Le théorème de concentration et la formule des points fixes de Lefschetz en géométrie d'Arakelov

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Le théorème de concentration et la formule des points fixes de Lefschetz en géométrie d'Arakelov

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Résumé. Dans les années quatre-vingts dix du siècle dernier, R. W. Thomason a démontré un théorème de concentration pour la K-théorie équivariante algébrique sur les schémas munis d'une action d'un groupe algébrique G diagonalisable. Comme d'habitude, un tel théorème entraîne une formule des points fixes de type Lefschetz qui permet de calculer la caractéristique d'Euler-Poincaré équivariante d'un G-faisceau cohérent sur un G-schéma propre en termes d'une caractéristique sur le sous-schéma des points fixes. Le but de cette thèse est de généraliser les résultats de R. W. Thomason dans le contexte de la géométrie d'Arakelov. Dans ce travail, nous considérons les schémas arithmétiques au sens de Gillet-Soulé et nous tout d'abord démontrons un analogue arithmétique du théorème de concentration pour les schémas arithmétiques munis d'une action du schéma en groupe diagonalisable associé à $\mathbb{Z}/n\mathbb{Z}$. La démonstration résulte du théorème de concentration algébrique joint à des arguments analytiques. Dans le dernier chapitre, nous formulons et démontrons deux types de formules de Lefschetz arithmétiques. Ces deux formules donnent une réponse positive à deux conjectures énoncées par K. Köhler, V. Maillot et D. Rössler.

Mots clefs : théorème de concentration, formule des points fixes de type Lefschetz, schéma arithmétique, géométrie d'Arakelov.

Concentration theorem and fixed point formula of Lefschetz type in Arakelov geometry

Abstract. In the nineties of the last century, R. W. Thomason proved a concentration theorem for the algebraic equivariant K-theory on the schemes which are endowed with an action of a diagonalisable group scheme G. As usual, such a concentration theorem induces a fixed point formula of Lefschetz type which can be used to calculate the equivariant Euler-Poincaré characteristic of a coherent G-sheaf on a proper G-scheme in terms of a characteristic on the fixed point subscheme. It is the aim of this thesis to generalize R. W. Thomason's results to the context of Arakelov geometry. In this work, we consider the arithmetic schemes in the sense of Gillet-Soulé and we first prove an arithmetic analogue of the concentration theorem for the arithmetic schemes endowed with an action of the diagonalisable group scheme associated to $\mathbb{Z}/n\mathbb{Z}$. The proof is a combination of the algebraic concentration theorem and some analytic arguments. In the last chapter, we formulate and prove two kinds of arithmetic Lefschetz formulae. These two formulae give a positive answer to two conjectures made by K. Köhler, V. Maillot and D. Rössler.

Keywords: concentration theorem, fixed point formula of Lefschetz type, arithmetic scheme, Arakelov geometry.

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Introduction

Le but principal de cette thèse est de démontrer deux types de formules des points fixes pour les schémas munis d'une action d'un schéma en groupe diagonalisable, dans le contexte de la géométrie d'Arakelov. Comme d'habitude, ces deux formules des points fixes peuvent être regardées comme des solutions de deux problèmes de Riemann-Roch. Tout d'abord, nous rappelons brièvement l'histoire de l'étude des formules des points fixes de Lefschetz et des problèmes de Lefschetz-Riemann-Roch relatifs.

Soit k un corps algébriquement clos et soit n un entier qui est premier à la caractéristique de k. Une k-variété projective X munie d'un automorphisme g d'ordre n s'appellera une variété équivariante. Un faisceau cohérent équivariant sur X est un faisceau cohérent F sur X avec un homomorphisme $\phi : g^*F \to F$. Il est clair que cet homomorphisme induit une famille d'endomorphismes $H^i(\phi)$ sur l'espaces des cohomologies $H^i(X, F)$.

Une formule des points fixes de Lefschetz classique est une formule qui donne une expression pour la somme alternée des traces des $H^i(\phi)$, en terme des contributions provenant de chaque composante de la sous-variété des points fixes X_g. D'autre part, de manière générale, un théorème de Lefschetz-Riemann-Roch est un diagramme commutatif qui décrit la compatibilité, dans la K-théorie équivariante, de l'application de restriction d'une variété équivariante à la sous-variété des points fixes et d'un morphisme équivariant entre deux variétés équivariantes. Ce diagramme commutatif peut être regardé comme une généralisation de type Grothendieck de la formule des points fixes. En effet, si l'on choisit un point comme la variété de base dans un tel diagramme commutatif, on peut obtenir la formule des points fixes de Lefschetz ordinaire.

Supposons que nous sommes dans le cadre ci-dessus. Soit X une variété équivariante. Si X n'est pas singulière, P. Donovan a démontré un tel théorème de Lefschetz-Riemann-Roch dans [Do]. Sa démonstration s'appuie sur quelque méthodes utilisées dans l'article de A. Borel et J. P. Serre sur le théorème de Grothendieck-Riemann-Roch (cf. [BS]). Dans [BFQ], P. Baum, W. Fulton et G. Quart ont généralisé le théorème de Donovan dans le cas où les variétés singulières sont considérées. L'étape clef dans leur démonstration s'appuie fortement sur une méthode élégante, qui s'appelle la déformation au cône normal. Désignons par $G_0(X, g)$ (resp. $K_0(X, g)$) le K-groupe algébrique de Quillen associé à la catégorie des faisceaux cohérents équivariants (resp. fibrés vectoriels des rangs finis) sur X. Il est bien connu que $K_0(\mathrm{Pt}, g)$ est isomorphe à l'anneau en groupe $\mathbb{Z}[k]$ et que $G_0(X, g)$ (resp. $K_0(X, g)$) a une structure de $K_0(\mathrm{Pt}, g)$-module (resp. $K_0(\mathrm{Pt}, g)$-

algèbre) naturelle. Soit f un morphisme projectif équivariant entre deux variétés équivariantes X et Y. La propreté du morphisme f implique qu'il y a une application f_* raisonable de $G_0(X,g)$ à $G_0(Y,g)$. Soit $\mathcal{R}$ une $K_0(\mathrm{Pt},g)$-algèbre plate dans laquelle $1-\zeta$ est inversible pour chaque racine n-ième non triviale de l'unité ζ dans k. Alors, le résultat principal de Baum, Fulton et Quart dit qu'il y a une famille d'homomorphismes de groupes $L.$ entre K-groupes pour lesquels nous avons le diagramme commutatif suivant :

$$\begin{array}{ccc} G_0(X,g) & \xrightarrow{L.} & G_0(X_g,g)\otimes_{\mathbb{Z}[k]}\mathcal{R} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_{g*}} \\ G_0(Y,g) & \xrightarrow{L.} & G_0(Y_g,g)\otimes_{\mathbb{Z}[k]}\mathcal{R} \end{array}$$

Supposons que Z est une variété équivariante nonsingulière telle qu'il y a une immersion fermée i équivariante de X à Z. Alors pour tout faisceau cohérent équivariant E sur X, l'homomorphisme $L.$ est exactement donné par la formule

$$L.(E) = \lambda_{-1}^{-1}(N_{Z/Z_g}^{\vee}) \cdot \sum_j (-1)^j \mathrm{Tor}^j_{\mathcal{O}_Z}(i_*E, \mathcal{O}_{Z_g})$$

où N_{Z/Z_g} est le fibré normal de Z_g dans Z et $\lambda_{-1}(N_{Z/Z_g}^{\vee}) := \sum (-1)^j \wedge^j (N_{Z/Z_g}^{\vee})$.

À la suite de Grothendieck, il est naturel de se demander comment généraliser le résultat de Baum, Fulton et Quart dans le cadre de la géométrie algèbrique schématique. Nous voulons souligner qu'il est toujours possible d'utiliser la déformation au cône normal pour le faire. Dans le cadre de cette généralisation, X et Y sont les schémas noethériens munis d'une action projective d'un schéma en groupe μ_n diagonalisable associé à $\mathbb{Z}/n\mathbb{Z}$. Notons qu'une action de μ_n sur un schéma X est une application $m_X : \mu_n \times X \to X$ qui satisfait quelque propriétés de compatibilité. Désignons par p_X la projection de $\mu_n \times X \to X$. Une action de μ_n sur un $\mathcal{O}_X$-module cohérent E est un isomorphisme $m_E : p_X^* E \to m_X^* E$ qui satisfait certains propriétés d'associativité. Nous référons à [Koe] et [KR1, Section 2] pour la théorie d'action de schéma en groupe nous parlons de ici.

Dans [Tho], R. W. Thomason a généralisé le résultat de Baum, Fulton et Quart au cadre des schémas en utilisant une méthode différente et il a supprimé la condition de projectivité. La stratégie de Thomason est de démontrer un théorème de concentration algébrique pour la suite de localisation de Quillen pour les K-groupes équivariants supérieurs. Plus précisément, soit D un anneau noethérien intègre et soit μ_n le schéma en groupe diagonalisable sur D associé à $\mathbb{Z}/n\mathbb{Z}$. Désignons l'anneau $K_0(\mathbb{Z})[\mathbb{Z}/n\mathbb{Z}] \cong \mathbb{Z}[T]/(1-T^n)$ par $R(\mu_n)$. Nous considérons l'idéal ρ premier dans $R(\mu_n)$ qui est le noyau du morphisme canonique suivant :

$$\mathbb{Z}[T]/(1-T^n) \to \mathbb{Z}[T]/(\Phi_n)$$

où Φ_n est le polynôme cyclotomique n-ième. Soit X un schéma séparé, de type fini et μ_n-équivariant sur D, donc le groupe $G_0(X,\mu_n)$ (resp. $K_0(X,\mu_n)$) a une structure de $R(\mu_n)$-module (resp. $R(\mu_n)$-algèbre) naturelle parce que $K_0(D,\mu_n) \cong K_0(D)[T]/(1-T^n)$.

Désignons par i l'immersion fermée de X_{μ_n} à X. Le théorème de concentration dit qu'il y a un homomorphisme de groupe i_* de $G_0(X_{\mu_n}, \mu_n)_\rho$ à $G_0(X, \mu_n)_\rho$ qui est un isomorphisme. Par ailleurs, si X est régulier, l'inverse de i_* est donné par $\lambda_{-1}^{-1}(N_{X/X_{\mu_n}}^\vee) \cdot i^*$ où $N_{X/X_{\mu_n}}$ est le fibré normal de X_{μ_n} dans X. Ce théorème de concentration peut être utilisé pour démontrer une formule des points fixes de Lefschetz singulière qui est une extension du résultat de Baum, Fulton et Quart en général. L'approche de Thomason n'a rien à voir avec la construction de la déformation au cône normal et la localisation dans le théorème de Thomason est légèrement plus faible que celle apparaissant dans le théorème de Baum, Fulton et Quart au sens où le complément de l'idéal ρ dans $R(\mu_n)$ n'est pas l'algèbre la plus petite dans laquelle touts éléments $1 - T^k$ $(k = 1, \ldots, n-1)$ sont inversibles. Si l'on choisit exactement le complément de l'idéal ρ dans $R(\mu_n)$ comme l'algèbre $\mathcal{R}$, alors ces deux localisations sont égales.

La géométrie d'Arakelov est une extension de la géométrie algébrique dans le cadre arithmétique, où on peut considérer les plongements d'un corps de nombres K dans les corps archimédiens $\mathbb{R}$ et $\mathbb{C}$ (i.e. les places à l'infini de K) sur le même plan que les idéaux premiers d'anneau des entiers $\mathcal{O}_K$ de K et que les plongements de K dans les corps p-adiques qui leur sont attachés.

Dans [KR1], K. Köhler et D. Rössler ont généralisé le cas régulier du résultat de Baum, Fulton et Quart dans le contexte de la géométrie d'Arakelov. À chaque schéma arithmétique X, μ_n-équivariant et régulier, ils ont associé un K_0-groupe $\widehat{K_0}(X, \mu_n)$ arithmétique équivariant qui contient une certaine classe des formes lisses comme donnée analytique. Ce K_0-groupe arithmétique équivariant a aussi une structure d'anneau et il peut être équipé d'une structure de $R(\mu_n)$-algèbre. Soit $\overline{N}_{X/X_{\mu_n}}$ le fibré normal à l'égard de l'immersion régulière $X_{\mu_n} \hookrightarrow X$ muni d'une métrique hermitienne μ_n-invariante, le théorème principal dans [KR1] dit que l'élément $\lambda_{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)$ est inversible dans $\widehat{K_0}(X_{\mu_n}, \mu_n) \otimes_{R(\mu_n)} \mathcal{R}$ et nous avons le diagramme commutatif suivant :

$$\begin{array}{ccc}
\widehat{K_0}(X, \mu_n) & \xrightarrow{\Lambda_R(f)^{-1} \cdot \tau} & \widehat{K_0}(X_{\mu_n}, \mu_n) \otimes_{R(\mu_n)} \mathcal{R} \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_{\mu_n *}} \\
\widehat{K_0}(D, \mu_n) & \xrightarrow{\iota} & \widehat{K_0}(D, \mu_n) \otimes_{R(\mu_n)} \mathcal{R}
\end{array}$$

où $\Lambda_R(f) := \lambda_{-1}(\overline{N}_{X/X_{\mu_n}}^\vee) \cdot (1 + R_g(N_{X/X_{\mu_n}}))$ et τ est l'application de la restriction. Ici $R_g(\cdot)$ est le R-genre équivariant, ces deux applications f_* et $f_{\mu_n *}$ sont définies via une donnée analytique très importante qui s'appelle la torsion analytique équivariante (cf. [Bi1]). La stratégie de Köhler et Rössler pour démontrer un tel théorème arithmétique de Lefschetz-Riemann-Roch est de démontrer un analogue de ce théorème pour les immersions fermées équivariantes par la construction de la déformation au cône normal. Après cela, ils décomposent le morphisme f comme $f = p \circ j$ où j est une immersion fermée de X dans un certain espace projectif $\mathbb{P}_D^r$ et p est un morphisme lisse de $\mathbb{P}_D^r$ à Spec(D). Donc le théorème dans la situation générale découle d'une étude du comportement du terme d'erreur sous les morphismes j et p.

Après l'extension de la torsion analytique équivariante à la forme de torsion analytique équivariante supérieure par X. Ma (cf. [Ma1]), dans [KR2], Köhler et Rössler ont conjecturé un analogue de [KR1, Theorem 4.4] dans le cadre relatif. Nous démontrerons cette conjecture dans cette thèse, c'est le premier résultat principal. Notre méthode est similaire à celle de Thomason et nous prouvons d'abord qu'il y a un théorème de concentration arithmétique en géométrie d'Arakelov. La formule des points fixes de Lefschetz découle de ce théorème de concentration arithmétique. Notre approche n'a rien non plus à voir avec la construction de la déformation au cône normal, mais elle est valide pour les schémas arithmétique réguliers seulement.

C'est une question naturelle de se demander s'il est possible de construire une $\widehat{G_0}$-théorie en général et de prouver une formule des points fixes de Lefschetz pour les schémas arithmétiques singulières qui est entièrement un analogue de la formule de Thomason dans le cadre de la géométrie d'Arakelov. La réponse est Oui, et c'est le deuxième résultat principal dans cette thèse. Pour le faire, on a besoin d'un théorème d'élimination dans la $\widehat{G_0}$-théorie qui peut être regardé comme une extension de la formule des points fixes pour l'immersions fermées de Köhler et Rössler au cas singulier. Soit X et Y deux schémas arithmétiques équivariants singuliers dont les fibres génériques sont lisses, et soit $f : X \to Y$ un morphisme qui est lisse sur le corps des nombres complexes. Supposons que la μ_n-action sur Y est triviale et que f a une décomposition $f = h \circ i$, où i est une immersion fermée équivariante de X dans un certain schéma arithmétique Z régulier et $h : Z \to Y$ est μ_n-équivariant et également lisse sur le corps des nombres complexes. Soit $\overline{\eta}$ un faisceau équivariant hermitien sur X. Nous référons au Chapitre V pour les définitions des notations ci-dessous. La formule des points fixes de Lefschetz pour les schémas arithmétiques éventuellement singuliers sur les fibres finies est l'égalité ci-dessous, vérifiée dans le groupe $\widehat{G_0}(Y, \mu_n, \emptyset)_\rho$:

$$\begin{aligned} f_*(\overline{\eta}) =& f^Z_{\mu_n *}(i^*_{\mu_n}(\lambda_{-1}^{-1}(\overline{N}^\vee_{Z/Z_{\mu_n}})) \cdot \sum_k (-1)^k \mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta}, \overline{\mathcal{O}}_{Z_{\mu_n}})) \\ &+ \int_{X_g/Y} \mathrm{Td}(Tf_g, \omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}}, \omega^X)\mathrm{Td}_g^{-1}(\overline{F}) \\ &- \int_{X_g/Y} \mathrm{Td}_g(Tf)\mathrm{ch}_g(\eta)R_g(N_{X/X_g}) \\ &+ \int_{X_g/Y} \widetilde{\mathrm{Td}}(Tf_g, \omega^X, \omega^Z_X)\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\mathrm{Td}_g^{-1}(\overline{F}). \end{aligned}$$

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Chapter I

Algebro-geometric preliminaries

In this chapter, for the convenience of the reader, we roughly recall some parts of the algebraic equivariant K-theory which were mainly developed by R. W. Thomason. We would like to use this as an opportunity to introduce the background of our work in this thesis. Until the end of this thesis, all schemes will be Noetherian and all vector bundles will be of finite rank.

1 Algebraic concentration theorem

Let D be an integral Noetherian ring. In this section we fix $S := \mathrm{Spec}(D)$ as the base scheme. Let n be a positive integer, we shall denote by μ_n the diagonalisable group scheme over S associated to the cyclic group $\mathbb{Z}/n\mathbb{Z}$. By a μ_n-equivariant scheme we understand a separable and of finite type scheme over S which admits a μ_n-action.

Let X be a μ_n-equivariant scheme, we consider the category of coherent $\mathcal{O}_X$-modules endowed with an action of μ_n which are compatible with the μ_n-structure of X. According to Quillen, to this category we may associate a graded abelian group $G_*(X, \mu_n)$ which is called the higher algebraic equivariant G-group. If one replaces the μ_n-equivariant coherent $\mathcal{O}_X$-modules by the μ_n-equivariant vector bundles, one gets the higher algebraic equivariant K-group $K_*(X, \mu_n)$. It is well known that the tensor product of μ_n-equivariant vector bundles induces a graded ring structure on $K_*(X, \mu_n)$ and a graded $K_*(X, \mu_n)$-module structure on $G_*(X, \mu_n)$. Notice that if X is regular, then the natural morphism from $K_*(X, \mu_n)$ to $G_*(X, \mu_n)$ is an isomorphism.

Denote by X_{μ_n} the fixed point subscheme of X under the action of μ_n, then the closed immersion $i : X_{\mu_n} \hookrightarrow X$ induces two group homomorphisms $i_* : G_*(X_{\mu_n}, \mu_n) \rightarrow G_*(X, \mu_n)$ and $i_* : K_*(X_{\mu_n}, \mu_n) \rightarrow K_*(X, \mu_n)$ which satisfy the projection formula. According to [SGA3, I 4.4], μ_n is the pull-back of a unique diagonalisable group scheme over $\mathbb{Z}$ associated to the same group, this group scheme will be still denoted by μ_n. Write $R(\mu_n)$ for the group $K_0(\mathbb{Z}, \mu_n)$ which is isomorphic to $\mathbb{Z}[T]/(1 - T^n)$. Let ρ be the prime ideal of $R(\mu_n)$ which is defined to be the kernel of the following canonical morphism

$$\mathbb{Z}[T]/(1 - T^n) \rightarrow \mathbb{Z}[T]/(\Phi_n)$$

where Φ_n stands for the n-th cyclotomic polynomial. The prime ideal ρ is chosen to satisfy the condition that the localization $R(\mu_n)_\rho$ is a $R(\mu_n)$-algebra in which the elements $1-T^k$ from $k=1$ to $n-1$ are all invertible. This condition plays a crucial role in the proof of the concentration theorem. If the μ_n-equivariant scheme X is regular, then X_{μ_n} is also regular. We shall write $\lambda_{-1}(N_{X/X_{\mu_n}}^\vee)$ for the alternating sum $\sum(-1)^j\wedge^j N_{X/X_{\mu_n}}^\vee$ where $N_{X/X_{\mu_n}}$ stands for the normal bundle associated to the regular immersion i. Then the algebraic concentration theorem in [Tho] can be described as the following.

Theorem I.1. *(Thomason) Let notations and assumptions be as above.*

- *The $R(\mu_n)_\rho$-module morphism $i_*: G_*(X_{\mu_n},\mu_n)_\rho\to G_*(X,\mu_n)_\rho$ is actually an isomorphism.*
- *If X is regular, then $\lambda_{-1}(N_{X/X_{\mu_n}}^\vee)$ is invertible in $G_*(X_{\mu_n},\mu_n)_\rho$ and the inverse map of i_* is given by $\lambda_{-1}^{-1}(N_{X/X_{\mu_n}}^\vee)\cdot i^*$.*

The proof of Thomason's algebraic concentration theorem can be split into three steps. The first step is to show that $G_*(U,\mu_n)_\rho\cong 0$ if U has no fixed point, then the claim that $i_*: G_*(X_{\mu_n},\mu_n)_\rho\to G_*(X,\mu_n)_\rho$ is an isomorphism follows from Quillen's localization sequence for higher equivariant K-theory, see [Tho, Théorème 2.1]. The second step is to show that $\lambda_{-1}(N_{X/X_{\mu_n}}^\vee)$ is invertible in $G_*(X_{\mu_n},\mu_n)_\rho$ if X is regular (cf. [Tho, Lemme 3.2]). The last step is a direct computation using the projection formula for equivariant K-theory to show that the inverse map of i_* is exactly $\lambda_{-1}^{-1}(N_{X/X_{\mu_n}}^\vee)\cdot i^*$ (cf. [Tho, Lemme 3.3]). The condition that the localization $R(\mu_n)_\rho$ is a $R(\mu_n)$-algebra in which the elements $1-T^k$ from $k=1$ to $n-1$ are all invertible was used in the first and the second step.

Since in the rest of this thesis we only consider G_0 and K_0-groups, it is helpful to introduce another definition of G_0 and K_0-groups due to Grothendieck.

Definition I.2. Let X be a μ_n-equivariant scheme. The Grothendieck group $G_0(X,\mu_n)$ (resp. $K_0(X,\mu_n)$) is the free abelian group generated by the isomorphism classes of μ_n-equivariant coherent $\mathcal{O}_X$-modules (resp. μ_n-equivariant vector bundles) on X, together with the relation :

- if $0\to E'\to E\to E''\to 0$ is a short exact sequence, then $E'-E+E''=0$.

Let A be a ring which is contained in $\mathbb{C}$, the following natural generalization of Definition I.2 is more useful in the arithmetic case.

Definition I.3. Let X be a μ_n-equivariant scheme. The Grothendieck group $G_{0,A}(X,\mu_n)$ (resp. $K_{0,A}(X,\mu_n)$) is the free A-module generated by the isomorphism classes of μ_n-equivariant coherent $\mathcal{O}_X$-modules (resp. μ_n-equivariant vector bundles) on X, together with the relation :

- if $0\to E'\to E\to E''\to 0$ is a short exact sequence, then $E'-E+E''=0$.

It is clear that $G_{0,A}(X,\mu_n)$ (resp. $K_{0,A}(X,\mu_n)$) is isomorphic to $G_0(X,\mu_n)\otimes_{\mathbb{Z}} A$ (resp. $K_0(X,\mu_n)\otimes_{\mathbb{Z}} A$), then the algebraic concentration theorem has an immediate corollary.

Corollary I.4. *Let notations and assumptions be as above.*

- *The $R(\mu_n)_\rho$-module morphism $i_*: G_{0,A}(X_{\mu_n},\mu_n)_\rho \to G_{0,A}(X,\mu_n)_\rho$ is actually an isomorphism.*
- *If X is regular, then $\lambda_{-1}(N^\vee_{X/X_{\mu_n}})$ is invertible in $G_{0,A}(X_{\mu_n},\mu_n)_\rho$ and the inverse map of i_* is given by $\lambda_{-1}^{-1}(N^\vee_{X/X_{\mu_n}})\cdot i^*$.*

2 Thomason's fixed point formulae of Lefschetz type

Let X and Y be two μ_n-equivariant schemes over S. Suppose that $f: X\to Y$ is a μ_n-equivariant proper morphism. Then the properness of the morphism f allows us to define a reasonable push-forward map $f_*: G_0(X,\mu_n)\to G_0(Y,\mu_n)$ which sends the class of a μ_n-equivariant coherent $\mathcal{O}_X$-module $[\mathcal{F}]$ to the alternating sum of its higher direct images $\sum(-1)^k[R^kf_*\mathcal{F}]$. This push-forward map is a well-defined group homomorphism and it satisfies the projection formula. We denote by f_{μ_n} the restriction of f to the fixed point subschemes. The first type of Thomason's fixed point formula can be described as follows.

Theorem I.5. *(Thomason) If the μ_n-equivariant schemes X and Y are both regular, then the identity*

$$f_*([\mathcal{F}]) = f_{\mu_n *}(\lambda_{-1}^{-1}(N^\vee_{X/X_{\mu_n}})\cdot[\mathcal{F}]\mid_{X_{\mu_n}})$$

holds in $K_0(Y,\mu_n)_\rho \cong K_0(Y_{\mu_n},\mu_n)_\rho$.

Proof. For simplicity, we denote by i the regular immersion $X_{\mu_n}\hookrightarrow X$. Then by the algebraic concentration theorem, in $K_0(Y,\mu_n)_\rho\cong K_0(Y_{\mu_n},\mu_n)_\rho$ we have

$$\begin{aligned} f_*([\mathcal{F}]) &= f_*i_*i_*^{-1}([\mathcal{F}]) \\ &= f_*i_*(\lambda_{-1}^{-1}(N^\vee_{X/X_{\mu_n}})\cdot i^*[\mathcal{F}]) \\ &= f_{\mu_n *}(\lambda_{-1}^{-1}(N^\vee_{X/X_{\mu_n}})\cdot[\mathcal{F}]\mid_{X_{\mu_n}}). \end{aligned}$$

which ends the proof. □

In the case where X and Y are not regular, we suppose that there exists a regular μ_n-equivariant scheme Z and a factorization $f = h\circ j$ such that $j: X\hookrightarrow Z$ is a μ_n-equivariant closed immersion and $h: Z\to Y$ is a μ_n-equivariant proper morphism. Then the second type of Thomason's fixed point formula is the following.

Theorem I.6. *(Thomason) Let notations and assumptions be as above. Then the identity*

$$f_*([\mathcal{F}]) = f_{\mu_n *}(j^*_{\mu_n}(\lambda_{-1}^{-1}(N^\vee_{Z/Z_{\mu_n}}))\cdot\sum(-1)^k[\mathrm{Tor}^k_{\mathcal{O}_Z}(j_*\mathcal{F},\mathcal{O}_{Z_{\mu_n}})])$$

holds in $G_0(Y,\mu_n)_\rho\cong G_0(Y_{\mu_n},\mu_n)_\rho$.

Proof. This is [Tho, Théorème 3.5]. □

We remark that Theorem I.5 is certainly a corollary of Theorem I.6 if one chooses Z to be X itself.

Chapter II

Differential-geometric preliminaries

In this chapter, we recall necessary definitions and results in differential geometry which are needed in our later arguments in Arakelov geometry. For the reason of terseness, most of the proofs will not be quoted from the original literature, we only give corresponding references.

1 Equivariant Chern-Weil theory

Let G be a compact Lie group and let M be a compact complex manifold which admits a holomorphic G-action. By an equivariant hermitian vector bundle on M, we understand a hermitian vector bundle on M which admits a G-action compatible with the G-structure of M and whose metric is G-invariant. Let $g \in G$ be an automorphism of M, we shall denote by $M_g = \{x \in M \mid g \cdot x = x\}$ the fixed point submanifold. M_g is also a compact complex manifold.

Now let $\overline{E}$ be an equivariant hermitian vector bundle on M, it is well known that the restriction of $\overline{E}$ to M_g splits as a direct sum

$$\overline{E} \mid_{M_g} = \bigoplus_{\zeta \in S^1} \overline{E}_\zeta$$

where the equivariant structure g^E of E acts on $\overline{E}_\zeta$ as multiplication by ζ. We often write $\overline{E}_g$ for $\overline{E}_1$ and call it the 0-degree part of $\overline{E} \mid_{M_g}$. As usual, $A^{p,q}(M)$ stands for the space of (p,q)-forms $\Gamma^\infty(M, \Lambda^p T^{*(1,0)}M \wedge \Lambda^q T^{*(0,1)}M)$, we define

$$\widetilde{A}(M) = \bigoplus_{p=0}^{\dim M} (A^{p,p}(M)/(\mathrm{Im}\partial + \mathrm{Im}\overline{\partial})).$$

We denote by $\Omega^{\overline{E}_\zeta} \in A^{1,1}(M_g)$ the curvature form associated to $\overline{E}_\zeta$. Let $(\phi_\zeta)_{\zeta \in S^1}$ be a family of $\mathbf{GL}(\mathbb{C})$-invariant formal power series such that $\phi_\zeta \in \mathbb{C}[[\mathbf{gl}_{\mathrm{rk}E_\zeta}(\mathbb{C})]]$ where $\mathrm{rk}E_\zeta$ stands for the rank of E_ζ which is a locally constant function on M_g. Moreover, let $\phi \in \mathbb{C}[[\bigoplus_{\zeta \in S^1} \mathbb{C}]]$ be any formal power series. We have the following definition.

Definition II.1. The way to associate a smooth form to an equivariant hermitian vector bundle $\overline{E}$ by setting

$$\phi_g(\overline{E}) := \phi((\phi_\zeta(-\frac{\Omega^{\overline{E}_\zeta}}{2\pi i}))_{\zeta\in S^1})$$

is called an g-equivariant Chern-Weil theory associated to $(\phi_\zeta)_{\zeta\in S^1}$ and ϕ. The class of $\phi_g(\overline{E})$ in $\widetilde{A}(M_g)$ is independent of the metric.

Write dd^c for the differential operator $\frac{\overline{\partial}\partial}{2\pi i}$. The theory of equivariant secondary characteristic classes is described in the following theorem.

Theorem II.2. *To every short exact sequence $\overline{\varepsilon}: 0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0$ of equivariant hermitian vector bundles on M, there is a unique way to attach a class $\widetilde{\phi}_g(\overline{\varepsilon}) \in \widetilde{A}(M_g)$ which satisfies the following three conditions :*

(i). $\widetilde{\phi}_g(\overline{\varepsilon})$ satisfies the differential equation

$$\mathrm{dd}^c\widetilde{\phi}_g(\overline{\varepsilon}) = \phi_g(\overline{E}' \oplus \overline{E}'') - \phi_g(\overline{E});$$

(ii). for every equivariant holomorphic map $f: M' \to M$, $\widetilde{\phi}_g(f^*\overline{\varepsilon}) = f_g^*\widetilde{\phi}_g(\overline{\varepsilon})$;

(iii). $\widetilde{\phi}_g(\overline{\varepsilon}) = 0$ if $\overline{\varepsilon}$ is equivariantly and orthogonally split.

Proof. Firstly note that one can carry out the principle of [BGS1, Section f.] to construct a new exact sequence of equivariant hermitian vector bundles

$$\overline{\widetilde{\varepsilon}}: \quad 0 \to \overline{E'(1)} \to \overline{\widetilde{E}} \to \overline{E}'' \to 0$$

on $M \times \mathbb{P}^1$ such that $i_0^*\overline{\widetilde{\varepsilon}}$ is isometric to $\overline{\varepsilon}$ and $i_\infty^*\overline{\widetilde{\varepsilon}}$ is equivariantly and orthogonally split. Here the projective line $\mathbb{P}^1$ carries the trivial G-action and the section i_0 (resp. i_∞) is defined by setting $i_0(x) = (x, 0)$ (resp. $i_\infty(x) = (x, \infty)$). Then one can show that an equivariant secondary characteristic class $\widetilde{\phi}_g(\overline{\varepsilon})$ which satisfies the three conditions in the statement of this theorem must be of the form

$$\widetilde{\phi}_g(\overline{\varepsilon}) = -\int_{\mathbb{P}^1} \phi_g(\widetilde{E}, h^{\widetilde{E}}) \cdot \log \mid z \mid^2 .$$

So the uniqueness has been proved. For the existence, one may take this identity as the definition of the equivariant secondary class $\widetilde{\phi}_g$, of course one should verify that this definition is independent of the choice of the metric $h^{\widetilde{E}}$ and really satisfies the three conditions above. The verification is totally the same as the non-equivariant case, one just add the subscript g to every corresponding notation.

