Contribution à la théorie mathématique du transport quantique dans les systèmes de Hall

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Contribution à la théorie mathématique du transport quantique dans les systèmes de Hall

Résumé

Dans ce travail, nous nous intéressons à l'étude mathématique du transport dans les systèmes de Hall quantiques en milieu désordonné. Plus précisément, nous commençons par étudier la théorie de la réponse linéaire dans le cas continu pour un opérateur de Schrödinger magnétique aléatoire. Nous exploitons le formalisme de l'intégration non commutative pour développer une théorie de la réponse linéaire adaptée au problème et obtenir une formule de Kubo-Středa. Dans un deuxième temps nous nous intéressons à la quantification des courants de bord créés par un mur magnétique modélisé par un Hamiltonien d'Iwatsuka. Nous démontrons la stabilité de cette quantification sous certaines perturbations magnétiques y compris aléatoire. Enfin nous achevons ce travail de thèse par une discussion plus approfondie sur le formalisme developpé dans la première partie, de manière à permettre une généralisation future de la théorie de la réponse linéaire aux modèles quasi-périodiques.

Abstract

In this work, we are interested in the mathematical study of the transport in the disordered quantum Hall systems. More precisely, we start by study the linear response theory in the continuum setting for a random magnetic Schrödinger operator. We exploit the formalism of the non-commutative integration theory to develop a linear response theory adapted to the problem and get a Kubo-Středa formula. Next, we are interested in the quantification of the edge current created by a magnetic wall modelized by an Iwatsuka Hamiltonian. We prove the stability of this quantification under some magnetic perturbations including random ones. We finish with a discussion on the formalism used in the first chapter to prepare an extension of the linear response theory to quasi-periodic models.

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Chapter 1

Introduction et Présentation

1 Introduction

L'effet Hall classique fut découvert expérimentalement en 1880 par E.Hall [H] qui entreprit l'expérience suivante. Considérons une feuille métallique (on suppose l'épaisseur suffisamment faible pour considérer du point de vue mathématiques la feuille comme une surface bi-dimensionelle). On applique un champ électromagnétique B perpendiculairement à la feuille. Les électrons sont alors soumis à la force de Lorentz perpendiculairement à leur vitesse et vont ainsi décrire des cercles d'un rayon inversement proportionel au module du champ magnétique soit $\sim \frac{1}{\sqrt{B}}$. Du point de vue physique, l'effet Hall possède un intérêt majeur qui est de classifier les conducteurs. A proprement parler, à chaque métal correspond un signe positif ou négatif dénommé "le signe du porteur de charge" qui caractérise sa propension à conduire l'énergie. L'or et le cuivre possèdent le signe (-), tandis que le fer le signe (+). L'effet Hall peut facilement se comprendre en écrivant l'équation d'équilibre du système. Soit N la densité des électrons ou des porteurs de charges. La contribution de champ électrique est alors donnée par NeE, où e est la charge élémentaire du porteur et E la différence de potentiel entre les deux bords de la surface. En quel cas, la force électromagnétique est le produit extérieur du courant, noté j, par le vecteur B représentant le champ magnétique perpendiculaire, alors l'équation d'équilibre s'écrit comme suit,

$$NeE + j \wedge B = 0 \ . \tag{1.1}$$

Cette équation explique l'existence d'une différence de potentiel dans le sens perpendiculaire au courant. Le conductance de Hall classique est alors définie comme le quotient entre la valeur absolue du courant et de la différence de potentiel.

Cent ans plus tard, en 1980, K.von-Klitzing, M.Paper ainsi que G.Dorda [K] ont mis au point un dispositif permettant de réaliser l'expérience de la conductivité de Hall en dehors du régime classique. La difficulté pour réaliser une telle expérience provenait du fait que, pour sortir du régime classique il était nécessaire de se placer à des températures très basses, tandis que la conductivité de Hall nécessite des champs électromagnétiques très grands. L'importance de cette expérience qui leur valut le prix Nobel s'explique par le fait suivant. Dans le cadre classique, on obtient une relation linéaire entre la conductivité et la densité d'électrons, tandis que dans le cas quantique on observe la formation de plateaux. Mais le fait le plus surprenant est que la mesure de la conductivité le long de ces plateaux en utilisant une certaine unité physique, $\frac{e^2}{h}$, se trouve être des entiers à 10^{-8} près. C'est ce que l'on nomme quantification de la conductance. Ce niveau de précision a conduit en première application à l'utiliser comme étalon de la résistivité.

Du point de vue théorique maintenant, l'étude de ce phénomène a aboutit à de profonds résultats et fut le théatre d'un exemple particulièrement saisissant de l'interpénétration de la physique théorique et des mathématiques fondamentales, justifiant ainsi l'enguouement particulier de la part des physicien-mathématiciens pour ce problème particulier. Le nombre de papiers publiés sur ce problème est impressionant de même que leurs transversalités. Ceci dans le sens où il y eut peu de domaine qui n'ait apporté sa pierre à l'édifice, de la géométrie aux probabilités en passant par la topologie. L'exemple le plus marquant en fut certainement la collaboration entre le physicien théoricien Jean Bellissard et le mathématicien Alain Connes "père" de la désormais célèbre géométrie non-commutative.
Historiquement, un premier pas fut accompli avec la conjecture de Laughlin [La] qui affirmait que la quantification avait des origines géométriques et relevait ainsi d'un phénomène général particulièrement robuste. Les premiers a avoir proposé des arguments afin de valider cette conjecture fut Thouless et al. [TKNN] pour un modèle périodique discret avec un champ magnétique ayant un flux rationel et une énergie de Fermi (le potentiel chimique à température nulle) dans une lacune spectrale. Quelques temps plus tard, Avron, Simon et Seiler [AvSS] prouvèrent que l'entier associé par Thouless et al. à la conductivité de Hall avait une interprétation topologique.
Mais l'explication de la quantification n'est pas suffisante en elle-même, l'autre aspect fondamental : l'existence de plateaux ainsi que l'absence de conductivité directe (i.e pas de transport non-transversal) pour les énergies se trouvant dans les régions où les plateaux apparaissent, n'étaient toujours pas justifiés. En 2008 eut lieu la célébration des cinquante ans de la découverte d'un autre phénomène étant précisement à l'origine de ce fait :

la localisation d'Anderson. Restant à un niveau intuitif, Anderson indiqua, il y a précisement cinquante ans, que la présence d'impuretés en densité suffisante dans un cristal donné, modélisées par des sauts de potentiel aléatoires, provoquait des interférences destructives dans le sens où la probabilité d'avoir la condition de Bragg nécessaire pour obtenir une interférence constructive pour l'onde quantique associée à la particule considérée tendait vers zero impliquant le fait que la particule est ainsi piégée dans les puits de potentiel. C'est ce que l'on appelle le phénomène de localisation d'Anderson. Aujourd'hui, ce phénomème est prouvé mathématiquement comme ayant lieu de façon universelle dès que l'on se donne un potentiel aléatoire ayant une constante de couplage assez grande ainsi qu'une mesure de probabilité non-dégénérée (pour un état de l'art voir [Kw]). La localisation d'Anderson implique l'existence d'états localisés lesquels ne participent pas au courant de Hall, en conséquence de quoi lorsque l'énergie de Fermi varie dans une région d'états localisés la conductance de Hall reste constante. Comment se fait-il alors que le courant de Hall ne soit pas nul lui aussi ? Les physiciens expliquent cela par le fait que le courant de Hall est transporté par les états de bords.

Revenons à présent aux mathématiques. Beaucoup de théoriciens comparent la théorie mathématique associée à ce problème à une sorte de théorème de Gauss-Bonnet du point de vue algèbrique. Le fait est que la quantification dans ce contexte est un phénomène particulièrement robuste dans le sens où il est invariant par des perturbations de natures très variées. Dans cet état d'esprit, alors que Connes était en train de développer la géométrie non-commutative, J.Bellissard formula l'interprétation théorique expliquant ce problème en colloboration avec ce dernier. Le premier pilier de la théorie est le calcul de la conductance de Hall par *la théorie de la réponse linéaire* et de *la formule de Kubo.* Ensuite l'étape importante est l'analyse de la structure algèbrique des observables associées au système physique. Le fait est que la conductance de Hall se révèle être un objet mathématique particulièrement intéressant. Lorsque l'on se place dans le contexte de la géométrie non-commutative, cet objet se trouve être un invariant propre à l'algèbre des observables, laquelle dans la théorie de A.Connes est vue comme une variété non-commutative tandis que l'objet associé à la conductance n'est autre que *le nombre de Chern* associé à cette variété. Dans ce formalisme, le fait que la conductance est un entier est prouvé via un théorème de l'indice démontré par A.Connes. Ce qui explique l'analogie faite entre ce résultat et le théorème de Gauss-Bonnet, dans le sens où celui-ci est le prototype des théorèmes de l'indice. On peut ainsi prouver que la conductance constitue un invariant topologique propre à l'algèbre des observables. Inversement ce problème de physique théorique fournit à A.Connes, de son propre aveu, une des premières intuitions de ce qui allait devenir plus tard la géométrie différentielle non-commutative : "Cet exemple était en fait le premier balbutiement de géométrie non-commutative, le premier qui ait montré qu'il se passait quelque chose", [C].

1.1 Présentation du travail de thèse

Nous décrivons et développons les objets mathématiques nécessaires à l'obtention d'une formule de Kubo via la théorie de la réponse linéaire pour les systèmes de Hall continus. Si l'intégration non-commutative a démontré sa pertinence conceptuelle dans les systèmes de Hall discrets [Bel], le travail de [BoGKS] n'exploite pas cette théorie et reconstruit "à la main" le cadre fonctionel utile au traitement des modèles discrets. Le premier sujet de cette thèse, et il en constitue le premier chapitre, consiste à réconcilier ces deux approches en exploitant les résultats de l'intégration non-commutative dans le cadre des systèmes de Hall continus. Nous tenterons de convaincre le lecteur du caractère naturel des espaces fonctionels construits à partir de l'algèbre de von-Neumann de référence associée au problème.

Après quoi, dans le deuxième chapitre, nous prouvons un nouveau résultat dans la théorie des systèmes de Hall. Nous prouvons l'existence d'un courant de bord pour un modèle de mur magnétique ainsi que la quantification de ce courant de bord. Ensuite nous démontrons un résultat général de stabilité de la conductance de bord sous des perturbations magnétiques ayant comme première conséquence la stabilité de la quantification sous ce type de perturbations magnétiques telles que des perturbations magnétiques aléatoires vivant dans un demi-plan pour le modéle de mur magnétique. Ces résultats ont d'autres conséquences non exposées dans cette thèse car encore en préparation [DGR1, DGR2].

Enfin nous terminons par une discussion sur la structure algèbrique des observables considérées dans le premier chapitre. Nous avons été au-delà des besoins propre au modèle considéré dans le premier chapitre dans le but d'obtenir une généralisation future de la théorie de la réponse linéaire à d'autres modèles tels que les structures quasi-périodiques.

Plus précisement, dans le premier chapitre nous nous sommes intéressés à la théorie de la réponse linéaire dans le cas continu multi-dimensionel et euclidien. Nous donnons une généralisation non-triviale ainsi qu'une clarification de ce problème. De nombreux points nécessitaient des approfondissements de même que certaines preuves manquaient dans la littérature. Notre travail se base essentiellement sur le travail récent [BoGKS]. Ce travail constitue

aujourd'hui la dérivation de la formule de Kubo la plus générale. Cela étant, le formalisme employé par [BoGKS] est empreint de nombreuses complications techniques rendant difficile toute géneralisation. En plus de cela, un des problèmes fortement lié à cette théorie, la formule de Mott, a reçu une validation mathématique trés récemment par [KLM]. Notre travail consiste donc à fournir une analyse plus naturelle grâce à la théorie de l'intégration non-commutative tout en intégrant le récent résultat obtenu par [KLM] ouvrant la voie à une meilleure compréhension et par voie de fait aux possibles généralisations.
On considère l'Hamiltonien associé au système physique à savoir l'opérateur de Schrödinger magnétique ergodique

$$H(\mathbf{A}_\omega, V_\omega) = (-i\nabla - \mathbf{A}_\omega(x))^2 + V_\omega(x) \,\text{agissant sur}\, \mathfrak{H} := L^2(\mathbb{R}^d) \;, \quad (1.2)$$

avec $\mathbf{A}_\omega$ le potentiel magnétique, $\mathbf{V}_\omega$ le potentiel scalaire associé au milieu et le paramètre aléatoire ω parcourant l'espace de probabilité $(\Omega, \mathbb{P})$. Une des propriétés importantes de ce système physique est qu'il possède des symétries encodées par les *translations magnétiques* $\{U_x, x \in \mathcal{Z}\}$ avec $\mathcal{Z}$ un groupe localement compact abélien associé à l'action sur $(\Omega, \mathbb{P})$ par les transformations préservant la mesure et agissant de façon ergodique $\{\tau(x); x \in \mathcal{Z}\}$, telle que la condition suivante soit respectée

$$U_x H_\omega U_x^* = H_{\tau(x)\omega} \;, \quad (1.3)$$

de tels opérateurs sont appelés *covariants.* Nous considèrons un état d'équilibre associé au système physique au temps $t = -\infty$ representé par la densité de matrice $\zeta_\omega = f(H_\omega)$ avec f une fonction positive régulière. Dans le cas où l'on suppose que la température est nulle, on a $\zeta_\omega = P^{E_F}(H_\omega)$, le projecteur de Fermi. Dans ce cas l'hypothèse essentielle, qui est garantie lorsque l'on considère que E_f, l'énergie de Fermi, se trouve dans une région de localisation spectrale [AG, GK1, GK2, GK3, AENSS], est la suivante

$$\mathbb{E}\left(\|[x_k, \zeta_\omega]\chi_0\|_2^2\right) < \infty \;. \quad (1.4)$$

Comme nous le montrerons, cette condition est équivalente au fait que ζ_ω appartienne à un analogue de l'espace de Sobolev du point de vue de l'analyse opératorielle, en respect d'une dérivation algèbrique notée, par analogie avec la dérivation usuelle, ∂_i. Ceci étant, nous prouvons que le courant à l'équilibre est nul comme cela est attendu du point de vue physique. Soit

$$\mathcal{T}(\partial_i(H_\omega)\zeta_\omega) = 0 \text{ pour tout } i = 1 \cdots d \;. \quad (1.5)$$

Notons que nous associons une trace $\mathcal{T}$ propre à l'algèbre des observables $\mathcal{K}_\infty$ à laquelle l'Hamiltonien magnétique aléatoire est affilié et que nous définirons précisement ainsi que les espaces d'intégrations non-commutatifs $L^p(\mathcal{K}_\infty)$. Après quoi nous perturbons adiabatiquement le système par un champ électrique dépendant du temps $\mathbf{E}(t,\eta)\cdot\mathbf{x}$ avec η le paramètre adabiatique. Dans ce cas, avec la jauge appropriée, la dynamique est générée par l'Hamiltonien ergodique dépendant du temps

$$H_\omega(t) = (-i\nabla - A_\omega - \mathbf{F}(t))^2 + V_\omega(x) = G(t)H_\omega G^*(t) \tag{1.6}$$

où

$$\mathbf{F}(t) = \int_{-\infty}^{t} \mathbf{E}(s)ds \ . \tag{1.7}$$

Ce choix de jauge permet comme c'est le cas dans [BoGKS] de travailler avec un Hamiltonien semi-borné inférieurement. Ainsi nous obtenons l'existence d'un propagateur $U_\omega(t,s)$ tel que $U_w(t,r)U_w(r,s) = U_\omega(t,s)$, $U_\omega(t,r)\mathcal{D} = \mathcal{D}$ et $i\partial_t U_\omega(t,r)\varphi = H(t)U_\omega(t,s)\varphi$ pour tout $\varphi \in \mathcal{D}$ avec $\mathcal{D}$ le domaine commun aux $H(t)$. Nous notons dans ce cas $\mathcal{U}(t,s)(A) = U(t,s)AU(s,t)$ pour A tel que le produit soit bien définit. Le premier point important est alors la résolution du problème de Cauchy associé à l'évolution. Dans ce cas nous obtenons

Theorem 1.1. *Soit $\eta > 0$ et supposons que $\int_{-\infty}^{t} \mathrm{e}^{\eta r}|\mathbf{E}(r)|\mathrm{d}r < \infty$ pour tout $t \in \mathbb{R}$. Le problème de Cauchy*

$$\begin{cases} i\partial_t \varrho(t) = [H(t), \varrho(t))] \\ \lim_{t\to-\infty} \varrho(t) = \zeta \end{cases} , \tag{1.8}$$

admet une unique solution dans $\mathrm{L}^p(\mathcal{K}_\infty)$, $p \geq 1$, donnée par

$$\begin{aligned} \varrho(t) &= \lim_{s\to-\infty} \mathcal{U}(t,s)\,(\zeta) && (1.9)\\ &= \lim_{s\to-\infty} \mathcal{U}(t,s)\,(\zeta(s)) && (1.10)\\ &= \zeta(t) - \int_{-\infty}^{t} \mathrm{d}r\, \mathrm{e}^{\eta r}\mathcal{U}(t,r)(\mathbf{E}(r)\cdot\nabla\zeta(r)). && (1.11) \end{aligned}$$

Nous avons aussi que

$$\varrho(t) = \mathcal{U}(t,s)(\varrho(s))\,, \ \ |||\varrho(t)|||_p = |||\zeta|||_p \ , \tag{1.12}$$

pour tout t, s. De plus, $\varrho(t)$ est non-négative, et si ζ est une projection alors il en est de même pour $\varrho(t)$ pour tout t.

Une fois le problème de Cauchy résolu, par la théorie de la réponse linéaire nous dérivons la formule de Kubo pour la conductance $\sigma(\eta, \zeta, t_0)$, soit

Theorem 1.2. *Soit $\eta > 0$. Le courant $\mathbf{J}(\eta, \mathbf{E}; \zeta, t_0)$ est différentiable selon $\mathbf{E}$ à $\mathbf{E} = 0$ et la dérivé $\sigma(\eta; \zeta, t_0)$ est donnée*

$$\sigma_{jk}(\eta; \zeta, t_0) \quad = \quad -\mathcal{T}\left\{2\int_{-\infty}^{t_0} \mathrm{d}r\, \mathrm{e}^{\eta r}\mathcal{E}(r)\mathbf{D}_j\, \mathcal{U}^{(0)}(t_0 - r)\,(\partial_k(\zeta))\right\}.$$

où $\mathbf{D}_j$ est définit comme l'opérateur vélocité dans la j-ième direction tandis que le terme $\mathcal{E}(r)$ est le terme associé à l'intensité du champ électrique. Enfin, pour finir nous considérons la température nulle, soit $\zeta_\omega = P_\omega^{E_F}$, et faisant tendre le paramètre adiabatique $\eta \to 0$ nous obtenons la désormais célèbre formule de Kubo-Středa.

Theorem 1.3. *Fixant $\mathcal{E}(t) = 1$ et $t_0 = 0$. Si $\zeta = P^{(E_F)}$ est un projecteur de Fermi satisfaisant à la condition (1.4)*

$$\sigma_{j,k}^{(E_F)} = -i\mathcal{T}\left\{P^{(E_F)}\left[\partial_j P^{(E_F)}, \partial_k P^{(E_F)}\right]\right\}, \tag{1.13}$$

pour tout $j, k = 1, 2, \ldots, d$. Comme conséquence, le tenseur de conductivité est antisymétrique ; En particulier $\sigma_{j,j}^{(E_F)} = 0$ pour $j = 1, 2, \ldots, d$.

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Dans le deuxième chapitre, nous considérons un modèle particulier, l'Hamiltonien d'Iwatsuka. Ce dernier a été introduit par A.Iwatsuka ([Iw]) afin de produire un exemple d'opérateur de Schrödinger magnétique avec un champ magnétique non-uniforme possédant du spectre absolument continu. Ce modèle est particulièrement intéressant en cela qu'il consiste en un opérateur de Schrödinger magnétique sur $\mathbb{R}^2$ usuel avec l'hypothèse que le champ magnétique ne dépend que d'une variable. En formulant quelques hypothèses supplémentaires que sont la monotonie du champ magnétique ainsi que le fait qu'il soit asymptotiquement constant, on obtient un modèle de *mur magnétique* créant un courant de bord en analogie avec les courants de bord étudiés où était considéré uniquement des murs électriques (un opérateur de Landau perturbé par un potentiel scalaire confinant ou un modèle de mur dur en considérant la restriction de l'opérateur de Landau à un demi-plan avec des conditions de Dirichlet au bord [CG, EGS, EG, DBP, CHS]). Dans notre contexte, nous prouvons que le courant de bord est, comme dans le cas électrique, quantifié en plus de posséder les même types d'invariance sous certaines perturbations.
Plus précisement, définissant la conductance de bord au travers de la fenêtre

spectrale I inclus dans une lacune spectrale de l'opérateur non-perturbé (l'opérateur de Landau) par

$$\sigma_e^{(I)}(H) := i\mathrm{tr}(g'(H)[H,\chi]) \tag{1.14}$$

avec g et χ des fonctions régulières dont les rôles seront précisés, nous obtenons

Proposition 1.4. *Soit $\mathcal{A}_{Iw}$ générant un (B_-,B_+)-champ magnétique. Supposant que l'interval I soit tel que $I\subset]B_-,B_+[$ et $I\subset](2n-1)B_-,(2n+1)B_-[$ pour un certain $n\in\mathbb{N}$. Alors nous avons*

$$\sigma_e^{(I)}(H(\mathcal{A}_{Iw})) = -n.$$

Où un (B_-,B_+)-champ magnétique est un champ magnétique d'Iwatsuka ayant B_- et B_+ comme limites asymptotiques en $-\infty$ et $+\infty$. Tandis que $\mathcal{A}_{Iw}$ est le potentiel magnétique d'Iwatsuka. Le second résultat qui nous intéresse est l'invariance de cette quantification sous certaines perturbations.

Theorem 1.5. *Soit $A\in\mathcal{C}^1(\mathbb{R}^2;\mathbb{R}^2)$. Supposons que $a\in\mathcal{C}^2(\mathbb{R}^2,\mathbb{R}^2)$ est un potentiel magnétique à support compact dans la première direction ainsi que polynomialement borné dans la seconde. Soit g une fonction régulière telle* $\mathrm{supp}g'\subset I$. *Alors*

$$\mathcal{K}(A,a) := g'(H(A+a)[H(A+a),\chi] - g'(H(A)))[H(A),\chi] \in \mathcal{T}_1 \tag{1.15}$$

De plus, si $g'(H(A)))[H(A),\chi]\in\mathcal{T}_1$, alors $\mathrm{tr}\mathcal{K}(A,a)=0$, tel que

$$\sigma_e^{(I)}(H(A+a)) = \sigma_e^{(I)}(H(A)). \tag{1.16}$$

En particulier, si $\mathcal{A}_{Iw}$ génère un (B_-,B_+)-champ magnétique , alors

$$\sigma_e^{(I)}(H(\mathcal{A}_{Iw}+a)) = n.$$

où l'intervalle I vérifie $I\subset]-\infty,B_+[$ ainsi que $I\subset](2n-1)B_-,(2n+1)B_-[$ pour un certain $n\in\mathbb{N}$

Ensuite, nous démontrons un résultat de perturbation plus général pour une perturbation vivant dans un demi-plan. Utilisant la notation suivante notation $H(A^{(1)},A^{(2)})$ pour $H(A^{(0)}+A^{(1)}+A^{(2)})$ où $\mathrm{supp}A^{(1)}$ (resp., $\mathrm{supp}A^{(2)}$), est inclus dans le demi-plan $x<R_1$ (resp., $x>R_2$), et $A^{(0)} := \left(-\frac{B_-y}{2},\frac{B_-x}{2}\right)$ avec $B_->0$. En particulier, $H(0,0)$ est l'Hamiltonien de Landau avec un champ magnétique scalaire uniforme B_-. Le résultat s'énonce comme suit.

Theorem 1.6. *Soit I un intervalle fermé tel que $I \cap \sigma(H(0,0)) = \emptyset$. Soit*

$$\mathcal{K}(a_\alpha, a_\beta) := g'(H(a_\alpha, a_\beta) i[H(a_\alpha, a_\beta), \chi] - g'(H(a_\alpha, 0) i[H(a_\alpha, 0), \chi] \quad (1.17)$$
$$- g'(H(0, a_\beta) i[H(0, a_\beta), \chi]. \quad (1.18)$$

Alors $\mathcal{K}(a_\alpha, a_\beta)$ est de classe trace. De plus, si deux des trois termes du membre de droite sont de classe trace, alors $\mathrm{tr}\mathcal{K}(a_\alpha, a_\beta) = 0$; en particulier

$$\sigma_e^{(I)}(H(a_\alpha, a_\beta)) = \sigma_e^{(I)}(H(a_\alpha, 0)) + \sigma_e^{(I)}(H(0, a_\beta)). \quad (1.19)$$

De plus, si $\mathcal{A}_{Iw}^{(L)}, \mathcal{A}_{Iw}^{(R)}$ sont des potentiels d'Iwatsuka à droite comme à gauche ainsi que $I \subset]-\infty, B_+[$ alors, si $\mathrm{supp} g' \subset]-\infty, B_+^{(L)}[$ et $\mathrm{supp} g' \subset]-\infty, B_+^{(R)}[$ le même résultat reste valide pour

$$\mathcal{K}'(a_\alpha, a_\beta) := g'(H(a_\alpha, a_\beta) i[H(a_\alpha, a_\beta), \chi] - g'(H(a_\alpha, \mathcal{A}_{Iw}^{(R)}) i[H(a_\alpha, \mathcal{A}_{Iw}^{(R)}), \chi] \quad (1.20)$$
$$- g'(H(\mathcal{A}_{Iw}^{(L)}, a_\beta) i[H(\mathcal{A}_{Iw}^{(L)}, a_\beta), \chi] \ ; \quad (1.21)$$

en particulier, si deux des trois termes sont de classe traces dans $\mathcal{K}'(a_\alpha, a_\beta)$, alors

$$\sigma_e^{(I)}(H(a_\alpha, a_\beta)) = \sigma_e^{(I)}(H(a_\alpha, \mathcal{A}_{Iw}^{(R)})) + \sigma_e^{(I)}(H(\mathcal{A}_{Iw}^{(L)}, a_\beta)) \ . \quad (1.22)$$

Nous terminons ce chapitre en considérant une perturbation magnétique aléatoire a_ω de type alliage mais vivant dans un demi-plan et prouvons que sous certaines hypothèses de localisation ainsi qu'en construisant une régularisation adaptée aux critères de localisation usuels tel que la sommabilité uniforme des fonctions propres (SULE) [GK3] que les résultats de stabilité restent valides dans cette situation.

Enfin, nous achevons ce travail de thèse par une discussion sur le formalisme développé dans le premier chapitre afin de traiter la réponse linéaire. Nous travaillons alors uniquement dans le formalisme des algèbres d'opérateurs sur un espace de Hilbert donné. Il s'agit d'une analyse constructive dans le sens où nous partons d'un nombre assez restreint d'axiomes. Ce faisant, notre formalisme met en avant de façon particulièrement claire que la structure des observables est complètement encodée par le groupe de symétries permettant ainsi de se détacher du modèle particulier considéré. Les caractéristiques du modèle impliqué dans la quantification ou son invariance, comme par exemple la K-théorie associée à l'algèbre des observables, sont insensibles au fait par exemple de passer du modèle discret au modèle continu par une équivalence de Morita [MM]. Ce fait a été mis en lumière par Mathai et Marcolli dans un cas particulier du plan hyperbolique où ils prouvent que les principaux résultats sont insensibles au passage du continu au discret alors que dans le même temps pour des problèmes assez différents Sunnada et Brünning prouvaient que certaines propriétés topologiques du spectre d'opérateur agissant sur des variétées étaient encodés par certaines propriétés des algèbres générées par la representation du groupe de symétries comme la positivité de la constante de Kadison [MM, BS]. Dans ce chapitre nous ne considérons que l'aspect purement algèbrique, nous concentrant sur le fait de donner une description qui soit la plus claire et générale possible. Le fait est que ce problème possède un intérêt mathématique propre. Dans un premier temps, nous détaillons la construction de l'algèbre de von-Neumann $\mathcal{K}_\infty$ de référence à laquelle tout les opérateurs considérés dans la théorie de la réponse linéaire sont affiliés. Après quoi, utilisant une analyse de Fourier abstraite, nous démontrons certains théorèmes de densité pour des sous-algèbres d'opérateurs possédant des propriétés de régularité importantes afin de pouvoir construire une analyse opératorielle sur $\mathcal{K}_\infty$. Ensuite, nous détaillons un peu plus la structure de l'algèbre en utilisant la structure de C^*-Hilbert module qui a connu un développement intensif ces dernière années. Grâce à cette structure particulière nous explicitons davantage le rôle joué par le groupe de symétries et obtenons un principe général de diagonalisation partielle des opérateurs. Cette discussion est largement inspirée des communications privées [G2, L] ainsi que des travaux [BC, BS, Gr, LPV].

Chapter 2

Théorie de la réponse linéaire pour un opérateur de Schrödinger magnétique en milieu désordonné et espace L^p non-commutatifs

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Résumé

Nous considérons un opérateur de Schrödinger ergodique avec un champ magnétique continue dans l'approximation de particule n'interagissant pas. Nous exploitons la théorie de l'intégration non commutative afin d'obtenir une dérivation plus naturelle de la théorie de la réponse linéaire et de la formule de Kubo-Středa.

Contents

1 Introduction

Dans ce premier chapitre nous traitons un des problèmes associés à la théorie du transport quantique en milieu désordonné, la théorie de la réponse

linéaire.
L'accent est mis sur une dérivation analytique détaillée de la formule de Kubo et plus particulièrement sur la formule de Kubo-Stˇreda.
Dans [BoGKS] les auteurs considèrent un opérateur de Schrödinger avec champ magnétique et donnent une dérivation contrôlée de la formule de Kubo pour le tenseur de conductivitée électrique validant ainsi la théorie de la réponse linéaire dans l'approximation où la particule est considérée comme n'interagissant pas avec d'autres. Pour un champ électrique appliqué adiabatiquement, ceux-ci retrouvent l'expression espérée pour la conductivitée quantique de Hall dès que l'énergie de Fermi se trouve dans une région de localisation de l'Hamiltonien de référence ou dans une lacune spectrale de celui-ci. Le but de ce travail est d'obtenir une preuve plus naturelle des résultats de [BoGKS] et de simplifier leur formalisme mathématique qui se révèle moins naturel et en conséquence responsable de complications dans la dérivation de la théorie de la réponse linéaire en plus de la rendre très difficilement généralisable à d'autres problèmes ou modèles. Pour cela nous utilisons le fait que l'algèbre des observables est naturellement une algèbre de von-Neumann et nous introduisons ainsi une théorie d'intégration non-commutative adaptée aux observables considérées. Nous mentionnons le fait que nous utilisons cette théorie afin de traiter de façon analytique ce problème dans le sens où nous n'utilisons aucun résultat de géométrie non-commutative dans ce travail. Il ne s'agit ici que de construire une analyse fonctionnelle naturellement adaptée au problème.
Le tenseur de conductivité électrique est habituellement exprimé en terme de "formule de Kubo" étant obtenue par la théorie de la réponse linéaire. Dans le contexte des milieux désordonnés, où la localisation d'Anderson est attendue (ou prouvée), la conductivitée électrique a reçue beaucoup d'attention venant des nombreuses perspectives qu'elle offre pour de nombreux problèmes en théorie du transport quantique. Pour des systèmes temporellement réversibles et à température nulle, l'annulation de la conductivitée directe est physiquement parlant une preuve évidente de la présence d'un régime de localisation [FS, AG]. Une autre direction d'intérêt est la connection entre la conductivitée directe et l'effet Hall quantique [TKNN, St, B, Ku, BES, AvSS, AG]. D'un autre côté le comportement de la conductivitée alternative à basse fréquence dans des régions de localisation est dicté par la célèbre formule de Mott [MD] (voir [KLP, KLM, KM] pour des développements récents). De même, en connection avec la conductivitée, la fonction de corrélation courant-courant a récemment été sujet d'une attention toute particulière (voir [BH, CGH]).
Durant ces deux dernières décennies nombres de travaux ont été dédiés à apporter une meilleure compréhension du point de vue mathématique de

ces dérivations e.g., [P, Ku, B, NB, AvSS, BES, SB1, SB2, AG, Na, ES, CoJM, CoNP]. Tandis qu'une grande partie de l'attention a été dédiée à l'obtention de la conductivitée à partir de la formule de Kubo (en particulier de la conductivitée quantique de Hall) ainsi qu'à l'étude de cette conductivité, peu ont concerné une dérivation controlée de la réponse linéaire ainsi que de la formule de Kubo elle-même, mis à part les récents travaux [SB2, ES, BoGKS, CoJM, CoNP].
Dans ce travail nous nous concentrons sur la dérivation de la réponse linéaire ainsi que des formules de Kubo et Kubo-Středa. Nous commençons par exposer la théorie de l'intégration non-commutative que nous utilisons ainsi que les propriétées particulières dont nous aurons besoin pour les différents calculs. Nous nous basons essentiellement sur [G, T]. Après cela nous décrivons le modèle particulier avec lequel nous allons travailler qui est l'opérateur de Schrödinger magnétique aléatoire agissant sur un espace Euclidien réel à dimension finie, puis nous rappelons certains résultats généraux sur l'opérateur de Schrödinger magnétique. Ceci étant fait, s'appuyant sur la théorie de l'intégration ainsi que sur les résultats généraux obtenus sur l'opérateur de Schrödinger magnétique, nous prouvons les estimations nécessaires dans ce contexte s'appuyant sur un calcul fonctionnel adapté. Enfin nous prouvons la validité de la théorie de la réponse linéaire en plusieurs étapes.
Premièrement nous décrivons l'équation de Liouville qui est associée à l'évolution temporelle de la matrice de densité associée à l'état du système sous l'action du champ éléctrique dépendant du temps (théorème 9.1). De façon standard, cette équation d'évolution peut s'écrire comme une équation intégrale par la formule de Duhamel.
Après quoi, nous calculons le courant par unité de volume induit par le champ électrique et prouvons qu'il est différentiable selon le champ électrique en l'origine. Cela nous donne en toute généralité la formule de Kubo désirée pour le tenseur de conductivitée électrique pour un paramètre adabiatique quelconque (théorème 9.5 and Corollaire 9.6).
Nous pouvons alors passer à la limite adabiatique afin d'obtenir la directe/ac conductivité à température nulle (Theoreme 9.7, Corollaire 9.8 et Remark 17). En particulier nous retrouvons l'expression espérée pour la conductivité quantique de Hall ainsi que la formule de Kubo-Středa comme dans [B, BES]. A température positive, nous notons que même si l'existence de la *mesure* de conductivité électrique peut-être facilement dérivée à partir de la formule de Kubo [KM], prouver que la conductivité elle-même, i.e sa densité, existe et est finie reste un problème ouvert.
Enfin généraliser la construction algèbrique au cas non d'un groupe mais d'un groupoïde permettrait de traiter le cas des modèles de Delone ou de pavage. Cette généralisation sera le sujet d'un travail encore en cours de rédaction.

Enfin nous précisons que récemment, dans un travail encore en préparation, [DGR2] prouvent un résultat de localisation pour un potentiel magnétique aléatoire vu comme une perturbation de l'Hamiltonien de Landau. Ainsi nous considérons naturellement dans ce travail un Hamiltonien général pouvant posséder à la fois un potentiel scalaire (électrique) aléatoire de même qu'un potentiel vectoriel (magnétique) aléatoire.

2 Assumptions and Usual formalism

We introduce the formalism and the main assumptions. We develop in more details the algebraic structure in the Ch3. Here, we are concerned only by the derivation of the linear response theory and consequently only the analytic point of view is of interest, but to construct an integration and differentiation calculus we need of some basic algebraic properties developed here. For further discussion on the algebraic structure we refer to the last chapter.
Let a separable Hilbert space, namely $\mathfrak{H}$ (in the physical context $\mathfrak{H} = L^2(X)$ with X a finite dimensional real vector space) and $\mathcal{Z}$ a locally compact discrete finitely generated group (in the most physical case $\mathcal{Z}$ is lattice of $\mathbb{R}^d$ or it is a discrete subgroup of the group of isometries on the manifold which acts cocompactly on X in the sense that $\mathcal{D} := X/\mathcal{Z}$ is a compact). We use an additive notation for convenience and note e or 0 the neutral element. Moreover we assume there exists a unitary projective representation on $\mathfrak{H}$ as it is defined below.
We mention that the most part of the computation is still valid for Delone set or tilling but the group structure must be remplaced by a groupoid one, a mathematical problem we shall investigate in a future work.

ζ-projective unitary representation

For the sake of completeness we recall some definitions we shall use. We refer to [EL], more precisely we follows the line of [BC]. Let $\mathbb{T}$ the one dimensional torus.

Definition 2.1. *A (normalized) 2-cocycle on $\mathcal{Z}$ with values in $\mathbb{T}$ is a map $\zeta : \mathcal{Z} \times \mathcal{Z} \mapsto \mathbb{T}$ such that the cocycle relation*

$$\zeta(x,y)\zeta(x+y,z) = \zeta(x,y+z)\zeta(y,z) \text{ for any } x,y,z \in \mathcal{Z}$$

holds and moreover we have

$$\zeta(x,\mathrm{e}) = \zeta(\mathrm{e},x) = 1 \tag{2.1}$$

$$\zeta(x,-x) = \zeta(-x,x) \text{ for any } x \in \mathcal{Z} \tag{2.2}$$

The set of 2-cocycle is denoted by $Z^2(\mathcal{Z}, \mathbb{T})$.

Remark 1. *We can without loss of generality suppose that the 2-cocycle is normalized in the sense that* $\zeta(\mathrm{e}, \mathrm{e}) = 1$. *As that explained in Ch3. the nature of the 2-cocycle depends only of the assumptions on the magnetic field. Here we assume that the deterministic part of the magnetic field is constant or periodic. As it is explained in [GI] we can define the analog structure for an arbitrary magnetic field but in this case the 2-cocycle takes its value not on* $U(1)$ *but in* $U(1)^X$ *(see [GI] for a detailed exposition of this context).*

Definition 2.2. *Let* $\zeta : \mathcal{Z} \times \mathcal{Z} \mapsto \mathbb{T}$. *A* ζ*-projective unitary representation of* $\mathcal{Z}$ *on a given Hilbert space* $\mathfrak{H}$ *is a map from* $\mathcal{Z}$ *into the group* $\mathcal{U}(\mathfrak{H})$ *of unitaries on* $\mathfrak{H}$ *such that the following holds*

$$U_x U_y = \zeta(x, y) U_{x+y} \text{ for any } x, y \in \mathcal{Z} \tag{2.3}$$

$$U_x^* = \overline{\zeta}(x, -x) U_{-x} \text{ for any } x \in \mathcal{Z}. \tag{2.4}$$

consequently we have that $U_{\mathrm{e}} = id_{\mathfrak{H}}$.

In fact for a given ζ-projective unitary representation on $\mathfrak{H}$ we can derive the cocycle relation from the associativity of the unitary map, i.e $U_x(U_y U_z) = (U_x U_y) U_z$.

Identity decomposition

We denote $\mathcal{P}(\mathcal{B}(\mathfrak{H}))$ the lattice set of projections belonging to $\mathcal{B}(\mathfrak{H})$. Now to encode the geometric symmetries, we assume the existence of a projection-valued map on $\mathcal{Z}$, namely χ.

Definition 2.3. *We call a projection-valued map on* $\mathcal{Z}$ *an identity decomposition with respect to the unitary representation an element of* $\mathcal{P}(\mathcal{B}(\mathfrak{H}))^{\mathcal{Z}}$ *such that the following holds*

$$\chi_x \chi_y = \delta_{x,y} \chi_x, \text{ for any } x, y \in \mathcal{Z} \tag{2.5}$$

$$\sum_{x \in \mathcal{Z}} \chi_x = id_{\mathfrak{H}} \tag{2.6}$$

$$U_x \chi_y U_x^* = \chi_{x+y} \text{ for all } x, y \in \mathcal{Z} \tag{2.7}$$

We define $\mathfrak{H}_c$ *as elements* φ *of* $\mathfrak{H}$ *such that there exists a finite subset* F *of* $\mathcal{Z}$ *such that* $\chi_F \varphi = \varphi$.

Where δ is the Dirac function which is equal to one if $x = y$ and zero otherwise.

Remark 2. *The orthogonality condition can be relaxed in many applications. It is actually sometimes useful to consider smooth functions χ_x. Then in this case we consider a smooth mollifer function u with support contained in the fundamental cell $\mathcal{D}$ and we define u_x as the translated by x of u but in the algebraic analysis the orthogonality plays an important role as we will see. The last relation provides us a decomposition of the Hilbert space with respect to the symmetry. As we will see this provides great simplifications and the possibility to construct an explicit decomposition of operators as in Ch3. Furthermore it provides a generalization which permits us to use this framework for many others physical models.*

Example 1. *When we consider $\mathfrak{H} = L^2(X)$ where X is a finite dimensional real vector space. The last condition 2.7 ensures the localness of the group-action in the sense that this implies the stability of $\mathfrak{H}_c$, i.e $U_x \mathfrak{H}_c U_x^* \subset \mathfrak{H}_c$ for any $x \in \mathcal{Z}$. If $\mathfrak{H} = \mathrm{L}^2(\mathbb{R}^d)$ there are functions with a compact support, U the magnetic translations and the set of orthogonal projections is defined as being the translation of characteristic functions on the fundamental cell $\mathcal{D}$ under the group action. Note that separability of the Hilbert space implies countability of the group.*
In the case where $X = \mathbb{H}$ is the hyperbolic plane with a Fuchsian group of isometries on $\mathbb{H}$, [MM] used of the notion of good orbifold. In the case of the Delone set or more generally for tilling model the Varonoï cells arise as the natural equivalent. The model of [MM] is used to treat fractional quantum Hall effect while [BBG] and [LPV] study the Delone model to modelize that the randomness comes from the space of configuration.

Randomness and Ergodic action

We work with random observables modelizing a disordered medium. Conceptually this is nothing else than a generalization of the periodic media which modelizess a perfect crystal.

Definition 2.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ a σ-Borel finite probability space with probability measure $\mathbb{P}$.*
A group $\{\tau_x\}_{x \in \mathcal{Z}}$ of measure preserving transformations τ on the Borel space Ω is a map from $\mathcal{Z}$ to the set of isomorphisms of Ω such that for any $x, y \in \mathcal{Z}$,

$$\begin{aligned} &\tau_x : \Omega \mapsto \Omega \text{ be a measure-preserving isomorphism, i.e} \frac{\mathrm{d}\mathbb{P} \circ \tau(x)}{\mathrm{d}\mathbb{P}} = 1 \\ &\tau_x \circ \tau_y = \tau_{x+y} \ , \ \tau_e = 1 \text{ and so } \tau_x^{-1} = \tau_{-x} \end{aligned} \tag{2.8}$$

Furthemore we assumed that τ acts ergodically on $(\Omega, \mathcal{F}, \mathbb{P})$.

3 Algebra of measurable covariant and locally bounded operators

In this section we recall some usual definitions and constructions for random operators and refer for more precise details to Ch3.

Measurable and locally bounded operators

Definition 3.1. *Let $\{\mathfrak{H}(\omega)\}_{\omega\in\Omega}$ a measurable field of Hilbert space, one defines then the direct integral of this field by $\tilde{\mathfrak{H}} = \int_\Omega^\oplus \mathfrak{H}(\omega)d\mathbb{P}(\omega)$ with the inner product : $\forall\ (\varphi,\psi) \in \tilde{\mathfrak{H}}^2$ defined as $\langle\varphi,\psi\rangle_{\tilde{\mathfrak{H}}} = \displaystyle\int_\Omega \langle\varphi(\omega),\psi(\omega)\rangle_{\mathfrak{H}(\omega)} d\mathbb{P}(\omega)\}$*

If $A_\omega \in L^\infty(\Omega,\mathbb{P},\mathcal{L})$ $(\mathcal{L} = \mathcal{L}(\mathfrak{H},\mathfrak{H}))$, then there is an unique decomposable operator such that $\forall\varphi \in \tilde{\mathcal{H}}$ one has $(\ \underline{A}\varphi)(\omega)) = A(\omega)\varphi(\omega)$. we note

$$\|A\|_\infty = \ ess - \sup_\Omega \|A(\omega)\|. \tag{3.1}$$

We call diagonal algebra, the algebra of multiplication operators, $T_f := \int_\Omega^\oplus f(\omega)\mathrm{d}\mathbb{P}$ with f a scalar-valued functions on Ω. It is usually denoted $\mathcal{A}$. We now define the first algebra of interest.

Definition 3.2. *An operator is called locally bounded (l.b) with respect to the map χ if the following holds*

$$\|\chi_x A_\omega\| < \infty \ \text{and}\ \|A_\omega \chi_x\| < \infty \ \text{for all}\ \omega \in \Omega,\ \text{and}\ x \in \mathcal{Z}$$

We then define

$$\mathcal{K}_{m,lb} = \{A \in \mathcal{L}(\tilde{\mathfrak{H}}) \cap \mathcal{A}' \ /A \ \text{is l.b}\ \}$$

and

$$\mathcal{K} = \{A \in \mathcal{L}(\tilde{\mathfrak{H}}) \cap \mathcal{A}' / \ \|A\|_\infty < \infty\}\ .$$

By construction, $\mathcal{K}$ is a von-Neumann algebra. We can now define the reference algebra which we use all over this work.

Definition 3.3. *If $A \in \mathcal{K}$ is covariant if and only if the following holds*

$$U_x A(\omega) U_x^* = A(\tau(x)\omega) \ \text{for all}\ x \in \mathcal{Z} \ \text{and}\ \omega \in \Omega\ , \tag{3.2}$$

where τ is ergodic group of measure preserving transformation defined previously.

We define a second ζ-projective unitary representation namely $\{U_x^\tau\}_{x\in\mathcal{Z}}$ on $\tilde{\mathfrak{H}}$ which we call derived representation and automorphism group defined as follows, U_x^τ acting on $\tilde{\mathfrak{H}}$ by

$$(U_x^\tau \varphi)(\omega) := U_x \varphi(\tau(-x)\omega) \text{ , for any } \varphi \in \tilde{\mathfrak{H}}$$

.

We refer to Ch3. for details.

Lemma 3.4. *We define the reference algebra as*

$$\mathcal{K}_\infty := \{A \in \mathcal{K}/A \text{ is covariant}\} \quad (3.3)$$

$$= \mathcal{K} \cap \{U_x^\tau\}'_{x\in\mathcal{Z}} \quad (3.4)$$

We then have that $\mathcal{K}_\infty$ is a $\mathcal{W}^\star$-algebra.

Proof. The proof is immediate and follows to the the fact that this algebra is the intersection of a von-Neumann algebra and the commutant of the unitary representation. For more details in this structure we refer to section 4 of the Ch3. □

4 Non-commutative L^p-spaces for random covariant operators

We begin with a short review of the theory of non-commutative L^p-space based on [G],[Te] and [T]. For the sake of completeness we recall some basic definitions and properties we shall need to implement the linear response theory. We denote as in Ch3. $\mathfrak{H}_0 := \chi_{\mathrm{e}}\mathfrak{H}$.

4.1 Trace on $\mathcal{K}_\infty$

The non-commutative L^p-space theory that we use is built on a von-Neumann algebra and a trace on it which must have some particular properties. We start to prove that the trace we consider has these properties.
Let the linear form defined by $\mathcal{T}(\cdot) \equiv \mathbb{E}\{tr(\chi_{\mathrm{e}} \cdot \chi_{\mathrm{e}})\}$. We have that $\mathcal{T}$ is naturally defined on $L^1(\Omega; \mathcal{L}_1(\mathfrak{H}_0))$. We refer to [Di] for the details on the ideals of definition of a trace on von-Neumann algebra. We recall that if the group is amenable, , by the ergodic theorem we have

$$\mathcal{T}(A) = \lim_{L\to\infty} \frac{1}{|\Lambda_L|} tr(\chi_{\Lambda_L} A_\omega \chi_{\Lambda_L}) \ \mathbb{P} - a.e$$

with Λ a Føelner net (see the Ch3.). When the Hilbert fibers are constant, we have

$$\mathcal{T}(|A|) = tr_{\tilde{\mathfrak{H}}}(\chi_{\mathrm{e}} \int_{\Omega}^{\oplus} |A_\omega| d\mathbb{P}(\omega) \chi_{\mathrm{e}}) = tr_{\tilde{\mathfrak{H}}}(\chi_{\mathrm{e}} |\underline{A}| \chi_{\mathrm{e}}) \text{ for any } A \text{ s.t } \mathcal{T}(|A|) < \infty \tag{4.1}$$

$$\mathcal{T}(AA^*) = tr_{\tilde{\mathfrak{H}}}(\chi_{\mathrm{e}} \underline{AA^*} \chi_{\mathrm{e}}) \text{ for any } A \text{ s.t } \mathcal{T}(AA^*) < \infty \tag{4.2}$$

We shall omit to specify the Hilbert space to which the trace is associated when no confusion can occur.

Lemma 4.1. *We have that* $\mathcal{T}(AA^*) = \mathcal{T}(A^*A)$ *and then* $\mathcal{T}$ *is a trace.*

Proof. We have $\underline{A^*A} = \sum U_x \underline{A^*} \chi_{\mathrm{e}} \underline{A} U_x$ because

$$\begin{aligned}
\underline{A^*A} &= \sum_{x \in \mathcal{Z}} \int_{\Omega}^{\oplus} A^*(\omega) \chi_x A(\omega) d\mathbb{P}(\omega) \\
&= \sum_{x \in \mathcal{Z}} U_x \int_{\Omega}^{\oplus} A^*(\tau(x)^{-1}\omega) \chi_{\mathrm{e}} A(\tau(x)^{-1}\omega) d\mathbb{P}(\omega) U_x^* \\
&= \sum_{x \in \mathcal{Z}} U_x \int_{\Omega}^{\oplus} A^*(\omega) \chi_{\mathrm{e}} A(\omega) d\mathbb{P}(\omega) U_x^* \\
&= \sum_{x \in \mathcal{Z}} U_x \underline{A^*} \chi_{\mathrm{e}} \underline{A} U_x^*.
\end{aligned}$$

as a consequence $\chi_{\mathrm{e}} \underline{A^*A} \chi_{\mathrm{e}} = \sum_{x \in \mathcal{Z}} U_x \chi_{-x} \underline{A^* \chi_{\mathrm{e}} A} \chi_{-x} U_x^*$.

Thus,

$$\begin{aligned}
\mathrm{tr}(\chi_{\mathrm{e}} \underline{A^*A} \chi_{\mathrm{e}}) &= \sum_{x \in \mathcal{Z}} \mathrm{tr}(U_x \chi_{-x} \underline{A^* \chi_{0\mathrm{e}} A} \chi_{-x} U_x^*) \\
&= \sum_{x \in \mathcal{Z}} \mathrm{tr}(\chi_x \underline{A^* \chi_{\mathrm{e}} \chi_{\mathrm{e}} A} \chi_x) \\
&= \sum_{x \in \mathcal{Z}} \mathrm{tr}\{(\chi_{\mathrm{e}} \underline{A} \chi_x)^* (\chi_{\mathrm{e}} \underline{A} \chi_x)\} \\
&= \sum_{x \in \mathcal{Z}} \mathrm{tr}\{(\chi_{\mathrm{e}} \underline{A} \chi_x)(\chi_{\mathrm{e}} \underline{A} \chi_x)^*\} \\
&= \mathrm{tr}(\chi_{\mathrm{e}} \underline{AA^*} \chi_{\mathrm{e}})
\end{aligned}$$

□

For the next result we will use of the following lemma.

