now look for the equation satisfied by $F_{2\pi}^0 := G$; we introduce the filtered function g_ε:

$$g_\varepsilon(t,x,v) = f_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v), \tag{3.5}$$

(meaning that we have removed the oscillations).

We easily compute the equation satisfied by g_ε:

$$\begin{aligned}\partial_t g_\varepsilon + v_\parallel.\nabla_x g_\varepsilon + \mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v).\nabla_x g_\varepsilon \\ +R(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v).\nabla_v g_\varepsilon = 0.\end{aligned} \tag{3.6}$$

Remark 3.1. Note here that g_ε 2-scale converges to G, and since it does not depend on τ, it also weakly converges to G.

Step 3: Getting some regularity on moments

From now on, the goal is to get some compactness for the moments of g_ε with respect to $v_\parallel$. The main tool we have in mind is the following averaging lemma proved by Bézard in [14], which is a refined version of the fundamental result of DiPerna, Lions and Meyer ([50]):

Theorem 3.1. *Let $1 < p \leq 2$. Let $f, g \in L^p(dt \otimes dx \otimes dv)$ be solutions of the following transport equation*

$$\partial_t f + v.\nabla_x f = (I - \Delta_{t,x})^{\tau/2}(I-\Delta_v)^{m/2} g, \tag{3.7}$$

with $m \in \mathbb{R}^+, \tau \in [0,1[$. Then $\forall \Psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\rho_\Psi(t,x) = \int f(t,x,v)\Psi(v)dv \in W^{s,p}_{t,x}(\mathbb{R}\times\mathbb{R}^d)$ where

$$s = \frac{1-\tau}{(1+m)p'}. \tag{3.8}$$

Moreover,

$$\|\rho_\Psi\|_{W^{s,p}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \le C\left(\|f\|_{L^p(dt\otimes dx\otimes dv)} + \|g\|_{L^p(dt\otimes dx\otimes dv)}\right) \tag{3.9}$$

(C is a positive constant independent of f and g)

Averaging lemmas are an important feature of transport equations: since the transport equation (3.7) is hyperbolic, one can obviously not expect the solution f to be more regular than the right hand side or the inital data. Nevertheless, if one considers the averaged quantity ρ_Ψ, one can actually notice a gain of regularity. This phenomenon was first observed independently by Golse, Perthame and Sentis ([71]) and Agoshkov ([1]) then was formulated in a precise way for the first time by Golse, Lions, Perthame and Sentis (see [70]); it is referred to as "velocity averaging". There exists many refined versions of these results and numerous interesting applications in kinetic theory, but we shall not dwell on that. We simply point out that this tool has been successfully applied to Vlasov equations, for instance to prove the existence of global weak solutions to the Vlasov-Maxwell system as it has been done by DiPerna and Lions ([51]).

These results have been proved for functions with values in $\mathbb{R}$. Here, for our purpose, we need a new version of L^p averaging lemma for functions with values in some Sobolev space $W^{\lambda,p}(\mathbb{R}^k)$ ($k\in\mathbb{N}^*$). We prove the following result, which is sufficient in our case (probably an analogous of Bézard's optimal result is also true):

Lemma 3.1. *Let $1<p<+\infty$ and $\lambda\in\mathbb{R}$. Let $f,g\in L^p_{t,x,v}(W^{\lambda,p}_y)$ be solutions of the following transport equation*

$$\partial_t f + v.\nabla_x f = (I-\Delta_v)^{m/2} g \tag{3.10}$$

with $m\in\mathbb{R}^+$. Then $\forall\Psi\in\mathcal{C}^\infty_c(\mathbb{R}^d)$, $\rho_\Psi(t,x)=\int f(t,x,v)\Psi(v)dv \in W^{s,p}_{t,x}(W^{\lambda,p}_y)$ for any s such that

$$s\le s_2=\frac{1}{2(1+m)} \text{ for } p=2 \tag{3.11}$$

and

$$s< s_p=\frac{1}{(1+m)p'} \text{ for } p\neq 2 \tag{3.12}$$

Moreover,

$$\|\rho_\Psi\|_{W^{s,p}_{t,x}(W^{\lambda,p}_y)} \le C\left(\|f\|_{L^p_{t,x,v}(W^{\lambda,p}_y)} + \|g\|_{L^p_{t,x,v}(W^{\lambda,p}_y)}\right) \tag{3.13}$$

(C is a positive constant independent of f and g)

Proof. We prove the result in the stationary case only:

$$v.\nabla_x f = (I-\Delta_v)^{m/2} g \tag{3.14}$$

By standard arguments (see [70]) the general case then follows.

The following estimate is obvious for $q=1$ or $q=+\infty$ (and actually we can not expect any smoothing effect) :

$$\|\rho_\Psi\|_{L^q_x(W^{\lambda,q}_y)} \le C\left(\|f\|_{L^q_{x,v}(W^{\lambda,q}_y)} + \|g\|_{L^q_{x,v}(W^{\lambda,q}_y)}\right). \tag{3.15}$$

For $p=2$, we prove the result as in Golse-Lions-Perthame-Sentis [70]. We denote by ξ (resp. η) the Fourier variable associated to x (resp. y).

The only point is to notice (using Fubini's inequality):

$$\|\rho_\Psi\|^2_{H^s_x(H^\lambda_y)} = \int (1+|\xi|^2)^{s/2}\int (1+|\eta|^2)^{\lambda/2}\left(\int \mathcal{F}_{\xi,\eta} f\Psi(v)dv\right)^2 d\eta d\xi$$

The proof is then identical and we get for $s = \frac{1}{2(1+m)}$:

$$\|\rho_\Psi\|_{H^s_x(H^\lambda_y)} \leq C\left(\|f\|_{L^2_{x,v}(H^\lambda_y)} + \|g\|_{L^2_{x,v}(H^\lambda_y)}\right). \tag{3.16}$$

Finally the general case $1 < p < +\infty$ is obtained by complex interpolation [29].

□

Equipped with this tool, we can now prove that moments in $v_\parallel$ are more regular with respect to t and $x_\parallel$ than the solution itself.

Proposition 3.1. *For each $\varepsilon > 0$, let g_ε be a function in $L^1_{x,v} \cap L^p_{x,v}$ (with $p > 3$) bounded uniformly with respect to ε and satisfying:*

$$\begin{aligned}\partial_t g_\varepsilon + v_\parallel.\nabla_x g_\varepsilon + \mathcal{R}(t/\varepsilon)E_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v).\nabla_x g_\varepsilon \\ + R(t/\varepsilon)E_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v).\nabla_v g_\varepsilon = 0,\end{aligned}$$

with E_ε the electric field uniformly bounded in $L^\infty_t(L^{3/2}_x)$.

Let $\Psi \in \mathcal{D}(\mathbb{R})$. Define

$$\eta_\varepsilon(t, x, v_\perp) = \int g_\varepsilon(t, x, v)\Psi(v_\parallel)dv_\parallel.$$

Then,

$$\eta_\varepsilon \text{ is uniformly bounded in } W^{s,\gamma}_{t,x_\parallel,loc}(W^{-1,\gamma}_{x_\perp,v_\perp,loc}), \tag{3.17}$$

for $\gamma \in]1; 2[$ defined by $\frac{1}{\gamma} = \frac{2}{3} + \frac{1}{p}$ and some $s \in]0; 1[$ (depending on γ).

Proof.

- The first step is to localize the equation. Let K be the cartesian product of compact sets:
$$K = [0, T] \times K_{x_\parallel} \times K_{x_\perp} \times K_{v_\parallel} \times K_{v_\perp}.$$
We now consider some positive smooth function $\Phi(t, x_\parallel, x_\perp, v_\parallel, v_\perp)$ which is $\mathcal{C}^\infty_c$ and which satisfies the condition:
$$\Phi \equiv 0 \text{ outside } K. \tag{3.18}$$
Noticing that:
$$\mathrm{div}_x\left(\mathcal{R}(t/\varepsilon)E_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v)\right) + \mathrm{div}_v\left(R(t/\varepsilon)E_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v)\right) = 0.$$
The equation satisfied by $g_\varepsilon\Phi$ is the following one:
$$\begin{aligned}\partial_t(g_\varepsilon\Phi) + v_\parallel.\nabla_x(g_\varepsilon\Phi) = -\underbrace{\nabla_x.(\mathcal{R}(t/\varepsilon)E_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v)g_\varepsilon\Phi)}_{(1)} \\ -\underbrace{\nabla_v.(R(t/\varepsilon)E_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v)g_\varepsilon\Phi)}_{(2)} - \partial_t(\Phi)g_\varepsilon - v_\parallel.\nabla_x(\Phi)g_\varepsilon \\ +\mathcal{R}(t/\varepsilon)E_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v).\nabla_x(\Phi)g_\varepsilon + R(t/\varepsilon)E_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v).\nabla_v(\Phi)g_\varepsilon.\end{aligned}$$
The idea is now to consider this equation as a kinetic equation with respect to the variables $(t, x_\parallel, v_\parallel)$ and with values in an abstract Banach space (which will be $W^{-1,\gamma}_{x_\perp,v_\perp}$). We then only study the first two terms of the right-hand side (noticing that the other terms have more regularity than these ones).

  From now on, for the sake of simplicity and readability, we will write L^p and $W^{s,p}$ norms without always specifying that they are taken on the compact support of Φ.

- **Estimate on the first term (1)**

Since E_ε does not depend on v, we have:

$$E_\varepsilon \in L^\infty_t(L^{3/2}_{x_\parallel}(L^\infty_v(L^{3/2}_{x_\perp}))).$$

In particular if we restrict to compact supports:

$$E_\varepsilon \in L^{3/2}_{t,x,v}.$$

The second point is that the differential operator applied in (1) involves only derivatives with respect to the $x_\perp$ variable and not in the parallel direction: this remark is fundamental for using the averaging lemma 3.1 (indeed, the case of a full derivative in $x_\parallel$ can not be handled).

Hölder's inequality simply implies that:

$$\|\mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v)g_\varepsilon\Phi\|_{L^\gamma_{x_\perp,v_\perp}} \leq \|E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v)\Phi\|_{L^{3/2}_{x_\perp,v_\perp}}\|g_\varepsilon\|_{L^p_{x_\perp,v_\perp}}, \tag{3.19}$$

where $\frac{1}{\gamma}=\frac{2}{3}+\frac{1}{p}$. Hence:

$$\|\nabla_x.(\mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v)g_\varepsilon\Phi)\|_{W^{-1,\gamma}_{x_\perp,v_\perp}} \leq \|E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v)\Phi\|_{L^{3/2}_{x_\perp,v_\perp}}\|g_\varepsilon\|_{L^p_{x_\perp,v_\perp}}. \tag{3.20}$$

Notice that the change of variables $(x,v)\mapsto(x+\mathcal{R}(s)v,v)$ has unit Jacobian for all $s\in\mathbb{R}$, so that:

$$\|E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v)\Phi\|_{L^{3/2}_{x_\perp,v_\perp}} = \|E_\varepsilon(t,x)\|_{L^{3/2}_{x_\perp,v_\perp}}. \tag{3.21}$$

So finally we have, after integrating in $t,x_\parallel,v_\parallel$ and thanks to Hölder's inequality:

$$\begin{aligned}&\|\nabla_x.\left(\mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v)g_\varepsilon\Phi\right)\|_{L^\gamma_{t,x_\parallel,v_\parallel}(W^{-1,\gamma}_{x_\perp,v_\perp})}\\ &\qquad\leq C\|E_\varepsilon(t,x)\|_{L^{3/2}_{t,x_\parallel,v_\parallel}(L^{3/2}_{x_\perp,v_\perp})}\|g_\varepsilon\|_{L^p_{t,x_\parallel,v_\parallel}(L^p_{x_\perp,v_\perp})},\end{aligned}$$

and C is a constant independent of ε.

Remark 3.2. The regularity of (1) with respect to $v_\perp$ is not optimal (since it involves no derivative in $v_\perp$ for g_ε). Nevertheless we are interested in the regularity of the whole right hand side, and we will see that the term (2) has this regularity in $v_\perp$.

- **Estimate on the second term (2)**

By the same method one gets:

$$\begin{aligned}&\|\nabla_v.\left(R(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v)g_\varepsilon\Phi\right)\|_{L^\gamma_{t,x_\parallel}(W^{-1,\gamma}_{v_\parallel}(W^{-1,\gamma}_{x_\perp,v_\perp}))}\\ &\qquad\leq C\|E_\varepsilon(t,x)\|_{L^{3/2}_{t,x_\parallel}(L^{3/2}_{x_\perp,v})}\|g_\varepsilon\Phi\|_{L^p_{t,x_\parallel}(L^p_{x_\perp,v})}.\end{aligned}$$

Finally we see that the right hand side is uniformly bounded in:

$$L^\gamma_{t,x_\parallel,\mathrm{loc}}(W^{-1,\gamma}_{v_\parallel,\mathrm{loc}}(W^{-1,\gamma}_{x_\perp,v_\perp,\mathrm{loc}})).$$

- **Regularity of the moments**

  By lemma 3.1 , for all $\Psi \in \mathcal{C}_c^\infty$, the moment:

$$\eta_\varepsilon(t,x,v_\perp) = \int g_\varepsilon(t,x,v)\Psi(v_\parallel)dv_\parallel$$

  is then uniformly bounded in the space $W^{s,\gamma}_{t,x_\parallel,\mathrm{loc}}(W^{-1,\gamma}_{x_\perp,v_\perp,\mathrm{loc}})$ for any $s > 0$ with $s < \frac{1}{2\gamma'}$.

□

We can now prove that the sequence of moments η_ε is compact in a space of distributions which is the dual of some space where the sequence (E_ε) is uniformly bounded.

Corollary 3.1. *There exists $\theta \in]0,1[$ and $\eta \in W^{s\theta,3}_{t,x_\parallel,loc}(W^{-\theta,3}_{x_\perp,v_\perp,loc})$ such that for all $\xi > 0$, up to a subsequence:*

$$\eta_\varepsilon \to \eta \text{ strongly in } L^3_{t,loc}(L^3_{x_\parallel,loc}(W^{-\theta-\xi,3}_{x_\perp,v_\perp,loc})). \tag{3.22}$$

Proof. By assumption on the initial data, there exists $q > 3$ such that $f_0 \in L^q_{x,v}$; thanks to the a priori L^q estimate, we get $g_\varepsilon \in L^\infty_t(L^q_{x,v})$. Define γ by:

$$\frac{1}{\gamma} = \frac{2}{3} + \frac{1}{q}$$

The previous lemma shows that for some $s > 0$:

$$\eta_\varepsilon \in W^{s,\gamma}_{t,x_\parallel \mathrm{loc}}(W^{-1,\gamma}_{x_\perp,v_\perp,\mathrm{loc}}) \text{ uniformly in } \varepsilon.$$

Since $g_\varepsilon \in L^q_{t,\mathrm{loc}}(L^q_{x,v})$ and Ψ has compact support, we get by Hölder's inequality:

$$\eta_\varepsilon \in L^q_{t,\mathrm{loc}}(L^q_{x_\parallel}(L^q_{x_\perp,v_\perp})).$$

Since $\frac{1}{\gamma} > \frac{2}{3} > \frac{1}{3}$ and $\frac{1}{q} < \frac{1}{3}$, there exists $\theta \in]0,1[$ such that

$$\frac{1}{3} = \frac{1-\theta}{q} + \frac{\theta}{\gamma}.$$

By interpolation ([29]) we deduce that:

$$\eta_\varepsilon \in W^{s\theta,3}_{t,x_\parallel \mathrm{loc}}(W^{-\theta,3}_{x_\perp,v_\perp,\mathrm{loc}})).$$

This implies that:

$$\eta_\varepsilon \in W^{s\theta,3}_{t,\mathrm{loc}}(L^3_{x_\parallel}(W^{-\theta,3}_{x_\perp,v_\perp,\mathrm{loc}})) \text{ uniformly in } \varepsilon.$$

$$\eta_\varepsilon \in L^3_{t,\mathrm{loc}}(W^{s\theta,3}_{x_\parallel,\mathrm{loc}}(W^{-\theta,3}_{x_\perp,v_\perp,\mathrm{loc}})) \text{ uniformly in } \varepsilon.$$

We then use the following refined interpolation result proved by Simon in [148], which is, roughly speaking, an anisotropic adaptation of the classical Riesz-Fréchet-Kolmogorov criterion for compactness in L^p:

Theorem 3.2. *Let $1 \le p \le \infty$ and $s > 0$. Let $T > 0$ and X, B, Y be three Banach Spaces such that $X \subset B \subset Y$ and with X compactly embedded in B. Let F be a bounded set of $L^p_t([0,T],X) \cap W^{s,p}_t([0,T],Y)$. Then F is relatively compact in $L^p_t([0,T],B)$.*

This entails, thanks to Sobolev's embeddings, that the sequence (η_ε) is strongly relatively compact in $L^3_{t,\mathrm{loc}}(L^3_{x_\parallel,\mathrm{loc}}(W^{-\theta-\xi,3}_{x_\perp,v_\perp,\mathrm{loc}}))$, for all $\xi > 0$.

$\square$

From now on, we consider ξ such that $\theta+\xi < 1$, which is of course possible since $\theta < 1$.

Remark 3.3. Following the remark in Step 2 and by uniqueness of the limit in the sense of distributions, we get:

$$\eta = \int G\Psi(v_\parallel)dv_\parallel.$$

Step 4: Passing to the weak limit

We will first need a technical lemma which is obtained directly from the 2-scale convergence of E_ε.

Lemma 3.2. *Up to a subsequence,*

- $\mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v)$ *weakly converges to* $\frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau \in L^\infty_t(L^\infty_{2\pi,\tau}(L^{3/2}_{x_\parallel}(W^{1,3/2}_{x_\perp}))$.

- $R(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v)$ *weakly converges to* $\frac{1}{2\pi}\int_0^{2\pi}R(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau \in L^\infty_t(L^\infty_{2\pi,\tau}(L^{3/2}_{x_\parallel}(W^{1,3/2}_{x_\perp}))$.

Proof. E_ε is uniformly bounded in $L^\infty_t(L^{3/2}_{x_\parallel}(W^{1,3/2}_{x_\perp}))$, so there exists $\mathcal{E}\in L^\infty_t(L^{3/2}_{x_\parallel}(W^{1,3/2}_{x_\perp})$ such that E_ε 2 scale converge to $\mathcal{E}$.

We take $\Psi(t,\tau,x)$ a 2π-periodic w.r.t. τ test function and use the 2 scale convergence of E_ε:

$$\begin{aligned}&\int \mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v).\Psi(t,t/\varepsilon,x)dtdx\\ =&\int E_\varepsilon(t,x).{}^t\mathcal{R}(t/\varepsilon)\Psi(t,t/\varepsilon,x-\mathcal{R}(-t/\varepsilon)v)dtdx\end{aligned}$$

$$\begin{aligned}\to\ & \frac{1}{2\pi}\int\int_0^{2\pi}\mathcal{E}(t,\tau,x).{}^t\mathcal{R}(\tau)\Psi(t,\tau,x-\mathcal{R}(-\tau)v)dtd\tau dx\\ =\ & \frac{1}{2\pi}\int\int_0^{2\pi}\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v).{}^t\mathcal{R}(\tau)\Psi(t,\tau,x)dtd\tau dx.\end{aligned}$$

The proof is the same for $R(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v)$.

$\square$

Now, we can write the weak formulation of the kinetic equation (3.6) against a smooth test function of the form $\Phi(t,x,v_\perp)\Psi(v_\parallel)$ with compact support. If we can pass to the limit for such test functions, then by density it will be also the case for all test functions.

Noticing that $\mathrm{div}_x\, v_\parallel = 0$ and that

$$\mathrm{div}_x\left(\mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v)\right)+\mathrm{div}_v\left(R(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v)\right)=0,$$

we get:

$$\int \Big(\partial_t(\Phi(t,x,v_\perp)\Psi(v_\parallel)) + v_\parallel.\nabla_x(\Phi\Psi) + \mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v).\nabla_x(\Phi\Psi)$$
$$+R(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v).\nabla_v(\Phi\Psi)\Big)g_\varepsilon dtdx_\perp dx_\parallel dv_\perp dv_\parallel$$
$$= -\int g_{\varepsilon,0}\Phi(0,x,v_\perp)\Psi(v_\parallel)dxdv.$$

We can easily take weak limits in the linear part $\partial_t g_\varepsilon + v_\parallel.\nabla_x g_\varepsilon$.
Consider now the "non linear" term:

$$\int \mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v).\nabla_x\Phi(t,x,v_\perp)g_\varepsilon\Psi(v_\parallel)dtdx_\perp dx_\parallel dv_\perp dv_\parallel =$$
$$\int \mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v).\nabla_x\Phi(t,x,v_\perp)\left(\int g_\varepsilon\Psi(v_\parallel)dv_\parallel\right)dtdx_\perp dx_\parallel dv_\perp.$$

The convergence of this term can be established by the strong/weak convergence principle. Nevertheless, we have to carefully use this technique to get the result and we will explicitly evaluate the difference:

$$\Big|\int \mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v).\nabla_x\Phi(t,x,v_\perp)\eta_\varepsilon dtdx_\perp dx_\parallel dv_\perp$$
$$-\int\frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau.\nabla_x\Phi\eta dtdxdv_\perp\Big|$$
$$\leq \Big|\int\Big(\mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v) - \frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\Big).\nabla_x\Phi\eta dtdxdv_\perp\Big|$$
$$+\Big|\int \mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v).\nabla_x\Phi(t,x,v_\perp)\left(\eta_\varepsilon-\eta\right)dtdx_\perp dx_\parallel dv_\perp\Big|.$$

The first term of the right hand side converges to zero because of the 2-scale convergence of E_ε (Lemma 3.2). We can control the second term by:

$$C\|E_\varepsilon.\nabla_x\Phi\|_{L_t^{3/2}(L_{x_\parallel}^{3/2}(W_{x_\perp,v_\perp}^{\theta+\xi,3/2}))}\|\eta_\varepsilon-\eta\|_{L_t^3(L_{x_\parallel}^3(W_{x_\perp,v_\perp}^{-\theta-\xi,3}))}. \tag{3.23}$$

(these norms are actually taken on the compact support of Φ but we do not write it for the sake of simplicity)

Using the fact that E_ε is uniformly bounded in $L_{t,\mathrm{loc}}^{3/2}(L_{x_\parallel}^{3/2}(W_{x_\perp}^{\theta+\xi,3/2}))$ (this is an easy consequence of Lemma 2.4) and Corollary 3.1,

$$\|\eta_\varepsilon-\eta\|_{L_t^3([0,T],L_{x_\parallel}^3(K_{x_\parallel},W_{x_\perp,v_\perp}^{-\theta-\xi,3}(K_{x_\perp}\times K_{v_\perp})))}\to 0,$$

we can deduce that

$$\left|\int \mathcal{R}(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v).\nabla_x\Phi(t,x,v_\perp)\left(\eta_\varepsilon-\eta\right)dtdx_\perp dx_\parallel dv_\perp\right|\to 0.$$

The proof is of course the same for the other non-linear term:

$$R(t/\varepsilon)E_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v).\nabla_v g_\varepsilon$$

To conclude let us compute the asymptotic equation satisfied by the 2-scale limit of V_ε denoted by V. We take $\Psi(t,\tau,x)$ a 2π-periodic w.r.t. τ test function. We write the weak formulation of the Poisson equation:

$$\begin{aligned}
&\int V_\varepsilon \nabla_{x_\parallel} \Psi(t,t/\varepsilon,x)dtdx + \\
\varepsilon^2 \int \nabla_{x_\parallel} V_\varepsilon \nabla_{x_\parallel} \Psi(t,t/\varepsilon,x)dtdx + &\int \nabla_{x_\perp} V_\varepsilon \nabla_{x_\perp} \Psi(t,t/\varepsilon,x)dtdx \\
= \int f_\varepsilon(t,x,v)\Psi(t,t/\varepsilon,x)dtdvdx - &\int \left(\int f_0 dvdx\right) \Psi(t,t/\varepsilon,x)dtdvdx.
\end{aligned}$$

We then pass to the 2 scale limit:

$$\begin{aligned}
&\frac{1}{2\pi}\int\int_0^{2\pi} V(t,\tau,x)\nabla_{x_\perp}\Psi(t,\tau,x)d\tau dtdx + 0 \\
+ \;&\frac{1}{2\pi}\int\int_0^{2\pi} \nabla_{x_\perp} V(t,\tau,x)\nabla_{x_\perp}\Psi(t,\tau,x)d\tau dtdx = \frac{1}{2\pi}\int\int_0^{2\pi} F(t,\tau,x,v)\Psi(t,\tau,x)dtdvdx \\
- \;&\frac{1}{2\pi}\int\int_0^{2\pi} \left(\int f_0 dvdx\right)\Psi(t,\tau,x)d\tau dvdx \\
= \;&\frac{1}{2\pi}\int\int_0^{2\pi} G(t,\tau,x+\mathcal{R}(\tau)v, R(\tau)v)\Psi(t,\tau,x)d\tau dvdx \\
- \;&\frac{1}{2\pi}\int\int_0^{2\pi} \left(\int f_0 dvdx\right)\Psi(t,\tau,x)d\tau dvdx,
\end{aligned}$$

from which we get the "Poisson" equation given in Theorem 1.4:

$$V - \Delta_\perp V = \int G(t,x+\mathcal{R}(\tau)v, R(\tau)v)dv - \int f_0 dvdx.$$

Moreover since $E_{\varepsilon,\perp} = -\nabla_{x_\perp} V_\varepsilon$ and thanks to Remark 1 following Lemma 2.4, we easily get if we pass to the two-scale limit:

$$\mathcal{E} = (-\nabla_{x_\perp} V, 0).$$

□

4 Concluding comments

4.1 Comments on the result

Finally we can see as in [60] (namely by performing the change of variables $x = x_c - v^\perp$ and looking at the new equations in the so-called gyro-variables (t, x_c, v)) that the drift involving the electric field in the asymptotic "kinetic" equation corresponds to the electric drift that we mentioned in the introduction and which was expected to appear. We also notice that the Poisson equation we get in the asymptotic system is the same than the one used in the numerical simulations of tokamak plasmas (see for example the GYSELA code in [78]). Nevertheless, physicists do not get it in the same formal way: they claim that it only expresses the quasineutrality of the plasma (there is no "real" Poisson equation involved) and the perpendicular laplacian happens to appear due from the so-called "polarization drift" ([78], see also [158] for a physical reference on the subject). It would be interesting to justify such a computation from a mathematical point of view.

At last, we wish to point out a really unpleasant feature of our model, which is that there is no parallel dynamics.

4.2 An alternative model

Let us give some comments on the gyrokinetic approximation of the system (4.2) which consists in considering a population of electrons in a fixed background of ions:

$$n^i_\varepsilon = \int f_0 dx dv. \tag{4.1}$$

Actually, quite surprisingly, this model gives to rise to rather subtle properties: in this case the parallel component of the electric field does not vanish but appears as a pressure in the end (which may bring difficulties both in the study of the asymptotic system and in numerical simulations).

$$\begin{cases} \partial_t f_\varepsilon + \frac{v_\perp}{\varepsilon}.\nabla_x f_\varepsilon + v_\parallel.\nabla_x f_\varepsilon + (E_\varepsilon + \frac{v\wedge B}{\varepsilon}).\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = (-\nabla_{x_\perp} V_\varepsilon, -\varepsilon \nabla_{x_\parallel} V_\varepsilon) \\ -\varepsilon^2 \Delta_{x_\parallel} V_\varepsilon - \Delta_{x_\perp} V_\varepsilon = \int f_\varepsilon dv - \int f_0 dv dx \\ f_{\varepsilon, t=0} = f_{\varepsilon,0}. \end{cases} \tag{4.2}$$

With the same computations as the present chapter, we get:

$$f_\varepsilon \quad \text{2-scale converges to} \quad F, \tag{4.3}$$

$$E_\varepsilon \quad \text{2-scale converges to} \quad \mathcal{E}. \tag{4.4}$$

In a formal sense, there exists a function G such that:

$$F(t,\tau,x,v) = G(t, x + \mathcal{R}(\tau)v, R(\tau)v) \tag{4.5}$$

and $(G, \mathcal{E})$ is solution to:

$$\partial_t G + v_\parallel.\nabla_x G + \frac{1}{2\pi}\left(\int_0^{2\pi} \mathcal{R}(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_x G$$
$$+\frac{1}{2\pi}\left(\int_0^{2\pi} R(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_v G = 0,$$

$$G_{|t=0} = f_0,$$

$$\mathcal{E} = (-\nabla_\perp V, \mathcal{E}_\parallel), \quad -\Delta_\perp V = \int G(t, x+\mathcal{R}(\tau)v, R(\tau)v)dv - \int f_0 dv dx,$$

still denoting by R and $\mathcal{R}$ the linear operators defined by:

$$R(\tau) = \begin{bmatrix} \cos\tau & -\sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathcal{R}(\tau) = \left(R(-\pi/2) - R(-\pi/2+\tau)\right).$$

The parallel component $\mathcal{E}_\parallel$ has to be seen as a pressure (or the Lagrange multiplier) associated to the "incompressibility" constraint $\int_{\mathbb{T}^2}\int G(t, x+\mathcal{R}(\tau)v, R(\tau)v)dv dx_\perp = \int f_0 dv dx$

Let us just give a few words on the difficulties that arise with this model. The Poisson equation can be restated as:

$$-\varepsilon^2 \Delta_{x_\parallel} V_\varepsilon - \Delta_{x_\perp} V_\varepsilon = \int f_\varepsilon dv - \int f_\varepsilon dv dx_\perp + \int f_\varepsilon dv dx_\perp - \int f_0 dv dx, \tag{4.6}$$

so that thanks to the linearity of the Poisson equation we can study separately two equations. The first one states:

$$-\varepsilon^2 \Delta_{x_\parallel} V_\varepsilon^1 - \Delta_{x_\perp} V_\varepsilon^1 = \int f_\varepsilon dv - \int f_\varepsilon dv dx_\perp. \tag{4.7}$$

For this part of the electric potential we get the same estimates as in lemma 2.4. Indeed, $\int \left(\int f_\varepsilon dv - \int f_\varepsilon dv dx_\perp \right) dx_\perp = 0$ so that we can use elliptic estimates on the torus $\mathbb{T}^2_{x_\perp}$. Consequently this electric potential does not give birth to any parallel dynamics, like in Theorem 1.4.

The second one is:

$$-\varepsilon^2 \Delta_{x_\parallel} V_\varepsilon^2 = \underbrace{\int f_\varepsilon dv dx_\perp - \int f_0 dv dx}_{\text{only depends on } x_\parallel}. \tag{4.8}$$

This equation, associated to the one giving the electric field $E^2_{\varepsilon,\parallel} = -\varepsilon \partial_{x_\parallel} V^2_\parallel$, is similar to the one studied by Grenier in [81], coupled to a Vlasov equation describing a quasineutral plasma. In this case it was shown that there exist plasma waves with both temporal and spatial oscillations with frequency $\frac{1}{\sqrt{\varepsilon}}$ and magnitude of order $\frac{1}{\sqrt{\varepsilon}}$. Because of these waves, it is much more difficult to pass to the limit in order to get a kinetic equation. Grenier managed to prove the convergence only for distribution functions with special form and got in the end a Euler-like system with an electric field interpreted as a Lagrange multiplier. Hence, in our case, this part of the electric field may engender an non-zero $\mathcal{E}_\parallel$.

This will be the matter of next chapter.

4.3 Prospects

An interesting issue would be to consider a “true” Boltzmann-Maxwell distribution for the electrons (not linearized like in this chapter) and perform an asymptotic analysis, maybe with a relative entropy method, as in Chapter 2.

Chapter 4

The three-dimensional finite larmor radius approximation: the case of electrons in a background of fixed ions

Article soumis.

Sommaire

Résumé : This chapter directly follows the previous one, where the three-dimensional analysis of a Vlasov-Poisson equation with finite Larmor radius scaling was led, corresponding to the case of ions with massless electrons whose density follows a linearized Maxwell-Boltzmann law. We now consider the case of electrons in a background of fixed ions, which was only sketched in [94]. Unfortunately, there is evidence that the formal limit is false in general. Nevertheless, we formally derive from the Vlasov-Poisson equation a fluid system for particular monokinetic data. We prove the local in time existence of analytic solutions and rigorously study the limit (when the intensity of the magnetic field and the Debye length vanish) to a new anisotropic fluid system. This is achieved thanks to Cauchy-Kovalevskaya type techniques, as introduced by Caflisch [34] and Grenier [81]. We finally show that this approach fails in Sobolev regularity, due to multi-fluid instabilities.

1 Introduction

1.1 Presentation of the problem

The main goal of this chapter is to derive some fluid model in order to understand the behaviour of a quasineutral gas of electrons in a neutralizing background of fixed ions and submitted to a strong magnetic field. For simplicity, we consider that the magnetic field has fixed direction and intensity. The density of the electrons is governed by the classical Vlasov-Poisson equation. We first introduce some notations:

Notations. *Let $(e_1, e_2, e_\parallel)$ be a fixed orthonormal basis of $\mathbb{R}^3$.*

- *The subscript $\perp$ stands for the orthogonal projection on the plane (e_1, e_2), while the subscript $\parallel$ stands for the projection on $e_\parallel$.*
- *For any vector $X = (X_1, X_2, X_\parallel)$, we define $X^\perp$ as the vector $(X_y, -X_x, 0) = X \wedge e_\parallel$.*
- *We define the differential operators $\Delta_{x_\parallel} = \partial^2_{x_\parallel}$ and $\Delta_{x_\perp} = \partial^2_{x_1} + \partial^2_{x_2}$.*

Then the magnetic field we consider can be taken as:

$$B = \overline{B} e_\parallel,$$

where $\overline{B} > 0$ is a constant. The scaling we consider (we refer to the appendix for physical explanations) leads to the study of the scaled Vlasov-Poisson system (for $t > 0, x \in \mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3, v \in \mathbb{R}^3$ and ε is a small positive constant):

$$\begin{cases} \partial_t f_\varepsilon + \frac{v_\perp}{\varepsilon}.\nabla_x f_\varepsilon + v_\parallel.\nabla_x f_\varepsilon + (E_\varepsilon + \frac{v \wedge e_\parallel}{\varepsilon}).\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = (-\nabla_{x_\perp} V_\varepsilon, -\varepsilon \nabla_{x_\parallel} V_\varepsilon) \\ -\varepsilon^2 \Delta_{x_\parallel} V_\varepsilon - \Delta_{x_\perp} V_\varepsilon = \int f_\varepsilon dv - \int f_\varepsilon dv dx \\ f_{\varepsilon, t=0} = f_{\varepsilon,0} \geq 0, \quad \int f_{\varepsilon,0} dv dx = 1. \end{cases} \tag{1.1}$$

The non-negative quantity $f_\varepsilon(t, x, v)$ is interpreted as the distribution function of the electrons: this means that $f_\varepsilon(t, x, v) dx dv$ is the probability of finding particles at time t with position x and velocity v; $V_\varepsilon(t, x)$ and $E_\varepsilon(t, x)$ are respectively the electric potential and force.

This corresponds to the so-called finite Larmor radius scaling for the Vlasov-Poisson equation, which was introduced by Frénod and Sonnendrücker in the mathematical literature [60]. The $2D$ version of the system (obtained when one restricts to the perpendicular dynamics) and the limit $\varepsilon \to 0$ were studied in [60] and more recently in [16] and [64]. A version of the full $3D$ system describing ions with massless electrons was studied by the author in [94]. In this work, we considered that the density of electrons follows a linearized Maxwell-Boltzmann law. This means that we studied the following Poisson equation for the electric potential:

$$V_\varepsilon - \varepsilon^2 \Delta_{x_\parallel} V_\varepsilon - \Delta_{x_\perp} V_\varepsilon = \int f_\varepsilon dv - \int f_\varepsilon dv dx. \tag{1.2}$$

In this case it was shown after some filtering that the number density f_ε weakly converges as $\varepsilon \to 0$ to some solution f to another kinetic system exhibiting the so-called $E \times B$ drift in the orthogonal plane, but with trivial dynamics in the parallel direction. This last feature seemed somehow disappointing.

We observed in [94] that in the case where the Poisson equation reads (which precisely corresponds to the case of (1.1)):

$$-\varepsilon^2 \Delta_{x_\parallel} V_\varepsilon - \Delta_{x_\perp} V_\varepsilon = \int f_\varepsilon dv - \int f_\varepsilon dv dx, \tag{1.3}$$

we could expect to make a pressure appear in the limit process $\varepsilon \to 0$, due to some incompressibility constraint. Indeed, passing formally to the limit $\varepsilon \to 0$ (and assuming that f_ε converges to f and V_ε converges to V in some sense), we obtain:

$$-\Delta_{x_\perp} V = \int f dv - \int f dv dx,$$

and integrating this equation with respect to $x_\perp$, we finally get the constraint:

$$\int f dv dx_\perp = \int f dv dx.$$

Unfortunately, we were not able to rigorously derive a kinetic limit or even a fluid limit from (1.1). This is not only due to technical mathematical difficulties. This is related to the existence of instabilities for the Vlasov-Poisson equation, such as the double-humped instabilities (see Guo and Strauss [89]) and their counterpart in the multi-fluid Euler equations, such as the two-stream instabilities (see Cordier, Grenier and Guo [38]). Such instabilities actually take over in the limit $\varepsilon \to 0$ and the formal limit is false in general, unless $f_{\varepsilon,0}$ does not depend on parallel variables, which corresponds to the $2D$ problem studied by Frénod and Sonnendrücker [60].

Actually, we can observe that if on the contrary the initial data $f_{\varepsilon,0}$ depends only on parallel variables, we obtain the one-dimensional quasineutral system:

$$\begin{cases} \partial_t f_\varepsilon + v_\parallel \partial_{x_\parallel} f_\varepsilon - \partial_{x_\parallel} V_\varepsilon \partial_{v_\parallel} f_\varepsilon = 0 \\ -\varepsilon^2 \partial_{x_\parallel}^2 V_\varepsilon = \int f_\varepsilon dv - \int f_\varepsilon dv dx_\parallel \\ f_{\varepsilon,t=0} = f_{\varepsilon,0} \geq 0, \quad \int f_{\varepsilon,0} dv dx_\parallel = 1. \end{cases} \tag{1.4}$$

The formal limit is easily obtained, by taking $\varepsilon = 0$:

$$\begin{cases} \partial_t f + v_\parallel \partial_{x_\parallel} f - \partial_{x_\parallel} V \partial_{v_\parallel} f = 0 \\ \int f dv = \int f dv dx_\parallel \\ f_{t=0} = f_0 \geq 0, \quad \int f_0 dv dx_\parallel = 1. \end{cases} \tag{1.5}$$

In [84], an explicit example of Grenier shows that the formal limit is false in general, because of the double-humped instability:

Theorem 1.1 (Grenier, [84]). *For any N and s in $\mathbb{N}$, and for any $\varepsilon < 1$, there exist for $i = 1, 2, 3, 4$, $v_i^\varepsilon(x) \in H^s(\mathbb{T})$ with $\|v_1^\varepsilon(x) + 1\|_{H^s} \le \varepsilon^N$, $\|v_2^\varepsilon(x) + 1/2\|_{H^s} \le \varepsilon^N$, $\|v_3^\varepsilon(x) - 1/2\|_{H^s} \le \varepsilon^N$, $\|v_4^\varepsilon(x) - 1\|_{H^s} \le \varepsilon^N$, such that the solution $f_\varepsilon(t, x, v)$ associated to the initial data defined by:*

$$\begin{aligned} f_{\varepsilon,0}(x,v) &= 1 \quad \textit{for} \quad v_1^\varepsilon(x) \le v \le v_2^\varepsilon(x) \textit{ and } v_3^\varepsilon(x) \le v \le v_4^\varepsilon(x) \\ &= 0 \quad \textit{elsewhere.} \end{aligned}$$

We also define f_0 by:

$$\begin{aligned} f_0(x,v) &= 1 \quad \textit{for} \quad -1 \le v \le -1/2 \textit{ and } 1/2 \le v \le 1 \\ &= 0 \quad \textit{elsewhere.} \end{aligned}$$

Then f_ε does not converge to f_0 in the following sense:

$$\liminf_{\varepsilon \to 0} \sup_{t \le T} \int |f_\varepsilon(t,x,v) - f_0(v)| v^2 dv dx > 0 \tag{1.6}$$

for any $T > 0$ and also for $T = \varepsilon^\alpha$, with $\alpha < 1/2$.

In order to overcome the effects of these instabilities for the usual quasineutral limit, there are two possibilities:

- One consists in restricting to particular initial profiles chosen in order to be stable (this would imply in particular some monotony conditions on the data, such as the Penrose condition [133]).
- The other one consists in considering data with analytic regularity, in which case the instabilities (which happen to be essentially of "Sobolev" nature) do not have any effect.

Here the situation is worst: by opposition to the usual quasineutral limit (see [27], [84]), restricting to stable profiles is not sufficient. This is due to the anisotropy of the problem and the dynamics in the perpendicular variables.

In this chapter, we illustrate this phenomenon by studying the following fluid system, formally derived from the kinetic system (1.1) by considering some physically relevant monokinetic data (we refer to the appendix for the detailed formal derivation).

$$\begin{cases} \partial_t \rho_\varepsilon + \nabla_\perp.(E_\varepsilon^\perp \rho_\varepsilon) + \partial_\parallel(v_{\parallel,\varepsilon}\rho_\varepsilon) = 0 \\ \partial_t v_{\parallel,\varepsilon} + \nabla_\perp.(E_\varepsilon^\perp v_{\parallel,\varepsilon}) + v_{\parallel,\varepsilon}\partial_\parallel(v_{\parallel,\varepsilon}) = -\varepsilon\partial_\parallel \phi_\varepsilon(t,x) - \partial_\parallel V_\varepsilon(t,x_\parallel) \\ E_\varepsilon^\perp = -\nabla^\perp \phi_\varepsilon \\ -\varepsilon^2 \partial_\parallel^2 \phi_\varepsilon - \Delta_\perp \phi_\varepsilon = \rho_\varepsilon - \int \rho_\varepsilon dx_\perp \\ -\varepsilon \partial_\parallel^2 V_\varepsilon = \int \rho_\varepsilon dx_\perp - 1, \end{cases} \tag{1.7}$$

where:

- $\rho_\varepsilon(t, x_\perp, x_\parallel) : \mathbb{R}^+ \times \mathbb{T}^3 \to \mathbb{R}_*^+$ can be interpreted as a charge density,
- $v_{\parallel,\varepsilon}(t, x_\perp, x_\parallel) : \mathbb{R}^+ \times \mathbb{T}^3 \to \mathbb{R}$ can be interpreted as a "parallel" current density.
- $\phi_\varepsilon(t, x_\parallel)$ and $V_\varepsilon(t, x)$ are electric potentials.

Although we have considerered monokinetic data, (1.7) is intrinsically a "multi-fluid" system, because of the dependence on $x_\perp$. Hence, we still have to face the two-stream instabilities ([38]): because of these, the limit is false in Sobolev regularity and we thus decide to study the associated Cauchy problem for analytic data.

We then prove the limit to a new fluid system which is strictly speaking compressible but also somehow "incompressible in average". This rather unusual feature is due to the anisotropy of the model. The fluid system is the following (obtained formally by taking $\varepsilon = 0$):

$$\begin{cases} \partial_t \rho + \nabla_\perp.(E^\perp \rho) + \partial_\parallel (v_\parallel \rho) = 0 \\ \partial_t v_\parallel + \nabla_\perp.(E^\perp v_\parallel) + v_\parallel \partial_\parallel (v_\parallel) = -\partial_\parallel p(t, x_\parallel) \\ E^\perp = \nabla^\perp \Delta_\perp^{-1} \left(\rho - \int \rho dx_\perp\right) \\ \int \rho dx_\perp = 1. \end{cases} \tag{1.8}$$

We observe that this system can be interpreted as an infinite system of Euler-type equations, coupled together through the "parameter" $x_\perp$ by the constraint:

$$\int \rho dx_\perp = 1.$$

It has some interesting features:

- This system is anisotropic in $x_\perp$ and $x_\parallel$ and it somehow combines two features of the incompressible Euler equations. The $2D$ part of the dynamics of the equation for ρ is nothing but the vorticity formulation of $2D$ incompressible Euler. Nevertheless, physically speaking, ρ should be interpreted here as a density rather than a vorticity. The dynamics in the parallel direction is similar to the dynamics of incompressible Euler written in velocity. We finally observe that the pressure p only depends on the parallel variable $x_\parallel$ and not on $x_\perp$.

- This does not strictly speaking describe an incompressible fluid, since $(E^\perp, v_\parallel)$ is not divergence free. Somehow, the fluid is hence compressible. But the constraint $\int \rho dx_\perp = 1$ can be interpreted as a constraint of "incompressibility in average" which allows one to recover the pressure law from the other unknowns. Indeed, we easily get, by integrating with respect to $x_\perp$ the equation satisfied by ρ:

$$\partial_{x_\parallel} \int \rho v_\parallel dx_\perp = 0. \tag{1.9}$$

  So by plugging this constraint in the equation satisfied by $\rho v_\parallel$, that is:

$$\partial_t(\rho v_\parallel) + \nabla_\perp(E^\perp \rho_\parallel v_\parallel) + \partial_\parallel(\rho v_\parallel^2) = -\partial_\parallel p(t, x_\parallel)\rho,$$

  we get the (one-dimensional) elliptic equation allowing to recover $-\partial_{x_\parallel} p$:

$$-\partial_\parallel^2 p(t, x_\parallel) = \partial_\parallel^2 \int \rho v_\parallel^2 dx_\perp,$$

  from which we get:

$$-\partial_\parallel p(t, x_\parallel) = \partial_\parallel \int \rho v_\parallel^2 dx_\perp. \tag{1.10}$$

- From the point of view of plasma physics, $E^\perp.\nabla_\perp$ corresponds to the so-called electric drift. By analogy with the so-called drift-kinetic equations [158], we can call this system a drift-fluid equation. To the best of our knowledge, this is the very first time such a model is exhibited in the literature.

From now on, when there is no risk of confusion, we will sometimes write v and v_ε instead of $v_\parallel$ and $v_{\parallel,\varepsilon}$.

1.2 Organization of the chapter

The outline of this chapter is as follows. In Section 2, we will state the main results of this chapter that are: the existence of analytic solutions to (1.7) locally in time but uniformly in ε (Theorem 2.1), the strong convergence to (1.8) with a complete description of the plasma oscillations (Theorem 2.2) and the existence and uniqueness of local analytic solutions to (1.8), in Proposition 2.1.

Section 3 is devoted to the proof of Theorem 2.1. First we recall some elementary features of the analytic spaces we consider (section 3.1), then we implement an approximation scheme for our Cauchy-Kovalesvkaya type existence theorem. The results are based on a decomposition of the electric field allowing for a good understanding of the so-called plasma waves (section 3.2).

In section 4, we prove Theorem 2.2, by using the uniform in ε estimates we have obtained in the previous theorem. The proof relies on another decomposition of the electric field, in order to exhibit the effects of the plasma waves as ε goes to 0.

Then, in section 5, we discuss the sharpness of our results:

- In sections 5.1 and 5.2, we discuss the analyticity assumption and explain why we can not lower down the regularity to Sobolev. In section 5.3, we explain why it is not possible to obtain global in time results. We obtain these results by considering some well-chosen initial data and using results of Brenier on multi-fluid Euler systems [25].

- Because of the two-stream instabilities, studying the limit with the relative entropy method is bound to fail. Nevertheless we found it interesting to try to apply the method and see at which point things get nasty: this is the object of section 5.4, where we study a kinetic toy model which retains the main unstable feature of system (1.7).

The two last sections are respectively a short conclusion and an appendix where we explain the scaling and the formal derivation of system (1.7).

2 Statement of the results

In order to prove both the existence of strong solutions to systems (1.7) and (1.8) and also prove the results of convergence, we follow the construction of Grenier [81], with some modifications adapted to our problem.

In [81], Grenier studies the quasineutral limit of the family of coupled Euler-Poisson systems:

$$\begin{cases} \partial_t \rho_\Theta^\varepsilon + \operatorname{div}(\rho_\Theta^\varepsilon v_\Theta^\varepsilon) = 0 \\ \partial_t v_\Theta^\varepsilon + v_\Theta^\varepsilon . \nabla (v_\Theta^\varepsilon) = E^\varepsilon \\ \operatorname{rot} E^\varepsilon = 0 \\ \varepsilon \operatorname{div} E^\varepsilon = \int_M \rho_\Theta^\varepsilon \mu(d\Theta) - 1, \end{cases} \tag{2.1}$$

with (M, Θ, μ) a probability space.

Following the proof of the Cauchy-Kovalevskaya theorem given by Caflisch [34], Grenier proved the local existence of analytic functions (with respect to x) uniformly with respect to ε and then, after filtering the fast oscillations due to the force field, showed the strong convergence to the system:

$$\begin{cases} \partial_t \rho_\Theta + \operatorname{div}(\rho_\Theta v_\Theta) = 0 \\ \partial_t v_\Theta + v^\varepsilon_\Theta . \nabla(v_\Theta) = E \\ \operatorname{rot} E = 0 \\ \int \rho_\Theta \mu(d\Theta) = 1. \end{cases} \tag{2.2}$$

We notice that the class of systems studied by Grenier is close to system (1.7), if we take $x = x_\parallel$, $\Theta = x_\perp$ and $(M, \mu) = (\mathbb{T}^2, dx_\perp)$, the main difference being that we have to deal with a dynamics in $\Theta = x_\perp$.

Hence, we introduce the same spaces of analytic functions as in [81], but this time depending also on $\Theta = x_\perp$.

Definition. *Let $\delta > 1$. We define B_δ the space of real functions ϕ on $\mathbb{T}^3$ such that*

$$|\phi|_\delta = \sum_{k \in \mathbb{Z}^3} |\mathcal{F}\phi(k)| \delta^{|k|} < +\infty, \tag{2.3}$$

where $\mathcal{F}\phi(k)$ is the k-th Fourier coefficient of ϕ defined by:

$$\mathcal{F}\phi(k) = \int_{\mathbb{T}^3} \phi(x) e^{-i2\pi k.x} dx.$$

The first theorem proves the existence of local analytic solutions of (1.7) with a life span uniform in ε.

Theorem 2.1. *Let $\delta_0 > 1$. Let $\rho_\varepsilon(0)$ and $v_\varepsilon(0)$ be two bounded families of B_{δ_0} such that $\int \rho_\varepsilon(0) dx = 1$ and:*

$$\left| \int \rho_\varepsilon(0) dx_\perp - 1 \right|_{\delta_0} \leq C\sqrt{\varepsilon}, \tag{2.4}$$

where $C > 0$ is some given universal constant. Then there exists $\eta > 0$ such that for every $\delta_1 \in]1, \delta_0[$, for any $\varepsilon > 0$, there exists a unique strong solution $(\rho_\varepsilon, v_\varepsilon)$ to (1.7) bounded uniformly in $\mathcal{C}([0, \eta(\delta_0 - \delta_1)[, B_{\delta_1})$ with initial conditions $(\rho_\varepsilon(0), v_\varepsilon(0))$. Moreover, $\sqrt{\varepsilon}\partial_\parallel V_\varepsilon$ is uniformly bounded in $\mathcal{C}([0, \eta(\delta_0 - \delta_1)[, B_{\delta_1})$.

Remark 2.1.
- The condition (2.4) implies that $\sqrt{\varepsilon}\partial_\parallel V_\varepsilon(0)$ is bounded uniformly in B_{δ_0} (this is the correct scale in view of the energy conservation).
- Note that for all $t \geq 0$, $\int \rho_\varepsilon dx = 1$. Hence the Poisson equation $-\varepsilon \partial_\parallel^2 V_\varepsilon = \int \rho_\varepsilon dx_\perp - 1$ can always be solved.
- As explained in the introduction, due to the two-streams instabilities, we have to restrict to data with analytic regularity: the Sobolev version of these results is false in general (see [38] and the discussion of Section 5).

We can then prove the convergence result:

Theorem 2.2. *Let $(\rho_\varepsilon, v_\varepsilon)$ be solutions to the system (1.7) for $0 \leq t \leq T$ satisfying for some $s > 7/2$ the following uniform estimates:*

$$(H) : \sup_{t \leq T, \varepsilon} \left(\|\rho_\varepsilon\|_{H^s_{x_\perp, x_\parallel}} + \|v_\varepsilon\|_{H^s_{x_\perp, x_\parallel}} + \|\sqrt{\varepsilon}\partial_{x_\parallel} V_\varepsilon\|_{H^s_{x_\parallel}} \right) < +\infty. \tag{2.5}$$

Then, up to a subsequence, we get the following convergences

$$\rho_\varepsilon \to \rho,$$

$$v_\varepsilon - \frac{1}{i}(E_+ e^{it/\sqrt{\varepsilon}} - E_- e^{-it/\sqrt{\varepsilon}}) \to v,$$

strongly respectively in $\mathcal{C}([0,T], H^{s'}_{x_\perp,x_\parallel})$ *and* $\mathcal{C}([0,T], H^{s'-1}_{x_\perp,x_\parallel})$ *for all* $s' < s$*, and*

$$-\sqrt{\varepsilon}\partial_{x_\parallel} V_\varepsilon - (E_+ e^{it/\sqrt{\varepsilon}} + E_- e^{-it/\sqrt{\varepsilon}}) \to 0,$$

strongly in $\mathcal{C}([0,T], H^{s'}_{x_\parallel})$ *for all* $s' < s-1$*, and where* (ρ, v) *is solution to the asymptotic system (1.8) on* $[0,T]$ *with initial conditions:*

$$\rho(0) = \lim_{\varepsilon\to 0} \rho_\varepsilon(0),$$

$$v(0) = \lim_{\varepsilon\to 0} \left(v_\varepsilon(0) - \int \rho_\varepsilon v_\varepsilon dx_\perp(0) \right)$$

and $E_+(t, x_\parallel), E_-(t, x_\parallel)$ *are gradient correctors which satisfy the transport equations:*

$$\partial_t E_\pm + \left(\int \rho v dx_\perp \right) \partial_{x_\parallel} E_\pm = 0,$$

with initial data:

$$E_+(0) = \lim_{\varepsilon\to 0} \frac{1}{2} \left(-\sqrt{\varepsilon}\partial_{x_\parallel} V_\varepsilon(0) + i \int \rho_\varepsilon v_\varepsilon dx_\perp(0) \right),$$

$$E_-(0) = \lim_{\varepsilon\to 0} \frac{1}{2} \left(-\sqrt{\varepsilon}\partial_{x_\parallel} V_\varepsilon(0) - i \int \rho_\varepsilon v_\varepsilon dx_\perp(0) \right).$$

Remark 2.2.

- It is clear that solutions built in Theorem 2.1 satisfy (H).
- If instead of (H) we make the stronger assumption, for $\delta > 1$

$$(H') : \sup_{t\le T,\varepsilon} \left(\|\rho_\varepsilon\|_{B_\delta} + \|v_\varepsilon\|_{B_\delta} + \|\sqrt{\varepsilon}\partial_{x_\parallel} V_\varepsilon\|_{B_\delta} \right) < +\infty, \tag{2.6}$$

  (which is still satisfied by the solutions built in Theorem 2.1), then we get the same strong convergences in $\mathcal{C}([0,T], B_{\delta'})$ for all $\delta' < \delta$.

  Using Lemma 3.1 $(ii), (iv)$, the proof under assumption (H') is the same as under assumption (H).

- The "well-prepared" case corresponds to the case when:

$$\lim_{\varepsilon\to 0} -\sqrt{\varepsilon}\partial_{x_\parallel} V_\varepsilon(0) = 0,$$

$$\lim_{\varepsilon\to 0} \int \rho_\varepsilon v_\varepsilon dx_\perp(0) = 0.$$

  Then there is no corrector.

With the same method used for Theorem 2.1, we can also prove a theorem of existence and uniqueness of analytic solutions to system (1.8).

Proposition 2.1. *Let $\delta_0 > 1$. For initial data $\rho(0), v(0) \in B_{\delta_0}$ satisfying*

$$\rho(0) \geq 0, \tag{2.7}$$

$$\int \rho(0) dx_\perp = 1 \tag{2.8}$$

and

$$\partial_\| \int \rho(0) v(0) dx_\perp = 0, \tag{2.9}$$

there exists $\eta > 0$ depending on δ_0 and on the initial conditions only such that there is a unique strong solution $(\rho, v_\|, p)$ to the system (1.8) with $\rho, v \in \mathcal{C}([0, \eta(\delta_0 - \delta_1)[, B_{\delta_1})$ for all $1 < \delta_1 < \delta_0$.

Remark 2.3. The uniqueness proved in Proposition 2.1 allows to say that the convergences of Theorem 2.2 hold without having to consider subsequences, provided that the whole sequences of initial data converge to some functions in B_{δ_0} satisfying the assumptions of Proposition 2.1.

3 Proof of Theorem 2.1

3.1 Functional analysis on B_δ spaces

First we define the time dependent analytic spaces we will work with.

Let β be an arbitrary constant in $]0, 1[$ (take for instance $\beta = 1/2$ to fix ideas) and $\eta > 0$ a parameter to be chosen later.

Definition. *Let $\delta_0 > 1$. We define the space $B^\eta_{\delta_0} = \{u \in \mathcal{C}^0([0, \eta(\delta_0 - 1)], B_{\delta_0 - t/\eta})\}$, endowed with the norm*

$$\|u\|_{\delta_0} = \sup_{\begin{cases} 1 < \delta \leq \delta_0 \\ 0 \leq t \leq \eta(\delta_0 - \delta) \end{cases}} \left(|u(t)|_\delta + \left(\delta_0 - \delta - \frac{t}{\eta}\right)^\beta |\nabla u(t)|_\delta \right),$$

where the norm $|u|_\delta$ was defined in (2.3):

$$|u|_\delta = \sum_{k \in \mathbb{Z}^3} |\mathcal{F}u(k)| \delta^{|k|},$$

We now gather from [81] a few elementary properties of these spaces, that we recall for the reader's convenience.

Lemma 3.1. *For all $\delta > 1$:*

(i) The spaces B_δ and B^η_δ are Banach algebra. More precisely, if $\phi_1, \phi_2 \in B_\delta$, and $\psi_1, \psi_2 \in B^\eta_\delta$ then:

$$|\phi_1 \phi_2|_\delta \leq |\phi_1|_\delta |\phi_2|_\delta,$$

$$\|\psi_1 \psi_2\|_\delta \leq \|\psi_1\|_\delta \|\psi_2\|_\delta.$$

(ii) If $\delta' < \delta$ then $B_\delta \subset B_{\delta'}$, the embedding being continuous and compact.

(iii) For all $s \in \mathbb{R}$, $B_\delta \subset H^s$, the embedding being continuous and compact.

(iv) For all $1 < \delta' < \delta$*, if* $\phi \in B_\delta$*,*

$$|\nabla \phi|_{\delta'} \leq \frac{\delta}{\delta - \delta'} |\phi|_\delta.$$

(v) If u *is in* $B^\eta_{\delta_0}$ *and if* $\delta' + t/\eta < \delta_0$ *then*

$$|\partial^2_{x_i,x_j} u(t)|_{\delta'} \leq 2\|u\|_{\delta_0} \delta_0 \left(\delta_0 - \delta' - \frac{t}{\eta}\right)^{-\beta-1}.$$

For further properties of these spaces we refer to the recent work of Mouhot and Villani [130], in which similar analytic spaces (and more sophisticated versions) are considered.

Proof of Lemma 3.1. For the reader's convenience, we briefly sketch the proof (more details can be found in [81]). Point (i) can be readily checked from the Fourier series caracterization. We give an elementary proof for (ii) which is not given in [81] . The embedding is obvious. We consider for $N \in \mathbb{N}$ the map i_N defined by:

$$i_N(\phi) = \sum_{|k| \leq N} \mathcal{F}\phi(k) e^{ixk}.$$

We then compute:

$$|(Id - i_N)\phi|_{\delta'} = \sum_{|k|>N} |\mathcal{F}\phi(k)| \delta'^{|k|} \leq \left(\frac{\delta'}{\delta}\right)^N \sum_{|k|>N} |\mathcal{F}\phi(k)| \delta^{|k|} \leq \left(\frac{\delta'}{\delta}\right)^N |\phi|_\delta.$$

So the embedding $B_\delta \subset B_{\delta'}$ is compact as the limit of finite rank operators. Point (iii) can be proved similarly. Point (iv) relies on the elementary estimate:

$$|k| \delta'^{|k|} \leq \frac{\delta}{\delta - \delta'} \delta^{|k|}.$$

For (v), consider $\delta = \delta' + \frac{\delta_0 - \delta' - t/\eta}{2}$ and apply (iv). ☐

We will also need the following elementary observation:

Remark 3.1. Let $\phi \in B_\delta$. Then:

$$\left| \int \phi dx_\perp \right|_\delta \leq |\phi|_\delta.$$

Proof. We simply compute:

$$\left| \int \phi dx_\perp \right|_\delta = \sum_{k_\perp = 0, k_\parallel \in \mathbb{N}} |\mathcal{F}(\phi)(k_\perp, k_\parallel)| \delta^{|k|} \leq \sum_{k \in \mathbb{N}^3} |\mathcal{F}(\phi)| \delta^{|k|} = |\phi|_\delta.$$

☐

3.2 Description of plasma oscillations

To simplify notations, we set $E_{\varepsilon,\parallel} = -\partial_{x_\parallel} V_\varepsilon(t, x_\parallel)$ (which has nothing to do with $E^\perp_\varepsilon$). In this paragraph, we want to understand the oscillatory behaviour of $E_{\varepsilon,\parallel}$. We will see that the dynamics in $x_\perp$ does not interfer too much with the equations on $E_{\varepsilon,\parallel}$, so that we get almost the same description of oscillations as in Grenier's paper [81].

First we differentiate twice with respect to time the Poisson equation satisfied by V_ε:

$$\varepsilon \partial_t^2 \partial_{x_\|} E_{\varepsilon,\|} = \partial_t^2 \int \rho_\varepsilon dx_\perp. \tag{3.1}$$

Integrating with respect to $x_\perp$ the equation satisfied by ρ_ε, we obtain:

$$\partial_t \int \rho_\varepsilon dx_\perp = \underbrace{-\int \nabla_\perp.(E_\varepsilon^\perp \rho_\varepsilon) dx_\perp}_{=0} - \partial_{x_\|} \int \rho_\varepsilon v_\varepsilon dx_\perp. \tag{3.2}$$

Then we integrate with respect to $x_\perp$ the equation satisfied by $\rho_\varepsilon v_\varepsilon$, that is:

$$\partial_t(\rho_\varepsilon v_\varepsilon) + \nabla_\perp.(E_\varepsilon^\perp \rho_\varepsilon v_\varepsilon) + \partial_{x_\|}(v_\varepsilon^2 \rho_\varepsilon) = -\rho_\varepsilon(\varepsilon \partial_{x_\|} \phi_\varepsilon(t,x) + \partial_{x_\|} V_\varepsilon(t, x_\|))$$

and we get:

$$-\partial_t \int \rho_\varepsilon v_\varepsilon dx_\perp = \partial_{x_\|} \int \rho_\varepsilon v_\varepsilon^2 dx_\perp - E_{\varepsilon,\|} \int \rho_\varepsilon dx_\perp + \int \rho_\varepsilon(\varepsilon \partial_{x_\|} \phi_\varepsilon) dx_\perp, \tag{3.3}$$

so that, combining (3.2) and (3.3):

$$\partial_t^2 \int \rho_\varepsilon dx_\perp = \partial_{x_\|}^2 \int \rho_\varepsilon v_\varepsilon^2 dx_\perp - \partial_{x_\|}(E_{\varepsilon,\|} \int \rho_\varepsilon dx_\perp) + \partial_{x_\|} \int \rho_\varepsilon(\varepsilon \partial_{x_\|} \phi_\varepsilon) dx_\perp. \tag{3.4}$$

Recall that by the Poisson equation:

$$\int \rho_\varepsilon dx_\perp = 1 + \partial_{x_\|} E_{\varepsilon,\|}.$$

Thus it comes by (3.1) and (3.4):

$$\varepsilon \partial_t^2 \partial_{x_\|} E_{\varepsilon,\|} + \partial_{x_\|} E_{\varepsilon,\|} = \partial_{x_\|}^2 \int \rho_\varepsilon v_\varepsilon^2 dx_\perp - \varepsilon \partial_{x_\|}[E_{\varepsilon,\|} \partial_{x_\|} E_{\varepsilon,\|}] + \partial_{x_\|} \int \rho_\varepsilon(\varepsilon \partial_{x_\|} \phi_\varepsilon) dx_\perp. \tag{3.5}$$

Equation (3.5) is the wave equation allowing to describe the essential oscillations. At least formally, this equation indicates that there are time oscillations with frequency $\frac{1}{\sqrt{\varepsilon}}$ and magnitude $\frac{1}{\sqrt{\varepsilon}}$ created by the right-hand side of the equation which acts like a source. We observe here that the source is expected to be of order $\mathcal{O}(1)$: indeed, by assumption on the data at $t = 0$, we can check that this quantity is bounded in a B_δ space.