Another way to show the existence is to use the non-equivariant secondary classes on M_g directly. We first split $\overline{\varepsilon}$ on M_g into a family of short exact sequences

$$\overline{\varepsilon}_\zeta: \quad 0 \to \overline{E}'_\zeta \to \overline{E}_\zeta \to \overline{E}''_\zeta \to 0$$

for all $\zeta \in S^1$. Using the non-equivariant secondary classes on X_g we define for $\zeta, \eta \in S^1$

$$(\widetilde{\phi_\zeta + \phi_\eta})(\overline{\varepsilon}_\zeta, \overline{\varepsilon}_\eta) := \widetilde{\phi}_\zeta(\overline{\varepsilon}_\zeta) + \widetilde{\phi}_\eta(\overline{\varepsilon}_\eta)$$

and

$$(\widetilde{\phi_\zeta \cdot \phi_\eta})(\overline{\varepsilon}_\zeta, \overline{\varepsilon}_\eta) := \widetilde{\phi}_\zeta(\overline{\varepsilon}_\zeta) \cdot \phi_\eta(\overline{E}_\eta) + \phi_\zeta(\overline{E}'_\zeta + \overline{E}''_\zeta) \cdot \widetilde{\phi}_\eta(\overline{\varepsilon}_\eta)$$

and similarly for other finite sums and products. With these notations we define $\widetilde{\phi}_g(\overline{\varepsilon}) := \phi((\widetilde{\phi_\zeta)_{\zeta\in S^1}})((\overline{\varepsilon}_\zeta)_{\zeta\in S^1})$. The equivariant secondary class $\widetilde{\phi}_g$ defined like this way satisfies the three conditions in the statement of this theorem, this fact follows from the axiomatic characterization of non-equivariant secondary classes. □

We remark that this theorem can be definitely generalized to the case of long exact sequences of equivariant hermitian vector bundles on M.

Now we give some examples of equivariant character forms and their corresponding secondary characteristic classes.

Example II.3. (i). The equivariant Chern character form $\mathrm{ch}_g(\overline{E}) := \sum_{\zeta\in S^1} \zeta \mathrm{ch}(\overline{E}_\zeta)$.

(ii). The equivariant Todd form $\mathrm{Td}_g(\overline{E}) := \frac{c_{\mathrm{rk}E_g}(\overline{E}_g)}{\mathrm{ch}_g(\sum_{j=0}^{\mathrm{rk}E}(-1)^j \wedge^j \overline{E}^\vee)}$. As in [Hir, Thm. 10.1.1] one can show that

$$\mathrm{Td}_g(\overline{E}) = \mathrm{Td}(\overline{E}_g) \prod_{\zeta\neq 1} \det(\frac{1}{1-\zeta^{-1} e^{\frac{\Omega^{\overline{E}_\zeta}}{2\pi i}}}).$$

(iii). Let $\overline{\varepsilon} : 0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0$ be a short exact sequence of hermitian vector bundles. The secondary Bott-Chern characteristic class is given by $\widetilde{\mathrm{ch}}_g(\overline{\varepsilon}) = \sum_{\zeta\in S^1} \zeta \widetilde{\mathrm{ch}}(\overline{\varepsilon}_\zeta)$.

(iv). If the equivariant structure g^ε has the eigenvalues $\zeta_1, \cdots, \zeta_m$, then the secondary Todd class is given by

$$\widetilde{\mathrm{Td}}_g(\overline{\varepsilon}) = \sum_{i=1}^m \prod_{j=1}^{i-1} \mathrm{Td}_g(\overline{E}_{\zeta_j}) \cdot \widetilde{\mathrm{Td}}(\overline{\varepsilon}_{\zeta_i}) \cdot \prod_{j=i+1}^m \mathrm{Td}_g(\overline{E}'_{\zeta_j} + \overline{E}''_{\zeta_j}).$$

Remark II.4. One can use Theorem II.2 to give a proof of the statements (iii) and (iv) in the examples above.

Let E be an equivariant hermitian vector bundle on M with two different hermitian metrics h_1 and h_2, we shall write $\widetilde{\phi}_g(E, h_1, h_2)$ for the equivariant secondary characteristic class associated to the exact sequence

$$0 \to (E, h_1) \to (E, h_2) \to 0 \to 0$$

where the map from (E, h_1) to (E, h_2) is the identity map.

The following proposition describes the additivity of equivariant secondary characteristic classes.

Proposition II.5. *Let*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{E}_1' & \longrightarrow & \overline{E}_1 & \longrightarrow & \overline{E}_1'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{E}_2' & \longrightarrow & \overline{E}_2 & \longrightarrow & \overline{E}_2'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{E}_3' & \longrightarrow & \overline{E}_3 & \longrightarrow & \overline{E}_3'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

be a double complex of equivariant hermitian vector bundles on M where all rows $\overline{\varepsilon}_i$ and all columns $\overline{\delta}_j$ are exact. Then we have

$$\widetilde{\phi}_g(\overline{\varepsilon}_1 \oplus \overline{\varepsilon}_3) - \widetilde{\phi}_g(\overline{\varepsilon}_2) = \widetilde{\phi}_g(\overline{\delta}_1 \oplus \overline{\delta}_3) - \widetilde{\phi}_g(\overline{\delta}_2).$$

Proof. We may have the corresponding diagram of hermitian vector bundles on $M \times \mathbb{P}^1$ by the first construction in the proof of Theorem II.2. Then

$$\begin{aligned}
\widetilde{\phi}_g(\overline{\varepsilon}_2) - \widetilde{\phi}_g(\overline{\varepsilon}_1 \oplus \overline{\varepsilon}_3) &= -\int_{\mathbb{P}^1} [\phi_g(\widetilde{E_2}, h^{\widetilde{E_2}}) - \phi_g(\widetilde{E_1} \oplus \widetilde{E_3}, h^{\widetilde{E_1}} \oplus h^{\widetilde{E_3}})] \cdot \log \mid z \mid^2 \\
&= \int_{\mathbb{P}^1} \mathrm{dd}^c \widetilde{\phi}_g(\overline{\widetilde{\delta}_2}) \cdot \log \mid z \mid^2 = \int_{\mathbb{P}^1} \widetilde{\phi}_g(\overline{\widetilde{\delta}_2}) \cdot \mathrm{dd}^c \log \mid z \mid^2 \\
&= i_0^* \widetilde{\phi}_g(\overline{\widetilde{\delta}_2}) - i_\infty^* \widetilde{\phi}_g(\overline{\widetilde{\delta}_2}) = \widetilde{\phi}_g(\overline{\delta}_2) - \widetilde{\phi}_g(\overline{\delta}_1 \oplus \overline{\delta}_3).
\end{aligned}$$

□

2 Equivariant analytic torsion forms

In [BK], J.-M. Bismut and K. Köhler extended the Ray-Singer analytic torsion to the higher analytic torsion form T for a holomorphic submersion. The purpose of making such an extension is that the differential equation on $\mathrm{dd}^c T$ gives a refinement of the Grothendieck-Riemann-Roch theorem. Later, in his article [Ma1], X. Ma generalized J.-M. Bismut and K. Köhler's results to the equivariant case. In this section, we shall briefly recall Ma's construction of the equivariant analytic torsion form. This construction is not very important for understanding the rest of this thesis, but the equivariant analytic torsion form itself will be used to define a reasonable push-forward morphism between equivariant arithmetic G_0-groups.

We first fix some notations and assumptions. Let $f : M \to B$ be a proper holomorphic submersion of complex manifolds, and let TM, TB be the holomorphic tangent bundle on M, B. Denote by J^{Tf} the complex structure on the real relative tangent bundle $T_{\mathbb{R}}f$. We assume that h^{Tf} is a hermitian metric on Tf which induces a Riemannian metric g^{Tf} on $T_{\mathbb{R}}f$. Let T^HM be a vector subbundle of TM such that $TM = T^HM \oplus Tf$, the following definition of Kähler fibration was given in [BGS2, Definition 1.4].

Definition II.6. The triple (f, h^{Tf}, T^HM) is said to define a Kähler fibration if there exists a smooth real $(1,1)-$form ω which satisfies the following three conditions :

(i). ω is closed ;

(ii). $T^H_{\mathbb{R}}M$ and $T_{\mathbb{R}}f$ are orthogonal with respect to ω ;

(iii). if $X, Y \in T_{\mathbb{R}}f$, then $\omega(X,Y) = \langle X, J^{Tf}Y\rangle_{g^{Tf}}$.

It was shown in [BGS2, Thm. 1.5 and 1.7] that for a given Kähler fibration, the form ω is unique up to addition of a form $f^*\eta$ where η is a real, closed $(1,1)$-form on B. Moreover, for any real, closed $(1,1)$-form ω on M such that the bilinear map $X, Y \in T_{\mathbb{R}}f \mapsto \omega(J^{Tf}X, Y) \in \mathbb{R}$ defines a Riemannian metric and hence a hermitian product h^{Tf} on Tf, we can define a Kähler fibration whose associated $(1,1)$-form is ω. In particular, for a given f, a Kähler metric on M defines a Kähler fibration if we choose T^HM to be the orthogonal complement of Tf in TM and ω to be the Kähler form associated to this metric.

Now we introduce the Bismut superconnection of a Kähler fibration. Let (ξ, h^ξ) be a hermitian holomorphic vector bundle on M. Let ∇^{Tf}, ∇^ξ be the holomorphic hermitian connections on (Tf, h^{Tf}) and (ξ, h^ξ). Let $\nabla^{\Lambda(T^{*(0,1)}f)}$ be the connection induced by ∇^{Tf} on $\Lambda(T^{*(0,1)}f)$. Then we may define a connection on $\Lambda(T^{*(0,1)}f)\otimes\xi$ by setting

$$\nabla^{\Lambda(T^{*(0,1)}f)\otimes\xi} = \nabla^{\Lambda(T^{*(0,1)}f)} \otimes 1 + 1 \otimes \nabla^\xi.$$

Let E be the infinite-dimensional bundle on B whose fibre at each point $b \in B$ consists of the C^∞ sections of $\Lambda(T^{*(0,1)}f)\otimes\xi\mid_{f^{-1}b}$. This bundle E is a smooth $\mathbb{Z}$-graded bundle. We define a connection ∇^E on E as follows. If $U \in T_{\mathbb{R}}B$, let U^H be the lift of U in $T^H_{\mathbb{R}}M$ so that $f_*U^H = U$. Then for every smooth section s of E over B, we set

$$\nabla^E_U s = \nabla^{\Lambda(T^{*(0,1)}f)\otimes\xi}_{U^H} s.$$

For $b \in B$, let $\overline{\partial}^{Z_b}$ be the Dolbeault operator acting on E_b, and let $\overline{\partial}^{Z_b*}$ be its formal adjoint with respect to the canonical hermitian product on E_b (cf. [Ma1, 1.2]). Let $C(T_{\mathbb{R}}f)$ be the Clifford algebra of $(T_{\mathbb{R}}f, h^{Tf})$, then the bundle $\Lambda(T^{*(0,1)}f)\otimes\xi$ has a natural $C(T_{\mathbb{R}}f)$-Clifford module structure. Actually, if $U \in Tf$, let $U' \in T^{*(0,1)}f$ correspond to U defined by $U'(\cdot) = h^{Tf}(U,\cdot)$, then for $U, V \in Tf$ we set

$$c(U) = \sqrt{2}U'\wedge, \quad c(\overline{V}) = -\sqrt{2}i_{\overline{V}}$$

where $i_{(\cdot)}$ is the contraction operator (cf. [BGV, Definition 1.6]). Moreover, if $U, V \in T_{\mathbb{R}}B$, we set $T(U^H, V^H) = -P^{Tf}[U^H, V^H]$ where P^{Tf} stands for the canonical projection from TM to Tf.

Definition II.7. Let $e_1, \ldots, e_{2m}$ be a basis of $T_{\mathbb{R}}B$, and let $e^1, \ldots, e^{2m}$ be the dual basis of $T^*_{\mathbb{R}}B$. Then the element

$$c(T) = \frac{1}{2} \sum_{1 \leq \alpha, \beta \leq 2m} e^\alpha \wedge e^\beta \widehat{\otimes} c(T(e^H_\alpha, e^H_\beta))$$

is a section of $(f^*\Lambda(T^*_{\mathbb{R}}B) \widehat{\otimes} \mathrm{End}(\Lambda(T^{*(0,1)}f) \otimes \xi))^{\mathrm{odd}}$.

Definition II.8. For $u > 0$, the Bismut superconnection on E is the differential operator

$$B_u = \nabla^E + \sqrt{u}(\overline{\partial}^Z + \overline{\partial}^{Z*}) - \frac{1}{2\sqrt{2u}} c(T)$$

on $f^*(\Lambda(T^*_{\mathbb{R}}B)) \widehat{\otimes} (\Lambda(T^{*(0,1)}f) \otimes \xi)$.

Definition II.9. Let N_V be the number operator on $\Lambda(T^{*(0,1)}f) \otimes \xi$ and on E, namely N_V acts as multiplication by p on $\Lambda^p(T^{*(0,1)}f) \otimes \xi$. For $U, V \in T_{\mathbb{R}}B$, set $\omega^{H\overline{H}}(U, V) = \omega^M(U^H, V^H)$ where ω^M is the closed form in the definition of Kähler fibration. Furthermore, for $u > 0$, set $N_u = N_V + \frac{i\omega^{H\overline{H}}}{u}$.

We now turn to the equivariant case. Let G be a compact Lie group, we shall assume that all complex manifolds, hermitian vector bundles and holomorphic morphisms considered above are G-equivariant and all metrics are G-invariant. We will additionally assume that the direct images $R^k f_* \xi$ are all locally free so that the G-equivariant coherent sheaf $R^{\cdot} f_* \xi$ is locally free and hence a G-equivariant vector bundle over B. [Ma1, 1.2] gives a G-invariant hermitian metric (the L^2-metric) $h^{R^{\cdot} f_* \xi}$ on the vector bundle $R^{\cdot} f_* \xi$.

For $g \in G$, let $M_g = \{x \in M \mid g \cdot x = x\}$ and $B_g = \{b \in B \mid g \cdot b = b\}$ be the fixed point submanifolds, then f induces a holomorphic submersion $f_g : M_g \to B_g$. Let Φ be the homomorphism $\alpha \mapsto (2i\pi)^{-\deg\alpha/2}$ of $\Lambda^{\mathrm{even}}(T^*_{\mathbb{R}}B)$ into itself. We put

$$\mathrm{ch}_g(R^{\cdot} f_* \xi, h^{R^{\cdot} f_* \xi}) = \sum_{k=0}^{\dim M - \dim B} (-1)^k \mathrm{ch}_g(R^k f_* \xi, h^{R^k f_* \xi})$$

and

$$\mathrm{ch}'_g(R^{\cdot} f_* \xi, h^{R^{\cdot} f_* \xi}) = \sum_{k=0}^{\dim M - \dim B} (-1)^k k \mathrm{ch}_g(R^k f_* \xi, h^{R^k f_* \xi}).$$

Definition II.10. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, let

$$\zeta_1(s) = -\frac{1}{\Gamma(s)} \int_0^1 u^{s-1} (\Phi \mathrm{Tr_s}[g N_u \exp(-B_u^2)] - \mathrm{ch}'_g(R^{\cdot} f_* \xi, h^{R^{\cdot} f_* \xi})) \mathrm{d}u$$

and similarly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) < \frac{1}{2}$, let

$$\zeta_2(s) = -\frac{1}{\Gamma(s)} \int_1^\infty u^{s-1} (\Phi \mathrm{Tr_s}[g N_u \exp(-B_u^2)] - \mathrm{ch}'_g(R^{\cdot} f_* \xi, h^{R^{\cdot} f_* \xi})) \mathrm{d}u.$$

X. Ma has proved that $\zeta_1(s)$ extends to a holomorphic function of $s \in \mathbb{C}$ near $s = 0$ and $\zeta_2(s)$ is a holomorphic function of s.

Definition II.11. The smooth form $T_g(\omega^M, h^\xi) := \frac{\partial}{\partial s}(\zeta_1 + \zeta_2)(0)$ on B_g is called the equivariant analytic torsion form.

Theorem II.12. *The form $T_g(\omega^M, h^\xi)$ lies in $\bigoplus_{p\geq 0} A^{p,p}(B_g)$ and satisfies the following differential equation*

$$\mathrm{dd}^c T_g(\omega^M, h^\xi) = \mathrm{ch}_g(R^\cdot f_*\xi, h^{R^\cdot f_*\xi}) - \int_{M_g/B_g} \mathrm{Td}_g(Tf, h^{Tf})\mathrm{ch}_g(\xi, h^\xi).$$

Here $A^{p,p}(B_g)$ stands for the space of smooth forms on B_g of type (p,p).

Proof. This is [Ma1, Theorem 2.12]. □

We define a secondary characteristic class

$$\widetilde{\mathrm{ch}}_g(R^\cdot f_*\xi, h^{R^\cdot f_*\xi}, h'^{R^\cdot f_*\xi}) := \sum_{k=0}^{\dim M - \dim B} (-1)^k \widetilde{\mathrm{ch}}_g(R^k f_*\xi, h^{R^k f_*\xi}, h'^{R^k f_*\xi})$$

such that it satisfies the following differential equation

$$\mathrm{dd}^c \widetilde{\mathrm{ch}}_g(R^\cdot f_*\xi, h^{R^\cdot f_*\xi}, h'^{R^\cdot f_*\xi}) = \mathrm{ch}_g(R^\cdot f_*\xi, h^{R^\cdot f_*\xi}) - \mathrm{ch}_g(R^\cdot f_*\xi, h'^{R^\cdot f_*\xi}),$$

then the anomaly formula can be described as follows.

Theorem II.13. *(Anomaly formula) Let ω' be the form associated to another Kähler fibration for $f : M \to B$. Let h'^{Tf} be the metric on Tf in this new fibration and let h'^ξ be another metric on ξ. The following identity holds in $\widetilde{A}(B_g) := \bigoplus_{p\geq 0}(A^{p,p}(B_g)/(\mathrm{Im}\partial + \mathrm{Im}\overline{\partial}))$:*

$$\begin{aligned} T_g(\omega^M, h^\xi) - T_g(\omega'^M, h'^\xi) =& \widetilde{\mathrm{ch}}_g(R^\cdot f_*\xi, h^{R^\cdot f_*\xi}, h'^{R^\cdot f_*\xi}) \\ &- \int_{M_g/B_g} [\widetilde{\mathrm{Td}}_g(Tf, h^{Tf}, h'^{Tf})\mathrm{ch}_g(\xi, h^\xi) \\ &+ \mathrm{Td}_g(Tf, h'^{Tf})\widetilde{\mathrm{ch}}_g(\xi, h^\xi, h'^\xi)]. \end{aligned}$$

In particular, the class of $T_g(\omega^M, h^\xi)$ in $\widetilde{A}(B_g)$ only depends on (h^{Tf}, h^ξ).

Proof. This is [Ma1, Theorem 2.13]. □

3 Equivariant Bott-Chern singular currents

The Bott-Chern singular current was defined by J.-M. Bismut, H. Gillet and C. Soulé in [BGS3] in order to generalize the usual Bott-Chern secondary characteristic class to the case where one considers the resolutions of hermitian vector bundles associated to the closed immersions of complex manifolds. In [Bi1], J.-M. Bismut generalized this topic to the equivariant case. We shall recall Bismut's construction of the equivariant Bott-Chern singular current in this section. Similar to the equivariant analytic torsion form, the construction is not very important for understanding our later arguments but the singular current itself will play a crucial role. Bismut's construction was realized via some current valued zeta function which involves the supertraces of Quillen's superconnections. This is similar to the non-equivariant case.

As before, let g be the automorphism corresponding to an element in a compact Lie group G. Let $i: Y \to X$ be an equivariant closed immersion of G-equivariant Kähler manifolds, and let $\overline{\eta}$ be an equivariant hermitian vector bundle on Y. Assume that $\overline{\xi}.$ is a complex (of homological type) of equivariant hermitian vector bundles on X which provides a resolution of $i_*\overline{\eta}$. We denote the differential of the complex $\xi.$ by v. Note that $\xi.$ is acyclic outside Y and the homology sheaves of its restriction to Y are locally free and hence they are all vector bundles. We write $H_n = \mathcal{H}_n(\xi. \mid_Y)$ and define a $\mathbb{Z}$-graded bundle $H = \bigoplus_n H_n$. For each $y \in Y$ and $u \in TX_y$, we denote by $\partial_u v(y)$ the derivative of v at y in the direction u in any given holomorphic trivialization of $\xi.$ near y. Then the map $\partial_u v(y)$ acts on H_y as a chain map, and this action only depends on the image z of u in N_y where N stands for the normal bundle of $i(Y)$ in X. So we get a chain complex of holomorphic vector bundles $(H, \partial_z v)$.

Let π be the projection from the normal bundle N to Y, then we have a canonical identification of $\mathbb{Z}$-graded chain complexes

$$(\pi^* H, \partial_z v) \cong (\pi^*(\wedge N^\vee \otimes \eta), i_z).$$

For this, one can see [Bi2, Section I.b]. Moreover, such an identification is an identification of G-bundles which induces a family of canonical isomorphisms $\gamma_n : H_n \cong \wedge^n N^\vee \otimes \eta$. Another way to describe these canonical isomorphisms γ_n is applying [GBI, Exp. VII, Lemma 2.4 and Proposition 2.5]. These two constructions coincide because they are both locally, on a suitable open covering $\{U_j\}_{j\in J}$, determined by any complex morphism over the identity map of $\eta \mid_{U_j}$ from $(\xi. \mid_{U_j}, v)$ to the minimal resolution of $\eta \mid_{U_j}$ (e.g. the Koszul resolution). The advantage of using the construction given in [GBI] is that it remains valid for arithmetic varieties over any base instead of the complex numbers. Later in [Bi1], for the use of normalization, J.-M. Bismut considered the automorphism of $N^\vee$ defined by multiplying a constant $-\sqrt{-1}$, it induces an isomorphism of chain complexes

$$(\pi^*(\wedge N^\vee \otimes \eta), i_z) \cong (\pi^*(\wedge N^\vee \otimes \eta), \sqrt{-1} i_z)$$

and hence

$$(\pi^* H, \partial_z v) \cong (\pi^*(\wedge N^\vee \otimes \eta), \sqrt{-1} i_z).$$

This identification induces a family of isomorphisms $\widetilde{\gamma_n} : H_n \cong \wedge^n N^\vee \otimes \eta$. By finite dimensional Hodge theory, for each $y \in Y$, there is a canonical isomorphism

$$H_y \cong \{f \in \xi._y \mid vf = 0, v^* f = 0\}$$

where v^* is the dual of v with respect to the metrics on $\xi.$. This means that H can be regarded as a smooth $\mathbb{Z}$-graded G-equivariant subbundle of ξ so that it carries an induced G-invariant metric. On the other hand, we endow $\wedge N^\vee \otimes \eta$ with the metric induced from $\overline{N}$ and $\overline{\eta}$. J.-M. Bismut introduced the following definition.

Definition II.14. We say that the metrics on the complex of equivariant hermitian vector bundles $\overline{\xi}.$ satisfy Bismut assumption (A) if the identification $(\pi^*H, \partial_z v) \cong (\pi^*(\wedge N^\vee \otimes \eta), \sqrt{-1}i_z)$ also identifies the metrics.

Proposition II.15. *There always exist G-invariant metrics on $\xi.$ which satisfy Bismut assumption (A) with respect to the equivariant hermitian vector bundles $\overline{N}$ and $\overline{\eta}$.*

Proof. This is [Bi1, Proposition 3.5]. □

From now on we always suppose that the metrics on a resolution satisfy Bismut assumption (A). Let ∇^ξ be the canonical hermitian holomorphic connection on $\xi.$, then for each $u > 0$, we may define a G-invariant superconnection

$$C_u := \nabla^\xi + \sqrt{u}(v + v^*)$$

on the $\mathbb{Z}_2$-graded vector bundle ξ. Moreover, let Φ be the map $\alpha \in \wedge(T^*_{\mathbb{R}}X_g) \to (2\pi i)^{-\deg\alpha/2}\alpha \in \wedge(T^*_{\mathbb{R}}X_g)$ and denote

$$(\mathrm{Td}_g^{-1})'(\overline{N}) := \frac{\partial}{\partial b}\mid_{b=0} (\mathrm{Td}_g(b \cdot \mathrm{Id} - \frac{\Omega^{\overline{N}}}{2\pi i})^{-1})$$

where $\Omega^{\overline{N}}$ is the curvature form associated to $\overline{N}$. We formulate as follows the construction of the equivariant singular current given in [Bi1, Section VI].

Lemma II.16. *Let N_H be the number operator on the complex $\xi.$ i.e. it acts on ξ_j as multiplication by j, then for $s \in \mathbb{C}$ and $0 < \mathrm{Re}(s) < \frac{1}{2}$, the current valued zeta function*

$$Z_g(\overline{\xi}.)(s) := \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}[\Phi\mathrm{Tr_s}(N_H g\exp(-C_u^2)) + (\mathrm{Td}_g^{-1})'(\overline{N})\mathrm{ch}_g(\overline{\eta})\delta_{Y_g}]\mathrm{d}u$$

is well-defined on X_g and it has a meromorphic continuation to the complex plane which is holomorphic at $s = 0$.

Definition II.17. The equivariant singular Bott-Chern current on X_g associated to the resolution $\overline{\xi}.$ is defined as

$$T_g(\overline{\xi}.) := \frac{\partial}{\partial s}\mid_{s=0} Z_g(\overline{\xi}.)(s).$$

Theorem II.18. *The current $T_g(\overline{\xi}.)$ is a sum of (p,p)-currents and it satisfies the differential equation*

$$\mathrm{dd}^c T_g(\overline{\xi}.) = i_{g*}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}) - \sum_k (-1)^k \mathrm{ch}_g(\overline{\xi}_k).$$

Moreover, the wave front set of $T_g(\overline{\xi}.)$ is contained in $N^\vee_{g,\mathbb{R}}$ where $N^\vee_{g,\mathbb{R}}$ stands for the underlying real bundle of the dual of N_g.

Proof. This follows from [Bi1, Theorem 6.7, Remark 6.8]. □

Finally, we recall a theorem concerning the relationship of equivariant Bott-Chern singular currents involved in a double complex. This theorem will be used to show that our definition of a general embedding morphism in equivariant arithmetic G_0-theory is reasonable.

Theorem II.19. *Let*

$$\overline{\chi}: \quad 0 \to \overline{\eta}_n \to \cdots \to \overline{\eta}_1 \to \overline{\eta}_0 \to 0$$

be an exact sequence of equivariant hermitian vector bundles on Y. *Assume that we have the following double complex consisting of resolutions of* $i_*\overline{\chi}$ *such that all rows are exact sequences.*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \overline{\xi}_{n,\cdot} & \longrightarrow & \cdots & \longrightarrow & \overline{\xi}_{1,\cdot} & \longrightarrow & \overline{\xi}_{0,\cdot} & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & i_*\overline{\eta}_n & \longrightarrow & \cdots & \longrightarrow & i_*\overline{\eta}_1 & \longrightarrow & i_*\overline{\eta}_0 & \longrightarrow & 0. \end{array}$$

For each k, *we write* $\overline{\varepsilon}_k$ *for the exact sequence*

$$0 \to \overline{\xi}_{n,k} \to \cdots \to \overline{\xi}_{1,k} \to \overline{\xi}_{0,k} \to 0.$$

Then we have the following equality in $\widetilde{\mathcal{U}}(X_g) := \bigoplus_{p\geq 0}(D^{p,p}(X_g)/(\mathrm{Im}\partial + \mathrm{Im}\overline{\partial}))$

$$\sum_{j=0}^{n}(-1)^j T_g(\overline{\xi}_{j,\cdot}) = i_{g*}\frac{\widetilde{\mathrm{ch}}_g(\overline{\chi})}{\mathrm{Td}_g(\overline{N})} - \sum_k (-1)^k \widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_k).$$

Here $D^{p,p}(X_g)$ *stands for the space of currents on* X_g *of type* (p,p).

Proof. This is [KR1, Theorem 3.14]. □

4 Bismut-Ma's immersion formula

In this section, we shall recall the Bismut-Ma's immersion formula which reflects the behaviour of the equivariant analytic torsion forms of a Kähler fibration under the composition of an immersion and a submersion. By translating to the equivariant arithmetic G_0-theoretic language, such a formula can be used to measure, in arithmetic G_0-theory, the difference between a push-forward morphism and the composition formed as an embedding morphism followed by a push-forward morphism. Although Bismut-Ma's immersion formula plays a very important role in this thesis, we shall not recall its proof since it is rather long and technical.

Let $i : Y \to X$ be an equivariant closed immersion of G-equivariant Kähler manifolds. Let S be a complex manifold with the trivial G-action, and let $f : Y \to S$, $l : X \to S$ be two equivariant holomorphic submersions such that $f = l \circ i$. Assume that

$\overline{\eta}$ is an equivariant hermitian vector bundle on Y and $\overline{\xi}.$ provides a resolution of $i_*\overline{\eta}$ on X whose metrics satisfy Bismut assumption (A). Let ω^Y, ω^X be the real, closed and G-invariant $(1,1)$-forms on Y, X which induce the Kähler fibrations with respect to f and l respectively. We additionally assume that ω^Y is the pull-back of ω^X so that the Kähler metric on Y is induced by the Kähler metric on X. As before, denote by N the normal bundle of $i(Y)$ in X. Consider the following exact sequence

$$\overline{\mathcal{N}}: \quad 0 \to \overline{Tf} \to \overline{Tl}\mid_Y \to \overline{N} \to 0$$

where N is endowed with the quotient metric, we shall write $\widetilde{\mathrm{Td}}_g(\overline{Tf}, \overline{Tl}\mid_Y)$ for $\widetilde{\mathrm{Td}}_g(\overline{\mathcal{N}})$, the equivariant Todd secondary characteristic class associated to $\overline{\mathcal{N}}$. It satisfies the following differential equation

$$\mathrm{dd}^c\widetilde{\mathrm{Td}}_g(\overline{Tf}, \overline{Tl}\mid_Y) = \mathrm{Td}_g(Tf, h^{Tf})\mathrm{Td}_g(\overline{N}) - \mathrm{Td}_g(Tl\mid_Y, h^{Tl}).$$

For simplicity, we shall suppose that in the resolution $\xi.$, ξ_j are all $l-$acyclic and moreover η is $f-$acyclic. By an easy argument of long exact sequence, we have the following exact sequence

$$\Xi: \quad 0 \to l_*(\xi_m) \to l_*(\xi_{m-1}) \to \ldots \to l_*(\xi_0) \to f_*\eta \to 0.$$

By the semi-continuity theorem, all the elements in the exact sequence above are vector bundles. In this case, we recall the definition of the L^2-metrics on direct images precisely as follows. We just take f_*h^η as an example. Note that the semi-continuity theorem implies that the natural map

$$(R^0f_*\eta)_s \to H^0(Y_s, \eta\mid_{Y_s})$$

is an isomorphism for every point $s \in S$ where Y_s stands for the fibre over s. We may endow $H^0(Y_s, \eta\mid_{Y_s})$ with a L^2-metric given by the formula

$$< u, v >_{L^2} := \frac{1}{(2\pi)^{d_s}} \int_{Y_s} h^\eta(u,v)\frac{{\omega^Y}^{d_s}}{d_s!}$$

where d_s is the complex dimension of the fibre Y_s. It can be shown that these metrics depend on s in a C^∞ manner (cf. [BGV, p.278]) and hence define a hermitian metric on $f_*\eta$. We shall denote it by f_*h^η.