Lemma 4.2. *Let $A \in \mathcal{K}_\infty$, then $A^\circ_\omega := A\chi_e \in L^\infty(\Omega, \mathbb{P}, \mathcal{B}(\mathfrak{H}_0, \mathfrak{H}))$ determines in unique way A.*

Proof. We have the following equality

$$A_\omega = \sum_{\mathcal{Z}} A_\omega \chi_x = \sum_{\mathcal{Z}} A_\omega \alpha_x(\chi_e) \tag{4.3}$$

$$= \sum_{\mathcal{Z}} U_x \alpha_x^{-1}(A_\omega)\chi_e U_x^* \tag{4.4}$$

$$= \sum_{\mathcal{Z}} U_x A^\circ_{\tau(-x)_\omega} \chi_e U_x^* \tag{4.5}$$

Thus we have that if $A\chi_e = 0$ implies that $A = 0$. □

To construct an integration on the von-Neumann algebra $\mathcal{K}_\infty$, it is necessary that the underlying trace enjoys of the following properties.

Theorem 4.3. *$\mathcal{T}$ is faithful, normal and semi-finite.*

Proof. Faithfulness: Let $A \in \mathcal{K}_\infty$ such that $\mathcal{T}(A) < \infty$

$$\mathcal{T}(A) = \int \|A(\omega)^{\frac{1}{2}} \chi_{\mathrm{e}}\|_2^2 d\mathbb{P}(\omega) \tag{4.6}$$

the result follows by the Lemma 4.2 .
Normality: We must suppose that $\{A_\alpha\}$ is a net of $\mathcal{K}_\infty$ such that $\sup_\alpha A_\alpha = A$ exists in $\mathcal{K}_\infty$. We then have, for (u_n) an orthonormal basis of $\mathfrak{H}_0$,

$$\mathcal{T}(A_\alpha) = \mathbb{E} \sum \langle u_n, |A_\alpha| u_n \rangle \tag{4.7}$$

but $S \to \langle u_n, |S| u_n \rangle$ is a normal form and therefore also

$$\phi : S \to \sum \langle u_n, |S| u_n \rangle \tag{4.8}$$

thus

$$\mathcal{T}(A_\alpha) = \int_\Omega \phi(A_\alpha(\omega)) d\mathbb{P}(\omega) = \langle \mathbf{1}, \phi \circ A_\alpha \cdot \mathbf{1} \rangle_{L^2(\Omega, \mathbb{P})} \tag{4.9}$$

and $\phi \circ A_\alpha \in L^\infty(\Omega, \mathbb{P})$ since $L^\infty(\Omega, \mathbb{P})$ is a $\mathcal{W}^*$algebra, we have that $\sup_\alpha \varphi \circ A_\alpha$ exists in $L^\infty(\Omega, \mathbb{P})$ and is equal to $\varphi \circ A \in L^\infty(\Omega, \mathbb{P})$ by the normality of ϕ. To finish, as $\langle 1| \cdot 1\rangle_{L^2(\Omega, \mathbb{P})}$ is normal, we obtain that

$$\sup_\alpha \mathcal{T}(A_\alpha) = \mathcal{T}(\sup_\alpha A_\alpha) = \mathcal{T}(A) \tag{4.10}$$

Semi-finiteness: Let a self-adjoint operator $B \in \mathcal{K}_\infty$ and such that $\mathcal{T}(|B|) < \infty$ we have that $\chi_{[-n,n]}(B) \in \mathcal{K}_\infty$ and $\mathcal{T}(\chi_{[-n,n]}(B)) < \infty$. As we have that

$$\chi_{[-n,n]}(B) \xrightarrow[n\to\infty]{s} id$$

then $\mathcal{T}$ is semi-finite. □

By this theorem we can construct non-commutative integration spaces.

4.2 $\mathcal{T}$-measure topology and Non-commutative L^p-space on $\mathcal{K}_\infty$

We introduce the measure topology with respect to the trace $\mathcal{T}$. The $\mathcal{T}$-measure topology is defined by the following neighborhood system:

$$N(\epsilon,\delta) \equiv \{A \in \mathcal{K}_\infty / \exists P \in \mathcal{P}(\mathcal{K}_\infty), \|AP\|_\infty < \epsilon \ , \mathcal{T}(P^\perp) < \delta\} \ . \qquad (4.11)$$

Where we note $\mathcal{P}(\mathcal{K}_\infty)$ as being the set of projectors of $\mathcal{K}_\infty$ which is a complete lattice. This definition arises naturally in the work of Murray and von-Neumann when they defined the generalization of the singular values called in the literature (see [FK]) the generalized s-number. To give an analogy with the usual trace ideals the usual singular values are defined by

$$\mu_n(A) = \inf\{\|AP_{\mathfrak{K}}\|, \mathfrak{K} \text{ is a closed subspace with } dim\mathfrak{K}^\perp < n\}$$

while the generalized-s numbers are defined by

$$\mu_t(A) = \inf\{\|AP\|; P \text{ is a projection in } \mathcal{K}_\infty \text{ with } \mathcal{T}(P^\perp) < t\}$$

for any $t > 0$.

Remark 3. *One has that*

$$A \in N(\epsilon,\delta) \Leftrightarrow \mathcal{T}(\chi_{]\epsilon,\infty[}(|A|)) \leq \delta \ . \qquad (4.12)$$

The last quantity is called the distribution function of A. As for the usual Shatten ideal we have a "minimax" characterization of the genralized s-number which allows to recover the usual inequality for the singular values. It is in this way that [FK] proved inequalities that we use as monotone congerence theorem, Fatou's lemma or Clarkson-McCarthy inequality. This "minimax" representation reads as follows for A a $\mathcal{T}$-measurable operator

$$\mu_t(A) = \inf_{P\in\mathcal{P}(\mathcal{K}_\infty) \ s.t \ \mathcal{T}(P^\perp)\leq t} \{ \sup_{\varphi\in P\mathfrak{H}, \|\varphi\|=1} \|A\varphi\|\} \ .$$

To give a concrete example if we consider the commutative algebra $L^\infty(X)$ with X a locally compact space with a measure m, then we have for any measurable function f that

$$\mu_t(f) = \inf\{s \geq 0 \ m(x \in X \ s.t \, |f(x)| > s) \leq t\}$$

In particular, the $\mathcal{T}$-measure topology can be generated by the following Frechet-norm [G]

$$\forall A \in \mathcal{K}_\infty \ , \ \parallel A \parallel_\tau = \inf_{P \in \mathcal{P}(\mathcal{K}_\infty)} \max\{|||AP|||_\infty , \mathcal{T}(P^\perp)\}$$

.

Definition 4.4. *Let $L^o(\mathcal{K}_\infty)$ the completion of $\mathcal{K}_\infty$ with respect to the $\mathcal{T}$-measure topology. The operators belonging to $L^o(\mathcal{K}_\infty)$ are called $\mathcal{T}$-measurable operators.*

We then have the following crucial proposition.

Proposition 4.5 ([Te])**.** *$L^o(\mathcal{K}_\infty)$ is a Hausdorff topological $*$-algebra in which $\mathcal{K}_\infty$ is dense.*

We can also define a $\mathcal{T}$-measure topology on $\mathfrak{H}$ by the same process. Set the following neighborhood system:

$$O(\epsilon, \delta) = \{\varphi \in \mathfrak{H} / \exists P \in \mathcal{P}(\mathcal{K}_\infty) \ \|P\varphi\| < \epsilon, \ \mathcal{T}(P^\perp) < \delta\} \tag{4.13}$$

and the associated map : $\mathfrak{H} \to [0, \infty[$ defined by

$$\parallel \varphi \parallel_\tau = \inf_{P \in \mathcal{P}(\mathcal{K}_\infty)} \max\{\|P\varphi\|, \mathcal{T}(P^\perp)\}$$

and consequently we have the following result which ensures the compatibility between this topology and the algebra structure.

Proposition 4.6. *[Te] There is a unique continous bilinear map $(A, u) \to Au$, maping $L^o(\mathcal{K}_\infty) \times L^o(\mathfrak{H})$ into $L^o(\mathfrak{H})$ which extends the natural map $\mathcal{K}_\infty \times \mathfrak{H}$ into $\mathfrak{H}$ and the corresponding representation of $L^o(\mathcal{K}_\infty)$ on $L^o(\mathfrak{H})$ is faithful.*

Remark 4. *Set A a densely defined closed self-adjoint operator and affiliated to $\mathcal{K}_\infty$ (we write $H\eta\mathcal{K}_\infty$) with A=$U|A|$, $|A| = \int_0^\infty \lambda d(e(\lambda))$ then we have A is $\mathcal{T}$-measurable is equivalent to $\lim_{\lambda\to\infty} \mathcal{T}(e(\lambda)^\perp) = 0$ or $\mathcal{T}(e(\lambda)) < \infty$ for λ arbitrary large.*

It is now possible to define natural and well-defined integration spaces on the von-Neumann algebra of reference $\mathcal{K}_\infty$. This is done by the following theorem.

Theorem 4.7. *For* $1 \leq p < \infty$ *and* $x \in L^o(\mathcal{K}_\infty)$*, set* $\|x\|_p \equiv \mathcal{T}(|x|^p)^{\frac{1}{p}} \in [0, +\infty]$*, define*

$$L^p(\mathcal{K}_\infty) \equiv \{x \in L^o(\mathcal{K}_\infty) / \|x\|_p < \infty\} \ . \tag{4.14}$$

$L^p(\mathcal{K}_\infty)$ is a Banach space in which $L^p_o(\mathcal{K}_\infty) := L^p(\mathcal{K}_\infty) \cap \mathcal{K}_\infty$ is dense. Furthermore a key property for the following is that $L^p(\mathcal{K}_\infty)$ is a $\mathcal{K}_\infty$-bimodule and that:

$$\|AB\|_p \leq \|A\|_\infty \|B\|_p \tag{4.15}$$

$$\|BA\|_p \leq \|B\|_p \|A\|_\infty \tag{4.16}$$

for any $A \in \mathcal{K}_\infty$ and $B \in L^p(\mathcal{K}_\infty)$. If $p^{-1} + q^{-1} = 1$, $(p \geq 1)$, then the product of $L^p(\mathcal{K}_\infty)$ and $L^q(\mathcal{K}_\infty)$ is on $L^1(\mathcal{K}_\infty)$ and obviously the Hölder inequality holds in the sense that:

$$|\mathcal{T}(AB)| \leq \|A\|_p \|B\|_q \ , \tag{4.17}$$

for $A \in L^p(\mathcal{K}_\infty)$ and $B \in L^q(\mathcal{K}_\infty)$.

Furthermore, we have that $L^p(\mathcal{K}_\infty)$ and $L^q(\mathcal{K}_\infty)$ are conjugate to each other. More generally we have $\prod_{p_i} L^{p_i}(\mathcal{K}_\infty) \subset L^r(\mathcal{K}_\infty)$ with $r^{-1} = \sum_i p_i^{-1}$.

Remark 5.

1. *We then have that $\{L^2(\mathcal{K}_\infty), \langle ., . \rangle_{L^2(\mathcal{K}_\infty)}\}$ with $\langle A, B \rangle_{L^2(\mathcal{K}_\infty)} = \mathcal{T}(A^*B)$ is an Hilbert space.*
2. *we note that we have also the following inequality that we shall use,*

$$\|A\|_p \leq \|A\|_1^{\frac{1}{p}} \|A\|_\infty^{\frac{1}{q}} \text{ for any } A \in L^1(\mathcal{K}_\infty) \tag{4.18}$$

coming from that $|A|^p = |A| \cdot |A|^{p-1} \leq |A| \, \|A\|_\infty$.

We note the important fact that unlike to the discrete case the Schrödinger operator, which we consider, is not a priori $\mathcal{T} - measurable$. This one of the main difficulties in our situation which forces us to go far away in the technics of the non-commutative integration theory.

4.3 Some properties of the $L^p(\mathcal{K}_\infty)$-spaces

We now give some properties of the underlying spaces. We recover some properties of the commutative L^p-spaces. We focus our attention, as it was done everywhere else, on the features needed for the computation of the linear response theory. The first result has been proved by Fack and Kosaky [FK]

Lemma 4.8. *[FK] With the above notations we have that*

1. *(Clarkson-McCarthy inequality) For all $p \in [2,\infty)$ with A and $B \in L^p(\mathcal{K}_\infty)$, we have that*

$$|||A+B|||_p^p + |||A-B|||_p^p \leq 2^{p-1}\{|||A|||_p^p + |||B|||_p^p\} \ . \tag{4.19}$$

2. *(monotone convergence theorem) if $A_n \leq A$(or simply if $\mu_t(A_n) \leq \mu_t(A)$) and A_n converges to A in $\mathcal{T}$-measure, then $\mathcal{T}(A) = \lim_{n\to\infty} \mathcal{T}(A_n)$.*

3. *(Fatou's lemma) $\mathcal{T}(A) \leq \lim \inf_{n\to\infty} \mathcal{T}(A_n)$.*

Remark 6. *We make the link between two Hilbert spaces, namely $L^2(\mathcal{K}_\infty)$ and $L^2(\Omega) \otimes \mathfrak{S}_2$. Where the second is the tensorial product between $L^2(\Omega)$ and $\mathfrak{S}_2$ the ideal of Hilbert-Schmidt operators. We have*

$$\|\!|A|\!\|_2^2 = \int_\Omega tr(\chi_e A^*(\omega)A(\omega)\chi_e)d\mathbb{P}(\omega) \tag{4.20}$$

$$= \int_\Omega \|A\chi_e\|_2^2 d\mathbb{P}(\omega) \tag{4.21}$$

$$= \sum_{x\in\mathcal{Z}} \int_\Omega \|\chi_x A\chi_e\|_2^2 d\mathbb{P}(\omega) \tag{4.22}$$

with $\|A\|_2$ the usual Hilbert-Schmidt norm, since $tr(\chi_x \underline{AA^*}\chi_x) = tr(\chi_e \underline{AA^*}\chi_e)$. We have in particular $\|\!|A_\omega|\!\|_2^2 = \sum_{y\in\mathcal{Z}} \mathbb{E}\left(\|A_{\omega,y}\|_2^2\right)$, where we have note $A_{w,y}$ the kernel obtained in the diagonalization as explained developped in Ch3. We get an isometry between $L^2(\mathcal{K}_\infty)$ and $L^2(\Omega\times\mathcal{Z};\mathfrak{S}_2(\mathfrak{H}_0))$. These operators are locally Hilbert-Schmidt operators in the sense that if $\mathcal{H} = L^2(\mathbb{R}^d)$ and χ_x is a characteristic function of a box centered on x and $K \subset \mathbb{R}^d$ a compact, then $\|\!|A|\!\|_2^2 < \infty$ implies that $A(\omega)\chi_K$ and $\chi_K A^*(\omega)$ are Hilbert-Schmidt operators.

Lemma 4.9. i) *Let $p, q \in [1,\infty)$ such that $p^{-1} + q^{-1} = 1$, then for $A \in L^p(\mathcal{K}_\infty)$ $B \in L^q(\mathcal{K}_\infty)$ we have that*

$$\mathcal{T}(AB) = \mathcal{T}(BA)$$

ii) *For any $A \in \mathrm{L}^p(\mathcal{K}_\infty), B \in \mathrm{L}^q(\mathcal{K}_\infty)$ and $C_\omega \in \mathcal{K}_\infty$, we have*

$$\mathcal{T}\{[C,A]B\} = \mathcal{T}\{C[A,B]\}. \tag{4.23}$$

iii) *For any $A, B \in \mathcal{K}_\infty$ and $C \in \mathrm{L}^1(\mathcal{K}_\infty)$, we have*

$$\mathcal{T}\{A[B,C]\} = \mathcal{T}\{[A,B]C\}. \tag{4.24}$$

Lemma 4.10. *Let $A \in L^p_o(\mathcal{K}_\infty)$, there exists $B \in L^q_o(\mathcal{K}_\infty)$ with $p \in [1,\infty)$ and $p^{-1} + q^{-1} = 1$, such that $\|A\|_p = \|AB\|_1$ and by adjonction the same thing is valid for right multiplication.*

Proof. We set for simplicity that $\|A\|_p = 1$. Let the polar decomposition $A = V|A|$, then we have that

$$|A^*|^p = V|A|^pV^* = V|A| \cdot |A|^{p-1}V^* = A|A|^{p-1}V^*$$

Therefore, if we set $B = |A|^{p-1}V^*$, and so $AB = |A^*|^p \in L^1_o(\mathcal{K}_\infty)$ and we have the desired property,

$$\mathcal{T}(AB) = \mathcal{T}(|A^*|^p) = \|A\|_p = 1 \tag{4.25}$$

On the other hand,

$$B^2 = BB^* = V|A|^{2(p-1)}V^* = |A^*|^{2(p-1)}$$

Hence $|B| = |A^*|^{(p-1)}$, as $q(p-1) = p$. We obtain that $|B|^q = |A^*|^p$, which implies the result as $\|B\|_q = 1$. □

As by product of this lemma we obtain a very useful corollary.

Corollary 4.11. *Set $A \in L^p_o(\mathcal{K}_\infty)$, with $p \in [1;\infty)$ and if we have for all $B \in L^q_0(\mathcal{K}_\infty)$, with q conjugate to p, that $\mathcal{T}(AB) = 0$, then $A = 0$ $\mathbb{P}$ a.e.*

This can be viewed more directly by the well-know identity (see [T])

$$\|A\|_p = \sup\{|\mathcal{T}(AB)|, B \in \mathfrak{m}_\mathcal{T}\ \|B\|_q \leq 1\} \tag{4.26}$$

where $\mathfrak{m}_\mathcal{T}$ is the *definition ideal* of $\mathcal{T}$ defined by the another subset

$$\mathfrak{n}_\mathcal{T} := \{A \in \mathcal{K}_\infty\ |\mathcal{T}(AA^*) < \infty\}$$

then

$$\mathfrak{m}_\mathcal{T} := \{AB\ |A, B \in \mathfrak{n}_\mathcal{T}\}\ .$$

We state a last technical result which we will use.

Lemma 4.12. *Let $A_n \in \mathcal{K}_\infty$ for all $n \in \mathbb{N}$ which converges strongly to $A \in \mathcal{K}_\infty$ and such that $\sup_n \| A_n \|_\infty < +\infty$. Then for any $p \geq 1$ and $B \in L^p(\mathcal{K}_\infty)$ A_nB converges to AB in $L^p(\mathcal{K}_\infty)$.*

Proof. We can without loss of generality assume that $A = 0$. We begin with $p = 2$. We have the equality

$$\| B \|_2^2 = \int_\Omega \|B_\omega \chi_0\|_2^2 d\mathbb{P}(\omega)$$

We use the fact that covariant operators are uniquely determined by their restriction on $\mathfrak{H}_0$ (Lemma 4.2). Let $B \in L_o^2(\mathcal{K}_\infty)$ and $\{\varphi_k\}_{k\in\mathbb{N}}$ a basis of $\mathfrak{H}_0$ we have

$$\mathbb{E}tr(\chi_0 B_\omega^* A_{\omega,n}^* A_{\omega,n} B_\omega \chi_0) = \mathbb{E}\sum_{k\in\mathbb{N}} \langle A_{\omega,n} B_\omega \varphi_k, A_{\omega,n} B_\omega \varphi_k\rangle_{\mathfrak{H}_0} \tag{4.27}$$

$$= \mathbb{E}\sum_{n\in\mathbb{N}} \|A_{\omega,n} B_\omega \varphi_k\|_{\mathfrak{H}_0}^2 \to 0 \tag{4.28}$$

where we use the dominated convergence theorem. Hence by the density of $L_o^2(\mathcal{K}_\infty)$ in $L^2(\mathcal{K}_\infty)$ we get the result for $p = 2$. The case $p = 1$ follows immediately by the fact that we can write an operator of $L^1(\mathcal{K}_\infty)$ as a product of two operators belonging to $L^2(\mathcal{K}_\infty)$ by the polar decomposition if $A \in L^1(\mathcal{K}_\infty)$ we can write $B = V|B|^{\frac{1}{2}}|B|^{\frac{1}{2}}$.
For $p > 2$ and $B \in \mathrm{L}_o^p(\mathcal{K}_\infty)$ we have that

$$\mathcal{T}((A_nB)^p) = \mathbb{E}\sum_{k\in\mathbb{N}} \langle \varphi_k, (B_\omega^* A_{\omega,n}^* A_{\omega,n} B_\omega)^{\frac{p}{2}} \varphi_k\rangle_{\mathfrak{H}_0} \tag{4.29}$$

$$= \mathbb{E}\sum_{k\in\mathbb{N}} \langle (B_\omega^* A_{\omega,n}^* A_{\omega,n} B_\omega)^{\frac{p}{2}-1} \varphi_k, B_\omega^* A_{\omega,n}^* A_{\omega,n} B_\omega \varphi_k\rangle_{\mathfrak{H}_0} \tag{4.30}$$

$$\leq C\mathbb{E}\sum_{k\in\mathbb{N}} \|A_{\omega,n} B_\omega \varphi_k\|_{\mathfrak{H}_0} \xrightarrow[n\to\infty]{} 0 \tag{4.31}$$

Now if we take $B \in \mathrm{L}^p(\mathcal{K}_\infty)$ we have that there exists $B_k \in L_o^p(\mathcal{K}_\infty)$ for any $k \in \mathbb{N}$ such that $\lim_{n\to\infty}^{(p)} B_k = B$ and by $\| A_n \|_\infty < C$. In the case $p \leq 2$ we use the following inequality

$$\| BA_n \|_p \leq \left\| V|B|^{\frac{1}{2}} \right\|_{2p} \left\| |B|^{\frac{1}{2}} A_n \right\|_{2p}$$

this finishes the proof. ∎

5 Functional analysis on $\mathcal{K}_\infty$ and trace estimates

In this section we define a differential calculus and some generalized commutators adapted to the framework of the non-commutative L^p-spaces. We assume that H is a self-adjoint operator on $\mathfrak{H}$, $H\eta\mathcal{K}_\infty$, i.e H is affiliated to the von-Neumann algebra $\mathcal{K}_\infty$, and is semi-bounded from below by a finite constant γ and $\mathfrak{H} = L^2(X)$ with X is a d-dimensional real vector space while we set $\mathcal{Z}$ a discrete abelian finitely generated (then automatically amenable with the Haagerup property c.f Ch3.) locally compact group associated to a lattice of X acting on it such that $X/\mathcal{Z}$ is compact.

5.1 Differential on L^p-spaces

We now introduce derivation to construct momentum spaces, analogously to the periodic media. We recall that $\mathcal{K}_\infty$ is a C^*-algebra, on which we can use the notion of $*$-derivation (see [Di]). We begin to define it on dense subspace of $\mathcal{K}_\infty$ and afterwhat we will define momentum-spaces.

Derivation on $\mathcal{K}_\infty$

We introduce the step operator, a useful tool that appears in the sequel as well as in related works.
Set

$$\Xi_i := \sum_{\mathcal{Z}} a_i \chi_a \text{ acting on } \mathfrak{H}_c \tag{5.1}$$

$$\Xi = (\Xi_1, \cdots, \Xi_d) \tag{5.2}$$

This operator arises quite naturally into computations. It can be viewed as an approximation of the position operator with respect to the group-structure. As the position operator it is range preserving in the sense that $\Xi\chi_x = \chi_x \Xi \chi_x$ and so it is locally bounded.
We give some motivations for using the step operator, note that the group action is more transparent in this formalism in the sense that it acts on the kernel-operator following the action of the group. As we shall see important quantities such as the conductance tensor $\sigma_{ij}^{E_F}(H_\omega)$ or the mean of the velocity operator $[H_\omega, \mathbf{x}]$ are invariant if we use the step operator Ξ in place of the usual position operator $\mathbf{x}$.

Care must be taken to the fact that, as for the position operator, the step operator is not covariant. Indeed we have for all $y \in \mathcal{Z}$ that

$$\alpha_y(\mathbf{x}) = \mathbf{x} - y \tag{5.3}$$

$$\alpha_y(\Xi) = \sum_{\mathcal{Z}} z\chi_{z+y} = \sum_{\mathcal{Z}} (z-y)\chi_z = \Xi - y \tag{5.4}$$

But this is not a problem since we shall only consider commutators with respect to these operators.

Definition 5.1. *Let*

$$\mathcal{K}_\infty^{\mathcal{F}} := \{A \in \mathcal{K}_\infty \mid \exists F \subset \mathcal{Z} \textit{finite s.t } \forall x, y \in \mathcal{Z}, x - y \notin F \Rightarrow \chi_x A \chi_y = 0\}$$

We recall that the $*$-algebra $\mathcal{K}_\infty^{\mathcal{F}}$ of finite range operator is a $*$-algebra and is a dense $*$-subalgebra of $\mathcal{K}_\infty$ by thm 5.8.

Definition 5.2. *for any $j \in \{1 \cdots d\}$ we define the $*$-maps that we note ∂_j and ∂_j^Ξ as the maps defined on $\mathcal{K}_\infty^{\mathcal{F}}$ by:*

$$\partial_j : A \in \mathcal{K}_\infty^{\mathcal{F}} \to i[\mathbf{x}_j, A] \in \mathcal{K}_\infty^{\mathcal{F}} \tag{5.5}$$

$$\partial_j^\Xi : A \in \mathcal{K}_\infty^{\mathcal{F}} \to i[\Xi_j, A] \in \mathcal{K}_\infty^{\mathcal{F}} \tag{5.6}$$

we further set $\nabla := (\partial_1 \cdots \partial_d)$, $\nabla^\Xi := (\partial_1^\Xi \cdots \partial_d^\Xi)$.

Remark 7. *We note that we have for $A \in \mathcal{K}_\infty^{\mathcal{F}}$ that*

$$\partial^\Xi(A) = i \sum_{x,y \in \mathcal{Z}} (x-y)\chi_x A \chi_y$$

This is reminiscent of the integral-kernel for the commutator with the usual position operator. More generally we must extend ∂_j on the dense $*$-algebra of rapid-decaying operators $H_L^\infty(\mathcal{K}_\infty)$ for group having sub-polynomial H-growth and on $\mathcal{E}(\mathcal{K}_\infty)$ for group having sub-exponentially H-growth, (see Ch3. section 5 for details).

These maps can be viewed as a differential form on the von-Neumann algebra $\mathcal{K}_\infty$ and it is a main object for the computation of the Kubo-Strěda formula. We recall that a $*$-derivation is a linear map.

Proposition 5.3. *∂_j^Ξ acts as $*$-covariant-derivation on a $*$-dense subalgebra of $\mathcal{K}_\infty$ and by natural extension the same is true for ∂_j.*

Remark 8. *We note that $\partial_i = \partial^\Xi + \delta_i$, where δ_i is the bounded derivation given by the commutator with respect to $\sum_{a \in \mathcal{Z}} (\mathbf{x} - a)\chi_a$.*

Proof. This is a simple consequence of the following facts.

With the above notation we have that

i) $\nabla^{\Xi}(AB) = \nabla^{\Xi}(A)B + A\nabla^{\Xi}(B)$

ii) $\nabla^{\Xi}(A + \lambda B) = \nabla^{\Xi}(A) + \lambda\nabla^{\Xi}(B)$

iii) $\nabla^{\Xi}(A^*) = \nabla^{\Xi}(A)^*$

iv) we have $[\alpha_a, \nabla^{\Xi}] = 0$ for all $a \in \mathcal{Z}$ and for all $A \in \mathcal{K}_\infty^{\mathcal{F}}$ in the sense that $\alpha_a \circ \nabla^{\Xi}(A) = \nabla^{\Xi} \circ \alpha_a(A)$

and the same is true for ∇. The proof is essentially trivial. For the stability of the covariance under the derivation we have for all $a \in \mathcal{Z}$ and for all $A \in \mathcal{K}_\infty^{\mathcal{F}}$

$$\begin{aligned} \alpha_z \circ \nabla_j^{\Xi}(A) &= iU_z \sum_{x,y\in\mathcal{Z}} (x-y)\chi_x A \chi_y U_z^* \\ &= i \sum_{x,y\in\mathcal{Z}} (x-y)\chi_{x+z} A \circ (\tau(z))\chi_{y+z} \\ &= \nabla_j^{\Xi} \circ \alpha_z(A) \end{aligned} \tag{5.7}$$

Moreover the $*$-covariant derivation ∇^{Ξ} is well defined on $\mathcal{K}_\infty^{\mathcal{F}}$ and its range belongs also to $\mathcal{K}_\infty^{\mathcal{F}}$. This is obvious by $\Xi A\chi_x = \chi_{x+\mathcal{F}}\Xi\chi_{x+\mathcal{F}}A\chi_x$ and $\nabla^{\Xi}(A) = \sum_{x\in F;y\in\mathcal{Z}} x\chi_{y+x}A\chi_y$. □

5.2 Weak differential calculus

We now define the non-commutative momentum space on the integration space on which we will working. This is reminiscent of the Sobolev space in the commutative case.

Definition 5.4. *We define the functional spaces*

1. $\mathcal{C}^k(\mathcal{K}_\infty) \equiv \{A \in \mathcal{K}_\infty / \partial_i^{\Xi^k}(A)) \in \mathcal{K}_\infty$ *for any* $i = 1 \cdots d\}$ *;*

2. $\mathcal{W}^{1,p}(\mathcal{K}_\infty) \equiv \{A \in L^p(\mathcal{K}_\infty)/\partial^{\Xi}(A) \in L^p(\mathcal{K}_\infty)\}$ *;*

3. $\mathcal{E}_\kappa^p(\mathcal{K}_\infty) := \{A \in L^p(\mathcal{K}_\infty)/ \left\|\mathcal{K}(x,y)\right\|_p \leq \mathrm{e}^{-\kappa|x-y|}\}$ *with* $\kappa \in \mathbb{R}$.

Remark 9. *The first set is clearly linked to the* $C^k(\mathbf{x})$ *as defined in the theory of* C_0*-group (see [ABG]) which we will use to define some commutators. Obviously the space of fundamental interest is the Hilbert space* $\mathcal{W}^{1,2}(\mathcal{K}_\infty)$.

Such spaces are particulary useful for the computation of linear response theory, the conductivity tensor and the Kubo-Stř eda formula. We call the last set, the set of RD-operators for the reason that it is the generalization of the rapid decaying functions on group (see Ch3. and [BC, MM]). We note that $\mathcal{E}_\kappa(\mathcal{K}_\infty)$ is dense and contains the $*$-sub-algebra of finite range operators.

We give some first basic properties which justify this point of view.

Lemma 5.5. *For any $i, j = 1 \cdots d$, we have the following equalities*

i) $\mathcal{T}(\partial_i(A)) = \mathcal{T}(\partial_i^{\Xi}(A)) = 0$ *for any* $A \in \mathcal{W}^{1,1}(\mathcal{K}_\infty)$

ii) $\mathcal{T}(\partial_i((\partial_i - \partial_i^{\Xi})(A))) = 0$ *for any* $A \in \mathcal{W}^{1,2}(\mathcal{K}_\infty)$

iii) $\mathcal{T}(\partial_i^{p,\Xi}(A)) = \mathcal{T}(\partial_i^{\Xi^p}(A))$ *for* $A \in \mathcal{W}^{1,p}(\mathcal{K}_\infty)$

iv) $\mathcal{T}(P\,[\partial_i(P), \partial_j(P)]) = \mathcal{T}(P[\partial_i^{\Xi}(P), \partial_j^{\Xi}(P)])$ *for any* $P \in \mathcal{P}(\mathcal{K}_\infty) \cap \mathcal{W}^{1,2}(\mathcal{K}_\infty)$

Proof. We recall that we have that $\partial_i = \partial^{\Xi} + \delta_i$ and by the lemma 4.9 we have that

$$\mathcal{T}(\delta(A)) = 0 \text{ for any } A \in L^1(\mathcal{K}_\infty)\ . \tag{5.8}$$

Denoting by $\Theta_j := i(\mathbf{x}_j - \Xi_j)$, the generator of the derivation δ_i, i.e $\delta_i(A) := ad_{\Theta_i}(A) := [\Theta_i, A]$. By the fact that Θ_i is locally bounded, it is bounded on $\mathfrak{H}_0$. Hence, for any $A \in \mathrm{L}^1(\mathcal{K}_\infty)$, $\mathcal{T}(\delta(A)) = \mathbb{E}\mathrm{tr}_{\mathfrak{H}_0}([\Theta_i, A_\omega]) = 0$. Secondly, we have by definition that for any $A \in \mathcal{W}^{1,1}(\mathcal{K}_\infty)$ that $\partial^{\Xi}(A) = \sum_{x,y \in \mathcal{Z}} (x-y)\chi_x A \chi_y$ and then $\mathcal{T}(\partial_i^{\Xi}(A)) = 0$.

The item ii) is immediate by i) and the fact that ∂_i^{Ξ} and δ_j commutes under the trace.

Using the Leibniz rule for any $A \in \mathcal{W}^{1,1}(\mathcal{K}_\infty)$, we get

$$\mathcal{T}(\partial_i(\delta_j(A)) = \mathcal{T}([\partial_i(\Theta_j), A]) + \mathcal{T}([\Theta_j, \partial_i(A)])$$

but by definition we have that $\partial_i(\Theta_j) = 0$ on $\mathfrak{H}_c$ and consequently that

$$\mathcal{T}((\delta_j(\partial_i(A))) = \mathcal{T}(\partial_i(\delta_j(A)))$$

iii) follows by the orthogonality of the decomposition of the identity, we have that $[\Xi_i, [\Xi_i^p, A]] = [\Xi^{p+1}, A]$.
Rest the item iv) which is of importance for the Linear response theory and few other transport problems. We can see this map as a two-points correlation function or physically as the *current-current corellation function.*

We first have that for $P \in \mathcal{W}^{1,p}(\mathcal{K}_\infty) \cap \mathcal{W}^{1,q}(\mathcal{K}_\infty)$ and $1 = \frac{1}{p} + \frac{1}{q}$, we have the well-definitness in the sense that there exits a postive constant C such that

$$|\mathcal{T}(P[\partial_i(P), \partial_j(P)])| \leq |||P[\partial_i(P), \partial_j(P)])|||_1 \leq C\, |||P|||_\infty\, |||\partial_i(P)|||_p\, |||\partial_j(P)|||_q \ .$$

Note that the assumption $P \in \mathcal{W}^{1,2}(\mathcal{K}_\infty)$ is physically meaningful. If P is the Fermi-Dirac projection, this assumption is holds whenever the Fermi-energy belongs to the region of dynamical localization (see [GK2]).
We use of the following relation

$$P\partial_i(P)P = 0 \tag{5.9}$$

coming from

$$\partial_i(P) = \partial_i(P^2) = \partial_i(P)P + P\partial_i(P) \ .$$

Hence,

$$P^\perp \partial_i(P)P + P\partial_i(P)P^\perp = \partial_i(P) \tag{5.10}$$

Writing

$$\begin{aligned} &\mathcal{T}(P[\partial_i(P), \partial_j(P)]) - \mathcal{T}(P[\partial_i^\Xi(P), \partial_j^\Xi(P)]) &(5.11)\\ &\quad = -\mathcal{T}(P[\delta_i(P), \partial_j(P)]) + \mathcal{T}(P[\partial_i(P), \delta_j(P)]) \ , &(5.12)\end{aligned}$$

and finally we get

$$\begin{aligned} \mathcal{T}(P[\delta_i(P), \partial_j(P)]) &= \mathcal{T}(P[\delta_i(P), P^\perp\partial_j(P)P + P\partial_j(P)P^\perp]) \qquad (5.13)\\ &= \mathcal{T}(P\delta_i(P)(P^\perp\partial_j(P)P + P\partial_j(P)P^\perp)) - \mathcal{T}(P\partial_j(P)P^\perp\delta_i(P))\\ &= \mathcal{T}(P\Theta_i P\partial_j(P)P^\perp) - \mathcal{T}(P\Theta_i(P^\perp\partial_j(P)P + P\partial_j(P)P^\perp))\\ &\qquad\qquad - \mathcal{T}(P\partial_j(P)P^\perp\Theta_i P)\\ &= -\left\{\mathcal{T}(P\Theta_i P^\perp\partial_j(P)P) + \mathcal{T}(P\partial_j(P)P^\perp\Theta_i P)\right\} \quad (5.14)\\ &= -\mathcal{T}(\partial_i(P)\Theta_i) = 0 \ . \qquad (5.15)\end{aligned}$$

In the equality (5.13) we use of the relation (5.10). To go from (5.14) to (5.15) we use the Lemma 4.9 and that Θ is locally bounded as follows

$$\begin{aligned} \mathcal{T}(P\Theta_i P^\perp\partial_j(P)P) &= \sum_{x\in\mathcal{Z}} \mathcal{T}(P\Theta_i\chi_x P^\perp\partial_j(P)P) &(5.16)\\ &= \sum_{x\in\mathcal{Z}} \mathcal{T}(\chi_x P^\perp\partial_j(P)P\Theta_i\chi_x) &(5.17)\\ &= \sum_{x\in\mathcal{Z}} \mathcal{T}(\chi_x P^\perp\partial_j(P)P\Theta_i\chi_x) \ . &(5.18)\end{aligned}$$

(5.19)

Using the covariance of the operator under the trace, that the transformation τ is measure-preserving and the centrality of the trace we get

$$\mathcal{T}(P\Theta_i P^{\perp}\partial_j(P)P) = \mathcal{T}(\Theta_i P^{\perp}\partial_j(P)P\Theta_i) \ .$$

In the same way we get

$$\mathcal{T}(P\partial_j(P)P^{\perp}\Theta_i P) = \mathcal{T}(P\partial_j(P)P^{\perp}\Theta_i) \ .$$

Finally the last equality (5.15) follows immediately by $\partial_i(\Theta_j) = 0$ on $\mathfrak{H}_c$, the Leibniz rule and the item i). the second term of the equation (5.12) is also equal to 0 by similar calculation and this finishes the proof.

□

5.3 Generalized L^p-commutators

By the previous considerations the following two generalized commutators are naturally well-defined.

i) Let $A, B \in L^0(\mathcal{K}_\infty)$ then $[A, B] \in L^0(\mathcal{K}_\infty)$

ii) Let $A \in L^p(\mathcal{K}_\infty)$ and $B \in L^q(\mathcal{K}_\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ then

$$[A, B] \in L^1(\mathcal{K}_\infty) \ .$$

Where we use the fact that $L^o(\mathcal{K}_\infty)$ is $*$-algebra while for the second it is a trivial consequence of the Hölder inequality. The main difficulty is when we want to define generalized commutators with operators which are only affiliated with $\mathcal{K}_\infty$ which turn the case with Schrödinger operzators.

Definition 5.6. *Let $A \in L^0(\mathcal{K}_\infty)$ and $H\eta\mathcal{K}_\infty$. We say that the commutator $[H, A]$ is defined and belongs to $L^p(\mathcal{K}_\infty)$, for some $1 \leq p \leq \infty$ if and only if there is a core $\mathfrak{D} \subset \mathcal{D}(H)$ of the operator H which is preserved by A i.e $A(\mathfrak{D}) \subset \mathcal{D}(H)$ and the operator $HA - AH$, initially defined on $\mathcal{D}(H)$, is closable, in which case the closure $\overline{HA - AH}$ belongs to $L^p(\mathcal{K}_\infty)$. In this case, the symbol $[H, A]$ stands for the closure of $HA - AH$.*

Remark 10. *In general, the commutator of two operators H and A is defined as quadratic form on $\mathcal{D}(H) \cap \mathcal{D}(A) \times \mathcal{D}(H) \cap \mathcal{D}(A)$ by*

$$Q(\varphi, \varphi) = \langle A\varphi, H\varphi\rangle - \langle H\varphi, A\varphi\rangle \ .$$

When $H\eta\mathcal{K}_\infty$ we use results on the C_0-group to define $[H, A]$.

When dealing with L^p-commutators it is natural to use the left-regular representation. We refer to the work [PS] for further precisions on this subject. We know that $L^2(\mathcal{K}_\infty)$ is an Hilbert space. We define the map $L : \mathcal{K}_\infty \mapsto \mathcal{B}(\mathrm{L}^2(\mathcal{K}_\infty))$ as follows, for any $A \in \mathcal{K}_\infty$ by $L_A(B) = AB$ for any $B \in L^2(\mathcal{K}_\infty)$. We note $\mathcal{K}_\infty^L$ the von-Neumann algebra acting on $L^2(\mathcal{K}_\infty)$ if we set $\mathcal{T}_L = \mathcal{T} \circ L^{-1}$, L becomes a trace preserving $*$-isomorphism. Then we identify the two algebras $\mathcal{K}_\infty^L$ and $\mathcal{K}_\infty$.
Then for $H\eta\mathcal{K}_\infty$ we define its (generalized) domain as

$$\mathfrak{D}(H) := \{A \in L^2(\mathcal{K}_\infty)/HA \in L^2(\mathcal{K}_\infty)\}$$

In the same way we can define the rigtht action on $L^2(\mathcal{K}_\infty)$ by the map $R_A(B) := B \cdot A$ for any $A \in \mathcal{K}_\infty$ and $B \in L^2(\mathcal{K}_\infty)$. In the same way we define the domain of *p-Liouvillian* by

$$\mathfrak{D}_H^p := \{A \in L^p(\mathcal{K}_\infty)/(L_H - R_H)(A) \in L^p(\mathcal{K}_\infty)\}$$

Therefore we define the *p-Liouvillian*, $\mathcal{L}_p$ by

$$\mathcal{L}_p(A) := [H, A] \text{ acting on } \mathfrak{D}_H^p$$

We can then define the associated quadratic forms for H_ω semi-bounded from below self-adjoint and affiliated to $\mathcal{K}_\infty$, denoting $Q(L_H) := \mathcal{D}(L_H)$ and $Q(R_H) := \mathcal{D}(R_H)$ their respectives form domains,

$$\mathbb{H}_L(A, B) := \langle A, L_H(B)\rangle_{L^2(\mathcal{K}_\infty)} \text{ for any } A_\omega, B_\omega \in Q(L_H) \tag{5.20}$$

$$\mathbb{H}_R(A, B) := \langle A, R_H(B)\rangle_{L^2(\mathcal{K}_\infty)} \text{ for any } A, B \in Q(R_H) \tag{5.21}$$

$$\mathbb{L}(A, B) := \mathbb{H}_L(A, B) - \mathbb{H}_R(A, B) \text{ for any } A, B \in Q := Q(L_H) \cap Q(R_H) \tag{5.22}$$

Proposition 5.7. *Let H be a self-adjoint operator semi-bounded from below by a constant γ such that $H\eta\mathcal{K}_\infty$. Then $\mathfrak{D}_H^p$ is a dense subset of $L^p(\mathcal{K}_\infty)$.*

Proof of proposition 5.7. We have by definition that $H\eta\mathcal{K}_\infty$ implies that for any bounded borel function g we have that $g(H) \in \mathcal{K}_\infty$. Hence if we take a spectral resolution of H, namely $f_n := \chi_{[\gamma,\, n]}$ for $n \in \mathbb{N}$, we have that $H\chi_{[\gamma,\, n]}(H) \in \mathcal{K}_\infty$ and $\chi_{[\gamma,\, n]}(H) \xrightarrow[n\to\infty]{s} id$. But we can not use the dominated convergence theorem for L^p-space because we need to an approximation which converges in $\mathcal{T}$-measure, then we need to use the lemma 4.12. Let $A \in \mathrm{L}^p(\mathcal{K}_\infty)$ and $A_n := f_n(H)Af_n(H) \in \mathfrak{D}_p(H)$, we have that $A_n \leq A$ and $A - A_t = (1 - f_t(H))Af_t(H) + f_t(H)A(1 - f_t(H))$ and by the lemma 4.12 we get that $\lim\limits_{n\to\infty}^{(p)} A - A_n = 0$ where we note $\lim^{(p)}$ the limit in $L^p(\mathcal{K}_\infty)$. □

Remark 11. *Considering the left-regular representation as defined above. Let* $A \in \mathcal{K}_\infty^L$ *such that* A^* *is domain-preserving for* H*, then in this case we have the following equality, non-immediate in other frameworks (as by example in [BoGKS])*
We have that $\langle AHx, y\rangle_{L^2(\mathcal{K}_\infty)} = \langle x, HA^*y\rangle_{L^2(\mathcal{K}_\infty)}$ *with* $x, y \in \mathfrak{D}(H)$ *where the symbol stands for the closure of the underlying operators. Furthermore we have that* $\mathfrak{D}(L_H)$ *is dense in* $L^2(\mathcal{K}_\infty)$ *using the same method than in the proposition 5.7.*

We then naturally define

$$|||A|||_{\mathfrak{D}_H^2} = |||HA|||_2 + |||HA^*|||_2 + |||A|||_2$$

with the obvious generalization to any $1 \leq p \leq \infty$. We can already state a simple lemma that we shall use intensively.

Lemma 5.8. *Let* $p, q \geq 1$ *be such that* $p^{-1} + q^{-1} = 1$*. For any* $A \in \mathcal{D}_H^p$*, resp.* $B \in \mathcal{D}_H^q$*, we have*

$$\mathcal{T}\left\{\mathcal{L}_p(A)B\right\} = -\mathcal{T}\left\{A\mathcal{L}_q(B)\right\}. \tag{5.23}$$

The strategy to get a suitable definition for the commutators we shall need in order to prove the linear response theory, namely $\nabla(f(H))$, $\nabla(H)$ and $[H, \nabla(f(H))]$ is the following. We first recall the Helffer-Söjstrand functional calculus. Adding to the Combes-Thomas property, we can derive general estimates for $\nabla(f(H))$. The second important point is the use of a trace estimate for the resolvent of the magnetic Hamiltonian coming from the well-know trace estimate of the Laplacian with scalar potential coupled with the diamagnetic inequality. Using these two estimates we can get general estimates for functions of the random magnetic Hamiltonian with respect to the L^p-norm. Using all these results, we achieve with the fundamental one that is $[H, \nabla(f(H))]$ by the fact that is the generator of the evolution operators applied to the solution of the Cauchy problem associated to the dynamics.

6 Magnetic Schrödinger operator and general estimates

6.1 The setting: magnetic Schrödinger operator

Magnetic Schrödinger operator

In this section we describe our background operators and recall from [BoGKS] the main properties we shall need in order to establish the Kubo formula, but within the framework of non-commutative integration when relevant (i.e. in Subsection 6.3). Throughout this chapter we shall consider Schrödinger operators of general form

$$H = H(\mathbf{A}, V) = (-i\nabla - \mathbf{A})^2 + V \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d), \tag{6.1}$$

where the magnetic potential $\mathbf{A}$ and the electric potential V satisfy the Leinfelder-Simader conditions:

- $\mathbf{A}(x) \in \mathrm{L}^4_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ with $\nabla \cdot \mathbf{A}(x) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$.
- $V(x) = V_+(x) - V_-(x)$ with $V_\pm(x) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$, $V_\pm(x) \geq 0$, and $V_-(x)$ relatively bounded with respect to Δ with relative bound < 1, i.e., there are $0 \leq \alpha < 1$ and $\beta \geq 0$ such that

$$\|V_- \psi\| \leq \alpha \|\Delta\psi\| + \beta\|\psi\| \quad \text{for all } \psi \in \mathcal{D}(\Delta). \tag{6.2}$$

Leinfelder and Simader have shown that $H(\mathbf{A}, V)$ is essentially self-adjoint on $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ [LS, Theorem 3] these conditions can be relaxed by the use of quadratic form (see [KK]) as descibed after. It has been checked in [BoGKS] that under these hypotheses $H(\mathbf{A}, V)$ is bounded from below:

$$H(\mathbf{A}, V) \geq -\frac{\beta}{(1-\alpha)} =: -\gamma + 1, \text{ so that } H + \gamma \geq 1. \tag{6.3}$$

We denote by x_j the multiplication operator in $\mathrm{L}^2(\mathbb{R}^d)$ by the j^{th} coordinate x_j, and $\mathbf{x} := (x_1, \cdots x_d)$. We want to implement the adiabatic switching of a time-dependent spatially uniform electric field $\mathbf{E}_\eta(t) \cdot \mathbf{x} = \mathrm{e}^{\eta t}\mathbf{E}(t) \cdot \mathbf{x}$ between time $t = -\infty$ and time $t = t_0$. Here $\eta > 0$ is the adiabatic parameter and we assume that

$$\int_{-\infty}^{t_0} \mathrm{e}^{\eta t} |\mathbf{E}(t)| \mathrm{d}t < \infty. \tag{6.4}$$

To do so we consider the time-dependent magnetic potential $\mathbf{A}(t) = \mathbf{A} + \mathbf{F}_\eta(t)$, with $\mathbf{F}_\eta(t) = \int_{-\infty}^{t} \mathbf{E}_\eta(s)\mathrm{d}s$. In other terms, the dynamics are generated by the time-dependent magnetic operator

$$H(t) = (-i\nabla - \mathbf{A} - \mathbf{F}_\eta(t))^2 + V(x) = H(\mathbf{A}(t), V)\,, \tag{6.5}$$

which is essentially self-adjoint on $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ with domain $\mathcal{D} := \mathcal{D}(H) = \mathcal{D}(H(t))$ for all $t \in \mathbb{R}$. One has (see [BoGKS, Proposition 2.5])

$$H(t) = H - 2\mathbf{F}_\eta(t) \cdot \mathbf{D}(\mathbf{A}) + \mathbf{F}_\eta(t)^2 \ \text{ on } \mathcal{D}(H), \tag{6.6}$$

where $\mathbf{D} = \mathbf{D}(\mathbf{A})$ is the closure of $(-i\nabla - \mathbf{A})$ as an operator from $\mathrm{L}^2(\mathbb{R}^d)$ to $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^d)$ with domain $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$. Each of its components $\mathbf{D}_j = \mathbf{D}_j(\mathbf{A}) = (-i\frac{\partial}{\partial x_j} - \mathbf{A}_j)$, $j = 1, \ldots, d$, is essentially self-adjoint on $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$.

To see that such a family of operators generates the dynamics of a quantum particle in the presence of the time-dependent spatially uniform electric field $\mathbf{E}_\eta(t) \cdot \mathbf{x}$, consider the gauge transformation

$$[G(t)\psi](x) := \mathrm{e}^{i\mathbf{F}_\eta(t)\cdot x}\psi(x) \,, \tag{6.7}$$

so that

$$H(t) \;=\; G(t)\left[(-i\nabla - \mathbf{A})^2 + V\right] G(t)^* \,. \tag{6.8}$$

Then if $\psi(t)$ obeys Schrödinger equation

$$i\partial_t \psi(t) = H(t)\psi(t), \tag{6.9}$$

one has, *formally*,

$$i\partial_t G(t)^*\psi(t) = \left[(-i\nabla - \mathbf{A})^2 + V + \mathbf{E}_\eta(t)\cdot x\right] G(t)^*\psi(t). \tag{6.10}$$

To summarize the action of the gauge transformation we recall the

Lemma 6.1. *[BoGKS, Lemma 2.6] Let $G(t)$ be as in (6.7). Then*

$$\begin{aligned} G(t)\mathcal{D} &= \mathcal{D}\,, & (6.11)\\ H(t) &= G(t)HG(t)^*\,, & (6.12)\\ \mathbf{D}(\mathbf{A}+\mathbf{F}_\eta(t)) &= \mathbf{D}(\mathbf{A}) - \mathbf{F}_\eta(t) = G(t)\mathbf{D}(\mathbf{A})G(t)^*. & (6.13)\end{aligned}$$

Moreover, $i[H(t), x_j] = 2\mathbf{D}(\mathbf{A}+\mathbf{F}_\eta(t))$ as quadratic forms on $\mathcal{D}\cap\mathcal{D}(x_j)$, $j = 1, 2, \ldots, d$.

The key observation is that the general theory of propagators with a time-dependent generator [Y, Theorem XIV.3.1] applies to $H(t)$. It thus yields the existence of a two parameters family $U(t,s)$ of unitary operators, jointly strongly continuous in t and s, that solves the Schrödinger equation.