In particular if we want to prove strong convergence results we will have to introduce non-trivial correctors in order to get rid of these oscillations. We notice also that (3.5) is very similar to the wave equation obtained in [81] (the only difference is a new term in the source), so that most of the calculations and estimates on $E_{\varepsilon,\|}$ we will need are done in [81].

We have just observed that $E_{\varepsilon,\|}$ roughly behaves like $\frac{1}{\sqrt{\varepsilon}} e^{\pm it/\sqrt{\varepsilon}}$. Hence if we consider the average in time:

$$G_\varepsilon = \int_0^t E_{\varepsilon,\|}(s, x_\|) ds, \tag{3.6}$$

we expect that G_ε is bounded uniformly with respect to ε in some functional space. We have the representation lemma which will be very useful to obtain a priori estimates:

Lemma 3.2. *The following identity holds:*

$$\mathcal{F}_{\|}G_\varepsilon(t,k_\|) = \int_0^t \left(\frac{1}{ik_\|}\left[1-\cos\left(\frac{t-s}{\sqrt{\varepsilon}}\right)\right]\mathcal{F}_{\|}g_\varepsilon(s,k_\|)\right)ds + \mathcal{F}_{\|}G_\varepsilon^0, \tag{3.7}$$

denoting by $\mathcal{F}_\|$ the Fourier transform with respect to the parallel variable only and $k_\|$ the Fourier variable and where:

$$g_\varepsilon = \partial_{x_\|}^2 \int \rho_\varepsilon v_\varepsilon^2 dx_\perp - \varepsilon\partial_{x_\|}[E_{\varepsilon,\|}\partial_{x_\|}E_{\varepsilon,\|}] + \partial_{x_\|}\int \rho_\varepsilon(\varepsilon\partial_{x_\|}\phi_\varepsilon)dx_\perp, \tag{3.8}$$

$$G_\varepsilon^0 = \sqrt{\varepsilon}E_{\varepsilon,\|}(0,x_\|)\sin\left(\frac{s}{\sqrt{\varepsilon}}\right) - \varepsilon\partial_t E_{\varepsilon,\|}(0,x_\|)\left(\cos\left(\frac{s}{\sqrt{\varepsilon}}\right)-1\right). \tag{3.9}$$

Proof of Lemma 3.2. We use Duhamel's formula for the wave equation (3.5) to get the following identity:

$$\mathcal{F}_\|E_\varepsilon(t,k_\|) = \frac{1}{\sqrt{\varepsilon}}\int_0^t \left(\frac{1}{ik_\|}\sin\left(\frac{t-s}{\sqrt{\varepsilon}}\right)\mathcal{F}_\|g_\varepsilon(s,k_\|)\right)ds + \mathcal{F}_\|E_\varepsilon^0, \tag{3.10}$$

with g_ε defined in (3.8) and

$$E_{\varepsilon,\|}^0 = E_{\varepsilon,\|}(0,x)\cos(\frac{s}{\sqrt{\varepsilon}}) + \sqrt{\varepsilon}\partial_t E_{\varepsilon,\|}(0,x)\sin(\frac{s}{\sqrt{\varepsilon}}). \tag{3.11}$$

Then we can integrate this formula to recover (3.7). □

We now introduce the translated current (which corresponds to some filtering of the time oscillations created by the electric field):

$$w_\varepsilon = v_\varepsilon - G_\varepsilon, \tag{3.12}$$

so that the transport equations of system (1.7) now read:

$$\begin{cases} \partial_t\rho_\varepsilon + \nabla_\perp.(E_\varepsilon^\perp\rho_\varepsilon) + \partial_\|((w_\varepsilon+G_\varepsilon)\rho_\varepsilon) = 0 \\ \partial_t w_\varepsilon + \nabla_\perp.(E_\varepsilon^\perp(w_\varepsilon+G_\varepsilon)) + (w_\varepsilon+G_\varepsilon)\partial_\|(v_\varepsilon+G_\varepsilon)) = -\varepsilon\partial_\|\phi_\varepsilon(t,x_\|). \end{cases} \tag{3.13}$$

3.3 Approximation scheme

To construct a solution, we use the usual approximation scheme for Cauchy-Kovalevskaya type of results ([34]). The principle is to define $\rho_\varepsilon^n, w_\varepsilon^n, G_\varepsilon^n, V_\varepsilon^n, \phi_\varepsilon^n$ by recursion:

Initialization First of all, for $0 < t < \eta(\delta_0 - 1)$, $G_\varepsilon^0(t)$ is given by formula (3.9); then for $0 < t < \eta(\delta_0-1)$, we can define:

$$\rho_\varepsilon^0(t) = \rho_\varepsilon(0),$$

$$w_\varepsilon^0(t) = v_\varepsilon(0) - G_\varepsilon^0(t),$$

$$-\varepsilon^2\partial_{x_\|}^2\phi_\varepsilon^0 - \Delta_{x_\perp}\phi_\varepsilon^0 = \rho_\varepsilon^0 - \int\rho_\varepsilon^0 dx_\perp,$$

$$E_\varepsilon^{\perp,0} = -\nabla^\perp\phi_\varepsilon^0,$$

and $-\partial_{x_\|}V_\varepsilon^0(t) = \partial_t G_\varepsilon^0(t)$.

Recursion For $0 < t < \eta(\delta_0 - 1)$, we define $\rho_\varepsilon^{n+1}, w_\varepsilon^{n+1}$ by the transport equations:

$$\begin{cases} \partial_t \rho_\varepsilon^{n+1} + \nabla_\perp.(E_\varepsilon^{\perp,n}\rho_\varepsilon^n) + \partial_\parallel((w_\varepsilon^n + G_\varepsilon^n)\rho_\varepsilon^n) = 0 \\ \partial_t w_\varepsilon^{n+1} + \nabla_\perp.(E_\varepsilon^{\perp,n}(w_\varepsilon^n + G_\varepsilon^n)) + (w_\varepsilon^n + G_\varepsilon^n)\partial_\parallel(v_\varepsilon^n + G_\varepsilon^n)) = -\varepsilon\partial_\parallel \phi_\varepsilon^n(t, x_\parallel), \end{cases} \tag{3.14}$$

with the initial conditions: $\rho_\varepsilon^{n+1}(0) = \rho_\varepsilon(0)$ and $w_\varepsilon^{n+1} = v_\varepsilon(0) - G_\varepsilon^0$.
Then we can define ϕ_ε^{n+1} as the solution to the Poisson equation:

$$-\varepsilon^2\partial_{x_\parallel}^2\phi_\varepsilon^{n+1} - \Delta_{x_\perp}\phi_\varepsilon^{n+1} = \rho_\varepsilon^{n+1} - \int \rho_\varepsilon^{n+1} dx_\perp.$$

$$E_\varepsilon^{\perp,n+1} = -\nabla^\perp\phi_\varepsilon^{n+1},$$

Furthermore, we can define $G_\varepsilon^{n+1}(t)$ by a variant of formula (3.7):

$$\mathcal{F}_\parallel G_\varepsilon^{n+1}(t,k_\parallel) = \int_0^t \left(\frac{1}{ik_\parallel}\left[1 - \cos(\frac{t-s}{\sqrt{\varepsilon}})\right] \mathcal{F}_\parallel g_\varepsilon^n(s,k_\parallel) \right) ds + \mathcal{F}_\parallel G_\varepsilon^0, \tag{3.15}$$

with $g_\varepsilon^n = \partial_{x_\parallel}^2 \int \rho_\varepsilon^n(v_\varepsilon^n + G_\varepsilon^n)^2 dx_\perp - \varepsilon\partial_{x_\parallel}[E_{\varepsilon,\parallel}^n \partial_{x_\parallel} E_{\varepsilon,\parallel}^n] + \partial_{x_\parallel}\int \rho_\varepsilon^n(\varepsilon\partial_{x_\parallel}\phi_\varepsilon^n)dx_\perp$.
Finally we define:

$$-\varepsilon\partial_{x_\parallel} V_\varepsilon^{n+1} = \partial_t G^{n+1}(t).$$

3.4 A priori estimates

Let $n \geq 0$. The goal is now to prove some a priori estimates for $G_\varepsilon^{n+1}, \rho_\varepsilon^{n+1}$ and w_ε^{n+1} (in terms of $G_\varepsilon^n, \rho_\varepsilon^n$ and w_ε^n). We are also able to get similar estimates on $E_\varepsilon^{\perp,n+1}$ and $\varepsilon\partial_{x_\parallel}\phi_\varepsilon^{n+1}$, thanks to the Poisson equation satisfied by ϕ_ε^{n+1}. Ultimately the goal is to prove that if the parameter η is chosen small enough, then all these sequences are Cauchy sequences in $B_{\delta_0}^\eta$.

3.4.1 Estimate on G_ε^{n+1} and $\sqrt{\varepsilon}E_{\varepsilon,\parallel}^{n+1}$

The first aim in this paragraph is to estimate $\|G_\varepsilon^{n+1}\|_{\delta_0}$, using (3.15). We have:

$$\begin{aligned} |G_\varepsilon^{n+1}|_\delta &\leq \left| \int_0^t \mathcal{F}_\parallel^{-1}\left(\frac{1}{ik_\parallel}[1 - \cos(\frac{t-s}{\sqrt{\varepsilon}})]\mathcal{F}_\parallel g_\varepsilon^n(s,k_\parallel) \right) ds \right|_\delta + |G_\varepsilon^0|_\delta \\ &\leq 2\int_0^t \left| \mathcal{F}_\parallel^{-1}\left(\frac{1}{ik_\parallel}\mathcal{F}_\parallel g_\varepsilon^n(s,k_\parallel)\right)\right|_\delta ds + |G_\varepsilon^0|_\delta, \end{aligned}$$

with:

$$\frac{1}{ik_\parallel}\mathcal{F}_\parallel g_\varepsilon^n = \mathcal{F}_\parallel\left(\partial_{x_\parallel}\int \rho_\varepsilon(w_\varepsilon^n + G_\varepsilon^n)^2 dx_\perp\right) - \varepsilon\mathcal{F}_\parallel\left(E_{\varepsilon,\parallel}^n\partial_{x_\parallel}E_{\varepsilon,\parallel}^n\right) + \mathcal{F}_\parallel\left(\int \rho_\varepsilon^n(\varepsilon\partial_{x_\parallel}\phi_\varepsilon^n)dx_\perp\right).$$

Thanks to Remark 3.1 and Lemma 3.1 , (i), we first estimate:

$$\begin{aligned} \left|\int \partial_{x_\parallel}(\rho_\varepsilon^n(w_\varepsilon^n + G_\varepsilon^n)^2)dx_\perp\right|_\delta \leq \left|\partial_{x_\parallel}(\rho_\varepsilon^n(w_\varepsilon^n + G_\varepsilon^n)^2)\right|_\delta &\leq (\delta_0 - \delta - \frac{s}{\eta})^{-\beta}\|\rho_\varepsilon^n(w_\varepsilon^n + G_\varepsilon^n)^2\|_{\delta_0} \\ &\leq (\delta_0 - \delta - \frac{s}{\eta})^{-\beta}\|\rho_\varepsilon^n\|_{\delta_0}\|w_\varepsilon^n + G_\varepsilon^n\|_{\delta_0}^2. \end{aligned} \tag{3.16}$$

Similarly, we prove:

$$
\begin{aligned}
\varepsilon \left| E^n_{\varepsilon,\|} \partial_{x_\|} E^n_{\varepsilon,\|} \right|_\delta \leq & \frac{1}{2} \left| \partial_{x_\|} (\sqrt{\varepsilon} E^n_{\varepsilon,\|})^2 \right|_\delta \\
\leq & \frac{1}{2} (\delta_0 - \delta - \frac{s}{\eta})^{-\beta} \left\| (\sqrt{\varepsilon} E^n_{\varepsilon,\|})^2 \right\|_{\delta_0} \\
\leq & \frac{1}{2} (\delta_0 - \delta - \frac{s}{\eta})^{-\beta} \| \sqrt{\varepsilon} E^n_{\varepsilon,\|} \|^2_{\delta_0},
\end{aligned}
\tag{3.17}
$$

$$
\left| \int \partial_{x_\|} \left(\rho^n_\varepsilon (\varepsilon \partial_{x_\|} \phi^n_\varepsilon) \right) dx_\perp \right|_\delta \leq (\delta_0 - \delta - \frac{s}{\eta})^{-\beta} \| \rho^n_\varepsilon \|_{\delta_0} \| \varepsilon \partial_{x_\|} \phi^n_\varepsilon \|_{\delta_0}.
$$

Thus, we finally obtain:

$$
|G^{n+1}_\varepsilon|_\delta \leq 2 \int_0^t (\delta_0 - \delta - \frac{s}{\eta})^{(-\beta)} (\|\rho_\varepsilon\|_{\delta_0} \|w^n_\varepsilon + G^n_\varepsilon\|^2_{\delta_0} + \|\sqrt{\varepsilon} E^n_{\varepsilon,\|}\|^2_{\delta_0} + \|\rho^n_\varepsilon\|_{\delta_0} \|\varepsilon \partial_{x_\|} \phi^n_\varepsilon\|_{\delta_0}) ds + |G^0_\varepsilon|_\delta.
$$

In what follows, $C(\delta_0, \beta)$ is a constant depending only on δ_0 and β that may change from one line to another. As before, one can show (this time we use lemma 3.1, (v)) that:

$$
\begin{aligned}
|\partial_{x_\|} G^{n+1}_\varepsilon|_\delta \leq C(\delta_0, \beta) \textstyle\int_0^t (\delta_0 - \delta - \frac{s}{\eta})^{(-\beta-1)} \Big(& \|\rho^n_\varepsilon\|_{\delta_0} \|w^n_\varepsilon + G^n_\varepsilon\|^2_{\delta_0} + \|\sqrt{\varepsilon} E^n_{\varepsilon,\|}\|^2_{\delta_0} \\
& + \|\rho^n_\varepsilon\|_{\delta_0} \|\varepsilon \partial_{x_\|} \phi^n_\varepsilon\|_{\delta_0} \Big) ds + |\partial_{x_\|} G^0_\varepsilon|_\delta.
\end{aligned}
$$

Hence using the elementary estimates

$$
\int_0^t \frac{ds}{(\delta_0 - \delta - \frac{s}{\eta})^\beta} \leq \eta \frac{2}{1-\beta} \delta_0^{1-\beta},
$$

$$
\int_0^t \frac{ds}{(\delta_0 - \delta - \frac{s}{\eta})^{\beta+1}} \leq \frac{2\eta}{\beta} (\delta_0 - \delta - \frac{t}{\eta})^{-\beta},
$$

we get:

$$
\|G^{n+1}_\varepsilon\|_{\delta_0} \leq \eta C(\delta_0, \beta) \Big((\|w^n_\varepsilon\|_{\delta_0} + \|G^n_\varepsilon\|_{\delta_0})^2 \|\rho^n_\varepsilon\|_{\delta_0} + \|\sqrt{\varepsilon} E^n_{\varepsilon,\|}\|^2_{\delta_0} + \|\rho^n_\varepsilon\|_{\delta_0} \|\varepsilon \partial_{x_\|} \phi^n_\varepsilon\|_{\delta_0} \Big) + \|G^0_\varepsilon\|_{\delta_0}.
\tag{3.18}
$$

Finally, we compare two solutions $(w^{n+1}_\varepsilon, \rho^{n+1}_\varepsilon, G^{n+1}_\varepsilon)$ and $(w^{n+2}, \rho^{n+2}, G^{n+2}_\varepsilon)$ (observe that these have the same initial data).

$$
|G^{n+2}_\varepsilon - G^{n+1}_\varepsilon|_\delta \leq \int_0^t \left| \mathcal{F}^{-1}_\| \left(\frac{1}{ik_\|} [1 - \cos(\frac{t-s}{\sqrt{\varepsilon}})] \left[\mathcal{F}_\| g^{n+1}_\varepsilon(s, k_\|) - \mathcal{F}_\| g^n_\varepsilon(s, k_\|) \right] \right) \right|_\delta ds,
\tag{3.19}
$$

We decompose the products appearing in $g^{n+1}_\varepsilon - g^n_\varepsilon$ in the following way:

$$
\rho^{n+1}_\varepsilon (w^{n+1}_\varepsilon)^2 - \rho^n_\varepsilon (w^n_\varepsilon)^2 = (\rho^{n+1}_\varepsilon - \rho^n_\varepsilon)(w^{n+1}_\varepsilon)^2 + (w^{n+1}_\varepsilon - w^n_\varepsilon)(w^{n+1}_\varepsilon + w^n_\varepsilon)\rho^n_\varepsilon,
$$

and we proceed likewise for the other terms. Then we obtain the following estimate with the same method as before:

$$
\begin{aligned}
\|G^{n+1}_\varepsilon - G^{n+2}_\varepsilon\|_{\delta_0} \leq \; & \eta C(\delta_0, \beta) \Big((\|w^{n+1}_\varepsilon - w^n_\varepsilon\|_{\delta_0} + \|G^{n+1}_\varepsilon - G^n_\varepsilon\|_{\delta_0}) \\
& \times (\|w^{n+1}_\varepsilon\|_{\delta_0} + \|w^n_\varepsilon\|_{\delta_0} + \|G^{n+1}_\varepsilon\|_{\delta_0} + \|G^n_\varepsilon\|_{\delta_0})(\|\rho^{n+1}_\varepsilon\|_{\delta_0} + \|\rho^n_\varepsilon\|_{\delta_0}) \\
+ \; & \|\rho^{n+1}_\varepsilon - \rho^n_\varepsilon\|_{\delta_0} (\|w^{n+1}_\varepsilon\|^2_{\delta_0} + \|w^n_\varepsilon\|^2_{\delta_0} + \|G^{n+1}_\varepsilon\|^2_{\delta_0} + \|G^n_\varepsilon\|^2_{\delta_0}) \\
+ \; & \|\rho^{n+1}_\varepsilon - \rho^n_\varepsilon\|_{\delta_0} (\|\varepsilon \partial_{x_\|} \phi^{n+1}_\varepsilon\|_{\delta_0} + \|\varepsilon \partial_{x_\|} \phi^n_\varepsilon\|_{\delta_0}) \\
+ \; & \|\varepsilon \partial_{x_\|} \phi^{n+1}_\varepsilon - \varepsilon \partial_{x_\|} \phi^n_\varepsilon\|_{\delta_0} (\|\rho^{n+1}_\varepsilon\|_{\delta_0} + \|\rho^n_\varepsilon\|_{\delta_0}) \\
+ \; & \|\sqrt{\varepsilon} E^{n+1}_{\varepsilon,\|} - \sqrt{\varepsilon} E^n_{\varepsilon,\|}\|_{\delta_0} (\|\sqrt{\varepsilon} E^{n+1}_{\varepsilon,\|}\|_{\delta_0} + \|\sqrt{\varepsilon} E^n_{\varepsilon,\|}\|_{\delta_0}) \Big).
\end{aligned}
\tag{3.20}
$$

Likewise we get the same kind of estimates for $\|\sqrt{\varepsilon}E^{n+1}_{\varepsilon,\|}\|_{\delta_0}$ since from (3.10) we have the formula:

$$\mathcal{F}_{\|}(\sqrt{\varepsilon}E^{n+1}_{\varepsilon,\|})(t,k_{\|}) = \int_0^t \left(\frac{1}{ik_{\|}}[\sin(\frac{t-s}{\sqrt{\varepsilon}})]\mathcal{F}_{\|}g^n_\varepsilon(s,k_{\|})\right)ds + \mathcal{F}_{\|}(\sqrt{\varepsilon}E^0_{\varepsilon,\|}), \tag{3.21}$$

3.4.2 Estimate on $E^{\perp,n+1}_\varepsilon$ and $\varepsilon\partial_{x_\|}\phi^{n+1}_\varepsilon$

We now use the scaled Poisson equation satisfied by ϕ^{n+1}_ε to get some similar a priori estimates. For the reader's convenience, we first recall this equation:

$$-\varepsilon^2\partial^2_{x_\|}\phi^{n+1}_\varepsilon - \Delta_\perp\phi^{n+1}_\varepsilon = \rho^{n+1}_\varepsilon - \int \rho^{n+1}_\varepsilon dx_\perp.$$

The principle here is to look at the symbols of the operators involved in the Poisson equation. Accordingly, we compute in Fourier variables:

$$\varepsilon^2 k_\|^2\mathcal{F}\phi^{n+1}_\varepsilon + |k_\perp|^2\mathcal{F}\phi^{n+1}_\varepsilon = \mathcal{F}\left(\rho^{n+1}_\varepsilon - \int \rho^{n+1}_\varepsilon dx_\perp\right). \tag{3.22}$$

Thus it comes:

$$\mathcal{F}\phi^{n+1}_\varepsilon = \frac{\mathcal{F}(\rho^{n+1}_\varepsilon - \int \rho^{n+1}_\varepsilon dx_\perp)}{\varepsilon^2 k_\|^2 + |k_\perp|^2}.$$

Since $\int(\rho^{n+1}_\varepsilon - \int \rho^{n+1}_\varepsilon dx_\perp)dx_\perp = 0$, we have for all $k_\| \in \mathbb{Z}$:

$$\mathcal{F}\left(\rho^{n+1}_\varepsilon - \int \rho^{n+1}_\varepsilon dx_\perp\right)(0,k_\|) = 0.$$

Thus we get, for all $k_\perp, k_\| \in \mathbb{Z}$:

$$|\mathcal{F}\phi^{n+1}_\varepsilon| \leq \frac{|\mathcal{F}(\rho^{n+1}_\varepsilon - \int \rho^{n+1}_\varepsilon dx_\perp)|}{|k_\perp|^2}.$$

In particular we easily get, using the relation $E^{\perp,n+1}_\varepsilon = -\nabla^\perp\phi^{n+1}_\varepsilon$:

$$|\mathcal{F}E^{\perp,n+1}_\varepsilon| \leq \frac{|\mathcal{F}(\rho^{n+1}_\varepsilon - \int \rho^{n+1}_\varepsilon dx_\perp)|}{|k_\perp|} \leq \left|\mathcal{F}\left(\rho^{n+1}_\varepsilon - \int \rho^{n+1}_\varepsilon dx_\perp\right)\right|.$$

Hence:

$$\|E^{\perp,n+1}_\varepsilon\|_{\delta_0} \leq 2\|\rho^{n+1}_\varepsilon\|_{\delta_0}. \tag{3.23}$$

Likewise, using the elementary inequality $ab \leq \frac{1}{2}(a^2+b^2)$ and $|k_\perp| \geq 1$:

$$|\mathcal{F}(\varepsilon\partial_{x_\|}\phi^{n+1}_\varepsilon)| \leq \frac{\varepsilon|k_\||\,|\mathcal{F}(\rho_\varepsilon - \int \rho_\varepsilon dx_\perp)|}{\varepsilon^2 k_\|^2 + |k_\perp|^2} \leq \frac{1}{2}|\mathcal{F}(\rho^{n+1}_\varepsilon - \int \rho^{n+1}_\varepsilon dx_\perp)|,$$

and consequently:

$$\|\varepsilon\partial_{x_\|}\phi^{n+1}_\varepsilon\|_{\delta_0} \leq \|\rho^{n+1}_\varepsilon\|_{\delta_0}. \tag{3.24}$$

Finally, if we compare two solutions at step $n+1$ and $n+2$:

$$\|E^{\perp,n+2}_\varepsilon - E^{\perp,n+1}_\varepsilon\|_{\delta_0} + \|\varepsilon\partial_{x_\|}\phi^{n+2}_\varepsilon - \varepsilon\partial_{x_\|}\phi^{n+1}_\varepsilon\|_{\delta_0} \leq 2\|\rho^{n+2}_\varepsilon - \rho^{n+1}_\varepsilon\|_{\delta_0}. \tag{3.25}$$

3.4.3 Estimate on ρ_ε^{n+1} and w_ε^{n+1}

We now use the conservation laws satisfied by ρ_ε^{n+1} and w_ε^{n+1} to get the appropriate estimates. We first recall that the density ρ_ε^{n+1} satisfies the equation:

$$\partial_t \rho_\varepsilon^{n+1} + \nabla_\perp.(E_\varepsilon^{\perp,n}\rho_\varepsilon^n) + \partial_\|((w_\varepsilon^n + G_\varepsilon^n)\rho_\varepsilon^n) = 0.$$

Writing $\rho_\varepsilon^{n+1} = \int_0^t \partial_t \rho_\varepsilon^{n+1} ds + \rho_\varepsilon(0)$, we get:

$$|\rho_\varepsilon^{n+1}|_\delta \leq \int_0^t |\partial_t \rho_\varepsilon^{n+1}|_\delta ds + |\rho_\varepsilon(0)|_\delta$$

With the same kind of computations as before and using estimate (3.23) we get:

$$|\nabla_\perp.(E_\varepsilon^{\perp,n}\rho_\varepsilon^n)|_\delta \leq (\delta_0 - \delta - \frac{s}{\eta})^{-\beta}\|E_\varepsilon^{\perp,n}\|_{\delta_0}\|\rho_\varepsilon^n\|_{\delta_0} \leq 2(\delta_0 - \delta - \frac{s}{\eta})^{-\beta}\|\rho_\varepsilon^n\|_{\delta_0}^2,$$

$$|\partial_\|((w_\varepsilon^n + G_\varepsilon^n)\rho_\varepsilon)|_\delta \leq (\delta_0 - \delta - \frac{s}{\eta})^{-\beta}\|w_\varepsilon^n + G_\varepsilon^n)\|_{\delta_0}\|\rho_\varepsilon^n\|_{\delta_0}.$$

As a consequence we obtain:

$$|\rho_\varepsilon^{n+1}|_\delta \leq |\rho_\varepsilon(0)|_\delta + C(\delta_0,\beta)\int_0^t (\delta_0 - \delta - \frac{s}{\eta})^{-\beta}\|\rho_\varepsilon^n\|_{\delta_0}(\|\rho_\varepsilon^n\|_{\delta_0} + \|w_\varepsilon^n\|_{\delta_0} + \|G_\varepsilon^n\|_{\delta_0})ds.$$

Similarly we estimate $|\partial_{x_i}\rho_\varepsilon^{n+1}|_\delta$ by differentiating with respect to x_i the equation satisfied by ρ_ε^{n+1}. Finally we get:

$$\|\rho_\varepsilon^{n+1}\|_{\delta_0} \leq \eta C(\delta_0,\beta)\|\rho_\varepsilon^n\|_{\delta_0}(\|\rho_\varepsilon^n\|_{\delta_0} + \|w_\varepsilon^n\|_{\delta_0} + \|G_\varepsilon^n\|_{\delta_0}) + \|\nabla\rho_\varepsilon(0)\|_{\delta_0}. \tag{3.26}$$

If we compare solutions at steps $n+1$ and $n+2$, we get likewise:

$$\begin{aligned}\|\rho_\varepsilon^{n+2} - \rho_\varepsilon^{n+1}\|_{\delta_0} &\leq \eta C(\delta_0,\beta)\Big((\|\rho_\varepsilon^{n+1}\|_{\delta_0} + \|\rho_\varepsilon^n\|_{\delta_0})(\|w_\varepsilon^{n+1} - w_\varepsilon^n\|_{\delta_0} + \|G_\varepsilon^{n+1} - G_\varepsilon^n\|_{\delta_0})\\ &+ (\|\rho_\varepsilon^{n+1}\|_{\delta_0} + \|\rho_\varepsilon^n\|_{\delta_0} + \|w_\varepsilon^{n+1}\|_{\delta_0} + \|w_\varepsilon^n\|_{\delta_0} + \|G_\varepsilon^{n+1}\|_{\delta_0} + \|G_\varepsilon^n\|_{\delta_0})\\ &\times (\|\rho_\varepsilon^{n+1} - \rho_\varepsilon^n\|_{\delta_0})\Big).\end{aligned} \tag{3.27}$$

In the same fashion, we recall that w_ε^{n+1} satisfies the following transport equation:

$$\partial_t w_\varepsilon^{n+1} + \nabla_\perp.(E_\varepsilon^{\perp,n}(w_\varepsilon^n + G_\varepsilon^n)) + (w_\varepsilon^n + G_\varepsilon^n)\partial_\|(v_\varepsilon^n + G_\varepsilon^n)) = -\varepsilon\partial_\|\phi_\varepsilon^n(t,x_\|),$$

and we can once again estimate the δ_0 norm of w_ε^{n+1}:

$$\|w_\varepsilon^{n+1}\|_{\delta_0} \leq \eta C(\delta_0,\beta)\left((\|w_\varepsilon^n\|_{\delta_0} + \|G_\varepsilon^n\|_{\delta_0})\|\rho_\varepsilon^n\|_{\delta_0} + (\|w_\varepsilon^n\|_{\delta_0} + \|G_\varepsilon^n\|_{\delta_0})^2 + \|\varepsilon\partial_\|\phi_\varepsilon^n\|_{\delta_0}\right), \tag{3.28}$$

and if we compare two solutions at steps $n+1$ and $n+2$:

$$\begin{aligned}\|w_\varepsilon^{n+2} - w_\varepsilon^{n+1}\|_{\delta_0} &\leq \eta C(\delta_0,\beta)\Big((\|\rho_\varepsilon^{n+1}\|_{\delta_0} + \|\rho_\varepsilon^n\|_{\delta_0})(\|w_\varepsilon^{n+1} - w_\varepsilon^n\|_{\delta_0} + \|G_\varepsilon^{n+1} - G_\varepsilon^n\|_{\delta_0})\\ &+ (\|w_\varepsilon^{n+1}\|_{\delta_0} + \|w_\varepsilon^n\|_{\delta_0} + \|G_\varepsilon^{n+1}\|_{\delta_0} + \|G_\varepsilon^n\|_{\delta_0})\\ &\quad \times(\|w_\varepsilon^{n+1} - w_\varepsilon^n\|_{\delta_0} + \|G_\varepsilon^{n+1} - G_\varepsilon^n\|_{\delta_0} + \|\rho_\varepsilon^{n+1} - \rho_\varepsilon^n\|_{\delta_0})\\ &\quad \|\varepsilon\partial_\|\phi_\varepsilon^{n+1} - \varepsilon\partial_\|\phi_\varepsilon^{n+1}\|_{\delta_0}\Big).\end{aligned} \tag{3.29}$$

3.5 Finding a fixed point

We are now in position to use our estimates to prove the existence and uniqueness of a fixed point.

First let C_1 defined by:

$$C_1 = \sup_{\eta \le 1} \left\{ \|\rho_\varepsilon(0)\|_{\delta_0}, \|w_\varepsilon(0)\|_{\delta_0}, \|G_\varepsilon(0)\|_{\delta_0}, \|\sqrt{\varepsilon}E_\varepsilon(0)\|_{\delta_0}, 1 \right\}$$

Let $C_2 = C_1 + 1$. It is possible to choose η small enough with respect to C_1 to propagate the following estimates by recursion (we refer to [81] for more details; more explicitly $\eta = \frac{1}{200C(\delta_0,\beta)C_2^3}$ is for instance convenient). At Step n ($n \ge 1$), the property reads:

(i)

$$\begin{cases} \|\rho_\varepsilon^n\|_{\delta_0} \le C_2, \\ \|w_\varepsilon^n\|_{\delta_0} \le C_2, \\ \|G_\varepsilon^n\|_{\delta_0} \le C_2, \\ \|\sqrt{\varepsilon}E_{\varepsilon,\|}^n\|_{\delta_0} \le C_2. \end{cases}$$

(ii)

$$\begin{cases} \|\rho_\varepsilon^n - \rho_\varepsilon^{n-1}\|_{\delta_0} \le \frac{C_2}{2^n}, \\ \|w_\varepsilon^n - w_\varepsilon^{n-1}\|_{\delta_0} \le \frac{C_2}{2^n}, \\ \|G_\varepsilon^n - G_\varepsilon^{n-1}\|_{\delta_0} \le \frac{C_2}{2^n}, \\ \|\sqrt{\varepsilon}E_{\varepsilon,\|}^n - \sqrt{\varepsilon}E_{\varepsilon,\|}^{n-1}\|_{\delta_0} \le \frac{C_2}{2^n}. \end{cases}$$

One first checks that (i) is satisfied for $n = 0$. In particular for the last condition, we use (2.4). As in [81], checking that (ii) is satisfied for $n = 1$ in fact needs a special treatment which is very similar to the general case, so we will not detail it.

To propagate these estimates for $n \ge 1$, we use the crucial estimates (3.20),(3.27),(3.29). Let us briefly explain the passage from Step $(n+1)$ to Step $(n+2)$ by examining the case of Property (ii) for G_ε^n (the other cases are treated similarly). Using (3.20) and the Properties (i) and (ii) at step $n+1$ we have:

$$\|G_\varepsilon^{n+1} - G_\varepsilon^{n+2}\|_{\delta_0} \le \eta C(\delta_0, \beta)\frac{C_2}{2^{n+1}} 30 C_2,$$

and with our choice of η, we notice that $\eta C(\delta_0,\beta)\frac{C_2}{2^{n+1}}30C_2^2 \le \frac{C_2}{2^{n+2}}$, which proves the property (ii) for G_ε at step $(n+2)$.

This proves that the sequences $\rho_\varepsilon^n, w_\varepsilon^n, G_\varepsilon^n, \sqrt{\varepsilon}E_\varepsilon, E_\varepsilon^{\perp,n}, \varepsilon\partial_{x_\|}\phi_\varepsilon^n$ are Cauchy sequences (with respect to n) in $B_{\delta_0}^\eta$, and consequently converge strongly in $B_{\delta_0}^\eta$, the estimates being uniform in ε. It is clear that the limit satisfies System (1.7). The requirement $\delta_1 < \delta_0$ and the explicit life span in Theorem 2.1 come directly from the definition of the $B_{\delta_0}^\eta$ spaces.

For the uniqueness part, one can simply notice that the estimates we have shown allow us to prove that the application $\mathfrak{F}$ defined by:

$$\mathfrak{F}(\rho_\varepsilon, w_\varepsilon) = \begin{pmatrix} \int_0^t (-\nabla_\perp.(E_\varepsilon^\perp \rho_\varepsilon) - \partial_\|((w_\varepsilon + G_\varepsilon)\rho_\varepsilon))ds \\ \int_0^t (-\nabla_\perp.(E_\varepsilon^\perp(w_\varepsilon + G_\varepsilon)) - (w_\varepsilon + G_\varepsilon)\partial_\|(v_\varepsilon + G_\varepsilon)) - \varepsilon\partial_\|\phi_\varepsilon(t, x_\|))ds \end{pmatrix},$$

is a contraction on the closed subset B of $B_{\delta_0} \times B_{\delta_0}$, defined by:

$$B = \{\rho, w \in B_{\delta_0}; \|\rho\|_{\delta_0} \le C, \|w\|_{\delta_0} \le C\},$$

with C large enough, provided that η is chosen small enough. The uniqueness of the analytic solution then follows.

3.6 Proof of Proposition 2.1

We can lead the same analysis as for the proof of Theorem 2.1, but even simpler since here we do not have to deal anymore with the fast oscillations in time. The only slightly different point is to estimate the norm of $\int_0^t -\partial_{\|} p ds = \int_0^t \partial_{\|} \int \rho v^2 dx_\perp ds$, which is straightforward:

$$\left\| \int_0^t \partial_{\|} p ds \right\|_{\delta_0} \leq \eta C \|\rho\|_{\delta_0} \|v\|_{\delta_0}^2.$$

Then as before, we can use a contraction argument to prove the proposition.

4 Proof of Theorem 2.2

Step 1: Another average in time for $E_{\varepsilon,\|}$

We have observed previously that the wave equation (3.5) describing the time oscillations of $E_{\varepsilon,\|}$ was the same as the one appearing in Grenier's work, except for a slight change in the source. Therefore the following decomposition taken from [81, Proposition 3.1.1] identically holds, since the proof only relies on the fact that the source g_ε is bounded in $L^\infty_t H^{s-1}_x$, which is still the case here.

Lemma 4.1. *Under assumption (H), there exist $E^{(1)}_\varepsilon, E^{(2)}_\varepsilon$ and W_ε such that $E_{\varepsilon,\|} = E^{(1)}_\varepsilon + E^{(2)}_\varepsilon$ and a positive constant C independent of ε such as:*

(i) $\|\sqrt{\varepsilon} E^{(1)}_\varepsilon\|_{L^\infty(H^{s-1}_{x_\|})} \leq C$.

(ii) $\partial_t W_\varepsilon = E^{(1)}_\varepsilon$, $\|W_\varepsilon\|_{L^\infty(H^{s-1}_{x_\|})} \leq C$ *and* $W_\varepsilon \rightharpoonup 0$ *in* L^2.

(iii) $W^\varepsilon(0) = -\varepsilon \partial_t E_{\varepsilon,\|}(0) = \int \rho_\varepsilon(0) v_\varepsilon(0) dx_\perp$.

(iv) $\|E^{(2)}_\varepsilon\|_{L^\infty(H^{s-1}_{x_\|})} \leq C$.

(v) $\int E^{(1)}_\varepsilon dx_\| = \int E^{(2)}_\varepsilon dx_\| = 0$.

Roughly speaking, this lemma allows to decompose $E_{\varepsilon,\|}$ into a oscillating part with magnitude $\frac{1}{\sqrt{\varepsilon}}$ that we will have to filter out and a bounded part that will give rise to the pressure term.

Step 2: Uniform bound on $E^\perp_\varepsilon$ and $\partial_{x_\|}\phi_\varepsilon$

Under hypothesis (H), using the Poisson equation satisfied by ϕ_ε, one can check that $E^\perp_\varepsilon$ and $\partial_{x_\|}\phi_\varepsilon$ are bounded in $L^\infty_t(H^{s-1})$ uniformly with respect to ε (we do not need any gain of elliptic regularity). Indeed, since:

$$\int \left(\rho_\varepsilon - \int \rho_\varepsilon dx_\perp \right) dx_\perp = 0,$$

we easily check that:

$$\|\phi_\varepsilon\|_{H^s_{x_\perp, x_\|}} \leq \left\| \rho - \int \rho dx_\perp \right\|_{H^s_{x_\perp, x_\|}}.$$

Hence the result holds.

Step 3: Passage to the limit

Let $w_\varepsilon = v_\varepsilon - W_\varepsilon$. According to assumption (H) and Lemma 4.1, w_ε is uniformly bounded in $L^\infty_t([0,T], H^{s-1})$. On the other hand, we have :

$$\partial_t w_\varepsilon + \nabla_\perp.(E_\varepsilon^\perp w_\varepsilon) + w_\varepsilon \partial_{x_\parallel} w_\varepsilon = -\varepsilon \partial_{x_\parallel}\phi_\varepsilon + E_\varepsilon^{(2)} - w_\varepsilon \partial_{x_\parallel} W_\varepsilon - W_\varepsilon \partial_{x_\parallel} w_\varepsilon - W_\varepsilon \partial_{x_\parallel} W_\varepsilon. \quad (4.1)$$

(Notice that $\nabla_\perp.(E_\varepsilon^\perp W_\varepsilon) = W_\varepsilon \nabla_\perp(E_\varepsilon^\perp) = 0$.)

Thus, using the uniform bounds of assumption (H) and the fact the H^{s-2}_x is an algebra, we can easily see that $\partial_t w_\varepsilon$ is bounded in $L^\infty_t([0,T], H^{s-2})$. Thanks to the Aubin-Lions lemma (see for instance [148], w_ε converges strongly (up to a subsequence) to some function w in $\mathcal{C}([0,T], H^{s'-1})$ for all $s' < s$.

According to Step 2, $\varepsilon \partial_{x_\parallel}\phi_\varepsilon \rightharpoonup 0$ in the distributional sense.

Since w_ε strongly converges in $\mathcal{C}([0,T], H^{s'-1})$, it also converges strongly in $L^2([0,T], L^2)$ and by Lemma 4.1, (ii), W_ε weakly converges in $L^2([0,T], L^2)$. Thus, the following convergence also holds in the sense of distributions:

$$-w_\varepsilon \partial_{x_\parallel} W_\varepsilon - W_\varepsilon \partial_{x_\parallel} w_\varepsilon \rightharpoonup 0,$$

and $-W_\varepsilon \partial_{x_\parallel} W_\varepsilon + E_\varepsilon^{(2)}$ weakly converges (up to a subsequence) to some function F since this term is uniformly bounded in $L^\infty([0,T], H^{s-2}_{x_\parallel})$.

Furthermore, we observe that:

$$\int \left(-W_\varepsilon \partial_{x_\parallel} W_\varepsilon + E_\varepsilon^{(2)}\right) dx_\parallel = \int \left(-\frac{1}{2}\partial_{x_\parallel} W_\varepsilon^2 + E_\varepsilon^{(2)}\right) dx_\parallel = 0,$$

using Lemma 4.1, (v). This implies that $\int F dx_\parallel = 0$, and thus there exists p such that $F = -\partial_{x_\parallel} p$.

Since $E_\varepsilon^\perp$ is uniformly bounded in $L^\infty_t([0,T], H^{s-1})$, it also weakly converges, up to a subsequence, to some function $E^\perp$.

We now use the strong limit of w_ε in $\mathcal{C}([0,T], H^{s'-1})$ in order to pass to the limit in the sense of distributions in the convection terms. As a consequence, we obtain, passing to the limit in the sense of distributions:

$$\partial_t w + \nabla_\perp.(E^\perp w) + w \partial_{x_\parallel} w = -\partial_{x_\parallel} p. \quad (4.2)$$

We recall now that the equation satisfied by ρ_ε is:

$$\partial_t \rho_\varepsilon + \nabla_\perp.(E_\varepsilon^\perp \rho_\varepsilon) + \partial_\parallel(w_\varepsilon \rho_\varepsilon) = -\partial_\parallel(W_\varepsilon \rho_\varepsilon).$$

Proceeding similarly, we infer that ρ_ε converges strongly, up to a subsequence, to ρ in $\mathcal{C}([0,T], H^{s'})$ for all $s' < s$, that satisfies the equation:

$$\partial_t \rho + \nabla_\perp.(E^\perp \rho) + \partial_\parallel(w\rho) = 0.$$

One can likewise take limits in the Poisson equations. We finally obtain (1.8).

Step 4: Equations for the correctors

The final step relies on the following lemma proved in Grenier's paper [81, Proposition 3.3.4] (the main point is to notice that the application $\varphi \mapsto e^{\pm it/\sqrt{\varepsilon}}\varphi$ is an isometry on $L^\infty(H^s)$ for any s).

Lemma 4.2. *There exist two correctors $E_+(t,x_\parallel)$ and $E_-(t,x_\parallel)$ in $\mathcal{C}([0,T],H^{s-1})$ such that, for all $s'<s$:*

- $\|\sqrt{\varepsilon}E_\varepsilon^{(1)} - e^{it/\sqrt{\varepsilon}}E_+ - e^{-it/\sqrt{\varepsilon}}E_-\|_{\mathcal{C}([0,T],H^{s'-1})} \to 0,$
- $\|W_\varepsilon - \frac{1}{i}\left(e^{it/\sqrt{\varepsilon}}E_+ - e^{-it/\sqrt{\varepsilon}}E_-\right)\|_{\mathcal{C}([0,T],H^{s'-1})} \to 0.$

In particular we can deduce that:

$$e^{-it/\sqrt{\varepsilon}}\sqrt{\varepsilon}E_\varepsilon^{(1)} \rightharpoonup E_+$$

(and similarly $e^{it/\sqrt{\varepsilon}}\sqrt{\varepsilon}E_\varepsilon^{(1)} \rightharpoonup E_-$).

Then, the idea is to use Lemmas 4.1 and 4.2 and the wave equation (3.5) in order to obtain the equations satisfied by $E_\pm$. Let us show how one can obtain the equation for E_- (the method being similar for E_+). Let us denote $F_\varepsilon = \sqrt{\varepsilon}e^{it/\sqrt{\varepsilon}}E_\varepsilon$. One can then observe that:

$$\varepsilon\partial_t^2 E_{\varepsilon,\parallel} + E_{\varepsilon,\parallel} = e^{-it/\sqrt{\varepsilon}}\left(\sqrt{\varepsilon}\partial_t^2 F_\varepsilon - 2i\partial_t F_\varepsilon\right).$$

Furthermore, by Lemmas 4.1 and 4.2, F_ε weakly converges (in the distributional sense) to E_-. Using (3.5), we obtain an equation satisfied by F_ε:

$$\begin{aligned}\sqrt{\varepsilon}\partial_t^2\partial_{x_\parallel}F_\varepsilon - 2i\partial_t\partial_{x_\parallel}F_\varepsilon &= e^{it/\sqrt{\varepsilon}}\partial_{x_\parallel}^2\int \rho_\varepsilon(w_\varepsilon+W_\varepsilon)^2 dx_\perp \\ +e^{it/\sqrt{\varepsilon}}\partial_{x_\parallel}\int \rho_\varepsilon(\varepsilon\partial_{x_\parallel}\phi_\varepsilon)dx_\perp &- e^{it/\sqrt{\varepsilon}}\varepsilon\partial_{x_\parallel}[E_{\varepsilon,\parallel}\partial_{x_\parallel}E_{\varepsilon,\parallel}].\end{aligned} \tag{4.3}$$

We first show that $\sqrt{\varepsilon}\partial_t^2\partial_{x_\parallel}F_{\varepsilon,\parallel}$ weakly converges to 0 in the distributional sense. For this purpose let $\Psi(t,x_\parallel)$ a smooth test function compactly supported in $\mathbb{R}^{+*}\times\mathbb{R}$. We have by integration by parts:

$$\begin{aligned}\int \sqrt{\varepsilon}\partial_t^2\partial_{x_\parallel}F_\varepsilon\Psi dtdx_\parallel &= -\int \sqrt{\varepsilon}\partial_t F_\varepsilon\partial_t\partial_{x_\parallel}\Psi dtdx_\parallel \\ &= \int \sqrt{\varepsilon}F_\varepsilon\partial_t^2\partial_{x_\parallel}\Psi dtdx_\parallel,\end{aligned}$$

and we can conclude that the contribution of this three term vanishes as ε vanishes since F_ε is uniformly bounded in $\mathcal{C}([0,T],H_{x_\parallel}^{s'-1})$ by Lemma 4.1. Likewise, we show that $-2i\partial_t F_\varepsilon$ converges in the distributional sense to $-2i\partial_t E_-$.

By Step 3, we recall that ρ_ε converges strongly (up to a subsequence) in $\mathcal{C}([0,T],H^{s'})$ (with $s'<s$). Let us show that $\varepsilon\partial_{x_\parallel}\phi_\varepsilon$ also converges strongly (up to a subsequence) in $\mathcal{C}([0,T],H^{s'})$. To that purpose, we rely once again on the Poisson equation satisfied by ϕ_ε, that we recall below:

$$-\varepsilon^2\partial_{x_\parallel}^2\phi_\varepsilon - \Delta_\perp\phi_\varepsilon = \rho_\varepsilon - \int\rho_\varepsilon dx_\perp.$$

By the same symbolic analysis as before, one can easily check, using assumption (H), that $\varepsilon\partial_{x_\parallel}\phi_\varepsilon$ is uniformly bounded in $L_t^\infty(H_x^s)$. Deriving the Poisson equation with respect to time, we obtain:

$$-\varepsilon^2\partial_{x_\parallel}^2\partial_t\phi_\varepsilon - \Delta_\perp\partial_t\phi_\varepsilon = \partial_t\rho_\varepsilon - \int\partial_t\rho_\varepsilon dx_\perp.$$

Using this time the uniform estimates on $\partial_t\rho_\varepsilon$, we deduce that $\varepsilon\partial_t\partial_{x_\parallel}\phi_\varepsilon$ is uniformly bounded in $L_t^\infty(H_x^{s-2})$.

Therefore, using the Aubin-Lions lemma, we have proved our claim.

We deduce that $\partial_{x_\parallel} \int \rho_\varepsilon(\varepsilon \partial_{x_\parallel} \phi_\varepsilon) dx_\perp$ converge strongly (up to a subsequence) in $\mathcal{C}([0,T], H^{s'-1}_{x_\parallel})$, so we can see that:

$$e^{it/\sqrt{\varepsilon}} \partial_{x_\parallel} \int \rho_\varepsilon(\varepsilon \partial_{x_\parallel} \phi_\varepsilon) dx_\perp \rightharpoonup 0$$

in the sense of distributions.

In order to take the limit in the other terms, we have to be a little more precise. By Lemmas 4.1 and 4.2, we can write:

$$\sqrt{\varepsilon} E_{\varepsilon,\parallel} = e^{it/\sqrt{\varepsilon}} E_+ + e^{-it/\sqrt{\varepsilon}} E_- + r_\varepsilon,$$

$$W_\varepsilon = \frac{1}{i}\left(e^{it/\sqrt{\varepsilon}} E_+ - e^{-it/\sqrt{\varepsilon}} E_-\right) + s_\varepsilon,$$

where r_ε and s_ε converge strongly to 0 in $\mathcal{C}([0,T], H^{s'-1}_{x_\parallel})$. Consequently we deduce that $e^{it/\sqrt{\varepsilon}} \varepsilon \partial_{x_\parallel}[E_{\varepsilon,\parallel} \partial_{x_\parallel} E_{\varepsilon,\parallel}]$ converges to 0 in the sense of distributions. Indeed, we have:

$$\begin{aligned} e^{it/\sqrt{\varepsilon}} \varepsilon \partial_{x_\parallel}[E_{\varepsilon,\parallel} \partial_{x_\parallel} E_{\varepsilon,\parallel}] =& \frac{1}{2} e^{it/\sqrt{\varepsilon}} \partial^2_{x_\parallel} \Big(r_\varepsilon^2 + e^{2it/\sqrt{\varepsilon}} E_+^2 + e^{-2it/\sqrt{\varepsilon}} E_-^2 \\ & + 2E_+E_- + 2e^{it/\sqrt{\varepsilon}} E_+ r_\varepsilon + 2e^{-it/\sqrt{\varepsilon}} E_- r_\varepsilon \Big) \end{aligned}$$

Thus, as r_ε converges strongly to 0 in $\mathcal{C}([0,T], H^{s'-1}_{x_\parallel})$, there is no resonance effect and this converges to 0 in the sense of distributions. Now we write:

$$\partial^2_{x_\parallel} \int \rho_\varepsilon (w_\varepsilon + W_\varepsilon)^2 dx_\perp = \partial^2_{x_\parallel} \int \rho_\varepsilon w_\varepsilon^2 dx_\perp + \partial^2_{x_\parallel} \left(\int \rho_\varepsilon dx_\perp \right) W_\varepsilon^2 + 2\partial^2_{x_\parallel} \int \rho_\varepsilon w_\varepsilon W_\varepsilon dx_\perp .$$

Since $\partial^2_{x_\parallel} \int \rho_\varepsilon w_\varepsilon^2 dx_\perp$ strongly converges in $\mathcal{C}([0,T], H^{s'-1}_{x_\parallel})$, the contribution of the first term, that is $e^{it/\sqrt{\varepsilon}} \partial^2_{x_\parallel} \int \rho_\varepsilon w_\varepsilon^2 dx_\perp$ vanishes. For the second term, we first notice that $\int \rho_\varepsilon dx_\perp$ is strongly convergent in $\mathcal{C}([0,T], H^{s'}_{x_\parallel})$. Then, we can check as before that there is no resonance effect and the contribution of $e^{it/\sqrt{\varepsilon}} \partial^2_{x_\parallel} \left(\int \rho_\varepsilon dx_\perp\right) W_\varepsilon^2$ vanishes. For the last term, $\rho_\varepsilon w_\varepsilon$ strongly converges to ρv in $\mathcal{C}([0,T], H^{s'-1}_{x})$; using once again the decomposition of W_ε, we obtain that the limit in the distributional sense of $e^{it/\sqrt{\varepsilon}} 2\partial^2_{x_\parallel} \int \rho_\varepsilon w_\varepsilon W_\varepsilon dx_\perp$ is $2i \left(\int \rho v dx_\perp\right) \partial_{x_\parallel}(\partial_{x_\parallel} E_-)$.

As a result, $\partial_{x_\parallel} E_\pm$ satisfy the transport equations:

$$\partial_t(\partial_{x_\parallel} E_\pm) + \left(\int \rho v dx_\perp \right) \partial_{x_\parallel}(\partial_{x_\parallel} E_\pm) = 0.$$

There remains to provide some initial data for these equations. This is achieved thanks to the strong convergences in Lemma 4.2 that hold in particular for $t = 0$.

The proof of the theorem is now complete.

5 Discussion on the sharpness of the results

5.1 On the analytic regularity

Let us recall that the multi-fluid system (2.2) is ill-posed in Sobolev spaces ([24]), because of the two-stream instabilities (remind that this is due to the coupling between the different phases of the fluid).

For system (1.8), we expect the situation to be similar. Due to the dependence on $x_\perp$ and the constraint $\int \rho dx_\perp = 1$, system (1.8) is by nature a coupled multi-fluid system. Nevertheless, one could maybe imagine that the dynamics in the $x_\perp$ variable could yield some mixing in $x_\perp$ and $x_\parallel$ (in the spirit of hypoellipticity results) and thus could perhaps bring stability. Here we explain why this is not the case.

The idea is to consider for (1.8) shear flows like initial data. This will allow to exactly recover the multi-fluid equations (2.2). Writing $x_\perp = (x_1, x_2)$, we take:

$$E_0^\perp = (0, \varphi(x_1, x_\parallel), 0),$$

and consequently since by definition:

$$\rho_0 = \operatorname{div}_x E_0 + 1,$$

we infer that $\rho_0 = \nabla_\perp \wedge E_0^\perp = -\varphi'(x_1, x_\parallel) + 1$. We also assume that $v_0(x_1, x_\parallel)$ does not depend on x_2.

Then we observe that:

$$\begin{aligned}
\nabla_\perp(E_0^\perp \rho_0) &= 0,\\
\nabla_\perp(E_0^\perp v_0) &= 0.
\end{aligned}$$

With such initial data, system (1.8) reduces to:

$$\left\{\begin{array}{l}
\partial_t \rho + \partial_\parallel(v_\parallel \rho) = 0\\
\partial_t v_\parallel + v_\parallel \partial_\parallel(v_\parallel) = -\partial_\parallel p(t, x_\parallel)\\
\int \rho dx_1 = 1,
\end{array}\right. \tag{5.1}$$

and we observe that there is no more dynamics in the $x_\perp$ variable. This is nothing but system (2.2) in dimension 1, with $M = [0, 1[$ and μ the Lebesgue measure.

Now, let us consider measure type of data in the x_1 variable for ρ and v (this corresponds to a "degenerate" version of the shear flows defined above). In particular if we choose:

$$\varphi = \frac{1}{2}\delta_{x_1 \leq \frac{1}{4}} \rho_{0,1}(x_\parallel) + \frac{1}{2}\delta_{x_1 \leq \frac{1}{2}} \rho_{0,2}(x_\parallel),$$

we get:

$$\begin{aligned}
\rho_0 &= \frac{1}{2}\delta_{x_1 = \frac{1}{4}} \rho_{0,1}(x_\parallel) + \frac{1}{2}\delta_{x_2 = \frac{1}{2}} \rho_{0,2}(x_\parallel),\\
v_0 &= \frac{1}{2}\delta_{x_1 = \frac{1}{4}} v_{0,1}(x_\parallel) + \frac{1}{2}\delta_{x_1 = \frac{1}{2}} v_{0,2}(x_\parallel)
\end{aligned} \tag{5.2}$$

and we obtain the following system for $\alpha = 1, 2$:

$$\left\{\begin{array}{l}
\partial_t \rho_\alpha + \partial_\parallel(v_\alpha \rho_\alpha) = 0\\
\partial_t v_\alpha + v_\alpha \partial_\parallel(v_\alpha) = -\partial_\parallel p(t, x_\parallel)\\
\rho_1 + \rho_2 = 1.
\end{array}\right. \tag{5.3}$$

This particular system was given as an example by Brenier in [24] to illustrate ill-posedness in Sobolev spaces of the multi-fluid equations. Indeed let us first denote $q = \rho_1 v_1$. Using the constraint $\rho_1 + \rho_2 = 1$, we easily obtain that

$$p_\parallel = -q^2 \left(\frac{1}{\rho_1} + \frac{1}{1 - \rho_1} \right).$$

We can then observe that the system:

$$\begin{cases} \partial_t \rho_1 + \partial_{\|} q = 0 \\ \partial_t q + \partial_{\|}(\frac{q^2}{\rho_1}) = -\rho_1 \partial_{\|} p(t, x_{\|}) \end{cases} \tag{5.4}$$

is elliptic in space-time, and consequently it is ill-posed in Sobolev spaces.

Actually this example is not completely satisfying, since it is singular in x_1. Nevertheless we can consider the convolution of this initial data with a standard mollifier, which yields the same qualitative behaviour.

5.2 On the analytic regularity in the perpendicular variable

We observe that if the initial datum $(\rho(0), v(0))$ does not depend on $x_{\|}$, then the fluid system (1.8) reduces to:

$$\begin{cases} \partial_t \rho + \nabla_{\perp}.(E^{\perp}\rho) = 0 \\ \partial_t v_{\|} + \nabla_{\perp}.(E^{\perp} v_{\|}) = 0 \\ E^{\perp} = \nabla^{\perp}\Delta_{\perp}^{-1}\left(\rho - \int \rho dx_{\perp}\right) \\ \int \rho dx_{\perp} = 1. \end{cases} \tag{5.5}$$

Thus, ρ satisfies $2D$ incompressible Euler system, written in vorticity formulation. This systems admits a unique global strong solution provided that $\rho(0) \in H^s(\mathbb{T}^2)$ (with $s > 1$), by a classical result of Kato [109] and even a unique global weak solution provided that $\rho(0) \in L^\infty(\mathbb{T}^2)$, by a classical result of Yudovic [160].

In the other hand, $v_{\|}$ satisfied a transport equation with the force field $E^{\perp}$. If we only assume for instance that v_0 is a positive Radon measure, then using the classical log-Lipschitz estimate on $E^{\perp}$ (we refer to [124, Chapter 8]), we get a unique global weak solution $v_{\|}$ by the method of characteristics. Let us give some details on this result:

One could think that it should be possible to build solutions to the final fluid system (1.8) with similar “weak” regularity in the $x_{\perp}$ variable (while keeping analyticity in the $x_{\|}$ variable). Actually this is not possible in general: this is related to the fact that $E^{\perp}$ depends also on $x_{\|}$ in general and this entails that we also need analytic regularity in the $x_{\perp}$ variable to get analytic regularity in the $x_{\|}$ variable (see estimations such as (3.26)).

5.3 On the local in time existence

In [25], Brenier considers potential velocity fields, that are velocity fields of the form $v_\Theta = \nabla_x \Phi_\Theta$, for the multi-fluid system:

$$\begin{cases} \Theta = 1, ..., M \quad M \in \mathbb{N}^* \\ \partial_t \rho_\Theta + \mathrm{div}(\rho_\Theta v_\Theta) = 0 \\ \partial_t v_\Theta + v_\Theta.\nabla(v_\Theta) = -\nabla_x p \\ \sum_{\Theta=1}^M \rho_\Theta = 1. \end{cases} \tag{5.6}$$

In this case the equation on the velocities becomes:

$$\partial_t \Phi_\Theta + \frac{1}{2}|\nabla_x \Phi_\Theta|^2 + p = 0. \tag{5.7}$$

It is proved in [25] that any strong solution satisfying

$$\inf_{\Theta,t,x} \rho_\Theta(t,x) > 0$$

can not be global in time unless the initial energy vanishes:

$$\sum_{\Theta=1}^{M} \int \rho_{\Theta,t=0} |u_{\Theta,t=0}|^2 dx = 0. \tag{5.8}$$

This striking result relies on a variational interpretation of these Euler equations. Using the same particular initial data as in section 5.1, this indicates that for system (1.8) also, there is no global strong solution, unless there is no dependence on $x_\perp$ or $x_\parallel$.

We observe that if the initial datum $(\rho(0), v(0))$ does not depend on $x_\perp$, the fluid system (1.8) does not make sense anymore (as for incompressible Euler in dimension 1). When the initial datum $(\rho(0), v(0))$ does not depend on $x_\parallel$, we have seen that we recover $2D$ incompressible Euler and there is indeed global existence (of strong or weak solutions).

5.4 The relative entropy method applied to a toy model : failure of the multi-current limit

5.4.1 The toy model

It seems very appealing to try to use the relative entropy method (which was introduced by Brenier [24] for Vlasov type of systems) to study the limit $\varepsilon \to 0$, as it would open the way to the study of the limit for solutions to the initial system (1.1) with low regularity. The only requirements would be that the initial data of (1.1) is closed in some sense (which will be made precise later) to a Dirac mass in velocity, and that the two first moments of the initial data are in a small neighborhood (say in L^2 topology) of the smooth initial data for the limit system (1.8). Nevertheless it is not possible to overcome the two-stream instabilities in this framework. We intend here to show why.

The toy model we consider in this paragraph is the following:

$$\begin{cases} \partial_t f_\varepsilon^\theta + v.\nabla_x f_\varepsilon^\theta + E_\varepsilon.\nabla_v f_\varepsilon^\theta = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\varepsilon \Delta_x V_\varepsilon = \int\int f_\varepsilon^\theta dv d\mu - 1 \\ f_\varepsilon^\theta(t=0) = f_{\varepsilon,0}^\theta, \quad \int\int f_\varepsilon^\theta dv dx d\theta = 1. \end{cases} \tag{5.9}$$

with $t > 0$, $x \in \mathbb{T}^3$, $v \in \mathbb{R}^3$ and where θ lies in $[0, 1]$ equipped with a probability measure μ which is:

- either a sum of Dirac masses with total mass 1, such as:

$$\mu = \sum_{i=0}^{N-1} \frac{1}{N} \delta_{\theta=i/N}.$$

  In this case, we model a plasma made of N phases.

- or the Lebesgue measure, in which case we model a continuum of phases.

Actually, we could have considered more general probability measures but we restrict to these cases for simplicity. This system can be seen as the kinetic counterpart of a simplified version of (1.7), which focuses on the unstable feature of the system. Of course we could have considered directly the fluid version, that is:

$$\begin{cases} \partial_t \rho_\varepsilon^\theta + \nabla_x.(\rho_\varepsilon^\theta u_\varepsilon^\theta) = 0 \\ \partial_t u_\varepsilon^\theta + u_\varepsilon^\theta.\nabla_x u_\varepsilon^\theta = E_\varepsilon \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\varepsilon \Delta_x V_\varepsilon = \int\int f_\varepsilon^\theta dv d\mu - 1 \end{cases} \tag{5.10}$$

but the proofs are essentially the same and the study of system (5.9) has some interests of its own.

One can observe that the energy associated to (5.9) is the following non-increasing (formally conserved) functional:

$$\mathcal{E}_\varepsilon(t) = \frac{1}{2}\int\int f_\varepsilon^\theta |v|^2 dvdxd\mu + \frac{1}{2}\varepsilon \int |\nabla_x V_\varepsilon|^2 dx. \tag{5.11}$$

We assume that there exists a constant $K > 0$ independent of ε, such as $\mathcal{E}_\varepsilon(0) \leq K$. We also assume that $f_0^\theta \in L_\theta^\infty L_{x,v}^1 \cap L_\theta^\infty L_{x,v}^\infty$, uniformly in ε. Then we can consider global weak solutions $(f_\varepsilon^\theta, V_\varepsilon)$ to (5.9), in the sense of Arsenev [4].That these solutions exist follows from a slight adaptation of the original proof in [4], which dealt with the usual Vlasov-Poison equation. These solutions satisfy that uniformly in ε, $f_\varepsilon^\theta \in L_{t,\theta}^\infty L_{x,v}^1 \cap L_{t,\theta}^\infty L_{x,v}^\infty$. In addition, for any ε and any $t \geq 0$:

$$\mathcal{E}_\varepsilon(t) \leq K. \tag{5.12}$$

Let (ρ^θ, u^θ) be the local strong solution, defined on $[0,T]$, to the system:

$$\begin{cases} \partial_t \rho^\theta + \nabla_x.(\rho^\theta u^\theta) = 0 \\ \partial_t u^\theta + u^\theta.\nabla_x u^\theta = -\nabla_x V \\ \int \rho^\theta d\mu = 1. \end{cases} \tag{5.13}$$

with inital data $(\rho_0^\theta, u_0^\theta)$ (which we actually have to take with analytic regularity in general). Observe here that the "incompressibility in average" constraint reads:

$$\nabla_x. \int \rho^\theta u^\theta d\mu = 0. \tag{5.14}$$

The case where u_0^θ genuinely depends on θ corresponds to the setting for two-stream instabilities [38]. In this case, as expected, we will not be able to conclude. On the contrary, when u_0^θ does not depend on θ, this precisely corresponds to the case where two-stream instabilities are avoided, and in that particular case, the relative entropy method will yield convergence: this is the result of Proposition 5.1.

5.4.2 The relative entropy method

Following the approach of Brenier [24] for the quasineutral limit of the Vlasov-Poisson equation with a single phase, we consider the relative entropy (built as a modulation of the energy $\mathcal{E}_\varepsilon$):

$$\mathcal{H}_\varepsilon(t) = \frac{1}{2}\int\int f_\varepsilon^\theta |v - u^\theta(t,x)|^2 dvdxd\mu + \frac{1}{2}\varepsilon \int |\nabla_x V_\varepsilon - \nabla_x V|^2 dx. \tag{5.15}$$

We assume that the system is well prepared in the sense that $\mathcal{H}_\varepsilon(0) \to 0$ when $\varepsilon \to 0$. The goal is to find some stability inequality in order to show that we also have $\mathcal{H}_\varepsilon(t) \to 0$ for $t \in [0,T]$.

