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Elena Veronica Belmega

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THÈSE DE DOCTORAT

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Ecole Doctorale « Sciences et Technologies de l'Information des Télécommunications et des Systèmes »

Présentée par :

Elena Veronica BELMEGA

Sujet :

Problèmes d'allocation de ressources dans les réseaux MIMO sans fil distribués

(On resource allocation problems in distributed MIMO wireless networks)

Soutenue le 14 Décembre 2010 devant les membres du jury:

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Contents

1 Introduction **13**
1.1 Background and Motivation 13
1.2 Non-cooperative Game Theoretical Concepts 15
1.3 Publications 18

2 Shannon-Rate Efficient Non-cooperative Power Allocation Games **23**
2.1 The MIMO Multiple Access Channel 24
2.1.1 Static Links 25
2.1.2 Fast Fading Links 29
2.2 The Parallel Interference Relay Channel 35
2.2.1 Decode-and-Forward (DF) Protocol 37
2.2.2 Estimate-and-Forward (EF) Protocol 39
2.2.3 Amplify-and-Forward (AF) Protocol 40
2.3 Conclusion and Open Issues 42

3 Energy-efficient Communications **45**
3.1 Static Links 46
3.2 Fast Fading Links 47
3.3 Slow Fading Links 48
3.4 Conclusion and Open Issues 50

4 Learning Algorithms in Resource Allocation Games **53**
4.1 Dynamical Systems and Stochastic Approximation 54
4.2 The Fast Fading MIMO Multiple Access Channel 57
4.3 The Slow Fading MIMO Channel 61
4.4 Conclusion and Open Issues 65

5 Conclusion and Perspectives **67**

A Publications on Shannon-Rate Efficient Non-Cooperative Power Allocation Games **71**
A.1 IEEE-TWC-2009 72
A.2 SPRINGER-TS-2010 84
A.3 Erratum 99

A.4 AJMAA-2010 104
A.5 ACM/ICST-VALUETOOLS-2009 110
A.6 EURASIP-JWCN-2010 121

B Publications on Energy-Efficient Communications 141
B.1 IEEE-PIMRC-2010 142
B.2 IEEE-TSP-2010 149

C Publication on Learning Algorithms 163
C.1 EURASIP-EUSIPCO-2010 164

D Résumé 171
D.1 Introduction 172
D.1.1 Contexte et Motivation 172
D.1.2 Élements de Théorie des Jeux Non-coopératifs 174
D.1.3 Publications 177
D.2 Jeux Non-coopératifs d'Allocation de Puissance Efficaces en Terme de Débit de Transmission 181
D.3 Communications Efficaces en Terme de Consommation d'Énergie 185
D.4 Les Algorithmes d'Apprentissage 187
D.5 Conclusions 189

Abstract

In this thesis manuscript, the main objective is to study the wireless networks where the node terminals are equipped with multiple antennas. Rising topics such as self-optimizing networks, green communications and distributed algorithms have been approached mainly from a theoretical perspective. To this aim, we have used a diversified spectrum of tools from Game Theory, Information Theory, Random Matrix Theory and Learning Theory in Games.

We start our analysis with the study of the power allocation problem in distributed networks. The transmitters are assumed to be autonomous and capable of allocating their powers to optimize their Shannon achievable rates. A non-cooperative game theoretical framework is used to investigate the solution to this problem. Distributed algorithms which converge towards the optimal solution, i.e. the Nash equilibrium, have been proposed. Two different approaches have been applied: iterative algorithms based on the best-response correspondence and reinforcement learning algorithms.

Another major issue is related to the energy-efficiency aspect of the communication. In order to achieve high transmission rates, the power consumption is also high. In networks where the power consumption is the bottleneck, the Shannon achievable rate is no longer suitable performance metric. This is why we have also addressed the problem of optimizing an energy-efficiency function.

Résumé

Dans ce manuscrit de thèse, lobjectif principal est d'étudier les réseaux sans fil dans lesquels les nuds terminaux sont équipés de plusieurs antennes. Plusieurs thèmes d'actualité, tels que les réseaux intelligents auto-optimisants, les communications dites green ou vertes et algorithmes distribués sont abordés. Dans ce but, nous utilisons une gamme diversifiée d'outils de la théorie des jeux, théorie de l'information, théorie des matrices aléatoires et théorie de l'apprentissage.

Nous commençons notre analyse par l'étude du problème d'allocation de puissance dans les réseaux MIMO distribués. Les émetteurs sont censés être autonomes et capables de gérer leurs puissances afin d'optimiser leur taux de Shannon atteignables. Le cadre des jeux non-coopératifs est utilisé pour étudier la solution de ce problème. Des algorithmes itératifs qui convergent vers la solution optimale donnée par léquilibre de Nash sont proposés. Deux approches différentes sont appliquées: des algorithmes basés sur les meilleures réponses et des algorithmes dapprentissage par renforcement.

Un autre problème majeur dans les réseaux sans fil est lié à la question de l'efficacité énergétique. Afin d'atteindre des débits de transmission élevés, la consommation d'énergie est également élevée. Dans les réseaux où la consommation d'énergie est une question critique, le débit de Shannon atteignable n'est plus une métrique de performance adaptée. C'est pourquoi nous abordons également le problème de l'optimisation d'une fonction d'efficacité énergétique.

Chapter 1

Introduction

This manuscript is focused on resource allocation problems in MIMO wireless networks which emerge in a communication context that is aiming towards intelligent or self-optimizing networks capable of operating in an energy-efficient or green regime.

1.1 Background and Motivation

Although game theory and information theory have both been extensively developed in the past sixty years, starting with the seminal contributions of J. von Neumann and O. Morgenstern [1], J. Nash [2] and C. E. Shannon [3], only recently the connections and interactions between the two theories started to be highlighted and exploited at a significant scale. However, the first application of game theoretical tools to reliable communications dates back to Mandelbrot's PhD dissertation [4] in 1952 and later in [5] and [6] where the communication between a transmitter and a receiver was modeled as a two player zero-sum game with a mutual information payoff function. The transmitter plays against a malicious nature that chooses the worst channel distribution in the sense of the mutual information. It turns out that the non-cooperative solution of this game is identical to the maximin worst-case capacity assuming that the transmitter has no knowledge about the channel conditions (noise and channel gains statistics).

The recent surge of interest in applying game theoretical tools to communications was due to the development of wireless communications. In this context, multiple transmitter and receiver devices share the same communication environment. Thus, the competition for public resources (such as frequency bands, time slots, space, transmit power or energy) naturally arises. These resources can be, a priori, optimized by a central authority. However, the centralized approach presents several inconveniences: i) it is unrealistic in an environment where multiple concurrent service providers exist; ii) the joint optimization problem over all the network parameters is, generally, a very complex non-convex optimization problem, involving high computational cost; iii) it is not scalable, i.e., a slight change in the network topology may lead to a much more complex or even intractable optimization problem; iv) it involves an important signaling cost when the network owner feedbacks the optimal allocation policies to each network

user; v) the centralized solution is not necessarily fair w.r.t. the quality of service (QoS) provided to the users; vi) in the context of autonomous and rational users, if the QoS of a user is unsatisfactory, it may deviate from the centralized allocation policy which may result in a very inefficient network operating point. For these reasons, a distributed solution may be desirable although its centralized counterpart is generally better from the overall network performance perspective. In distributed environments, the competition for resources gives rise to interactive situations. Game theory seems to be the suitable mathematical framework to study such interactive situations.

In this context, one of our main objectives is to study non-cooperative resource allocation games in multiple-input multiple-output (MIMO) wireless communication networks. The motivation behind the choice of MIMO channels is two-fold: i) MIMO channels model a variety of real-world communication channels; ii) they provide an elegant mathematical framework (i.e., the compact matrix notation). To be more precise, the agents or game players are autonomous transmitter devices. These devices are capable of sensing the environment and to decide their individual actions, i.e., their own power allocation (PA) policies. Assuming they are rational and selfish devices, the action chosen are the ones maximizing their individual payoffs, i.e., their Shannon transmission rates. There are a lot of reasons why this type of payoffs has been often considered in the literature. Here, we will mention only the most important ones. First, Shannon transmission rates characterize the performance limits of a communication system and allow us to study distributed networks where good coding schemes are implemented. Second, the direct relationship between the achievable transmission rate of a user and his signal-to-interference plus noise ratio (SINR) allows us to optimize performance metrics like the SINR or related quantities of the same type (e.g., the carrier-to-interference ratio). Third, from a mathematical perspective, Shannon rates have many desirable properties (e.g., concavity properties), which allow one to conduct thorough performance analyses. Therefore they provide useful insights and concepts that are exploitable for a practical design of decentralized networks.

However, during the past decade, energy consumption has become an increasingly important issue in wireless networks. For instance, in the current cellular networks, the mobile terminals are equipped with relatively large screens, required to offer more and more functionalities and they need to operate at higher transmission rates for a longer period of time. Furthermore, in sensor networks where changing the batteries of devices is highly impractical or, in some cases, even impossible, the power consumption becomes a critical issue. In these scenarios, optimizing the Shannon achievable rates without considering the cost incurred is no longer a suitable performance metric. In order to take into account the consumed transmit power to achieve these rates, a new energy-efficient metric has been proposed in the literature [7] [8]. This metric, which we will also investigate in our scenario of MIMO networks, measures the number of bits that can be reliably conveyed through the channel per units of energy consumed.

From a practical point of view, the computation of the solution concepts of both, the Shannon rate-efficient or energy-efficient power allocation problems, often involves the implementation of complex algorithms at the transmitter level, a high amount of signaling among the transmitters, transmitter rationality assumption. In this context,

we will study learning algorithms which can be used to model adaptive decision making of the devices. They are low complexity algorithms that can be implemented with no rationality assumption. The devices choose their action based on a simple feed-back from the environment (i.e., scores for their choices). It turns out that, in the long term, they can improve their performance while operating in an unknown environment and converge to desirable network operating points.

The present manuscript is organized in three parts corresponding to the three main chapters.

We start by analyzing the multi-user Shannon-rate efficient power allocation problem in Chapter 2. This problem is formulated as a non-cooperative game. The existence and multiplicity of the Nash equilibrium (NE) solution will be investigated for two different network models. The multiple access channel (MAC) [9], [10], where multiple transmitters send their messages to a common receiver is studied in Section 2.1. The interference relay channel (IRC) [11], [12], which consists of an interference channel [13] (where several transmitter-receiver pairs coexist in a common environment creating mutual interference) and several relaying nodes that can be used by the transmitters to improve the performance of their communications will be studied in Section 2.2. We also evaluate the performance obtained at the NE operating points via numerical simulations and water-filling or best-response based iterative algorithms.

Optimizing the Shannon achievable rate is not always the best policy, especially in networks where the terminal devices are equipped with batteries of limited capacity. This is why, in Chapter 3, we study an energy-efficiency metric that takes into account both, the achievable transmission rates and the consumed power to achieve these rates. Because of the difficulties encountered, the energy-efficient power allocation problem is studied only for the single-user MIMO channel. The multi-user scenario is left as a useful extension of this work. One of the main difficulties encountered lies in the fact that the outage probability optimization in slow fading MIMO channels is still an open problem [14].

Finally, in Chapter 4, we consider learning algorithms. These algorithms are shown to provide an alternative way for the users to converge to some desirable operating points, such as the Nash equilibrium of the games in Chapter 2 or the optimal power allocation that optimizes the outage probability or energy-efficiency metric given in Chapter 3. They are simple adaptive algorithms that require little knowledge of the environment and no rationality assumptions.

In Chapter 5, we conclude our analysis with several remarks and open issues.

1.2 Non-cooperative Game Theoretical Concepts

We will briefly review hereunder some basic game-theoretical concepts which will be used throughout the manuscript. By definition, game theory is the mathematical framework dedicated to the study of interactive situations among decision makers or agents. We will consider the *rationality assumption* of the game players in the sense that a player chooses its best strategy to maximize the benefit [1]. The mathematical characterization

of a strategic form game is given by the following definition.

Definition 1.2.1 *A strategic form game is a triplet* $\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{u_k\}_{k\in\mathcal{K}})$*, where* $\mathcal{K} = \{1, \ldots, K\}$ *represents the* set of players*,* $\mathcal{A}_k$ *represents the* set of strategies *or actions that player* $k \in \mathcal{K}$ *can take and* $u_k : \times_{\ell\in\mathcal{K}}\mathcal{A}_\ell \to \mathbb{R}$ *represents the benefit or payoff function of user* k *which is a measure of its satisfaction.*

In the case of non-cooperative games, in which the players act in a selfish and independent manner, the Nash equilibrium (NE) introduced in [2] represents a solution concept of the game. The NE has been extensively studied in resource allocation problems because it is a very important concept to network designers. It represents an operating point which is, both, predictable and robust to unilateral deviations (which is realistic considering the fact that the players are assumed to be non-cooperative and act in an isolated manner). This means that once the system is operating in this state, no user has any incentive to deviate because it will lose in terms of its own benefit. The mathematical definition of the NE is as follows:

Definition 1.2.2 *A strategy profile* $(\underline{a}_1^{\mathrm{NE}}, \ldots, \underline{a}_K^{\mathrm{NE}}) \in \times_{\ell\in\mathcal{K}}\mathcal{A}_\ell$ *is a Nash equilibrium if for all* $k \in \mathcal{K}$ *and for all* $\underline{a}'_k \in \mathcal{A}_k$*,* $u_k(\underline{a}_k^{\mathrm{NE}}, \underline{a}_{-k}^{\mathrm{NE}}) \geq u_k(\underline{a}'_k, \underline{a}_{-k}^{\mathrm{NE}})$*, where* $\underline{a}_{-k} = (\underline{a}_1, \ldots, \underline{a}_{k-1}, \underline{a}_{k+1}, \ldots, \underline{a}_K)$ *denotes the set of the other players' actions.*

Based on the game structure, the topological properties of the strategy sets and the payoff functions, the main issues to be solved are: i) existence of an NE; ii) multiplicity of the NE; iii) design distributed algorithms that allow the users to converge to a NE state using only local knowledge of the environment; iv) determine the network performance of the NE. Concerning the design of distributed algorithms the NE has another appealing feature. As we will see in Chapter 4, the NE can be observed as the outcome of simple iterative adapting rules, i.e., learning algorithms. What is remarkable is that these iterative algorithms necessitate very little knowledge from the environment. In particular they don't require neither the knowledge of the game structure nor the players rationality assumption.

In general, the NE performance is suboptimal with respect to the overall network performance, which can be measured for example by the sum of individual user payoffs $u(\underline{a}) = \sum_{k\in\mathcal{K}} u_k(\underline{a}_k, \underline{a}_{-k})$. Moreover, it is not a fair state with respect to the users' payoffs in general. Desirable operating points that cope with these issues are the so-called Pareto-optimal states. A state of the system is Pareto-optimal if there is no other state that wouldn't be preferred by all the network users.

Definition 1.2.3 *Let* $\underline{a}$ *and* $\underline{a}'$ *be two different strategy profiles in* $\times_{\ell\in\mathcal{K}}\mathcal{A}_\ell$*. Then, if*

$$\forall k \in \mathcal{K},\ \ u_k(\underline{a}_k, \underline{a}_{-k}) \geq u_k(\underline{a}'_k, \underline{a}'_{-k}), \tag{1.1}$$

with strict inequality for at least one player, the strategy $\underline{a}$ *is Pareto-superior to* $\underline{a}'$*. If there exists no other strategy that is Pareto-superior to a strategy profile* $\underline{a}^{\mathrm{PO}}$ *then* $\underline{a}^{\mathrm{PO}}$ *is Pareto-optimal.*

However, a Pareto-optimal state is not necessarily a stable state in a selfish non-cooperative user environment. Furthermore the Pareto-optimal solution is rather a centralized type solution because global network information is needed to compute these states of the system.

There exist different techniques that can be used to improve the performance of the NE. In general, they involve the interference of a centralized authority or some kind of user cooperation. There is always a trade-off between the performance obtained at the equilibrium state of the network and the signaling cost it involves. A more detailed study of the non-cooperative game theoretic concepts can be found in the specialized literature [15], [16]. For a thorough analysis on the methodologies for analyzing NE in wireless games in general, the reader is referred to [17] and [18].

An important class of games are potential games introduced in [19].

Definition 1.2.4 *A strategic form game* $\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{u_k\}_{k\in\mathcal{K}})$ *is an* exact potential game *if there exists a potential function* $V : \mathcal{A} \to \mathbb{R}_+$ *such that, for all* $k \in \mathcal{K}$ *and every* $\underline{a}, \underline{a}' \in \mathcal{A}$

$$u_k(\underline{a}_k, \underline{a}_{-k}) - u_k(\underline{a}'_k, \underline{a}_{-k}) = V(\underline{a}_k, \underline{a}_{-k}) - V(\underline{a}'_k, \underline{a}_{-k}). \tag{1.2}$$

This definition translates the fact that all users have the same incentives in changing their actions which leads the system from state $\underline{a}$ to state $\underline{a}'$. For example, congestion or routing games are potential games. Following [19], the local maxima of the potential function are NE points of the game. Thus, every potential game has at least one NE. Furthermore, in finite games, the iterative best-response type algorithms converge to one of the NE states (see the finite improvement path (FIP) property in [19]) depending on the initial point.

In order to tackle the existence and uniqueness issues for Nash equilibria, we will often exploit the properties of concave games and the results of Rosen [20]. These results are stated here below and are valid for the case where the actions of players are vectors.

Theorem 1.2.5 *[20] Let* $\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{u_k\}_{k\in\mathcal{K}})$ *be a game where* $\mathcal{K} = \{1, ..., K\}$ *is the set of players,* $\mathcal{A}_1, ..., \mathcal{A}_K$ *the corresponding sets of strategies and* $u_1, ..., u_k$ *the utilities of the different players. If the following three conditions are satisfied: (i) each* u_k *is continuous in the all the strategies* $\underline{a}_j \in \mathcal{A}_j, \forall j \in \mathcal{K}$*; (ii) each* u_k *is concave in* $\underline{a}_k \in \mathcal{A}_k$*; (iii)* $\mathcal{A}_1, ..., \mathcal{A}_K$ *are compact and convex sets; then* $\mathcal{G}$ *has at least one NE.*

Theorem 1.2.6 *[20] Consider the* K*-player concave game of Theorem D.1.5. If the following (diagonally strict concavity) condition is met: for all* $k \in \mathcal{K}$ *and for all* $(\underline{a}'_k, \underline{a}''_k) \in \mathcal{A}_k^2$ *such that there exists at least one index* $j \in \mathcal{K}$ *for which* $\underline{a}'_j \neq \underline{a}''_j$, $\sum_{k=1}^{K} (\underline{a}''_k - \underline{a}'_k)^T \left[\nabla_{\underline{a}_k} u_k(\underline{a}'_k, \underline{a}'_{-k}) - \nabla_{\underline{a}_k} u_k(\underline{a}''_k, \underline{a}''_{-k})\right] > 0$*; then the uniqueness of the NE is insured.*

These theorems prove to be particularly useful in Shannon rate-efficient power allocation problems where the users' payoffs are generally concave w.r.t. the transmit power.

Other non-cooperative game solutions that generalize the concept of pure strategy NE are: the mixed strategy NE and the correlated equilibrium. A mixed strategy for user k is a probability distribution over its own action set $\mathcal{A}_k$. Let $\Delta(\mathcal{A}_k)$ denote the set of probability measures over the set $\mathcal{A}_k$. The mixed NE is defined similarly to pure-strategy NE by replacing the pure strategies with the mixed strategies. The existence of the mixed NE has been proven in [2] for all discrete games. If the action spaces are discrete finite sets, then $\underline{p}_k \in \Delta(\mathcal{A}_k)$ denotes the probability vector such that $p_{k,j}$ represents the probability that user k chooses a certain action $\underline{a}_k^{(j)} \in \mathcal{A}_k$ and $\sum_{\underline{a}_k^{(j)} \in \mathcal{A}_k} p_{k,j} = 1$.

We also define the concept of correlated equilibrium [21] which can be viewed as the NE of a game where the players receive some private signaling or playing recommendation from a common referee or mediator. The mathematical definition is as follows:

Definition 1.2.7 *A joint probability distribution $\underline{q} \in \Delta(\mathcal{A})$ is a correlated equilibrium if for all $k \in \mathcal{K}$ and all $\underline{a}_k^{(j)}, \underline{a}_k^{(i)} \in \mathcal{A}_k$*

$$\sum_{\underline{a} \in \mathcal{A}: \underline{a}_k = \underline{a}_k^{(j)}} q_{\underline{a}} \left[u_k(\underline{a}_k^{(j)}, \underline{a}_{-k}) - u_k(\underline{a}_k^{(i)}, \underline{a}_{-k}) \right] \geq 0, \tag{1.3}$$

where $q_{\underline{a}}$ denotes the probability associated to the action profile $\underline{a} \in \mathcal{A}$.

At the CE, User k has no incentive in deviating from the mediator's recommandation to play $\underline{a}_k^{(j)} \in \mathcal{A}_k$ knowing that all the other players follow as well the mediator's recommendation $(\underline{a}_{-k})$. Notice that the set of mixed NE is included in the set of CE by considering independent p.d.f's. Similarly, the set of pure strategy NE is included in the set of mixed strategy NE by considering degenerate probability distributions (i.e. $p_{k,j} \in \{0, 1\}$) over the action sets of users.

1.3 Publications

The research work conducted during the three years of the thesis has lead to several publications. The papers are classified hereunder in function of the related topics.

Shannon-Rate Efficient Non-Cooperative Power Allocation Games

The contributions on the non-cooperative power allocation games for the ***fast fading MIMO multiple access channel*** have been published in four journal papers, among which two of them are mathematics journals, and one conference paper:

- E. V. Belmega, S. Lasaulce, M. Debbah, M. Jungers, and J. Dumont, "Power allocation games in wireless networks of multi-antenna terminals", *Springer Telecommunications Systems Journal*, DOI: 10.1007/s11235-010-9305-3, May 2010.

- E. V. Belmega, S. Lasaulce, and M. Debbah, "Power allocation games for MIMO multiple access channels with coordination", *IEEE Trans. on Wireless Communications*, vol. 8, no. 6, pp. 3182–3192, Jun. 2009.

- E. V. Belmega, M. Jungers, and S. Lasaulce, "A generalization of a trace inequality for positive definite matrices", *The Australian Journal of Mathematical Analysis and Applications (AJMAA)*, to appear, 2010.
- E. V. Belmega, S. Lasaulce, and M. Debbah, "A trace inequality for positive definite matrices", *Journal of Inequalities in Pure and Applied Mathematics (JIPAM)*, vol. 10, no. 1, pp. 1-4, 2009.
- E. V. Belmega, S. Lasaulce, and M. Debbah "Power Control in Distributed Multiple Access Channels with Coordination", *IEEE/ACM Proc. of the Intl. Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks and Workshops (WIOPT)*, Berlin, Germany, 1–8 Apr. 2008.

The study of the power allocation problem for the ***static parallel multiple access channel***, formulated as a non-cooperative routing game, resulted in two conference papers:

- S. Medina Perlaza, E. V. Belmega, S. Lasaulce, and M. Debbah, "On the base station selection and base station sharing in self-configuring networks", *International Conference on Performance Evaluation Methodologies and Tools (VALUETOOLS)*, Pisa, Italy, invited paper, Oct. 2009.
- E. V. Belmega and S. Lasaulce, "Information theoretic congestion games in heterogeneous wireless networks", *Game Theory for Analysis and Optimization of Computer Systems (GAMECOMP)*, Grenoble, France, invited talk, May 2008.
- E. V. Belmega, S. Lasaulce, and M. Debbah, "Decentralized handovers in cellular networks with cognitive terminals", *in the IEEE Proc. of the 3rd International Symposium on Communications, Control and Signal Processing (ISCCSP)*, St Julians, Malta, invited paper, 12–14 Mar. 2008.

The non-cooperative power allocation game for the ***static parallel interference relay channel*** has been investigated in one journal paper, currently under review, and four conference papers:

- E. V. Belmega, B. Djeumou, and S. Lasaulce, "Power allocation games in interference relay channels: Existence analysis of Nash equilibria", *EURASIP Journal on Wireless Communications and Networking (JWCN)*, **accepted for publication**, Nov. 2010.
- E. V. Belmega, B. Djeumou, and S. Lasaulce "Resource allocation games in interference relay channels", *IEEE Intl. Conference on Game Theory for Networks (Gamenets)*, Istanbul, Turkey, invited paper, May 2009.
- E. V. Belmega, B. Djeumou, and S. Lasaulce "What happens when cognitive terminals compete for a relay node?", *IEEE Intl. Conference on Acoustics, Speech and Signal Processing (ICASSP)*, Taipei, Taiwan, 1–4 Apr. 2009.

- E. V. Belmega, B. Djeumou, and S. Lasaulce "Jeux d'allocation de puissance pour les canaux à interférence à relais", *GRETSI*, Dijon, France, Sep. 2009.
- B. Djeumou, E. V. Belmega, and S. Lasaulce, "Régions de taux atteignables pour le canal à interférence à relais", *GRETSI*, Dijon, France, Sep. 2009.

The state of the art with respect to this topic is to be published in the following book chapter:

- E. V. Belmega, S. Lasaulce, and M. Debbah, "Shannon rate efficient power allocation games", *Game Theory for Wireless Communications and Networking*, Auerbach Publications, Taylor and Francis Group, CRC Press, **accepted for publication**, 2009.

Energy-Efficient Communications

The power allocation problem, in the sense of maximizing an energy-efficient performance metric, i.e., the number of bits that can be reliably transmitted over the channel per unit of consumed energy, has been investigated in the ***single-user MIMO channel***. The main results are presented in one submitted journal paper and three conference papers:

- E. V. Belmega, and S. Lasaulce, "Energy-efficient precoding for multiple-antenna terminals", *IEEE Trans. on Signal Processing*, **accepted for publication**, Sep. 2010.
- E. V. Belmega, S. Lasaulce, and M. Debbah, "A survey on energy-efficient communications", *IEEE Intl. Symp. on Personal, Indoor and Mobile Radio Communications (PIMRC 2010)*, Istanbul, Turkey, Sep. 2010.
- E. V. Belmega, and S. Lasaulce, "An information-theoretic look at MIMO energy-efficient communications", *International Conference on Performance Evaluation Methodologies and Tools (VALUETOOLS)*, Pisa, Italy, Oct. 2009.
- E. V. Belmega, S. Lasaulce, M. Debbah, and A. Hjørungnes "A new energy efficiency function for quasi-static MIMO channels", *International Wireless Communications and Mobile Computing Conference (IWCMC)*, Leipzig, Germany, invited paper, Jun. 2009.

Learning Algorithms

The study of learning algorithms, that enable the transmitters to converge to desirable network states with little knowledge on the communication environment and under no rationality assumptions, has lead to two conference papers:

- E. V. Belmega, H. Tembine, and S. Lasaulce, "Learning to precode in outage minimization games over MIMO interference channels", *IEEE Asilomar Conf. on Signals, Systems, and Computers*, Pacific Grove, CA, USA, invited paper, Nov. 2010.

- E. V. Belmega, S. Lasaulce, M. Debbah, and A. Hjørungnes, "Learning Distributed Power Allocation Policies in MIMO Channels", *European Signal Processing Conference (EUSIPCO)*, Aalborg, Denmark, invited paper, Aug. 2010.

This manuscript represents a summary guide to the main contributions published in the aforementioned papers. The most relevant papers have been added in the Appendix sections. They will be used as references to the missing analysis details and mathematical proofs.

Other contributions which will not be discussed in this manuscript have been obtained and/or published in two journal papers, one book chapter and three conference papers:

- E. Altman, Y. Hayel, and E. V. Belmega, "Modelling competition between Hawks and Doves in a dynamic framework", *Special Issue on Evolutionary Games: Dynamic Games and Applications, Birkhauser-Springer*, **to be submitted**, Jan. 2011.

- E. V. Belmega, B. Djeumou, and S. Lasaulce, "Gaussian broadcast channels with an orthogonal and bidirectionnal cooperation link", *EURASIP J. on Wireless Communications and Networking (JWCN)*, pp.1–16, doi:10.1155/2008/341726, 2008.

- E. V. Belmega, S. Lasaulce, and M. Debbah, "Capacity of cooperative channels: three terminals case study", *Cooperative Wireless Communication, ISBN 142006469X*, Auerbach Publications, Taylor and Francis Group, CRC Press, Oct. 2008.

- P. Mertikopoulos, E. V. Belmega, A. L. Moustakas, and S. Lasaulce, "Dynamic Power Allocation Games in Parallel Multiple Access Channels", *Valuetools 2011, Paris, France*, **submitted, arXiv:1011.5610v1**, Nov. 2010.