Theorem 6.2. *[BoGKS, Theorem 2.7] The time-dependant Hamiltoinian $H(t)$ has a unique unbitary propagator $U(t,s)$, i.e there exists a unique two-parameter family $U(t,s)$ of unitary operators, jointly strongly continuous in t and s, such that*

$$\begin{aligned} U(t,r)U(r,s) &= U(t,s) & (6.14)\\ U(t,t) &= I & (6.15)\\ U(t,s)\mathcal{D} &= \mathcal{D}\,, & (6.16)\\ i\partial_t U(t,s)\psi &= H(t)U(t,s)\psi \;\; \textit{for all } \psi\in\mathcal{D}\,, & (6.17)\\ i\partial_s U(t,s)\psi &= -\,U(t,s)H(s)\psi \;\; \textit{for all } \psi\in\mathcal{D}\,. & (6.18)\end{aligned}$$

In addition $W(t,s) = (H(t)+\gamma)U(t,s)(H(s)+\gamma)^{-1}$ is a bounded operator jointly strongly continuous in t and s.
The operators $U(t,s)(H(t)+\gamma)^{-1}$ and $(H(t)+\gamma)^{-1}U(t,s)$ are jointly conitnuous in t and s in operator norm and

$$i\partial_t\{U(t,s)(H(s)+\gamma)^{-2}\} = H(t)U(t,s)(H(s)+\gamma)^{-2} \tag{6.19}$$
$$i\partial_s\{(H(s)+\gamma)^{-2}U(t,s)\} = -(H(t)+\gamma)^{-2}U(t,s)H(s) \tag{6.20}$$

We refer to [BoGKS, Theorem 2.7] for other relevant properties.

To compute the linear response, we shall make use of the following "Duhamel formula". Let $U^{(0)}(t) = \mathrm{e}^{-itH}$. For all $\psi \in \mathcal{D}$ and $t,s \in \mathbb{R}$ we have [BoGKS, Lemma 2.8]

$$U(t,s)\psi = U^{(0)}(t-s)\psi + i\int_s^t U^{(0)}(t-r)(2\mathbf{F}_\eta(r)\cdot\mathbf{D}(\mathbf{A}) - \mathbf{F}_\eta(r)^2)U(r,s)\psi \,\mathrm{d}r\,. \tag{6.21}$$

Moreover,

$$\lim_{|\mathbf{E}|\to 0} U(t,s) = U^{(0)}(t-s) \ \text{ strongly}\,. \tag{6.22}$$

Recently [KK] have proved a weaker version of this result and an extension of the Yosida's theorem under weaker assumption than the Leinfelder-Simader. More precisely they only assume that

1. $\mathbf{A}(x) \in L^2_{loc}(\mathbb{R}^d;\mathbb{R}^d)$
2. $V(x) = V_+(x) - V_-(x)$ with $V_\pm(x) \in L^1_{loc}(\mathbb{R}^d)$, $V_\pm(x) \geq 0$ is relatively form bounded with respect to ∇ with relative bound < 1

Noting that these condition are sufficient to get the diamagnetic inequality.

6.2 General estimates for magnetic Hamiltonian

Firstly we use the so-called diamagnetic inequality to get a general trace estimate on the resolvent. Then we derive an estimate on the velocity operator. Let $H = H(A,V)$ with $(\mathbf{A},V)$ respecting the assumptions defined previously. We have the diamagnetic inequality [Si2]

$$\left|\mathrm{e}^{-tH(\mathbf{A},V)}\psi\right| \leq \mathrm{e}^{-tH(0,V)}|\psi| \text{ for all } \psi \in \mathrm{L}^2(\mathbb{R}^d) \text{ and } t > 0$$

and by the equality

$$\int_0^\infty t^q \mathrm{e}^{-t(x+\lambda)}\mathrm{d}t = \Gamma(q)(x+\lambda)^{-q}$$

we get easily that

$$\left|(H(\mathbf{A},V)+\lambda)^{-q}\,\psi\right| \le (H(0,V)+\lambda)^{-q}\,|\psi| \tag{6.23}$$

for all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$, $\lambda > \frac{\beta}{(1-\alpha)}$, and $q > 0$.

An important consequence is that the usual trace estimates for $-\Delta + V$ are valid for the magnetic Schrödinger operator $H(\mathbf{A},V)$, with bounds independent of $\mathbf{A}$ and depending on V only through α and β. We state them as in [GK3, Lemma A.4].

Proposition 6.3. *Let $\nu > \frac{d}{4}$. There is a finite constant $\mathcal{T}_{\nu,d,\alpha,\beta}$, depending only on the indicated constants, such that*

$$\operatorname{tr}\left\{\langle x\rangle^{-2\nu}\,(H(\mathbf{A},V)+\gamma)^{-2[[\frac{d}{4}]]}\,\langle x\rangle^{-2\nu}\right\} \le \mathcal{T}_{\nu,d,\alpha,\beta}\,, \tag{6.24}$$

where $[[\frac{d}{4}]]$ is the smallest integer bigger than $\frac{d}{4}$ and γ is the constant defined in 6.3.

Proof. The proposition follows once the estimate (6.23) is converted into an estimate on traces, because then the well known trace estimates for $-\Delta+V$, e.g., [GK3, Lemma A.4], finish the argument. Hence (6.24) follows from the following lemma, with

$$\begin{aligned} A &= \langle x\rangle^{-2\nu}\,(H(\mathbf{A},V)+\gamma)^{-2[[\frac{d}{4}]]}\,\langle x\rangle^{-2\nu}\,, \\ B &= \langle x\rangle^{-2\nu}\,(H(0,V)+\gamma)^{-2[[\frac{d}{4}]]}\,\langle x\rangle^{-2\nu}\,, \end{aligned} \tag{6.25}$$

using the fact that the operator $(H(0,V)+\gamma)^{-2[[\frac{d}{4}]]}$ is positivity preserving. and that if A and B be bounded positive operators on $\mathrm{L}^2(\mathbb{R}^d)$, with B a positivity preserving operator, such that

$$\langle\psi, A\psi\rangle \le \langle|\psi|, B|\psi|\rangle \quad \text{for all } \psi \in \mathrm{L}^2(\mathbb{R}^d). \tag{6.26}$$

Then $\operatorname{tr} A \le \operatorname{tr} B$. □

We now give an estimate on the velocity operator which is nothing else that $\mathbf{v} = i[H,\mathbf{x}]$, where $\mathbf{x}$ is the operator from $\mathrm{L}^2(\mathbb{R}^d)$ to $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^d)$.
We note that

$$i[H,\mathbf{x}] = 2(-i\nabla - \mathbf{A}) \ \text{ on } C_c^\infty(\mathbb{R}^d)\,. \tag{6.27}$$

We let $\mathbf{D} = \mathbf{D}(\mathbf{A})$ be the closure of the $(-i\nabla - \mathbf{A})$ as an operator from $\mathrm{L}^2(\mathbb{R}^d)$ to $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^d)$ with domain $C_c^\infty(\mathbb{R}^d)$.

Each of its components $\mathbf{D}_j = \mathbf{D}_j(\mathbf{A}) = (-i\frac{\partial}{\partial x_j} - \mathbf{A}_j)$, $j = 1, \ldots, d$, is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$ since $\mathbf{A}(x) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ (see [Si1, Lemma 2.5]).

Proposition 6.4. *We have*
(i) $\mathcal{D}(\sqrt{H+\gamma}) \subset \mathcal{D}(\mathbf{D})$. *In fact there exists* $C_{\alpha,\beta} < \infty$ *such that*

$$\left\| \mathbf{D}\,(H+\gamma)^{-\frac{1}{2}} \right\| \;\le\; C_{\alpha,\beta}. \tag{6.28}$$

(ii) *For all* $\chi \in C_c^\infty(\mathbb{R}^d)$ *we have* $\chi\mathcal{D}(H) \subset \mathcal{D}(H)$ *and*

$$H\chi\psi = \chi H\psi - (\Delta\chi)\psi - 2i(\nabla\chi)\cdot\mathbf{D}\psi \quad \text{for all } \psi \in \mathcal{D}(H). \tag{6.29}$$

Proof. To prove (i), note that $\mathbf{D}^*\mathbf{D} = (-i\nabla - \mathbf{A})^2$ and by the a.e pointwise inequality [BeG]

$$|\nabla(|\varphi|)| \le |(i\nabla - \mathbf{A})\varphi| \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^d) \tag{6.30}$$

we have for all $\alpha' > \alpha$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ using (6.2) that

$$\begin{aligned}\langle\varphi, V_-\varphi\rangle &\le \alpha'\|\nabla|\varphi|\|^2 + \frac{\alpha'}{\alpha'-\alpha}\beta\|\varphi\|^2 \\ &\le \alpha'\langle\varphi, H(\mathbf{A}, V_+)\varphi\rangle + \frac{\alpha'}{\alpha'-\alpha}\beta\|\varphi\|^2\end{aligned}$$

(see for details [LS, BoGKS]), then we get that

$$\delta\alpha'\mathbf{D}^*\mathbf{D} \;\le\; (1+\delta)\alpha'(-i\nabla - \mathbf{A})^2 - V_- + \frac{\alpha'}{\alpha-\alpha'}\beta \;\le\; H + \frac{\alpha'}{\alpha-\alpha'}\beta \tag{6.31}$$

for $\alpha' \in (\alpha, 1)$ and δ such that $(1+\delta)\alpha' < 1$. Choosing α' and δ such that

$$\frac{\alpha'}{\alpha-\alpha'}\beta = \gamma \quad \text{and} \quad (1+\delta)\alpha' = 1\,, \tag{6.32}$$

we have

$$(1-\alpha')\mathbf{D}^*\mathbf{D} \le\; H + \gamma \tag{6.33}$$

as quadratic forms. Since $\alpha' = \alpha'(\alpha, \beta)$ is strictly less than one, it follows that $\mathcal{D}(\mathbf{D}) \subset \mathcal{D}(\sqrt{H+\gamma})$ and furthermore

$$(H+\gamma)^{-\frac{1}{2}}\,\mathbf{D}^*\mathbf{D}\,(H+\gamma)^{-\frac{1}{2}} \;\le\; \frac{1}{1-\alpha'}\,, \tag{6.34}$$

which gives (6.28) with $C_{\alpha,\beta} = \sqrt{\frac{1}{1-\alpha'}}$.

□

6.3 Random magnetic Hamiltonian

Let $(\Omega, \mathbb{P})$ be a probability space equipped with an ergodic group $\{\tau_a;\ a \in \mathbb{Z}^d\}$ of measure preserving transformations.

Throughout the rest of this chapter we shall use the material of the above sections with $\mathfrak{H} = \mathrm{L}^2(\mathbb{R}^d)$ and $\mathcal{Z} = \mathbb{Z}^d$.
The projective representation of $\mathbb{Z}^d$ on $\mathfrak{H}$ is given by magnetic translations

$$(U(a)\psi)(x) = e^{-ia\cdot Sx}\psi(x-a)$$

S being a given $d \times d$ real matrix. We then have that the 2-cocycle is given by $\zeta(x,y) = e^{-iy\cdot Sx}$. The projection χ_a is the characteristic function of the unit cube of $\mathbb{R}^d$ centered at $a \in \mathbb{Z}^d$.
We state the technical assumptions on our reference Hamiltonian H_ω.

Assumptions 1. *The ergodic Hamiltonian $\omega \mapsto H_\omega$ is a measurable map from the probability space $(\Omega, \mathbb{P})$ to self-adjoint operators on $\mathfrak{H}$ such that*

$$H_\omega = H(\mathbf{A}_\omega, V_\omega) = (-i\nabla - \mathbf{A}_\omega)^2 + V_\omega\ , \tag{6.35}$$

almost surely, where $\mathbf{A}_\omega$ (V_ω) are vector (scalar) potential valued random variables which satisfy the Leinfelder-Simader conditions almost surely. It is furthermore assumed that H_ω is covariant with respect to the projective unitary representation $\{U_x\}_{x\in\mathbb{Z}^d}$ and the measure preserving group of transformation τ, set

$$U_x H_\omega U_x^* = H_{\tau(x)\omega} \text{ for all } x \in \mathbb{Z}^d$$

We denote by H the operator $(H_\omega)_{\omega\in\Omega}$ acting on $\tilde{\mathfrak{H}}$.

We have therefore in such case that H_ω is affiliated to $\mathcal{K}_\infty$ and then the above formalism can be used.

Remark 12. *The definition of the magnetic translations implies a restriction on the magnetic potential considered, but this can be relaxed considering more general magnetic field as for example a perturbation of a periodic magnetic field by a random magnetic field. We give more details in the section 1.1 of the Ch3.*

7 Functional analysis on $\mathcal{K}_\infty$ and trace estimates

We establich some estimates needed for the computation of the linear response theory. Some natural consequences of the previous formalism are given at the begining. The linear response theory deals with computation of some particular commutators. Then one subtle points is the definiteness for commutators of unbounded operators which are automatically bounded in the discrete case.

7.1 Functional calculus for Schrödinger operators

RD-operators and derivation

We first define a class of functions, called symbols in the literature, which can be support this functional calculus. We refer to Amrein-Georgescu and Hunziker-Sigal [AG, HS] (in the context of the N-body model) for a detailed treatment of Helffer-Söjstrand functional calculus and commutators expansion while the Kernel estimate of function of *Generalized*-Schrödinger operators we refer to the paper of Bouclet-Germinet-Klein [BoGK].

Definition 7.1. $f : \mathbb{R} \to \mathbb{C}$ *is called slow smooth decreasing function of order if there derivatives are polynomially controlled in the sense that there exists a negative constant* γ *such that we have*

$$|g^{(j)}(u)| \leq c_j \langle u \rangle^{\gamma - j} \text{ for all } j$$

We note $\mathcal{A}^\gamma$ *this set of functions and* $\mathcal{A} := \cup_{\gamma<0}\mathcal{A}^\gamma$. *We can adjoint the following norm*

$$\{\{g\}\}_n := \sum_{j=0}^{n} \int_{\mathbb{R}} |g^{(j)}(u)| \langle u \rangle^{j-1} du \tag{7.1}$$

Furthermore we have that $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{A}$ and the completion of $\{\mathcal{A}, \{\{\cdot\}\}_n\}$ is a Banach space.

We note that for the following the sole property required for the decaying etimates is the so-called Combes-thomas estimates, we doing as in [BoGK], we state this as definition.

Definition 7.2. *Let a self adjoint operator* H *on the Hilbert space* $L^2(\mathbb{R}^d;\mathbb{C}^k)$, *we said that this operator has the CT-property if for any* $z \in \rho(H)$ *the following holds,*

$$|||\mathcal{K}_z(x,y)|||_\infty \leq \frac{C}{\eta_z} exp(-\frac{C\eta_z}{1+\eta_z+|z|}|x-y|)$$

where $R(z) = (H-z)^{-1}$ *and* $\eta_z = dist(z, \sigma(H))$.

In the random case, it suffices to state that the *CT-property* holds uniformly with repect to ω and as we use the ess-supremum of the usual operator norm the *CT-property* is hold for $\mathcal{K}_\infty$-norm. It is well known that this property is satisfied for a wide class of random or deterministic operators, for example the class of generalized Schrödinger operators defined in [BoGK] satisfy this property. In particular, the Hamiltonian of interest for us, namely the random magnetic Hamiltonian $H(\mathbf{A}, \mathbf{V})$ enjoys the *CT-property.* Noting that the operators having the CT-property have RD-decaying resolvent in the sense that for each $z \in \rho(H)$ there exists $\kappa(z)$ such that $R_\omega(z) \in \mathcal{E}^\infty_{\kappa(z)}$. We now recall the Helffer-Söjstrand functional calculus which is the key step to express $g(H)$ in term of resolvent. Let g a function in $C^{n+1}(\mathbb{R})$ then we define $\tilde{g}_n$ its quasi-analytic extension of order n by

$$\tilde{g}(u+iv) = \rho(u,v) \sum_{k=0}^{n} g^{(k)}(u) \frac{(iv)^k}{k!}$$

where $\rho(u,v) = \varphi(u)\tau(\frac{v}{\langle u \rangle})$, with $\tau \in C_0^\infty(\mathbb{R})$ a smooth function such that $\tau = 1$ on a neighborhood of 0 and $\varphi = 1$ near $\sigma(H)$ and supported on I. In this case for $\overline{\partial} := \partial_x + i\partial_y$, the Cauchy operator, a simple computation then gives

$$\overline{\partial}\tilde{g}_n(z) = \overline{\partial}\rho(u,v) \sum_{k=0}^{n} g^{(k)}(u) \frac{(iv)^k}{k!} + \frac{\rho(u,v)}{2} g^{n+1}(u) \frac{(iv)^n}{(n)!} \; . \tag{7.2}$$

With this in hand, we obtain the following estimate which justifies the introduction of the previous norm 7.1

$$\int |\overline{\partial}(\tilde{g}_n)| |\Im z|^{-1} \leq C \{\{g\}\}_n \; . \tag{7.3}$$

In the following we will also use that

$$\int |\overline{\partial}\tilde{g}(z)| \frac{\langle \Re z \rangle^{p-1}}{|\Im z|^p} \leq c_p \{\{g\}\}_m \; . \tag{7.4}$$

for any $m \geq p+1$. Using the spectral functional calculus coupled with Cauchy integral represention and others topics in complex analysis such as the Green theorem we can obtain the so-called Helffer-Söjstrand formula. Given $g \in \mathcal{A}$ and H a semi-bounded self-adjoint operator we have

$$g(H) = \int \overline{\partial}\tilde{g}(u+iv) R(u+iv) du dv \; . \tag{7.5}$$

At the same time we give two very useful technical lemmas, we refer to [HS] for a complete demonstration.

Lemma 7.3 (HS). *Let $f \in \mathcal{A}$, then,*

$$g^{(p)}(H) = \int \overline{\partial}\tilde{g}(u+iv)R^p(u+iv)dudv \ .$$

Similarly, we can work with a commutator of any order. We note

$$ad_A^k(H) := [ad_A^{k-1}(H), A] \ ,$$

and

$$ad_A^k(g(H)) := \int \overline{\partial}\tilde{g}(u+iv)ad_A^k(R(z))dudv \ .$$

We can now give the decay estimate of the operator kernel as we have defined above. We note $\mathcal{K}_{n,z}$ the operator kernel of the resolvent at power n while $\mathcal{K}_g$ stands for the operator kernel of $g(H)$. We want to obtain estimates for the operator kernel of

$$\nabla^{\Xi}(g(H)) = \sum_{\mathcal{Z}^2}(x-y)\chi_x g(H)\chi_y$$

and more generally $\nabla^{\Xi^p}(g(H)) = \sum_{\mathcal{Z}^2}(x-y)^p\chi_x g(H)\chi_y$.

We can then use of the following adaptation to our formalism of a result of Gemninet-Klein ([GK2] Thm2.).

Theorem 7.4. *Let H be a (random) Schrödinger operator densely defined semi-bounded self-adjoint operator with the CT property on $L^2(\mathbb{R}^d, \mathbb{C}^n)$ and $z \in \rho(H)$ then*

i) $\|\mathcal{K}_{n,z}(x,y)\|_\infty \leq C_{d,z,n}e^{-m_{z,n}|x-y|}$

ii) if $g \in \mathcal{A}$, $\|\mathcal{K}_g(x,y)\|_\infty \leq C_{d,k,\theta}\frac{\{\{g\}\}_{\langle k+2\rangle}}{\langle x-y\rangle^k}$

Corollary 7.5. *If $g \in \mathcal{A}$ and $p \leq k$,* $\|\mathcal{K}_{d^p g}(x,y)\|_\infty \leq C_{d,k,\theta}\frac{\{\{g\}\}_{\langle k+2\rangle}}{\langle x-y\rangle^{k-p}}$

We now turn on the magnetic Hamiltonian $H = (i\nabla - A)^2 + V$ applying the results of the first section with the Helffer-Söjstrand formula we get the following estimates. We denote by $\|\cdot\|^{\Phi} := \|\cdot\Phi\|$ the weighted norm with $\Phi \in L^\infty(\mathbb{R})$.

Proposition 7.6. *Let $H(A,V)$ the magnetic Hamiltonian as defined previously. Then*

(i) *Let*

$$\Phi_{d,\alpha,\beta}(E) = \chi_{\left[-\frac{\beta}{1-\alpha},\infty\right)}(E)\,(E+\gamma)^{2[[\frac{d}{4}]]}\ , \tag{7.6}$$

we have

$$\operatorname{tr}\left(\langle x\rangle^{-2\nu} f(H)\,\langle x\rangle^{-2\nu}\right) \leq \mathcal{T}_{\nu,d,\alpha,\beta}\|f\|_{\infty}^{\Phi_{d,\alpha,\beta}} < \infty \tag{7.7}$$

for every Borel measurable function $f \geq 0$ on the real line.

(ii) *Let*

$$\widetilde{\Phi}_{d,\alpha,\beta}(E) \ := \ (E+\gamma)^{\frac{1}{2}}\Phi_{d,\alpha,\beta}(E) \ = \ \chi_{\left[-\frac{\beta}{1-\alpha},\infty\right)}(E)\,(E+\gamma)^{2[[\frac{d}{4}]]+\frac{1}{2}}\ . \tag{7.8}$$

If f is Borel measurable function on the real line with $\|f\|_{\infty}^{\widetilde{\Phi}} < \infty$, the bounded operator $|\mathbf{D}f(H)| = \left\{\overline{f}(H)\mathbf{D}^*\mathbf{D}f(H_\omega)\right\}^{\frac{1}{2}}$ satisfies

$$\operatorname{tr}\left\{\langle x\rangle^{-2\nu}\,|\mathbf{D}f(H)|\,\langle x\rangle^{-2\nu}\right\} \ \leq \ \widetilde{\mathcal{T}}_{\nu,d,\alpha,\beta}, \tag{7.9}$$

where $\widetilde{\mathcal{T}}_{\nu,d,\alpha,\beta} < \infty$ for $\nu > d/4$ and depends only on the indicated constants.

Proof. part (i) is a direct consequence of the proposition 6.3 and the Helffer-Söjstrand formula. Part (ii) follows from (6.28), since the identity holds for $\psi \in C_c^\infty$. To get part (iii) we combine Proposition 6.3 with estimate

$$|\mathbf{D}f(H)| \ \leq \ C_{\alpha,\beta}(H+\gamma)^{\frac{1}{2}}|f|(H)\ , \tag{7.10}$$

which follows from (6.33) and monotonicity of the square root. □

To provide a definition of $[\mathbf{x}, H]$ that is compatible with the non-commutative functional analysis, we developped in section, we shall resort to the theory of C_0-group, we refer to [ABG]
Let H and A be two self-adjoint operators. An operator H is said to be $C^k(A)$ if the following map $\mathrm{e}^{-itA}R(z)\mathrm{e}^{itA}$ is C^k for all $\varphi \in \mathfrak{H}$ and $z \in \rho(H)$. We have the following well-known result(see [ABG]) which claims that if

1. There exists $C > 0$ such that

$$\langle A\varphi, H\varphi\rangle - \langle H\varphi, A\varphi\rangle \leq C(\|H\varphi\|^2 + \|\varphi\|^2) \text{ for any } \varphi \in \mathcal{D}(A)$$

.

2. $\{\varphi \in \mathcal{D}(A)/R(z)\varphi \in \mathcal{D}(A) \text{ and } R(\overline{z})\varphi \in \mathcal{D}(A)\}$ is an core for A.

Then the limit $\lim_{t\to 0} \frac{e^{-itA}R(z)e^{itA} - R(z)}{t}$ exists as operator belonging to $\mathcal{B}(\mathfrak{H})$ is equal to $[R(z), A]$ and the following equality holds in $\mathcal{B}(\mathfrak{H})$

$$[R(z), A] = R(z)[H, A]R(z) \text{ for any } z \in \rho(H)$$

Using this result we get that

Proposition 7.7. *Let H the magnetic Hamiltonian and $f \in C^\infty(\mathbb{R})$ with $|||f|||_3 < \infty$. Then, on $\mathcal{B}(\mathfrak{H})$,*

$$[R(z), \mathbf{x}] = R(z)[H, \mathbf{x}]R(z) \text{ for any } z \in \rho(H)$$

Moreover the operator $[\mathbf{x}, f(H)]$ is well-defined on $\mathfrak{H}_c$ and has a bounded closure: there exists a constant $C_{\alpha,\beta} < \infty$ such that

$$\left\| \overline{[\mathbf{x}, f(H)]} \right\| \leq C_{\alpha,\beta} |||f|||_3 \,. \tag{7.11}$$

Proof. By the inequality 6.28 we have that $\mathbf{v} = 2\mathrm{D}$ is H-bounded. The rest of the proof comes from the Combes-Thomas estimate which ensures that $R(z)\mathfrak{H}_c \subset \mathcal{D}(\mathbf{x})$, with $R(z) = (H - z)^{-1}$, whenever $\mathrm{Im}\, z \neq 0$. In fact, we have $R(z)\mathfrak{H}_c \subset \mathcal{D}(e^{\mu(z)|\mathbf{x}|})$ with the explicit estimate

$$\left\| e^{\mu(z)|\mathbf{x}-y|} R(z)\chi_y \right\| \;\leq\; C_{\alpha,\beta} \frac{1}{|\mathrm{Im}\, z|} \,, \qquad \text{for every unit cube } \chi_y, \tag{7.12}$$

where $\mu(z) = C_{\alpha,\beta} \,|\mathrm{Im}\, z|/(\langle \mathrm{Re}\, z\rangle + |\mathrm{Im} z|)$. We conclude that

$$\|\mathbf{x}R(z)\chi_y\| \;\leq\; C_{\alpha,\beta,y} \frac{1}{\mu(z)|\mathrm{Im}\, z|} \;\leq\; C_{\alpha,\beta,y} \begin{cases} \frac{\langle \mathrm{Re}\, z\rangle}{|\mathrm{Im}\, z|^2} \,, & |\mathrm{Im}\, z| \leq \langle \mathrm{Re}\, z\rangle \,, \\ \frac{1}{|\mathrm{Im}\, z|} \,, & |\mathrm{Im}\, z| \geq \langle \mathrm{Re}\, z\rangle \,, \end{cases} \tag{7.13}$$

Then we have that $[\mathbf{x}, R(z)]$ is a bounded operator with

$$[\mathbf{x}, R(z)] \;=\; 2iR(z)\mathbf{D}(\mathbf{A})R(z) \,. \tag{7.14}$$

Specifically we have

$$\|[\mathbf{x}, R(z)]\| \;\leq\; 2 \left\| R(z)\sqrt{H+\gamma} \right\| \cdot \left\| \sqrt{H+\gamma}^{-1} \mathbf{D}(\mathbf{A}) \right\| \cdot \|R(z)\| \,, \tag{7.15}$$

with the middle factor bounded by Proposition 6.4(iii), and the first and last factors bounded by $\sqrt{|z+\gamma|}/|\mathrm{Im} z|$ and $1/|\mathrm{Im} z|$ respectively. Plugging these bounds into the Helffer-Sjöstrand formula (7.3), and using (7.4), we find

$$\|[\mathbf{x}, f(H)]\| \;\leq C_{\alpha,\beta} \int |d\tilde{f}(z)| \frac{\sqrt{|z|+\gamma}}{|\mathrm{Im} z|^2} \;< \infty \,. \tag{7.16}$$

and we note that as $[\mathbf{x}, H]$ is H-bounded we have also that

$$\|[\mathbf{x}, H]\, f(H)\| \ \leq C_{\alpha,\beta} \int |d\tilde{f}(z)| \frac{\sqrt{|z| + \gamma}}{|\mathrm{Im} z|} \tag{7.17}$$

□

We note that the last bound could be obtained using the bound on $\|\partial^{\Xi}(f(H)\|$ and writing $\nabla = \nabla^{\Xi} + \delta$, but is useful to have a weak definition of ∇H for other computations.

7.2 Trace estimates for random magnetic Hamiltonian

We provide some estimates on the magnetic random Hamiltonian and thus come back to the forma lism of non-commutative L^p-space.

Proposition 7.8. *Let f be a bounded Borel measurable function on the real line such that f belongs to $L^{\infty}_{\Phi}(\mathbb{R})$. Then we have $f(H_\omega) \in L^1(\mathcal{K}_\infty)$, and consequently $f(H_\omega) \in L^p(\mathcal{K}_\infty)$ for all $p \geq 1$.*

Proof. This comes from the fact that $H\eta\mathcal{K}_\infty$ and the Helffer-Sojstrand functional calculus used in the previous estimate 7.7. □

In the following we get trace estimates on $\nabla^{\Xi}(f(H))$.

Proposition 7.9.

(i) Let $f \in \mathcal{S}(\mathbb{R})$, then we have $f(H_\omega) \in \mathcal{W}^{1,p}(\mathcal{K}_\infty)$ $\forall p \geq 1$, $j = 1, 2, \ldots, d$.

(ii) Let f be such that $f(H_\omega) = g(H_\omega)h(H_\omega)$ with $g \in \mathcal{S}(\mathbb{R})$ and h a Borel measurable function such that h^2 belongs to $L^{\Phi}_{\infty}(\mathbb{R})$, and assume $h(H_\omega) \in \mathcal{W}^{1,p}(\mathcal{K}_\infty)$ for some $p \geq 1$.
Then we also have $f(H_\omega) \in \mathcal{W}^{1,1}(\mathcal{K}_\infty) \cap \mathcal{W}^{1,p}(\mathcal{K}_\infty)$.

Proof. The proof comes from two key facts. First that as proved in coral-lary 7.5 we have

$$\partial_i^{\Xi}(f(H_\omega)) \in \mathcal{K}_\infty \text{ for any } f \in \mathcal{S}(\mathbb{R})$$

for any $i \in \{1 \cdots d\}$. Second, by the functional analysis and [GK3, Eq. (3.8)] we have that

$$\|\chi_x f(H_\omega)\chi_0\|_2^2 \leq C\|f\|_{\infty}^{\Phi} \frac{\{\{g\}\}_{k+2}}{\langle x\rangle^{k-2\nu}} \tag{7.18}$$

where we note $C = C_{d,\alpha,\beta,\nu,k}$ a constant depending only on this parameter and $\nu > \frac{d}{4}$. We begin to treat i) for $p = 2$. We recall that we have

$$\left\|\left|\partial_j^{\Xi}(f(H_\omega))\right|\right\|_2^2 = \mathbb{E}\left\|\partial_j^{\Xi} f(H_\omega)\chi_0\right\|_2^2 = \sum_{a\in\mathcal{Z}} |a_j|^2 \mathbb{E}\|\chi_a f(H_\omega)\chi_0\|_2^2$$

and from (7.18) with sufficiently large k we have that

$$\sum_{a\in Z^d} |a_j|^2 \left\|\chi_a f(H_\omega)\chi_0\right\|_2^2 \leq C \, \|f\|_\infty^\Phi \{\{g\}\}_{k+2} \sum_{a\in Z^d} |a_j|^2 \langle a\rangle^{-k+2\nu}$$

We finally have that $f(H_\omega) \in \mathcal{W}^{1,2}(\mathcal{K}_\infty)$ for any $f \in \mathcal{S}(\mathbb{R})$. Before to prove for any p we need to prove the item ii).
If we consider two functions g and h such that $g \in \mathcal{S}(\mathbb{R})$ and h is a Borel function belonging to $L_\infty^\Phi(\mathbb{R})$ and $h(H_\omega) \in \mathcal{W}^{1,p}(\mathcal{K}_\infty)$. Then by the Leibniz rule we get that

$$\nabla^{\Xi}(g(H)h(H)) = \nabla^{\Xi}(g(H))h(H) + g(H)\nabla^{\Xi}(h(H)) \tag{7.19}$$

and by Hölder inequality

$$\left\|\left|\nabla^{\Xi}(g(H)h(H))\right|\right\|_1 \leq \left\|\left|\nabla^{\Xi}(g(H))\right|\right\|_q \left\|\left|h(H)\right|\right\|_p + \left\|\left|g(H)\right|\right\|_q \left\|\left|\nabla^{\Xi}(h(H))\right|\right\|_p .$$

Care must be taken to the fact that $h(H)g(H)$ is not a priori in the domain of the derivation and then the left-hand side of (7.19) does not make sense, but we can use a limiting argument to approximate $h(H_\omega)$ and the dominated convergence theorem. Then $h(H_\omega)g(H_\omega)$ belongs to $\mathcal{W}^{1,1}(\mathcal{K}_\infty)$.

To prove that if $f \in \mathcal{S}(\mathbb{R})$ then $f(H_\omega) \in \mathcal{W}^{1,1}(\mathcal{K}_\infty)$. We use the last result after writing $f(x) = (f(x)\langle x+\gamma\rangle^t)\langle x+\gamma\rangle^{-t}$ for a large enought t.

After for prove item i) for any $p \geq 1$ we use the inequalities (4.18)

$$\left\|\left|\partial_i^{\Xi}(f(H_\omega))\right|\right\|_p \leq \left\|\left|\partial_i^{\Xi}(f(H_\omega))\right|\right\|_1^{\frac{1}{p}} \left\|\left|\partial_i^{\Xi}(f(H_\omega))\right|\right\|_\infty^{\frac{1}{q}} < \infty$$

with $1 = \frac{1}{p} + \frac{1}{q}$.
Then i) is proved. It remains to prove that $g(H)h(H) \in \mathcal{W}^{1,p}(\mathcal{K}_\infty)$ this is done by the inequality

$$\left\|\left|\nabla^{\Xi}(g(H)h(H))\right|\right\|_p \leq \left\|\left|\nabla^{\Xi}(g(H))\right|\right\|_p \left\|\left|h(H)\right|\right\|_\infty + \left\|\left|g(H)\right|\right\|_\infty \left\|\left|\nabla^{\Xi}(h(H))\right|\right\|_p$$

this finishes the proof. □

Remark 13. *We note that the we have a natural pairing,*

$$\mathcal{W}^{1,p}(\mathcal{K}_\infty) \times \mathcal{W}^{1,q}(\mathcal{K}_\infty) \to \mathbb{C} \tag{7.20}$$
$$(A, B) \to \langle A|B\rangle_{L^2(\mathcal{K}_\infty)} + \langle \partial_i^{\Xi}(A)|\partial_i^{\Xi}(B)\rangle_{L^2(\mathcal{K}_\infty)} \tag{7.21}$$

As we will deal often with projection, namely the Fermi-Dirac projection, we give a corollary about trace estimate of derivation of projection considered here.

Corollary 7.10. *Let* $P_\omega^{(E)} = \chi_{(-\infty,E]}(H_\omega)$ *a spectral resolution of* H*. Then* $P_\omega^E \in L^p(\mathcal{K}_\infty)$ *for any* $p \geq 1$*. Moreover, Assume that* $P_\omega^{(E)} \in \mathcal{W}^{1,p}(\mathcal{K}_\infty)$ *for any* $p \leq 1$*, then we have also that*

$$P_\omega^{(E)} \in \mathcal{W}^{1,1}(\mathcal{K}_\infty)$$

To sum up : we have provided a suitable definition of the commutators $[\mathbf{x}, R(z)]$ and $[\mathbf{x}, f(H)]$, for f smooth enough, in the strong sense and proved they belong to the L^p-spaces. Using the theory of C_0-groups we have obtained a weak definition to commutators $[\mathbf{x}, H]$ as operators which can be extended to operators belonging to $\mathcal{B}(\mathcal{D}(H), \mathcal{D}(H)^*)$ where the second space is the dual in the graph-topology (see [ABG]) which gives a meaning to the equality, $[\mathbf{x}, R(z)] = R(z)[\mathbf{x}, H]R(z)$ for any $z \in \rho(H)$, understood as the composition of the three operators $R(z) : \mathfrak{H} \mapsto \mathcal{D}(H)$, $[\mathbf{x}, H] : \mathcal{D}(H) \mapsto \mathcal{D}(H)^*$ and $R(z) : \mathcal{D}(H)^* \mapsto \mathfrak{H}$.
To complete this discussion on (generalized) L^p-commutators we have to prove that $\nabla(f(H)) \in \mathfrak{D}_H^p$ for smooth enough f.
We begin to study the definiteness in a strong sense. Note first that we have *formally* the following equality for $f \in \mathcal{S}(\mathbb{R})$

$$[H, [\mathbf{x}, f(H)]] = [[H, \mathbf{x}], f(H)] \ , \tag{7.22}$$

whereas we know, as $[H, \mathbf{x}]$ is H-bounded, that the right hand-side of (7.22) is a bounded operator. While the commutator equality (7.22) is only formal, since we know that $R(z) : \mathfrak{H} \mapsto \mathcal{D}(H)$ with $\mathcal{D}(H)$ dense in $\mathfrak{H}$, if we note $A = [H, [\mathbf{x}, f(H)]]$ and $B = [[H, \mathbf{x}], f(H)]$ and we have for any $f, g \in \mathfrak{H}$ that

$$\langle R(\overline{z})f, AR(z)g\rangle = \langle R(\overline{z})f, BR(z)g\rangle$$

with B bounded and since $H\partial(f(H)), \partial(f(H))H$ are bounded or can be extended to bounded operators, therefore it is enough to prove that for some $z \in \rho(H)$ that we can write in the strong sense that

$$R(z)[H, \partial(f(H))]R(z) = -R(z)[\partial(H), f(H)]R(z) \tag{7.23}$$

This is done as follows. Using the Leibniz rule we get

$$[\partial(R(z)), f(H)] + [R(z), \partial(f(H))] = \partial([R(z), f(H)]) = 0$$

hence

$$R(z)[H, \partial(f(H))]R(z) = -[R(z), \partial(f(H))] \tag{7.24}$$
$$= [\partial(R(z)), f(H)] \tag{7.25}$$
$$= -R(z)[\partial(H), f(H)]R(z) \tag{7.26}$$

which proves the equality 7.23.

We now look at the L^p-bounds. We know that for any $f \in \mathcal{S}(\mathbb{R})$, that $\partial_i^{\Xi}(f(H_\omega)) \in \mathrm{L}^p(\mathcal{K}_\infty)$ for any $i = 1 \cdots d$. By the previous discussion we can write on $\mathfrak{H}_c$

$$\partial_i^{\Xi}(H_\omega f(H_\omega)) = \partial_i^{\Xi}(H_\omega) f(H_\omega) + H_\omega \partial_i^{\Xi}(f(H_\omega)) \ .$$

The term of the left hand side belongs to any $L^p(\mathcal{K}_\infty)$ since $xf(x) \in \mathcal{S}(\mathbb{R})$ for any $f(x) \in \mathcal{S}(\mathbb{R})$. As for the first term of the right hand side, we know there exists $C > 0$ such that

$$\left\| \partial_i^{\Xi}(H_\omega) f(H_\omega) \right\|_p \leq C \left\| \partial_i(H_\omega) R_w(-\gamma) \right\|_\infty \times \left\| (H_\omega + \gamma) f(H_\omega) \right\|_p < \infty \ .$$

We get finally

Lemma 7.11. *Let $f \in \mathcal{S}(\mathbb{R})$ then $\partial_i^{\Xi}(H_\omega) f(H_\omega)$ and $H_\omega \partial_i^{\Xi}(f(H_\omega))$ belong to $L^p(\mathcal{K}_\infty)$ for any $1 \leq p \leq \infty$ and consequently $\left\| \partial_i^{\Xi}(f(H)) \right\|_{\mathfrak{D}_p(H)} < \infty$ and consequently we have also that $\left\| \partial_i(f(H)) \right\|_{\mathfrak{D}_p(H)} < \infty$.*

We turn to the following

Proposition 7.12. *Let H as defined above and $f \in \mathcal{S}(\mathbb{R})$. Then the following holds*

$$\mathcal{T}(\partial_i(f(H))) = \mathcal{T}(f'(H)\partial_i(H)) \ ,$$

for any $i = 1 \cdots d$

Proof. The first step is to show this equality in a weaker sense. We show that for any f and $g \in \mathcal{S}(\mathbb{R})$, $\langle \partial_i(f(H)), g(H) \rangle_{L^2(\mathcal{K}_\infty)} = \langle f'(H)\partial_i(H), g(H) \rangle_{L^2(\mathcal{K}_\infty)}$

$$\langle \partial_i(f(H)), g(H) \rangle_{L^2(\mathcal{K}_\infty)} = \frac{1}{\pi} \int \overline{\partial} \tilde{f} \mathcal{T}(\partial_i(R(z)) g(H)) du dv \tag{7.27}$$
$$= -\frac{1}{\pi} \int \overline{\partial} \tilde{f} \mathcal{T}(R(z)\partial_i(H)R(z) g(H)) du dv \tag{7.28}$$
$$= -\frac{1}{\pi} \int \overline{\partial} \tilde{f} \mathcal{T}(R(z)^2 \partial_i(H) g(H)) du dv \tag{7.29}$$
$$= \mathcal{T}(f'(H)\partial_i(H) g(H)) \ , \tag{7.30}$$

where we have used Lemma 7.3. Take $g_n \in C_c^\infty(\mathbb{R})$ such that $|g_n| \leq 1$ and $g_n = 1$ on $[-n; n]$. Taking the limits $n \to \infty$, using the dominated convergence theorem and Lemma 7.11 we get the result. □

In the sequel, we shall rather use the following corollary.

Corollary 7.13. *Let $f \in L^\Phi_\infty(\mathbb{R})$, then for any $i = 1 \cdots d$,*

$$\mathcal{T}(f(H)\partial_i(H)) = 0$$

Proof. It is a straightforward consequence of proposition 7.12. It is sufficient to take $f_n \in \mathcal{S}(\mathbb{R})$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} f_n = f$ and $f_n \in L^{\tilde{\Phi}}_\infty(\mathbb{R})$. Then we have by definition and by the equation 7.9 that $\tilde{\Phi}(H)(f_n(H) - f(H)) \in \mathcal{K}_\infty$ and finally $(\partial_i(H)R(-\gamma))\Phi(H)^{-1}(\tilde{\Phi}(H)(f_n(H) - f(H))) \in L^1(\mathcal{K}_\infty)$ and by the lemma 4.12 the convergence in $L^1(\mathcal{K}_\infty)$ follows. It remains to consider $F_n \in \mathcal{S}(\mathbb{R})$ such that $F_n'(x) = f_n(x)$ and apply Lemma 5.5. □

8 Dynamics on the L^p-spaces

8.1 Propagators on the L^p-spaces

L^p-space being naturally Banach space we can use the notion of Bochner-integral and get suitable definitions of time-dependent observables (see [DS] for precise defintion of the Bochner integrability).
For $\mathbb{P}$-a.e.ω we define $U_\omega(t, s)$ as the unitary propagator given by the usual construction done in the Theorem 6.2.

Let

$$\mathcal{U}(t, s)(A) = \int_\Omega^\oplus U_\omega(t, s)A_\omega U_\omega(s, t)d\mathbb{P}(\omega) \quad \text{for} \quad A \in \mathrm{L}^o(\mathcal{K}_\infty) \; , \tag{8.1}$$

which is well defined since $L^o(\mathcal{K}_\infty)$ as well as the L^p-space are $\mathcal{K}_\infty$-bimodules.

Remark 14. *As pointed out in [BoGKS] the main issue is to prove measurability in ω. But this follows by construction of the $U_\omega(t, s)$ as a limit of Riemmann products in the sense of multiplicative Riemmann sums, each of which being measurable since it is a product of finitely many propagators (see [BoGKS]).*

Proposition 8.1. *$\mathcal{U}(t, s)$ is a linear operator on $L^o(\mathcal{K}_\infty)$ leaving invariant the L^p-space. Moreover it is an isometry in $L^p(\mathcal{K}_\infty)$ for all $p \geq 1$, unitary on the Hilbert space $L^2(\mathcal{K}_\infty)$ and it is jointly strongly continuous in t and s on $\mathrm{L}^p(K_\infty)$ for all $p \geq 1$.*

We have the following identities

$$\begin{aligned} \mathcal{U}(t,r)\mathcal{U}(r,s) &= \mathcal{U}(t,s)\,, &(8.2)\\ \mathcal{U}(t,t) &= I\,, &(8.3)\\ \{\mathcal{U}(t,s)(A)\}^* &= \mathcal{U}(t,s)(A^*)\,. &(8.4)\end{aligned}$$

Proof. The proposition follows also from the fact that $L^o(\mathcal{K}_\infty)$ and L^p-spaces are $\mathcal{K}_\infty$-bimodules. Using (8.2) and (8.3)

$$\|\!|\mathcal{U}(t,s)(A_\omega)|\!\|_p \leq \|\!|A_\omega|\!\|_p \leq \|\!|\mathcal{U}(t,s)(A_\omega)|\!\|_p \tag{8.5}$$

for any $p \geq 1$, where we used $A_\omega = \mathcal{U}(s,t)\left(\mathcal{U}(t,s)(A_\omega)\right)$. The joint strong continuity of $\mathcal{U}(t,s)$ on $L^p(\mathcal{K}_\infty)$ follows from the joint strong continuity of $U_\omega(t,s)$ on $\mathfrak{H}$ and the fact that multiplication on $L^o(\mathcal{K}_\infty)$ is uniformly continuous. □

We need to prove a condition of domain-preserving in the following sense.

Proposition 8.2. *With the previous notations we have that $\mathfrak{D}^p_{H(r)} = \mathfrak{D}^p_H$ for any $p \geq 1$ and $r \in (-\infty, t_0]$. Moreover the propagator is domain-preserving for the p-Liouvillian in the following sense $\mathcal{U}(t,r)(\mathfrak{D}^p_H) \subset \mathfrak{D}^p_H$.*

Proof. The first point follows immediately from 6.6 and 6.4 which implies for $A \in L^p(\mathcal{K}_\infty)$ that $\mathbf{D}_j A \in L^p(\mathcal{K}_\infty)$ if $HA \in L^p(\mathcal{K}_\infty)$. Now set $A \in \mathfrak{D}^p_H$, we then have that $HA \in L^o(\mathcal{K}_\infty)$, then HA is $\mathcal{T}$-almost surely bounded in the sense that for all $\varepsilon > 0$ there exists a projection $P \in \mathcal{K}_\infty$ such that $\mathcal{T}(P^\perp) < \varepsilon$ with $HAP \in \mathcal{K}_\infty$ and $AP\mathfrak{H} \subset \mathcal{D}$. Since $U_\omega(t,r)$ is domain-preserving, the operator $H_\omega(t)U_\omega(t,r)A_\omega$ is well-defined on $P\mathfrak{H}$. Then as operators in $L^o(\mathcal{K}_\infty)$ we can write

$$H_\omega(t)U_\omega(t,r)A_\omega = (W_\omega(t,r) - \gamma U_\omega(t,r)(H_\omega(r)+\gamma)^{-1})(H_\omega(r)+\gamma)A_\omega$$

where

$$W_\omega(t,r) = (H_\omega(t)+\gamma)U_\omega(t,r)(H_\omega(r)+\gamma)^{-1} \in \mathcal{K}_\infty$$

for any $t, r \in (-\infty, t_0]$ by (6.2), which implies

$$\|\!|(H_\omega(t)+\gamma)U_\omega(t,r)A_\omega|\!\|_p \leq \|\!|W_\omega(t,r)|\!\|_\infty \|\!|(H_\omega(r)+\gamma)A_\omega|\!\|_p\ . \tag{8.6}$$

Finally,

$$\|\!|[H_\omega(t), \mathcal{U}_\omega(t,r)(A_\omega)]|\!\|_p \leq C_\gamma \|\!|W_\omega(t,r)|\!\|_\infty \|\!|A|\!\|_{\mathfrak{D}^p_H} \tag{8.7}$$

□

We now state a result on differentiability of the propagators.

Proposition 8.3. *Let $A \in \mathfrak{D}^p_H$ The maps $r \to \mathcal{U}(t,r)(A)$ and $t \to \mathcal{U}(t,r)(A)$ are differentiable in $L^p(\mathcal{K}_\infty)$ with*

$$i\partial_r \mathcal{U}(t,r)(A_\omega) = -\mathcal{U}(t,r)([H_\omega(r), A_\omega])\,,$$

$$i\partial_t \mathcal{U}(t,r)(A) = [H_\omega(t), \mathcal{U}(t,r)_\omega(A_\omega)])\,,$$

Moreover, we have that there exists $C > 0$ such that

$$|||i\partial_t \mathcal{U}(t,r)(A_\omega)|||_p \leq C\ |||A_\omega|||_{\mathcal{D}_p(H)}\ . \tag{8.8}$$

As consequence $\partial_t \mathcal{U}(r,t)$ is a bounded map densely defined from $\mathfrak{D}^p_H(H)$ to $L^p(\mathcal{K}_\infty)$.

Proof. Fix p and ω. All the expressions make sense as elements of $L^p(\mathcal{K}_\infty)$. As in [BoGKS] we use of the following decomposition with the difference that all the products are automatically well defined.

$$(\mathcal{U}(t,r+h)(A_\omega) - \mathcal{U}(t,r)(A_\omega)) \tag{8.9}$$
$$= \ (U_\omega(t,r+h) - U_\omega(t,r))\, A_\omega U_\omega(r+h,t) \tag{8.10}$$
$$+\, U_\omega(t,r) A_\omega\, (U_\omega(r+h,t) - U_\omega(r,t)) \quad . \tag{8.11}$$

by Theorem 6.2, we have that

$$\frac{1}{h}\,(U_\omega(t,r+h) - U_\omega(t,r))\,(H_\omega(r)+\gamma)^{-1} \to iU_\omega(t,r)H_\omega(r)(H_\omega(r)+\gamma)^{-1}$$

strongly with uniformly bounded norm, as $h \to 0$.
By the uniform continuity of the $\mathcal{K}_\infty$-multiplication on $L^p(\mathcal{K}_\infty)$ and the strong continuity of $U_\omega(r,t)$ in r, we get by inserting $(H_\omega(r)+\gamma)^{-1}(H_\omega(r)+\gamma)$ and then removing it,

$$\lim_{h\to 0} \frac{i}{h}\,(U_\omega(t,r+h) - U_\omega(t,r))\, A_\omega U_\omega(r+h,t) \tag{8.12}$$
$$= -U_\omega(t,r)H_\omega(r)A_\omega U_\omega(r,t).$$

To treat (8.11) we apply the same method using that the adjonction $A_\omega \to A^*_\omega$ is an isometry on $L^p(\mathcal{K}_\infty)$ and we write

$$A_\omega\,(U_\omega(r+h,t) - U_\omega(r,t)) = (((U_\omega(t,r+h) - U_\omega(t,r))\, A^*_\omega)^*\ .$$

Hence

$$\lim_{h\to 0} U_\omega(t,r)A_\omega \frac{i}{h}\,(U_\omega(t,r+h) - U_\omega(t,r)) = U_\omega(t,r)(H_\omega(r)A^*_\omega)^* U_\omega(r,t). \tag{8.13}$$

The limit being uniform on ω, the limit makes sense for the direct integral. We now prove the differentiability with respect to the first variable. As previously on account of Theorem 6.2 and $U_\omega(t,r)\mathfrak{D}_H^p \subset \mathfrak{D}_H^p$, we have

$$\lim_{h\to 0} \frac{i}{h}\left(U_\omega(t+h,r) - U_\omega(t,r)\right) A_\omega U_\omega(r,t+h) = (H_\omega(t)U_\omega(t,r)A_\omega)U_\omega(r,t) \ . \tag{8.14}$$

Next using again (8.14)

$$\lim_{h\to 0} U_\omega(t,r)A_\omega \frac{i}{h}\left(U_\omega(r,t+h) - U_\omega(r,t)\right) \tag{8.15}$$

$$= U_\omega(t,r)\left(\lim_{h\to 0}\frac{i}{h}\left(U_\omega(t+h,r) - U_\omega(t,r)\right) A_\omega^*\right)^* \tag{8.16}$$

$$= U_\omega(t,r)(H_\omega(t)U_\omega(t,r)A_\omega^*)^* \tag{8.17}$$

$$= U_\omega(t,r)A_\omega U_\omega(t,r)H_\omega(t) \tag{8.18}$$

by proposition 8.2 we finishes the proof.

□

We are now in position to define the *p-Liouvillian* . Let

$$\mathcal{U}^{(0)}(t)(A) = \int_\Omega^\oplus U_\omega^{(0)}(t)A_\omega U_\omega^{(0)}(-t)\mathrm{d}\mathbb{P}(\omega) \quad \text{for} \quad A_\omega \in L^o(\mathcal{K}_\infty)\,,$$

where $U_\omega^{(0)}(t) = \mathrm{e}^{-itH_\omega}$ as in (6.22).