In order to understand the statement of the Bismut-Ma's immersion formula, we still have to introduce an important concept defined by J.-M. Bismut, the equivariant R-genus. Let W be a G-equivariant complex manifold, and let $\overline{E}$ be an equivariant hermitian vector bundle on W. For $\zeta \in S^1$ and $s > 1$ consider the zeta function

$$L(\zeta, s) = \sum_{k=1}^{\infty} \frac{\zeta^k}{k^s}$$

and its meromorphic continuation to the whole complex plane. Define the formal power series in x

$$\widetilde{R}(\zeta, x) := \sum_{n=0}^{\infty} \Big(\frac{\partial L}{\partial s}(\zeta, -n) + L(\zeta, -n)\sum_{j=1}^{n}\frac{1}{2j}\Big)\frac{x^n}{n!}.$$

Definition II.20. The Bismut equivariant R-genus of an equivariant hermitian vector bundle $\overline{E}$ with $\overline{E}\mid_{X_g}=\sum_\zeta \overline{E}_\zeta$ is defined as

$$R_g(\overline{E}):=\sum_{\zeta\in S^1}\big(\mathrm{Tr}\widetilde{R}(\zeta,-\frac{\Omega^{\overline{E}_\zeta}}{2\pi i})-\mathrm{Tr}\widetilde{R}(1/\zeta,\frac{\Omega^{\overline{E}_\zeta}}{2\pi i})\big)$$

where $\Omega^{\overline{E}_\zeta}$ is the curvature form associated to $\overline{E}_\zeta$. Actually, the class of $R_g(\overline{E})$ in $\widetilde{A}(X_g)$ is independent of the metric and we just write $R_g(E)$ for it. Furthermore, the class $R_g(\cdot)$ is additive.

We remark that if the automorphism g is the identity, then R_g reduces to the usual R-genus which was defined by H. Gillet and C. Soulé.

Theorem II.21. *(Bismut-Ma's immersion formula) Let notations and assumptions be as above. Then the equality*

$$\begin{aligned}&\sum_{i=0}^m(-1)^iT_g(\omega^X,h^{\xi_i})-T_g(\omega^Y,h^\eta)+\widetilde{\mathrm{ch}}_g(\Xi,h^{L^2})\\ =&\int_{X_g/S}\mathrm{Td}_g(Tl,h^{Tl})T_g(\overline{\xi}.)+\int_{Y_g/S}\frac{\widetilde{\mathrm{Td}}_g(\overline{Tf},\overline{Tl}\mid_Y)}{\mathrm{Td}_g(\overline{N})}\mathrm{ch}_g(\overline{\eta})\\ &+\int_{X_g/S}\mathrm{Td}_g(Tl)R_g(Tl)\sum_{i=0}^m(-1)^i\mathrm{ch}_g(\xi_i)-\int_{Y_g/S}\mathrm{Td}_g(Tf)R_g(Tf)\mathrm{ch}_g(\eta)\end{aligned}$$

holds in $\widetilde{A}(S)$.

Proof. This is the combination of [BM, Theorem 0.1 and 0.2], the main theorems in that paper. □

Chapter III

A vanishing theorem for equivariant closed immersions

In this chapter, we formulate and prove a vanishing theorem for equivariant closed immersions in the analytic setting. In the last chapter of this thesis, this vanishing theorem will be translated to an arithmetic G_0-theoretic version which is the kernel of the proof of the second type of our arithmetic Lefschetz fixed point formula.

1 The statement

By a projective manifold we shall understand a compact complex manifold which is projective algebraic, that means a projective manifold is the complex analytic space $X(\mathbb{C})$ associated to a smooth projective variety X over $\mathbb{C}$ (cf. [Har, Appendix B]). Let μ_n be the diagonalisable group variety over $\mathbb{C}$ associated to $\mathbb{Z}/n\mathbb{Z}$. We say X is μ_n-equivariant if it admits a μ_n-projective action (cf. [KR1, Section 2]), this means the associated projective manifold $X(\mathbb{C})$ admits an action by the group of complex n-th roots of unity. Denote by X_{μ_n} the fixed point subvariety of X, by GAGA principle, $X_{\mu_n}(\mathbb{C})$ is equal to $X(\mathbb{C})_g$ where g is the automorphism on $X(\mathbb{C})$ corresponding to a fixed primitive n-th root of unity. If no confusion arises, we shall not distinguish between X and $X(\mathbb{C})$ as well as X_{μ_n} and X_g. Since the classical arguments of locally free resolutions may not be compatible with the equivariant setting, we summarize some crucial facts we need as follows. We shall use the language of schemes rather than the language of manifolds for the later use.

(i). Every μ_n-equivariant coherent sheaf on a scheme with μ_n-projective action is a μ_n-equivariant quotient of a μ_n-equivariant locally free coherent sheaf.

(ii). Every μ_n-equivariant coherent sheaf on a scheme with μ_n-projective action admits a μ_n-equivariant locally free resolution. It is finite if the scheme is regular.

(iii). An exact sequence of μ_n-equivariant coherent sheaves on a scheme with μ_n-projective action admits an exact sequence of μ_n-equivariant locally free resolutions.

(iv). Any two μ_n-equivariant locally free resolutions of a μ_n-equivariant coherent sheaf on a scheme with μ_n-projective action can be dominated by a third one.

Now let $i : Y \to X$ be a μ_n-equivariant closed immersion of μ_n-equivariant projective manifolds with normal bundle N. Let S be a projective manifold with the trivial μ_n-action and let $h : X \to S$ be an equivariant holomorphic submersion whose restriction $f : Y \to S$ is also an equivariant holomorphic submersion. According to our assumptions, we may define a Kähler fibration with respect to h by choosing a $\mu_n(\mathbb{C})$-invariant Kähler form ω^X on X. By restricting ω^X to Y we obtain a Kähler fibration with respect to f. The same thing goes to $h_g : X_g \to S$ and $f_g : Y_g \to S$. Let $\overline{\eta}$ be a μ_n-equivariant hermitian holomorphic vector bundle on Y, assume that $(\overline{\xi.}, v)$ is a complex of μ_n-equivariant hermitian vector bundles on X which provides a resolution of $i_*\overline{\eta}$, whose metrics satisfy Bismut assumption (A).

Write N_g for the 0-degree part of $N\mid_{Y_g}$ which is isomorphic to the normal bundle of $i_g(Y_g)$ in X_g and denote by F the orthogonal complement of N_g. According to [GBI, Exp. VII, Lemma 2.4 and Proposition 2.5], we know that there exists a canonical isomorphism from the homology sheaf $H(\xi.\mid_{X_g})$ to $i_{g*}(\wedge^{\cdot} F^\vee \otimes \eta\mid_{Y_g})$ which is equivariant. Then the restriction of $(\xi., v)$ to X_g can always split into a series of short exact sequences in the following way :

$$(*) : \quad 0 \to \mathrm{Im} \to \mathrm{Ker} \to i_{g*}(\wedge^{\cdot} F^\vee \otimes \eta\mid_{Y_g}) \to 0$$

and

$$(**) : \quad 0 \to \mathrm{Ker} \to \xi.\mid_{X_g} \to \mathrm{Im} \to 0.$$

Suppose that $\wedge^{\cdot} F^\vee \otimes \eta\mid_{Y_g}$ and $\xi.\mid_{X_g}$ are all acyclic (higher direct images vanish). Then according to an easy argument of long exact sequence, these short exact sequences $(*)$ and $(**)$ induce a series of short exact sequences of direct images :

$$\mathcal{H}(*) : \quad 0 \to R^0 h_{g*}(\mathrm{Im}) \to R^0 h_{g*}(\mathrm{Ker}) \to R^0 f_{g*}(\wedge^{\cdot} F^\vee \otimes \eta\mid_{Y_g}) \to 0$$

and

$$\mathcal{H}(**) : \quad 0 \to R^0 h_{g*}(\mathrm{Ker}) \to R^0 h_{g*}(\xi.\mid_{X_g}) \to R^0 h_{g*}(\mathrm{Im}) \to 0.$$

By semi-continuity theorem, all elements in the exact sequences above are vector bundles. We endow $R^0 h_{g*}(\xi.\mid_{X_g})$ and $R^0 f_{g*}(\wedge^{\cdot} F^\vee \otimes \eta\mid_{Y_g})$ with the L^2-metrics which are induced by the metrics on $\xi.$, η and F. Here the normal bundle N admits the quotient metric induced from the exact sequence

$$0 \to Tf \to Th\mid_Y \to N \to 0$$

and the bundle F admits the metric induced by the metric on N. Moreover, we endow $R^0 h_{g*}(\mathrm{Im})$ and $R^0 h_{g*}(\mathrm{Ker})$ with the metrics induced by the L^2-metrics of $R^0 h_{g*}(\xi.\mid_{X_g})$ so that $\mathcal{H}(*)$ and $\mathcal{H}(**)$ become short exact sequences of equivariant hermitian vector bundles. Denote by $\widetilde{\mathrm{ch}}_g(\overline{\xi.}, \overline{\eta})$ the alternating sum of the equivariant secondary Bott-Chern characteristic classes of $\mathcal{H}(*)$ and $\mathcal{H}(**)$ such that it satisfies the following differential equation

$$\mathrm{dd}^c\widetilde{\mathrm{ch}}_g(\overline{\xi.},\overline{\eta}) = \sum_j(-1)^j\mathrm{ch}_g(R^0f_{g*}(\wedge^j\overline{F}^\vee\otimes\overline{\eta}\mid_{Y_g})) - \sum_j(-1)^j\mathrm{ch}_g(R^0h_{g*}(\overline{\xi}_j\mid_{X_g})).$$

Now the difference

$$\begin{aligned}\delta(i,\overline{\eta},\overline{\xi}.) :=& \widetilde{\mathrm{ch}}_g(\overline{\xi.},\overline{\eta}) - \sum_k(-1)^kT_g(\omega^{Y_g},h^{\wedge^kF^\vee\otimes\eta|_{Y_g}}) \\ &+ \sum_k(-1)^kT_g(\omega^{X_g},h^{\xi_k|_{X_g}}) - \int_{X_g/S}T_g(\overline{\xi.})\mathrm{Td}(\overline{Th_g}) \\ &- \int_{Y_g/S}\mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g) \\ &- \int_{Y_g/S}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Th_g}\mid_{Y_g})\end{aligned}$$

makes sense and it is an element in $\bigoplus_{p\geq 0}A^{p,p}(S)/(\mathrm{Im}\partial+\mathrm{Im}\overline{\partial})$. Here the symbols $T_g(\cdot)$ in the summations stand for the equivariant analytic torsion forms introduced in Chapter II, Section 2, the symbol $T_g(\overline{\xi.})$ in the integral is the equivariant Bott-Chern singular current introduced in Chapter II, Section 3.

Then the vanishing theorem for equivariant closed immersions can be formulated as the following.

Theorem III.1. *Let $i: Y\to X$ be an equivariant closed immersion of μ_n-equivariant projective manifolds, and let S be a projective manifold with the trivial μ_n-action. Assume that we are given two equivariant holomorphic submersions $f: Y\to S$ and $h: X\to S$ such that $f = h\circ i$. Then X admits a μ_n-equivariant hermitian very ample invertible sheaf $\overline{\mathcal{L}}$ relative to the morphism h, and for any equivariant hermitian resolution $0\to\overline{\xi}_m\to\cdots\overline{\xi}_1\to\overline{\xi}_0\to i_*\overline{\eta}\to 0$ we have*

$$\delta(i,\overline{\eta}\otimes i^*\overline{\mathcal{L}}^{\otimes n},\overline{\xi.}\otimes\overline{\mathcal{L}}^{\otimes n}) = 0 \qquad \text{for} \quad n\gg 0.$$

Here the metrics on the resolution are supposed to satisfy Bismut assumption (A).

2 Deformation to the normal cone

To prove the vanishing theorem for closed immersions, we use a geometric construction called the deformation to the normal cone which allows us to deform a resolution of hermitian vector bundle associated to a closed immersion of projective manifolds to a simpler one. The δ-difference of this new simpler resolution is much easier to compute.

Let $i: Y\hookrightarrow X$ be a closed immersion of projective manifolds with normal bundle $N_{X/Y}$. For a vector bundle E on X or Y, the notation $\mathbb{P}(E)$ will stand for the projective space bundle $\mathrm{Proj}(\mathrm{Sym}(E^\vee))$.

Definition III.2. The deformation to the normal cone $W(i)$ of the immersion i is the blowing up of $X \times \mathbb{P}^1$ along $Y \times \{\infty\}$. We shall just write W for $W(i)$ if there is no confusion about the immersion.

There are too many geometric objects and morphisms appearing in the construction of the deformation to the normal cone, we have to fix various notations in a clear way. We denote by p_X (resp. p_Y) the projection $X \times \mathbb{P}^1 \to X$ (resp. $Y \times \mathbb{P}^1 \to Y$) and by π the blow-down map $W \to X \times \mathbb{P}^1$. We also denote by q_X (resp. q_Y) the projection $X \times \mathbb{P}^1 \to \mathbb{P}^1$ (resp. $Y \times \mathbb{P}^1 \to \mathbb{P}^1$) and by q_W the composition $q_X \circ \pi$. It is well known that the map q_W is flat and for $t \in \mathbb{P}^1$, we have

$$q_W^{-1}(t) \cong \begin{cases} X \times \{t\}, & \text{if } t \neq \infty, \\ P \cup \widetilde{X}, & \text{if } t = \infty, \end{cases}$$

where $\widetilde{X}$ is isomorphic to the blowing up of X along Y and P is isomorphic to the projective completion of $N_{X/Y}$ i.e. the projective space bundle $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$. Denote the canonical projection from $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$ to Y by π_P, then the morphism $\mathcal{O}_Y \to N_{X/Y} \oplus \mathcal{O}_Y$ induces a canonical section $i_\infty : Y \hookrightarrow \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$ which is called the zero section embedding. Moreover, let $j : Y \times \mathbb{P}^1 \to W$ be the canonical closed immersion induced by $i \times \mathrm{Id}$, then the component $\widetilde{X}$ doesn't meet $j(Y \times \mathbb{P}^1)$ and the intersection of $j(Y \times \mathbb{P}^1)$ with P is exactly the image of Y under the section i_∞.

The advantage of the construction of the deformation to the normal cone is that we may control the rational equivalence class of the fibres $q_W^{-1}(t)$. More precisely, in the language of line bundles, we have the isomorphisms $\mathcal{O}(X) \cong \mathcal{O}(P + \widetilde{X}) \cong \mathcal{O}(P) \otimes \mathcal{O}(\widetilde{X})$ which is an immediate consequence of the isomorphism $\mathcal{O}(0) \cong \mathcal{O}(\infty)$ on $\mathbb{P}^1$.

On $P = \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$, there exists a tautological exact sequence

$$0 \to \mathcal{O}(-1) \to \pi_P^*(N_{X/Y} \oplus \mathcal{O}_Y) \to Q \to 0$$

where Q is the tautological quotient bundle. This exact sequence and the inclusion $\mathcal{O}_P \to \pi_P^*(N_{X/Y} \oplus \mathcal{O}_Y)$ induce a section $\sigma : \mathcal{O}_P \to Q$ which vanishes along the zero section $i_\infty(Y)$. By duality we get a morphism $Q^\vee \to \mathcal{O}_P$, and this morphism induces the following exact sequence

$$0 \to \wedge^n Q^\vee \to \cdots \to \wedge^2 Q^\vee \to Q^\vee \to \mathcal{O}_P \to i_{\infty *}\mathcal{O}_Y \to 0$$

where n is the rank of Q. Note that i_∞ is a section of π_P i.e. $\pi_P \circ i_\infty = \mathrm{Id}$, the projection formula implies the following definition.

Definition III.3. For any vector bundle η on Y, the following complex of vector bundles

$$0 \to \wedge^n Q^\vee \otimes \pi_P^*\eta \to \cdots \to \wedge^2 Q^\vee \otimes \pi_P^*\eta \to Q^\vee \otimes \pi_P^*\eta \to \pi_P^*\eta \to 0$$

provides a resolution of $i_{\infty *}\eta$ on P. This complex is called the Koszul resolution of $i_{\infty *}\eta$ and will be denoted by $\kappa(\eta, N_{X/Y})$. If the normal bundle $N_{X/Y}$ admits some hermitian metric, then the tautological exact sequence induces a hermitian metric on Q. If, moreover, the bundle η also admits a hermitian metric, then the Koszul resolution is a complex of hermitian vector bundles and will be denoted by $\overline{\kappa}(\overline{\eta}, \overline{N}_{X/Y})$.

Now, assume that X is a μ_n-equivariant projective manifold and E is an equivariant locally free sheaf on X. Then according to [Koe, (1.4) and (1.5)], $\mathbb{P}(E)$ admits a canonical μ_n-equivariant structure such that the projection map $\mathbb{P}(E) \to X$ is equivariant and the canonical bundle $\mathcal{O}(1)$ admits an equivariant structure. Moreover, let $Y \to X$ be an equivariant closed immersion of projective manifolds, according to [Koe, (1.6)] the action of μ_n on X can be extended to the blowing up $\mathrm{Bl}_Y X$ such that the blow-down map is equivariant and the canonical bundle $\mathcal{O}(1)$ admits an equivariant structure. So by endowing $\mathbb{P}^1$ with the trivial μ_n-action, the construction of the deformation to the normal cone described above is compatible with the equivariant setting.

For the use of our later arguments, the Kähler metric chosen on W should be well controlled on the fibres of the deformation. For this purpose, it is necessary to introduce the following definition.

Definition III.4. (Rössler) A metric h on W is said to be normal to the deformation if

(a). it is invariant and Kähler ;

(b). the restriction $h\mid_{j_{g*}(Y_g\times\mathbb{P}^1)}$ is a product $h'\times h''$, where h' is a Kähler metric on Y_g and h'' is a Kähler metric on $\mathbb{P}^1$;

(c). the intersections of $X\times\{0\}$ with $j_*(Y\times\mathbb{P}^1)$ and of P with $j_*(Y\times\mathbb{P}^1)$ are orthogonal at the fixed points.

Lemma III.5. *For any μ_n-invariant Kähler metric h^X on X which induces an invariant Kähler metric h^Y on Y, there exists a metric h^W on W which is normal to the deformation and the restriction of h^W to $X\cong X\times\{0\}$ (resp. $Y\cong Y\times\{\infty\}$) is exactly h^X (resp. h^Y). Moreover, we may require that the hermitian normal bundles $\overline{N}_{Y\times\mathbb{P}^1/Y\times\{0\}}$ and $\overline{N}_{Y\times\mathbb{P}^1/Y\times\{\infty\}}$ are both isometric to the trivial bundles with trivial metrics.*

Proof. The existence of the metric which is normal to the deformation is the content of [KR1, Lemma 6.13] and [Roe, Lemma 6.14], such a metric is constructed via the Grassmannian graph construction. Roughly speaking, according to another description of the deformation to the normal cone via the Grassmannian graph construction, we have an embedding $W\to X\times\mathbb{P}^r\times\mathbb{P}^1$ and the metric h^W is the μ_n-average of the restriction of a product metric on $X\times\mathbb{P}^r\times\mathbb{P}^1$ (cf. [Roe, Lemma 6.14]). When we endow X in the product with the metric h^X, the requirements on restrictions are automatically satisfied since h^X is μ_n-invariant. To fulfill the requirements on hermitian normal bundles, we may just choose the Fubini-Study metric on $\mathbb{P}^1$. □

We summarize some very important results about the application of the deformation to the normal cone as follows. Their proofs can be found in [KR1, Section 2 and 6.2].

Theorem III.6. *Let $i: Y\to X$ be an equivariant closed immersion of equivariant projective manifolds, and let $W=W(i)$ be the deformation to the normal cone of i. Assume that $\overline{\eta}$ is an equivariant hermitian vector bundle on Y. Then* *(i). there exists an equivariant hermitian resolution of* $j_*p_Y^*(\overline{\eta})$ *on* W*, whose metrics satisfy Bismut assumption (A) and whose restriction to* $\widetilde{X}$ *is equivariantly and orthogonally split;*

(ii). the natural morphism from the deformation to the normal cone $W(i_g)$ *to the fixed point submanifold* $W(i)_g$ *is a closed immersion, this closed immersion induces the closed immersions* $\mathbb{P}(N_{X_g/Y_g}\oplus\mathcal{O}_{Y_g})\to\mathbb{P}(N_{X/Y}\oplus\mathcal{O}_Y)_g$ *and* $\widetilde{X_g}\to\widetilde{X}_g$*;*

(iii). the fixed point submanifold of $\mathbb{P}(N_{X/Y}\oplus\mathcal{O}_Y)$ *is the disjoint union of* $\mathbb{P}(N_{X_g/Y_g}\oplus\mathcal{O}_{Y_g})$ *and* $\coprod_{\zeta\neq 1}\mathbb{P}((N_{X/Y})_\zeta)$*;*

(iv). the closed immersion $i_{\infty,g}$ *factors through* $\mathbb{P}(N_{X_g/Y_g}\oplus\mathcal{O}_{Y_g})$ *and the other components* $\mathbb{P}((N_{X/Y})_\zeta)$ *don't meet* Y*. Hence the complex* $\kappa(\mathcal{O}_Y,N_{X/Y})_g$*, obtained by taking the* 0*-degree part of the Koszul resolution, provides a resolution of* $\mathcal{O}_{Y_g}$ *on* $\mathbb{P}(N_{X/Y}\oplus\mathcal{O}_Y)_g$*.*

3 Proof of the vanishing theorem

We shall first prove the first part of the vanishing theorem for closed immersions i.e. the existence of an equivariant hermitian very ample invertible sheaf on X which is relative to the morphism $h:X\to S$. Generally speaking, such an invertible sheaf can be constructed rather easily since X admits a μ_n-projective action and the μ_n-action on S is supposed to be trivial, but for the whole proof of the vanishing theorem we would like to construct a special one which is the pull-back of some equivariant hermitian very ample invertible sheaf on $W(i)$ under the identification $X\cong X\times\{0\}$. Our starting point is the following.

Definition III.7. Let M be a μ_n-projective manifold, and let $\mathbb{P}^n_M$ be some relative projective space over M. A μ_n-action on $\mathbb{P}^n_M$ arising from some μ_n-action on the free sheaf $\mathcal{O}_M^{\oplus n+1}$ via the functorial properties of the Proj symbol will be called a global μ_n-action.

The advantage of considering global μ_n-action is that on a projective space which admits a global μ_n-action the twisting line bundle $\mathcal{O}(1)$ is naturally μ_n-equivariant.

Lemma III.8. *The morphism* $h:X\to S$ *factors though some relative projective space* $\mathbb{P}^r_S$ *which admits a global* μ_n*-action.*

Proof. By assumption, X admits a μ_n-projective action. Then [KR1, Lemma 2.4 and 2.5] imply that there exists an equivariant closed immersion from X to some projective space $\mathbb{P}^r$ endowed with a global action. By using the universal property of the fibre product, we obtain a morphism from X to $\mathbb{P}^r_S=S\times\mathbb{P}^r$ which is equivariant. Moreover, this morphism is clearly a closed immersion. Since the action on S is trivial, the induced action on the fibre product $S\times\mathbb{P}^r$ is still global. So we are done. □

Lemma III.9. *Let* $l:W(i)\to S$ *be the composition* $h\circ p_X\circ\pi$*. Then* $W(i)$ *admits an equivariant very ample invertible sheaf* $\mathcal{L}$ *which is relative to* l*.*

Proof. By Lemma III.8, $h : X \to S$ factors through some relative projective space $\mathbb{P}^r_S$ which admits a global μ_n-action. So X admits an equivariant very ample invertible sheaf relative to h. Since the μ_n-action on S is supposed to be trivial, $\mathbb{P}^1_X = X \times \mathbb{P}^1 \cong X \times_S \mathbb{P}^1_S$ also admits an equivariant very ample invertible sheaf relative to the morphism $h \circ p_X$ which is denoted by $\mathcal{G}$. Moreover, by construction, $W(i)$ admits a very ample invertible sheaf $\mathcal{O}_W(1) \otimes \pi^*\mathcal{G}^{\otimes b}$ for some $b \geq 0$ which is relative to the blow-down map π (cf. [Har, II. Proposition 7.10]). Assume that $\mathbb{P}^1_X \times_S \mathbb{P}^m_S$ is the relative projective space associated to $\mathcal{O}_W(1) \otimes \pi^*\mathcal{G}^{\otimes b}$, and that $\mathbb{P}^n_S$ is the relative projective space associated to $\mathcal{G}$. Then the very ample invertible sheave on $\mathbb{P}^1_X \times_S \mathbb{P}^m_S$ with respect to the embedding

$$\mathbb{P}^1_X \times_S \mathbb{P}^m_S \hookrightarrow \mathbb{P}^n_S \times_S \mathbb{P}^m_S$$

can be written as $\mathcal{G} \boxtimes \mathcal{O}_{\mathbb{P}^m_S}(1)$ whose restriction to $W(i)$ is equal to $\mathcal{O}_W(1) \otimes \pi^*\mathcal{G}^{\otimes b+1}$. Therefore, $\mathcal{O}_W(1) \otimes \pi^*\mathcal{G}^{\otimes b+1}$ is a very ample invertible sheaf on $W(i)$ relative to $l : W(i) \to S$, this invertible sheaf is clearly equivariant. □

From now on, we shall fix the equivariant very ample invertible sheaf $\mathcal{L}$ constructed in Lemma III.9. We also fix a μ_n-invariant hermitian metric on $\mathcal{L}$, note that this metric always exists according to an argument of partition of unity. When we deal with the tensor product of a coherent sheaf $\mathcal{F}$ with some power $\mathcal{L}^{\otimes n}$, we just write it as $\mathcal{F}(n)$ for simplicity. Before we give the proof of the rest of the vanishing theorem, we shall introduce the concept of equivariant standard complex and some technical results.

Definition III.10. Let S be a compact complex manifold and let $\overline{\xi}.$ be a bounded complex of hermitian vector bundles on S. We say $\overline{\xi}.$ is a standard complex if the homology sheaves of $\overline{\xi}.$ are all locally free and they are endowed with some hermitian metrics. We shall write a standard complex as $(\overline{\xi}., h^H)$ to emphasize the choice of the metrics on the homology sheaves. Endow the kernel and the image of every differential with the induced metrics from $\overline{\xi}.$. We say that a standard complex $(\overline{\xi}., h^H)$ is homologically split if the following short exact sequences

$$0 \to \overline{\mathrm{Im}} \to \overline{\mathrm{Ker}} \to \overline{H}_* \to 0$$

and

$$0 \to \overline{\mathrm{Ker}} \to \overline{\xi}_* \to \overline{\mathrm{Im}} \to 0$$

of hermitian vector bundles are all orthogonally split.

In [Ma2], X. Ma proved the following uniqueness theorem for standard complexes.

Theorem III.11. *Let S be a compact complex manifold, then to each standard complex of hermitian vector bundles $(\overline{\xi}., h^H)$ on S there is a unique way to associate an element $M(\overline{\xi}., h^H) \in \widetilde{A}(S)$ satisfying the following conditions.*

(i). $\mathrm{dd}^c M(\overline{\xi}., h^H) = \sum(-1)^i \mathrm{ch}(\overline{H}_i) - \sum(-1)^j \mathrm{ch}(\overline{\xi}_j)$.

(ii). For any holomorphic morphism $f : S' \to S$, we have $M(f^*\overline{\xi}.f^*h^H) = f^*M(\overline{\xi}., h^H)$.

(iii). If $(\overline{\xi}., h^H)$ is homologically split, then $M(\overline{\xi}., h^H) = 0$.

The definition of standard complex and Ma's uniqueness theorem can be easily generalized to the equivariant case. We summarize these generalizations as follows.

Definition III.12. Let S be a compact complex manifold which admits a holomorphic action of a compact Lie group G. Fix an element $g \in G$. An equivariant standard complex on S is a bounded complex of G-equivariant hermitian vector bundles on S whose restriction to S_g is standard and the metrics on the homology sheaves are g-invariant. Again we shall write an equivariant standard complex as $(\overline{\xi}.,h^H)$ to emphasize the choice of the metrics on the homology sheaves.

Theorem III.13. *Let S be a compact complex manifold which admits a holomorphic action of a compact Lie group G. Fix an element $g \in G$. Then to each equivariant standard complex $(\overline{\xi}.,h^H)$ on S, there is a unique additive way to associate an element $M_g(\overline{\xi}.,h^H) \in \widetilde{A}(S_g)$ satisfying the following conditions.*

(i). $\mathrm{dd}^c M_g(\overline{\xi}.,h^H)=\sum(-1)^i\mathrm{ch}_g(\overline{H_i}(\xi.\mid_{S_g}))-\sum(-1)^j\mathrm{ch}_g(\overline{\xi}_j)$.

(ii). For any holomorphic equivariant morphism $f: S'\to S$, we have

$$M_g(f^*\overline{\xi}.,f^*h^H)=f_g^*M_g(\overline{\xi}.,h^H)$$

(iii). If $(\overline{\xi}.\mid_{S_g},h^H)$ is homologically split, then $M_g(\overline{\xi}.,h^H)=0$.

Proof. The complex $\overline{\xi}.$ splits on S_g orthogonally into a series of standard complexes $\overline{\xi}_\zeta.$ for all $\zeta \in S^1$. Using the non-equivariant Bott-Chern-Ma classes on S_g, we define

$$M_g(\overline{\xi}.,h^H)=\sum_{\zeta\in S^1}\zeta M(\overline{\xi}_\zeta.,h^{H_\zeta}).$$

Then the axiomatic characterization follows from the non-equivariant one in Theorem III.11 and the definition of ch_g. For the uniqueness, first note that by the condition (ii), the relation $M_g(\overline{\xi}.,h^H)=M_g(\overline{\xi}.\mid_{S_g},h^H)$ should be satisfied, then we may reduce our proof to the case where S is equal to S_g. Since M_g is required to be additive, we only have to show that for every $\zeta \in S^1$, $M_g(\overline{\xi}_\zeta.,h^{H_\zeta})=\zeta M(\overline{\xi}_\zeta.,h^{H_\zeta})$. This follows from Theorem III.11 since every compact complex manifold can be regarded as an equivariant compact complex manifold (with the trivial action), on which any standard complex can be endowed with a g-structure as multiplication by ζ. Such an approach is similar to the proof of [KR1, Theorem 3.4]. □

Remark III.14. (i). The condition of compactness in Definition III.10 and Theorem III.11 is not necessary, we just add this limitation for the proof of Theorem III.13 given above.

(ii). If one directly generalizes the proof of Theorem III.11 to the equivariant case (by trivially adding the subscript g to every notation), then the limitation of additivity in Theorem III.13 can be removed. Actually the additivity is a byproduct of such a proof.

To emphasize that it is a kind of equivariant Bott-Chern secondary characteristic class, we often write $\widetilde{\mathrm{ch}}_g(\overline{\xi.}, h^H)$ for $M_g(\overline{\xi.}, h^H)$.

Now, let $0 \to \overline{\xi.}' \to \overline{\xi.} \to \overline{\xi.}'' \to 0$ be a short exact sequence of equivariant standard complexes on S. Then by restricting to the fixed point submanifold S_g, we get a short exact sequence of standard complexes $0 \to \overline{\xi.}' \mid_{S_g} \to \overline{\xi.} \mid_{S_g} \to \overline{\xi.}'' \mid_{S_g} \to 0$. Hence we obtain a long exact sequence of homology sheaves of these three standard complexes. We shall make a stronger assumption. Suppose that for any $j \geq 0$, we have short exact sequence $0 \to H_j(\overline{\xi.}' \mid_{S_g}) \to H_j(\overline{\xi.} \mid_{S_g}) \to H_j(\overline{\xi.}'' \mid_{S_g}) \to 0$ which is denoted by $\overline{\chi}_j$. Moreover, for any $j \geq 0$, denote by $\overline{\varepsilon}_j$ the short exact sequence $0 \to \overline{\xi}'_j \to \overline{\xi}_j \to \overline{\xi}''_j \to 0$.