- B. Djeumou, E. V. Belmega, and S. Lasaulce, "Recombinaison de signaux décodés et transférés pour le canal à relais à division fréquentielle", *Actes du GRETSI*, Troyes, France, 1–4 Sep. 2007.

- E. V. Belmega, B. Djeumou, and S. Lasaulce, "Performance analysis for the AF-based frequency division cooperative broadcast channel", *in the IEEE Proceedings of the Signal Processing Advances in Wireless Communications conference (SPAWC)*, Helsinki, Finland, 1–5 Jun. 2007.

Chapter 2

Shannon-Rate Efficient Non-cooperative Power Allocation Games

In this chapter, our objective is to study distributed or self-optimizing wireless networks. In such networks the transmitter terminals are able to manage their own resources with little or, ideally, no intervention from a central authority. The transmitters are assumed to be rational, selfish and capable of allocating their own power allocation policies to maximize their communications' performance. The performance of a communication is measured in terms of achievable transmission rates. The mutual interference created by the transmitters sharing the same communication environment induces an interaction among the transmitters. This interaction is studied using a non-cooperative game theoretical framework. Unless otherwise specified, the components of the non-cooperative game can be identified as follows. The players are the transmitter devices. The payoff functions are their achievable transmission rates. The action sets are their precoding schemes.

Three different steps can be identified in our approach. First, we study the one-shot non-cooperative game where the users have perfect knowledge of the game structure. We investigate the existence and multiplicity of the Nash equilibrium solution concept. Several issues arise. The perfect knowledge of the structure of the game at the transmitter level is often unrealistic. Furthermore, in general, closed-form expressions of the NE points are not available. Moreover, if more than one such state exist, there is no reason to assume that rational transmitters should expect the same outcome of the game. In this situation, the network may operate at a non-equilibrium state. To cope with these issues, one possible solution is to consider iterative algorithms. Therefore, as a following step in our approach, we investigate iterative algorithms based on the best-response correspondences. As we will see, these algorithms are identical to the iterative water-filling algorithms. Furthermore, they can allow the users to converge to one of the NE of the one-shot game. These algorithms are distributed in the sense that they require less information about the structure of the game. Another drawback of the NE

concept is the fact that it is generally an inefficient operating point (w.r.t. the overall network performance but also the individual users' performance), thus, as a last step, we also discuss the Stackelberg formulation [22] to improve the performance of the NE point. This will involve a certain cost in terms of centralized signaling from the system authority.

The Shannon-rate efficient non-cooperative power allocation game is studied for two basic network models: the multiple access channels (MAC) and the interference relay channels (IRC). Before stating our main contributions, we will discuss hereunder several important assumptions and differences between these two network models. In terms of channel coherence time, three cases can be distinguished. The channel gains can be: i) deterministic constants, i.e., static links; ii) random variables independently changing with every channel use, i.e., fast fading links; iii) random variables fixed for the whole transmission duration, i.e., slow fading links. In the third case, the Shannon achievable rates are strictly equal to zero [14] [23]. Thus, to study the power allocation game, one has to consider a different metric to measure the users' satisfaction. For example, as we will see in Chapter 3, one could consider a performance metric depending on the outage probability [24]. However, this is out of the scope here and will not be considered in this chapter. For the MAC channel, we will briefly discuss the static links case and then focus on the more challenging case of fast fading links. From an information-theoretical point of view, the analysis of the IRC is much more difficult, this is why only the static case will be considered. For the same reason, the general multiple-input multiple-output (MIMO) channel will be studied only for the MAC. Whereas for the IRC, we will restrict our attention to the parallel sub-channels case. Another inherent difference between the two network models consists in the decoding technique. For the MAC, the receiver has to decode all the messages from the transmitters and, thus, is assumed to know the code-books used by all the transmitters. In this situation, two decoding techniques can be used: i) single user decoding (SUD), i.e., the transmitters' messages are decoded simultaneously (when decoding one transmitter's message, the other transmitters' signals are considered as additive noise); ii) successive interference cancellation (SIC), i.e., the message of the transmitters are decoded sequentially (when decoding one transmitter's message, the previous decoded messages are taken into account to reduce the level of interference). For the IRC, each decoder only knows the code-books employed by its own transmitter. Therefore, the SIC technique, even though appealing in terms of transmission rate, will not be studied.

2.1 The MIMO Multiple Access Channel

In this section, we consider the multiple access channel (MAC) consisting of several transmitter nodes and a common destination. The receiver node decodes the incoming messages from all the transmitters. We assume that the terminals are equipped with

multiple antennas. The received base-band signal can be written as:

$$\underline{Y}(\tau) = \sum_{k=1}^{K} \mathbf{H}_k(\tau)\underline{X}_k(\tau) + \underline{Z}(\tau) \tag{2.1}$$

where $\underline{X}_k(\tau)$ is the $n_{t,k}$-dimensional column vector of symbols transmitted by user k at time τ, $\mathbf{H}_k \in \mathbb{C}^{n_r \times n_{t,k}}$ is the channel matrix of user k and $\underline{Z}(\tau)$ is a n_r-dimensional complex white Gaussian noise distributed as $\mathcal{N}(\underline{0}, \sigma^2 \mathbf{I}_{n_r})$, assuming a number of $n_{t,k}$ antennas at the transmitter $k \in \{1, \ldots, K\}$ and n_r antennas at the common receiver. For the sake of clarity we will omit the time index τ from our notations.

The transmitters are assumed to use good codes in the sense of their Shannon achievable rates. Also, we assume coherent communications, i.e., the receiver has perfect channel state information and decodes perfectly all its intended messages. Two different decoding techniques will be studied: single user decoding (SUD) and successive interference cancellation (SIC). The SIC decoding technique involves the existence of a coordination signal representing the decoding order at the receiver. The coordination signal is assumed to be perfectly known at the transmitter side. If the coordination signal is generated by the receiver, its distribution can be optimized but it induces a certain cost in terms of downlink signaling. On the other hand, if the coordination signal comes from an external source, e.g., an FM transmitter, the transmitter nodes can acquire the coordination signal for free in terms of downlink signaling. However this generally involves a certain sub-optimality in terms of uplink rate. In both cases, the coordination signal will be represented by a random variable denoted by $S \in \mathcal{S} \triangleq \{1, ..., K!\}$. In a real wireless system, the frequency with which the realizations would be drawn would be roughly proportional to the reciprocal of the channel coherence time (i.e., $1/T_{\text{coh}}$). Note that the proposed coordination mechanism is suboptimal because it does not depend on the realizations of the channel matrices. We will see that the corresponding performance loss is in fact very small.

In what follows, first we will briefly review the static case and then proceed with the more challenging case of fast fading links.

2.1.1 Static Links

In this section, the channel gains are assumed to be deterministic constants.

Let us start the analysis with a toy example: the two user single-input single-output (SISO) MAC as in [25]. This example gives useful insights on the general case because it can be solved in closed-form. When SUD is assumed, transmitting at full power strictly dominates all the other strategies. Thus, the unique NE corresponds to the state where all the transmitters saturate their powers. This is a very inefficient operating point because of the high interference level. One way to reduce the interference is to consider SIC decoding. The decoding order is dictated by flipping an unfair coin (Bernoulli random variable of parameter $q \in [0, 1]$). The existence and uniqueness of the NE solution can be proved using Theorems D.1.5 and Theorem D.1.6. It turns out that, for any value of $q \in [0, 1]$, the performance of the system at the NE achieves the sum-capacity of the SISO MAC channel. In conclusion, the receiver node can choose a certain

operating point on the Pareto-optimal frontier by tuning the parameter q. Another way to combat multi-user interference was proposed in [26]. The authors introduced a pricing function as a penalty for the interference the users create in the network. In the remaining of this section, we will focus on the general MIMO MAC.

Single user decoding

We assume that the receiver applies a SUD scheme. The strategy of User k consists in choosing the input covariance matrix, $\mathbf{Q}_k = \mathbb{E}[\underline{x}_k \underline{x}_k^H]$ in the set:

$$\mathcal{A}_k^{\text{SUD}} = \{\mathbf{Q} \in \mathbb{C}^{n_{t,k} \times n_{t,k}} : \mathbf{Q} \succeq \mathbf{0}, \text{Tr}(\mathbf{Q}) \leq \overline{P}_k\}, \tag{2.2}$$

to maximize its achievable rate:

$$\mu_k^{\text{SUD}}(\mathbf{Q}_k, \mathbf{Q}_{-k}) = \log_2 \left| \mathbf{I}_{n_r} + \rho \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H + \rho \sum_{\ell \neq k}^{K} \mathbf{H}_\ell \mathbf{Q}_\ell \mathbf{H}_\ell^H \right| - \log_2 \left| \mathbf{I}_{n_r} + \rho \sum_{\ell \neq k} \mathbf{H}_\ell \mathbf{Q}_\ell \mathbf{H}_\ell^H \right|, \tag{2.3}$$

where $\rho = \frac{1}{\sigma^2}$. The existence of the NE follows from Theorem D.1.5. As opposed to the SISO case, the NE is not unique in general. Sufficient conditions that ensure the uniqueness of the NE are: i) $\text{rank}(\mathbf{H}^H \mathbf{H}) = \sum_{k=1}^{K} n_{t,k}$ where $\mathbf{H} = [\mathbf{H}_1, \ldots, \mathbf{H}_K]$; ii) $\sum_{k=1}^{K} n_{t,k}^2 \leq n_r^2 + K$. The proof follows from [27] (see Appendix A.2, Appendix A.3) by considering that the static links case corresponds to a single realization of the fast fading links case.

The users converge to one of the Nash equilibrium points by applying an iterative algorithm based on the best-response correspondences:

$$\text{BR}_k(\mathbf{Q}_{-k}) = \arg \max_{\mathbf{Q}_k \in \mathcal{Q}_k} \mu_k(\mathbf{Q}_k, \mathbf{Q}_{-k}). \tag{2.4}$$

The best-response correspondence of User k is identical to the well known single-user water-filling solution given in [28]. Using the following algorithm, the users converge to one of the NE points in a distributed manner:

- Initialization: for all $\ell \in \mathcal{K}, \mathbf{Q}_\ell^{[1]} \leftarrow \frac{\overline{P}_\ell}{n_{t,\ell}} \mathbf{I}_{n_{t,\ell}}$ and $k \leftarrow 1$
- At iteration step $t > 1$, only User k updates its covariance matrix: $\mathbf{Q}_k^{[t]} \leftarrow \text{BR}_k\left(\mathbf{Q}_{-k}^{[t-1]}\right)$, $\mathbf{Q}_{-k}^{[t]} \leftarrow \mathbf{Q}_{-k}^{[t-1]}$. If $k = K$ then $k \leftarrow 1$, else $k \leftarrow k+1$.
- Repeat the previous step ($t \leftarrow t+1$) until convergence is reached.

The players sequentially update their covariance matrices in a certain specified order (e.g., round-robin order). In [28], this algorithm was proven to converge to a point maximizing the sum-capacity of the MIMO Gaussian MAC channel from any initial

point. However, because of the interference terms, the sum-capacity of the Gaussian MIMO MAC is not achieved at the NE point, similarly to the SISO case.

Parallel MAC
Before moving on to the SIC decoding scheme, we discuss another special, but interesting, particular case: the parallel MAC. The channel matrices are square diagonal matrices where $\forall k \in \mathcal{K}$, $n_r = n_{t,k} = B$ and $\mathbf{H}_k = \mathbf{diag}(h_{k,1}, \ldots, h_{k,B})$. This model describes for example the uplink communication of a multi-cellular wireless network studied in [29], [30] (see Appendix A.5). The network is composed of K mobile terminals (MT) and B base-stations (BS) operating in orthogonal frequency bands. In this case, the user action sets are reduced to vector sets instead of matrix sets (i.e., the users choose their power allocation policies among the available frequencies).

What is interesting about this case is that two different power allocation games can be defined. The difference lies in the choice of the users' action sets: i) the users share their available powers among the BS (i.e., Soft-Handover or BS Sharing Game); ii) the users are restricted to transmit to a single BS, (i.e., Hard-Handover or BS Selection Game).

For the BS Sharing Game, all the results discussed in the MIMO case apply here. As we have argued in the general MIMO case, the NE is not unique in general. The sufficient conditions ensuring the uniqueness of the NE simply become: i) $\text{rank}(\mathbf{H}^H\mathbf{H}) = K \times B$; ii) $K \times B \leq K + B$. However, these conditions are very restrictive and are met only in some particular cases: a) $K = 1$ and $B \geq 1$ the problem is reduced to a simple optimization solved by water-filling; b) $K \geq 1$ and $B = 1$ where the dominant strategy for every user is to transmit with full power at the unique BS available; c) $K = 2$ and $B = 2$ which is somehow more complicated. Recently, we have proved in [31] that the NE is unique with probability one for any $K \geq 1$ and $B \geq 1$.

For the BS Selection Game, none of the previous results can be applied. The action sets are not compact and convex sets but discrete sets. Therefore, we cannot use the properties of concave games. The discrete power allocation problem is studied from a routing game theoretical perspective. The game is proven to be an exact potential one where the potential function is the network achievable sum-rate. This implies directly the existence of at least one pure-strategy NE. Based on elements of non-oriented graph theory, the exact number of equilibrium points is provided. Also, an iterative algorithm based on the best-response correspondences is shown to converge to one of the NE. The proof is based on random walks and directed-graph theory. The algorithm is fully distributed since it requires only local channel state information and the overall interference plus noise power.

When comparing the performance in terms of network sum-rate of the two games, a remarkable observation was made. Assuming $K \geq B$, the performance of the BS Selection Game is superior to BS Sharing Game. This characterizes a Braess-type paradox, i.e., increasing the possible choices of players results in a degeneration of the global network performance.

Successive Interference Cancellation

In what follows, we assume that SIC decoding scheme is applied at the receiver. The strategy of User k, consists in choosing the best vector of precoding matrices $\mathbf{Q}_k = \left(\mathbf{Q}_k^{(1)}, \mathbf{Q}_k^{(2)}, ..., \mathbf{Q}_k^{(K!)}\right)$ where $\mathbf{Q}_k^{(s)} = \mathbb{E}\left[\underline{X}_k^{(s)} \underline{X}_k^{(s),H}\right]$, for $s \in \mathcal{S}$. For clarity sake, we introduce a notation which will be used to replace the realization s of the coordination signal with no loss of generality. We denote by $\mathcal{P}_K$ the set of all possible permutations of K elements, such that $\pi \in \mathcal{P}_k$ denotes a certain decoding order for the K users and $\pi(k)$ denotes the rank of user $k \in \mathcal{K}$ and $\pi^{-1} \in \mathcal{P}_K$ denotes the inverse permutation (i.e. $\pi^{-1}(\pi(k)) = k$) such that $\pi^{-1}(r)$ denotes the index of the user that is decoded with rank $r \in \mathcal{K}$. We denote by $p_\pi \in [0, 1]$ the probability that the receiver implements the decoding order $\pi \in \mathcal{P}_K$, which means that $\sum_{\pi \in \mathcal{P}_K} p_\pi = 1$. In the static case, p_π represents the fraction of time when the receiver applies the decoding order π. The vector of precoding matrices can be denoted by $\mathbf{Q}_k = \left(\mathbf{Q}_k^{(\pi)}\right)_{\pi \in \mathcal{P}_K}$. The payoff function of User k is given by:

$$\mu_k^{\text{SIC}}(\mathbf{Q}_k, \mathbf{Q}_{-k}) = \sum_{\pi \in \mathcal{P}_K} p_\pi R_k^{(\pi)}(\mathbf{Q}_k^{(\pi)}, \mathbf{Q}_{-k}^{(\pi)}) \tag{2.5}$$

where

$$\begin{aligned} R_k^{(\pi)}(\mathbf{Q}_k^{(\pi)}, \mathbf{Q}_{-k}^{(\pi)}) &= \log_2 \left| \mathbf{I}_{n_r} + \rho \mathbf{H}_k \mathbf{Q}_k^{(\pi)} \mathbf{H}_k^H + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \mathbf{H}_\ell \mathbf{Q}_\ell^{(\pi)} \mathbf{H}_\ell^H \right| - \\ & \log_2 \left| \mathbf{I}_{n_r} + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \mathbf{H}_\ell \mathbf{Q}_\ell^{(\pi)} \mathbf{H}_\ell^H \right| \end{aligned} \tag{2.6}$$

where $\mathcal{K}_k^{(\pi)} = \{\ell \in \mathcal{K} | \pi(\ell) \geq \pi(k)\}$ represents the subset of users that will be decoded after User k in the decoding order π. An important point to mention here is the power constraint under which the utilities are maximized. Indeed, three different power allocation games can be defined in function of the power constraint:

- Spatial power allocation (SPA) game

$$\mathcal{A}_k^{\text{SIC,SPA}} = \left\{ \mathbf{Q}_k = \left(\mathbf{Q}_k^{(\pi)}\right)_{\pi \in \mathcal{P}_K} \quad | \quad \forall \pi \in \mathcal{P}_K, \mathbf{Q}_k^{(\pi)} \succeq 0, \text{Tr}(\mathbf{Q}_k^{(\pi)}) \leq n_t \overline{P}_k \right\}. \tag{2.7}$$

  Here, the users are restricted to uniformly allocate their powers over time (independently from the decoding order).

- Temporal power allocation (TPA) game

$$\mathcal{A}_k^{\text{SIC,TPA}} = \left\{ \mathbf{Q}_k = \left(\alpha_k^{(\pi)} P_k \mathbf{I}_{n_t}\right)_{\pi \in \mathcal{P}_K} \quad | \quad \forall \pi \in \mathcal{P}_K, \alpha_k^{(\pi)} \geq 0, \sum_{\vartheta \in \mathcal{P}_K} p_\vartheta \alpha_k^{(\vartheta)} \leq 1 \right\}. \tag{2.8}$$

Here, the users are restricted to uniformly allocate their powers over their transmit antennas.

- Space-time power allocation (STPA) game

$$\mathcal{A}_k^{\text{SIC,STPA}} = \left\{ \mathbf{Q}_k = \left(\mathbf{Q}_k^{(\pi)}\right)_{\pi \in \mathcal{P}_K} \quad | \quad \forall \pi \in \mathcal{P}_K, \mathbf{Q}_k^{(\pi)} \succeq 0, \sum_{\vartheta \in \mathcal{P}_K} p_\vartheta \text{Tr}(\mathbf{Q}_k^{(\vartheta)}) \leq n_t \overline{P}_k \right\}. \tag{2.9}$$

This is a generalization of the previous cases where the users are free to allocate their powers in time and over the transmit antennas.

The analysis of the NE follows the same lines as the fast fading MIMO MAC studied in [32] and [27]. The existence of the NE is ensured based on the properties of concave games. For the particular cases of the SPA and TPA games, the NE is proved to be unique [32](see Appendix A.1). This is no longer true for the space-time power allocation game [27] (see Appendix A.2). The sufficient conditions ensuring the uniqueness of the NE are: i) $\forall, k \in \mathcal{K}$, $\text{rank}(\mathbf{H}_k^H \mathbf{H}_k) = n_{t,k}$; ii) $n_{t,k}^2 \leq n_r^2 + 1$. In order to determine the covariance matrices at the NE point, we can again apply an iterative water-filling algorithm similar to the SUD case. Convergence results for a sequential updating rule can be found in [33].

2.1.2 Fast Fading Links

Similarly to Section 2.1.1, we start by summarizing the two-player SISO case in [34]. Assuming SUD scheme at the receiver, the authors of [34] proved the existence and uniqueness of the NE point. What is interesting is that at the NE point, only the user with the strongest channel will transmit while the other remains silent. The interference is completely cancelled. Furthermore, as opposed to the static links case, the system sum-rate at the NE achieves the sum-capacity point of the achievable rate region. For the SIC decoding, the authors propose a scheme where the decoding order depends on the fading coefficients. The NE is proved to exist and to be unique. However, only the two corner points of the achievable rate region can be achieved. In order to achieve the other Pareto-optimal points, the authors propose a repeated game formulation.

Now, we will focus on the general fast fading MIMO case we have analyzed in [27] (see Appendix A.2) and [32] (see Appendix A.1). In [32], only the special cases of TPA and SPA games were investigated assuming SIC decoding. In [27], both decoding techniques were considered (i.e., SUD and SIC decoding). When SIC was assumed, the general STPA game was studied.

In order to take into account the effects of antenna correlation, we will assume the channel matrices to be structured according to the unitary-independent-unitary model introduced in [35]:

$$\forall k \in \{1, ..., K\}, \ \mathbf{H}_k = \mathbf{V}_k \tilde{\mathbf{H}}_k \mathbf{W}_k, \tag{2.10}$$

where $\mathbf{V}_k$ and $\mathbf{W}_k$ are deterministic unitary matrices that allow one to take into consideration the correlation effects at the receiver and transmitters, respectively. The

channel matrix $\tilde{\mathbf{H}}_k$ is an $n_r \times n_{t,k}$ random matrix whose entries are zero-mean independent complex Gaussian variables with an arbitrary variance profile, such that $\mathbb{E}|\tilde{\mathbf{H}}_k(\ell,c)|^2 = \frac{\sigma_k^2(\ell,c)}{n_{t,k}}$. The Kronecker propagation model, for which the channel transfer matrices factorizes as $\mathbf{H}_k = \mathbf{R}_k^{1/2}\mathbf{\Theta}_k\mathbf{T}_k^{1/2}$, is a special case of the UIU model. The variance profile is separable i.e., $\sigma_k^2(\ell,c) = \frac{d_k^{(T)}(\ell)d_k^{(\mathrm{R})}(c)}{n_t}$, $\mathbf{V}_k = \mathbf{U}_{\mathbf{R}_k}$, $\mathbf{W}_k = \mathbf{U}_{\mathbf{T}_k}$ with $\mathbf{\Theta}_k$ is a random matrix with zero-mean i.i.d. entries. The matrices $\mathbf{T}_k$, $\mathbf{R}_k$ represent the transmitter and receiver antenna correlation matrices such that $d_k^{(\mathrm{T})}(\ell)$, $d_k^{(\mathrm{R})}(c)$ are their associated eigenvalues and $\mathbf{U}_{\mathbf{T}_k}$, $\mathbf{U}_{\mathbf{R}_k}$ are their associated eigenvectors matrices.

It turns out that, for both decoding techniques, the existence of a unique NE can be proved. When determining the NE state, the main difficulty is that there are no closed-form expressions of the ergodic rates and the optimal power allocation policies. Indeed, the optimal eigenvalues of the transmit covariance matrices are not easy to find. They might be found using extensive numerical techniques. Our approach consists of approximating these ergodic rates to obtain expressions that are easier to interpret and to optimize. In order to do this, two extra assumptions will be made: i) $\mathbf{V}_k = \mathbf{V}, \forall k \in \mathcal{K}$, which means that the receive antenna correlation matrices $\mathbf{R}_k$ (Kronecker model) decompose to the same eigenvector basis $\mathbf{V}$; ii) $n_{t,k} = n_t$, i.e., the transmitters have the same number of antennas. Our approach consists in two steps. First, we determine the optimal eigenvectors. Then, based on this result, we will approximate the different transmission rates by their large-system (i.e., $n_r \to \infty$, $n_t \to \infty$, $\frac{n_r}{n_t} \to \beta$) deterministic equivalents. We will also exploit the results provided in [35] for the single-user MIMO channels to obtain the optimal eigenvalues of the covariance matrices. The corresponding approximates can be found to be accurate even for relatively small numbers of antennas (see e.g., [36][37] for more details).

Single User Decoding

When the SUD is assumed at the receiver, each user has to choose the best precoding matrix $\mathbf{Q}_k = \mathbb{E}\left[\underline{X}_k\underline{X}_k^H\right]$, in the sense of his payoff function:

$$u_k^{\mathrm{SUD}}(\mathbf{Q}_1,\mathbf{Q}_2) = \mathbb{E}\log\left|\mathbf{I}_{n_r} + \rho\mathbf{H}_k\mathbf{Q}_k\mathbf{H}_k^H + \rho\sum_{\ell\neq k}\mathbf{H}_\ell\mathbf{Q}_\ell\mathbf{H}_\ell^H\right| - \mathbb{E}\log\left|\mathbf{I}_{n_r} + \rho\sum_{\ell\neq k}\mathbf{H}_\ell\mathbf{Q}_\ell\mathbf{H}_\ell^H\right| \tag{2.11}$$

The strategy set of user k is given in (2.2).

Theorem 2.1.1 *The space power allocation game described by:* $\mathcal{G}^{\mathrm{SUD}} = \left(\mathcal{K}, \{\mathcal{A}_k^{\mathrm{SUD}}\}_{k\in\mathcal{K}}, \{u_k^{\mathrm{SUD}}\}_{k\in\mathcal{K}}\right)$, *where the payoff functions* $u_k^{\mathrm{SUD}}(\mathbf{Q}_k,\mathbf{Q}_{-k})$ *are given by (2.11) has a unique pure-strategy Nash equilibrium.*

The proof of the existence of a NE is based on the properties of concave games in [20]. Proving the diagonal strict concavity condition of Rosen [20] is sufficient to ensure the uniqueness of the NE point (see Appendix A.2 and Appendix A.3). However, extending Theorem D.1.6 is not trivial and the proof is given in Appendix A.2.

In order to find the optimum covariance matrices, we proceeded in the same way as described in [38]. First we determine the optimum eigenvectors and then the optimum eigenvalues by approximating the payoff functions under the large system assumption.

Since we have assumed $\mathbf{V}_k = \mathbf{V}$, we can exploit the results in [35] [39] for single-user MIMO channels, assuming the asymptotic regime in terms of the number of antennas: $n_r \to \infty$, $n_t \to \infty$, $\frac{n_r}{n_t} \to \beta$. It turns out, that there is no loss of optimality by choosing the covariance matrices $\mathbf{Q}_k = \mathbf{W}_k \mathbf{P}_k \mathbf{W}_k^H$, where $\mathbf{W}_k$ is the same unitary matrix as in (2.10) and $\mathbf{P}_k$ is the diagonal matrix containing the eigenvalues of $\mathbf{Q}_k$. Although this result is easy to obtain, it is instrumental in our context for two reasons. First, the search of the optimum precoding matrices boils down to the search of the eigenvalues of these matrices. Second, as the optimum eigenvectors are known, available results in random matrix theory can be applied to find an accurate approximation of these eigenvalues [35] [39]. The corresponding approximated payoff for user k is:

$$\begin{aligned}
\tilde{u}_k^{\mathrm{SUD}}(\mathbf{P}_k, \mathbf{P}_{-k}) \;=\; & \frac{1}{n_r} \sum_{k=1}^{K} \sum_{j=1}^{n_t} \log_2(1 + K\rho P_k(j)\gamma_k(j)) + \\
& \frac{1}{n_r} \sum_{i=1}^{n_r} \log_2 \left(1 + \frac{1}{K n_t} \sum_{k=1}^{K} \sum_{j=1}^{n_t} \sigma_k(i,j)\delta_k(j) \right) - \\
& \frac{1}{n_r} \sum_{k=1}^{K} \sum_{j=1}^{n_t} \gamma_k(j)\delta_k(j) \log_2 e - \\
& \frac{1}{n_r} \sum_{\ell \neq k} \sum_{j=1}^{n_t} \log_2(1 + (K-1)\rho P_\ell(j)\phi_\ell(j)) - \\
& \frac{1}{n_r} \sum_{i=1}^{n_r} \log_2 \left(1 + \frac{1}{(K-1)n_t} \sum_{\ell \neq k} \sum_{j=1}^{n_t} \sigma_\ell(i,j)\psi_\ell(j) \right) + \\
& \tfrac{1}{n_r} \sum_{\ell \neq k} \sum_{j=1}^{n_r} \phi_\ell(j)\psi_\ell(j) \log_2 e
\end{aligned} \tag{2.12}$$

where the parameters $\gamma_k(j)$ and $\delta_k(j)$ $\forall j \in \{1, \ldots, n_t\}$, $k \in \{1, 2\}$ are solution of the system:

$$\begin{cases}
\forall j \in \{1, \ldots, n_t\}, k \in \mathcal{K}: \\
\gamma_k(j) \;=\; \dfrac{1}{K n_t} \displaystyle\sum_{i=1}^{n_r} \dfrac{\sigma_k(i,j)}{1 + \frac{1}{K n_t} \displaystyle\sum_{\ell=1}^{K} \sum_{m=1}^{n_t} \sigma_\ell(i,m)\delta_\ell(m)} \\
\delta_k(j) \;=\; \dfrac{K\rho P_k(j)}{1 + K\rho P_k(j)\gamma_k(j)}
\end{cases} \tag{2.13}$$

and $\phi_\ell(j)$, $\psi_\ell(j)$, $\forall j \in \{1,\dots,n_t\}$ are the unique solutions of the following system:

$$\begin{cases} & \forall j \in \{1,\dots,n_t\}, \ell \in \mathcal{K} \setminus \{k\} : \\ \phi_\ell(j) & = \dfrac{1}{(K-1)n_t} \displaystyle\sum_{i=1}^{n_r} \dfrac{\sigma_\ell(i,j)}{1 + \frac{1}{(K-1)n_t} \displaystyle\sum_{r \neq k} \sum_{m=1}^{n_t} \sigma_r(i,m)\psi_r(m)} \\ \psi_\ell(j) & = \dfrac{(K-1)\rho P_\ell(j)}{1 + (K-1)\rho P_\ell(j)\phi_\ell(j)}. \end{cases} \tag{2.14}$$

The optimal eigenvalues are given by the water-filling solutions:

$$P_k^{\mathrm{NE}}(j) = \left[\frac{1}{n_r \lambda_k \ln 2} - \frac{1}{K\rho\gamma_k(j)} \right]^+, \tag{2.15}$$

where $\lambda_k \geq 0$ is the Lagrangian multiplier tuned in order to meet the power constraint: $\displaystyle\sum_{j=1}^{n_t} \left[\frac{1}{n_r \lambda_k \ln 2} - \frac{1}{K\rho\gamma_k(j)} \right]^+ = n_t \overline{P}_k$. Notice that, in order to solve the system of equations given above, we can use the same iterative power allocation algorithm as the one described in [38]. The transmitters are assumed to have perfect knowledge of all the channels' distributions. As for the efficiency of the NE point, the SUD decoding technique is sub-optimal w.r.t. the centralized case and the sum-capacity is not reached at the NE similarly to the static MIMO MAC channel.