Definition-Proposition 1. *$\mathcal{U}^{(0)}(t)$ is a one-parameter group of operators on $L^o(\mathcal{K}_\infty)$, preserving $L^p(\mathcal{K}_\infty)$ and $\mathfrak{D}_H^p$ for $1 \le p \le \infty$.*
Moreover $\mathcal{U}^{(0)}(t)$ is strongly continuous and unitary on the Hilbert space $L^2(\mathcal{K}_\infty)$ and an isometry on the Banach spaces $L^p(\mathcal{K}_\infty)$.
Then we define the p-Liouvillian as the infinitesimal generators of $\mathcal{U}^0(t)$ acting on $\mathfrak{D}_H^p$ such that

$$\mathcal{U}^{(0)}(t) = \mathrm{e}^{-it\mathcal{L}_p} \quad \textit{for all } t\in\mathbb{R} \textit{ on } \mathfrak{D}_H^p \ . \tag{8.19}$$

and

$$\mathcal{L}_p(A_\omega) = [H_\omega, A_\omega] \quad \textit{for all } A_\omega \in \mathfrak{D}_H^p, \quad p = [1, \ldots\infty\ ),. \tag{8.20}$$

Proof. The result follows from Propositions 8.1, 8.3 and Stone Theorem for the Hilbert space $L^2(\mathcal{K}_\infty)$ and the Hille-Yosida Theorem for the Banach space $L^p(\mathcal{K}_\infty)$. □

Remark 15. *In the same way, under the weaker assumption of [KK] we get that for $A, B \in Q(L_H)$*

$$i\partial_t\langle A_\omega, \mathcal{U}(t,r)(B_\omega)\rangle_{L^2(\mathcal{K}_\infty)} = \mathbb{L}_t(A_\omega, \mathcal{U}(t,r)(B_\omega) \tag{8.21}$$

$$i\partial_r\langle A_\omega, \mathcal{U}(t,r)(B_\omega)\rangle_{L^2(\mathcal{K}_\infty)} = -\mathbb{L}_t(\mathcal{U}(r,t)(A_\omega), B_\omega) \tag{8.22}$$

8.2 Gauge transformations in the L^p-spaces

We now prove some results that we will need on the map

$$\mathcal{G}(t)(A) = G(t)AG(t)^* \,, \tag{8.23}$$

with $G(t) = \mathrm{e}^{i\int_{-\infty}^{t} \mathbf{E}(s)\cdot\mathbf{x}}$. Using $[G(t), \chi_x] = 0$ we have that $G(t)$ is an isometry on $L^p(\mathcal{K}_\infty)$.

Lemma 8.4. *The map $\mathcal{G}(t)$ is strongly continuous on $L^p(\mathcal{K}_\infty)$ $\forall p \in [1;\infty]$ and*

$$\lim_{t\to-\infty} \mathcal{G}(t) = I \quad \textit{strongly on } L^p(\mathcal{K}_\infty)$$

Moreover $\mathcal{G}(t)(A)$ is continuously differentiable in $L^p(\mathcal{K}_\infty)$ for any $A \in \mathcal{W}^{1,p}(\mathcal{K}_\infty)$.

$$\partial_t \mathcal{G}(t)(A) = \mathbf{E}(t)\cdot\nabla(\mathcal{G}(t)(A)) \tag{8.24}$$

Proof. The proof uses exploits the Hölder inequality and Clarkson-McCarthy inequality for the L^p-spaces.
Let $A \in L^p(\mathcal{K}_\infty)$ with $p \geq 2$

$$\mathcal{G}(t+h)(A) - \mathcal{G}(t)(A) = \mathcal{G}(t)(\mathcal{G}(t+h)\mathcal{G}(-t) - 1)(A)\,. \tag{8.25}$$

Therefore continuity follows if we show that

$$\lim_{h\to 0} \left|\!\left|\!\left| (\mathcal{G}_t(h) - 1)(A) \right|\!\right|\!\right|_p = 0\,, \tag{8.26}$$

where $\mathcal{G}_t(h)(A) = G_t(h)(A)G_t(h)^*$, with $G_t(h) = G(t+h)G(-t)$ being the unitary operator given by multiplication by the function $\mathrm{e}^{-i\int_t^{t+h}\mathbf{E}(s)\cdot\mathbf{x}\,\mathrm{d}s}$. Thus

$$(\mathcal{G}_t(h) - 1)(A_\omega) = G_t(h)\,[A_\omega, (G_t(h)^* - 1)] \tag{8.27}$$

By the Clarkson inequality

$$\left|\!\left|\!\left| (\mathcal{G}_t(h) - 1)(A) \right|\!\right|\!\right|_p^p \leq 2^{p-1} \left|\!\left|\!\left| (1 - G_t(h)^*)A_\omega \right|\!\right|\!\right|_p^p + \left|\!\left|\!\left| A_\omega(G_t(h)^* - 1) \right|\!\right|\!\right|_p^p .$$

Some care must be given to the fact that $G_t(h)^*$ is not covariant but using the fact that $[G(t), \chi_x] = 0$ for any x and using the same method as in the proof of lemma 4.12.

Using the Hölder inequality in $L^2(\mathcal{K}_\infty)$ implies the result on $L^1(\mathcal{K}_\infty)$ by unitarity of $\mathcal{G}(t)$ on $L^2(\mathcal{K}_\infty)$. To prove differentiability and (8.24) it remains only to show that

$$\frac{1}{h}(\mathcal{G}_t(h) - 1)\,(A) \xrightarrow[h\to 0]{} \mathbf{E}(t)\cdot\nabla(A) \text{ on } L^p(\mathcal{K}_\infty) \tag{8.28}$$

Let $A \in \mathcal{W}^{1,p}(\mathcal{K}_\infty)$ and we define the L^p-valued Bochner integral

$$\Phi(h) = \frac{i}{h}\int_0^h \mathrm{d}u\, \mathcal{G}_t(u)\left(\mathbf{E}(t+u)\cdot\nabla(A)\right) \tag{8.29}$$

Therefore we get that $\lim_{h\to 0}\Phi(h) = i\,\mathbf{E}(t)\cdot\nabla(A)$. Secondly, taking the derivative of the kernel-operator $h\chi_x\Phi(h)\chi_y$ we get the equality with the kernel-operator of $(\mathcal{G}_t(h)-1)\chi_x A\chi_y$ using again that $[G(t),\chi_x]=0$ adding to the fact that the two operators vanish at $h=0$. We then get the desired equality. □

9 Linear response theory and Kubo formula

9.1 Adiabatic switching of the electric field

We now fix an initial equilibrium state of the system, i.e., we specify a density matrix ζ_ω whith $[H_\omega,\zeta_\omega]=0$. In physical applications, we would generally take $\zeta_\omega = f(H_\omega)$ with f the Fermi-Dirac distribution at inverse temperature $\beta\in(0,\infty]$ and *Fermi energy* $E_F\in\mathbb{R}$, i.e., $f(E)=\frac{1}{1+\mathrm{e}^{\beta(E-E_F)}}$ if $\beta<\infty$ and $f(E)=\chi_{(-\infty,E_F]}(E)$ if $\beta=\infty$; explicitly

$$\zeta_\omega = \begin{cases} F_\omega^{(\beta,E_F)} := \frac{1}{1+\mathrm{e}^{\beta(H_\omega-E_F)}}\,, & \beta<\infty\,,\\ P_\omega^{(E_F)} := \chi_{(-\infty,E_F]}(H_\omega)\,, & \beta=\infty\,. \end{cases} \tag{9.1}$$

However we note that our analysis allows for fairly general functions f [BoGKS]. We set $\zeta=(\zeta_\omega)_{\omega\in\Omega}\in\mathcal{K}_\infty$ but shall also write ζ_ω instead of ζ. That f is the Fermi-Dirac distribution plays no role in the derivation of the linear response. However computing the Hall conductivity itself (once the linear response is performed) we shall restrict our attention to the zero temperature case with the *Fermi projection* $P_\omega^{(E_F)}$.

The system is described by the ergodic time dependent Hamiltonian $H_\omega(t)$, as in (6.5). Assuming the system was in equilibrium at $t=-\infty$ with the density matrix $\varrho_\omega(-\infty)=\zeta_\omega$, the time dependent density matrix $\varrho_\omega(t)$ is the solution of the Cauchy problem for the Liouville equation. Since we shall solve the evolution equation in $\mathrm{L}^p(\mathcal{K}_\infty)$, we work with $H(t)=(H_\omega(t))_{\omega\in\Omega}$, as in Assumption 1.

The electric field $\mathbf{E}_\eta(t)\cdot\mathbf{x} = \mathrm{e}^{\eta t}\mathbf{E}(t)\cdot\mathbf{x}$ is switched on adiabatically between $t=-\infty$ and $t=t_0$ (typically $t_0=0$). Depending on which conductivity we are interested in we may consider different forms for $\mathbf{E}(t)$. In particular $\mathbf{E}(t)=\mathbf{E}$ leads to the direct conductivity, while $\mathbf{E}(t)=cos(\nu t)\mathbf{E}$ leads to the

AC-conductivity at frequency ν [1]. The first one is relevant when studying the Quantum Hall effect (see subsection 9.4), while the second enters the Mott's formula [KLP, KLM].

Assumptions 2. *The initial equilibrium state ζ_ω is non-negative, i.e. $\zeta_\omega \geq 0$. For any $p \in [2, \infty)$ $\zeta_\omega \in \mathcal{W}^{1,1}(\mathcal{K}_\infty) \cap \mathcal{W}^{1,p}(\mathcal{K}_\infty)$ and $\nabla\zeta_\omega \in \mathfrak{D}^p_H$.*

Remark 16. *By Proposition 7.8(ii) and Prop. 7.8(iii) this assumption is satisfied if $\zeta_\omega = g(H_\omega)$ with $g \in \mathcal{S}(\mathbb{R})$, or $\zeta = g(H)h(H)$ with $g \in \mathcal{S}(\mathbb{R})$ and $h \in L^\Phi_\infty(\mathbb{R})$ and $h(H) \in \mathcal{W}^{1,p}$.*

It has been proved that the assumption holds whenever the Fermi energi belongs to localization region as proved in [GK5]. The existence of localization region is well know for a random scalar potential. In [DGR2] the existence of localization region for random magnetic potential is proved.

As explained in [BoGKS] the Fermi-Dirac distribution at all temperature satisfies the above assumptions.

The electric field $\mathbf{E}_\eta(t)\cdot\mathbf{x} = \mathrm{e}^{\eta t}\mathbf{E}(t)\cdot\mathbf{x}$ is switched on adiabatically between $t = -\infty$ and $t = t_0$ (typically $t_0 = 0$).

We write

$$\zeta(t) = G(t)\zeta G(t)^* = \mathcal{G}(t)(\zeta), \quad \text{i.e.,} \quad \zeta(t) = f(H(t)). \tag{9.2}$$

and the p-Liouvillian at time t:

$$\mathcal{L}_p(t) = \mathcal{G}(t)\mathcal{L}_p\mathcal{G}(t)^*, \quad p = [1; \infty)\,. \tag{9.3}$$

Theorem 9.1. *Let $\eta > 0$ and assume that $\int_{-\infty}^t \mathrm{e}^{\eta r}|\mathbf{E}(r)|\mathrm{d}r < \infty$ for all $t \in \mathbb{R}$. The Cauchy problem*

$$\begin{cases} i\partial_t\varrho(t) = [H(t), \varrho(t))] \\ \lim_{t\to-\infty}\varrho(t) = \zeta \end{cases}, \tag{9.4}$$

has a unique solution in $\mathrm{L}^p(\mathcal{K}_\infty)$, that is given by

$$\begin{aligned} \varrho(t) &= \lim_{s\to-\infty} \mathcal{U}(t,s)\,(\zeta) && (9.5)\\ &= \lim_{s\to-\infty} \mathcal{U}(t,s)\,(\zeta(s)) && (9.6)\\ &= \zeta(t) - \int_{-\infty}^t \mathrm{d}r\, \mathrm{e}^{\eta r}\mathcal{U}(t,r)(\mathbf{E}(r)\cdot\nabla\zeta(r)). && (9.7) \end{aligned}$$

1. The AC-conductivity may be better defined using the from (9.23) as argued in [KLM].

We also have

$$\varrho(t) = \mathcal{U}(t,s)(\varrho(s))\,, \quad \|\varrho(t)\|_p = \|\zeta\|_p\,, \tag{9.8}$$

for all t, s. *Furthermore,* $\varrho(t)$ *is non-negative, and if* ζ *is a projection, then so is* $\varrho(t)$ *for all* t.

Proof of Theorem 9.1. Let us first define

$$\varrho(t,s) := \mathcal{U}(t,s)(\zeta(s)). \tag{9.9}$$

We get, as operators in $\mathrm{L}^o(\mathcal{K}_\infty)$,

$$\begin{aligned}\partial_s \varrho(t,s) &= i\mathcal{U}(t,s)\left([H(s),\zeta(s)]\right) + \mathcal{U}(t,s)\left(\mathbf{E}_\eta(s)\cdot\nabla\zeta(s)\right)\\ &= \mathcal{U}(t,s)\left(\mathbf{E}_\eta(s)\cdot\nabla\zeta(s)\right)\,,\end{aligned} \tag{9.10}$$

where we used (8.3) and Lemma 8.4. As a consequence, with $\mathbf{E}_\eta(r) = \mathrm{e}^{\eta r}\mathbf{E}(r)$,

$$\varrho(t,t) - \varrho(t,s) = \int_s^t \mathrm{d}r\, \mathrm{e}^{\eta r}\mathcal{U}(t,r)(\mathbf{E}(r)\cdot\nabla\zeta(r)). \tag{9.11}$$

Since $\|\mathcal{U}(t,r)(\mathbf{E}(r)\cdot\nabla(\zeta(r))\|_p \le c_d|\mathbf{E}(r)|\ \|\nabla\zeta\|_p < \infty$, the integral is absolutely convergent by hypothesis on $\mathbf{E}_\eta(t)$, and the limit as $s \to -\infty$ can be performed in $\mathrm{L}^p(\mathcal{K}_\infty)$. It yields the equality between (9.6) and (9.7). Equality of (9.5) and (9.6) follows from Lemma 8.4 which gives

$$\zeta = \lim_{s\to-\infty} \zeta(s) \ \text{ in } \mathrm{L}^p(\mathcal{K}_\infty). \tag{9.12}$$

Since $\mathcal{U}(t,s)$ are isometries on $\mathrm{L}^p(\mathcal{K}_\infty)$, it follows from (9.5) that $\|\varrho(t)\|_p = \|\zeta\|_p$. We also get $\varrho(t) = \varrho(t)^*$. Moreover, (9.5) with the limit in $\mathrm{L}^p(\mathcal{K}_\infty)$ implies that $\varrho(t)$ is nonnegative.

Furthermore, if $\zeta = \zeta^2$ then $\varrho(t)$ can be seen to be a projection as follows. Note that convergence in L^p implies convergence in $\mathrm{L}^o(\mathcal{K}_\infty)$, so that,

$$\begin{aligned}\varrho(t) &= \lim_{s\to-\infty}{}^{(\tau)}\, \mathcal{U}(t,s)\,(\zeta) = \lim_{s\to-\infty}{}^{(\tau)}\, \mathcal{U}(t,s)\,(\zeta)\,\mathcal{U}(t,s)\,(\zeta)\\ &= \left\{\lim_{s\to-\infty}{}^{(\tau)}\, \mathcal{U}(t,s)\,(\zeta)\right\}\left\{\lim_{s\to-\infty}{}^{(\tau)}\, \mathcal{U}(t,s)\,(\zeta)\right\} \;=\; \varrho(t)^2\,.\end{aligned} \tag{9.13}$$

where we denote $\lim^{(\tau)}$ the limit in the topological algebra $\mathrm{L}^o(\mathcal{K}_\infty)$. We have choosen the lim in measurable operator for generality but we could only work in the $L^p(\mathcal{K}_\infty)$ spaces using the Hölder inequality for the product of limits.

To see that $\varrho(t)$ is a solution of (9.4) in $\mathrm{L}^p(\mathcal{K}_\infty)$, we differentiate the expression (9.7) using lemma 8.3 and Lemma 8.4. We get

$$\begin{aligned} i\partial_t\varrho(t) &= -\int_{-\infty}^{t} \mathrm{d}r\, \mathrm{e}^{\eta r}\, [H(t), \mathcal{U}(t,r)\, (\mathbf{E}(r)\cdot d\zeta(r))] && (9.14) \\ &= -\left[H(t), \left\{\int_{-\infty}^{t} \mathrm{d}r\, \mathrm{e}^{\eta r}\mathcal{U}(t,r)\, (\mathbf{E}(r)\cdot d\zeta(r))\right\}\right] && (9.15) \\ &= \left[H(t), \left\{\zeta(t) - \int_{-\infty}^{t} \mathrm{d}r\, \mathrm{e}^{\eta r}\mathcal{U}(t,r)\, (\mathbf{E}(r)\cdot d\zeta(r))\right\}\right] \\ &= [H(t), \varrho(t)]\,. && (9.16) \end{aligned}$$

The integral being Bochner integral in the Banach space $L^p(\mathcal{K}_\infty)$, in (9.14) converges since (8.7). Then we justify going from (9.14) to (9.15) by inserting a resolvent $(H(t)+\gamma)^{-1}$ and making use of (8.6) and then remove it.
It remains to show that the solution of (9.4) is unique in $\mathrm{L}^p(\mathcal{K}_\infty)$. It suffices to show that if $\nu(t)$ is a solution of (9.4) with $\zeta = 0$ then $\nu(t) = 0$ for all t. We define $\tilde{\nu}^{(s)}(t) = \mathcal{U}(s,t)(\nu(t))$ and proceed by duality. Since $p \geq 1$, with pick q s.t. $p^{-1} + q^{-1} = 1$. If $A \in \mathcal{D}_H^q$, we have, using the lemma 5.8

$$\begin{aligned} &i\partial_t\mathcal{T}\left\{A\tilde{\nu}^{(s)}(t)\right\} = i\partial_t\mathcal{T}\left\{\mathcal{U}(t,s)(A)\nu(t)\right\} && (9.17) \\ &\quad = \mathcal{T}\left\{\mathcal{L}_q(\mathcal{U}(t,s)(A))\ \nu(t)\right\} + \mathcal{T}\left\{\mathcal{U}(t,s)(A)\mathcal{L}_p(t)(\nu(t))\right\} \\ &\quad = -\mathcal{T}\left\{\mathcal{U}(t,s)(A)\mathcal{L}_p(t)(\nu(t))\right\} + \mathcal{T}\left\{\mathcal{U}(t,s)(A)\ \mathcal{L}_p(t)(\nu(t))\right\} = 0\,. \end{aligned}$$

We conclude that for all t and $A \in \mathcal{D}_H^q$ we have

$$\mathcal{T}\left\{A\tilde{\nu}^{(s)}(t)\right\} = \mathcal{T}\left\{A\tilde{\nu}^{(s)}(s)\right\} = \mathcal{T}\left\{A\nu(s)\right\}. \tag{9.18}$$

Thus $\tilde{\nu}^{(s)}(t) = \nu(s)$ by Lemma 4.11 that is, $\nu(t) = \mathcal{U}(t,s)(\nu(s))$. Since by hypothesis $\lim_{s\to-\infty}\nu(s) = 0$, we obtain that $\nu(t) = 0$ for all t. □

9.2 The current and the conductivity

We recall that the velocity operator $\mathbf{v}$ is defined as

$$\mathbf{v} = \mathbf{v}(\mathbf{A}) = 2\mathbf{D}(\mathbf{A}), \tag{9.19}$$

where $\mathbf{D} = \mathbf{D}(\mathbf{A})$ is defined below (6.6). Recall that $\mathbf{v} = 2(-i\nabla - \mathbf{A}) = i[H, \mathbf{x}]$ on $C_c^\infty(\mathbb{R}^d)$. We also set $\mathbf{D}(t) = \mathbf{D}(\mathbf{A} + \mathbf{F}_\eta(t))$ as in (6.13), and $\mathbf{v}(t) = 2\mathbf{D}(t)$.

From now on $\varrho(t)$ will denote the unique solution to (9.4), given explicitly in (9.7). If $H(t)\varrho(t) \in L^p(\mathcal{K}_\infty)$ then clearly $\mathbf{D}_j(t)(\varrho(t))$ can be defined as well by

$$\mathbf{D}_j(t)\varrho(t) = \left(\mathbf{D}_j(t)(H(t)+\gamma)^{-1}\right)((H(t)+\gamma)\varrho(t))\,, \tag{9.20}$$

since $\mathbf{D}_j(t)(H(t)+\gamma)^{-1} \in \mathcal{K}_\infty$, and thus $\mathbf{D}_j(t)\varrho(t) \in \mathrm{L}^p(\mathcal{K}_\infty)$.

Definition 9.2. *Starting with a system in equilibrium in state ζ, the net current (per unit volume), $\mathbf{J}(\eta, \mathbf{E}; \zeta, t_0) \in \mathbb{R}^d$, generated by switching on an electric field $\mathbf{E}$ adiabatically at rate $\eta > 0$ between time $-\infty$ and time t_0, is defined as*

$$\mathbf{J}(\eta, \mathbf{E}; \zeta, t_0) = \mathcal{T}\left(\mathbf{v}(t_0)\varrho(t_0)\right) - \mathcal{T}\left(\mathbf{v}\zeta\right). \tag{9.21}$$

As it is well known, the current is null at equilibrium: As straightforward consequence of the lemma 7.13 we have

Lemma 9.3. *One has for all $j = 1, \cdots, d$, and thus $\mathcal{T}\left(\mathbf{v}_i\zeta\right) = 0$.*

Assume that the electric field has the form

$$\mathbf{E}(t) = \mathcal{E}(t)\mathbf{E}, \tag{9.22}$$

where $\mathbf{E} \in \mathbb{C}^d$ gives the intensity of the electric in each direction while $|\mathcal{E}(t)| = \mathcal{O}(1)$ modulates this intensity as time varies. As pointed out above, the two cases of particular interest are $\mathcal{E}(t) = 1$ and $\mathcal{E}(t) = \cos(\nu t)$. We may however, as in [KLM], use the more general form

$$\mathcal{E}(t) = \int_{\mathbb{R}} \cos(\nu t)\hat{\mathcal{E}}(\nu)\mathrm{d}\nu, \tag{9.23}$$

for suitable $\hat{\mathcal{E}}(\nu)$ (see [KLM]).

It is useful to rewrite the current (9.21), using (9.7) and Lemma 9.3, as

$$\begin{aligned} \mathbf{J}(\eta, \mathbf{E}; \zeta, t_0) &= \mathcal{T}\left\{2\mathbf{D}(0)\left(\varrho(t_0) - \zeta(t_0)\right)\right\} \qquad (9.24)\\ &= -\mathcal{T}\left\{2\int_{-\infty}^{t_0} \mathrm{d}r\, \mathbf{D}(0)\,\mathcal{U}(t_0, r)\left(\mathbf{E}(r)\cdot\nabla\zeta(r)\right)\right\}. \\ &= -\mathcal{T}\left\{2\int_{-\infty}^{t_0} \mathrm{d}r\, \mathrm{e}^{\eta r}\mathcal{E}(r)\mathbf{D}(0)\,\mathcal{U}(t_0, r)\left(\mathbf{E}\cdot\partial\zeta(r)\right)\right\}. \end{aligned}$$

The conductivity tensor $\sigma(\eta; \zeta, t_0)$ is defined as the derivative of the function $\mathbf{J}(\eta, \mathbf{E}; \zeta, t_0)\colon \mathbb{R}^d \to \mathbb{R}^d$ at $\mathbf{E} = 0$. Note that $\sigma(\eta; \zeta, t_0)$ is a $d \times d$ matrix $\{\sigma_{jk}(\eta; \zeta, t_0)\}$.

Definition 9.4. *For $\eta > 0$ and $t_0 \in \mathbb{R}$, the conductivity tensor $\sigma(\eta; \zeta, t_0)$ is defined as*

$$\sigma(\eta; \zeta, t_0) = \partial_{\mathbf{E}}(\mathbf{J}(\eta, \mathbf{E}; \zeta, t_0))_{|\mathbf{E}=0}\,, \tag{9.25}$$

if it exists. The conductivity tensor $\sigma(\zeta, t_0)$ is defined by

$$\sigma(\zeta, t_0) := \lim_{\eta \downarrow 0} \sigma(\eta; \zeta, t_0)\,, \tag{9.26}$$

whenever the limit exists.

9.3 Computing the linear response: a Kubo formula for conductivity

The next theorem gives a "Kubo formula" for conductivity at positive adiabatic parameter and positive temperature. The Kubo formula gives the component of the conductance tensor as an integral of the response function which is the *current-current correlation function.*

Theorem 9.5. *Let $\eta > 0$. Under the hypotheses of Theorem 9.1 for $p = 1$, the current $\mathbf{J}(\eta, \mathbf{E}; \zeta, t_0)$ is differentiable with respect to $\mathbf{E}$ at $\mathbf{E} = 0$ and the derivative $\sigma(\eta; \zeta)$ is given by*

$$\sigma_{jk}(\eta; \zeta, t_0) \quad = \quad -\mathcal{T}\left\{2\int_{-\infty}^{t_0} \mathrm{d}r\, \mathrm{e}^{\eta r}\mathcal{E}(r)\mathbf{D}_j\,\mathcal{U}^{(0)}(t_0 - r)\left(\partial_k(\zeta)\right)\right\}. \tag{9.27}$$

The analogue of [BES, Eq. (41)] and [SB2, Theorem 1] then holds:

Corollary 9.6. *Assume that $\mathcal{E}(t) = \Re \mathrm{e}^{i\nu t}$, $\nu \in \mathbb{R}$, then the conductivity $\sigma_{jk}(\eta; \zeta; \nu)$ at frequency ν is given by*

$$\begin{aligned} \sigma_{jk}(\eta; \zeta; \nu; 0) \quad &= \quad -\mathcal{T}\left\{2\mathbf{D}_j\left(i\mathcal{L}_1 + \eta + i\nu\right)^{-1}\left(\partial_k\zeta\right)\right\} & (9.28) \\ &= \quad -\langle 2\mathbf{D}_j | (i\mathcal{L}_1 + \eta + i\nu)^{-1}\partial_k\zeta\rangle_{L^2(\mathcal{K}_\infty)} & (9.29) \end{aligned}$$

Proof of corollary 9.6. Recall 9.12 , in particular $\zeta = \zeta^{\frac{1}{2}}\zeta^{\frac{1}{2}}$. It follows that $\sigma(\eta; \nu; \zeta; 0)$ in (4.27) is real (for arbitrary $\zeta = f(H)$ write $f = f_+ - f_-$). As a consequence ,

$$\sigma(\eta; \nu; \zeta; 0) = -\Re\mathcal{T}\left\{2\int_{-\infty}^{t_0} \mathrm{d}r\, \mathrm{e}^{\eta r}\mathrm{e}^{i\nu r}\mathbf{D}_j\,\mathcal{U}^{(0)}(t_0 - r)\left(\partial_k(\zeta)\right)\right\}. \tag{9.30}$$

Integrating over r yields the result. □

Proof of Theorem 9.5. For clarity, in this proof we display the argument $\mathbf{E}$ in all functions which depend on $\mathbf{E}$. From (9.24) and $\mathbf{J}_j(\eta, 0; \zeta, t_0) = 0$ (Lemma 9.3), we have

$$\sigma_{jk}(\eta; \zeta, t_0) = -\lim_{E\to 0} 2\mathcal{T}\left\{\int_{-\infty}^{t_0} \mathrm{d}r\, \mathrm{e}^{\eta r}\mathcal{E}(r)\mathbf{D}_{\mathbf{E},j}(0)\mathcal{U}(\mathbf{E}, 0, r)\left(\partial_k\zeta(\mathbf{E}, r)\right)\right\}. \tag{9.31}$$

First understand we can interchange integration and the limit $\mathbf{E} \to 0$, and get

$$\sigma_{jk}(\eta; \zeta, t_0) \;=\; -2\int_{-\infty}^{t_0} \mathrm{d}r\, \mathrm{e}^{\eta r}\mathcal{E}(r)\lim_{E\to 0}\mathcal{T}\left\{\mathbf{D}_j(\mathbf{E}, 0)\mathcal{U}(\mathbf{E}, 0, r)\left(\partial_k\zeta(\mathbf{E}, r)\right)\right\}. \tag{9.32}$$

The latter can easily be seen by inserting a resolvent $(H(t)+\gamma)^{-1}$ and making use of (8.7), the fact that $H\nabla\zeta \in \mathrm{L}^1(\mathcal{K}_\infty)$, the inequality : $|\mathcal{T}(A)| \le \mathcal{T}(|A|)$ and dominated convergence theorem.

Next we just recall that for any s we have

$$\lim_{E\to 0} \mathcal{G}(\mathbf{E}, s) = I \text{ strongly in } \mathrm{L}^1(\mathcal{K}_\infty)\,, \tag{9.33}$$

which can be proven by an argument similar to the one used to prove Lemma 8.4. Along the same lines, for $B \in \mathcal{K}_\infty$ we have

$$\lim_{E\to 0} \mathcal{G}(\mathbf{E}, s)(B_\omega) = B_\omega \text{ strongly in } \mathfrak{H}, \text{ with } \|\mathcal{G}(\mathbf{E}, s)(B)\|_\infty = \|B\|_\infty\,. \tag{9.34}$$

Recalling that $\mathbf{D}_{j,\omega}(\mathbf{E}, 0) = \mathbf{D}_{j,\omega} - \mathbf{F}_j(0)$ and that $\|\partial_k\zeta(\mathbf{E}, r)\|_1 = \|\partial_k\zeta\|_1 < \infty$, using Lemma 4.12,

$$\begin{aligned}
&\lim_{E\to 0} \mathcal{T}\{\mathbf{D}_j(\mathbf{E}, 0)\mathcal{U}(\mathbf{E}, 0, r)(\partial_k\zeta(\mathbf{E}, r))\} \\
&= \lim_{E\to 0} \mathcal{T}\left\{\mathbf{D}_j(\mathbf{E}, 0)(H_\omega(\mathbf{E}, 0) + \gamma)^{-1}\right\} \\
&\qquad \times (H_\omega(\mathbf{E}, 0) + \gamma)\mathcal{U}(\mathbf{E}, 0, r)(H_\omega(\mathbf{E}, r) + \gamma)^{-1} \\
&\qquad\qquad \times (H_\omega(\mathbf{E}, r) + \gamma)\left(\partial_k\zeta(\mathbf{E}, r)\right)\}
\end{aligned} \tag{9.35}$$

Taking in account 8.6 we can then write that

$$\begin{aligned}
\lim_{E\to 0}\mathcal{T}\left\{\mathbf{D}_j(\mathbf{E}, 0)\mathcal{U}(\mathbf{E}, 0, r)\left(\partial_k\zeta(\mathbf{E}, r)\right)\right\} &= \lim_{\mathbf{E}\to 0} \mathcal{T}\left\{\mathbf{D}_j U(\mathbf{E}, 0, r)(\partial_k\zeta)U(\mathbf{E}, r, 0)\right\} \\
&= \lim_{\mathbf{E}\to 0} \mathcal{T}\left\{\mathbf{D}_j U(\mathbf{E}, 0, r)(\partial_k\zeta)U^{(0)}(r)\right\},
\end{aligned} \tag{9.36}$$

To proceed it is convenient to introduce a cutoff so that we can deal with $\mathbf{D}_j$ as if it were in $\mathcal{K}_\infty$. Thus we pick $f_n \in C_c^\infty(\mathbb{R})$, real valued, $|f_n| \le 1$, $f_n = 1$ on $[-n, n]$, so that $f_n(H)$ converges strongly to 1. We have

$$\begin{aligned}
&\mathcal{T}\left\{\mathbf{D}_j U(\mathbf{E}, 0, r)(\partial_k\zeta)U^{(0)}(r)\right\} = \lim_{n\to\infty} \mathcal{T}\left\{f_n(H)\mathbf{D}_j U(\mathbf{E}, 0, r)(\partial_k\zeta)U^{(0)}(r)\right\} \\
&= \lim_{n\to\infty} \mathcal{T}\left\{U(\mathbf{E}, 0, r)\left((\partial_k\zeta)(H + \gamma)\right)U^{(0)}(r)(H + \gamma)^{-1} f_n(H)\mathbf{D}_j\right\} \\
&= \mathcal{T}\left\{U(\mathbf{E}, 0, r)\left((\partial_k\zeta)(H + \gamma)\right)\left(U^{(0)}(r)(H + \gamma)^{-1}\mathbf{D}_j\right)\right\}\,,
\end{aligned} \tag{9.37}$$

where we used the centrality of the trace, the fact that $(H + \gamma)^{-1}$ commutes with $U^{(0)}$ and then that $(H + \gamma)^{-1}\mathbf{D}_j \in \mathcal{K}_\infty$ in order to remove to limit $n \to \infty$. Finally, combining (9.36) and (9.37), we get

$$\lim_{E\to 0}\mathcal{T}\left\{\mathbf{D}_j(\mathbf{E}, 0)\mathcal{U}(\mathbf{E}, 0, r)\left(\partial_k\zeta(\mathbf{E}, r)\right)\right\} \tag{9.38}$$

$$\begin{aligned}
&= \mathcal{T}\left\{U^{(0)}(-r)\;\left((\partial_k\zeta)(H + \gamma)\right)U^{(0)}(r)(H + \gamma)^{-1}\mathbf{D}_j\right\} \\
&= \mathcal{T}\left\{\mathbf{D}_j\mathcal{U}^{(0)}(-r)(\partial_k\zeta)\right\}.
\end{aligned} \tag{9.39}$$

The Kubo formula (9.27) now follows from (9.32) and (9.39). □

9.4 The Kubo-Středa formula for the Hall conductivity

Following [BES, AG], we now recover the well-known Kubo-Středa formula for the Hall conductivity at zero temperature (see however Remark 17 for AC-conductivity). To that aim we consider the case $\mathcal{E}(t) = 1$ and $t_0 = 0$. Recall Definition 9.4. We write

$$\sigma_{j,k}^{(E_f)} = \sigma_{j,k}(P^{(E_F)}, 0) \ , \text{ and } \ \sigma_{j,k}^{(E_f)}(\eta) = \sigma_{j,k}(\eta; P^{(E_F)}, 0) \ . \tag{9.40}$$

Theorem 9.7. *Take $\mathcal{E}(t) = 1$ and $t_0 = 0$. If $\zeta = P^{(E_F)}$ is a Fermi projection satisfying the hypotheses of Theorem 9.1 with $p = 2$, we have*

$$\sigma_{j,k}^{(E_F)} = -i\mathcal{T}\left\{P^{(E_F)}\left[\partial_j P^{(E_F)}, \partial_k P^{(E_F)}\right]\right\}, \tag{9.41}$$

for all $j, k = 1, 2, \ldots, d$. As a consequence, the conductivity tensor is antisymmetric; in particular $\sigma_{j,j}^{(E_F)} = 0$ for $j = 1, 2, \ldots, d$.

Clearly the direct conductivity vanishes, $\sigma_{jj}^{(E_F)} = 0$. Note that, if the system is time-reversible the off diagonal elements are zero in the region of localization, as expected.

Corollary 9.8. *Under the assumptions of Theorem 9.7, if $\mathbf{A} = 0$ (no magnetic field), we have $\sigma_{j,k}^{(E_F)} = 0$ for all $j, k = 1, 2, \ldots, d$.*

This follows from the fact that we can define an antiunitary complex conjugaison on $\mathfrak{H}$ by $J\varphi = \overline{\varphi}$ and so on $\mathcal{K}_\infty$ by $\Theta(A) = JAJ$. Then if the magnetic field is zero we have that the Hamiltonian is invariant under Θ. But Θ respect the following properties, $\mathcal{T}(\Theta(A)) = \mathcal{T}(A)$ and $\Theta(\partial_i(P_\omega)) = -\partial_i(P_\omega)$. Then by the above theorem and this remark we have that $\sigma_{j,k}^{(E_F)} = -\sigma_{j,k}^{(E_F)}$ and the result follows immediately.

We have the crucial following lemma for computing the Kubo-Středa formula, which already appears in [BES] (and then in [AG]).

Lemma 9.9. *Let $P \in \mathcal{K}_\infty$ be a projection such that $\partial_k P \in \mathrm{L}^p(\mathcal{K}_\infty)$, then as operators in $\mathrm{L}^o(\mathcal{K}_\infty)$ (and thus in $\mathrm{L}^p(\mathcal{K}_\infty)$),*

$$\partial_k P = [P, [P, \partial_k P]] \, . \tag{9.42}$$

Proof. Here we note that $\partial_k P = \partial_k P^2 = P\partial_k P + (\partial_k P)P$ so that multiplying left and right both sides by P implies that $P(\partial_k P)P = 0$. We then have, in $\mathrm{L}^p(\mathcal{K}^\infty)$,

$$\begin{aligned}\partial_k P &= P\partial_k P + (\partial_k P)P = P\partial_k P + (\partial_k P)P - 2P(\partial_k P)P \\ &= P(\partial_k P)(1-P) + (1-P)(\partial_k P)P \\ &= [P,[P,\partial_k P]]\,.\end{aligned}$$

□

Note that Lemma (9.9) heavily relies on the fact that P is a projection. We shall apply it to the situation of zero temperature, i.e. when the initial density matrix is the orthogonal projection $P^{(E_F)}$. The argument would not go through at positive temperature.

Proof of Theorem 9.7. We again regularize the velocity $\mathbf{D}_{j,\omega}$ with a smooth function $f_n \in \mathcal{C}_c^\infty(\mathbb{R})$, $|f_n| \leq 1$, $f_n = 1$ on $[-n,n]$, but this time we also require that $f_n = 0$ outside $[-n-1,n+1]$, so that $f_n\chi_{[-n-1,n+1]} = f_n$. Thus $\mathbf{D}_j f_n(H) \in \mathrm{L}^p_o(\mathcal{K}_\infty)$, $0 < p \leq \infty$. Moreover

$$f_n(H)(2\mathbf{D}_j)f_n(H) = f_n(H)P_n(2\mathbf{D}_j)P_n f_n(H) = -f_n(H)\partial_j(P_nH)f_n(H) \tag{9.43}$$

where $P_n = P_n^2 = \chi_{[-n-1,n+1]}(H)$ so that HP_n is a bounded operator. We have, using the centrality of the trace $\mathcal{T}$, that

$$\begin{aligned}\widetilde{\sigma}_{jk}^{(E_F)}(r) &:= -\mathcal{T}\left\{2\mathbf{D}_{j,\omega}\mathcal{U}^{(0)}(-r)(\partial_k P^{(E_F)})\right\} &(9.44)\\ &= -\lim_{n\to\infty}\mathcal{T}\left\{\mathcal{U}^{(0)}(r)(f_n(H)2\mathbf{D}_{j,\omega}f_n(H))\partial_k P^{(E_F)}\right\}. &(9.45)\end{aligned}$$

Using Lemma 5.8 and applying Lemma 9.9 applied to $P = P^{(E_F)}$, it follows that

$$\begin{aligned}&\mathcal{T}\left\{\mathcal{U}^{(0)}(r)(f_n(H)2\mathbf{D}_j f_n(H))\partial_k P^{(E_F)}\right\} &(9.46)\\ &\quad= \mathcal{T}\left\{\mathcal{U}^{(0)}(r)(f_n(H)2\mathbf{D}_j f_n(H))\left[P^{(E_F)},\left[P^{(E_F)},\partial_k P^{(E_F)}\right]\right]\right\}\\ &\quad= \mathcal{T}\left\{\mathcal{U}^{(0)}(r)\left(\left[P^{(E_F)},\left[P^{(E_F)},f_n(H)2\mathbf{D}_j f_n(H)\right]\right]\right)\partial_k P^{(E_F)}\right\},\\ &\quad= -\mathcal{T}\left\{\mathcal{U}^{(0)}(r)\left(\left[P^{(E_F)},f_n(H)\left[P^{(E_F)},\partial_j(HP_n)\right]f_n(H)\right]\right)\partial_k P^{(E_F)}\right\},\end{aligned}$$

where we used that $P^{(E_F)}$ commutes with $U^{(0)}$ and $f_n(H)$, and (9.43). Now, as elements in $L^o(\mathcal{K}^\infty)$,

$$\left[P^{(E_F)},\partial_j HP_n\right] = \left[HP_n,\partial_j P^{(E_F)}\right]. \tag{9.47}$$

Since $[H, \partial_j P^{(E_F)}]]$ is well defined by hypothesis, $f_n(H)\left[HP_n, \partial_j P^{(E_F)}\right] f_n(H)$ converges in L^p to the latter as n goes to infinity. Combining (9.45), (9.46), and (9.47), we get after taking $n \to \infty$,

$$\widetilde{\sigma}_{jk}^{(E_F)}(r) \;=\; -\mathcal{T}\left\{\mathcal{U}^{(0)}(r)\left(\left[P^{(E_F)},\left[H,\partial_j P^{(E_F)}\right]\right]\right)\partial_k P^{(E_F)}\right\}\,. \tag{9.48}$$

Next,

$$\left[P^{(E_F)},\left[H,\partial_j P^{(E_F)}\right]\right] = \left[H,\left[P^{(E_F)},\partial_j P^{(E_F)}\right]\right], \tag{9.49}$$

so that, recalling Proposition 1,

$$\begin{aligned}\widetilde{\sigma}_{jk}^{(E_F)}(r) \;&=\; -\mathcal{T}\left\{\mathcal{U}^{(0)}(r)\left(\left[H,\left[P^{(E_F)},\partial_j P^{(E_F)}\right]\right]\right)\partial_k P^{(E_F)}\right\}\\ &=\; -\left\langle \mathrm{e}^{-ir\mathcal{L}}\mathcal{L}_2\left(\left[P^{(E_F)},\partial_j P^{(E_F)}\right]\right),\partial_k P^{(E_F)}\right\rangle_{\mathrm{L}^2},\end{aligned} \tag{9.50}$$

where we note $\langle A, B\rangle_{\mathrm{L}^2} = \mathcal{T}(A^*B)$ instead of $\langle\langle A, B\rangle\rangle$ to make the computation lighter. Combining (9.27), (9.44), and (9.50), we get

$$\sigma_{jk}^{(E_F)}(\eta) = -\left\langle i\left(\mathcal{L}_2 + i\eta\right)^{-1}\mathcal{L}_2\left(\left[P^{(E_F)},\partial_j P^{(E_F)}\right]\right),\partial_k P^{(E_F)}\right\rangle_{\mathrm{L}^2}. \tag{9.51}$$

It follows from the spectral theorem (applied to $\mathcal{L}_2$) that

$$\lim_{\eta\to 0}\left(\mathcal{L}_2 + i\eta\right)^{-1}\mathcal{L}_2 = P_{(\mathrm{Ker}\,\mathcal{L}_2)^\perp} \text{ strongly in } \mathrm{L}^2(\mathcal{K}_\infty)\,, \tag{9.52}$$

where $P_{(\mathrm{Ker}\,\mathcal{L}_2)^\perp}$ is the orthogonal projection onto $(\mathrm{Ker}\,\mathcal{L}_2)^\perp$. Moreover, as in [BoGKS] one can prove that

$$\left[P^{(E_F)},\partial_j P^{(E_F)}\right] \in (\mathrm{Ker}(\mathcal{L}_2))^\perp\,. \tag{9.53}$$

by the fact $A \in \mathrm{Ker}(\mathcal{L}_2)$ is equivalent to $[f(H_\omega), A_\omega] = 0$ and also with $f(H_\omega) = P^{(E_F)}$. Then immediately,

$$\langle A, [P^{(E_F)}, \partial_i P^{(E_F)}]\rangle_{L^2} = \langle [P^{(E_F)}, A], \partial_i P^{(E_F)}\rangle_{L^2} = 0$$

Combining (9.51), (9.52), (9.53), and Lemma 5.8, we get

$$\sigma_{j,k}^{(E_F)} = i\left\langle\left[P^{(E_F)},\partial_j P^{(E_F)}\right],\partial_k P^{(E_F)}\right\rangle_{\mathrm{L}^2} = -i\mathcal{T}\left\{P^{(E_F)}\left[\partial_j P^{(E_F)},\partial_k P^{(E_F)}\right]\right\}\,,$$

which is just (9.41). □

Remark 17. *If one is interested in the AC-conductivity, then the proof above is valid up to* (9.51)*. In particular, with* $\mathcal{E}(t) = \Re\mathrm{e}^{i\nu t}$*, one obtains*

$$\sigma_{jk}^{(E_F)}(\eta) = -\Re\left\langle i\left(\mathcal{L}_2 + \nu + i\eta\right)^{-1}\mathcal{L}_2\left(\left[P^{(E_F)},\partial_j P^{(E_F)}\right]\right),\partial_k P^{(E_F)}\right\rangle_{\mathrm{L}^2}. \tag{9.54}$$

The limit $\eta \to 0$ *can still be performed as in [KLM, Corollary 3.4]. It is the main achievement of [KLM] to be able to investigate the behaviour of this limit as* $\nu \to 0$ *in connection with Mott's formula.*

Chapter 3

Quantification et invariance des courants de bord pour un mur magnétique

Co-écrit avec François Germinet et Georgi Raikov

Titre original :Quantization of the edge conductance for magnetic perturbation of Iwatsuka Hamiltonians.

Résumé

Nous étudions la conductance de bord d'une particule soumise à un champ magnétique de type Iwatsuka, jouant le rôle d'un confinement magnétique. Après quoi, nous considérons des perturbations magnétiques de cette barrière magnétique et prouvons, dans l'esprit de [CG], la stabilitée de la valeure quantifée de la conductance de bord. Enfin nous prouvons un principe de décomposition de la conductance de bord similaire à celle obtenue dans le cas d'un barrière électrique dans [CG].

Contents

1 Introduction

Dans ce chapitre, nous continuons notre travail concernant l'étude de la conductance par un résultat portant sur un autre aspect récemment développé de la théorie des systèmes de Hall, la conductance de bord.
L'étude de la quantification des états de bord en dimension 2 est apparue dans le contexte de l'effet Hall quantique avec le travail fondateur d'Halperin [Ha]. L'existence d'états quantiques se déplaçant le long du bord a été étudiée mathématiquement dans plusieurs travaux récents, e.g [DBP, FGW, FM1, FM2, CHS]. Dans tous ces travaux, le bord est modélisé par un potentiel confinant électrique ou par un mur dur avec des conditions de bord de Dirichlet, lequel peut être interprété comme un potentiel électrique infini vivant dans un demi-plan.
L'intérêt que suggère ce problème provient de multiples raisons. Premièrement, cette quantification de bord apporte un regard différent et complémentaire sur les systèmes de Hall ainsi que sur leur compréhension. D'un autre côté la stabilité de cette quantification prouvée récemment dans le cas d'un mur électrique met en lumière le fait que les courants de bords relevaient d'un phénomène structurel, mathématiquement parlant: d'un invariant. Notre travail fournit un argument supplémentaire dans cette direction par le fait que nous prouvons que le courant de bord ne dépend pas de la nature physique du mur. Nous démontrons en effet que la quantification ainsi que la stabilité demeure avec un mur magnétique. A cela s'ajoute le fait que la quantification du courant de bord et du courant de Hall habituel s'effectue de façon

simultanée dans le sens où les conductances sont égales rejoignant ainsi les résultats prouvés dans le cas électrique sur l'égalité entre la conductance de bord et de la conductance de Hall prouvée dans [SBKR, EG, EGS, CG]. Résultat que nous retrouvons dans le cas d'un mur magnétique. Enfin la quantification de la conductance de bord est stable sous certaines perturbations. A cela s'ajoute certaines propriétées propres au modèle que nous étudions.
Dans un travail encore en préparation nous prouvons l'existence d'un courant en présence de deux murs magnétiques, ce qui n'a pas lieu dans le cas électrique [CG]. Imposant certaines conditions particulières sur le champ magnétique nous prouvons l'existence d'un courant quantifié dans un guide d'onde magnétique avec une valeur de la conductance double de la valeur usuelle. Ce phénomène ne semble pas avoir été observé jusqu'à maintenant dans la littérature physique-mathématiques. Dans ce chapitre les états de bord sont produits par une variation du champ magnétique.
Le champ magnétique non-perturbé est supposé dépendre seulement de la première coordonnée x ce qui permet de diagonaliser par une transformation de Fourier partielle l'Hamiltonien. Nous supposons que le profil du champ magnétique $B(x)$ possède deux limites disctinctes en $+\infty$ et $-\infty$, modélisant ainsi en quelque sorte une perturbation d'un champ magnétique uniforme. Le mur est ainsi modélisé par un champ magnétique d'Iwatsuka [Iw].

On peut se représenter shématiquement le phénomène en imaginant deux demi-plans dont chacun est soumis à un champ magnétique uniforme de module différent B_- pour $x > 0$ ainsi que B_+ pour $x < 0$ avec $B_+ - B_-$ assez grand. La particule se trouvant dans le demi-plan $x > 0$ sera localisée le long du cercle de rayon $\frac{1}{\sqrt{B_-}}$ tandis que sur l'autre demi-plan sur un cercle de rayon $\frac{1}{\sqrt{B_+}}$. Ainsi lorsque la particule traverse la frontière son rayon cyclotronique va se modifier, la contraignant ainsi à se déplacer le long de la frontière (voir figure 6.1 [CyFKS]). Au niveau de l'analyse spectrale ce fait peut s'interpréter par l'absolue continuité du spectre comme prouvé dans [Iw, MP]. Cela étant, ceci ne rend pas compte de l'existence d'un courant quantifié le long du bord. Nous commençons donc ce travail par confirmer cette intuition et prouvons sa quantification en accord avec la physique et l'argument d'Halperin. Nous faisons remarquer que l'existence d'états de bord, dans le cas électrique a été prouvé dans [DBP, FGW] en combinant l'existence de courbes de dispersion avec la théorie de Mourre, ce qui ne semble pas aussi simple avec ce modèle. Les perturbations que nous considérons sont aussi de natures magnétiques. Afin de justifier l'étude de telle perturbation, nous mentionnons le fait que dans le contexte de l'effet Hall quantique les perturbations pertinentes sont les perturbations aléatoires

modélisant des impuretés et ceci pour la raison que les états localisés qu'elles engendrent sont responsables des célèbres plateaux de la conductance de Hall. L'existence d'états localisés provenant de perturbations magnétiques en dimension 2 en relation avec en physique a été intensivement étudié lors de ces 15 dernières années (voir e.g [AHK, BSK, Fu, V]). Mathématiquement, l'existence de localisation d'Anderson due à un potentiel magnétique aléatoire n'est pas un problème simple et seuls quelques résultats préliminaires existent : récemment Ghribi, Hislop et Klopp [GrHK] ont prouvé la localisation pour une perturbation donnée un potentiel magnétique aléatoire d'un potentiel magnétique périodique utilisant les méthodes propres au cas périodique (voir aussi [KNNY] pour un modèle discret particulier). Ueki [U] a prouvé l'existence de localisation pour une perturbation magnétique aléatoire de l'Hamiltonien de Landau dans le fond du spectre. Sous les premiers niveaux de Landau, dans un travail connexe nous donnons un exemple pertinent pour la théorie de l'effet Hall quantique, soit une perturbation magnétique aléatoire de l'Hamiltonien de Landau avec des états localisés aux bords des N premières bandes [DGR2].
Dans ce chapitre, nous commençons par démontrer qu'une perturbation à support compact n'affecte pas le courant de bord. Nous considérons alors une perturbation portée par une bande et enfin une perturbation vivant dans un demi-plan. Nous donnons un résultat de décomposition de la conductance de bord similaire à celle donnée dans [CG]. A savoir la conductance de bord du système perturbé est la somme de la conductance de bord du potentiel magnétique confinant et de la conductance de bord du système sans mur magnétique définit par l'Hamiltonien de Landau avec un champ magnétique de module B_- perturbé par le potentiel magnétique. Cela nous permet de calculer la conductance de bord de l'hamiltonien perturbé avec des énergies se trouvant dans une lacune spectrale de l'Hamiltonien de Landau. Pour considérer les énergies se trouvant dans des régions d'états localisés nous devons aller un peu plus loin encore en considérant une régularisation de la conductance de bord en utilisant les critères de localisation pour régulariser la trace définissant la conductance de bord (voir [CG, EGS]). Il suit évidemment des présents résultats ainsi que ceux de [CG] qu'une combinaison de potentiels électrique et magnétique pourrait se traiter en utilisant les techniques mises en oeuvres ici.
Dans la section 2. nous exposons les principaux résultats. Dans la section 3 nous considérons le cas d'une perturbation magnétique à support compact et prouvons la stabilité de la conductance de bord. Dans la section 4 nous considérons une perturbation vivant dans un demi-plan et prouvons la loi de décomposition de la conductance de bord. Enfin dans la section 5 nous discutons la régularisation de la conductance de bord en présence de désordre.