Lemma III.15. *Let notations and assumptions be as above. The identity*

$$\widetilde{\mathrm{ch}}_g(\overline{\xi.}', h^H) - \widetilde{\mathrm{ch}}_g(\overline{\xi.}, h^H) + \widetilde{\mathrm{ch}}_g(\overline{\xi.}'', h^H) = \sum (-1)^j \widetilde{\mathrm{ch}}_g(\overline{\chi}_j) - \sum (-1)^j \widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j)$$

holds in $\bigoplus_{p\geq 0} A^{p,p}(S_g)/(\mathrm{Im}\partial + \mathrm{Im}\overline{\partial})$.

Proof. On S_g, every equivariant standard complex $(\overline{\xi.}, h^H)$ splits into a series of short exact sequences of equivariant hermitian vector bundles in the following way

$$0 \to \overline{\mathrm{Im}} \to \overline{\mathrm{Ker}} \to \overline{H.} \to 0$$

and

$$0 \to \overline{\mathrm{Ker}} \to \overline{\xi.} \mid_{S_g} \to \overline{\mathrm{Im}} \to 0.$$

According to Theorem III.13, $\widetilde{\mathrm{ch}}_g(\overline{\xi.}, h^H)$ is equal to the alternating sum of the equivariant Bott-Chern secondary characteristic classes of the short exact sequences above. Now since we have supposed that $0 \to H_j(\overline{\xi.}' \mid_{S_g}) \to H_j(\overline{\xi.} \mid_{S_g}) \to H_j(\overline{\xi.}'' \mid_{S_g}) \to 0$ are all exact, by using Snake lemma, we know that $0 \to \mathrm{Im}(\overline{\xi.}' \mid_{S_g}) \to \mathrm{Im}(\overline{\xi.} \mid_{S_g}) \to \mathrm{Im}(\overline{\xi.}'' \mid_{S_g}) \to 0$ and $0 \to \mathrm{Ker}(\overline{\xi.}' \mid_{S_g}) \to \mathrm{Ker}(\overline{\xi.} \mid_{S_g}) \to \mathrm{Ker}(\overline{\xi.}'' \mid_{S_g}) \to 0$ are also all exact sequences. Then the identity in the statement of this lemma immediately follows from the construction of $\widetilde{\mathrm{ch}}_g(\overline{\xi.}, h^H)$ and the double complex formula for the equivariant Bott-Chern secondary characteristic classes. This double complex formula is an immediate consequence of Theorem II.19 if one considers the identity map and resolutions of the zero bundle. □

Corollary III.16. *Let* $0 \to \overline{\xi.}^{(m)} \to \cdots \to \overline{\xi.}^{(1)} \to \overline{\xi.}^{(0)} \to 0$ *be an exact sequence of equivariant standard complexes on* S *such that for every* $j \geq 0$, $0 \to H_j(\overline{\xi.}^{(m)} \mid_{S_g}) \to \cdots \to H_j(\overline{\xi.}^{(1)} \mid_{S_g}) \to H_j(\overline{\xi.}^{(0)} \mid_{S_g}) \to 0$ *is exact. Then the identity*

$$\sum_{k=0}^{m} (-1)^k \widetilde{\mathrm{ch}}_g(\overline{\xi.}^{(k)}, h^H) = \sum (-1)^j \widetilde{\mathrm{ch}}_g(\overline{\chi}_j) - \sum (-1)^j \widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j)$$

holds in $\bigoplus_{p\geq 0} A^{p,p}(S_g)/(\mathrm{Im}\partial + \mathrm{Im}\overline{\partial})$.

Proof. We claim that for every $1 \leq k \leq m$, the kernel of the complex morphism $\overline{\xi.}^{(k)} \to \overline{\xi.}^{(k-1)}$ is still an equivariant standard complex on S. It is clear that we only need to prove this for $k=1$. Firstly, the kernel of $\overline{\xi.}^{(1)} \to \overline{\xi.}^{(0)}$ is a complex of equivariant hermitian vector bundles, let's denote it by $\overline{K}$. By restricting to S_g and using an argument of long exact sequence, we know that the homology sheaves of $\overline{K} \mid_{S_g}$ are all equivariant hermitian vector bundles since for any $j \geq 0$ the bundle morphism $H_j(\overline{\xi.}^{(1)} \mid_{S_g}) \to H_j(\overline{\xi.}^{(0)} \mid_{S_g})$ is already surjective. Therefore, the assumption of exactness on homologies implies that we can split $0 \to \overline{\xi.}^{(m)} \to \cdots \to \overline{\xi.}^{(1)} \to \overline{\xi.}^{(0)} \to 0$ into a series of short exact sequences of equivariant standard complexes, so the identity in the statement of this corollary follows from Lemma III.15. □

Remark III.17. A generalized version of Corollary III.16, in which the exact sequence of (equivariant) standard complexes is replaced by an (equivariant) double standard complex was obtained in Xiaonan Ma's Ph.D thesis (cf. [Ma2]) where more discussions concerning spectral sequences were involved. Anyway, for arithmetical reason, we only need these special versions as in Lemma III.15 and Corollary III.16.

Now we turn back to our proof of the vanishing theorem. As before, let $W = W(i)$ be the deformation to the normal cone associated to an equivariant closed immersion of projective manifolds $i: Y \to X$. For simplicity, denote by P_g^0 the projective space bundle $\mathbb{P}(N_{X_g/Y_g} \oplus \mathcal{O}_{Y_g})$. Moreover, given an invariant Kähler metric on X, we fix an invariant Kähler metric on W which is constructed in Lemma III.5. In this situation, all normal bundles appearing in the construction of the deformation to the normal cone will be endowed with the quotient metrics. We recall the following lemma.

Lemma III.18. *Over $W(i_g)$, there are hermitian metrics on $\mathcal{O}(X_g)$, $\mathcal{O}(P_g^0)$ and $\mathcal{O}(\widetilde{X_g})$ such that the isometry $\overline{\mathcal{O}}(X_g) \cong \overline{\mathcal{O}}(P_g^0) \otimes \overline{\mathcal{O}}(\widetilde{X_g})$ holds and such that the restriction of $\overline{\mathcal{O}}(X_g)$ to X_g yields the metric of $N_{W(i_g)/X_g}$, the restriction of $\overline{\mathcal{O}}(\widetilde{X_g})$ to $\widetilde{X_g}$ yields the metric of $N_{W(i_g)/\widetilde{X_g}}$ and the restriction of $\overline{\mathcal{O}}(P_g^0)$ to P_g^0 induces the metric of $N_{W(i_g)/P_g^0}$.*

Proof. This is [KR1, Lemma 6.15]. □

Definition III.19. Let $\overline{\eta}$ be an equivariant hermitian vector bundle on Y, we say that a resolution

$$\overline{\Xi}: \quad 0 \to \overline{\widetilde{\xi}}_m \to \cdots \to \overline{\widetilde{\xi}}_0 \to j_* p_Y^*(\overline{\eta}) \to 0$$

satisfies the condition (T) if

(i). the metrics on $\widetilde{\xi}.$ satisfy Bismut assumption (A);

(ii). the restriction of $\overline{\Xi}$ to $\widetilde{X}$ is an equivariantly and orthogonally split exact sequence;

(iii). the restrictions of $\overline{\Xi}_\nabla$ to $W(i_g)$, X_g, P_g^0, $\widetilde{X_g}$ and $P_g^0 \cap \widetilde{X_g}$ are complexes with l-acyclic elements and l-acyclic homologies, here $\overline{\Xi}_\nabla$ is the complex of hermitian vector bundles obtained by omitting the last term $j_* p_Y^*(\overline{\eta})$ in $\overline{\Xi}$;

(iv). the tensor products $\overline{\overline{\Xi}}_\nabla\mid_{W(i_g)}\otimes\overline{\mathcal{O}}(-X_g)$, $\overline{\overline{\Xi}}_\nabla\mid_{W(i_g)}\otimes\overline{\mathcal{O}}(-P_g^0)$ and $\overline{\overline{\Xi}}_\nabla\mid_{W(i_g)}\otimes\overline{\mathcal{O}}(-\widetilde{X_g})$ are complexes with l-acyclic elements and l-acyclic homologies.

From Theorem III.6 (i), we already know that there always exists a resolution of $j_*p_Y^*(\overline{\eta})$ which satisfies the conditions (i) and (ii) in Definition III.19. Let $\overline{\overline{\Xi}}$ be such a resolution, we have the following.

Proposition III.20. *For $n\gg 0$, $\overline{\overline{\Xi}}(n)$ satisfies the condition (T).*

Proof. The reason is that $W(i_g)$, X_g, P_g^0, $\widetilde{X_g}$ and $P_g^0\cap\widetilde{X_g}$ are all closed submanifolds of W. □

It is well known that both two squares in the following deformation diagram

$$\begin{array}{ccccc}
Y\times\{0\} & \xrightarrow{s_0} & Y\times\mathbb{P}^1 & \xleftarrow{s_\infty} & Y\times\{\infty\} \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i_\infty} \\
X\times\{0\} & \longrightarrow & W & \longleftarrow & \mathbb{P}(N_{X/Y}\oplus N_{\mathbb{P}^1/\infty})
\end{array}$$

are Tor-independent. Moreover, according to our choices of the Kähler metrics, we may identify $Y\times\{0\}$ with Y, $X\times\{0\}$ with X, $Y\times\{\infty\}$ with Y and $\mathbb{P}(N_{X/Y}\oplus N_{\mathbb{P}^1/\infty})$ with $P=\mathbb{P}(N_{X/Y}\oplus\mathcal{O}_Y)$. So if $\overline{\overline{\Xi}}$ is a resolution of $j_*p_Y^*(\overline{\eta})$ on W, then the restriction of $\overline{\overline{\Xi}}$ to X (resp. P) provides a resolution of $i_*\overline{\eta}$ (resp. $i_{\infty *}\overline{\eta}$). The following theorem is the kernel of the whole proof of the vanishing theorem.

Theorem III.21. *(Deformation theorem) Let $\overline{\overline{\Xi}}$ be a resolution of $j_*p_Y^*(\overline{\eta})$ on W which satisfies the condition (T), then we have $\delta(\overline{\overline{\Xi}}\mid_X)=\delta(\overline{\overline{\Xi}}\mid_P)$.*

Proof. Consider the following tensor product of $\overline{\overline{\Xi}}_\nabla\mid_{W(i_g)}$ with the Koszul resolution associated to the immersion $X_g\hookrightarrow W(i_g)$

$$0\to\overline{\overline{\Xi}}_\nabla\mid_{W(i_g)}\otimes\overline{\mathcal{O}}(-X_g)\to\overline{\overline{\Xi}}_\nabla\mid_{W(i_g)}\otimes\overline{\mathcal{O}}_{W(i_g)}\to\overline{\overline{\Xi}}_\nabla\mid_{W(i_g)}\otimes i_{X_g*}\overline{\mathcal{O}}_{X_g}\to 0.$$

We have to caution the reader that here the tensor product is not the usual tensor product of two complexes, precisely our resulting sequence is a double complex and we don't take its total complex. Since we have assumed that $\overline{\overline{\Xi}}$ satisfies the condition (T), this tensor product induces a short exact sequence of equivariant standard complexes on S by taking direct images. For $j\geq 0$, its j-th row is the following short exact sequence

$$\overline{\varepsilon}_j:0\to R^0l^0_{g*}(\overline{\mathcal{O}}(-X_g)\otimes\overline{\overline{\widetilde{\xi}}}_j\mid_{W(i_g)})\to R^0l^0_{g*}(\overline{\overline{\widetilde{\xi}}}_j\mid_{W(i_g)})\to R^0h_{g*}(\overline{\overline{\widetilde{\xi}}}_j\mid_{X_g})\to 0$$

where l_g^0 is the composition of the inclusion $W(i_g)\hookrightarrow W$ with the morphism l.

Note that the j-th homology of $\overline{\overline{\Xi}}_\nabla\mid_{W(i_g)}$ is equal to $j_{g*}(\wedge^j\overline{\overline{\widetilde{F}}}^\vee\otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mid_{W(i_g)}$ where $\widetilde{F}$ is the non-zero degree part of the normal bundle associated to the immersion j. Actually j_g factors through $j_g^0:Y_g\times\mathbb{P}^1\hookrightarrow W(i_g)$, then the j-th homology of $\overline{\overline{\Xi}}_\nabla\mid_{W(i_g)}$

can be rewritten as $j^0_{g*}(\wedge^j\overline{\widetilde{F}}^\vee\otimes p^*_{Y_g}\overline{\eta}\mid_{Y_g})$. Write $Y_{g,0}:=Y_g\times\{0\}$ for simplicity. Using the fact that $j^{0*}_g\mathcal{O}(-X_g)$ is isomorphic to $\mathcal{O}(-Y_{g,0})$, we deduce from the short exact sequence

$$0\to j^0_{g*}(\overline{\mathcal{O}}(-Y_{g,0})\otimes\wedge^j\overline{\widetilde{F}}^\vee\otimes p^*_{Y_g}\overline{\eta}\mid_{Y_g})\to j^0_{g*}(\overline{\mathcal{O}}_{Y_g\times\mathbb{P}^1}\otimes\wedge^j\overline{\widetilde{F}}^\vee\otimes p^*_{Y_g}\overline{\eta}\mid_{Y_g})$$
$$\to j^0_{g*}(i_{Y_g*}\overline{\mathcal{O}}_{Y_g}\otimes\wedge^j\overline{\widetilde{F}}^\vee\otimes p^*_{Y_g}\overline{\eta}\mid_{Y_g})\to 0$$

that the j-th homologies of the induced short exact sequence of equivariant standard complexes form a short exact sequence

$$\overline{\chi}_j:0\to R^0u_{g*}(\overline{\mathcal{O}}(-Y_{g,0})\otimes\wedge^j\overline{\widetilde{F}}^\vee\otimes p^*_{Y_g}\overline{\eta}\mid_{Y_g})\to R^0u_{g*}(\wedge^j\overline{\widetilde{F}}^\vee\otimes p^*_{Y_g}\overline{\eta}\mid_{Y_g})$$
$$\to R^0f_{g*}(\wedge^j\overline{F}^\vee\otimes\overline{\eta}\mid_{Y_g})\to 0$$

where u_g is the composition of the inclusion $Y_g\times\mathbb{P}^1\hookrightarrow W(i_g)$ with the morphism l^0_g.

The main idea of this proof is that the equivariant Bott-Chern secondary characteristic class of the quotient term of the induced short exact sequence of equivariant standard complexes is nothing but $\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_X,h^H)$ which appears in the expression of $\delta(\overline{\Xi}\mid_X)$ and the equivariant secondary characteristic classes of $\overline{\chi}_j,\overline{\varepsilon}_j$ can be computed by Bismut-Ma's immersion formula.

Precisely, denote by g_{X_g} the Euler-Green current associated to X_g which was constructed by Bismut, Gillet and Soulé in [BGS4, Section 3. (f)], it satisfies the differential equation $\mathrm{dd}^cg_{X_g}=\delta_{X_g}-c_1(\overline{\mathcal{O}}(X_g))$. We write $\mathrm{Td}(\overline{X_g})$ for $\mathrm{Td}(\overline{\mathcal{O}}(X_g))$, [BGS4, Theorem 3.17] implies that $\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}$ is equal to the singular Bott-Chern current of the Koszul resolution associated to $X_g\hookrightarrow W(i_g)$ modulo $\mathrm{Im}\partial+\mathrm{Im}\overline{\partial}$. Moreover, write $\overline{\xi}.$ for the restriction $\overline{\Xi}_\nabla\mid_X$. Then for any $j\geq 0$, we compute

$$\begin{aligned}\widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j)=&T_g(\omega^{X_g},h^{\xi_j|_{X_g}})-T_g(\omega^{W(i_g)},h^{\widetilde{\xi}_j|_{W(i_g)}})\\&+T_g(\omega^{W(i_g)},h^{\mathcal{O}(-X_g)\otimes\widetilde{\xi}_j|_{W(i_g)}})\\&+\int_{W(i_g)/S}\mathrm{ch}_g(\overline{\widetilde{\xi}}_j)\mathrm{Td}(\overline{Tl^0_g})\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\&+\int_{X_g/S}\mathrm{ch}_g(\overline{\xi}_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl^0_g}\mid_{X_g})\\&+\int_{X_g/S}\mathrm{ch}_g(\xi_j)R(N_{W(i_g)/X_g})\mathrm{Td}(Th_g).\end{aligned}$$

Here, one should note that to simplify the last two terms in the right-hand side of Bismut-Ma's immersion formula, we have used an Atiyah-Segal-Singer type formula for immersion

$$i_{g*}(\mathrm{Td}_g^{-1}(N)\mathrm{ch}_g(x))=\mathrm{ch}_g(i_*(x)).$$

This formula is the content of [KR1, Theorem 6.16]. Similarly, for any $j \geq 0$, we compute

$$\begin{aligned}\widetilde{\mathrm{ch}}_g(\overline{\chi}_j) =& T_g(\omega^{Y_g}, h^{\wedge^j F^\vee \otimes \eta|_{Y_g}}) - T_g(\omega^{Y_g\times\mathbb{P}^1}, h^{\wedge^j \widetilde{F}^\vee \otimes p_{Y_g}^*\eta|_{Y_g}}) \\ &+ T_g(\omega^{Y_g\times\mathbb{P}^1}, h^{\mathcal{O}(-Y_{g,0})\otimes\wedge^j \widetilde{F}^\vee \otimes p_{Y_g}^*\eta|_{Y_g}}) \\ &+ \int_{Y_g\times\mathbb{P}^1/S} \mathrm{ch}_g(\wedge^j \overline{\widetilde{F}}^\vee \otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\mathrm{Td}^{-1}(\overline{Y_{g,0}})g_{Y_{g,0}} \\ &+ \int_{Y_g/S} \mathrm{ch}_g(\wedge^j \overline{F}^\vee \otimes \overline{\eta}\mid_{Y_g})\mathrm{Td}^{-1}(\overline{N}_{Y_g\times\mathbb{P}^1/Y_{g,0}})\widetilde{\mathrm{Td}}(\overline{Tf_g}, \overline{Tu_g}\mid_{Y_{g,0}}) \\ &+ \int_{Y_g/S} \mathrm{ch}_g(\wedge^j F^\vee \otimes \eta\mid_{Y_g})R(N_{Y_g\times\mathbb{P}^1/Y_{g,0}})\mathrm{Td}(Tf_g).\end{aligned}$$

Denote by $\overline{\Omega}(W(i_g))$ (resp. $\overline{\Omega}(-X_g)$) the middle (resp. sub) term of the induced short exact sequence of equivariant standard complexes. According to Lemma III.15, we have

$$\begin{aligned}&\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_X, h^H) - \widetilde{\mathrm{ch}}_g(\overline{\Omega}(W(i_g)), h^H) + \widetilde{\mathrm{ch}}_g(\overline{\Omega}(-X_g), h^H) \\ =& \sum(-1)^j \widetilde{\mathrm{ch}}_g(\overline{\chi}_j) - \sum(-1)^j \widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j) \\ =& \sum(-1)^j T_g(\omega^{Y_g}, h^{\wedge^j F^\vee\otimes\eta|_{Y_g}}) - \sum(-1)^j T_g(\omega^{Y_g\times\mathbb{P}^1}, h^{\wedge^j \widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}}) \\ &+ \sum(-1)^j T_g(\omega^{Y_g\times\mathbb{P}^1}, h^{\mathcal{O}(-Y_{g,0})\otimes\wedge^j \widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}}) \\ &+ \int_{Y_g\times\mathbb{P}^1/S} \sum(-1)^j \mathrm{ch}_g(\wedge^j \overline{\widetilde{F}}^\vee \otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\mathrm{Td}^{-1}(\overline{Y_{g,0}})g_{Y_{g,0}} \\ &+ \int_{Y_g/S} \sum(-1)^j \mathrm{ch}_g(\wedge^j \overline{F}^\vee \otimes \overline{\eta}\mid_{Y_g})\mathrm{Td}^{-1}(\overline{N}_{Y_g\times\mathbb{P}^1/Y_{g,0}})\widetilde{\mathrm{Td}}(\overline{Tf_g}, \overline{Tu_g}\mid_{Y_{g,0}}) \\ &+ \int_{Y_g/S} \sum(-1)^j \mathrm{ch}_g(\wedge^j F^\vee \otimes \eta\mid_{Y_g})R(N_{Y_g\times\mathbb{P}^1/Y_{g,0}})\mathrm{Td}(Tf_g) \\ &- \sum(-1)^j T_g(\omega^{X_g}, h^{\xi_j|_{X_g}}) + \sum(-1)^j T_g(\omega^{W(i_g)}, h^{\widetilde{\xi}_j|_{W(i_g)}}) \\ &- \sum(-1)^j T_g(\omega^{W(i_g)}, h^{\mathcal{O}(-X_g)\otimes\widetilde{\xi}_j|_{W(i_g)}}) \\ &- \int_{W(i_g)/S} \sum(-1)^j \mathrm{ch}_g(\overline{\widetilde{\xi}}_j)\mathrm{Td}(\overline{Tl_g^0})\mathrm{Td}^{-1}(\overline{X_g})g_{X_g} \\ &- \int_{X_g/S} \sum(-1)^j \mathrm{ch}_g(\overline{\xi}_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\widetilde{\mathrm{Td}}(\overline{Th_g}, \overline{Tl_g^0}\mid_{X_g}) \\ &- \int_{X_g/S} \sum(-1)^j \mathrm{ch}_g(\xi_j)R(N_{W(i_g)/X_g})\mathrm{Td}(Th_g).\end{aligned} \tag{3.0.1}$$

Similarly, we consider the tensor products of $\overline{\Xi}_\nabla\mid_{W(i_g)}$ with the following three Koszul resolutions

$$0 \to \overline{\mathcal{O}}(-P_g^0) \to \overline{\mathcal{O}}_{W(i_g)} \to i_{P_g^0 *}\overline{\mathcal{O}}_{P_g^0} \to 0,$$

$$0 \to \overline{\mathcal{O}}(-\widetilde{X_g}) \to \overline{\mathcal{O}}_{W(i_g)} \to i_{\widetilde{X_g}*}\overline{\mathcal{O}}_{\widetilde{X_g}} \to 0,$$

and

$$0 \to \overline{\mathcal{O}}(-\widetilde{X_g}) \otimes \overline{\mathcal{O}}(-P_g^0) \to \overline{\mathcal{O}}(-\widetilde{X_g})\oplus\overline{\mathcal{O}}(-P_g^0) \to \overline{\mathcal{O}}_{W(i_g)} \to i_{\widetilde{X_g}\cap P_g^0 *}\overline{\mathcal{O}}_{\widetilde{X_g}\cap P_g^0} \to 0.$$

We shall still denote by $\overline{\chi.}$ (resp. $\overline{\varepsilon.}$) the exact sequences consisting of homologies (resp. elements) in the induced exact sequences of equivariant standard complexes.

For the first one, denote by $g_{P_g^0}$ the Euler-Green current associated to P_g^0 and write $\overline{\xi.}^\infty$ for the restriction $\overline{\Xi}_\nabla\mid_P$. Moreover, denote by $\overline{\Omega}(-P_g^0)$ the sub term of the induced short exact sequence of equivariant standard complexes and denote by b_g the composition of the inclusion $P_g^0 \hookrightarrow W(i_g)$ with the morphism l_g^0. According to Lemma III.15, we have

$$\begin{aligned}
&\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla \mid_{P_g^0}, h^H) - \widetilde{\mathrm{ch}}_g(\overline{\Omega}(W(i_g)), h^H) + \widetilde{\mathrm{ch}}_g(\overline{\Omega}(-P_g^0), h^H)\\
=&\sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\chi}_j) - \sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j)\\
=&\sum(-1)^jT_g(\omega^{Y_g}, h^{\wedge^j F_\infty^\vee\otimes\eta|_{Y_g}}) - \sum(-1)^jT_g(\omega^{Y_g\times\mathbb{P}^1}, h^{\wedge^j \widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})\\
&+\sum(-1)^jT_g(\omega^{Y_g\times\mathbb{P}^1}, h^{\mathcal{O}(-Y_{g,\infty})\otimes\wedge^j \widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})\\
&+\int_{Y_g\times\mathbb{P}^1/S}\sum(-1)^j\mathrm{ch}_g(\wedge^j\overline{\widetilde{F}}^\vee\otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\mathrm{Td}^{-1}(\overline{Y_{g,\infty}})g_{Y_{g,\infty}}\\
&+\int_{Y_g/S}\{\sum(-1)^j\mathrm{ch_g}(\wedge^j\overline{F}_\infty^\vee\otimes\overline{\eta}\mid_{Y_g})\mathrm{Td}^{-1}(\overline{N}_{Y_g\times\mathbb{P}^1/Y_{g,\infty}})\\
&\qquad\qquad\cdot\widetilde{\mathrm{Td}}(\overline{Tf_g}, \overline{Tu_g}\mid_{Y_{g,\infty}})\}\\
&+\int_{Y_g/S}\sum(-1)^j\mathrm{ch}_g(\wedge^jF_\infty^\vee\otimes\eta\mid_{Y_g})R(N_{Y_g\times\mathbb{P}^1/Y_{g,\infty}})\mathrm{Td}(Tf_g)\\
&-\sum(-1)^jT_g(\omega^{P_g^0}, h^{\xi_j^\infty|_{P_g^0}}) + \sum(-1)^jT_g(\omega^{W(i_g)}, h^{\widetilde{\xi}_j|_{W(i_g)}})\\
&-\sum(-1)^jT_g(\omega^{W(i_g)}, h^{\mathcal{O}(-P_g^0)\otimes\widetilde{\xi}_j|_{W(i_g)}})\\
&-\int_{W(i_g)/S}\sum(-1)^j\mathrm{ch}_g(\overline{\widetilde{\xi}}_j)\mathrm{Td}(\overline{Tl_g^0})\mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0}\\
&-\int_{P_g^0/S}\sum(-1)^j\mathrm{ch_g}(\overline{\xi}_j^\infty)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0})\widetilde{\mathrm{Td}}(\overline{Tb_g}, \overline{Tl_g^0}\mid_{P_g^0})\\
&-\int_{P_g^0/S}\sum(-1)^j\mathrm{ch}_g(\xi_j^\infty)R(N_{W(i_g)/P_g^0})\mathrm{Td}(Tb_g)
\end{aligned}\tag{3.0.2}$$

where $\overline{F}_\infty$ is the non-zero degree part of the hermitian normal bundle $\overline{N}_\infty$ associated to i_∞.

For the second one, denote by $g_{\widetilde{X_g}}$ the Euler-Green current associated to $\widetilde{X_g}$ and denote by $\overline{\Omega}(-\widetilde{X_g})$ the sub term of the induced short exact sequence of equivariant standard complexes. Since the restriction of $\overline{\Xi}$ to the component $\widetilde{X}$ is equivariantly and orthogonally split, we know that $\widetilde{\mathrm{ch}}_g(\overline{\Xi}\mid_{\widetilde{X_g}}, h^H)$ is equal to 0 and the summation $\sum(-1)^j\mathrm{ch}_g(\overline{\widetilde{\xi}}_j)$ vanishes on $\widetilde{X_g}$. Using again Lemma III.15, we obtain

$$\begin{aligned}
&-\widetilde{\mathrm{ch}}_g(\overline{\Omega}(W(i_g)),h^H)+\widetilde{\mathrm{ch}}_g(\overline{\Omega}(-\widetilde{X_g}),h^H)\\
=&\sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\chi}_j)-\sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j)\\
=&-\sum(-1)^jT_g(\omega^{Y_g\times\mathbb{P}^1},h^{\wedge^j\widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})\\
&+\sum(-1)^jT_g(\omega^{Y_g\times\mathbb{P}^1},h^{j_g^{0*}\mathcal{O}(-\widetilde{X_g})\otimes\wedge^j\widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})\\
&-\int_{Y_g\times\mathbb{P}^1/S}\{\sum(-1)^j\mathrm{ch}_g(\wedge^j\overline{\widetilde{F}}^\vee\otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\\
&\qquad\qquad\cdot\widetilde{\mathrm{ch}}(j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X_g}),\overline{O}_{Y_g\times\mathbb{P}^1})\}\\
&+\sum(-1)^jT_g(\omega^{W(i_g)},h^{\widetilde{\xi}_j|_{W(i_g)}})\\
&-\sum(-1)^jT_g(\omega^{W(i_g)},h^{\mathcal{O}(-\widetilde{X_g})\otimes\widetilde{\xi}_j|_{W(i_g)}})\\
&-\int_{W(i_g)/S}\sum(-1)^j\mathrm{ch}_g(\overline{\widetilde{\xi}}_j)\mathrm{Td}(\overline{Tl_g^0})\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}}.
\end{aligned}\tag{3.0.3}$$

Here the element $\widetilde{\mathrm{ch}}(j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X_g}),\overline{O}_{Y_g\times\mathbb{P}^1})$ is the equivariant secondary characteristic class of the following short exact sequence

$$0\to0\to j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X_g})\to\overline{O}_{Y_g\times\mathbb{P}^1}\to0.$$

We now consider the last one. This is also a Koszul resolution because $\widetilde{X_g}$ and P_g^0 intersect transversally. By [BGS4, Theorem 3.20], the Euler-Green current associated to $\widetilde{X_g}\cap P_g^0$ is the current $c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}+\delta_{\widetilde{X_g}}g_{P_g^0}$. Then, by using the isometry $\overline{\mathcal{O}}(X_g)\cong\overline{\mathcal{O}}(P_g^0)\otimes\overline{\mathcal{O}}(\widetilde{X_g})$ and Corollary III.16, we get

$$\begin{aligned}
&-\widetilde{\mathrm{ch}}_g(\overline{\Omega}(W(i_g)),h^H)+\widetilde{\mathrm{ch}}_g(\overline{\Omega}(-\widetilde{X_g}),h^H)\\
&\qquad\qquad+\widetilde{\mathrm{ch}}_g(\overline{\Omega}(-P_g^0),h^H)-\widetilde{\mathrm{ch}}_g(\overline{\Omega}(-X_g),h^H)\\
=&\sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\chi}_j)-\sum(-1)^j\widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_j)\\
=&-\sum(-1)^jT_g(\omega^{Y_g\times\mathbb{P}^1},h^{\wedge^j\widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})\\
&+\sum(-1)^jT_g(\omega^{Y_g\times\mathbb{P}^1},h^{j_g^{0*}\mathcal{O}(-\widetilde{X_g})\otimes\wedge^j\widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})\\
&+\sum(-1)^jT_g(\omega^{Y_g\times\mathbb{P}^1},h^{\mathcal{O}(-Y_{g,\infty})\otimes\wedge^j\widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})\\
&-\sum(-1)^jT_g(\omega^{Y_g\times\mathbb{P}^1},h^{\mathcal{O}(-Y_{g,0})\otimes\wedge^j\widetilde{F}^\vee\otimes p_{Y_g}^*\eta|_{Y_g}})\\
&-\int_{Y_g\times\mathbb{P}^1/S}\sum(-1)^j\mathrm{ch}_g(\wedge^j\overline{\widetilde{F}}^\vee\otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\widetilde{\mathrm{ch}}(\overline{\Theta})
\end{aligned}$$

$$
\begin{aligned}
&+\sum(-1)^jT_g(\omega^{W(i_g)},h^{\widetilde{\xi}_j|_{W(i_g)}})\\
&-\sum(-1)^jT_g(\omega^{W(i_g)},h^{\mathcal{O}(-\widetilde{X}_g)\otimes\widetilde{\xi}_j|_{W(i_g)}})\\
&-\sum(-1)^jT_g(\omega^{W(i_g)},h^{\mathcal{O}(-P^0_g)\otimes\widetilde{\xi}_j|_{W(i_g)}})\\
&+\sum(-1)^jT_g(\omega^{W(i_g)},h^{\mathcal{O}(-X_g)\otimes\widetilde{\xi}_j|_{W(i_g)}})\\
&-\int_{W(i_g)/S}\{\sum(-1)^j\mathrm{ch}_g(\overline{\widetilde{\xi}}_j)\mathrm{Td}(\overline{Tl^0_g})\mathrm{Td}^{-1}(\overline{\widetilde{X}_g})\mathrm{Td}^{-1}(\overline{P^0_g})\\
&\qquad\qquad\cdot[c_1(\overline{\mathcal{O}}(P^0_g))g_{\widetilde{X}_g}+\delta_{\widetilde{X}_g}g_{P^0_g}]\}.
\end{aligned}
$$

(3.0.4)

Here the element $\widetilde{\mathrm{ch}}(\overline{\Theta})$ is the equivariant secondary characteristic class of the following short exact sequence

$$\overline{\Theta}:\quad 0\to\overline{\mathcal{O}}(-Y_{g,0})\to j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X}_g)\oplus\overline{\mathcal{O}}(-Y_{g,\infty})\to\overline{O}_{Y_g\times\mathbb{P}^1}\to 0.$$

Since $s_0:Y\times\{0\}\to Y\times\mathbb{P}^1$ and $s_\infty:Y\times\{\infty\}\to Y\times\mathbb{P}^1$ are sections of smooth morphism, the normal sequences

$$0\to\overline{Tf_g}\to\overline{Tu_g}\mid_{Y_{g,0}}\to\overline{N}_{Y_g\times\mathbb{P}^1/Y_{g,0}}\to 0$$

and

$$0\to\overline{Tf_g}\to\overline{Tu_g}\mid_{Y_{g,\infty}}\to\overline{N}_{Y_g\times\mathbb{P}^1/Y_{g,\infty}}\to 0$$

are orthogonally split so that $\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tu_g}\mid_{Y_{g,0}})$ and $\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tu_g}\mid_{Y_{g,\infty}})$ are both equal to 0. Moreover, the normal bundles $N_{Y_g\times\mathbb{P}^1/Y_{g,0}}$ and $N_{Y_g\times\mathbb{P}^1/Y_{g,\infty}}$ are isomorphic to trivial bundles so that $R(N_{Y_g\times\mathbb{P}^1/Y_{g,0}})$ and $R(N_{Y_g\times\mathbb{P}^1/Y_{g,\infty}})$ are both equal to 0. Furthermore, we may drop all the terms where an integral is taken over $\widetilde{X}_g$ because $\sum(-1)^j\mathrm{ch}_g(\overline{\widetilde{\xi}}_j)$ vanishes on $\widetilde{X}_g$.