An important remark has to be made. The analysis of the NE (i.e., existence and uniqueness issues) has been analysed in the finite setting (exact game). The determination of the NE is performed using numerical methods (required to implement water-filling type algorithms) and the approximated utilities in the asymptotic regime. This is motivated by the fact that the ergodic rates are very difficult to be numerically optimized. Furthermore, it turns out that the large system approximations of ergodic rates have the same properties as their exact counterparts, as shown recently by [40]. However, the analysis of the NE for the approximated game is an interesting issue which is left as an extension of this work.

Another rising issue could be the characterization of the correlated equilibria of the game. The results in [41] can be applied here directly, since $\mathcal{G}^{\mathrm{SUD}}$ is an exact potential game with a strict concave potential function, i.e., the achievable network sum-rate. The author of [41] proved that for strict concave potential games, the CE is unique and consists in playing with probability one the unique pure NE.

Successive Interference Cancellation

As we have mentioned in Section 2.1.1, three different power allocation games can be defined in function of the action sets: SPA, TPA and the joint STPA, for which the action sets are given by (2.7),(2.8) and (2.9), respectively. In the remaining of this section, we will only focus on the general STPA game. The results for the special cases

follow directly. The main difference with the static case, consists in the payoff function of users which are given by the ergodic achievable rates:

$$u_k^{\mathrm{SIC}}(\mathbf{Q}_k, \mathbf{Q}_{-k}) = \sum_{\pi \in \mathcal{P}_K} p_\pi R_k^{(\pi)}(\mathbf{Q}_k^{(\pi)}, \mathbf{Q}_{-k}^{(\pi)}) \tag{2.16}$$

where

$$\begin{aligned} R_k^{(\pi)}(\mathbf{Q}_k^{(\pi)}, \mathbf{Q}_{-k}^{(\pi)}) \quad = \quad & \mathbb{E}\log_2 \left| \mathbf{I}_{n_r} + \rho \mathbf{H}_k \mathbf{Q}_k^{(\pi)} \mathbf{H}_k^H + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \mathbf{H}_\ell \mathbf{Q}_\ell^{(\pi)} \mathbf{H}_\ell^H \right| - \\ & \mathbb{E}\log_2 \left| \mathbf{I}_{n_r} + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \mathbf{H}_\ell \mathbf{Q}_\ell^{(\pi)} \mathbf{H}_\ell^H \right| \end{aligned} \tag{2.17}$$

with the same notation as in Sec. 2.1.1 As opposed to the static case, for the fast fading one the uniqueness is guaranteed for the general joint space-time power allocation game. The obtained results are stated in the following theorem.

Theorem 2.1.2 *The joint space-time power allocation game described by:* $\mathcal{G}^{\mathrm{SIC}} = \left(\mathcal{K}, \left\{\mathcal{A}_k^{\mathrm{SIC,STPA}}\right\}_{k\in\mathcal{K}}, \left\{u_k^{\mathrm{SIC}}\right\}_{k\in\mathcal{K}}\right)$*, where the payoff functions* $u_k^{\mathrm{SIC}}(\mathbf{Q}_k, \mathbf{Q}_{-k})$ *are given by (2.16), has a unique pure-strategy Nash equilibrium.*

The proof is similar to the SUD case and exploits the extended results of Rosen [20] and is given in Appendix A.2. The difficulty here lies in proving a matrix trace inequality (see Appendix A.2 and Appendix A.3). This inequality is instrumental in proving that the diagonal strict concavity condition holds. Its proof is given in [42] for $K = 2$ and in [43] for arbitrary $K \geq 2$ (see Appendix A.4).

In order to find the optimal covariance matrices we proceed in the same way as described in Section 2.1.2. The optimal eigenvectors of the covariance matrix $\mathbf{Q}_k^{(\pi)}$ are given by $\mathbf{U}_{Q_k^{(\pi)}} = \mathbf{W}_k$. And, the optimal eigenvalues, $\mathbf{P}_k^{(\pi)}$, can be found using the large-system assumptions. The approximated payoff for User k is $\tilde{u}_k^{\mathrm{SIC}}(\{\mathbf{P}_k^{(\pi)}\}_{k\in\mathcal{K},\pi\in\mathcal{P}_K}) =$

$\sum_{\pi\in\mathcal{P}_K} p_\pi \tilde{R}_k^{(\pi)}(\mathbf{P}_k^{(\pi)},\mathbf{P}_{-k}^{(\pi)})$ where

$$\begin{aligned}
\tilde{R}_k^{(\pi)}(\mathbf{P}_k^{(\pi)},\mathbf{P}_{-k}^{(\pi)}) \;=\; & \frac{1}{n_r}\sum_{\ell\in\mathcal{K}_k^{(\pi)}\cup\{k\}}\sum_{j=1}^{n_t}\log_2\left(1+(N_k^{(\pi)}+1)\rho P_\ell^{(\pi)}(j)\gamma_\ell^{(\pi)}(j)\right)+\\
& \frac{1}{n_r}\sum_{i=1}^{n_r}\log_2\left(1+\frac{1}{(N_k^{(\pi)}+1)n_t}\sum_{\ell\in\mathcal{K}_k^{(\pi)}\cup\{k\}}\sum_{j=1}^{n_t}\sigma_\ell(i,j)\delta_\ell^{(\pi)}(j)\right)-\\
& \frac{1}{n_r}\sum_{\ell\in\mathcal{K}_k^{(\pi)}\cup\{k\}}\sum_{j=1}^{n_t}\gamma_\ell^{(\pi)}(j)\delta_\ell^{(\pi)}(j)\log_2 e-\\
& \frac{1}{n_r}\sum_{\ell\in\mathcal{K}_k^{(\pi)}}\sum_{j=1}^{n_t}\log_2\left(1+N_k^{(\pi)}\rho P_\ell^{(\pi)}(j)\phi_\ell^{(\pi)}(j)\right)-\\
& \frac{1}{n_r}\sum_{i=1}^{n_r}\log_2\left(1+\frac{1}{N_k^{(\pi)}n_t}\sum_{\ell\in\mathcal{K}_k^{(\pi)}}\sum_{j=1}^{n_t}\sigma_\ell(i,j)\psi_\ell^{(\pi)}(j)\right)+\\
& \tfrac{1}{n_r}\sum_{\ell\in\mathcal{K}_k^{(\pi)}}\sum_{j=1}^{n_r}\phi_\ell^{(\pi)}(j)\psi_\ell^{(\pi)}(j)\log_2 e
\end{aligned} \tag{2.18}$$

where $N_k^{(\pi)}=|\mathcal{K}_k^{(\pi)}|$ and the parameters $\gamma_k^{(\pi)}(j)$ and $\delta_k^{(\pi)}(j)$ $\forall j\in\{1,\dots,n_t\}$, $k\in\mathcal{K}$, $\pi\in\mathcal{P}_K$ are the solutions of:

$$\begin{cases}
\forall j\in\{1,\dots,n_t\},\ell\in\mathcal{K}_k^{(\pi)}\cup\{k\}: \\
\gamma_\ell^{(\pi)}(j) \;=\; \dfrac{1}{(N_k^{(\pi)}+1)n_t}\displaystyle\sum_{i=1}^{n_r}\frac{\sigma_\ell(i,j)}{1+\frac{1}{(N_k^{(\pi)}+1)n_t}\displaystyle\sum_{r\in\mathcal{K}_k^{(\pi)}\cup\{k\}}\sum_{m=1}^{n_t}\sigma_r(i,m)\delta_r^{(\pi)}(m)} \\
\delta_\ell^{(\pi)}(j) \;=\; \dfrac{(N_k^{(\pi)}+1)\rho P_\ell^{(\pi)}(j)}{1+(N_k^{(\pi)}+1)\rho P_\ell^{(\pi)}(j)\gamma_\ell^{(\pi)}(j)},
\end{cases} \tag{2.19}$$

and $\phi_\ell^{(\pi)}(j)$, $\psi_\ell^{(\pi)}(j)$, $\forall j\in\{1,\dots,n_t\}$ and $\pi\in\mathcal{P}_K$ are the unique solutions of the following system:

$$\begin{cases}
\forall j\in\{1,\dots,n_t\},\ell\in\mathcal{K}_k^{(\pi)}: \\
\phi_\ell^{(\pi)}(j) \;=\; \dfrac{1}{N_k^{(\pi)}n_t}\displaystyle\sum_{i=1}^{n_r}\frac{\sigma_\ell(i,j)}{1+\frac{1}{N_k^{(\pi)}n_t}\displaystyle\sum_{r\in\mathcal{K}_k^{(\pi)}}\sum_{m=1}^{n_t}\sigma_r(i,m)\psi_r^{(\pi)}(m)} \\
\psi_\ell^{(\pi)}(j) \;=\; \dfrac{N_k^{(\pi)}\rho P_\ell^{\pi)}(j)}{1+N_k^{(\pi)}\rho P_\ell^{(\pi)}(j)\phi_\ell^{(\pi)}(j)}.
\end{cases} \tag{2.20}$$

The corresponding water-filling solution is:

$$P_k^{(\pi),\mathrm{NE}}(j) = \left[\frac{1}{n_r \lambda_k \ln 2} - \frac{1}{N_k^{(\pi)} \rho \gamma_k^{(\pi)}(j)}\right]^+, \tag{2.21}$$

where $\lambda_k \geq 0$ is the Lagrangian multiplier tuned in order to meet the power constraint:

$$\sum_{\pi \in \mathcal{P}_K} \sum_{j=1}^{n_t} p_\pi \left[\frac{1}{n_r \lambda_k \ln 2} - \frac{1}{N_k^{(\pi)} \rho \gamma_k^{(\pi)}(j)}\right]^+ = n_t \overline{P}_k.$$

In order to measure the efficiency of the decentralized network w.r.t. its centralized counterpart, we introduce the following quantity:

$$\mathrm{SRE} = \frac{R_{\mathrm{sum}}^{\mathrm{NE}}}{C_{\mathrm{sum}}} \leq 1, \tag{2.22}$$

where SRE stands for sum-rate efficiency; the quantity $R_{\mathrm{sum}}^{\mathrm{NE}}$ represents the sum-rate of the decentralized network at the Nash equilibrium, which is achieved for certain choices of coding and decoding strategies; the quantity C_{sum} corresponds to the sum-capacity of the centralized network, which is reached only if the optimum coding and decoding schemes are known. Note that this is the case for the MAC but not for other channels like the interference channel. Obviously, the efficiency measure we introduce here is strongly connected to the price of anarchy introduced in [44] (PoA). The SRE measures the gap between the sum-rate at the NE and the network sum-rate obtained with the optimal decoding technique, whereas the PoA does not consider the optimal decoding technique. In our context, information theory provides us with fundamental physical limits on the social welfare (network sum-capacity) while in general no such upper bound is available.

We have proved that, both, in the low and high SNR regimes the SRE tends to one. This means that, in the extreme SNR regimes, the sum-capacity of the fast fading MAC is achieved at the NE point, in spite of the sub-optimal coordination mechanism applied at the receiver. For non-extreme SNR regimes, a closed-form expression of the SRE cannot be found. Numerical simulations have been provided to asses this optimality gap. When SIC is assumed, for the three power allocation games (TPA, SPA, STPA), the sum-rate efficiency at the NE is close to one. Quite surprisingly, the NE of the STPA game performs a little worse than its purely spatial counterpart. This highlights another Braess-type paradox as in Section 2.1.1 (see [27] in Appendix A.2).

2.2 The Parallel Interference Relay Channel

In this section, we study a different network model, the parallel interference relay channel. The Shannon-rate efficient power allocation game fr the interference channel has been extensively studied in the literature: [45] [46] for SISO frequency selective channels, [47] [48] for the static parallel interference channel and [49] [50] [51] [52] [53] for the

static MIMO channels. The difference in our work is that we allow the transmitters to exploit the existence of some relaying nodes to improve the communication performance.

As opposed to the previous section, we will focus here only on the particular case of static parallel IRC [54] (see Appendix A.6).

The system under investigation is composed of two source nodes $\mathcal{S}_1$, $\mathcal{S}_2$, transmitting their private messages to their respective destination nodes $\mathcal{D}_1$, $\mathcal{D}_2$. To this end, each source can exploit Q non-overlapping frequency bands (the notation (q) will be used to refer to band $q \in \{1, \ldots, Q\}$) which are assumed to be of normalized bandwidth. The signals transmitted by $\mathcal{S}_1$ and $\mathcal{S}_2$ in band (q), denoted by $X_1^{(q)}$ and $X_2^{(q)}$ respectively, are assumed to be independent and power constrained:

$$\forall k \in \{1, 2\}, \ \sum_{q=1}^{Q} \mathbb{E}|X_k^{(q)}|^2 \leq \overline{P}_k. \tag{2.23}$$

For $k \in \mathcal{K} \triangleq \{1, 2\}$, we denote by $\theta_k^{(q)}$ the fraction of power that is used by $\mathcal{S}_k$ for transmitting in band (q) that is, $\mathbb{E}|X_k^{(q)}|^2 = \theta_k^{(q)} \overline{P}_k$. Therefore, the set of possible power allocation policies for User k can be defined as:

$$\mathcal{A}_k = \left\{ \underline{\theta}_k \in [0, 1]^Q \left| \sum_{q=1}^{Q} \theta_k^{(q)} \leq 1 \right. \right\} \tag{2.24}$$

Additionally, we assume that there exists a multi-band relay $\mathcal{R}$. With these notations, the signals received by $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{R}$ in band (q) express as:

$$\left\{ \begin{array}{lcl} Y_1^{(q)} & = & h_{11}^{(q)} X_1^{(q)} + h_{21}^{(q)} X_2^{(q)} + h_{r1}^{(q)} X_r^{(q)} + Z_1^{(q)} \\ Y_2^{(q)} & = & h_{12}^{(q)} X_1^{(q)} + h_{22}^{(q)} X_2^{(q)} + h_{r2}^{(q)} X_r^{(q)} + Z_2^{(q)} \\ Y_r^{(q)} & = & h_{1r}^{(q)} X_1^{(q)} + h_{2r}^{(q)} X_2^{(q)} + Z_r^{(q)} \end{array} \right. \tag{2.25}$$

where $Z_k^{(q)} \sim \mathcal{N}(0, N_k^{(q)})$, $k \in \{1, 2, r\}$, represents the Gaussian complex noise on band (q) and, for all $(k, \ell) \in \mathcal{K}^2$, $h_{k\ell}^{(q)}$ is the channel gain between $\mathcal{S}_k$ and $\mathcal{D}_\ell$, $h_{kr}^{(q)}$ is the channel gain between $\mathcal{S}_k$ and $\mathcal{R}$, $h_{rk}^{(q)}$ is the channel gain between $\mathcal{R}$ and $\mathcal{D}_k$ in band (q). The channel gains are considered to be static. Concerning channel state information (CSI), we will always assume coherent communications for each transmitter-receiver pair $(\mathcal{S}_k, \mathcal{D}_k)$ whereas, at the transmitters, the information assumptions will be context-depending. The single-user decoding (SUD) will always be assumed at $\mathcal{D}_1$ and $\mathcal{D}_2$.

At the relay, the implemented reception scheme will depend on the protocol assumed. The expressions of the signals transmitted by the relay, $X_r^{(q)}, q \in \{1, ..., Q\}$, will also depend on the relay protocol and will therefore also be explained in the corresponding sections. So far, we have not mentioned any power constraint on the signals $X_r^{(q)}$. We also assume that the relay implements a fixed power allocation policy between the Q available bands ($\mathbb{E}|X_r^{(q)}|^2 = P_r^{(q)}, q \in \{1, ..., Q\}$). As in [12][11][55], the relay is assumed to operate in the full-duplex mode.

In what follows, we investigate the existence of an NE solution for the non-cooperative power allocation game where the transmitters are assisted by several relaying nodes. We analyse three different games depending on the relaying protocol assumed, which can be: estimate-and-forward (EF), decode-and-forward (DF) or amplify-and-forward. In general, the multiplicity of the NE points is an intractable problem. However, in the special case of fixed amplification gain AF protocol, we completely characterize the number of NE and prove the convergence of the best-response based iterative algorithm.

2.2.1 Decode-and-Forward (DF) Protocol

We start with the decode-and-forward protocol. The basic idea behind this protocol is as follows. From each message intended for the destination, the source builds a coarse and a fine message. With these two messages, the source superposes two codewords. The rates associated with these codewords (or messages) are such that the relay can reliably decode both of them while the destination can only decode the coarse message. After decoding this message, the destination can subtract the corresponding signal and try to decode the fine message. To help the destination to do so, the relay cooperates with the source by sending some information about the fine message. Mathematically, this translates as follows. The signal transmitted by $\mathcal{S}_k$ in band (q) is structured as $X_k^{(q)} = X_{k,0}^{(q)} + \sqrt{\frac{\tau_k^{(q)}}{\nu_k^{(q)}} \frac{\theta_k^{(q)} P_k}{P_r^{(q)}}} X_{r,k}^{(q)}$ where: the signals $X_{k,0}^{(q)}$ and $X_{r,k}$ are independent and precisely correspond to the coarse and fine messages respectively; the parameter $\nu_k^{(q)}$ represents the fraction of transmit power the relay allocates to user i, hence we have $\nu_1^{(q)} + \nu_2^{(q)} \leq 1$; the parameter $\tau_k^{(q)}$ represents the fraction of transmit power $\mathcal{S}_k$ allocates to the cooperation signal (conveying the fine message). The transmitted signal by the relay $\mathcal{R}$ writes as: $X_r^{(q)} = X_{r,1}^{(q)} + X_{r,2}^{(q)}$. From [11], and for a given allocation policy $\underline{\theta}_k = \left(\theta_k^{(1)}, ..., \theta_k^{(Q)}\right)$, the source-destination pair $(\mathcal{S}_k, \mathcal{D}_k)$ achieves the transmission rate $\sum_{q=1}^{Q} R_k^{(q),\mathrm{DF}}$ where

$$\left\{ \begin{array}{lcl} R_1^{(q),\mathrm{DF}} & = & \min\left\{R_{1,1}^{(q),\mathrm{DF}}, R_{1,2}^{(q),\mathrm{DF}}\right\} \\ R_2^{(q),\mathrm{DF}} & = & \min\left\{R_{2,1}^{(q),\mathrm{DF}}, R_{2,2}^{(q),\mathrm{DF}}\right\} \end{array} \right. , \tag{2.26}$$

$$\begin{cases} R_{1,1}^{(q),\mathrm{DF}} &= C\left(\frac{\left|h_{1r}^{(q)}\right|^2\overline{\tau_1^{(q)}}\theta_1^{(q)}P_1}{\left|h_{2r}^{(q)}\right|^2\overline{\tau_2^{(q)}}\theta_2^{(q)}P_2+N_r^{(q)}}\right) \\ R_{2,1}^{(q),\mathrm{DF}} &= C\left(\frac{\left|h_{2r}^{(q)}\right|^2\overline{\tau_2^{(q)}}\theta_2^{(q)}P_2}{\left|h_{1r}^{(q)}\right|^2\overline{\tau_1^{(q)}}\theta_1^{(q)}P_1+N_r^{(q)}}\right) \\ R_{1,2}^{(q),\mathrm{DF}} &= C\left(\frac{\left|h_{11}^{(q)}\right|^2\theta_1^{(q)}P_1+\left|h_{r1}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}+2\mathrm{Re}\left(h_{11}^{(q)}h_{r1}^{(q),*}\right)\sqrt{\tau_1^{(q)}\theta_1^{(q)}P_1\nu^{(q)}P_r^{(q)}}}{\left|h_{21}^{(q)}\right|^2\theta_2^{(q)}P_2+\left|h_{r1}^{(q)}\right|^2\overline{\nu^{(q)}}P_r^{(q)}+2\mathrm{Re}\left(h_{21}^{(q)}h_{r1}^{(q),*}\right)\sqrt{\tau_2^{(q)}\theta_2^{(q)}P_2\overline{\nu^{(q)}}P_r^{(q)}}+N_1^{(q)}}\right) \\ R_{2,2}^{(q),\mathrm{DF}} &= C\left(\frac{\left|h_{22}^{(q)}\right|^2\theta_2^{(q)}P_2+\left|h_{r2}^{(q)}\right|^2\overline{\nu^{(q)}}P_r^{(q)}+2\mathrm{Re}\left(h_{22}^{(q)}h_{r2}^{(q),*}\right)\sqrt{\tau_2^{(q)}\theta_2^{(q)}P_2\overline{\nu^{(q)}}P_r^{(q)}}}{\left|h_{12}^{(q)}\right|^2\theta_1^{(q)}P_1+\left|h_{r2}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}+2\mathrm{Re}\left(h_{12}^{(q)}h_{r2}^{(q),*}\right)\sqrt{\tau_1^{(q)}\theta_1^{(q)}P_1\nu^{(q)}P_r^{(q)}}+N_2^{(q)}}\right) \end{cases} \tag{2.27}$$

and $(\nu^{(q)}, \tau_1^{(q)}, \tau_2^{(q)})$ is a given triple of parameters in $[0,1]^3$, $\tau_1^{(q)} + \tau_2^{(q)} \leq 1$ and $C(x) \triangleq \log_2(1+x)$ denotes the capacity function for complex signals.

The achievable transmission rate of User k is given by:

$$\mu_k^{\mathrm{DF}}(\underline{\theta}_k, \underline{\theta}_{-k}) = \sum_{q=1}^{Q} R_k^{(q),\mathrm{DF}}(\theta_k^{(q)}, \theta_{-k}^{(q)}). \tag{2.28}$$

The one-shot game is defined by the triplet $\mathcal{G}^{\mathrm{DF}} = \left(\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{\mu_k^{\mathrm{DF}}\}_{k\in\mathcal{K}}\right)$. Although this setup might seem to be demanding in terms of CSI at the source nodes, it turns out that the equilibria predicted in such a framework can be effectively observed in more realistic frameworks where each player observes the strategy played by the other players and reacts accordingly by maximizing his payoff iteratively.

The existence theorem for the DF protocol is given hereunder.

Theorem 2.2.1 *If the channel gains satisfy the condition $\mathcal{R}e(h_{kk}^{(q)}h_{rk}^{(q)*}) \geq 0$, for all $k \in \mathcal{K}$ and $q \in \{1,\ldots,Q\}$ the game defined by $\mathcal{G}^{\mathrm{DF}} = \left(\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{\mu_k^{\mathrm{DF}}\}_{k\in\mathcal{K}}\right)$ has always at least one pure-strategy NE.*

This theorem shows that for the pathloss channel model, where $h_{k\ell} > 0$, $(k,\ell) \in \{1,2,r\}^2$, there always exists an equilibrium. As a consequence, if some relays are added in the network, the transmitters will adapt their PA policies accordingly and, whatever the locations of the relays, an equilibrium will be observed. This is a nice property for the system under investigation. As the PA game with DF is concave it is tempting to try to verify whether the sufficient condition for uniqueness of [20] is met here. It turns out that the diagonally strict concavity condition of [20] is not trivial to be checked. Additionally, it is possible that the game has several equilibria as it is proven to be the case for the AF protocol.

In a context of decentralized networks, each source $\mathcal{S}_k$ has to optimize the parameter τ_k in order to maximize its transmission rate R_k. In the rate region above, one can observe that this choice is not independent of the choice of the other source. Therefore, each source finds its optimal strategy by optimizing its rate w.r.t. $\tau_k^*(\tau_\ell)$. In order to do that, each source has to make some assumptions on the value τ_ℓ used by the

other source. This is precisely a non-cooperative game where each player makes some assumptions on the other player's behaviour and maximizes its own payoff. Interestingly, we see that, even in the single-band case, the DF protocol introduces a power allocation game through the parameter τ_k representing the cooperation degree between the source $\mathcal{S}_k$ and relay. For more details on the game induced by the cooperation degrees the reader is referred to [56].

2.2.2 Estimate-and-Forward (EF) Protocol

The quality of the links from the sources to the relay represents the bottleneck of the DF protocol. In some situations, the presence of the relays may even degrade the transmission performance. For example, if the relay is situated far away from the sources such that the destinations are in better reception condition. We will now consider a protocol that always improves the performance of the transmission, the estimate-and-forward. The principle of the EF protocol for the standard relay channel is that the relay sends an approximated version of its observed signal to the receiver. In our setup, we have two different receivers. The relay can either create a single quantized version of its observation, common to both receivers, or two quantized versions, one for each destination (see [57]). Here, we consider that the relay uses two resolution levels to compress its observation signal, each of these levels being adapted to the considered destination; we call the corresponding version of the EF protocol, "bi-level compression EF". We make the same assumptions as in Section 2.2.1 concerning the reception schemes and PA policies at the relays: we assume that each node $\mathcal{R}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ implements single-user decoding and the PA policy at each relay i.e., $\underline{\nu} = \left(\nu^{(1)}, ..., \nu^{(Q)}\right)$ is fixed. The payoff for User $k \in \mathcal{K}$ can be expressed as follows

$$\mu_k^{\mathrm{EF}}(\underline{\theta}_k, \underline{\theta}_{-k}) = \sum_{q=1}^{Q} R_k^{(q),\mathrm{EF}} \tag{2.29}$$

where

$$\begin{aligned} R_1^{(q),\mathrm{EF}} &= C\left(\frac{\left(\left|h_{2r}^{(q)}\right|^2\theta_2^{(q)}P_2+N_r^{(q)}+N_{wz,1}^{(q)}\right)\left|h_{11}^{(q)}\right|^2\theta_1^{(q)}P_1+\left(\left|h_{21}^{(q)}\right|^2\theta_2^{(q)}P_2+\left|h_{r1}^{(q)}\right|^2\overline{\nu^{(q)}}P_r^{(q)}+N_1^{(q)}\right)\left|h_{1r}^{(q)}\right|^2\theta_1^{(q)}P_1}{\left(N_r^{(q)}+N_{wz,1}^{(q)}\right)\left(|h_{21}^{(q)}|^2\theta_2^{(q)}P_2+|h_{r1}^{(q)}|^2\overline{\nu^{(q)}}P_r^{(q)}+N_1^{(q)}\right)+|h_{2r}^{(q)}|^2\theta_2^{(q)}P_2\left(|h_{r1}^{(q)}|^2\overline{\nu^{(q)}}P_r^{(q)}+N_1^{(q)}\right)}\right) \\ R_2^{(q),\mathrm{EF}} &= C\left(\frac{\left(|h_{1r}|^2\theta_1^{(q)}P_1+N_r^{(q)}+N_{wz,2}^{(q)}\right)\left|h_{22}^{(q)}\right|^2\theta_2^{(q)}P_2+\left(\left|h_{12}^{(q)}\right|^2\theta_1^{(q)}P_1+\left|h_{r2}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}+N_2^{(q)}\right)\left|h_{2r}^{(q)}\right|^2\theta_2^{(q)}P_2}{\left(N_r^{(q)}+N_{wz,2}^{(q)}\right)\left(\left|h_{12}^{(q)}\right|^2\theta_1^{(q)}P_1+\left|h_{r2}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}+N_2^{(q)}\right)+\left|h_{1r}^{(q)}\right|^2\theta_1^{(q)}P_1\left(\left|h_{r2}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}+N_2^{(q)}\right)}\right) \end{aligned} \tag{2.30}$$

$$\begin{aligned} N_{wz,1}^{(q)} &= \frac{\left(\left|h_{11}^{(q)}\right|^2\theta_1^{(q)}P_1+\left|h_{21}^{(q)}\right|^2\theta_2^{(q)}P_2+\left|h_{r1}^{(q)}\right|^2\overline{\nu^{(q)}}P_r^{(q)}+N_1^{(q)}\right)A^{(q)}-\left|A_1^{(q)}\right|^2}{\left|h_{r1}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}} \\ N_{wz,2}^{(q)} &= \frac{\left(\left|h_{22}^{(q)}\right|^2\theta_2^{(q)}P_2+\left|h_{12}^{(q)}\right|^2\theta_1^{(q)}P_1+\left|h_{r2}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}+N_2^{(q)}\right)A^{(q)}-\left|A_2^{(q)}\right|^2}{\left|h_{r2}^{(q)}\right|^2\overline{\nu^{(q)}}P_r^{(q)}} \end{aligned}, \tag{2.31}$$

$\nu^{(q)} \in [0,1]$, $A^{(q)} = |h_{1r}^{(q)}|^2\theta_1^{(q)}P_1 + |h_{2r}^{(q)}|^2\theta_2^{(q)}P_2 + N_r^{(q)}$, $A_1^{(q)} = h_{11}^{(q)}h_{1r}^{(q),*}\theta_1^{(q)}P_1 + h_{21}^{(q)}h_{2r}^{(q),*}\theta_2^{(q)}P_2$ and $A_2^{(q)} = h_{12}^{(q)}h_{1r}^{(q),*}\theta_1^{(q)}P_1 + h_{22}^{(q)}h_{2r}^{(q),*}\theta_2^{(q)}P_2$. What is interesting with this EF protocol is that, here again, one can prove that the payoff function is concave for every user. This is the purpose of the following theorem.