Enfin en appendice nous rappelons les estimations de trace que nous utilisons tout au long de ce travail.

2 Models and main Results

2.1 The Iwatsuka Hamiltonian

Let $A \in L^2_{\rm loc}(\mathbb{R}^2;\mathbb{R}^2)$. Define

$$H(A) := (i\nabla + A)^2$$

as the self-adjoint operator generated in $L^2(\mathbb{R}^2)$ by the closure of the quadratic form

$$\int_{\mathbb{R}^2} |i\nabla u + Au|^2 dx, \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^2).$$

If $A \in L^4_{\rm loc}(\mathbb{R}^2;\mathbb{R}^2)$ and $\operatorname{div} A \in L^2_{\rm loc}(\mathbb{R}^2;\mathbb{R})$, then $H(A)$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^2)$ (see [LS]).

We will say that the magnetic potential $A = (A_1, A_2)$ generates the magnetic field $B : \mathbb{R}^2 \to \mathbb{R}$ if

$$\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = B(x,y), \quad (x,y) \in \mathbb{R}^2. \tag{2.1}$$

Assume that the magnetic field $B \in \mathcal{C}^1(\mathbb{R}^2;\mathbb{R})$ depends only on the first coordinate, i.e. $B(x,y) = B(x)$. Suppose in addition that B is an increasing function of x. Finally, assume that there exist two numbers B_- and B_+ such that

$$0 < B_- < B_+ < \infty,$$

and

$$\lim_{x\to\pm\infty} B(x) = B_\pm.$$

We will call such a magnetic field a (B_-, B_+)–magnetic field. Introduce the magnetic potential $A = (A_1, A_2)$ with

$$A_1 = 0, \quad A_2 = \beta(x) := \int_0^x B(s)ds, \quad x \in \mathbb{R}.$$

Obviously, A generates B. We will call this particular magnetic Hamiltonian *the Iwatsuka Hamiltonian*, since it has been introduced by Akira Iwatsuka in [Iw]. Denote by $\mathcal{F}$ the partial Fourier transform with respect to y, i.e.

$$(\mathcal{F}u)(x,k) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-iky} u(x,y)dy, \quad u \in \mathrm{L}^2(\mathbb{R}^2).$$

Then the Iwatsuka Hamiltonian is unitarily equivalent to a direct integral of operators with discrete spectrum, i.e.

$$\mathcal{F}H(A)\mathcal{F}^* = \int_{\mathbb{R}}^{\oplus} h(k)dk$$

where

$$h(k) := -\frac{d^2}{dx^2} + (k - \beta(x))^2, \quad k \in \mathbb{R}.$$

The spectrum of $h(k)$ with k fixed, is discrete and simple. Denote by $\{\omega_j(k)\}_{j=1}^{\infty}$ the increasing sequence of the eigenvalues of the operator $h(k)$, $k \in \mathbb{R}$. In the next proposition we summarize for further references several spectral properties of the Iwatsuka Hamiltonian.

Proposition 2.1. [Iw, Lemmas 2,3, 4.1] *Let B be a (B_-, B_+)–magnetic field, and $\{\omega_j(k)\}_{j=1}^{\infty}$ be the eigenvalues defined above. Then we have*

$$(2j-1)B_- \leq \omega_j(k) \leq (2j-1)B_+, \quad j \in \mathbb{N}^* := \{1, 2, \ldots\}, \quad k \in \mathbb{R}, \quad (2.2)$$

and

$$\lim_{k\to\pm\infty} \omega_j(k) = (2j-1)B_\pm, \quad j \in \mathbb{N}^*. \quad (2.3)$$

As a consequence,

$$\sigma(H(A)) = \cup_{j=1}^{\infty}[(2j-1)B_-, (2j-1)B_+]. \quad (2.4)$$

2.2 Edge conductance

We will call $f : \mathbb{R} \to [0,1]$ *a switch function* if $f \in C^\infty(\mathbb{R})$ is increasing, with derivative compactly supported, and $f \equiv 1$, (resp., $f \equiv 0$), on the right (resp., on the left) of suppf'. Following [SBKR, EG, EGS, CG], the edge conductance related to a switch function g is defined as follows.

Definition 2.2. *Let $\chi \in \mathcal{C}^\infty(\mathbb{R}^2)$ be a x-translation invariant switch function with* supp $\chi' \subset \mathbb{R}\times[-\frac{1}{4}, \frac{1}{4}]$, *and $g \in \mathcal{C}^\infty(\mathbb{R})$ be a switch function with* supp $g' \subset$ *$I = [a, b]$, a compact interval. The edge conductance of a Hamiltonian H in the interval I is defined as*

$$\sigma_e^{(I)}(H) := -\mathrm{tr}\,(g'(H)i[H, \chi]), \quad (2.5)$$

whenever it exists.

We note that although the above definition depends a priori on the choice of g and χ, results will not as long as $\operatorname{supp} g' \subset I$ does not meet $(2\mathbb{N}+1)B_-$. To alleviate notations, we shall thus drop these two parameters and simply write $\sigma_e^{(I)}(H)$.

For the Iwatsuka Hamiltonian and some intervals I, the edge conductance can be explicitly computed.

Proposition 2.3. *Let $\mathcal{A}_{Iw}$ be a magnetic potential generating a (B_-, B_+)-magnetic field. Assume that the interval I satisfies $I \subset]B_-, B_+[$ and $I \subset](2n-1)B_-, (2n+1)B_-[$ for some $n \in \mathbb{N}$. Then we have*

$$\sigma_e^{(I)}(H(\mathcal{A}_{Iw})) = -n.$$

Proof. Arguing as in the proof of [CG, Proposition 1], and taking into account (2.3), we get

$$\begin{aligned} -\operatorname{tr}(g'(H(\mathcal{A}_{Iw}))i[H(\mathcal{A}_{Iw}), \chi]) &= -\sum_{j\in\mathbb{N}} \int_{\mathbb{R}} g'(\omega_j(k))\omega_j'(k)dk && (2.6)\\ &= \sum_{j\in\mathbb{N}} g((2j-1)B_-) - g((2j-1)B_+) && (2.7)\\ &= -n. && (2.8) \end{aligned}$$

□

2.3 Statement of the main results

We now state our main two theorems, that are results concerning the stability of the edge conductance under pure magnetic perturbations. The first one asserts that magnetic perturbations supported on a strip in the y direction do no affect the edge conductance.

Theorem 2.4. *Let $A \in \mathcal{C}^1(\mathbb{R}^2; \mathbb{R}^2)$. Assume that $a \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^2)$ is a magnetic potential compactly supported in the x-direction and polynomially bounded in the y-direction. Let g be as in Definition 2.2, with* $\operatorname{supp} g' \subset I$. *Then*

$$g'(H(A+a)[H(A+a), \chi] - g'(H(A)))[H(A), \chi] \in \mathcal{T}_1 \tag{2.9}$$

Moreover, if $g'(H(A)))[H(A), \chi] \in \mathcal{T}_1$, then

$$\sigma_e^{(I)}(H(A+a)) = \sigma_e^{(I)}(H(A)). \tag{2.10}$$

In particular, if $\mathcal{A}_{Iw}$ generates a (B_-, B_+)-magnetic field, then

$$\sigma_e^{(I)}(H(\mathcal{A}_{Iw} + a)) = -n$$

where the interval I satisfies $I \subset]-\infty, B_+[$ and $I \subset](2n-1)B_-, (2n+1)B_-[$ for some $n \in \mathbb{N}$.

Remark 18. *In [CG, Theorem 1], the analog of the difference* (2.9) *is not only trace class but its trace automatically vanishes. This is not the case here for the velocity operators defined for $H(A)$ and $H(A+a)$ differ. Indeed,*

$$\begin{aligned} &g'(H(A+a)[H(A+a),\chi] - g'(H(A)))[H(A),\chi] \qquad (2.11)\\ &= (g'(H(A+a) - g'(H(A)))\,[H(A),\chi] + g'(H(A+a)))[H(A+a) - H(A),\chi]. \end{aligned}$$

The second term on the r.h.s. is due to the magnetic nature of the perturbation, and may lead to a non trivial contribution to the current since a direct computation yields $2ig'(H(A+a))a_2\chi' \in \mathcal{T}_1$. To cancel this extra term, we shall introduce a suitable gauge transform that will make a_2 vanish. To perform that gauge transform, we assume a bit more than in [CG], namely, we assume that $g'(H(A)))[H(A),\chi]$ (or equivalently $g'(H(A+a)))[H(A+a),\chi]$) is trace class.

We get the following corollary, which is the analog of [CG, Corollary 1]. The magnetic nature of the wall forces us to use non trivial results on the spectrum of magnetic operators.

Corollary 2.5. *Let $\mathcal{A}_{\mathrm{strip}} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$ generate a magnetic field $B \in \mathcal{C}^2(\mathbb{R}^2; \mathbb{R})$ satisfying $B(x,y) = B(x) \geq B_0$ for all $(x,y) \in \{|x| \geq R_0\}$, $R_0 > 0$. Then for any magnetic potential $a \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^2)$ supported on $\{|x| \geq R_0\}$, and for any closed interval $I \subset]-\infty, B_0[$, we have $\sigma_e^{(I)}(H(\mathcal{A}_{\mathrm{strip}} + a)) = 0$.*

Remark 19. *(i) If we perturb the operator $H(\mathcal{A}_{\mathrm{strip}})$ by a* magnetic field *supported on a strip S, then Proposition 3.2 below implies that there exists a potential a which generates this magnetic field, and vanishes outside S, so that the hypotheses of Corollary 2.5 are satisfied.*
(ii)Magnetic potentials $\mathcal{A}_{\mathrm{strip}}$ of Corollary 2.5 can be produced by the superposition of two Iwatsuka-type potentials $\mathcal{A}_{Iw}^{(L)}, \mathcal{A}_{Iw}^{(R)}$, generating respectively a decreasing magnetic field $B^{(L)} = B^{(L)}(x)$ with upper limit $B_+^{(L)} \geq B_0$, and an increasing magnetic field $B^{(R)} = B^{(R)}(x)$ with upper limit $B_+^{(R)} \geq B_0$. Particles are then trapped in a magnetic strip created by these two magnetic barriers and thus can only travel along the direction of the strip. Corollary 2.5 asserts that no net current can flow in such a strip, whatever the potential inside the strip is.

Our second theorem provides a sum rule similar to [CG, Theorem 2]. We use the convenient notation $H(A^{(1)}, A^{(2)})$ for $H(A^{(0)} + A^{(1)} + A^{(2)})$ where $\text{supp}A^{(1)}$ (resp., $\text{supp}A^{(2)}$), is included in the half-plane $x < R_1$ (resp., $x > R_2$), and $A^{(0)} := \left(-\frac{B_- y}{2}, \frac{B_- x}{2}\right)$ with $B_- > 0$. In particular, $H(0,0)$ is the Landau Hamiltonian with scalar constant magnetic field B_-.

Theorem 2.6. *Let I be a closed interval so that $I \cap \sigma(H(0,0)) = \emptyset$. Set*

$$\mathcal{K}(a_\alpha, a_\beta) := g'(H(a_\alpha, a_\beta) i[H(a_\alpha, a_\beta), \chi] - g'(H(a_\alpha, 0) i[H(a_\alpha, 0), \chi] \quad (2.12)$$
$$- g'(H(0, a_\beta) i[H(0, a_\beta), \chi]. \quad (2.13)$$

Then $\mathcal{K}(a_\alpha, a_\beta)$ is trace class. Moreover, if two out of the three terms of the r.h.s. are trace class, then $\text{tr}\mathcal{K}(a_\alpha, a_\beta) = 0$; in particular

$$\sigma_e^{(I)}(H(a_\alpha, a_\beta)) = \sigma_e^{(I)}(H(a_\alpha, 0)) + \sigma_e^{(I)}(H(0, a_\beta)). \quad (2.14)$$

Moreover, if $\mathcal{A}_{Iw}^{(L)}, \mathcal{A}_{Iw}^{(R)}$ are left and right Iwatsuka-type potentials described in Remark 19, and $I \subset]-\infty, B_+[$, then if, with obvious notations, $\text{supp}g' \subset]-\infty, B_+^{(L)}[$ *and* $\text{supp}g' \subset]-\infty, B_+^{(R)}[$*, then same result holds for*

$$\mathcal{K}'(a_\alpha, a_\beta) := g'(H(a_\alpha, a_\beta) i[H(a_\alpha, a_\beta), \chi] - g'(H(a_\alpha, \mathcal{A}_{Iw}^{(R)}) i[H(a_\alpha, \mathcal{A}_{Iw}^{(R)}), \chi] \quad (2.15)$$
$$- g'(H(\mathcal{A}_{Iw}^{(L)}, a_\beta) i[H(\mathcal{A}_{Iw}^{(L)}, a_\beta), \chi]. \quad (2.16)$$

In particular, if two out of the three terms in $\mathcal{K}'(a_\alpha, a_\beta)$ are trace class, then

$$\sigma_e^{(I)}(H(a_\alpha, a_\beta)) = \sigma_e^{(I)}(H(a_\alpha, \mathcal{A}_{Iw}^{(R)})) + \sigma_e^{(I)}(H(\mathcal{A}_{Iw}^{(L)}, a_\beta)) \quad (2.17)$$

As a consequence of Proposition 2.3 and Theorem 2.6, we obtain a quantization of the edge conductance for magnetic perturbations of the Iwatsuka Hamiltonian, which in its turn implies the existence of edge states flowing in the y direction.

Corollary 2.7. *Let $\mathcal{A}$ generate a (B_-, B_+)-magnetic field with $B_+ \geq 3B_-$. Let $a \in \mathcal{C}^2(\{x \leq R_1\} \times \mathbb{R})$ for some $R_1 < \infty$ be so that $\|a\|_\infty \leq K_1\sqrt{B_-}$ and $\|\text{div}a\|_\infty \leq K_2 B_-$, then there exists $0 < K_0 < \infty$ such that*

$$\sigma_e^{(I)}(H(a, \mathcal{A})) = -n, \quad n \in \mathbb{N},$$

for B_- large enough and an interval I satisfying

$$I \subset](2n-1)B_- + K_0 d_n(a, B_-), (2n+1)B_- - K_0 d_n(a, B_-)[\quad (2.18)$$

if $n \in \mathbb{N}^*$, or

$$I \subset]-\infty, B_- - K_0 d_n(a, B_-)[\quad (2.19)$$

if $n = 0$, where $d_n(a, B_-) = \max(\|\text{div}a\|_\infty, \|a\|_\infty \sqrt{(n+1)B_-})$.

If the interval I does not lie in a gap of $\sigma(H(0,0))$ anymore, we have to introduce a regularization of the edge conductance, as in [CG, CGH].

As a concluding remark we note that similar results can be obtained for Hamiltonians mixing the point of view of [CG] with pure electric potentials (wall and perturbation), and the one of this work that is pure magnetic potentials (wall and perturbation). We can indeed perturb an Iwatsuka Hamiltonian by an electric potential, or perturb a Hamiltonian with an electric confining potential by a magnetic potential. Proofs are then similar to those of [CG] and those of the present article, the most technical case being the purely magnetic model.

3 Perturbation by a magnetic potential supported on a strip

Since in full generality $[H(A+a),\chi] \neq [H(A),\chi]$, we introduce a suitable gauge transform to recover equality, as in the electric potential case. This is a new feature compared to [CG], which is due to the magnetic nature of the perturbation.

3.1 More on magnetic fields and magnetic potentials

. This subsection contains well-known facts about the possibility to construct magnetic potentials $A:\mathbb{R}^d \to \mathbb{R}^d$ with prescribed properties, generating given magnetic fields $B:\mathbb{R}^2 \to \mathbb{R}$.
In the first proposition we define a magnetic potential A in the so-called *Poincaré gauge.*

Proposition 3.1. [T, Eq. (8.154)] *Let $B \in \mathcal{C}^1(\mathbb{R}^2;\mathbb{R})$. Then the potential*

$$A=(A_1,A_2)=\left(-y\int_0^1 sB(sx,sy)ds,\ x\int_0^1 sB(sx,sy)ds\right),\quad (x,y)\in\mathbb{R}^2, \tag{3.1}$$

generates the magnetic field B.

Proposition 3.2. *Let $B \in \mathcal{C}^k(\mathbb{R}^2;\mathbb{R})$, $k \in \mathbb{N}^*$, satisfy $B(\mathbf{x}) = 0$ for $\mathbf{x} = (x,y) \in \mathbb{R}^2$ with $|x| \geq R_0$, $R_0 > 0$. Then there exists a magnetic potential $\mathcal{A} \in \mathcal{C}^k(\mathbb{R}^2;\mathbb{R}^2)$ which generates B, and vanishes identically on $\{(x,y) \in \mathbb{R}^2 \,|\, |x| \geq R_0\}$.*

Proof. Pick any magnetic potential $A \in \mathcal{C}^k(\mathbb{R}^2;\mathbb{R}^2)$ which generates B (say, the potential appearing in (3.1)). Set

$$S_- := \{(x,y)\in\mathbb{R}^2 \,|\, x < -R_0\},\quad S_+ := \{(x,y)\in\mathbb{R}^2 \,|\, x > R_0\}.$$

Since $S_\pm$ are simply connected domains, and B identically vanishes on them, there exist functions $F_\pm \in \mathcal{C}^{k+1}(\overline{S}_\pm;\mathbb{R})$ such that

$$\nabla F_- = A \quad \text{on} \quad S_-, \quad \nabla F_+ = A \quad \text{on} \quad S_+.$$

Then there exists an extension $\mathcal{F} \in \mathcal{C}^{k+1}(\mathbb{R}^2;\mathbb{R})$ such that

$$\mathcal{F} = F_- \quad \text{on} \quad S_-, \quad \mathcal{F} = F_+ \quad \text{on} \quad S_+.$$

On $\mathbb{R}^2$ define $\mathcal{A} := A - \nabla\mathcal{F}$. Evidently, the magnetic potential $\mathcal{A} \in \mathcal{C}^k(\mathbb{R}^2;\mathbb{R}^2)$ generates B, and $\mathcal{A}(\mathbf{x}) = 0$ for $\mathbf{x} \in S_- \cup S_+$. ☐

3.2 Proof of Theorem 2.4

Lemma 3.3. *Let $A \in \mathcal{C}^2(\mathbb{R}^2;\mathbb{R}^2)$. Assume that $a \in \mathcal{C}^1(\mathbb{R}^2;\mathbb{R}^2)$ is supported on the strip $[-x_0, x_0] \times \mathbb{R}$ and admits the bound $|a(x,y)| \le C_a \langle y\rangle^k$, for some $k \ge 0$ and $C_a < \infty$. Set*

$$F(x,y) = -\int_0^y a_2(x,s)ds, \quad (x,y) \in \mathbb{R}^2. \tag{3.2}$$

Then we have

$$[H(A + a + \nabla F), \chi] = [H(A), \chi]. \tag{3.3}$$

Proof. Note that if $\tilde{a} = (\tilde{a}_1, \tilde{a}_2) \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$, we have

$$H(A + \tilde{a}) - H(A) = 2\tilde{a} \cdot (i\nabla + A) + i\mathrm{div}\,\tilde{a} + |\tilde{a}|^2. \tag{3.4}$$

Since $\partial_x \chi = 0$, a direct computation shows that

$$[H(A + \tilde{a}), \chi] - [H(A), \chi] = 2i\tilde{a} \cdot \nabla\chi = 2i\tilde{a}_2\chi'. \tag{3.5}$$

Therefore,

$$[H(A + \tilde{a}), \chi] - [H(A), \chi] = 0 \Longleftrightarrow \tilde{a}_2\chi' = 0. \tag{3.6}$$

Applying (3.5) – (3.6) with $\tilde{a} = a + \nabla F$, and taking into account that in this case $\tilde{a}_2 = a_2 + \partial_y F = 0$ by (3.2), we obtain (3.3). ☐

Remark 20. *Note that $a + \nabla F$ is supported on $[-x_0, x_0] \times \mathbb{R}$, and $|a + \nabla F| \le C_a \langle y\rangle^{k+1}$, $(x,y) \in \mathbb{R}^2$.*

Proposition 3.4. *Let A, a be as in Lemma 3.3. Then*

$$\left(g'(H(A + a)) - g'(H(A))\right)[H(A), \chi] \in \mathcal{T}_1,$$

and

$$\mathrm{tr}\left(g'(H(A + a)) - g'(H(A))\right)[H(A), \chi] = 0. \tag{3.7}$$

Assuming for the moment the validity of Proposition 3.4, we provide the proof of Theorem 2.4.

Proof of Theorem 2.4. The fact that the operator defined in (2.9) is trace class follows from the decomposition (2.11) of Remark 18, Proposition 3.4 and (3.6). To prove that the trace is zero we introduce a gauge transform $\exp(iF)$, where F is given by Lemma 3.3. By Proposition 3.4 applied to the perturbation $a + \nabla F$, the operator

$$g'(H(A+a+\nabla F))[H(A),\chi] = g'(H(A+a+\nabla F))[H(A+a+\nabla F),\chi]$$

is trace-class since $g'(H(A))[H(A),\chi]$ is by hypothesis.

Now, since F and χ commute, we have

$$g'(H(A+a))i[H(A+a),\chi] \tag{3.8}$$
$$= e^{-iF}(e^{iF}g'(H(A+a))e^{-iF}e^{iF}i[H(A+a),\chi]e^{-iF})e^{iF} \tag{3.9}$$
$$= e^{-iF}(g'(H(A+a+\nabla F))i[H(A+a+\nabla F),\chi])e^{iF}, \tag{3.10}$$

which is trace class, so that $g'(H(A+a))i[H(A+a),\chi] \in \mathcal{T}_1$. It follows, using the cyclicity of the trace, that

$$\mathrm{tr} g'(H(A+a)i[H(A+a),\chi] - \mathrm{tr} g'(H(A))i[H(A),\chi] \tag{3.11}$$
$$= \mathrm{tr}(g'(H(A+a+\nabla F)) - g'(H(A)))i[H(A),\chi] \tag{3.12}$$
$$= 0 \tag{3.13}$$

□

The rest of the section is devoted to the proof of Proposition 3.4.

Proof of Proposition 3.4. As in the proof of [CG, Theorem 1], we first prove Proposition 3.4 for compactly supported perturbations, and then extend the result to general perturbations supported on a strip.
Let the smooth cut-off function $\varphi_r : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy $\varphi_r(x,y) = \varphi_r(y)$, $\varphi_r = 1$ for $|y| \leq r-1$ and $\varphi_r = 0$ for $|y| \geq r$, $r > 1$. We decompose $a = a\varphi_r + a(1-\varphi_r) := a_{\leq r} + a_{\geq r}$. Then we have

$$\begin{aligned} &\mathrm{tr}\,(g'(H(A+a)) - g'(H(A)))\,[H(A),\chi] = \\ &\mathrm{tr}\,(g'(H(A+a)) - g'(H(A+a_{\leq r})))\,[H(A),\chi] + \\ &\mathrm{tr}\,(g'(H(A+a_{\leq r})) - g'(H(A)))\,[H(A),\chi]. \end{aligned} \tag{3.14}$$

First we will show that

$$g'(H(A+a_{\leq r})) - g'(H(A)[H(A),\chi] \in \mathcal{T}_1, \tag{3.15}$$

and

$$\operatorname{tr}\left(g'(H(A+a_{\leq r}))-g'(H(A))\right)[H(A),\chi]=0. \tag{3.16}$$

After that we will show that there exists $C<\infty$ and $p\geq 1$ such that

$$\|(g'(H(A+a))-g'(H(A+a_{\leq r})))\,[H(A),\chi]\|_1\leq Cr^{-p} \tag{3.17}$$

for r large enough.
Let us now prove (3.14). By the Helffer-Sjöstrand functional calculus (see e.g. [HS, Lemma B.2]), applied to function $G(x):=\int_{-\infty}^x g(s)ds$, we have

$$g'(H(A+a_{\leq r}))-g'(H(A))=-\frac{2}{\pi}\int_{\mathbb{R}^2}\overline{\partial}\tilde{G}(u+iv)(R_1^3-R_2^3)dudv, \tag{3.18}$$

where $R_1=(H_j-z)^{-1}$, $j=1,2$, and $H_1:=H(A)$, $H_2:=H(A+a_{\leq r})$. Put

$$\mathcal{W}:=H(A+a_{\leq r})-H(A)=2a_{\leq r}\cdot(i\nabla+A)+i\operatorname{div}a_{\leq r}+|a_{\leq r}|^2.$$

Due to the fact that $\mathcal{W}$ is a first-order differential operator, we need one extra power of the resolvents in comparison to [CG]. We are thus left with $\operatorname{tr}(R_1^3-R_2^3)[H(A),\chi]$ that we have to develop. It is easy to check

$$2(R_1^3-R_2^3)=R_1^2\mathcal{W}R_2R_1+R_1\mathcal{W}R_2^2R_1+R_1\mathcal{W}R_2R_1^2 \tag{3.19}$$
$$+R_2^2R_1\mathcal{W}R_2+R_2R_1^2\mathcal{W}R_2+R_2R_1\mathcal{W}R_2^2. \tag{3.20}$$

The formula itself can actually be guessed by formally computing $\partial_z(R_1^2-R_2^2)=\partial_z(R_1\mathcal{W}R_2R_1+R_2R_1\mathcal{W}R_2)$. Since $[\mathcal{W},\chi]=[H_2,\chi]-[H_1,\chi]=0$, we have

$$\begin{gathered}2(R_1^3-R_2^3)[H_1,\chi]=\\ \left(R_1^2\mathcal{W}R_2R_1+R_1\mathcal{W}R_2^2R_1+R_1\mathcal{W}R_2R_1^2\right)[H_1,\chi]+\\ \left(R_2^2R_1\mathcal{W}R_2+R_2R_1^2\mathcal{W}R_2+R_2R_1\mathcal{W}R_2^2\right)[H_2,\chi].\end{gathered} \tag{3.21}$$

Applying Corollary 6.5 and Lemma 6.1 (ii), and bearing in mind that the operators $R_j[H_j,\chi]$, $j=1,2$, are bounded, we find that all the terms on the r.h.s. of (3.21) are trace-class.

Next, we prove (3.16). Using the identities $R_j[H_j,\chi]R_j=[\chi,R_j]$, j=1,2, undoing the commutators, and introducing obvious notations, we get

$$\operatorname{tr}(R_1^2\mathcal{W}R_2R_1)[H_1,\chi]=\operatorname{tr}[\chi,R_1]R_1WR_2=:I_1+I_2, \tag{3.22}$$

$$\operatorname{tr}(R_1\mathcal{W}R_2^2R_1)[H_1,\chi]=\operatorname{tr}[\chi,R_1]WR_2^2=:II_1+II_2, \tag{3.23}$$

$$\operatorname{tr}(R_1\mathcal{W}R_2R_1^2)[H_1,\chi]=\operatorname{tr}[\chi,R_1]WR_2R_1=:III_1+III_2, \tag{3.24}$$

$$\operatorname{tr}(R_2^2R_1\mathcal{W}R_2)[H_2,\chi] = \operatorname{tr}[\chi,R_2]R_2R_1W =: IV_1 + IV_2, \tag{3.25}$$

$$\operatorname{tr}(R_2R_1^2\mathcal{W}R_2)[H_2,\chi] = \operatorname{tr}[\chi,R_2]R_1^2W =: V_1 + V_2, \tag{3.26}$$

$$\operatorname{tr}(R_2R_1\mathcal{W}R_2^2)[H_2,\chi] = \operatorname{tr}[\chi,R_2]R_1WR_2 =: VI_1 + VI_2. \tag{3.27}$$

Let $0 \le \zeta_j \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, $j = 0,1$, satisfy $\zeta_0 a_{\le r} = a_{\le r}$ and $\zeta_1\zeta_0 = \zeta_0$ on $\mathbb{R}^2$. Rearranging the terms (3.22) – (3.27) of $2\operatorname{tr}(R_1^2 - R_2^3)[H_1,\chi]$ (see (3.21)), and applying Lemma 6.1 and Lemma 6.2, we get

$$I_1 + V_2 =$$

$$\operatorname{tr}[\chi R_1^2W, R_2] = \operatorname{tr}[\chi R_1^2W, \zeta_0R_2] = 0,$$

$$II_1 + V_2 =$$

$$\operatorname{tr}[\chi R_1WR_2, R_2] = \operatorname{tr}\left([\chi R_1WR_2, \zeta_0R_2] - [\chi R_1WR_2[H_2,\zeta_0]R_2, R_2]\right) = 0,$$

$$III_1 + I_2 =$$

$$\operatorname{tr}[\chi R_1WR_2, R_1] = \operatorname{tr}\left([\chi R_1WR_2, \zeta_0R_1] - [\chi R_1WR_2[H_2,\zeta_0]R_2, R_1]\right) = 0,$$

$$IV_1 + II_2 =$$

$$\operatorname{tr}[\chi R_2^2, R_1W\chi] = \operatorname{tr}\big([\chi R_2\zeta_0R_2, R_1W\chi] - [\zeta_0R_2[H_2,\zeta_1]\chi R_2^2, R_1W\chi]\big) = 0,$$

$$V_1 + III_2 =$$

$$\operatorname{tr}[\chi R_2R_1, R_1W\chi] = \operatorname{tr}\left([\chi R_2\zeta_0R_1, R_1W\chi] - [\zeta_0R_2[H_2,\zeta_1]\chi R_2R_1, R_1W\chi]\right) = 0,$$

$$VI_1 + IV_2 =$$

$$\operatorname{tr}[\chi R_2R_1W, R_2] = \operatorname{tr}[\chi R_2R_1W, \zeta_0R_2] = 0.$$

Therefore,

$$\operatorname{tr}(R_1^3 - R_2^3)[H_1,\chi] = 0. \tag{3.28}$$

Now (3.28) and (3.18) imply (3.16).

Finally, we prove (3.17). Due to the Helffer-Sjöstrand formula (3.18), we have to control

$$\|(R_1^3 - R_2^3)[H(A),\chi]\|_1,$$

where we use the notations $R(z) := (H(A) - z)^{-1}$ and $R_r := (H(A + a_{\le r}) - z)^{-1}$. The resolvent identity yields

$$R^3 - R_r^3 = R^3\mathcal{W}_rR + R^2\mathcal{W}_rR_r^2 + R\mathcal{W}_rR_r^3, \tag{3.29}$$

where

$$\mathcal{W}_r := H(A + a_{\le r}) - H(A + a) = -\big(2a_{\ge r}\cdot(i\nabla + A) + i\operatorname{div}(a_{\ge r}) + |a|^2 - |a_{\le r}|^2\big).$$

Note that $\mathcal{W}_r \equiv 0$ whenever $|y| \leq r-1$, so that we write

$$\mathcal{W}_r = \sum_{(x_1,y_1)\in\mathbb{Z}^2\cap([-x_0-1,x_0+1]\times[-r+2,r-2]^c)} \mathbf{1}_{(x_1,y_1)}\mathcal{W}_r \tag{3.30}$$

$$= \sum_{(x_1,y_1)\in\mathbb{Z}^2\cap([-x_0-1,x_0+1]\times[-r+2,r-2]^c)} \mathcal{W}_r\mathbf{1}_{(x_1,y_1)}, \tag{3.31}$$

where $\mathbf{1}_{(x,y)}$ stands for a smooth characteristic function of the cube of length one and centered at $(x,y)\in\mathbb{R}^2$ such that $\sum_{(x,y)\in\mathbb{Z}^2}\mathbf{1}_{(x,y)} = 1$. Similarly,

$$[H(A),\chi] = \sum_{x_2\in\mathbb{Z}}[H(A),\chi]\mathbf{1}_{(x_2,0)}. \tag{3.32}$$

For the moment fix $x_1, y_1, x_2 \in \mathbb{Z}$, and introduce the short-hand notations $\zeta_0 := \mathbf{1}_{(x_1,y_1)}$, and $\zeta := \mathbf{1}_{(x_2,0)}$. Let ζ_j be non-negative smooth compactly supported functions such that $\zeta_j\zeta_{j-1} = \zeta_{j-1}$ on $\mathbb{R}^2$, $j = 1,2,3$. Then we have

$$\|R^3\mathbf{1}_{(x_1,y_1)}\mathcal{W}_r R_r[H(A),\chi]\mathbf{1}_{x_2}\|_1 \leq$$
$$\|R^3\zeta_0\mathcal{W}_r\|_1\|\zeta_1 R_r[H(A),\chi]\zeta\|, \tag{3.33}$$
$$\|R^2\mathbf{1}_{(x_1,y_1)}\mathcal{W}_r R_r^2[H(A),\chi]\mathbf{1}_{x_2}\|_1 \leq$$
$$\|R^2\zeta_0\mathcal{W}_r R_r\|_1\left(\|\zeta_1 R_r[H(A),\chi]\zeta\| + \|\zeta_2[H(A),\zeta_1]R_r^2[H(A),\chi]\zeta\|\right) \tag{3.34}$$
$$\|R\mathbf{1}_{(x_1,y_1)}\mathcal{W}_r R_r^3[H(A),\chi]\mathbf{1}_{x_2}\|_1 \leq$$
$$\|R\zeta_0\mathcal{W}_r R_r^2\|_1\left(\|\zeta_1 R_r[H(A),\chi]\zeta\| + \|\zeta_2[H(A),\zeta_1]R_r^2[H(A),\chi]\zeta\|\right) +$$
$$\|R\zeta_0\mathcal{W}_r R_r[H(A),\zeta_1]R_r\|_1\left(\|\zeta_2 R_r^2[H(A),\chi]\zeta\| + \|\zeta_3[H(A),\zeta_2]R_r^3[H(A),\chi]\zeta\|\right). \tag{3.35}$$

Assume now that z is in a compact subset of $\mathbb{C}$, and $\Im z \neq 0$. Applying Proposition 6.4 and estimate (6.20), we find that there exists a constant c_1 independent of $x_1, y_1, x_2 \in \mathbb{Z}$, and z, such that the trace-class norms

$$\|R^3\zeta_0\mathcal{W}_r\|_1, \quad \|R^2\zeta_0\mathcal{W}_r R_r\|_1, \quad \|R\zeta_0\mathcal{W}_r R_r[H(A),\zeta_1]R_r\|_1,$$

appearing on the r.h.s. of (3.33) – (3.35) are upper bounded by $c_1|\Im z|^{-3}$. On the other hand, making use of estimates of Combes-Thomas type (see [CT, **?**]), we find that there exists a constant $c_2 > 0$ independent of $x_1, y_1, x_2 \in \mathbb{Z}$, and z such that the operator norms

$$\|\zeta_1 R_r[H(A),\chi]\zeta\|, \quad \|\zeta_2[H(A),\zeta_1]R_r^2[H(A),\chi]\zeta\|,$$

$$\|\zeta_2 R_r^2[H(A),\chi]\zeta\|, \quad \|\zeta_3[H(A),\zeta_2]R_r^3[H(A),\chi]\zeta\|,$$

appearing on the r.h.s. of (3.33) – (3.35) are upper bounded by

$$c_2|\Im z|^{-1}\exp\left(-c_2|\Im z|(|x_1-x_2|+|y_1|)\right).$$

Taking into account these estimates, bearing into mind the representations (3.32) and (3.31), and arguing as in the proof of [CG, Lemma 2], we easily obtain (3.16). □

Proof of Corollary 2.5. We introduce a modified strip confining potential $\tilde{\mathcal{A}}_{\rm strip}$ generating a magnetic field $\tilde{B}\in\mathcal{C}^2(\mathbb{R}^2;\mathbb{R})$ which satisfies $\tilde{B}(x,y)\geq B_0$ for all $(x,y)\in\mathbb{R}^2$ and $\tilde{B}(x)=B(x)$ on $\{|x|\geq R_0\}$. Since the operator $H(\tilde{\mathcal{A}}_{\rm strip})-\tilde{B}$ is non-negative, we have $\inf\sigma(H(\tilde{\mathcal{A}}_{\rm strip}))\geq B_0$ (see e.g. [E]). As a consequence $\sigma_e^{(I)}(H(\tilde{\mathcal{A}}_{\rm strip}))=0$. Since the magnetic field $B-\tilde{B}$ is supported on a strip, Proposition 3.2 implies the existence a magnetic potential $A\in\mathcal{C}^2(\mathbb{R}^2;\mathbb{R}^2)$ which generates the magnetic field $B-\tilde{B}$, and is supported on a strip. Applying Theorem 2.4, we find that

$$\sigma_e^{(I)}(H(\tilde{\mathcal{A}}_{\rm strip}+A))=\sigma_e^{(I)}(H(\tilde{\mathcal{A}}_{\rm strip}))=0.$$

Since the potentials $\tilde{\mathcal{A}}_{\rm strip}+A$ and $\mathcal{A}_{\rm strip}$ generate the same magnetic field B, the operators $H(\tilde{\mathcal{A}}_{\rm strip}+A)$ and $H(\mathcal{A}_{\rm strip})$ are unitarily equivalent under an appropriate gauge transform. Therefore,

$$\sigma_e^{(I)}(H(\mathcal{A}_{\rm strip}))=\sigma_e^{(I)}(H(\tilde{\mathcal{A}}_{\rm strip}+A))=0.$$

Finally, applying Theorem 2.4 once more, we find that

$$\sigma_e^{(I)}(H(\mathcal{A}_{\rm strip}+a))=\sigma_e^{(I)}(H(\mathcal{A}_{\rm strip}))=0.$$

□

4 Sum rule for magnetic perturbations

The aim of this section is to prove Theorem 2.6 and Corollary 2.7.

Proof of Theorem 2.6. It is enough to prove the first part of the statement (the one concerning $\mathcal{K}(a_\alpha,a_\beta)$), for the second part will follow from the relation

$$\mathcal{K}'(\mathcal{A}_{Iw},a_\alpha,a_\beta)=\mathcal{K}(a_\alpha,a_\beta)-\mathcal{K}(a_\alpha,\mathcal{A}_{Iw})-\mathcal{K}(\mathcal{A}_{Iw},a_\beta)+\mathcal{K}(\mathcal{A}_{Iw},\mathcal{A}_{Iw}),\quad(4.1)$$

where we used that $\mathrm{tr}g'(H(\mathcal{A}_{Iw},\mathcal{A}_{Iw}))[H(\mathcal{A}_{Iw},\mathcal{A}_{Iw}),\chi]=0$ by Corollary 2.5.

We set $\mathcal{K}:=\mathcal{K}(a_\alpha,a_\beta)$ and $\mathcal{K}_r:=\mathcal{K}(a_\alpha,a_{\beta_r})$, where $a_{\beta_r}:=a_\beta\varphi_r$ and φ_r is a smooth characteristic function of the region $x\leq r$. By Theorem 2.4,

$\mathcal{K}_r \in \mathcal{T}_1$ and $\mathrm{tr}\mathcal{K}_r = 0$. It is thus enough to prove polynomial decay in r of $\|\mathcal{K} - \mathcal{K}_r\|_1$. We have, setting

$$\mathcal{Q}^{\chi}_{(a,b)} = i[H(a,b), \chi], \tag{4.2}$$

$$\begin{aligned}
\mathcal{K} - \mathcal{K}_r &= g'(H(a_\alpha, a_\beta)\mathcal{Q}^{\chi}_{(a_\alpha,a_\beta)} - g'(H(a_\alpha, a_{\beta_r})\mathcal{Q}^{\chi}_{(a_\alpha,a_{\beta_r})} && (4.3)\\
&\quad - g'(H(0, a_\beta)\mathcal{Q}^{\chi}_{(0,a_\beta)} + g'(H(0, a_{\beta_r})\mathcal{Q}^{\chi}_{(0,a_{\beta_r})} && (4.4)\\
&= (g'(H(a_\alpha, a_\beta) - g'(H(a_\alpha, a_{\beta_r})))\mathcal{Q}^{\chi}_{(a_\alpha,a_{\beta_r})} && (4.5)\\
&\quad - (g'(H(0, a_\beta) - g'(H(0, a_{\beta_r})))\mathcal{Q}^{\chi}_{(0,a_{\beta_r})} && (4.6)\\
&\quad + (g'(H(a_\alpha, a_\beta) - g'(H(0, a_\beta)))\mathcal{Q}^{\chi}_{(0,a_\beta-a_{\beta_r})} && (4.7)\\
&= (g'(H(a_\alpha, a_\beta) - g'(H(a_\alpha, a_{\beta_r})) - g'(H(0, a_\beta) + g'(H(0, a_{\beta_r})))\mathcal{Q}^{\chi}_{(a_\alpha,a_{\beta_r})} && (4.8)\\
&\quad - (g'(H(0, a_\beta) - g'(H(0, a_{\beta_r})))\mathcal{Q}^{\chi}_{(a_\alpha,0)} && (4.9)\\
&\quad + (g'(H(a_\alpha, a_\beta) - g'(H(0, a_\beta)))\mathcal{Q}^{\chi}_{(0,a_\beta-a_{\beta_r})}, && (4.10)
\end{aligned}$$

where we used that $\mathcal{Q}^{\chi}_{(a,b)} - \mathcal{Q}^{\chi}_{(a',b')} = \mathcal{Q}^{\chi}_{(a-a',b-b')}$ for arbitrary a, b, a', b' in $\mathcal{C}^1$. The term in (4.8) is evaluated as in [CG], while (4.9) and (4.10) are new terms coming from the magnetic nature of the perturbation.

We first show that (4.9) and (4.10) satisfy a bound of the type (3.17). Notice, from the very definition of commutators $\mathcal{Q}^{\chi}_{(a,b)}$, that if a, b given are supported on some closed region Γ, then so is the operator $\mathcal{Q}^{\chi}_{(a,b)}$, in the sense that $(1 - \chi_\Gamma)\mathcal{Q}^{\chi}_{(a,b)} = (\mathcal{Q}^{\chi}_{(a,b)})_{|\mathbb{R}^2\setminus\Gamma} \equiv 0$, χ_Γ being the characteristic function of Γ.

Let us first consider (4.9). We use the Helffer-Sjöstrand formula (3.18) plus the resolvent identity (3.29), and then estimate the resulting terms as in the final part of the proof of Proposition 3.4 (see Section 3), with

$$\begin{aligned}
\mathcal{W}_r &= H(0, a_{\beta_r}) - H(0, a_\beta) && (4.11)\\
&= \sum_{(x_1,y_1)\in\mathbb{Z}^2\cap\{x_1\geq r-1\}} \mathbf{1}_{(x_1,y_1)}\mathcal{W}_r && (4.12)\\
&= \sum_{(x_1,y_1)\in\mathbb{Z}^2\cap\{x_1\geq r-1\}} \mathcal{W}_r\mathbf{1}_{(x_1,y_1)} && (4.13)
\end{aligned}$$

instead of (3.31), and

$$\mathcal{Q}^{\chi}_{(a_\alpha,0)} = \sum_{x_2\in\mathbb{Z}\cap\{x_2\leq 1\}} \mathcal{Q}^{\chi}_{(a_\alpha,0)}\mathbf{1}_{(x_2,0)} \tag{4.14}$$

instead of (3.32). The proof of (4.10) is similar with $\mathcal{W}_r = H(a_\alpha, a_\beta) - H(0, a_\beta)$ which we decompose over boxes centered at points $(x_1, y_1) \in \mathbb{Z}^2 \cap \{x_1 \leq 1\}$, while we decompose $\mathcal{Q}^\chi_{(0, a_\beta - a_{\beta_r})}$ over boxes centered at points $(x_2, y_2) \in \mathbb{Z}^2 \cap \{x_2 \geq r-1, y_2 = 0\}$.

We turn now to (4.8). Again, in order to estimate

$$\left\| \left\{ g'(H(a_\alpha, a_\beta) - g'(H(a_\alpha, a_{\beta_r})) - g'(H(0, a_\beta) + g'(H(0, a_{\beta_r})) \right\} \mathcal{Q}^\chi_{(a_\alpha, a_{\beta_r})} \right\|_1, \tag{4.15}$$

we bound, with obvious notations for the resolvents, the norm

$$\left\| \left(R^3_{(a_\alpha, a_\beta)} - R^3_{(a_\alpha, a_{\beta_r})} \right) - \left(R^3_{(0, a_\beta)} - R^3_{(0, a_{\beta_r})} \right) \mathcal{Q}^\chi_{(a_\alpha, a_{\beta_r})} \right\|_1. \tag{4.16}$$

To do so, we utilize the resolvent identity (3.29) together with

$$H(a_\alpha, a_\beta) - H(a_\alpha, a_{\beta_r}) = H(0, a_\beta) - H(0, a_{\beta_r}) =: -\mathcal{W}_r, \tag{4.17}$$

and get

$$\left(R^3_{(a_\alpha, a_\beta)} - R^3_{(a_\alpha, a_{\beta_r})} \right) - \left(R^3_{(0, a_\beta)} - R^3_{(0, a_{\beta_r})} \right) \tag{4.18}$$

$$= \left(R^3_{(a_\alpha, a_\beta)} \mathcal{W}_r R_{(a_\alpha, a_{\beta_r})} + R^2_{(a_\alpha, a_\beta)} \mathcal{W}_r R^2_{(a_\alpha, a_{\beta_r})} + R_{(a_\alpha, a_\beta)} \mathcal{W}_r R^3_{(a_\alpha, a_{\beta_r})} \right) \tag{4.19}$$

$$- \left(R^3_{(0, a_\beta)} \mathcal{W}_r R_{(0, a_{\beta_r})} + R^2_{(0, a_\beta)} \mathcal{W}_r R^2_{(0, a_{\beta_r})} + R_{(0, a_\beta)} \mathcal{W}_r R^3_{(0, a_{\beta_r})} \right) \tag{4.20}$$

$$= \left(R^3_{(a_\alpha, a_\beta)} \mathcal{W}_r R_{(a_\alpha, a_{\beta_r})} - R^3_{(0, a_\beta)} \mathcal{W}_r R_{(0, a_{\beta_r})} \right) \tag{4.21}$$

$$+ \left(R^2_{(a_\alpha, a_\beta)} \mathcal{W}_r R^2_{(a_\alpha, a_{\beta_r})} - R^2_{(0, a_\beta)} \mathcal{W}_r R^2_{(0, a_{\beta_r})} \right) \tag{4.22}$$

$$+ \left(R_{(a_\alpha, a_\beta)} \mathcal{W}_r R^3_{(a_\alpha, a_{\beta_r})} - R_{(0, a_\beta)} \mathcal{W}_r R^3_{(0, a_{\beta_r})} \right). \tag{4.23}$$

Next, with $\mathcal{W}_{a_\beta} = H(0, a_\beta) - H(a_\alpha, a_\beta)$ and $\mathcal{W}_{a_{\beta_r}} = H(0, a_{\beta_r}) - H(a_\alpha, a_{\beta_r})$, we rewrite (4.21) using the resolvent identity:

$$R^3_{(a_\alpha, a_\beta)} \mathcal{W}_r R_{(a_\alpha, a_{\beta_r})} - R^3_{(0, a_\beta)} \mathcal{W}_r R_{(0, a_{\beta_r})} \tag{4.24}$$

$$= R_{(0, a_\beta)} \mathcal{W}_{a_\beta} R^3_{(a_\alpha, a_\beta)} \mathcal{W}_r R_{(a_\alpha, a_{\beta_r})} + R^2_{(0, a_\beta)} \mathcal{W}_{a_\beta} R^2_{(a_\alpha, a_\beta)} \mathcal{W}_r R_{(a_\alpha, a_{\beta_r})} \tag{4.25}$$

$$+ R^3_{(0, a_\beta)} \mathcal{W}_{a_\beta} R_{(a_\alpha, a_\beta)} \mathcal{W}_r R_{(a_\alpha, a_{\beta_r})} + R^3_{(0, a_\beta)} \mathcal{W}_r R_{(a_\alpha, a_{\beta_r})} \mathcal{W}_{a_{\beta_r}} R_{(0, a_{\beta_r})}. \tag{4.26}$$

As previously, $\mathcal{W}_r$ is decomposed over boxes centered at $(x_2, y_2) \in \mathbb{Z}^2$ such that $x_2 \geq r - 1$, while both $\mathcal{W}_{a_\beta}$ and $\mathcal{W}_{a_{\beta_r}}$ are decomposed over integers (x_3, y_3)'s such that $x_3 \leq 1$. Proceeding as above yields the result. We then apply the same argument as for (4.22) and (4.23). □

Corollary 2.7 is a direct consequence of Theorem 2.6 and Lemma 4.1, for it is enough to prove that $I \cap \sigma(H(a,0)) = \emptyset$ (which readily implies that $\sigma_e^{(I)}(H(\mathcal{A}_{Iw}, a_\beta)) = 0$).

Lemma 4.1. *Let $H(A^{(0)})$ be the Landau Hamiltonian with constant magnetic filed B_-. Let $a \in \mathcal{C}^1(\mathbb{R}^2)$ be such that $\|a\|_\infty \leq K_1\sqrt{B_-}$ and $\|\text{div}a\|_\infty \leq K_2 B_-$. Then there exists a constant $0 < K_0 < \infty$ such that we have*

$$I \cap \sigma(H(A_0 + a)) = \emptyset$$

for B_- large enough and for any interval I satisfying

$$I \subset](2n-1)B_- + K_0 d_n(a, B_-), (2n+1)B_- - K_0 d_n(a, B_-)[, \tag{4.27}$$

if $n \in \mathbb{N}^*$, or

$$I \subset] - \infty, B_- - K_0 d_0(a, B_-)[\tag{4.28}$$

if $n = 0$, where $d_n(a, B_-) = \max(\|\text{div}a\|_\infty, \|a\|_\infty \sqrt{(n+1)B_-})$, $n \in \mathbb{N}$.

Proof. We denote by R_a, resp. R_0, the resolvent of $H(A_0 + a)$, resp. $H(A_0)$, and set $\mathcal{W}_a = H(A_0 + a) - H(A_0)$, We start with by resolvent identity

$$R_0(E) = R_a(E)(\text{id} - \mathcal{W}_a R_0(E)) \tag{4.29}$$

with $E \in I \subset \mathbb{R} \setminus \sigma(H(A_0))$. To show that E belongs to the resolvent set of $H(A^{(0)} + a)$, it is enough to show that $\|\mathcal{W}_a R_0(E)\| < 1$ so that $\text{id} - \mathcal{W}_a R_0(E)$ is invertible. We have

$$\|\mathcal{W}_a R_0(E)\| \leq \|(\text{div}a + |a|^2)R_0(E)\| + \|a \cdot (i\nabla + A^{(0)})R_0(E)\| \tag{4.30}$$

$$\leq (\|\text{div}a\|_\infty + \|a\|_\infty^2)\|R_0(E)\| + \|a\|_\infty \|(i\nabla + A^{(0)})R_0(E)\|. \tag{4.31}$$

We set $d = \text{dist}(E, \sigma(H(0,0)))$. We have, noting that $d \leq \max(|E|, B_-)$,

$$\|(i\nabla + A^{(0)})R_0(E)\|^2 = \|R_0(E)H(A_0)R_0(E)\| \leq \frac{1}{d} + \frac{\max(|E|, 0)}{d^2} \leq \frac{2\max(|E|, B_-)}{d^2}, \tag{4.32}$$

so that

$$\|\mathcal{W}_a R_0(E)\| \leq (\|\text{div}a\|_\infty + \|a\|_\infty^2)\frac{1}{d} + \|a\|_\infty \frac{\sqrt{2\max(|E|, B_-)}}{d}. \tag{4.33}$$

Assuming that E belongs to the nth band, $n \in \mathbb{N}$, and d satisfies both $d > 8\|a\|_\infty \sqrt{(n+1)B_-}$ and $d > \frac{1}{2}(\|\text{div}a\|_\infty + \|a\|_\infty^2)$, we find that $\|\mathcal{W}_a R_0(E)\| < 1$. □

5 Regularization and disordered systems

Quantum Hall effect actually deals with disordered systems, for its famous plateaux are consequences of the existence of localized states. As noticed in [CG, EGS], when adding a random potential the definition of the edge conductance requires a regularization to make sense. This regularization encodes the localization properties of the disordered systems, killing possible spurious currents.