We have, since the energy is non-increasing:

$$\begin{aligned} \frac{d}{dt}\mathcal{H}_\varepsilon(t) \leq & \int\int \partial_t f_\varepsilon^\theta \left(\frac{1}{2}|u^\theta|^2 - v.u^\theta\right) dvdxd\mu + \int\int f_\varepsilon^\theta \partial_t \left(\frac{1}{2}|u^\theta|^2 - v.u^\theta\right) dvdxd\mu \\ & + \frac{1}{2}\varepsilon \int \partial_t |\nabla_x V|^2 dx - \varepsilon \int \nabla_x V_\varepsilon.\partial_t \nabla_x V dx - \varepsilon \int \partial_t \nabla_x V_\varepsilon.\nabla_x V dx. \end{aligned} \tag{5.16}$$

We clearly have $\varepsilon \int \partial_t |\nabla_x V|^2 dx = \mathcal{O}(\varepsilon)$. Moreover, we get, by Cauchy-Schwarz inequality:

$$\varepsilon \Big| \int \nabla_x V_\varepsilon . \partial_t \nabla_x V dx \Big| \leq \sqrt{\varepsilon} \| \sqrt{\varepsilon} \nabla_x V_\varepsilon \|_{L^\infty_t L^2_x} \| \partial_t \nabla_x V \|_{L^\infty_t L^2_x},$$

which is of order $\mathcal{O}(\sqrt{\varepsilon})$ by the conservation of energy.

For the last term of (5.16), we compute with successive integrations by parts:

$$\begin{aligned} -\varepsilon \int \partial_t \nabla_x V_\varepsilon . \nabla_x V dx &= \varepsilon \int \partial_t \Delta_x V_\varepsilon V dx \\ &= - \int \partial_t \left(\int f_\varepsilon^\theta dv d\mu \right) V dx \\ &= \int \nabla_x . \left(\int f_\varepsilon^\theta v \, dv d\mu \right) V dx \\ &= - \int \left(\int f_\varepsilon^\theta v \, dv d\mu \right) . \nabla_x V dx. \end{aligned} \tag{5.17}$$

In this computation we have used the Poisson equation as well as the local conservation of mass (obtained by integrating the Vlasov equation in (5.9) against v):

$$\partial_t \int f_\varepsilon^\theta dv + \nabla_x . \left(\int v f_\varepsilon^\theta dv \right) = 0.$$

In the other hand we can compute:

$$\begin{aligned} &\int \int \partial_t f_\varepsilon^\theta \left(\frac{1}{2} |u^\theta|^2 - v.u^\theta \right) dv dx d\mu + \int \int f_\varepsilon^\theta \partial_t \left(\frac{1}{2} |u^\theta|^2 - v.u^\theta \right) dv dx d\mu \\ &= - \int \int (v.\nabla_x f_\varepsilon^\theta + E_\varepsilon . \nabla_v f_\varepsilon^\theta) \left(\frac{1}{2} |u^\theta|^2 - v.u^\theta \right) dv dx d\mu + \int \int f_\varepsilon^\theta (u^\theta - v). \partial_t u^\theta dv dx d\mu \\ &= - \int \int f_\varepsilon^\theta v.((u^\theta - v).\nabla_x u^\theta) dv dx d\mu - \int f_\varepsilon^\theta E_\varepsilon . u^\theta dv dx d\mu + \int \int f_\varepsilon^\theta (u^\theta - v).\partial_t u^\theta dv dx d\mu \\ &= \int \int f_\varepsilon^\theta (u^\theta - v).((u^\theta - v).\nabla_x u \theta) dv dx d\mu + \int \int f_\varepsilon^\theta (u^\theta - v).(\partial_t u^\theta + u^\theta . \nabla_x u^\theta) dv dx d\mu \\ &- \int f_\varepsilon^\theta E_\varepsilon . u^\theta dv dx d\mu. \end{aligned} \tag{5.18}$$

All the trouble comes from the last term:

$$\int f_\varepsilon^\theta E_\varepsilon . u^\theta dv dx d\mu.$$

When no assumption is made on u^θ, it can be of order $\mathcal{O}(1/\sqrt{\varepsilon})$. This wild term can be interpreted as the appearance of the two-stream instabilities. Therefore we have to make an additional assumption in order to avoid this instability. This is done by assuming that u^θ initially does not depend on θ (which yields that u^θ does not depend on θ by uniqueness), in which case we can write:

$$u^\theta = u$$

and consequently, we have

$$- \int f_\varepsilon^\theta E_\varepsilon . u \, dv dx d\mu = \int (\varepsilon \Delta_x V_\varepsilon - 1) \, E_\varepsilon . u dx. \tag{5.19}$$

We first compute:

$$\begin{aligned} -\int \varepsilon \int \Delta_x V_\varepsilon \nabla_x V_\varepsilon . u dx &= -\varepsilon \int \nabla_x : (\nabla_x V_\varepsilon \otimes \nabla_x V_\varepsilon) u dx + \varepsilon \int \frac{1}{2} \nabla_x |\nabla_x V_\varepsilon|^2 u dx \\ &= \varepsilon \int D(u) : (\nabla_x V_\varepsilon \otimes \nabla_x V_\varepsilon) dx - \varepsilon \int \frac{1}{2} |\nabla_x V_\varepsilon|^2 \operatorname{div}_x u dx, \end{aligned}$$

with $D(u) = \frac{1}{2}\left(\partial_{x_i} u_j + \partial_{x_j} u_i\right)_{i,j}$.

In addition, the incompressibility constraint (5.14) becomes $\nabla_x . u = 0$, and thus:

$$\int E_\varepsilon . u dx = \int V_\varepsilon \nabla_x . u dx = 0.$$

Gathering all pieces together, we obtain:

$$\begin{aligned} &\mathcal{H}_\varepsilon(t) \le \mathcal{H}_\varepsilon(0) + R_\varepsilon(t) + C \int_0^t \|\nabla_x u\| \mathcal{H}_\varepsilon(s) ds \\ &+ \int_0^t \int\int f_\varepsilon^\theta (u - v)(\partial_t u + u.\nabla_x u) d\mu dv dx ds - \int_0^t \int\int f_\varepsilon^\theta v.\nabla_x V d\mu dv dx ds, \end{aligned} \tag{5.20}$$

where $C > 0$ is a universal constant, $R_\varepsilon(t) \to 0$ as ε goes to 0. Furthermore, we remark that:

$$\int \left(\int f_\varepsilon^\theta dv d\mu\right) u.\nabla_x V dv = \int u.\nabla_x V - \varepsilon \int \Delta_x V_\varepsilon u.\nabla_x V \tag{5.21}$$

The first term is equal to 0 according to the incompressibility constraint, while the second is of order $\mathcal{O}(\sqrt{\varepsilon})$, by the energy inequality. We finally get the stability inequality:

$$\begin{aligned} &\mathcal{H}_\varepsilon(t) \le \mathcal{H}_\varepsilon(0) + \tilde{R}_\varepsilon(t) + C \int_0^t \|\nabla_x u\| \mathcal{H}_\varepsilon(s) ds \\ &+ \int_0^t \int\int f_\varepsilon^\theta (u - v)(\partial_t u + u.\nabla_x u + \nabla_x V) d\mu dv dx ds, \end{aligned} \tag{5.22}$$

where $C > 0$ is a universal constant, $\tilde{R}_\varepsilon(t) \to 0$ as ε goes to 0 and the last term is 0 by definition of (u, V).

As as result, by Gronwall's inequality, we infer that $\mathcal{H}_\varepsilon(t) \to 0$, uniformly locally in time. To conclude, by a classical interpolation argument using the fact that $f_\varepsilon|v|^2$ is uniformly in $L^\infty_t(L^1_{x,v,\theta})$ and that f_ε is uniformly in $L^\infty_{t,\theta,x,v}$, we infer that $\rho_\varepsilon^\theta := \int f_\varepsilon^\theta dv$ and $J_\varepsilon^\theta := \int f_\varepsilon^\theta v dv$ are uniformly bounded in $L^\infty_t(L^p_{\theta,x})$ for some $p > 1$. Thus, up to a subsequence, there exist ρ^θ and J^θ (at least in $L^\infty_t(L^1_{\theta,x})$) such that ρ_ε^θ weakly converges in L^1 to ρ^θ (resp. J_ε^θ to J^θ). Passing to the limit in the local conservation of charge, which reads:

$$\partial_t \rho_\varepsilon^\theta + \nabla_x J_\varepsilon^\theta = 0,$$

we obtain:

$$\partial_t \rho^\theta + \nabla_x J^\theta = 0.$$

The goal is now to prove that $J^\theta = \rho^\theta u$.

By a simple use of Cauchy-Schwarz inequality, we have:

$$\int\int \frac{|\rho_\varepsilon^\theta u - J_\varepsilon^\theta|}{\rho_\varepsilon^\theta} dx d\mu \le \int\int f_\varepsilon^\theta |v - u|^2 dv dx d\mu. \tag{5.23}$$

Using a classical convexity argument due to Brenier [27], one can prove that the functional $(\rho, J) \mapsto \int \frac{|\rho u - J|}{\rho} dx d\mu$ is lower semi-continuous with respect to the weak convergence of measures. We finally obtain by passing to the limit that:

$$J^\theta = \rho^\theta u.$$

By uniqueness of the solution to the limit system, provided that the whole sequence $(\rho^\theta_{\varepsilon,0})$ weakly converges to ρ^θ_0, we obtain the convergences without having to extract subsequences.

Finally we have proved the result:

Proposition 5.1. *Let $(f^\theta_\varepsilon, V_\varepsilon)$ be a global weak solution in the sense of Arsenev to (5.9). Assume that for some functions (ρ^θ_0, u_0) in $(L^1_{\theta,x} \times H^s_x)$, with $s > 5/2$, (we emphasize on the fact that u_0 does not depend on θ, in order to avoid two-stream instabilities) satisfying:*

$$\begin{cases} \int \rho^\theta_0 d\mu = 1, \\ \nabla_x . u_0 = 0, \end{cases} \tag{5.24}$$

we initially have:

$$\frac{1}{2} \int \int f^\theta_\varepsilon (t=0) |v - u_0(x)|^2 dv dx d\mu + \frac{1}{2} \varepsilon \int |\nabla_x V_\varepsilon(0) - \nabla_x V|^2 dx \to 0 \tag{5.25}$$

and $\int f^\theta_\varepsilon dv \rightharpoonup \rho^\theta_0$ in the weak L^1 sense.

Let (u, V) is the (unique) local strong solution (defined on $[0, T]$) to the incompressible Euler system:

$$\begin{cases} \partial_t u + u.\nabla_x u = -\nabla_x V \\ \nabla_x u = 0, \end{cases} \tag{5.26}$$

with initial data $u(t = 0) = u_0$. Then,

$$\frac{1}{2} \int \int f^\theta_\varepsilon |v - u(t,x)|^2 dv dx d\mu + \frac{1}{2} \varepsilon \int |\nabla_x V_\varepsilon - \nabla_x V|^2 dx \to 0, \tag{5.27}$$

where (u, V) is the local strong solution to the incompressible Euler system:

$$\begin{cases} \partial_t u + u.\nabla_x u = -\nabla_x V \\ \nabla_x u = 0. \end{cases} \tag{5.28}$$

Moreover, $\rho^\theta_\varepsilon := \int f^\theta_\varepsilon dv$ converges in the weak L^1 sense to ρ^θ the unique solution to:

$$\partial_t \rho^\theta + u.\nabla_x \rho^\theta = 0, \tag{5.29}$$

with $\rho^\theta(t = 0) = \rho^\theta_0$ and $J^\theta_\varepsilon := \int f^\theta_\varepsilon v dv$ converges in the weak L^1 sense to $\rho^\theta u$.

6 Conclusion

In this work, we have provided a first analysis of the mathematical properties of the three-dimensional finite Larmor radius approximation (FLR), for electrons in a fixed background of ions. We have shown that the limit is unstable in the sense that we have to restrict to data with both particular profiles and analytic data. In particular, we have pointed out that the analytic assumption is not only a mere technical assumption, but is necessary if one choses to consider strong solutions. In addition, the results are only local-in-time.

On the other hand, we proved in [94] that the FLR approximation for ions with massless electrons is by opposition very stable, in the sense that we can deal with initial data with no prescribed profile and weak (that is in a Lebesgue space) regularity.

This rigorously justifies why physicists rather consider the equations on ions rather than those on electrons, especially for numerical experiments (we refer for instance to Grandgirard et al. [78]).

7 Appendix: Formal derivation of the drift-fluid problem

Scaling of the Vlasov equation

Let us recall that our purpose is to describe the behaviour of a gas of electrons in a neutralizing background of ions at thermodynamic equilibrium, submitted to a large magnetic field. For simplicity, we consider a magnetic field with a fixed direction $e_\parallel$ (also denoted by e_z) and a fixed large magnitude $\bar{B}$.

Because of the strong magnetic field, the dynamics of particles in the parallel direction $e_\parallel$ is completely different to their dynamics in the orthogonal plane. We therefore consider anisotropic characteristic spatial lengths in order to consider dimensionless quantities:

$$\tilde{x}_\perp = \frac{x_\perp}{L_\perp}, \quad \tilde{x}_\perp = \frac{x_\parallel}{L_\parallel},$$

$$\tilde{t} = \frac{t}{\tau} \quad \tilde{v} = \frac{v}{v_{th}},$$

$$f(t, x_\perp, x_\parallel, v) = \bar{f}\tilde{f}(\tilde{t}, \tilde{x}_\perp, \tilde{x}_\parallel, \tilde{v}) \quad V(t, x_\perp, x_\parallel) = \bar{V}\tilde{V}(\tilde{t}, \tilde{x}_\perp, \tilde{x}_\parallel) \quad E(t, x_\perp, x_\parallel) = \bar{E}\tilde{E}(\tilde{t}, \tilde{x}_\perp, \tilde{x}_\parallel).$$

This yields:

$$\begin{cases} \partial_{\tilde{t}}\tilde{f}_\varepsilon + \frac{v_{th}\tau}{L_\perp}\tilde{v}_\perp.\nabla_{\tilde{x}_\perp}\tilde{f}_\varepsilon + \frac{v_{th}\tau}{L_\parallel}\tilde{v}_\parallel.\nabla_{\tilde{x}_\parallel}\tilde{f}_\varepsilon + \left(\frac{e\bar{E}\tau}{mv_{th}}\tilde{E}_\varepsilon + \frac{e\bar{B}}{m}\tau\tilde{v}\wedge e_\parallel\right).\nabla_{\tilde{v}}\tilde{f}_\varepsilon = 0 \\ \frac{\bar{E}}{\bar{V}}\tilde{E}_\varepsilon = \left(-\frac{1}{L_\perp}\nabla_{\tilde{x}_\perp}\tilde{V}_\varepsilon, -\frac{1}{L_\parallel}\nabla_{\tilde{x}_\parallel}\tilde{V}_\varepsilon\right) \\ -\frac{\varepsilon_0\bar{V}}{L_\perp^2}\Delta_{\tilde{x}_\perp}\tilde{V}_\varepsilon - \frac{\varepsilon_0\bar{V}}{L_\parallel^2}\Delta_{\tilde{x}_\parallel}\tilde{V}_\varepsilon = e\bar{f}v_{th}^3\left(\int \tilde{f}_\varepsilon d\tilde{v} - 1\right) \\ \tilde{f}_{\varepsilon,|\tilde{t}=0} = \tilde{f}_{0,\varepsilon}, \quad \bar{f}L_\perp^2 L_\parallel v_{th}^3 \int \tilde{f}_{0,\varepsilon}d\tilde{v}d\tilde{x} = 1. \end{cases} \tag{7.1}$$

In order to keep normalization, it is first natural to set $\bar{f}L_\perp^2 L_\parallel v_{th}^3 = 1$.

We set now $\Omega = \frac{e\bar{B}}{m}$: this is the cyclotron frequency (also referred to as the gyrofrequency). We also consider the so-called electron Larmor radius (or electron gyroradius) r_L defined by:

$$r_L = \frac{v_{th}}{\Omega} = \frac{mv_{th}}{e\bar{B}} \tag{7.2}$$

This quantity can be physically understood as the typical radius of the helix around axis $e_\parallel$ described by the particles, due to the intense magnetic field.

We also introduce the so-called Debye length:

$$\lambda_D = \frac{\varepsilon_0\bar{V}}{e\bar{f}v_{th}^3},$$

which is interpreted as the typical length above which the plasma can be interpreted as being neutral.

The Vlasov equation now reads:

$$\partial_{\tilde{t}}\tilde{f}_\varepsilon + \frac{r_L}{L_\perp}\Omega\tau\tilde{v}_\perp.\nabla_{\tilde{x}_\perp}\tilde{f}_\varepsilon + \frac{r_L}{L_\parallel}\Omega\tau\tilde{v}_\parallel.\nabla_{\tilde{x}_\parallel}\tilde{f}_\varepsilon + \left(\frac{\bar{E}}{\bar{B}v_{th}}\Omega\tau\tilde{E}_\varepsilon + \Omega\tau\tilde{v}\wedge e_\parallel\right).\nabla_{\tilde{v}}\tilde{f}_\varepsilon = 0.$$

The strong magnetic field ordering consists in:

$$\Omega\tau = \frac{1}{\varepsilon}, \quad \frac{\bar{E}}{\bar{B}v_{th}} = \varepsilon,$$

with $\varepsilon > 0$ is a small parameter.

The spatial scaling we perform is the so-called finite Larmor radius scaling (see Frénod and Sonnendrucker [60] for a reference in the mathematical literature): basically the idea is to consider the typical perpendicular spatial length $L_\perp$ with the same order as the so-called electron Larmor radius. This allows to describe the turbulent behaviour of the plasma at fine scales, see [114]. On the contrary, the parallel observation length $L_\parallel$ is taken much larger:

$$\frac{r_L}{L_\perp} = 1, \quad \frac{r_L}{L_\parallel} = \varepsilon. \tag{7.3}$$

This is typically an anisotropic situation.

This particular scaling allows, at least in a formal sense, to observe more precise effects in the orthogonal plane than with the isotropic scaling (studied for instance in [72]):

$$\frac{r_L}{L_\perp} = \varepsilon, \quad \frac{r_L}{L_\parallel} = \varepsilon.$$

In particular we wish to observe the so-called electric drift $E^\perp$ (also referred to as the $E \times B$ drift) whose effect is of great concern in tokamak physics (see [92] for instance).

The quasineutral ordering we adopt is the following:

$$\frac{\lambda_D}{L_\parallel} = \sqrt{\varepsilon}. \tag{7.4}$$

After straightforward calculations (we refer to [60] for details), we get the following Vlasov-Poisson system in dimensionless form, for $t \geq 0, x = (x_\perp, x_\parallel) \in \mathbb{T}^2 \times \mathbb{T}, v = (v_\perp, v_\parallel) \in \mathbb{R}^2 \times \mathbb{R}$:

$$\begin{cases} \partial_t f_\varepsilon + \frac{v_\perp}{\varepsilon}.\nabla_x f_\varepsilon + v_\parallel.\nabla_x f_\varepsilon + (E_\varepsilon + \frac{v\wedge e_z}{\varepsilon}).\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = (-\nabla_{x_\perp}V_\varepsilon, -\varepsilon\nabla_{x_\parallel}V_\varepsilon) \\ -\varepsilon^2\Delta_{x_\parallel}V_\varepsilon - \Delta_{x_\perp}V_\varepsilon = \int f_\varepsilon dv - \int f_\varepsilon dv dx \\ f_{\varepsilon,t=0} = f_{\varepsilon,0}. \end{cases} \tag{7.5}$$

Remark 7.1. It seems physically relevant to consider scalings such as:

$$\lambda_D/L_\parallel \sim \varepsilon^\alpha, \tag{7.6}$$

with $\alpha \geq 1$. However with such a scaling, the systems seem too degenerate with respect to ε and we have not been able to handle this situation. The scaling we study is nevertheless relevant for some extreme magnetic regimes in tokamaks .

Hydrodynamic equations

In order to isolate this quasineutral problem, thanks to the linearity of the Poisson equation, we split the electric field into two parts:

$$\begin{cases} E = E^1_\varepsilon + E^2_\varepsilon, \\ E^1_\varepsilon = (-\nabla_{x_\perp} V_\varepsilon, -\varepsilon \nabla_{x_\parallel} V_\varepsilon), \\ -\varepsilon^2 \Delta_{x_\parallel} V^1_\varepsilon - \Delta_{x_\perp} V^1_\varepsilon = \int f_\varepsilon dv - \int f_\varepsilon dv dx_\perp, \\ E^2_\varepsilon = -\partial_{x_\parallel} V^2_\varepsilon, \\ -\varepsilon \Delta_{x_\parallel} V^2_\varepsilon = \int f_\varepsilon dv dx_\perp - \int f_\varepsilon dv dx. \end{cases} \tag{7.7}$$

In order to make the fast oscillations in time due to the singularly penalized operator $\frac{v_\perp}{\varepsilon}.\nabla_x$ disappear, we perform the same change of variables as in [64], to get the so-called gyro-coordinates:

$$x_g = x_\perp + v^\perp, v_g = v_\perp. \tag{7.8}$$

We easily compute the equation satisfied by the new distribution function $g_\varepsilon(t, x_g, v_g, v_\parallel) = f_\varepsilon(t, x, v)$.

$$\partial_t g_\varepsilon + v_\parallel \partial_{x_\parallel} g_\varepsilon + E^1_{\varepsilon,\parallel}(t, x_g - v_g^\perp)\partial_{v_\parallel} g_\varepsilon + E^2_\varepsilon(t, x_{g,\parallel})\partial_{v_\parallel} g_\varepsilon$$
$$+E^1_{\varepsilon,\perp}(t, x_g - v_g^\perp).(\nabla_{v_g} g_\varepsilon - \nabla^\perp_{x_g} g_\varepsilon) + \frac{1}{\varepsilon} v_g^\perp.\nabla_{v_g} g_\varepsilon = 0.$$

Notice here that in the process, the so-called electric drift $E^\perp$ appears since:

$$-E^1_{\varepsilon,\perp}(t, x_g - v_g^\perp).\nabla^\perp_{x_g} g_\varepsilon = E^{1,\perp}_\varepsilon(t, x_g - v_g^\perp).\nabla_{x_g} g_\varepsilon.$$

The equation satisfied by the charge density $\rho_\varepsilon = \int g_\varepsilon dv$ states:

$$\partial_t \rho_\varepsilon + \partial_{x_\parallel} \int v_\parallel g_\varepsilon dv + \nabla^\perp_{x_g} \int E^1_{\varepsilon,\perp}(t, x_g - v_g^\perp) g_\varepsilon dv = 0, \tag{7.9}$$

One can observe that since $E^1_{\varepsilon,\perp}$ is a gradient:

$$\mathrm{div}_{v_g} E^1_{\varepsilon,\perp}(t, x_g - v_g^\perp) = 0.$$

Thus, integrating the equation satisfied by g_ε against $(v_g, v_\parallel)$, we deduce that the one satisfied by the current density $J_\varepsilon = \int g_\varepsilon v dv \left(= \begin{pmatrix} \int g_\varepsilon v_\perp dv \\ \int g_\varepsilon v_\parallel dv \end{pmatrix} \right)$ is the following:

$$\partial_t J_\varepsilon + \partial_{x_\parallel} \int v_\parallel \begin{pmatrix} v_g \\ v_\parallel \end{pmatrix} g_\varepsilon dv + \nabla^\perp_{x_g} \int E^1_{\varepsilon,\perp}(t, x_g - v_g^\perp) \begin{pmatrix} v_g \\ v_\parallel \end{pmatrix} g_\varepsilon dv$$
$$= \int \begin{pmatrix} E^1_{\varepsilon,\perp}(t, x_g - v_g^\perp) \\ 0 \end{pmatrix} g_\varepsilon dv + \int \begin{pmatrix} 0 \\ E^1_{\varepsilon,\parallel}(t, x_g - v_g^\perp) \end{pmatrix} g_\varepsilon dv$$
$$+ \begin{pmatrix} 0 \\ E^2_\varepsilon(t, x_{g,\parallel})\rho_\varepsilon \end{pmatrix} + \frac{J^\perp_\varepsilon}{\varepsilon}. \tag{7.10}$$

We now assume that we deal with special monokinetic data of the form:

$$g_\varepsilon(t, x, v) = \rho_\varepsilon(t, x)\delta_{v_\parallel = v_{\parallel,\varepsilon}(t,x)}\delta_{v_g=0}. \tag{7.11}$$

This assumption is nothing but the classical "cold plasma" approximation together with the assumption that the transverse particle velocities are isotropically distributed (which is physically relevant, see [150]) : in other words, the average motion of particles in the perpendicular plane is only due to the advection by the electric drift $E^{\perp}$.

For the sake of readability, we denote by now $\nabla_{x_g} = \nabla_{\perp}$ and $\nabla_{x_{\parallel}} = \nabla_{\parallel}$. Note in particular that with these monokinetic data, we have in particular $J_{\varepsilon}^{\perp} = 0$. Then we get formally the hydrodynamic model:

$$\begin{cases} \partial_t \rho_\varepsilon + \nabla_\perp (E_\varepsilon^\perp \rho_\varepsilon) + \partial_\parallel (v_{\parallel,\varepsilon} \rho_\varepsilon) = 0 \\ \partial_t (\rho_\varepsilon v_{\parallel,\varepsilon}) + \nabla_\perp (E_\varepsilon^\perp \rho_\varepsilon v_{\parallel,\varepsilon}) + \partial_\parallel (\rho_\varepsilon v_{\parallel,\varepsilon}^2) = -\varepsilon \partial_\parallel \phi_\varepsilon(t,x) \rho_\varepsilon - \partial_\parallel V_\varepsilon(t, x_\parallel) \rho_\varepsilon \\ E_\varepsilon^\perp = -\nabla^\perp \phi_\varepsilon \\ -\varepsilon^2 \partial_\parallel^2 \phi_\varepsilon - \Delta_\perp \phi_\varepsilon = \rho_\varepsilon - \int \rho_\varepsilon dx_\perp \\ -\varepsilon \partial_\parallel^2 V_\varepsilon = \int \rho_\varepsilon dx_\perp - 1 \end{cases} \tag{7.12}$$

One can use the first equation to simplify the second one (the systems are equivalent provided that we work with regular solutions and that $\rho_\varepsilon > 0$):

$$\begin{cases} \partial_t \rho_\varepsilon + \nabla_\perp (E_\varepsilon^\perp \rho_\varepsilon) + \partial_\parallel (v_{\parallel,\varepsilon} \rho_\varepsilon) = 0 \\ \partial_t v_{\parallel,\varepsilon} + \nabla_\perp (E_\varepsilon^\perp v_{\parallel,\varepsilon}) + v_{\parallel,\varepsilon} \partial_\parallel (v_{\parallel,\varepsilon}) = -\varepsilon \partial_\parallel \phi_\varepsilon(t,x) - \partial_\parallel V_\varepsilon(t, x_\parallel) \\ E_\varepsilon^\perp = -\nabla^\perp \phi_\varepsilon \\ -\varepsilon^2 \partial_\parallel^2 \phi_\varepsilon - \Delta_\perp \phi_\varepsilon = \rho_\varepsilon - \int \rho_\varepsilon dx_\perp \\ -\varepsilon \partial_\parallel^2 V_\varepsilon = \int \rho_\varepsilon dx_\perp - 1. \end{cases} \tag{7.13}$$

Remarks 7.1. i. Notice here that we do not deal with the usual charge density and current density, since these ones are taken within the gyro-coordinates.

ii. We mention that we could have considered the more general case:

$$g_\varepsilon(t,x,v) = \int_M \rho_\varepsilon^\Theta(t,x) \delta_{v_\parallel = v_{\parallel,\varepsilon}^\Theta(t,x)} \nu(d\Theta) \delta_{v_g = 0} \tag{7.14}$$

where (M, Θ, ν) is a probability space which allows to model more realistic plasmas than "cold plasmas" and covers many interesting physical data, like multi-sheet electrons or water-bags data (we refer for instance to [13] and references therein). We will not do so for the sake of readability but we could deal with it with exactly the same analytic framework: the analogues of Theorems 2.1 and 2.2 identically hold. We get in the end the system:

$$\begin{cases} \partial_t \rho^\Theta + \nabla_\perp (E^\perp \rho^\Theta) + \partial_\parallel (v_\parallel^\Theta \rho^\Theta) = 0 \\ \partial_t v_\parallel^\Theta + \nabla_\perp (E^\perp v_\parallel^\Theta) + v_\parallel^\Theta \partial_\parallel (v_\parallel^\Theta) = -\partial_\parallel p(t, x_\parallel) \\ E^\perp = \nabla^\perp \Delta_\perp^{-1} \left(\int \rho^\Theta d\nu - \int \rho^\Theta dx_\perp d\nu \right) \\ \int \rho^\Theta(t,x) dx_\perp d\nu = 1. \end{cases} \tag{7.15}$$

As before, the equations are coupled through $x_\perp$ and here also through the new parameter Θ.

iii. Actually, the choice:

$$g_\varepsilon(t,x,v) = \rho_\varepsilon(t,x) \delta_{v = v_\varepsilon(t,x)} \tag{7.16}$$

leads to an ill-posed system. Indeed, we have to solve in this case equations of the form $v_\varepsilon^\perp = v_{\varepsilon,\perp}(t, x - v_\varepsilon^\perp)$ where $v_{\varepsilon,\perp}$ is the unknown. We can not say if this relation is invertible, even locally.

Chapter 5

Effect of the Polarization drift in a strongly magnetized plasma

Article soumis.

Sommaire

Résumé : We consider a strongly magnetized plasma described by a Vlasov-Poisson system with a large external magnetic field. The finite Larmor radius scaling allows to describe its behaviour at very fine scales. We give a new interpretation of the asymptotic equations obtained by Frénod and Sonnendrücker in [60], when the intensity of the magnetic field goes to infinity. We introduce the so-called polarization drift and show that its contribution is not negligible in the limit, contrary to what is usually said. This is due to the non linear coupling between the Vlasov and Poisson equations.

1 Introduction

1.1 The finite Larmor radius scaling for the Vlasov-Poisson equation

Consider a hot plasma made of negatively charged particles (electrons) and positively charged particles (ions). For simplicity, we make the hypothesis that the plasma is made of only one species of ions. We assume that the temperature is so high that collisions can be neglected. Then, the motion of a charged particle (of mass m and charge q) is described by the Newton equations:

$$\begin{cases} \frac{dX}{dt} = V, \\ \frac{dV}{dt} = \frac{q}{m}\left(E(t,x) + V \wedge B(t,x)\right), \end{cases} \tag{1.1}$$

where X denotes the position and V the velocity of the particle. The fields E and B are respectively the electric and magnetic fields, which are created collectively by the charged particles themselves. Let T and L be the characteristic time and length of observation. Let c be the speed of light. We assume that:

$$\frac{L}{Tc} \gg 1,$$

in which case the electrostatic approximation is relevant: this means that we can consider a Poisson equation instead of the full Maxwell equations, in order to compute the electric field created by the plasma, and assume that the magnetic field is an exterior stationary field that satisfies the divergence free condition:

$$\mathrm{div}_x B = 0$$

For simplicity we restrict to the most simple geometric case where the magnetic field has constant direction and modulus. An orthonormal basis (e_1, e_2, e_3) of $\mathbb{R}^3$ being fixed, we set:

$$B(x) = \overline{B} e_3,$$

with $\overline{B}$ constant and uniform. For any vector $A = (A_1, A_2, A_3)$, we denote $A^\perp = (A_2, -A_1, 0)$ and therefore we can write $v \wedge B(x) = |B| v^\perp$.

In this work, in order to describe the plasma, we adopt a kinetic point of view, which means that we do not solve Newton equations for all the particles but rather give a statistical description of their motion. For simplicity, we assume that the particles evolve in the periodic torus $\mathbb{T}^3$, which allows to confine them without having to deal with boundary effects. Given one type of charged particles, we introduce the so-called distribution function $f(t,x,v)$, which describes their statistical repartition in the phase space (x,v). The quantity $f(t,x,v)\, dx\, dv$ is interpreted as the probability of finding an electron at time $t \in \mathbb{R}^+$ at position $x \in \mathbb{T}^3$ and velocity $v \in \mathbb{R}^3$. Then transport is described by the following Vlasov equation:

$$\begin{cases} \partial_t f + v.\nabla_x f + (E + v \wedge B).\nabla_v f = 0, \\ f_{t=0} = f_0, \end{cases} \tag{1.2}$$

with $E = -\nabla_x V$ and the potential V is computed with a Poisson equation (whose precise form depends on the nature of the particles we have chosen to describe; we will come back to this point soon).

We are particularly interested in strongly magnetized plasmas, that are submitted to an intense magnetic field. Such plasmas are nowadays intensively studied in view of energy production by fusion (we refer to [158] and references therein).

In this chapter, we discuss and revisit the finite Larmor radius scaling for the Vlasov-Poisson equation with a strong external magnetic field. This scaling was introduced and first studied in the mathematical literature by Frénod and Sonnendrücker [60].

It is well-known that with a large magnetic field, the typical motion of a charged particle is an helix whose axis has the same direction as the magnetic field. The radius of the helix is usually called the Larmor radius; it vanishes as the intensity of the magnetic field goes to infinity. We refer to Section 2 for some detailed computations. The finite Larmor radius scaling is an anisotropic scaling whose basic principle is to distinguish between the typical observation length in the plane (e_1, e_2) (that is orthogonal to the magnetic field) and the typical observation length in the line (e_3) (that is parallel to the magnetic field). The idea is then to consider a perpendicular observation length with the same order as the Larmor radius. This allows to give a description of the behaviour of the plasma at fine scales. This is physically relevant, since it was observed experimentally that the plasma undergoes a "turbulent" behaviour at such scales [114]. Let us finally mention that recently, this regime was investigated in [16, 64, 94, 93, 55]. The scaled Vlasov equation reads[1]:

$$\begin{cases} \partial_t f_\varepsilon + \frac{v_\perp}{\varepsilon}.\nabla_{x_\perp} f_\varepsilon + v_\parallel \partial_{x_\parallel} f_\varepsilon + (E_\varepsilon + \frac{v^\perp}{\varepsilon}).\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -(\nabla_{x_\perp} V_\varepsilon, \varepsilon \partial_{x_\parallel} V_\varepsilon) \\ f_{\varepsilon,t=0} = f_{0,\varepsilon}. \end{cases} \tag{1.3}$$

In this system, ε is a small positive parameter (roughly, $\varepsilon \sim \frac{1}{B}$) that we intend to let go to zero, in order to obtain a somehow simplified asymptotic equation.

As mentioned before, this system is coupled with a Poisson equation, which depends on the nature of the charged particles (ions or electrons) that we intend to describe. The main point is to observe that the mass of an ion is much larger than the mass of an electron, so that their dynamics is completely different.

- If we describe the electrons by the distribution function f_ε, we can consider that their mass is of order 1 and that the mass of the ions is equal to $+\infty$. Thus we can assume that the ions are motionless, and for simplicity we assume that their density n_i is uniform, equal to 1. As a result, the Poisson equation in this case reads:

$$-\varepsilon^2 \partial_{x_\parallel}^2 V_\varepsilon - \Delta_{x_\perp} V_\varepsilon = \int f_\varepsilon dv - 1. \tag{1.4}$$

- If we describe the ions by the distribution function f_ε, we can consider that their mass is of order 1 and that the mass of the electrons is equal to 0. Therefore, the electrons instantaneously reach their thermodynamic equilibrium. It is usually assumed that their density n_e follows a Maxwell-Boltzmann law, that is:

$$n_e = e^{V_\varepsilon}.$$

[1] We refer to Appendix 4 for some details on the scaling.

We make the simplifying assumption (commonly done in plasma physics) that this law can be linearized, so that:

$$n_e = 1 + V_\varepsilon.$$

Thus the Poisson equation reads in this case:

$$V_\varepsilon - \varepsilon^2 \partial^2_{x_\parallel} V_\varepsilon - \Delta_{x_\perp} V_\varepsilon = \int f_\varepsilon dv - 1. \tag{1.5}$$

Although (1.5) may seem to be a harmless modification of (1.4), the additional term V_ε actually makes a huge difference in the analysis of the full three-dimensional problem, as it has been discussed in [94, 93].

In this chapter, we restrict the so-called $2D$ problem, which means that we restrict to initial data which do not depend on $x_\parallel$ and $v_\parallel$. Actually in the $2D$ setting, considering (1.5) or (1.4) does not make any difference, and thus for simplicity we will describe a gas of electrons and consider the $2D$ version of (1.4). Then, the scaled Vlasov-Poisson system reads:

$$\begin{cases} \partial_t f_\varepsilon + \frac{v}{\varepsilon}.\nabla_x f_\varepsilon + (E_\varepsilon + \frac{v^\perp}{\varepsilon}).\nabla_v f_\varepsilon = 0, \quad t \geq 0, x \in \mathbb{T}^2, v \in \mathbb{R}^2, \\ E_\varepsilon = -\nabla_{x_\perp} V_\varepsilon, \\ -\Delta_x V_\varepsilon = \int f_\varepsilon dv - 1, \\ f_{\varepsilon,t=0} = f_{0,\varepsilon}. \end{cases} \tag{1.6}$$

This is the system we will focus on throughout this chapter.

1.2 Short review of the $2D$ problem and presentation of the issue

In [60], Frénod et Sonnendrücker showed the following result :

Theorem 1.1 (Frénod-Sonnendrücker)**.** *Let $(f_{0,\varepsilon})$ a sequence of initial data satisfying the following conditions:*

(H1) For any ε, $f_{0,\varepsilon} \geq 0$ and $\int f_{0,\varepsilon}\, dv\, dx = 1$.

(H2) There exists $C > 0$, for any ε, $\int f_{0,\varepsilon} |v|^2\, dv\, dx \leq C$.

(H3) There exists $q > 2$, such that $(f_{0,\varepsilon})$ is uniformly bounded in $L^q(\mathbb{T}^2 \times \mathbb{R}^2)$.

We assume that $(f_{\varepsilon,0})$ weakly converges in $L^2_{x,v}$ to some $f_0 \in L^1_{x,v} \cap L^q_{x,v}$. Let (f_ε) the global weak solutions in the sense of Arsenev [4] of (1.6) with initial conditions $(f_{0,\varepsilon})$. Then, up to a subsequence, (f_ε) converges in the sense of distributions to f defined by:

$$f = \int_0^{2\pi} G(t, x + \mathcal{R}(\tau) v, R(\tau) v) d\tau \tag{1.7}$$

with G solution to the equation :

$$\begin{cases} \partial_t G + \frac{1}{2\pi} \left(\int_0^{2\pi} \mathcal{R}(\tau) \mathcal{E}(t, \tau, x + \mathcal{R}(-\tau) v) d\tau \right).\nabla_x G \\ + \frac{1}{2\pi} \left(\int_0^{2\pi} R(\tau) \mathcal{E}(t, \tau, x + \mathcal{R}(-\tau) v) d\tau \right).\nabla_v G = 0 \\ \mathcal{E} = -\nabla \Phi, \quad -\Delta \Phi = \int G(t, x + \mathcal{R}(\tau) v, R(\tau) v) dv - 1, \\ G_{|t=0} = f_0, \end{cases} \tag{1.8}$$

denoting by R and $\mathcal{R}$ the rotation operators defined by

$$R(\tau) = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix}, \mathcal{R}(\tau) = \left(R(\pi/2) - R(\pi/2 + \tau) \right). \tag{1.9}$$

One can observe that it is possible to write equation (1.8) in a (slightly) more tractable way, by using the change of variables:

$$\tilde{G}(t,\tau,x,v) := G(t,\tau,x-v^\perp,v). \tag{1.10}$$

The equation satisfied by $\tilde{G}$ then reads:

$$\begin{cases} \partial_t \tilde{G} + \frac{1}{2\pi}\left(\int_0^{2\pi}\mathcal{E}^\perp(t,\tau,x-R(\pi/2+\tau)v)d\tau\right).\nabla_x\tilde{G} \\ +\frac{1}{2\pi}\left(\int_0^{2\pi}R(\tau)\mathcal{E}(t,\tau,x-R(\pi/2+\tau)v)d\tau\right).\nabla_v\tilde{G}=0 \\ \mathcal{E}=-\nabla\Phi, \quad -\Delta\Phi=\int\tilde{G}(t,x-R(\pi/2+\tau)v,R(\tau)v)dv-1, \\ \tilde{G}_{|t=0}=f_0(x-v^\perp,v), \end{cases} \tag{1.11}$$

We can observe that the term:

$$\frac{1}{2\pi}\left(\int_0^{2\pi}\mathcal{E}^\perp(t,\tau,x-R(-\tau-\pi/2)v)d\tau\right).\nabla_x \tag{1.12}$$

corresponds the classical electric drift (we refer to Section 2 for some physical insight about the drifts arising in presence of a strong magnetic field):

$$v_E=\frac{E\wedge B}{|B|}=E^\perp,$$

in an averaged form (usually called *gyro-averaged*). This corresponds to an average around a circle of rotation in the orthogonal plane. For any function $F(x)$, we call gyroaverage of F the function:

$$< F > (x,v)=\frac{1}{2\pi}\left(\int_0^{2\pi}F(x-R(-\tau-\pi/2)v)d\tau\right).$$

That the gyroaveraged electric drift appears instead of its "standard" form is related to the finite Larmor radius scaling: any variation within a length of a few Larmor radius is not negligible, and thus this average is natural.

With such an approach, we observe that there is another term, which has a priori no physical meaning:

$$\frac{1}{2\pi}\left(\int_0^{2\pi}R(\tau)\mathcal{E}(t,\tau,x-R(-\tau-\pi/2)v)d\tau\right).\nabla_v\tilde{G}. \tag{1.13}$$

The questions we have in mind are the following: what is the physical meaning of this term? Is it possible to make it vanish?

It is interesting to see that the equation used by plasma physicists for numerical simulations, see for instance the GYSELA code ([78]), is a simplified version of this equation, the term (1.13) being neglected (actually there are other drifts, due to geometric effects, but those are not considered here).

For convenience, in this chapter, we will denote $e^{i\theta}$ for the vector $\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$.

$$\begin{cases} \partial_t f+\frac{1}{2\pi}\int_0^{2\pi}E^\perp(t,x-|v|e^{i\theta+i\pi/2})d\theta.\nabla_x f=0, \\ E=-\nabla_x V, \\ -\Delta_x V=\int f(t,x-v^\perp,v)dv-\int f dvdx. \end{cases} \tag{1.14}$$

Lately there has been some effort to obtain equation (1.14) from (1.6): we refer to the works of Bostan [16], Ghendrih, Hauray and Nouri [64] and Frénod and Mouton [55].

In [16, Theorem 5.1], under some well-prepared assumptions on the initial data, the author is able to derive (1.14) from (1.6):

Theorem 1.2 (Bostan)**.** *Let $(f_{0,\varepsilon})$ a sequence of initial conditions satisfying the following conditions:*

(H1) For any ε, $f_{0,\varepsilon} \geq 0$ and $\int f_{0,\varepsilon}\, dv\, dx = 1$.

(H2) There exists a bounded nonincreasing function $F_0 \in L^\infty(\mathbb{R}^+) \cap L^(\mathbb{R}^+, rdr)$ such that:

$$f_0^\varepsilon(x,p) \leq F_0(|p|).$$

(H3) There exists $q > 2$, such that $(f_{0,\varepsilon})$ is uniformly bounded in $W^{2,q}(\mathbb{T}^2 \times \mathbb{R}^2)$.

(H4) There exists a subsequence (ε_k) such that $\varepsilon_k \to 0$ and a function f_0 in $W^{2,q}(\mathbb{T}^2 \times \mathbb{R}^2)$, compactly supported in velocity, and satisfying:

$$v.\nabla_x f_0 + v^\perp f_0 = 0, \tag{1.15}$$

such that

$$f_{0,\varepsilon_k} \to f_0,$$

strongly in $L^2(\mathbb{T}^2 \times \mathbb{R}^2)$.

Let (f_{ε_k}) the global weak solutions in the sense of Arsenev [4] of (1.6) with initial conditions (f_{0,ε_k}). Then there exists $T > 0$ such that (f_{ε_k}) converges strongly in $L^\infty(]0,T[, L^2(\mathbb{T}^2 \times \mathbb{R}^2)$ to f solution of (1.14) with initial data f_0.

We can compare the assumptions of the two previous theorems. The assumptions (H2) and (H3) in Theorem 1.2 are more restrictive than those of Theorem 1.1, the main reason is that one needs much more control on high velocities and also some stronger stability estimates for proving Theorem 1.2. Actually, the most restricting condition corresponds to the assumption (H4): the condition (1.15) means that f_0 belongs to the kernel of the singular penalization operator $v.\nabla_x + v^\perp.\nabla_v$. This a well-prepared assumption. Unfortunately, in "real life" plasmas, this does not seem relevant, since it is in practice impossible to impose it or even to check that it is satisfied. By opposition, somehow, Theorem 1.1 therefore corresponds to an ill-prepared situation, since there is not such assumption on the initial data.

In [55], the authors try to derive (1.14) from (1.6) in such an ill-prepared situation. Unfortunately, the goal is reached only by assuming the additional assumption (called "unphysical" in [55]) that $E_\varepsilon \to E$ strongly in time and space.

As a matter of fact, as we shall see afterwards, the electric field E_ε displays oscillations in time of frequency of order $\mathcal{O}(1/\varepsilon)$. This is due to the nonlinear coupling between the Vlasov and Poisson equations. This means that in general, E_ε never converges strongly, but only weakly. It is well known that weak convergence behaves nastily with respect to multiplication: resonance effects may appear and produce a non-trivial mean transport. As a result, the formal analysis may be false.

In this chapter, we consider "general" initial data which do not satisfy the well-prepared assumption. The approach that we carry out allow us to provide a clarification and a better understanding to the gyrokinetic equation (1.8), in particular to the terms (1.13). Our goal is to give evidence that (1.8) actually describes more effects than the mere electric drift, which is the usual interpretation. More specifically, we will exhibit effects due to the so-called polarization drift, which is usually considered as a higher order term and thus neglected, in the physics literature. The contribution is due precisely to the fast oscillations in time of the electric field.

This means that if one wants to give an accurate numerical simulation of strongly magnetized plasmas (for instance for industrial purposes), system (1.8) is more relevant than system (1.14).

1.3 Organization of the chapter

The chapter is organized as follows: in Section 2, we first give some physical heuristics in order to understand the expected dynamics for the charged particles. In Section 3, we give and we prove the main results of this chapter; our approach is based on a simple non linear change of frame, inspired by the heuristics, which allows to interpret the mysterious terms (1.13) as the effect of the polarization drift (we refer in particular to Theorem 3.3). Finally, we gather in two appendices some elements on the finite Larmor radius scaling and we propose a new physical model that may be more accurate than (1.14).

2 The electric drift, and the polarization drift

Let us qualitatively describe the dynamics of a charged particle (of mass m and charge q, assumed to be positive for simplicity), under the influence of a given electric field E and a given magnetic field B. As before, we restrict to the simplest geometric case: B is uniform and constant. We focus on the dynamics in the perpendicular (to the magnetic field) plane. The main feature of the magnetic field is that it is assumed to be large, so that $|B| \sim \frac{1}{\varepsilon}$, with $\varepsilon \ll 1$ (while $|E| \sim 1$). We now perform a multi-scale expansion of the solutions to the Newton equations, that we recall here:

$$\begin{cases} \frac{dX}{dt} = V, \\ m\frac{dV}{dt} = q\left(E + V \wedge B\right). \end{cases} \tag{2.1}$$

At leading order, the motion is dominated by the influence of the magnetic field and we can neglect the electric field. The equation satisfied by V is then $\frac{dV}{dt} = qV \wedge B$. By some straightforward and elementary computations, we can easily obtain that the motion is circular:

$$V = R(t\Omega_c)V^0, \quad X = X^0 + r_L \mathcal{R}(t\Omega_c)\frac{V^0}{|V^0|},$$

with R and $\mathcal{R}$ the rotation operators defined in (1.9) and we denote by r_L the so-called Larmor radius defined by:

$$r_L = \frac{q|V^0|}{m|B|},$$

and Ω_c the cyclotron frequency:

$$\Omega_c = \frac{q|B|}{m}.$$

As explained in the introduction, the finite Larmor radius (FLR) scaling consists precisely of considering a typical space length with the same order as r_L, so that at this scale, neglecting the electric field is not relevant, since higher order effets are no more negligible. This is the matter of the next paragraphs.

Let us first suppose that the electric field E does not depend on time. We reconsider the equation:

$$m\frac{dV}{dt} = qE + qV \wedge B$$

and set $\tilde{V} = V - \frac{E \wedge B}{|B|^2}$. Then, the velocity field $\tilde{V}$ satisfies the equation:

$$m\frac{d\tilde{V}}{dt} = q\tilde{V} \wedge B.$$

We thus observe that $\tilde{V}$ thus satisfies the same equation as (2.1) but without electric field. As a consequence, we have:

$$V_\perp = \frac{E \wedge B}{|B|^2} + R(t\Omega_c)\tilde{V}_\perp^0.$$

The motion of the particle is then given by:

$$X = X^0 + t\frac{E \wedge B}{|B|^2} + \frac{q}{m|B|}\mathcal{R}(t\Omega_c)\left(V_\perp^0 - \frac{E \wedge B}{|B|^2}\right).$$

We call "electric drift" (also called $E \times B$-drift) the drift:

$$v_E = \frac{E \wedge B}{|B|^2}.$$

We observe that $|v_E| \sim \varepsilon$, and consequently, in the FLR regime, this becomes an effect of order $\mathcal{O}(1)$ (in other words this a leading order term in this regime).

Suppose now that the electric field depends on time (which is the case in most situations). In this case, the equation satisfied by $\tilde{V}$ is the following :

$$\frac{d\tilde{V}}{dt} = \tilde{V} \wedge B - \frac{1}{|B|^2}\frac{dE}{dt} \wedge B.$$

We can set, as in the previous case, $\bar{V} = \tilde{V} + \frac{1}{|B|^2}\frac{dE_\perp}{dt}$. Then the equation satisfied by $\bar{V}$ is the following one:

$$\frac{d\tilde{V}}{dt} = q\tilde{V} \wedge B - \frac{1}{|B|^2}\frac{d^2E_\perp}{dt^2}.$$

We neglect the term $-\frac{1}{|B|^2}\frac{d^2E}{dt^2}$: indeed, assuming that $|\frac{d^2E}{dt^2}|$ is of order 1, this is seemingly a higher order term in ε, which gives rise to higher order terms.

If we proceed as in the previous case, this results in another drift, called "polarization drift":

$$v_p = \frac{1}{|B|^2}\frac{dE}{dt}. \tag{2.2}$$

We can notice that if $|\frac{dE}{dt}|$ is of order 1, then we have $|v_p| \sim \varepsilon^2$. With an observation length of order ε, this is consequently a term of order $\mathcal{O}(\varepsilon)$. As a result, one usually considers that the polarization drift is indeed a higher order term than the electric drift, so it shouldn't have any influence on the asymptotic equation. As first explained in the introduction, we point out that this is likely to be a wrong belief. Indeed, due to some resonance phenomena, there are oscillations in time for the electric field, which entails that $\varepsilon\partial_t E_\varepsilon$ is actually of order one. Consequently, the scaling analysis is likely to be false and the electric and polarization drifts can be of the same order.

Even worse, the resonance phenomena entail that $-\frac{1}{|B|^2}\frac{d^2E}{dt^2}$ is actually not a higher order term, so it shouldn't be neglected in order to describe the motion.

3 Effects of the polarization drift in the finite Larmor radius approximation

3.1 A dynamical change of frame

Our goal is now to rigorously justify (in the framework of the kinetic Vlasov-Poisson system) the discussion of the previous section.

Recall that we a priori expect to obtain a drift $\varepsilon E_\varepsilon^\perp$. Thus, we consider the new distribution function f'_ε defined by the non linear change of frame:

$$f'_\varepsilon(t,x,v) = f_\varepsilon(t,x,v+\varepsilon E_\varepsilon^\perp), \tag{3.1}$$

where f_ε satisfies (1.6). We straightforwardly obtain the equation satisfied by f'_ε:

$$\begin{aligned}\partial_t f'_\varepsilon + &\frac{v}{\varepsilon}.\nabla_x f'_\varepsilon + E_\varepsilon^\perp.\nabla_x f'_\varepsilon \\ &+\Big(\underbrace{-\varepsilon\partial_t E_\varepsilon^\perp - \frac{v+\varepsilon E_\varepsilon^\perp}{\varepsilon}.\nabla_x(\varepsilon E_\varepsilon^\perp)}_{:=F_\varepsilon}\Big).\nabla_v f'_\varepsilon + \frac{v^\perp}{\varepsilon}.\nabla_v f'_\varepsilon = 0.\end{aligned} \tag{3.2}$$

We call the force F_ε the "polarization force".

Let us now give two approaches (based on filtering techniques) which will help us to let appear the influence of the polarization drift.

i. The first is the approach used in [64]. This one is somehow the easiest, and makes both the electric and the polarization drift in a very clear and explicit way.

   Unfortunately, one can not handle the full Vlasov-Poisson system with this approach, since the oscillations due to magnetic field will remain after the change of variables. Consequently there is a lack of compactness in time if we try to pass to the limit. Nevertheless, this approach will work when the electric field is a chosen external force field.

   The change of variables consists of:

$$g_\varepsilon(t,x,v) = f'_\varepsilon(t,x-v^\perp,v).$$

   We then obtain the equation satisfied by g_ε:

$$\begin{aligned}\partial_t g_\varepsilon + &\underbrace{E_\varepsilon^\perp(t,x-v^\perp)}_{\text{electric drift}}.\nabla_x g_\varepsilon + \underbrace{F_\varepsilon^\perp(t,x-v^\perp,v)}_{\text{polarization drift}}.\nabla_x g_\varepsilon \\ &+F_\varepsilon(t,x-v^\perp,v).\nabla_v g_\varepsilon + \frac{v^\perp}{\varepsilon}.\nabla_v g_\varepsilon = 0.\end{aligned} \tag{3.3}$$

   Observe here that $F_\varepsilon^\perp$ indeed corresponds to the polarization drift defined in (2.2). More explicitly, we have:

$$F_\varepsilon^\perp = \varepsilon\partial_t E_\varepsilon + v.\nabla_x E_\varepsilon + \varepsilon E_\varepsilon^\perp.\nabla_x E_\varepsilon. \tag{3.4}$$

ii. The second approach is that of [60], that we have evoked in the first section of the chapter. It consists of filtering out all the fast time oscillations. We set h_ε defined by:

$$h_\varepsilon(t,x,v) = f'_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v), \tag{3.5}$$

   where R and $\mathcal{R}$ are the rotation operators defined in (1.9).

   After some straightforward computations we get:

$$\begin{aligned}\partial_t h_\varepsilon + &E_\varepsilon^\perp(t,x+\mathcal{R}(-t/\varepsilon)v).\nabla_x h_\varepsilon \\ &+\mathcal{R}(t/\varepsilon)F_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v).\nabla_x h_\varepsilon \\ &+R(t/\varepsilon)F_\varepsilon(t,x+\mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v).\nabla_v h_\varepsilon = 0.\end{aligned} \tag{3.6}$$

3.2 External electric field case

Let us consider the simple case when $E_\varepsilon := E = -\nabla_x V$ is an external (smooth) electric field, in $\mathcal{C}^1_{t,x} \cap H^1_{t,x,loc}$. In this case, since the electric field does not oscillate in time with a frequency of order $\frac{1}{\varepsilon}$, we do not expect to observe any polarization effect in the limit.

Actually, we recover the same results[2] as in [64]. It is nevertheless interesting to explain how the contribution of the polarization drift vanishes in our framework.

Theorem 3.1. *Let $(g_{\varepsilon,0})$ a sequence of initial data, uniformly bounded in $L^1_{x,v} \cap L^2_{x,v}$. We assume that $(g_{\varepsilon,0})$ weakly converges in $L^2_{x,v}$ to some $g_0 \in L^1_{x,v} \cap L^2_{x,v}$. Then up to a subsequence, the sequence (g_ε) of solutions to (3.3) with initial data $(g_{\varepsilon,0})$ converges in the sense of distributions to g solution to:*

$$\begin{cases} \partial_t g + \frac{1}{2\pi}\int_0^{2\pi} E^\perp(t, x - |v|e^{i\theta + i\pi/2})d\theta . \nabla_x g = 0, \\ g_{|t=0} = \frac{1}{2\pi}\int_0^{2\pi} g_0(x, |v|e^{i\theta})d\theta. \end{cases} \tag{3.7}$$

We recall that we use the notation $e^{i\theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$.

Remark 3.1. Until now, we have restricted to the 2D setting, but in this case the 3D results would easily follow. Indeed, in the three-dimensional setting we obtain for the polarization force:

$$F_\varepsilon = -\varepsilon\partial_t E^\perp - v_\perp.\nabla_{x_\perp} E^\perp - \varepsilon E^\perp.\nabla_{x_\perp} E^\perp - \varepsilon v_\| \partial_{x_\|} E^\perp. \tag{3.8}$$

The only additional term is the last one, and since E is external it clearly vanishes in the limit. That being said, we restrict to the $2D$ case for the sake of readability.

Before proving the theorem, we state a technical lemma which explains that due to symmetry reasons, the contributions appearing because of polarization cancel in the limit.

Lemma 3.1. *For any $r \geq 0$, we have the cancellations:*

$$\int_0^{2\pi} e^{i\theta}.\nabla_x E(x - re^{i\theta+i\pi/2})d\theta = 0,$$

$$\int_0^{2\pi} e^{i\theta}.\nabla_x E^\perp(x - re^{i\theta+i\pi/2}).e^{i\theta}d\theta = 0.$$

Proof of Lemma 3.1. We observe that:

$$\int_0^{2\pi} e^{i\theta}.\nabla_x E(x - re^{i\theta+i\pi/2}).d\theta = \int_0^{2\pi} \partial_\theta[E(x - re^{i\theta+i\pi/2})]d\theta = 0.$$

$$\begin{aligned} \int_0^{2\pi} e^{i\theta}.\nabla_x E^\perp(x - re^{i\theta+i\pi/2}).e^{i\theta}d\theta &= \int_0^{2\pi} \partial_\theta[E^\perp(x - re^{i\theta+i\pi/2})].e^{i\theta}d\theta \\ &= -\int_0^{2\pi} [E^\perp(x - re^{i\theta+i\pi/2})].\partial_\theta e^{i\theta}d\theta \\ &= \int_0^{2\pi} \sin\theta E_2(x - re^{i\theta+i\pi/2}) + \cos\theta E_1(x - re^{i\theta+i\pi/2})d\theta \\ &= \int_0^{2\pi} \partial_\theta[V(x - re^{i\theta+i\pi/2})]d\theta = 0. \end{aligned}$$

□

[2]However, we need a little more regularity on E in order to give sense to the terms of the polarization force and drift. Nevertheless, in our approach, all terms have a clear physical interpretation, which is not the case in the approach of [64].

Proof of Theorem 3.1. To prove this result, we can closely follow the proof in [64], except for some new terms involving the polarization force and drift.

We observe that $\mathrm{div}_v(F_\varepsilon) = 0$. This entails that:

$$\mathrm{div}_x\, F_\varepsilon^\perp(t, x - v^\perp, v) + \mathrm{div}_v\, F_\varepsilon(t, x - v^\perp, v) = 0 \tag{3.9}$$

By Liouville's theorem, this means that all $L^p_{x,v}$ norms of (g_ε) are conserved. Thus, up to a subsequence, g_ε admits a weak limit in the sense of distributions, denoted by g, that we should characterize now.

Multiplying (3.3) by ε and passing to the limit $\varepsilon \to 0$, we easily get:

$$v^\perp.\nabla_v g = 0. \tag{3.10}$$

This means that g only depends on the modulus of the velocity variable, and not on the angle.

As in [72, 64], the idea is now to consider smooth test functions $\Psi(t, x, r)$ with a radial dependance in the velocity variable. We test the Vlasov equation against such a function, compactly supported in $\mathbb{R}_t^+ \times \mathbb{R}_x^2 \times \mathbb{R}_r^+$.

$$\begin{aligned}\int g_\varepsilon \Big(\partial_t \Psi + &E^\perp(t, x - v^\perp).\nabla_x \Psi + F_\varepsilon^\perp(t, x - v^\perp, v).\nabla_x \Psi \\ &+ F_\varepsilon(t, x - v^\perp, v).\nabla_v \Psi + v^\perp.\nabla_v \Psi\Big) dtdxdv = - \int g_{0,\varepsilon} \Psi dxdv.\end{aligned} \tag{3.11}$$

In the following, we denote $v = re^{i\theta}$. We first have, since Ψ and g do not depend on θ, as $\varepsilon \to 0$:

$$\begin{aligned}&\int g_\varepsilon \Big(\partial_t \Psi + E^\perp(t, x - v^\perp).\nabla_x \Psi\Big) dtdxdv \\ \to 2\pi \int g \partial_t \Psi r dr dt dx + &\int \int_0^{2\pi} E^\perp(t, x - |v| e^{i\theta + i\pi/2}) d\theta.\nabla_x \Psi g r dr dt dx.\end{aligned}$$

We also observe that $v^\perp.\nabla_v \Psi = 0$. Furthermore, in $L_t^\infty L_x^2$, both terms $\varepsilon \partial_t E$, $\varepsilon E^\perp.\nabla_x E^\perp$ vanish in the limit $\varepsilon \to 0$.

Passing to the limit $\varepsilon \to 0$, we have

$$\int g_\varepsilon \left(v.\nabla_x E^\perp(t, x - v^\perp)\right).\nabla_v \Psi dtdxdv \to \int g \left(v.\nabla_x E^\perp(t, x - v^\perp)\right).\nabla_v \Psi dtdxdv.$$

We recall that g and Ψ have only a radial dependance in the velocity variable. We have:

$$\int g \left(v.\nabla_x E^\perp(t, x - v^\perp)\right).\nabla_v \Psi dv = \int \left(\int \left(e^{i\theta}.\nabla_x E^\perp(t, x - re^{i\theta + i\pi/2})\right).e^{i\theta} d\theta\right) rg\partial_r \Psi dr.$$

Likewise, we have:

$$\begin{aligned}\int g_\varepsilon \left(v.\nabla_x E(t, x - v^\perp)\right).\nabla_x \Psi dtdxdv &\to \int g \left(v.\nabla_x E(t, x - v^\perp)\right).\nabla_x \Psi dtdxdv \\ &= \int g \left(\int e^{i\theta}.\nabla_x E(t, x - re^{i\theta + i\pi/2}) d\theta\right).\nabla_x \Psi r^2 dtdxdr.\end{aligned}$$

We finally use Lemma 3.1 to prove that the contributions of both of these terms vanish.

Concerning the initial data, we get:

$$\int g_{0,\varepsilon}\Psi dxdv \to \int \left(\int g_0 d\theta\right) rdxdr.$$

Gathering all pieces together, this proves the Theorem.

□

Remark 3.2. Let us now assume that E_ε is an external electric field, with some time oscillations of frequency $\frac{1}{\varepsilon}$. Then we can easily see that the polarization terms do not vanish in the limit $\varepsilon \to 0$. For instance we may consider:

$$E_\varepsilon = \left(1 - \sin\frac{t}{\varepsilon}\right) e_2.$$

One can observe that E_ε is uniformly in $L^\infty_t H^s_x$ for any $s \in \mathbb{R}$. For the initial data, we take $g_{\varepsilon,0} = g_0$ in L^∞_v which does not depend on x. Then the polarization force is given by $F_\varepsilon = \cos\frac{t}{\varepsilon}e_1 \neq 0$ and (3.3) reads:

$$\partial_t g_\varepsilon + \cos\frac{t}{\varepsilon}\partial_{v_1} g_\varepsilon + \frac{v^\perp}{\varepsilon}.\nabla_v g_\varepsilon = 0.$$

Setting $h_\varepsilon(t,x,v) := g_\varepsilon(t,x,R(-t/\varepsilon)v)$, one can show that h_ε satisfies the equation:

$$\partial_t h_\varepsilon + \begin{pmatrix} \cos^2\frac{t}{\varepsilon} \\ \frac{1}{2}\sin 2\frac{t}{\varepsilon}\end{pmatrix}.\nabla_v h_\varepsilon = 0.$$

Denoting by h a weak limit in $L^\infty_{t,x,v}$ of h_ε, one can show by some elementary computations that h satisfies the limit equation:

$$\partial_t h + \frac{1}{2}\partial_{v_1} h = 0.$$

The acceleration term $\frac{1}{2}\partial_{v_1}$ stems from the polarization force.