Theorem 2.2.2 *The game defined by* $\mathcal{G}^{\mathrm{EF}} = \left(\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{\mu_k^{\mathrm{EF}}\}_{k\in\mathcal{K}}\right)$ *has always at least one pure-strategy NE.*

The proof is similar to the proof of Theorem 2.2.1. To be able to apply Theorem D.1.5 of Rosen, we have to prove that the payoff μ_k^{EF} is concave w.r.t. $\underline{\theta}_k$. The problem is less simple than for DF because the compression noise $N_{wz,k}^{(q)}$ which appears in the denominator of the capacity function in Eq. (2.30) depends on the strategy $\underline{\theta}_k$ of transmitter k. It turns out that it is still possible to prove the desired result as shown in Appendix A.6.

2.2.3 Amplify-and-Forward (AF) Protocol

In this section, we assume that the the relay implements an analog amplifier which does not introduce any delay on the relayed signal. The signal transmitted by the relay writes as $X_r = a_r Y_r$ where a_r corresponds to the relay amplification gain. We call the corresponding protocol the zero-delay scalar amplify-and-forward (ZDSAF). One of the nice features of the ZDSAF protocol is that relays are very easy to be deployed since they can be used without any change on the existing (non-cooperative) communication system. The amplification gain for the relay on band (q) will be denoted by $a_r^{(q)}$. Here, we consider that the amplification gain is such that the relay exploits all the available power on each band. The achievable transmission rate is given by

$$\mu_k^{\mathrm{AF}}(\underline{\theta}_k, \underline{\theta}_{-k}) = \sum_{q=1}^{Q} R_k^{(q),\mathrm{AF}}(\theta_k^{(q)}, \theta_{-k}^{(q)}) \tag{2.32}$$

where $R_k^{(q),\mathrm{AF}}$ is the rate user k obtains by using band (q) when the ZDSAF protocol is used by the relay $\mathcal{R}$.

$$\forall k \in \mathcal{K}, R_k^{(q),\mathrm{AF}} \quad = \quad C\left(\frac{\left|a_r^{(q)}\ h_{kr}^{(q)}h_{rk}^{(q)} + h_{kk}^{(q)}\right|^2\theta_k^{(q)}\rho_k}{\left|a_r^{(q)}\ h_{\ell r}^{(q)}h_{rk}^{(q)} + h_{\ell k}^{(q)}\right|^2\rho_\ell\theta_\ell^{(q)}\frac{N_\ell^{(q)}}{N_k^{(q)}} + \left(a_r^{(q)}\right)^2\left|h_{rk}^{(q)}\right|^2\frac{N_r^{(q)}}{N_k^{(q)}} + 1}\right) \tag{2.33}$$

where $a_r^{(q)} = \tilde{a}_r^{(q)}(\theta_1^{(q)}, \theta_2^{(q)}) \triangleq \sqrt{\frac{P_r}{\left|h_{1r}^{(q)}\right|^2 P_1 + \left|h_{2r}^{(q)}\right|^2 P_2 + N_r}}$ and $\rho_k^{(q)} = \frac{P_k}{N_k^{(q)}}$. Without loss of generality and for the sake of clarity we will assume in Sec. 2.2.3 that $\forall (k,q) \in \{1,2,r\} \times \{1,\ldots,Q\}, N_k^{(q)} = N$, $P_r^{(q)} = \overline{P}_r$ and we introduce the quantities $\rho_k = \frac{\overline{P}_k}{N}$. In this setup the following existence theorem can be proven.

Theorem 2.2.3 *If one of the following conditions is met: i)* $\left|a_r^{(q)} h_{kr}^{(q)} h_{rk}^{(q)}\right| \gg \left|h_{kk}^{(q)}\right|$ *and* $\left|a_r^{(q)} h_{\ell r}^{(q)} h_{rk}^{(q)}\right| \gg \left|h_{\ell k}^{(q)}\right|$ *(i.e., the direct links* $\mathcal{S}_k - \mathcal{D}_k$ *are negligible and the communication is possible only through the relay node), ii)* $\left|h_{kk}^{(q)}\right| \gg \left|a_r^{(q)} h_{kr}^{(q)} h_{rk}^{(q)}\right|$ *and* $\left|h_{\ell k}^{(q)}\right| \gg \left|a_r^{(q)} h_{\ell r}^{(q)}\right| \min\left\{1, \left|h_{rk}^{(q)}\right|\right\}$ *(i.e., the links* $\mathcal{R} - \mathcal{D}_k$ *are negligible), iii)* $a_r^{(q)} = A_r^{(q)} \in [0, \tilde{a}_r^{(q)}(1,1)]$ *(i.e., the amplification gain is constant), there exists at least one pure-strategy NE in the PA game* $\mathcal{G}^{\mathrm{AF}} = \left(\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{\mu_k^{\mathrm{AF}}\}_{k\in\mathcal{K}}\right)$.

The proof is similar to the proof of Theorem 2.2.1. The sufficient conditions ensure the concavity of the function $R_k^{(q),\mathrm{AF}}$ w.r.t. $\theta_k^{(q)}$. Notice that, under the last two conditions, the analysis is very similar to that of a parallel IC for which the existence of the NE is always guaranteed [47] [48].

We have seen that, under certain sufficient conditions on the channel gains, the existence of the NE point is ensured for all three protocols investigated. The multiplicity and the convergence of best-response algorithms is not trivial in general. However, in [54] (see Appendix A.6), we have thoroughly studied a particular case of the AF protocol: $Q = 2$ and constant amplification factors $\forall q \in \{1, 2\}$, $a_r^{(q)} = A_r^{(q)} \in [0, \tilde{a}_r(1,1)]$. It turns out that, for this particular case, a complete characterization of the number of NE can be made based on the best-response function analysis. The best-response correspondences are piece-wise affine functions thus the network can have either one, two, three or an infinity number of NE. Based on the "Cournot duopoly" [58], the iterative algorithm based on the best-response functions is guaranteed to converge to one of the NE points.

In Appendix A.6, we prove that, using a time-sharing technique, the existence of an NE can always be ensured irrespective of the relaying protocol or channel gains. The basic idea is that, assuming that the transmitters could be coordinated, the achievable rates become concave by using time-sharing techniques.

A strong motivation for studying IRCs is to be able to introduce relays in a network with non-coordinated and interfering pairs of terminals. For example, relays could be introduced by an operator aiming at improving the performance of the communications of his customers. In such a scenario, the operator acts as a player and more precisely as a game leader in the sense of Stackelberg [22]. In this context, the game leader is the operator/engineer/relay who chooses the parameters of the relays. The followers are the adaptive/cognitive transmitters that adapt their PA policy to what they observe. In the preceding sections we have mentioned some of these parameters: the location of each relay; the amplification gain of each relay in the case of AF; while in the case of DF and EF, the power allocation policy between the two cooperative signals at each relay i.e., the parameter $\nu^{(q)}$. Therefore, the relay can be thought as a player minimizing its own payoff. This payoff can be either the individual payoff of a given transmitter (picture one WiFi subscriber wanting to increase his downlink throughput by locating his cellular phone somewhere in his apartment while his neighbour can also exploit the same spectral resources) or the network sum-rate (in the case of an operator). The Stackelberg formulation was studied through numerical simulations for the particular case of $Q = 2$

and $K = 2$. Several interesting observations were made. For the ZDSAF with constant amplification gain, it is not necessarily optimal to saturate the relay transmit power. For the general ZDSAF, the optimal relay node position w.r.t the system sum-rate lies exactly on the segment between $\mathcal{S}_k$ and $\mathcal{D}_k$ of the better receiver. This is due to the fact that the selfish behaviour of the users leads to an operating point where there is no or little interference. Assuming the DF and EF protocols, the optimal power allocation policy at the relay is to allocates all its available power to the better receiver.

2.3 Conclusion and Open Issues

In this chapter, we have studied the non-cooperative power allocation games in wireless networks where the transmitters chose their best power allocation policies that maximize their Shannon achievable rates. Two different network models were investigated: the MIMO multiple access channel and the parallel interference relay channel.

Our contributions can be summarized as follows:

- *The MIMO MAC* channel.
  - We have extended the results in [20] to the case where the action sets of users are matrix sets instead of vector sets.
  - Based on this, we have investigated the existence and uniqueness of the NE state in the one-shot game. The existence of at least one NE is guaranteed. For the static links case, the NE is not generally unique and sufficient conditions ensuring the uniqueness were provided. For the fast fading links case, the NE is proven to be unique.
  - For the fast-fading case, determining the NE point it is not trivial because the ergodic achievable rates have no closed-form expressions. First, we have determined the optimal eigenvectors. Then, we have used tools from random matrix theory to approximate the ergodic rates with their deterministic equivalents. At last, we propose an iterative water-filling type algorithm that converges to the optimal eigenvalues.
  - The power allocation game was analysed for two decoding techniques: SUD and SIC. The SUD decoding is easier to implement but it turns out to be inefficient w.r.t. achievable network sum-rate. For the SIC decoding we have proposed a sub-optimal coordination signal which dictates the decoding order at the receiver. This signal has to be known at the transmitter side and, thus, it involves a certain signaling cost.
  - To asses the network performance at the NE point, we have introduced the sum-rate efficiency. This measure translates the gap between the achievable sum-rate at the NE point and the sum-capacity of the fast-fading MIMO MAC. In the high and low SNR regimes, assuming SIC decoding, this gap tends to zero. For arbitrary SNR, this gap was evaluated through numerical simulations. It turns out that, assuming SIC decoding, the performance gap

is very small, even if the coordination signal doesn't depend on the fading coefficients. As we have predicted, SUD decoding is less efficient. An interesting Braess paradox was highlighted: if the users are restricted to allocate their power uniformly in time (and irrespective of the coordination signal) the sum-rate at the NE point is greater than the general case.

  - An interesting particular channel model is the static parallel MAC assuming SUD decoding. This case was studied from a routing game perspective. Two different games were investigated: i) the transmitters share their powers among the available sub-channels; ii) the transmitters are restricted to chose a single sub-channel. The first game is a particular case of the general concave game discussed so far. The second game, is a discrete game where the actions sets of users are finite discrete sets. Since the results of the concave games cannot be applied here, existence, multiplicity of the NE and convergence of fully distributed iterative algorithms were proved using the exact potential property of the game.

- *The parallel IRC*

  - The power allocation game was studied for three different relaying protocols: Decode-and-Forward (DF), Estimate-and-Forward (EF) and Amplify-and-Forward (AF).
  - Sufficient conditions ensuring the existence of at least one NE were provided for DF and AF. In the EF case, the existence of a NE is always guaranteed.
  - Based on a time-sharing technique, the existence of the NE can always be guaranteed irrespective of the relaying protocol. However, this involves a certain level of coordination among the transmitters.
  - Analyzing the multiplicity of the NE is not trivial. For the particular case of AF with constant amplification gain, the complete analysis characterizing the number of NE points in function of the channel parameters was provided. Furthermore, based on the "Cournot duopoly", the iterative best-response algorithms was proven to converge to one of the NE points.
  - Numerical simulations were used to evaluate the performance at the NE. When comparing the three relaying protocols w.r.t. the achievable sum-rate, similar observations as the classic relay channel were made: DF is optimal if the relay is close to the sources (very good source-relays links), while EF is optimal if the relay is close to the destinations (very good relays-destinations links). Several Stackelberg formulations, where the system owner tunes the parameters of the relays (i.e., spatial location, amplification factor for AF, power allocation for DF and EF) have been evaluated using numerical simulations.

Several open issues and interesting extensions are given hereunder:

- *The MIMO MAC:* An interesting extension would be to study the case of imperfect channel knowledge at the receiver. Also, an interesting open issue is to determine the optimal eigenvectors and eigenalues of the covariance matrices when the constraint $\mathbf{V}_k = \mathbf{V}$ for all $k \in \mathcal{K}$ is relaxed. In this sense, recent advances on random matrix theory in [59] can be used. An interesting open issue is to mathematically prove the convergence of the iterative water-filling algorithms proposed.

- *The parallel IRC:* An extension to this work is to consider more efficient coding-decoding schemes and relaying protocols such as those of [60] and related works. It is also important to fully determine the number or the topology of the set of Nash equilibria and derive convergent iterative distributed power allocation algorithms. We have also seen that additional power allocation problems come into play, and need to be considered, in a general non-cooperative game: to allocate transmit power between different bands at the sources (AF,DF,EF); to choose the cooperation degrees at the sources (DF); to allocate the power between the cooperation signals at the relay (EF and DF); to allocate the transmit power over time.

In this chapter, we considered that the transmitters allocate their powers to maximize their Shannon achievable rates with no consideration on the power consumption. The power consumption has a direct impact on the battery life of devices. There are applications (e.g., sensor networks) where the battery life of the devices plays a crucial role and maximizing the transmission rate is no longer of primary importance. For this kind of applications, a different performance measure has to be considered. In Chapter 3, we will study the energy-efficiency metric which measures the number of bits that can be reliably conveyed through the channel per unit of energy consumed.

Another important consideration regards the iterative water-filling type algorithms. There are several rising issues. First of all, it involves rationality of the users and perfect knowledge of their own payoff functions or the best-response correspondences. Also, it generally requires perfect global channel state (for the static case) or distribution (for the fast fading case) information at the transmitter level. Moreover, it involves a lot of signaling among users. At each iteration, the user updating its choice has to send this information to all the other users. All these assumptions can be regarded as being unrealistic in many applications. One possible solution is provided by the learning theory in games which will be introduced in Chapter 4.

Chapter 3

Energy-efficient Communications

In the previous chapter, the performance of the communication was measured in terms of achievable transmission rates. The cost of the communication, i.e., the transmit power used to achieve the corresponding transmission rate, has not been taken into account. In cellular or sensor networks, where the mobiles or sensor terminals are equipped with batteries of limited capacity, maximizing the battery life is more important than the transmission rate maximization. Therefore, this cost has to be taken into consideration. In this chapter, we will consider a different information-theoretic performance metric, i.e., an energy-efficiency measure. This measure is defined as the ratio between the benefit of the transmission (i.e., number of reliable transmitted bits per channel use) and the transmission cost (i.e., transmit power).

As we have discussed in Chapter 1, the energy-efficiency power allocation problem has been studied from two different perspectives: an information theoretic and a pragmatic approach. The research on this topic has been focused on two main approaches: a pragmatic approach based on practical modulations, coding-decoding schemes, electronics (see [8], [61], [62], [63]), and an information theoretical approach based on the capacity per unit cost introduced in [7]. A detailed discussion reviewing the relevant literature on the two approaches is given in [64] (see Appendix B.1). Most of this research is centered on networks composed of single antenna terminals. It is well known that, for a point-to-point communication, using multiple antenna terminals [65][66][14] in full diversity mode (i.e., all the transmit antennas are used to send the same information over the channel) allows one to decrease the transmit power while ensuring a fixed quality of transmission (e.g., the bit error rate). Therefore, we will investigate the energy-efficient power allocation policy in MIMO channels. Also, in what follows, we will focus only on the power allocation problem from an information theoretical point of view, similarly to the previous chapter. An important assumption is that only the transmit power at the output of the RF circuits (or the transmit power for reliable data) is considered. Even if this assumption is unrealistic, it allows us to characterize the upper bound of the performance that can be achieved in practice.

Our initial objective was to study the non-cooperative power allocation game for the MIMO multiple access channel, as described in Section 2.1. The difference is that

the players, the transmitters, chose their best input covariance matrices to maximize their energy-efficiency function instead of the Shannon transmission rates. However, the problem turned out to be very difficult. Because of this, we were restricted to study only the particular case of the single-user MIMO channel (see Appendix B.2). Notice that under this assumption, the game is reduced to an optimization problem which will prove to be generally intractable.

We consider a point-to-point communication with multiple antenna terminals. The signal at the receiver is modelled by:

$$\underline{Y}(\tau) = \mathbf{H}(\tau)\underline{X}(\tau) + \underline{z}(\tau), \tag{3.1}$$

where $\mathbf{H}$ is the $n_r \times n_t$ channel transfer matrix and n_t (n_r) the number of transmit (receive) antennas. The vector $\underline{X}$ is the n_t-dimensional column vector of transmitted symbols and $\underline{z}$ is an n_r-dimensional complex white Gaussian noise distributed as $\mathcal{N}(\underline{0}, \sigma^2\mathbf{I}_{n_r})$. We denote by $\mathbf{Q} = \mathbb{E}[\underline{X}\underline{X}^H]$ the input covariance matrix. The corresponding total power constraint is $\mathrm{Tr}(\mathbf{Q}) \leq \overline{P}$.

The matrix $\mathbf{H}$ is assumed to be perfectly known at the receiver (coherent communication assumption) whereas only the statistics of $\mathbf{H}$ are available at the transmitter. Three cases will be studied depending on the channel coherence time: i) the static links; ii) fast fading links; iii) slow fading links. For the first two cases, the benefit of the transmission will be measured in terms of achievable transmission rate. For slow fading channels, as we have argued in Chapter 2, this is no longer a suitable performance measure and a different function based on the outage probability will be considered. We will see that, in the first two cases, the solution is trivial and corresponds to the transmitters remaining silent. However, this is no longer the case for slow fading channels. In this case, the solution to the optimization problem is provided only for the particular case of MISO (the receiver is equipped with a single antenna). For the MIMO case, the optimal solution is conjectured and validated through numerical simulations.

3.1 Static Links

By definition, in the static links case, the frequency at which the channel matrix varies is strictly zero. In other words, $\mathbf{H}$ is a constant matrix. In this particular context, both the transmitter and receiver are assumed to know this matrix. We are exactly in the same framework as [14]. Thus, for a given precoding scheme $\mathbf{Q}$, the transmitter can send reliably to the receiver $\log_2 \left|\mathbf{I}_{n_r} + \rho\mathbf{H}\mathbf{Q}\mathbf{H}^H\right|$ bits per channel use (bpcu) with $\rho = \frac{1}{\sigma^2}$. Let us define the energy-efficiency of this communication by:

$$G_{\text{static}}(\mathbf{Q}) = \frac{\log_2 \left|\mathbf{I}_{n_r} + \rho\mathbf{H}\mathbf{Q}\mathbf{H}^H\right|}{\mathrm{Tr}(\mathbf{Q})}. \tag{3.2}$$

The energy-efficiency $G_{\text{static}}(\mathbf{Q})$ corresponds to an achievable rate per unit cost for the MIMO channel as defined in [7] under the assumption that the input alphabet does not contain any zero-cost symbols (i.e., silence at the transmitter does not convey

information). It turns out that the result obtained in [7] for the single-input single-output channel extends to the MIMO channel.

Proposition 3.1.1 *The energy-efficiency of a MIMO communication over a static channel, measured by G_{static}, is maximized when $\mathbf{Q} = \mathbf{0}$ and this maximum is*

$$G^*_{\text{static}} = \frac{1}{\ln 2} \frac{\text{Tr}(\mathbf{H}\mathbf{H}^H)}{n_t \sigma^2}. \tag{3.3}$$

The proof can be found in [67] in Appendix B.2. We see that, for static MIMO channels, the energy-efficiency defined in Eq. (3.2) is maximized by transmitting at very low powers. This kind of scenario occurs for example, when deploying sensors in the ocean to measure a temperature field (which varies very slowly). In some applications however, the rate obtained by using such a scheme can be insufficient. In this case, the benefit to cost ratio can turn out to be an irrelevant measure and other performance metrics have to be considered (e.g., minimize the transmit power under a rate constraint).

3.2 Fast Fading Links

In this section, the frequency with which the channel matrix varies is the reciprocal of the symbol duration ($\underline{X}(\tau)$ being a symbol). This means that it can be different for each channel use. Therefore, the channel varies over a transmitted codeword (or packet) and, more precisely, each codeword sees as many channel realizations as the number of symbols per codeword. In this framework, let us define energy-efficiency by:

$$G_{\text{fast}}(\mathbf{Q}) = \frac{\mathbb{E}_{\mathbf{H}}\left[\log_2 \left|\mathbf{I}_{n_r} + \rho \mathbf{H}\mathbf{Q}\mathbf{H}^H\right|\right]}{\text{Tr}(\mathbf{Q})}. \tag{3.4}$$

The proof in Section 3.1 can be applied for any channel realization and thus the trivial solution is obtained irrespective of the channel distribution.

Proposition 3.2.1 *The energy-efficiency of a MIMO communication over a fast fading channel, measured by G_{fast}, is maximized when $\mathbf{Q} = \mathbf{0}$ and this maximum is*

$$G^*_{\text{fast}} = \frac{1}{\ln 2} \frac{\text{Tr}(\mathbb{E}\left[\mathbf{H}\mathbf{H}^H\right])}{n_t \sigma^2}. \tag{3.5}$$

We see that, for fast fading MIMO channels, maximizing energy-efficiency also amounts to transmitting at low power. Interestingly, in slow fading MIMO channels, where outage events are unavoidable, we will see that the answer can be different.

Before studying the slow fading channel, we make the following remark w.r.t. the non-cooperative power allocation game for the MIMO multiple access channel. For any decoding technique (single user decoding or successive interference cancellation, see Chapter 2), the trivial solution $\mathbf{Q}_k = \mathbf{0}$ represents a strictly dominating strategy for transmitter k. Thus, at the energy-efficient NE point, none of the transmitters will send any information.

3.3 Slow Fading Links

In this section, the channel remains constant over a codeword and varies from block to block. As a consequence, the Shannon achievable rate is equal to zero. A suitable performance metric that measures the benefit of the transmission in slow-fading channels is the probability of an outage for a given transmission rate target R given in [24]. This metric allows one to quantify the probability that the rate target R is not reached by using a good channel coding scheme and is defined as follows:

$$\mathrm{P}_{\mathrm{out}}(\mathbf{Q}, R) = \Pr\left[\log_2 \left|\mathbf{I}_{n_r} + \rho \mathbf{H}\mathbf{Q}\mathbf{H}^H\right| < R\right]. \tag{3.6}$$

For the sake of simplicity, we will restrict our attention to the case where the entries of $\mathbf{H}$ are i.i.d. zero-mean unit-variance complex Gaussian random variables. In terms of information assumptions, here again, it can be checked that only the second-order statistics of $\mathbf{H}$ are required to optimize the precoding matrix $\mathbf{Q}$. In this framework, we propose to define the energy-efficiency as follows:

$$\Gamma(\mathbf{Q}, R) = \frac{R[1 - \mathrm{P}_{\mathrm{out}}(\mathbf{Q}, R)]}{\mathrm{Tr}(\mathbf{Q})}. \tag{3.7}$$

In other words, the energy-efficiency or goodput-to-power ratio (GPR) is defined as the ratio between the expected throughput (see [68],[69] for details) and the average transmit power. The expected throughput can be seen as the average system throughput over many transmissions. In contrast with static and fast fading channels, energy-efficiency is not necessarily maximized at low transmit powers. Thus, a non-trivial solution may exist to the optimization of GPR. In the remaining of this section, we study both, the determination of the optimal covariance matrix maximizing the GPR and the quasi-concavity property of the GPR. The latter issue is not only useful to study the maximum of the GPR but is also an attractive property in the multi-user scenario. For example, by considering MIMO multiple access channels with SUD at the receiver, the distributed power allocation game where the transmitters' utilities are their GPR is guaranteed to have a pure Nash equilibrium (see Debreu-Fan-Glicksberg theorem in [15]).

Finding the optimal covariance matrix is not trivial. Indeed, even the outage probability minimization problem w.r.t. $\mathbf{Q}$ is still an open problem [14], [70], [71]. The result is conjectured as follows.

Conjecture 3.3.1 *There exists a power threshold $\overline{P}_0$ such that:*

- *if $\overline{P} \leq \overline{P}_0$ then $\mathbf{Q}^* \in \arg\min_{\mathbf{Q}} P_{\mathrm{out}}(\mathbf{Q}, R) \;\Rightarrow\; \mathbf{Q}^* \in \arg\max_{\mathbf{Q}} \Gamma(\mathbf{Q}, R)$;*
- *if $\overline{P} > \overline{P}_0$ then $\Gamma(\mathbf{Q}, R)$ has a unique maximum in $\mathbf{Q}^* = \frac{p^*}{n_t}\mathbf{I}_{n_t}$ where $p^* \leq \overline{P}$.*

This conjecture states that, if the available transmit power is less than a threshold, maximizing the GPR is equivalent to minimizing the outage probability. If it is above the threshold, the uniform power allocation is optimal. However using all the available power is generally suboptimal in terms of energy-efficiency. Regarding the optimization

problem associated with (3.7) several comments are in order. First, there is no loss of optimality by restricting the search for optimal precoding matrices to diagonal matrices: for any eigenvalue decomposition $\mathbf{Q} = \mathbf{U}\mathbf{D}\mathbf{U}^H$ with $\mathbf{U}$ unitary and $\mathbf{D} = \mathbf{Diag}(\underline{p})$ with $\underline{p} = (p_1, \ldots, p_{n_t})$, both the outage and trace are invariant w.r.t. the choice of $\mathbf{U}$. The energy-efficiency can be written as:

$$\Gamma(\mathbf{D}, R) = \frac{R[1 - \mathrm{P}_{\mathrm{out}}(\mathbf{D}, R)]}{\sum_{i=1}^{n_t} p_i}. \tag{3.8}$$

Second, the GPR is generally not quasi-concave w.r.t. $\mathbf{D}$. In [67] Appendix B.2, we give a counter-exemple for which the GPR is proven not to be quasi-concave. Third, the conjecture was validated using Monte-Carlo numerical simulations for the 2×2 case where both the transmitter and receiver are equipped with two antennas. Fourth, the conjecture 3.3.1 was rigorously solved for MISO channels where the receiver is equipped with a single antenna (see [67] Appendix B.2 for details).