Following [CG], a family $\{J_r\}_{r>0}$ will be called a regularization for an Hamiltonian H and an interval I if the following conditions hold true

$$C1 := \|J_r\| = 1, \forall r > 0 \text{ and } \forall \varphi \in E_H(I)\mathrm{L}^2(\mathbb{R}^2), \lim_{r\to\infty} J_r\varphi = \varphi \tag{5.1}$$

$$C2 := g'(H)i[H,\chi]J_r \in \mathcal{T}_1, \forall r > 0 \text{ and } \lim_{r\to\infty} \mathrm{tr}(g'(H)i[H,\chi]J_r) \text{ converges} \tag{5.2}$$

For such a regularization we can define the regularized edge conductance by

$$\sigma_e^{\mathrm{reg},I} := -\lim_{r\to\infty} \mathrm{tr}(g'(H)i[H,\chi]J_r) \tag{5.3}$$

Consider now a pair $(H(\mathcal{A}_{Iw},a),\, H(0,a))$. As an immediate consequence of Theorem 2.6, if J_R regularizes one operator of this pair then it regularizes the second one, and one has

$$\sigma_e^{\mathrm{reg},I}(H(\mathcal{A}_{Iw},a)) = -n + \sigma_e^{\mathrm{reg},I}(H(0,a)). \tag{5.4}$$

In particular, if we can show that $\sigma_e^{\mathrm{reg},I}(H(0,a)) = 0$, for instance under some localization property, then the edge quantization for $H(\mathcal{A}_{Iw},a)$ satisfies, for any $I \subset](2n-1)B_-, (2n+1)B_-[\cap] - \infty, B_+[$:

$$\sigma_e^{\mathrm{reg},I}(H(\mathcal{A}_{Iw},a)) = -\lim_{R\to\infty} \mathrm{tr}(g'(H(\mathcal{A}_{Iw},a))i[H(\mathcal{A}_{Iw},a),\chi]J_R) = -n. \tag{5.5}$$

The value n is of course in agreement with the value of the (bulk) Hall conductance, as argued by Halperin [Ha]. Indeed, by extending [GKS1] of [**?**] to random magnetic perturbations, the Hall conductance can be defined and computed for Fermi energies lying in the localized states region, and shown to be equal to then highest Landau level below the Fermi level.

Several possible regularizations have been introduced in [CG, CGH] within this context, each of them based on a specific localization property of the interface random Hamiltonian, where the randomness in only located on the half plane where the energy barrier is not effective. Thess properties are knwon to hold in the region of complete localization [GK4, GK5]. It is worth pointing out that this non ergodic random Hamiltonian is the relevant operator within our context where we deal with interface issues. In particular in some situations, it is possible to observe edge currents "without edges", meaning edge currents created by an interface random potential, as shown in [CG, CGH] [1]. Such regularization are thus designed to study the interface problem, and compute directly the edge conductance. The equality with the bulk conductance is then a by-product of this computation if by other means the bulk conductance could be computed.

If the focuss is rather put on the equality-bulk edge, then a similar regularization to the ones above, but involving the localization properties of the $\mathbb{Z}^2$ ergodic bulk Hamiltonian is needed. Such an analysis is pulled through in [EGS] where the authors are able to reconciliate a priori the edge and bulk points of view, showing that their regularized edge conductance and the Hall conductance match. It is very likely that such an analysis can be carried over to the context of the present work. However it would require to extend the analysis of [EGS] to the continuous setting and to random magnetic potentials.

To illustrate the discussion of this section, let us consider the random magnetic field

$$a_{\lambda,\omega}(x,y) = \lambda \sum_{j=(j_1,j_2)\in\mathbb{Z}^2,\ j_1\geq 0} \omega_j u_j(x,y),$$

with $u_j(x,y) = (u_1(x-j_1), u_2(y-j_2))$, u_1, u_2 being two given L^∞ compactly supported functions, and ω_j independent and identically distributed random variables supported on $[-1,1]$, with common density

$$\rho_\eta(s) = C_\eta \eta^{-1} \exp(-s\eta^{-1}) \chi_{[-1,1]}(s) \ ,$$

$\eta > 0$, and C_η et so that $\int \rho_\eta = 1$. The support of ρ_η is $[-1,1]$ for all $\eta > 0$, but as η goes to zero the disorder becomes weaker, for most j the coupling ω_j will be almost zero. This model is the half-plane perturbation version of the model considered in [DGR2].

We denote by $H_{B_-,\lambda,\omega}$ the corresponding random operator $(-i\nabla + A_0 + a_{\lambda,\omega})^2$, where A_0 generates a constant magnetic field of strength B_- in the perpendicular direction. By Lemma 4.1, for λ small enough, the spectrum

1. Playing with the sum rule it is actually possible to show the quantization of the regularized edge conductance for models considered in [EKSS], namely two different random electric potentials on the left and right half spaces, provided the disorder difference is large. It can be extended to a high disorder electric potential and a small disorder magnetic potential. We cannot yet prove such a phenomenon for pure random magnetic potentials

of $H_{B_-,\lambda,\omega}$ is contained in disjoint intervals centered at the Landau levels. Thanks to the ergodicity in the y direction, the spectrum is almost surely deterministic (see e.g. [EKSS, Theorem 2], which can be extended to random perturbation of order 1 as we consider). In [DGR2], we show that for the $\mathbb{Z}^2$-ergodic version of $H_{B_-,\lambda,\omega}$, there exists $\lambda^* > 0$ and for any $\lambda \in]0, \lambda^*]$ some $\eta^*(\lambda)$ such that for any $\eta \in]0, \eta^*(\lambda)]$, the full picture of localization as described in [**?**, **?**] is valid at the edge of the spectrum. The same analysis holds true for the half-plane version of the randomness as well for the Wegner estimate of [HK] holds the same (the same vector field can be used), and the initial condition is verified the same way uniformly for all boxes of the initial scale. This comes from the fact that within the region where the magnetic perturbation is zero localization holds for free, at a given distance of the Landau levels.

The remaining issue is to make sure that the spectrum is not empty where localization can be proven. In [DGR2] specific perturbations $a_{\lambda,\omega}$ are explicitely constructed where for a given integer J, the J^{th} first Landau levels of the Landau Hamiltonian $H(B_-)$ split into non trivial intervals as λ is turned on. It then follows from [EKSS, Theorem 2] that these intervals are also contained in the spectrum of the corresponding $H_{B_-,\lambda,\omega}$ [2].

6 Appendix: trace estimates

Let $\mathcal{H}$ be a given separable Hilbert space. Denote by $\mathcal{B}$ the class of bounded linear operators with norm $\|\cdot\|$, acting in $\mathcal{H}$, and by $\mathcal{T}_p$, $p \in [1, \infty)$, the Schatten-von Neumann class of compact operators acting in $\mathcal{H}$. We recall that $\mathcal{T}_p$ is a Banach space with norm $\|T\|_p := \left(\mathrm{tr}\,(T^*T)^{p/2}\right)^{1/p}$. In particular, in coherence with our previous notation, $\mathcal{T}_1$ is the trace class, and $\mathcal{T}_2$ is the Hilbert-Schmidt class. The following lemma contains some well-known properties of the Schatten-von Neumann spaces, used systematically in the proofs of our results.

Lemma 6.1. [Si] *(i) Let $T \in \mathcal{T}_p$, $p \in [1, \infty)$. Then $T^* \in \mathcal{T}_p$ and we have*

$$\|T\|_p = \|T^*\|_p. \tag{6.1}$$

(ii) Let $T \in \mathcal{T}_p$, $p \in [1, \infty)$, and $Q \in \mathcal{B}$. Then $TQ \in \mathcal{T}_p$, and we have

$$\|TQ\|_p \le \|T\|_p \|Q\|. \tag{6.2}$$

2. The rational for that result is that if E belongs to the almost sure spectrum of the $\mathbb{Z}^2$-ergodic model, it is approximated, in term of a Weyl sequence, by eigenvalues of a large volume Hamiltonian with a specific configuration of the random variables; then almost every potential, even defined on the half space will exhibit somewhere the same pattern, thus creating the same eigenvalue.

(iii) Let $p_j \in [1,\infty)$, $j = 1\dots,n$, $p \in [1,\infty)$, *and* $\sum_{j=1}^n p_j^{-1} = p^{-1}$. *Assume that* $T_j \in \mathcal{T}_{p_j}$, $j = 1\dots,n$. *Then* $T := T_1 \dots T_n \in \mathcal{T}_p$, *and we have*

$$\|T\|_p \leq \prod_{j=1}^n \|T\|_{p_j}. \tag{6.3}$$

Lemma 6.2. [Si] *(i) Let* $T \in \mathcal{T}_1$, $Q \in \mathcal{B}$. *Then we have*

$$\operatorname{tr} TQ = \operatorname{tr} QT. \tag{6.4}$$

(ii) Let $p \in [1,\infty)$, $q \in [1,\infty)$, $p^{-1} + q^{-1} = 1$. *Assume that* $T \in \mathcal{T}_p$, $Q \in \mathcal{T}_q$. *Then* (6.4) *holds true again.*

Our next lemma contains a simple condition which guarantees the inclusion $T \in \mathcal{T}_p$ for operators of the form $T = f(x)g(-i\nabla)$.

Lemma 6.3. [Si, Theorem 4.1] *Let* $d \geq 1$, $p \in [2,\infty)$, $f, g \in L^p(\mathbb{R}^d)$. *Set* $T := f(x)g(-i\nabla)$. *Then we have* $T \in \mathcal{T}_p$, *and*

$$\|T\|_p \leq (2\pi)^{-d/p} \|f\|_{L^p} \|g\|_{L^p}. \tag{6.5}$$

Assume that

$$\beta \in L^\infty(\mathbb{R}^2;\mathbb{C}^2), \quad \operatorname{div}\beta \in L^\infty(\mathbb{R}^2). \tag{6.6}$$

Define the operator

$$\mathcal{L}_\beta u := \beta \cdot \nabla u, \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^2),$$

and then close it in $L^2(\mathbb{R}^2)$.

Proposition 6.4. *Let* $A \in \mathcal{C}^1(\mathbb{R}^2;\mathbb{R}^2)$, $z \in \mathbb{C}\setminus[0,\infty)$. *Set* $R_A(z) := (H(A)-z)^{-1}$.
(i) Assume that $\alpha \in L^2(\mathbb{R}^2)$. *Then we have*

$$\alpha R_A(z) \in \mathcal{T}_2, \quad R_A(z)\alpha \in \mathcal{T}_2. \tag{6.7}$$

Moreover, there exists a constant C_1 *independent of* z, *such that*

$$\|\alpha R_A(z)\|_2 = \|R_A(z)\alpha\|_2 \leq C_1 C_0(z) \tag{6.8}$$

where

$$C_0(z) := \sup_{\lambda\in[0,\infty)} \frac{\lambda+1}{|\lambda - z|}. \tag{6.9}$$

(ii) Assume that β *is compactly supported and satisfies* (6.6). *Then we have*

$$\mathcal{L}_\beta R_A(z) \in \mathcal{T}_4, \quad R_A(z)\mathcal{L}_\beta \in \mathcal{T}_4. \tag{6.10}$$

Moreover, there exists a constant C_2 *independent of* z, *such that*

$$\|\mathcal{L}_\beta R_A(z)\|_4 \leq C_2 C_0(z), \quad \|R_A(z)\mathcal{L}_\beta\|_4 \leq C_2 C_0(z). \tag{6.11}$$

Proof. (i) By (6.1) it suffices to prove only the first inclusion in (6.7). This inclusion follows immediately from

$$\|\alpha R_A(z)\|_2 = \|\alpha R_A(-1)(H(A)+1)R_A(z)\|_2 \le C_0(z)\|\alpha R_A(-1)\|_2 \le$$

$$C_0(z)\|\alpha(-\Delta+1)^{-1}\|_2 = \frac{C_0(z)}{2\pi}\left(\int_{\mathbb{R}^2}|\alpha|^2dx\int_{\mathbb{R}^2}\frac{d\xi}{(|\xi|^2+1)^2}\right)^{1/2} < \infty. \quad (6.12)$$

Note that the second inequality is a special case of the diamagnetic inequality of Hilbert-Schmidt operators (see e.g. [Si, Theorem 2.13]), and the last equality just follows from the Parseval identity.
(ii) Since we have

$$(R_A(z)\mathcal{L}_\beta)^* = (-\mathcal{L}_{\overline{\beta}} - \operatorname{div}\overline{\beta})R_A(\overline{z})$$

and $\operatorname{div}\overline{\beta}R_A(\overline{z}) \in \mathcal{T}_2 \subset \mathcal{T}_4$ by (6.7), again it suffices to check only the first inclusion in (6.10). As in the proof of (6.7) we have

$$\|\mathcal{L}_\beta R_A(z)\|_4 \le C_0(z)\|\mathcal{L}_\beta R_A(-1)\|_4.$$

Further,

$$\mathcal{L}_\beta R_A(-1) = i\beta\cdot(-i\nabla - A)R_A(-1) + i\beta\cdot AR_A(-1), \quad (6.13)$$

and $i\beta\cdot AR_A(-1) \in \mathcal{T}_2 \subset \mathcal{T}_4$ by (6.7). Let $0 \le \zeta_j \in C_0^\infty(\mathbb{R}^2)$, $j = 0, 1$, satisfy $\zeta_0\beta = \beta$, $\zeta_1\zeta_0 = \zeta_0$ on $\mathbb{R}^2$. Since $[R_A(-1), \zeta_0] = -R_A(-1)[H(A), \zeta_0]R_A(-1)$, we have

$$i\beta\cdot(-i\nabla - A)R_A(-1) =$$

$$i\beta\cdot(-i\nabla - A)R_A(-1)\zeta_0 - i\beta\cdot(-i\nabla - A)R_A(-1)\zeta_1[H(A),\zeta_0]R(-1). \quad (6.14)$$

Note that the operator

$$[H(A),\zeta_0]R_A(-1) = 2\nabla\zeta_0\cdot(-\nabla + iA)R_A(-1) - \Delta\zeta_0 R_A(-1)$$

is bounded. Therefore, it follows from (6.13), (6.14), (6.2), that it suffices to check

$$\beta\cdot(-i\nabla - A)R_A(-1)\zeta \in \mathcal{T}_4 \quad (6.15)$$

with $0 \le \zeta \in C_0^\infty(\mathbb{R}^2)$. The mini-max principle implies

$$\|\beta\cdot(-i\nabla - A)R_A(-1)\zeta\|_4 \le \|\beta\|_{L^\infty}\|R_A(-1)^{1/2}\zeta\|_4. \quad (6.16)$$

On the other hand,

$$\|R_A(-1)^{1/2}\zeta\|_4 = \|\zeta R_A(-1)^{1/2}\|_4 \quad (6.17)$$

by (6.1). The diamagnetic inequality for $\mathcal{T}_4$-operators (see e.g. [Si, Theorem 2.13]) entails

$$\|\zeta R_A(-1)^{1/2}\|_4 \leq \|\zeta(-\Delta+1)^{-1/2}\|_4, \tag{6.18}$$

and by (6.5) we obtain

$$\|\zeta(-\Delta+1)^{-1/2}\|_4 \leq (2\pi)^{-1/2}\|\zeta\|_{L^4}\left(\int_{\mathbb{R}^2}\frac{d\xi}{(|\xi|^2+1)^2}\right)^{1/4}. \tag{6.19}$$

Putting together (6.16) - (6.19), we obtain (6.15), and hence (6.10) - (6.11). □

Remark 21. *If $\Im z \neq 0$, then the constant $C_0(z)$ defined in* (6.9) *admits the estimate*

$$C_0(z) \leq \frac{(\Re z+1)_+}{|\Im z|}+1. \tag{6.20}$$

Corollary 6.5. *Let $A^{(j)} \in \mathcal{C}^1(\mathbb{R}^2;\mathbb{R}^2)$, $H_j := H(A^{(j)})$, $R_j := (H_j - z)^{-1}$, $z \in \mathbb{C}\setminus[0,\infty)$, $j = 1,2,3$. Assume that $\alpha \in L^2(\mathbb{R}^2)$, β satisfies* (6.6)*, and α and β are compactly supported. Then the operators*

$$(\mathcal{L}_\beta+\alpha)R_jR_kR_l, \quad R_j(\mathcal{L}_\beta+\alpha)R_kR_l, \quad R_jR_k(\mathcal{L}_\beta+\alpha)R_l, \quad R_jR_kR_l(\mathcal{L}_\beta+\alpha) \tag{6.21}$$

with $j,k,l = 1,2,3$, are trace-class.

Proof. By (6.1) it suffices to consider only the first two operators in (6.21). Introduce three functions $0 \leq \zeta_j(\mathbb{R}^2) \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, $j = 0,1,2$, such that $\zeta_0\alpha = \alpha$, $\zeta_0\beta = \beta$, $\zeta_j\zeta_{j-1} = \zeta_{j-1}$, $j = 1,2$. Note that $[R_j,\zeta_k] = -R_j[H_j,\zeta_k]R_j$, and

$$[H_j,\zeta_k] = 2\nabla\zeta_k\cdot(-\nabla+iA^{(j)}) - \Delta\zeta_k \tag{6.22}$$

with $j = 1,2,3$, and $k = 0,1,2$. Then we have

$$(\mathcal{L}_\beta+\alpha)R_jR_kR_l =$$

$$(\mathcal{L}_\beta+\alpha)R_j\zeta_0R_k\zeta_1R_l-(\mathcal{L}_\beta+\alpha)R_j\zeta_0R_k[H_k,\zeta_1]R_kR_l-(\mathcal{L}_\beta+\alpha)R_j[H_j,\zeta_0]R_j\zeta_1R_kR_l+$$
$$(\mathcal{L}_\beta+\alpha)R_j[H_j,\zeta_0]R_j[H_j,\zeta_1]R_j\zeta_2R_kR_l-$$
$$(\mathcal{L}_\beta+\alpha)R_j[H_j,\zeta_0]R_j[H_j,\zeta_1]R_j[H_j,\zeta_2]R_jR_kR_l, \tag{6.23}$$

$$R_j(\mathcal{L}_\beta+\alpha)R_kR_l =$$
$$R_j\zeta_0(\mathcal{L}_\beta+\alpha)R_k\zeta_1R_l - R_j\zeta_0(\mathcal{L}_\beta+\alpha)R_k[H_k,\zeta_1]R_kR_l. \tag{6.24}$$

Taking into account Proposition 6.5 (6.22), as well as (6.3) with $p = 1$ and (6.2), we find that (6.23) and (6.24) imply that the operators in (6.21) are trace-class. □

Chapter 4

Opérateurs projectivement covariants aléatoires

Résumé

Nous discutons plus en détail le formalisme developpé pour prouver la réponse linéaire. Ceci dans le but d'obtenir une généralisation future. Nous commençons dans un premier temps par prouver un théorème de densité, alors que, dans un deuxième temps, nous mettons en avant le rôle joué par le groupe associé aux symétries du systèmes physique dans la structure de l'algèbre des observables.

Contents

1 Introduction

Dans ce dernier chapitre nous proposons une discussion plus appronfondie au sujet de la structure de l'algèbre des opérateurs projectivement covariants aléatoires impliqués dans la théorie de la réponse linéaire. Nous mettons en évidence sa structure interne. Dans un deuxième temps, motivé par ces premiers résultats nous entrons plus en détail dans le rôle joué par le groupe de structure ainsi qu'à la possible diagonalisation partielle des opérateurs de l'algèbre. Physiquement, les opérateurs que nous considérons sont associés à l'analyse de l'équation de Schrödinger magnétique aléatoire, laquelle modèlise le comportement d'une particule soumise à un champ électromagnétique dans un milieu désordonné. Notre première motivation était de développer le formalisme adapté au traitement de la réponse linéaire pour un opérateur de Schrödinger $\mathbb{Z}^d$-ergoqique. Si dans le premier chapitre nous nous en sommes tenus à une description la plus succinte possible des outils nécessaires, nous avons néanmoins passé sous silence de nombreux aspects de la structure algèbrique qui soutend la compréhension (et la richesse) de ces modèles et phénomèmes. Nous revenons sur ce formalisme dans ce chapitre

de manière à dégager les outils d'une généralisation future de la théorie de la réponse linéaire pour des opérateurs de Schrödinger associés construits sur des ensembles de Delone. Nous reprenons les concepts rapidement introduits dans le premier chapitre. Un premier élément de contexte vient du fait que les symétries de l'opérateur de Schrödinger magnétique sont implémentées par les translations magnétiques. Mais contrairement au cas périodique, ces translations ne commutent pas. Ceci provient d'un facteur de phase ζ qui dépend du flux magnétique. Ce facteur de phase est un *2-cocycle*. En conséquence, la représentation unitaire sur l'espace de Hilbert $\mathfrak{H}$ associée à ces translations forment une représentation *projective* du groupe $\mathcal{Z}$ associée aux symétries.
Dans un deuxième temps les opérateurs que nous considérons sont aléatoires. Historiquement, ce type d'opérateurs peut être considéré comme une généralisation des opérateurs périodiques qui modélisent un cristal parfait. Ce type d'opérateurs modélise la présence d'impuretées dans le cristal. Nous supposons tout de même l'existence d'un groupe de transformations préservant la mesure τ provenant de l'action du groupe $\mathcal{Z}$ sur l'espace de probabilité $(\Omega, \mathbb{P})$. Par suite les symétries sont encodées par le groupe $\mathcal{Z}$, le *2-cocycle* ζ ainsi que par le groupe de transformation τ. Nous avons donc une algèbre d'opérateurs avec des symétries plus riches que dans le cas des opérateurs périodiques, mais nous montrons que nous pouvons construire tout de même une base de diagonalisation adaptées à cette situation.
D'un autre côté, certains des modèles qui ont émergé ces derniers temps possèdent un groupe de symétries non-abelien. Dans ce cas précis l'analyse de Fourier ne peut plus être utilisée telle quelle du fait que la dualité de Pontrjagin n'a plus de sens. Mais grâce à l'utilisation de structures adapatées nous pouvons passer outre cette difficultée. C'est ce que nous proposons notament de mettre en lumière dans ce chapitre
Nous résumons les principales avancées qui nous utilisons. Dans un premier temps, Bédos et Conti [BC] ont développé une analyse de Fourier abstraite pour l'algèbre réduite des groupes *tordus*. Ils construisent une généralisation des notions introduites par *Haagerup* dans ces travaux fondateurs [HI, HII, HK]. Nous utilisons leur construction et la généralisons au cas de l'algèbre produit-croisés. Ceci étant assez naturel dans le sens où l'algèbre réduite de groupe est un cas trivial d'algèbre de produit-croisé.
Dans un deuxième temps l'autre clé se trouve dans le choix de la bonne structure algèbrique afin d'obtenir une diagonalisation partielle. Cette structure est le C^*-module Hilbertien qui a connu un développement intensif ces dernières années. Les premiers travaux sur le sujet ont été ceux de I.Kaplansky, W.Paschke and M.Rieffel. Nous nous référons dans cet exposé essentiellement à la monographie de V.M. Manuilov and E.V Troitsky [MT].

Comme nous allons le montrer, cette structure est la structure naturelle associée aux opérateurs considérés.
La structure de l'exposition est la suivante. Nous commençons par faire une première analyse globale de l'algèbre des opérateurs projectivement covariants aléatoires. Nous démontrons certains résultats de densités. Aprés quoi nous reformulons la théorie de Floquet usuelle dans le cadre naturel des C^*-modules Hilbertien. Ceci étant fait nous pouvons l'appliquer au contexte qui nous intéresse. Entre autres choses, nous obtenons une diagonalisation partielle qui respecte la structure du groupe des symétries de ces opérateurs. Cette discussion est largement inspirée de [G, L2, BS, Gr, LPV].

1.1 Preliminaries on physical symmetries

In this section, we summarize the physical origins of the formalism we interested in. We consider the Schrödinger equation

$$i\partial_t \varphi_t = H_\omega \varphi_t \ ,$$

where H_ω is the associated random magnetic Hamiltonian with ω runs over a probability space $(\Omega, \mathbb{P})$. In the study of such physical systems one of the main tools are the symmetries of the underlying system. These symmetries characterize strongly the transport and spectral properties of the observables. Here we are interested in the consequence on the algebraic structure of the symmetries of the physical system. We refer essentially to [GI] for general magnetic field in the Euclidian case and [S, BS, G] for uniform and periodic magnetic field in the Riemannian case. In [GI], the authors are essentially interested in the N-body problem and the spectral properties of the observables affiliated to a certain algebra arising from this problem. But they provide a constructive description of the magnetic translations which we reproduce here. In [BS] the authors consider the spectral theory of the magnetic Schrödinger operators on manifold with a uniform or periodic magnetic field. In the same way we consider here only uniform constant or periodic magnetic field. Note that this restriction concerns most of the studied models. We begin to give a general construction of the magnetic translation following [GI].

Let X be a finite dimensional real vector space, then the magnetic Hamiltonian is defined by

$$H = H(\mathbf{A}, V) = (-i\nabla - \mathbf{A})^2 + V \quad \text{on} \quad \mathrm{L}^2(X), \tag{1.1}$$

where $\mathbf{A}$ is the magnetic potentiel and V electric potentiel. We assume that they respect the Leinfelder-Simader conditions as described in Ch1.
Following [GI], we now give a description of the magnetic translations which encode the symmetries as defined by Zak [Z].
To each magnetic field we can associate a mathematical object called a *2-cocycle* as shown in [GI]. The first important point is that the 2-cocycle depends only on the magnetic field and not on the magnetic potential.
A magnetic potentiel is by definition a differential 1-form on X, $A : X \mapsto X^*$ where X^* is the dual of X. Then by the usual definition of integral of a 1-form A along a path γ, we have

$$\Gamma^A(\gamma) := \int_\gamma A = \int \langle \partial_t \gamma(t), A(\gamma(t)) \rangle dt \ .$$

Noting $[x, y]$ a segment between the points x et y, we can then define

$$\xi(x, y) := \mathrm{e}^{-i\Gamma^A([x,y])}$$

as continuous function, $\xi : X \times X \mapsto U(1)$. Recall that the physical equation between the magnetic potential and magnetic field is given by $\mathrm{d}A = B$. Note that this equation is hightly degenerated. Using the Stokes formula we get that

$$\int_{[x,y,z]} A = \int_{\langle x,y,z \rangle} dA = \int_{\langle x,y,z \rangle} B$$

where $[x, y, z] = [x, y] \cup [y, z] \cup [z, x]$ and $\langle x, y, z \rangle$ the surface delimited by $[x, y, z]$. The last term is called the magnetic flux through the triangle $\langle x, y, z \rangle$ and obviously depends only on the magnetic field.
We note $p = -i\nabla$ the momentum observable and q the position operator. In this context [GI] associate to each magnetic field a 2-cocycle ζ by

$$\zeta(x, y)(z) = \xi(z, z + x)\xi(z + x, z + x + y)\xi(z + x + y, z) \tag{1.2}$$

$$= \exp\left(-i \int_{\langle z,z+x,z+x+y \rangle} B \right) \tag{1.3}$$

and get a general construction of the magnetic translation

Proposition 1.1. *[GI] Let A, ξ and ζ be defined as above. The magnetic translations $\{U_x\}_{x \in X}$ are unitary operators acting on $L^2(X)$ by*

$$U_x = \xi(q, q + x)\mathrm{e}^{i\langle x,D \rangle}$$

and for any $x, y \in X$ the following relation holds

$$U_x U_y = \zeta(x, y) U_{x+y}$$

Where $D = -i\nabla$ is the momentum observable. We have then by construction that

$$U_x q U_x^* = q - x \tag{1.4}$$
$$\text{and } U_x H(\mathbf{A}, V) U_x^* = H(A, V) \tag{1.5}$$

In the following, we only consider constant 2-cocycle in the sense that in the equation (1.2) the 2-cocycle depend only of thge two first variables. This assumption provides a particular algebraic structure of interest. Furthermore in the quantum Hall systems the most part of models studied are covered by this assumption.

We now give a short description of the random Schrödinger operator which is concerned by this discussion. It allows for random magnetic potrential in addition to the usual random electric potential (like Anderson). We refer to [DGR1, DGR2] for details on the assumptions and localization results for purely magnetic fields.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a measured space , the magnetic random Schrödinger operator is defined by

$$H_\omega := H(\mathbf{A}_\omega, V_\omega) = (-i\nabla - \mathbf{A}_\omega)^2 + V_\omega \text{ acting on } L^2(X) \ ,$$

where $A_\omega = A + \mathbf{a}_\omega$ with A the magnetic potential considered above and

$$V_\omega(q) = \sum_{x \in \mathcal{Z}} \eta_x(\omega) u(q - x) \ ,$$

$$\mathbf{a}_\omega(q) = \sum_{x \in \mathcal{Z}} \kappa_x(\omega) v(q - x) \ ,$$

where

1. $u \in C^\infty(X)$ (respectively $v \in C^\infty(X, X)$) with $supp(u) \subset \Delta_1(0)$ (respc. $supp(v) \subset \Delta_1(0)$) the unit box centered in the origine.

2. $\{\eta_x\}_{x \in \mathcal{Z}}$ and $\{\kappa_x\}_{x \in \mathcal{Z}}$ are independant and identically distribued random variables. -

3. we assume a covariance condition in the following sense. We consider an ergodic action τ of $\mathcal{Z}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that the following relations hold
$$U_x \mathbf{a}_\omega U_x^* = \mathbf{a}_{\tau(x)\omega} \text{ and } U_x V_\omega U_x^* = V_{\tau(x)\omega} \ , \tag{1.6}$$

more generally we assume that

$$U_x H_\omega U_x^* = H_{\tau(x)\omega} \ . \tag{1.7}$$

4. we assume that for any $\omega \in \Omega$ that H_ω is essentially self-adjoint on $C_c^\infty(X)$.

2 General assumptions and Usual formalism

For the convenience of the reader we recall the basic definitions and the main assumptions.
Let a separable Hilbert space, namely $\mathfrak{H}$ and $\mathcal{Z}$ a locally compact discrete finitely generated group. We use additive notation for the convenience and denote by e the neutral element. Other assumptions will be made when needed. Moreover we assume there exists a unitary projective representation on $\mathfrak{H}$ as defined in what follows.

ζ-projective unitary representation

For the sake of completeness we recall all the definitions which we shall use. We refer to [EL] which is the reference on the subject, more precisely we follows the line of [BC].

Definition 2.1. *A (normalized) 2-cocycle on $\mathcal{Z}$ with values in $\mathbb{T}$ (the one dimensional torus) is a map $\zeta : \mathcal{Z} \times \mathcal{Z} \mapsto \mathbb{T}$ such that the cocycle relation*

$$\zeta(x,y)\zeta(x+y,z) = \zeta(x,y+z)\zeta(y,z) \text{ for any } x,y,z\mathcal{Z}$$

holds and moreover we have that

$$\zeta(x,\mathrm{e}) = \zeta(\mathrm{e},x) = 1 \tag{2.1}$$

$$\zeta(x,-x) = \zeta(-x,x) \text{ for any } x \in \mathcal{Z} \tag{2.2}$$

The set of 2-cocycle is denoted by $Z^2(\mathcal{Z},\mathbb{T})$.

Remark 22. *We can without loss of generality suppose that the normalized condition which simply means that $\zeta(\mathrm{e},\mathrm{e}) = 1$.*

Definition 2.2. *Let $\zeta : \mathcal{Z} \times \mathcal{Z} \mapsto \mathbb{T}$ a 2-cocycle. A ζ-projective unitary representation of $\mathcal{Z}$ on a given Hilbert space $\mathfrak{H}$ is a map from $\mathcal{Z}$ into the group $\mathcal{U}(\mathfrak{H})$ of unitaries on $\mathfrak{H}$ such that the following holds*

$$U_x U_y = \zeta(x,y) U_{x+y} \text{ for any } x,y \in \mathcal{Z} \tag{2.3}$$

$$U_x^* = \overline{\zeta}(x,-x) U_{-x} \text{ for any } x \in \mathcal{Z}. \tag{2.4}$$

consequently we have that $U_\mathrm{e} = id_{\mathfrak{H}}$.

In fact for a given ζ-projective unitary representation on $\mathfrak{H}$ we can derive the cocycle relation from the associativity of the unitary map, i.e $U_x(U_yU_z) = (U_xU_y)U_z$.

Identity decomposition

We denote by $\mathcal{P}(\mathcal{B}(\mathfrak{H}))$ the lattice of projection belonging to $\mathcal{B}(\mathfrak{H})$. Now to encode the geometric symmetries, we assume the existence of a projection-valued map on $\mathcal{Z}$.

Definition 2.3. *We call a projection-valued map on $\mathcal{Z}$ an identity decomposition with respect to the unitary representation an element of $\mathcal{P}(\mathcal{B}(\mathfrak{H}))^{\mathcal{Z}}$ such that the following holds*

$$\chi_x\chi_y = \delta_{x,y}\chi_x, \text{ for any } x, y \in \mathcal{Z} \tag{2.5}$$

$$\sum_{x\in\mathcal{Z}} \chi_x = id_{\mathfrak{H}} \tag{2.6}$$

$$U_x\chi_yU_x^* = \chi_{x+y} \text{ for all } x, y \in \mathcal{Z} \tag{2.7}$$

Where δ is the Dirac function which is equal to one if $x = y$ and zero otherwise. We define $\mathfrak{H}_c$ as elements φ of $\mathfrak{H}$ such that there exists a finite subset F of $\mathcal{Z}$ such that $\chi_F\varphi = \varphi$.

Randomness and Ergodic action

We work with random observables modelizing a disordered medium. Conceptually this is nothing else than a generalization of the periodic media which modelizes a perfect crystal. Therefore it is fundamental to be able to define "random symmetries" by an another group-action but on the underlying probability space.

Definition 2.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ a σ-Borel finite probability space with probability measure $\mathbb{P}$.*
A group $\{\tau_x\}_{x\in\mathcal{Z}}$ of measure preserving transformations τ on the Borel space Ω is a map from $\mathcal{Z}$ to the set of isomorphisms of Ω such that for any $x, y \in \mathcal{Z}$,

$\tau_x : \Omega \mapsto \Omega$ *Be a measure-preserving transformation, i.e* $\frac{\mathrm{d}\mathbb{P}\circ\tau(x)}{\mathrm{d}\mathbb{P}} = 1$
$\tau_x \circ \tau_y = \tau_{x+y}$, $\tau_e = 1$ *and so* $\tau_x^{-1} = \tau_{-x}$

Moreover we assume that τ acts ergodically on $(\Omega, \mathcal{F}, \mathbb{P})$.

Remark 23. *The ergodicity is equivalent to both these properties:*

- *The only stable sets are of measure 0 or 1,*
- *if a measurable function f is τ-invariant then f is constant*
- *the unitary group on $L^2(\Omega, \mathbb{P})$ inducted by $\{\tau_a\}$ has one for eigenvalue with multiplicity one.*

3 Dual group, representation and action

In the following we begin to assume that $\mathcal{Z}$ is abelian. We consider non-abelian group only in last section 7.1 and 7.2. A great part of our analysis takes place on Fourier analysis on group and twisted group. We recall the basic results and notations about the Fourier analysis on groups. In the abelian case we can deal with the Pontrjagin duality and use Fourier analysis on groups.

3.1 Fourier's analysis on group

Preliminaries on group theory

We begin with general definitions and assumptions on groups that we will implicitely use in the sequel. Before using some regularity notions we define what we mean by convergence to zero at infinity for a function on $\mathcal{Z}$. We define general decaying properties which will be generalized to the twisted case following the definitions introduced in [BC]. We denote $\mathcal{K}(\mathcal{Z})$ the set of functions on $\mathcal{Z}$ with finite support and for a function f on $\mathcal{Z}$ we write $f_F = f\mathbf{1}_F$, with $\mathbf{1}_F$ the characteristic function of the subset F. An important notion which is deeply linked with the existence of identity approximation or existence of ergodic theorem is the notion of *amenability* of a group.

Definition 3.1. *A group is called amenable if it has a left translation invariant state on $l^\infty(\mathcal{Z})$.*

The first important consequence (Thm2.17[BC]) of this property is the existence of a Følner net which provides an exhaustion procedure that enables us to use the ergodic theorem or the existence of limits of the density of state (see [LPV]). In our case the first interest is the existence of identity approximation but for the sake of completeness we reproduce here the definition.
A sequence $\{F_\alpha\}$ of finite non-empty subset of $\mathcal{Z}$ is called a Føelner net if

for any $x \in \mathcal{Z}$ we have that $\frac{|xF_\alpha \Delta F_\alpha|}{|F_\alpha|} \underset{\alpha}{\to} 0$. We now define what we mean by regular functions on the group.

Definition 3.2. *Let $\mathcal{G}$ a subset of $l^2(\mathcal{Z})$ which contains $\mathcal{K}(\mathcal{Z})$, $\|\cdot\|_0$ a norm on it and $f \in \mathcal{G}$. We say that $f \to 0$ at infinity with respect to the norm $\|\cdot\|_0$ if for every $\varepsilon > 0$ there exists a finite subset F of $\mathcal{Z}$ such that $\|f_K\|_0 < \varepsilon$ for all $K \subset F^c$. We denote by $C_0(\mathcal{Z})$ the set of functions which tend to zero at infinity with respect to $|\cdot|_\infty := \sup_{K \subset \mathcal{Z} finite} |\cdot|$.*

We want now to ensure the existence of approximation of the identity for a general locally compact group. This comes from the *amenability* and from the *Haagerup* property as described below. There exists several other descriptions of the amenability, see [BC]. In our case we need a more refined notion than just *amenability*, which has been introduced in the untwisted case by Haagerup [HI, HII] which ensures the existence of an approximation of identity and convergence of Fourier series.

Definition-Proposition 2 ([BC])**.** *A group $\mathcal{Z}$ is said to have the Haagerup property or (a-T-menable) if there exists a net φ_n of normalized positive definite functions such that $\varphi_n \in \mathcal{C}_0(\mathcal{Z})$ and $\varphi_n \xrightarrow[n\to\infty]{} 1$ pointwise.*
A countable group has the Haagerup property if and only if the group has a Haagerup length function L which is a proper function such that

$$L : \mathcal{Z} \mapsto [0, \infty) \text{ such that } L(\mathrm{e}) = 0$$
$$L(-x) = L(x) \text{and } L(x + y) \leq L(x) + L(y)$$
$$L^{-1}([0, t]) \text{ is finite for all } t \geq 0$$

We now introduce an important tool that we shall use intensively namely the algebra realization of the group. It is well-known that elements of a group can be represented as operators acting on $l^2(\mathcal{Z})$ through translations.

Definition 3.3. *The left regular representation of $\mathcal{Z}$ on $l^2(\mathcal{Z})$ is defined by the operators $\{\lambda(x)\}_{x\in\mathcal{Z}}$ acting on $l^2(\mathcal{Z})$ by*

$$(\lambda(x)\varphi)(y) = \varphi(y - x) \text{ for any } x, y \in \mathcal{Z} \ .$$

Fourier's analysis

We recall that the dual group of $\mathcal{Z}$ is defined as the set of homomorphisms on $\mathcal{Z}$ to the unit circle of $\mathbb{C}$. It is compact and separable ($\mathcal{Z}$ is assumed to be discrete and countable). We denote it by $\mathcal{Z}^*$. Let $\theta \in L^1(\mathcal{Z}^*)$, the formal Fourier transform is given by

$$\widehat{\theta}(x) = \int_{\mathcal{Z}^*} \overline{r(x)}\theta(r)dr \ ,$$

where dr is the natural Haar measure on the compact separable space $\mathcal{Z}^*$. We first give the classical character's relations that we will often use in the following.

i) $\displaystyle\int_{\mathcal{Z}^*} r(x)\overline{r}(y)dr = \delta_{x,y}$

ii) $\displaystyle\sum_{\mathcal{Z}} \overline{r}(x)r'(x) = \delta_{r,r'}$

iii) $\displaystyle\int_{\mathcal{Z}^*} r(x)dr = \delta_{\mathrm{e}}(x)$

We mention that the second equality must be obviously understood in the distributional sense, namely for f belonging to $L^1(\mathcal{Z}^*)$ we have that $\sum_{\mathcal{Z}} \int_{\mathcal{Z}^*} f(r')\overline{r}(x)r'(x)dr = f(r')$. This relation provides the inverse Fourier formulae.

To emphasize the duality, in the literature, it is often written $\langle r, x\rangle$ instead of $r(x)$. We note that by the so-called *Riemann-Lebesgue lemma* we have for $\theta \in L^1(\mathcal{Z}^*)$ that $\widehat{\theta} \in C_0(\mathcal{Z})$. For any $\theta \in L^2(\mathcal{Z}^*)$ we define $\varepsilon_x : r \mapsto r(x)$ then the following holds

$$\theta = \sum_{\mathcal{Z}} |\varepsilon_x\rangle\langle\varepsilon_x|\theta\rangle = \sum_{\mathcal{Z}} \varepsilon_x \int_{\mathcal{Z}^*} \overline{r}(x)\theta\ (r)dr \tag{3.1}$$

Then using the character relation $ii)$ we get that

$$\left(\sum_{\mathcal{Z}} \varepsilon_x \int_{\mathcal{Z}^*} \overline{r}(x)\theta\ (r)dr\right)(r_0) = \int_{\mathcal{Z}^*} \sum_{\mathcal{Z}} \overline{r}(x)r_0(x)\theta\ (r)dr = \theta(r_0)$$

Let now for any $f \in l^1(\mathcal{Z})$, we set $\check{f}(r) := \sum_{\mathcal{Z}} r(x)f(x)$, we then have that $\theta(r) = \check{\widehat{\theta}}(r)$. What proves the inverse Fourier formula in the general context of group. In a general setting we can also get the Plancherel formula. For any $f \in L^1(\mathcal{Z}) \cap L^2(\mathcal{Z})$ as $\|\mathcal{F}(f)\|_2 = \|f\|_2$, where we write $\mathcal{F}(f)(r) = \sum_{\mathcal{Z}} \overline{r}(x)f(x)dx$. Then in the abelian case we recover lot of the usual features of the Fourier's analysis. We now show that it is the natural tool in order to construct a analysis for covariant operators.

3.2 Action of the dual group on operators

We start by giving some basic constructions. Afterwhat we will show that is in fact a first example of more general structural features of the underlying operators. But it is interesting since it provides a good illustration of the pertinence of the adopted point of view.

Definition 3.4. *Let $\varphi \in \mathcal{B}(\mathcal{Z})$, the $\mathcal{W}^*$-algebra of bounded functions on $\mathcal{Z}$. Then one defines the following map from the $\mathcal{W}^*$-algebra of bounded operator on $\mathcal{Z}$ into the $\mathcal{W}^*$-algebra of bounded operators on $\mathfrak{H}$ by*

$$V : \mathcal{B}(\mathcal{Z}) \to \mathcal{B}(\mathfrak{H}) \tag{3.2}$$

$$\varphi \to \sum_{\mathcal{Z}} \varphi(x)\chi_x \tag{3.3}$$

a unitary and faithful $*$ -morphism.
Furthermore, if $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ boundedly then $V(\varphi_n\ ) \xrightarrow[n\to\infty]{} V(\varphi\ )$

Proposition 3.5. *Let V_r with $r \in \mathcal{Z}^*$, then $\{V_r\}_{r\in\mathcal{Z}^*}$ is a unitary projective representation of $\mathcal{Z}^*$ on $\mathfrak{H}$ such that $\{U, V, \mathfrak{H}\}$ is a covariant representation on $\mathfrak{H}$.*

Proof. Unitarity follows from

$$V_r V_r^* := (\sum_{x\in\mathcal{Z}} r(x)\chi_x)(\sum_{y\in\mathcal{Z}} \overline{r}(y)\chi_y) = \sum_{x\in\mathcal{Z}} \chi_x = id_{\mathfrak{H}}$$

Next, we have the following commutation rule
$U_x V(\varphi) U_x^* = V(\varphi^x)$, where $\varphi^x(y) = (\lambda_x\varphi)(y) = \varphi(y-x)$ by

$$U_x V(\varphi) U_x^* = \sum_{\mathcal{Z}} \varphi(y) U_x \chi_y U_x^* = \sum_{\mathcal{Z}} \varphi(y)\chi_{y+x} = \sum_{\mathcal{Z}} \varphi(y-x)\chi_y \tag{3.4}$$

As a consequence we have the Heisenberg-Weyl commutation relation

$$U_x V_r = \overline{r}(x) V_r U_x \tag{3.5}$$

for all $x \in \mathcal{Z}$ and $r \in \mathcal{Z}^*$(also called CCR in the literature). Denoting $\alpha_x(\cdot) = U_x \ \cdot\ U_x^*$ the associated automorphism group we have that, $\alpha_x(V_r) = \overline{r(x)} V_r$ which proves the covariance of the pair of representation U and V. □

Remark 24. *The converse holds true as well, if $\{V_r\}_{r\in\mathcal{Z}^*}$ is a representation of $\mathcal{Z}^*$ we recover the projections χ_x. In fact we have that $V_r = \sum_{\mathcal{Z}} r(x)\chi_x$, and with $\mathrm{d}r$ the canonical Haar measure of $\mathcal{Z}^*$(normalized), then $\epsilon_x = \{r \mapsto r(x)\}$ gives an orthonomalized basis of $\mathrm{L}^2(\mathcal{Z}^*)$. Using the character relation $\int_{\mathcal{Z}^*} \overline{r(x)} r(y) \mathrm{d}r = \delta_{xy}$ we get that $\int_{\mathcal{Z}^*} \overline{r(x)} V_r dr = \chi_x$.*

We know that a covariant representation is unique up to some Hilbert space in the following sense. If we define the canonical covariant representation as

$$(\lambda(x)f)(y) = f(y-x) \text{ for any} f \in l^2(\mathcal{Z}) \text{ and } x, y \in \mathcal{Z} \tag{3.6}$$

$$(\mu(r)f)(y) = \overline{r(x)}f(y) \text{ for any} f \in l^2(\mathcal{Z}) \text{ and } r \in \mathcal{Z}^* \tag{3.7}$$

Then there exists (see [TII, prop2.2]) a Hilbert space $\mathfrak{H}_0$ such that we have

$$\{\lambda \otimes id, \mu \otimes id, l^2(\mathcal{Z}) \otimes \mathfrak{H}_0\} \simeq \{U, V, \mathfrak{H}\} \ .$$

In the sense that the algebra generated by the pair $\{U, V\}$ acting on $\mathfrak{H}$ is isomorphic to the algebra generated by the pair $\{\lambda \otimes id, \mu \otimes id, l^2(\mathcal{Z}) \otimes \mathfrak{H}_0\}$ acting on $l^2(\mathcal{Z}) \otimes \mathfrak{H}_0$.

4 Algebra of measurable covariant and locally bounded operators

Before using the group-action, we give a precise description of the observables considered and construct the algebra involved in physical applications. This is the first step toward constructing operatorial analysis. In Ch1. [DG], to treat the linear response theory we use the non-commutative L^p-space on that algebra that we disctut in the next section and all Hamiltonians considered in the following will be affiliated with it. We recall some usual definitions and constructions for random operators. After that we construct our reference algebra on which we work for the rest of the discussion. Some of the following definitions are well-known but for the sake of completeness we recall them. The main reference on the subject is the book of Dixmier [Di].

Measurable and locally bounded operators

Definition 4.1. *Let $\{\mathfrak{H}(\omega)\}_{\omega\in\Omega}$ be a measurable field of Hilbert space. One defines then the direct integral of this field by*

$$\tilde{\mathfrak{H}} = \int_\Omega^\oplus \mathfrak{H}(\omega) d\mathbb{P}(\omega) \tag{4.1}$$

with the natural inner product : $\forall\ (\varphi, \psi) \in \tilde{\mathfrak{H}}^2$ defined as

$$\langle \varphi, \psi \rangle_{\tilde{\mathfrak{H}}} = \int_\Omega \langle \varphi(\omega), \psi(\omega) \rangle_{\mathfrak{H}(\omega)} d\mathbb{P}(\omega) \tag{4.2}$$

and with the norm

$$\|\varphi\|_{\tilde{\mathfrak{H}}} = \{\int_\Omega \|\varphi(\omega)\|^2_{\mathfrak{H}(\omega)} d\mathbb{P}(\omega)\}^{\frac{1}{2}}. \tag{4.3}$$

We refer to [Di] for a complete description of this structure.

Remark 25. *We note that in the case (which is in physical applications very frequent) where the Hilbert-fibers are constants in the sense that for all* $\omega \in \Omega, \mathfrak{H}(\omega) \simeq \mathfrak{H}$ *for a fixed* $\mathfrak{H}$*, we have trivially that*

$$\int_\Omega^\oplus \mathfrak{H}(\omega) \equiv \mathrm{L}^2(\Omega, \mathbb{P}, \mathfrak{H}) \simeq \mathrm{L}^2(\Omega, \mathbb{P}) \otimes \mathfrak{H} \ .$$

One can define formally $\mathcal{L}(\tilde{\mathfrak{H}})$ as $\prod_\Omega \mathcal{L}(\mathfrak{H}(\omega), \mathfrak{H}(\omega))$ in the sense that for all measurable vector fields $\{\varphi(\omega), \omega \in \Omega\}$ then for any $A \in \mathcal{L}(\tilde{\mathfrak{H}})$ we have that $\{A(\omega)\varphi(\omega), \omega \in \Omega\}$ is still a measurable vector field. These operators are called essentially-bounded if $\|A_\omega\|_{\mathcal{L}(\mathcal{H})}$ is an element of $L^\infty(\Omega, \mathbb{P})$. To simplify the notations we write $\underline{A} = \int_\Omega^\oplus A(\omega) d\mathbb{P}(\omega)$ and when no confusion occurs we omit the bar. We call *diagonal algebra*, the algebra of multiplication operators, $T_f := \int_\Omega^\oplus f(\omega) \mathrm{d}\mathbb{P}$ with f a scalar-valued functions on Ω. It is usually denoted $\mathcal{A}$.

More precisely one has the following results (cf [Di]).

Theorem 4.2. *[Di]* $A \in \mathcal{L}(\tilde{\mathfrak{H}})$ *is decomposable if and only if* $A \in \mathcal{A}'$*.*

In consequence we have that for any $A_\omega \in L^\infty(\Omega, \mathbb{P}, \mathcal{L})$ $(\mathcal{L} = \mathcal{L}(\mathfrak{H}, \mathfrak{H}))$ there is an unique decomposable operator such that $\forall \varphi \in \tilde{\mathcal{H}}$ one has ($\underline{A}\varphi)(\omega)) = A(\omega)\varphi(\omega)$.
We note

$$|\!|\!| A |\!|\!|_\infty = \ ess - \sup_\Omega \|A(\omega)\|. \tag{4.4}$$

We now define the first algebra of interest.

Definition 4.3. *An operator* A *is called locally bounded (l.b) with respect to the map* χ *if the following holds*

$$\|\chi_x A_\omega\| < \infty \ \textit{and} \ \|A_\omega \chi_x\| < \infty \ \textit{for all} \ \omega \in \Omega, \ \textit{and} \ x \in \mathcal{Z}$$

We then define

$$\mathcal{K}_{m,lb} = \{A \in \mathcal{A}' \ / A \ \textit{is l.b} \ \}$$

and

$$\mathcal{K} = \mathcal{A}' \cap \{A / \ |\!|\!| A |\!|\!|_\infty < \infty\}$$

Construction of the reference algebra

We can now define the reference algebra which we use all over this work.