3.3 Analysis of the limit in the Poisson case

We now investigate the case of (3.6) coupled with the Poisson equation, in which case the coupled system reads:

$$\begin{cases} \partial_t h_\varepsilon + E_\varepsilon^\perp(t, x+\mathcal{R}(-t/\varepsilon)v).\nabla_x h_\varepsilon + \mathcal{R}(t/\varepsilon)F_\varepsilon(t, x+\mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v).\nabla_x h_\varepsilon \\ +R(t/\varepsilon)F_\varepsilon(t, x+\mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v).\nabla_v h_\varepsilon = 0, \\ E_\varepsilon = -\nabla_x \Phi_\varepsilon, \\ -\Delta_x \Phi_\varepsilon = \int h_\varepsilon(t, x+\mathcal{R}(t/\varepsilon)v, R(t/\varepsilon)v)dv - 1, \end{cases} \tag{3.12}$$

with $F_\varepsilon = -\varepsilon\partial_t E_\varepsilon^\perp - v + \varepsilon E_\varepsilon^\perp.\nabla_x(E_\varepsilon^\perp)$.

Before stating the proposition, we gather in the following lemma some useful uniform estimates for the solutions (f_ε) of (1.6) and the solutions (h_ε) of (3.12). At fist, let us give a definition which will be useful to state some of the results.

Definition. *Let $u \in \mathcal{D}'(\mathbb{T}^2_x \times \mathbb{R}^2_v)$. Let $s \in \mathbb{R}, p \in [1,+\infty]$.*

We say that $u \in W^{s,p}_{x,v,loc}$ if for any function ξ in $\mathcal{D}(\mathbb{T}^2_x \times \mathbb{R}^2_v)$, $\xi u \in W^{s,p}_{x,v}(\mathbb{T}^2_x \times \mathbb{R}^2_v)$ (where $W^{s,p}_{x,v}(\mathbb{T}^2_x \times \mathbb{R}^2_v)$ is the usual Sobolev space).

Lemma 3.2. *i. The energy defined by the functional:*

$$\mathcal{E}_\varepsilon(t) = \int f_\varepsilon |v|^2 dv dx + \varepsilon \int |\nabla_x V_\varepsilon|^2 dx, \tag{3.13}$$

where f_ε is solution to (1.6), is non-increasing.

All L^p norms of f_ε are conserved:

$$\|f_\varepsilon\|_{L^\infty_t L^p_{x,v}} \leq \|f_{0,\varepsilon}\|_{L^p_{x,v}}.$$

ii. The charge $\rho_\varepsilon := \int f_\varepsilon dv$ is uniformly bounded in $L^\infty_t L^{3/2}_x$. The current $J_\varepsilon := \int f_\varepsilon v dv$ is uniformly bounded in $L^\infty_t L^{5/4}_x$.

iii. Let

$$S_\varepsilon = \mathcal{R}(t/\varepsilon) F_\varepsilon(t, x+\mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v).\nabla_x h_\varepsilon + R(t/\varepsilon) F_\varepsilon(t, x+\mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v).\nabla_v h_\varepsilon.$$

Then S_ε is uniformly bounded in $L^\infty_t W^{-1,6/5}_{x,v,loc}$.

iv. The sequence h_ε is strongly relatively compact in $L^\infty_{t,loc} W^{-1,3/2}_{x,v,loc}$.

Sketch of proof. i. The property of the energy follows from an explicit computation of $\frac{d\mathcal{E}_\varepsilon}{dt}$. We refer for instance to [60] or [94].

The conservation of L^p norms comes from Liouville's theorem (since the force field is divergence free in v).

ii. The estimates come from a by now very classical principle of real interpolation. It relies on the facts that f_ε is in $L^\infty_{t,x,v}$ uniformly in ε and $f_\varepsilon |v|^2$ is in $L^\infty_t L^1_{x,v}$ uniformly in ε. The principle is to decompose the velocity space in two parts:

$$\int f_\varepsilon dv = \int_{|v| \leq R} f_\varepsilon dv + \int_{|v|>R} f_\varepsilon dv$$

and optimize in R.

iii. We first recall that:

$$F_\varepsilon = -\varepsilon \partial_t E_\varepsilon^\perp - v.\nabla_x (E_\varepsilon^\perp) + \varepsilon E_\varepsilon^\perp.\nabla_x (E_\varepsilon^\perp).$$

Therefore, one can readily check that $\operatorname{div}_v F_\varepsilon = 0$, so that after the change of variables, using the chain rule, we have:

$$\begin{aligned} &\operatorname{div}_x \Big[\mathcal{R}(t/\varepsilon) F_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v)\Big] \\ &+ \operatorname{div}_v \Big[R(t/\varepsilon) F_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v)\Big] = 0. \end{aligned}$$

Therefore we can deduce that S_ε can be rewritten as:

$$\begin{aligned} &\operatorname{div}_x \Big[\mathcal{R}(t/\varepsilon) F_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v) h_\varepsilon\Big] \\ &+ \operatorname{div}_v \Big[R(t/\varepsilon) F_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v) h_\varepsilon\Big]. \end{aligned}$$

We claim that $F_\varepsilon h_\varepsilon \in L_t^\infty(L^{6/5}_{x,v,loc})$ uniformly in ε. First, using the local conservation of charge (obtained by integrating the transport equation in (1.6)):

$$\partial_t \int f_\varepsilon dv + \frac{1}{\varepsilon} \operatorname{div}_x \int f_\varepsilon v dv = 0,$$

and integrating with respect to time the Poisson equation in (1.6):

$$-\Delta_x \partial_t V_\varepsilon = \partial_t \int f_\varepsilon dv,$$

we obtain:

$$\varepsilon \partial_t E_\varepsilon = \nabla_x \Delta_x^{-1} \operatorname{div}_x \int f_\varepsilon v dv. \tag{3.14}$$

As a result, using point 2 of the lemma and by elliptic regularity, we have:

$$-\varepsilon \partial_t E_\varepsilon^\perp \in L_t^\infty L_x^{5/4},$$

uniformly in ε.

By point 2 of the lemma, $\rho_\varepsilon \in L_t^\infty L_x^{3/2}$. Consequently, by elliptic regularity, we get $E_\varepsilon \in L_t^\infty(W_x^{1,3/2})$, and thus by Sobolev embedding in $2D$, we have $E_\varepsilon \in L_t^\infty(L_x^6)$ (uniformly in ε). By Hölder's inequality, we have:

$$E_\varepsilon^\perp . \nabla_x E_\varepsilon^\perp \in L_t^\infty L_x^{6/5}.$$

This means that F_ε is uniformly in $L_t^\infty L^{6/5}_{x,v,loc}$.

Since all L^p norms of h_ε are conserved, we have $h_\varepsilon \in L_t^\infty L^\infty_{x,v}$ uniformly in ε and thus $F_\varepsilon h_\varepsilon \in L_t^\infty(L^{6/5}_{x,v,loc})$.

iv. We use the transport equation satisfied by h_ε in (3.12) and the uniform bound on S_ε to show that $\partial_t h_\varepsilon$ is uniformly in $L_t^\infty W^{-1,6/5}_{x,v,loc}$. In addition, we have $h_\varepsilon \in L_t^\infty L^\infty_{x,v,loc}$. Then, the compactness property relies on the Aubin-Lions compactness lemma (see for instance [60] for more details).

□

We now prove the following proposition:

Proposition 3.1. *Let $(f_{\varepsilon,0})$ be a sequence of initial data uniformly bounded in $L^1_{x,v} \cap L^\infty_{x,v}$ and with uniformly bounded energy $\mathcal{E}_\varepsilon(0)$.*

We assume that $(h_{\varepsilon,0})$ weakly converges in $L^2_{x,v}$ to some $h_0 \in L^1_{x,v} \cap L^2_{x,v}$. Then there exists $S \in L_t^\infty(W^{-1,q}_{x,v,loc})$ such that h_ε solution to (3.12) with initial data $h_{\varepsilon,0}$ weakly converges in the sense of distributions, up to a subsequence, to h, solution to the non linear transport equation:

$$\begin{cases} \partial_t h + \frac{1}{2\pi}\int_0^{2\pi} E^\perp(t,\tau,x+\mathcal{R}(-\tau)v)d\tau . \nabla_x h = -S \\ E = -\nabla_x \Phi, \\ -\Delta_x \Phi = \int h(t, x+\mathcal{R}(\tau)v, R(\tau)v)dv - 1 \\ h_{|t=0} = h_0 \end{cases} \tag{3.15}$$

For the moment, this result is not very satisfactory since we are not able to compute or characterize the "source" S, which comes from the polarization drift and force that we have introduced earlier. In the next subsection, we will nevertheless explain how to compute S (by another way).

The proof of Proposition 3.1 relies on two-scale convergence notions, that we recall for the sake of readability. This notion is useful in this context since it is more precise than weak convergence, insofar as it can capture the oscillating behaviour of the sequence of the functions under study.

Definition. *Let X be a separable Banach space, X' be its topological dual space and $(.,.)$ the duality bracket between X' and X. For all $\alpha > 0$, denote by $\mathcal{C}_\alpha(\mathbb{R}, X)$ (respectively $L^{q'}_\alpha(\mathbb{R}; X')$) the space of α-periodic continuous (respectively $L^{q'}$) functions on $\mathbb{R}$ with values in X. Let $q \in [1; \infty[$.*

Given a sequence (u_ε) of functions belonging to the space $L^{q'}(0, t; X')$ and a function $U^0(t, \theta) \in L^{q'}(0, T; L^{q'}_\alpha(\mathbb{R}; X'))$ we say that

$$u_\varepsilon \text{ 2-scale converges to } U^0$$

if for any function $\Psi \in L^q(0, T; \mathcal{C}_\alpha(\mathbb{R}, X))$ we have:

$$\lim_{\varepsilon \to 0} \int_0^T \left(u_\varepsilon(t), \Psi\left(t, \frac{t}{\varepsilon}\right)\right) dt = \frac{1}{\alpha} \int_0^T \int_0^\alpha \left(U^0(t, \tau), \Psi(t, \tau)\right) d\tau dt. \tag{3.16}$$

The main theorem we use is the following, due to Nguetseng [132] and Allaire [2].

Theorem 3.2 (NGuetseng and Allaire)**.** *Given a sequence (u_ε) bounded in $L^{q'}(0, t; X')$, there exists for all $\alpha > 0$ a function $U^0_\alpha \in L^{q'}(0, T; L^{q'}_\alpha(\mathbb{R}; X'))$ such that up to a subsequence,*

$$u_\varepsilon \text{ 2-scale converges to } U^0_\alpha.$$

The profile U^0_α is called the α-periodic two scale limit of u_ε and the link between U^0_α and the weak- limit u of u_ε is given by:*

$$\frac{1}{\alpha} \int_0^\alpha U^0 d\tau = u. \tag{3.17}$$

Proof of Proposition 3.1. We have the following convergences in the sense of distributions:

$$h_\varepsilon \rightharpoonup h,$$

$$E_\varepsilon \rightharpoonup E,$$

since these terms are uniformly bounded in some L^p space.

Using two-scale convergence tools and the compactness obtained in point 4 of Lemma we can show:

$$E^\perp_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v).\nabla_x h_\varepsilon \rightharpoonup \frac{1}{2\pi} \int_0^{2\pi} E^\perp(t, \tau, x + \mathcal{R}(-\tau)v) d\tau.\nabla_x h.$$

Furthermore, by point 3 of Lemma 3.2, S_ε is uniformly bounded in $L^\infty_t W^{-1,\alpha}_{x,v,loc}$ and up to a subsequence it converges (in the sense of distributions) to some S in $L^\infty_t W^{-1,\alpha}_{x,v,loc}$

Passing to the limit $\varepsilon \to 0$ in the sense of distributions, we finally get:

$$\partial_t h + \frac{1}{2\pi} \int_0^{2\pi} E^\perp(t, \tau, x + \mathcal{R}(-\tau)v) d\tau.\nabla_x h = -S.$$

Finally, one can also pass to the limit in the Poisson equation, using two-scale convergence tools. We refer to [60] or [94] for more details.

□

3.4 Relevancy and a new interpretation of Frénod and Sonnendrücker's asymptotic equation

Let $\varepsilon > 0$ and f_ε a solution to (1.6). If we directly set:

$$\tilde{h}_\varepsilon = f_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v), \tag{3.18}$$

we recall that is [60], it is shown that up to a subsequence, $\tilde{h}_\varepsilon$ converges in the sense of distributions to $\tilde{h}$ solution to:

$$\begin{cases} \partial_t \tilde{h} + \frac{1}{2\pi}\left(\int_0^{2\pi} \mathcal{R}(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_x \tilde{h} \\ +\frac{1}{2\pi}\left(\int_0^{2\pi} R(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_v \tilde{h} = 0 \\ \mathcal{E} = -\nabla\Phi, \quad -\Delta\Phi = \int \tilde{h}(t, x+\mathcal{R}(\tau)v, R(\tau)v)dv - 1, \\ \tilde{h}_{|t=0} = f_0 \end{cases}$$

We recall now that we have considered in this work the following distribution function:

$$h_\varepsilon = f'_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v). \tag{3.19}$$

In Proposition 3.1, we have proved that any weak limit h satisfies (3.15). In the definition (3.1) of f'_ε, we observe that because of the conservation of energy in Lemma 3.2, $\sqrt{\varepsilon}E_\varepsilon$ is bounded in $L^\infty_t L^2$ and therefore $\varepsilon E^\perp_\varepsilon$ vanishes as ε goes to 0, so we should have $h = \tilde{h}$. This is the matter of the next Theorem, in which we are able to compute S.

Theorem 3.3. *Let $(f_{0,\varepsilon})$ be a sequence of non-negative initial data, uniformly bounded in $L^1_{x,v} \cap L^\infty_{x,v}$.*

i. Up to a subsequence, we have the convergence, in the sense of distributions:

$$h_\varepsilon - \tilde{h}_\varepsilon \rightharpoonup 0. \tag{3.20}$$

ii. Assume that the whole sequence $f_{0,\varepsilon}$ converges to some f_0 satisfying:

$$\begin{aligned} f_0 &\in W^{1,1}_{x,v}, \\ \|(1+|v|^4)f_0\|_{L^\infty_{x,v}} &< \infty, \\ \|(1+|v|^4)Df_0\|_{L^\infty_{x,v}} &< \infty. \end{aligned} \tag{3.21}$$

Then (3.20) holds without extraction. Furthermore, with the notations of Proposition 3.1, we have:

$$\begin{aligned} S =& \frac{1}{2\pi}\left(\int_0^{2\pi} \begin{pmatrix} \sin\tau & -\cos\tau \\ \cos\tau & \sin\tau \end{pmatrix} \mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_x h \\ &+\frac{1}{2\pi}\left(\int_0^{2\pi} R(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau\right).\nabla_v h. \end{aligned} \tag{3.22}$$

This theorem rigorously justifies our previous heuristic statements: equation (1.8) actually implicitly takes into account the effects of the polarization drift (and actually even more, see Section 2). In other words, the formerly mysterious terms (1.13) are a direct consequence of polarization effects.

Proof. i. Let $\Psi(t,x,v)$ be a smooth test function in $\mathcal{D}(\mathbb{R}_t^+ \times \mathbb{T}_x^2 \times \mathbb{R}_v^2)$. We can evaluate, by definition of h_ε (in (3.19)) and $\tilde{h}_\varepsilon$ (in (3.18)):

$$
\begin{aligned}
&\int (h_\varepsilon(t,x,v) - \tilde{h}_\varepsilon(t,x,v))\Psi(t,x,v)\,dt\,dx\,dv \\
&= \int (f'_\varepsilon(t, x+\mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v) - f_\varepsilon(t, x+\mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v))\Psi\,dt\,dx\,dv \\
&= \int \Big(f_\varepsilon\big(t, x+\mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v + \varepsilon E_\varepsilon^\perp(t, x+\mathcal{R}(-t/\varepsilon)v)\big) \\
&-f_\varepsilon\big(t, x+\mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v\big)\Big)\Psi\,dt\,dx\,dv \\
&= \int f_\varepsilon(t,x',v')\Big(\Psi\big(t, x'+\mathcal{R}(t/\varepsilon)v' - \varepsilon\mathcal{R}(t/\varepsilon)E_\varepsilon^\perp(t,x'), R(t/\varepsilon)v' - \varepsilon E_\varepsilon^\perp(t,x')\big) \\
&-\Psi\big(t, x'+\mathcal{R}(t/\varepsilon)v', R(t/\varepsilon)v'\big)\Big)\,dt\,dx'\,dv'.
\end{aligned}
$$

In the last line of this computation, we have performed the changes of variables (each has unit Jacobian):

$$
\begin{cases} x' = x + \mathcal{R}(-t/\varepsilon)v, \\ v' = R(-t/\varepsilon)v + \varepsilon E_\varepsilon^\perp(t,x'), \end{cases}
$$

and

$$
\begin{cases} x' = x, \\ v' = R(-t/\varepsilon)v. \end{cases}
$$

Hence, the following bound holds, after using Taylor's formula and Cauchy-Schwarz inequality:

$$
\begin{aligned}
\Big|\int (h_\varepsilon(t,x,v) - &\tilde{h}_\varepsilon(t,x,v))\Psi(t,x,v)dtdxdv\Big| \\
&\leq \sqrt{\varepsilon}C(\Psi)\|f_\varepsilon\|_{L_t^\infty L_{x,v}^2}\|\sqrt{\varepsilon}E_\varepsilon\|_{L_t^\infty L_x^2}\|\Psi\|_{W_{t,x,v}^{1,\infty}}
\end{aligned}
$$

Using the energy estimate (point 1 of Lemma 3.2), we recall that $\sqrt{\varepsilon}E_\varepsilon$ is uniformly bounded in $L_t^\infty L_x^2$, and therefore this proves our claim.

ii. Under the additional smoothness assumption (3.21), it was observed in a Remark of [94] that there is uniqueness of a solution $\tilde{h}$ to (1.8). Such a statement is classical for the "usual" Vlasov-Poisson system. For the system under study, the result is very similar. This relies on some fixed point arguments in the spirit of [151] or [45]. This implies that $\tilde{h}_\varepsilon \rightharpoonup \tilde{h}$ without having to take a subsequence. This means also that the whole sequence h_ε converges in the sense of distributions to $\tilde{h}$.

In Proposition 3.1, we have at the same time proved that h_ε (up to a subsequence) converges in the sense of distributions to h, solution to (3.15). Consequently, we infer that $\tilde{h} = h$, which proves our claim.

□

Remark 3.3. We believe it is possible to deal with weaker assumptions than (3.21), maybe if we use the approach due to Lions and Perhame [119] for the uniqueness problem for the Vlasov-Poisson system.

4 Appendix: Finite Larmor radius regime

The scaling is by now classical since it was studied in many papers, see for instance [60, 64, 94, 93]. We recall the main lines in the $2D$ setting, for the sake of completeness.

We first introduce the dimensionless variables and quantities:

$$\tilde{x} = \frac{x}{L}, \quad \tilde{t} = \frac{t}{\tau}, \quad \tilde{v} = \frac{v}{v_{th}}$$
$$f(t,x,v) = \bar{f}\tilde{f}(\tilde{t},\tilde{x},\tilde{v}) \quad V(t,x,x) = \bar{V}\tilde{V}(\tilde{t},\tilde{x},\tilde{x}) \quad E(t,x) = \bar{E}\tilde{E}(\tilde{t},\tilde{x}).$$

This yields the Vlasov-Poisson system:

$$\begin{cases} \partial_{\tilde{t}}\tilde{f}_\varepsilon + \frac{v_{th}\tau}{L}\tilde{v}.\nabla_{\tilde{x}}\tilde{f}_\varepsilon + + \left(\frac{e\bar{E}\tau}{mv_{th}}\tilde{E}_\varepsilon + \frac{e\bar{B}}{m}\tau\tilde{v}^\perp\right).\nabla_{\tilde{v}}\tilde{f}_\varepsilon = 0 \\ \frac{\bar{E}}{\bar{V}}\tilde{E}_\varepsilon = -\frac{1}{L}\nabla_{\tilde{x}}\tilde{V}_\varepsilon \\ -\frac{\varepsilon_0\bar{V}}{L^2}\Delta_{\tilde{x}}\tilde{V}_\varepsilon = e\bar{f}v_{th}^2\left(\int \tilde{f}_\varepsilon d\tilde{v} - 1\right) \\ \tilde{f}_{\varepsilon,|\tilde{t}=0} = \tilde{f}_{0,\varepsilon}, \quad \bar{f}L^2v_{th}^2\int \tilde{f}_{0,\varepsilon}d\tilde{v}d\tilde{x} = 1. \end{cases} \tag{4.1}$$

In order to have normalized distributions, it is first natural to set $\bar{f}L_\perp^2 L_\| v_{th}^3 = 1$.

As in Section 2, we set $\Omega = \frac{e\bar{B}}{m}$: this is the cyclotron frequency (also referred to as the gyrofrequency) We also consider the so-called electron Larmor radius (or electron gyroradius) r_L defined by:

$$r_L = \frac{v_{th}}{\Omega} = \frac{mv_{th}}{e\bar{B}} \tag{4.2}$$

As explained in Section 2, this can be physically understood as the typical radius of the helix around axis $e_\|$ described by the particles, due to the intense magnetic field. We also introduce the so-called Debye length:

$$\lambda_D = \frac{\varepsilon_0\bar{V}}{e\bar{f}v_{th}^3},$$

which is interpreted as the typical length above which the plasma can be interpreted as being neutral.

The Vlasov equation now reads:

$$\partial_{\tilde{t}}\tilde{f}_\varepsilon + \frac{r_L}{L}\Omega\tau\tilde{v}.\nabla_{\tilde{x}}\tilde{f}_\varepsilon + + \left(\frac{\bar{E}}{\bar{B}v_{th}}\Omega\tau\tilde{E}_\varepsilon + \Omega\tau\tilde{v}^\perp\right).\nabla_{\tilde{v}}\tilde{f}_\varepsilon = 0.$$

The strong magnetic field ordering (roughly speaking it corresponds to $\bar{B} \to +\infty$) consists in taking:

$$\Omega\tau = \frac{1}{\varepsilon}, \quad \frac{\bar{E}}{\bar{B}v_{th}} = \varepsilon.$$

The spatial scaling we perform is the so-called finite Larmor radius scaling: basically the idea is to consider the typical spatial length L with the same order as the so-called electron Larmor radius.

$$\frac{r_L}{L} = 1 \tag{4.3}$$

The quasineutral ordering we adopt is the following:

$$\frac{\lambda_D}{L} = 1/\sqrt{\varepsilon}. \tag{4.4}$$

After straightforward computations, we get the following Vlasov-Poisson system in dimensionless form, for $t \geq 0, x \in \mathbb{T}^2, v \in \mathbb{R}^2$:

$$\begin{cases} \partial_t f_\varepsilon + \frac{v}{\varepsilon}.\nabla_x f_\varepsilon + (E_\varepsilon + \frac{v^\perp}{\varepsilon}).\nabla_v f_\varepsilon = 0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\Delta_x V_\varepsilon = \int f_\varepsilon dv - \int f_\varepsilon dv dx \\ f_{\varepsilon,t=0} = f_{\varepsilon,0}. \end{cases} \tag{4.5}$$

5 Appendix: Formal derivation of a new gyrokinetic model

We consider that our study mainly indicates that two-scale numerical methods should be more and more developed. Lately, there have been improvements and recent advances for the numerical simulation of two-scale models. We refer to the works of Frénod, Mouton and Sonnendrücker [56], Frénod, Salvarani and Sonnendrücker [58], Mouton [131] and references therein. Nevertheless for the time being, such models remain difficult to simulate and are expensive, due to the introduction of a new variable (namely the fast time variable τ).

In this appendix, the aim is to formally derive a new model which may be easier to handle in numerical simulations than (1.8), but which would keep some polarization effects. We are first able to compute the weak limit of F_ε:

Proposition 5.1. *In the Vlasov-Poisson case of (3.12), the polarization force F_ε converges in the sense of distributions to:*

$$F := \nabla^\perp \Delta^{-1} \operatorname{div} \left(\int f'(t,x,v) v \right) - v.\nabla_x E^\perp, .$$

where f' is a weak limit of f'_ε and E a weak limit of E_ε.

Proof. We have $F_\varepsilon = -\varepsilon \partial_t E_\varepsilon^\perp - (v + \varepsilon E_\varepsilon^\perp).\nabla_x (E_\varepsilon^\perp)$. Recalling (3.14), we can compute:

$$\begin{aligned} -\varepsilon \partial_t E_\varepsilon^\perp =& \nabla^\perp \Delta^{-1} \operatorname{div}_x \int f_\varepsilon(t,x,v) v dv \\ =& \nabla^\perp \Delta^{-1} \operatorname{div} \left(\int f'_\varepsilon(t,x,v) v dv + \varepsilon E_\varepsilon^\perp \int f'_\varepsilon(t,x,v) dv \right) \\ =& \nabla^\perp \Delta^{-1} \operatorname{div} \left(\int f'_\varepsilon(t,x,v) v dv + \varepsilon E_\varepsilon^\perp \rho_\varepsilon \right). \end{aligned}$$

Now we can observe that:

$$\nabla^\perp \Delta^{-1} \operatorname{div} E_\varepsilon^\perp \rho_\varepsilon = E_\varepsilon^\perp.\nabla_x E_\varepsilon^\perp.$$

Indeed, taking the rotational of each quantity, this is equivalent to prove that:

$$\operatorname{div} E_\varepsilon^\perp \rho_\varepsilon = \operatorname{rot}(E_\varepsilon^\perp.\nabla_x E_\varepsilon^\perp).$$

Recalling the Poisson equation:

$$\rho_\varepsilon - 1 = \operatorname{div} E_\varepsilon,$$

one can easily check that each quantity is equal to:

$$\partial_1(E_2 \partial_2 E_2 + E_2 \partial_1 E_1) - \partial_2(E_1 \partial_2 E_2 + E_1 \partial_1 E_1).$$

Finally, we have:

$$F_\varepsilon = \nabla^\perp \Delta^{-1} \operatorname{div} \left(\int f'_\varepsilon(t,x,v) v dv \right) - v.\nabla_x E_\varepsilon.$$

In the sense of distributions we obtain:

$$v.\nabla_x (E_\varepsilon^\perp) \rightharpoonup v.\nabla_x E^\perp$$

Denote by f' a weak limit of f'_ε. Then we have in the distributional sense:

$$\int f'_\varepsilon(t,x,v) v dv \rightharpoonup \int f'(t,x,v) dx dv.$$

Finally we have shown that F_ε weakly converges to F defined by:

$$F = \nabla^\perp \Delta^{-1} \operatorname{div} \left(\int f'(t,x,v) v dv \right) - v.\nabla_x E^\perp \tag{5.1}$$

□

Following the computations of Theorem 3.1 and Proposition 5.1 , we can pass formally to the limit in (3.3) and thus we can propose the following new model which displays some effects of the polarization drift:

$$\begin{cases} \partial_t g + \frac{1}{2\pi}\int_0^{2\pi} E^\perp(t, x - re^{i\theta + i\pi/2}) d\theta.\nabla_x g \quad t \geq 0, x \in \mathbb{T}^2, r \geq 0 \\ -\frac{1}{2\pi}\int_0^{2\pi} \tilde{J}(t, x - re^{i\theta+i\pi/2}) d\theta.\nabla_x g + \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta}.\tilde{J}^\perp(t, x - re^{i\theta+i\pi/2}) d\theta \partial_r g = 0, \\ E = -\nabla_x \Phi, \\ -\Delta_x \Phi = \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}^+} g(t, x - re^{i\theta+i\pi/2}, r) r dr d\theta - 1 \\ \tilde{J} = \nabla \Delta^{-1} \operatorname{div} \left(\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}^+} g(t, x - re^{i\theta+i\pi/2}, r) r^2 e^{i\theta} dr d\theta \right) \end{cases} \tag{5.2}$$

However, it seems really difficult to study this system from the mathematical point of view, since the field $\tilde{J}$ only a priori belongs to some Lebesgue space, so there is a lack of regularity for this field. We can not even use an averaging lemma to overcome this feature, since the free transport operator has disappeared in the process.

Chapter 6

On the confinement of a tokamak plasma

Article paru à SIAM Journal on Mathematical Analysis (2010).

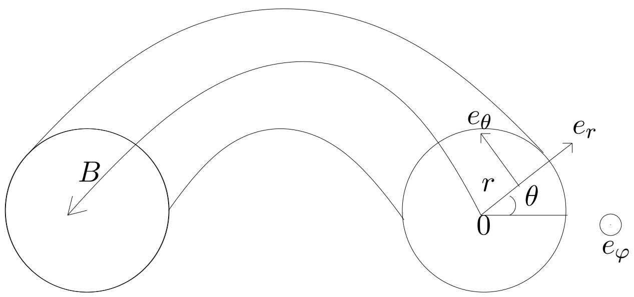
Sommaire

Résumé : The goal of this chapter is to understand from a mathematical point of view the magnetic confinement of plasmas for fusion. Following Frénod and Sonnendrücker [60], we first use two-scale convergence tools to derive a gyrokinetic system for a plasma submitted to a large magnetic field with a slowly spatially varying intensity. We formally derive from this system a simplified bi-temperature fluid system. We then investigate the behaviour of the plasma in such a regime and we prove nonlinear stability or instability depending on which side of the tokamak we are looking at. In our analysis, we will also point out that there exists a temperature gradient threshold beyond which one can expect stability, even in the "bad" side : this corresponds to the so-called H-mode.

1 Introduction

1.1 Magnetic confinement for plasmas

Fusion is undoubtly one of the most promising research fields in order to find new sources of energy. For the time being, magnetic confinement fusion represents one of the two main approaches (the other one being inertial confinement fusion). The principle consists basically in using a magnetic field in order to confine the very high temperature plasma. Good confinement is absolutely compulsory since the plasma could otherwise damage the surrounding materials.



A first step towards confinement is to use a tokamak[1], i.e. a torus-shaped box and consider a large purely toroidal magnetic field B, in other words $B = \frac{\mathcal{B}}{\varepsilon} e_\varphi$ with $\varepsilon > 0$ small. One can formally show that at leading order in ε, particles oscillate around the magnetic field lines. The drawback of this technique is that there are in fact many drifts appearing at higher order, some due to the geometry of B and one we are specifically concerned with, which is called the electric drift or $E \times B$ drift:

$$v_E = \frac{E \wedge B}{|B|^2},$$

where E denotes the electric field.

Since the electric field is induced by the plasma itself, one can not precisely predict its qualitative behaviour and thus this drift may prevent us from getting a good confinement property : if we wait long enough, particles may stop to perfectly turn around the torus and start drifting toward the edge of the tokamak. In order to overcome the effects of the electric drift, the idea is basically to take advantage of the other drifts due to the geometry of B.

[1] Actually there are other possibilities, like stellarators. These kinds of devices are much more difficult to study from the mathematical viewpoint, since they have a very complex structure.

In the present chapter, we make the assumption that the ions of the plasma are at thermodynamic equilibrium and we describe the distribution of electrons by a kinetic equation. For the sake of simplicity, we restrict to the $2D$ problem in the plane orthogonal to B, in order to understand the behaviour of the particles in the slice. We take a magnetic field given by

$$B = \frac{\mathcal{B}}{\varepsilon} e_\varphi,$$

with $\varepsilon > 0$ a small parameter and $\mathcal{B}$ to be fixed later. We consider the Finite Larmor Radius scaling (see [60] for a reference in the mathematical literature) which consists in considering a characteristic spatial length with the same order as the Larmor radius (which is of order ε). This scaling allows for a better description of the orthogonal motion and is expected to make the electric drift appear in the limit $\varepsilon \to 0$. The density $f_\varepsilon(t, x, v)$ (with $t > 0, x \in \mathbb{T}^2, v \in \mathbb{R}^2$) of the electrons is then given by the following dimensionless Vlasov Poisson system :

$$\begin{cases} \partial_t f_\varepsilon + \frac{v}{\varepsilon}.\nabla_x f_\varepsilon + (E_\varepsilon + v^\perp \frac{\mathcal{B}}{\varepsilon}).\nabla_v f_\varepsilon = 0 \\ f_{\varepsilon,|t=0} = f_0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\Delta_x V_\varepsilon = \displaystyle\int f_\varepsilon dv - 1. \end{cases} \tag{1.1}$$

We denote $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in the local orthogonal basis (see figure 6.1). For any vector $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, we denote $A^\perp = \begin{pmatrix} A_2 \\ -A_1 \end{pmatrix}$.

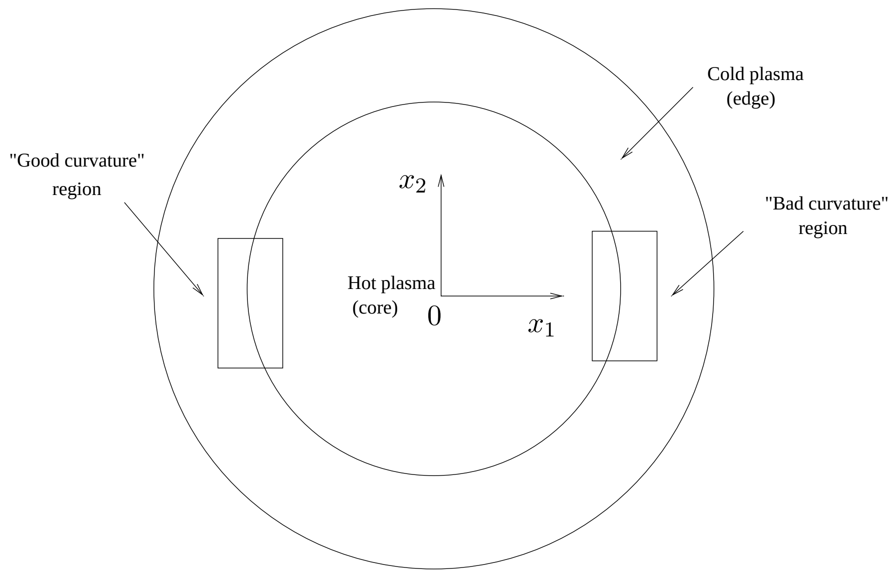


Figure 6.1: A slice of tokamak

Following Grandgirard et al. ([78]), we consider the explicit formula for $\mathcal{B}$:

$$\mathcal{B} = \frac{R_0}{R_0 + \varepsilon r \cos\theta} = \frac{R_0}{R_0 + \varepsilon x_1}, \tag{1.2}$$

denoting by R_0 the small radius of the torus. (We recall that the characteristic spatial length is of order ε)

We consider that $R_0 \sim 1$; consequently at first order in ε we get:

$$\mathcal{B} = 1 - \varepsilon x_1, \tag{1.3}$$

leading to the following system:

$$\begin{cases} \partial_t f_\varepsilon + \frac{v}{\varepsilon}.\nabla_x f_\varepsilon + (E_\varepsilon + \frac{v^\perp}{\varepsilon} - x_1 v^\perp).\nabla_v f_\varepsilon = 0 \\ f_{\varepsilon,|t=0} = f_0 \\ E_\varepsilon = -\nabla_x V_\varepsilon \\ -\Delta_x V_\varepsilon = \displaystyle\int f_\varepsilon dv - 1. \end{cases} \tag{1.4}$$

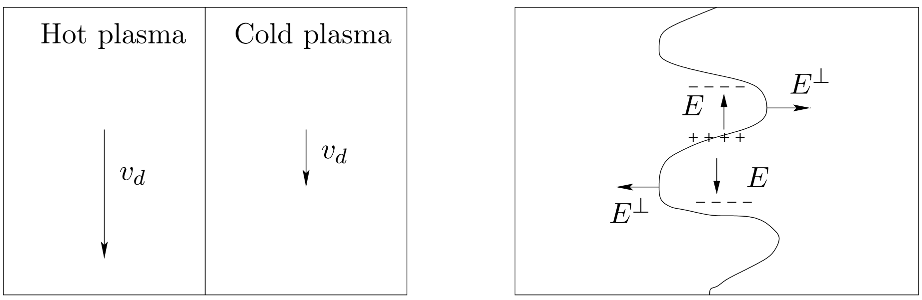
We will see that taking an inhomogeneous intensity for the magnetic field, even at order 1 in ε, leads to a quite different behaviour for the plasma.

Indeed, in the limit $\varepsilon \to 0$, we can derive rigorously another kinetic system which is qualitatively close to the following one (see sections 2 and 3):

$$\begin{cases} \partial_t f - \frac{1}{2}|v|^2 \partial_{x_2} f + E^\perp.\nabla_x f = 0 \\ f_{|t=0} = f_0 \\ E = -\nabla_x V \\ -\Delta_x V = \displaystyle\int f dv - 1. \end{cases} \tag{1.5}$$

Observe here that $E^\perp$ corresponds to the electric drift $E \times B$ that we mentioned earlier; the additional drift $v_d = -\frac{1}{2}v^2 e_2$ is due to the inhomogeneity of the magnetic field intensity. The remarkable point is that this drift has a fixed direction; it makes the particles "fall" toward the "bottom" of the slice. At this point of the modeling, we now have to distinguish between the plasma-core and the plasma edge (see figure 1), the only difference between the two we are concerned with, being that the core is much hotter than the edge. This means from a kinetic point of view that the velocities are much smaller in the edge.

We now divide the slice into two areas: we denote the part $x_1 > 0$ the **"bad curvature"** side and the part $x_1 < 0$ the **"good curvature"** side: indeed, we expect the plasma in the "good curvature" side to be well confined, while the plasma in the "bad curvature" region is badly confined. This behaviour can be easily predicted with the following heuristic study in the "bad curvature" side:



Particles in the hot plasma drift faster (left figure), so if there is any perturbation (right figure), there appears a separation of charge creating an electric field E, which entails a drift $E^\perp$ that accentuates the perturbation: in other words, the equilibrium is unstable. This discussion is part of the folklore in plasma physics for tokamaks and this instability is recognized to be one of the main sources of disruption for the plasma.

In the other hand one can lead the same qualitative analysis in the "good curvature" side and show in this case stability.

1.2 Objectives and results of this chapter

In this chapter, following the previous heuristic argument, we will particularly focus on system (1.6), which is a kind of simplified "bi-fluid" version of (1.5). We consider that the plasma is made of two mixable phases, one being the hot plasma (with constant temperature T^+ and density $\rho^+(t,x)$) and the other the cold plasma (with constant temperature T^- and density $\rho^-(t,x)$) . Of course, hot and cold means that $T^+ > T^-$.

$$\begin{cases} \partial_t \rho^+ - T^+ \partial_{x_2} \rho^+ + E^\perp . \nabla_x \rho^+ = 0 \\ \partial_t \rho^- - T^- \partial_{x_2} \rho^+ + E^\perp . \nabla_x \rho^- = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \rho^+ + \rho^- - 1 \\ V = 0 \text{ on } x_1 = 0, L \\ (\rho^+, \rho^-)_{|t=0} = (\rho_0^+, \rho_0^-) \text{ with } \int \rho_0^+ + \rho_0^- = 1, \end{cases} \tag{1.6}$$

for $t \geq 0, x \in [0,L] \times \mathbb{R}/L\mathbb{Z}$ and L is the size of the box.

The temperature $T(t,x)$ of the plasma is given by:

$$T(t,x) = \frac{\rho^+(t,x)T^+ + \rho^-(t,x)T^-}{\rho^+(t,x) + \rho^-(t,x)}. \tag{1.7}$$

(We refer to Section 3 for more details.)

Unfortunately, we were able to derive this system only formally from system (1.4) and had to make some physical and mathematical approximations. These are precisely explained in Section 3.

We observe that this system shares structural similarities with 2D Euler equations in vorticity form, which describe an incompressible inviscid fluid.

$$\text{Plasma} \quad \leftrightarrow \quad \text{Fluid}$$

$$\text{density} \quad \rho - 1 \quad \leftrightarrow \quad \text{vorticity} \quad \omega$$

$$\text{(rotated) electric field} \quad E^\perp = \nabla^\perp \Delta^{-1}(\rho - 1) \quad \leftrightarrow \quad \text{velocity} \quad u = \nabla^\perp \Delta^{-1} \omega$$

Hence, our system can be seen somehow as a Euler system with two kinds of vorticities.

Such an analogy between strongly magnetized plasmas and bi-dimensional ideal fluids has been observed for a long time by physicists (for instance, see [96]). We mention that the convergence towards $2D$ Euler in strong magnetic fields regimes (but different from the one studied here) was rigorously established by Golse and Saint-Raymond [72] and Brenier [27].

Let us now define precisely the stability and instability notions that we will work on until the end of the chapter. One should be aware that for such infinite-dimensional dynamical systems, the choice of the norm is particularly important.

Definition. *Let ξ be a solution to (1.6). This solution is said to be stable with respect to the X norm if for any $\eta > 0$, there exists $\delta > 0$ such that: for any solution ρ to (1.6), the initial control $\|\rho(0) - \xi(0)\|_X \leq \delta$ implies that for any $t \geq 0$, $\|\rho(t) - \xi(t)\|_X \leq \eta$.*

Otherwise, the solution ξ is said to be unstable with respect to the X norm.

Of course, instability will then be interpreted as bad confinement, and stability as good confinement.

We will investigate stability and instability around the following steady states, modeling the good and bad curvature sides:

$$\mu^{bad}(x_1) = \left(\mu^{bad,+} = 1 - \frac{x_1}{L}, \mu^{bad,-} = \frac{x_1}{L}\right), \tag{1.8}$$

$$\mu^{good}(x_1) = \left(\mu^{good,+} = \frac{x_1}{L}, \mu^{good,-} = 1 - \frac{x_1}{L}\right), \tag{1.9}$$

which actually model a linear transition between the hot and the cold plasma. Indeed, for μ^{bad}, the temperature of the plasma is given by:

$$T(t,x) = T^{-}\frac{x_1}{L} + T^{+}\left(1 - \frac{x_1}{L}\right),$$

whereas for μ^{good}, it is given by:

$$T(t,x) = T^{+}\frac{x_1}{L} + T^{-}\left(1 - \frac{x_1}{L}\right).$$

Hence the profile of the temperature is a straight line with a slope equal to the so-called temperature gradient $\frac{T^+ - T^-}{L}$. We mention that such linear profiles seem physically relevant in the edge of the tokamak (according to the graphs in [79] or [158]).

Despite this rather rough model, our predictions will qualitatively correspond to observations made by physicists.

Our first objective will be to confirm the linear scenario exposed in the heuristic study by exhibiting a growing mode with maximal growth for the linearized operator in the bad-curvature area (but only when the temperature gradient $\frac{T^+ - T^-}{L}$ is not too large) and by showing that there is no such growing mode in the good-curvature area.

Then our aim is to show that nonlinear instability also holds. As system (1.6) looks a lot like 2D Euler, it is not so surprising that techniques allowing to pass from spectral instability to nonlinear instability for 2D Euler may apply here. On the topic of stability and instability of ideal plane flows, we mention some recent developments; let us nevertheless emphasize that this list is by no means exhaustive. In [85], Grenier proved instability in the L^2 velocity norm for some shear flows with zero Lyapunov exponent. In [31], Bardos, Guo and Strauss, following a method introduced in [86], proved instability in the L^2 vorticity norm around some steady states. It is assumed that the linearized operator has a growth exceeding the Lyapunov exponent of the steady flow. We also mention the paper of Vishik and Friedlander [155] where instability in the L^2 velocity norm is proved under the same assumptions on the steady states. The best result available by now is due to Lin [112]. Under rather general assumptions on the steady states (in particular, there is no assumption on the growth of the linearized operator), he showed nonlinear instability in the L^2 vorticity norm and in the same time, that velocity grows exponentially in the L^2 norm. In this work, we will obtain similar results to those of Lin.

On the other hand, let us emphasize that in our linear analysis, when the temperature gradient $\frac{T^+ - T^-}{L}$ exceeds a threshold, there is no growing mode in the bad curvature side. We will show that the nonlinear equations inherit this linear property. This rather unexpected stability phenomenon can be interpreted as the so-called **High Confinement mode** (**H-mode** for short), by opposition to the "standard" regime, referred to as the Low Confinement mode (L-mode for short). The H-mode is a high-confinement regime obtained by heating of the plasma and triggered when the heating power exceeds some threshold. It has been experimentally observed by physicists for a long time: it was discovered in the ASDEX tokamak [157], we also refer to [79], [104] and ([158], Section 4.13). We can also remark that the H-mode is accompanied with an increase of the gradient of temperature ([79], [158]). These experimental observations fit very well with our qualitative results.

There exists a huge literature in physics on this particular topic. Nevertheless, despite a huge amount of works, the H-mode is still rather mysterious. Its understanding, especially the mechanism of transition from L-mode to H-mode is crucial for fusion research. To the very best of our knowledge, the H-mode has never been rigorously justified at the nonlinear or even at the linear level, with such a simple model.

It is sometimes believed that the formation of confining transport barriers is due to a sheared $E \times B$ flow. In some sense, our model shares similarities with linear shear flows (the linearized equations are similar); thus our study is not in contradiction with these considerations.

We finally mention that in the physics papers, the H-mode is most of the time numerically investigated with more complicated models (including more physics, such as the effect of collisions, friction, energy sources), we refer to [53] and references therein. Our model can be seen as a two-temperature caricature of the model of [53]. The transition to the H-mode is also numerically investigated in [105], where the existence of thresholds is shown. Unfortunately, we were not able to find any analytical formulae for those thresholds that we could have compared with ours.

The main results proved in this chapter (Corollary 5.1 and Theorem 6.1) are gathered in the following theorem:

Theorem 1.1. *For system (1.6):*

i. (Nonlinear stability)

The equilibrium μ^{good} is nonlinearly stable with respect to the L^2 norm.

If the temperature gradient $\frac{T^+-T^-}{L}$ satisfies:

$$\frac{T^+ - T^-}{L} > \frac{1}{\pi^2}, \tag{1.10}$$

then the equilibrium μ^{bad} is nonlinearly stable with respect to the L^2 norm.

ii. (Nonlinear instability)

If the temperature gradient satisfies:

$$\frac{T^+ - T^-}{L} < \frac{4}{5\pi^2}, \tag{1.11}$$

then there exist constants $\delta_0, \eta_0 > 0$ such that for any $0 < \delta < \delta_0$ and any $s \geq 0$ there exists a solution ρ to (1.6) with $\|\rho(0) - \mu^{bad}\|_{H^s} \leq \delta$ but such that:

$$\|E(t_\delta)\|_{L^2} \geq \eta_0, \tag{1.12}$$

denoting $E(t_\delta) = \nabla\Delta^{-1}(\rho^+(t_\delta) + \rho^-(t_\delta) - 1)$ the electric field at time $t_\delta = O(|\log \delta|)$. In particular, the equilibrium μ^{bad} is nonlinearly unstable with respect to the L^2 norm.

1.3 Organization of the chapter

The present chapter is organized as follows: section 2 is devoted to the study of the limit $\varepsilon \to 0$ for the system (1.4). In section 3 we present the simplified bi-fluid model we study in order to investigate stability and instability for the plasma. Section 4 is dedicated to the study of the linearized system around the steady states μ^{good} and μ^{bad} ; in particular we show the existence of dominant growing mode in the "bad curvature" region, provided that the gradient of temperatures is not too large. If the temperature gradient exceeds some threshold, then there is linear stability. In section 5, we are concerned with the nonlinear stability property for the "good curvature" region and for the "bad curvature" region for large enough temperature gradients (referred to as the high confinement mode in plasma physics), which will be achieved by exhibiting remarkable energies around the steady states. In section 6, for small enough temperature gradients we pass from linear spectral instability to nonlinear instability in the L^2 vorticity norm, using a high order approximation method introduced by Grenier. Then we prove that the electric field also grows exponentially in the L^2 norm, by using the energy exhibited in the previous section.

2 Gyrokinetic derivation of the equations

Following Frénod and Sonnendrücker ([60]), we can use two-scale convergence tools in order to derive the gyrokinetic equation we are interested in. We shall not dwell on the rigorous derivation of this system since the justifications in two dimensions are essentially done in [60].

First of all, let us recall precisely the two-scale convergence notions (due to Nguetseng [132] and Allaire [2]) we will use in this section.

Definition. *Let X be a separable Banach space, X' be its topological dual space and $(.,.)$ the duality bracket between X' and X. For all $\alpha > 0$, denote by $\mathcal{C}_\alpha(\mathbb{R}, X)$ (respectively $L^{q'}_\alpha(\mathbb{R}; X')$) the space of α-periodic continuous (respectively $L^{q'}$) functions on $\mathbb{R}$ with values in X. Let $q \in [1; \infty[$.*

Given a sequence (u_ε) of functions belonging to the space $L^{q'}(0, t; X')$ and a function $U^0(t, \theta) \in L^{q'}(0, T; L^{q'}_\alpha(\mathbb{R}; X'))$ we say that

$$u_\varepsilon \text{ 2-scale converges to } U^0$$

if for any function $\Psi \in L^q(0, T; \mathcal{C}_\alpha(\mathbb{R}, X))$ we have:

$$\lim_{\varepsilon \to 0} \int_0^T \left(u_\varepsilon(t), \Psi\left(t, \frac{t}{\varepsilon}\right) dt \right) = \frac{1}{\alpha} \int_0^T \int_0^\alpha \left(U^0(t, \tau), \Psi(t, \tau)\right) d\tau dt. \tag{2.1}$$

The new variable τ has to be understood as a "fast-time variable" which describes the fast time oscillations. As for weak-star convergence in L^p spaces, one can show that boundedness implies $2-$scale convergence in L^p spaces.

Theorem 2.1. *([132], [2])*

Given a sequence (u_ε) bounded in $L^{q'}(0, t; X')$, there exists for all $\alpha > 0$ a function $U^0_\alpha \in L^{q'}(0, T; L^{q'}_\alpha(\mathbb{R}; X')$ such that up to a subsequence,

$$u_\varepsilon \text{ 2-scale converges to } U^0_\alpha.$$

The profile U^0_α is called the α-periodic two scale limit of u_ε and the link between U^0_α and the weak- limit u of u_ε is given by:*

$$\frac{1}{\alpha} \int_0^\alpha U^0_\alpha d\tau = u. \tag{2.2}$$

For the reader's sake we recall the main arguments and refer to [2] for the complete proof.

Sketch of proof. Let $\alpha > 0$. We can consider φ_{u_ε} :

$$\Psi \in L^q(0,T;\mathcal{C}_\alpha(\mathbb{R},X)) \mapsto \int_0^T u_\varepsilon(t)\Psi\left(t,\frac{t}{\varepsilon}\right)dt$$

and show that it is a continuous linear form on $L^q(0,T;\mathcal{C}_\alpha(\mathbb{R},X))$, so that it can be identified with a unique U_ε in $L^q(0,T;\mathcal{C}_\alpha(\mathbb{R},X))'$. Then we can show that U_ε is uniformly bounded in $L^q(0,T;\mathcal{C}_\alpha(\mathbb{R},X))'$; thus, since $L^q(0,T;\mathcal{C}_\alpha(\mathbb{R},X))$ is a separable Banach space, U_ε weakly-* converges up to a subsequence to some U in $L^q(0,T;\mathcal{C}_\alpha(\mathbb{R},X))'$. Using Riesz's representation theorem, one can show that it can be identified with some $U^0_\alpha \in L^{q'}(0,T;L^{q'}_\alpha(\mathbb{R};X'))$ and that u_ε two-scale converges to U_0.

□

We can now state the main result of this section:

Proposition 2.1. *For each ε, let f_ε be a global weak solution to (1.4) in the sense of Arsenev .*

Then, up to an extraction, f_ε 2-scale converges to a function F:

$$F(t,\tau,x,v) = G(t,x+\mathcal{R}(\tau)v, R(\tau)v) \tag{2.3}$$

and G satisfies:

$$\begin{cases} \partial_t G + \left(\frac{1}{2\pi}\int_0^{2\pi}\mathcal{R}(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau + \begin{pmatrix} -v_1(v_2-x_1) \\ v_2(x_1-v_2)-\frac{1}{2}(v_1^2+v_2^2)\end{pmatrix}\right).\nabla_x G \\ + \left(\frac{1}{2\pi}\int_0^{2\pi} R(\tau)\mathcal{E}(t,\tau,x+\mathcal{R}(-\tau)v)d\tau + \begin{pmatrix} v_2(-x_1+v_2) \\ -v_1(-x_1+v_2)\end{pmatrix}\right).\nabla_v G = 0 \\ G_{|t=0} = f_0 \\ E = -\nabla_x V \\ -\Delta V = \displaystyle\int G(t,x+\mathcal{R}(\tau)v, R(\tau)v)dv - 1, \end{cases} \tag{2.4}$$

denoting by R and $\mathcal{R}$ the linear operators defined by :

$$R(\tau) = \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix}, \quad \mathcal{R}(\tau) = (R(-\pi/2) - R(-\pi/2+\tau)).$$

Proof. We do not wish to develop the very beginning of the proof since it is strictly identical to the one given in [60].

The first step consists in deriving the so-called constraint equation. To this end, let $\Psi(t,\tau,x,v)$ be a 2π-periodic oscillating test function in τ and define:

$$\Psi^\varepsilon \equiv \Psi\left(t,\frac{t}{\varepsilon},x,v\right).$$

Then we can write the weak formulation of the Vlasov equation against Ψ^ε and pass to the two-scale limit. We find that the two-scale limit of $f_\varepsilon(t,x,v)$, denoted by $F(t,\tau,x,v)$, satisfies the following equation:

$$\partial_\tau F + v_\perp.\nabla_x F + v \wedge e_z.\nabla_v F_\alpha = 0. \tag{2.5}$$

As a consequence, F is constant along the characteristics meaning that there exists a profile G with:

$$F(t,\tau,x,v) = G(t,x+\mathcal{R}(\tau)v, R(\tau)v), \tag{2.6}$$

where R and $\mathcal{R}$ are defined in the proposition.

The next step is to determine the profile G. We introduce the filtered function g_ε:

$$g_\varepsilon(t,x,v) = f_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v, R(-t/\varepsilon)v), \tag{2.7}$$

which represents the number density from which we have removed the essential oscillations. Notice that this function is chosen so that g_ε two-scale converges, as well as weakly-* converges to G.

We easily obtain the equation satisfied by g_ε:

$$\begin{aligned}
\partial_t g_\varepsilon + &\mathcal{R}(t/\varepsilon)E_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v).\nabla_x g_\varepsilon \\
&+ R(t/\varepsilon)E_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v).\nabla_v g_\varepsilon \\
&- \mathcal{R}(t/\varepsilon)\left((x + \mathcal{R}(-t/\varepsilon)v)_1.(R(-t/\varepsilon)v)^\perp\right).\nabla_x g_\varepsilon \\
&- R(t/\varepsilon)\left((x + \mathcal{R}(-t/\varepsilon)v)_1.(R(-t/\varepsilon)v)^\perp\right).\nabla_v g_\varepsilon = 0.
\end{aligned} \tag{2.8}$$

We now pass to the limit in the sense of distributions. We can prove that the following convergence holds for the nonlinear terms (using elliptic regularity for the electric field to gain some compactness):

$$\mathcal{R}(t/\varepsilon)E_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v).\nabla_x g_\varepsilon \rightharpoonup \frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}(\tau)\mathcal{E}(t,\tau, x + \mathcal{R}(-\tau)v)d\tau.\nabla_x G, \tag{2.9}$$

$$R(t/\varepsilon)E_\varepsilon(t, x + \mathcal{R}(-t/\varepsilon)v).\nabla_v g_\varepsilon \rightharpoonup \frac{1}{2\pi}\int_0^{2\pi} R(\tau)\mathcal{E}(t,\tau, x + \mathcal{R}(-\tau)v)d\tau.\nabla_v G. \tag{2.10}$$

Likewise, we have the following convergences for the last two terms (here there is basically nothing to justify since these are linear quantities):

$$\begin{aligned}
-\mathcal{R}(t/\varepsilon)&\left((x + \mathcal{R}(-t/\varepsilon)v)_1.(R(-t/\varepsilon)v)^\perp\right).\nabla_x g_\varepsilon \\
&\rightharpoonup -\frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}(\tau)\big((x + \mathcal{R}(-\tau)v)_1.(R(-\tau)v)^\perp\big)d\tau.\nabla_x G,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
-R(t/\varepsilon)&\left((x + \mathcal{R}(-t/\varepsilon)v)_1.(R(-t/\varepsilon)v)^\perp\right).\nabla_v g_\varepsilon \\
&\rightharpoonup -\frac{1}{2\pi}\int_0^{2\pi} R(\tau)\big((x + \mathcal{R}(-\tau)v)_1.(R(-\tau)v)^\perp\big)d\tau.\nabla_v G.
\end{aligned} \tag{2.12}$$

We then compute the following quantities:

$$-\frac{1}{2\pi}\int_0^{2\pi} R(\tau)\big((x + \mathcal{R}(-\tau)v)_1 \times (R(-\tau)v)^\perp\big)d\tau = \begin{pmatrix} v_2(-x_1+v_2) \\ -v_1(-x_1+v_2) \end{pmatrix}, \tag{2.13}$$

$$-\frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}(\tau)\big((x + \mathcal{R}(-\tau)v)_1 \times (R(-\tau)v)^\perp\big)d\tau = \begin{pmatrix} -v_1(-x_1+v_2) \\ v_2(x_1-v_2) - \frac{1}{2}(v_1^2+v_2^2) \end{pmatrix}. \tag{2.14}$$

This concludes the proof.

□

Qualitative interpretation of the gyrokinetic system

The influence of the variations of $\mathcal{B}$ is given by the drift/acceleration terms:

$$\begin{pmatrix} -v_1(v_2 - x_1) \\ v_2(x_1 - v_2) - \frac{1}{2}(v_1^2 + v_2^2) \end{pmatrix} .\nabla_x G + \begin{pmatrix} v_2(-x_1 + v_2) \\ -v_1(-x_1 + v_2) \end{pmatrix} .\nabla_v G.$$

Let us imagine that there is no electric field in the asymptotic equation (2.4). Then, the characteristics are given by the following ODEs:

$$\begin{cases} \frac{dx}{dt} = \begin{pmatrix} -v_1(v_2 - x_1) \\ v_2(x_1 - v_2) - \frac{1}{2}(v_1^2 + v_2^2) \end{pmatrix} \\ \frac{dv}{dt} = \begin{pmatrix} v_2(-x_1 + v_2) \\ -v_1(-x_1 + v_2) \end{pmatrix}. \end{cases} \tag{2.15}$$

At first sight, this dynamical system seems a bit complicated with some unpleasant quadratic terms. Actually, this system has some nice invariants.

First, notice that

$$\frac{d}{dt}(x_1 - v_2) = 0.$$

This means that $x_1 = v_2 + C_1$ (with $C_1 = x_1(0) - v_2(0)$). The equation for v can now be written in the simple form:

$$\frac{dv}{dt} = \begin{pmatrix} -C_1 v_2 \\ C_1 v_1 \end{pmatrix}. \tag{2.16}$$

The velocity is thus periodic (and we could compute it easily). Notice also that

$$\frac{d}{dt}(v_1^2 + v_2^2) = 0,$$

so that $v_1^2 + v_2^2 = C_2$ (with $C_2 = v_1^2(0) + v_2^2(0)$).

We get as well a periodic motion for x_1 (since $x_1 = v_2 + C_1$). Finally we notice for x_2:

$$\frac{d}{dt}(x_2 + v_1) = -\frac{1}{2}(v_1^2 + v_2^2) = -\frac{1}{2}C_2. \tag{2.17}$$

The motion along the e_2 direction is hence a sum of a periodic motion plus a fall which only depends on the initial velocity of the particles (and not on their position). Such a drift of the particles "in the bottom" of the tokamak and depending only on the square of their velocity is predicted by physicists and is often referred to as the $\nabla\mathcal{B}$ drift ([158], Section 2.6). To support our discussion we give some graphs of the characteristic curves (figures 6.2 and 6.3).

Likewise, Frénod and Sonnendrücker introduced in [60] the new variables:

$$x_c = x - v^\perp = \begin{pmatrix} x_1 - v_2 \\ x_2 + v_1 \end{pmatrix}$$

the so-called guiding center variable and

$$w = -v^\perp$$

the so-called Larmor radius variable. With these, they showed that the terms in (2.4) involving the electric field were qualitatively close to the drift

$$\frac{1}{2\pi}\int_0^{2\pi} \mathcal{E}^\perp(t, \tau, x_c + R(\tau)w)d\tau.\nabla_{x_c},$$

which corresponds to the gyroaveraged electric drift ([158], Section 2.11).

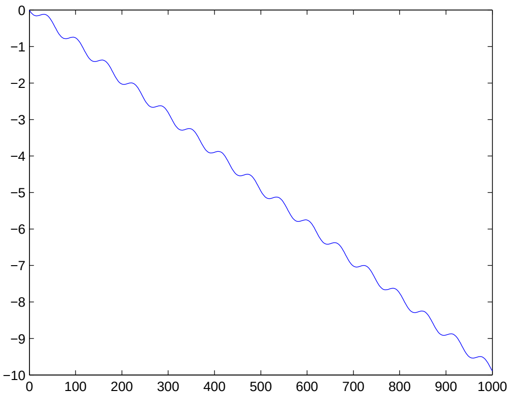


Figure 6.2: $\nabla\mathcal{B}$ drift (in the x-axis: time and in the y-axis: x_2)

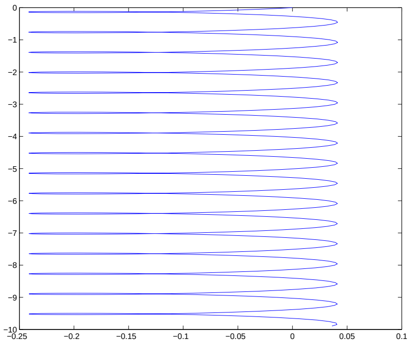


Figure 6.3: Motion of a particle starting at (0,0) in the slice of the tokamak (in the x-axis: x_1 and in the y-axis: x_2)

3 The simplified mathematical model

To be completely rigorous, system (2.4) is the system we would have to study in order to investigate good or bad confinement. Nevertheless, at least at first sight, its algebraic structure seems to be too complicated. Morevore, it is not so clear how to choose a steady state describing the physical situation we want to study.

Consequently we will make several approximations (some of them being quite rough) on (2.4) in order to get a more tractable model.

3.1 A drift-kinetic system

A first step is to obtain a simplified kinetic system, whose dynamics is close to system (2.4). We therefore consider the following drift-kinetic equation, which is actually a classical physical model ([158], Section 2.11). It is commonly used for numerical simulations (see for instance the GYSELA code [78]):

$$\begin{cases} \partial_t f - \frac{1}{2}|v|^2 \partial_{x_2} f + E^\perp . \nabla_x f = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \displaystyle\int f dv - 1. \end{cases} \tag{3.1}$$

This system can be heuristically derived from Newton equations with some elementary physical considerations (see for instance [158], Section 2.6); unfortunately we were not able to derive it rigorously from (1.4) or (2.4). Nevertheless, considering the qualitative study of last paragraph, this seems to be a reasonable model, if we make the following approximations:

- We neglect the oscillations in time, which amounts to get rid of the explicit dependance on the fast time variable τ for the electric field. This can be justified if we consider well-prepared initial data: we refer to the work of Bostan [16].

- We neglect the gyroaverage operators:

$$\frac{1}{2\pi}\int_0^{2\pi} E^\perp(t, x_c + R(\tau)w)d\tau \to E^\perp(t, x_c).$$

which is reasonable if we consider that the variation of the electric field across a Larmor radius is negligeable.

3.2 The bi-temperature drift-fluid system

In order to get a simplified fluid model, we assume that the plasma is made of two phases, one being the cold plasma (with low velocities, low temperature T^- and density ρ^-) and the other the hot plasma (with large velocities, large temperature T^+ and density ρ^+). Of course, we take $T^- < T^+$.

Hence, we assume that the solution to (3.1) takes the form:

$$f(t,x,v) = \rho^+(t,x)\nu^+(v) + \rho^-(t,x)\nu^-(v), \tag{3.2}$$

where ρ^+ (resp. ρ^-) is a positive density such that the total mass is equal to 1, that is:

$$\int (\rho^+ + \rho^-)dx = 1.$$

Furthermore, ν^+ and ν^- are measures defined by:

$$\nu^+ = \frac{1}{2\pi\sqrt{2T^+}}\mathbb{1}_{|v|=\sqrt{2T^+}},$$

$$\nu^- = \frac{1}{2\pi\sqrt{2T^-}}\mathbb{1}_{|v|=\sqrt{2T^-}}.$$

Considering that transverse particle velocities are isotropically distributed is physically relevant for such magnetized plasmas, as indicated in [150].

We observe that $\int d\nu^\pm = 1$ and $\int v d\nu^\pm = 0$. Thus, the charge and current densities are given by:

$$\rho(t,x) := \int f dv = \rho^+(t,x) + \rho^-(t,x)$$

and

$$u(t,x) := \frac{\int f v dv}{\int f dv} = 0.$$

In addition, we have:

$$T^+ = \frac{1}{2}\int |v|^2 d\nu^+(w),$$

$$T^- = \frac{1}{2}\int |v|^2 d\nu^-(w).$$

The kinetic temperature $T(t,x)$ of the plasma is then given by:

$$\begin{aligned} T(t,x) :=& \frac{1}{2}\frac{\int f(v-u(t,x))^2 dv}{\rho(t,x)} \\ =& \frac{\rho^+(t,x)T^+ + \rho^-(t,x)T^-}{\rho^+(t,x)+\rho^-(t,x)}. \end{aligned} \tag{3.3}$$

We moreover assume we can decouple the transport equations satisfied by ρ^+ and ρ^-. We get in the end the macroscopic system:

$$\begin{cases} \partial_t \rho^+ - T^+ \partial_{x_2} \rho^+ + E^\perp . \nabla_x \rho^+ = 0 \\ \partial_t \rho^- - T^- \partial_{x_2} \rho^+ + E^\perp . \nabla_x \rho^- = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \rho^+ + \rho^- - 1 \\ (\rho^+, \rho^-)_{|t=0} = (\rho_0^+, \rho_0^-) \text{ with } \int \rho_0^+ + \rho_0^- = 1, \end{cases} \tag{3.4}$$

with $x \in [0, L] \times \mathbb{R}/L\mathbb{Z}$.