Now, we compute (3.0.1)−(3.0.2)−(3.0.3)+(3.0.4) which is

$$
\begin{aligned}
&\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_X,h^H)-\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_{P^0_g},h^H)+\sum(-1)^jT_g(\omega^{X_g},h^{\xi_j|_{X_g}})\\
&-\sum(-1)^jT_g(\omega^{P^0_g},h^{\xi^\infty_j|_{P^0_g}})-\sum(-1)^jT_g(\omega^{Y_g},h^{\wedge^jF^\vee\otimes\eta|_{Y_g}})\\
&\qquad\qquad+\sum(-1)^jT_g(\omega^{Y_g},h^{\wedge^jF^\vee_\infty\otimes\eta|_{Y_g}})\\
=&\int_{Y_g\times\mathbb{P}^1/S}\{\sum(-1)^j\mathrm{ch}_g(\wedge^j\overline{\widetilde{F}}^\vee\otimes p^*_{Y_g}\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\cdot[\mathrm{Td}^{-1}(\overline{Y_{g,0}})g_{Y_{g,0}}\\
&\qquad-\mathrm{Td}^{-1}(\overline{Y_{g,\infty}})g_{Y_{g,\infty}}+\widetilde{\mathrm{ch}}(j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X}_g),\overline{O}_{Y_g\times\mathbb{P}^1})-\widetilde{\mathrm{ch}}_g(\overline{\Theta})]\}\\
&-\int_{W(i_g)/S}\sum(-1)^j\mathrm{ch}_g(\overline{\widetilde{\xi}}_j)\mathrm{Td}(\overline{Tl^0_g})\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\
&-\int_{X_g/S}\sum(-1)^j\mathrm{ch}_g(\overline{\xi}_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl^0_g}\mid_{X_g})
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{X_g/S}\sum(-1)^j\mathrm{ch}_g(\xi_j)R(N_{W(i_g)/X_g})\mathrm{Td}(Th_g)\\
&+\int_{W(i_g)/S}\sum(-1)^j\mathrm{ch}_g(\overline{\overline{\xi}}_j)\mathrm{Td}(\overline{Tl^0_g})\mathrm{Td}^{-1}(\overline{P^0_g})g_{P^0_g}\\
&+\int_{P^0_g/S}\sum(-1)^j\mathrm{ch}_{\mathrm{g}}(\overline{\xi}^\infty_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P^0_g})\widetilde{\mathrm{Td}}(\overline{Tb_g},\overline{Tl^0_g}\mid_{P^0_g})\\
&+\int_{P^0_g/S}\sum(-1)^j\mathrm{ch}_g(\xi^\infty_j)R(N_{W(i_g)/P^0_g})\mathrm{Td}(Tb_g)\\
&+\int_{W(i_g)/S}\sum(-1)^j\mathrm{ch}_g(\overline{\overline{\xi}}_j)\mathrm{Td}(\overline{Tl^0_g})\mathrm{Td}^{-1}(\overline{\widetilde{X}_g})g_{\widetilde{X}_g}\\
&-\int_{W(i_g)/S}\sum(-1)^j\mathrm{ch}_g(\overline{\overline{\xi}}_j)\mathrm{Td}(\overline{Tl^0_g})\mathrm{Td}^{-1}(\overline{\widetilde{X}_g})\mathrm{Td}^{-1}(\overline{P^0_g})c_1(\overline{\mathcal{O}}(P^0_g))g_{\widetilde{X}_g}.
\end{aligned}
$$

Denote by i_X (resp. i_P) the inclusion from X to $W(i)$ (resp. P to $W(i)$). We may use the Atiyah-Segal-Singer type formula for immersions and the projection formula in cohomology to compute

$$
\begin{aligned}
&i_{X_g*}\big(\sum(-1)^j\mathrm{ch}_g(\xi_j)R(N_{W(i_g)/X_g})\mathrm{Td}(Th_g)\big)\\
=&i_{X_g*}\big(R(N_{W(i_g)/X_g})\mathrm{Td}(Th_g)i_{g*}(\mathrm{Td}_g^{-1}(N_{X/Y})\mathrm{ch}_g(\eta))\big)\\
=&(i_{X_g}\circ i_g)_*\big(R(N_{W(i_g)/X_g})\mathrm{Td}(Th_g)\mathrm{Td}_g^{-1}(N_{X/Y})\mathrm{ch}_g(\eta)\big).
\end{aligned}
$$

Note that the restriction of $N_{W(i_g)/X_g}$ to Y_g is trivial so that the last expression vanishes. An entirely analogous reasoning implies that

$$
i_{P_g*}\big(\sum(-1)^j\mathrm{ch}_g(\xi^\infty_j)R(N_{W(i_g)/P^0_g})\mathrm{Td}(Tb_g)\big)=0.
$$

Thus, we are left with the equality

$$
\begin{aligned}
&\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_X,h^H)-\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_{P^0_g},h^H)+\sum(-1)^jT_g(\omega^{X_g},h^{\xi_j|_{X_g}})\\
&-\sum(-1)^jT_g(\omega^{P^0_g},h^{\xi^\infty_j|_{P^0_g}})-\sum(-1)^jT_g(\omega^{Y_g},h^{\wedge^jF^\vee\otimes\eta|_{Y_g}})\\
&\qquad\qquad+\sum(-1)^jT_g(\omega^{Y_g},h^{\wedge^jF^\vee_\infty\otimes\eta|_{Y_g}})\\
=&\int_{Y_g\times\mathbb{P}^1/S}\{\sum(-1)^j\mathrm{ch}_g(\wedge^j\overline{\widetilde{F}}^\vee\otimes p^*_{Y_g}\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\cdot[\mathrm{Td}^{-1}(\overline{Y_{g,0}})g_{Y_{g,0}}\\
&\qquad-\mathrm{Td}^{-1}(\overline{Y_{g,\infty}})g_{Y_{g,\infty}}+\widetilde{\mathrm{ch}}(j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X}_g),\overline{O_{Y_g\times\mathbb{P}^1}})-\widetilde{\mathrm{ch}}_g(\overline{\Theta})]\}\\
&-\int_{W(i_g)/S}\{\sum(-1)^j\mathrm{ch}_g(\overline{\overline{\xi}}_j)\mathrm{Td}(\overline{Tl^0_g})\cdot[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}-\mathrm{Td}^{-1}(\overline{P^0_g})g_{P^0_g}\\
&\qquad-\mathrm{Td}^{-1}(\overline{\widetilde{X}_g})g_{\widetilde{X}_g}+\mathrm{Td}^{-1}(\overline{\widetilde{X}_g})\mathrm{Td}^{-1}(\overline{P^0_g})c_1(\overline{\mathcal{O}}(P^0_g))g_{\widetilde{X}_g}]\}\\
&-\int_{X_g/S}\sum(-1)^j\mathrm{ch}_{\mathrm{g}}(\overline{\xi}_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl^0_g}\mid_{X_g})\\
&+\int_{P^0_g/S}\sum(-1)^j\mathrm{ch}_{\mathrm{g}}(\overline{\xi}^\infty_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P^0_g})\widetilde{\mathrm{Td}}(\overline{Tb_g},\overline{Tl^0_g}\mid_{P^0_g}).
\end{aligned}
$$

Using the differential equation which $T_g(\overline{\overline{\xi}}.)$ satisfies, we compute

$$\begin{aligned}
&-\int_{W(i_g)/S}\{\sum(-1)^j\mathrm{ch}_g(\overline{\overline{\xi}}_j)\mathrm{Td}(\overline{Tl_g^0})\cdot[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}-\mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0}\\
&\qquad -\mathrm{Td}^{-1}(\overline{\overline{\widetilde{X}_g}})g_{\widetilde{X}_g}+\mathrm{Td}^{-1}(\overline{\overline{\widetilde{X}_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X}_g}]\}\\
&=\int_{W(i_g)/S}\{\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\overline{\xi}}.)\cdot[\mathrm{Td}^{-1}(\overline{X_g})\delta_{X_g}-\mathrm{Td}^{-1}(\overline{P_g^0})\delta_{P_g^0}\\
&\qquad -\mathrm{Td}^{-1}(\overline{\overline{\widetilde{X}_g}})\delta_{\widetilde{X}_g}+\mathrm{Td}^{-1}(\overline{\overline{\widetilde{X}_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))\delta_{\widetilde{X}_g}]\}\\
&-\int_{W(i_g)/S}\{\mathrm{Td}(\overline{Tl_g^0})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbb{P}^1})\delta_{Y_g\times\mathbb{P}^1}\cdot[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\
&-\mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0}-\mathrm{Td}^{-1}(\overline{\overline{\widetilde{X}_g}})g_{\widetilde{X}_g}+\mathrm{Td}^{-1}(\overline{\overline{\widetilde{X}_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X}_g}]\}.
\end{aligned}\tag{3.0.5}$$

Here we have used the equation

$$\begin{aligned}
&\mathrm{Td}^{-1}(\overline{X_g})c_1(\overline{\mathcal{O}}(X_g))-\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))-\mathrm{Td}^{-1}(\overline{\overline{\widetilde{X}_g}})c_1(\overline{\mathcal{O}}(\widetilde{X}_g))\\
&\qquad +\mathrm{Td}^{-1}(\overline{\overline{\widetilde{X}_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))c_1(\overline{\mathcal{O}}(\widetilde{X}_g))=0
\end{aligned}\tag{3.0.6}$$

which is [KR1, (23)].

Again using the fact that $\overline{\overline{\xi}}.$ is equivariantly and orthogonally split on $\widetilde{X}$, the first integral in the right-hand side of (3.0.5) is equal to

$$\int_{X_g/S}\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\xi}.)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})-\int_{P_g^0/S}\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\xi}.^{\infty})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0}).$$

According to the normal sequence $0\to\overline{Th_g}\to\overline{Tl_g^0}\mid_{X_g}\to\overline{N}_{W(i_g)/X_g}\to 0$, we may write

$$\mathrm{Td}(\overline{Tl_g^0})=\mathrm{Td}(\overline{Th_g})\mathrm{Td}(\overline{N}_{W(i_g)/X_g})-\mathrm{dd}^c\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl_g^0}\mid_{X_g}).$$

So we get

$$\begin{aligned}
&\int_{X_g/S}\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\xi}.)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\\
&=\int_{X_g/S}\mathrm{Td}(\overline{Th_g})T_g(\overline{\xi}.)\\
&-\int_{X_g/S}\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl_g^0}\mid_{X_g})\delta_{Y_g}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\\
&+\int_{X_g/S}\sum(-1)^j\mathrm{ch}_g(\overline{\xi}_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl_g^0}\mid_{X_g}).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\int_{P_g^0/S}\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\xi.}^{\infty})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0})\\
=&\int_{P_g^0/S}\mathrm{Td}(\overline{Tb_g})T_g(\overline{\xi.}^{\infty})\\
&-\int_{P_g^0/S}\widetilde{\mathrm{Td}}(\overline{Tb_g},\overline{Tl_g^0}\mid_{P_g^0})\delta_{Y_g}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_\infty)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0})\\
&+\int_{P_g^0/S}\sum(-1)^j\mathrm{ch_g}(\overline{\xi}_j^{\infty})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0})\widetilde{\mathrm{Td}}(\overline{Tb_g},\overline{Tl_g^0}\mid_{P_g^0}).
\end{aligned}$$

Note that the normal sequence of $\overline{Th_g}$ in $\overline{Tl_g^0}$ (resp. $\overline{Tb_g}$ in $\overline{Tl_g^0}$) is orthogonally split on $Y_g\times\{0\}$ (resp. $Y_g\times\{\infty\}$), then $\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl_g^0}\mid_{X_g})\delta_{Y_g}$ and $\widetilde{\mathrm{Td}}(\overline{Tb_g},\overline{Tl_g^0}\mid_{P_g^0})\delta_{Y_g}$ are both equal to 0. Combining these computations above we get

$$\begin{aligned}
&\int_{X_g/S}\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\xi.})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\\
&\qquad\qquad-\int_{P_g^0/S}\mathrm{Td}(\overline{Tl_g^0})T_g(\overline{\xi.}^{\infty})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0})\\
=&\int_{X_g/S}\mathrm{Td}(\overline{Th_g})T_g(\overline{\xi.})\\
&+\int_{X_g/S}\sum(-1)^j\mathrm{ch_g}(\overline{\xi}_j)\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/X_g})\widetilde{\mathrm{Td}}(\overline{Th_g},\overline{Tl_g^0}\mid_{X_g})\\
&-\int_{P_g^0/S}\mathrm{Td}(\overline{Tb_g})T_g(\overline{\xi.}^{\infty})\\
&-\int_{P_g^0/S}\sum(-1)^j\mathrm{ch_g}(\overline{\xi}_j^{\infty})\mathrm{Td}^{-1}(\overline{N}_{W(i_g)/P_g^0})\widetilde{\mathrm{Td}}(\overline{Tb_g},\overline{Tl_g^0}\mid_{P_g^0}).
\end{aligned}$$

(3.0.7)

We now compute the second integral in the right-hand side of (3.0.5). According to the normal sequence

$$0\to\overline{Tu_g}\to\overline{Tl_g^0}\mid_{Y_g\times\mathbb{P}^1}\to\overline{N}_{W(i_g)/Y_g\times\mathbb{P}^1}\to 0,$$

we may write

$$\mathrm{Td}(\overline{Tl_g^0})=\mathrm{Td}(\overline{Tu_g})\mathrm{Td}(\overline{N}_{W(i_g)/Y_g\times\mathbb{P}^1})-\mathrm{dd}^c\widetilde{\mathrm{Td}}(\overline{Tu_g},\overline{Tl_g^0}\mid_{Y_g\times\mathbb{P}^1}).$$

Hence

$$\begin{aligned}
&-\int_{W(i_g)/S}\{\mathrm{Td}(\overline{Tl_g^0})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbb{P}^1})\delta_{Y_g\times\mathbb{P}^1}\cdot[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\
&-\mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0}-\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}}+\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}]\}\\
=&-\int_{Y_g\times\mathbb{P}^1/S}\{\mathrm{Td}(\overline{Tu_g})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{\widetilde{F}})\cdot j_g^{0*}[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\
&-\mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0}-\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}}+\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}]\}\\
&+\int_{Y_g\times\mathbb{P}^1/S}\{\widetilde{\mathrm{Td}}(\overline{Tu_g},\overline{Tl_g^0}\mid_{Y_g\times\mathbb{P}^1})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbb{P}^1})\\
&\cdot[\mathrm{Td}^{-1}(\overline{X_g})(\delta_{X_g}-c_1(\overline{\mathcal{O}}(X_g)))-\mathrm{Td}^{-1}(\overline{P_g^0})(\delta_{P_g^0}-c_1(\overline{\mathcal{O}}(P_g^0)))\\
&-\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})(\delta_{\widetilde{X_g}}-c_1(\overline{\mathcal{O}}(\widetilde{X_g})))\\
&+\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))(\delta_{\widetilde{X_g}}-c_1(\overline{\mathcal{O}}(\widetilde{X_g})))]\}.
\end{aligned}$$

By our choices of the metrics, we have $\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbb{P}^1})\mid_{Y_{g,0}}=\mathrm{Td}_g^{-1}(\overline{N})$, $\mathrm{Td}(\overline{X_g})\mid_{Y_{g,0}}=1$ and $\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbb{P}^1})\mid_{Y_{g,\infty}}=\mathrm{Td}_g^{-1}(\overline{N}_\infty)$, $\mathrm{Td}(\overline{P_g^0})\mid_{Y_{g,\infty}}=1$. Furthermore, by replacing all tangent bundles by relative tangent bundles, on can carry through the proof given in [KR1, P. 378-379] to show that

$$\widetilde{\mathrm{Td}}(\overline{Tu_g},\overline{Tl_g^0}\mid_{Y_g\times\mathbb{P}^1})\mid_{Y_{g,0}}=\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Th_g}\mid_{Y_g})$$

and

$$\widetilde{\mathrm{Td}}(\overline{Tu_g},\overline{Tl_g^0}\mid_{Y_g\times\mathbb{P}^1})\mid_{Y_{g,\infty}}=\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tb_g}\mid_{Y_g}).$$

So combining with the equation (3.0.6), we get

$$\begin{aligned}
&-\int_{W(i_g)/S}\{\mathrm{Td}(\overline{Tl_g^0})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_{W/Y\times\mathbb{P}^1})\delta_{Y_g\times\mathbb{P}^1}\cdot[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\
&-\mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0}-\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}}+\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}]\}\\
=&-\int_{Y_g\times\mathbb{P}^1/S}\{\mathrm{Td}(\overline{Tu_g})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{\widetilde{F}})\cdot j_g^{0*}[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}\\
&-\mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0}-\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}}+\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}]\}\\
&+\int_{Y_g/S}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Th_g}\mid_{Y_g})\\
&-\int_{Y_g/S}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_\infty)\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tb_g}\mid_{Y_g}).
\end{aligned}$$

(3.0.8)

At last, using the fact that the intersections in the deformation diagram are transversal and the fact that $j_g^0(Y_g\times\mathbb{P}^1)$ has no intersection with $\widetilde{X_g}$, we can compute

$$\begin{aligned}&\int_{Y_g\times\mathbb{P}^1/S}\{\sum(-1)^j\mathrm{ch}_g(\wedge^j\overline{\widetilde{F}}^\vee\otimes p_{Y_g}^*\overline{\eta}\mid_{Y_g})\mathrm{Td}(\overline{Tu_g})\cdot[\mathrm{Td}^{-1}(\overline{Y_{g,0}})g_{Y_{g,0}}\\&\qquad-\mathrm{Td}^{-1}(\overline{Y_{g,\infty}})g_{Y_{g,\infty}}+\widetilde{\mathrm{ch}}(j_g^{0*}\overline{\mathcal{O}}(-\widetilde{X_g}),\overline{O}_{Y_g\times\mathbb{P}^1})-\widetilde{\mathrm{ch}}_g(\overline{\Theta})]\}\\=&\int_{Y_g\times\mathbb{P}^1/S}\{\mathrm{Td}(\overline{Tu_g})\mathrm{ch}_g(p_Y^*\overline{\eta})\mathrm{Td}_g^{-1}(\overline{\widetilde{F}})\cdot j_g^{0*}[\mathrm{Td}^{-1}(\overline{X_g})g_{X_g}-\mathrm{Td}^{-1}(\overline{P_g^0})g_{P_g^0}\\&\qquad-\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})g_{\widetilde{X_g}}+\mathrm{Td}^{-1}(\overline{\widetilde{X_g}})\mathrm{Td}^{-1}(\overline{P_g^0})c_1(\overline{\mathcal{O}}(P_g^0))g_{\widetilde{X_g}}]\}.\end{aligned}\tag{3.0.9}$$

Gathering (3.0.5), (3.0.7), (3.0.8) and (3.0.9) we finally get

$$\begin{aligned}&\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_X,h^H)-\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_{P_g^0},h^H)+\sum(-1)^jT_g(\omega^{X_g},h^{\xi_j|_{X_g}})\\&-\sum(-1)^jT_g(\omega^{P_g^0},h^{\xi_j^\infty|_{P_g^0}})-\sum(-1)^jT_g(\omega^{Y_g},h^{\wedge^jF^\vee\otimes\eta|_{Y_g}})\\&\qquad+\sum(-1)^jT_g(\omega^{Y_g},h^{\wedge^jF_\infty^\vee\otimes\eta|_{Y_g}})\\=&\int_{X_g/S}\mathrm{Td}(\overline{Th_g})T_g(\overline{\xi.})-\int_{P_g^0/S}\mathrm{Td}(\overline{Tb_g})T_g(\overline{\xi.}^\infty)\\&+\int_{Y_g/S}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Th_g}\mid_{Y_g})\\&-\int_{Y_g/S}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_\infty)\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tb_g}\mid_{Y_g}).\end{aligned}\tag{3.0.10}$$

On the other hand, by definition, we have

$$\begin{aligned}\delta(\overline{\Xi}\mid_P):=&\widetilde{\mathrm{ch}}_g(\overline{\xi.}^\infty,\overline{\eta})-\sum_k(-1)^kT_g(\omega^{Y_g},h^{\wedge^kF_\infty^\vee\otimes\eta|_{Y_g}})\\&+\sum_k(-1)^kT_g(\omega^{P_g},h^{\xi_k^\infty|_{P_g}})\\&-\int_{Y_g/S}\mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\\&-\int_{P_g/S}T_g(\overline{\xi.}^\infty)\mathrm{Td}(\overline{Tb'_g})\\&-\int_{Y_g/S}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_\infty)\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tb'_g}\mid_{Y_g})\end{aligned}$$

where $b':P\to S$ is the composition of the inclusion $P\hookrightarrow W(i)$ and the morphism l. Note that P_g^0 is an open and closed submanifold of P_g and $\overline{\xi.}^\infty$ is orthogonally split on the other components since they all belong to $\widetilde{X}_g$, then we can rewrite $\delta(\overline{\Xi}\mid_P)$ as

$$\begin{aligned}\delta(\overline{\Xi}\mid_P) =&\widetilde{\mathrm{ch}}_g(\overline{\Xi}_\nabla\mid_{P_g^0},h^H)-\sum_k(-1)^kT_g(\omega^{Y_g},h^{\wedge^kF_\infty^\vee\otimes\eta|_{Y_g}})\\&+\sum_k(-1)^kT_g(\omega^{P_g^0},h^{\xi_k^\infty|_{P_g^0}})\\&-\int_{Y_g/S}\mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\\&-\int_{P_g^0/S}T_g(\overline{\xi.}^\infty)\mathrm{Td}(\overline{Tb_g})\\&-\int_{Y_g/S}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N}_\infty)\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tb_g}\mid_{Y_g}).\end{aligned}$$

Comparing with the definition of $\delta(\overline{\Xi}\mid_X)$, the equality (3.0.10) implies that

$$\delta(\overline{\Xi}\mid_X)-\delta(\overline{\Xi}\mid_P)=0$$

which completes the whole proof of this deformation theorem. □

Now we consider the zero section imbedding $i_\infty : Y\to P=\mathbb{P}(N_\infty\oplus\mathcal{O}_Y)$. Here we use the fact that N_∞ is isomorphic to $N_{X/Y}$, we caution the reader that this is not necessarily an isometry since $\overline{N}_\infty$ carries the quotient metric induced by the Kähler metric on P but $N_{X/Y}$ carries the quotient metric induced by the Kähler metric on X. We recall that on P we have a tautological exact sequence

$$0\to\mathcal{O}(-1)\to\pi_P^*(N_\infty\oplus\mathcal{O}_Y)\to Q\to 0.$$

The equivariant section $\sigma:\mathcal{O}_P\to\pi_P^*(N_\infty\oplus\mathcal{O}_Y)\to Q$ induces the following Koszul resolution

$$0\to\wedge^{\mathrm{rk}Q}Q^\vee\to\cdots\to Q^\vee\to\mathcal{O}_P\to i_{\infty*}\mathcal{O}_Y\to 0.$$

Since σ is equivariant, the image of $\mathcal{O}_{P_g}$ under $\sigma\mid_{P_g}$ is contained in Q_g. This means that $\sigma\mid_{P_g}$ induces a Koszul resolution on P_g of the following form

$$0\to\wedge^{\mathrm{rk}Q_g}Q_g^\vee\to\cdots\to Q_g^\vee\to\mathcal{O}_{P_g}\to i_{\infty,g*}\mathcal{O}_{Y_g}\to 0.$$

Proposition III.22. *Let $\overline{\kappa}:=\overline{\kappa}(\overline{\eta},\overline{N}_\infty)$ be a hermitian Koszul resolution on P defined in Definition III.3. Then for $n\gg 0$, we have $\delta(\overline{\kappa}(n))=0$.*

Proof. Denote the non-zero degree part of $Q\mid_{P_g}$ by $Q_\perp$, then we have the following isometry

$$\wedge^i\overline{Q}^\vee\mid_{P_g}=\wedge^i(\overline{Q}_g^\vee\oplus\overline{Q}_\perp^\vee)\cong\bigoplus_{t+s=i}(\wedge^t\overline{Q}_g^\vee\otimes\wedge^s\overline{Q}_\perp^\vee).$$

Consider the following complex of equivariant hermitian vector bundles on P_g

$$\begin{aligned}0\to\wedge^{\mathrm{rk}Q_g}\overline{Q}_g^\vee\otimes(\wedge^k\overline{Q}_\perp^\vee\otimes\pi_{P_g}^*\overline{\eta}\mid_{Y_g})\to\cdots\to\overline{Q}_g^\vee\otimes(\wedge^k\overline{Q}_\perp^\vee\otimes\pi_{P_g}^*\overline{\eta}\mid_{Y_g})\\\to\wedge^k\overline{Q}_\perp^\vee\otimes\pi_{P_g}^*\overline{\eta}\mid_{Y_g}\to 0\end{aligned}$$

which provides a resolution of $i_{\infty,g*}(\wedge^k\overline{F}_\infty^\vee\otimes\overline{\eta}\mid_{Y_g})$ where F_∞, as before, is the non-zero degree part of the normal bundle N_∞ associated to i_∞. We denote this resolution by $\overline{\kappa}^{(k)}$, then according to the arguments given before this proposition we have a decomposition of complexes $\overline{\kappa}_\nabla\mid_{P_g}\cong\bigoplus_{k\geq 0}\overline{\kappa}_\nabla^{(k)}[-k]$ where $\overline{\kappa}_\nabla^{(k)}[-k]$ is obtained from $\overline{\kappa}_\nabla^{(k)}$ by shifting degree. Replacing $\overline{\kappa}$ by $\overline{\kappa}(n)$ for big enough n, we may assume that all elements in $\overline{\kappa}$ and $\overline{\kappa}^{(k)}$ are acyclic. Therefore, by Bisumt-Ma's immersion formula we have the following equality

$$\begin{aligned}\widetilde{\mathrm{ch}}_g(b'_{g*}\overline{\kappa}^{(k)}) =& T_g(\omega^{Y_g},h^{\wedge^kF_\infty^\vee\otimes\eta|_{Y_g}})-\sum_{i=0}^{\mathrm{rk}Q_g}(-1)^iT_g(\omega^{P_g},h^{\wedge^iQ_g^\vee\otimes\wedge^kQ_\perp^\vee\otimes\pi_{P_g}^*\eta|_{Y_g}})\\ &+\int_{Y_g/S}\mathrm{ch}_g(\wedge^kF_\infty^\vee\otimes\eta\mid_{Y_g})R(N_{\infty,g})\mathrm{Td}(Tf_g)\\ &+\int_{P_g/S}\mathrm{Td}(\overline{Tb'_g})T_g(\overline{\kappa}^{(k)})\\ &+\int_{Y_g/S}\mathrm{ch}_g(\wedge^k\overline{F}_\infty^\vee\otimes\overline{\eta}\mid_{Y_g})\mathrm{Td}^{-1}(\overline{N}_{\infty,g})\widetilde{\mathrm{Td}}(\overline{Tf_g},\overline{Tb'_g}\mid_{Y_g}).\end{aligned}$$

It is easily seen from the decomposition $\overline{\kappa}_\nabla\mid_{P_g}=\bigoplus_{k\geq 0}\overline{\kappa}_\nabla^{(k)}[-k]$ that the secondary characteristic class $\widetilde{\mathrm{ch}}_g(\overline{\kappa})$ appearing in the definition of $\delta(\overline{\kappa})$ is exactly $\sum(-1)^k\widetilde{\mathrm{ch}}_g(b'_{g*}\overline{\kappa}^{(k)})$. So taking the alternating sum of both two sides of the equality above and using the fact that equivariant analytic torsion form is additive for direct sum of acyclic bundles, we know that to prove $\delta(\overline{\kappa})=0$, we are left to show that $\sum(-1)^kT_g(\overline{\kappa}^{(k)})$ is equal to $T_g(\overline{\kappa})$. In fact, by using [KR1, Lemma 3.15], we have modulo $\mathrm{Im}\partial+\mathrm{Im}\overline{\partial}$

$$\begin{aligned}\sum(-1)^kT_g(\overline{\kappa}^{(k)}) &= \sum(-1)^k\mathrm{ch}_g(\wedge^k\overline{Q}_\perp^\vee)\mathrm{ch}_g(\pi_{P_g}^*\overline{\eta}\mid_{Y_g})T_g(\overline{\wedge^\cdot Q_g^\vee})\\ &= \mathrm{Td}_g^{-1}(\overline{Q}_\perp)\mathrm{ch}_g(\pi_{P_g}^*\overline{\eta}\mid_{Y_g})T_g(\overline{\wedge^\cdot Q_g^\vee})\\ &= \mathrm{Td}_g^{-1}(\overline{Q})\mathrm{ch}_g(\pi_{P_g}^*\overline{\eta}\mid_{Y_g})T_g(\overline{\wedge^\cdot Q_g^\vee})\mathrm{Td}(\overline{Q}_g)\\ &= \mathrm{ch}_g(\pi_{P_g}^*\overline{\eta}\mid_{Y_g})T_g(\overline{\wedge^\cdot Q^\vee})\\ &= T_g(\overline{\kappa}).\end{aligned}$$

So we are done. □

It's now ready to finish the proof of the vanishing theorem for equivariant closed immersions. Let $\overline{\eta}$ be an equivariant hermitian vector bundle on Y, assume that

$$\overline{\Psi}:\quad 0\to\overline{\xi}_m\to\cdots\to\overline{\xi}_1\to\overline{\xi}_0\to i_*\overline{\eta}\to 0$$

is a resolution of $i_*\overline{\eta}$ by equivariant hermitian vector bundles on X which satisfies Bismut assumption (A). We need to prove that for $n\gg 0$, $\delta(\overline{\Psi}(n))=0$.