Proposition 3.3.2 *For all $\ell \in \{1, ..., n_t - 1\}$, let c_ℓ be the unique solution of the equation (in x) $\Pr\left[\frac{1}{\ell+1}\sum_{i=1}^{\ell+1}|X_i|^2 \leq x\right] - \Pr\left[\frac{1}{\ell}\sum_{i=1}^{\ell}|X_i|^2 \leq x\right] = 0$ where X_i are i.i.d. zero-mean Gaussian random variables with unit variance. By convention $c_0 = +\infty$, $c_{n_t} = 0$. Let ν_{n_t} be the unique solution of the equation (in y) $\frac{y^{n_t}}{(n_t-1)!} - \sum_{i=0}^{n_t-1}\frac{y^i}{i!} = 0$. Then the optimum precoding matrices have the following form:*

$$\mathbf{D}^* = \left| \begin{array}{ll} \frac{\overline{P}}{\ell}\boldsymbol{Diag}(\underline{e}_\ell) & \text{if } \overline{P} \in \left[\frac{c}{c_{\ell-1}}, \frac{c}{c_\ell}\right) \\ \min\left\{\frac{\sigma^2(2^R-1)}{\nu_{n_t}}, \frac{\overline{P}}{n_t}\right\}\mathbf{I}_{n_t} & \text{if } \overline{P} \geq \frac{c}{c_{n_t-1}} \end{array} \right. \tag{3.9}$$

where $c = \sigma^2(2^R - 1)$ and $\underline{e}_\ell \in \mathcal{S}_\ell$.

Similarly to the optimal precoding scheme for the outage probability minimization [71], the solution maximizing the GPR consists in sharing the available power uniformly among a subset of $\ell \leq n_t$ antennas. As i.i.d entries are assumed for $\mathbf{H}$, the choice of these antennas does not matter. What matters is the number of antennas selected, which depends on the available transmit power $\overline{P}$: the higher the transmit power, the higher the number of used antennas. The difference between the outage probability minimization and GPR maximization problems appears when the transmit power is greater than the threshold $\frac{c}{c_{n_t-1}}$. In this regime, saturating the power constraint is suboptimal for the GPR optimization. The conjecture 3.3.1 has also been solved for the SIMO channel where the transmitter is equipped with a single antenna, and also for the MIMO channel assuming the extreme SNR regimes (low and high SNR regimes).

A special case of interest is the case of uniform power allocation (UPA): $\mathbf{D} = \frac{p}{n_t}\mathbf{I}_{n_t}$ where $p \in [0, \overline{P}]$ and $\Gamma_{\mathrm{UPA}}(p, R) \triangleq \Gamma\left(\frac{p}{n_t}\mathbf{I}_{n_t}, R\right)$. One of the reasons for studying this

case is the famous conjecture of Telatar in [14]. This conjecture states that, depending on the channel parameters and target rate (i.e., σ^2, R), the power allocation (PA) policy minimizing the outage probability is to spread all the available power uniformly over a subset of $\ell^* \in \{1, \ldots, n_t\}$ antennas. If this can be proved, then it is straightforward to show that the covariance matrix $\mathbf{D}^*$ that maximizes the GPR is $\frac{p^*}{\ell^*}\mathbf{Diag}(\underline{e}_{\ell^*})$, where $\underline{e}_{\ell^*} \in \mathcal{V}_{\ell^*}$[1]. Thus, $\mathbf{D}^*$ has the same structure as the covariance matrix minimizing the outage probability except that using all the available power is not necessarily optimal, $p^* \in [0, \overline{P}]$. In conclusion, solving Conjecture 3.3.1 reduces to solving Telatar's conjecture and also the UPA case.

The main difficulty in studying the outage probability and/or the energy-efficiency function is the fact that the probability distribution function of the mutual information is generally intractable. In the literature, the outage probability is often studied by assuming an UPA policy over all the antennas and also using the Gaussian approximation of the p.d.f. of the mutual information. This approximation is valid in the asymptotic regime of large number of antennas. However, simulations show that it also quite accurate for reasonable small MIMO systems [72], [73] (for e.g., assuming four-antenna terminals, the approximation is very good and assuming eight-antenna terminals, the error is negligible).

Under the UPA policy assumption, the GPR $\Gamma_{\text{UPA}}(p, R)$ is conjectured to be quasi-concave w.r.t. p.

Conjecture 3.3.3 *Assume that* $\mathbf{D} = \frac{p}{n_t}\mathbf{I}_{n_t}$. *Then* $\Gamma_{\text{UPA}}(p, R)$ *is quasi-concave w.r.t.* $p \in \left[0, \overline{P}\right]$.

This conjecture was proved for the special cases of MISO and SIMO. Furthermore, it was proved for the general MIMO case assuming the large system approach for three cases: $n_t < +\infty$ and $n_r \to +\infty$; $n_t \to +\infty$ and $n_r < +\infty$; $n_t \to +\infty$, $n_r \to +\infty$ with $\lim\limits_{n_i \to +\infty, i \in \{t,r\}} \dfrac{n_r}{n_t} = \beta < +\infty$. Numerical simulations were provided to validate the conjecture for finite number of antennas. Furthermore, the numerical simulations show that the optimal value of the energy-efficiency function is increasing with the number of antennas.

3.4 Conclusion and Open Issues

In this chapter, we have analysed the energy-efficiency power allocation problem for the single-user MIMO channel. For the static and fast fading links, the solution is proved to be trivial. In order to be energy-efficient, the transmitter sends data at fading transmit power which imply fading data rates. For the slow fading links, the optimization problem is more difficult and the solution is proven to be non-trivial in general. The contributions are as follows:

[1]The set $\mathcal{V}_\ell = \left\{ \underline{v} \in \{0,1\}^{n_t} | \sum_{i=1}^{n_t} v_i = \ell \right\}$ represents the set of n_t-dimensional vectors containing ℓ ones and $n_t - \ell$ zeros, for all $\ell \in \{1, \ldots, n_t\}$.

- We conjecture the result for the general problem and solve it for special cases: in the high and low SNR regimes and in n the particular cases where one of the terminals is equipped with a single antenna (MISO and SIMO channels).

- An interesting case of study is the uniform power allocation case where the energy-efficiency function is conjectured to be quasi-concave with respect to the transmit power and it has been proved to be quasi-concave using the large system assumption.

For the static and fast fading cases, the energy-efficiency function seems to be a non-suitable performance metric. The trivial solution that maximizes the energy-efficiency function can be explained by the fact that the circuitry energy consumption is not taken into account. An important extension would be to consider the circuitry power as well. In this case, if the transmitter remains silent, it won't be at zero-cost and the trivial solution will no longer be optimal. Also, having multiple antennas at the terminals may turn out to be suboptimal w.r.t. the single-antenna case. Another way to avoid the trivial solution is to consider minimal QoS constraints (e.g., minimal data-rates).

The fact that the solution is no longer trivial for the slow fading case can be explained by the fact that the benefit of the transmission is inherently different. For the static and fast-fading channels the transmission is constrained to be asymptotically reliable (with zero-error probability). This constraint turns out to be too stringent and energy-efficient communication is not possible at non-zero transmit powers. For the static channel, there is an optimal trade-off between the outage probability and the transmit rate that allows energy-efficient communication at non-zero transmit powers.

Many open problems are introduced by the proposed performance metric, here we just mention some of them:

- First of all, the conjecture of the optimal precoding schemes for general MIMO channels needs to be proven.

- The quasi-concavity of the goodput-to-power ratio when uniform power allocation is assumed remains to be proven in the finite setting.

- A more general channel model should be considered. We have considered i.i.d. channel matrices but considering non zero-mean matrices with arbitrary correlation profiles appears to be a challenging problem for the goodput-to-power ratio.

- The connection between the proposed metric and the diversity-multiplexing trade-off at high SNR has not been explored.

- Only single-user channels have been considered. Clearly, multi-user MIMO channels such as multiple access or interference channels should be considered. The problem of distributed multi-user channels and the non-cooperative power allocation game. is an interesting issue. In this respect, only one result is mentioned here: the existence of a pure Nash equilibrium in distributed MIMO multiple access channels assuming uniform power allocation transmit policy.

The main difficulty lies in the fact that the optimization problem of the outage probability is still an open problem. We have seen that to solve the general problem it suffices to prove Telatar's conjecture and the uniform power allocation case. Optimizing the outage probability is a difficult problem even from a numerical point of view. This is why, in the following chapter, we will see whether, using simple learning algorithms, the transmitter may converge to the solution maximizing the outage probability.

Chapter 4

Learning Algorithms in Resource Allocation Games

In Chapter 2, we have studied one-shot non-cooperative power allocation games where the transmitters choose their optimal power allocation policy to maximize their achievable rates. Computing the Nash equilibria required rationality at the transmitter level and perfect knowledge of the game structure. Furthermore, player rationality was assumed to be common knowledge. The iterative best-response algorithms which allowed the transmitters to compute the equilibria were as well very demanding in terms of knowledge assumptions and computational capabilities.

In this chapter, we study an alternative way of explaining how the players may converge to an equilibrium point of the one-shot game. This alternative is offered by the theory of learning in games [74]. Learning algorithms are long-run processes in which players, with less restrictive knowledge and assumptions, try to optimize their payoffs relying on simple updating rules. We will mainly study a reinforcement learning algorithm similar to [75]. In this framework, the users are simple automata capable of choosing their actions from a finite set of actions. Their choices are based on past results and feed-back from the environment. Thus, they can improve their performance over time while operating in an almost unknown environment.

Two different scenarios are considered. First, we study a similar power allocation game to the one described in Section 2.1.2. The difference consists in the action sets of players which are discrete and finite sets. Because of this difference, the analysis conducted in Section 2.1.2 is no longer valid. Therefore, we must analyse the Nash equilibrium for the one-shot non-cooperative game. Then, we will see that, using simple adaptive rules, the players converge to one of the Nash equilibrium points. Second, we focus on the slow fading MIMO channel in Section 3.3. We see that the optimal precoding matrix optimizing the outage probability can be computed applying a similar reinforcement algorithm.

4.1 Dynamical Systems and Stochastic Approximation

Before analysing the aforementioned scenarios, we will discuss the basic mathematical tools used to analyse the asymptotic behaviour of the learning algorithms: dynamical systems and stochastic approximation algorithms.

Dynamical systems are mathematical formalizations used to capture the evolution in time of systems. In general, they consist of three components:

- *System state*, $\underline{x}(t)$, representing the parameters which characterizes the system at time t;
- *State space*, $\mathcal{U}$, representing all the possible states of the system;
- *State transition function* $\Phi : \underline{U} \times [0, +\infty) \to \underline{U}$, which is a flow defining the change in the system state from one moment to the other (i.e., $\Phi_\tau(\underline{x}_0) = \underline{x}$ translates the fact that the system is in state $\underline{x}$ at time $t = \tau$ starting from $\underline{x}_0$ at $t = 0$). If this function is differentiable, then it can be characterized by the solution of an autonomous ordinary differential equation (ODE):

$$\frac{\mathrm{d}\underline{x}}{\mathrm{d}t} = v(\underline{x}(t)), \tag{4.1}$$

  where $v(\underline{x}(t))$ is the vector field describing the speed of evolution of the system state and which does not depend explicitly on time. The existence of a uniqueness solution of this ODE, for any initial state $\underline{x}_0 \in \mathcal{U}$ is ensured if the vector field is Lipschitz continuous, i.e., there exists a constant $L > 0$ such that, for all $\underline{x}$, $\underline{y}$: $\|v(\underline{x}) - v(\underline{y})\| \leq L\|\underline{x} - \underline{y}\|$.

In this chapter, we will be interested in characterizing the asymptotic behaviour of the dynamical system. To this aim, we define the following notions: stationary states, stable states, asymptotically stable states.

Definition 4.1.1 *A system state $\tilde{\underline{x}} \in \mathcal{U}$ such that $v(\tilde{\underline{x}}) = 0$ is called a* stationary state *or* equilibrium state*. The equilibrium states correspond to fixed points of the flow: $\Phi_t(\tilde{\underline{x}}) = \tilde{\underline{x}}$.*

However, the equilibrium regime of dynamical systems does not necessarily consist of isolated points and can consist of a whole subspace of $\mathcal{U}$ (e.g. cycles, periodic orbits). The general concept is that of invariant set which is defined here below.

Definition 4.1.2 *Let $\mathcal{W}$ be a sub-space of $\mathcal{U}$. $\mathcal{W}$ is called an invariant (respectively positive invariant) set, if, for all $t \in \mathbb{R}$ (respectively in $\mathbb{R}_+$), $\Phi_t(\mathcal{W}) \subseteq \mathcal{W}$. It is said to be "internally chain transitive" if, for any $\underline{x}, \underline{y} \in \mathcal{W}$ and any $\varepsilon > 0$, $T > 0$, there exists $n \geq 1$ and $\underline{x}^{[0]} = \underline{x}, \underline{x}^{[1]}, \ldots, \underline{x}^{[n]} = \underline{y}$ in $\mathcal{W}$ such that the trajectory in (4.1), initiated at $\underline{x}^{[m]}$ meets with the ε-neighbourhood of $\underline{x}^{[m+1]}$ for $0 \leq m \leq n$ after a time $\geq T$. Furthermore, if $\underline{x} = \underline{y}$, the set is said to be "internally chain recurrent".*

The stability issue of the equilibrium regime answers to questions such as: If the system is moved away from an equilibrium state, will the system return to this equilibrium? Can a small deviation, which slightly moves the system away from an equilibrium state, have important consequences and be amplified in time?

Definition 4.1.3 *An equilibrium point $\underline{\tilde{x}}$ is stable in the sense of Lyapunov if, for any $\varepsilon > 0$, it exists $\eta > 0$ such that, for all $\underline{y} \in \mathcal{U}$ verifying $\|\underline{y} - \underline{\tilde{x}}\| \leq \eta$, then $\|\Phi_t(\underline{y}) - \underline{\tilde{x}}\| \leq \varepsilon$ for all $t > 0$.*

Definition 4.1.4 *An equilibrium point $\underline{\tilde{x}}$, is asymptotically stable in the sense of Lyapunov if it is stable in the sense of Lyapunov and, for any $\underline{y} \in \mathcal{U}$ sufficiently close to $\underline{\tilde{x}}$, $\lim_{t \to +\infty} \Phi_t(\underline{y}) = \underline{\tilde{x}}$.*

Sufficient conditions that ensures the stability or asymptotic stability are given in the following theorem.

Theorem 4.1.5 *If $\underline{\tilde{x}}$ is an equilibrium point and if exists a differentiable function $V : \mathcal{U} \to \mathbb{R}_+$ with continuous derivative such that: $V(\underline{\tilde{x}}) = 0$, $V(\underline{y}) > 0$ for all $\underline{y} \neq \underline{\tilde{x}}$ and $\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0$ (i.e., V is decreasing along all trajectories). Then, $\underline{\tilde{x}}$ is stable in the sense of Lyapunov. If $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$ for all $\underline{y} \neq \underline{\tilde{x}}$, then $\underline{\tilde{x}}$ is asymptotically stable in the sense of Lyapunov. Furthermore, if $V(\underline{x})$ goes to infinity when $\underline{x}$ approaches infinity, then all trajectories tend to $\underline{\tilde{x}}$. In this case, $\underline{\tilde{x}}$ is called globally asymptotically stable state.*

For further details on dynamical systems and their associated asymptotic behavior, the reader is referred to [76][77][78].

The stochastic approximation theory is used to study discrete-time stochastic processes that can be written as:

$$\underline{X}^{[n+1]} = \underline{X}^{[n]} + \gamma^{[n+1]} \left(f(\underline{X}^{[n]}) + \underline{Z}^{[n+1]} \right) \tag{4.2}$$

where $\underline{X}^{[n]}$ is a vector in an Euclidean space which is updated based on a noisy observation, $f(\cdot)$ is a deterministic vector field and $\underline{Z}^{[n+1]}$ is a random noise. The idea is to approximate $\underline{X}^{[n]}$ with a certain continuous-time interpolation process, e.g., the following piecewise linear function given by:

$$\underline{\widehat{X}}(t) \quad = \quad \underline{X}^{[n]} + \tfrac{t - \tau^{[n]}}{\tau^{[n+1]} - \tau^{[n]}} (\underline{X}^{[n+1]} - \underline{X}^{[n]}), \quad \text{if} \quad t \in [\tau^{[n]}, \tau^{[n+1]}), \tag{4.3}$$

where $\tau^{[0]} = 0$ and $\tau^{[n]} = \sum_{m=0}^{n} \gamma^{[m]}$ (i.e., the parameter that covers the time axis).

In the asymptotic regime, i.e., $n \to +\infty$, and under certain conditions on the quantization step $\gamma^{[n]}$, on the vector field $f(\cdot)$ and on the noise process $\underline{Z}^{[n]}$, this interpolated process follows the solution of the deterministic ODE:

$$\frac{d\underline{x}}{dt} = f(\underline{x}(t)). \tag{4.4}$$

Therefore, using the stochastic approximation approach, the study of the iterative process $\underline{X}^{[n]}$ amounts to the study of the deterministic continuous-time ODE in (4.4). Notice that the discrete process described by (4.2) can be viewed as a noisy Euler scheme for numerically approximating the trajectory of the ODE in (4.4).

Two different approaches can be distinguished as a function of the step size: i) variable step-size ($\gamma^{[n]}$ changes at each iteration); ii) constant step-size ($\gamma^{[n]} = \varepsilon, \forall n$). For the case where the step size is variable, the main convergence result is given in [79] and a clear proof is presented in [80]. The author of [79] proved the almost sure convergence (convergence with probability one) in the asymptotic regime ($n \to +\infty$) under the following conditions:

[H1] The learning steps satisfy:

$$\begin{cases} \gamma^{[n]} & \geq & 0, \forall n \\ \lim\limits_{n\to+\infty} \gamma^{[n]} & = & 0 \\ \lim\limits_{n\to+\infty} \sum\limits_{m=0}^{n} \gamma^{[m]} & = & +\infty \\ \lim\limits_{n\to+\infty} \sum\limits_{m=0}^{n} \left(\gamma^{[m]}\right)^2 & < & +\infty. \end{cases} \tag{4.5}$$

The parameters $\left\{\gamma^{[n]}\right\}$ correspond to the quantization steps and, thus, small values are desirable to suppress the quantization errors. Too small values however imply a long convergence time of the algorithm. In conclusion, relative large values of $\gamma^{[n]}$ are desirable at the initial steps ($n = 1, 2, ...$) but, as n grows large, $\gamma^{[n]}$ should become very small. Since the discrete process will be approximated by the continuous-time ODE, the discrete steps must cover the entire time axis, $\sum\limits_{n\geq 0} \gamma^{[n]} = +\infty$. The errors introduced by the noise must also be asymptotically suppressed. The condition $\sum\limits_{n\geq 0} \left(\gamma^{[n]}\right)^2 < +\infty$ is necessary to this purpose.

[H2] The vector field $f(\cdot)$ is Lipschitz continuous.

[H3] The discrete process remains bounded with probability one, i.e., $\sup_n \|\underline{X}^{[n]}\| < +\infty$.

[H4] The noise sequence $\left\{\underline{Z}^{[n]}\right\}$ is a Martingale difference sequence such that with probability one $\mathbb{E}\left[\underline{Z}^{[n+1]} \middle| \underline{Z}^{[m]}, m \leq n\right] = \underline{0}$.

[H5] The noise sequence $\left\{\underline{Z}^{[n]}\right\}$ is square integrable, i.e., $\mathbb{E}\left[\left\|\underline{Z}^{[n+1]}\right\|^2 \middle| \underline{Z}^{[m]}, m \leq n\right] < +\infty$. This condition, together with $\sum\limits_{n\geq 0} \left(\gamma^{[n]}\right)^2 < +\infty$, asymptotically suppresses the overall contribution of the noise. Thus, the discrete process converges asymptotically to the mean behaviour given by the deterministic ODE (4.4).

Before stating the main results, we define $\underline{X}^s(t), s \leq t$ as the trajectory that starts at time $s > 0$ in the point $\widehat{\underline{X}}(s)$:

$$\left\{ \begin{array}{lcl} \frac{\mathrm{d}\underline{X}^s}{\mathrm{d}t}(t) & = & f(\underline{X}^s(t)) \\ \underline{X}^s(s) & = & \widehat{\underline{X}}(s) \end{array} \right. \tag{4.6}$$

Also, $\underline{X}_s(t), s \geq t$ is the trajectory that ends at time $s > 0$ in the point $\widehat{\underline{X}}(s)$ defined as:

$$\left\{ \begin{array}{lcl} \frac{\mathrm{d}\underline{X}_s}{\mathrm{d}t}(t) & = & f(\underline{X}_s(t)) \\ \underline{X}_s(s) & = & \widehat{\underline{X}}(s) \end{array} \right. \tag{4.7}$$

The following convergence results for stochastic approximation are due to Benaïm [79].

Theorem 4.1.6 *[79] Assuming that the hypothesis **[H1]-[H5]** are met, we have that, for all $T > 0$:*

$$\left\{ \begin{array}{lcl} \Pr\left[\lim\limits_{s\to+\infty} \sup\limits_{t\in[s,s+T]} \|\widehat{\underline{X}}(t) - \underline{X}^s(t)\| = 0\right] & = & 1 \\ \Pr\left[\lim\limits_{s\to+\infty} \sup\limits_{t\in[s-T,s]} \|\widehat{\underline{X}}(t) - \underline{X}_s(t)\| = 0\right] & = & 1. \end{array} \right. \tag{4.8}$$

Furthermore, the discrete process $\{\underline{X}^{[n]}\}$ in (4.2) converges almost surely, when $n \to +\infty$, to a (possibly path dependent) compact connected internally chain transitive set of the ODE (4.4).

Intuitively this means that, in the asymptotic regime (i.e., $n \to +\infty$), the interpolation process converges almost surely to the solution of the deterministic ODE in (4.4).

For the constant-step size case, $\gamma^{[n]} = \gamma$ for all n, only weak convergence results (convergence in distribution) can be proved in the asymptotic regime ($\gamma \to 0$ and $n \to +\infty$) [81]. The corresponding theorems can be found in [81]. The sufficient conditions ensuring the weak convergence are less restrictive than the conditions ensuring the almost sure convergence. Thus, if the conditions [H2]-[H5] are satisfied the weak convergence is guaranteed. For a more detailed discussion, the reader is referred to the specialized books [80], [81].

In conclusion, the asymptotic study of the iterative process $\underline{X}^{[n]}$ is reduced to the asymptotical study of the ordinal differential equation (4.4).

4.2 The Fast Fading MIMO Multiple Access Channel

In this section, we study the power allocation game in fast fading MIMO multiple access channels, similarly to Section 2.1.2. We will restrict our attention to the case where single user decoding is used at the receiver side. The same notations will be used in this section. However, the power allocation game we study here differs from the game

in Section 2.1.2 in two respects. First, the action sets of the players are discrete and finite sets, as opposed to the convex and compact set of positive semi-definite matrices of constrained trace. Second, the channel matrices are restricted to live in compact sets of bounded support: $|\mathbf{H}_k(i,j)| \leq h_{\max} < +\infty$ (no other constraints are made on the channel matrices). The analysis of the Nash equilibrium for the discrete game is completely different than in Section 2.1.2. For example, the existence of a pure strategy Nash equilibrium is generally not guaranteed for this type of games. Also, the concavity property of the payoff functions and the results in [20] cannot be used here. We propose the following action set for user k which is a simple quantized version of $\mathcal{A}_k^{\mathrm{SUD}}$ in (2.2):

$$\mathcal{D}_k = \left\{ \frac{\overline{P}_k}{\ell} \mathbf{Diag}(\underline{e}_\ell) \middle| \ell \in \{1,\ldots,n_t\}, \underline{e}_\ell \in \{0,1\}^{n_t}, \sum_{i=1}^{n_t} e_\ell(i) = \ell \right\}. \tag{4.9}$$

Notice that $\mathcal{D}_k$ represents the set of diagonal matrices consisting in uniformly spreading the available power over a subset of $\ell \in \{1,\ldots,n_t\}$ eigen-modes. The components of the game are $\mathcal{G}_{\mathrm{D}}^{\mathrm{SUD}} = \left(\mathcal{K}, \{u_k^{\mathrm{SUD}}\}_{k\in\mathcal{K}}, \{\mathcal{D}_k\}_{k\in\mathcal{K}}\right)$ where the payoff function is given by the achievable ergodic rate in (2.11). As discussed in Section 2.1.2, the discrete game is an exact potential game. The potential function is given by the achievable system sum-rate:

$$V(\mathbf{Q}_1,\ldots,\mathbf{Q}_K) = \mathbb{E}\log_2 \left| \mathbf{I}_{n_r} + \rho \sum_{k=1}^{K} \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H \right|. \tag{4.10}$$

Thus, the game $\mathcal{G}_{\mathrm{D}}^{\mathrm{SUD}}$ has at least one pure-strategy Nash equilibrium. However, the uniqueness property of the NE is lost in general. In [82] (see Appendix C.1), we prove that for full-rank channels the NE is unique whereas for unit-rank channels all the strategy profiles are NE points. Knowing that the game is an exact potential game, by the Finite Improvement Property [19], the iterative algorithms based on the best-response dynamics converge to one of the possible pure strategy NE depending on the starting point.

In the remaining of this section, we will study a reinforcement learning algorithm similarly to [75]. As opposed to the best-response type algorithm, the users are no longer rational devices but simple automata that know only their own action sets. They start at a completely naive state choosing randomly their action (following the uniform distribution over the action sets, for example). After the play, each users obtains a certain feed-back from the nature (e.g., the realization of a random variable, the value of its own payoff). Based only on this value, each user applies a simple updating rule on its own discrete probability distribution or mixed strategy over its action set. It turns out that, in the long-run, the updating rules converge to some desirable system states. It is important to notice that the transmitters don't even need to know the structure of the game or that a game is played at all. The price to pay for the lack of knowledge and rationality will be reflected in the longer convergence time.

Let us index the possible actions that User k can take as follows: $\mathcal{D}_k = \{\mathbf{D}_k^{(1)},\ldots,\mathbf{D}_k^{(M_k)}\}$ with $M_k = \mathrm{Card}(\mathcal{D}_k)$ (i.e., the cardinality of $\mathcal{D}_k$). At step $n > 0$ of the iterative process, User k randomly chooses a certain action $\mathbf{Q}_k^{[n]} \in \mathcal{D}_k$ based on the probability distribu-

tion $\underline{p}_k^{[n-1]}$ from the previous iteration. As a consequence, it obtains the realization of a random variable, which is, in our case, the normalized instantaneous mutual information $\mu_k^{[n]} = \frac{\mu_k\left(\mathbf{Q}_k^{[n]},\mathbf{Q}_{-k}^{[n]}\right)}{I_{\max}} \in [0,1]$ such that:

$$\mu_k(\mathbf{Q}_k,\mathbf{Q}_{-k}) = \log_2\left|\mathbf{I}_{n_r} + \rho\mathbf{H}_k\mathbf{Q}_k\mathbf{H}_k^H + \rho\sum_{\ell\neq k}^{K}\mathbf{H}_\ell\mathbf{Q}_\ell\mathbf{H}_\ell^H\right| - \log_2\left|\mathbf{I}_{n_r} + \rho\sum_{\ell\neq k}\mathbf{H}_\ell\mathbf{Q}_\ell\mathbf{H}_\ell^H\right|$$

and $I_{\max}$ is the maximum value of the mutual information. Under the assumption that the channel takes values in a compact set we have $I_{\max} < +\infty$. Based on this feed-back, User k updates its own probability distribution as follows:

$$p_{k,j_k}^{[n]} = p_{k,j_k}^{[n-1]} - \gamma^{[n]}\mu_k^{[n]}p_{k,j_k}^{[n-1]} + \gamma^{[n]}\mu_k^{[n]}\mathbb{1}_{\left(\mathbf{Q}_k^{[n]}=\mathbf{D}_k^{(j_k)}\right)} \tag{4.11}$$

where $0 < \gamma^{[n]} < 1$ is the step-size and $p_{k,j_k}^{[n]}$ represents the probability that user k chooses $\mathbf{D}_k^{(j_k)}$ at iteration n. Notice that the assumption on the channel matrices w.r.t. their bounded support is required for the normalization of the mutual information. In order to make sure that the discrete probability distribution in (4.11) is well defined, the random payoff must be bounded as follows $0 \leq \mu_k^{[n]} \leq 1$. An intuition behind this simple reinforcement algorithm is that the frequency with which an action is played depends on its score (i.e., the payoff obtained by playing this action) such that the actions which perform well are increased in probability.