As noticed in the introduction, this systems looks like $2D$ incompressible Euler, but with two kinds of vorticities.

Here, the constant $L > 0$ stands for the size of the box. The periodicity with respect to x_2 is physically justified if we consider that L is small enough with respect to the size of the tokamak, so that we can decompose it in many identical cells of size L (see Figure 1).

We now have to impose some relevant boundary conditions on $x_1 = 0, L$:

- For the Poisson equation, we opt for the perfect conductor assumption on $x_1 = 0, L$ (which is the ideal case for plasma physics models).

$$E^\perp . n = \pm E_2 = \partial_{x_{c2}} V = 0. \tag{3.5}$$

To this end, we can impose the following Dirichlet boundary condition on $x_1 = 0, L$:

$$V = 0. \tag{3.6}$$

From the fluid mechanics point of view, we observe this corresponds to the classical no slip condition.

- For the transport equation, we actually do not need any boundary condition. There is indeed no entering or leaving trajectories, since the linear "drift" operator only entails a motion along the e_2 direction, and $E_2 = 0$ on the boundaries $x_1 = 0, L$.

Following classical works on the Cauchy problem for the $2D$ incompressible Euler system (we refer for instance to the book of Majda and Bertozzi [124]), we get the following global existence and uniqueness result of strong and weak solutions to (3.4):

Theorem 3.1. *Let $\rho_0 = (\rho_0^+, \rho_0^-) \in (L^1((0, L) \times \mathbb{R}/L\mathbb{Z}))^2$ with ρ_0^+, ρ_0^- non-negative and $\int (\rho_0^+ + \rho_0^-) dx = 1$.*

i. (Kato, [109]) If ρ_0 is H^s (with $s > 1$) then there exists a unique classical solution ρ to (3.4) in $\mathcal{C}_t^0([0, \infty[, H^s) \cap \mathcal{C}_t^1([0, \infty[, H^{s-1})$ with initial data ρ_0.

ii. (Yudovic, [160]) If $\rho_0 \in L^\infty$, then there exists a unique global non-negative weak solution $\rho \in L_t^\infty(L^1 \cap L^\infty)$ to (3.4) with initial data ρ_0.

Sketch of proof. i. The existence of a global strong solution follows from a fixed point argument, as in the classical work on 2D Euler by Kato [109]. Actually, dealing with two "vorticities" only slightly modifies the main lines; therefore, for the sake of brevity, we only recall here the main arguments of the proof.

We may consider the map $F : (\xi^+, \xi^-) \mapsto (\rho^+, \rho^-)$ where (ρ^+, ρ^-) is solution to:

$$\begin{cases} \partial_t \rho^+ + \left(E^\perp - T^+ e_2\right) . \nabla_x \rho^+ = 0 \\ \partial_t \rho^- + \left(E^\perp - T^- e_2\right) . \nabla_x \rho^- = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \xi^+ + \xi^- - 1 \\ V = 0 \text{ on } x_1 = 0, L \\ (\rho^+, \rho^-)_{|t=0} = (\rho_0^+, \rho_0^-) \text{ with } \int \rho_0^+ + \rho_0^- = 1. \end{cases} \tag{3.7}$$

which is well-defined if (ξ^+, ξ^-) is smooth enough thanks to the characteristics' method. As in Kato's proof, for any $T > 0$, one can show that F is continuous on some convex compact S of $(\mathcal{C}([0,T] \times [0,L] \times \mathbb{R}/L\mathbb{Z}))^2$ and that $F(S) \subset S$. The existence of a fixed point is finally a consequence of Schauder's theorem. The crucial points are:

- Establishing some log-lipschitz estimate on the electric field (in this case with a constant involving the L^∞ norms of the two vorticities), which is obtained exactly in the same way as for Euler's equation.
  If $\xi^+, \xi^- \in L^\infty$ and E is solution to the elliptic problem

$$\begin{cases} E = -\nabla_x V \\ -\Delta_x V = \xi^+ + \xi^- - 1 \\ V = 0 \text{ on } x_1 = 0, L \end{cases} \tag{3.8}$$

  then there exists C depending on L but independent of ξ^+, ξ^- such that for all $(x, y) \in ([0, L] \times \mathbb{R}/L\mathbb{Z})$, we have:

$$|E(x) - E(y)| \leq C \left(\|\xi^+\|_{L^\infty} + \|\xi^-\|_{L^\infty}\right) |x - y| \log^+(|x - y|), \tag{3.9}$$

  where $\log^+(s) = \begin{cases} 1 - \log s & \text{if} \quad s \leq 1 \\ 0 \quad \text{if} \quad s > 1. \end{cases}$
- The L^∞ norm of each "vorticity" ρ^+ and ρ^- is conserved by the transport equations, since $\mathrm{div}_x \left(E^\perp - T^\pm e_2\right) = 0$.

Finally uniqueness is obtained by an energy estimate argument, exactly as in Kato's proof.

ii. For the bi-dimensional Euler equations in a general domain, the result is due to Yudovic [160]. His main arguments can be easily adapted and reproduced in our case, with a crucial use of the two above points. The result also follows by a simple adaptation of the alternative proof given in ([124], Theorem 8.1), which consists in regularizing the initial data, solve the smoothed Cauchy problem and then pass to the weak limit thanks to a crucial compactness result ([124], Proposition 8.2).

□

3.3 Modeling of the plasma equilibria

We now consider the following steady states, in order to model what is happening in the "good curvature" or the "bad curvature" side of the plasma, the only difference being the relative position between the hot and the cold plasma:

- in the "bad curvature" region:

$$\mu^{bad}(x_1) = \left(\underbrace{\mu^{bad,+} = 1 - \frac{x_1}{L}}_{\text{hot plasma}}, \underbrace{\mu^{bad,-} = \frac{x_1}{L}}_{\text{cold plasma}} \right), \tag{3.10}$$

- in the "good curvature" region:

$$\mu^{good}(x_1) = \left(\mu^{good,+} = \frac{x_1}{L}, \mu^{good,-} = 1 - \frac{x_1}{L} \right). \tag{3.11}$$

These are steady states of (3.4) and the associated electric field is zero.

We observe here for μ^{good} the temperature $T(t,x)$ is:

$$T(t,x) = T^+\frac{x_1}{L} + T^-\left(1-\frac{x_1}{L}\right). \tag{3.12}$$

Such linear transitions between the cold and hot plasma are the most simple model one can think of. We observe that the slope of the line is equal to $\frac{T^+-T^-}{L}$ which is referred to as the temperature gradient in this chapter.

We now investigate stability and instability for μ^{bad} and μ^{good}.

4 Linear instability in the "bad curvature" region

We first consider the case of the "bad curvature" region, for which we expect to obtain instability. The equilibrium writes:

$$\mu^{bad}(x_1) = \begin{pmatrix} 1-\frac{x_1}{L} \\ \frac{x_1}{L}. \end{pmatrix}$$

The first step before trying to prove any instability property for the nonlinear transport equations consists in investigating the problem of instability for the linearized operator around μ^{bad}. We accordingly consider the following linearized system (for $t>0$, $x \in [0,L]\times\mathbb{R}/L\mathbb{Z}$):

$$\begin{cases} \partial_t\rho^+ - T^+\partial_{x_2}\rho^+ - \frac{E_2}{L} = 0 \\ \partial_t\rho^- - T^-\partial_{x_2}\rho^- + \frac{E_2}{L} = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \rho^+ + \rho^- \ , \ V = 0 \text{ on } x_1 = 0, L \\ (\rho^+,\rho^-)_{|t=0} = (\rho_0^+,\rho_0^-) \text{ with } \int \rho_0^+ + \rho_0^- = 0. \end{cases} \tag{4.1}$$

4.1 Looking for unstable eigenfunctions

We look for special solutions under the form $\rho_k(t,x) = \begin{pmatrix} h_k^+(t)g_k(x) \\ h_k^-(t)g_k(x) \end{pmatrix}$. The equations (4.1) can be restated as:

$$\begin{cases} \partial_t(h_k^+(t)g_k(x)) - T^+\partial_{x_2}(h_k^+(t)g_k(x)) - \frac{E_2}{L} = 0 \\ \partial_t(h_k^-(t)g_k(x)) - T^-\partial_{x_2}(h_k^-(t)g_k(x)) + \frac{E_2}{L} = 0 \\ E = -\nabla_x V \\ -\Delta_x V = (h_k^-(t) + h_k^+(t))g_k(x) \ , \ V = 0 \text{ on } x_1 = 0, L. \end{cases} \tag{4.2}$$

We take g_k with the particular form $g_k(x) = \sin\left(\frac{k_1}{L}\pi x_1\right)e^{i2\pi\frac{k_2}{L}x_2}$ (with $k_1,k_2 \in \mathbb{Z}^*$), so that g_k is an eigenfunction for the laplacian with the considered boundary conditions. It satisfies indeed:

$$\Delta g_k(x) = -\pi^2\left(\frac{k_1^2}{L^2} + 4\frac{k_2^2}{L^2}\right)g_k(x) \tag{4.3}$$

and also $g_k = 0$ on $x_1 = 0, L$.

The solution to the Poisson equation is then given by:

$$V_k = \frac{1}{\pi^2\left(\frac{k_1^2}{L^2} + 4\frac{k_2^2}{L^2}\right)}(h_k^-(t) + h_k^+(t))g_k(x)$$

and thus we have

$$E_2 = \frac{-i2k_2/L}{\pi\left(\frac{k_1^2}{L^2}+4\frac{k_2^2}{L^2}\right)}(h_k^-(t)+h_k^+(t))g_k(x),$$

which leads us to study the following first order ordinary differential equation:

$$\partial_t \begin{pmatrix} h_k^+(t) \\ h_k^-(t) \end{pmatrix} + 1/L \begin{pmatrix} -2i\pi T^+ k_2 + \frac{i2k_2}{\pi\left(\frac{k_1^2}{L}+4\frac{k_2^2}{L}\right)} & \frac{i2k_2}{\pi\left(\frac{k_1^2}{L}+4\frac{k_2^2}{L}\right)} \\ -\frac{i2k_2}{\pi\left(\frac{k_1^2}{L}+4\frac{k_2^2}{L}\right)} & -2i\pi T^- k_2 - \frac{i2k_2}{\pi\left(\frac{k_1^2}{L}+4\frac{k_2^2}{L}\right)} \end{pmatrix} \begin{pmatrix} h_k^+(t) \\ h_k^-(t) \end{pmatrix} = 0. \tag{4.4}$$

We want to compute the eigenvalues of the matrix; its characteristic polynomial states:

$$X^2 + 2i\pi k_2\left(T^+ + T^-\right)X - 4\pi^2 k_2^2 T^+ T^- - \frac{4k_2^2}{\frac{k_1^2}{L}+4\frac{k_2^2}{L}}\left(T^+ - T^-\right)$$

and its discriminant:

$$\begin{aligned} \Delta &= -4\pi^2 k_2^2 (T^+ - T^-)^2 + \frac{16k_2^2 L}{k_1^2+4k_2^2}\left(T^+ - T^-\right) \\ &= -4\pi^2 k_2^2 (T^+ - T^-)\left((T^+ - T^-) - \frac{4L}{\pi^2(k_1^2+4k_2^2)}\right). \end{aligned} \tag{4.5}$$

We recall that by definition,

$$T^+ - T^- > 0.$$

We can now distinguish between two cases:

- First case:

$$\frac{4}{5\pi^2} > \frac{T^+ - T^-}{L}. \tag{4.6}$$

  In the case where the gradient of temperature is not too large, then there exist k_1, k_2 such that $\Delta > 0$. We consequently obtain two complex roots, one of which has a stricly negative real part equal to $-\frac{\sqrt{\Delta}}{2}$. In other words, this shows the existence of an unstable mode.

- Second case:

$$\frac{4}{5\pi^2} \leq \frac{T^+ - T^-}{L}. \tag{4.7}$$

  In the opposite case, we always have $\Delta \leq 0$ and consequently, we are not able to find a growing mode !

  This phenomenon may at first sight look like a mathematical artifact due to the periodicity constraint in the x_2 direction. This is not the case, as we could have considered an infinite cylinder instead of studying periodic conditions in x_2, and the results would have been similar.

  Furthermore as explained in the introduction, the existence of such a threshold is well known in plasma physics, beyond which one can expect tremendous confinement properties. In very rough terms: heating brings stability. The stable mode is referred to as the H-mode, by opposition to the L-mode.

***Remark* 4.1.** In the “good curvature” region, that is around μ_{good}, the linearized system states:

$$\begin{cases} \partial_t \rho^+ - T^+ \partial_{x_2} \rho^+ + \frac{E_2}{L} = 0 \\ \partial_t \rho^- - T^- \partial_{x_2} \rho^- - \frac{E_2}{L} = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \rho^+ + \rho^- \ , \ V = 0 \text{ on } x_1 = 0, L. \end{cases} \tag{4.8}$$

With the same method, we obtain the following ordinary differential equation:

$$\partial_t \begin{pmatrix} h_k^+(t) \\ h_k^-(t) \end{pmatrix} + 1/L \begin{pmatrix} -2i\pi T^+ k_2 - \frac{i2k_2}{\pi\left(\frac{k_1^2}{L} + 4\frac{k_2^2}{L}\right)} & -\frac{i2k_2}{\pi\left(\frac{k_1^2}{L} + 4\frac{k_2^2}{L}\right)} \\ \frac{i2k_2/L}{\pi\left(\frac{k_1^2}{L} + 4\frac{k_2^2}{L}\right)} & -2i\pi T^- + k_2 \frac{i2k_2}{\pi\left(\frac{k_1^2}{L} + 4\frac{k_2^2}{L}\right)} \end{pmatrix} \begin{pmatrix} h_k^+(t) \\ h_k^-(t) \end{pmatrix} = 0. \tag{4.9}$$

We consequently have to look for the roots to the polynomial:

$$X^2 + 2i\pi k_2 v^2 X - 4\pi^2 T^+ T^- k_2^2 + \frac{4k_2^2}{\frac{k_1^2}{L} + 4\frac{k_2^2}{L}} (T^+ - T^-).$$

In this case, one can check as before that the discriminant is always stricly negative, so that the roots always have a vanishing real part. As a result, we do not find any unstable mode by this method. Note that we only consider this fact as a good and encouraging indication for stability around μ^{good}. Actually, we will never use it when proving nonlinear stability in section 5.

***Remark* 4.2.** We finally mention that the quest for unstable modes seems more difficult in the kinetic case, since one has to deal with the continuous velocity space. A very famous criterion in the Vlasov-Poisson case was given by Penrose [133] and rigorously studied later on by Guo and Strauss [89].

4.2 On the spectrum of the linearized operator around μ^{bad} on $H^s([0, L] \times \mathbb{R}/\mathbb{Z})$, $s \geq 0$

We assume here the existence of unstable modes for the linearized operator around μ^{bad}, that is in the situation where we have:

$$\frac{4}{5\pi^2} > \frac{T^+ - T^-}{L}.$$

The main tool we use now is a variant of a classical theorem by Weyl, stated for instance in the paper of Guo and Strauss [89] and proved by Vidav in [152]. Basically, it gives informations on the spectrum of some compact perturbation of a linear operator which has no spectrum in the half-plane $\{\operatorname{Re} z > 0\}$. It entails the existence of a dominant unstable eigenvalue provided the existence of at least one growing mode.

Theorem 4.1 (Weyl). *Let Y be a Banach space and A be a linear operator that generates a strongly continuous semigroup on Y such that $\|e^{-tA}\| \leq M$ for all $t \geq 0$. Let K be a compact operator from Y to Y. Then $(A + K)$ generates a strongly continuous semigroup $e^{-t(A+K)}$ and $\sigma(-A - K)$ consists of a finite number of eigenvalues of finite multiplicity in $\{\operatorname{Re} \lambda > \delta\}$ for every $\delta > 0$. These eigenvalues can be labeled by:*

$$\operatorname{Re} \lambda_1 \geq \operatorname{Re} \lambda_2 \geq \ldots \operatorname{Re} \lambda_N \geq \delta.$$

Furthermore, for any $\gamma > \operatorname{Re}\lambda_1$, there exists some constant C_γ such that

$$\|e^{-t(A+K)}\|_{Y\to Y} \leq C_\gamma e^{t\gamma}. \tag{4.10}$$

Corollary 4.1. *Let $s \geq 0$ and*

$$Y = \{y_1, y_2 \in H^s([0,L]\times\mathbb{R}/\mathbb{Z})^2, \int (y_1+y_2)dx = 0\}.$$

Let M be the linear operator defined by:

$$g = \begin{pmatrix} g^+ \\ g^- \end{pmatrix} \in Y \mapsto M\begin{pmatrix} g^+ \\ g^- \end{pmatrix} = \begin{pmatrix} -T^+\partial_{x_2}g^+ - \frac{E_2}{L} \\ -T^-\partial_{x_2}g^- + \frac{E_2}{L} \end{pmatrix}, \tag{4.11}$$

with $E_2 = -\partial_{x_2}V$, $-\Delta V = (g^+ + g^-)$ and with $V = 0$ on $x_1 = 0, L$.

Then there exists an eigenvalue λ with a non-vanishing and maximal real part associated to a C^∞ eigenvector. Furthermore for any $\gamma > \operatorname{Re}\lambda$, there is a constant $C(\gamma, s)$ such that for all $t \geq 0$:

$$\|e^{-tM}\|_{H^s\to H^s} \leq C(\gamma, s)e^{t\gamma} \tag{4.12}$$

Proof. The linear operator A, defined by

$$A : g \mapsto \begin{pmatrix} -T^+\partial_{x_2}g^+ \\ -T^-\partial_{x_2}g^- \end{pmatrix}$$

is clearly an isometry on Y (indeed we know how to explicitly solve the semi-group). The operator K is defined by

$$K : g \mapsto \begin{pmatrix} -\frac{E_2}{L} \\ \frac{E_2}{L} \end{pmatrix}.$$

This operator is compact on Y thanks to standard elliptic estimates.

Moreover, we have shown in the last paragraph the existence of an unstable eigenfunction for the linearized operator that belongs to any H^s. We can therefore apply Weyl's theorem which gives the existence of an eigenfunction associated to an eigenvalue with a non-vanishing and maximal real part. At last, the estimate in the corollary follows directly from the estimate given in Weyl's theorem.

□

In the following, we denote for any $h \in L^2$ with $\int h dx = 0$, $\Delta^{-1}h$ the unique solution u in H^1 to the problem:

$$\begin{cases} -\Delta u = h \\ u = 0 \text{ on } x_1 = 0, L \end{cases}$$

We now prove that any eigenvector associated to a non vanishing eigenvalue for the linearized operator any L^2 actually has $\mathcal{C}^\infty$ regularity.

Lemma 4.1. *Let $\rho = (\rho^+, \rho^-) \in (L^2)^2$ with $\int(\rho^+ + \rho^-)dx = 0$ and $\lambda \neq 0$ such that:*

$$\begin{aligned} -T^+\partial_{x_2}\rho^+ - \frac{1}{L}\partial_{x_2}\Delta^{-1}(\rho^+ + \rho^-) &= \lambda\rho^+, \\ -T^-\partial_{x_2}\rho^- - \frac{1}{L}\partial_{x_2}\Delta^{-1}(\rho^+ + \rho^-) &= \lambda\rho^-, \end{aligned}$$

then $(\rho^+, \rho^-) \in \mathcal{C}^\infty([0,L]\times\mathbb{R}/\mathbb{Z})$.

Proof. The principle of the proof is to show by recursion that $\rho \in H^k$, for any $k \in \mathbb{N}^*$.

For $k = 1$, we can observe, thanks to elliptic estimates, that $\partial_{x_2}\rho \in L^2$. Indeed, we have the identity:

$$-T^+\partial_{x_2}\rho^+ = \frac{1}{L}\partial_{x_2}\Delta^{-1}(\rho^+ + \rho^-) + \lambda\rho^+. \tag{4.13}$$

Hence, $\partial_{x_2}\rho^+ \in L^2$. Likewise, $\partial_{x_2}\rho^- \in L^2$.

We can apply the differential operator ∂_{x_1} to the equation satisfied by ρ^+, which entails:

$$-T^+\partial_{x_1}\partial_{x_2}\rho^+ - \frac{1}{L}\partial_{x_1}\partial_{x_2}\Delta^{-1}(\rho^+ + \rho^-) \quad = \quad \lambda\partial_{x_1}\rho^+.$$

Then we multiply by $\partial_{x_1}\rho^+$ and integrate with respect to x:

$$\lambda\|\partial_{x_1}\rho^+\|_{L^2}^2 \quad = \quad \int -T^+\partial_{x_1}\partial_{x_2}\rho^+\partial_{x_1}\rho^+dx - \frac{1}{L}\int \partial_{x_1}\partial_{x_2}\Delta^{-1}(\rho^+ + \rho^-)\partial_{x_1}\rho^+dx.$$

Thanks to the periodicity with respect to x_2, we get:

$$\int \partial_{x_1}\partial_{x_2}\rho^+\partial_{x_1}\rho^+dx = 1/2\int \partial_{x_2}\left(\partial_{x_1}\rho^+\right)^2 dx = 0.$$

Then using Cauchy-Schwarz inequality:

$$\lambda\|\partial_{x_1}\rho^+\|_{L^2}^2 \le \frac{1}{L}\|\partial_{x_1}\partial_{x_2}\Delta^{-1}(\rho^+ + \rho^-)\|_{L^2}\|\partial_{x_1}\rho^+\|_{L^2}.$$

As a result we showed that:

$$\lambda^2\|\partial_{x_1}\rho^+\|_{L^2} \le \frac{1}{L}\|\partial_{x_1}\partial_{x_2}\Delta^{-1}(\rho^+ + \rho^-)\|_{L^2}. \tag{4.14}$$

By standard elliptic estimates the right-hand side is finite since ρ^+ et ρ^- belong to L^2. As a result we have proved $\rho \in H^1$.

We can then conclude by recursion. Let us assume that $\rho \in H^k$, for some $k \in \mathbb{N}^*$; we prove that $\rho \in H^{k+1}$.

Let $\alpha, \beta \in \mathbb{N}$ such that $\alpha + \beta = k$. We set $\partial_k = \partial_{x_1}^\alpha\partial_{x_2}^\beta$. Then $\partial_k\rho$ belongs to L^2 and satisfies the equation:

$$\begin{aligned}
-T^+\partial_{x_2}\partial_k\rho^+ - \frac{1}{L}\partial_k\partial_{x_2}\Delta^{-1}(\rho^+ + \rho^-) \quad &= \quad \lambda\partial_k\rho^+, \\
-T^-\partial_{x_2}\partial_k\rho^- - \frac{1}{L}\partial_k\partial_{x_2}\Delta^{-1}(\rho^+ + \rho^-) \quad &= \quad \lambda\partial_k\rho^-.
\end{aligned}$$

By elliptic regularity, $\partial_k\partial_{x_2}\Delta^{-1}(\rho^+ + \rho^-) \in H^1$. Thus, we are in the same case as for $L^2 \to H^1$, which entails that $\rho \in H^{k+1}$.

□

5 On nonlinear stability

Let $\rho = \begin{pmatrix}\rho^+ \\ \rho^-\end{pmatrix}$ a solution to the nonlinear transport equation (3.4), that we recall here:

$$\begin{cases}
\partial_t\rho^+ - T^+\partial_{x_2}\rho^+ + E^\perp.\nabla_x\rho^+ = 0 \\
\partial_t\rho^- - T^-\partial_{x_2}\rho^+ + E^\perp.\nabla_x\rho^- = 0 \\
E = -\nabla_x V \\
-\Delta_x V = \rho^+ + \rho^- - 1 \\
V = 0 \text{ on } x_1 = 0, L \\
(\rho^+, \rho^-)_{|t=0} = (\rho_0^+, \rho_0^-) \text{ with } \int \rho_0^+ + \rho_0^- = 1.
\end{cases}$$

We begin with a very simple observation in the limit case $T = T^+ = T^-$. In this situation, setting $\tilde{\rho} = \rho^+ + \rho^-$, ρ satisfies the usual $2D$ Euler equation in vorticity form, with some linear drift term:

$$\partial_t \tilde{\rho} + E^\perp.\nabla_x \tilde{\rho} - T\partial_{x_2}\tilde{\rho} = 0,$$
$$E^\perp = \nabla^\perp \Delta^{-1}(\tilde{\rho} - 1).$$

We investigate stability around the steady state $\tilde{\mu} = \mu^+ + \mu^- = 1$. It is well-known that any L^p norm of the "vorticity" $\tilde{\rho} - 1$ is non-increasing, that is:

$$\|\tilde{\rho}(t) - 1\|_{L^p} \leq \|\tilde{\rho}(0) - 1\|_{L^p}, \tag{5.1}$$

which clearly entails nonlinear L^p stability.

The idea in the general case $T^+ > T^-$ in order to show nonlinear stability is to obtain some similar nice energy identity.

Theorem 5.1. *For any inital data $\rho_0 \in L^\infty$ with $\int(\rho_0^+ + \rho_0^-)dx = 1$, the solution ρ to (3.4) satisfies the following statements.*

- *Around the "good-curvature" steady state, the following functional is non-increasing:*

$$\mathcal{E}(t) = \|\rho - \mu^{good}\|_{L^2}^2 + \frac{1}{L(T^+ - T^-)}\int |\nabla V|^2 dx \leq \mathcal{E}(0), \tag{5.2}$$

with $\|\rho - \mu\|_{L^2}^2 = \|\rho^+ - \mu^+\|_{L^2}^2 + \|\rho^- - \mu^-\|_{L^2}^2$ and $\nabla V = \nabla\Delta^{-1}(\rho^+ + \rho^- - 1)$.

- *Around the "bad-curvature" steady state, the following functional is non-increasing:*

$$\mathcal{F}(t) = \|\rho - \mu^{bad}\|_{L^2}^2 - \frac{1}{L(T^+ - T^-)}\int |\nabla V|^2 dx \leq \mathcal{F}(0), \tag{5.3}$$

Remark 5.1. We observe that in the functionals, the first term corresponds to the enstrophy in fluid mechanics (in a modulated form adapted to our needs). The second term can be interpreted as the kinetic energy of the fluid (whereas from the plasma physics point of view, this is the electric energy).

As an immediate consequence of this theorem, we obtain L^2 stability, in the "good-curvature" side, but also in the "bad-curvature" side, for large enough temperature gradients, like for the linearized equations.

Corollary 5.1. *The equilibrium μ^{good} is nonlinearly stable with respect to the L^2 norm.*

If the temperature gradient $\frac{T^+ - T^-}{L}$ satisfies:

$$\frac{T^+ - T^-}{L} > \frac{1}{\pi^2}, \tag{5.4}$$

then the equilibrium μ^{bad} is nonlinearly stable with respect to the L^2 norm.

Proof of the corollary. Thanks to the energy identity (5.2) and to the Poisson equation:

$$\begin{aligned}
\|\rho - \mu^{good}\|_{L^2}^2 &\leq \|\rho - \mu^{good}\|_{L^2}^2 + \frac{1}{L(T^+ - T^-)}\int |\nabla V|^2 dx \\
&\leq \|\rho(0) - \mu^{good}\|_{L^2}^2 + \frac{1}{L(T^+ - T^-)}\int |\nabla V(0)|^2 dx \\
&\leq \|\rho(0) - \mu^{good}\|_{L^2}^2 + C\frac{1}{L(T^+ - T^-)}\|\rho(0) - \mu^{good}\|_{L^2}^2.
\end{aligned} \tag{5.5}$$

This means that for any $\eta > 0$ there exists $\delta > 0$ such that if $\|\rho(0) - \mu^{good}\|_{L^2} \leq \delta$ then for any $t \geq 0$, $\|\rho - \mu^{good}\|_{L^2} \leq \eta$. In other words, μ^{good} is nonlinearly stable for the L^2 norm.

Around the bad-curvature steady state, we have shown that the following quantity is non-increasing:

$$\mathcal{F}(t) = \|\rho - \mu^{bad}\|_{L^2}^2 - \frac{1}{L(T^+ - T^-)} \int |\nabla V|^2 dx. \tag{5.6}$$

We can easily prove with the help of Fourier variables the Poincaré-like inequality:

$$\int |\nabla V|^2 dx \leq \frac{L^2}{\pi^2} \|\rho - \mu^{bad}\|_{L^2}^2. \tag{5.7}$$

So we get:

$$\begin{aligned} \|\rho - \mu^{bad}\|_{L^2}^2 &\leq \mathcal{F}(0) + \frac{1}{L(T^+ - T^-)} \int |\nabla V|^2 dx \\ &\leq \mathcal{F}(0) + \frac{1}{L(T^+ - T^-)} \frac{L^2}{\pi^2} \|\rho - \mu^{bad}\|_{L^2}^2. \end{aligned}$$

Hence,

$$\left(1 - \frac{L}{\pi^2 (T^+ - T^-)}\right) \|\rho - \mu^{bad}\|_{L^2}^2 \leq \|\rho(0) - \mu^{bad}\|_{L^2}^2. \tag{5.8}$$

As a consequence, there is L^2 nonlinear stability in the bad curvature side, provided that

$$\frac{T^+ - T^-}{L} > \frac{1}{\pi^2}.$$

Otherwise, we can not deduce anything.

□

Remark 5.2. Note also that in the linear discussion, there was stability provided that

$$\frac{T^+ - T^-}{L} \geq \frac{4}{5\pi^2}. \tag{5.9}$$

We do not know what happens for $\frac{T^+ - T^-}{L} \in [\frac{4}{5\pi^2}, \frac{1}{\pi^2}]$ in the nonlinear case. Maybe one could expect to observe some bifurcation phenomenon.

Remark 5.3. Actually the decrease of $\mathcal{E}(t)$ tells us a little more than just L^2 stability. Indeed, there exists $C > 0$ such that for any $\delta > 0$, if $\|\rho(0) - \mu^{good}\|_{L^2}^2 \leq \delta$, then for any $t > 0$,

$$\|\rho(t) - \mu^{good}\|_{L^2}^2 \leq \left(1 + C \frac{1}{L(T^+ - T^-)}\right) \delta.$$

This means in particular that for large values of $T^+ - T^-$, better confinement is obtained, which is qualitatively in agreement with experimental observations ([79]).

We now prove Theorem 5.1. We first give a technical lemma in which will help for the proof.

Lemma 5.1. *For $\mu = \mu^{good}$ or μ^{bad}, we have for any $t > 0$,*

$$\int E_2 \left(\rho^+ - \mu^+\right) dx = - \int E_2 \left(\rho^- - \mu^-\right) dx. \tag{5.10}$$

Proof. In order to prove this identity, one can simply compute:

$$\begin{aligned}\int E_2\left((\rho^- - \mu^-) - (\rho^+ - \mu^+)\right)dx &= \int E_2\left(\rho^+ + \rho^- - \mu^+ - \mu^- - 2(\rho^+ - \mu^+)\right)dx \\ &= \int E_2\left(\Delta V - 2(\rho^+ - \mu^+)\right)dx \\ &= -2\int E_2\left(\rho^+ - \mu^+\right)dx.\end{aligned}$$

Indeed, thanks to periodicity with respect to x_2 and since $\partial_{x_2} V = 0$ on $x_1 = 0, L$, we get:

$$\begin{aligned}\int \partial_{x_2} V \Delta V dx &= -\int \partial_{x_2} \nabla V . \nabla V dx + \underbrace{\int \operatorname{div}(\partial_{x_2} V \nabla V) dx}_{=0} \\ &= \int \partial_{x_2}\left(\frac{|\nabla V|^2}{2}\right)dx = 0.\end{aligned}$$

If we make the same computation, by symmetry, we can also observe that:

$$\int E_2\left((\rho^- - \mu^-) - (\rho^+ - \mu^+)\right)dx = 2\int E_2\left(\rho^- - \mu^-\right)dx.$$

□

Proof of Theorem 5.1. We will only focus on the proof of the conservation of $\mathcal{E}(t)$, the proof being very similar for $\mathcal{F}(t)$. For the sake of readibility, we write μ instead of μ^{good} until the end of the proof.

We observe that the equation satisfied by $(\rho - \mu)$ reads in this case:

$$\partial_t(\rho - \mu) - \begin{pmatrix} T^+\partial_{x_2}(\rho^+ - \mu^+) \\ T^-\partial_{x_2}(\rho^- - \mu^-) \end{pmatrix} + E^\perp . \nabla_x(\rho - \mu) = \begin{pmatrix} -\frac{E_2}{L} \\ \frac{E_2}{L} \end{pmatrix} \tag{5.11}$$

and $E = -\nabla_x V$ with $-\Delta V = \rho^+ + \rho^- - \mu^+ - \mu^-$ and with Dirichlet conditions on the boundaries $x_1 = 0$ and $x_1 = 1$.

Taking the scalar product with $(\rho - \mu)$ in the transport equation and integrating with respect to x entails:

$$\frac{d}{dt}\|\rho - \mu\|_{L^2}^2 = \int -\frac{E_2}{L}(\rho^+ - \mu^+)dx + \int \frac{E_2}{L}(\rho^- - \mu^-). \tag{5.12}$$

Indeed, thanks to the periodicity with respect to x_2, we first have:

$$\int \partial_{x_2}(\rho^+ - \mu^+)(\rho^+ - \mu^+)dx = \int \frac{1}{2}\partial_{x_2}(\rho^+ - \mu^+)^2 dx = 0.$$

Similarly we have:

$$\int \partial_{x_2}(\rho^- - \mu^-)(\rho^- - \mu^-)dx = 0.$$

In the same fashion, with Green's Formula, and using $\operatorname{div} E^\perp = 0$, we have:

$$\begin{aligned}\int E^\perp . \nabla_x(\rho - \mu)(\rho - \mu)dx &= \frac{1}{2}\int E^\perp . \nabla_x(\rho - \mu)^2 dx \\ &= \frac{1}{2}\int \operatorname{div}(E^\perp(\rho - \mu)^2)dx \\ &= \frac{1}{2}\left(\int_{x_1=0} E^\perp(\rho - \mu)^2.(-e_1)dx_2 + \int_{x_1=1} E^\perp(\rho - \mu)^2.e_1 dx_2\right) \\ &= 0,\end{aligned}$$

since $E_2 = -\partial_{x_2} V = 0$ on $x_1 = 0, L$.

Now, by Lemma 5.1 we get:

$$\begin{aligned}
\frac{d}{dt}\|\rho-\mu\|^2_{L^2} &= \int -\frac{E_2}{L}(\rho^+ - \mu^+)dx + \int \frac{E_2}{L}(\rho^- - \mu^-)dx \\
&= -2\int \frac{E_2}{L}\left(\rho^+ - \mu^+\right)dx \left(= 2\int \frac{E_2}{L}(\rho^- - \mu^-)\right).
\end{aligned}$$

We have:

$$\begin{aligned}
-\int E_2\left(\rho^+ - \mu^+\right)dx &= -\int V\partial_{x_2}\left(\rho^+ - \mu^+\right)dx + \underbrace{\int \operatorname{div}(V\left(\rho^+ - \mu^+\right)e_2)dx}_{=0} \\
&= \frac{1}{T^+}\int -V\left(\partial_t(\rho^+ - \mu^+) + E^\perp.\nabla_x(\rho^+ - \mu^+) + \frac{E_2}{L}\right)dx \\
&= \frac{1}{T^+}\int V\left(-\partial_t(\rho^+ + \rho^- - \mu^+ - \mu^-) + E^\perp.\nabla_x(\rho^+ + \rho^- - \mu^+ - \mu^-)\right)dx \\
&+ \frac{1}{T^+}\int -T^- V\partial_{x_2}(\rho^- - \mu^-)dx \\
&= \frac{1}{T^+}\int V\left(\partial_t\Delta V - E^\perp.\nabla_x\Delta V\right)dx + \frac{T^-}{T^+}\int E_2(\rho^- - \mu^-)dx,
\end{aligned}$$

where we have plugged in the equation satisfied by $(\rho^+ - \mu^+)$, by $(\rho^- - \mu^-)$, and also plugged in the Poisson equation.

Finally we have:

$$\left(1 - \frac{T^-}{T^+}\right)\frac{d}{dt}\|\rho-\mu\|^2_{L^2} = \frac{2}{LT^+}\int V\left(\partial_t\Delta V - E^\perp.\nabla_x\Delta V\right)dx.$$

So that:

$$\frac{d}{dt}\|\rho-\mu\|^2_{L^2} = \frac{2}{L(T^+ - T^-)}\int V\left(\partial_t\Delta V - E^\perp.\nabla_x\Delta V\right)dx.$$

To conclude the proof, we observe, using the Dirichlet boundary conditions and the periodicity:

$$\int V\partial_t\Delta V dx = -\frac{d}{dt}\frac{1}{2}\left(\int |\nabla V|^2 dx\right) \tag{5.13}$$

and

$$\begin{aligned}
\int VE^\perp.\nabla\Delta V dx &= \int V\operatorname{div}(E^\perp\Delta V)dx \\
&= \underbrace{-\int \nabla V.E^\perp\Delta V dx}_{=0} + \underbrace{\int \operatorname{div}(VE^\perp\Delta V)dx}_{=0}.
\end{aligned}$$

The first term is equal to zero since $E.E^\perp = 0$, the second one thanks to the boundary condition on $x_1 = 0, L$ and to the periodicity with respect to x_2.

As a result we have proved that

$$\frac{d}{dt}\mathcal{E}(t) = 0.$$

Actually the computations we have made are rigorously valid only for smooth solutions to (3.4). Nevertheless, these can be justified by smoothing the initial data, and then passing to the weak limit, which entails (5.2). □

Remark 5.4. We observe that we have also proved that:

$$\frac{d}{dt}\|\rho^+(t)-\mu^+\|_{L^2}^2 = \frac{d}{dt}\|\rho^-(t)-\mu^-\|_{L^2}^2, \tag{5.14}$$

which yields:

$$\|\rho^+(t)-\mu^+\|_{L^2}^2 - \|\rho^-(t)-\mu^-\|_{L^2}^2 = \|\rho^+(0)-\mu^+\|_{L^2}^2 - \|\rho^-(0)-\mu^-\|_{L^2}^2. \tag{5.15}$$

This will be useful for the study of nonlinear instability.

Remark 5.5. Let us add that the explicit form of the equilibria is crucial in the proof of the theorem. It would not work similarly if we had taken an equilibrium of the form:

$$\mu(x_1) = \begin{pmatrix} \Phi(x_1) \\ 1-\Phi(x_1) \end{pmatrix}$$

with Φ a smooth function. In this case it should be maybe more relevant to use the general Lyapunov functionals method of Arnold [3].

Likewise one can notice that the proof would have not worked if we had chosen any other boundary condition than Dirichlet.

6 On nonlinear instability

What we intend to show now is a property of nonlinear instability in the "bad curvature" region when the physical parameters satisfy:

$$\frac{T^+ - T^-}{L} < \frac{4}{5\pi^2}.$$

This can be interpreted as a bad confinement property. We first recall that the equilibrium in this case is the following:

$$\mu^{bad}(x_1) = \begin{pmatrix} 1-\frac{x_1}{L} \\ \frac{x_1}{L} \end{pmatrix}.$$

Thanks to the existence of an eigenvalue with maximal positive real part for the linearized operator around μ^{bad}, we can prove a nonlinear instability result. Indeed, using the method introduced by Grenier [85], we are able to pass from the linear spectral instability to the nonlinear instability in the L^2 norm. Grenier's method was originally used to prove instability in the L^2 velocity norm for Euler ; we show here that this technique can also be adapted to show instability in the L^2 vorticity norm.

The drawback of this method is that it requires high regularity on an eigenfunction associated to the dominant eigenvalue, which could be difficult to check in more complicated cases. In our case, we were able to prove such a smoothness in Lemma 4.1.

For the sake of readability we will write μ instead of μ^{bad} since there is no risk of confusion.

Theorem 6.1. *There exist constants $\delta_0, \eta_1, \eta_2 > 0$ such that for any $0 < \delta < \delta_0$ and any $s \geq 0$ there exists a solution (ρ, E) to (3.4) with $\|\rho(0)-\mu\|_{H^s} \leq \delta$ but such that:*

$$\|\rho(t_\delta)-\mu\|_{L^2} \geq \eta_1 \tag{6.1}$$

and:

$$\|E(t_\delta)\|_{L^2} \geq \eta_2, \tag{6.2}$$

with $t_\delta = O(|\log\delta|)$.

In particular, μ is unstable with respect to the L^2 norm.

Remark 6.1. This instability result is complementary to the stability result proved in Corollary 5.1, since they involve the same L^2 norms.

Remark 6.2. This instability result can also be obtained by techniques similar to those used by Bardos, Guo and Strauss in [31] for the 2D incompressible Euler system, which consist in proving that the solution ρ remains "close" in some norm to a growing mode associated to the maximal growth rate of the linearized operator M.

Using some bootstrap argument introduced by Bardos, Guo and Strauss [31], and a idea of Lin [112] consisting in:

- estimating $\|E(t_\delta)\|_{L^2}$ with Duhamel's formula
- then studying the semi-group e^{-tM} (where M is the linearized operator) on H^{-1},

we can also prove the exponential growth of the L^2 norm of the electric field.

We refer to the paper of Lin [113] which could be adapted to our case with some minor modifications.

Here we will provide an alternative proof by using Grenier's method (which consists in proving that the solution ρ remains 'close" in some norm to the growing mode plus some high order correction) to prove (6.1) and finally by using the energy of Theorem 5.1 to prove (6.2).

Proof. We begin with some preliminaries on the linearized operator. Using the same notations as in paragraph 4.2, we consider M the linearized operator around μ on

$$Y = \{y_1, y_2 \in H^s([0,L] \times \mathbb{R}/\mathbb{Z})^2, \int (y_1 + y_2)dx = 0\}, \quad \text{for } s \geq 0.$$

For $s = 0$, by Corollary 4.1, we know the existence of an eigenfunction R associated to an eigenvalue λ with maximal real part $\mathrm{Re}\,\lambda$. In addition, by Lemma 4.1, R belongs to any H^s, $s \geq 0$. For the sake of simplicity we will assume that R is real and associated to the eigenvalue $\mathrm{Re}\,\lambda$. In the general case, since the linearized operator is real, the conjugate of λ is also an eigenvalue so that one can consider by linearity real-valued growing modes and the following of the proof remains the same.

We recall also that by Corollary 4.1, for any $\gamma > \mathrm{Re}\,\lambda$, there is a constant $C(\gamma, s)$ such that for all $s \geq 0$:

$$\|e^{-tM}\|_{H^s \to H^s} \leq C(\gamma, s)e^{t\gamma}. \tag{6.3}$$

Basically the idea of Grenier is to construct a high order approximation of the nonlinear equation, that is a more precise approximation than the "usual" linearized equation. Indeed instead of showing that $f - \mu$ is close to a well chosen eigenfunction, we show that it is close to the high order asymptotic expansion:

$$\rho_{app}^{(N)} = \delta u_1 + \sum_{i=2}^{N} \delta^i u_i, \tag{6.4}$$

where $u_1 = Re^{\mathrm{Re}\,\lambda t}$. Note than for any $s > 0$, we have:

$$\|u_1\|_{H^s} \leq Ce^{\mathrm{Re}\,\lambda t}.$$

The approximated density ρ_{app} is constructed in order to have the following high order approximation:

$$\partial_t \rho_{app} + M\rho_{app} + E_{app}^{\perp}.\nabla_x \rho_{app} = R_{app}, \tag{6.5}$$

where $E_{app} = \nabla\Delta^{-1}(\rho^1_{app} + \rho^2_{app})$ and R_{app} is a remainder satisfying the estimate:

$$\|R_{app}\|_{H^{L-2N-1}} \leq C\delta^{N+1}\exp\left((N+1)\operatorname{Re}\lambda t\right).$$

Let $N \in \mathbb{N}^*$ to be chosen later and take any $S > 0$ such that $S > 2N+1$. We choose also $\theta < 1$ (to be fixed later) such that $\frac{1}{2} \geq \frac{\theta}{1-\theta}$ and define t_δ such that $\theta = \delta\exp(\operatorname{Re}\lambda t_\delta)$.

Now we can construct the $u_j = \begin{pmatrix} u_j^+ \\ u_j^- \end{pmatrix}$ by recursion; we will ensure that for all $1 \leq j \leq N$, $\int (u_j^+ + u_j^-)dx = 0$ and

$$\|u_j\|_{H^{S-j}} \leq C\exp(j\operatorname{Re}\lambda t).$$

Suppose we have u_j for $j \leq k$. Then we define u_{k+1} as the solution of the linear equation:

$$\partial_t u_{k+1} + Mu_{k+1} + \sum_{j=1}^{k} E_j^\perp.\nabla_x u_{k+1-j} + E_{k+1-j}^\perp.\nabla_x u_j = 0, \tag{6.6}$$

with $E_j = \nabla\Delta^{-1}(u_j^+ + u_j^-)$ and $u_{k+1}(0,x) = 0$ as initial condition. Intuitively, u_{k+1} is chosen in order to counterbalance the non-linear interaction between the previous terms of the expansion.

Thanks to Corollary 4.1 with $\gamma \in]\operatorname{Re}\lambda, 2\operatorname{Re}\lambda[$, we get the following estimate:

$$\begin{aligned}
\|u_{k+1}\|_{H^{S-(k+1)}} &\leq \int_0^t \|e^{M(t-s)}(\sum_{j=1}^{k} E_j^\perp.\nabla_x u_{k+1-j} + E_{k+1-j}^\perp.\nabla_x u_j)\|_{H^{L-(k+1)}}ds \\
&\leq C\int_0^t e^{\gamma(t-s)}(\sum_{j=1}^{k} \|E_j^\perp\|_{H^{S-(k+1)}}\|u_{k+1-j}\|_{H^{S-k}} + \|E_{k+1-j}^\perp\|_{H^{S-(k+1)}}\|u_j\|_{H^{S-k}})ds \\
&\leq C\int_0^t e^{\gamma(t-s)}\exp\left((k+1)\operatorname{Re}\lambda s\right)ds \\
&\leq C\exp\left((k+1)\operatorname{Re}\lambda t\right).
\end{aligned}$$

Note also that since $\frac{d}{dt}\int (u_{k+1}^+ + u_{k+1}^-)dx = 0$, we clearly have

$$\int (u_{k+1}^+ + u_{k+1}^-)dx = 0.$$

Now we can see that:

$$\partial_t\rho_{app} + M\rho_{app} + E_{app}^\perp.\nabla_x\rho_{app} = R_{app}, \tag{6.7}$$

with $R_{app} = \sum_{2N \geq j+j' > N} \delta^{j+j'} E_j^\perp.\nabla_x u_{j'}$. Then, noticing that for all $t \leq t_\delta$:

$$\delta\exp(\operatorname{Re}\lambda t) \leq \theta < 1,$$

the following estimate follows:

$$\|R_{app}\|_{H^{S-2N-1}} \leq C_N\delta^{N+1}\exp\left((N+1)\operatorname{Re}\lambda t\right), \tag{6.8}$$

where C_N is a constant depending only on N.

Now we consider the solution ρ to (3.4) such that $\rho(0) - \mu = \rho_{app}(0)$; the equation satisfied by $w = \rho - \mu - \rho_{app}$ is the following:

$$\partial_t w - \begin{pmatrix} T^+\partial_{x_2}w^+ \\ T^-\partial_{x_2}w^- \end{pmatrix} + E_w^\perp.\nabla_x w + E_{app}^\perp.\nabla_x w + E_w^\perp.\nabla_x\rho_{app} = -E_w^\perp.\nabla_x\mu - R_{app}, \tag{6.9}$$

with $E_w = \nabla\Delta^{-1}(w^+ - w^-)$.

Then we want to estimate $\|w\|_{L^2}$ by using some modulated energy inequality. To this end, we multiply by w and integrate with respect to x:

$$\begin{aligned}
\frac{d}{dt}\|w\|_{L^2}^2 &\leq \int |E_w^\perp.\nabla_x\rho_{app}w|dx + \int |E_w^\perp.\nabla_x\mu w|dx + \|R_{app}\|_{L^2}\|w\|_{L^2} \\
&\leq (\|\nabla_x\rho_{app}\|_{L^\infty} + \|\nabla_x\mu\|_{L^\infty})\|E_w^\perp\|_{L^2}\|w\|_{L^2} + \frac{1}{2}\|w\|_{L^2}^2 + \frac{1}{2}\|R_{app}\|_{L^2}^2 \\
&\leq C\left((1+\|\nabla_x\rho_{app}\|_{L^\infty})\|w\|_{L^2}^2 + \|R_{app}\|_{L^2}^2\right) \\
&\leq C\Big((1+\|\nabla_x\rho_{app}\|_{L^\infty})\|w\|_{L^2}^2 + C_N\delta^{2(N+1)}\exp\left(2(N+1)\operatorname{Re}\lambda t\right)\Big).
\end{aligned}$$

But for $\alpha > 0$ such that $2+\alpha < S-N$ and for $t \leq t_\delta$, we can control the Lipschitz norm of ρ_{app}:

$$\begin{aligned}
\|\nabla_x\rho_{app}\|_{L^\infty} &\leq \|\rho_{app}\|_{H^{2+\alpha}} \\
&\leq \sum_{i=1}^N \delta^i\|u_i\|_{H^{2+\alpha}} \leq \sum_{i=1}^N \delta^i\|u_i\|_{H^{S-i}} \\
&\leq \sum_{i=1}^N \delta^i \exp\left(i\operatorname{Re}\lambda t\right) \\
&\leq \sum_{i=1}^N \theta^i \leq \frac{\theta}{1-\theta} \leq \frac{1}{2}.
\end{aligned}$$

Now choose N such that:

$$N+1 > \frac{3C}{4\operatorname{Re}\lambda}. \tag{6.10}$$

By Gronwall's lemma we consequently get:

$$\|\rho - \mu - \rho_{app}\|_{L^2} = \|w\|_{L^2} \leq C_N\delta^{N+1}\exp\left((N+1)\operatorname{Re}\lambda t\right) \leq C_N\theta^{N+1}. \tag{6.11}$$

On the other hand we have a bound from below for the L^2 norm of ρ_{app}, for $t = t_\delta$:

$$\begin{aligned}
\|\rho_{app}\|_{L^2} &\geq \delta\|u_1\|_{L^2} - \sum_{i=2}^N \delta^i\|u_i\|_{L^2} \\
&\geq \delta\exp\left(\operatorname{Re}\lambda t_\delta\right) - \sum_{i=2}^N \delta^i\exp\left(i\operatorname{Re}\lambda t_\delta\right) \\
&= \theta - \sum_{i=2}^N \theta^i \\
&\geq \frac{1}{2}\theta.
\end{aligned}$$

Finally we have, for $t = t_\delta$:

$$\begin{aligned}
\|\rho - \mu\|_{L^2} &\geq \|\rho_{app}\|_{L^2} - \|\rho - \mu - \rho_{app}\|_{L^2} \\
&\geq \frac{1}{2}\theta - C_N\delta^{N+1}\exp\left((N+1)\operatorname{Re}\lambda t\right) \geq \frac{1}{2}\theta - C_N\theta^{N+1} \\
&\geq \frac{1}{4}\theta := \eta_1 > 0,
\end{aligned}$$

if θ is chosen small enough with respect to N. This proves the expected instability result (6.1).

Now, in order to prove the exponential growth of the electric field, we use the conservation of the energy proved in Theorem 5.1. We have for any $t \geq 0$:

$$\|\rho(t)-\mu\|_{L^2}^2 - \frac{1}{L(T^+-T^-)}\int |\nabla V(t)|^2 dx \leq \|\rho(0)-\mu\|_{L^2}^2 - \frac{1}{L(T^+-T^-)}\int |\nabla V(0)|^2 dx, \tag{6.12}$$

which implies that:

$$\begin{aligned}\eta_1^2 \leq \|\rho(t_\delta)-\mu\|_{L^2}^2 &\leq \frac{1}{L(T^+-T^-)}\int |\nabla V(t_\delta)|^2 dx + \|\rho(0)-\mu\|_{L^2}^2 \\ &\leq \frac{1}{L(T^+-T^-)}\int |\nabla V(t_\delta)|^2 dx + \delta^2.\end{aligned}$$

We can consider that $\delta < \delta_0 < \eta_1/2$, so that:

$$\int |\nabla V(t_\delta)|^2 dx \geq L(T^+-T^-)(\eta_1^2-\delta_0^2) := \eta_2^2. \tag{6.13}$$

This proves (6.2). □

By Remark 5.4 we can be a little more precise on the growth of the densities.

Remark 6.3. With the same notations as in the previous theorem, there exists $\eta_3 > 0$ such that:

$$\|\rho^+(t_\delta)-\mu^+\|_{L^2} \geq \eta_3 \tag{6.14}$$

and

$$\|\rho^-(t_\delta)-\mu^-\|_{L^2} \geq \eta_3, \tag{6.15}$$

which means that both the hot and the cold plasma are unstable.

Proof. According to Remark 5.4, we have

$$\|\rho^-(t_\delta)-\mu^-\|_{L^2} - \|\rho^+(t_\delta)-\mu^+\|_{L^2} = \|\rho^-(0)-\mu^-\|_{L^2} - \|\rho^+(0)-\mu^+\|_{L^2}, \tag{6.16}$$

which implies that

$$\big|\|\rho^-(t_\delta)-\mu^-\|_{L^2} - \|\rho^+(t_\delta)-\mu^+\|_{L^2}\big| \leq \delta_0.$$

By Theorem 6.1, we have

$$\|\rho^-(t_\delta)-\mu^-\|_{L^2} + \|\rho^+(t_\delta)-\mu^+\|_{L^2} \geq \eta_1.$$

Assuming as in the previous proof that $\delta_0 \leq \varepsilon_1/2$, this clearly implies that

$$\|\rho^-(t_\delta)-\mu^-\|_{L^2}, \|\rho^+(t_\delta)-\mu^+\|_{L^2} \geq \eta_1/4 := \eta_3,$$

which proves our claim. □

7 Conclusion

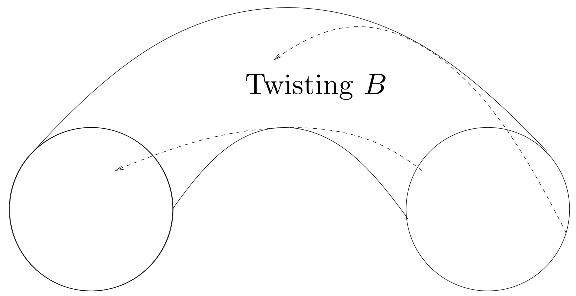
We have finally managed to provide a mathematical explanation of stability in the "good curvature" region and instability in the "bad curvature" region with our simplified nonlinear model. In our analysis we have pointed out that large temperature gradients brought nonlinear stability even in the bad curvature region. In other terms, if there is enough heating, there is good confinement: this is the H-mode.

A first natural extension to this work would be to generalize the stability/instability result to the kinetic model (3.1):

$$\begin{cases} \partial_t f - \frac{1}{2}|v|^2 \partial_{x_2} f + E^{\perp}.\nabla_x f = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \displaystyle\int f dv - 1. \end{cases}$$

This shall be the object of a future work. Another important issue is to understand the influence of the gyroaverage operator, that we have neglected in this work.

In "real" tokamaks, the next step towards confinement consists in considering a magnetic field with a variable direction, i.e. $B = B_0 e_\varphi + B_1 e_\theta$. At leading order, particles still follow the magnetic field lines: consequently, with such a twisting field, particles from the "bad curvature" region travel every now and then to the "good curvature" region. We accordingly expect overall confinement for the plasma.



A very challenging and interesting problem would be to prove overall confinement with such a twisting magnetic field. In this case, the accurate parameter to consider is the so-called safety factor, which stands for the number of times the magnetic field lines twist around the torus the long way for each time they twist around the short way. In "real" tokamaks, it has to be chosen with precaution in order to get good confinement properties (see [158]). But it seems to be much more complicated, since one has to deal with many drifts due to the geometry of the magnetic field.

Finally let us conclude by mentioning that the analysis of confinement provided in this chapter is very naive since it is now well known that there is a loss of confinement in tokamak plasmas, referred to as anomalous transport. Many models have been proposed and intensively studied to justify these phenomena: it would be very interesting to study some of them from the mathematical point of view.

Chapitre 7

On the controllability of the Vlasov-Poisson system in the presence of external force fields

Article soumis, en collaboration avec Olivier Glass.

Sommaire

Résumé : In this chapter, we are interested in the controllability of Vlasov-Poisson systems in the presence of an external force field (namely a bounded force field or a magnetic field), by means of a local interior control. We are able to extend the results of [65], where the only present force was the self-consistent electric field.

1 Introduction and main results

We consider the controllability of the Vlasov-Poisson system in the periodic domain $\mathcal{T}^n$ (where n is the space dimension), which describes the evolution of a population of electrons in a neutralizing background of fixed ions, under the influence of a self-generated electric field. The control questions are addressed by means of an interior control located in an open set ω of the domain, which is a priori arbitrary. We assume in this chapter that the charged particles evolve with the influence of an additional *fixed* external force, denoted by $F(t,x,v)$ (at least with Lipschitz regularity and a sublinear growth at infinity in velocity). The equations read:

$$\partial_t f + v.\nabla_x f + F(t,x,v).\nabla_v f + \nabla_x \Phi.\nabla_v f = \mathbb{1}_\omega G, \quad x \in \mathbb{T}^n, \quad v \in \mathbb{R}^n \tag{1.1}$$

$$\Delta_x \Phi = \int_{\mathbb{R}^n} f dv - \int_{\mathcal{T}^n \times \mathbb{R}^n} f dv dx, \tag{1.2}$$

$$f_{|t=0} = f_0. \tag{1.3}$$

In these equations, $f(t,x,v)$ is the so-called distribution function, which describes the density of particles at time $t \in \mathbb{R}^+$, at position $x \in \mathcal{T}^n$ and velocity $v \in \mathbb{R}^n$. The initial density distribution $f_0(x,v)$ is a non-negative integrable function. The right-hand side of the transport equation $\mathbb{1}_\omega G$ is a source term describing emission and absorption of particles, supported in ω. Moreover, to preserve global neutrality, G has to satisfy the following constraint:

$$\forall t \in \mathbb{R}^+, \quad \int_{\mathcal{T}^n \times \mathbb{R}^n} \mathbb{1}_\omega G \, dv \, dx = 0.$$

We normalize here the torus so that its Lebesgue measure is 1.

The controllability problem is the following. Let $f_1(x,v)$ be another non-negative integrable function satisfying $f_1 \geq 0$ and

$$\int f_1 dv dx = \int f_0 dv dx,$$

and let $T > 0$ a fixed time. The question is: is it possible to find a control G such that:

$$f(T,x,v) = f_1(x,v). \tag{1.4}$$

When the only acting force is the self-consistent electric field (that is when $F = 0$), the first author provided in [65] some positive answers to the question. More specifically, two kinds of results were obtained: first local controllability (which means that f_0 and f_1 are small in some weighted L^∞ norm) were obtained in two dimensions, for an arbitrary control zone ω. Global controllability results (without restriction on the size of f_0 and f_1) in any dimension was also obtained, provided that the control zone ω contains the image of a hyperplane of $\mathbb{R}^n$ by the canonical surjection (which is called a hyperplane of the torus in [65]). The proofs of these results relied on the nice geometry of free transport in the torus: we shall recall their principle in a subsequent paragraph.

When one considers a non-trivial external force F, the underlying dynamical system is more complicated; thus the characteristics can have a complex geometry, making the generalization not straightforward from the case $F = 0$.

In this chapter, we are able to extend results of [65] for the two following classes of force fields:

- The case of bounded force fields $F \in L^\infty_t W^{1,\infty}_{x,v}$.
- In two dimensions, the case of Lorentz forces for magnetic fields with a fixed direction $F(x,v) = b(x)(v_2, -v_1)$ with b satisfying a certain geometric condition (which will be precisely described later).

As we will see later on, the treatment of these two cases are rather different (in particular for what concerns high velocities) and involve different strategies. As a matter of fact, we were not able to find a general strategy which would allow to treat all forces F which are Lipschitz with a sublinear growth at infinity in velocity.

Let us now briefly review the existing results on the Cauchy theory for the Vlasov-Poisson equation posed in the whole space $\mathbb{R}^n$ or in the torus $\mathcal{T}^n$. In this work, we will only focus on strong solutions (at least with a $\mathcal{C}^1$ regularity in all variables); in the case where $F = 0$, the first results for such solutions are due Ukai and Okabe [151] who have proved global in time existence in two dimensions and local in time existence in three dimensions, in the whole space setting. One can readily check that the proof is the same for the torus case. In three dimensions, in the whole space, global in time results were proved independently by Pfaffelmoser [135] and Lions and Perthame [119]. The results of Pfaffelmoser were adapted to the torus case by Batt and Rein [11]. Concerning global weak solutions, the main result is due to Arsenev [4]. One can observe that all these results can be easily adapted to incorporate an additional external force F (with F satisfying the previous regularity assumptions).

We will only rely on the construction due to Ukai and Okabe in the following. We are now in position to precisely state the main results proved in this chapter.

1.1 Results in the bounded external field case

We first consider the case where $F \in L^\infty_t W^{1,\infty}_{x,v}$. In this case, we are able to exactly extend those for $F = 0$, that are a local and a global controllability results. The local result concerns only the dimension $n = 2$, but is valid for any control zone ω. On the contrary, the global result is valid for any n, but requires a stronger geometric assumption on the control zone ω.

Theorem 1.1 (Local result). *Let $n = 2$. Let $F(t,x,v) \in L^\infty_t W^{1,\infty}_{x,v}$. Let $\gamma > 2$ and $T > 0$. There exist $\kappa, \kappa' > 0$ small enough such that the following holds. Let f_0 and f_1 be two functions in $C^1(\mathcal{T}^2 \times \mathbb{R}^2) \cap W^{1,\infty}(\mathcal{T}^2 \times \mathbb{R}^2)$, satisfying the condition that for any $(x,v) \in \mathcal{T}^2 \times \mathbb{R}^2$ and $i \in \{0,1\}$,*

$$\begin{cases} |f_i(x,v)| \le \kappa(1+|v|)^{-\gamma-1}, \\ |\nabla_x f_i| + |\nabla_v f_i| \le \kappa'(1+|v|)^{-\gamma}, \end{cases} \tag{1.5}$$

and

$$\int_{\mathcal{T}^n \times \mathbb{R}^n} f_0 = \int_{\mathcal{T}^n \times \mathbb{R}^n} f_1. \tag{1.6}$$

Then there exists a control $G \in C^0([0,T] \times \mathcal{T}^2 \times \mathbb{R}^2)$, such that the solution of (1.1)-(1.2) and (1.3) exists, is unique, and satisfies (1.4).

Theorem 1.2 (Global result)**.** *Let $\gamma > n$ and $\kappa, \kappa' > 0$. Suppose that the regular open set ω contains the image of a hyperplane in $\mathbb{R}^n$ by the canonical surjection, supposed to be closed. Let f_0 and f_1 be two functions in $C^1(\mathcal{T}^n \times \mathbb{R}^n)$, satisfying the conditions*

$$\begin{cases} |f_i(x,v)| \leq \kappa(1+|v|)^{-\gamma-2}, \\ |\nabla_x f_i| + |\nabla_v f_i| \leq \kappa'(1+|v|)^{-\gamma}, \end{cases} \tag{1.7}$$

and (1.6). Then there exists a control $G \in C^0([0,T] \times \mathcal{T}^n \times \mathbb{R}^n)$, such that the solution of (1.1)-(1.2) and (1.3) exists, is unique, and satisfies (1.4).

1.2 Results in the magnetic field case

Let us now state our result when F represents an external magnetic field. For all results dealing with this case, we will systematically assume that the space dimension $n = 2$. First, let us explain the physical meaning of the system under consideration. In the physical space $\mathbb{R}^3$, let (e_1, e_2, e_3) a fixed orthonormal base. We consider the stationary magnetic field B, with fixed direction e_3:

$$B(x) = b(x)e_3,$$

where b is a Lipschitz function on $\mathcal{T}^3$. Since B has to satisfy the divergence free condition, this implies that b only depends on x_1 and x_2. The associated Lorentz force writes:

$$F = v \wedge B(x) = b(x)v^{\perp},$$

denoting $v^{\perp} = (v_2, -v_1, 0)$. We then restrict to distribution functions which do not depend on x_3 and v_3, so that we can restrict the study of the dynamics to the bidimensional plane (e_1, e_2). For the sake of readability, we rewrite the Vlasov-Poisson system that we study:

$$\partial_t f + v.\nabla_x f + b(x)v^{\perp}.\nabla_v f + \nabla_x \Phi.\nabla_v f = \mathbb{1}_\omega G, \quad x \in \mathbb{T}^2, \quad v \in \mathbb{R}^2 \tag{1.8}$$

$$\Delta_x \Phi = \int_{\mathbb{R}^2} f dv - \int_{\mathcal{T}^2 \times \mathbb{R}^2} f dv dx, \tag{1.9}$$

$$f_{|t=0} = f_0. \tag{1.10}$$

We now precisely state the geometric assumption we have to make on b.