Definition 4.4. *$A \in \mathcal{K}$ is covariant if and only if the following holds*

$$U_x A(\omega) U_x^* = A(\tau(x)\omega) \text{ for all } x \in \mathcal{Z} \text{ and } \omega \in \Omega \tag{4.5}$$

Where τ is ergodic group of measure preserving transformation coming from Definition 2.4.
Noting that $\alpha_x(A) = U_x A\, U_x^* : \mathcal{K} \to \mathcal{K}$, we have that $\alpha : \mathcal{Z} \to \mathrm{Aut}(\mathcal{K})$ is a one parameter group of automorphism on $\mathcal{K}$. The stability is immediate. Using the 2-cocycle relation we have that $\alpha_x \circ \alpha_y = \alpha_{x+y}$. The first step to understanding the structure of an algebra is to study its commutant. We therefore define a second unitary representation and so one parameter group of automorphism namely $\{U_x^\tau\}_{x\in\mathcal{Z}}$ which we call a derived representation of $\mathcal{Z}$.

Definition 4.5. *We call random translations (as opposed to the magnetic translations) the unitary group inducted by the measure preserving group of isomorphism on Ω, namely τ, defined for $f \in L^2(\Omega)$ by*

$$(\mathcal{W}(x)f)(\omega) := f(\tau^{-1}(x)\omega) \text{ for any } \omega \in \Omega \text{ and } x \in \mathcal{Z}$$

The map U^τ from $\mathcal{Z}$ into $\mathcal{U}(\tilde{\mathfrak{H}})$ defined as $U_x^\tau := \mathcal{W}(x) \otimes U_x$ acting on $\tilde{\mathfrak{H}}$ by

$$(U_x^\tau \varphi)(\omega) := U_x \varphi(\tau(-x)\omega) \text{ , for any } \varphi \in \tilde{\mathfrak{H}} \text{ ,}$$

is called a derived ζ-projective unitary representation on $\tilde{\mathfrak{H}}$.

Remark 26. *It is immediate that $\{U_x^\tau\}_{x\in\mathcal{Z}}$ is still a unitary ζ-projective representation of $\mathcal{Z}$ by construction, but on $\tilde{\mathfrak{H}}$ with $U_x^\tau U_y^\tau = \zeta(x,y) U_{x+y}^\tau$.*

Definition-Proposition 3. *Let $\alpha_x^\tau(A) := U_x^\tau A U_x^{\tau *}$ the derived one parameter automorphism group.*
Then the reference algebra is given by

$$\begin{aligned}
\mathcal{K}_\infty &= \{A \in \mathcal{K} / \forall a \in \mathcal{Z}, [A, U_a^\tau] = 0\} && (4.6)\\
&= \{A \in \mathcal{K} / \forall x \in \mathcal{Z}, \alpha_x^\tau(A) = A\} && (4.7)\\
&= \mathcal{K} \cap \{U_x^\tau | x \in \mathcal{Z}\}' && (4.8)
\end{aligned}$$

By construction $\mathcal{K}_\infty$ is a $\mathcal{W}^\star$-algebra.

5 Dual action and regularization of covariant operators

After having defined the action of $\mathcal{Z}$ on $\mathcal{K}_\infty$, we now define the dual action of $\mathcal{Z}^*$ on $\mathcal{K}_\infty$ by

$$\widehat{\alpha}_r := A \to V_r A V_r^* \text{ for any } r \in \mathcal{Z}^* \text{ and } A \in \mathcal{K}_\infty \ . \tag{5.1}$$

We have seen that it is a well-defined one parameter group of automorphism. The triplet $\{\mathcal{K}_\infty, \widehat{\alpha}, \mathcal{Z}^*\}$ is called the dual covariant system of $\{\mathcal{K}_\infty, \alpha, \mathcal{Z}\}$. This one parameter group has a natural regularization action on operators.

Definition 5.1. *We call the regularized with respect to $\theta \in L^1(\mathcal{Z}^*)$ of $A \in \mathcal{K}_\infty$ (or simply belonging to $\mathcal{B}(\mathfrak{H})$ or $\mathcal{B}(\tilde{\mathfrak{H}})$) the operator defined by*

$$A_\theta := \int_{\mathcal{Z}^*} \widehat{\alpha}_r(A)\theta(r)dr \text{ for any } \theta \in L^1(\mathcal{Z}^\star) \ .$$

We note this is in some sense the operatorial analogue of a summation process for Fourier series. This made even more explicit by the following result. For an operator A , we denote by $\mathcal{K}_A(x,y) = \chi_x A \chi_y$ its operator-kernel.

Proposition 5.2. *Let $A \in \mathcal{K}_\infty$ (or simply belonging to $\mathcal{B}(\mathfrak{H})$ or $\mathcal{B}(\tilde{\mathfrak{H}})$) and $\theta \in L^1(\mathcal{Z}^*)$, then*

$$A_\theta = \sum_{x,y\in\mathcal{Z}} \widehat{\theta}(x-y)\mathcal{K}_A(x,y) \tag{5.2}$$

Proof.

$$\begin{aligned}
A_\theta &= \int_{\mathcal{Z}^\star} \widehat{\alpha}_r(A)\theta(r)dr && (5.3)\\
&= \int_{\mathcal{Z}^*} V_r A V_r^* \theta(r)dr && (5.4)\\
&= \int_{\mathcal{Z}^*} \left(\sum_x r(x)\chi_x\right) A \left(\sum_y \overline{r}(y)\chi_y\right)\theta(r)dr && (5.5)\\
&= \sum_{x,y} \left(\int_{\mathcal{Z}^*} \overline{r}(y-x)\theta(r)dr\right)\chi_x A \chi_y && (5.6)\\
&= \sum_{x,y} \widehat{\theta}(y-x)\chi_x A\chi_y && (5.7)
\end{aligned}$$

□

Then the *Riemann-Lebesgue lemma* implies that for any θ of $L^1(\mathcal{Z}^*)$ we have $\widehat{\theta} \in C_0(\mathcal{Z})$. This is a first example of the principle which consists of transposing results from Fourier analysis on general group into results for covariant operators.
In particular we have that $\widehat{\theta} \to 0$ at infinity implies that for any $\varepsilon > 0$ there exists some finite subset F_ε of $\mathcal{Z}$ such that for any other finite subset F disjoint of F_ε we have that $\sup_{x\in F} |\widehat{\theta_F}(x)| \leq \varepsilon$. We can without loss of generality assume that e $\in F_\varepsilon$. Using the Haagerup length function L, we can conclude that

Lemma 5.3. *For any $A \in \mathcal{B}(\tilde{\mathfrak{H}})$ and $\varepsilon > 0$ there exists a constant $C(\varepsilon,\theta)$ such that for any $x, y \in \mathcal{Z}$*

$$L(x-y) \geq C_\varepsilon \text{ implies } \|\mathcal{K}_{A_\theta}(x,y)\| \leq \varepsilon$$

Proof. The proof is immediate and consequence of the above discussion. It suffices to see that by definition there exists C_ε such that for a fixed $y \in \mathcal{Z}$ if $L(x-y) \geq C_\varepsilon$ then $(x-y) \notin F_\varepsilon$. □

One has that the regularization by the dual-action has the effect of decoupling the regions which are quite far apart with respect to the Haagerup lenght function.
In the abelian case we can simply use the fact that $\mathcal{Z}^*$ is a compact separable space and use the usual approximation of identity. Note however that not all group of physical interest are abelian, in which case $\mathcal{Z}^*$ does not make sense.

Proposition 5.4. *Let $\{\theta_n\}$ be an approximation of the identity on $\mathrm{L}^1(\mathcal{Z}^*)$, i.e*

$$\int_{\mathcal{Z}^*} \theta_n = 1, \theta_n \geq 0 \ , \ \int_{\mathcal{Z}^*} |\theta_n| < +\infty \text{ and } \mathrm{supp}(\theta_n) \to \{\widehat{e}\}$$

(we note $\widehat{e}$ the neutral element of $\mathcal{Z}^*$).
Then $A_{\theta_n} \to_s A$. Furthermore we have that if $A \in \mathcal{K}_\infty$ then $A_\theta \in \mathcal{K}_\infty$.

Proof. In fact we take θ_n such that $\lim_{n\to\infty} \theta_n = \delta_{\widehat{e}}$ in the distributional sens i.e $\lim_{n\to\infty} \int_{\mathcal{Z}^*} \theta_n(r) f(r) dr = f(\widehat{e})$. By the explicit definition of A_{θ_n} the result follows immediately. Furthermore we have that $\|A_\theta\|_\infty \leq \|A\|_\infty$. Whereas the *Heisenberg-Weyl commutation relation* we have that $[\alpha, \widehat{\alpha}] = 0$ in the sense that $\alpha_x \circ \widehat{\alpha}_r(A) = \widehat{\alpha}_r(A \circ \tau_x)$ for all $x \in \mathcal{Z}$ and $r \in \mathcal{Z}^*$ then $A_\theta \in \mathcal{K}_\infty$. □

One then obtains a first density theorem for $\mathcal{K}_\infty$. For the rest of the exposition we call approximation of the identity a sequence of maps $\varphi_n \in C_0(\mathcal{Z})$ converging pointwise to 1. As explained above the existence of such a

net is equivalent to the fact that the group has the Haagerup property, while the fact that the net is countable comes from the countability of the group (see [BC]).
We define the first space of regular observables.

Theorem 5.5. *Let*

$C_0(\mathcal{K}_\infty) := \{A \in \mathcal{K}_\infty /$ *there exists* $f \in C_0(\mathcal{Z})$ *s.t* $\|\mathcal{K}_A(x,y)\|_\infty \leq |f(x-y)|\}$

Then for any $A \in \mathcal{K}_\infty$ *there exists* $A_n \in C_0(\mathcal{K}_\infty)$ *such that*

$$A_n \xrightarrow[n\to\infty]{s} A \text{ and } A_n^* \xrightarrow[n\to\infty]{s} A^* \text{ on } \mathfrak{H}_c$$

Proof. Let $A \in \mathcal{K}_\infty$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ an approximation of the identity. Then if we define $A_n := \sum_{a\in\mathcal{Z}} V_{\varphi_n^y} A\chi_y$ we have that $A_n \in \mathcal{K}_\infty$ and $A_n \xrightarrow[n\to\infty]{s} A$ where $V_{\varphi_n^y} = \alpha_y(V_{\varphi_n})$, or more generally $\varphi_n^y(x) = (\lambda(y)\varphi_n)(x)$. One has immediately that

$$A_n := \sum_{x\in\mathcal{Z}} V_{\varphi_n^x} A\chi_x \tag{5.8}$$
$$= \sum_{x\in\mathcal{Z}} U_x V_{\varphi_n} U_x^* A\chi_x \tag{5.9}$$
$$= \sum_{x\in\mathcal{Z}} U_x \left(\sum_{y\in\mathcal{Z}} \varphi_n(y)\chi_y\right) U_x^* A\chi_x \tag{5.10}$$
$$= \sum_{x\in\mathcal{Z}} \sum_{y\in\mathcal{Z}} \varphi_n(y)\chi_{y+x} A\chi_x \tag{5.11}$$
$$= \sum_{x\in\mathcal{Z}} \sum_{y\in\mathcal{Z}} \varphi_n(y-x)\chi_y A\chi_x \tag{5.12}$$

then we have that for any $\psi \in \mathfrak{H}_c$ that $A_n\psi \xrightarrow[n\to\infty]{s} A\psi$. □

We will see now that this is a particular case of a more general density theorem. We introduce the operatorial analogue of compactly supported functions, namely *the finite range operators.*

Remark 27. *We note that the last result, as some others, does not only work for operators belonging to* $\mathcal{K}_\infty$*. The technics developed provide a description of this algebra.*

Algebra of finite range operator

Definition 5.6. *Let $A \in \mathcal{B}(\mathfrak{H})$ or $\mathcal{B}(\tilde{\mathfrak{H}})$. We say that A is a finite range operator (we note F.R) if there exists $F \subset \mathcal{Z}$ which is finite and such that $A\chi_x = \chi_{a+F}A\chi_x$ for any $x \in \mathcal{Z}$ or equivalently, $\chi_a A \chi_b$ is zero if and only if $b-a \notin F$.*

Remark 28. *Since we assumed that $\mathcal{Z}$ has the Haagerup property we can formulate the finite range property using the length function: an operator A is finite-range if there exists a positive constant R_A such that for any $x, y \in \mathcal{Z}$ if $L(x-y) > R_A$ then $\chi_x A \chi_y = 0$.*

We denote by $\mathcal{K}_\infty^{\mathcal{F}}$ the set of finite range operators.

Proposition 5.7. *$\mathcal{K}_\infty^{\mathcal{F}}$ is a $*$-algebra.*

Proof. Let $\underline{A}$ of *finite range* F, then for all $\omega \in \Omega$, $A(\omega) = \sum_{x \in \mathcal{Z}} \chi_{x+F} A(\omega) \chi_x$ and for all $\varphi \in \mathfrak{H}$

$$\langle A^*(\omega)\chi_x\varphi, \varphi\rangle = \sum_{x\in\mathcal{Z}} \langle \varphi, \chi_x A(\omega)\varphi\rangle \tag{5.13}$$

$$= \sum_{x\in\mathcal{Z}} \langle \varphi, \chi_x A(\omega)\chi_{x-F}\varphi\rangle \tag{5.14}$$

$$= \sum_{x\in\mathcal{Z}} \langle \chi_{x-F} A^*(\omega)\chi_x\varphi, \varphi\rangle \ . \tag{5.15}$$

Therefore A^* is of finite range $-F$. While if A of range F and B of range G, then

$$AB\chi_x = A(\chi_{x+G}B\chi_x) = \chi_{x+F+G}A\chi_{X+G}B\chi_x \tag{5.16}$$

Then AB is F.R $F+G$. □

Note we can always choose an approximation of the identity $\{\varphi_n\}_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ we have $\varphi_n \in \mathcal{K}(\mathcal{Z})$ with $\text{supp}(\varphi_n) \subset K_n$ a finite subset of $\mathcal{Z}$. Using the above construction we have the following density theorem consequence of the theorem 5.5.

Theorem 5.8. *The $*$-algebra $\mathcal{K}_\infty^{\mathcal{F}}$ of finite range operators of $\mathcal{K}_\infty$ is a $*$-subalgebra (sequentially) $*$ -strongly-dense in $\mathcal{K}_\infty$.*

Some other examples of dense spaces of RD-operators

We can construct the analogue of weighted spaces thanks to the length function. Let $\kappa : \mathcal{Z} \mapsto [1, \infty)$ then for any $1 \le p \le \infty$ we define

$$l^p_\kappa(\mathcal{Z}) := \{f : \mathcal{Z} \mapsto \mathbb{C},\ f\kappa \in l^p(\mathcal{Z})\}$$

which is a Banach space with the norm $\|f\|_{p,\kappa} = \|f\kappa\|_p$. Then a natural weight is gives by $\kappa_s = (1+L)^s$ for some $s > 0$ and L the length function or again $\kappa_{\exp} = \exp(tL^2)$. Hence we can define the Frechet space given by

$$H_L^\infty = \bigcap_{s>0} l^2_{\kappa_s}(\mathcal{Z}) \ .$$

as explained in [BC], this is a dense $*$- subalgebra of $C^*_r(\mathcal{Z})$ called algebra of rapid decaying functions (RD). In the case where the group has a subpolynomial H-growth that we will define after in the section 7.1. In the case where the group has a subexponential H-growth $\mathcal{E}(\mathcal{Z}) := l^2_{\kappa_{\exp}}(\mathcal{Z})$ is also a dense space.

In analogy to $C_0(\mathcal{K}_\infty)$ we can define the $*$-dense sub-algebras of RD-operators given by

$$H_L^\infty(\mathcal{K}_\infty) := \{A \in \mathcal{K}_\infty / \text{ there exists } f \in H_L^\infty \text{ s.t } \|\mathcal{K}_A(x,y)\|_\infty \le |f(x-y)|\}$$
$$\mathcal{E}(\mathcal{K}_\infty) := \{A \in \mathcal{K}_\infty / \text{ there exists } f \in \mathcal{E}(\mathcal{Z}) \text{ s.t } \|\mathcal{K}_A(x,y)\|_\infty \le |f(x-y)|\}$$

Remark 29. *An important tool in K-theroy and many other problems as the quantization of the conductance of the Hall-effect or the Baum-Connes conjecture is the fact that there exists an isomorphism between the dense algebra H_L^∞ and the reduced group algebra as introduced below.*

First step in the analysis of the structure of $\mathcal{K}_\infty$

For simplicity we assume that for any $\omega \in \Omega$, $\mathfrak{H}(\omega) \simeq \mathfrak{H}$. This means that we consider here, to exhibit more clearly the internal structure of $\mathcal{K}_\infty$, that the Hilbert bundle is trivial in the sense that $\tilde{\mathfrak{H}} \simeq \mathrm{L}^2(\Omega) \otimes \mathfrak{H}$. Finally we set $\mathfrak{H}_0 = \chi_e \mathfrak{H}$ and more generally one set $\mathfrak{H}_x = \chi_x \mathfrak{H}$.

Lemma 5.9. *Let $A \in \mathcal{K}_\infty$, then $A^\circ_\omega := A\chi_e \in L^\infty(\Omega, \mathbb{P}, \mathcal{B}(\mathfrak{H}_0, \mathfrak{H}))$ determines in unique way A.*

Proof. We have the following equality

$$A_\omega = \sum_{\mathcal{Z}} A_\omega \chi_x = \sum_{\mathcal{Z}} A_\omega \alpha_x(\chi_e) \tag{5.17}$$
$$= \sum_{\mathcal{Z}} U_x \alpha_x^{-1}(A_\omega) \chi_e U_x^* \tag{5.18}$$
$$= \sum_{\mathcal{Z}} U_x A^\circ_{\tau(-x)_\omega} \chi_e U_x^* \tag{5.19}$$

Thus we have that $A\chi_e = 0$ implies that $A = 0$. ☐

We state now a theorem which justifies in some sense our framework.

Theorem 5.10. *Let F a finite subset of $\mathcal{Z}$. There exists a bijective correspondance between the finite range operators $A^\circ \in L^\infty(\Omega, \mathbb{P}, \mathcal{B}(\mathfrak{H}_0, \mathfrak{H}_F))$ and the operators of finite range F of $\mathcal{K}_\infty$.*

Proof. Set A$\in \mathcal{K}_\infty^{\mathcal{F}}$ with finite range F. One then sets, as above, that $A\chi_0 = A^\circ$. One has again that $A^\circ \in L^\infty(\Omega, \mathbb{P}, \mathcal{B}(\mathfrak{H}_0, \mathfrak{H}_F))$, where one sets that $\mathfrak{H}_F = \sum_{a\in F}^{\oplus} \mathfrak{H}_a$ and by the previous proposition we have that

$$A = \sum_{\mathcal{Z}} U_x (A^\circ \circ \tau(x)^{-1}) U_x^* \tag{5.20}$$

.
Conversely, set $F \subset \mathcal{Z}$ a finite subset and $A^\circ \in L^\infty(\Omega, \mathbb{P}, \mathcal{B}(\mathfrak{H}_0, \mathfrak{H}_F))$ arbitraly chosen. If we set

$$A \equiv \sum_{\mathcal{Z}} U_a (A^\circ \circ \tau(a)^{-1}) U_a^\star \tag{5.21}$$

The proof is achieved by

Lemma 5.11. *A as operators sum is well defined on $\tilde{\mathfrak{H}} \cong L^2(\Omega, \mathbb{P}) \otimes \mathfrak{H}$ and belongs to $L^\infty(\Omega, \mathbb{P}, \mathfrak{H})$.*

□

To prove the first lemma we use a another very general result. For the sake of completeness we provide its proof.

Lemma 5.12. *Let $\mathfrak{K} \subset \mathfrak{H}$ be a countable subset. Then there exists a map $\mathbf{n} : \mathfrak{K} \in \varphi \mapsto \mathbf{n}(\varphi) \in \mathbb{N}$ such that,*

$$\|\sum_{\varphi\in\mathfrak{K}} \varphi\|^2 \leq \sum_{\varphi\in\mathfrak{K}} \mathbf{n}(\varphi) \|\varphi\|^2 \ .$$

Proof of lemma 5.12. The proof use an induction argument. Let $\eta \in \mathfrak{K}$. We denote by $\alpha = \{\varphi, \psi\}$ a couple of elements which are differents to each other and by $\theta_\alpha = \Re\langle\varphi, \psi\rangle$. Then $\|\sum_{\varphi\in\mathfrak{K}} \varphi\|^2 = \sum_{\varphi\in\mathfrak{K}} \|\varphi\|^2 + \sum_{\alpha} \theta_\alpha$.
If we note $\varphi \sim \psi$ if $\varphi \neq \psi$ and $\langle\varphi, \psi\rangle \neq 0$ Then

$$\|\sum_{\varphi\in\mathfrak{K}} \varphi\|^2 = \sum_{\varphi\in\mathfrak{K}} \|\varphi\|^2 + \sum_{\{\alpha|\alpha\ni\eta\}} \theta_\alpha + \sum_{\{\alpha|\eta\notin\alpha\}} \theta_\alpha$$

Let $\tilde{\mathfrak{K}} = \mathfrak{K}/\{\eta\}$ then trivially we have that $\sum\limits_{\alpha\ni\eta} \theta_\alpha \leq (\mathbf{n}(\eta)-1)\|\eta\|^2 + \sum\limits_{\varphi\sim\eta} \|\varphi\|^2$.
Hence

$$\|\sum_{\varphi\in\mathfrak{K}}\varphi\|^2 \leq \mathbf{n}(\eta)\|\eta\|^2 + \sum_{\varphi\sim\eta}\|\varphi\|^2 + \sum_{\varphi\in\tilde{\mathfrak{K}}}\|\varphi\|^2 + \sum_{\alpha\subset\tilde{\mathfrak{K}}} 2\theta_\alpha \tag{5.22}$$

$$= \mathbf{n}(\eta)\|\eta\|^2 + \sum_{\varphi\sim\eta}\|\varphi\|^2 + \|\sum_{\varphi\in\tilde{\mathfrak{K}}}\varphi\|^2 \tag{5.23}$$

By induction the lemma is proved. □

Proof of lemma 5.11. Formally,

$$A = \sum_{\mathcal{Z}} U_x A^\circ \circ \tau(x)^{-1} U_x^* \tag{5.24}$$

$$= \sum_{\mathcal{Z}} U_x \chi_F A^\circ \circ \tau(x)^{-1} \chi_e U_x^* \tag{5.25}$$

$$= \sum_{\mathcal{Z}} \chi_{x+F} U_x A^\circ \circ \tau(x)^{-1} U_x^* \chi_x \ . \tag{5.26}$$

If A° is of finite range F then A is still a finite range operator of range F and sum are convergent. Now using the Lemma 5.12 we have that if $f \in \tilde{\mathfrak{H}}$ such that $\chi_G f = f$ with G finite, then Af is a finite sum, then it is well defined thus

$$\|Af\| \leq \sum_{x\in\mathcal{Z}} \mathbf{n}_x \|\chi_{x+F} U_x A^\circ \circ \tau(x)^{-1} U_x^* \chi_x f\|^2 \tag{5.27}$$

$$\leq \mathbf{n}_0 \|A^\circ\|^2_{\mathcal{B}(\tilde{\mathcal{H}})} \sum \|\chi_x f\|^2 \tag{5.28}$$

$$\leq \mathbf{n}_0 \|A^\circ\|^2_{\mathcal{B}(\tilde{\mathfrak{H}})} \|f\|^2_{\tilde{\mathfrak{H}}} \tag{5.29}$$

Where we have use the fact that two terms are non-orthogonals only if the components in x and y are such that $x \in y + 2F$ this implies that $\sup\limits_{x\in\mathcal{Z}} n_x :=$ $\mathbf{n}_0 \leq 2|F|$. Then it is finite and therefore $A \in L^\infty(\Omega, \mathbb{P}, \mathcal{B}(\mathfrak{H}))$.
Furthermore,

$$\alpha_x(A) = \sum_{y\in\mathcal{Z}} U_x U_y A^\circ \circ \tau(y)^{-1} U_y^* U_x^* \tag{5.30}$$

$$= \sum_{y\in\mathcal{Z}} \zeta(x,y) U_{x+y} A^\circ \circ \tau(y)^{-1} \overline{\zeta(x,y)} U_{x+y}^* \tag{5.31}$$

$$= \sum_{y\in\mathcal{Z}} U_y A^\circ \circ \tau(y-x)^{-1} U_y^* \tag{5.32}$$

$$= A \circ \tau(x) \tag{5.33}$$

Thus A is covariant. □

To end this section we prove that the reference algebra is naturally the algebra of affiliation of elements of $\mathcal{K}_{mc,lb}$.

Proposition 5.13. *$\mathcal{K}_\infty$ is the algebra of affiliation of $\mathcal{K}_{mc,lb}$.*

Proof. The only subtle point is the measurability. We need to prove the measurability of function of the type $(1 + |A_\omega|)^{-1}$. We proceed as in [PF, BoGKS] by an approximation argument. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ a basis of $\mathfrak{H}_0$ and $\{\varphi_n^x\}_{x\in\mathcal{Z}}$ the family of its translated. The system $\{\varphi_n^x\}_{(n\in\mathbb{N},\ x\in\mathcal{Z})}$ is a basis for $\mathfrak{H}$. Then let $M_\omega^n = (A_\omega P_n)^* A_\omega P_n$ where P_n is the projection on the n first elements of the above basis. Then M_ω^n is measurable as bounded operator. It follows that $(|M_\omega^n| + 1)^{-1}$ is also measurable roughtly speaking by the fact that the matrix elements of the inverse can be expressed as ratio of the determinant. It remains only to prove that $(|M_\omega^n| + 1)^{-1} \to (|A_\omega| + 1)^{-1}$ weakly. For $\varepsilon > 0$, let $\phi \in \mathcal{D}(\overline{A_\omega})$ and $\varphi \in \mathfrak{H}_c$ such that the graph norm of the difference of the two functions are bounded by ε i.e $\|\phi - \varphi\| + \|\overline{A}_\omega(\phi - \varphi)\| < \varepsilon$ and using the equality

$$\langle A_\omega\varphi, A_\omega(M_\omega^n + 1)^{-1}\psi\rangle + \langle\varphi, (M_\omega^n + 1)^{-1}\psi\rangle = \langle\varphi, \psi\rangle$$

for n large enough, we get that

$$|\langle\overline{A}_\omega(\phi - \varphi), A_\omega(M_\omega^n + 1)^{-1}\psi\rangle + \langle\phi-\varphi, (M_\omega^n + 1)^{-1}\psi\rangle - \langle\phi - \varphi, \psi\rangle \\ \leq 3\varepsilon\|\psi\|$$

Then for n large enough we have that

$$\lim_{n\to\infty}\langle\overline{A}_\omega\phi, A_\omega(M_\omega^n + 1)^{-1}\psi\rangle + \langle\phi, (M_\omega^n + 1)^{-1}\psi\rangle = \langle\phi, \psi\rangle$$

Taking $\eta = (|\overline{A}_\omega|^2 + 1)\phi$ we get $\lim_{n\to\infty}\langle\eta, (M_\omega^n + 1)^{-1}\psi\rangle = \langle(|\overline{A}_\omega|^2 + 1)^{-1}\eta, \psi\rangle$ After what we know that in its polar decomposition $A_\omega = U_\omega|A_\omega|$ and

$$U_\omega = \lim_{\varepsilon\to 0}\overline{A}_\omega(\overline{A}_\omega + \varepsilon)^{-1} \text{ strongly on } \mathfrak{H}$$

We have then that $U_\omega \in \mathcal{K}_\infty$. This finishes the proof. □

6 Diagonalization

We now construct a general and explicit structure naturally adapted to $\mathcal{K}_\infty$. We will show that the adapted algebraic structure comes from the Hilbert C^*-module as presented below. We will see that we can define a generalized-Floquet type transform, for operators considered in the context of transport theory in magnetic and disordered media. To do so we use harmonic analysis with the Pontrjagin duality supposing first commutativity of the group structure of the algebra in the first time. Afterwards that we generalize it to the twisted and random context which represents the *magnetic random media*. This generalization becomes possible by the fact that we decompose the operators on fibers not along the dual group itself but along the reduced C^*-algebra which still makes sense even if the group is not abelian nor twisted. The origin of this formalism comes from the periodic media which modelized perfect crystals. Many important results has been done in this direction. In the last decade some consequent progress has been done by considering transport along fibers, as for example the work of Gerard and Nier which develop general Mourre theory for analytic fibered Hamiltonians [GN]. In spectral theory, Sunada, Bruning and Grüber [BS, Gr] have shown general results on fibered periodic elliptic operators on smooth manifolds using the group of symmetries. It is an alternative to Bellisard's theory which is based on the construction of the Hull associated to the particular Hamiltonian considered whereas in this theory the main point is to find a natural decomposition of the Hilbert space with respect to symmetries of the observables considered exactly as the original Floquet theory. Bruning and Sunada open this way of investigation for spectral theory of operator on smooth manifolds with some theorems about the topological nature of the spectrum [BS]. It is thus natural to give an algebraic description of the theory developed by [BS, Gr]. To obtain an meaningful description we need to introduce some additional algebraic structures which exactly encodes the symmetries of the observables.

6.1 Hilbert C^*-module and operators

Hilbert C^*-module

In this section we recall the definition of structure adapted to the fibration on the symmetry group. The essential difference between the above structure is that we need to work on the continuous setting and not on the measurable one in the sense that we now work on C^*-algebra instead of von-Neumann algebra as we have used to define non-commutative L^p-space that needed for the linear response theory in Ch1. Our main reference about this structure is the monography [MT].
We start by giving the definition of the analogue of a measurable field but in the continuous setting. For a precise description we refer to the Dixmier-Douady monography [DD].

Definition 6.1 (Continous fields of Banach and Hilbert spaces)**.** *Let B a* *topological space,* $(E(z)))_{z\in B}$ *a family of Banach spaces. The linear space* $\Pi := \Pi_{z\in B}E(z)$ *is called space of all vectors fields. A continuity structure on* Π *is defined by a subspace* $\Lambda \subset \Pi$ *such that*

1. Λ *is a* $C_\infty(B)-$*module of* Π
2. $\forall z \in B : \forall \zeta \in E(z) \exists x \in \Gamma : x(z) = \zeta$
3. $\forall x \in \Lambda : (z \mapsto \|x(z)\|) \in C_\infty(B)$
4. $\forall x \in \pi : ((\forall \varepsilon > 0 : \forall z \in B : \exists x' \in \Lambda,$ *neighborhood* $U \ni z :$ $\forall z' \in U : \|x(z') - x'(z')\| < \varepsilon) \Rightarrow x \in \Lambda$

We now introduce now the framework in which the fibration takes place. It can be viewed as a suitable decomposition of the Hilbert space with respect to the symmetry of the considered observables.

Definition 6.2 ((right)-pre-Hilbert module)**.** *Let* $\mathcal{A}$ *a* C^**-algebra.* $\mathcal{M}$ *is called (right) pre-Hilbert module if it is equiped with a map* $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \mapsto \mathcal{A}$ *with the following properties:*

i) $\langle x|x \rangle \geq 0$ *for any* $x \in \mathcal{M}$

ii) $\langle x|x \rangle = 0 \Leftrightarrow x = 0$

iii) $\langle x|y + z \rangle = \langle x|y \rangle + \langle x|z \rangle$ *for any* $x, y, z \in \mathcal{M}$

iv) $\langle x|\lambda y \rangle = \lambda \langle x|y \rangle$ *for any* $x, y \in \mathcal{M}$ *and* $\lambda \in \mathbb{C}$

v) $\langle x|a \cdot y \rangle = \langle x|y \rangle a$*for any*$x, y \in \mathcal{M}$ *and* $a \in \mathcal{A}$

vi) $\langle y|x \rangle = \langle x|y \rangle^*$

Let $\mathcal{M}$ a pre-Hilbert-module. Set $\|x\|_{\mathcal{M}} := \|\langle x|x \rangle\|_{\mathcal{A}}$.

Proposition 6.3. $\|\cdot\|_{\mathcal{M}}$ *is a norm and enjoy the following properties*

i) $\|x \cdot a\|_{\mathcal{M}} \leq \|x\|_{\mathcal{M}} \|a\|_{\mathcal{A}}$ *for any* $x \in \mathcal{M}, ; a \in \mathcal{A}$

ii) $\langle x, y \rangle \langle y, x \rangle \leq \|y\|_{\mathcal{M}} \langle x|x \rangle$ *for any* $x, y \in \mathcal{M}$

iii) $\|\langle x|y \rangle\|_{\mathcal{A}} \leq \|x\|_{\mathcal{M}} \|y\|_{\mathcal{M}}$ *for any* $x, y \in \mathcal{M}$

Definition 6.4. *A pre-Hilbert-module* $\mathcal{M}$ *is called a Hilbert* C^**-module if it is complete with respect to the norm* $\|\cdot\|_{\mathcal{M}}$.

Moreover a lemma of [DD] proves that the correspondance between continuous field of Hilbert space over a topological space B and Hilbert $C_0(B)$-module is one-to-one. To end this section we recall a last fact about the link between this structure and the GNS representation. This is a simple but important feature which serves to make a correspondance between the decomposition following the Hilbert C^*-module where we can diagonalize and the original Hilbert space where the spectral analysis and another Hilbertian features can be performed. Let $\mathcal{A}$ be a C^*-algebra, μ a state on it and $\mathcal{M}$ a Hilbert $\mathcal{A}$-module. Then we define the following scalar product, $\langle x|y\rangle_\mu = (\mu\langle x|, y\rangle)$ for $x, y \in \mathcal{M}$. Denoting by $\mathbf{N}_\mu := \{x \in \mathcal{M} | \langle x|x\rangle = 0\}$ the null space. Then the GNS representation $\mathcal{M}_\mu$ is given by the completion of $\mathcal{M}/\mathbf{N}_\mu$ with respect to the preceding scalar product.

6.2 Floquet transform

We now define a Floquet transform adapted to our formalism. For the reader convenience we cut in several steps which follow the level of complexity of the symmetries associated with the considered observable . To sum up, the content of the exposition is the following.

1. $\mathcal{Z}$ is assumed abelian, the 2-cocycle identically equal to 1 and the operators are deterministic, i.e $\zeta \equiv 1$ and the observable is non-random. We call this case the commutative deterministic case. (Here commutative have two meaning, $\mathcal{Z}$ is abelian and the unitary representaion is commutative).
2. We convert this framework to the C^*-module formalism which appear to be the natural formalism for a generalization.
3. After that we can treat the projectively covariant case using the theory of twisted groups.
4. Finally time we treat the random case.

Commutative case

We start with the simplest case where operators are deterministic and the 2-cocycle $\zeta \equiv 1$. We recall that the aim idea is to find an adapted decomposition of the physical space. We said adapted in the sense that it provides a diagonalization basis for the observable. Therefore the goal is to decompose the Hilbert space with respect to the diagonalization basis of the commutant of $\mathcal{K}_\infty$.

We work here in the Hilbertian setting and group theory. Using the results of the previous section, the first step is to define a map between $\mathfrak{H}$ and $l^2(\mathcal{Z}; \mathfrak{H}_0)$ which respects the unitary representation encoding symmetries.

To ensure the convergence of all the sums of operators, we work on $\mathfrak{H}_c \simeq \mathcal{K}(\mathcal{Z}) \otimes \mathfrak{H}_0$ with operators belonging to $\mathcal{K}_\infty^{\mathcal{F}}$.

We denote this map by Π and defined as follows

$$\mathfrak{H} \to l^2(\mathcal{Z};\mathfrak{H}_0) \tag{6.1}$$

$$\varphi \to (x \to \chi_{\mathrm{e}} U_x^*(\varphi)) \tag{6.2}$$

For simplicity, we denote the range under Π of φ by $\tilde{\varphi}_\bullet$. We recall that we have the isomorphism $l^2(\mathcal{Z};\mathfrak{H}_0) \simeq l^2(\mathcal{Z}) \otimes \mathfrak{H}_0$.

The inverse transform from $l^2(\mathcal{Z};\mathfrak{H}_0)$ into $\mathfrak{H}$ is given by

$$\Pi^*(\tilde{\varphi}_x) := \sum_{\mathcal{Z}} \chi_x U_x(\tilde{\varphi}_x) = \varphi \ , \tag{6.3}$$

moreover we have that

$$\langle \Pi^*(\varphi), \Pi^*(\varphi)\rangle_{\mathfrak{H}} = \sum_{\mathcal{Z}} \langle \chi_x \Pi^*(\varphi), \chi_x \Pi^*(\varphi)\rangle_{\mathfrak{H}} \tag{6.4}$$

$$= \sum_{\mathcal{Z}} \langle \varphi_x, \varphi_x\rangle_{\mathfrak{H}_0} \tag{6.5}$$

$$= \langle \varphi, \varphi\rangle_{l^2(\mathcal{Z})\otimes\mathfrak{H}_0} \ , \tag{6.6}$$

where we have used the definition the fact that χ_x is a projection, the unitarity of U_x and the relation between χ and U 2.7. In the same way, we get

$$\langle \Pi(\varphi), \Pi(\varphi)\rangle_{l^2(\mathcal{Z})\otimes\mathfrak{H}_0} = \sum_{\mathcal{Z}} \langle \Pi(\varphi)_x, \Pi(\varphi)_x\rangle_{\mathfrak{H}_0} = \langle \varphi, \varphi\rangle_{\mathfrak{H}} \tag{6.7}$$

Where we use that $\Pi^* \circ \Pi = \sum_{\mathcal{Z}} \chi_x = 1$. As a consequence we obtain a unitary map between $\mathfrak{H}$ and $l^2(\mathcal{Z}) \otimes \mathfrak{H}_0$.
We get a direct integral decomposition of $\mathfrak{H}$ along $\mathcal{Z}$:

$$\mathfrak{H} \simeq \int_{\mathcal{Z}}^{\oplus} \mathfrak{H}_z dz$$

where $\mathfrak{H}_z := \Pi_z(\mathfrak{H}) \simeq \mathfrak{H}_0$, the Hilbert-sections and dz the Haar measure associated to $\mathcal{Z}$.

Remark 30. *Let $A \in \mathcal{K}_\infty^{\mathcal{F}}$. We set some notations $\Gamma_x := \chi_x U_x$, $\Gamma_x^* := \chi_{\mathrm{e}} U_x^*$ and $A(x,y) := \Gamma_x^* A \Gamma_y$ in that case we have that*

$$\Pi(A\varphi)_z = \Pi(\sum_{\mathcal{Z}^2} \Gamma_x A(x,y) \Gamma_y^* \varphi) \tag{6.8}$$

$$= \sum_{y\in\mathcal{Z}} A(z,y) \Pi(\varphi)_y \tag{6.9}$$

and therefore that if we note $A_x := \Pi_x(A\Pi^*)$

$$A_x\varphi_x = \sum_{y\in\mathcal{Z}} A(x,y)\varphi_y \tag{6.10}$$

which is clearly reminiscent of the usual integral-kernel or more naturaly to a matricial action.

Now we can define the analogue of the Floquet transform. We note $\widehat{\cdot}$ the usual discrete Fourier transform on $\mathcal{Z}$ defined for $f \in l^1(\mathcal{Z})$ by

$$l^2(\mathcal{Z}) \to L^2(\mathcal{Z}^*) \tag{6.11}$$

$$\varphi \mapsto \sum_{\mathcal{Z}} \overline{\langle r, x\rangle} f(x) \tag{6.12}$$

We simply define the Floquet transform in this particular case as the composition of the map Π with the Fourier transform.
Let,

$$\mathcal{F}(\varphi)(r) := \widehat{\Pi(\varphi)}(r) \ ,$$

more precisely,

$$\mathfrak{H} \mapsto L^2(\mathcal{Z}^*) \otimes \mathfrak{H}_0 \tag{6.13}$$

$$\varphi \mapsto \sum_{\mathcal{Z}} \overline{\langle r, x\rangle} \chi_0 U_{-x}(\varphi) \ . \tag{6.14}$$

Then we have that $\mathcal{F}$ map $\mathfrak{H}$ into $L^2(\mathcal{Z}^*; \mathfrak{H}_0)$. For simplicity we write $\tilde{\varphi}_r := \widehat{\Pi}(\varphi)_r$.
To give a concrete example, if we consider $\mathfrak{H} = L^2(\mathbb{R}^d)$ and $\mathcal{Z} = \mathbb{Z}^d$ we get that

$$\mathcal{F}\varphi := \sum_{\mathbb{Z}^d} e^{-ik\cdot x} U_{-x}(\chi_x \varphi)$$

and more precisely as we are in the commutative-deterministic case we recover the usual *Zak* transform used for periodic operators.

Secondly we define the adjoint of the transform by the following map

$$\mathcal{F}^*(\varphi) := \sum_{\mathcal{Z}} \Gamma_x \int_{\mathcal{Z}^*} \langle r, x\rangle(\varphi) dr \ . \tag{6.15}$$

It is nothing else than the composition of the inverse Fourier transform and the adjoint Π. Then we have that $\mathcal{F} \circ \mathcal{F}^* = id_{L^2(\mathcal{Z}^*)\otimes\mathfrak{H}_0}$ and $\mathcal{F}^* \circ \mathcal{F} = id_{\mathfrak{H}}$.

We have an isometry.

$$\|\tilde{\varphi}_r\|^2_{L^2(\mathcal{Z}^*)\otimes\mathfrak{H}_0} = \int_{\mathcal{Z}^*} \langle\tilde{\varphi}_r|\tilde{\varphi}_r\rangle_{\mathfrak{H}_0} dr \tag{6.16}$$

$$= \int_{\mathcal{Z}^*} \left\langle \left(\sum_x \overline{r}(x)\tilde{\varphi}_x\right) | \left(\sum_y \overline{r}(y)\tilde{\varphi}_y\right)\right\rangle_{\mathfrak{H}_0} dr \tag{6.17}$$

$$= \int_{\mathcal{Z}^*} \sum_{x,y} \overline{r}(x)r(y) \langle\tilde{\varphi}_x|\tilde{\varphi}_y\rangle_{\mathfrak{H}_0} dr \tag{6.18}$$

$$= \sum_{x,y} \delta_{x,y} \langle\tilde{\varphi}_x|\tilde{\varphi}_y\rangle_{\mathfrak{H}_0} dr \tag{6.19}$$

$$= \|\varphi\|^2_{l^2(\mathcal{Z})\otimes\mathfrak{H}_0} \tag{6.20}$$

$$= \|\varphi\|^2_{\mathfrak{H}} . \tag{6.21}$$

Where we used the character relation $\int_{\mathcal{Z}^*} \overline{r}(x)r(y)dr = \delta_{x,y}$ and the above computation proving that $\mathfrak{H} \simeq l^2(\mathcal{Z}) \otimes \mathfrak{H}_0$. We finally get

Lemma 6.5. *The mapping $\mathcal{F}$ is an isometry between $\mathfrak{H}$ and $\int^{\oplus}_{\mathcal{Z}^*} \mathfrak{H}_r dr$.*

As it can be seen, one of the main interests of the above formalism comes from the fact that we can perform all our computations staying in the strict Hilbertian formalism. This is exactly what provides us a good way to get an adapted and explicit generalization to more complex cases.
We first note that the Floquet transform sends translations on the mutiplications by the associated character in the usal way. We start by showing how the conjugation by Π sends the unitary representation into the usual translations or more generally on the regular representation of $\mathcal{Z}$ on $l^2(\mathcal{Z})$. Let $f \in l^2(\mathcal{Z}) \otimes \mathfrak{H}_0$ and $x, y \in \mathcal{Z}$ then

$$\Pi(U_z\Pi^* f)(y) = \Pi(U_z \sum_{x\in\mathcal{Z}} \chi_x U_x f(x))(y) \tag{6.22}$$

$$= \Pi(\sum_{x\in\mathcal{Z}} \chi_{z+x} U_{z+x} f(x))(y) \tag{6.23}$$

$$= \Pi(\sum_{x\in\mathcal{Z}} \chi_x U_x f)(x - z))(y) \tag{6.24}$$

$$= f(y - z) = (\lambda(z) \otimes id_{\mathfrak{H}_0} f)(y) \tag{6.25}$$

As it is well-known the Fourier transform diagonalizes translations in the sense that for any $f \in l^1(\mathcal{Z})$ we have that $(\widehat{\lambda(z)f})(r) = \overline{r}(z)\widehat{f}(r)$, then we have that $\mathcal{F}(U_z\mathcal{F}^*\varphi)_r = \overline{r}(z)\varphi_r$. We have only to perform a change of variable

in the sum. We recover a fibered structure as in the usual Floquet theory. At each character r we associate a Hilbert section. This only means that the Floquet transform diagonalizes translations by an element $z \in \mathcal{Z}$ with the eigenvalue $\overline{r}(z)$.
To sum up we draw the following commutative diagram

$$\begin{array}{ccccc} \mathfrak{H} & \longrightarrow & l^2(\mathcal{Z}) \otimes \mathfrak{H}_0 & \xrightarrow{\mathcal{F}} & L^2(\mathcal{Z}^*) \otimes \mathfrak{H}_0 \\ \downarrow{\scriptstyle U_x} & & \downarrow{\scriptstyle \lambda(x)\otimes id_{\mathfrak{H}_0}} & & \downarrow{\scriptstyle \overline{\langle \cdot, x\rangle}\otimes id_{\mathfrak{H}_0}} \\ \mathfrak{H} & \longrightarrow & l^2(\mathcal{Z}) \otimes \mathfrak{H}_0 & \xrightarrow{\mathcal{F}} & L^2(\mathcal{Z}^*) \otimes \mathfrak{H}_0 \end{array}$$

Matricial realization with respect to the decomposition

Before diagonalizing the observables, we show that the decomposition of the Hilbert space with repect to the symmetries induces also a decompostion on the von-Neumamm algebra of bounded operators on $\mathfrak{H}$. We have proved previously in the Lemma 5.9 that an operator A which belongs to $\mathcal{K}_\infty$ is completely determined by $A\chi_{\mathrm{e}}$. More generally we prove that an operator which belongs to $\mathcal{K}_\infty$ is completely determined by a kernel which is an operator on $\mathfrak{H}_0$. To prove this rigorously we use of the notion of matrix unit [T].

Definition 6.6. *We call a family of operators, $\{e_{i,j}\}_{(i,j)\in I^2}$ of a von-Neumann algebra, a matrix unit if the following conditions hold*

1. $e^*_{x,y} = e_{y,x}$
2. $e_{x,y}e_{w,z} = \delta_{y,w}e_{x,z}$
3. $\sum_{\mathcal{Z}} e_{x,x} = id_{\mathfrak{H}}$

It is natural to consider in our context the following matrix identity. Let $\Gamma_x := \chi_x U_x$, then we consider the family composed by the $e_{x,y} := \Gamma_x \Gamma^*_y$. In this case we have

Lemma 6.7. *The family $\{e_{x,y}\}_{(x,y)\in\mathcal{Z}^2}$ is a matrix identity for the von-Neumann alegbra $\mathcal{B}(\mathfrak{H})$.*

Proof. We have that $\sum\limits_{\mathcal{Z}} e_{x,x} = \sum_{\mathcal{Z}} \chi_x = id_{\mathfrak{H}}$ and by construction the first point. Next,

$$e_{x,y}e_{w,z} = \chi_x U_x U^*_y \chi_y \chi_w U_w U^*_z \chi_z \tag{6.26}$$
$$= \delta_{(y,w)} \chi_x U_x U^*_y \chi_y U_y U^*_z \chi_z \tag{6.27}$$
$$= \delta_{(y,w)} \chi_x U_x \chi_e U^*_z \chi_z \tag{6.28}$$
$$= \delta_{(y,w)} \chi_x \chi_x U_x U^*_z \chi_z \tag{6.29}$$
$$= \delta_{(y,w)} e_{x,z} \ . \tag{6.30}$$

□

We then have the following lemma.

Lemma 6.8. *We have*

$$\{\mathcal{B}(\mathfrak{H}), \mathfrak{H}\} \simeq \{\mathcal{K}_0, \mathfrak{H}_0\} \otimes \{\mathcal{B}(l^2(\mathcal{Z})), l^2(\mathcal{Z})\} \tag{6.31}$$

where $\mathcal{K}_0 = \chi_e \mathcal{B}(\mathfrak{H})\chi_e \simeq \mathcal{B}(\mathfrak{H}_0)$

We provide a short proof for the sake of completeness.

Proof. Let $\{\delta_x\}_{x\in\mathcal{Z}}$ the canonical basis of $l^2(\mathcal{Z})$. For each $\sum \zeta_x \otimes \delta_x \in \mathfrak{H}_0 \otimes l^2(\mathcal{Z})$ we define

$$U\left(\sum \zeta_x \otimes \delta_x\right) = \sum \Gamma_x \zeta_x \ .$$

We then have as explained previously that

$$\left\langle U\left(\sum \zeta_x \otimes \delta_x\right) | U\left(\sum \eta_y \otimes \delta_y\right)\right\rangle = \left\langle \sum \Gamma_x \zeta_x | \sum \Gamma_y \eta_y \right\rangle \tag{6.32}$$

$$= \sum_{x,y\in\mathcal{Z}} \langle \Gamma_y^* \Gamma_x \zeta_x | \eta_y \rangle \tag{6.33}$$

$$= \sum_{x\in\mathcal{Z}} \langle \zeta_x | \eta_x \rangle \tag{6.34}$$

$$= \left\langle \sum_{x\in\mathcal{Z}} \zeta_x \otimes \delta_x | \sum_{y\in\mathcal{Z}} \eta_y \otimes \delta_y \right\rangle \ . \tag{6.35}$$

Then U define an isometry of $\mathfrak{H}_0 \otimes l^2(\mathcal{Z})$ into $\mathfrak{H}$. Moreover we have that the range of U contains all $\Gamma_x \mathfrak{H}_0 = e_{x,x}\mathfrak{H}$ and then U is surjective since $\sum e_{x,x} = 1$.
Setting $\mathcal{K}_0 := \chi_e \mathcal{B}(\mathfrak{H}_0)\chi_e$. Let $A = (A_{x,y}) \in \mathcal{K}_0 \otimes \mathcal{B}(l^2(\mathcal{Z}))$, we have that

$$UAU^*\zeta = UA\left(\sum_{y\in\mathcal{Z}} \Gamma_y^* \zeta \otimes \delta_y\right) \tag{6.36}$$

$$= U \sum_{x,y\in\mathcal{Z}} A_{x,y} \Gamma_y^* \zeta \otimes \delta_y \ ) \tag{6.37}$$

$$= \sum_{x,y\in\mathcal{Z}} \Gamma_x A_{x,y} \Gamma_y^* \zeta \ . \tag{6.38}$$

and $UAU^* \in \mathcal{B}(\mathfrak{H})$. Conversely if $A \in \mathcal{B}(\mathfrak{H})$

$$U^*AU\left(\sum \zeta_y \otimes \delta_y\right) = U^*A\sum \Gamma_y \zeta_y \tag{6.39}$$

$$= U^*(\sum A\Gamma_y \zeta_y) \tag{6.40}$$

$$= \sum_{x,y\mathcal{Z}} (\Gamma_x^* A \Gamma_y \zeta_y)\ . \tag{6.41}$$

Then $(U^*AU)_{x,y} = \Gamma_x^* A \Gamma_y \in \mathcal{K}_0$ such that $U^*AU \in \mathcal{K}_0 \otimes \mathcal{B}(l^2(\mathcal{Z}))$.

□

Remark 31. *We will see that $\mathcal{K}_\infty$ can be characterized by this decomposition in the sense that the operators which belong to $\mathcal{K}_\infty$ are operators which have matricial components with some invariance properties.*

Diagonalization of operators

As it is well-known we have that for a group $\mathcal{Z}$ and $L^\infty(\mathcal{Z}, \mathcal{B}(\mathfrak{H}))$, the set of decomposable operators along $\mathcal{Z}$, we have that

$$L^\infty(\mathcal{Z}, \mathcal{B}(\mathfrak{H})) \simeq L^\infty(\mathcal{Z}, \mathbb{C})'\ .$$

We then use a classical trick to go from a measurable structure to a continuous one. We know by the von-Neumann bicommutant theorem that commutants are weakly closed and secondly that $C(\mathcal{Z}, \mathbb{C})$ is weakly dense in $L^\infty(\mathcal{Z}, \mathbb{C})$, then $(C(\mathcal{Z}, \mathbb{C}))' = (L^\infty(\mathcal{Z}, \mathbb{C}))'$. Using this fact we get the following result.