In order to study the asymptotic behaviour and convergence properties of the discrete stochastic process in (4.11), the objective is to prove that it can be approximated with the solution of the following deterministic ODE:

$$\frac{\mathrm{d}p_{k,j_k}}{\mathrm{d}t} = p_{k,j_k}\sum_{i_k=1}^{M_k} p_{k,i_k}[h_{k,j_k}(\underline{p}_{-k}) - h_{k,i_k}(\underline{p}_{-k})], \tag{4.12}$$

where

$$h_{k,j_k}(\underline{p}_{-k}) = \sum_{\underline{i}_{-k}} u_k^{(\mathrm{SUD})}\left(D_k^{(j_k)}, D_{-k}^{(i_{-k})}\right)\prod_{\ell\neq k} p_{\ell,i_\ell}$$

The idea is to apply the stochastic approximation results and, in particular, to check whether Theorem 4.1.6 applies or not in our context. To this aim, we have to verify that the conditions [H2]-[H5] in Section 4.1 are satisfied. The vector field of the ODE in (4.12) is $f(\underline{p}) = \left\{f_{k,j_k}(\underline{p})\right\}_{k\in\mathcal{K},j_k\in\{1,\ldots,M_k\}}$ with:

$$f_{k,j_k}(\underline{p}) = p_{k,j_k}\sum_{i_k=1}^{M_k} p_{k,i_k}[h_{k,j_k}(\underline{p}_{-k}) - h_{k,i_k}(\underline{p}_{-k})],$$

where $\underline{p}$ represents the concatenation vector of the discrete probability distributions of all the players. It is easy to see that $f(\underline{p})$ is a class C^1 vector field (i.e., differentiable with

continuous partial derivatives). This implies that $f(\underline{p})$ is Lipschitz continuous ([H2]). Since $\underline{p} \in \prod_k \Delta(\mathcal{D}_k)$, the condition [H3] is straight forward. The noise process is given by:

$$Z_{k,j_k}^{[n]} = -\mu_k^{[n]} p_{k,j_k}^{[n-1]} + \mu_k^{[n]} \mathbb{1}_{\left(\mathbf{Q}_k^{[n]} = \mathbf{D}_k^{(j_k)}\right)} - f_{k,j_k}(\underline{p}^{[n-1]})$$

We can observe that $\left\{\underline{p}^{[n]}, (\mathbf{Q}^{[n-1]}, \underline{\mu}^{[n-1]})\right\}_n$ is a Markov process where $\mathbf{Q}^{[n-1]} = \{\mathbf{Q}_k^{[n-1]}\}_{k\in\mathcal{K}}$ and $\underline{\mu}^{[n-1]} = \{\mu_k^{[n-1]}\}_{k\in\mathcal{K}}$. Also, for time invariant distributions $\underline{p}^{[n-1]} = \underline{p}$, the sequence $(\mathbf{Q}^{[n]}, s^{[n]})_n$ is an i.i.d. sequence which implies that condition [H4] is met. Also, condition [H5] can be verified easily since $\underline{p}^{[n-1]} \in \prod_k \Delta(\mathcal{D}_k)$ and $\mu_k^{[n]} \in [0,1]$. Therefore, depending on the choice of the step-size (either constant or verifying [H1]) the almost sure or weak convergence are guaranteed.

Therefore, in order to study the stochastic process $p_{k,j_k}^{[n]}$, we can focus on the study of the deterministic ODE that captures its average behaviour. Notice that the ODE (4.12) is similarly to the replicator dynamics [83]. The mixed and pure-strategy NE are rest points of this kind of dynamics. However, all the pure-strategy profiles, even those which are not NE are also rest points. Moreover, the border of the domain where the vector of probability distributions $\underline{p}$ lives is an invariant set. Using the fact that the game is an exact potential game the pure-strategy NE points can be proved to be stable in the sense of Lyapunov.

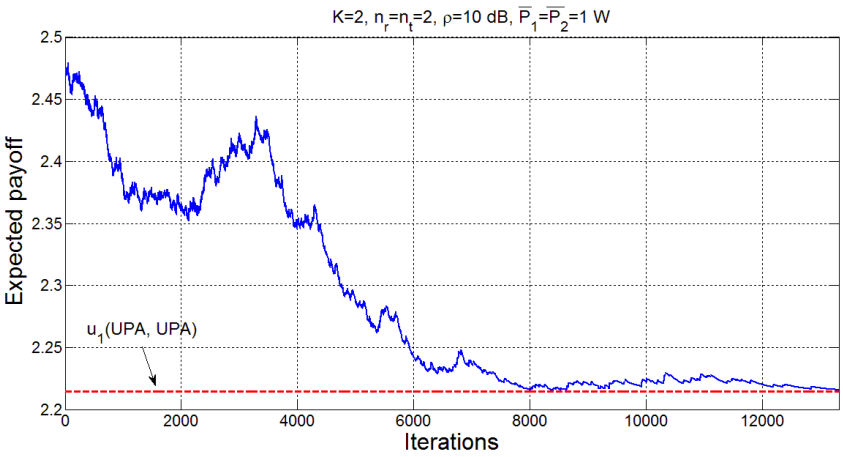
Numerical simulations.

We consider the following scenario: $K = 2$, $n_r = n_t = 2$, the entries of the channel matrices are drawn independently following a truncated complex Gaussian distribution, i.e., $|\mathbf{H}_k(i,j)| \leq h_{\max} = 1$ and $\rho = 10$ dB, $\overline{P}_1 = \overline{P}_2 = 1$ W. The actions the users can take are: $\mathbf{D}_k^{(1)} = \overline{P}_k \mathbf{diag}(0,1)$, $\mathbf{D}_k^{(2)} = \overline{P}\mathbf{diag}(1,0)$, $\mathbf{D}_k^{(3)} = \frac{\overline{P}}{2}\mathbf{diag}(1,1)$. The beam-forming strategies are identical in terms of payoff and the users can be considered as having only two strategies: beam-forming (BF) (either $\mathbf{D}_k^{(1)}$ or $\mathbf{D}_k^{(2)}$) and uniform power allocation (UPA) ($\mathbf{D}_k^{(3)}$). The payoff matrix for user 1 is given by its ergodic achievable rate:

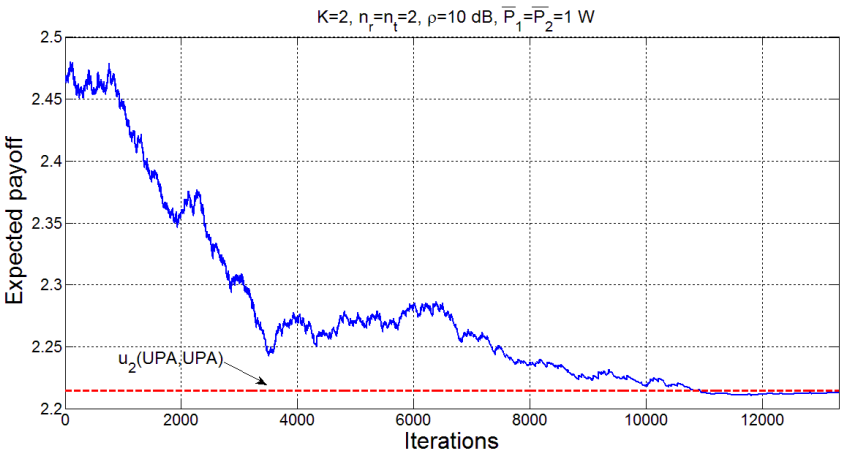
$$\mathbf{ER}_1 = \begin{pmatrix} 2.6643 & 1.9271 \\ 3.0699 & 2.2146 \end{pmatrix}$$

and since the example is symmetric we have $\mathbf{ER}_2 = \mathbf{ER}_1^T$. The elements of these matrices correspond to the payoffs of the two users in the following situations: $\mathbf{ER}_k(1,1)$ when both players choose BF; $\mathbf{ER}_k(1,2)$ when User k chooses BF while the opponent plays UPA; $\mathbf{ER}_k(2,1)$ when User k chooses UPA while the opponent plays BF; $\mathbf{ER}_k(2,2)$ when both players choose UPA. In this case, we observe that the unique NE consists in playing the uniform power allocation by both users.

We apply the reinforcement algorithm considering $I_{\max} = 8.7846$ bpcu (i.e., the maximum single-user instantaneous mutual information under the assumption that $h_{max} = 1$). In Fig. 4.1, we plot the expected payoff depending on the probability distribution over the action sets at every iteration for User 1 in Fig. 4.1(a) and for



(a) User 1.



(b) User 2.

Figure 4.1: Expected payoff vs. iteration number for $K = 2$ users.

User 2 in Fig. 4.1(b) assuming $\overline{P}_1 = \overline{P}_2 = 5$ W. We observe that the users converge to the Nash equilibrium after approximatively 1.3×10^4 iterations while, by using a best-response algorithm, the convergence is almost instantaneous (only 2 or 3 iterations). However, in the latter case, the rationality of the players and the perfect knowledge of the game structure are required.

4.3 The Slow Fading MIMO Channel

We will now study the single-user slow fading MIMO channel, under the same assumptions as in Section 3.3. As we have already discussed, when slow fading is assumed, the mutual information is a random variable, varying from block to block, and thus it is not possible to guarantee that it is above a certain threshold. In this case, the achievable transmit rate in the sense of Shannon is zero. A more suitable performance metric is the

probability of an outage for a given transmission rate target [24]. This metric allows one to quantify the probability that the rate target is not reached by using a good channel coding scheme and is defined as follows:

$$\mathrm{P}_{\mathrm{out}}(\mathbf{Q}, R) = \Pr\left[\mu(\mathbf{Q}) < R\right]. \tag{4.13}$$

Here, $\mu(\mathbf{Q})$ denotes the instantaneous mutual information:

$$\mu(\mathbf{Q}) = \log_2 \left|\mathbf{I}_{n_r} + \rho \mathbf{H}\mathbf{Q}\mathbf{H}^H\right|. \tag{4.14}$$

The idea is to implement a reinforcement algorithm that allows the transmitter to compute its best precoding matrix minimizing the outage probability. We start with the same simple action set given in Section 4.2:

$$\mathcal{D} = \left\{ \frac{\overline{P}}{\ell}\mathbf{Diag}(\underline{e}_\ell) \middle| \ell \in \{1, \ldots, n_t\}, \underline{e}_\ell \in \{0,1\}^{n_t}, \sum_{i=1}^{n_t} e_\ell(i) = \ell \right\}. \tag{4.15}$$

The choice of this set is motivated by two reasons: i) for Rayleigh fading, the optimal covariance matrix is diagonal; ii) Telatar [14] conjectured that the optimal covariance matrix is to uniformly allocate the power on a subset of antennas.

We assume that the only feed-back the transmitter receives at iteration n is an ACK/NACK bit denoted $s^{[n]}$, i.e., the realization of the following random variable: $S = 0$ if $\mu(\mathbf{Q}) \leq R$ otherwise $S = 1$. Therefore, if an outage occurs at time n the receiver feed-backs $s^{[n]} = 0$, otherwise $s^{[n]} = 1$. Notice that the random variable S follows a Bernoulli distribution of parameter $q = 1 - \mathrm{P}_{\mathrm{out}}(\mathbf{Q}, R)$. Its expected value is equal to $1 - \mathrm{P}_{\mathrm{out}}(\mathbf{Q}, R)$ and, therefore, if the instantaneous payoff is $s^{[n]}$, then its expected payoff is exactly the success probability: $1 - \mathrm{P}_{\mathrm{out}}(\mathbf{Q}, R)$.

Based only on $s^{[n]}$, the user applies a simple updating rule over its own probability distribution over the action space. Let us index the elements of $\mathcal{D} = \{\mathbf{D}^{(1)}, \ldots, \mathbf{D}^{(M)}\}$ with $M = \mathrm{Card}(\mathcal{D})$ (i.e., the cardinality of $\mathcal{D}$). We want to find out whether using a simple reinforcement learning algorithm will allow us to solve the open problem:

$$u_{\max} = \max_{j \in \{1,\ldots,M\}} u(\mathbf{D}^{(j)}), \tag{4.16}$$

where $u(\mathbf{D}^{(j)}) = 1 - P_{\mathrm{out}}(\mathbf{D}^{(j)}, R)$ represents the success probability. At step $n > 0$ of the iterative process, the transmitter randomly chooses a certain action $\mathbf{Q}^{[n]} \in \mathcal{D}$ based on the probability distribution $\underline{p}^{[n-1]}$ from the previous iteration. As a consequence, it obtains the realization of a random variable, which is, in our case, $s^{[n]}$. Based on this value, the transmitter updates its own probability distribution as follows:

$$p_j^{[n]} = p_j^{[n-1]} - \gamma^{[n]} s^{[n]} p_j^{[n-1]} + \gamma^{[n]} s^{[n]} \mathbb{1}_{\left(\mathbf{Q}^{[n]} = \mathbf{D}^{(j)}\right)} \tag{4.17}$$

where $0 < \gamma^{[n]} < 1$ is a step size and $p_j^{[n]}$ represents the probability that the transmitter choses $\mathbf{D}^{(j)}$ at iteration n. Notice that, as opposed to the previous section (Sec. 4.2), no assumptions on the channel matrix have to be made since $s^{[n]} \in \{0, 1\}$.

The sequence $\underline{p}^{[n]}$ can be approximated in the asymptotic regime with the solution of the following ODE:

$$\frac{\mathrm{d}p_j}{\mathrm{d}t} = p_j \left[u(\mathbf{D}^{(j)}) - \sum_{i=1}^{M} p_i u(\mathbf{D}^{(i)}) \right], \tag{4.18}$$

for all $j \in \{1, \ldots, M\}$.

Similarly to the previous section, the conditions [H2]-[H5] can be proven to be satisfied. First, we observe that the vector field $f(\underline{p})$, with the components given by $f_j(\underline{p}) = p_j \left[u(\mathbf{D}^{(j)}) - \sum_{i=1}^{M} p_i u(\mathbf{D}^{(i)}) \right]$ for all $j \in \{1, \ldots, M\}$, is a class $\mathcal{C}^1$ function. This implies that $f(\underline{p})$ is Lipschitz continuous and [H2] is met. Since the updated process corresponds to the discrete probability distribution, $\underline{p}^{[n]} \in \Delta(\mathcal{D})$, it is always bounded ([H3]). The noise here is given by:

$$Z_j^{[n]} = -s^{[n]} p_j^{[n-1]} + s^{[n]} \mathbb{1}_{(\mathbf{Q}^{[n]} = \mathbf{D}^{(j)})} - f_j(\underline{p}^{[n-1]})$$

Here as well, by construction, we have that $\left\{\underline{p}^{[n]}, (\mathbf{Q}^{[n-1]}, s^{[n-1]})\right\}_n$ is a Markov process and for a time invariant distribution $\underline{p}^{[n-1]} = \underline{p}$ $(\mathbf{Q}^{[n-1]}, s^{[n-1]})_n$ is an i.i.d. sequence, which implies that condition [H4] is met. Also, condition [H5] can be easily verified.

We can observe that only the corner points of the simplex $\Delta(\mathcal{D})$ (i.e., the pure-actions) are stationary states of this ODE. It turns out that, the only stable states are solutions to the optimization problem in (4.16).

Numerical simulations.

Consider the simple case of i.i.d. channel matrix of complex standard Gaussian entries, $n_t = n_r = 2$, $R = 1$ bpcu, $\overline{P} = 0.1$ W, $\rho = 10$ dB. In this case, the user can choose between beam-forming and the uniform power allocation: $\mathbf{D}^{(1)} = \overline{P}\mathbf{diag}(0, 1)$, $\mathbf{D}^{(2)} = \overline{P}\mathbf{diag}(1, 0)$, $\mathbf{D}^{(3)} = \frac{\overline{P}}{2}\mathbf{diag}(1, 1)$. The success probability is given by $u(\mathbf{D}^{(1)}) = u(\mathbf{D}^{(2)}) = 0.7359$, $u(\mathbf{D}^{(3)}) = 0.8841$. Notice that, the positions of the active antennas do not matter and only the number of active modes has an influence on the success probability. In Fig. 4.2, we trace the expected payoff $\sum_{j=1}^{M} p_j^{[n]} u(\mathbf{D}^{(j)})$ in function of the iterations. We assume $\gamma^{[n]} = \gamma = 0.01$ (constant step-size) and observe that the optimal solution is reached after 2554 iterations. However, the performance of the algorithm depends on the choice of the learning parameter: the larger γ, the smaller the convergence time. The problem with large steps is that the algorithm may converge to a corner of the simplex which is not a maximizer of the success probability. In Tab. 4.1, the same scenario is investigated. Here, we summarize the results of the reinforcement algorithm obtained after 1000 experiments in terms of average number of iterations and convergence to the maximum point. We observe that there is a trade-off between the convergence time and the convergence to the optimal point. This trade-off can be controlled by tuning the learning step-size γ.

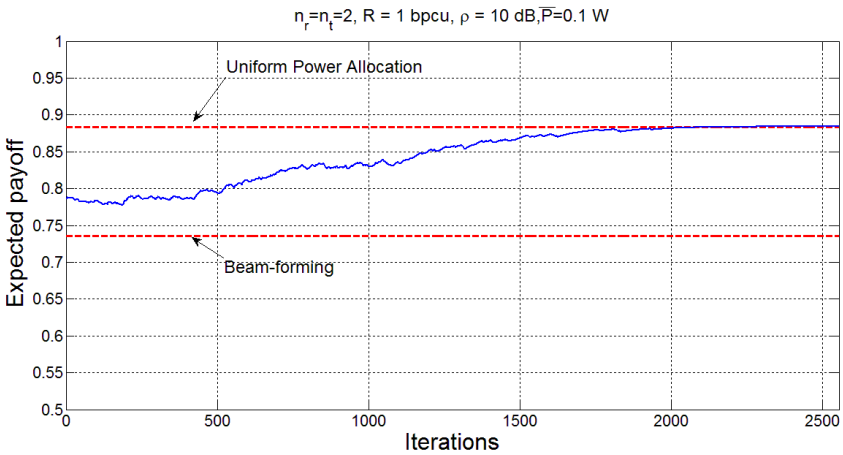


Figure 4.2: Average payoff vs. number of iterations.

Table 4.1: Trade-off between the convergence time and the convergence to the optimal point

γ	Time [nb. iterations]	Convergence to optimum [%]
0.001	3755	100
0.1	261	71
0.5	27	45
0.9	9	39

4.4 Conclusion and Open Issues

In this chapter, we have investigated reinforcement learning algorithms that allow the transmitters to converge to desirable network states: NE or other system operating points. These algorithms have several appealing features. They are completely distributed, adaptive and low complexity algorithms in which the users update the probabilities of choosing their actions based on a certain feed-back from the environment. The updating rule does not require any other knowledge on the environment (e.g., the network topology, channel state information) nor any assumption on users' rationality. However, all these benefits come at the cost of large convergence time. Moreover, the algorithms are stochastic in nature and only asymptotic convergence in probability can be ensured. In practice, this translates the fact that a very careful choice of the learning step-size has to be made to ensure a good performance. We have seen that there is a trade-off between the probability (frequency) of convergence and the convergence time. These problems are due to the inherent properties of reinforcement learning and replicator dynamics. Reinforcement learning algorithms allow the users to converge to the solutions of the ODE describing the replicator dynamics. All the pure-strategies, even those which are not optimal, are stationary points of this dynamics. This is one of the reasons for the numerical methods to fail in converging to the optimal points. One possibility to overcome these issues, is to consider other learning techniques such as Boltzman-Gibbs learning and Q learning [84].

In order to relax the constraint on the channel matrices in Sec. 4.2, the projected dynamics and stochastic algorithms [81] [80] dedicated to processes constrained to bounded sets can be applied. In Sec. 4.3, we have seen that learning algorithms can be used to numerically compute the optimal precoding matrix minimizing the outage probability for the single-user MIMO channel. This analysis can be extended to the energy-efficiency problem defined in Section 3.3. Furthermore, another interesting extension is to study the general case of multi-user scenario of MIMO multiple-access channels. At last, a rising issue is to evaluate the performance gap between the continuous case studied in the previous chapters (Chapter 2 and Chapter 3) and the discrete case studied in this chapter. Notice that for the standard Rayleigh channel there is no gap of optimality between the two approaches. However, for general channel models the problem is not trivial and a deep mathematical analysis is required.

Chapter 5

Conclusion and Perspectives

In this manuscript, our primary objective was to study decentralized wireless networks in which the terminal nodes are equipped with multiple antennas. Rising topics, such as self-optimizing networks, green communications and distributed algorithms have been approached mainly from a theoretical perspective. To this aim, we have used a diversified spectrum of tools from Game Theory, Information Theory, Random Matrix Theory and Learning Theory. Although there is still a large gap to be filled in order to make these studies relevant from a real-world point of view, their importance lies in the fact that they represent the limits of performance that can be achieved in practice.

We started our analysis with the study of the power allocation problem in distributed networks. The transmitters were assumed to be autonomous and capable of allocating their powers to optimize their Shannon achievable rates. A non-cooperative game theoretical framework was used to investigate the solution to this problem. Iterative algorithms based on the best-response functions were implemented to compute the Nash equilibrium solutions. Two different models were considered: the MIMO multiple access channel and the SISO parallel interference relay channel.

The first model was characterized by the fact that a more complex decoding technique could be implemented at the receiver: the successive interference cancellation. We have seen that, using a simple coordination signal dictating the decoding order at the receiver, the system sum-rate at the Nash equilibrium is fairly close to the centralized network solution. Furthermore, the distribution of this public signal can be manipulated in a centralized way to control the network operating point. Assuming fast fading links, random matrix theory was used to determine the Nash equilibrium point. An interesting particular case is the case of static parallel MAC, which was studied from a routing game perspective. Several Braess paradoxes have also been highlighted. The second model was characterized by the presence of the supplementary relaying nodes. These nodes could have been exploited by the transmitters to improve their network performance. Three different relaying protocols were investigated and their performance compared via numerical simulations: amplify-and-forward, decode-and-forward and estimate-and-forward. The parameters of the relaying nodes can be manipulated by the owner of the system to control the operating point of the network. Several interesting open issues

arose: the study of the general game involving all the freedom degrees of the transmitters, the multiplicity of the Nash equilibria, the Stackelberg formulation.

Several major issues appear in the one-shot non-cooperative games and with the iterative best-response algorithms. First, the transmitters are assumed to be strictly rational devices. Second, the information needed at the transmitter side with respect to his own payoff function and channel state or statistics is often unavailable in realistic scenarios. Third, the best-response based iterations involve an arguable large amount of network signaling. Fourth, proving the convergence of these algorithms is generally a difficult problem.

The learning theory in games appears to be a candidate solution to all these issues. We have seen that using simple updating rules (e.g., reinforcement learning), each user converges to one of the Nash equilibria of the initial game even though, in this framework, the users are no longer rational devices but simple automata. Furthermore, the only needed knowledge of the environment is a feed-back (e.g., the payoff value) that scores the choices of the users. It turned out that, based on this feedback, the users adapt and learn in time their optimal strategies. However, these algorithms are stochastic in nature and only probabilistic convergence can be guaranteed. Also, the convergence in practice involves a relatively long time.

Another major issue that has been considered is related to the energy-efficiency aspect of the communication. Indeed, in order to achieve high transmission rates, the used power has to be high as well. In networks where the power consumption is the bottleneck, the Shannon achievable rate is no longer suitable performance metric. This is why we have addressed the problem of optimizing an energy-efficiency function. This performance metric translates the average number of bits that can be conveyed through the channel per unit of energy consumed. Because of the encountered difficulties, our work was limited to the MIMO single-user channel. From an information-theoretic perspective, if no error probability is tolerated at the receiver, then energy-efficient transmission is not possible at non-zero transmit power and non-zero data rate. However, assuming a fixed transmission rate, if a certain error probability is tolerated, then energy-efficient communication is possible at a non-zero transmit power.

For the slow fading MIMO case, the general problem of finding the optimal precoding matrix optimizing the proposed goodput-to-power ratio is still an open issue. We have seen that it amounts to solving the particular case of uniform power allocation over the antennas, and to proving Telatar's conjecture [14]. However, finding the optimal precoding matrix minimizing the outage probability turned out to be a challenging issue. As we have seen, learning theory provides effective numerical algorithms to compute this optimal solution. Also, once that the aforementioned conjectures are solved, an important extension of our work would be the study of the multi-user scenario (e.g., the multiple access channel or the interference channel) from the energy-efficient perspective.

Appendix A

Publications on Shannon-Rate Efficient Non-Cooperative Power Allocation Games

A.1 IEEE-TWC-2009

E. V. Belmega, S. Lasaulce, and M. Debbah, "Power allocation games for MIMO multiple access channels with coordination", *IEEE Trans. on Wireless Communications*, vol. 8, no. 6, pp. 3182–3192, Jun. 2009.

Power Allocation Games for MIMO Multiple Access Channels with Coordination

Elena-Veronica Belmega, *Student Member, IEEE,* Samson Lasaulce, *Member, IEEE,* and Merouane Debbah, *Senior Member, IEEE*

***Abstract*—A game theoretic approach is used to derive the optimal decentralized power allocation (PA) in fast fading multiple access channels where the transmitters and receiver are equipped with multiple antennas. The players (the mobile terminals) are free to choose their PA in order to maximize their individual transmission rates (in particular they can ignore some specified centralized policies). A simple coordination mechanism between users is introduced. The nature and influence of this mechanism is studied in detail. The coordination signal indicates to the users the order in which the receiver applies successive interference cancellation and the frequency at which this order is used. Two different games are investigated: the users can either adapt their temporal PA to their decoding rank at the receiver or optimize their spatial PA between their transmit antennas. For both games a thorough analysis of the existence, uniqueness and sum-rate efficiency of the network Nash equilibrium is conducted. Analytical and simulation results are provided to assess the gap between the decentralized network performance and its equivalent virtual multiple input multiple output system, which is shown to be zero in some cases and relatively small in general.**

***Index Terms*—Game theory, large systems, MAC, MIMO, Nash equilibrium, power allocation games, random matrix theory.**

I. Introduction

We consider a special case of decentralized or distributed wireless networks, the decentralized multiple access channel (MAC). In this context, the MAC consists of a network of several mobile stations (MS) and one base station (BS). In the present work, the network is said to be decentralized in the sense that each user can choose freely his power allocation (PA) policy in order to selfishly maximize a certain individual performance criterion. This means that, even if the BS broadcasts some specified policies, every (possibly cognitive) user is free to ignore the policy intended for him if the latter does not maximize his performance criterion.

The problem of decentralized PA in wireless networks is not new and has been properly formalized for the first time in [2], [3]. Interestingly, this problem can be formulated quite naturally as a non-cooperative game with different performance criteria (utilities) such as the carrier-to-interference ratio [4], aggregate throughput [5] or energy efficiency [6], [7]. In this paper, we assume that the users want to maximize information-theoretic utilities and more precisely their Shannon transmission rates. Many reasons why this kind of utilities is often considered are provided in the literature related to the problem under investigation (some references are provided further). Here we will just mention three of them. First, Shannon transmission rates allow one to characterize the performance limits of a communication system and study the behavior of (selfish) users in a network where good coding schemes are implemented. As there is a direct relationship between the achievable transmission rate of a user and his signal-to-interference plus noise ratio (SINR), they also allow one to optimize performance metrics like the SINR or related quantities of the same type (e.g., the carrier-to-interference ratio).

From the mathematical point of view, Shannon rates have many desirable properties (e.g., concavity properties), which allows one to conduct deep performance analyses. Therefore they provide useful insights and concepts that are exploitable for a practical design of decentralized networks. Indeed, the point of view adopted here is close to the one proposed by the authors of [8] for DSL (digital subscriber lines) systems, which are modeled as a parallel interference channel; [9] for the single input single output (SISO) and single input multiple output (SIMO) fast fading MACs with global CSIR and global CSIT (Channel State Information at the Receiver/Transmitters); [10] for MIMO (Multiple Input Multiple Output) MACs with global CSIR, channel distribution information at the transmitters (global CDIT) and single-user decoding (SUD) at the receivers; [11], [12] for Gaussian MIMO interference channels with global CSIR and local CSIT and, by definition of the conventional interference channel [13], SUD at the receivers. Note that reference [14] where the authors considered Gaussian MIMO MACs with neither CSIT nor CDIT differs from our approach and that of [8], [9], [10], [11], [12] because in [14] the MIMO MAC is seen as a two-player zero-sum game where the first player is the group of transmitters and the second player is the set of MIMO sub-channels. In the list of the aforementioned references, [9] seems to be the closest work to ours. However, our approach differs from [9] on several technical key points. First of all, not only the BS but also the MSs can be equipped with multiple antennas. This is an important technical difference since the

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power control problem of [9] becomes a PA problem for which the precoding matrix of each user has to be determined. Also the issues regarding the existence and uniqueness of the network equilibrium are more complicated to be dealt with, as it will be seen. Specifically, random matrix theory will be exploited to determine the optimum eigenvalues of the precoding matrices. In [9], several assumptions made, especially the one involving the knowledge of all the instantaneous channels at each MS can be argued in some contexts. One of our objectives is to decrease the amount of signaling needed from the BS. This is why we assume that the BS can only send to the users sufficient training signals for them to know the statistics of the different channels and a simple and common coordination signal. The underlying coordination mechanism is simple because it consists in periodically sending the realization of a $K!$-state random signal, where K is the number of active users. As it will be seen in detail, such a mechanism is mandatory because, in contrast with [10], we assume here successive interference cancellation (SIC) at the BS. Thus each user needs to know his decoding rank in order to adapt his PA policy to maximize the transmission rate. The coordination signal precisely indicates to all the users the decoding order employed by the receiver. Therefore the proposed formulation can be seen from two different standpoints. If the distribution of the coordination signal is fixed, then the addressed problem can be regarded as a non-cooperative game where the BS is imposed to follow the realizations of the random coordination signal. In this case the respective signal can be generated by any device (and not necessarily by the BS), in order to select the decoding order. On the other hand, if the distribution of the coordination signal can be optimized, the problem can be addressed as a Stackelberg game. Here the BS is the game leader and chooses his best mixed strategy (namely a distribution over the possible decoding orders) in order to maximize a certain utility, which will be chosen to be the network uplink sum-rate.