- **Fixed sign.** We assume that b has a fixed (say non-negative) sign.
- **Geometric control condition.** We assume that there exists K a compact set of $\mathbb{T}^2$ on which $b > 0$ and which satisfies the geometric control condition:

$$\text{For any } x \in \mathbb{T}^2 \text{ and any direction } e \in \mathbb{S}^1, \quad \text{there exists } y \in \mathbb{R}^+ \text{ such that } x + ye \in K. \tag{1.11}$$

One can notice that the geometric control condition corresponds to the geometric control condition of Bardos, Lebeau and Rauch [8] for the controllability of the wave equation. Let us underline however that here this condition concerns the magnetic field only, and not the control zone ω. As we will see, this condition assures that the particles are sufficiently influenced by the magnetic field.

Examples. Let us give some examples, where this geometric assumption is satisfied.

i. The most simple example that one can have in mind is the case where b is positive on $\mathcal{T}^2$. Then taking $K = \mathcal{T}^2$, the geometric assumption is satisfied. Obviously, this includes the case where b is a positive constant.

ii. Assume that b is non-negative and has finite number N of zeros $x_1, ..., x_N \in \mathcal{T}^2$. Then there is r small enough such that $K = \mathcal{T}^2 \backslash \cup_{i=1}^N B(x_i, r)$ is appropriate. One could also extend this consideration to the case where the zeros of b are given by a sequence $(x_i)_{i \in \mathcal{N}}$ with a finite number of cluster points.

iii. We can consider some b which is identically equal to 0 in a large set of the torus, provided the existence of some K satisfying the geometric control condition. For instance, if we identify $\mathcal{T}^2$ with $[0,1]^2$ with periodic conditions, a subset K containing $(\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$ satisfies the geometric assumption.

With these particular magnetic fields, we are able to prove a local controllability result, which is similar to Theorem 1.1 (but we emphasize once again that the proofs will be rather different).

Theorem 1.3. *Let b satisfying the geometric assumption* (1.11). *Let $\gamma > 2$ and $T > 0$. There exist $\kappa, \kappa' > 0$ such that the following holds. Let f_0 and f_1 be two functions in $C^1(\mathcal{T}^2 \times \mathbb{R}^2) \cap W^{1,\infty}(\mathcal{T}^2 \times \mathbb{R}^2)$, satisfying the condition that for any $(x, v) \in \mathcal{T}^2 \times \mathbb{R}^2$ and $i \in \{0, 1\}$,*

$$\begin{cases} |f_i(x,v)| \leq \kappa(1+|v|)^{-\gamma-1}, \\ |\nabla_x f_i| + |\nabla_v f_i| \leq \kappa'(1+|v|)^{-\gamma}, \end{cases} \tag{1.12}$$

and

$$\int_{\mathcal{T}^n \times \mathbb{R}^n} f_0 = \int_{\mathcal{T}^n \times \mathbb{R}^n} f_1. \tag{1.13}$$

Then there exists a control $G \in C^0([0,T] \times \mathcal{T}^2 \times \mathbb{R}^2)$, such that the solution of (1.8)-(1.9) and (1.10) exists, is unique, and satisfies $f(T, x, v) = f_1$.

1.3 Organization of the chapter

The chapter is organized as follows: first, in Section 2, we will recall some considerations on the Vlasov-Poisson equation and will explain the general strategy of the proofs. Then, we prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4, for what concerns the bounded external field case. Finally, in Section 5, we prove Theorem 1.3 on the local controllability in the external magnetic field case.

2 Strategy of the proofs

2.1 Notations

For $T > 0$, we denote $Q_T := [0,T] \times \mathcal{T}^n \times \mathbb{R}^n$, and $\Omega_T := [0,T] \times \mathcal{T}^n$. For a domain Ω, we write also $C_b^l(\Omega)$, for $l \in \mathcal{N}$, for the set $C^l(\Omega) \cap W^{l,\infty}(\Omega)$. All the same, $C_b^{l+\sigma}(\Omega)$ for $\sigma \in (0,1)$ stands for the set of C^l functions with bounded σ-Hölder l-th derivatives. Also, $C_b^{\sigma,l+\sigma'}(\Omega_T)$ (resp. $C_b^{\sigma,l+\sigma'}(Q_T)$), for $l \in \mathcal{N}$, $\sigma, \sigma' \in [0,1)$ is the set of continuous functions in Ω_T (resp. Q_T), which are C^l with respect to x (resp. to (x,v)), and which l-th derivatives are all C_b^σ with respect to t and $C_b^{\sigma'}$ with respect to x (resp. to (x,v)).

For x in $\mathcal{T}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball with center x and radius r, and by $S(x, r)$ the corresponding sphere. The radii will always be chosen small enough in order that $S(x, r)$ does not intersect itself (that is $r < 1/2$ in the standard torus).

2.2 The case $F = 0$, obstructions to controllability

In this paragraph,we focus on the case $F = 0$, following [65]. Let us consider the linearized equation around the trivial state $(\overline{f}, \overline{\Phi}) = (0, 0)$. The linearized equation happens to be the free transport equation, which simply reads:

$$\partial_t f + v.\nabla_x f = \mathbb{1}_\omega g.$$

By Duhamel's formula, we obtain the explicit representation for f:

$$f(t, x, v) = f_0(x - tv, v) + \int_0^t (\mathbb{1}_\omega g)(s, x - (t - s)v, v) ds, \tag{2.1}$$

from which one can observe that there are two types of obstruction to controllability:

- (small velocities) The second obstruction concerns the small velocities. The velocity of a particle can have a good direction, but if it is not high enough, then it will not be able to reach zone in the desired time, see Figure 7.1.
- (large velocities, wrong direction) The first obstruction is of geometric control type as in [8] for what concerns the wave equation: if a particle has initially a wrong direction, then it will never reach the control zone, and thus we cannot influence its trajectory, see again Figure 7.1.

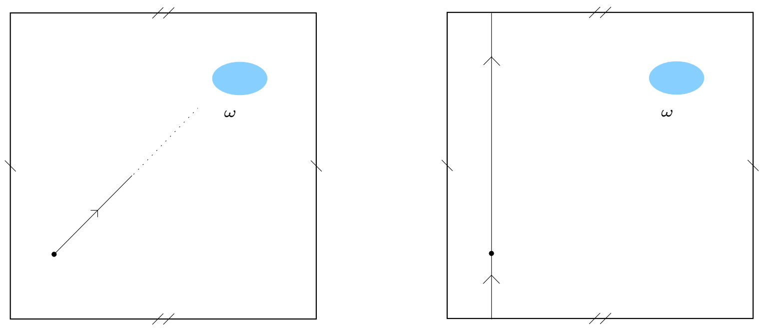


Figure 7.1: Obstructions for small and large velocities

It follows that in general, the linearized equation fails to be controllable.

2.3 The return method

In order to overcome these obstructions, the idea is to use the return method, which was introduced by Coron in [40] for the study of the stabilization of finite-dimensional systems, and then used in the context of the control of PDEs by Coron in [41] for the control of the two-dimensional Euler equation for perfect incompressible fluids. It has been used since in many different contexts of PDE control: we refer to the monograph of Coron [42] for several illustrations and references for this method. The principle is to build a reference solution $(\overline{f}, \overline{\Phi})$ starting from $(0, 0)$ and reaching $(0, 0)$ in some fixed time, and around which the linearized equation enjoys nice controllability properties. Such a construction can be delicate, and crucially depends on the structure of the studied equation.

Here, the problem is more or less equivalent to find solutions $(\overline{f}, \overline{\Phi})$ (starting from $(0,0)$ and reaching $(0,0)$) and such that the characteristics associated to $\nabla\overline{\Phi}$ satisfy:

$$\forall x \in \mathbb{T}^n, \quad \forall v \in \mathbb{R}^n, \quad \exists t \in [0,T], \quad X(t,0,x,v) \in \omega. \tag{2.2}$$

(As a matter of fact, the characteristics will not be quite associated to $\nabla\overline{\Phi}$ inside the control zone.)

When no exterior force is present, the existence of such a reference solution $\overline{f}$ was proved by the first author in [65] in two dimensions, for an arbitrary control set ω. This is achieved using complex analysis tools by building harmonic potentials outside ω, which allow to sufficiently influence the trajectories, so that the two previous obstructions are circumvented. This strategy distinguishes between high and low velocities, for which the relevant potentials are different.

2.4 On the scaling properties of Vlasov-Poisson equations

We notice that (1.1)-(1.2) is "invariant" by some change of scales. More precisely, when f is a solution of (1.1)-(1.2) in $[0,T] \times \mathcal{T}^n \times \mathbb{R}^n$, then for $\lambda \neq 0$, the function

$$f^\lambda(t,x,v) := |\lambda|^{2-n} f(\lambda t, x, v/\lambda), \tag{2.3}$$

is still a solution of (1.1)-(1.2), in $[0,T/\lambda] \times \mathcal{T}^n \times \mathbb{R}^n$ for the following potential

$$\varphi^\lambda(t,x) := \lambda^2 \varphi(\lambda t, x). \tag{2.4}$$

and the external force

$$F^\lambda(t,x,v) := \lambda^2 F(\lambda t, x, v/\lambda). \tag{2.5}$$

The choice of some particular parameters λ will be of great help for the controllability problem.

The choice $\lambda = -1$. Using (2.3) with $\lambda = -1$, we observe that in order to prove Theorems 1.1 and 1.3, it is sufficient to prove the result for the case where $f_1 = 0$ in $[\mathcal{T}^n \backslash \omega] \times \mathbb{R}^n$. Indeed, we observe that after imposing (2.3) with $\lambda = -1$, the corresponding external field remains in the same class, that is, if F is bounded, then $F^{\lambda=-1}$ is still bounded (resp. if F corresponds to a magnetic field satisfying the geometric condition, then $F^{\lambda=-1}$ still corresponds to a magnetic field satisfying the fixed sign and the geometric conditions).

Then one can follow the procedure that we detail below:

- Take f_0 as initial value and 0 (in $(\mathcal{T}^n \backslash \omega) \times \mathbb{R}^n$) as the final one,
- Take $(x,v) \mapsto f_1(x,-v)$ as initial value and again 0 as the final one within the force field $F(T-t,x,-v)$.

each in time $T/3$. We obtain two functions $\hat{f}_0$ and $\hat{f}_1$. Now we may consider the function $\hat{f}$ partially defined in Q_T by

$$\begin{cases} \hat{f}(t,x,v) = \hat{f}_0(t,x,v), \quad \text{in } [0,T/3] \times \mathcal{T}^n \times \mathbb{R}^n, \\ \hat{f}(t,x,v) = 0, \quad \text{in } [T/3, 2T/3] \times [\mathcal{T}^n \backslash \omega] \times \mathbb{R}^n, \\ \hat{f}(t,x,v) = \hat{f}_1(T-t,x,-v) \quad \text{in } [2T/3, T] \times \mathcal{T}^n \times \mathbb{R}^n. \end{cases}$$

Then we can complete in a regular manner $\hat{f}$ inside $[T/3, 2T/3] \times \omega \times \mathbb{R}^n$, taking care to preserve for any t the value of $\int_{\mathcal{T}^n \times \mathbb{R}^n} \hat{f}(t, x, v) dx dv$. Finally we get a relevant solution f. For this reason, we will systematically assume that $f_1 = 0$ in $[\mathcal{T}^n \backslash \omega]$ for all controllability results discussed in this work.

The choice $0 < \lambda \ll 1$: The choice of the parameters in such a range is useful to prove global controllability results. As in [65], it will help us in particular to prove Theorem 1.2 in the bounded external field case. The principle is that when λ is chosen small enough then $\nabla \Phi^\lambda$ has a small L^∞ small (and this is also the case for F^λ), so that we can expect characteristics for f^λ to be close to those of some well chosen relevant reference solution. This will allow us to get rid of the smallness assumption on f_0. Nevertheless in order to avoid concentration effects, we will need some assumptions on the characteristics associated to the reference solution.

In the magnetic field case, we observe that $F(x, v) = b(x) v^\perp$ and thus $F^\lambda(x, v) = \lambda b(x) v^\perp$. For this reason, due to our treatment of high velocities for this case, this will not allow us to prove a global result.

2.5 General strategy for external force fields F

Following [65] the main steps for proving local controllability results will be:

Step 1. Build a reference solution $(\overline{f}, \overline{\Phi})$ of (1.1)-(1.2) with a certain control $\overline{G}$, starting from $(0, 0)$ and arriving at $(0, 0)$, such that the characteristics associated to $F - \nabla \overline{\Phi}$ satisfy (2.2).

Step 2. Build a solution (f, Φ) close to $(\overline{f}, \overline{\Phi})$, taking into account the initial condition (f_0, Φ_0) and still arriving at $(0, 0)$ (outside ω). This is achieved using a fixed point operator involving an absorption process in the control zone. This is where we use the smallness assumption on f_0.

The treatment of Step 2. will be quite similar to that in [65], although a bit more technical since we will have to take into account the geometry due to F. The main difference is the treatment of Step 1., for which we have to propose new ideas. The strategy is the following:

Bounded force field. Our strategy relies on the fact that for short times, the dynamics with the external force F is well approximated by the dynamics with $F = 0$. We recall that in [65], the reference solution can be constructed for any time (which can be arbitrarily small) and any control zone in the torus. Thus, we use the construction in the case $F = 0$, for very short times and a small subset of the control zone ω, and using the approximation of the dynamics, this will give us a relevant reference solution.

Magnetic field. The strategy in this case can be understood in the most simple case, that is when b is a positive constant. In this case, the characteristics associated to the magnetic field can be explicitly computed: these are circles, whose radius is proportional to the norm of the velocity (which is a conserved quantity). We make two crucial observations:

- When the velocity is very large, the curvature of the circles are close to zero, and at least locally (that is for small times), the trajectory is well approximated by the straight lines of the free transport case.
- The magnetic field has "mixing features", in other words it makes the velocities of particles take every value of $\mathcal{S}^1$, which removes the above obstruction concerning

high velocities. Hence, due to this effect, at high velocity, we do not need to create any additional force field to make the particles cross the control zone.

This means that at high velocity any subset ω of the torus automatically satisfies the geometric condition (1.11) for the caracteristics associated to the magnetic field.

In the general case, the geometric condition on b allows us to make sure that the particles are sufficiently influenced by the magnetic field, so that the previous considerations will still hold.

2.6 On the uniqueness of the solution

In this paragraph, we briefly discuss the uniqueness question included in the above results.

The first point is that, if we drop the uniqueness from the conclusions of the above theorems, we can replace the assumption

$$|\nabla_x f_i| + |\nabla_v f_i| \leq \kappa'(1+|v|)^{-\gamma},$$

by the weaker one

$$|\nabla_x f_i| + |\nabla_v f_i| \leq \kappa'.$$

This is easily seen when reading the proofs below.

Hence the assumptions is of ∇f_i belonging to some weighted space is only useful for the uniqueness issue. Let us explain how one can show uniqueness under this assumption. The main point is that in this case the solution described above satisfies

$$|\nabla_{x,v} f(t,x,v)| \leq C(f_0, f_1)(1+|v|)^{-\gamma},$$

for all t. This follows from the construction described below, and from the estimates on ∇f in the proof. Once these estimates are obtained, the proof of uniqueness is exactly the one of Ukai-Okabe. It consists in making the difference of two potential solutions; this difference satisfies a certain transport equation with source. Then one performs an $L^1 \cap L^\infty$ estimate on the solution of this equation and uses a Gronwall argument. In our case, the source term disappears when we make this difference, so one can follow [151] without change.

This gives the uniqueness among the solutions satisfying

$$f \in C^1([0,T] \times \mathcal{T}^n \times \mathbb{R}^n), \ |f| + |\nabla_{x,v} f(t,x,v)| \leq C(1+|v|)^{-\gamma} \ \text{ and } \ \nabla\varphi \in L^\infty(0,T; W^{1,\infty}(\mathcal{T}^n)).$$

3 Bounded external field case

In this section, we prove Theorem 1.1. As already explained, the main difficulty is to build the reference solution. Then one can use the same absorption process, that was proposed in [65], and find a solution to the non-linear system by a similar fixed-point argument.

3.1 Design of the reference solution for the bounded field case

We begin with the construction of the reference solution. Accordingly to the previous strategy, we distinguish between high and low velocities.

For the large velocities, we prove the following proposition:

Proposition 3.1. *Let $\tau > 0$ and $H \in L^\infty((0,\tau)\times \mathcal{T}^2;\mathbb{R}^2)$. Given x_0 in $\mathbb{T}^2$ and r_0 a small positive number, there exist $\varphi \in \mathcal{C}^\infty([0,\tau]\times\mathbb{T}^2;\mathbb{R})$ and $\underline{m} > 0$ such that*

$$\Delta\varphi = 0 \quad in \quad [0,\tau]\times\left[\mathbb{T}^2\backslash\overline{B}(x_0, r_0/10)\right] \tag{3.1}$$

$$\operatorname{Supp}\varphi \subset (0,\tau)\times\mathbb{T}^2 \tag{3.2}$$

and such that, if one consider the characteristics $(\overline{X},\overline{V})$ associated to the force field $H+\nabla\varphi$ then for all $m \geq \underline{m}$:

$$\forall x \in \mathbb{T}^2, \forall v \in \mathbb{R}^2 \text{ such that } |v| \geq m,\ \exists t \in (\tau/3, 2\tau/3),$$
$$\text{such that } \overline{X}(t,0,x,v) \in B(x_0, r_0/4) \text{ and } |\overline{V}(t,0,x,v)| \geq \frac{m}{2}. \tag{3.3}$$

Proof of Proposition 3.1. In the case $H = 0$, this proposition was already proved in [65, Proposition 1, p. 340]. We fix $x_0' = x_0$, $r_0' = r_0/2$. Applying this result for $\tau = 1$, we thus obtain the existence of $\varphi_1 \in \mathcal{C}^\infty([0,1]\times\mathbb{T}^d;\mathbb{R})$ and $m' \in \mathbb{R}^{+*}$ with compact support in time in $(0,1)$, satisfying:

$$\Delta\varphi = 0 \quad \text{in}\,[0,1]\times\left[\mathbb{T}^2\backslash\overline{B}(x_0,r_0/20)\right], \tag{3.4}$$

$$\operatorname{Supp}\varphi \subset (0,1)\times\mathbb{T}^2, \tag{3.5}$$

and such that, if one consider the characteristics $(\tilde{X}^1,\tilde{V}^1)$ associated to the force field $\nabla\varphi_1$ then:

$$\forall x \in \mathbb{T}^d, \forall v \in \mathbb{R}^d, \text{ such that } |v| \geq m, \exists t \in (1/4, 3/4), \tilde{X}^1(t,0,x,v) \in B(x_0, r_0/8). \tag{3.6}$$

Let $\tau' < \tau$ to be fixed later. For this given τ', we can construct $\varphi_{\tau'}$ by rescaling φ_1 as follows:

$$\varphi_{\tau'}(t,x) := \frac{1}{(\tau')^2}\varphi_1\left(\frac{t}{\tau'}, x\right), \tag{3.7}$$

which corresponds to follow the characteristics with time $\frac{t}{\tau'}$.

Now let us consider the shifted in time potential φ defined by:

$$\varphi(t,x) = \varphi_{\tau'}\left(t - \frac{\tau-\tau'}{2}, x\right). \tag{3.8}$$

We extend φ by 0 in $(0,\tau)\setminus\left(\frac{\tau-\tau'}{2}, \frac{\tau+\tau'}{2}\right)$.

We define the characteristics $(\tilde{X},\tilde{V})$ associated to the force field $\nabla\varphi$, which satisfy by construction:

$$\forall x \in \mathbb{T}^d, \forall v \in \mathbb{R}^d, \text{ such that } |v| \geq m, \exists t \in (\frac{\tau-\tau'}{2}, \frac{\tau+\tau'}{2}), \tilde{X}(t,0,x,v) \in B(x_0, r_0/8). \tag{3.9}$$

Let us now compare $(\tilde{X},\tilde{V})$ and $(\overline{X},\overline{V})$, which is associated to the force field $H+\nabla\varphi$ on $(0,\tau)$. By Taylor's formula we have:

$$\begin{aligned}|\overline{X}(t,\frac{\tau-\tau'}{2},x,v) - \tilde{X}(t,\frac{\tau-\tau'}{2},x,v)| \leq \int_{\frac{\tau-\tau'}{2}}^t (t-s)\Big[&|\nabla\varphi(s,\tilde{X}(s,\frac{\tau-\tau'}{2},x,v)) - \nabla\varphi(s,\overline{X}(s,\frac{\tau-\tau'}{2},x,v))| \\ &+|H(s,\tilde{X}(s,\frac{\tau-\tau'}{2},x,v),\tilde{V}(s,\frac{\tau-\tau'}{2},x,v))|\Big]\,ds.\end{aligned} \tag{3.10}$$

By Gronwall lemma we deduce for $t \in \left(\frac{\tau-\tau'}{2}, \frac{\tau+\tau'}{2}\right)$:

$$\begin{aligned} |\overline{V}(t, \frac{\tau-\tau'}{2}, x, v) - \tilde{V}(t, \frac{\tau-\tau'}{2}, x, v)| &\leq \tau' \|H\|_{L^\infty_{t,x,v}} e^{\frac{\tau'^2}{2}\|\nabla^2\varphi\|_{L^\infty((0,\tau)\times\mathbb{T}^d)}}, \\ |\overline{X}(t, \frac{\tau-\tau'}{2}, x, v) - \tilde{X}(t, \frac{\tau-\tau'}{2}, x, v)| &\leq \frac{\tau'^2}{2} \|H\|_{L^\infty_{t,x,v}} e^{\frac{\tau'^2}{2}\|\nabla^2\varphi\|_{L^\infty((0,\tau)\times\mathbb{T}^d)}}. \end{aligned} \tag{3.11}$$

The crucial point is now to observe that φ described above satisfies:

$$\|\nabla^2\varphi\|_{L^\infty((0,\tau)\times\mathbb{T}^d)} = \mathcal{O}\left(\frac{1}{\tau'^2}\right) \quad \text{as } \tau' \to 0,$$

as it can be seen from (3.7).

Thus for τ' small enough we infer that $\overline{X}(t, 0, x, v)$ meets $B(x_0, r_0/4)$ for some $t \in \left(\frac{\tau-\tau'}{2}, \frac{\tau+\tau'}{2}\right) \subset (\tau/3, 2\tau/3)$, for all x and v, provided that $|\overline{V}(\frac{\tau-\tau'}{2}, 0, x, v)|$ is large enough. This is ensured if $|v| \geq \underline{m}$ is chosen large enough, thanks to the inequality:

$$|\overline{V}(\frac{\tau-\tau'}{2}, 0, x, v)| \geq |v| - \frac{\tau-\tau'}{2}\|H\|_{L^\infty_{t,x,v}}.$$

$\square$

Remark 3.1. In this proof, this is crucial that $H \in L^\infty((0,\tau)\times\mathcal{T}^2; \mathbb{R}^2)$. Thus this approach will fail for the magnetic field case.

The above proposition shows that with a suitable electric potential, all particles having a sufficiently high velocity will eventually reach ω. The following proposition explains how one can accelerate all particles in order to make all the remaining ones also reach ω. This will also rely on the construction in the case $F = 0$.

Proposition 3.2. *Let $\tau > 0$, $M > 0$ and $H \in L^\infty((0,\tau) \times \mathcal{T}^2; \mathbb{R}^2)$. Given x_0 in $\mathbb{T}^2$ and r_0 a small positive number, there exists $\tilde{M} > 0$, $\mathcal{E} \in C^\infty([0,\tau] \times \mathcal{T}^2; \mathbb{R}^2)$ and $\varphi \in C^\infty([0,\tau] \times \mathcal{T}^2; \mathbb{R})$ satisfying*

$$\mathcal{E} = \nabla\varphi \text{ in } [0,\tau] \times (\mathcal{T}^2 \backslash B(x_0, r_0)), \tag{3.12}$$

$$\mathit{Supp}(\mathcal{E}) \subset (0,\tau) \times \mathcal{T}^2, \tag{3.13}$$

$$\Delta\varphi = 0 \text{ in } [0,\tau] \times (\mathcal{T}^2 \backslash B(x_0, r_0)), \tag{3.14}$$

such that if (X, V) are the characteristics corresponding the force

$$\mathcal{I} := \mathcal{E} + H, \tag{3.15}$$

then

$$\forall (x, v) \in \mathcal{T}^2 \times B(0, M),\ V(\tau, 0, x, v)) \in B(0, \tilde{M}) \setminus B(0, M+1). \tag{3.16}$$

Proof of Proposition 3.2. By [65, Lemma 3, p. 356], there exists $\theta \in C^\infty(\mathcal{T}^2; \mathbb{R})$ such that

$$\begin{aligned} \Delta\theta = 0 &\text{ in } \mathcal{T}^2 \setminus B(x_0, r_0), \\ |\nabla\theta(x)| > 0 &\text{ in } \mathcal{T}^2 \setminus B(x_0, r_0). \end{aligned}$$

From the second condition, one sees that $\mathrm{Ind}_{S(x_0,r_0)}(\nabla\theta) = 0$, so that $\nabla\theta_{|\mathcal{T}^2\setminus B(x_0,r_0)}$ can be extended to $\mathcal{T}^2$ as a smooth non-vanishing vector field, let us say W. Call $\Lambda \in C_0^\infty((0,1); \mathbb{R})$

a nonnegative function with $\int_0^1 \Lambda = 1$. We claim that for sufficiently small $\tau' < \tau$, and sufficiently large $\mathcal{C} > 0$,

$$\mathcal{E}(t,x) := \frac{\mathcal{C}}{\tau'}\Lambda\left(\frac{t}{\tau'}\right) W(x),$$

is convenient. Then all properties above but (3.16) are clear.

Call $(\overline{X}, \overline{V})$ the characteristics associated to $\mathcal{E}$ only. We see that for all $(x,v) \in \mathcal{T}^2 \times B(0,M)$ and $t \in [0,\tau']$,

$$|\overline{V}(t,0,x,v) - v| \leq \mathcal{C}\|\mathcal{E}\|_\infty, \ |\overline{X}(t,0,x,v) - x| \leq \tau'(\mathcal{C}\|\mathcal{E}\|_\infty + M),$$

so

$$|\overline{V}(\tau',0,x,v) - v + \mathcal{C}\mathcal{E}(x)| \leq \tau'\|\mathcal{E}\|_\sigma[\tau'(\mathcal{C}\|\mathcal{E}\|_\infty + M)].$$

Noting that, due to the time support of $\mathcal{E}$, $\overline{V}(\tau,0,x,v) = \overline{V}(\tau',0,x,v)$ and using that $|\mathcal{E}| \geq c > 0$ on $\mathcal{T}^2$, one sees that one can choose $\mathcal{C}$ and then τ' such that

$$\forall (x,v) \in \mathcal{T}^2 \times B(0,M), \ \overline{V}(\tau,0,x,v) \in \mathbb{R}^2 \setminus B(0, M + 2 + \tau\|H\|_\infty).$$

We now consider the characteristics (X,V) associated to $\mathcal{E} + H$ and evaluate:

$$|\overline{X}(t,0,x,v) - X(t,0,x,v)| \leq \int_0^t |\overline{V}(s,0,x,v) - V(s,0,x,v)| ds$$

$$\begin{aligned}|\overline{V}(t,0,x,v) - V(t,0,x,v)| \leq & \int_0^t \Big(|\mathcal{E}(s,\overline{X}(s,0,x,v)) - \mathcal{E}(s,X(s,0,x,v))| \\ & + |H(t,X(s,0,x,v),V(s,0,x,v))| \Big) ds \\ \leq & \|\nabla\mathcal{E}\|_{L^\infty((0,\tau')\times\mathbb{T}^2)} \int_0^t (t-s)|\overline{V}(s,0,x,v) - V(s,0,x,v)| ds + t\|H\|_{L^\infty_{t,x,v}}.\end{aligned} \tag{3.17}$$

By Gronwall's inequality:

$$|\overline{V}(t,0,x,v) - V(t,0,x,v)| \leq t\|H\|_{L^\infty_{t,x,v}} e^{\frac{t^2}{2}\|\nabla\mathcal{E}\|} \tag{3.18}$$

We observe that we have:

$$\frac{\tau'^2}{2}\|\nabla\mathcal{E}\|_{L^\infty((0,\tau')\times\mathbb{T}^2)} = \mathcal{O}(\tau') \quad \text{as } \tau' \to 0. \tag{3.19}$$

Taking τ' small enough, using $t = \tau'$ in (3.18), and observing that

$$|V(\tau,0,x,v) - V(\tau',0,x,v)| \leq |\tau - \tau'|\|H\|_\infty,$$

allow us to prove our claim. The existence of $\tilde{M}$ is a matter of compactness of $\mathcal{T}^2 \times \overline{B}(0, M + 2 + \tau\|H\|_\infty)$. □

Remark 3.2. We can observe that there is some "margin" in the previous proof, in the sense that if we only had

$$\frac{\tau'^2}{2}\|\nabla\mathcal{E}\|_{L^\infty((0,\tau')\times\mathbb{T}^2)} = \mathcal{O}(1) \quad \text{as } \tau' \to 0,$$

the proof would still follow. However, that (3.19) holds will actually be crucial in the proof of the equivalent lemma in the magnetic field case, and this time this will be sharp.

The reference solution. Now we are able to define the reference solution. Consider x_0 in ω and r_0 a small positive number such that

$$B(x_0, 2r_0) \subset \omega.$$

We first define a reference potential $\overline{\varphi} : [0,T] \times \mathcal{T}^2 \to \mathbb{R}$ as follows. We apply Proposition 3.1 with $\tau = T/3$, $H = F_{|[0,T/3]}$, we obtain $\overline{\varphi}_1$ and some $\underline{m}_1 > 0$ such that (3.3) is satisfied.

Let

$$\alpha = \max\left(\frac{600 r_0}{T}, C_{r_0}(1 + \|F\|_\infty + \|\overline{\varphi}_1\|_\infty + \|\overline{\varphi}_3\|_\infty)\right), \tag{3.20}$$

$$M_1 = \max(\underline{m}_1, 2\alpha) + \frac{T}{3}\left(\|\nabla\overline{\varphi}_1\|_\infty + \|F\|_\infty\right), \quad M_2 = \max(\underline{m}_3, 2\alpha), \quad M = \max(M_1, M_2). \tag{3.21}$$

Above C_{r_0} is a positive geometric constant depending only on r_0, and which will be described later.

We also use Proposition 3.1 again with $\tau = T/3$, $H(t,x) = F(t + \frac{2T}{3}, x)$ for $t \in [0, T/3]$, we obtain $\overline{\varphi}_3$ and some $\underline{m}_3 > 0$ such that (3.3) is satisfied. Then we apply Proposition 3.2 with $\tau = T/3$, $H(t,x) = F(t + \frac{T}{3}, x)$ for $t \in [0, T/3]$, and M described above. We obtain $\overline{\mathcal{E}}_2$ $\overline{\varphi}_2$ and some $\tilde{M}$.

Finally we set:

$$\overline{\varphi}(t,\cdot) = \begin{cases} \overline{\varphi}_1(t,\cdot) \text{ for } t \in [0, \frac{T}{3}], \\ \overline{\varphi}_2(t - \frac{T}{3}, \cdot) \text{ for } t \in [\frac{T}{3}, \frac{2T}{3}], \\ \overline{\varphi}_3(t - \frac{2T}{3}, \cdot) \text{ for } t \in [\frac{2T}{3}, T], \end{cases}$$

and

$$\overline{\mathcal{E}}(t,\cdot) = \begin{cases} \nabla\overline{\varphi}_1(t,\cdot) \text{ for } t \in [0, \frac{T}{3}], \\ \overline{\mathcal{E}}_2(t - \frac{T}{3}, \cdot) \text{ for } t \in [\frac{T}{3}, \frac{2T}{3}], \\ \nabla\overline{\varphi}_3(t - \frac{2T}{3}, \cdot) \text{ for } t \in [\frac{2T}{3}, T]. \end{cases}$$

Let us now introduce $\overline{f}$. Consider a function $\mathcal{Z} \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ satisfying the following constraints

$$\begin{cases} \mathcal{Z} \geq 0 \text{ in } \mathbb{R}^n, \\ \text{Supp } \mathcal{Z} \subset B_{\mathbb{R}^n}(0,1), \\ \displaystyle\int_{\mathbb{R}^n} \mathcal{Z} = 1. \end{cases} \tag{3.22}$$

We introduce $\overline{f} = \overline{f}(t,x,v)$ as

$$\overline{f}(t,x,v) := \mathcal{Z}(v)\Delta\overline{\varphi}(t,x). \tag{3.23}$$

Of course, $\overline{f}$ satisfies (1.1) in $[0,T] \times \mathcal{T}^2 \times \mathbb{R}^2$, with source term

$$\overline{G}(t,x,v) := \partial_t \overline{f} + v.\nabla_x \overline{f} + (F + \nabla\overline{\varphi}).\nabla_v \overline{f}, \tag{3.24}$$

which is supported in $[0,T] \times B(x_0, r_0) \times \mathbb{R}^2$. Up to an additive function of t, the function φ satisfies the equation (1.2) corresponding to $\overline{f}$ (with $\overline{f}(0,\cdot,\cdot) \equiv 0$). We denote

$$\overline{\rho}(t,x) := \int_{\mathbb{R}^2} \overline{f}(t,x,v)\, dv = \Delta\overline{\varphi}(t,x).$$

3.2 Fixed point operator

To prove Theorem 1.1, we construct directly the solution f starting at f_0 and reaching 0 in $\mathcal{T}^2 \setminus \omega$ at time T, provided that f_0 is suitably small. This is done by a fixed-point procedure. In this subsection, we describe the operator; in the next ones, we will find a solution to our controllability problem as a fixed point of this operator.

Let $\varepsilon \in (0,1)$. We first define the domain $\mathcal{S}_\varepsilon$ of V_ε by

$$\mathcal{S}_\varepsilon := \left\{ g \in C_b^{\delta_2}(Q_T) \;\Big/ \right.$$
$$\begin{array}{ll} \textbf{a.} & \| \int_{\mathbb{R}^2} (g - \overline{f})\, dv \|_{C^{\delta_1}(\Omega_T)} \leq \varepsilon, \\ \textbf{b.} & \|(1+|v|)^\gamma (g - \overline{f})\|_{L^\infty(Q_T)} \leq c_1 \left[\|f_0\|_{C_b^1(\mathcal{T}^2\times\mathbb{R}^2)} + \|(1+|v|)^\gamma f_0\|_{C_b^0(\mathcal{T}^2\times\mathbb{R}^2)} \right], \\ \textbf{c.} & \|g - \overline{f}\|_{C_b^{\delta_2}(Q_T)} \leq c_2 \left[\|f_0\|_{C_b^1(\mathcal{T}^2\times\mathbb{R}^2)} + \|(1+|v|)^\gamma f_0\|_{C_b^0(\mathcal{T}^2\times\mathbb{R}^2)} \right], \\ \textbf{d.} & \forall t \in [0,T],\ \int_{\mathcal{T}^2\times\mathbb{R}^2} g(t,x,v)\, dx\, dv = \int_{\mathcal{T}^2\times\mathbb{R}^2} f_0(x,v) dx dv \left. \right\}, \end{array} \tag{3.25}$$

with c_1, c_2 depending only on γ, T, ω (and hence on $(\overline{f}, \overline{\varphi})$) and F, but not on ε. The indices $\delta_1 < \delta_2$ in $(0,1)$ are fixed as follows

$$\delta_1 := \frac{\gamma - n}{2(\gamma+1)} \text{ and } \delta_2 := \frac{\gamma}{\gamma+1}. \tag{3.26}$$

For fixed c_1 and c_2 large enough depending only on $(\overline{f}, \varphi)$, and f_0 small enough, one has

$$\left| \int f_0 dv dx \right| \leq \varepsilon,$$

and consequently, in this case $f_0 + \overline{f} \in \mathcal{S}_\varepsilon$, so $\mathcal{S}_\varepsilon \neq \emptyset$. From now, this is systematically supposed to be the case.

Now we introduce the following subsets of $S(x_0, r_0) \times \mathbb{R}^2$:

$$\gamma^- := \left\{ (x,v) \in S(x_0,r_0) \times \mathbb{R}^2 \ / \ |v| > \frac{1}{2} \text{ and } v.\nu(x) < -\frac{1}{10}|v| \right\}, \tag{3.27}$$

$$\gamma^{2-} := \left\{ (x,v) \in S(x_0,r_0) \times \mathbb{R}^2 \ / \ |v| \geq 1 \text{ and } v.\nu(x) \leq -\frac{1}{8}|v| \right\}. \tag{3.28}$$

$$\gamma^{3-} := \left\{ (x,v) \in S(x_0,r_0) \times \mathbb{R}^2 \ / \ |v| \geq 2 \text{ and } v.\nu(x) \leq -\frac{1}{5}|v| \right\}, \tag{3.29}$$

$$\gamma^+ := \left\{ (x,v) \in S(x_0,r_0) \times \mathbb{R}^2 \ / \ v.\nu(x) \geq 0 \right\}, \tag{3.30}$$

where $\nu(x)$ stands for the unit outward normal to the sphere $S(x_0, r_0)$ at point x. It can be easily seen that

$$\operatorname{dist}([S(x_0,r_0) \times \mathbb{R}^2] \backslash \gamma^{2-}; \gamma^{3-}) > 0.$$

We introduce a $C^\infty \cap C_b^1$ regular function $U : S(x_0,r_0) \times \mathbb{R}^2 \to \mathbb{R}$, satisfying

$$\begin{cases} 0 \leq U \leq 1, \\ U \equiv 1 \text{ in } [S(x_0,r_0) \times \mathbb{R}^2] \backslash \gamma^{2-}, \\ U \equiv 0 \text{ in } \gamma^{3-}. \end{cases} \tag{3.31}$$

We also introduce a function $\Upsilon : \mathbb{R}^+ \to \mathbb{R}^+$, of class C^∞, such that

$$\Upsilon = 0 \text{ in } \left[0, \frac{T}{48}\right] \cup \left[\frac{47T}{48}, T\right] \text{ and } \Upsilon = 1 \text{ in } \left[\frac{T}{24}, \frac{23T}{24}\right]. \tag{3.32}$$

Now, given $g \in \mathcal{S}_\varepsilon$, we associate φ^g on $[0,T] \times \mathcal{T}^2$ by

$$\begin{cases} \Delta\varphi^g(t,x) = \int_{\mathbb{R}^n} g(t,x,v)\,dv - \int_{\mathcal{T}^n\times\mathbb{R}^n} g(t,x,v)\,dv\,dx \text{ in } [0,T]\times\mathcal{T}^n, \\ \int_{\mathcal{T}^n} \varphi^g(t,x)\,dx = 0 \text{ in } [0,T]. \end{cases} \tag{3.33}$$

Then, we define $\tilde{V}(g) := f$ to be the solution of the following system

$$\begin{cases} f(0,x,v) = f_0 \text{ on } \mathcal{T}^2 \times \mathbb{R}^2, \\ \partial_t f + v.\nabla_x f + (F + \nabla\varphi^g + \overline{\mathcal{E}} - \nabla\overline{\varphi}).\nabla_v f = 0 \text{ in } [0,T] \times [(\mathcal{T}^n \times \mathbb{R}^n)\backslash\gamma^-], \\ f(t,x,v) = [1-\Upsilon(t)]f(t^-,x,v) + \Upsilon(t)U(x,v)f(t^-,x,v) \text{ on } [0,T]\times\gamma^-. \end{cases} \tag{3.34}$$

To explain the last equation, we introduce the characteristics (X,V) associated to the force field $F + \nabla\varphi^g + \overline{\mathcal{E}} - \nabla\overline{\varphi}$. In the previous writing, $f(t^-,x,v)$ is the limit value of f on the characteristic $(X,V)(s,t,x,v)$ as the time s goes to t^-. (For times before t, but close to t, the corresponding characteristic is not in γ^-.) When the characteristics (X,V) meet γ^- at time t, then the value of f at time t^+ is fixed according to the last equation in (3.34). One can see the function $\Upsilon(t)U(x,v)$ as an opacity factor which varies according to time and to the incidence of the characteristic on $S(x_0,r_0)$. In this process a part of f is absorbed on γ^-, which varies from the totality of f to no absorption according to the angle of incidence, the modulus of the velocity and the time.

The set of times when a characteristic meets γ^- is discrete. Indeed, if $(X,V)(t,0,x,v) \in \gamma^-$ and $(X,V)(t',0,x,v) \in \gamma^-$, then there exists $s \in (t,t')$ for which $(X,V)(s,0,x,v) \in \gamma^+$. The conclusion follows from $\mathrm{dist}(\gamma^+,\gamma^-) > 0$.

We now consider a continuous linear extension operator $\overline{\pi} : C^0(\mathcal{T}^2\backslash B(x_0,2r_0);\mathbb{R}) \to C^0(\mathcal{T}^2;\mathbb{R})$, and which has the property that each C^α-regular function is continuously mapped to a C^α-regular function, for any $\alpha \in [0,1]$.

From this operator, we deduce a new one $\tilde{\pi} : C^0((\mathcal{T}^n\backslash B(x_0,2r_0))\times\mathbb{R}^n) \to C^0(\mathcal{T}^n\times\mathbb{R}^n)$ according to the rule:

$$(\pi f)(x,v) := [\overline{\pi} f(\cdot,v)](x). \tag{3.35}$$

Then we modify this operator in order to get the further property that for any integrable $f \in C^0((\mathcal{T}^n\backslash B(x_0,2r_0))\times\mathbb{R}^n)$, one has

$$\int_{\mathcal{T}^n\times\mathbb{R}^n} \pi(f)\,dv\,dx = \int_{\mathcal{T}^n\times\mathbb{R}^n} f_0(x,v)\,dv\,dx. \tag{3.36}$$

This condition can easily be obtained by considering a regular, compactly supported, non-negative function u with integral 1 in $B(x_0,r_0)\times\mathbb{R}^n$, and adding to $\pi(f)$ the function

$$\left[\int_{\mathcal{T}^n\times\mathbb{R}^n} f_0 - \int_{(\mathcal{T}^n\backslash\omega)\times\mathbb{R}^n} f\right] u.$$

We obtain a continuous affine operator π satisfying that for some constant c_π, one has for any integrable $f \in C^1(\mathcal{T}^2 \backslash B(x_0, 2r_0))$, one has

$$\|\pi(f)\|_{C^1_b} \leq c_\pi \|f\|_{C^1_b} + \left| \int_{(\mathcal{T}^n \backslash \omega) \times \mathbb{R}^n} f - \int_{\mathcal{T}^n \times \mathbb{R}^n} f_0 \, dv \, dx \right|,$$

$$\|\pi(f)\|_{L^\infty} \leq c_\pi \|f\|_{L^\infty}. + \left| \int_{(\mathcal{T}^n \backslash \omega) \times \mathbb{R}^n} f - \int_{\mathcal{T}^n \times \mathbb{R}^n} f_0 \, dv \, dx \right|.$$

Due to the compact support of u, π continuously sends $L^\infty((\mathcal{T}^n \setminus \omega) \times \mathbb{R}^n; (1+|v|)^\gamma \, dx)$ into $L^\infty(\mathcal{T}^n \times \mathbb{R}^n; (1+|v|)^\gamma \, dx)$, with estimates as above.

It is convenient to introduce another truncation in time function $\tilde{\Upsilon}$ such that:

$$\tilde{\Upsilon} = 0 \text{ in } \left[0, \frac{T}{100}\right] \quad \text{and} \quad \tilde{\Upsilon} = 1 \text{ in } \left[\frac{T}{48}, T\right]. \tag{3.37}$$

Finally, we introduce the operator $\Pi : C^0(([0,T] \times \left[\mathcal{T}^2 \backslash B(x_0, 2r_0)\right] \times \mathbb{R}^2) \cup ([0, T/48] \times \mathcal{T}^2 \times \mathbb{R}^2)) \to C^0([0,T] \times \mathcal{T}^2 \times \mathbb{R}^2)$ given by:

$$(\Pi f)(t,x,v) := (1 - \tilde{\Upsilon}(t)) f(t,x,v) + \tilde{\Upsilon}(t)[\pi f(t,\cdot,\cdot)](x,v). \tag{3.38}$$

We finally define $\mathcal{V}[g]$ by:

$$\mathcal{V}[g] := \overline{f} + \Pi(f_{|([0,T] \times [\mathcal{T}^2 \backslash B(x_0, 2r_0)] \times \mathbb{R}^2) \cup ([0,T/48] \times \mathcal{T}^2 \times \mathbb{R}^2)}) \text{ in } [0,T] \times \mathcal{T}^2 \times \mathbb{R}^2. \tag{3.39}$$

3.3 Existence of a fixed point

The goal of this paragraph is to prove the existence of a fixed point for small values of ε, which corresponds to the following lemma.

Lemma 3.1. *There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, there exists a fixed point of $\mathcal{V}$ in $\mathcal{S}_\varepsilon$.*

The proof is almost the same as in [65, Section 3.3]. In order to avoid to repeat it, we only give the main arguments and refer to it for the details. We only focus on the main differences.

We endow the domain $\mathcal{S}_\varepsilon$ with the norm of $C^0([0,T] \times \mathcal{T}^2 \times \mathbb{R}^2)$. The existence of a fixed point of $\mathcal{V}$ on $\mathcal{S}_\varepsilon$ relies on Schauder's theorem. Accordingly, we have to prove that $\mathcal{S}_\varepsilon$ is a convex compact subset of $C^0([0,T] \times \mathcal{T}^2 \times \mathbb{R}^2)$, that $\mathcal{V}$ is continuous on $\mathcal{S}_\varepsilon$ for this topology, and finally that $\mathcal{V}(\mathcal{S}_\varepsilon) \subset \mathcal{S}_\varepsilon$.

That $\mathcal{S}_\varepsilon$ is convex is clear; that it is compact follows from Ascoli's theorem, using both uniform Hölder estimates and the uniform weighted estimates.

Now let us discuss the continuity of $\mathcal{V}$. Here the proof of [65, Section 3.3] actually holds without further modification. Let us briefly explain the argument. Due to the compactness of $\mathcal{S}_\varepsilon$, it is sufficient to prove that if $f_n \to f$ in $\mathcal{S}_\varepsilon$, then $\mathcal{V}[f_n] \to \mathcal{V}[f]$ pointwise. Let us fix $(x,v) \in \mathcal{T}^2 \times \mathbb{R}^2$. Call (X^n, V^n) and (X, V) the characteristics associated to the force $F + \nabla\varphi^{f_n}$ and $F + \nabla\varphi^f$, respectively. By Gronwall's lemma, (X^n, V^n) converges to (X, V) uniformly on compacts.

If there was no absorption (that is, if we took $U = 0$), then the convergence

$$\mathcal{V}[f_n](t,x,v) \to \mathcal{V}[f](t,x,v)$$

would follow from $\nabla\varphi^{f_n} \to \nabla\varphi^f$ uniformly on $[0,T] \times \mathcal{T}^2$ and Gronwall's lemma. The difficulty comes from the fact that we have to take into account in $\mathcal{V}[f](t,x,v)$ the various times of absorption on γ^-. But from the convergence of (X^n, V^n) to (X, V) (uniformly on compacts), one can deduce that for n large enough, $(X^n, V^n)(\cdot, 0, x, v)$ meets γ^- the same number of times as $(X, V)(\cdot, 0, x, v)$, and that the intersection points of $(X^n, V^n)(\cdot, 0, x, v)$ and γ^- converge towards those of $(X, V)(\cdot, 0, x, v)$. Then the continuity of $\mathcal{V}$ follows.

The main point in the proof is to establish that $\mathcal{V}(\mathcal{S}_\varepsilon) \subset \mathcal{S}_\varepsilon$. The crucial estimate here is the following.

Lemma 3.2. *Let $g \in \mathcal{S}_\varepsilon$, and (X, V) the characteristics associated to $F + \nabla\varphi^g$. Then one has*

$$\big||v| - |V(t, 0, x, v)|\big| \leq 1 + t\|F + \nabla\varphi^g\|_\infty. \tag{3.40}$$

This lemma is trivial in the case under view, even with $|v - V(t, 0, x, v)|$ on the left hand side. But since the estimate with $|v - V(t, 0, x, v)|$ on the left hand side is not valid in the presence of a magnetic field, we prefer to use (3.40).

Let $g \in \mathcal{S}_\varepsilon$. That the point **d.** is true for $\mathcal{V}[g]$ comes from the construction, in particular from the choice of the operator Π (see (3.36)).

Let us explain why the point **b.** is satisfied by $f := \tilde{\mathcal{V}}[g]$. From the construction, on γ^- one has $|f(t^+, x, v)| \leq |f(t^-, x, v)|$. It follows that

$$|f(t, x, v)| \leq |f_0[(X, V)(0, t, x, v)]|.$$

Now,

$$\begin{aligned} |f(t, x, v)| &\leq \|(1 + |v|)^\gamma f_0\|_{L^\infty} \Big[1 + \big||v| - \big(|v| - |V(0, t, x, v)|\big)\big|\Big]^{-\gamma} \\ &\leq \|(1 + |v|)^\gamma f_0\|_{L^\infty} \left(\frac{1 + \big||v| - |V(0, t, x, v)|\big|}{1 + |v|}\right)^\gamma, \end{aligned}$$

where we used

$$(1 + |x - x'|)^{-1} \leq \frac{1 + |x|}{1 + |x'|}.$$

Note that $\|F + \nabla\varphi^g\|_\infty \leq \|F\|_\infty + \varepsilon \leq \|F\|_\infty + 1$. With Lemma 3.2, we deduce that for some $C > 0$ independent of f_0 and ε:

$$|(1 + |v|)^\gamma f(t, x, v)| \leq C\|(1 + |v|)^\gamma f_0\|_{L^\infty}.$$

Then the fact that $\mathcal{V}[g]$ also satisfies **b.** follows from the construction of the operator Π.

Let us now explain the point **c.** We have the following lemma:

Lemma 3.3. *For $g \in \mathcal{S}_\varepsilon$, one has $\tilde{\mathcal{V}}[g] \in C^1(Q_T \backslash \Sigma_T)$, with $\Sigma_T := [0, T] \times \gamma^-$. Moreover, for any (t, x, v) and (t', x', v') in $[0, T] \times [\mathcal{T}^2 \backslash \omega] \times \mathbb{R}^2$, with $|v - v'| \leq 1$, one has,*

$$\begin{aligned} |\tilde{\mathcal{V}}[g](t, x, v) - \tilde{\mathcal{V}}[g](t', x', v')| \leq C[\|f_0\|_{C^1_b(\mathcal{T}^2 \times \mathbb{R}^2)} + \|(1 + |v|)^\gamma f_0\|_{L^\infty(\mathcal{T}^2 \times \mathbb{R}^2)}] \\ \times (1 + |v|)|(t, x, v) - (t', x', v')|, \end{aligned} \tag{3.41}$$

and also

$$|\tilde{\mathcal{V}}[g](t, x, v) - \tilde{\mathcal{V}}[g](t, x', v')| \leq C[\|f_0\|_{C^1_b(\mathcal{T}^2 \times \mathbb{R}^2)} + \|(1 + |v|)^\gamma f_0\|_{L^\infty(\mathcal{T}^2 \times \mathbb{R}^2)}]|(x, v) - (x', v')|, \tag{3.42}$$

the constant C being independent of f_0.

This lemma is rather technical. Actually without absorption, this estimate follows from Gronwall's lemma and the regularity of $\tilde{\mathcal{V}}[g]$ follows from the fact that f_0 and the characteristics are of class C^1. But here at each passage in γ^-, there is a jump between $\nabla\tilde{\mathcal{V}}[g](t^+,x,v)$ and $\nabla\tilde{\mathcal{V}}[g](t^-,x,v)$. One can see by using an explicit computation based on the last equation in (3.34) that

$$|\nabla\tilde{\mathcal{V}}[g](t^+,x,v)| \leq |\nabla\tilde{\mathcal{V}}[g](t^-,x,v)| + C|\tilde{\mathcal{V}}[g](t^-,x,v)|,$$

where ∇ is either ∇_x or ∇_v.

The main point is that the number $n(x,v)$ of times a characteristic $(X,V)(t,0,x,v)$ can cross γ^- is estimated as follows. Using $\mathrm{dist}(\gamma^-,\gamma^+)>0$ and Lemma 3.2, we infer that

$$n(x,v) \leq C(1+\max_t |V(t,0,x,v)|) \leq C(1+|v|).$$

This allows to bound $\nabla\tilde{\mathcal{V}}[g]$ using to the uniform estimates on $(1+|v|)^\gamma\tilde{\mathcal{V}}[g]$.

Finally, point **a.** is a consequence of points **b.**, **c.** and an easy interpolation argument between weighted Hölder spaces, provided that f_0 is small enough.

3.4 A fixed point is relevant

Let us prove that, provided that ε is small enough, the fixed point that we constructed is indeed a solution f starting at f_0 and reaching 0 in $\mathcal{T}^2\setminus\omega$ at time T. For this we show that $\tilde{V}[g](T)=0$ in $\mathcal{T}^2\times\mathbb{R}^2$.

Call again (X,V) the characteristics associated to $F+\nabla\varphi^f-\nabla\overline{\varphi}+\overline{\mathcal{E}}$.

Due to the construction, it is enough to prove the following lemma.

Lemma 3.4. *There exists $\varepsilon_1>0$ such that for any $0<\varepsilon<\varepsilon_1$, all the characteristics (X,V) meet γ^{3-} for some time in $[\frac{T}{24},\frac{23T}{24}]$.*

Proof of Lemma 3.4. We denote by $(\overline{X},\overline{V})$ the characteristics associated to $F+\overline{\mathcal{E}}$.

1. We first prove that for all $(x,v)\in\mathcal{T}^2\times\mathbb{R}^2$, there exists $\sigma\in[\frac{T}{12},\frac{3T}{12}]\cup[\frac{9T}{12},\frac{11T}{12}]$ such that

$$\overline{X}(\sigma,0,x,v)\in\gamma^{4-} := \left\{(x,v)\in S(x_0,r_0)\times\mathbb{R}^2 \;/\; |v|\geq\frac{5}{2} \text{ and } v.\nu(x)\leq -\frac{1}{4}|v|\right\}. \quad (3.43)$$

Let $(x,v)\in\mathcal{T}^2\times\mathbb{R}^2$. We claim that that there exists $t\in[\frac{T}{9},\frac{2T}{9}]\cup[\frac{7T}{9},\frac{8T}{9}]$ such that

$$\overline{X}(t,0,x,v)\in B(x_0,r_0/4), \quad (3.44)$$

and

$$|\overline{V}(t,0,x,v))|\geq\alpha. \quad (3.45)$$

We discuss this according to the modulus of $V(T/3,0,x,v)$:

- If $|V(T/3,0,x,v)|\geq M\geq M_1$, then one can observe that $|v|\geq\max(\underline{m}_1,2\alpha)$, using the characteristics equation. Then by Proposition 3.1, the claim is proved for some $t\in[\frac{T}{9},\frac{2T}{9}]$.
- If $|V(T/3,0,x,v)|<M$, then by Proposition 3.2, $|V(2T/3,0,x,v)|\geq M+1\geq M_2$, and one can once again apply Proposition 3.1, to prove the claim for some $t\in[\frac{7T}{9},\frac{8T}{9}]$.

Now, one can easily see that for some $s > 0$ with $s < \frac{3r_0}{\alpha} \leq \frac{T}{200}$,

$$\overline{X}(t,0,x,v) - s\overline{V}(t,0,x,v) \in S(x_0,r_0) \text{ with } \overline{V}(t,0,x,v).\nu \leq -\frac{\sqrt{3}}{2}|\overline{V}(t,0,x,v)|, \quad (3.46)$$

because a straight line arising from $B(x_0, r_0/2)$ cuts $S(x_0, r_0)$ with angle to the normal ν at the circle of value at most $\pi/6$. The same argument shows that:

$$\overline{X}(t,0,x,v) - 2s\overline{V}(t,0,x,v) \notin B(x_0, 3r_0/2).$$

Now it is clear that,

$$|\overline{V}(\tau,0,x,v) - \overline{V}(t,0,x,v)| \leq 2s[\|F\|_\infty + \|\nabla\overline{\varphi}_1\|_\infty + \|\nabla\overline{\varphi}_3\|_\infty] \text{ for } \tau \in [t-2s,t], \quad (3.47)$$

$$\begin{aligned}|\overline{X}(\tau,0,x,v) - \overline{X}(t,0,x,v) + (t-\tau)\overline{V}(t,0,x,v)|& \\ \leq 2s^2[\|F\|_\infty + \|\nabla\overline{\varphi}_1\|_\infty + \|\nabla\overline{\varphi}_3\|_\infty] \text{ for } \tau \in [t-2s,t]. \quad (3.48)\end{aligned}$$

In the other hand, if C_{r_0} is large enough, we have the estimate:

$$|\overline{X}(t-2s,0,x,v) - \overline{X}(t,0,x,v) + 2s\overline{V}(t,0,x,v)| \leq \frac{r_0}{2}.$$

Therefore by the intermediate value theorem that there exists $\sigma \in [t - \frac{T}{100}, t]$, such that $\overline{X}(\tau,0,x,v) \in S(x_0,r_0)$. Using (3.46), (3.47) and (3.48), and provided that C_{r_0} is large enough (in terms of r_0 only), we deduce that for this σ, (3.43) applies.

2. Now to prove that all the characteristics meet γ^{3-} during $[\frac{T}{12}, \frac{11T}{12}]$, let us compare $(\overline{X},\overline{V})$ and (X,V). Using point **a.** in the definition of $\mathcal{S}_\varepsilon$, we deduce by Gronwall's lemma and elliptic estimates that

$$|(X,V) - (\overline{X},\overline{V})| \leq C\varepsilon.$$

Proceeding as previously, we deduce that if ε is small enough, then for all $(x,v) \in \mathcal{T}^2 \times \mathbb{R}^2$, there exists $t \in [\frac{T}{24}, \frac{23T}{24}]$, such that

$$(X,V)(t,0,x,v) \in \gamma^{3-}.$$

□

We can now gather all the ingredients to prove Theorem 1.1.

Proof of Theorem 1.1. Using Lemma 3.1, we deduce the existence of some fixed point $f = \mathcal{V}[f]$. Using Lemma 3.4, and (3.31), (3.32) and (3.34), we see that, provided that ε is small enough, $\tilde{\mathcal{V}}[f](T) = 0$. Hence f satisfies $\mathrm{Supp}\,[f(T,\cdot,\cdot)] \subset \omega \times \mathbb{R}^2$.

It remains to prove that f satisfies (1.1). This comes from the fact that, due to (3.12) and (3.34), one has

$$\partial_t f + v.\nabla_x f + (F + \nabla\varphi^f).\nabla_v f = 0 \text{ in } [0,T] \times [\mathcal{T}^n \setminus \omega] \times \mathbb{R}^n.$$

Since f is C^1, one has

$$\partial_t f + v.\nabla_x f + (F + \nabla\varphi^f).\nabla_v f = G \text{ in } [0,T] \times \mathcal{T}^n \times \mathbb{R}^n,$$

for some continuous function G. This concludes the proof of Theorem 1.1. □

4 Global controllability for the bounded external field case

In this section, we prove Theorem 1.2.

We call H a hyperplane in $\mathbb{R}^n$ such that its image $\mathcal{H}$ by the canonical surjection $s : \mathbb{R}^n \to \mathcal{T}^n$ is included in ω. We recall that $\mathcal{H}$ is supposed to be closed. We call n_H a unit vector, orthogonal to $\mathcal{H}$. For $l > 0$, we denote

$$\mathcal{H}_l := \mathcal{H} + [-l, l] n_H.$$

Since $\mathcal{H}$ is closed in $\mathcal{T}^n$, we can define $d \in \mathbb{R}^{+*}$ such that

$$\mathcal{H}_{2d} \subset \omega,$$

and such that $4d$ is less than the distance between two different hyperplanes in $s^{-1}(\mathcal{H})$.

4.1 Design of the reference solution

The reference solution is not quite the same as in Section 3. In order to get a global result, as explained in Section 2, we will need the following property, refered to as a "non concentration property" for the characteristics (X, V) associated to $\overline{\varphi}$ (up to a slight modification *inside the control zone*): there exist $c > 0$ such that

$$\forall x, y \in \mathcal{T}^n, \ |X(t, 0, x, 0) - X(t, 0, y, 0)| \geq c|x - y|.$$

The assumption on the control zone ω is motivated by the fact that in this case we can atually construct a reference solution whose characteristics satisfy this condition.

To construct $(\overline{\varphi}, \overline{f})$, we start with the following lemma.

Lemma 4.1. *There exists $\varphi \in C^\infty(\mathcal{T}^n; \mathbb{R})$ such that*

$$\Delta \varphi = 0 \ \text{on } \mathcal{T}^n \setminus \mathcal{H}_d, \tag{4.1}$$

and

$$\nabla \varphi = n_H \ \text{on } \mathcal{T}^n \setminus \mathcal{H}_d. \tag{4.2}$$

Proof of Lemma 4.1. In the domain $\mathcal{T}^n \backslash \mathcal{H}_d$, $x \mapsto n_H$ coincides with the gradient of a harmonic function. Call φ a function in $C^\infty(\mathcal{T}^n; \mathbb{R})$, whose gradient coincides in $\mathcal{H}_d$ with n_H; this function is automatically harmonic in $\mathcal{H}_d$. □

Now given such a φ, we can construct $\overline{\varphi}$ and $\overline{f}$. Consider a function $\mathcal{Y} \in C_0^\infty(0, T)$ satisfying

$$\begin{cases} \text{Supp } \mathcal{Y} \subset (\frac{T}{3}, \frac{2T}{3}), \\ \mathcal{Y} \geq 0, \\ \displaystyle\int_{[0,T]} \mathcal{Y} = 1. \end{cases} \tag{4.3}$$

Set

$$\overline{\varphi}(t, \cdot) = \begin{cases} 0 \text{ for } t \in \left[0, \dfrac{T}{3}\right] \cup \left[\dfrac{2T}{3}, T\right], \\ \mu \mathcal{Y}(t) \varphi(\cdot) \text{ for } t \in \left[\dfrac{T}{3}, \dfrac{2T}{3}\right], \end{cases}$$

$$\overline{\mathcal{E}}(t,\cdot) = \begin{cases} 0 \text{ for } t \in \left[0, \dfrac{T}{3}\right] \cup \left[\dfrac{2T}{3}, T\right], \\ \mu \mathcal{Y}(t) n_H \text{ for } t \in \left[\dfrac{T}{3}, \dfrac{2T}{3}\right], \end{cases}$$

where μ is a positive parameter depending on ω, T and F only, according to the following lemma.

Lemma 4.2. *Given ω as above, $T > 0$ and F, there exists $\mu > 0$ such that all the characteristics associated to $\overline{\mathcal{E}}$ meet*

$$\gamma^{3-} := \{(x,v) \in \partial\mathcal{H}_d \times \mathbb{R}^n \ / \ |v| \geq 2 \text{ and } v.\nu \leq -2\}, \tag{4.4}$$

for some time in $[\frac{T}{6}, \frac{5T}{6}]$, where $\nu = \pm n_H$ is the outward unit vector on $\partial\mathcal{H}_d$.

Once defined $\overline{\varphi}$, we define $\overline{f} : [0,T] \times \mathcal{T}^2 \times \mathbb{R}^2$ as previously by (3.22)-(3.23).

Proof of Lemma 4.2. Let $(x,v) \in \mathcal{T}^n \times \mathbb{R}^n$. Call $(\overline{X}, \overline{V})$ the characteristics associated to $\overline{\mathcal{E}}$. We discuss according to $V(\frac{T}{6}, 0, x, v) \cdot n_H$.

- If $V(\frac{T}{6}, 0, x, v) \cdot n_H$ is large enough, say larger than $c > 0$, then one sees easily using the characteristic equation that there exists $t \in [\frac{T}{6}, \frac{T}{4}]$ such that $(\overline{X}, \overline{V})(t, 0, x, v) \in \gamma^{3-}$.
- For the other (x,v), one can find $\mu > 0$ such that $V(\frac{2T}{3}, 0, x, v) \cdot n_H \geq c$. Then there exists $t \in [\frac{2T}{3}, \frac{5T}{6}]$ such that $(\overline{X}, \overline{V})(t, 0, x, v) \in \gamma^{3-}$.