Proposition 6.9. *The Floquet transform defines an isomorphism between the decomposable operators and the covariant operators. Let*

$$L^\infty(\mathcal{Z}^*, \mathcal{B}(\mathfrak{H}_0)) \simeq \{A \in \mathcal{K} \mid \forall x \in \mathcal{Z}, [U_x, A] = 0\}$$

Proof. By the well-known theorem ([Di] Ch2.2.Thm1) an operator is decomposable if and only if it commutes with the *diagonal algebra*. As explained above this is equivalent to commutation with respect to the action generated by the continuous functions on the underlying group.
Set $T_\theta = \int_{\mathcal{Z}^*}^{\oplus} \theta(r)dr$ with $\theta \in C(\mathcal{Z}^*)$ and $\widehat{A}(r) := \mathcal{F}(A\mathcal{F}^*)(r)$. These follow the commutation rules: for all $r \in \mathcal{Z}^*$,

$$\begin{aligned} A \in L^\infty(\mathcal{Z}^*, \mathcal{B}(\mathfrak{H}_0)) &\Leftrightarrow [A, T_\theta] = 0 \text{ for all } \theta \in C(\mathcal{Z}^*) \\ &\Leftrightarrow [\widehat{A}(r), \theta(r)] = 0 \text{ for all } r \in \mathcal{Z}^* \text{ and } \theta \in C(\mathcal{Z}^*) \\ &\Leftrightarrow [A, (\mathcal{F}^*\theta(r)\mathcal{F})] = 0\ . \end{aligned}$$

it remains to prove that the latter is equivalent to $[A, U_x] = 0$ for all $x \in \mathcal{Z}$.

Lemma 6.10. *The action of $C(\mathcal{Z})$ and $C(\mathcal{Z}^*)$ on $\mathfrak{H}$ are defined by the following operators,*

$$\lambda[f] = \sum_{x\in\mathcal{Z}} f(x)\chi_x \text{ , for } f \in C(\mathcal{Z}) \text{ ,} \tag{6.42}$$

and

$$\widehat{\lambda}[\theta] = \sum_{x,y\in\mathcal{Z}} \widehat{\theta}(y-x)e_{x,y} \text{ , for } \theta \in C(\mathcal{Z}^*) \text{ .} \tag{6.43}$$

Proof.

$$\Pi^*(f(x)\Pi_x) = \sum_{\mathcal{Z}} \chi_x U_x f(x)\chi_0 U_{-x} \tag{6.44}$$

$$= \sum_{\mathcal{Z}} \chi_x U_x U_{-x} f(x)\chi_x \tag{6.45}$$

$$= \sum_{\mathcal{Z}} f(x)\chi_x \tag{6.46}$$

Whereas

$$\mathcal{F}^*(\theta(r)\mathcal{F}_r) = \sum_{x\in\mathcal{Z}} \chi_x U_x \int_{\mathcal{Z}^*} \langle r,x\rangle\theta(r) \sum_{y\in\mathcal{Z}} \overline{\langle r,y\rangle}\chi_0 U_{-y} dr \tag{6.47}$$

$$= \sum_{x,y\in\mathcal{Z}} \chi_x U_x \int_{\mathcal{Z}^*} \langle r,x-y\rangle\theta(r)\chi_0 U_{-y} dr \tag{6.48}$$

$$= \sum_{x,y\in\mathcal{Z}} \widehat{\theta}(x-y)e_{x,y} \tag{6.49}$$

□

We start by proving the first implication that is, A is covariant implies A decomposable. We have proved above that the Floquet transform sends the translation into multiplication by a character in the sense that

$$(\mathcal{F}U_z\mathcal{F}^*\varphi)(r) = \overline{r}(z)\varphi(r) \text{ .}$$

Then $[A, U_x] = 0$ implies that $\widehat{A}(r)$ commutes with the functions $x(r) := r(x) \in C(\mathcal{Z}^*)$ and then also for linear combination of such functions of the form $f(r) = \sum_{i\in\mathbb{N}} \lambda_i x_i(r)$ with $\lambda_i \in \mathbb{C}$. But it is well know (Stone-Weierstrass theorem) that such functions are dense in $C(\mathcal{Z}^*)$. This proves the first implication. The converse follows if we prove that $[A, \widehat{\lambda}[\theta]] = 0$ implies the covariance of A. But we have that what implies that $[A, \sum_{k,l\in\mathcal{Z}} e_{k,l}] = 0$ and after a basic change of variable under the sum and using the covariance of the projector-valued map χ we get easily that A commutes with $\widehat{\lambda}[\theta]$ implies that A commutes with $\sum_{x\in\mathcal{Z}} U_x$ which is clearly equivalent to that A is covariant. $\square$

We note that this framework permits us to give an explicit expression for the diagonalized operators. We have that

$$\mathcal{F}A\mathcal{F}^*(r)\varphi(r) = \sum_{x\in\mathcal{Z}} \overline{\langle r, x\rangle}\chi_0 U_x^* A \sum_{y\in\mathcal{Z}} \chi_y U_y \int_{\mathcal{Z}^*} \langle r', y\rangle \varphi(r')dr' \tag{6.50}$$

$$= \sum_{x,y\in\mathcal{Z}} \overline{\langle r, x-y\rangle}(\chi_0 A U_{y-x}\chi_0) \int_{\mathcal{Z}^*} \langle rr', y\rangle \varphi(r')dr' \tag{6.51}$$

$$= \widehat{A}(r)\varphi(r)\ , \tag{6.52}$$

where $\widehat{A}(r) = \sum_{x\in\mathcal{Z}} \overline{\langle r, x\rangle}\chi_0 A U_x^* \chi_0$. We have only used the character relation $\sum_{\mathcal{Z}} r(y)r'(y) = \delta_{rr'}$ and a change of variable.

Remark 32. *We note that we have by the Heisenberg-Weyl commutation relation described previously.*

$$U_x V_r = \overline{r}(x) V_r U_x\ .$$

So, if φ is such that $U_x\varphi = \overline{r}(x)\varphi$ then we have $V_r\varphi$ is invariant under the action of U_x.

7 C^*-Fibration

In this section we show that the formalism of the Hilbert C^*-module is a natural structure arising when inspecting the fibration of covariant operators and appears as the adapted formulation of the Floquet theory and consequently of its possible generalizations. We denote μ te state associated to the Haar measure dr.

Proposition 7.1. *The Floquet transform $\mathcal{F}$ defines a pre-Hilbert $C(\mathcal{Z}^*)$-module structure by the $C(\mathcal{Z}^*)$-valued scalar product*

$$\langle \varphi, \phi\rangle_r := \langle \mathcal{F}(\varphi)_r | \mathcal{F}(\phi)_r\rangle_{\mathfrak{H}_0}\ . \tag{7.1}$$

Let $\mathcal{M}$, the completion of $\mathfrak{H}_c$ under the norm $\|\cdot\|_{ess-\mu}$ is a Hilbert $C^*(\mathcal{Z})$-module and the GNS representation with respect to the state μ corresponds exactly with $\mathfrak{H}$. Consequently the $C(\mathcal{Z}^*)$-module defines a continuous field of Hilbert field.

Proof. Let $\varphi, \phi \in \mathfrak{H}_c$,

$$\langle \varphi, \phi \rangle_r = \langle \mathcal{F}(\varphi)_r | \mathcal{F}(\phi)_r \rangle_{\mathfrak{H}_0} \tag{7.2}$$
$$= \sum_{x,y \in \mathcal{Z}} \langle \overline{\langle r, x \rangle} \Pi_x(\varphi) | \overline{\langle r, y \rangle} \Pi_y(\phi) \rangle_{\mathfrak{H}_0} \tag{7.3}$$
$$= \sum_{x,y \in \mathcal{Z}} \overline{\langle r, x - y \rangle} \langle \varphi | \Pi_x^* \Pi_y \phi \rangle_{\mathfrak{H}_0} \tag{7.4}$$
$$= \sum_{x,y \in \mathcal{Z}} \overline{\langle r, x - y \rangle} \langle \varphi | e_{x,y} \phi \rangle_{\mathfrak{H}_0} \ . \tag{7.5}$$

This equality makes explicit the dependence of the scalar product with respect to the character which is therefore r-continuous. The $C(\mathcal{Z}^*)$-linearity comes from that for all $f \in C(\mathcal{Z}^*)$, we have

$$\langle \varphi, \widehat{\lambda}[\theta] \phi \rangle_r = \sum_{x,y \in \mathcal{Z}} \overline{\langle r, x - y \rangle} \langle \varphi | \left(\sum_{\mathcal{Z}} \widehat{\theta}(w - z) e_{w,z} \right) e_{x,y} \phi \rangle_{\mathfrak{H}_0} \tag{7.6}$$
$$= \sum_{x,y,w,z \in \mathcal{Z}} \overline{\langle r, x - y \rangle \widehat{\theta}}(w - z) \langle \varphi | \delta_{z,x} e_{w,y} \phi \rangle_{\mathfrak{H}_0} \tag{7.7}$$
$$= \sum_{x,y,w \in \mathcal{Z}} \overline{\langle r, x - y \rangle \widehat{\theta}}(w - x) \langle \varphi | e_{w,y} \phi \rangle_{\mathfrak{H}_0} \tag{7.8}$$
$$= \sum_{y,w \in \mathcal{Z}} \sum_{x} (\overline{\langle r, x - w \rangle \widehat{\theta}}(w - x)) \overline{\langle r, w - y \rangle} \langle \varphi | e_{w,y} \phi \rangle_{\mathfrak{H}_0} \tag{7.9}$$
$$= \sum_{y,w \in \mathcal{Z}} \theta(r) \overline{\langle r, w - y \rangle} \langle \varphi | e_{w,y} \phi \rangle_{\mathfrak{H}_0} \tag{7.10}$$
$$= \theta(r) \langle \varphi, \phi \rangle_r \ . \tag{7.11}$$

Where we have used the matrix identity relation,$e_{a,b} e_{c,d} = \delta_{b,c} e_{a,d}$ and the fact that the Fourier transform sends a translation on a multiplication by a character. Furthermore using that $e_{i,j}^* = e_{i,j}$ and the character relation $\overline{\langle r, x \rangle} = \langle r, -x \rangle$, we have that

$$\langle \varphi, \phi \rangle_r^* = \sum_{x,y \in \mathcal{Z}} \langle r, x - y \rangle \langle e_{x,y} \phi | \varphi \rangle_{\mathfrak{H}_0} \tag{7.12}$$
$$= \sum_{x,y \in \mathcal{Z}} \overline{\langle r, y - x \rangle} \langle \phi | e_{y,x} \varphi \rangle_{\mathfrak{H}_0} \tag{7.13}$$
$$= \langle \phi, \varphi \rangle_r \tag{7.14}$$

Finally using the state defined by the Haar measure on $\mathcal{Z}^*$ denoted by μ, we get a Hilbert C^*-module which corresponds exactly to the GNS representation defined with respect to this state. Using that the group is the dual of a discrete group and is consequently compact and as by definition the Haar measure does not have open set of measure zero this state is faithful.
We then define the following norm,

$$\|\varphi\|_{ess-\mu} = \sup_{r \in \mathcal{Z}^*} \langle \varphi | \varphi \rangle (r)$$

Furthermore we recall that in this case, as explained in the above section, we have that

$$\begin{aligned}
\langle \varphi | \phi \rangle_\mu &\equiv \mu(\langle \varphi | \phi \rangle_r) && (7.15)\\
&= \int_{\mathcal{Z}^*} \sum_{x,y \in \mathcal{Z}} \langle r, x-y \rangle \langle \varphi | e_{x,y} \phi \rangle_{\mathfrak{H}_0} && (7.16)\\
&= \sum_{x,y \in \mathcal{Z}} \delta_{x,y} \langle \varphi | e_{x,y} \phi \rangle_{\mathfrak{H}_0} && (7.17)\\
&= \langle \varphi | \phi \rangle_{\mathfrak{H}} && (7.18)
\end{aligned}$$

This shows that the GNS representation space associated to μ is exactly $\mathfrak{H}$. □

We have shown that the Hilbert C^*-module is the natural formalism for the Floquet theory. Now we can construct a generalization of this theory for situation when the classical Floquet theory fails. An important physical case where the classical Floquet theory is when the group is not abelian then the von-Kampen-Pontrjagin duality cannot be used and then the dual group $\mathcal{Z}^*$ do not make sense. This is the case by example for some random Schrödinger operators on manifold (see [LPV]). The second, which is our main motivation, the case where the translations do not commute as it is the case in presence of a magnetic field.

7.1 C^*-fibration for projectively covariant operators

In his seminal paper Haagerup (see [HI, HII, HK]) opened a new way of performing some analysis on reduced C^*-group algebra. One of the most important tool is the introduction of the so-called *Haagerup property* that we have defined in definition 2 to prove the density of some $*$-subalgebra. This way had provide to study the convergence of some abstract Fourier series. Recently $[BC]$ have constructed a generalization of Haagerup's methods to

the twisted case. Our work is strongly inspired by their papers which permit us to generalize this framework to the general setting of algebra of magnetic random operators on Hilbert space.
We work in the general setting of the twisted reduced C^*-algebra $C_r^*(\mathcal{Z},\zeta)$, the untwisted case being a particular case (it suffices to take $\zeta \equiv 1$ to get the untwisted framework). When it has some importance we emphasing the dependency on the 2-cocycle.

Analysis on twisted group alegbras

We refer to the following papers on this matter [BC, GI, T]. We recall that we consider discrete locally compact finitely generated by amenable group. Assumptions are compatible with the physics of random magnetic media.

Definition 7.2. *Let $\zeta \in Z^2(\mathcal{Z},\mathbb{T})$ a normalized 2-cocycle . The left regular ζ-projective unitary representation $\Lambda_\zeta(x)$ of $\mathcal{Z}$ on $l^2(\mathcal{Z})$ is defined by*

$$(\Lambda_\zeta(x)f)(y) = \zeta(x, y-x)f(y-x) \text{ for any } x,y \in \mathcal{Z} \text{ and } f \in l^2(\mathcal{Z}) \ .$$

The closure in the operator norm (resp. weak operator) topology of the $*$-algebra generated by the left regular ζ-projective representation, is called the reduced twisted group C^*-algebra $C_r^*(\mathcal{Z},\zeta)$ (resp. the twisted group von-Neumann algebra $vN(\mathcal{Z},\zeta)$)

Remark 33. *1. As claimed above we see that if we take $\zeta = 1$ we recover the left regular representation, i.e $\Lambda_1 = \lambda$, and the usual C^*-reduced group algebra $C_r^*(\mathcal{Z})$.*

2. the right ζ-regular representation is given by

$$(\mathcal{R}_\zeta(x)f)(y) = \zeta(x,y)f(y-x) \text{ for any } x,y \in \mathcal{Z} \text{ and } f \in l^2(\mathcal{Z}) \ .$$

The important fact is that the left ζ-regular representation commutes with the right $\overline{\zeta}$-right representation. (and the right ζ-regular representation commutes with the left $\overline{\zeta}$-regular representation.)

Setting $\{\delta_x\}_{x\in\mathcal{Z}}$ the canonical basis of $l^2(\mathcal{Z})$, we have that,

$$\Lambda_\zeta(x)\delta_y = \zeta(x,y)\delta_{x+y} \text{ for any } x,y \in \mathcal{Z} \ .$$

In particular, we have that $\Lambda_\zeta(x)\delta = \delta_x$ for any $x \in \mathcal{Z}$. The algebra structure of $C_r^*(\mathcal{Z},\zeta)$ is defined by the following operations.
Let f and g belonging to $l^2(\mathcal{Z})$,

$$f *_\zeta g(x) = \sum_{y\in\mathcal{Z}} f(y)\zeta(y,x-y)g(x-y) \text{ for any } x \in \mathcal{Z} \ .$$

This operation is called the ζ-convolution of f and g and is well defined in the sense that $\|f *_\zeta g\|_\infty \leq \|f\|_2 \|g\|_2$. With the following adjonction

$$f^*(x) = \overline{\zeta}(x, -x)\overline{f}(-x) \ ,$$

the involutive Banach algebra is denoted by $l^1(\mathcal{Z}, \zeta)$. It is possible to define a faithful $*$-representation π_ζ of $l^1(\mathcal{Z}, \zeta)$ on $l^2(\mathcal{Z})$ by

$$\pi_\zeta(f)g = f *_\zeta g \text{ for } f \in l^1(\mathcal{Z}) \text{ and } g \in l^2(\mathcal{Z}) \ .$$

In fact there exists a bijective correspondance between the ζ-projective unitary representation U of $\mathcal{Z}$ and the non-degenerate $*$-representation of $l^1(\mathcal{Z}, \zeta)$ by
$U \mapsto \pi_U(f) = \sum_{\mathcal{Z}} f(x)U_x$, for $f \in l^1(\mathcal{Z})$. The inverse map is given by $U_\pi(g) = \pi(\delta_g)$.
Then $C_r^*(\mathcal{Z}, \zeta)$ is the closure in norm topology of $\pi_\zeta(l^1(\mathcal{Z}))$. We note that as we assume that $\mathcal{Z}$ is amenable, the extension of π_ζ is faithful and we can identify $C_r^*(\mathcal{Z}, \zeta)$ with the full twisted group C*-algebra $C^*(\mathcal{Z}, \zeta)$ what is the enveloping C^*-algebra of $l^1(\mathcal{Z}, \zeta)$(see [ZM]).

This representation permits us to define an abstract Fourier transform as well as abstract Fourier series.
It is well-known that the relation $\Lambda_\zeta(x)\delta = \delta_x$ provides an isometric map between $vN(\mathcal{Z}, \zeta)$ and $l^2(\mathcal{Z})$ which is used to get the GNS representation. We summarize this result in the following proposition.
.

Definition-Proposition 4. *With the above notations we have that*

1. *The functional ν on $vN(\mathcal{Z}, \zeta)$ defined for any $a \in vN(\mathcal{Z}, \zeta)$ by*

$$\nu(a) := \langle a\delta, \delta\rangle_{l^2(\mathcal{Z})}$$

   is a faithful tracial state and the inducted norm is denoted by $\|a\|_{(\zeta,\ 2)} := \nu(a^*a)^{\frac{1}{2}}$.

2. *The map defined by $\widehat{a} := a\delta$ is a linear isometry from $(vN(\mathcal{Z}, \zeta), \|\cdot\|_{(\zeta,\ 2)})$ into $(l^2(\mathcal{Z}), \|\cdot\|_2)$ and the complex number $\widehat{a}(g)$ is called the Fourier coefficient of a.*

We note that

$$\pi_\zeta(f) = \sum_{x \in \mathcal{Z}} f(x)\Lambda_\zeta(x) \text{ for any } f \in l^1(\mathcal{Z}) \ .$$

Then we see that π_ζ is a twisted analogue of the Fourier series used above, namely $\sum_{x\in\mathcal{Z}}\langle r,x\rangle f(x)$. As in the above case we recover the usual identities of the Fourier analysis. We have for any $f\in l^1(\mathcal{Z})$ and $a\in vN(\mathcal{Z},\zeta)$ that

$$\widehat{\pi_\zeta(f)}=\pi_\zeta(f)\delta=\sum_{\mathcal{Z}}f(x)\Lambda_\zeta(x)\delta=\sum_{\mathcal{Z}}f(x)\delta_x=f$$

and

$$\pi_\zeta(\widehat{a})=\sum_{x\in\mathcal{Z}}\widehat{a}(x)\Lambda_\zeta(x)=\sum_{x\in\mathcal{Z}}\langle a\delta,\delta_x\rangle\Lambda_\zeta(x)=\sum_{x\in\mathcal{Z}}\langle a\delta,\Lambda_\zeta(x)\delta\rangle\Lambda_\zeta(x)=a$$

Remark 34. *As consequence we have that the natural GNS representation of $C_r^*(\mathcal{Z},\zeta)$ is exactly $l^2(\mathcal{Z})$. Then it seems natural to identify δ_x with $\Lambda_\zeta(x)$ for the left ζ-representation or δ_{-x} with $\overline{\zeta}(x,-x)R_\zeta(x)$ in the right ζ-representation.Therefore we get by the last identifications that $l^1(\mathcal{Z})\subset C_r^*(\mathcal{Z},\zeta)\subset l^2(\mathcal{Z})$. The first inclusion by the $*$-representation π_ζ while the second by the map $\widehat{\cdot}$.*

In this abstract and general setting we need to state some definitions and results to ensure the convergence of abstract Fourier series.

Definition 7.3. *Let G a subspace of $l^2(\mathcal{Z})$ which contains the finitely supported functions on $\mathcal{Z}$, $\mathcal{K}(\mathcal{Z})$. We say that $(\mathcal{Z},\zeta)$ has the G-decay property with respect to $\|\cdot\|'$ if the following condition holds*

1. *for each $f\in G$ we have that $f\to 0$ at infinity w.r.t $\|\cdot\|'$.*
2. *The map $f\mapsto\pi_\zeta(f)$ from $(\mathcal{K}(\mathcal{Z}),\|\cdot\|')$ to $(C_r^*(\mathcal{Z},\zeta),\|\cdot\|)$ is bounded.*

In such a case $(\mathcal{Z},\zeta)$ is said to be κ-decaying if it is $\mathcal{L}^2_\kappa$-decaying with $\kappa:\mathcal{Z}\mapsto[0,\infty]$ and $\mathcal{L}^2_\kappa$ as defined previously in section 5. We recall that the Haagerup content is given by

$$c(F):=\sup\{\|\pi_1(f)\|\ |f\in\mathcal{K}(\mathcal{Z}),\mathrm{supp}(f)\subset F,\|f\|_2=1\}\ .$$

When $\mathcal{Z}$ is amenable then $c(F)=|F|^{\frac{1}{2}}$. Set $B_{r,L}:=\{x\in\mathcal{Z}|L(x)\leq r\}$ for any $r\in\mathbb{R}^+$. Then $\mathcal{Z}$ is said to have a polynomial H-growth if there exists $C,p>0$ such that $c(B_{r,L})\leq C(1+r)^p$ for all $r\in\mathbb{R}^+$, while it is said to have subexponential H-growth if for any $k>1$ there exists r_0 such that $c(B_{r,L})\leq k^r$ for all $r\geq r_0$. By Th3.13 of [BC] if we assume that $\mathcal{Z}$ has polynomial H-growth then there exists s_0 such that $(\mathcal{Z},\zeta)$ is $(1+L)^{s_0}$-decaying or if $\mathcal{Z}$ have an subexponential H-growth then $(\mathcal{Z},\zeta)$ is a^L-decaying for all $a>1$.

In this context [BC] introduces the notion of ζ-multiplier. A function φ :

$\mathcal{Z} \to \mathbb{C}$ is called a ζ-multiplier if the map M_φ defined by $M_\varphi(\pi_\zeta(f)) = \pi_\zeta(\varphi f)$ is bounded in operator norm for all $f \in \mathcal{K}(\mathcal{Z})$. We note that the map M_φ is uniquely determined by its range on the generator $\Lambda_\zeta(x)$, $x \in \mathcal{Z}$. Denoting the set of ζ-mutiplier by $MA(\mathcal{Z}, \zeta)$. This notion is strongly linked to the Schur-multiplier(see [BC]). Then they define the set $MCF(\mathcal{Z}, \zeta) = \{\varphi \in \ MA(\mathcal{Z}, \zeta) | M_\varphi$ maps $C_r^*(\mathcal{Z}, \zeta)$ into $CF(\mathcal{Z}, \zeta)\}$, where $CF(\mathcal{Z}, \zeta)$ is the set $a \in C_r^*(\mathcal{Z}, \zeta)$ such that $\pi_\zeta(\widehat{a})$ is convergent in operator norm. Next *an approximate multiplier* is a net $\varphi_\alpha \in MA(\mathcal{Z}, \zeta)$ such that $M_{\varphi_\alpha}(a) \to a$ in operator norm for every $a \in C_r^*(\mathcal{Z}, \zeta)$. Since $\mathcal{Z}$ countable we can always get an countably approximate multiplier from any approximate multiplier.
A *Fourier summing net* is then an approximate multiplier such that for all $\alpha, \varphi_\alpha \in MCF(\mathcal{Z}, \zeta)$.
Finally we say that $(\mathcal{Z}, \zeta)$ has the *Fejer metric property* if there exists a Fourier summing net φ_α such that φ_α converges pointwise to 1, is bounded in $MA(\mathcal{Z}, \zeta)$ and $\|M_{\varphi_\alpha}\| = 1$ for all α.
Finally we state the following result which we will use implicitly.

Proposition 7.4. *[BC] With the above notations we have that*

1. *Assume that $\mathcal{Z}$ is amenable then it has the metric Fejer property, with the Fourier summing net given by $\varphi_\alpha(x) = \langle \lambda(x)\xi_\alpha, \xi_\alpha \rangle$, with $\xi_\alpha := |F_\alpha| \chi_{F_\alpha}$ with F_α the Foelner net and χ_{F_α} the characteristic function on it.*
2. *Assume that $\mathcal{Z}$ is countable and has the Haagerup property. If there exists a Haagerup length function such that $\mathcal{Z}$ has a sub-exponential H-growth, then has the metric Fejer property.*

Note that, as explained above, for the random case we must consider one least amenable group to use ergodic theorem. But some group of interest in non-random case are not amenable. For example in several models studied by Mathai et al. [MM] the underlying group is not amenable.

To summarize, we have defined an abstract analogue of the Fourier series on the twisted reduced group algebra. We have given a suitable definition of the Fourier series and ensured its convergences for general group in the sens that if the group respects some growth condition then the Fourier series are well defined for a large enough set of elements of the algebra. We then now use these recent results to get the natural basis of diagonalization for *the projectively covariant random operators* mimicking the untwisted and abelian context.

Projectively covariant operators

We now can give a constructive and abstract Floquet theory for operators with symmetries that are more rich than the ones of the abelian group of translation.

The obstruction to the usual diagonalization can be seen by the following computation. Let A belongs to $\mathcal{K}^{\mathcal{F}}_{\infty}$, the *finite range* operator and $\varphi \in \mathfrak{H}_c$.

$$\Pi(A\Pi^*\varphi)(x) = \sum_{y\in\mathcal{Z}} \chi_{\mathrm{e}} U_x^* A U_y \chi_{\mathrm{e}} \varphi(y) \tag{7.19}$$

$$= \sum_{y\in\mathcal{Z}} \overline{\zeta}(x,-x)\zeta(-x,y)\chi_{\mathrm{e}}\ A U_{-x+y}\chi_{\mathrm{e}}\varphi(y) \tag{7.20}$$

$$= \sum_{y\in\mathcal{Z}} \overline{\zeta}(x,y)A(y)\varphi(x+y) \tag{7.21}$$

$$= \sum_{y\in\mathcal{Z}} A(y)(\mathcal{R}_{\overline{\zeta}}(y)\varphi)(x) \tag{7.22}$$

with $A(y) = \chi_{\mathrm{e}} A U_y \chi_{\mathrm{e}}$. We have used that $U_x^* U_y = \overline{\zeta}(x,-x)\zeta(-x,y)U_{-x+y}$ and by the cocycle relation we have that $\zeta(x,-x+y)\zeta(-x,y) = \zeta(x,-x)$. After a simple change of variable in the sum we get the result. To break this obstruction we consider a twisted tranformation. We then reconsider the action of the group and therefore structure-group of the underlying operators.

Let $f \in \mathfrak{H}_0 \otimes l^2(\mathcal{Z})$ then

$$(\Pi U_x \Pi^* f)(y) = \chi_0 U_y^* U_x \sum_{k\in\mathcal{Z}} \chi_k U_k f(k) \tag{7.23}$$

$$= \sum_{k\in\mathcal{Z}} \chi_0 U_y^* \chi_{k+x}\zeta(x,k)U_{k+x} f(k) \tag{7.24}$$

$$= \sum_{k\in\mathcal{Z}} \chi_0 U_y^* \chi_k \zeta(x,k-x)U_k f(k-x) \tag{7.25}$$

$$= \chi_0 U_y^* \chi_y \zeta(x,y-x)U_y f(y-x) \tag{7.26}$$

$$= \zeta(x,y-x)f(y-x) \tag{7.27}$$

$$= (\Lambda_\zeta(x)f)(y)\ . \tag{7.28}$$

Then we have that the conjugaison by Π sends the unitary projections on the generators of the ζ-left regular representation of $(\mathcal{Z},\zeta)$ on $l^2(\mathcal{Z})$ which are, as explained above, the generators of the twisted reduced group algebra. Then the group of symmetries is not as previously the algebra $C(\mathcal{Z}^*)$ but the twisted-reduced group C^*-algebra $C_r^*(\mathcal{Z}, \zeta)$. The above computation shows that an operator A is covariant if and only if $\Pi A \Pi^*$ commutes with the $C_r^*(\mathcal{Z}, \zeta)$-action. We then have the following diagram.

$$\begin{array}{ccc} \mathfrak{H} & \xrightarrow{\Pi} & \mathfrak{H}_0 \otimes l^2(\mathcal{Z}) \\ \downarrow{\scriptstyle U_x} & & \downarrow{\scriptstyle id_{\mathfrak{H}_0} \otimes \Lambda_\zeta(x)} \\ \mathfrak{H} & \xrightarrow{\Pi} & \mathfrak{H}_0 \otimes l^2(\mathcal{Z}) \end{array}$$

In fact it is not surprising by the fact that our operator are in some sense generalizations from discrete to continuous setting of the so-called Harper operator and it is well-known that it can be defined by the right ζ-regular representation. Moreover we know that the $\overline{\zeta}$-right regular representation commutes with the ζ-left regular representation. Furthermore by definition the *twisted group von-Neumann algebra* is generated by the ζ-left regular representation. We finally get

Lemma 7.5. *We have the following isomorphism*

$$\mathcal{K}_\infty \simeq \mathcal{B}(\mathfrak{H}_0) \otimes vN(\mathcal{Z}, \zeta)'$$

To generalize the previous construction we need to construct a new Hilbert C^*-module adapted to this context.
We first note that we can write for $\varphi \in \mathfrak{H}$ that

$$\Pi(\varphi) = \sum_{\mathcal{Z}} \Pi_x(\varphi) \otimes \delta_x \in \mathfrak{H}_0 \otimes l^2(\mathcal{Z})$$

We can then use the transform defined above π_ζ which will provide a natural diagonalization.
Set $\varphi \in \mathfrak{H}_0 \otimes l^2(\mathcal{Z})$, then defining $\pi_\zeta(\varphi) := \sum_{x \in \mathcal{Z}} \varphi(x) \Lambda_\zeta(x)$ we have that $\pi_\zeta : \mathfrak{H}_0 \otimes l^2(\mathcal{Z}) \to \mathfrak{H}_0 \otimes C_r^*(\mathcal{Z}, \zeta)$ is well defined by the previous considerations.
We then define the Floquet transform as the composition of π_ζ and Π.

$$\mathcal{F}^\zeta(\varphi) := \sum_{\mathcal{Z}} \Pi_x(\varphi) \otimes \Lambda_\zeta(x) \in \mathfrak{H}_0 \otimes C_r^*(\mathcal{Z}, \zeta)$$

We define the Hilbert right C^*- module $\mathcal{M} := \mathfrak{H}_0 \otimes C_r^*(\mathcal{Z}, \zeta)$ and then

$$\langle \varphi, \phi \rangle_\zeta = \langle \mathcal{F}^\zeta(\varphi), \mathcal{F}^\zeta(\phi) \rangle_{\mathcal{M}} \ .$$

We must give care to the fact that here we must work on $\mathcal{M}^o$ the *opposite* Hilbert C^*-module which is isomorphic to $\mathcal{M}$ (see [T, Di] for details) with the associated scalar product.

$$\begin{aligned}&\mathcal{M} \times \mathcal{M} \mapsto C_r^*(\mathcal{Z}, \zeta)^o \\ &f, g \mapsto \langle g, f \rangle^o\end{aligned}$$

and for $a^o, b^o \in C_r^*(\mathcal{Z}, \zeta)^{op}$ we have the opposite product defined by $(ab)^o = b^o a^o$. and obviously the associated GNS representation with repsect to the state ν given by $\langle \varphi, \phi \rangle_\nu^o = \nu(\langle \phi, \varphi \rangle_{\mathcal{M}})$.
This comes from the particularity of the Hilbert-module that we consider here which is tensorial Hilbert-module. Let $\mathcal{A}$ a C^*-algebra and $\mathfrak{H}$ a Hilbert space then the $\mathcal{A}$-inner product is naturally defined by the following relation

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{\mathfrak{H} \otimes \mathcal{A}} = \langle x_2, x_1 \rangle_{\mathfrak{H}} y_1^* y_2$$

(see [MT] for details). Here as the algebraic structures are isomorphic and as no confusion occurs we alleviate the notation of the opposed product. We have in a second time, that contrary to the untwisted case $e_{x,y}$ is not equal to e_{x-y}, we have here a supplementary phase factors which enables us to get that:

$$\begin{aligned}\langle \varphi, \phi \rangle_\zeta &= \sum_{x,y \in \mathcal{Z}} \langle \Pi_y(\phi), \Pi_x(\varphi) \rangle_{\mathfrak{H}_0} \Lambda_\zeta^*(x) \Lambda_\zeta(y) \\ &= \sum_{x,y \in \mathcal{Z}} \langle e_{x,y} \phi, \varphi \rangle_{\mathfrak{H}_0} \Lambda_\zeta^*(x) \Lambda_\zeta(y) \\ &= \sum_{x,y \in \mathcal{Z}} \langle \phi, e_{y,x} \varphi \rangle_{\mathfrak{H}_0} \Lambda_\zeta^*(x) \Lambda_\zeta(y) \\ &= \sum_{x,y \in \mathcal{Z}} \langle \phi, e_{y-x} \varphi \rangle_{\mathfrak{H}_0} \overline{\zeta(y, -x)} \zeta(x, -x) \Lambda_\zeta^*(x) \Lambda_\zeta(y) \\ &= \sum_{x,y \in \mathcal{Z}} \langle \phi, e_{y-x} \varphi \rangle_{\mathfrak{H}_0} \Lambda_\zeta(y - x) \ \in C_r^*(\mathcal{Z}, \zeta) \ . \end{aligned} \tag{7.29}$$

where we have used the definition of the opposite product and the antilinearity in the second variable of the Hilbert scalar product. We then show that the GNS representation of the right-Hilbert C^*-module coincide with $\mathfrak{H}$.

$$\nu(\langle \varphi, \phi \rangle_{\mathcal{M}}) = \sum_{x,y \in \mathcal{Z}} \langle \phi, e_{y-x} \varphi \rangle_{\mathfrak{H}_0} \nu(\Lambda_\zeta(y - x)) = \langle \varphi, \phi \rangle_{\mathfrak{H}} \ .$$

We note that the sums (7.29) are well-defined for a dense set of elements by the theorem 7.4 of the last section. Using the Cauchy-Schwartz inequality the sum (7.29) is convergent if $\pi_\zeta(\|\Pi_x(\varphi)\|_{\mathfrak{H}_0})$ is convergent in operator norm but we know that is done for a dense subset containing the compact supported functions.
Finally we get

Proposition 7.6. *The scalar product $\langle \cdot, \cdot \rangle_\Lambda$ defines a structure of right pre-Hilbert C^*-module which the completion with respect to the associated norm is isomorphic to $\mathcal{M}$ and the GNS representation coincide with $\mathfrak{H}$.*

We then have an explicit diagonalization. We note that as explained above that we can write $\varphi = \pi_\zeta(\widehat{\varphi})$ and $\widehat{\pi_\zeta(\phi)} = \phi$. For any $\varphi \in C^*_r(\mathcal{Z}, \zeta)$ and $f \in l^2(\mathcal{Z})$. Let $A \in \mathcal{K}^{\mathcal{F}}_\infty$ and $\varphi \in \mathfrak{H}_0 \otimes C^*_r(\mathcal{Z}, \zeta)$.

$$\mathcal{F}^\zeta(A\mathcal{F}^{\zeta,*}f) = \sum_{x,y\in\mathcal{Z}} \overline{\zeta(x, -x+y)} A(-x+y)\widehat{f}(y) \otimes \Lambda_\zeta(x) \tag{7.30}$$

$$= \sum_{x,y\in\mathcal{Z}} A(y)(\mathcal{R}_\zeta(y)\widehat{f})(x) \otimes \Lambda_\zeta(x) \ . \tag{7.31}$$

Using the cocycle relation we get that

$$\sum_{x,y\in\mathcal{Z}} \overline{\zeta(x, -x+y)} A(-x+y)\widehat{f}(y) \otimes \Lambda_\zeta(x) \tag{7.32}$$

$$= \sum_{x,y\in\mathcal{Z}} K(x,y)A(-x+y)\widehat{f}(y) \otimes \Lambda_\zeta(y)\Lambda^*_\zeta(-x+y) \ , \tag{7.33}$$

where $K(x,y) = \overline{\zeta(x, -x+y)\zeta(y, -y+x)}\zeta(-x+y, -y+x)$. Performing a change of variable in the sum over x we get

$$\sum_{x,y\in\mathcal{Z}} K(x,y)A(-x+y)\widehat{f}(y) \otimes \Lambda_\zeta(y)\Lambda^*_\zeta(-x+y) \tag{7.34}$$

$$= \sum_{x,y\in\mathcal{Z}} K(y+x,y)A(-x)\widehat{f}(y) \otimes \Lambda_\zeta(y)\Lambda^*_\zeta(-x) \ . \tag{7.35}$$

But using the cocycle relation we have that

$$K(y+x,y) = \overline{\zeta(y+x,-x)\zeta(y,x)}\zeta(-x,-x) \tag{7.36}$$

$$= \overline{\zeta(y+x,\mathrm{e})\zeta(-x,-x)}\zeta(-x,-x) = 1 \ . \tag{7.37}$$

We finally get that

$$\mathcal{F}^\zeta(A\mathcal{F}^{\zeta,*}f) = \sum_{x,y\in\mathcal{Z}} A(-x)\widehat{f}(y) \otimes \Lambda_\zeta(y)\Lambda^*_\zeta(-x) \tag{7.38}$$

$$= (\sum_{x\in\mathcal{Z}} A(-x) \otimes \Lambda^*_\zeta(-x))(\sum_{y\in\mathcal{Z}} \widehat{f}(y) \otimes \Lambda_\zeta(y)) \tag{7.39}$$

$$= \widehat{A}f \ , \tag{7.40}$$

where we have used the definition of the opposite product. We thus get an explicit diagonalization of the projectively covairant operators as claimed previously.

To end this section we make some comments. First we note that if we consider the discrete setting, i.e $\mathfrak{H}_0 = \mathbb{C}$ we have that the operators are of the form $\sum_{y\in\mathcal{Z}} \mathcal{R}^{-}_{\overline{\zeta}}(y)$ which is linked to the random walk operator or more generally to the Harper operator.
Another comment is that behind this structure we have in some sense a crossed product algebra. This last structure is subject to lot of work, see by example [GI]. In fact we have that $C^*_r(\mathcal{Z}) = \mathbb{C} \ltimes \mathcal{Z}$ while for the twisted verion $C^*_r(\mathcal{Z}, \zeta) = \mathbb{C} \ltimes^{\zeta} \mathcal{Z}$. This gives a confirmation to the fact that this generalization is quite natural in the sense that if $\mathcal{Z}$ is abelian we have (see [GI]) that when the automorphism group is trivial then $\mathbb{C} \ltimes \mathcal{Z} = \mathbb{C} \otimes C_0(\mathcal{Z}^*)$.

7.2 C^*-fibration for Random-Projectively Covariant operators

The disordered media has been subjected to a lot of study over the last years. Historically we can consider this model as a generalisation of the periodic one. This is why it looks natural to try to exlpoit a generalization of the periodic formalism. Here $\mathcal{K}_\infty$ stands for random covariant operators as defined in the first section of this chapter. Whereas the approach is different we end up with a structure that is analogous to the one developed by Bellissard. But we point out that our approach provide a different point of view closer to the spirit of the work of Sunada, Brüning and Grüber [BS, Gr]. We do not use the notion of Hull as introduced by Bellissard [BES] and we work with more general group and Hilbert space.
Let $A \in \mathcal{K}^{\mathcal{F}}_\infty$ and $\varphi \in \tilde{\mathfrak{H}}_0 \otimes l^2(\mathcal{Z})$, we then have

$$\Pi A_\omega \Pi^*(\varphi)(x,\omega) = \sum_{y\in\mathcal{Z}} A(y, \tau^{-1}(x)\omega)\varphi(x+y,\omega) \ .$$

An obstruction comes from the relation

$$U_x A(\omega) U_x^* = A(\tau(x)\omega) \ .$$

As in the previous section we work now on $\tilde{\mathfrak{H}} = \int_\Omega^\oplus \mathfrak{H}(\omega) \mathrm{d}\mathbb{P}(\omega)$, and use the derived unitary representation as in the definition 4.5

$$(U_x^\tau(f))(y,\omega) = (U_x f)(y, \tau^{-1}(x)\omega)$$

Here as we will see we can construct an adapted Hilbert C^*-module using this structure coming from the discrete and euclidian case (see Bellissard et al in the 2-dimensional euclidian and discrete setting [BES]).
We recall how to construct the crossed-product group C^*-algebra. We first consider the product $\Omega \times \mathcal{Z}$ and consider the continious compactly supported functions $\mathcal{K}(\Omega \times \mathcal{Z})$. After that we consider the following product between elements of this functional space: for $f, g \in \mathcal{K}(\Omega \times \mathcal{Z})$ we define the product

$$f *^{\tau} g(\omega, x) := \sum_{y \in \mathcal{Z}} f(\omega, y) g(\tau^{-1}(y)\omega, x - y)$$

and the involution $f^*(\omega, x) := \overline{f}(\tau^{-1}(x)\omega, -x)$. The closure of continuous compactly supported functions with respect to the supremum norm of this algebra is by definition the crossed-product C^*-group algebra $C(\Omega) \ltimes_{\tau} \mathcal{Z}$.
When working in the twisted case we have

$$f *^{\tau}_{\zeta} g(\omega, x) := \sum_{y \in \mathcal{Z}} f(\omega, y) \zeta(y, x - y) g(\tau^{-1}(y)\omega, x - y)$$

and the involution $f^*(\omega, x) := \overline{\zeta}(x, -x)\overline{f}(\tau^{-1}(x)\omega, -x)$. We note that similarly to the above construction we can use $*$-representation to define this algebra. Let the random translation as defined previously

$$(\mathcal{W}(x)f)(\omega)) := f(\tau^{-1}(x)\omega) \ .$$

Define, for $f \in L^1(\Omega \times \mathcal{Z})$,

$$\pi^{\tau}_{\zeta}(f) = \sum_{x \in \mathcal{Z}} f(\omega, x) \mathcal{W}(x) \otimes \Lambda_{\zeta}(x) \in C(\Omega) \ltimes^{\zeta}_{\tau} \mathcal{Z} \ .$$

Then we have (exactly as in the description of the twisted group algebra) that $\pi^{\zeta}_{\tau}(f)g = f *^{\tau}_{\zeta} g$.
In that case we see that for $f \in \mathfrak{H}_0 \otimes L^2(\Omega \times \mathcal{Z})$ we have

$$\Pi(U^{\tau}_{z} \Pi^* f)(y, \omega) = (\mathcal{W}(x) \otimes \Lambda_{\zeta}(x) f)(\omega, y) \qquad (7.41)$$

Writing $\Lambda^{\tau}_{\zeta}(x) := \mathcal{W}(x) \otimes \Lambda_{\zeta}(x)$, we have the following diagram

$$\begin{array}{ccc} \tilde{\mathfrak{H}} & \longrightarrow & \mathfrak{H}_0 \otimes L^2(\Omega \times \mathcal{Z}) \\ \downarrow U_x & & \downarrow id_{\mathfrak{H}_0} \otimes \Lambda^{\tau}_{\zeta}(x) \\ \tilde{\mathfrak{H}} & \longrightarrow & \mathfrak{H}_0 \otimes L^2(\Omega \times \mathcal{Z}) \end{array}$$

We denote by $\mathcal{R}^\tau_\zeta(x) = \mathcal{W}(x) \otimes \mathcal{R}_\zeta(x)$ the right analogue. We have only to remplace the crossed product algebra on $\mathbb{C}$ (which is nothing else that $C^*_r(\mathcal{Z},\zeta)$) by the crossed product algebra on $C(\Omega)$. This diagram suggests to a new conjugation. We denote Π_τ the usual map but where we interwine the usual U_x with the derived one U^τ_x. In that case we get for $A \in \mathcal{K}^{\mathcal{F}}_\infty$ and $\varphi \in L^2(\Omega \times \mathcal{Z})$ that

$$\Pi^\tau(A\Pi^{\tau\ *}(\varphi))(\omega, x) = \sum_{y\in\mathcal{Z}} A(\omega, y)(\mathcal{R}^\tau_\zeta(y)\varphi)(\omega, x)$$

Remark 35. *In the discrete case (that is $\mathfrak{H}_0 = \mathbb{C}$) we recover the structure studied by Bellissard et al . [BES].*

We note that in the abelian untwisted case we recover the diagonalization along $C(\mathcal{Z}^*)$. We define as previously $\langle\varphi, \phi\rangle(r) := \mathbb{E}\langle\mathcal{F}(\varphi)(r, \omega), \mathcal{F}(\phi)(r, \omega)\rangle$ and we have that

$$\mathbb{E}\langle\mathcal{F}(\varphi)(r, \omega), \mathcal{F}(\phi)(r, \omega)\rangle = \sum_{x,y\in\mathcal{Z}} \langle r, x - y\rangle\mathbb{E}\langle\varphi(\omega), \mathcal{W}_{x-y} \otimes e_{x-y}\phi(\omega)\rangle_{\mathfrak{H}_0}\ .$$

Using the character relations we get

$$\int_{\mathcal{Z}^*} \langle\varphi, \phi\rangle(r) = \langle\varphi, \phi\rangle_{\tilde{\mathfrak{H}}}$$

, and consequently the

Proposition 7.7. *Assume that $\mathcal{Z}$ is abelian and the unitary representation is commutative (in the sense of untwisted). Then the conjugation by Π^τ is a unitary isomorphism which sends random covariant operators on the decomposables operators on $\int^\oplus_{\mathcal{Z}^*} \tilde{\mathfrak{H}}(r)\mathrm{d}r$.*

We can now easily adapt the scheme of the previous section to the case where the group is non-abelian and the representation is twisted.
We set the following notations $\mathcal{A} := C(\Omega) \ltimes^\zeta_\tau \mathcal{Z}$, $\mathcal{M} = \mathfrak{H}_0 \otimes \mathcal{A}$ and $\tilde{\delta}_x := \mathbf{1}_\Omega \otimes \delta_x$ for any $x \in \mathcal{Z}$ and we recall that we assume that Ω is compact and that the group have the Haagerup property with subexponentially H-growth. These assumptions ensure the convergence of the series for a dense subset containing the compactly supported functions. Then for $a \in \mathcal{A}$, we define the following map $\widehat{a} := a\ \tilde{\delta}$.
We then recover the relation $\Lambda^\tau_\zeta(x)\tilde{\delta} = \mathbf{1}_\Omega(\tau^{-1}(x)\omega) \otimes \delta_x = \tilde{\delta}_x$.

For $f \in \mathrm{L}^2(\Omega \times \mathcal{Z})$ we have that

$$\widehat{\pi_\zeta^\tau(f)} = \sum_{x\in\mathcal{Z}} f(\omega,x)\Lambda_\zeta^\tau(x)\delta^\tau = \sum_{x\in\mathcal{Z}} f(\omega,x)\mathbf{1}_\Omega \otimes \delta_x = f \ , \tag{7.42}$$

$$\pi_\zeta^\tau(\widehat{a}) = \sum_{x\in\mathcal{Z}} \widehat{a}(x)\Lambda_\zeta^\tau(x) = \sum_{x\in\mathcal{Z}} \langle a\tilde{\delta}, \tilde{\delta}_x\rangle \Lambda_\zeta^\tau(x) = \sum_{x\in\mathcal{Z}} \langle a\tilde{\delta}, \Lambda_\zeta^\tau(x)\tilde{\delta}\rangle \Lambda_\zeta^\tau(x) = a \ . \tag{7.43}$$

The second relation must be understood in the sense that if the series is convergent then it is equal to a.
Then we define for $\varphi \in \tilde{\mathfrak{H}}$, $\mathcal{F}_\tau^\zeta(\varphi) = \sum_{x\in\mathcal{Z}} \Pi_x^\tau(\varphi) \otimes \Lambda_\zeta^\tau$. We recover all the results than in the non-random case, namely the Hilbert-module structure and the explicit diagonalization.
We have

$$\begin{aligned} \langle \varphi, \phi\rangle_\zeta^\tau &:= \langle \mathcal{F}_\zeta^\tau(\varphi), \mathcal{F}_\zeta^\tau(\phi)\rangle_{\mathcal{M}} \\ &= \sum_{x,y\in\mathcal{Z}} \langle \phi(\omega), e_{y-x}\varphi(\omega)\rangle_{\mathfrak{H}_0} \Lambda_\zeta^\tau(y-x) \ \in C(\Omega) \ltimes_\tau^\zeta \mathcal{Z} \end{aligned} \tag{7.44}$$

We define the associated state as

$$\nu^\tau(\cdot) := \langle \ \cdot \ \tilde{\delta}, \tilde{\delta}\rangle_{L^2(\Omega\times\mathcal{Z})}$$

We have that

1. $\nu^\tau(id) = \mathbb{E}(\mathbf{1}_\Omega) \sum_{x\in\mathcal{Z}} \delta(x) = \mathbb{P}(\Omega) = 1$
2. $\nu^\tau(\pi_\zeta^\tau(f)) = \mathbb{E}(f(\omega,0))$

Therefore ν^τ define a faithful tracial state and moreover we have

$$\nu^\tau(\langle \varphi, \phi\rangle_\zeta^\tau) = \langle \varphi, \phi\rangle_{\tilde{\mathfrak{H}}}$$

In conclusion, we finishe the study of the structure of random projectively covariant operators by the

Theorem 7.8. *The inner product $\langle \cdot, \cdot\rangle_{\mathcal{M}}$ define a right pre-Hilbert $\mathcal{A}$-module and its closure under the associated norm is isomorphic to the right Hilbert-module $\mathcal{M}$ by the conjuguation $\mathcal{F}_\zeta^\tau$ such that the GNS representation is isomorphic to $\tilde{\mathfrak{H}}$. Moreover we have obviously that*

$$\mathcal{F}_\zeta^\tau(A_\omega)\mathcal{F}_\zeta^{\tau\ *} = \left(\sum_{x\in\mathcal{Z}} A_\omega(-x) \otimes \Lambda_\zeta^{\tau\ *}(-x)\right) \tag{7.45}$$

$$\textit{and } \mathcal{K}_\infty \simeq \mathcal{B}(\mathfrak{H}_0) \otimes \{C(\Omega) \ltimes_\tau^\zeta \mathcal{Z}\}' \tag{7.46}$$

Concluding remark

The main goal of this discussion was to extend the linear response linear theory to a more general situation than in the Ch1. ([DG]).
In a work in preparation we extend this analysis to the case of twisted groupoïd in place of twisted group structure. This provides a generalization to the quasi-periodic structure where exactly the same analysis takes place. Using the work of [R3] on the twisted groupoid which appears as a central object between the algebraic and the dynamic point of view.
The pinciple of diagonalization appears as essential in many transport problems. The study of transport and spectral problem associated to it is an another motivation coming from this discussion.
Secondly we will study the spectral flow associated to this diagonalization using the recent advances in this domain in the algebraic contextas in for example [ACS] to get spectral results.

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