In the described framework, one of our objectives is to know how well a non-cooperative but weakly coordinated system performs in terms of overall sum-rate w.r.t. its centralized counterpart (by "centralized" we mean that the users are imposed to follow the BS PA policies) when SIC is used at the BS. In this setting, several interesting questions arise. When the users' utility functions are chosen to be their individual transmission rates, is there a Nash equilibrium (NE) in the corresponding game and is it unique? What is the optimum way for a selfish user to allocate (spatially or temporally) his transmit power? How to choose the coordination signal that maximizes the network sum-rate? What is the performance loss of the decentralized network w.r.t. the equivalent virtual MIMO network?

This paper is structured as follows. After presenting the system model (Sec. II), we study in detail two PA games. In the first case (Sec. III), each MS is imposed to share his power uniformly between his transmit antennas but can freely allocate his power over time. In the second case (Sec. IV), we assume that the temporal PA is uniform and thus our objective is to derive the best spatial PA scheme. For each of these frameworks the existence, uniqueness, determination and sum-rate efficiency of the NE is investigated. Numerical results are provided in Sec. V to illustrate our theoretical analysis and in particular to better assess the sum-rate efficiency of the different games considered. We conclude the paper by several remarks and possible extensions of our work in Sec. VI.

II. System Model

Throughout the paper $\underline{v}$, $\mathbf{M}$, $(.)^T$ and $(.)^H$ will stand for vector, matrix, transpose and transpose conjugate, respectively. For simplicity and without loss of generality, we will assume a MAC with $K = 2$ users. Note that the type of multiple access technique assumed corresponds to the one considered in the standard definition of the Gaussian MAC by [15],[16]: all transmitters send at once and at different rates over the entire bandwidth. In this (information theoretic) context, very long codewords can be used and the receiver is not limited in terms of complexity. Thus the codewords of the different transmitters can be decoded jointly using a maximum likelihood decoding procedure (see [16] for more details). Interestingly, the transmission rates of the capacity region corresponding to the coding-decoding procedure just mentioned, can also be achieved, as discussed in [16], by using perfect SIC at the receiver. In this paper we also adopt this decoding scheme, which means that not only the different channel matrices are perfectly known to the receiver but also that the codewords of all the users are decoded reliably. The case of imperfect CSIR and error propagation in the SIC procedure is thus seen as a useful extension of this paper. Since we assume SIC at the BS and that the users want to maximize their individual transmission rates, it is necessary for them to know the decoding order used by the BS. This is why we assume the existence of a source broadcasting a discrete coordination signal to all the terminals in presence. If this source is the BS itself, this induces a certain cost in terms of downlink signaling but the distribution of the coordination signal can then be optimized. On the other hand, if the coordination signal comes from an external source, e.g., an FM transmitter, the MSs can acquire their coordination signal for free in terms of downlink signaling. However this generally involves a certain sub-optimality in terms of uplink rate. Analyzing this kind of tradeoffs is precisely one of the goals of this paper. In both cases, the coordination signal will be represented by a Bernouilli random variable denoted with $S \in \mathcal{S}$. Since we study the $2-$user MAC, $\mathcal{S} = \{1, 2\}$ is a binary alphabet and S is distributed as $\Pr[S = 1] = p$, $\Pr[S = 2] = 1 - p \triangleq \overline{p}$. Without loss of generality we assume that when the realization of S is 1, user 1 is decoded in the second place and therefore sees no multiple access interference; in a real wireless system the frequency at which the realizations would be drawn is roughly proportional to the reciprocal of the channel coherence time (T_{coh}). Note that the proposed coordination mechanism is suboptimal in the sense that the coordination signal does not depend on the realizations of the channel matrices. We will see that the corresponding performance loss is in fact very small.

We will further consider that each MS is equipped with n_t antennas whereas the BS has n_r antennas. In our analysis, the flat fading channel matrices of the different links vary from symbol vector to symbol vector. We assume that the receiver knows all the channel matrices whereas the transmitters have

only access to the statistics of the different channels. At this point, the authors would like to re-emphasize their point of view:

- On the one hand, we think that in some contexts our approach can be interesting in terms of signaling cost. We have seen that S lies in a $K!-$element alphabet and the realizations are drawn approximatively at $\frac{1}{T_{\text{coh}}}$[Hz], therefore the coordination mechanism requires at most $\frac{\log_2(K!)}{T_{\text{coh}}}$ bps from the BS and 0 bps if it is built from an external source. Another source of signaling cost is the acquisition of the knowledge of the statistics of the uplink channels at the MSs. For example, in the context of coherent communications where the BS regularly sends some data to the MSs and channel reciprocity assumption is valid (e.g., in time division duplex systems) the corresponding cost can be reasonable. In general, this cost will have to be compared to the cost of the centralized system where the BS has to send accurate enough quantized versions of the (possibly large) precoding matrices at a certain frequency.
- On the other hand, even if our approach is not interesting in terms of signaling, it can be very useful in contexts where terminals are autonomous and may have some selfish reasons to deviate from the centralized policies. In such scenarios, the concept of network equilibrium is of high importance.

The equivalent baseband signal received by the BS can be written as:

$$\underline{y}^{(s)}(\tau) = \sum_{k=1}^{K} \boldsymbol{H}_k(\tau)\underline{x}_k^{(s)}(\tau) + \underline{z}^{(s)}(\tau), \tag{1}$$

where $\underline{x}_k^{(s)}(\tau)$ is the n_t-dimensional column vector of symbols transmitted by user k at time τ for the realization $s \in \mathcal{S}$ of the coordination signal, $\mathbf{H}_k(\tau) \in \mathbb{C}^{n_r \times n_t}$ is the channel matrix (stationary and ergodic process) of user k and $\underline{z}^{(s)}(\tau)$ is an n_r-dimensional complex white Gaussian noise distributed as $\mathcal{N}(\underline{0}, \sigma^2\mathbf{I}_{n_r})$; for sake of clarity we will omit the time index τ from our notations. As [17] we assume that, for each $s \in \mathcal{S}$, the data streams of user k are multiplexed in the eigen-directions of the matrix $\mathbf{Q}_k^{(s)} = \mathbb{E}\left[\underline{x}_k^{(s)}\underline{x}_k^{(s),H}\right] \triangleq \mathbf{V}_k^{(s)}\mathbf{P}_k^{(s)}\mathbf{V}_k^{(s),H}$. Finding the optimal eigen-values $\mathbf{P}_k^{(s)}$ and coordinate systems $\mathbf{V}_k^{(s)}$ that maximize the transmission rate of user k is one of the main issues we will solve in the next two sections. In order to take into account the antenna correlation effects at the transmitters and receiver, we assume the different channel matrices to be structured according to the Kronecker propagation model [18] with common receive correlation [19]:

$$\forall k \in \{1, ..., K\},\ \mathbf{H}_k = \mathbf{R}^{\frac{1}{2}}\boldsymbol{\Theta}_k\mathbf{T}_k^{\frac{1}{2}} \tag{2}$$

where $\mathbf{R}$ is the receive antenna correlation matrix, $\mathbf{T}_k$ is the transmit antenna correlation matrix for user k and $\boldsymbol{\Theta}_k$ is an $n_r \times n_t$ matrix whose entries are zero-mean independent and identically distributed complex Gaussian random variables with variance $\frac{1}{n_t}$. The motivation for assuming a channel model with common receive correlation is twofold. First, there exist some situations where this MIMO MAC model is realistic, the most simple situation being the case of no receive correlation i.e., $\mathbf{R} = \mathbf{I}$ (see e.g., [20]). Although it is not explicitly stated in [19] the second feature of this model is that the overall channel matrix $\mathbf{H} = [\mathbf{H}_1 ... \mathbf{H}_K]$ can also be factorized as a Kronecker model, which will allow us to re-exploit existing results from the random matrix theory literature. Therefore the case where the overall channel matrix is not separable can be seen as a possible extension of this paper that can be dealt with by using the results in [21].

In this paper we study in detail two special but useful cases of decentralized PA problems. In the first case (Game 1), we assume (for instance because of practical technical/complexity constraints) that each user is imposed to share his power uniformly between his transmit antennas but can freely allocate his power over time; this problem will be referred to as temporal PA game (Sec. III). In the second case (Game 2), for every realization of the coordination signal, each user is assumed to transmit with the same total power (denoted by P_k) but can freely share it between his antennas; this problem will be referred to as spatial PA game (Sec. IV). For both games the *strategy* of user $k \in \{1, 2\}$ consists in choosing the distribution of $\underline{x}_k^{(s)}$, for each $s \in \mathcal{S}$ in order to maximize his *utility* function which is given by:

$$u_k(\mathbf{Q}_1^{(1)}, \mathbf{Q}_1^{(2)}, \mathbf{Q}_2^{(1)}, \mathbf{Q}_2^{(2)}) = \sum_{s=1}^{2} \Pr[S = s] R_k^{(s)}(\mathbf{Q}_1^{(s)}, \mathbf{Q}_2^{(s)}) \tag{3}$$

where

$$R_k^{(s)}(\mathbf{Q}_1^{(s)}, \mathbf{Q}_2^{(s)}) = \begin{cases} \mathbb{E}\log|\mathbf{I} + \eta\mathbf{H}_k\mathbf{Q}_k^{(s)}\mathbf{H}_k^H| & \text{if } k = s \\ \mathbb{E}\log|\mathbf{I} + \eta\sum_{k=1}^{2}\mathbf{H}_k\mathbf{Q}_k^{(s)}\mathbf{H}_k^H| & \\ \quad -\mathbb{E}\log|\mathbf{I} + \eta\mathbf{H}_{-k}\mathbf{Q}_{-k}^{(s)}\mathbf{H}_{-k}^H| & \text{if } k \neq s \end{cases} \tag{4}$$

with $\eta \triangleq \frac{1}{\sigma^2}$ and the usual notation for $-k$, which stands for the other user than k. Note that we implicitly assume Gaussian codebooks for the two users since this choice is optimum in terms of their individual Shannon transmission rates (see e.g., [22]). This is why the strategy of a user boils down to choosing the best pair of covariance matrices $(\mathbf{Q}_k^{(1)}, \mathbf{Q}_k^{(2)})$. The corresponding maximization is performed under the following transmit power constraint for each MS: $\text{Tr}\left(\sum_{s=1}^{2}\Pr[S = s]\mathbf{Q}_k^{(s)}\right) \leq n_t P_k$. The main difference between Games 1 and 2 relies precisely on how this general power constraint is specialized. In Game 1, the precoding matrices are imposed to have the following structure: $\forall k \in \{1, 2\}, \forall s \in \{1, 2\}, \mathbf{Q}_k^{(s)} = \alpha_k^{(s)} P_k \mathbf{I}_{n_t}$, which amounts to rewriting the total power constraint as follows

$$\sum_{s=1}^{2} \Pr[S = s]\alpha_k^{(s)} \leq 1. \tag{5}$$

On the other hand, in Game 2, the power constraint expresses as

$$\forall k \in \{1, 2\}, \forall s \in \{1, 2\}, \text{Tr}(\mathbf{Q}_k^{(s)}) \leq n_t P_k. \tag{6}$$

In both game frameworks, an important issue for a wireless network designer/owner is to know whether by leaving the users decide their PA by themselves, the network is going to operate at a given and predictable state. This precisely corresponds to the notion of a network equilibrium, a state

from which no user has interest to deviate. The main issue is to know if there exists an equilibrium point, whether it is unique, how to determine the corresponding strategies and characterize the efficiency of this equilibrium in terms of network sum-rate.

III. Temporal Power Allocation Game

As mentioned above, in the temporal power allocation (TPA) game, the strategy of user $k \in \{1,2\}$ merely consists in choosing the best pair $(\alpha_k^{(1)}, \alpha_k^{(2)})$. Since each transmission rate is a concave and non-decreasing function of the $\alpha_k^{(s)}$'s, each user will saturate the power constraint (5) i.e., $\sum_{s=1}^{2} \Pr[S = s]\alpha_k^{(s)} = 1$, which leads to optimizing a single parameter $\alpha_k^{(1)}$ or $\alpha_k^{(2)}$. From now on, for sake of clarity we will use the notations $\alpha_1^{(1)} = \alpha_1$, $\alpha_2^{(2)} = \alpha_2$. Indeed, it is easy to verify that the power constraints are characterized completely, for the first user by $\alpha_1^{(2)} = \frac{1-p\alpha_1}{1-p}$ with $\alpha_1 \in \mathcal{A}_1^{\mathrm{TPA}} \triangleq \left[0, \frac{1}{p}\right]$, and for the second user by $\alpha_2^{(1)} = \frac{1-(1-p)\alpha_2}{p}$ with $\alpha_2 \in \mathcal{A}_2^{\mathrm{TPA}} \triangleq \left[0, \frac{1}{1-p}\right]$. Thus the strategy of user $k \in \{1,2\}$ consists in choosing the best fraction α_k from the action set $\mathcal{A}_k^{\mathrm{TPA}}$. Our main goal is to investigate if there exists an NE and determine the corresponding profile of strategies $(\alpha_1^{\mathrm{NE}}, \alpha_2^{\mathrm{NE}})$. It turns out that the issues of the existence and uniqueness of an NE can be properly dealt with by applying Theorems 1 and 2 of [23] in our context. For making this paper sufficiently self-contained, we review here these two theorems (Theorem 2 is given for the $2-$user case for simplicity and because it is sufficient under our assumptions).

Theorem 1: [23] *Let $\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{u_k\}_{k\in\mathcal{K}})$ be a game where $\mathcal{K} = \{1, ..., K\}$ is the set of players, $\mathcal{A}_1, ..., \mathcal{A}_K$ the corresponding sets of strategies and $u_1, ..., u_k$ the utilities of the different players. If the following three conditions are satisfied: (i) each u_k is continuous in the vector of strategies $(a_1, ..., a_K) \in \prod_{k=1}^{K} \mathcal{A}_k$; (ii) each u_k is concave in $a_k \in \mathcal{A}_k$; (iii) $\mathcal{A}_1, ..., \mathcal{A}_K$ are compact and convex sets; then $\mathcal{G}$ has at least one NE.*

Theorem 2: [23] *Consider the K-player concave game of Theorem 1 with $K = 2$. If the following (diagonally strict concavity) condition is met: for all $(a_1', a_1'') \in \mathcal{A}_1^2$ and $(a_2', a_2'') \in \mathcal{A}_2^2$ such that $(a_1', a_2') \neq (a_1'', a_2''), (a_1'' - a_1')\left[\frac{\partial u_1}{\partial a_1}(a_1', a_2') - \frac{\partial u_1}{\partial a_1}(a_1'', a_2'')\right] + (a_2'' - a_2')\left[\frac{\partial u_2}{\partial a_2}(a_1', a_2') - \frac{\partial u_2}{\partial a_2}(a_1'', a_2'')\right] > 0$; then the uniqueness of the NE is insured.*

At this point we can state the first main result of this paper, which is provided in the following theorem. For sake of clarity we will also use the notations: $p_k \triangleq p$ if $k = 1$ or $p_k \triangleq \overline{p}$ if $k = 2$.

Theorem 3 (Existence and uniqueness of an NE in Game 1): the temporal PA game described by: the set of players $\mathcal{K} = \{1,2\}$; the sets of actions $\mathcal{A}_k^{\mathrm{TPA}} = \left[0, \frac{1}{p_k}\right]$ and utilities $u_k(\alpha_k, \alpha_{-k}) = pR_k^{(1)}(\alpha_k, \alpha_{-k}) + \overline{p}R_k^{(2)}(\alpha_k, \alpha_{-k})$, where the rates $R_k^{(s)}$ follow from Eq. (4) has a unique NE.

Proof:

Existence of an NE. It is guaranteed by the geometrical and topological properties of the utility functions and the strategy sets of the users (over which the maximization is performed). Indeed, we can apply [23] in our matrix case. Without loss of generality, let us consider user 1. The utility of user 1 comprises two terms corresponding to the two coordination signal realizations: $u_1(\alpha_1, \alpha_2) = pR_1^{(1)}(\cdot,\cdot) + \overline{p}R_1^{(2)}(\cdot,\cdot)$. Using the fact that $\frac{d^2}{dt^2}\log|\mathbf{X} + t\mathbf{Y}\mathbf{Y}^H| = -\mathrm{Tr}\left[\mathbf{Y}^H(\mathbf{X}+t\mathbf{Y}\mathbf{Y}^H)^{-1}\mathbf{Y}\mathbf{Y}^H(\mathbf{X}+t\mathbf{Y}\mathbf{Y}^H)^{-1}\mathbf{Y}\right]$ it is easy to verify that $\frac{\partial^2 R_1^{(1)}}{\partial \alpha_1^2}(\alpha_1, \alpha_2) = -\mathbb{E}\mathrm{Tr}[\mathbf{B}\mathbf{B}^H] < 0$ and $\frac{\partial^2 R_1^{(2)}}{\partial \alpha_1^2}(\alpha_1, \alpha_2) = -\mathbb{E}\mathrm{Tr}[\mathbf{C}\mathbf{C}^H] < 0$ where $\mathbf{B} = \rho_1\mathbf{H}_1^H(\mathbf{I} + \rho_1\alpha_1\mathbf{H}_1\mathbf{H}_1^H)^{-1}\mathbf{H}_1$, $\mathbf{C} = \frac{p}{\overline{p}}\rho_1\mathbf{H}_1^H\left(\mathbf{I} + \rho_1\frac{1-p\alpha_1}{\overline{p}}\mathbf{H}_1\mathbf{H}_1^H + \rho_2\alpha_2\mathbf{H}_2\mathbf{H}_2^H\right)^{-1}\mathbf{H}_1$ and $\rho_1 = \eta P_1$, $\rho_2 = \eta P_2$ correspond to the signal-to-noise ratios of the users. Thus for every user k, the utility u_k is strictly concave w.r.t. to α_k. Also it is continuous in (α_1, α_2) over the convex and compact strategy sets $\mathcal{A}_k^{\mathrm{TPA}}$. Therefore the existence of at least one NE is guaranteed. Interestingly, we observe that for a fixed game rule, which is the value of the parameter p, there will always be an equilibrium. The users adapt their strategies to the rule of the game in order to optimize their individual transmission rates.

Uniqueness of the NE. We always apply [23] in our matrix case (see Appendix A) and prove that the diagonally strict concavity condition is actually met. The key of the proof is the following Lemma which is proven in Appendix B.

Lemma 1: Let $\mathbf{A}'$, $\mathbf{A}''$, $\mathbf{B}'$ and $\mathbf{B}''$ be Hermitian and non-negative matrices such that either $\mathbf{A}' \neq \mathbf{A}''$ or $\mathbf{B}' \neq \mathbf{B}''$. Assume that the classical matrix order $\succeq$ is total for each of the pairs of matrices $(\mathbf{A}', \mathbf{A}'')$ and $(\mathbf{B}', \mathbf{B}'')$ i.e., either $\mathbf{A}' \succeq \mathbf{A}''$ (resp. $\mathbf{B}' \succeq \mathbf{B}''$) or $\mathbf{A}'' \succ \mathbf{A}'$ (resp. $\mathbf{B}'' \succ \mathbf{B}'$). Then we have $\mathrm{Tr}(\mathbf{M} + \mathbf{N}) \geq 0$ with $\mathbf{M} = (\mathbf{A}'' - \mathbf{A}')\left[(\mathbf{I} + \mathbf{A}')^{-1} - (\mathbf{I} + \mathbf{A}'')^{-1}\right]$, $\mathbf{N} = (\mathbf{B}'' - \mathbf{B}')\left[(\mathbf{I} + \mathbf{B}' + \mathbf{A}')^{-1} - (\mathbf{I} + \mathbf{B}'' + \mathbf{A}'')^{-1}\right]$.

It can be shown (see Appendix A for more details) that the diagonally strict concavity condition writes in our setup as $p\mathcal{T}^{(1)} + \overline{p}\mathcal{T}^{(2)} > 0$ where $\forall s \in \{1,2\}$, $\mathcal{T}^{(s)}$ is defined by $\mathcal{T}^{(s)} = \mathrm{Tr}(\mathbf{M}^{(s)} + \mathbf{N}^{(s)})$ where the matrices $\mathbf{M}^{(s)}$, $\mathbf{N}^{(s)}$ have exactly the same structure as $\mathbf{M}$, $\mathbf{N}$ in the above Lemma. For example, if we consider two pairs of parameters $(\alpha_1', \alpha_1'') \in \left(\mathcal{A}_1^{\mathrm{TPA}}\right)^2$ and $(\alpha_2', \alpha_2'') \in \left(\mathcal{A}_2^{\mathrm{TPA}}\right)^2$ such that either $\alpha_1' \neq \alpha_1''$ or $\alpha_2' \neq \alpha_2''$ as in Theorem 2, $\mathcal{T}^{(1)}$ can be obtained by using the following matrices $\mathbf{A}' = \rho_1\alpha_1'\mathbf{H}_1\mathbf{H}_1^H$, $\mathbf{A}'' = \rho_1\alpha_1''\mathbf{H}_1\mathbf{H}_1^H$, $\mathbf{B}' = \rho_2\frac{1-\overline{p}\alpha_2'}{p}\mathbf{H}_2\mathbf{H}_2^H$, $\mathbf{B}'' = \rho_2\frac{1-\overline{p}\alpha_2''}{p}\mathbf{H}_2\mathbf{H}_2^H$. The term $\mathcal{T}^{(2)}$ has a similar form as $\mathcal{T}^{(1)}$ thus, applying Lemma 1 twice and considering the special structure of the four matrices ($\mathbf{A}'$, $\mathbf{A}''$, $\mathbf{B}'$, $\mathbf{B}''$), one can prove that the term $p\mathcal{T}^{(1)} + \overline{p}\mathcal{T}^{(2)}$ is strictly positive. Therefore the unconditional uniqueness of the NE is guaranteed. ■

Determination of the NE. In order to determine the selfish PA of the users at the NE, we now exploit the large system approach derived in [24] for single-user fading MIMO channels. This will lead us to simple approximations of the utility functions which are much easier to optimize. From now on, we assume the asymptotic regime in terms of the number of antennas: $n_t \longrightarrow \infty$, $n_r \longrightarrow \infty$, and $\lim_{n_t\to\infty, n_r\to\infty} \frac{n_t}{n_r} = c < \infty$. In this asymptotic regime, references [24], [25], [26] provide an equivalent of the ergodic capacity of single-user MIMO

channels, which corresponds exactly to the situation seen by user 1 (resp. 2) when $S = 1$ (resp. $S = 2$); this gives directly the approximation of the rates $R_1^{(1)}$ and $R_2^{(2)}$; see Eq. (4). From Eq. (4) we also see that the rates $R_1^{(2)}$ and $R_2^{(1)}$ correspond to the difference between the sum-rate of the equivalent $Kn_t \times n_r$ virtual MIMO system and an $n_t \times n_r$ single-user MIMO system, therefore the results of [24], [25], [26] can also be applied directly. The corresponding approximates can then be easily checked to be:

$$\begin{aligned}
\tilde{R}_1^{(1)}(\alpha_1,\alpha_2) &= \sum_{i=1}^{n_t}\log_2\left[1+\eta\alpha_1 P_1 d_1^{(\mathrm{T})}(i)\gamma_1\right] \\
&\quad + \sum_{j=1}^{n_r}\log_2\left[1+\eta d^{(\mathrm{R})}(j)\delta_1\right] \\
&\quad - n_t\eta\gamma_1\delta_1\log_2 e \\
\tilde{R}_2^{(1)}(\alpha_1,\alpha_2) &= \sum_{i=1}^{n_t}\log_2\left[1+2\eta\alpha_1 P_1 d_2^{(\mathrm{T})}(i)\gamma_2\right] \\
&\quad + \sum_{i=1}^{n_t}\log_2\left[1+2\eta\frac{1-\overline{p}\alpha_2}{p}P_2 d_2^{(\mathrm{T})}(i)\gamma_2\right] \\
&\quad + \sum_{j=1}^{n_r}\log_2\left[1+2\eta d^{(\mathrm{R})}(j)\delta_2\right] \\
&\quad - 4n_t\eta\gamma_2\delta_2\log_2 e - \tilde{R}_1^{(1)}(\alpha_1,\alpha_2).
\end{aligned} \tag{7}$$

where $\forall k \in \{1,2\}, d_k^{(\mathrm{T})}(i)$, $i \in \{1,...,n_t\}$ are the eigenvalues of the transmit correlation matrices $\mathbf{T}_k$ (see Eq. (2)), $d^{(\mathrm{R})}(j)$, $j \in \{1,...,n_r\}$, are the eigenvalues of the receive correlation matrix $\mathbf{R}$ and the parameters γ_i, δ_j are the unique solutions of the following systems of 2−degree equations:

$$\begin{cases} \gamma_1 = \dfrac{1}{n_t}\displaystyle\sum_{j=1}^{n_r}\dfrac{d^{(\mathrm{R})}(j)}{1+\eta d^{(\mathrm{R})}(j)\delta_1} \\ \delta_1 = \dfrac{1}{n_t}\displaystyle\sum_{i=1}^{n_t}\dfrac{\alpha_1 P_1 d_1^{(\mathrm{T})}(i)}{1+\eta\alpha_1 P_1 d_1^{(\mathrm{T})}(i)\gamma_1} \end{cases} \tag{8}$$

$$\begin{cases} \gamma_2 = \dfrac{1}{2n_t}\displaystyle\sum_{j=1}^{n_r}\dfrac{d^{(\mathrm{R})}(j)}{1+2\eta d^{(\mathrm{R})}(j)\delta_2} \\ \delta_2 = \dfrac{1}{2n_t}\left[\displaystyle\sum_{i=1}^{n_t}\dfrac{\alpha_1 P_1 d_1^{(\mathrm{T})}(i)}{1+2\eta\alpha_1 P_1 d_1^{(\mathrm{T})}(i)\gamma_2}\right. \\ \qquad \left. + \displaystyle\sum_{i=1}^{n_t}\dfrac{\frac{1-\overline{p}\alpha_2}{p}P_2 d_2^{(\mathrm{T})}(i)}{1+2\eta\frac{1-\overline{p}\alpha_2}{p}P_2 d_2^{(\mathrm{T})}(i)\gamma_2}\right]. \end{cases} \tag{9}$$

The approximate functions $\tilde{R}_1^{(2)}(\cdot,\cdot)$ and $\tilde{R}_2^{(2)}(\cdot)$ can be obtained in a similar way and the approximated utility of user $k \in \{1,2\}$ follows: $\tilde{u}_k(\alpha_1,\alpha_2) = p\tilde{R}_k^{(1)}(\alpha_1,\alpha_2) + \overline{p}\tilde{R}_k^{(2)}(\alpha_1,\alpha_2)$. Now, in order to solve the constrained optimization problem, we introduce the Lagrange multipliers $(\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22}) \in [0,+\infty)^4$ and define for $k \in \{1,2\}$ the function $\mathcal{L}_k(\alpha_1,\alpha_2,\lambda_{k1},\lambda_{k2}) = -\tilde{u}_k(\alpha_1,\alpha_2) + \lambda_{k1}\left(\alpha_k - \frac{1}{p_k}\right) - \lambda_{k2}\alpha_k$. The Kuhn-Tucker optimality conditions follow. Therefore, the optimum selfish PAs, $(\alpha_1^{\mathrm{NE}},\alpha_2^{\mathrm{NE}})$, can be obtained by using a fixed-point method and an iterative algorithm, following the same idea as in [10] for non-coordinated MIMO MACs with single-user decoding. At this point we have to make an important technical comment. Our proof for the existence and uniqueness of the NE holds for the exact game. For the approximated game, we need the approximated utilities to have the same properties as their exact counterparts. It turns out that the large system approximation of the ergodic mutual information can be shown to have the desired properties [27]. In particular, the results of [27] show that the approximated utilities are strictly concave and that if the iterative PA algorithm converges, it converges towards the global maximum.