□

4.2 Definition of the fixed-point operator

For $\lambda \in (0,1]$, we define again a subset $\mathcal{S}_\varepsilon^\lambda$ of $C_b^{\delta_2}(Q_T)$ on which we will define the operator $\mathcal{V}$ (which actually depends on λ):

$$\mathcal{S}_\varepsilon^\lambda := \Big\{ \ g \in C_b^{\delta_2}(Q_T) \ \Big/$$
$$\begin{array}{ll} \textbf{a.} & \| \int_{\mathbb{R}^n} (g - \overline{f}) dv \|_{C^{\delta_1}(\Omega_T)} \leq \varepsilon, \\ \textbf{b.} & \|(1+|v|)^\gamma (g - \overline{f})\|_{L^\infty(Q_T)} \leq c_1 \left[\|f_0^\lambda\|_{C_b^1(\mathcal{T}^n \times \mathbb{R}^n)} + \|(1+|v|)^\gamma f_0^\lambda\|_{C_b^0(\mathcal{T}^n \times \mathbb{R}^n)} \right], \\ \textbf{c.} & \|g - \overline{f}\|_{C_b^{\delta_2}(Q_T)} \leq c_2 \left[\|f_0^\lambda\|_{C_b^1(\mathcal{T}^n \times \mathbb{R}^n)} + \|(1+|v|)^\gamma f_0^\lambda\|_{C_b^0(\mathcal{T}^n \times \mathbb{R}^n)} \right], \\ \textbf{d.} & \forall t \in [0,T], \ \int_{\mathcal{T}^n \times \mathbb{R}^n} g(t,x,v) dx dv = \int_{\mathcal{T}^n \times \mathbb{R}^n} f_0^\lambda(x,v) dx dv \ \Big\}, \end{array} \tag{4.5}$$

with c_1, c_2 to be fixed later depending only on γ, T and ω (and hence on $(\overline{f}, \varphi)$), but not on λ; here, δ_1 and δ_2 are fixed as follows

$$\delta_1 = \frac{\gamma - n}{2(n+1)(\gamma+1)} \text{ and } \delta_2 = \frac{\gamma}{\gamma+1}.$$

For fixed c_1 and c_2 large enough depending only on $(\overline{f}, \varphi)$, one has for λ small enough depending on ε that

$$\left| \int f_0^\lambda dv dx \right| \leq \varepsilon,$$

see (2.3). Hence in that case $g(t,x,v) = f_0^\lambda(x,v) + \overline{f}(t,x,v)$ belongs to $\mathcal{S}_\varepsilon^\lambda$ for $\lambda < \mu(\varepsilon)$, so $\mathcal{S}_\varepsilon^\lambda \neq \emptyset$. From now, we suppose that this is the case.

We write $\Gamma_1 := \mathcal{H} - dn_H$, $\Gamma_2 := \mathcal{H} + dn_H$ and $\Gamma := \Gamma_1 \cup \Gamma_2$. Let $\nu = -n_h$ on Γ_1 and $\nu = n_h$ on Γ_2. We define

$$\gamma^- := \Big\{(x,v) \in \Gamma \times \mathbb{R}^n \ / \ v.\nu(x) < -1\Big\}, \tag{4.6}$$

$$\gamma^{2-} := \Big\{(x,v) \in \Gamma \times \mathbb{R}^n \ / \ |v| \geq 1 \text{ and } v.\nu(x) \leq -3/2\Big\}, \tag{4.7}$$

$$\gamma^+ := \Big\{(x,v) \in \Gamma \times \mathbb{R}^n \ / \ v.\nu(x) \geq 0\Big\}. \tag{4.8}$$

Note that γ^{3-} defined in (4.4) can be reformulated as

$$\gamma^{3-} = \Big\{(x,v) \in \Gamma \times \mathbb{R}^n \ / \ |v| \geq 2 \text{ and } v.\nu(x) \leq -2\Big\}.$$

Again, we observe that

$$\text{dist}((\Gamma \times \mathbb{R}^n)\backslash\gamma^-; \gamma^{2-}) > 0.$$

We introduce a $C^\infty \cap C^1_b$ regular function U from $\Gamma \times \mathbb{R}^n$ to $\mathbb{R}$ the same way as previously, by

$$\left\{\begin{array}{l} 0 \leq U \leq 1, \\ U \equiv 1 \text{ in } (\Gamma \times \mathbb{R}^n)\backslash\gamma^-, \\ U \equiv 0 \text{ in } \gamma^{2-}. \end{array}\right. \tag{4.9}$$

The function Υ is again introduced by (3.32). As in Section 3, we define π as a continuous affine extension operator $\overline{\pi}$ from $C^0(\mathcal{H}_{2d};\mathbb{R})$ to $C^0(\mathcal{T}^n;\mathbb{R})$, and which has the same property that each C^α-regular function is continuously mapped to a C^α-regular function, for any $\alpha \in [0,1]$. Moreover, we manage again in order that for any $f \in C^0(\mathcal{H}_{2d};\mathbb{R})$, (3.36) occurs. The operator Π is given by (3.38).

Now, given $g \in \mathcal{S}^\lambda_\varepsilon$, we first define φ^g by (3.33).
Then we introduce $f = \tilde{\mathcal{V}}[g]$ as the solution of the following system:

$$\left\{\begin{array}{l} f(0,x,v) = f_0^\lambda \text{ on } \mathcal{T}^n \times \mathbb{R}^n, \\ \partial_t f + v.\nabla_x f + (F^\lambda + \nabla(\varphi^g - \overline{\varphi}) + \mu\mathcal{Y}(t)n_H).\nabla_v f = 0 \text{ in } [0,T] \times [(\mathcal{T}^n \times \mathbb{R}^n)\backslash\gamma^-], \\ f(t,x,v) = [1-\Upsilon(t)]f(t^-,x,v) + \Upsilon(t)U(x,v)f(t^-,x,v) \text{ on } [0,T] \times \gamma^-. \end{array}\right. \tag{4.10}$$

The meaning of this equation is the same one as in Section 3 (and $\mu\mathcal{Y}(t)n_H$ plays the same role as $\mathcal{E}$ in Section 3). Recall that F^λ was defined in (2.5).
Then, as for Section 3, we define $\mathcal{V}[g]$ by

$$\mathcal{V}[g] := \overline{f} + \Pi(f_{|[0,T]\times\mathcal{H}_{2d}\times\mathbb{R}^n\cup[0,T/48]\times\mathcal{T}^n\times\mathbb{R}^n}) \text{ in } [0,T] \times \mathcal{T}^n \times \mathbb{R}^n. \tag{4.11}$$

Again, $f_{|[0,T]\times\mathcal{H}_{2d}\times\mathbb{R}^n\cup[0,T/48]\times\mathcal{T}^n\times\mathbb{R}^n}$ is C^1 regular, and, together with the construction of Π, it will follow that $\mathcal{V}[g]$ is in $C^1([0,T] \times \mathcal{T}^n \times \mathbb{R}^n)$.

Considering the form of (4.10), the characteristics that we consider in the sequel are (X^g, V^g) associated to $F^\lambda + \nabla(\varphi^g - \overline{\varphi}) + \mu\mathcal{Y}(t)n_H$, which coincide with the ones associated to $F^\lambda + \nabla\varphi^g$ outside the control zone, but not necessarily inside.

4.3 Existence of a fixed point

Now our goal is to prove the following lemma.

Lemma 4.3. *For any small $\varepsilon > 0$, there exists $\overline{\lambda}(\varepsilon) > 0$ such that for any positive $\lambda < \overline{\lambda}(\varepsilon)$, the operator $\mathcal{V}$ has a fixed point in $\mathcal{S}_\varepsilon^\lambda$.*

Proof of Lemma 4.3. We prove Lemma 4.3 by checking the assumptions for Schauder's fixed point Theorem on $\mathcal{V}$. We will sometimes forget the indices and exponents ε and λ.

1. Again, $\mathcal{S}$ is a convex compact subset of $C^0(Q_T)$.

2. The continuity of $\mathcal{V}$ can be proven in the same way as in Section 3.

3. The difficulty is to check that for λ small, one has $\mathcal{V}(\mathcal{S}_\varepsilon^\lambda) \subset \mathcal{S}_\varepsilon^\lambda$. Accordingly, we have to check the points **a.**, **b.**, **c.** and **d.** for $\mathcal{V}[g]$.

That $\mathcal{V}[g]$ satisfies **d.** comes directly from the construction. That $\tilde{\mathcal{V}}[g]$ and consequently $\mathcal{V}[g]$ satisfies estimates as **b.** is not difficult and proven as in Section 3. In particular Lemma 3.2 is still satisfied.

For what concerns point **c.** we have as previously (see also [65, Lemma 4, p. 370])

Lemma 4.4. *For $g \in \mathcal{S}_\varepsilon^\lambda$, one has $\tilde{\mathcal{V}}[g] \in C^1(Q_T \backslash \Sigma_T)$, with $\Sigma_T := [0,T] \times \gamma^-$. Moreover, for any (t,x,v) and (t',x',v') in $[0,T] \times [\mathcal{T}^2 \backslash \omega] \times \mathbb{R}^2$, with $|v-v'| \leq 1$, one has,*

$$|\tilde{\mathcal{V}}[g](t,x,v) - \tilde{\mathcal{V}}[g](t',x',v')| \leq C[\|f_0\|_{C^1_b(\mathcal{T}^2\times\mathbb{R}^2)} + \|(1+|v|)^{\gamma+2} f_0\|_{L^\infty(\mathcal{T}^2\times\mathbb{R}^2)}] \times (1+|v|)|(t,x,v)-(t',x',v')|, \quad (4.12)$$

and also

$$|\tilde{\mathcal{V}}[g](t,x,v) - \tilde{\mathcal{V}}[g](t,x',v')| \leq C[\|f_0\|_{C^1_b(\mathcal{T}^2\times\mathbb{R}^2)} + \|(1+|v|)^{\gamma+2} f_0\|_{L^\infty(\mathcal{T}^2\times\mathbb{R}^2)}]|(x,v)-(x',v')|, \quad (4.13)$$

the constant C being independent from f_0.

The central part is point **a.**, where the smallness of λ and the non concentration property of $\overline{\varphi}$ are used. We begin by a lemma which asserts that the non concentration property is preserved by a small perturbation. Recall that (X^g, V^g) are associated to $F^\lambda + \nabla(\varphi^g - \overline{\varphi}) + \mu\mathcal{Y}(t)n_H$.

Lemma 4.5. *There exists $c > 0$ such that for any λ small enough (in terms of T, ω and F), for any $g \in \mathcal{S}_\varepsilon^\lambda$, one has*

$$\forall (x,y) \in (\mathcal{T}^n)^2,\ \forall t \in [0,T], \quad c^{-1}|x-y| \leq |X^g(t,0,x,0) - X^g(t,0,y,0)| \leq c|x-y|. \quad (4.14)$$

Proof of Lemma 4.5. Define $(\overline{X}, \overline{V})$ as the characteristics associated to the force $\mu\mathcal{Y}(t)n_H$. It is clear that $(\overline{X}, \overline{V})$ satisfy the non concentration property:

$$\forall (x,y) \in (\mathcal{T}^n)^2,\ \forall t \in [0,T], \quad |\overline{X}(t,0,x,0) - \overline{X}(t,0,y,0)| \geq |x-y|. \quad (4.15)$$

(This is actually an equality!) Now, it follows from Gronwall's inequality that for a constant C depending only on μ, $\mathcal{Y}$ and F, one has

$$\|(X^g, V^g) - (\overline{X}, \overline{V})\|_{C^0_b([0,T]^2\times\mathcal{T}^n\times\mathbb{R}^n)} \leq C(\varepsilon + \lambda^2). \quad (4.16)$$

One can get a further inequality in the following way (when it is not explicit, the norm considered is the L^∞ one)

$$\begin{aligned}
&\frac{d}{dt^+}\|\nabla(X^g,V^g)(t,s,x,v)-\nabla(\overline{X},\overline{V})(t,s,x,v)\| \\
&\qquad \le \|\nabla V^g(t,s,x,v)-\nabla\overline{V}(t,s,x,v)\| \\
&\qquad + \|\nabla_x E_g(t,X^g(t,s,x,v))\nabla X^g(t,s,x,v)-\nabla_x E_{\overline{f}}(t,\overline{X}(t,s,x,v))\nabla\overline{X}(t,s,x,v)\| \\
&\qquad + \|\nabla_{x,v}F^\lambda(t,X^g(t,s,x,v),V^g(t,s,x,v))\nabla(X^g,V^g)(t,s,x,v) \\
&\qquad\qquad - \nabla_{x,v}F^\lambda(t,\overline{X}(t,s,x,v),\overline{V}(t,s,x,v))\nabla(\overline{X},\overline{V})(t,s,x,v)\|
\end{aligned}$$

where ∇ stands either for ∇_x or for ∇_v. Now the second term is bounded as follows

$$\|\nabla_x E_g(t,X^g(t,s,x,v))\nabla X^g(t,s,x,v)-\nabla_x E_{\overline{f}}(t,\overline{X}(t,s,x,v))\nabla\overline{X}(t,s,x,v)\| \le A_1+A_2+A_3,$$

with

$$\left\{\begin{array}{l}
A_1 = \|\nabla_x E_g(t,X^g(t,s,x,v))\nabla X^g(t,s,x,v)-\nabla_x E_g(t,X^g(t,s,x,v))\nabla\overline{X}(t,s,x,v)\|,\\
A_2 = \|\nabla_x E_g(t,X^g(t,s,x,v))\nabla\overline{X}(t,s,x,v)-\nabla_x E_{\overline{f}}(t,X^g(t,s,x,v))\nabla\overline{X}(t,s,x,v)\|,\\
A_3 = \|\nabla_x E_{\overline{f}}(t,X^g(t,s,x,v))\nabla\overline{X}(t,s,x,v)-\nabla_x E_{\overline{f}}(t,\overline{X}(t,s,x,v))\nabla\overline{X}(t,s,x,v)\|.
\end{array}\right.$$

Now

$$\begin{array}{rcl}
A_1 & \le & \|\nabla_x E_g\|_{C^0_b(\Omega_T)}\|\nabla X^g(t,s,x,v)-\nabla\overline{X}(t,s,x,v)\|_{C^0_b([0,T]^2\times\mathcal{T}^n\times\mathbb{R}^n)},\\
A_2 & \le & \|\nabla_x E_g-\nabla_x E_{\overline{f}}\|_{C^0_b(\Omega_T)}\|\nabla\overline{X}\|_{C^0_b([0,T]^2\times\mathcal{T}^n\times\mathbb{R}^n)},\\
A_3 & = & 0.
\end{array}$$

Hence we obtain

$$\begin{aligned}
&\|\nabla_x E_g(t,X^g(t,s,x,v))\nabla X^g(t,s,x,v)-\nabla_x E_{\overline{f}}(t,\overline{X}(t,s,x,v))\nabla\overline{X}(t,s,x,v)\| \\
&\qquad\qquad \le C(\varepsilon+\|\nabla X^g(t,s,x,v)-\nabla\overline{X}(t,s,x,v)\|_{C^0_b([0,T]^2\times\mathcal{T}^n\times\mathbb{R}^n)}).
\end{aligned}$$

We treat the term concerning F^λ in the same way and obtain

$$\begin{aligned}
&\|\nabla_{x,v}F^\lambda(t,X^g(t,s,x,v),V^g(t,s,x,v))\nabla(X^g,V^g)(t,s,x,v) \\
&\qquad - \nabla_{x,v}F^\lambda(t,\overline{X}(t,s,x,v),\overline{V}(t,s,x,v))\nabla(\overline{X},\overline{V})(t,s,x,v)\| \\
&\qquad \le C(\lambda+\|\nabla(X^g,V^g)(t,s,x,v)-\nabla(\overline{X},\overline{V})(t,s,x,v)\|_{C^0_b([0,T]^2\times\mathcal{T}^n\times\mathbb{R}^n)}).
\end{aligned}$$

It follows then by Gronwall's lemma that for a certain constant C, one has

$$\|(X^g,V^g)-(\overline{X},\overline{V})\|_{L^\infty([0,T];C^1_b(\mathcal{T}^n\times\mathbb{R}^n))} \le C(\varepsilon+\lambda).$$

Hence, if ε and λ are small enough, then (4.15) is still valid when replacing $(\overline{X},\overline{V})$ by (X^g,V^g), up to a multiplicative constant. This gives (4.14). □

Let us come back to the proof of point **a**. Let us treat the L^∞-norm; the C^{δ_1} one will follow by interpolation. From (4.14), we deduce that $X^g(t,0,\cdot,0):\mathcal{T}^n\to\mathcal{T}^n$ is invertible; call $(X^g_t)^{-1}$ its inverse, and define the function $W^g_t:[0,T]\times\mathcal{T}^n\to\mathbb{R}^n$ by

$$W^g_t(x) := V^g(t,0,(X^g_t)^{-1}(x),0).$$

One can describe $(X^g_t)^{-1}(x)$ as the initial position of a particle, which starting with velocity 0, reaches x at time t; then $W^g_t(x)$ is its velocity at time t.

Let us give an estimate on $v - W_t^g(x)$. First,

$$v - W_t^g(x) = V^g(0, t, X^g(t, 0, x, v), V^g(t, 0, x, v)) - V^g(t, 0, (X_t^g)^{-1}(x), 0).$$

By using Gronwall's lemma on $V(0, t, \cdot, \cdot)$, we deduce that for some constant independent of $\lambda \in (0, 1]$

$$|v - W_t^g(x)| \leq C \left(|X^g(t, 0, x, v) - (X_t^g)^{-1}(x)| + |V^g(t, 0, x, v)| \right).$$

That the constant is independent of λ comes from the fact that we have uniform Lipschitz estimates on $F^\lambda + \nabla(\varphi^g - \overline{\varphi}) + \mu \mathcal{Y}(t) n_H$ for $\lambda \in (0, 1]$.

To estimate the first term, we first notice that the non-concentration property (4.14) gives

$$\begin{aligned}
(c')^{-1} |(X_t^g)^{-1}(x) - X^g(0, t, x, v)| &\leq |X^g(t, 0, (X_t^g)^{-1}(x), 0) - X^g(t, 0, X^g(0, t, x, v), 0)| \\
&= |x - X^g(t, 0, X^g(0, t, x, v), 0)| \\
&= |X^g(t, 0, X^g(0, t, x, v), V^g(0, t, x, v)) - X^g(t, 0, X^g(0, t, x, v), 0)|
\end{aligned}$$

where the first equality comes from the definition of $(X_t^g)^{-1}$, and the second one of the flow property.

Now Gronwall's lemma for $X^g(t, 0, \cdot, \cdot)$, we deduce that for some constant $C > 0$ independent of $\lambda \in (0, 1]$ one has

$$|X^g(t, 0, X^g(0, t, x, v), V^g(0, t, x, v)) - X^g(t, 0, X^g(0, t, x, v), 0)| \leq C |V^g(0, t, x, v)|.$$

Finally we deduce that for some constant $K > 0$ independent of λ, one has, for any $\lambda \in (0, 1]$ and any $g \in \mathcal{S}_\varepsilon^\lambda$,

$$|v - W_t^g(x)| \leq K |V^g(0, t, x, v)|. \tag{4.17}$$

Now, one has

$$\begin{aligned}
|f(t, x, v)| &\leq |f_0^\lambda \left[(X^g, V^g)(0, t, x, v) \right]| \\
&\leq \lambda^{2-n} \|f_0 (1 + |v|)^\gamma\|_{L^\infty(\mathcal{T}^n \times \mathbb{R}^n)} \left(1 + \frac{1}{\lambda} |V^g(0, t, x, v)| \right)^{-\gamma}.
\end{aligned}$$

Using (4.17), we get that

$$|f(t, x, v)| \leq \lambda^{2-n} \|f_0 (1 + |v|)^\gamma\|_{L^\infty(\mathcal{T}^n \times \mathbb{R}^n)} \left(1 + \frac{1}{K\lambda} |v - W_t(x)| \right)^{-\gamma}.$$

It follows that

$$\left| \int_{\mathbb{R}^n} f(t, x, v) dv \right| \leq \lambda^{2-n} \|f_0 (1 + |v|)^\gamma\|_{L^\infty(\mathcal{T}^n \times \mathbb{R}^n)} \int_{\mathbb{R}^n} \left(1 + \frac{1}{K\lambda} |v - W_t^g(x)| \right)^{-\gamma} dv.$$

We deduce that

$$\left| \int_{\mathbb{R}^n} \tilde{\mathcal{V}}[g](t, x, v) dv \right| \leq \kappa \lambda^{2-n} \|f_0 (1 + |v|)^\gamma\|_{L^\infty(\mathcal{T}^n \times \mathbb{R}^n)} K^n \lambda^n.$$

One deduces from the construction of $\mathcal{V}$ that

$$\left\| \int (\mathcal{V}[g] - \overline{f})(t, x, v) dv \right\|_{L^\infty(\Omega_T)} \leq C \lambda^{2-n} \|f_0 (1 + |v|)^\gamma\|_{L^\infty(\mathcal{T}^n \times \mathbb{R}^n)} \lambda^n \leq C(f_0) \lambda^2. \tag{4.18}$$

Now we turn to the Hölder estimate. It follows by interpolation between points **b** and **c**, that for a certain constant C independent from λ, and for $\tilde{\gamma} = \frac{n+\gamma}{2}$ and $\delta = \gamma/(\gamma+1)$ one has
$$|\mathcal{V}[g] - \overline{f}|_{\delta}^{\tilde{\gamma}} \leq C \left[\|f_0^\lambda\|_{C_b^1(\mathcal{T}^n \times \mathbb{R}^n)} + \|(1+|v|)^\gamma f_0^\lambda\|_{C^0(\mathcal{T}^n \times \mathbb{R}^n)} \right].$$
We deduce that, for $\lambda \leq 1$ and another constant C (depending on f_0 but not on λ),
$$\| \int (\mathcal{V}[g] - \overline{f}) dv \|_{C^\delta(\Omega_T)} \leq C\lambda^{1-n}.$$
Now we interpolate again this inequality with (4.18). We get that for δ_1, one has
$$\| \int (\mathcal{V}[g] - \overline{f}) dv \|_{C^{\delta_1}(\Omega_T)} \leq K'\lambda.$$
which concludes the point **a**, for it is sufficient to find a proper λ. This finally proves $\mathcal{V}(\mathcal{S}_\varepsilon^\lambda) \subset \mathcal{S}_\varepsilon^\lambda$.

□

4.4 A fixed point is relevant

Now we can prove that the characteristics associated to the fixed point are relevant:

Lemma 4.6. *There exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon < \varepsilon_1$, all the characteristics (X, V) meet γ^{2-} for some time in $[\frac{T}{24}, \frac{23T}{24}]$.*

Proof of Lemma 4.6. We recall that by the scaling $F^\lambda = \lambda^2 F(\lambda t, x, \frac{v}{\lambda})$, so that $\|F^\lambda\|_{L^\infty_{t,x,v}} \leq \lambda^2 \|F\|_{L^\infty_{t,x,v}}$. As for Lemma 3.4, the proof follows, recalling the Gronwall's estimate (4.16), and the fact that the characteristics associated to the reference solution $\overline{f}$ meet $\gamma^{3-} \times [\frac{T}{6}, \frac{5T}{6}]$. (We recall that μ was defined when we have constructed the reference solution $\overline{f}$.)
□ Finally, we can conclude the proof of the theorem.

Proof of Theorem 1.2. Using Lemma 4.3, we deduce the existence of some fixed point g for λ sufficiently small. Using Lemma 4.6, and (3.32), (4.9) and (4.10), we see that it satisfies $\operatorname{Supp}[g(T,\cdot,\cdot)] \subset \omega \times \mathbb{R}^2$. Now, we define $f(t,x,v) = g(\frac{t}{\lambda}, x, \lambda v)$, which satisfies the conclusions of Theorem 1.2. The fact that (1.1) is satisfied for some G supported in ω is done as in Section 3. □

5 External magnetic field case

In this section, we prove Theorem 1.3, that is the local controllability result for the external magnetic field case.

5.1 Rephrasing the geometric assumption

We begin by transforming the geometric assumption (1.11) in a way that is easier to handle in the sequel. For K a compact subset of $\mathcal{T}^2$ and $r > 0$ we denote
$$K_r := \{x \in \mathcal{T}^2 \ / \ d(x,K) \leq r\}. \tag{5.1}$$

The geometric assumption can be reinterpreted with the help of the folllowing lemma.

Lemma 5.1. *Let $K \subset \mathcal{T}^2$ such that $b > 0$ on K and satisfying* (1.11). *Then there exists $\underline{b} > 0$, $d > 0$ and $D > 0$ such that*

$$b \geq \underline{b} \quad \text{on } K_{2d}, \tag{5.2}$$

$$\forall x \in \mathcal{T}^2,\ \forall e \in \mathcal{S}^1,\ \exists t \in [0, D],\ \forall s \in \left[t, t + \frac{d}{2}\right],\ x + se \in K_d. \tag{5.3}$$

Proof of Lemma 5.1. An easy argument relying on the compactness of K shows that for $d > 0$ suitably small, one has (5.2).

To prove (5.3), we use the compactness of $\mathcal{T}^2 \times \mathcal{S}^1$. For any $(x, e) \in \mathcal{T}^2 \times \mathcal{S}^1$, there exists $t \in \mathbb{R}^+$ such that $x + te \in K$. One deduces that for (x', e') in an open neighborhood of (x, e) in $\mathcal{T}^2 \times \mathcal{S}^1$, one has $x' + te' \in K_{d/2}$.

Hence by compactness of $\mathcal{T}^2 \times \mathcal{S}^1$, there exists a maximal time D such that for any $(x, e) \in \mathcal{T}^2 \times \mathcal{S}^1$, there exists $t \in [0, D]$ for which $x + te \in K_{d/2}$. Now if $x + te \in K_{d/2}$ and $x + t'e \notin K_d$, then one has $|t - t'| \geq d/2$, since $\mathrm{dist}(K_{d/2}, \mathcal{T}^2 \setminus K_d) \geq d/2$. The conclusion (5.3) follows. $\square$

5.2 Design of the reference solution

The first step consists in building the reference solution, once again distinguishing between high and low velocities. We first treat the case of large velocities. We prove that with the geometric assumption on b, high velocity particles spontaneously reach the arbitrary open set. One can observe that this is very different to the case of bounded force fields. Actually we can prove a stronger result than announced, since we can add to the Lorentz force any additional bounded force field. Such a generalization will be actually crucial for the proof of Lemma 5.3.

Proposition 5.1. *Let $T > 0$ and $r_0 > 0$. Let b satisfy the geometric condition* (1.11). *There exists $\underline{m} \in \mathbb{R}^{+*}$ large enough depending only on b, T and ω such that for all $\mathfrak{F} \in L^\infty(0, T; W^{1,\infty}(\mathcal{T}^2 \times \mathbb{R}^2))$ satisfying $\|\mathfrak{F}\|_{L^\infty} \leq 1$, the characteristics $(\overline{X}, \overline{V})$ associated to $b(x)v^\perp + \mathfrak{F}$ satisfy:*

$$\begin{aligned}\forall x \in \mathbb{T}^2, \forall v \in \mathbb{R}^2 \text{ such that } |v| \geq \underline{m}, \exists t \in (T/4, 3T/4),\ \overline{X}(t, 0, x, v) \in B(x_0, r_0/2) \\ \text{and for all } s \in [0, T],\quad \frac{|v|}{2} \leq |\overline{V}(s, 0, x, v)| \leq 2|v|.\end{aligned} \tag{5.4}$$

Proof of Proposition 5.1. We prove Proposition 5.1 in several cases of increasing complexity. In a first time (Cases 1–3), we suppose that $\mathfrak{F} = 0$. In Case 4, we explain how to take $\mathfrak{F}$ into account.

In all cases, we define

$$\overline{b} := \max_{x \in \mathcal{T}^2} b(x). \tag{5.5}$$

1. *An enlightening case: constant magnetic field modulus.* Let us first suppose b constant; for readability we assume here that $b(x) := 1$.

As noticed in [65, Appendix A, p. 373-374], there are only a finite number of direction in $\mathbb{S}^1$ (identifying $\mathbb{S}^1$ with $[0, 2\pi[$, we denote them $\alpha_1, ..., \alpha_N \in [0, 2\pi[$) for which there exists a half-line in $\mathbb{T}^2$ which does not intersect $B(x_0, r_0/8)$. Indeed if the slope is irrational, then each corresponding half-line is dense in the torus, and consequently meets $B(x_0, r_0/8)$. If the slope is rational, say p/q with $p \in \mathbb{Z}$, $q \in \mathcal{N} \setminus \{0\}$ and $\gcd(p, q) = 1$, then these half-lines L are closed periodic lines in $\mathcal{T}^2$. Due to Bézout's theorem, the distance between to consecutive lines in $s^{-1}(L)$ is less than $\min(\frac{1}{|p|}, \frac{1}{q})$, and the conclusion follows.

We introduce the neighborhoods of α_i:

$$\mathcal{V}_i = (\alpha_i - \beta_i/2, \alpha_i + \beta_i/2),$$

as follows. Let $\beta_i > 0$ and $\tau \leq T$ small enough such that

$$\beta_i < \frac{\tau}{4} \text{ and } \frac{\tau}{4} < \min_{i \neq j} d(\mathcal{V}_i, \mathcal{V}_j).$$

By a compactness argument, there exists a length $L > 0$ such that for any $x \in \mathbb{T}^2$, $\forall a_i \in \mathbb{S}^1 \setminus \cup_{i=1}^N \mathcal{V}_i$, any particle starting from x with a direction a_i has to travel at most a distance L to meet $B(x_0, r_0/8)$.

We fix m large enough such that:

$$T_m := \frac{L}{m} < \tau/4.$$

This is the time "free" particles with velocity m take to cover the distance L. We observe that for any $|v| \geq m$, we have $T_{|v|} := \frac{L}{|v|} \leq T_m$.

Now let $x \in \mathbb{T}^2, v \in \mathbb{R}^2$ with $|v| \geq m$. Let us discuss according to the direction of v.

- First case : $\frac{v}{|v|} \in \mathbb{S}^1 \setminus \cup_{i=1}^N \mathcal{V}_i$.

  We denote $(X^{\#}, V^{\#})$ the characteristics associated to free transport.

  We have, for any $t < T_{|v|}$,

  $$|X^{\#}(t + T/4, T/4, x, v) - \overline{X}(t + T/4, T/4, x, v)| \leq |v| \frac{T_{|v|}^2}{2} = \frac{L^2}{2|v|} \leq \frac{L^2}{2m}.$$

  We can impose m large enough such that $\frac{L^2}{2m} < r_0/8$. As a result:

  $$\exists t \in (T/4, T/2], \overline{X}(t, 0, x, v) \in B(x_0, r_0/4),$$

  and (5.4) is trivial here since $|\overline{V}(t, 0, x, v)|$ is conserved.

- Second case : $\frac{v}{|v|} \in \cup_{i=1}^N \mathcal{V}_i$, say $\mathcal{V}_j$.

  The idea is to simply wait for a time $\tau/4$. Let us consider

  $$(x', v') := (\overline{X}((T + \tau)/4, T/4, x, v), \overline{V}((T + \tau)/4, T/4, x, v)).$$

  We observe that because of the "rotation" induced by the magnetic field and due to the choice of β_i,

  $$\frac{v'}{|v'|} \in \mathbb{S}^1 \setminus \cup_{i=1}^N \mathcal{V}_i,$$

  and thus we are in the same case as before.

Consequently we have proven that:

$$\exists t \in (T/4, 3T/4], \overline{X}(t, 0, x, v) \in B(x_0, r_0/4).$$

2. *Positive magnetic field modulus.* Here we suppose that $b > 0$ on $\mathcal{T}^2$.

We are in the case where in Lemma 5.1, we can take $K = K_d = \mathcal{T}^2$ and

$$\underline{b} = \inf_{x \in \mathbb{T}^2} b.$$

Keeping the same notations as before, we set $\tau \in (0, T]$ and $\beta_i > 0$ in order that

$$\beta_i < \underline{b}\frac{\tau}{4} < \min_{i \neq j} d(\mathcal{V}_i, \mathcal{V}_j).$$

The proof is very similar to the previous one. Indeed, the following estimate still holds:

$$|X^{\#}(t, T/4, x, v) - \overline{X}(t, T/4, x, v)| \leq \frac{L^2}{2m}\overline{b}. \tag{5.6}$$

Let $\tilde{x} \in \mathbb{T}^2, \tilde{v} \in \mathbb{R}^2$. We distinguish as before between two possibilities. Using the previous inequality (5.6), the first case holds identically for m large. For the second case just have to check that with this magnetic field, the velocity is rotated by an angle at least equal to β_i after some time $t \in (0, \frac{\tau}{4})$.

We use the following computation for general (x, v). Denote by $\theta(t)$ the angle (modulo 2π) between $v^\perp$ and $\overline{V}(t, 0, x, v)$. Taking the scalar product with $\overline{V}(t, 0, x, v)$ in:

$$\frac{d\overline{V}(t, 0, x, v)}{dt} = b(\overline{X}(t, 0, x, v))\overline{V}(t, 0, x, v)^\perp,$$

we obtain that $|\overline{V}(t, 0, x, v)| = |v|$. Then, taking the scalar product with $v^\perp$, we obtain:

$$\sin\theta(t)\theta'(t) = b(\overline{X}(t, 0, x, v))\sin\theta(t),$$

so that

$$\theta'(t) = b(\overline{X}(t, 0, x, v)), \tag{5.7}$$

(even if $\sin\theta(t) = 0$ in which case one considers the scalar product with v.) We deduce that $\theta'(t) \geq \underline{b}$.

Thus going back to $(\tilde{x}, \tilde{v})$, by the intermediate value theorem and the definition of the neighborhoods $\mathcal{V}_i$, there is a time T_0 less or equal to $\tau/4$ for which we have:

$$\overline{V}(T_0 + \frac{T}{4}, \frac{T}{4}, \tilde{x}, \tilde{v}) \in \mathbb{S}^1 \setminus \cup_{i=1}^N \mathcal{V}_i,$$

and we conclude as previously.

3. *Magnetic field modulus satisfying the geometric condition.* Let us consider the general case for b, but without the additional force $\mathfrak{F}$.

Given K satisfying the geometric condition (1.11), we introduce d and D as in Lemma 5.1. Let

$$U := \mathcal{T}^2 \setminus K_d,$$

where we recall the notation (5.1). We assume here that $\tau \in (0, T]$ and β_i are such that

$$\beta_i < \frac{\underline{b}}{2}\inf(\frac{\tau}{4}, \frac{\tau d}{32D}) < \min_{i \neq j} d(\mathcal{V}_i, \mathcal{V}_j).$$

We denote by $(X^{\#}, V^{\#})$ the characteristics associated to free transport, while $(\overline{X}, \overline{V})$ corresponds to those associated to the magnetic field.

Let $x \in \mathbb{T}^2, v \in \mathbb{R}^2$. We once again distinguish between the two possibilities. As before the first case is still similar since (5.6) is still valid. We have to give a new argument for the second case.

We will assume that m is large enough so that $T_m < \frac{\tau}{8}$. We distinguish between several sub-cases:

a. Assume that $\overline{X}(t, 0, x, v) \in K_d$ for some t in a time interval of length at least equal to $\frac{T}{4}$ inside $[\frac{T}{4}, \frac{3T}{4}]$. Then one can apply the positive magnetic modulus case (case 2).

b. Assume more generally that $\mathcal{L}^1(\{t \in [\frac{T}{4}, \frac{3T}{4}], \overline{X}(t, 0, x, v) \in K_d\}) \geq T/4$. On U, one has $b \geq 0$, so the angle of $V(t, 0, x, v)$ with v is non decreasing over time. It follows that we can apply (5.7) to each passage of the particle in K_d and we conclude as before.

c. We assume now that the previous cases do not hold. Then $\overline{X}(t, 0, x, v)$ remains in $\mathbb{T}^2 \backslash K_d$ at least during a time $\frac{T}{4}$ in $(\frac{T}{4}, \frac{3T}{4})$.

   By (5.3), each passage in $\mathbb{T}^2 \backslash K_d$ of $X^{\#}(t, 0, x, v)$ lasts at most $D/|v|$. Actually, in U, the characteristics $\overline{X}$ are not straight lines since they are modified by the magnetic field. Let us prove nevertheless that if $|v|$ is large enough, then the particle can remain at most during a time $D/|v|$ in U.

   Let $x \in U$, and $\frac{v}{|v|} \in \mathbb{S}_1$, let $\sigma \in (\frac{T}{4}, \frac{3T}{4})$. By Lemma 5.1, there exists $s < \frac{D}{|v|}$ such that $X^{\#}(\sigma + s, \sigma, x, v) \in K$. Now we can evaluate as for a previous computation:

$$\left| X^{\#}(\sigma + s, \sigma, x, v) - \overline{X}(\sigma + s, \sigma, x, v) \right| \leq \overline{b} \frac{D^2}{2|v|}.$$

   We can choose m large enough such that for any $|v| \geq m$, $\overline{X}(\sigma + s, \sigma, x, v) \in K_d$. Hence at each passage of $X(t, 0, x, v)$ in $\mathbb{T}^2 \backslash K_d$ lasts at most during a time $D/|v|$, which proves the claim.

   This involves that there are at least $\lfloor \frac{T|v|}{4D} \rfloor - 1$ passages in U, and therefore there are also at least $\lfloor \frac{T|v|}{4D} \rfloor - 2$ passages in K_d. This is larger than $\frac{T|v|}{8D}$ for $|v|$ large enough.

   Now we denote by t' a time for which $\overline{X}(t', 0, x, v) \in K_d$, with $\overline{X}(t, 0, x, v) \notin K_d$ for $t < t'$ and t close to t'. Let us show that $\overline{X}(t' + s, 0, x, v)$ remains in K_d for $s \leq \frac{1}{4} \frac{d}{|v|}$, if the velocity is large enough. We have for all $s \in [0, \frac{1}{4} \frac{d}{|v|}]$,

$$|X^{\#}(t' + s, t', x, v) - \overline{X}(t' + s, t', x, v)| \leq \overline{b} |v| \frac{\left(\frac{1}{4} \frac{d}{|v|} \right)^2}{2}.$$

   On the other hand, by (5.3), each passage of $X^{\#}$ in $K_{d/2}$ lasts at least $\frac{d}{4|v|}$. Hence we can choose m large enough such that for any $|v| \geq m$, $\overline{X}(t' + s, t', x, v) \in K_d$ for $s \in [0, \frac{1}{4} \frac{d}{|v|}]$.

   Consequently, $X(t, 0, x, v)$ remains in K_d during a time $\frac{Td}{32D}$ inside $(\frac{T}{4}, \frac{3T}{4})$, and we conclude as before.

4. *With a nontrivial additional force $\mathfrak{F}$.*

Let us finally explain how one can take $\mathfrak{F}$ into account. First, we consider the equations for $|\overline{V}|$ and θ, where θ is the angle between v and $\overline{V}(t,0,x,v)$. The following computations are valid for v large so that $|\overline{V}(t,0,x,v)|$ does not vanish and for a time interval where $\theta \in [-\pi/2, \pi/2]$.

- For what concerns $|V|$, it suffices to take the scalar product with $\overline{V}(t,0,x,v)$ of the equation of $\overline{V}$. We infer
$$\frac{d}{dt}|\overline{V}(t,0,x,v)|^2 = 2\mathfrak{F}\cdot\overline{V}(t,0,x,v),$$
so that
$$\frac{d}{dt}|\overline{V}(t,0,x,v)| = \frac{\mathfrak{F}\cdot\overline{V}(t,0,x,v)}{|\overline{V}(t,0,x,v)|}. \tag{5.8}$$
In particular, for m large enough, one has for all $(x,v)\in\mathbb{T}^2\times\mathbb{R}^2$ with $|v|\geq m$,
$$\frac{|v|}{2} \leq |\overline{V}(t,0,x,v)| \leq 2|v|. \tag{5.9}$$

- For what concerns θ, taking the scalar product of the equation of $\overline{V}$ with v we deduce
$$\begin{aligned}\left(\frac{d}{dt}|\overline{V}(t,0,x,v)|\right)|v|\cos\theta(t) - |\overline{V}(t,0,x,v)||v|\theta'(t)\sin\theta(t)&\\ = b(\overline{X}(t,0,x,v))\overline{V}^{\perp}(t,0,x,v)\cdot v + \mathfrak{F}\cdot v&.\end{aligned}$$
Hence
$$\begin{aligned}&|\overline{V}(t,0,x,v)||v|\theta'(t)\sin\theta(t)\\ &\quad= b(\overline{X}(t,0,x,v))|\overline{V}(t,0,x,v)||v|\sin(\theta(t)) - \mathfrak{F}\cdot\left(v - \frac{\overline{V}(t,0,x,v)|v|}{|\overline{V}(t,0,x,v)|}\cos\theta(t)\right).\end{aligned}$$
We notice that
$$v - \frac{\overline{V}(t,0,x,v)|v|}{|\overline{V}(t,0,x,v)|}\cos\theta(t) = v - \frac{\overline{V}(t,0,x,v)\cdot v}{|\overline{V}(t,0,x,v)|^2}\overline{V}(t,0,x,v) = \mathrm{p}_{\{\overline{V}(t,0,x,v)\}^{\perp}}(v),$$
where $\mathrm{p}_{\{\overline{V}(t,0,x,v)\}^{\perp}}(v)$ denotes the orthogonal projection of v on $\{\overline{V}(t,0,x,v)\}^{\perp}$. So
$$\theta'(t) = b(\overline{X}(t,0,x,v)) + \frac{1}{|\overline{V}(t,0,x,v)|}\mathfrak{F}\cdot\frac{\mathrm{p}_{\{\overline{V}(t,0,x,v)\}^{\perp}}(v)}{|v|\sin\theta(t)}. \tag{5.10}$$
Note that
$$|\mathrm{p}_{\{\overline{V}(t,0,x,v)\}^{\perp}}(v)| = |v|\,|\sin(\theta(t))|,$$
so that:
$$\frac{1}{|\overline{V}(t,0,x,v)|}\left|\mathfrak{F}\cdot\frac{\mathrm{p}_{\{\overline{V}(t,0,x,v)\}^{\perp}}(v)}{|v|\sin\theta(t)}\right| \geq -\frac{1}{|\overline{V}(t,0,x,v)|}\|\mathfrak{F}\|_{\infty}.$$

Now let us revisit the three sub-cases of Case 3 to include $\mathfrak{F}$.

a. Assume that $\overline{X}(t,0,x,v) \in K_d$ for all t in a time interval of length at least equal to $\frac{bT}{4}$. Then using (5.9) and (5.10) we deduce

$$\theta'(t) \geq \underline{b} - 2\frac{\|\mathfrak{F}\|_\infty}{m}, \tag{5.11}$$

so one can conclude as in the positive magnetic modulus case.

b. Assume more generally that $\mathcal{L}^1(\{t \in [\frac{T}{4}, \frac{3T}{4}], \overline{X}(t,0,x,v) \in K_d\}) \geq T/4$. On U, one has $b \geq 0$, so the angle of $V(t,0,x,v)$ with v satisfies

$$\theta'(t) \geq -\frac{2}{m}\|\mathfrak{F}\|_\infty, \tag{5.12}$$

and (5.11) when $\overline{X}(t,0,x,v) \in K_d$. In total the variation of θ is no less than $\frac{bT}{4} - \frac{T}{2m}\|\mathfrak{F}\|_\infty$, so one can conclude as previously (taking m large enough).

c. We assume now that the previous cases do not hold. Then $X(t,0,x,v)$ remains in $\mathbb{T}^2\backslash K_d$ at least during a time $\frac{T}{4}$ inside $(\frac{T}{4}, \frac{3T}{4})$. Let us compare the characteristics (X,V) associated to $\mathfrak{F} + b(x)v^\perp$ with the characteristics $(\overline{X}, \overline{V})$ associated to the magnetic field $b(x)v^\perp$ alone.

   Let $x \in U$, and $\frac{v}{|v|} \in \mathbb{S}_1$, and let let $\sigma \in (\frac{T}{4}, \frac{3T}{4})$. Using the analysis of case 3, there exists $t' < \frac{D}{|v|}$ such that $\overline{X}(\sigma + t', \sigma, x, v) \in K_d$. Now comparing $(\overline{X}, \overline{V})$ and (X,V) and using Gronwall's inequality we deduce

$$\left\{ \begin{array}{l} |V(\sigma+t',\sigma,x,v) - \overline{V}(\sigma+t',\sigma,x,v)| \leq \|\mathfrak{F}\|_\infty \exp(\|b\|_{W^{1,\infty}}(1+2|v|)t'), \\ |X(\sigma+t',\sigma,x,v) - \overline{X}(\sigma+t',\sigma,x,v)| \leq t'\|\mathfrak{F}\|_\infty \exp(\|b\|_{W^{1,\infty}}(1+2|v|)t'). \end{array} \right. \tag{5.13}$$

   Using that $|v|t'$ is of order 1 and taking m large enough, we see that for any $|v| \geq m$, $X(\sigma+t',\sigma,x,v) \in K_{3d/2}$. Hence each passage of $X(t,0,x,v)$ in $\mathbb{T}^2\backslash K_{3d/2}$ lasts at most $D/|v|$. We deduce as previously that there are at least $\lfloor \frac{T|v|}{4D} \rfloor - 2$ passages of $X(t,0,x,v)$ in $K_{3d/2}$ during $(\frac{T}{4}, \frac{3T}{4})$.

   Now reasoning as in Case 3, using Gronwall's estimate (5.13), we see that if $X(\sigma,0,x,v) \in K_{3d/2}$, and m is large enough, then $X(\sigma+t',0,x,v)$ remains in K_{2d} for all times $t' < \frac{Td}{64D}$, and we conclude as before.

□

Now let us turn to the case of low velocities. This time we proceed as in the case of bounded force fields and prove that an analogue of Proposition 3.2 holds:

Proposition 5.2. *Let $\tau > 0$ and $M > 0$. There exists $\tilde{M} > 0$, $\mathcal{E} \in C^\infty([0,\tau] \times \mathcal{T}^2; \mathbb{R}^2)$ and $\varphi \in C^\infty([0,\tau] \times \mathcal{T}^2; \mathbb{R})$ satisfying*

$$\mathcal{E} = -\nabla\varphi \text{ in } [0,\tau] \times (\mathcal{T}^2\backslash B(x_0,r_0)), \tag{5.14}$$

$$\mathit{Supp}(\mathcal{E}) \subset (0,\tau) \times \mathcal{T}^2, \tag{5.15}$$

$$\Delta\varphi = 0 \text{ in } [0,\tau] \times (\mathcal{T}^2\backslash B(x_0,r_0)), \tag{5.16}$$

such that, for any $\mathfrak{F} \in L^\infty(0,T; W^{1,\infty}(\mathcal{T}^2 \times \mathbb{R}^2))$ satisfying $\|\mathfrak{F}\|_{L^\infty} \leq 1$, if (X,V) are the characteristics corresponding the force $\mathfrak{F} + \mathcal{E} + b(x)v^\perp$,

$$\forall (x,v) \in \mathcal{T}^2 \times B(0,M),\ V(\tau,0,x,v) \in B(0,\tilde{M}) \setminus B(0,M+1). \tag{5.17}$$

Proof of Proposition 5.2. Again, we introduce θ and $\mathcal{E}$ as in the proof of Proposition 3.2. Again, one can choose $\mathcal{C}$ and then τ' such that

$$\forall (x,v) \in \mathcal{T}^2 \times B(0,M),\ \overline{V}(\tau,0,x,v) \in \mathbb{R}^2 \setminus B(0, M+2+\tau\|\mathfrak{F}\|_\infty).$$

Let us denote by $(\overline{X}, \overline{V})$ the characteristics corresponding to the force $\mathcal{E}$ alone.

We first observe that we have:

$$\frac{d}{dt}|V|^2 = (\mathfrak{F}(s,X,V) + \mathcal{E}(s,X)) \cdot V.$$

Thus, using Cauchy-Schwarz and Gronwall's estimates, we obtain:

$$|V|^2 \leq \max\left(1, |v|^2 e^{t(\|\mathfrak{F}\|_\infty + \|\mathcal{E}\|_\infty)}\right).$$

We evaluate:

$$|X(t,0,x,v) - \overline{X}(t,0,x,v)| \leq \int_0^t |V(s,0,x,v) - \overline{V}(s,0,x,v)|\, ds,$$

$$\begin{aligned}|V(t,0,x,v) - \overline{V}(t,0,x,v)| \leq \int_0^t \Big[& |\mathcal{E}(s, X(s,0,x,v)) - \mathcal{E}(s, \overline{X}(s,0,x,v))| \\ & + |\mathfrak{F}(t,X,V)| + b|V(s,0,x,v)^\perp| \Big] ds \\ \leq \int_0^t & \|\nabla \mathcal{E}\|_\infty (t-s) |V(s,0,x,v) - \overline{V}(s,0,x,v)|\, ds \\ & + \max\left(T, \frac{2M}{\|\mathcal{E}\|_\infty + \|\mathfrak{F}\|_\infty}(e^{\frac{t}{2}(\|\mathcal{E}\|_\infty + \|\mathfrak{F}\|_\infty)} - 1)\right).\end{aligned}$$

By Gronwall's inequality:

$$|V(t,0,x,v) - \overline{V}(t,0,x,v)| \leq \max\left(T/2, \frac{2M}{\|\mathcal{E}\|_\infty + \|\mathfrak{F}\|_\infty}(e^{\frac{t}{2}(\|\mathcal{E}\|_\infty + \|\mathfrak{F}\|_\infty)} - 1)\right) e^{\frac{t^2}{2}\|\nabla^2 \varphi\|_\infty}. \tag{5.18}$$

For $t = \tau'$, we have:

$$\|\nabla \mathcal{E}\|_\infty = \frac{C}{\tau'}, \quad \|\mathcal{E}\|_\infty = \frac{C'}{\tau'},$$

where C and C' depend only on ω, M, and the conclusion follows as previously since

$$\big|\, |V(\tau,0,x,v)| - |V(\tau',0,x,v)| \,\big| \leq |\tau - \tau'| \|\mathfrak{F}\|_\infty.$$

□

The reference solution. Let us now describe the reference solution. Consider x_0 in ω and $r_0 > 0$ such that $B(x_0, 2r_0) \subset \omega$. We define the reference potential $\overline{\varphi} : [0,T] \times \mathcal{T}^2 \to \mathbb{R}$ as follows. We apply Proposition 5.1 with $\tau = T/3$, we obtain some $\underline{m} > 0$ such that (5.4) is satisfied. Then we apply Proposition 5.2 with $\tau = T/3$ and

$$M = \max\left(\underline{m} + \frac{T}{3}, 100, \frac{800 r_0}{T}, 32 r_0(\overline{b}+1)\right), \tag{5.19}$$

and obtain some $\overline{\varphi}_2$, $\overline{\mathcal{E}}_2$ and some $\tilde{M} > 0$ such that (5.17) is satisfied. We set

$$\overline{\varphi}(t,\cdot) = \begin{cases} 0 \text{ for } t \in [0, \frac{T}{3}] \cup [\frac{2T}{3}, T], \\ \overline{\varphi}_2(t - \frac{T}{3}, \cdot) \text{ for } t \in [\frac{T}{3}, \frac{2T}{3}], \end{cases}$$

and

$$\overline{\mathcal{E}}(t,\cdot) = \begin{cases} 0 \text{ for } t \in [0, \frac{T}{3}] \cup [\frac{2T}{3}, T], \\ \overline{\mathcal{E}}_2(t - \frac{T}{3}, \cdot) \text{ for } t \in [\frac{T}{3}, \frac{2T}{3}]. \end{cases}$$

Then once defined $\overline{\varphi}$, we define $\overline{f} : [0,T] \times \mathcal{T}^2 \times \mathbb{R}^2$ as previously by (3.22)-(3.23).

5.3 Proof of Theorem 1.3

We consider $\mathcal{S}_\varepsilon$ the same convex set as in the proof of Theorem 1.1, and $\mathcal{V}$ the same fixed point operator with $F = b(x)v^\perp$. As before, the proof consists in proving first the existence of a fixed point, and in a second time in proving that such a fixed point is relevant.

For what concerns the existence of a fixed point we have:

Lemma 5.2. *There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, there exists a fixed point of $\mathcal{V}$ in $\mathcal{S}_\varepsilon$.*

Proof of Lemma 5.2. The proof of Lemma 5.2 is exactly the same as the one of Lemma 3.1 and is therefore omited. Note in particular that a variant of the crucial Lemma 3.2 is still valid here, using (5.8). $\square$

In the second part of the proof we show that a fixed point is relevant. In this part lies the main difference with Theorem 1.1. This is given by the following lemma.

Lemma 5.3. *There exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon < \varepsilon_1$, all the characteristics (X, V) associated to $b(x)v^\perp + \overline{\mathcal{E}} - \nabla\overline{\varphi} + \nabla\varphi^f$, where f is a fixed point of $\mathcal{V}$ in $\mathcal{S}_\varepsilon$, meet γ^{3-} for some time in $[\frac{T}{12}, \frac{11T}{12}]$.*

Proof of Lemma 5.3.
We begin by noticing that $\nabla\varphi^f - \nabla\overline{\varphi}$ satisfies

$$\|\nabla\varphi^f - \nabla\overline{\varphi}\|_\infty \leq 1, \tag{5.20}$$

provided that ε is small enough, which we suppose from now. Consequently we can apply Propositions 5.1 and 5.2 to $\mathfrak{F} := \nabla\varphi^f - \nabla\overline{\varphi}$.

It follows that any $(x, v) \in \mathcal{T}^2 \times \mathbb{R}^2$ is (at least) in one of the following situations:

- If $|V(\frac{T}{3}, 0, x, v)| \geq M$, then using (5.20), we deduce $|v| \geq \underline{m}$. Hence there exists $\tau \in [\frac{T}{12}, \frac{3T}{12}]$ such that
$$X(\tau, 0, x, v) \in B(x_0, r_0/2), \tag{5.21}$$
and reasoning as for (5.9) we deduce that for all $s \in [0, \frac{T}{3}]$ one has
$$|V(s, 0, x, v))| \geq \frac{M}{2}, \tag{5.22}$$
where M was defined in (5.19).

- Or $|V(\frac{T}{3}, 0, x, v)| < M$, so $|V(\frac{2T}{3}, 0, x, v)| \geq M + 1$, and there exists $\tau \in [\frac{9T}{12}, \frac{11T}{12}]$ such that (5.21) is true and (5.22) is valid for all $s \in [\frac{2T}{3}, T]$.

Let us consider (x, v) in the first situation, the reasoning being identical for the second situation. As in the proof of Lemma 3.4, we deduce the existence of some $s > 0$ with $s < \frac{4r_0}{|v|} \leq \frac{T}{100}$,

$$X(\tau, 0, x, v) - sV(\tau, 0, x, v) \in S(x_0, \frac{3r_0}{2}) \text{ with } V(\tau, 0, x, v).\nu \leq -\frac{\sqrt{3}}{2}|V(\tau, 0, x, v)|. \tag{5.23}$$

Let us show that this involves for $|v|$ large enough the existence of $\tau_* \in [\tau, t]$ such that

$$x_* := X(\tau, 0, x, v) - (\tau_* - \tau)V(\tau, 0, x, v) \in S(x_0, r_0).$$

We have for $\sigma \in [\tau - s, \tau]$:

$$\frac{M}{2} \leq |V(\sigma, 0, x, v)| \leq 2|v|, \tag{5.24}$$

$$\left| \frac{V(\sigma, 0, x, v)}{|V(\sigma, 0, x, v)|} - \frac{V(\tau, 0, x, v)}{|V(\tau, 0, x, v)|} \right| \leq s \Big[\overline{b} + \frac{2\|\nabla \varphi^f\|_\infty}{M} \Big], \tag{5.25}$$

$$|X(\sigma, 0, x, v) - X(\tau, 0, x, v) + (\tau - \sigma) V(\tau, 0, x, v)| \leq \frac{s^2}{2}(2|v| + \|\nabla \varphi^f\|_\infty). \tag{5.26}$$

Estimate (5.25) comes from the identity

$$\frac{d}{d\sigma}\left(\frac{V(\sigma, 0, x, v)}{|V(\sigma, 0, x, v)|} \right) = \frac{\frac{dV}{d\sigma}(\sigma, 0, x, v)}{|V(\sigma, 0, x, v)|} + \frac{\nabla \varphi^f(\sigma, x, v) \cdot V(\sigma, 0, x, v)}{|V(\sigma, 0, x, v)|^3} V(\sigma, 0, x, v).$$

Let us check that this involves the existence of $t \in [\tau, \tau - s]$ such that $(X(t, 0, x, v), V(t, 0, x, v)) \in \gamma^{3-}$. The existence of of $t \in [\tau, \tau - s]$ such that $X(t, 0, x, v) \in S(x_0, r_0)$ follows from (5.26) and

$$\frac{s^2}{2}(2|v| + \|\nabla \varphi^f\|_\infty) \leq \frac{8r_0}{|v|^2}(2|v| + 1) \leq 8r_0 \frac{2M+1}{M^2} \leq \frac{24 r_0}{M} \leq \frac{r_0}{4}.$$

At such a t, from (5.24), we have $|V(t, 0, x, v)| \geq 2$ since $M \geq 4$.

The fact that at such a moment t, one has $V(t, 0, x, v).\nu(X(t, 0, x, v)) \leq -\frac{1}{5}|V(t, 0, x, v,)|$ comes from

$$\begin{aligned}
\Big| \frac{V(t, 0, x, v)}{|V(t, 0, x, v)|} \cdot \nu(X(t, 0, x, v)) - \frac{V(\tau, 0, x, v)}{|V(\tau, 0, x, v)|} \cdot \nu(x_*) \Big| \\
&\leq \Big| \frac{V(t, 0, x, v)}{|V(t, 0, x, v)|} - \frac{V(\tau, 0, x, v)}{|V(\tau, 0, x, v)|} \Big| + \Big| \nu(X(t, 0, x, v)) - \nu(x_*) \Big| \\
&\leq (\overline{b} + \frac{2}{M}) \frac{4r_0}{|v|} + \frac{1}{r_0}|X(t, 0, x, v) - x_*| \\
&\leq (\overline{b} + 1)\frac{4r_0}{M} + \frac{24}{M} \leq \frac{1}{4},
\end{aligned}$$

and from (5.23). This concludes the proof of Lemma 5.3.

□

Let us finally gather all the pieces to prove Theorem 1.3.

Proof of Theorem 1.3. Using Lemma 5.2, we deduce the existence of some fixed point f of $\mathcal{V}$ in $\mathcal{S}_\varepsilon$. Using Lemma 5.3 we can again use the definitions (3.31), (3.32) and (3.34) to deduce that $\operatorname{Supp}[f(T, \cdot, \cdot)] \subset \omega \times \mathbb{R}^2$ and one checks that f satisfies the equation for some G as previously. This concludes the proof of Theorem 1.3.

□

Chapitre 8

L^1 averaging lemma for transport equations with Lipschitz force fields

Article paru à Kinetic and Related Models (2010).

Sommaire

Résumé : The purpose of this chapter is to extend the L^1 averaging lemma of Golse and Saint-Raymond [73] to the case of a kinetic transport equation with a force field $F(x) \in W^{1,\infty}$. To this end, we will prove a local in time mixing property for the transport equation $\partial_t f + v.\nabla_x f + F.\nabla_v f = 0$.

Introduction

Let $d \in \mathbb{N}^*$ and $1 < p < +\infty$. We consider $\mathbb{R}^d$ equipped with the Lebesgue measure. Let $f(x,v)$ and $g(x,v)$ be two measurable functions in $L^p(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying the transport equation:

$$v.\nabla_x f = g. \tag{0.1}$$

Although transport equations are of hyperbolic nature (and thus there is a priori no regularizing effect), it was first observed for by Golse, Perthame and Sentis in [71] and then by Golse, Lions, Perthame and Sentis [70] (see also Agoshkov [1] for related results obtained independently) that the velocity average (or moment) $\rho(x) = \int f\Psi(v)dv$ with $\Psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ is smoother than f and g : more specifically it belongs to some Sobolev space $W^{s,p}(\mathbb{R}^d)$ with $s > 0$. These kinds of results are referred to as "velocity averaging lemma". The analogous results in the time-dependent setting also hold, that is for the equation:

$$\partial_t f + v.\nabla_x f = g. \tag{0.2}$$

Refined results with various generalizations (like derivatives in the right-hand side, functions with different integrability in x and v...) were obtained in [50], [14], [134], [108]. There exist many other interesting contributions. We refer to Jabin [107] which is a rather complete review on the topic.

Velocity averaging lemmas are tools of tremendous importance in kinetic theory since they provide some strong compactness which is very often necessary to study non-linear terms (for instance when one considers an approximation scheme to build weak solutions, or for the study of asymptotic regimes). There are numerous applications of these lemmas; two emblematic results are the existence of renormalized solutions to the Boltzmann equation [52] and the existence of global weak solutions to the Vlasov-Maxwell system [51]. Both are due to DiPerna and Lions.

The limit case $p = 1$ is actually of great interest. In general, for a sequence (f_n) uniformly bounded in $L^1(dx \otimes dv)$ with $v.\nabla_x f_n$ also uniformly bounded in $L^1(dx \otimes dv)$, the sequence of velocity averages $\rho_n = \int f_n \Psi(v)dv$ is not relatively compact in $L^1(dx)$ (we refer to [70] for an explicit counter-example). This lack of compactness is due to the weak compactness pathologies of L^1. Indeed, as soon as we add some weak compactness to the sequence (or equivalently some equiintegrability in x and v in view of the classical Dunford-Pettis theorem), then we recover some strong compactness in L^1 for the moments (see Proposition 3 of [70] or Proposition 1.2 below).

We recall precisely the notion of equiintegrability which is central in this chapter.

Definition 1. *i. (Local equiintegrability in x and v)*

Let (f_ε) be a bounded family of $L^1_{loc}(dx \otimes dv)$. It is said locally equiintegrable in x and v if and only if for any $\eta > 0$ and for any compact subset $K \subset \mathbb{R}^d \times \mathbb{R}^d$, there exists $\alpha > 0$ such that for any measurable set $A \subset \mathbb{R}^d \times \mathbb{R}^d$ with $|A| < \alpha$, we have for any ε :

$$\int_A \mathbb{1}_K(x,v)|f_\varepsilon(x,v)|dvdx \leq \eta. \tag{0.3}$$

ii. (Local equiintegrability in v)
Let (f_ε) be a bounded family of $L^1_{loc}(dx \otimes dv)$. It is said locally equiintegrable in v if and only if for any $\eta > 0$ and for any compact subset $K \subset \mathbb{R}^d \times \mathbb{R}^d$, there exists $\alpha > 0$ such that for each family $(A_x)_{x \in \mathbb{R}^d}$ of measurable sets of $\mathbb{R}^d$ satisfying $\sup_{x \in \mathbb{R}^d} |A_x| < \alpha$, we have for any ε :

$$\int \left(\int_{A_x} \mathbb{1}_K(x,v)|f_\varepsilon(x,v)|dv \right) dx \leq \eta. \tag{0.4}$$

We observe that local equiintegrability in (x, v) always implies local equiintegrability in v, whereas the converse is false in general.

The major improvement of the paper of Golse and Saint-Raymond [73] is to show that actually, only equiintegrability in v is needed to obtain the L^1 compactness for the moments. This observation was one of the key arguments of their outstanding paper [75] which establishes the convergence of renormalized solutions to the Boltzmann equation in the sense of DiPerna-Lions to weak solutions to the Navier-Stokes equation in the sense of Leray.

More precisely, the result they prove is Theorem 0.1 stated afterwards, with $F = 0$ (free transport case). The aim of this chapter is to show that the result also holds if one adds some force field $F(x) = (F_i(x))_{1 \leq i \leq d}$ with $F \in W^{1,\infty}(\mathbb{R}^d)$:

Theorem 0.1. *Let (f_ε) be a family bounded in $L^1_{loc}(dx \otimes dv)$ locally equiintegrable in v and such that $v.\nabla_x f_\varepsilon + F.\nabla_v f_\varepsilon$ is bounded in $L^1_{loc}(dx \otimes dv)$. Then :*

*i. (f_ε) is locally equiintegrable in both variables x **and** v.*

ii. For all $\Psi \in \mathcal{C}^1_c(\mathbb{R}^d)$, the family $\rho_\varepsilon(x) = \int f_\varepsilon(x,v)\Psi(v)dv$ is relatively compact in $L^1_{loc}(dv)$.

One key ingredient of the proof for $F = 0$ is the nice dispersion properties of the free transport operator. We will show in Section 2 that an analogue also holds for small times when $F \neq 0$:

Proposition 0.3. *Let $F(x)$ be a Lipschitz vector field. There exists a maximal time $\tau > 0$ (depending only on $\|\nabla_x F\|_{L^\infty}$) such that, if f is the solution to the transport equation:*

$$\begin{cases} \partial_t f + v.\nabla_x f + F.\nabla_v f = 0, \\ f(0,.,.) = f^0 \in L^p(dx \otimes dv), \end{cases}$$

Then:

$$\forall |t| \leq \tau, \|f(t)\|_{L^\infty_x(L^1_v)} \leq \frac{2}{|t|^d} \|f^0\|_{L^1_x(L^\infty_v)}. \tag{0.5}$$

Let us also mention that the main theorem generalizes to the time-dependent setting, for transport equations of the form (0.2). The usual trick to deduce such a result from the stationary case is to enlarge the phase space. Indeed we can consider $x' = (t, x)$ in $\mathbb{R}^{d+1}$ endowed with the Lebesgue measure, and $v' = (t, v)$ in $\mathbb{R}^{d+1}$ endowed with the measure $\mu = \delta_{t=1} \otimes \text{Leb}$ (where δ is the dirac measure). Then such a measure μ satisfies property (2.1) of [70]. As a consequence, all the results of Section 1 will still hold.