Proof. (of Theorem III.1) We first construct a resolution of $p_Y^*\overline{\eta}$ on $W(i)$ as

$$\overline{\Xi}:\quad 0\to\overline{\widetilde{\xi}}_m\to\cdots\to\overline{\widetilde{\xi}}_0\to\overline{\widetilde{\xi}}_0\to j_*p_Y^*(\overline{\eta})\to 0$$

which satisfies the condition (i) and (ii) in Definition III.19. Then the restriction of $\overline{\Xi}$ to X (resp. P) provides a resolution of $i_*\overline{\eta}$ (resp. $i_{\infty *}\overline{\eta}$). Over X, we can find a third resolution $\overline{\Phi}$ of $i_*\overline{\eta}$ which dominates $\overline{\Psi}$ and $\overline{\Xi}\mid_X$. Namely we get short exact sequences of exact sequences

$$0 \to \overline{\mathrm{Ker}}(n) \to \overline{\Phi}(n) \to \overline{\Psi}(n) \to 0$$

and

$$0 \to \overline{\mathrm{Ker}}'(n) \to \overline{\Phi}(n) \to \overline{\Xi}(n)\mid_X \to 0.$$

Then after omitting $i_*\overline{\eta}$ their restrictions to X_g become two exact sequences of complexes. Since $n \gg 0$ we may assume that all elements and homologies in the induced double complexes are acyclic, so that by taking direct images we get two exact sequences of equivariant standard complexes on S. These two short exact sequences of equivariant standard complexes clearly satisfy the assumptions in Lemma III.15. Therefore, using Lemma III.15, Bismut-Ma's immersion formula and the double complex formula of equivariant Bott-Chern singular currents (cf. Theorem II.19), we obtain that

$$\begin{aligned}\widetilde{\mathrm{ch}}_g(\overline{\Psi}(n)) - \widetilde{\mathrm{ch}}_g(\overline{\Phi}(n)) + \widetilde{\mathrm{ch}}_g(\overline{\mathrm{Ker}}(n)) \\ + T_g(\omega^{X_g}, h^{\Psi(n)_\bigtriangledown}) - T_g(\omega^{X_g}, h^{\Phi(n)_\bigtriangledown}) + T_g(\omega^{X_g}, h^{\mathrm{Ker}(n)_\bigtriangledown}) \\ = \int_{X_g/S} [T_g(\overline{\Psi}(n)_\bigtriangledown) - T_g(\overline{\Phi}(n)_\bigtriangledown) + T_g(\overline{\mathrm{Ker}}(n)_\bigtriangledown)] \cdot \mathrm{Td}(\overline{Th_g})\end{aligned}$$

which implies that

$$\delta(\overline{\Phi}(n)) = \delta(\overline{\Psi}(n)) + \delta(\overline{\mathrm{Ker}}(n)).$$

By applying Bismut-Ma's immersion formula to the case where the immersion is the identity map and $\overline{\eta}$ is equal to the zero bundle, we get $\delta(\overline{\mathrm{Ker}}(n)) = 0$ so that $\delta(\overline{\Phi}(n)) = \delta(\overline{\Psi}(n))$. Similarly, we have $\delta(\overline{\Phi}(n)) = \delta(\overline{\Xi}(n)\mid_X)$ and hence $\delta(\overline{\Psi}(n)) = \delta(\overline{\Xi}(n)\mid_X)$. An entirely analogous reasoning implies that $\delta(\overline{\kappa}(n)) = \delta(\overline{\Xi}(n)\mid_P)$. Then the vanishing of $\delta(\overline{\Psi}(n))$ follows from Theorem III.21 and Proposition III.22. □

Chapter IV

Arithmetic concentration theorem

In this chapter, we shall define the equivariant arithmetic Grothendieck groups with fixed wave front sets for equivariant arithmetic schemes. This kind of Grothendieck group is bigger than the one considered in [KR1], for the use of defining a reasonable embedding morphism in our later arguments. Then we shall formulate and prove an arithmetic concentration theorem which is an analog of Thomason's result in the context of Arakelov geometry.

1 Equivariant arithmetic Grothendieck groups

By an arithmetic ring D we understand a regular, excellent, Noetherian integral ring, together with a set $\mathcal{S}$ of embeddings $D \hookrightarrow \mathbb{C}$, which is invariant under a conjugate-linear involution F_∞ (cf. [GS1, Def. 3.1.1]). Denote by μ_n the diagonalisable group scheme over D associated to $\mathbb{Z}/n\mathbb{Z}$. A μ_n-equivariant arithmetic scheme over D is a separated scheme of finite type, endowed with a μ_n-projective action over D (cf. [KR1, Section 2]). Let X be a μ_n-equivariant arithmetic scheme whose generic fibre is smooth, then $X(\mathbb{C})$, the set of complex points of the variety $\coprod_{\sigma\in\mathcal{S}} X\times_D \mathbb{C}$, is a disjoint union of projective manifolds. This manifold admits an action of the group of complex n-th roots of unity and an anti-holomorphic involution induced by F_∞ which is still denoted by F_∞. It was shown in [Tho, Prop. 3.1] that if X is regular, then the fixed point subscheme X_{μ_n} is also regular. Fix a primitive n-th root of unity ζ_n and denote its corresponding holomorphic automorphism on $X(\mathbb{C})$ by g, by GAGA principle we have a natural isomorphism $X_{\mu_n}(\mathbb{C})\cong X(\mathbb{C})_g$.

Definition IV.1. An equivariant hermitian sheaf (resp. vector bundle) $\overline{E}$ on X is a coherent sheaf (resp. vector bundle) E on X, assumed locally free on $X(\mathbb{C})$, endowed with a μ_n-action which lifts the action of μ_n on X and a hermitian metric h on the associated bundle $E_{\mathbb{C}}$, which is invariant under F_∞ and g.

Remark IV.2. Let $\overline{E}$ be an equivariant hermitian sheaf (resp. vector bundle) on X, the restriction of $\overline{E}$ to the fixed point subscheme X_{μ_n} has a natural $\mathbb{Z}/n\mathbb{Z}$-grading structure

$\overline{E}\mid_{X_{\mu_n}}\cong\oplus_{k\in\mathbb{Z}/n\mathbb{Z}}\overline{E}_k$. We shall often write $\overline{E}_{\mu_n}$ for $\overline{E}_0$. It is clear that the associated bundle of $\overline{E}_{\mu_n}$ over $X(\mathbb{C})$ is exactly equal to $\overline{E}_g$.

Over a complex manifold M, we may consider the current space which is the continuous dual of the space of smooth complex valued differential forms (cf. [deRh, Chapter IX]). The wave front set $\mathrm{WF}(\omega)$ of a current ω over M is a closed conical subset of the cotangent bundle $T^*_{\mathbb{R}}M_0:=T^*_{\mathbb{R}}M\backslash\{0\}$. This conical subset measures the singularities of ω, actually the projection of $\mathrm{WF}(\omega)$ in M is equal to the singular locus of the support of ω. It also allows us to define certain products and pull-backs of currents. We refer to [Hoer, Chapter VIII] for the definition and various properties of wave front set.

Now let X be a μ_n-equivariant arithmetic scheme with smooth generic fibre and let S be a conical subset of $T^*_{\mathbb{R}}X(\mathbb{C})_{g,0}$, denote by $D^{p,p}(X(\mathbb{C})_g,S)$ the set of currents ω of type (p,p) on $X(\mathbb{C})_g$ which satisfy $F^*_\infty\omega=(-1)^p\omega$ and whose wave front sets are contained in S, we shall write $\widetilde{\mathcal{U}}(X_{\mu_n},S)$ for the current class

$$\widetilde{\mathcal{U}}(X(\mathbb{C})_g,S):=\bigoplus_{p\geq 0}(D^{p,p}(X(\mathbb{C})_g,S)/(\mathrm{Im}\partial+\mathrm{Im}\overline{\partial})).$$

Let $\overline{E}$ be an equivariant hermitian sheaf or vector bundle on X. Following the same notations and definitions as in Chapter II, Section 1, we write $\mathrm{ch}_g(\overline{E})$ for the equivariant Chern character form $\mathrm{ch}_g((E_{\mathbb{C}},h))$ associated to the hermitian holomorphic vector bundle $(E_{\mathbb{C}},h)$ on $X(\mathbb{C})$. Similarly, we have the equivariant Todd form $\mathrm{Td}_g(\overline{E})$. Furthermore, let $\overline{\varepsilon}:0\to\overline{E}'\to\overline{E}\to\overline{E}''\to 0$ be an exact sequence of equivariant hermitian sheaves or vector bundles on X, we can associate to it an equivariant Bott-Chern secondary characteristic class $\widetilde{\mathrm{ch}}_g(\overline{\varepsilon})\in\widetilde{\mathcal{U}}(X_{\mu_n},\emptyset)$ which satisfies the differential equation

$$\mathrm{dd}^c\widetilde{\mathrm{ch}}_g(\overline{\varepsilon})=\mathrm{ch}_g(\overline{E}')-\mathrm{ch}_g(\overline{E})+\mathrm{ch}_g(\overline{E}'').$$

Definition IV.3. Let A be a ring contained in $\mathbb{C}$. Let X be a μ_n-equivariant arithmetic scheme with smooth generic fibre and let S be a conical subset of $T^*_{\mathbb{R}}X(\mathbb{C})_{g,0}$, we define the equivariant arithmetic Grothendieck group $\widehat{G_{0,A}}(X,\mu_n,S)$ (resp. $\widehat{K_{0,A}}(X,\mu_n,S)$) with respect to X and S as the free A-module generated by the elements of $\widetilde{\mathcal{U}}(X_{\mu_n},S)$ and by the equivariant isometry classes of equivariant hermitian sheaves (resp. vector bundles) on X, together with the relations

(i). for every exact sequence $\overline{\varepsilon}$ as above, $\widetilde{\mathrm{ch}}_g(\overline{\varepsilon})=\overline{E}'-\overline{E}+\overline{E}''$;

(ii). for $\lambda\in A$ and for $\alpha',\alpha''\in\widetilde{\mathcal{U}}(X_{\mu_n},S)$, the equality

$$\lambda\cdot(0,\alpha')+(0,\alpha'')=(0,\lambda\cdot\alpha'+\alpha'')$$

holds in $\widehat{G_{0,A}}(X,\mu_n,S)$ (resp. $\widehat{K_{0,A}}(X,\mu_n,S)$).

From now on, we shall fix such a ring A and, for the reason of terseness, we shall remove the symbol A from the notations of the equivariant arithmetic Grothendieck groups $\widehat{G_{0,A}}(X,\mu_n,S)$ and $\widehat{K_{0,A}}(X,\mu_n,S)$.

Remark IV.4. (i). If $S' \subset S$, then the natural map from $\widetilde{\mathcal{U}}(X_{\mu_n}, S')$ to $\widetilde{\mathcal{U}}(X_{\mu_n}, S)$ is injective. This follows from a $\partial\overline{\partial}$-lemma for the spaces of currents with fixed wave front sets. (cf. [BL, Corollary 4.7])

(ii). If X is regular, then one can carry out the proof of [KR1, Proposition 4.2] to show that the natural morphism from $\widehat{K_0}(X, \mu_n, S)$ to $\widehat{G_0}(X, \mu_n, S)$ is an isomorphism.

(iii). The definition of the equivariant arithmetic Grothendieck group implies that there are exact sequences

$$\widetilde{\mathcal{U}}(X_{\mu_n}, S) \xrightarrow{a} \widehat{G_0}(X, \mu_n, S) \xrightarrow{\pi} G_0(X, \mu_n) \longrightarrow 0$$

and

$$\widetilde{\mathcal{U}}(X_{\mu_n}, S) \xrightarrow{a} \widehat{K_0}(X, \mu_n, S) \xrightarrow{\pi} K_0(X, \mu_n) \longrightarrow 0$$

where a is the natural map which sends $\alpha \in \widetilde{\mathcal{U}}(X_{\mu_n}, S)$ to the class of α in $\widehat{G_0}(X, \mu_n, S)$ (resp. $\widehat{K_0}(X, \mu_n, S)$) and π is the forgetful map. Here the group $G_0(X, \mu_n)$ is the Grothendieck group of μ_n-equivariant coherent sheaves which are locally free on $X(\mathbb{C})$, by a theorem of Quillen (cf. [Qui, Thm. 3 Cor. 1]) we know that it is isomorphic to the ordinary Grothendieck group of μ_n-equivariant coherent sheaves.

Now we introduced the A-algebra structure of $\widehat{K_0}(X, \mu_n, \emptyset)$. We consider the generators of the A-module $\widehat{K_0}(X, \mu_n, \emptyset)$, for two equivariant hermitian vector bundles $\overline{E}$, $\overline{E}'$ on X and two elements α, α' in $\widetilde{\mathcal{U}}(X_{\mu_n}, \emptyset)$, we define the rules of the product $\cdot$ as $\overline{E} \cdot \overline{E}' := \overline{E} \otimes \overline{E}'$, $\overline{E} \cdot \alpha = \alpha \cdot \overline{E} := \mathrm{ch}_g(\overline{E}) \wedge \alpha$ and $\alpha \cdot \alpha' := \mathrm{dd}^c\alpha \wedge \alpha'$. Note that α and α' are both smooth, so $\alpha \cdot \alpha'$ is well-defined and it is commutative in $\widetilde{\mathcal{U}}(X_{\mu_n}, \emptyset)$. It is easy to verify that our definition is compatible with the two generating relations in Definition IV.3, we leave the verification to the reader.

Since we may have a well-defined product of two currents if their wave front sets have no intersection, and the wave front set is invariant under the operation of multiplying a smooth current, we know that the Grothendieck group $\widehat{K_0}(X, \mu_n, S)$ has a $\widehat{K_0}(X, \mu_n, \emptyset)$-module structure. The same thing goes to $\widehat{G_0}(X, \mu_n, S)$. Furthermore, recall that $R(\mu_n) = \mathbb{Z}[T]/(1 - T^n)$. Let $\overline{I}$ be the μ_n-equivariant hermitian D-module whose term of degree 1 is D endowed with the trivial metric and whose other terms are 0. Then we may make $\widehat{K_0}(D, \mu_n, \emptyset)$ an $R(\mu_n)$-algebra under the ring morphism which sends T to $\overline{I}$. By doing pull-backs, we may endow every arithmetic Grothendieck group we defined with an $R(\mu_n)$-module structure. Notice finally that there is a well-defined map from $\widehat{G_0}(X, \mu_n, \emptyset)$ (resp. $\widehat{K_0}(X, \mu_n, \emptyset)$) to the space of complex closed differential forms, which is defined by the formula $\mathrm{ch}_g(\overline{E} + \alpha) := \mathrm{ch}_g(\overline{E}) + \mathrm{dd}^c\alpha$ where $\overline{E}$ is an equivariant hermitian sheaf (resp. vector bundle) and $\alpha \in \widetilde{\mathcal{U}}(X_{\mu_n}, \emptyset)$.

Now we investigate the wave front set of a current after doing push-forward. Let f be a holomorphic map of compact complex manifolds, we may define a push-forward f_* on current space which is the dual map of the pull-back of smooth forms. When f is smooth, the push-forward f_* extends the integration of smooth forms over the fibre. Assume that we are given a smooth morphism $f: U \to V$ of compact complex manifolds, then f_* induces a current K over the product space $V \times U$ defined as

$$K(\alpha \otimes \beta) = (f_*\beta)(\alpha)$$

where α and β are smooth forms over V and U respectively. Define

$$M = \{(v,u) \in V \times U \mid f(u) = v\}$$

which is a submanifold in $V \times U$. From the fact that $f_*\beta$ is just the integration of smooth forms over the fibre, it is easily seen that the current $K \in D^*(V \times U)$ is exactly the object $\mathrm{d}S_M$ in [Hoer, Theorem 8.1.5]. Then by that theorem, the wave front set of K is equal to

$$\mathrm{WF}(K) = \{(v,u,\xi,-f^*(\xi)) \in T^*_{\mathbb{R}}V \times T^*_{\mathbb{R}}U \mid f(u) = v, \xi \neq 0\}.$$

Let S be a conical subset of $T^*_{\mathbb{R}}U_0$, we fix some notations as follows.

$$\begin{aligned}\mathrm{WF}(K)_V =& \{(v,\xi) \in T^*_{\mathbb{R}}V_0 \mid \exists u \in U, (v,u,\xi,0) \in \mathrm{WF}(K)\}\\ \mathrm{WF}'(K)_U =& \{(u,\eta) \in T^*_{\mathbb{R}}U_0 \mid \exists v \in V, (v,u,0,-\eta) \in \mathrm{WF}(K)\}\\ \mathrm{WF}'(K)_V \circ S =& \{(v,\xi) \in T^*_{\mathbb{R}}V_0 \mid \exists (u,\eta) \in S, (v,u,\xi,-\eta) \in \mathrm{WF}(K)\}.\end{aligned}$$

Theorem IV.5. *Let notations and assumptions be as above. Assume that ω is a current over U whose wave front set is contained in S with $S \cap \mathrm{WF}'(K)_U = \emptyset$, then the wave front set of $f_*\omega$ is contained in*

$$S' := \mathrm{WF}(K)_V \cup \mathrm{WF}'(K) \circ S.$$

Proof. This follows from [Hoer, Theorem 8.2.12 and 8.2.13]. □

Remark IV.6. (i) In our situation, the condition $S \cap \mathrm{WF}'(K)_U = \emptyset$ is always satisfied because by definition we have $\mathrm{WF}'(K)_U = \emptyset$.

(ii). In our situation, S' is always equal to $\mathrm{WF}'(K) \circ S$ because $\mathrm{WF}(K)_V = \emptyset$.

(iii). If S is the empty set, then S' is also empty. This is compatible with the push-forward of smooth forms.

(iv). Assume that the restriction of f to a closed submanifold W is also smooth. Denote by $N_{U/W}$ the normal bundle of W in U. If $S = N^\vee_{U/W,\mathbb{R}} \setminus \{0\}$, then $S' = \emptyset$.

We now turn to the arithmetic case. Let X, Y be two μ_n-equivariant arithmetic schemes with smooth generic fibres, and let $f: X \to Y$ be an equivariant morphism over D which is smooth on the complex numbers. Fix a $\mu_n(\mathbb{C})$-invariant Kähler metric on $X(\mathbb{C})$ so that we get a Kähler fibration with respect to the holomorphic submersion $f_{\mathbb{C}}: X(\mathbb{C}) \to Y(\mathbb{C})$. Let $\overline{E}$ be an f-acyclic μ_n-equivariant hermitian sheaf on X, we know that the direct image f_*E is locally free on $Y(\mathbb{C})$ and it can be endowed with a natural equivariant structure and the L^2-metric. Let $\widehat{G_0}^{\mathrm{ac}}(X,\mu_n,S)$ be the A-module generated by f-acyclic equivariant hermitian sheaves on X and the elements of $\widetilde{\mathcal{U}}(X_{\mu_n},S)$, with the same relations as in Definition IV.3. A theorem of Quillen (cf. [Qui, Cor.3 P. 111]) for the algebraic analogs of these groups implies that the natural map $\widehat{G_0}^{\mathrm{ac}}(X,\mu_n,S) \to \widehat{G_0}(X,\mu_n,S)$ is an isomorphism. So the following definition does make sense.

Definition IV.7. Let notations and assumptions be as above. The push-forward morphism $f_*:\widehat{G_0}(X,\mu_n,S)\to\widehat{G_0}(Y,\mu_n,S')$ is defined in the following way.

(i). For every f-acyclic μ_n-equivariant hermitian sheaf $\overline{E}$ on X, $f_*\overline{E}=(f_*E,f_*h^E)-T_g(\omega^X,h^E)$.

(ii). For every element $\alpha\in\widetilde{\mathcal{U}}(X_{\mu_n},S)$, $f_*\alpha=\int_{X_g/Y_g}\mathrm{Td}_g(Tf,h^{Tf})\alpha\in\widetilde{\mathcal{U}}(Y_{\mu_n},S')$.

Remark IV.8. If Y is regular, by Remark IV.4 (ii) we know that $\widehat{K_0}(Y,\mu_n,S')$ is naturally isomorphic to $\widehat{G_0}(Y,\mu_n,S')$ so that (f_*E,f_*h^E) admits a finite equivariant hermitian resolution; if the morphism f is flat and Y is reduced, then (f_*E,f_*h^E) is locally free when E is so. Therefore in both two cases above, one can also define a reasonable push-forward morphism $f_*:\widehat{K_0}(X,\mu_n,S)\to\widehat{K_0}(Y,\mu_n,S')$.

Theorem IV.9. *The push-forward morphism f_* is a well-defined A-module homomorphism.*

Proof. We have to prove that our definition for f_* is compatible with the two generating relations of the equivariant arithmetic Grothendieck group. Indeed, assume that we are given a short exact sequence

$$\overline{\varepsilon}:\quad 0\to\overline{E}'\to\overline{E}\to\overline{E}''\to 0$$

of f-acyclic equivariant hermitian sheaves on X. We apply Bismut-Ma's immersion formula to the case where the fibration is with respect to $f_{\mathbb{C}}:f_{\mathbb{C}}^{-1}(Y_g)\to Y_g$ and the closed immersion is the identity map, then the equality

$$T_g(\omega^X,h^{E'})-T_g(\omega^X,h^E)+T_g(\omega^X,h^{E''})-\widetilde{\mathrm{ch}}_g(f_*\overline{\varepsilon})=-\int_{X_g/Y_g}\mathrm{Td}_g(Tf,h^{Tf})\widetilde{\mathrm{ch}}_g(\overline{\varepsilon})$$

holds in $\widetilde{\mathcal{U}}(Y_{\mu_n},\emptyset)$. Then by the definition of the push-forward map, we get

$$f_*\overline{E}'-f_*\overline{E}+f_*\overline{E}''=\int_{X_g/Y_g}\mathrm{Td}_g(Tf,h^{Tf})\widetilde{\mathrm{ch}}_g(\overline{\varepsilon}).$$

This final expression means that the push-forward morphism f_* is compatible with the first generating relation of the equivariant arithmetic Grothendieck groups. For the second one, it is rather clear from the definition. So we are done. □

Lemma IV.10. *(Projection formula) For any elements $y\in\widehat{K_0}(Y,\mu_n,\emptyset)$ and $x\in\widehat{G_0}(X,\mu_n,S)$, the identity $f_*(f^*y\cdot x)=y\cdot f_*x$ holds in $\widehat{G_0}(Y,\mu_n,S')$.*

Proof. Assume that $y=\overline{E}$ is an equivariant hermitian vector bundle and $x=\overline{F}$ is an f-acyclic equivariant hermitian sheaf, then $f^*y\cdot x=f^*\overline{E}\otimes\overline{F}$. By projection formula for direct images and the definition of the L^2-metric, we know that $f_*(f^*\overline{E}\otimes\overline{F})$ is isometric to $\overline{E}\otimes f_*\overline{F}$. Moreover, concerning the analytic torsion form, we have $T_g(\omega^X,h^{f^*E\otimes F})=\mathrm{ch}_g(\overline{E})T_g(\omega^X,h^F)$. So the projection formula $f_*(f^*y\cdot x)=y\cdot f_*x$ holds in this case.

Assume that $y=\overline{E}$ is an equivariant hermitian vector bundle and $x=\alpha$ is represented by some singular current. We write f_g^* and f_{g_*} for the pull-back and push-forward of currents respectively, then

$$\begin{aligned} f_*(f^*y\cdot x) =& f_*(f_g^*\mathrm{ch}_g(\overline{E})\alpha)=f_{g_*}(f_g^*\mathrm{ch}_g(\overline{E})\alpha\mathrm{Td}_g(\overline{Tf}))\\ =&\mathrm{ch}_g(\overline{E})f_{g_*}(\alpha\mathrm{Td}_g(\overline{Tf}))\\ =&\mathrm{ch}_g(\overline{E})\int_{X_g/Y_g}\alpha\mathrm{Td}_g(\overline{Tf})=y\cdot f_*x. \end{aligned}$$

Here we have used an extension of projection formula of smooth forms $p_*(p^*\alpha_1\wedge\alpha_2)=\alpha_1\wedge p_*\alpha_2$ (cf. [GHV, Prop. IX p. 303]) to the case where the second variable α_2 is replaced by a singular current. The fact that this extension is valid follows from the definition of p_* and the definition of the product of smooth form and singular current.

Assume that $y=\beta$ is represented by some smooth form and $x=\overline{E}$ is an f-acyclic hermitian sheaf, then

$$\begin{aligned} f_*(f^*y\cdot x) =& f_*(f_g^*(\beta)\mathrm{ch}_g(\overline{F}))=f_{g_*}(f_g^*(\beta)\mathrm{ch}_g(\overline{F})\mathrm{Td}_g(\overline{Tf}))\\ =&\beta f_{g_*}(\mathrm{ch}_g(\overline{E})\mathrm{Td}_g(\overline{Tf}))\\ =&\beta\int_{X_g/Y_g}\mathrm{ch}_g(\overline{E})\mathrm{Td}_g(\overline{Tf})=\beta(\mathrm{ch}_g(\overline{f_*F})-\mathrm{dd}^cT_g(\omega^X,h^F)) \end{aligned}$$

which is exactly $y\cdot f_*x$.

Finally, assume that $y=\beta$ is represented by some smooth form and $x=\alpha$ is represented by some singular current, then

$$\begin{aligned} f_*(f^*y\cdot x) =& f_*(f_g^*(\beta)\mathrm{dd}^c\alpha)=f_{g_*}(f_g^*(\beta)\mathrm{dd}^c\alpha\mathrm{Td}_g(\overline{Tf}))\\ =&\beta\mathrm{dd}^cf_{g_*}(\alpha\mathrm{Td}_g(\overline{Tf})) \end{aligned}$$

which is also equal to $y\cdot f_*x$.

Since f_* and f^* are both A-module homomorphisms, we may conclude the projection formula by linear extension. □

Remark IV.11. Lemma IV.10 implies that f_* is a homomorphism of $R(\mu_n)$-modules, and hence it induces a push-forward morphism after taking localization.

To end this section, we recall an important lemma which will be used frequently in our later arguments.

Lemma IV.12. *Let X be a regular μ_n-equivariant arithmetic scheme and let $\overline{E}$ be an equivariant hermitian vector bundle on X_{μ_n} such that $\overline{E}_{\mu_n}=0$. Then the element $\lambda_{-1}(\overline{E})$ is invertible in $\widehat{K_0}(X_{\mu_n},\mu_n,S)_\rho$.*

Proof. This immediately follows from [KR1, Lemma 4.5]. □

2 Concentration theorem for $\widehat{K_0}$-groups

Let X be a μ_n-equivariant arithmetic scheme with smooth generic fibre, we consider a special equivariant closed immersion $i : X_{\mu_n} \hookrightarrow X$ where X_{μ_n} stands for the fixed point subscheme of X. We shall first construct a well-defined A-module homomorphism i_* between equivariant arithmetic G_0-groups as in the algebraic case. To construct i_*, some analytic datum, which is the equivariant Bott-Chern singular current, should be involved. Precisely speaking, let $\overline{\eta}$ be a μ_n-equivariant hermitian sheaf on X_{μ_n} and let $\overline{\xi}.$ be a bounded complex of μ_n-equivariant hermitian sheaves which provides a resolution of $i_*\overline{\eta}$ on X. Such a resolution always exists since the generic fibre of X is supposed to be smooth. Then we may have an equivariant Bott-Chern singular current $T_g(\overline{\xi}.) \in \widetilde{\mathcal{U}}(X_{\mu_n})$. Note that on the complex numbers the 0-degree part of the normal bundle $N := N_{X/X_g}$ vanishes (cf. [KR1, Prop. 2.12]) so that the wave front set of $T_g(\overline{\xi}.)$ is the empty set. This fact means that the following definition does make sense.

Definition IV.13. Let notations and assumptions be as above. Let S be a conical subset of $T^*_{\mathbb{R}}X(\mathbb{C})_{g,0}$. The embedding morphism

$$i_* : \quad \widehat{G_0}(X_{\mu_n}, \mu_n, S) \to \widehat{G_0}(X, \mu_n, S)$$

is defined in the following way.

(i). For every μ_n-equivariant hermitian sheaf $\overline{\eta}$ on X_{μ_n}, suppose that $\overline{\xi}.$ is a resolution of $i_*\overline{\eta}$ on X whose metrics satisfy Bismut assumption (A),

$$i_*[\overline{\eta}] = \sum_k (-1)^k[\overline{\xi}_k] + T_g(\overline{\xi}.).$$

(ii). For every $\alpha \in \widetilde{\mathcal{U}}(X_{\mu_n}, S)$, $i_*\alpha = \alpha \mathrm{Td}_g^{-1}(\overline{N})$.

Theorem IV.14. *The embedding morphism i_* is a well-defined A-module homomorphism.*

Proof. We have to prove that our definition for i_* is well-defined and it is compatible with the two generating relations of the arithmetic G_0-groups. Indeed, assume that we are given a short exact sequence

$$\overline{\chi} : \quad 0 \to \overline{\eta}' \to \overline{\eta} \to \overline{\eta}'' \to 0$$

of equivariant hermitian sheaves on X_{μ_n}. As in Theorem II.19, let $\overline{\xi}.'$, $\overline{\xi}.$ and $\overline{\xi}.''$ be resolutions on X of $\overline{\eta}'$, $\overline{\eta}$ and $\overline{\eta}''$ which fit the following double complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{\xi}.' & \longrightarrow & \overline{\xi}. & \longrightarrow & \overline{\xi}.'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & i_*\overline{\eta}' & \longrightarrow & i_*\overline{\eta} & \longrightarrow & i_*\overline{\eta}'' & \longrightarrow & 0 \end{array}$$

such that all rows are exact. For each k, we write $\overline{\varepsilon}_k$ for the exact sequence

$$0 \to \overline{\xi}'_k \to \overline{\xi}_k \to \overline{\xi}''_k \to 0.$$

Then Theorem II.19 implies that the equality

$$T_g(\overline{\xi}.') - T_g(\overline{\xi}.) + T_g(\overline{\xi}.'') = \frac{\widetilde{\mathrm{ch}}_g(\overline{\chi})}{\mathrm{Td}_g(\overline{N})} - \sum_k (-1)^k \widetilde{\mathrm{ch}}_g(\overline{\varepsilon}_k)$$

holds in $\widetilde{\mathcal{U}}(X_{\mu_n}, \emptyset)$. This means $i_*[\overline{\eta}'] - i_*[\overline{\eta}] + i_*[\overline{\eta}''] = 0$ in the group $\widehat{G_0}(X, \mu_n, S)$ according to its generating relations. Note that if $\overline{\xi}.$ is an exact sequence then $T_g(\overline{\xi}.)$ is equal to $-\widetilde{\mathrm{ch}}_g(\overline{\xi}.)$, so we have $i_*[0] = 0$. Moreover, any two resolutions of $i_*\overline{\eta}$ are dominated by a third one, then our arguments above also show that $i_*[\overline{\eta}]$ is independent of the choice of the resolution. Therefore, the embedding morphism i_* is well-defined and it is compatible with the first generating relation of the arithmetic G_0-groups. On the other hand, the compatibility with the second relation is trivial. So we are done. □

Lemma IV.15. *(Projection formula) For any elements $x \in \widehat{K_0}(X, \mu_n, \emptyset)$ and $y \in \widehat{G_0}(X_{\mu_n}, \mu_n, S)$, the identity $i_*(i^*x \cdot y) = x \cdot i_*y$ holds in $\widehat{G_0}(X, \mu_n, S)$.*

Proof. Assume that $x = \overline{E}$ is an equivariant hermitian vector bundle and $y = \overline{F}$ is an equivariant hermitian sheaf. Let $\overline{\xi}.$ be a resolution of $i_*\overline{F}$ on X, then $\overline{E} \otimes \overline{\xi}.$ provides a resolution of $i_*(i^*\overline{E} \otimes \overline{F})$. By definition we have

$$i_*(i^*x \cdot y) = \sum (-1)^k [\overline{\xi}_k \otimes \overline{E}] + \mathrm{ch}_g(\overline{E}) T_g(\overline{\xi}.)$$

which is exactly $x \cdot i_*y$. Assume that $x = \alpha$ is represented by some smooth form and $y = \overline{F}$ is an equivariant hermitian sheaf. Again let $\overline{\xi}.$ be a resolution of $i_*\overline{F}$ on X, then

$$i_*(i^*x \cdot y) = \alpha \mathrm{Td}_g^{-1}(\overline{N}_{X/X_g}) \mathrm{ch}_g(\overline{F}) = \alpha[\mathrm{dd}^c T_g(\overline{\xi}.) + \sum (-1)^k \mathrm{ch}_g(\overline{\xi}_k)]$$

which is exactly $x \cdot i_*y$. Now assume that $x = \overline{E}$ is an equivariant hermitian vector bundle and $y = \alpha$ is represented by some singular current, then

$$i_*(i^*x \cdot y) = i_*(\mathrm{ch}_g(\overline{E})\alpha) = \mathrm{ch}_g(\overline{E}) \alpha \mathrm{Td}_g^{-1}(\overline{N}_{X/X_g})$$

which is exactly $x \cdot i_*y$. Finally, if x is represented by some smooth form and y is represented by some singular current then their product is well-defined and $i_*(i^*x \cdot y)$ is obviously equal to $x \cdot i_*y$. Note that i_* and i^* are A-module homomorphisms, so we may conclude the projection formula from its correctness on generators. This completes the whole proof. □

Remark IV.16. Lemma IV.15 implies that i_* is even a homomorphism of $R(\mu_n)$-modules so that it induces a homomorphism between arithmetic G_0-groups after taking localization.

With Remark IV.16, we may formulate the arithmetic concentration theorem as follows.

Theorem IV.17. *The embedding morphism $i_*: \widehat{K_0}(X_{\mu_n},\mu_n,S)_\rho \to \widehat{K_0}(X,\mu_n,S)_\rho$ is an isomorphism if X is regular. In this case, the inverse morphism of i_* is given by $\lambda_{-1}^{-1}(\overline{N}^\vee_{X/X_{\mu_n}})\cdot i^*$ where $N_{X/X_{\mu_n}}$ is the normal bundle of $i(X_{\mu_n})$ in X.*

The proof of this concentration theorem relies on the following crucial lemma.

Lemma IV.18. *Let $\overline{\eta}$ be an equivariant hermitian vector bundle on X_{μ_n}. Assume that $\overline{\xi.}$ is an equivariant hermitian resolution of $i_*\overline{\eta}$ on X whose metrics satisfy Bismut assumption (A). Then the equality*

$$\lambda_{-1}(\overline{N}^\vee_{X/X_{\mu_n}})\cdot\overline{\eta}-\sum_j(-1)^j i^*(\overline{\xi}_j)=T_g(\overline{\xi}.)$$

holds in the group $\widehat{K_0}(X_{\mu_n},\mu_n,S)$.

Before we give a proof of this lemma, we would like to make the following analytic preliminaries.

To every equivariant standard complex $\overline{\xi.}$ of equivariant hermitian vector bundles on an equivariant complex manifold X, we may associated a new canonical equivariant standard complex in which the metrics on homology bundles $H_*(\xi.\mid_{X_g})$ are induced by the metrics on $\xi.$ (see Chapter II, Section 3). This special choice of metrics will be denoted by $h^H_{\rm ind}$. It is easy to compute the difference of $\widetilde{\rm ch}_g(\overline{\xi}.,h^H)$ and $\widetilde{\rm ch}_g(\overline{\xi}.,h^H_{\rm ind})$. It is the alternating sum of secondary characteristic classes

$$\sum(-1)^i\widetilde{\rm ch}_g(H_i(\xi.\mid_{X_g}),h^H,h^H_{\rm ind}).$$

Now we define another equivariant secondary class associated to $(\overline{\xi}.,h^H_{\rm ind})$ by using the supertraces of Quillen's superconnections as follows.

For $s\in\mathbb{C}$ with ${\rm Re}(s)>1$, let

$$\zeta_1(s)=-\frac{1}{\Gamma(s)}\int_0^1 u^{s-1}\{\Phi{\rm Tr_s}[Ng{\rm exp}(-A_u^2)]-\Phi{\rm Tr_s}[Ng{\rm exp}(-\nabla^{H(\overline{\xi}.),2})]\}{\rm d}u$$

and similarly for $s\in\mathbb{C}$ with ${\rm Re}(s)<\frac{1}{2}$, let

$$\zeta_2(s)=-\frac{1}{\Gamma(s)}\int_1^\infty u^{s-1}\{\Phi{\rm Tr_s}[Ng{\rm exp}(-A_u^2)]-\Phi{\rm Tr_s}[Ng{\rm exp}(-\nabla^{H(\overline{\xi}.),2})]\}{\rm d}u.$$

We define $\zeta(\overline{\xi}.,h^H_{\rm ind}):=\frac{\partial}{\partial s}(\zeta_1+\zeta_2)(0)$. This is just a generalization of [Ma2, Définition 10.3] in the equivariant case, we refer to that paper for the explanation of the notations appearing in the definition of the zeta-functions above. We thank X. Ma for his comment that an argument similar to the one in [BGS1, Cor. 1.30] can be used to prove the following lemma.

Lemma IV.19. *Define* $\zeta(\overline{\xi}.,h^H):=\zeta(\overline{\xi}.,h^H_{\rm ind})+\sum(-1)^i\widetilde{\rm ch}_g(H_i(\xi.\mid_{X_g}),h^H,h^H_{\rm ind})$*. Then* $\zeta(\overline{\xi}.,h^H)$ *determines an element in* $\widetilde{A}(X_g)$ *which satisfies the three conditions in Theorem III.13 and hence we have* $\zeta(\overline{\xi}.,h^H)=\widetilde{\rm ch}_g(\overline{\xi}.,h^H)$*.*

Proof. Actually, according to [Ma2, Proposition 10.4], one just need to add the subscript g to every step in the argument given in [BGS1, Cor. 1.30] and nothing else should be changed. Here, we roughly describe that why $\zeta(\overline{\xi}.,h^H_{\rm ind})=0$ when $(\overline{\xi}.\mid_{X_g},h^H)$ is homologically split (note that in this case h^H should be equal to $h^H_{\rm ind}$). This can be seen from the following argument. If $(\overline{\xi}.\mid_{X_g},h^H)$ is homologically split, then up to isometries we may write $\overline{E}_k:=\overline{\xi}_k\mid_{X_g}\cong\overline{F}_k\oplus\overline{H}_k\oplus\overline{F}_{k-1}$ where $\{\overline{F}_k\}$ is a family of hermitian vector bundles on X_g. Moreover, the differential v is given by $(v_1,v_2,v_3)\mapsto(v_3,0,0)$. So we compute directly that $A_u^2\mid_{\overline{E}_k}=\nabla^2+u({\rm Id}_{\overline{F}_k}\oplus{\rm Id}_{\overline{F}_{k-1}})$. This equality implies that

$$\begin{aligned}{\rm Tr_s}[Ng{\rm exp}(-A_u^2)] &= \sum_k(-1)^k\{{\rm Tr}\mid_{\overline{F}_k}[kg{\rm exp}(-\nabla^2-u{\rm Id})]\\ &\quad+{\rm Tr}\mid_{\overline{H}_k}[kg{\rm exp}(-\nabla^2)]+{\rm Tr}\mid_{\overline{F}_{k-1}}[kg{\rm exp}(-\nabla^2-u{\rm Id})]\}\end{aligned}$$

and hence

$$\zeta_1(s)+\zeta_2(s)=-\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}e^{-u}\{\Phi{\rm Tr_s}[Ng{\rm exp}(-\nabla^2)]\}{\rm d}u=-\Phi{\rm Tr_s}[Ng{\rm exp}(-\nabla^2)]$$

which has nothing to do with s. So we get $\zeta(\overline{\xi}.,h^H)=\frac{\partial}{\partial s}(\zeta_1+\zeta_2)(0)=0$. □

Corollary IV.20. *We have* $\zeta(\overline{\xi}.,h^H_{\rm ind})=\widetilde{\rm ch}_g(\overline{\xi}.,h^H_{\rm ind})$ *in* $\widetilde{A}(X_g)$*.*

Now we go back to the arithmetic case. Let notations and assumptions be as in Lemma IV.18, the complex $\overline{\xi}_{\mathbb{C}}.$ is naturally an equivariant standard complex such that h^H is equal to $h^H_{\rm ind}$.

Proof. (Proof of Lemma IV.18) According to our remark given before the definition of Bismut assumption (A) in Chapter II, Section 3, we can split $\overline{\xi}.\mid_{X_{\mu_n}}$ into the following series of exact sequences of equivariant hermitian vector bundles

$$0\rightarrow\overline{\rm Im}\rightarrow\overline{\rm Ker}\rightarrow\wedge^*\overline{N}^\vee_{X/X_{\mu_n}}\otimes\overline{\eta}\rightarrow 0$$

and

$$0\rightarrow\overline{\rm Ker}\rightarrow\overline{\xi}_*\mid_{X_{\mu_n}}\rightarrow\overline{\rm Im}\rightarrow 0.$$

Then by the definition of the arithmetic K_0-theory, $\lambda_{-1}(\overline{N}^\vee_{X/X_{\mu_n}})\cdot\overline{\eta}-\sum_j(-1)^ji^*(\overline{\xi}.)$ is nothing but $\widetilde{\rm ch}_g(\overline{\xi}.,h^H)$ or $\widetilde{\rm ch}_g(\overline{\xi}.,h^H_{\rm ind})$ in $\widehat{K_0}(X_{\mu_n},\mu_n,S)$.

Comparing with the construction of the equivariant Bott-Chern singular current introduced in Chapter II, Section 3 or with more details in [Bi1, Section VI], we claim that in our special situation $T_g(\overline{\xi}.)$ is equal to $\zeta(\overline{\xi}.,h^H_{\rm ind})$ defined before this proof. Actually since $\xi.$ is supposed to admit the metrics satisfying Bismut assumption (A), the superconnection A_u in the definition of $\zeta(\overline{\xi}.,h^H_{\rm ind})$ is exactly the superconnection C_u in the definition of $T_g(\overline{\xi}.)$. Moreover, since $(H_*(\xi.\mid_{X_{\mu_n}}),h^H_{\rm ind})$ are isometric to $\wedge^*\overline{N}_{X/X_g}^\vee \otimes \overline{\eta}_{\mathbb{C}}$ the supertrace $\mathrm{Tr_s}[Ng\exp(-\nabla^{H(\overline{\xi}.),2})]$ in the definition of $\zeta(\overline{\xi}.,h_{\mathrm{ind}}^H)$ is equal to $-(\mathrm{Td}_g^{-1})'(\overline{N})\mathrm{ch}_g(\overline{\eta})$ in the definition of $T_g(\overline{\xi}.)$, this can be seen directly from the computation [Bi1, (6.26)]. So according to Corollary IV.20 we have $\widetilde{\mathrm{ch}}_g(\overline{\xi}.,h_{\mathrm{ind}}^H)=\zeta(\overline{\xi}.,h_{\mathrm{ind}}^H)=T_g(\overline{\xi}.)$ in $\widetilde{\mathcal{U}}(X_{\mu_n},\emptyset)$ and hence they are equal in the group $\widehat{K_0}(X_{\mu_n},\mu_n,S)$. This implies the equality in the statement of this lemma. □

We are now ready to give a complete proof of our arithmetic concentration theorem.

Proof. (of Theorem IV.17) Denote by U the complement of X_{μ_n} in X, then $j:U\hookrightarrow X$ is a μ_n-equivariant open subscheme of X whose fixed point set is empty. We consider the following double complex

$$\begin{array}{ccccccc}
\widetilde{\mathcal{U}}(X_{\mu_n},S)_\rho & \xrightarrow{i_*} & \widetilde{\mathcal{U}}(X_{\mu_n},S)_\rho & \xrightarrow{j^*} & \widetilde{\mathcal{U}}(U_{\mu_n},\emptyset)_\rho & \longrightarrow & 0\\
\downarrow a & & \downarrow a & & \downarrow a & & \\
\widehat{K_0}(X_{\mu_n},\mu_n,S)_\rho & \xrightarrow{i_*} & \widehat{K_0}(X,\mu_n,S)_\rho & \xrightarrow{j^*} & \widehat{K_0}(U,\mu_n,\emptyset)_\rho & \longrightarrow & 0\\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \\
K_0(X_{\mu_n},\mu_n)_\rho & \xrightarrow{i_*} & K_0(X,\mu_n)_\rho & \xrightarrow{j^*} & K_0(U,\mu_n)_\rho & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

whose first and second columns are both exact sequences according to Remark IV.4 (iii). For the third column, $K_0(U,\mu_n)_\rho$ is equal to 0 by [Tho, (2.1.3)], $\widetilde{\mathcal{U}}(U_{\mu_n},\emptyset)_\rho$ is also equal to 0 since U_{μ_n} is empty. Then from Remark IV.4 (iii) we know that $\widehat{K_0}(U,\mu_n,\emptyset)_\rho$ is equal to 0. We claim that $i_*:\widehat{K_0}(X_{\mu_n},\mu_n,S)_\rho\to\widehat{K_0}(X,\mu_n,S)_\rho$ is surjective. Indeed, for any element $x\in\widehat{K_0}(X,\mu_n,S)_\rho$ we may find an element $y\in\widehat{K_0}(X_{\mu_n},\mu_n,S)_\rho$ such that $i_*\pi(y)=\pi(x)$ because the third line is exact. This means $x-i_*(y)$ is in the kernel of π, so there exists an element $\alpha\in\widetilde{\mathcal{U}}(X_{\mu_n},S)_\rho$ such that $\alpha=x-i_*(y)$ in $\widehat{K_0}(X,\mu_n,S)_\rho$. Set $\beta=\alpha\mathrm{Td}_g(\overline{N})$, we get $i_*(y+\beta)=i_*(y)+\alpha=x$ in $\widehat{K_0}(X,\mu_n,S)_\rho$. Hence, i_* is surjective.

We now prove that the embedding morphism $i_*:\widehat{K_0}(X_{\mu_n},\mu_n,S)_\rho\to\widehat{K_0}(X,\mu_n,S)_\rho$ is really an isomorphism by constructing its inverse morphism. Let ω be an element in $\widetilde{\mathcal{U}}(X_{\mu_n},S)$, by definition we have

$$\begin{aligned}
\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)\cdot i^*i_*(\omega) =&\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)\cdot\omega\mathrm{Td}_g^{-1}(\overline{N}_{X/X_g})\\
=&\mathrm{ch}_g(\lambda_{-1}^{-1}(\overline{N}_{X/X_g}^\vee))\omega\mathrm{Td}_g^{-1}(\overline{N}_{X/X_g})\\
=&\omega.
\end{aligned}$$

Let $\overline{\eta}$ be an equivariant hermitian vector bundle on X_{μ_n} and assume that $\overline{\xi}.$ is an equivariant hermitian resolution of $i_*\overline{\eta}$ on X whose metrics satisfy Bismut assumption

(A), then by the definition of the embedding morphism i_* and Lemma IV.18 we have

$$\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)\cdot i^*i_*(\overline{\eta}) = \lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)\cdot i^*(\sum_k (-1)^k\overline{\xi}_k + T_g(\overline{\xi}.)) = \overline{\eta}.$$

So the inverse morphism of i_* is of the form $\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)\cdot i^*$ and we are done. $\square$

Chapter V

Arithmetic fixed point formulae of Lefschetz type

In this chapter, we shall first translate the vanishing theorem and the Bismut-Ma's immersion formula to their $\widehat{G_0}$-theoretic versions. Then we use these two results together with the arithmetic concentration theorem to prove two kinds of fixed point formulae of Lefschetz type in the context of Arakelov geometry.

1 Technical preliminaries

Let $f : X \to Y$ be a μ_n-equivariant morphism between two arithmetic schemes with smooth generic fibres, which is smooth on the complex numbers. This morphism f is automatically projective and hence proper, according to the definition of equivariant arithmetic scheme. Suppose that f factors through some regular equivariant arithmetic scheme Z. More precisely, our assumption is that there exist an equivariant closed immersion $i : X \hookrightarrow Z$ and an equivariant morphism $h : Z \to Y$ such that $f = h \circ i$ and h is also smooth on the complex numbers. Moreover, we shall assume that the μ_n-action on Y is trivial.

Let η be an equivariant coherent sheaf on X, then there exists a bounded complex of equivariant vector bundles which provides a resolution of $i_*\eta$ on Z because Z is regular. Since any two equivariant resolution of $i_*\eta$ can be dominated by a third one, the symbol $\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\eta, \mathcal{O}_{Z_{\mu_n}})$ does make sense.

We choose arbitrary μ_n-invariant Kähler forms ω^Z and ω^X on $Z(\mathbb{C})$ and $X(\mathbb{C})$ respectively, the Kähler form ω^X is not necessarily the Kähler form induced by ω^Z. The Kähler form on $X(\mathbb{C})$ induced by ω^Z will be denoted by ω^Z_X. Denote by N the normal bundle of $i_{\mathbb{C}}(X(\mathbb{C}))$ in $Z(\mathbb{C})$, we endow it with the quotient metric provided that $TX(\mathbb{C})$ carries the Kähler metric corresponding to ω^Z_X. Let $\overline{F}$ be the non-zero degree part of $\overline{N}$, then by [GBI, Exp. VII, Lem. 2.4 and Prop. 2.5] for any equivariant hermitian sheaf $\overline{\eta}$ on X there exists a canonical isomorphism on X_g

$$\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\eta, \mathcal{O}_{Z_{\mu_n}})_{\mathbb{C}} \cong \wedge^k F^\vee \otimes \eta_{\mathbb{C}} \mid_{X_g}$$

which is equivariant. This means we may endow $\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\eta, \mathcal{O}_{Z_{\mu_n}})_{\mathbb{C}}$ with a hermitian metric induced by the metrics on F and η so that it becomes an equivariant hermitian sheaf on X_{μ_n}.

The push-forward homomorphism from the arithmetic G_0-group $\widehat{G_0}(X, \mu_n, \emptyset)$ to $\widehat{G_0}(Y, \mu_n, \emptyset)$ with respect to the Kähler form ω^X is denoted by f_* as usual. The push-forward homomorphism from $\widehat{G_0}(X_{\mu_n}, \mu_n, \emptyset)$ to $\widehat{G_0}(Y, \mu_n, \emptyset)$ with respect to the Kähler form ω_X^Z will be denoted by $f^Z_{\mu_n *}$.

Moreover, we write $\widetilde{\mathrm{Td}}((Tf_g, \omega_X^Z), \overline{Th_g}\mid_{X_g})$ for the equivariant secondary Todd class of the following exact sequence

$$0 \to (Tf_g, \omega_X^Z) \to \overline{Th_g}\mid_{X_g} \to \overline{N}_g \to 0.$$

Then the $\widehat{G_0}$-theoretic vanishing theorem is the following.

Theorem V.1. *Let notations and assumptions be as above. Let $\overline{\eta}$ be an equivariant hermitian sheaf on X, and let*

$$\overline{\Psi}: \quad 0 \to \overline{\xi}_m \to \cdots \to \overline{\xi}_1 \to \overline{\xi}_0 \to i_*\overline{\eta} \to 0$$

be a resolution of $i_*\overline{\eta}$ by equivariant hermitian vector bundles on Z. Denote by $h_{\mu_n *}$ the push-forward homomorphism from $\widehat{K_0}(Z_{\mu_n}, \mu_n, N^\vee_{g,\mathbb{R}} \setminus \{0\})$ to $\widehat{G_0}(Y, \mu_n, \emptyset)$ with respect to the Kähler form ω^Z. Then the formula

$$\begin{aligned} & f^Z_{\mu_n *}(\sum(-1)^k \mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta}, \overline{\mathcal{O}}_{Z_{\mu_n}})) - h_{\mu_n *}(\sum(-1)^k(\overline{\xi}_k\mid_{Z_{\mu_n}})) \\ = & \int_{Z_g/Y} T_g(\overline{\xi}.)\mathrm{Td}(\overline{Th_g}) + \int_{X_g/Y} \mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g) \\ & + \int_{X_g/Y} \mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}((Tf_g, \omega_X^Z), \overline{Th_g}\mid_{X_g}) \end{aligned}$$

holds in $\widehat{G_0}(Y, \mu_n, \emptyset)$.

Proof. Following the same arguments given in the proof of Lemma III.9, we may show that the deformation to the normal cone $W(i)$ admits an equivariant hermitian very ample invertible sheaf $\overline{\mathcal{L}}$ which is relative to the morphism $l: W(i) \to Y$. By Theorem III.1 and the fact that $\mathcal{L}$ is very ample, we conclude that there exists an integer $k_0 > 0$ such that for $n \geq k_0$, $\mathcal{L}^{\otimes n}$ is l-acyclic and $\delta(\overline{\Psi}(n)_{\mathbb{C}}) = 0$. Then l factors through an equivariant projective space bundle $\mathbb{P}(\mathcal{E}^\vee)$ where $\mathcal{E}$ is locally free of rank $r+1$ on Y and $l_*\mathcal{L}^{\otimes k_0}$ is an equivariant quotient of $\mathcal{E}$. Denote by $p: \mathbb{P}(\mathcal{E}^\vee) \to Y$ the canonical projection. On $P := \mathbb{P}(\mathcal{E}^\vee)$, we have a canonical exact sequence

$$\mathcal{H}: \quad 0 \to \mathcal{O}_P \to p^*(\mathcal{E}^\vee)(1) \to \cdots \to p^*(\wedge^{r+1}\mathcal{E}^\vee)(r+1) \to 0.$$

Restricting this sequence to Z, we obtain an exact sequence of exact sequences

$$0 \to \Psi \to \Psi \otimes h^*(\mathcal{E}^\vee)(1) \to \cdots \to \Psi \otimes h^*(\wedge^{r+1}\mathcal{E}^\vee)(r+1) \to 0.$$

Endow $\mathcal{E}$ with any $\mu_n(\mathbb{C})$-invariant hermitian metric. We claim that the assumption that Theorem V.1 holds for $\overline{\Psi}\otimes h^*(\wedge^n\overline{\mathcal{E}}^\vee)(n)$ with $n\geq 1$ implies that it holds for $\overline{\Psi}$. In fact, since $\mathcal{H}$ is an exact sequence of flat modules, for any $k\geq 0$ we have the following exact sequence on X_{μ_n}

$$0\to \mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})\to \mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})\otimes f^*_{\mu_n}(\overline{\mathcal{E}}^\vee)(1)\to\cdots$$
$$\to \mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})\otimes f^*_{\mu_n}(\wedge^{r+1}\overline{\mathcal{E}}^\vee)(r+1)\to 0.$$

We compute

$$\begin{aligned}&f^Z_{\mu_n *}(\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}}))\\ =&f^Z_{\mu_n *}(-\sum_{j=1}^{r+1}(-1)^j\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})\otimes f^*_{\mu_n}(\wedge^j\overline{\mathcal{E}}^\vee)(j))\\ &+\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^Z_X)\mathrm{ch}_g(\wedge^k\overline{F}^\vee)\mathrm{ch}_g(\overline{\eta})(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=0}^m(-1)^k h_{\mu_n *}(\overline{\xi}_k\mid_{Z_{\mu_n}})\\ =&\sum_{k=0}^m(-1)^k h_{\mu_n *}(-\sum_{j=1}^{r+1}(-1)^j\overline{\xi}_k\mid_{Z_{\mu_n}}\otimes h^*_{\mu_n}(\wedge^j\overline{\mathcal{E}}^\vee)(j))\\ &+\sum_{k=0}^m(-1)^k\int_{Z_g/Y}\mathrm{Td}(\overline{Th_g})\mathrm{ch}_g(\overline{\xi}_k)(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}}).\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\int_{X_g/Y}\mathrm{Td}(Tf_g)\mathrm{ch}_g(\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\eta,\mathcal{O}_{Z_{\mu_n}}))R(N_g)\\ =&\int_{X_g/Y}-\sum_{j=1}^{r+1}(-1)^j\mathrm{ch}_g(\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\eta,\mathcal{O}_{Z_{\mu_n}})\otimes f^*_{\mu_n}(\wedge^j\mathcal{E}^\vee)(j))R(N_g)\mathrm{Td}(Tf_g)\end{aligned}$$

and

$$\begin{aligned}&\int_{Z_g/Y}T_g(\overline{\xi}.)\mathrm{Td}(\overline{Th_g})\\ =&\int_{Z_g/Y}\mathrm{Td}(\overline{Th_g})\{\delta_{X_g}\mathrm{Td}_g^{-1}(\overline{N})\mathrm{ch}_g(\overline{\eta})(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\\ &-\sum_{k=0}^m(-1)^k\mathrm{ch}_g(\overline{\xi}_k)(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})-\sum_{j=1}^{r+1}(-1)^jT_g(\overline{\xi}.)\mathrm{ch}_g(h^*_{\mu_n}(\wedge^j\overline{\mathcal{E}}^\vee)(j))\}\end{aligned}$$

by the double complex formula of equivariant Bott-Chern singular currents. At last, we also have

$$\begin{aligned}
&\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g})\\
=&\int_{X_g/Y}\{\mathrm{dd}^c(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\mathrm{ch}_g(\overline{\eta})-\sum_{j=1}^{r+1}(-1)^j\mathrm{ch}_g(\overline{\eta}\otimes f^*(\wedge^j\overline{\mathcal{E}}^\vee)(j))\}\\
&\qquad\cdot\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g})\\
=&-\int_{X_g/Y}(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\mathrm{ch}_g(\overline{\eta})\cdot\{\mathrm{Td}_g^{-1}(\overline{N})\mathrm{Td}(\overline{Th_g})\\
&\qquad-\mathrm{Td}(Tf_g,\omega_X^Z)\mathrm{Td}_g^{-1}(\overline{F})\}\\
&-\int_{X_g/Y}\sum_{j=1}^{r+1}(-1)^j\mathrm{ch}_g(\overline{\eta}\otimes f^*(\wedge^j\overline{\mathcal{E}}^\vee)(j))\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g}).
\end{aligned}$$

Gathering all these computations above and using our assumption, we get

$$\begin{aligned}
&f^Z_{\mu_n*}(\sum(-1)^k\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}}))-h_{\mu_n*}(\sum(-1)^k\overline{\xi}_k\mid_{Z_{\mu_n}})\\
&-\int_{Z_g/Y}T_g(\overline{\xi}.)\mathrm{Td}(\overline{Th_g})-\int_{X_g/Y}\mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\\
&\qquad-\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g})\\
=&\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega_X^Z)\mathrm{Td}_g^{-1}(\overline{F})\mathrm{ch}_g(\overline{\eta})(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\\
&-\sum_{k=0}^m(-1)^k\int_{Z_g/Y}\mathrm{Td}(\overline{Th_g})\mathrm{ch}_g(\overline{\xi}_k)(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\\
&-\int_{X_g/Y}\mathrm{Td}(\overline{Th_g})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{ch}_g(\overline{\eta})(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\\
&+\sum_{k=0}^m(-1)^k\int_{Z_g/Y}\mathrm{Td}(\overline{Th_g})\mathrm{ch}_g(\overline{\xi}_k)(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\\
&+\int_{X_g/Y}(-1)^{r+1}\widetilde{\mathrm{ch}}_g(\overline{\mathcal{H}})\mathrm{ch}_g(\overline{\eta})\{\mathrm{Td}_g^{-1}(\overline{N})\mathrm{Td}(\overline{Th_g})\\
&\qquad-\mathrm{Td}(Tf_g,\omega_X^Z)\mathrm{Td}_g^{-1}(\overline{F})\}
\end{aligned}$$

which vanishes. This ends the proof of our claim.

By the construction of the projective space bundle P, we have already known that $\delta(\overline{\Psi}(n)_{\mathbb{C}})$ vanishes from $n=1$ to $n=r+1$. Moreover, according to the projection formula of higher direct images, the operation of tensoring with the element $l^*(\wedge^n\overline{\mathcal{E}}^\vee)$ doesn't change the property of l-acyclicity. Hence we also have $\delta(\overline{\Psi}\otimes h^*(\wedge^n\overline{\mathcal{E}}^\vee)(n)_{\mathbb{C}})=0$. By the generating relations and the definition of push-forward morphisms of arithmetic

G_0-groups, this is equivalent to say that Theorem V.1 holds for $\overline{\Psi}\otimes h^*(\wedge^n\overline{\mathcal{E}}^\vee)(n)$. Therefore the equality in the statement of this theorem follows from our claim before. □

Corollary V.2. *Let notations and assumptions be as in Theorem V.1, and let x be any element in $\widehat{K_0}(Z,\mu_n,\emptyset)_\rho$. Then the formula*

$$\begin{aligned}
&f^Z_{\mu_n *}(i^*x\mid_{X_{\mu_n}}\cdot\sum(-1)^k\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}}))\\
&\qquad\qquad - h_{\mu_n *}(x\mid_{Z_{\mu_n}}\cdot\sum(-1)^k(\overline{\xi}_k\mid_{Z_{\mu_n}}))\\
&=\int_{Z_g/Y}T_g(\overline{\xi}.)\mathrm{Td}(\overline{Th_g})\mathrm{ch}_g(x)\\
&\quad+\int_{X_g/Y}\mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\mathrm{ch}_g(i^*x)\\
&\quad+\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{ch}_g(i^*x)\widetilde{\mathrm{Td}}((Tf_g,\omega^Z_X),\overline{Th_g}\mid_{X_g})
\end{aligned}$$

holds in $\widehat{G_0}(Y,\mu_n,\emptyset)_\rho$.

Proof. If $x=\overline{E}$ is an equivariant hermitian vector bundle on Z, then $\overline{\xi}.\otimes\overline{E}$ provides a resolution of $i_*(\overline{\eta}\otimes i^*\overline{E})$. Hence the formula follows from Theorem V.1 in this case. If $x=\alpha$ is represented by some smooth form, then

$$\begin{aligned}
&f^Z_{\mu_n *}(i^*x\mid_{X_{\mu_n}}\cdot\sum(-1)^k\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}}))\\
&=\int_{Z_g/Y}\mathrm{Td}(Tf_g,\omega^Z_X)\mathrm{Td}_g^{-1}(\overline{F})\mathrm{ch}_g(\overline{\eta})\delta_{X_g}\alpha
\end{aligned}$$

and

$$h_{\mu_n *}(x\mid_{Z_{\mu_n}}\cdot\sum(-1)^k(\overline{\xi}_k\mid_{Z_{\mu_n}}))=\int_{Z_g/Y}\mathrm{Td}(\overline{Th_g})\alpha\sum(-1)^k\mathrm{ch}_g(\overline{\xi}_k).$$

Moreover, by the definition of $\mathrm{ch}_g(x)$ we have

$$\begin{aligned}
\int_{Z_g/Y}T_g(\overline{\xi}.)\mathrm{Td}(\overline{Th_g})\mathrm{ch}_g(x)&=\int_{Z_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\delta_{X_g}\mathrm{Td}(\overline{Th_g})\alpha\\
&\quad-\int_{Z_g/Y}\sum(-1)^k\mathrm{ch}_g(\overline{\xi}_k)\mathrm{Td}(\overline{Th_g})\alpha
\end{aligned}$$

and

$$\int_{X_g/Y}\mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\mathrm{ch}_g(i^*x)=0.$$

Finally, using the definition of $\widetilde{\mathrm{Td}}$ we compute

$$\begin{aligned}&\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{ch}_g(i^*x)\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g})\\ =&\int_{Z_g/Y}\mathrm{Td}(Tf_g,\omega_X^Z)\mathrm{Td}_g^{-1}(\overline{F})\mathrm{ch}_g(\overline{\eta})\delta_{X_g}\alpha\\ &\qquad\qquad -\int_{Z_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\delta_{X_g}\mathrm{Td}(\overline{Th_g})\alpha.\end{aligned}$$

Gathering all computations above, we know that the formula still holds for x which is represented by smooth form. Since both two sides are additive, we are done. □

Corollary V.3. *Let notations and assumptions be as in Theorem V.1, and let y be any element in $\widehat{K_0}(Z_{\mu_n},\mu_n,\emptyset)_\rho$. Then the formula*

$$\begin{aligned}&f^Z_{\mu_n*}(i^*_{\mu_n}y\cdot\sum(-1)^k\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}}))-h_{\mu_n*}(y\cdot\sum(-1)^k(\overline{\xi}_k\mid_{Z_{\mu_n}}))\\ =&\int_{Z_g/Y}T_g(\overline{\xi}.)\mathrm{Td}(\overline{Th_g})\mathrm{ch}_g(y)\\ &+\int_{X_g/Y}\mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\mathrm{ch}_g(i^*_{\mu_n}y)\\ &+\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{ch}_g(i^*_{\mu_n}y)\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g})\end{aligned}$$

holds in $\widehat{G_0}(Y,\mu_n,\emptyset)_\rho$.