Nevertheless, we observe that our key local in time mixing estimate (0.5) seems to not hold when $\mathbb{R}^{d+1}$ is equipped with the new measure μ (the main problem being that the only speed associated to the first component of v' is 1). For this reason, we can not prove that equiintegrability in v implies equiintegrability in t. One result (among other possible variants) is the following:

Theorem 0.1. *Let $F(t,x) \in \mathcal{C}^0(\mathbb{R}^+, W^{1,\infty}(\mathbb{R}^d))$. Let (f_ε) be a family bounded in $L^1_{loc}(dt \otimes dx \otimes dv)$ locally equiintegrable in v and such that $\partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + F.\nabla_v f_\varepsilon$ is bounded in $L^1_{loc}(dt \otimes dx \otimes dv)$. Then :*

i. (f_ε) is locally equiintegrable in the variables x and v (but not necessarily with respect to t).

ii. For all $\Psi \in \mathcal{C}^1_c(\mathbb{R}^d)$, the family $\rho_\varepsilon(t,x) = \int f_\varepsilon(t,x,v)\Psi(v)dv$ is relatively compact with respect to the x variable in $L^1_{loc}(dt \otimes dx)$, that is, for any compact $K \subset \mathbb{R}^+_t \times \mathbb{R}^d_x$:

$$\lim_{\delta \to 0} \sup_{\varepsilon} \sup_{|x'| \leq \delta} \|(\mathbb{1}_K \rho_\varepsilon)(t, x+x') - (\mathbb{1}_K \rho_\varepsilon)(t,x)\|_{L^1(dt \otimes dx)} = 0. \tag{0.6}$$

Another possibility is to assume that f_ε is bounded in $L^\infty_{t,loc}(L^1_{x,v,loc})$, in which case we will get equiintegrability in t, x and v and thus compactness for ρ_ε in t and x. We refer to [75], Lemma 3.6, for such a statement in the free transport case.

***Remarks* 0.1.** i. Since the result of Theorem 0.1 is essentially of local nature, we could slightly weaken the assumption on F:

For any $R > 0$,

$$\exists M(R), \forall |x_1|, |x_2| \leq R, \quad |F(x_1) - F(x_2)| \leq M(R)|x_1 - x_2|. \tag{0.7}$$

In other words we can deal with $F \in W^{1,\infty}_{loc}$.

ii. With the same proof, we can treat the case of force fields $F(x,v) \in W^{1,\infty}_{x,v,loc}$ with zero divergence in v :

$$\operatorname{div}_v F = 0.$$

Typically we may think of the Lorentz force $v \wedge B$ where B is a smooth magnetic field.

iii. We can handle a family of force fields (F_ε) depending on ε as soon as (F_ε) is uniformly bounded in $W^{1,\infty}(\mathbb{R}^d)$.

The following of the chapter is devoted to the proof of Theorem 0.1. In Section 1, we prove that a family satisfying the assumptions of Theorem 0.1 and in addition locally equiintegrable in x and v, has moments which are relatively strongly compact in L^1. In Section 2, we investigate the local in time mixing properties of the transport equation $\partial_t f + v.\nabla_x f + F.\nabla_v f = 0$. Finally in the last section, thanks to the mixing properties we establish, we show by an interpolation argument that equiintegrability in v provides some equiintegrability in x.

1 A first step towards L^1 compactness

The first step is to show that under the assumptions of Theorem 0.1, point 1 implies point 2. Using classical averaging lemma in L^2 ([51], [50]), we first prove the following L^2 averaging lemma.

Lemma 1.1. *Let $f, g \in L^2(dx \otimes dv)$ satisfy the transport equation:*

$$v.\nabla_x f + F.\nabla_v f = g. \tag{1.1}$$

Then for all $\Psi \in \mathcal{C}_c^1(\mathbb{R}^d)$, $\rho(x) = \int f(x,v)\Psi(v)dv \in H_x^{1/4}$. Moreover,

$$\|\rho\|_{H_x^{1/4}} \leq C\left(\|F\|_{L_x^\infty}\|f\|_{L^2_{x,v}} + \|g\|_{L^2_{x,v}}\right). \tag{1.2}$$

(C is a constant depending only on Ψ.)

Proof. The standard idea is to consider $-F.\nabla_v f + g$ as a source. Then, since $\operatorname{div}_v F(x) = 0$, we have :

$$-F.\nabla_v f + g \quad = \quad -\sum_{i=1}^{d} \frac{\partial}{\partial_{v_i}}(F_i f) + g.$$

We conclude by applying the L^2 averaging lemma of [51], Theorem 3. $\square$

We recall now in Proposition 1.1 an elementary and classical representation result, obtained by the method of characteristics.

Let $b = (v, F)$, $Z = (X, V)$. Since $F \in W^{1,\infty}$, b satisfies the hypotheses of the global Cauchy-Lipschitz theorem. We therefore consider the trajectories defined by:

$$\begin{cases} Z'(t; x_0, v_0) = b(Z(t; x_0, v_0)) \\ Z(0; x_0, v_0) = (x_0, v_0). \end{cases} \tag{1.3}$$

For all time, the application $(x_0, v_0) \mapsto Z(t; x_0, v_0) = (X(t; x_0, v_0), V(t; x_0, v_0))$ is well-defined and is a C^1 diffeomorphism. Moreover, since b does not depend explicitly on time, it is also classical that $Z(t)$ is a group. The inverse is thus given by $(x, v) \mapsto Z(-t; x, v)$.

Remark 1.1. Since $\operatorname{div}(b) = 0$, Liouville's theorem shows that the volumes in the phase space are preserved (the jacobian determinant of Z is equal to 1).

Proposition 1.1. *i. The time-dependent Cauchy problem :*

$$\begin{cases} \partial_t f + v.\nabla_x f + F.\nabla_v f = 0, \\ f(0,.,.) = f^0 \in L^p(dx \otimes dv) \end{cases} \tag{1.4}$$

has a unique solution (in the distributional sense) represented by

$$f(t,x,v) = f^0(X(-t;x,v)), V(-t;x,v)) \in L^p(dx \otimes dv).$$

ii. For any $\lambda > 0$, the transport equation

$$\lambda f(x,v) + v.\nabla_x f + F.\nabla_v f = g \in L^p(dx \otimes dv) \tag{1.5}$$

has a unique solution (in the distributional sense) represented by:

$$R_\lambda : g(x,v) \mapsto f(x,v) = \int_0^{+\infty} e^{-\lambda s} g(X(-s;x,v), V(-s;x,v))ds \in L^p(dx \otimes dv).$$

In addition, R_λ is a linear continuous map on L^p with a norm equal to $\frac{1}{\lambda}$.

Using Rellich's compactness theorem, we straightforwardly have the following corollary:

Corollary 1.1. *The linear continuous map* $T_{\lambda,\Psi}$ *:*

$$L^2(dx \otimes dv) \to L^2_{loc}(dx)$$
$$g \mapsto \rho = \int R_\lambda(g)(.,v)\Psi(v)dv$$

is compact for all $\Psi \in \mathcal{C}^1_c(\mathbb{R}^d)$ *and all* $\lambda > 0$.

Proof. Using Lemma 1.1 and Proposition 1.1, we have:

$$\|T_{\lambda,\Psi}(g)\|_{H^{1/4}_x} \leq C\,(1 + \|F\|_{L^\infty})\,\|g\|_{L^2_{x,v}}.$$

The conclusion follows. □

Using this compactness property, as in Proposition 3 of [70], we can show the next result:

Proposition 1.2. *Let* $\mathcal{K}$ *be a bounded subset of* $L^1(dx \otimes dv)$ *equiintegrable in* x *and* v *(in view of the Dunford-Pettis theorem, it means in other words that* $\mathcal{K}$ *is weakly compact in* L^1*), then* $T_{\lambda,\Psi}(\mathcal{K})$ *is relatively strongly compact in* $L^1_{loc}(dx)$.

Proof. We recall the proof of this result for the sake of completeness.

The proof is based on a real interpolation argument. We fix a parameter $\eta > 0$. For any $g \in \mathcal{K}$ and any $\alpha > 0$, we may write :

$$g = g_1^\alpha + g_2^\alpha,$$

with

$$\begin{aligned} g_1^\alpha &= \mathbb{1}_{\{|g(x,v)|>\alpha\}} g, \\ g_2^\alpha &= \mathbb{1}_{\{|g(x,v)|\leq\alpha\}} g. \end{aligned}$$

Then, by linearity of $T_{\lambda,\Psi}$, we write $u = T_{\lambda,\Psi}(g) = u_1 + u_2$, with $u_1 = T_{\lambda,\Psi}(g_1^\alpha)$ and $u_2 = T_{\lambda,\Psi}(g_2^\alpha)$.

Let K be a fixed compact set of $\mathbb{R}^d_x$.

We clearly have, since $T_{\lambda,\Psi}$ is linear continuous on $L^1(K)$:

$$\|u_1\|_{L^1_x(K)} \leq C\|g_1^\alpha\|_1.$$

We notice that:

$$|\{(x,v), |g(x,v)| > \alpha\}| \leq \frac{1}{\alpha}\|g\|_{L^1} \leq \frac{1}{\alpha}C.$$

Since $\mathcal{K}$ is equiintegrable, there exists $\alpha > 0$ such that for any $g \in \mathcal{K}$:

$$\int |g\mathbb{1}_{\{|g(x,v)|>\alpha\}}|dxdv \leq \frac{\eta}{C}.$$

Consequently for α large enough, we have:

$$\|u_1\|_{L^1_x(K)} \leq \eta.$$

The parameter α being fixed, we clearly see that $\{g_2^\alpha, g \in \mathcal{K}\}$ is a bounded subset of $L^1_{x,v} \cap L^\infty_{x,v}$, and consequently of $L^2_{x,v}$. Because of Corollary 1.1, $\{u_2, u_2 = T_{\lambda,\Psi}(g_2^\alpha), g \in \mathcal{K}\}$ is relatively compact in $L^2_{loc}(dx)$. In particular it is relatively compact in $L^1_{loc}(dx)$.

As a result, we have shown that for any $\eta > 0$, there exists $\mathcal{K}_\eta \subset L^1_x(K)$ compact, such that $T_{\lambda,\Psi}(\mathcal{K}) \subset \mathcal{K}_\eta + B(0,\eta)$. So this family is precompact and consequently it is compact since $L^1_x(K)$ is a Banach space.

□ We deduce the preliminary result (which means that the first point implies the second in Theorem 0.1):

Theorem 1.1. *Let (f_ε) a family of $L^1_{loc}(dx \otimes dv)$ locally equiintegrable in x and v such that $(v.\nabla_x f_\varepsilon + F.\nabla_v f_\varepsilon)$ is a bounded family of $L^1_{loc}(dx \otimes dv)$. Then for all $\Psi \in \mathcal{C}^1_c(\mathbb{R}^d)$, the family $\rho_\varepsilon(x) = \int f_\varepsilon(x,v)\Psi(v)dv$ is relatively compact in $L^1_{loc}(dx)$.*

Proof. Let $\Psi \in \mathcal{C}^1_c(\mathbb{R}^d)$. Let $R > 0$ be a large number such that $\operatorname{Supp}\Psi \subset B(0,R)$; we intend to show that $(\mathbb{1}_{B(0,R)}(x)\rho_\varepsilon(x))$ is compact in $L^1(B(0,R))$. First of all, we can assume that the f_ε are compactly supported in the same compact set $K \subset \mathbb{R}^d \times \mathbb{R}^d$, with $B(0,R) \times B(0,R) \subset \mathring{K}$. Indeed we can multiply the family by a smooth function χ such that:

$$\begin{gathered}\operatorname{supp}\chi \subset K,\\ \chi \equiv 1 \text{ on } B(0,R) \times B(0,R).\end{gathered}$$

We observe that :

$$\begin{aligned} v.\nabla_x(\chi f_\varepsilon) &= \chi(v.\nabla_x f_\varepsilon) + f_\varepsilon(v.\nabla_x \chi),\\ F.\nabla_v(\chi f_\varepsilon) &= \chi F.\nabla_v(f_\varepsilon) + f_\varepsilon(F.\nabla_v \chi). \end{aligned}$$

Thus the family (χf_ε) satisfies the same L^1 boundedness properties as (f_ε). The equiintegrability property is also clearly preserved. Furthermore, for any x in $B(0,R)$, we have :

$$\int f_\varepsilon(x,v)\Psi(v)dv = \int f_\varepsilon(x,v)\chi(v)\Psi(v)dv$$

Consequently we are now in the case of functions supported in the same compact set.

We have for all $\varepsilon > 0, \lambda > 0$, by linearity of the resolvent R_λ defined in Proposition 1.1 :

$$\begin{aligned}\int f_\varepsilon(x,v)\Psi(v)dv =& \int R_\lambda(\lambda f_\varepsilon + v.\nabla_x f_\varepsilon + F.\nabla_v f_\varepsilon)\Psi(v)dv\\ =&\lambda\int (R_\lambda f_\varepsilon)(x,v)\Psi(v)dv + \int (R_\lambda(v.\nabla_x f_\varepsilon + F.\nabla_v f_\varepsilon))(x,v)\Psi(v)dv.\end{aligned}$$

Let $\eta > 0$. We take $\lambda = \sup_\varepsilon \dfrac{\|(v.\nabla_x f_\varepsilon + F.\nabla_v f_\varepsilon)\|_{L^1_{x,v}}\|\Psi\|_{L^\infty}}{\eta}$.

Then we have by Proposition 1.1 :

$$\begin{aligned}\left\|\int (R_\lambda(v.\nabla_x f_\varepsilon + F.\nabla_v f_\varepsilon))(x,v)\Psi(v)dv\right\|_{L^1_x} \leq& \|R_\lambda(v.\nabla_x f_\varepsilon + F.\nabla_v f_\varepsilon)\|_{L^1_{x,v}}\|\Psi\|_{L^\infty_v}\\ \leq&\frac{1}{\lambda}\|(v.\nabla_x f_\varepsilon + F.\nabla_v f_\varepsilon)\|_{L^1_{x,v}}\|\Psi\|_{L^\infty_v}\\ \leq&\eta.\end{aligned}$$

Moreover, since (f_ε) is bounded in $L^1(dx \otimes dv)$ and equiintegrable in x and v, Proposition 1.2 implies that the family $(\int R_\lambda(f_\varepsilon)\Psi(v)dv)$ is relatively compact in $L^1_x(B(0,R))$. Finally we can argue as for the end of the proof of Proposition 1.2: for all $\eta > 0$, there exists $K_\eta \subset L^1_x(B(0,R))$ compact, such that $(\rho_\varepsilon) \subset K_\eta + B(0,\eta)$. So this family is precompact and consequently it is compact since $L^1(B(0,R))$ is a Banach space.

□

2 Mixing properties of the operator $v.\nabla_x + F.\nabla_v$

2.1 Free transport case

In the case when $F = 0$, Bardos and Degond in [7] proved a mixing result (also referred to as a dispersion result for large time asymptotics) which is a key argument in the proof of Theorem 0.1 (with $F = 0$) by Golse and Saint-Raymond [73]. This kind of estimate was introduced for the study of classical solutions of the Vlasov-Poisson equation in three dimensions and for small initial data.

Lemma 2.1. *Let f be the solution to:*

$$\begin{cases} \partial_t f + v.\nabla_x f = 0, \\ f(0,.,.) = f^0. \end{cases} \tag{2.1}$$

Then for all $t > 0$:

$$\|f(t)\|_{L^\infty_x(L^1_v)} \leq \frac{1}{|t|^d}\|f^0\|_{L^1_x(L^\infty_v)}. \tag{2.2}$$

For further results and related questions (Strichartz estimates...), we refer to Castella and Perthame [35] and Salort [142], [143], [144].

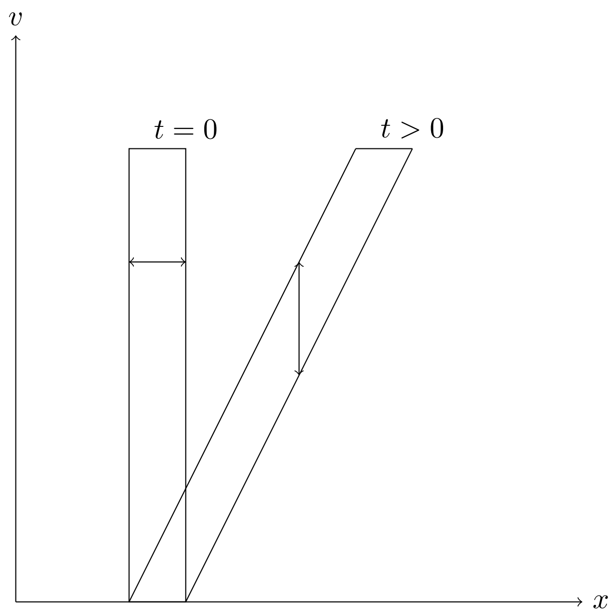


Figure 8.1: Mixing property for free transport

When f^0 is the indicator function of a set with "small" measure with respect to x, then the previous estimate (2.2) asserts that for $t > 0$, $f(t)$ is for any fixed x the indicator function of a set with a "small" measure in v (at least that we may estimate): this property is crucial for the following. In (2.2), there is blow-up when $t \to 0$, which is intuitive, but this does not matter since we have nevertheless a control of the left-hand side for any positive time.

Actually, for our purpose, parameter t is an artificial time (it does not have the usual physical meaning). It appears as an interpolation parameter in Lemma 3.1, and can be taken rather small. This is the reason why local in time mixing is sufficient. We will consequently look for local in time mixing properties. Anyway, the explicit study in Example 2 below shows that the dispersion inequality is in general false for large times when $F \neq 0$.

Proof of Lemma 2.1. The proof of this result is based on the explicit solution to (2.1), which is:

$$f(t,x,v) = f^0(x - tv, v)$$

We now evaluate:

$$\begin{aligned}
\|f(t)\|_{L^\infty_x(L^1_v)} &= \sup_x \int f^0(x - tv, v)dv \\
&= \sup_x \int f^0(z, \frac{x-z}{t})|t|^{-d}dz \\
&\leq |t|^{-d} \int \|f^0(z,.)\|_\infty dz \\
&\leq |t|^{-d}\|f^0\|_{L^1_x(L^\infty_v)}.
\end{aligned}$$

The key argument is the change of variables $x - tv \mapsto z$, the jacobian of which is equal to t^{-d}. □

We intend to do the same in the more complicated case when f is the solution of a transport equation with $F \neq 0$. Let us mention that in [7], Bardos and Degond actually prove the dispersion result for non zero force fields but with a polynomial decay in time. Here, this is not the case (the field F does not even depend on time t), but we will prove that the result holds anyway for small times.

2.2 Study of two examples

In the following examples, f is the explicit solution to the transport equation (1.4) with an initial condition f^0 and a force deriving from a potential.

Example 1
Force $F = -\nabla_x V$, with $V = -|x|^2/2$.
Let f be the solution to:

$$\left\{ \begin{array}{l} \partial_t f + v.\nabla_x f + x.\nabla_v f = 0, \\ f(0,.,.) = f^0. \end{array} \right. \tag{2.3}$$

The effect of such a potential will be to make the particles escape faster to infinity. So we expect to have results very similar to those of lemma 2.1.

After straightforward computations we get :

$$f(t,x,v) = f^0\left(x\left(\frac{e^t+e^{-t}}{2}\right) + v\left(\frac{e^{-t}-e^t}{2}\right), x\left(\frac{e^{-t}-e^t}{2}\right) + v\left(\frac{e^t+e^{-t}}{2}\right)\right),$$

which allows to show the same dispersion estimate with a factor $\frac{e^t-e^{-t}}{2}$ instead of t.
For all $t > 0$, we have:

$$\|f(t)\|_{L^\infty_x(L^1_v)} \leq \frac{2^d}{(e^t - e^{-t})^d}\|f^0\|_{L^1_x(L^\infty_v)}. \tag{2.4}$$

Example 2
(Harmonic potential) Force $F = -\nabla_x V$, with $V = |x|^2/2$.

Let f be the solution to:

$$\begin{cases} \partial_t f + v.\nabla_x f - x.\nabla_v f = 0, \\ f(0,.,.) = f^0. \end{cases} \tag{2.5}$$

With such a potential, particles are expected to be confined and consequently do not drift to infinity. For this reason, it is hopeless to prove the analogue of Lemma 2.1 for large times (here there is no dispersion). As mentioned before, it does not matter since we only look for a result valid for small times. We expect that there is enough mixing in the phase space to prove the result.

After straightforward computations we explicitly have :

$$f(t,x,v) = f^0(x\cos t - v\sin t, v\cos t + x\sin t).$$

We observe here that the solution f is periodic with respect to time. Thus, as expected, it is not possible to prove any decay when $t \to +\infty$; nevertheless we can prove a mixing estimate with a factor $|\sin t|$ instead of t.

For all $t > 0$:

$$\|f(t)\|_{L^\infty_x(L^1_v)} \leq \frac{1}{|\sin t|^d}\|f^0\|_{L^1_x(L^\infty_v)}. \tag{2.6}$$

Of course, this estimate is useless when $t = k\pi, k \in \mathbb{N}^*$.

Remark 2.1. We notice that $\frac{e^t - e^{-t}}{2} \sim_0 t$ and $\sin(t) \sim_0 t$, which seems encouraging.

2.3 General case : F with Lipschitz regularity

The study of these two examples suggests that at least for small times, the mixing estimate is still satisfied, maybe with a corrector term which does not really matter.

One nice heuristic way to understand this is to see that since F is quite smooth, the dynamics associated to the operator $v.\nabla_x + F.\nabla_v$ is expected to be close to those of free transport, at least for small times.

Let $X(t;x,v)$ and $V(t;x,v)$ be the diffeomorphisms introduced in the method of characteristics in Section 1 and defined in (1.3).

Using Taylor's formula, we get by definition of X and V :

$$X(-t;x,v) = x - tv + \int_0^t (t-s)F(X(-s;x,v))ds.$$

We recall Rademacher's theorem which asserts that $W^{1,\infty}$ functions are almost everywhere derivable. Hence, using Lebesgue domination theorem, we get:

$$\partial_v[X(-t;x,v)] = -tId + \int_0^t (t-s)\nabla_x F(X(-s;x,v))\partial_v[X(-s;x,v)]ds. \tag{2.7}$$

We deduce the estimate :

$$\|\partial_v[X(-t;x,v)]\|_\infty \leq t + \int_0^t (t-s)\|\nabla_x F\|_\infty \|\partial_v[X(-s;x,v))]\|_\infty ds.$$

Gronwall's lemma implies then that:

$$\|\partial_v[X(-t;x,v)]\|_\infty \leq t e^{\frac{t^2}{2}\|\nabla_x F\|_\infty}. \tag{2.8}$$

We can also take the determinant of identity (2.7) :

$$\det(\partial_v[X(-t;x,v)]) = \\ (-t)^d \det\left(Id - \frac{1}{t}\int_0^t (t-s)\nabla_x F(X(-s;x,v))\partial_v[X(-s;x,v)]ds\right). \tag{2.9}$$

The right-hand side is the determinant of a matrix of the form $Id + A(t)$ where A is a matrix whose L^∞ norm is small for small times t (one can use estimate (2.8) to ensure that $\|A(t)\|_\infty = o(t)$). Consequently, in a neighborhood of 0, for any fixed x, $\partial_v[X(-t;x,v)]$ is invertible. Furthermore the map $v \mapsto X(-t;x,v))$ is injective for small positive times. Indeed, let $v \neq v'$. We compare:

$$X(-t;x,v') - X(-t;x,v) = t(v-v') + \int_0^t (t-s)[F(X(-s;x,v')) - F(X(-s;x,v))]ds.$$

Consequently we have:

$$|X(-t;x,v') - X(-t;x,v)| \leq t|v-v'| + \int_0^t (t-s)\|\nabla_x F\|_{L^\infty}|X(-s;x,v') - X(-s;x,v)|ds.$$

Thus, by Gronwall inequality we obtain:

$$|X(-t;x,v') - X(-t;x,v)| \leq t|v-v'|e^{\frac{t^2}{2}\|\nabla_x F\|_{L^\infty}}.$$

Finally we observe that:

$$\begin{aligned} |X(-t;x,v') - X(-t;x,v)| \geq & t|v-v'| - \left|\int_0^t (t-s)[F(X(-s;x,v')) - F(X(-s;x,v))]ds\right| \\ \geq & t|v-v'| - \int_0^t (t-s)\|\nabla_x F\|_{L^\infty}|X(-s;x,v') - X(-s;x,v)|ds \\ \geq & |v-v'|\left(t - \int_0^t (t-s)se^{\frac{s^2}{2}\|\nabla_x F\|_{L^\infty}}\|\nabla_x F\|_{L^\infty}ds\right). \end{aligned}$$

Consequently, there is a maximal time $\tau_0 > 0$, depending only on $\|\nabla_x F\|_{L^\infty}$ such that for any $|t| \leq \tau_0$, we have :

$$|X(-t;x,v') - X(-t;x,v)| \geq \frac{t}{2}|v-v'|.$$

This proves our claim.

Thus, by the local inversion theorem, this map is a $\mathcal{C}^1$ diffeomorphism on its image.

We have now the following elementary quantitative estimate :

Lemma 2.2. *Let $t \mapsto A(t)$ be a continuous map defined on a neighborhood of 0, such that $\|A(t)\|_\infty = o(t)$. Then for small times:*

$$\det(Id + A(t)) \geq 1 - d!\|A(t)\|_\infty.$$

We recall that d is the space dimension and $d! = 1 \times 2 \times ... \times d$.

We apply this lemma to (2.9), which allows us to say that there exists a maximal time $\tau > 0$ such that for any $|t| \leq \tau$, we have :

$$|\det(\partial_v[X(-t;x,v)]|^{-1} \leq 2|t|^{-d}. \tag{2.10}$$

We have proved that $v \mapsto X(-t;x,v)$ is a $\mathcal{C}^1$ diffeomorphism such that the jacobian of its inverse satisfies (2.10) in a neighborhood of $t = 0$. We can consequently conclude as in the proof of Lemma 2.1 (by performing the change of variables $X(-t;x,v) \mapsto v$).

As a result we have proved the proposition :

Proposition 2.1. *Let $F(x)$ be a Lipschitz vector field. There exists a maximal time $\tau > 0$ (depending only on $\|\nabla_x F\|_{L^\infty}$) such that, if f is the solution to the transport equation:*

$$\begin{cases} \partial_t f + v.\nabla_x f + F.\nabla_v f = 0, \\ f(0,.,.) = f^0 \in L^p(dx \otimes dv). \end{cases} \tag{2.11}$$

Then:

$$\forall |t| \leq \tau, \|f(t)\|_{L^\infty_x(L^1_v)} \leq \frac{2}{|t|^d}\|f^0\|_{L^1_x(L^\infty_v)}. \tag{2.12}$$

Remark 2.2. If one writes down more explicit estimates, it can be easily shown that τ is bounded from below by T defined as the only positive solution to the equation:

$$\frac{d!}{3}\|\nabla_x F\|_\infty T^2 e^{\|\nabla_x F\|_\infty \frac{T^2}{2}} = 1. \tag{2.13}$$

Remark 2.3. Of course, one can replace the factor 2 in the mixing estimate by any $q > 1$ (and the maximal time τ will depend also on q).

3 From local equiintegrability in velocity to local equiintegrability in position and velocity

In this section, we finally proceed as in [73], with some slight modifications adapted to our case. We start from the following Green's formula :

Lemma 3.1. *Let $f \in L^1(dx \otimes dv)$ with compact support such that $v.\nabla_x f + F.\nabla_v f \in L^1(dx \otimes dv)$. Then for all $\Phi^0 \in L^\infty(dx \otimes dv)$, we have for all $t \in \mathbb{R}_+^*$:*

$$\begin{aligned} \int f(x,v)\Phi^0(x,v)dxdv = &\int f(x,v)\Phi(t,x,v)dxdv \\ &- \int_0^t \int \Phi(s,x,v)(v.\nabla_x f + F.\nabla_v f)dsdxdv, \end{aligned} \tag{3.1}$$

where Φ is the solution to:

$$\begin{cases} \partial_t \Phi + v.\nabla_x \Phi + F.\nabla_v \Phi = 0 \\ \Phi_{|t=0} = \Phi_0. \end{cases} \tag{3.2}$$

Proof. We have for all $t > 0$, $\int_\Omega f(x,v)(\partial_t + v.\nabla_x + F.\nabla_v)\Phi(s,x,v)dsdxdv = 0$, where $\Omega =]0,t[\times \mathbb{R}^d \times \mathbb{R}^d$. We first have:

$$\int_\Omega f(x,v)\partial_t \Phi(s,x,v)dsdxdv = \int\int f(x,v)\Phi(t,x,v)dxdv - \int\int f(x,v)\Phi^0(x,v)dxdv.$$

Finally, by Green's formula we obtain:

$$\int_0^t \int f(x,v)(v.\nabla_x + F.\nabla_v)\Phi(s,x,v)dsdxdv$$
$$= -\int_0^t \int \Phi(s,x,v)(v.\nabla_x + F.\nabla_v)f(x,v)dsdxdv.$$

There is no contribution from the boundaries since f is compactly supported. $\square$

Lemma 3.2. *Let (f_ε) a bounded family of $L^1_{loc}(dx \otimes dv)$ locally integrable in v such that $(v.\nabla_x f_\varepsilon + F.\nabla_v f_\varepsilon)$ is a bounded family of $L^1_{loc}(dx \otimes dv)$. Then for all $\Psi \in \mathcal{C}^1_c(\mathbb{R}^d)$, such that $\Psi \geq 0$, the family $\rho_\varepsilon(x) = \int |f_\varepsilon(x,v)|\Psi(v)dv$ is locally equiintegrable.*

Proof. Let K_1 be a compact subset of $\mathbb{R}^d$. We want to prove that $(\mathbb{1}_{K_1}\rho_\varepsilon(x))$ is equiintegrable. Without loss of generality, we can assume as previously that the f_ε are supported in the same compact support $K = K_1 \times K_2$ and that Ψ is compactly supported in K_2. Furthermore, the formula $\nabla|f_\varepsilon| = \mathrm{sign}(f_\varepsilon)\nabla f_\varepsilon$ shows that the $|f_\varepsilon|$ satisfy the same assumptions of equiintegrability and L^1 boundedness as the family (f_ε). For the sake of readability, we will thus assume that f_ε are almost everywhere non-negative instead of considering $|f_\varepsilon|$. Finally we may assume that $\|\Psi\|_\infty = 1$ (multiplying by a constant does not change the equiintegrability property).

The idea of the proof is to show that thanks to the mixing properties established previously, the equiintegrability in v provides some equiintegrability in x.

Let $\eta > 0$. By definition of the local equiintegrability in v, we obtain a parameter $\alpha > 0$ associated to K and η. We also consider parameters $\alpha' > 0$ and $t \in]0, \tau[$ (where τ is the maximal time in Proposition 2.1) to be fixed ultimately. We mention that t will be chosen only after α' is fixed.

Let A a bounded mesurable subset included in K_1 with $|A| \leq \alpha'$. We consider $\Phi^0(x,v) = \mathbb{1}_A(x)$ and Φ the solution of the transport equation (3.2) with Φ^0 as initial data.

Observe now that we have $\|\Phi^0\|_{L^1_x(L^\infty_v)} = |A|$. Moreover, since Φ^0 takes its values in $\{0,1\}$, it is also the case for Φ (this is a plain consequence of the transport of the data).

We define for all $s > 0$ and for all $x \in \mathbb{R}^d$, the set $A(s)_x = \{v \in \mathbb{R}^d, \Phi(s,x,v) = 1\}$. At this point of the proof, we make a crucial use of the mixing property stated in Proposition 2.1 :

$$\begin{aligned}
\sup_x |A(t)_x| &= \sup_x \int \Phi(t,x,v)dv \\
&= \|\Phi(t,.,.)\|_{L^\infty_x(L^1_v)} \\
&\leq 2|t|^{-d} \underbrace{\|\Phi^0\|_{L^1_x(L^\infty_v)}}_{|A| \leq \alpha'} \\
&\leq \alpha,
\end{aligned}$$

if we choose α' satisfying $\alpha' < \frac{1}{2}t^D\alpha$.

Thanks to Lemma 3.1:

$$\int f(x,v)\Psi(v)\Phi^0(x,v)dxdv = \int f(x,v)\Psi(v)\Phi(t,x,v)dxdv$$
$$- \int_0^t \int \Phi(s,x,v)(v.\nabla_x + F.\nabla_v)(f_\varepsilon(x,v)\Psi(v))dxdvds.$$

In other words, the operator $\partial_t + v.\nabla_x + F.\nabla_v$ has transported the indicator function and has transformed a subset small in x into a subset small in v.

By definition of ρ_ε, we have:

$$\int f(x,v)\Psi(v)\Phi^0(x,v)dxdv = \int \mathbb{1}_A(x)\rho_\varepsilon(x)dx.$$

By definition of $A(t)_x$ we also have:

$$\int f(x,v)\Psi(v)\Phi(t,x,v)dxdv = \int \left(\int_{A(t)_x} f_\varepsilon(x,v)\Psi(v)dv\right)dx.$$

Thus, since (f_ε) are locally equiintegrable in v we may evaluate:

$$\begin{aligned}\int \left(\int_{A(t)_x} f_\varepsilon(x,v)\Psi(v)dv\right)dx \leq& \int\int_{A(t)_x} |f_\varepsilon|\mathbb{1}_K \underbrace{\|\Psi\|_\infty}_{=1} dxdv\\ \leq& \eta.\end{aligned}$$

Finally we have:

$$\begin{aligned}\int \mathbb{1}_A(x)\rho_\varepsilon(x)dx =& \int \left(\int_{A(t)_x} f_\varepsilon(x,v)\Psi(v)dv\right)dx\\ &- \int_0^t \int \Phi(s,x,v)(v.\nabla_x + F.\nabla_v)(f_\varepsilon(x,v)\Psi(v))dsdxdv\\ \leq& \eta + \int_0^t \int |\Phi(s,x,v)||(v.\nabla_x + F.\nabla_v)(f_\varepsilon(x,v)\Psi(v))|dsdxdv\\ \leq& \eta + t\left[\underbrace{\|\Psi\Phi\|_\infty}_{\leq 1}\|v.\nabla_x f_\varepsilon + F.\nabla_v f_\varepsilon\|_1 + \underbrace{\|\Phi\|_\infty}_{=1}\|F.\nabla_v\Psi(v)\|_\infty\|f_\varepsilon\|_1\right]\\ \leq& 2\eta,\end{aligned}$$

by taking t sufficiently small:

$$t < \frac{\eta}{\sup_\varepsilon \|v.\nabla_x f_\varepsilon + F.\nabla_v f_\varepsilon\|_1 + \|F.\nabla_v\Psi(v)\|_\infty\|f_\varepsilon\|_1}.$$

This finally proves that (ρ_ε) is locally equiintegrable in x. □

Lemma 3.3. *Let (g_ε)a bounded family of $L^1_{loc}(dx \otimes dv)$ locally equiintegrable in v. If for all $\Psi \in \mathcal{C}^1_c(\mathbb{R}^d)$ such that $\Psi \geq 0$, $x \mapsto \int |g_\varepsilon(x,v)|\Psi(v)dv$ is locally equiintegrable (in x), then (g_ε) is locally equiintegrable in x and v.*

Proof. Let K be a compact subset of $\mathbb{R}^d \times \mathbb{R}^d$. We want to prove that $(\mathbb{1}_K g_\varepsilon)$ is equiintegrable in x and v. As before, we can clearly assume that the g_ε are compactly supported in K.

Let $\eta > 0$. By definition of the local equiintegrability in v for (g_ε), we obtain $\alpha_1 > 0$ associated to η and K.

Let $\Psi \in \mathcal{C}^1_c(\mathbb{R}^d)$ a smooth non-negative and compactly supported function such that $\Psi \equiv 1$ on $p_v(K)$ (where $p_v(K)$ is the projection of K on $\mathbb{R}^d_v$). By assumption, there exists $\alpha_2 > 0$ such that for any $A \subset \mathbb{R}^d$ measurable set satisfying $|A| \leq \alpha_2$,

$$\int_A \left(\int |g_\varepsilon|\Psi dv\right)dx < \eta.$$

Let B a measurable subset of $\mathbb{R}^d \times \mathbb{R}^d$ such that $|B| < \inf(\alpha_1^2, \alpha_2^2)$. We define for all $x \in \mathbb{R}^d$, $B_x = \{v \in \mathbb{R}^d, (x,v) \in B\}$.

We consider now $E = \{x \in \mathbb{R}^d, |B_x| \leq |B|^{1/2}\}$: this is the subset of x for which there exist few v such that $(x,v) \in B$. Consequently for this subset, we can use the local equiintegrability in v.

Concerning $B \backslash E$, on the contrary, we can not use this property, but thanks to Chebychev's inequality we show that this subset is of small measure, which allows us to use this time the local equiintegrability in x of $\int |g_\varepsilon(x,v)|\Psi(v)dv$:

$$\begin{aligned} |E^c| &= |\{x \in \mathbb{R}^d, |B_x| > |B|^{1/2}\}| \\ &\leq \frac{|B|}{|B|^{1/2}} \\ &\leq \alpha_2. \end{aligned}$$

Hence we have :

$$\begin{aligned} \int \mathbb{1}_B |g_\varepsilon| dx dv &\leq \int_E \left(\int_{B_x} |g_\varepsilon| dv \right) dx + \int_{E^c} \left(\int |g_\varepsilon| dv \right) dx \\ &\leq \eta + \int_{E^c} \left(\int |g_\varepsilon| \Psi(v) dv \right) dx \\ &\leq 2\eta. \end{aligned}$$

This shows the expected result. □

We are now able to conclude the proof of Theorem 0.1.

End of the proof of Theorem 0.1. If we successively apply Lemmas 3.2 and 3.3, we deduce that the family (f_ε) is locally equiintegrable in x and v.

Finally we have shown in Section 1 that the first point implies the second.

□

Références

[1] Agoshkov, V. Spaces of functions with differential-difference characteristics and the smoothness of solutions of the transport equation. *Dokl. Akad. Nauk SSSR 276*, 6 (1984), 1289–1293.

[2] Allaire, G. Homogenization and Two-scale Convergence. *SIAM J. Math. Anal. XXIII*, 6 (1992), 1482–1518.

[3] Arnold, V. I. *Mathematical methods of classical mechanics*, second ed., vol. 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein.

[4] Arsenev, A. Existence in the large of a weak solution of Vlasov's system of equations. *Z. Vychisl. Mat. Mat. Fiz 15* (1975), 136–147.

[5] Asano, K., and Ukai, S. On the Vlasov-Poisson limit of the Vlasov-Maxwell equation. In *Patterns and waves*, vol. 18 of *Stud. Math. Appl.* North-Holland, Amsterdam, 1986, pp. 369–383.

[6] Bardos, C. Problèmes aux limites pour les équations aux dérivées partielles du premier ordre à coefficients réels ; théorèmes d'approximation ; application à l'équation de transport. *Ann. Sci. École Norm. Sup. (4) 3* (1970), 185–233.

[7] Bardos, C., and Degond, P. Global existence for the Vlasov-Poisson equation in 3 space variables with small initial data. *Ann. Inst. H. Poincaré Anal. Non Linéaire 2*, 2 (1985), 101–118.

[8] Bardos, C., Lebeau, G., and Rauch, J. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim. 30*, 5 (1992), 1024–1065.

[9] Batt, J., Morrison, P., and Rein, G. Linear stability of stationary solutions of the Vlasov-Poisson system in three dimensions. *Arch. Rational Mech. Anal. 130*, 2 (1995), 163–182.

[10] Batt, J., and Rein, G. Global classical solutions of the periodic Vlasov-Poisson system in three dimensions. *C. R. Acad. Sci. Paris Sér. I Math. 313*, 6 (1991), 411–416.

[11] Batt, J., and Rein, G. A rigorous stability result for the Vlasov-Poisson system in three dimensions. *Ann. Mat. Pura Appl. (4) 164* (1993), 133–154.

[12] Ben Abdallah, N. Weak solutions of the initial-boundary value problem for the Vlasov-Poisson system. *Math. Methods Appl. Sci. 17*, 6 (1994), 451–476.

[13] Besse, N., Berthelin, F., Brenier, Y., and Bertrand, P. The multi-water-bag equations for collisionless kinetic modeling. *Kinet. Relat. Models 2*, 1 (2009), 39–80.

[14] Bézard, M. Régularité L^p précisée des moyennes dans les équations de transport. *Bull. Soc. math. France 122* (1994), 29–76.

[15] Boltzmann, L. Weitere Studien über das Wärmegleichgewich unter Gasmolekülen. *Sitzungsberichte der Akademie der Wissenschaften, Wien 66* (1872), 275–370.

[16] Bostan, M. The Vlasov-Poisson system with strong external magnetic field. Finite Larmor radius regime. *Asymptot. Anal. 61*, 2 (2009), 91–123.

[17] Bostan, M. Gyrokinetic Vlasov Equation in Three Dimensional Setting. Second Order Approximation. *Multiscale Modeling & Simulation 8*, 5 (2010), 1923–1957.

[18] Bostan, M. Transport equations with disparate advection fields. Application to the gyrokinetic models in plasma physics. *J. Differential Equations 249*, 7 (2010), 1620–1663.

[19] Bouchut, F. Global weak solution of the Vlasov-Poisson system for small electrons mass. *Comm. Partial Differential Equations 16* (1991), 1337–1365.

[20] Bouchut, F., Golse, F., and Pallard, C. Classical solutions and the Glassey-Strauss theorem for the 3D Vlasov-Maxwell system. *Arch. Ration. Mech. Anal. 170*, 1 (2003), 1–15.

[21] Bouchut, F., Golse, F., and Pulvirenti, M. *Kinetic Equations and Asymptotic Theory*. Series in Applied Mathematics. Gauthier-Villars, 2000.

[22] Braasch, P., Rein, G., and Vukadinović, J. Nonlinear stability of stationary plasmas—an extension of the energy-Casimir method. *SIAM J. Appl. Math. 59*, 3 (1999), 831–844 (electronic).

[23] Brenier, Y. A Vlasov-Poisson type formulation of the Euler equations for perfect incompressible fluids. *Rapport de recherche INRIA* (1989).

[24] Brenier, Y. Some conservation laws given by kinetic models. *Journées EDP* (1995), 1–13.

[25] Brenier, Y. A homogenized Model for Vortex Sheets. *Arch. for Rational Mech. and Anal. 138* (1997), 319–353.

[26] Brenier, Y. Homogeneous hydrostatic flows with convex velocity profiles. *Nonlinearity 12*, 3 (1999), 495–512.

[27] Brenier, Y. Convergence of the Vlasov-Poisson system to the incompressible Euler equations. *Comm. Partial Differential Equations 25* (2000), 737–754.

[28] Brenier, Y., and Grenier, E. Limite singuliere du systeme de Vlasov-Poisson dans le regime de quasi neutralite : le cas independant du temps. *C. R. Acad. Sci. Paris Ser. I Math. 318*, 2 (1994), 121–124.

[29] Brenier, Y., and Loeper, G. A geometric approximation to the Euler equations : the Vlasov-Monge-Ampère system. *Geom. Funct. Anal. 14*, 6 (2004), 1182–1218.

[30] Brenier, Y., Mauser, N., and Puel, M. Incompressible Euler and e-MHD as scaling limits of the Vlasov-Maxwell system. *Commun. Math. Sci. 1*, 3 (2003), 437–447.

[31] Brezis, H., Golse, F., and Sentis, R. Analyse asymptotique de l'équation de Poisson couplée à la relation de Boltzmann. Quasi-neutralité des plasmas. *C. R. Acad. Sci. Paris Sér. I Math. 321*, 7 (1995), 953–959.

[32] Brizard, A., and Hahm, T. Foundations of nonlinear gyrokinetic theory. *Reviews Of Modern Physics*, 79 (2007).

[33] Cáceres, M. J., Carrillo, J. A., and Dolbeault, J. Nonlinear stability in L^p for a confined system of charged particles. *SIAM J. Math. Anal. 34*, 2 (2002), 478–494 (electronic).

[34] Caflisch, R. A simplified version of the abstract Cauchy-Kowalewski theorem with weak singularities. *Bulletin (New Series) of the American Mathematical Society 23*, 2 (1990), 495–500.

[35] Castella, F., and Perthame, B. Estimations de Strichartz pour les équations de transport cinétique. *C. R. Acad. Sci. Paris Sér. I Math. 322*, 6 (1996), 535–540.

[36] Chemin, J.-Y., Desjardins, B., Gallagher, I., and Grenier, E. *Mathematical geophysics*, vol. 32 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, Oxford, 2006. An introduction to rotating fluids and the Navier-Stokes equations.

[37] Cheverry, C., Gallagher, I., Paul, T., and Saint-Raymond, L. Trapping Rossby waves. *C. R. Math. Acad. Sci. Paris 347*, 15-16 (2009), 879–884.

[38] Cordier, S., Grenier, E., and Guo, Y. Two-stream instabilities in plasmas. *Methods Appl. Anal. 7*, 2 (2000), 391–405.

[39] Cordier, S., and Grenier, E. Quasineutral limit of an Euler-Poisson system arising from plasma physics. *Comm. Partial Differential Equations 25*, 5 (2000), 1099–1113.

[40] Coron, J.-M. Global asymptotic stabilization for controllable systems without drift. *Math. Control Signals Systems 5*, 3 (1992), 295–312.

[41] Coron, J.-M. On the controllability of 2-D incompressible perfect fluids. *J. Math. Pures Appl. (9) 75*, 2 (1996), 155–188.

[42] Coron, J.-M. *Control and nonlinearity*, vol. 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[43] Dafermos, C. M. The second law of thermodynamics and stability. *Arch. Rational Mech. Anal. 70*, 2 (1979), 167–179.

[44] Degond, P. Global existence of smooth solutions for the Vlasov-Fokker-Planck equations in 1 and 2 space dimensions. *Ann. Sci. Ecole Norm. Sup. 19* (1986), 519–542.

[45] Degond, P. Local existence of solutions of the Vlasov-Maxwell equations and convergence to the Vlasov-Poisson equations for infinite light velocity. *Math. Methods Appl. Sci. 8*, 4 (1986), 533–558.

[46] Delort, J.M. Existence de nappes de tourbillon en dimension deux. *J. Amer. Math. Soc. 4*, 3 (1991), 553–586.

[47] Desjardins, B., and Grenier, E. Linear instability implies nonlinear instability for various types of viscous boundary layers. *Ann. Inst. H. Poincaré Anal. Non Linéaire 20*, 1 (2003), 87–106.

[48] Dewar, R., and Abdullatif, R. Zonal flow generation by modulational instability. *Proceedings of the CSIRO/COSNet Workshop on Turbulence and Coherent Structures, Canberra, Australia World Scientific, eds. J.P. Denier and J.S. Frederiksen* (2006).

[49] Diperna, R., and Lions, P. Solutions globales d'equations du type Vlasov-Poisson. *C. R. Acad. Sci. Paris Ser. I Math. 307*, 12 (1988), 655–658.

[50] DiPerna, R., Lions, P., and Meyer, Y. L^p regularity of velocity averages. *Annales de l'I.H.P, section C 8*, 3-4 (1991), 271–287.

[51] DiPerna, R. J., and Lions, P.-L. Global weak solutions of Vlasov-Maxwell systems. *Comm. Pure Appl. Math. 42*, 6 (1989), 729–757.

[52] DiPerna, R. J., and Lions, P.-L. On the Cauchy problem for Boltzmann equations : global existence and weak stability. *Ann. of Math. (2) 130*, 2 (1989), 321–366.

[53] Figarella et al., C. Transport reduction by rotation shear in Tokamak-Edge Turbulence. *Phys. Rev. Let. 90*, 1 (2003).

[54] Fitzpatrick, R. The Physics of Plasmas. *available at http ://farside.ph.utexas.edu/teaching/plasma/380.pdf*.

[55] Frénod, E., and Mouton, A. Two-dimensional Finite Larmor Radius approximation in canonical gyrokinetic coordinates. *Journal of Pure and Applied Mathematics : Advances and Applications 4*, 2 (2010), 135–166.

[56] Frénod, E., Mouton, A., and Sonnendrücker, E. Two-scale numerical simulation of the weakly compressible 1D isentropic Euler equations. *Numer. Math. 108*, 2 (2007), 263–293.

[57] Frénod, E., Raviart, P., and Sonnendrücker, E. Asymptotic expansion of the Vlasov equation in a large external magnetic field. *J. Math. Pures et Appl. 80*, 8 (815-843), 2001.

[58] Frénod, E., Salvarani, F., and Sonnendrücker, E. Long time simulation of a beam in a periodic focusing channel via a two-scale PIC-method. *Math. Models Methods Appl. Sci. 19*, 2 (2009), 175–197.

[59] Frénod, E., and Sonnendrücker, E. Homogenization of the Vlasov equation and of the Vlasov-Poisson system with a strong external magnetic field. *Asymp. Anal. 18*, 3-4 (1998), 193–214.

[60] Frénod, E., and Sonnendrücker, E. The Finite Larmor Radius Approximation. *SIAM J. Math. Anal. 32*, 6 (2001), 1227–1247.

[61] Gallaire, F., Gérard-Varet, D., and Rousset, F. Three-dimensional instability of planar flows. *Arch. Ration. Mech. Anal. 186*, 3 (2007), 423–475.

[62] Gasser, I., Jabin, P.-E., and Perthame, B. Regularity and propagation of moments in some nonlinear Vlasov systems. *Proc. Roy. Soc. Edinburgh Sect. A 130*, 6 (2000), 1259–1273.

[63] Gérard-Varet, D. Oscillating solutions of incompressible magnetohydrodynamics and dynamo effect. *SIAM J. Math. Anal. 37*, 3 (2005), 815–840 (electronic).

[64] Ghendrih, P., Hauray, M., and Nouri, A. Derivation of a gyrokinetic model. Existence and uniqueness of specific stationary solutions. *Kinet. and Relat. Models 2*, 4 (2009), 707–725.

[65] Glass, O. On the controllability of the Vlasov-Poisson system. *J. Differential Equations 195*, 2 (2003), 332–379.

[66] Glassey, R., and Schaeffer, J. The "two and one-half-dimensional" relativistic Vlasov Maxwell system. *Comm. Math. Phys. 185*, 2 (1997), 257–284.

[67] Glassey, R. T., and Schaeffer, J. The relativistic Vlasov-Maxwell system in two space dimensions. I, II. *Arch. Rational Mech. Anal. 141*, 4 (1998), 331–354, 355–374.

[68] Glassey, R. T., and Schaeffer, J. W. Global existence for the relativistic Vlasov-Maxwell system with nearly neutral initial data. *Comm. Math. Phys. 119*, 3 (1988), 353–384.

[69] Glassey, R. T., and Strauss, W. A. Singularity formation in a collisionless plasma could occur only at high velocities. *Arch. Rational Mech. Anal. 92*, 1 (1986), 59–90.

[70] Golse, F., Lions, P., Perthame, B., and Sentis, R. Regularity of the moments of the solution of a transport equation. *J. Funct. Anal. 26* (1988), 110–125.

[71] Golse, F., Perthame, B., and Sentis, R. Un résultat de compacité pour les equations de transport et application au calcul de la limite de la valeur propre principale d'un opérateur de transport. *C.R. Acad. Sci. Paris Sér. I Math. 301*, 7 (1985), 341–344.

[72] Golse, F., and Saint-Raymond, L. The Vlasov-Poisson system with strong magnetic field. *J. Math. Pures. Appl. 78* (1999), 791–817.

[73] Golse, F., and Saint-Raymond, L. Velocity averaging in L^1 for the transport equation. *C. R. Math. Acad. Sci. Paris 334*, 7 (2002), 557–562.

[74] Golse, F., and Saint-Raymond, L. The Vlasov-Poisson system with strong magnetic field in quasineutral regime. *Mathematical Models and Methods in Applied Sciences 13*, 5 (2003), 661–714.

[75] Golse, F., and Saint-Raymond, L. The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels. *Invent. Math. 155*, 1 (2004), 81–161.

[76] Goudon, T., Jabin, P., and Vasseur, A. Hydrodynamic limit for the Vlasov-Navier-Stokes equations. II. Fine particles regime. *Indiana Univ. Math. J. 53*, 6 (2004), 1517–1536.

[77] Grad, H. The guiding center plasma. *Proc. Sympos. Appl. Math. Amer. Math. Soc. 18* (1967), 162–248.

[78] Grandgirard et al, V. Global full-f gyrokinetic simulations of plasma turbulence. *Plasma Phys. Control. Fusion 49* (2007), 173–182.

[79] Greenwald et al., M. H Mode Confinement in Alcator C-Mod. *Nuclear Fusion 37*, 6 (1997), 793–807.

[80] Grenier, E. Defect measures of the Vlasov-Poisson system in the quasineutral regime. *Comm. Partial Differential Equations 20*, 7-8 (1995), 1189–1215.

[81] Grenier, E. Oscillations in quasineutral plasmas. *Comm. Partial Differential Equations 21*, 3-4 (1996), 363–394.

[82] Grenier, E. Oscillatory perturbations of the Navier-Stokes equations. *J. Math. Pures Appl. 76*, 9 (1997), 477–498.

[83] Grenier, E. Pseudo-Differential Energy Estimates of Singular Perturbations. *Comm. Pure App. Math. 50* (1997), 0821–0865.

[84] Grenier, E. Limite quasineutre en dimension 1. In *Journees "Equations aux Derivées Partielles" (Saint-Jean-de-Monts, 1999).* Univ. Nantes, Nantes, 1999, pp. Exp. No. II, 8.

[85] Grenier, E. Nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math. 53* (2000), 1067–1091.

[86] Grillakis, M., Shatah, J., and Strauss, W. Stability theory of solitary waves in the presence of symmetry. II. *J. Funct. Anal. 94*, 2 (1990), 308–348.

[87] Guo, Y. Global weak solutions of the Vlasov-Maxwell system with boundary conditions. *Comm. Math. Phys. 154*, 2 (1993), 245–263.

[88] Guo, Y. Regularity for the Vlasov equations in a half-space. *Indiana Univ. Math. J. 43*, 1 (1994), 255–320.

[89] Guo, Y., and Strauss, W. Nonlinear instability of double-humped equilibria. *Annales de l'I.H.P., section C 12*, 3 (1995), 339–352.

[90] Guo, Y., and Strauss, W. A. Instability of periodic BGK equilibria. *Comm. Pure Appl. Math. 48*, 8 (1995), 861–894.

[91] Guo, Y., and Strauss, W. A. Magnetically created instability in a collisionless plasma. *J. Math. Pures Appl. (9) 79*, 10 (2000), 975–1009.

[92] Han-Kwan, D. On the confinement of a tokamak plasma. *SIAM J. Math. Anal 42*, 6 (2010), 2337–2367.

[93] Han-Kwan, D. On the three-dimensional finite Larmor radius approximation : the case of electrons in a fixed background of ions. *Submitted* (2010).

[94] Han-Kwan, D. The three-dimensional finite Larmor radius approximation. *Asymptot. Anal. 66*, 1 (2010), 9–33.

[95] Hasegawa, A., and Mima, K. Pseudo-three-dimensional turbulence in magnetized nonuniform plasma. *Phys. Fluids 21*, 1 (1978), 87–92.

[96] Hasegawa, A., and Mima, K. Pseudo-three-dimensional turbulence in magnetized nonuniform plasma. *Phys. Fluids 21*, 1 (1978), 87–92.

[97] Hauray, M., and Jabin, P.E. N-particles approximation of the Vlasov equations with singular potential. *Arch. Ration. Mech. Anal. 183*, 3 (2007), 489–524.

[98] Horst, E. On the asymptotic growth of the solutions of the Vlasov-Poisson system. *Math. Methods Appl. Sci. 16*, 2 (1993), 75–86.

[99] Horst, E., and Hunze, E. Weak solutions of the initial value problem for the unmodified nonlinear Vlasov equation. *Math. Methods Appl. Sci. 6* (1982), 262–279.

[100] Hwang, H. Regularity for the Vlasov-Poisson system in a convex domain. *SIAM J. Math. Anal. 36*, 1 (2004), 121–171 (electronic).

[101] Hwang, H.J., and Guo, Y. On the dynamical Rayleigh-Taylor instability. *Arch. Ration. Mech. Anal. 167*, 3 (2003), 235–253.

[102] Hwang, H.J., and Velázquez, J. J. L. On global existence for the Vlasov-Poisson system in a half space. *J. Differential Equations 247*, 6 (2009), 1915–1948.

[103] Hwang, H.J., and Velázquez, J. J. L. Global existence for the Vlasov-Poisson system in bounded domains. *Arch. Ration. Mech. Anal. 195*, 3 (2010), 763–796.

[104] Ido et al., T. Observation of the Fast Potential Change at L-H Transition by a Heavy-Ion-Beam Probe on JFT-2M. *Phys. Rev. Let. 88*, 5 (2002).

[105] Itoh, S., and Itoh, K. Change of transport at L- and H-Mode Transition. *Journal of The Physical Society of Japan 59*, 11 (1990), 3815–3818.

[106] Jabin, P.E. The Vlasov-Poisson system with infinite mass and energy. *J. Statist. Phys. 103*, 5-6 (2001), 1107–1123.

[107] Jabin, P.E. Averaging lemmas and dispersion estimates for kinetic equations. *Riv. Mat. Univ. Parma (8) 1* (2009), 71–138.

[108] Jabin, P.-E., and Vega, L. A real space method for averaging lemmas. *J. Math. Pures Appl. (9) 83*, 11 (2004), 1309–1351.

[109] Kato, T. On classical solutions of the two-dimensional nonstationary Euler equation. *Arch. Rational Mech. Anal. 25* (1967), 188–200.

[110] Klainerman, S., and Staffilani, G. A new approach to study the Vlasov-Maxwell system. *Commun. Pure Appl. Anal. 1*, 1 (2002), 103–125.

[111] Lieberman, M., and Lichtenberg, A. Principles of plasma discharge and materials processing. *New-York, Wiley* (1994).

[112] Lin, Z. Nonlinear instability of ideal plane flows. *Int. Math. Res. Not.*, 41 (2004), 2147–2178.

[113] Lin, Z. Some recent results on instability of ideal plane flows. In *Nonlinear partial differential equations and related analysis*, vol. 371 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2005, pp. 217–229.

[114] Lin, Z., Ethier, S., Hahm, T. S., and Tang, W. M. Size Scaling of Turbulent Transport in Magnetically Confined Plasmas. *Physical Review Letters 88*, 19 (May 2002), 195004–1–195004–4.

[115] Lin, Z., and Strauss, W. A. A sharp stability criterion for the Vlasov-Maxwell system. *Invent. Math. 173*, 3 (2008), 497–546.

[116] Lions, P.L. Régularité optimale des moyennes en vitesses. *C. R. Acad. Sci. Paris Sér. I Math. 320*, 8 (1995), 911–915.

[117] Lions, P. *Mathematical topics in fluid mechanics. Vol. 1*, vol. 3 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1996. Incompressible models, Oxford Science Publications.

[118] Lions, P.L. Régularité optimale des moyennes en vitesses. II. *C. R. Acad. Sci. Paris Sér. I Math. 326*, 8 (1998), 945–948.

[119] Lions, P.L., and Perthame, B. Propagation of moments and regularity for the three-dimensional Vlasov-Poisson system. *Invent. Math. 105* (1991), 415–430.

[120] Littlejohn, R. A guiding center Hamiltonian : A new approach. *J. Math. Phys. 20*, 12 (1979), 2445–2458.

[121] Littlejohn, R. Hamiltonian formulation of guiding center motion. *Phys. Fluids 24*, 9 (1981), 1730–1749.

[122] Loeper, G. Uniqueness of the solution to the Vlasov-Poisson system with bounded density. *J. Math. Pures Appl. 86* (2006), 68–79.

[123] Majda, A. Compressible fluid flow and systems of conservation laws in several space variables. *Springer-Verlag* (1984).

[124] Majda, A. J., and Bertozzi, A. L. *Vorticity and incompressible flow*, vol. 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[125] Masmoudi, N. From Vlasov-Poisson system to the incompressible Euler system. *Comm. Partial Differential Equations 26*, 9 (2001), 1913–1928.

[126] Maxwell, J. On the dynamical theory of gases. *Phil. Trans. Roy. Soc. 175* (1866).

[127] Maxwell, J. *Electricity and Magnetism*. 1873.

[128] Mischler, S. On the initial boundary value problem for the Vlasov-Poisson-Boltzmann system. *Comm. Math. Phys. 210*, 2 (2000), 447–466.

[129] Mischler, S. Kinetic equations with Maxwell boundary condition. *Annales scientifiques de l'ENS 43*, 5 (2010), 719–760.

[130] Mouhot, C., and Villani, C. On Landau damping. *To appear in Acta Mathematica* (2011).

[131] Mouton, A. Two-scale semi-Lagrangian simulation of a charged particle beam in a periodic focusing channel. *Kinet. Relat. Models 2*, 2 (2009), 251–274.

[132] Nguetseng, G. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal. 20*, 3 (1989), 608–623.

[133] Penrose, O. Electrostatic instability of a uniform non-Maxwellian plasma. *Phys. Fluids 3* (1960), 258–265.

[134] Perthame, B., and Souganidis, P. E. A limiting case for velocity averaging. *Ann. Sci. École Norm. Sup. (4) 31*, 4 (1998), 591–598.

[135] Pfaffelmoser, K. Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. *J. Diff. Equations. 95* (1992), 281–303.

[136] Puel, M., and Saint-Raymond, L. Quasineutral limit for the relativistic Vlasov-Maxwell system. *Asymptot. Anal. 40*, 3-4 (2004), 303–352.

[137] Rein, G. Non-linear stability for the Vlasov-Poisson system - the energy-Casimir method. *Math. Meth. Appl. Sci. 17*, 14 (1994), 1129–1140.

[138] Rousset, F., and Tzvetkov, N. Transverse nonlinear instability for two-dimensional dispersive models. *Ann. Inst. H. Poincaré Anal. Non Linéaire 26*, 2 (2009), 477–496.

[139] Saint-Raymond, L. The gyrokinetic approximation for the Vlasov-Poisson system. *Math. Mod. Meth. Appl. Sci. 10*, 9 (2000), 1305–1332.

[140] Saint-Raymond, L. Control of large velocities in the two-dimensional gyrokinetic approximation. *J. Math. Pures Appl. 81* (2002), 379–399.

[141] Saint-Raymond, L. Hydrodynamic limits : some improvements of the relative entropy method. *Ann. Inst. H. Poincaré Anal. Non Linéaire 26*, 3 (2009), 705–744.

[142] Salort, D. Weighted dispersion and Strichartz estimates for the Liouville equation in 1D. *Asymptot. Anal. 47*, 1-2 (2006), 85–94.

[143] Salort, D. Dispersion and Strichartz estimates for the Liouville equation. *J. Differential Equations 233*, 2 (2007), 543–584.

[144] Salort, D. Transport equations with unbounded force fields and application to the Vlasov-Poisson equation. *Math. Models Methods Appl. Sci. 19*, 2 (2009), 199–228.

[145] Schaeffer, J. The classical limit of the relativistic Vlasov-Maxwell system. *Comm. Math. Phys. 104*, 3 (1986), 403–421.

[146] Schaeffer, J. Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. *Comm. Partial Differential Equations 16*, 8-9 (1991), 1313–1335.

[147] Schochet, S. Fast singular limits of hyperbolic PDEs. *J. Differential Equations 114* (1994), 476–512.

[148] Simon, J. Compact sets in $L^p(0,T;B)$. *Ann. Mat. Pura. Appl. 146* (1987), 65–96.

[149] Spohn, H. Large scale dynamics of interacting particles. *Springer-Verlag Berlin* (1991).

[150] Sulem, P.L. Introduction to the guiding center theory. *Topics in kinetic theory, Fields Inst. Commun., Amer. Math. Soc. 46* (2005), 109–149.

[151] Ukai, S., and Okabe, T. On classical solutions in the large in time of two-dimensional Vlasov's equation. *Osaka J. Math. 15*, 2 (1978), 245–261.

[152] Vidav, I. Spectra of perturbed semigroups with applications to transport theory. *J. Math. Anal. Appl. 30* (1970), 264–279.

[153] Villani, C. A review of mathematical topics in collisional kinetic theory. In *Handbook of mathematical fluid dynamics, Vol. I.* North-Holland, Amsterdam, 2002, pp. 71–305.

[154] Villani, C. Hypocoercivity. *Mem. Amer. Math. Soc. 202*, 950 (2009), iv+141.

[155] Vishik, M., and Friedlander, S. Nonlinear instability in two dimensional ideal fluids : the case of a dominant eigenvalue. *Comm. Math. Phys. 243*, 2 (2003), 261–273.

[156] Vlasov, A. On the oscillation properties of an electron gas. *Zh. Eksper. Teoret. Fiz. 8* (1938), 291–318.

[157] Wagner, F. et al. Regime of Improved Confinement and High Beta in Neutral-Beam-Heated Divertor Discharges of the ASDEX Tokamak. *Phys. Rev. Lett. 49*, 19 (1982), 1408 – 1412.

[158] Wesson, J. Tokamaks. *Clarendon Press-Oxford* (2004).

[159] Yau, H. Relative entropy and hydrodynamics of Ginzburg-Landau models. *Lett. Math. Phys. 22*, 1 (1991), 63–80.

[160] Yudovič, V. I. Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat. i Mat. Fiz. 3* (1963), 1032–1066.