Proof. Provided Corollary V.2, it is enough to prove that for any $y\in\widehat{K_0}(Z_{\mu_n},\mu_n,\emptyset)_\rho$ there exists an element $x\in\widehat{K_0}(Z,\mu_n,\emptyset)_\rho$ such that $i_Z^*x=y$. Here i_Z stands for the inclusion $Z_{\mu_n}\hookrightarrow Z$. Actually, set $x=i_{Z*}(\lambda_{-1}^{-1}(\overline{N}^\vee_{Z/Z_{\mu_n}})\cdot y)$, we have

$$i_Z^*x=i_Z^*i_{Z*}(\lambda_{-1}^{-1}(\overline{N}^\vee_{Z/Z_{\mu_n}})\cdot y)=\lambda_{-1}(\overline{N}^\vee_{Z/Z_{\mu_n}})\cdot\lambda_{-1}^{-1}(\overline{N}^\vee_{Z/Z_{\mu_n}})\cdot y=y.$$

This follows from our arithmetic concentration theorem. □

The following is the $\widehat{G_0}$-theoretic version of Bismut-Ma's immersion formula.

Theorem V.4. *Let notations and assumptions be as in Theorem V.1. Then the equality*

$$\begin{aligned}f^Z_*(\overline{\eta})-\sum_{j=0}^m(-1)^jh_*(\overline{\xi}_j)=&\int_{X_g/Y}\mathrm{ch}_g(\eta)R_g(N)\mathrm{Td}_g(Tf)\\ &+\int_{Z_g/Y}T_g(\overline{\xi}.)\mathrm{Td}_g(\overline{Th})\\ &+\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g((Tf,\omega_X^Z),\overline{Th}\mid_X)\mathrm{Td}_g^{-1}(\overline{N})\end{aligned}$$

holds in $\widehat{G_0}(Y,\mu_n,\emptyset)$.

Proof. We first suppose that η and $\xi.$ are all acyclic, then the verification follows rather directly from the generating relations of arithmetic G_0-theory. In fact

$$\begin{aligned} f_*^Z(\overline{\eta}) - \sum_{j=0}^{m}(-1)^j h_*(\overline{\xi}_j) =& \overline{f_*\eta} - T_g(\omega_X^Z, h^\eta) - (\sum_{j=0}^{m}(-1)^j(\overline{h_*\xi_j} - T_g(\omega^Z, h^{\xi_j}))) \\ =& \widetilde{\mathrm{ch}}_g(h_*\Xi) - T_g(\omega_X^Z, h^\eta) + \sum_{j=0}^{m}(-1)^j T_g(\omega^Z, h^{\xi_j}). \end{aligned}$$

And the right-hand side of the last equality is exactly the left-hand side of Bismut-Ma's immersion formula. We emphasize again that to simplify the right-hand side of Bismut-Ma's immersion formula, we have used an Atiyah-Segal-Singer type formula for immersion

$$i_{g*}(\mathrm{Td}_g^{-1}(N)\mathrm{ch}_g(x)) = \mathrm{ch}_g(i_*(x)).$$

To remove the condition of acyclicity, one can use the argument which is essentially the same as in the proof of Theorem V.1. Since it doesn't use any new techniques, we omit it here. □

2 Regular case : the first type of the fixed point formula

Let $f : X \to Y$ be a μ_n-equivariant morphism between two regular arithmetic schemes, which is smooth on the complex numbers. In this section, we remove the limitation that the μ_n-action on Y is trivial, but we additionally suppose that f is flat and the fibre product $f^{-1}(Y_{\mu_n})$ is also regular. We shall naturally endow $X_{\mu_n}(\mathbb{C})$ and $f^{-1}(Y_{\mu_n}(\mathbb{C}))$ with the Kähler metrics induced by the Kähler metric of $X(\mathbb{C})$.

We consider the following Cartesian square

$$\begin{array}{ccc} f^{-1}(Y_{\mu_n}) & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y_{\mu_n} & \longrightarrow & Y \end{array}$$

whose rows are both closed immersions of regular μ_n-equivariant arithmetic schemes and whose columns are both flat. Then the normal bundle $N_{X/f^{-1}(Y_{\mu_n})}$ is isomorphic to the pull-back of normal bundle $f^*N_{Y/Y_{\mu_n}}$. Therefore by restricting to X_{μ_n} we get a short exact sequence

$$\overline{\mathcal{N}}: \quad 0 \to \overline{N}_{f^{-1}(Y_{\mu_n})/X_{\mu_n}} \to \overline{N}_{X/X_{\mu_n}} \to f^*\overline{N}_{Y/Y_{\mu_n}} \to 0$$

of equivariant hermitian vector bundles. Here all metrics on these normal bundles are the quotient metrics. Define

$$\begin{aligned} M(f) := \big(\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)\lambda_{-1}(f^*\overline{N}_{Y/Y_{\mu_n}}^\vee) + \widetilde{\mathrm{Td}}_g(\overline{\mathcal{N}})\mathrm{Td}_g^{-1}(f^*\overline{N}_{Y/Y_{\mu_n}})\big) \\ \cdot\big(1 - R_g(N_{X/X_{\mu_n}}) + R_g(f^*N_{Y/Y_{\mu_n}})\big). \end{aligned}$$

The first type of the fixed point formula can be formulated as follows.

Theorem V.5. *Let notations and assumptions be as above. Then the following diagram*

$$\begin{array}{ccc} \widehat{K_0}(X,\mu_n,\emptyset) & \xrightarrow{M(f)\cdot\tau} & \widehat{K_0}(X_{\mu_n},\mu_n,\emptyset)_\rho \\ \downarrow f_* & & \downarrow f_{\mu_n *} \\ \widehat{K_0}(Y,\mu_n,\emptyset) & \xrightarrow{\tau} & \widehat{K_0}(Y_{\mu_n},\mu_n,\emptyset)_\rho \end{array}$$

commutes, where τ stands for the restriction map.

Proof. Since f is flat, the Cartesian square introduced before this theorem induces a commutative diagram in arithmetic K_0-theory

$$\begin{array}{ccc} \widehat{K_0}(X,\mu_n,\emptyset) & \xrightarrow{\tau} & \widehat{K_0}(f^{-1}(Y_{\mu_n}),\mu_n,\emptyset) \\ \downarrow f_* & & \downarrow f_* \\ \widehat{K_0}(Y,\mu_n,\emptyset) & \xrightarrow{\tau} & \widehat{K_0}(Y_{\mu_n},\mu_n,\emptyset). \end{array}$$

From the exact sequence

$$\overline{\mathcal{N}}: \quad 0\to \overline{N}_{f^{-1}(Y_{\mu_n})/X_{\mu_n}}\to \overline{N}_{X/X_{\mu_n}}\to f^*\overline{N}_{Y/Y_{\mu_n}}\to 0$$

we know that

$$\lambda_{-1}^{-1}(\overline{N}_{f^{-1}(Y_{\mu_n})/X_{\mu_n}}^\vee)\lambda_{-1}^{-1}(f^*\overline{N}_{Y/Y_{\mu_n}}^\vee)-\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)=\widetilde{\mathrm{Td}}_g(\overline{\mathcal{N}})$$

according to [KR1, Lemma 7.1] and that

$$R_g(N_{f^{-1}(Y_{\mu_n})/X_{\mu_n}})+R_g(f^*N_{Y/Y_{\mu_n}})=R_g(N_{X/X_{\mu_n}})$$

since the equivariant R-genus is additive. Therefore, we may reduce our proof to the case where the μ_n-action on the base scheme Y is trivial. In this case, denote by i the canonical closed immersion $X_{\mu_n}\to X$. Then for any element $x\in\widehat{K_0}(X,\mu_n,\emptyset)_\rho$, we have

$$f_*(x)=f_*i_*i_*^{-1}(x)=f_*i_*(\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee)\cdot\tau(x)).$$

Consider the factorization $f_{\mu_n}=f\circ i$, we have to compute the difference $f_{\mu_n *}-f_*i_*$ in arithmetic K_0-theory. Indeed, this difference can be measured by the Bismut-Ma's immersion formula. By applying Theorem V.4 to the closed immersion i, for any equivariant hermitian vector bundle $\overline{\eta}$ on X_{μ_n} we have

$$f_{\mu_n *}(\overline{\eta})-f_*i_*(\overline{\eta})=\int_{X_g/Y}\mathrm{ch}_g(\eta)R_g(N_{X/X_{\mu_n}})\mathrm{Td}_g(Tf_{\mu_n})=f_{\mu_n *}(\mathrm{ch}_g(\eta)R_g(N_{X/X_{\mu_n}})).$$

The first equality holds because the exact sequence

$$0\to\overline{Tf_g}\to\overline{Tf}\mid_{X_g}\to\overline{N}_{X/X_g}\to 0$$

is orthogonally split on X_g so that $\widetilde{\mathrm{Td}}_g(\overline{Tf_g}, \overline{Tf}\mid_{X_g}) = 0$. The second equality follows from [KR1, Lemma 7.3] and the fact that $\mathrm{ch}_g(\eta)R_g(N_{X/X_{\mu_n}})$ is dd^c-closed. On the other hand, let α be an element in $\widetilde{\mathcal{U}}(X_{\mu_n}, \emptyset)$, we have

$$\begin{aligned} f_{\mu_n *}(\alpha) - f_* i_*(\alpha) &= \int_{X_g/Y} \alpha \mathrm{Td}_g(\overline{Tf_{\mu_n}}) - \int_{X_g/Y} \alpha \mathrm{Td}_g^{-1}(\overline{N}_{X/X_{\mu_n}}) \mathrm{Td}_g(\overline{Tf}) \\ &= \int_{X_g/Y} \alpha \mathrm{Td}_g^{-1}(\overline{N}_{X/X_{\mu_n}}) \mathrm{dd}^c \widetilde{\mathrm{Td}}_g(\overline{Tf_g}, \overline{Tf}\mid_{X_g}) = 0. \end{aligned}$$

Combing these two computations, by the A-algebra structure of $\widehat{K_0}(\cdot)$, we know that the equality

$$f_{\mu_n *}(y) - f_* i_*(y) = f_{\mu_n *}(y \cdot R_g(N_{X/X_{\mu_n}}))$$

holds for any element $y \in \widehat{K_0}(X_{\mu_n}, \mu_n, \emptyset)$ since both two sides are additive. Now continue our computation for $f_*(x)$, we obtain that

$$\begin{aligned} f_*(x) &= f_{\mu_n *}(\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee) \cdot \tau(x)) - f_{\mu_n *}(\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee) \cdot \tau(x) \cdot R_g(N_{X/X_{\mu_n}})) \\ &= f_{\mu_n *}(\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}}^\vee) \cdot (1 - R_g(N_{X/X_{\mu_n}})) \cdot \tau(x)). \end{aligned}$$

The last thing should be indicated is that the following square

$$\begin{array}{ccc} \widehat{K_0}(X, \mu_n, \emptyset) & \xrightarrow{\iota} & \widehat{K_0}(X, \mu_n, \emptyset)_\rho \\ \downarrow f_* & & \downarrow f_* \\ \widehat{K_0}(Y, \mu_n, \emptyset) & \xrightarrow{\iota} & \widehat{K_0}(Y, \mu_n, \emptyset)_\rho \end{array}$$

is naturally commutative. Here ι is the natural morphism from a ring or a module to its localization which sends an element e to $\frac{e}{1}$. So the fixed point formula holds in the case where Y admits the trivial μ_n-action. By the observation given at the beginning of this proof, it is enough to conclude the statement in our theorem. □

Remark V.6. (i). Theorem V.5 is a natural generalization of Theorem I.5 in the context of Arakelov geometry.

(ii). The condition that $f^{-1}(Y_{\mu_n})$ is regular can be satisfied if the μ_n-action on the base scheme is trivial or the morphism f is not only smooth over the complex numbers but also smooth everywhere. This is already enough for the applications in practice. For instance, our main result implies various formulae stated as conjectural in [MR2], in particular Proposition 2.3 in that article.

(iii). The condition of flatness on the morphism f is only used in the reduction of the general case to the case where the μ_n-action on the base scheme Y is trivial. If the μ_n-action on Y is trivial, one can certainly remove the condition of flatness.

3 Singular case : the second type of the fixed point formula

In this section, we shall fix the same setting as in Section 1 of this chapter. Namely, the equivariant arithmetic schemes X and Y are not necessarily regular and the μ_n-action on Y is supposed to be trivial.

Recall that $\overline{F}$ is the non-zero degree part of $\overline{N}$, where $\overline{N}$ is the normal bundle of $i_{\mathbb{C}}(X(\mathbb{C}))$ in $Z(\mathbb{C})$ endowed with the quotient metric provided that $TX(\mathbb{C})$ carries the Kähler metric corresponding to ω_X^Z. It is well known that the hermitian vector bundle $\overline{F}$ fits the following exact sequence

$$(\overline{\mathcal{F}},\omega^X):\quad 0\to\overline{N}_{X/X_g}\to\overline{N}_{Z/Z_g}\to\overline{F}\to 0$$

where N_{Z/Z_g} admits the quotient metric associated to ω^Z and N_{X/X_g} admits the quotient metric associated to ω^X. Similarly, we shall denote by $(\overline{\mathcal{F}},\omega_X^Z)$ the hermitian exact sequence $\overline{\mathcal{F}}$ whose metric on N_{X/X_g} is induced by ω_X^Z.

Write $\widetilde{\mathrm{Td}}(Tf_g,\omega^X,\omega_X^Z)$ for the secondary characteristic class of the exact sequence

$$0\longrightarrow(Tf_g,\omega^X)\xrightarrow{\mathrm{Id}}(Tf_g,\omega_X^Z)\longrightarrow 0\longrightarrow 0$$

where the middle term carries the metric induced by ω_X^Z and the sub term carries the metric induced by ω^X. Then the second type of the fixed point formula can be formulated as follows.

Theorem V.7. *Let notations and assumptions be as above. Then for any equivariant hermitian sheaf $\overline{\eta}$ on X, the equality*

$$\begin{aligned} f_*(\overline{\eta}) =& f^Z_{\mu_n *}(i^*_{\mu_n}(\lambda_{-1}^{-1}(\overline{N}^\vee_{Z/Z_{\mu_n}}))\cdot\sum_k(-1)^k\mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}}))\\ &+\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^X)\mathrm{Td}_g^{-1}(\overline{F})\\ &-\int_{X_g/Y}\mathrm{Td}_g(Tf)\mathrm{ch}_g(\eta)R_g(N_{X/X_g})\\ &+\int_{X_g/Y}\widetilde{\mathrm{Td}}(Tf_g,\omega^X,\omega_X^Z)\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\mathrm{Td}_g^{-1}(\overline{F}) \end{aligned}$$

holds in the group $\widehat{G_0}(Y,\mu_n,\emptyset)_\rho$.

Definition V.8. The inclusion $i: X\hookrightarrow Z$ induces an embedding morphism

$$i_*:\quad \widehat{G_0}(X,\mu_n,\emptyset)\to\widehat{K_0}(Z,\mu_n,N^\vee_{g,\mathbb{R}}\setminus\{0\})$$

which is defined as follows.

(i). For every μ_n-equivariant hermitian sheaf $\overline{\eta}$ on X, suppose that $\overline{\xi.}$ is a resolution of $i_*\overline{\eta}$ on Z whose metrics satisfy Bismut assumption (A),

$$i_*[\overline{\eta}]=\sum_k(-1)^k[\overline{\xi}_k]+T_g(\overline{\xi.}).$$

(ii). For every $\alpha \in \widetilde{\mathcal{U}}(X_{\mu_n}, \emptyset)$, $i_*\alpha = \alpha \mathrm{Td}_g^{-1}(\overline{N})\delta_{X_g}$.

Remark V.9. Similar to Theorem IV.14 and Lemma IV.15, one can prove that the embedding morphism is a well-defined homomorphism of $R(\mu_n)$-modules.

Proof. (of Theorem V.7) We first prove that this fixed point formula holds when ω^X is equal to ω_X^Z, namely the Kähler metric on $X(\mathbb{C})$ is induced by the Kähler metric on $Z(\mathbb{C})$. By Theorem V.4 and Definition V.8, we have the following equality

$$\begin{aligned} f_*^Z(\overline{\eta}) = & h_* i_*(\overline{\eta}) + \int_{X_g/Y} \mathrm{ch}_g(\eta) R_g(N) \mathrm{Td}_g(Tf) \\ & + \int_{X_g/Y} \mathrm{ch}_g(\overline{\eta}) \widetilde{\mathrm{Td}}_g((Tf, \omega_X^Z), \overline{Th} \mid_X) \mathrm{Td}_g^{-1}(\overline{N}) \end{aligned}$$

which holds in $\widehat{G_0}(Y, \mu_n, \emptyset)$. Now we claim that for any element $y \in \widehat{K_0}(Z_{\mu_n}, \mu_n, N_{g,\mathbb{R}}^\vee \setminus \{0\})_\rho$, we have

$$h_{\mu_n *}(y) - h_* i_{Z*}(y) = h_{\mu_n *}(y \cdot R_g(N_{Z/Z_{\mu_n}})).$$

Since all morphisms are homomorphisms of $R(\mu_n)$-modules, we can only consider the generators of $\widehat{K_0}(Z_{\mu_n}, \mu_n, N_{g,\mathbb{R}}^\vee \setminus \{0\})$. Indeed, by applying Theorem V.4 to the closed immersion i_Z, for any equivariant hermitian vector bundle $\overline{E}$ on Z_{μ_n} we have

$$\begin{aligned} h_{\mu_n *}(\overline{E}) - h_* i_{Z*}(\overline{E}) = & \int_{Z_g/Y} \mathrm{ch}_g(E) R_g(N_{Z/Z_{\mu_n}}) \mathrm{Td}_g(Th_g) \\ = & h_{\mu_n *}(\mathrm{ch}_g(E) R_g(N_{Z/Z_{\mu_n}})). \end{aligned}$$

The first equality holds because the exact sequence

$$0 \to \overline{Th_g} \to \overline{Th} \mid_{Z_g} \to \overline{N}_{Z/Z_g} \to 0$$

is orthogonally split on Z_g so that $\widetilde{\mathrm{Td}}_g(\overline{Th_g}, \overline{Th} \mid_{Z_g}) = 0$. The second equality follows from [KR1, Lemma 7.3] and the fact that $\mathrm{ch}_g(E) R_g(N_{Z/Z_{\mu_n}})$ is dd^c-closed. On the other hand, let α be an element in $\widetilde{\mathcal{U}}(Z_{\mu_n}, N_{g,\mathbb{R}}^\vee \setminus \{0\})$, we have

$$\begin{aligned} h_{\mu_n *}(\alpha) - h_* i_{Z*}(\alpha) = & \int_{Z_g/Y} \alpha \mathrm{Td}_g(\overline{Th_g}) - \int_{Z_g/Y} \alpha \mathrm{Td}_g^{-1}(\overline{N}_{Z/Z_{\mu_n}}) \mathrm{Td}_g(\overline{Th}) \\ = & \int_{Z_g/Y} \alpha \mathrm{Td}_g^{-1}(\overline{N}_{Z/Z_{\mu_n}}) \mathrm{dd}^c \widetilde{\mathrm{Td}}_g(\overline{Th_g}, \overline{Th} \mid_{Z_g}) = 0. \end{aligned}$$

Combing these two computations, we get our claim by linear extension.

Now using arithmetic concentration theorem, we compute

$$\begin{aligned}
h_*i_*(\overline{\eta}) =& h_*i_{Z*}i_{Z*}^{-1}i_*(\overline{\eta}) \\
=& h_{\mu_n*}(i_{Z*}^{-1}i_*(\overline{\eta})\cdot(1-R_g(N_{Z/Z_{\mu_n}}))) \\
=& h_{\mu_n*}(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)\cdot i_Z{}^*i_*(\overline{\eta})\cdot(1-R_g(N_{Z/Z_{\mu_n}}))) \\
=& h_{\mu_n*}(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)\cdot\{\sum_k(-1)^k(\overline{\xi}_k\mid_{Z_{\mu_n}}) \\
& \qquad\qquad + T_g(\overline{\xi}.)\}\cdot(1-R_g(N_{Z/Z_{\mu_n}}))) \\
=& h_{\mu_n*}(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)\cdot\sum_k(-1)^k(\overline{\xi}_k\mid_{Z_{\mu_n}})) \\
& + h_{\mu_n*}(\mathrm{Td}_g(\overline{N}_{Z/Z_{\mu_n}})T_g(\overline{\xi}.)) \\
& - h_{\mu_n*}(\mathrm{Td}_g(N_{Z/Z_{\mu_n}})\sum_k(-1)^k\mathrm{ch}_g(\xi_k)R_g(N_{Z/Z_{\mu_n}})).
\end{aligned}$$

According to Corollary V.3, by setting $y=\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)$, we compute

$$\begin{aligned}
& h_{\mu_n*}(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee)\cdot\sum_k(-1)^k(\overline{\xi}_k\mid_{Z_{\mu_n}})) \\
=& f_{\mu_n*}^Z(i_{\mu_n}^*(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee))\cdot\sum(-1)^k\mathrm{Tor}_{\mathcal{O}_Z}^k(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})) \\
& -\int_{Z_g/Y}T_g(\overline{\xi}.)\mathrm{Td}(\overline{Th_g})\mathrm{Td}_g(\overline{N}_{Z/Z_g}) \\
& -\int_{X_g/Y}\mathrm{Td}(Tf_g)\mathrm{Td}_g^{-1}(F)\mathrm{ch}_g(\eta)R(N_g)\mathrm{Td}_g(N_{Z/Z_g}) \\
& -\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g}) \\
=& f_{\mu_n*}^Z(i_{\mu_n}^*(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee))\cdot\sum(-1)^k\mathrm{Tor}_{\mathcal{O}_Z}^k(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}})) \\
& -\int_{Z_g/Y}T_g(\overline{\xi}.)\mathrm{Td}_g(\overline{Th})-\int_{X_g/Y}\mathrm{Td}_g(Tf)\mathrm{ch}_g(\eta)R(N_g) \\
& -\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\widetilde{\mathrm{Td}}((Tf_g,\omega_X^Z),\overline{Th_g}\mid_{X_g}).
\end{aligned}$$

Here we have used various relations of character forms or characteristic classes arising from the following double complex

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (Tf_g, \omega_X^Z) & \longrightarrow & \overline{Th_g} & \longrightarrow & \overline{N}_g & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (Tf, \omega_X^Z) & \longrightarrow & \overline{Th} & \longrightarrow & \overline{N} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (N_{X/X_g}, \omega_X^Z) & \longrightarrow & \overline{N}_{Z/Z_g} & \longrightarrow & \overline{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

whose columns are all orthogonally split. Also, for this double complex, one may use Example II.3 (iv) to compute that

$$
\begin{aligned}
\widetilde{\mathrm{Td}}_g((Tf, \omega_X^Z), \overline{Th}\mid_X) =& \widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}}, \omega_X^Z)\mathrm{Td}(\overline{N}_g)\mathrm{Td}(Tf_g, \omega_X^Z) \\
& + \widetilde{\mathrm{Td}}((Tf_g, \omega_X^Z), \overline{Th_g}\mid_{X_g})\mathrm{Td}_g(\overline{N}_{Z/Z_g}).
\end{aligned} \tag{3.0.1}
$$

We deduce from (3.0.1) that

$$
\begin{aligned}
& \int_{X_g/Y} \mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g((Tf, \omega_X^Z), \overline{Th}\mid_X)\mathrm{Td}_g^{-1}(\overline{N}) \\
=& \int_{X_g/Y} \mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}((Tf_g, \omega_X^Z), \overline{Th_g}\mid_{X_g})\mathrm{Td}_g^{-1}(\overline{N})\mathrm{Td}_g(\overline{N}_{Z/Z_g}) \\
& + \int_{X_g/Y} \mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}})\mathrm{Td}_g^{-1}(\overline{F})\mathrm{Td}(Tf_g, \omega_X^Z).
\end{aligned} \tag{3.0.2}
$$

Moreover, we have

$$
\begin{aligned}
h_{\mu_n *}(\mathrm{Td}_g(\overline{N}_{Z/Z_{\mu_n}})T_g(\overline{\xi}.)) =& \int_{Z_g/Y} T_g(\overline{\xi}.)\mathrm{Td}(\overline{Th_g})\mathrm{Td}_g(\overline{N}_{Z/Z_g}) \\
=& \int_{Z_g/Y} T_g(\overline{\xi}.)\mathrm{Td}_g(\overline{Th})
\end{aligned} \tag{3.0.3}
$$

and

$$\begin{aligned}
&h_{\mu_n *}(\mathrm{Td}_g(N_{Z/Z_{\mu_n}})\sum_k(-1)^k\mathrm{ch}_g(\xi_k)R_g(N_{Z/Z_{\mu_n}}))\\
&=\int_{Z_g/Y}\mathrm{Td}_g(N_{Z/Z_{\mu_n}})\delta_{X_g}\mathrm{ch}_g(\eta)\mathrm{Td}_g^{-1}(N)R_g(N_{Z/Z_g})\mathrm{Td}(Th_g)\\
&=\int_{X_g/Y}\mathrm{Td}_g(N_{X/X_g})\mathrm{Td}_g(F)\mathrm{ch}_g(\eta)\mathrm{Td}_g^{-1}(N)\\
&\qquad\cdot[R_g(N_{X/X_g})+R_g(N)-R(N_g)]\mathrm{Td}(Tf_g)\mathrm{Td}(N_g)\\
&=\int_{X_g/Y}\mathrm{Td}_g(Tf)\mathrm{ch}_g(\eta)[R_g(N_{X/X_g})+R_g(N)-R(N_g)]. \qquad (3.0.4)
\end{aligned}$$

Gathering (3.0.2), (3.0.3) and (3.0.4) we finally get

$$\begin{aligned}
f_*^Z(\overline{\eta}) =& f_{\mu_n *}^Z(i_{\mu_n}^*(\lambda_{-1}^{-1}(\overline{N}_{Z/Z_{\mu_n}}^\vee))\cdot\sum_k(-1)^k\mathrm{Tor}_{\mathcal{O}_Z}^k(i_*\overline{\eta},\overline{\mathcal{O}}_{Z_{\mu_n}}))\\
&+\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega_X^Z)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega_X^Z)\mathrm{Td}_g^{-1}(\overline{F})\\
&-\int_{X_g/Y}\mathrm{Td}_g(Tf)\mathrm{ch}_g(\eta)R_g(N_{X/X_g})
\end{aligned}$$

which completes the proof of Theorem V.7 in the case where the Kähler metric on $X(\mathbb{C})$ is induced by the Kähler metric on $Z(\mathbb{C})$.

Next, in analogy with the notation $\widetilde{\mathrm{Td}}(Tf_g,\omega^X,\omega_X^Z)$, we write $\widetilde{\mathrm{Td}}_g(N_{X/X_g},\omega^X,\omega_X^Z)$ for the secondary characteristic class of the exact sequence

$$0\longrightarrow N_{X/X_g}\xrightarrow{\mathrm{Id}}N_{X/X_g}\longrightarrow 0\longrightarrow 0$$

where the middle term carries the metric induced by ω_X^Z and the sub term carries the metric induced by ω^X. Similarly, we have the notation $\widetilde{\mathrm{Td}}_g(Tf,\omega^X,\omega_X^Z)$. Then by applying the argument in the proof of (3.0.1) to the double complex

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (Tf_g,\omega^X) & \longrightarrow & (Tf_g,\omega_X^Z) & \longrightarrow & 0 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (Tf,\omega^X) & \longrightarrow & (Tf,\omega_X^Z) & \longrightarrow & 0 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (N_{X/X_g},\omega^X) & \longrightarrow & (N_{X/X_g},\omega_X^Z) & \longrightarrow & 0 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We get

$$\begin{aligned}\widetilde{\mathrm{Td}}_g(Tf,\omega^X,\omega^Z_X)=&\widetilde{\mathrm{Td}}(Tf_g,\omega^X,\omega^Z_X)\mathrm{Td}_g(N_{X/X_g},\omega^Z_X)\\&+\widetilde{\mathrm{Td}}_g(N_{X/X_g},\omega^X,\omega^Z_X)\mathrm{Td}(Tf_g,\omega^X).\end{aligned}$$

Moreover, by Proposition II.5, we obtain that

$$\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^X)=\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^Z_X)+\widetilde{\mathrm{Td}}_g(N_{X/X_g},\omega^X,\omega^Z_X)\mathrm{Td}_g(\overline{F}).$$

With these two comparison formulae, we can compute

$$\begin{aligned}&\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^X)\mathrm{Td}_g^{-1}(\overline{F})\\&\qquad-\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^Z_X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^Z_X)\mathrm{Td}_g^{-1}(\overline{F})\\
=&\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^X)\mathrm{Td}_g^{-1}(\overline{F})\\
&-\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^Z_X)\mathrm{Td}_g^{-1}(\overline{F})\\
&+\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^Z_X)\mathrm{Td}_g^{-1}(\overline{F})\\
&-\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^Z_X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}},\omega^Z_X)\mathrm{Td}_g^{-1}(\overline{F})\\
=&\int_{X_g/Y}\widetilde{\mathrm{Td}}(Tf_g,\omega^X,\omega^Z_X)\mathrm{ch}_g(\overline{\eta})\\
&\cdot[\mathrm{Td}_g(N_{X/X_g},\omega^Z_X)\mathrm{Td}_g(\overline{F})-\mathrm{Td}_g(\overline{N}_{Z/Z_g})]\mathrm{Td}_g^{-1}(\overline{F})\\
&+\int_{X_g/Y}\mathrm{Td}(Tf_g,\omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(N_{X/X_g},\omega^X,\omega^Z_X)\\
=&\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(Tf,\omega^X,\omega^Z_X)\\
&-\int_{X_g/Y}\widetilde{\mathrm{Td}}(Tf_g,\omega^X,\omega^Z_X)\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\mathrm{Td}_g^{-1}(\overline{F}).\end{aligned}$$

At last, using [KR1, Lemma 7.3], we get the equality

$$f_*(\overline{\eta})-f_*^Z(\overline{\eta})=\int_{X_g/Y}\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(Tf,\omega^X,\omega^Z_X).$$

Together with the fact that the other two terms have nothing to do with the choice of the metric ω^X, we finally obtain that

$$\begin{aligned} f_*(\overline{\eta}) =& f^Z_{\mu_n *}(i^*_{\mu_n}(\lambda_{-1}^{-1}(\overline{N}^\vee_{Z/Z_{\mu_n}})) \cdot \sum_k (-1)^k \mathrm{Tor}^k_{\mathcal{O}_Z}(i_*\overline{\eta}, \overline{\mathcal{O}}_{Z_{\mu_n}})) \\ &+ \int_{X_g/Y} \mathrm{Td}(Tf_g, \omega^X)\mathrm{ch}_g(\overline{\eta})\widetilde{\mathrm{Td}}_g(\overline{\mathcal{F}}, \omega^X)\mathrm{Td}_g^{-1}(\overline{F}) \\ &- \int_{X_g/Y} \mathrm{Td}_g(Tf)\mathrm{ch}_g(\eta)R_g(N_{X/X_g}) \\ &+ \int_{X_g/Y} \widetilde{\mathrm{Td}}(Tf_g, \omega^X, \omega^Z_X)\mathrm{ch}_g(\overline{\eta})\mathrm{Td}_g(\overline{N}_{Z/Z_g})\mathrm{Td}_g^{-1}(\overline{F}) \end{aligned}$$

which ends the proof of Theorem V.7. □

Remark V.10. (i). Theorem V.7 is a natural extension of Theorem I.6 in the context of Arakelov geometry.

(ii). Let Y be an affine arithmetic scheme Spec(D), and choose ω^X to be the induced Kähler form ω^Z_X. Then the formula in Theorem V.7 is the content of [MR1, Conjecture 5.1].

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