Sum-rate efficiency of the NE. Now, let us focus on the sum-rate of the decentralized network and compare it with the optimal sum-rate of its centralized counterpart. The centralized network sum-rate, denoted by $R_{\mathrm{sum}}^{(\mathrm{C})}$, is by definition obtained by jointly maximizing the sum-rate over all the pairs of power fractions $(\alpha_1,\alpha_2) \in [0,1]^2$: $R_{\mathrm{sum}}^{(\mathrm{C})} \triangleq \max_{(\alpha_1,\alpha_2)} u_1(\alpha_1,\alpha_2) + u_2(\alpha_1,\alpha_2)$. Knowing that $\log|\cdot|$ is a concave function, one can easily verify that the maximum is obtained for $(\alpha_1^*,\alpha_2^*) = (1,1)$ and that $R_{\mathrm{sum}}^{(C)} = \mathbb{E}\log\left|\mathbf{I}+\rho_1\mathbf{H}_1\mathbf{H}_1^H+\rho_2\mathbf{H}_2\mathbf{H}_2^H\right|$. As the optimum precoding matrices are proportional to the identity matrix, it can be checked that the network sum-rate at the NE (denoted by $R_{\mathrm{sum}}^{\mathrm{NE}}$) is equal to the centralized network sum-rate for $p = 0$ and $p = 1$: $R_{\mathrm{sum}}^{\mathrm{NE}}(0) = R_{\mathrm{sum}}^{\mathrm{NE}}(1) = R_{\mathrm{sum}}^{(\mathrm{C})}$. Indeed, let us consider that $p = 1$. In this case, user 1 is always decoded in the second place ($\Pr[S = 1] = 1$). This means that there is no temporal power allocation game here and each user always allocates all of his available power for the case where $S = 1$: $(\alpha_1^{\mathrm{NE}},\alpha_2^{\mathrm{NE}}) = (1,0)$. Replacing in Eq. (4) the corresponding correlation matrices: $\mathbf{Q}_1^{(1)} = \mathbf{I}_{n_t}$, $\mathbf{Q}_2^{(1)} = \mathbf{I}_{n_t}$ and $\mathbf{Q}_1^{(2)} = \mathbf{O}_{n_t}$ (the square zero matrix), $\mathbf{Q}_2^{(2)} = \mathbf{O}_{n_t}$ we obtain that $R_{\mathrm{sum}}^{\mathrm{NE}}(1) = R_{\mathrm{sum}}^{(\mathrm{C})}$.

In the *high SNR regime*, where $\eta \to \infty$, we obtain from (8),(9) that $\eta\delta_1 \to \frac{1}{\gamma_1}$, $\eta\delta_2 \to \frac{1}{2\gamma_2}$ and thus γ_1 and γ_2 are the unique solutions of the following equations: $\frac{1}{n_t}\sum_{j=1}^{n_r}\frac{d^{(\mathrm{R})}(j)}{\gamma_1+d^{(\mathrm{R})}(j)} = 1$, $\frac{1}{2n_t}\sum_{j=1}^{n_r}\frac{d^{(\mathrm{R})}(j)}{\gamma_2+d^{(\mathrm{R})}(j)} = 1$. The approximated utilities become:

$$\begin{aligned}
\tilde{R}_1^{(1)}(\alpha_1,\alpha_2) &= \sum_{i=1}^{n_t}\log_2\left[1+\eta\alpha_1 P_1 d_1^{(\mathrm{T})}(i)\gamma_1\right] \\
&\quad + \sum_{j=1}^{n_r}\log_2\left[1+\frac{d^{(\mathrm{R})}(j)}{\gamma_1}\right] - n_t\log_2 e \\
\tilde{R}_2^{(1)}(\alpha_1,\alpha_2) &= \sum_{i=1}^{n_t}\log_2\left[1+2\eta\alpha_1 P_1 d_1^{(\mathrm{T})}(i)\gamma_2\right] \\
&\quad + \sum_{i=1}^{n_t}\log_2\left[1+2\eta\frac{1-\overline{p}\alpha_2}{p}P_2 d_2^{(\mathrm{T})}(i)\gamma_2\right] \\
&\quad + \sum_{j=1}^{n_r}\log_2\left[1+\frac{d^{(\mathrm{R})}(j)}{\gamma_2}\right] \\
&\quad - 2n_t\log_2 e - \tilde{R}_1^{(1)}(\alpha_1,\alpha_2).
\end{aligned} \tag{10}$$

By setting the derivatives of $\tilde{u}_1(\cdot,\cdot)$ w.r.t. α_1 and $\tilde{u}_2(\cdot,\cdot)$ w.r.t. α_2 to zero, we obtain that, for each user, the PA at the NE is the uniform PA $(\alpha_1^{\mathrm{NE}},\alpha_2^{\mathrm{NE}}) = (1,1)$, regardless of the distribution of the coordination signal $p \in [0,1]$. Therefore,

at the equilibrium, we have that

$$\begin{aligned} R_{\text{sum}}^{\text{NE}}(p) &= pR_1^{(1)}(\alpha_1^{\text{NE}},\alpha_2^{\text{NE}}) + \overline{p}R_1^{(2)}(\alpha_1^{\text{NE}},\alpha_2^{\text{NE}}) \\ &\quad + pR_2^{(1)}(\alpha_1^{\text{NE}},\alpha_2^{\text{NE}}) + \overline{p}R_2^{(2)}(\alpha_1^{\text{NE}},\alpha_2^{\text{NE}}) \\ &= p\mathbb{E}\log|\mathbf{I}+\rho_1\mathbf{H}_1\mathbf{H}_1^H| \\ &\quad + \overline{p}\mathbb{E}\log|\mathbf{I}+\rho_1\mathbf{H}_1\mathbf{H}_1^H+\rho_2\mathbf{H}_2\mathbf{H}_2^H| \\ &\quad - \overline{p}\mathbb{E}\log|\mathbf{I}+\rho_2\mathbf{H}_2\mathbf{H}_2^H| \\ &\quad + p\mathbb{E}\log|\mathbf{I}+\rho_1\mathbf{H}_1\mathbf{H}_1^H+\rho_2\mathbf{H}_2\mathbf{H}_2^H| \\ &\quad - p\mathbb{E}\log|\mathbf{I}+\rho_1\mathbf{H}_1\mathbf{H}_1^H| \\ &\quad + \overline{p}\mathbb{E}\log|\mathbf{I}+\rho_2\mathbf{H}_2\mathbf{H}_2^H| \\ &= R_{\text{sum}}^{(\text{C})}. \end{aligned} \tag{11}$$

Knowing that the uniform spatial PA is optimal in the high SNR regime [17], [10], the centralized network sum-rate coincides with the sum-capacity of the centralized MAC channel, $R_{\text{sum}}^{(\text{C})} = C_{\text{sum}}$.

In the *low SNR regime*, where $\eta \to 0$, we obtain from (8), (9) that $\eta\delta_1 \to 0$, $\eta\delta_2 \to 0$ and thus $\gamma_1 = \frac{1}{n_t}\sum_{j=1}^{n_r} d^{(\text{R})}(j)$, $\gamma_2 = \frac{1}{2n_t}\sum_{j=1}^{n_r} d^{(\text{R})}(j)$. Approximating $\ln(1+x) \approx x$ for $x << 1$, the achievable rates become:

$$\begin{aligned} \tilde{R}_1^{(1)}(\alpha_1) &= \frac{\eta P_1\alpha_1}{n_t}\sum_{j=1}^{n_r} d^{(\text{R})}(j)\sum_{i=1}^{n_t} d_1^{(\text{T})}(i)\log_2 e \\ \tilde{R}_2^{(1)}(\alpha_1,\alpha_2) &= \frac{\eta P_2}{n_t}\frac{1-\overline{p}\alpha_2}{p}\sum_{j=1}^{n_r} d^{(\text{R})}(j)\sum_{i=1}^{n_t} d_1^{(\text{T})}(i)\log_2 e \end{aligned} \tag{12}$$

We see that the utilities $\tilde{u}_k(\alpha_1,\alpha_2) = \frac{1}{n_t}\eta P_k\sum_{j=1}^{n_r} d^{(\text{R})}(j)\sum_{i=1}^{n_t} d_k^{(\text{T})}(i)\log_2 e$ converge and the network sum-rate at the NE coincides here again with the centralized network sum-rate: $R_{\text{sum}}^{(\text{C})} = \frac{1}{n_t}\sum_{j=1}^{n_r} d^{(\text{R})}(j)\left(\eta P_1\sum_{i=1}^{n_t} d_1^{(\text{T})}(i) + \eta P_2\sum_{i=1}^{n_t} d_2^{(\text{T})}(i)\right)\log_2 e$. In this case also, the price of anarchy [28] is minimal for any distribution of the coordination signal.

To sum up we have seen that there is no loss of optimality in terms of sum-rate by decentralizing the PA procedure in at least four special cases: 1) $p = 0$; 2) $p = 1$; 3) when $\eta \to \infty$ for any $p \in [0,1]$; 4) when $\eta \to 0$ for any $p \in [0,1]$. Additionally, in case 3), since there is no loss by imposing the spatially uniform PA [17], [10], the centralized (and cooperative) MAC sum-capacity is achieved. If we further assume that there is no correlation among the transmit antennas, $\mathbf{T}_k = \mathbf{I}$, the uniform spatial PA is optimal [17] for any η. Thus, the centralized sum-rate is always identical to the sum-capacity of the centralized MAC channel, $R_{\text{sum}}^{(\text{C})} = C_{\text{sum}}$. This means that if the BS chooses to use a completely unfair SIC-based decoding scheme, the selfish behavior of the users will always lead to achieving the centralized sum-capacity. This result is in agreement with [9], where the authors have proposed a water-filling game for the fast fading SISO MAC (assuming perfect CSIT and CSIR) and shown that the equilibrium sum-rate is equal to the maximum sum-rate point of the capacity region. However, as opposed to the SISO MAC with the proposed coordination mechanism [1], the decentralized MIMO MAC with coordination does not achieve the sum-rate of the equivalent virtual MIMO network for any value of p and for an arbitrary noise level at the BS. In particular, the fair choice $p = \frac{1}{2}$ is not optimal. We will quantify the corresponding performance gap through simulation results. Furthermore, in the low and high SNR regimes, the centralized sum-capacity is also achieved for any value of p. The consequence of these results is that any binary coordination signal can be used without loss of global optimality.

IV. Spatial Power Allocation Game

In this section, we assume that the users are free to share their transmit power between their antennas but for each realization of the coordination signal the transmit power is constrained by Eq. (6). In other words we assume that the users cannot distribute their power over time: they cannot decide the amount of power they dedicate to a given realization of the coordination signal. As a consequence of this power constraint (Eq. (6)), the two precoding matrices that each user needs to choose can be optimized independently and each of them does not depend on p. Consider for example user 1. Its objective is to maximize its own payoff (Eq. (3)):

$$\begin{aligned} &\max_{\mathbf{Q}_1^{(1)},\mathbf{Q}_1^{(2)}} u_1(\mathbf{Q}_1^{(s)},\mathbf{Q}_2^{(s)}) \\ &= \max_{\mathbf{Q}_1^{(1)},\mathbf{Q}_1^{(2)}} \left\{pR_1^{(1)}(\mathbf{Q}_1^{(1)}) + (1-p)R_1^{(2)}(\mathbf{Q}_1^{(2)},\mathbf{Q}_2^{(2)})\right\} \\ &= p\max_{\mathbf{Q}_1^{(1)}} R_1^{(1)}(\mathbf{Q}_1^{(1)}) + (1-p)\max_{\mathbf{Q}_1^{(2)}} R_1^{(2)}(\mathbf{Q}_1^{(2)},\mathbf{Q}_2^{(2)}), \end{aligned} \tag{13}$$

where the last inequality follows directly form the power constraint (Eq. (6)). The strategy set of user k in the spatial PA (SPA) game is:

$$\begin{aligned} \mathcal{A}_k^{\text{SPA}} = \Big\{&\mathbf{Q}_k = (\mathbf{Q}_k^{(1)},\mathbf{Q}_k^{(2)})\,|\mathbf{Q}_k^{(1)} \succeq 0, \mathbf{Q}_k^{(1)} = \mathbf{Q}_k^{(1)H}, \\ &\text{Tr}(\mathbf{Q}_k^{(1)}) \le n_tP_k, \mathbf{Q}_k^{(2)} \succeq 0, \mathbf{Q}_k^{(2)} = \mathbf{Q}_k^{(2)H}, \text{Tr}(\mathbf{Q}_k^{(2)}) \le n_tP_k\Big\}. \end{aligned} \tag{14}$$

Theorem 4 (Existence and uniqueness of an NE in Game 2): The SPA game defined by the set of players $\mathcal{K} = \{1,2\}$, the strategy sets $\mathcal{A}_k^{(\text{SPA})}$ and utilities $u_k(\alpha_k,\alpha_{-k})$ given by Eq. (3), has a unique NE.

Proof: The main feature of the game under the aforementioned power constraint is that there exists a unique NE in each sub-game defined by the realization of the coordination signal. The proof is much simpler than that of the time PA problem since the use of Rosen's Theorem [23] is not required. Without loss of generality assume that $S = 1$. Whatever the strategy of user 2, user 1 sees no interference. Therefore he can choose $\mathbf{Q}_1^{(1)}$ independently of user 2. Because $R_1^{(1)}(\mathbf{Q}_1^{(1)},\mathbf{Q}_2^{(1)})$ is a strictly concave function to be maximized over a convex set, there is a unique optimum strategy for user 1. As we assume a game with complete information and rational users, user 2 knows the utility of user 1 and thus the precoding matrix he will choose. The same concavity argument can be used for $R_2^{(1)}(\mathbf{Q}_1^{(1)},\mathbf{Q}_2^{(1)})$ and therefore guarantees that user 2 employs a unique precoding matrix. ■

Determination of the NE. In order to find the optimum covariance matrices, we proceed in the same way as described in [10]. First we focus on the optimum eigenvectors and then we determine the optimum eigenvalues by approximating the

utility functions under the large system assumption. In order to determine the optimum eigenvectors, the proof in [20] can be applied in our context to assert that there is no loss of optimality by restricting the search for the optimum covariance matrix when imposing the structure $\mathbf{Q}_k^{(s)} = \mathbf{U}_k \mathbf{P}_k^{(s)} \mathbf{U}_k^H$, where $\mathbf{U}_k$ is a unitary matrix coming from the spectral decomposition of transmit correlation matrix $\mathbf{T}_k = \mathbf{U}_k \mathbf{D}_k \mathbf{U}_k^H$ defined in Eq. (2) and the diagonal matrix $\mathbf{P}_k^{(s)} = \mathrm{Diag}(P_k^{(s)}(1), ..., P_k^{(s)}(n_t))$ represents the powers user k allocates to the different eigenvectors. As a consequence, we can exploit once again the results of [24], [25], [26] assuming the asymptotic regime in terms of the number of antennas. The new approximated rates are:

$$\begin{aligned}
\tilde{R}_1^{(1)}(\mathbf{P}_1^{(1)}) &= \sum_{i=1}^{n_t} \log_2 \left[1 + \eta P_1^{(1)}(i) d_1^{(\mathrm{T})}(i) \gamma_1\right] \\
&\quad + \sum_{j=1}^{n_r} \log_2 \left[1 + \eta d^{(\mathrm{R})}(j) \delta_1\right] \\
&\quad - n_t \eta \gamma_1 \delta_1 \log_2 e \\
\tilde{R}_2^{(1)}(\mathbf{P}_1^{(1)}, \mathbf{P}_2^{(1)}) &= \sum_{\ell=1}^{2} \sum_{i=1}^{n_t} \log_2 \left[1 + 2\eta P_\ell^{(1)}(i) d_\ell^{(\mathrm{T})}(i) \gamma_2\right] \\
&\quad + \sum_{j=1}^{n_r} \log_2 \left[1 + 2\eta d^{(\mathrm{R})}(j) \delta_2\right] \\
&\quad - 4 n_t \eta \gamma_2 \delta_2 \log_2 e - \tilde{R}_1^{(1)}(\mathbf{P}_1^{(1)})
\end{aligned} \tag{15}$$

where $\forall k \in \{1,2\}, d_k^{(\mathrm{T})}(i)$, $i \in \{1, ..., n_t\}$ are always the eigenvalues of the transmit correlation matrices $\mathbf{T}_k$, $d^{(\mathrm{R})}(j)$, $j \in \{1, ..., n_r\}$, are the eigenvalues of the receive correlation matrix $\mathbf{R}$ and the parameters γ_i, δ_j are the unique solutions of the following systems of equations:

$$\begin{cases} \gamma_1 = \dfrac{1}{n_t} \displaystyle\sum_{j=1}^{n_r} \dfrac{d^{(\mathrm{R})}(j)}{1 + \eta d^{(\mathrm{R})}(j) \delta_1} \\ \delta_1 = \dfrac{1}{n_t} \displaystyle\sum_{i=1}^{n_t} \dfrac{P_1^{(1)}(i) d_1^{(\mathrm{T})}(i)}{1 + \eta P_1^{(1)}(i) d_1^{(\mathrm{T})}(i) \gamma_1} \end{cases} \tag{16}$$

$$\begin{cases} \gamma_2 = \dfrac{1}{2n_t} \displaystyle\sum_{j=1}^{n_r} \dfrac{d^{(\mathrm{R})}(j)}{1 + 2\eta d^{(\mathrm{R})}(j) \delta_2} \\ \delta_2 = \dfrac{1}{2n_t} \displaystyle\sum_{\ell=1}^{2} \sum_{i=1}^{n_t} \dfrac{P_\ell^{(1)}(i) d_\ell^{(\mathrm{T})}(i)}{1 + 2\eta P_\ell^{(1)}(i) d_\ell^{(\mathrm{T})}(i) \gamma_2}. \end{cases} \tag{17}$$

Then, optimizing the approximated rates $\tilde{R}_k^{(1)}(\cdot)$ w.r.t. $P_k^{(1)}(i)$ leads to the following water-filling equations:

$$\forall k \in \{1,2\},\ P_k^{(1),\mathrm{NE}}(i) = \left[\frac{1}{\ln 2 \lambda_k^{(1)}} - \frac{1}{\eta d_k^{(\mathrm{T})}(i) \gamma_k}\right]^+ \tag{18}$$

where $\lambda_k^{(1)} \geq 0$, $k \in \{1,2\}$, are the Lagrangian multipliers tuned in order to meet the power constraints given in (6): $\sum_{i=1}^{n_t} P_k^{(1),\mathrm{NE}}(i) = n_t P_k$. We use the same iterative PA algorithm as the one described in [10]. Under the large systems assumption, in this game also, the approximated utilities have the same properties as the exact utilities.

Sum-rate efficiency of the NE. Unlike the temporal PA game, we have not assumed a particular structure for the precoding matrices and thus the centralized solution coincides with the sum-capacity of the virtual MIMO network, $R_{\mathrm{sum}}^{(\mathrm{C})} = C_{\mathrm{sum}}$. Another important point to notice here is that the equilibrium precoding matrices do not depend on p. This considerably simplifies the BS's choice for the sum-rate optimal value for p. Indeed, as we have already mentioned, the precoding matrices do no depend on p and therefore the sum-rate $R_{\mathrm{sum}}(p)$ is merely a linear function of p: $R_{\mathrm{sum}}^{\mathrm{NE}}(p) = ap + b$ where

$$\begin{aligned}
a &= \mathbb{E} \log |\mathbf{I} + \eta \mathbf{H}_1 \mathbf{Q}_1^{(1),\mathrm{NE}} \mathbf{H}_1^H + \eta \mathbf{H}_2 \mathbf{Q}_2^{(1),\mathrm{NE}} \mathbf{H}_2^H| \\
&\quad - \mathbb{E} \log |\mathbf{I} + \eta \mathbf{H}_1 \mathbf{Q}_1^{(2),\mathrm{NE}} \mathbf{H}_1^H + \eta \mathbf{H}_2 \mathbf{Q}_2^{(2),\mathrm{NE}} \mathbf{H}_2^H| \\
b &= \mathbb{E} \log |\mathbf{I} + \eta \mathbf{H}_1 \mathbf{Q}_1^{(2),\mathrm{NE}} \mathbf{H}_1^H + \eta \mathbf{H}_2 \mathbf{Q}_2^{(2),\mathrm{NE}} \mathbf{H}_2^H|.
\end{aligned} \tag{19}$$

Depending on the sign of a, if the BS wants to maximize the sum-rate, it will choose either $p = 0$ or $p = 1$. If it wants a fair game it will choose $p = \frac{1}{2}$ and accept a certain loss of global optimality. Note that even for $p \in \{0, 1\}$ the sum-capacity is not reached in general: this is because the matrix $\mathbf{Q}_1^{(1),\mathrm{NE}}$ (resp. $\mathbf{Q}_2^{(2),\mathrm{NE}}$) does not coincide with the first (resp. second) component of the pair of precoding matrices that maximizes the (strictly concave) network sum-rate. However, as we did for the temporal PA game, in the low and high SNR regimes one can show that the decentralized MIMO MAC has the same performance (w.r.t. the sum-rate) as its equivalent $Kn_t \times n_r$ virtual MIMO network.

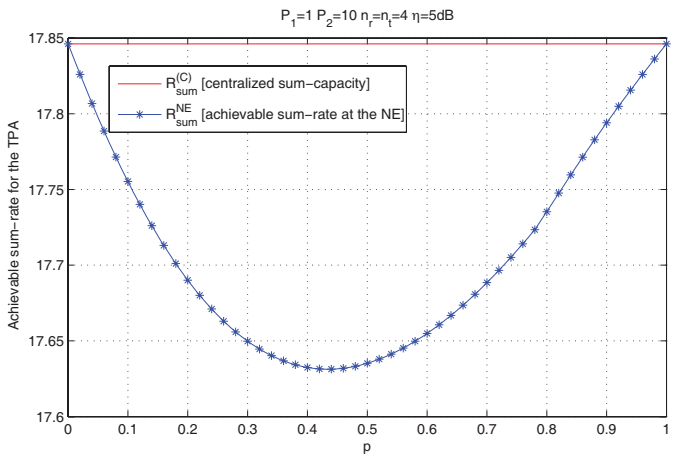

Fig. 1. Temporal PA game. Achievable network sum-rate versus p for $P_1 = 1$, $P_2 = 10$, $n_r = n_t = 4$, $\eta = 5$ dB. The sum-capacity of fading MIMO MACs is reached for both unfair SIC decoding schemes ($p_1^* = 0$ and $p_2^* = 1$) and is very close to this upper bound for any distribution of the coordination signal, $\forall p \in (0, 1)$.

V. Simulation Examples

All the results will be provided by assuming the asymptotic regime in the numbers of antennas. We know, from many contributions (see e.g., [10], [27], [29], [30]) that large-system approximates of ergodic rates are accurate even for relatively small systems. We also assume that $\mathbf{R} = \mathbf{I}$.

For the TPA problem, we look at the case where there is no transmit correlation, $\mathbf{T}_k = \mathbf{I}$. We have seen that the performance of decentralized MAC depends on the rule of the game i.e., the value of p. This is exactly what Fig. 1 depicts for the following scenario: $P_1 = 1$, $P_2 = 10$, $\eta = 5$ dB, $n_t = n_r = 4$. First, we see that the MAC sum-rate is a convex function of p and the maximum of $R_{\mathrm{sum}}^{\mathrm{NE}}(p)$ is reached for $p \in \{0, 1\}$. In these points, which correspond

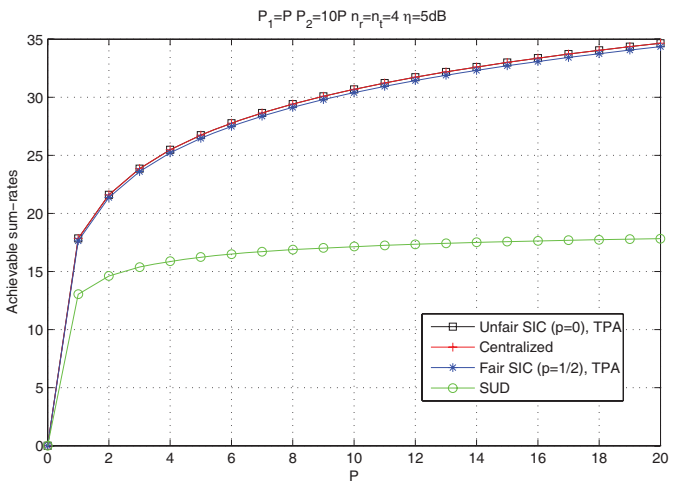

Fig. 2. Temporal PA game. MAC sum-rate versus the transmit power P for $P_1 = P$, $P_2 = 10P$, $n_r = n_t = 4$, $\eta = 5$ dB. Comparison between the fair SIC decoding scheme ($p = \frac{1}{2}$), the unfair SIC scheme ($p = 0$), and SUD decoding scheme.

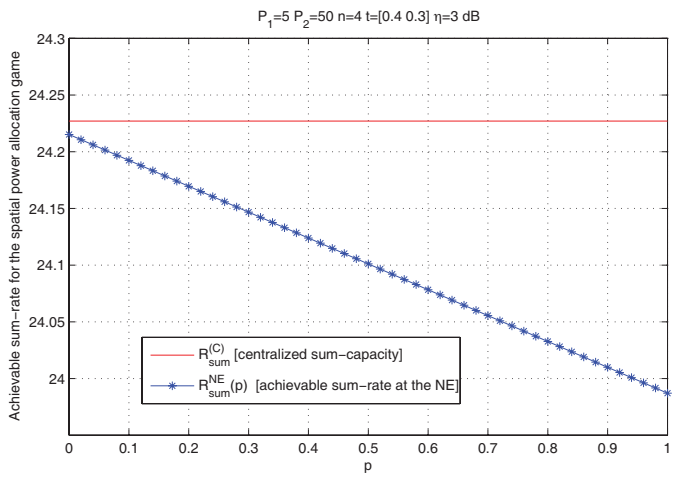

Fig. 3. Spatial PA game. MAC sum-rate versus p for $P_1 = 5$, $P_2 = 50$, $n_r = n_t = 4$, $\eta = 3$ dB, $t_1 = 0.4$, $t_2 = 0.3$. The achievable network sum-rate of fading MIMO MACs is linear w.r.t. $p \in [0, 1]$ and is very close to the centralized upper bound. The optimal distribution obtained with the Stackelberg game is $p^* = 0$.

to the most unfair decoding schemes (either user 1 or 2 is always decoded first) the centralized sum-capacity of the MAC is achieved. One important observation to be made is that the minimum and maximum only differ by about 1%. Many other simulations have confirmed this observation. This shows that whatever the value of p, the gap between the sum-rate of a decentralized MIMO MAC with selfish users and the sum-capacity of the equivalent cooperative MAC (virtual MIMO network) is in fact very small. Now, we want to evaluate the benefits brought by using a SIC instead of single-user decoding [10]. For the scenario where $P_1 = P$, $P_2 = 10P$ with $P \in [0, 20]$, $n_r = n_t = 4$ and $\eta = 5$ dB, Fig. 2 shows the achievable network sum-rate at the NE versus the available power at the first transmitter P. For the SUD scheme, the users are decoded simultaneously at the receiver. In this case both users see all the interference coming from the others. We see that the SIC scheme performs much better than the proposed SUD scheme, regardless of the distribution of the coordination signal: this comparison makes sense especially for the point $p = \frac{1}{2}$ since both decoding schemes are fair.

From now on, we consider the SPA problem. In this case we assume an exponential correlation profile for $\mathbf{T}_k$ such that $\mathbf{T}_k(i, j) = t_k^{|i-j|}$ (note that $\mathrm{Tr}(\mathbf{T}_k) = n_t$), where $0 \leq t_k \leq 1$ is the corresponding correlation coefficient [31], [32]. We already know that the sum-rate is a linear function of p and therefore is maximized when either $p = 0$ or $p = 1$. It turns out that this slope has a small value. Furthermore, it has been observed to be even 0 for a symmetric MAC, i.e., $P_1 = P_2$ and $t_1 = t_2$. These observations have been confirmed by many simulations. In Fig. 3 we have plotted the sum-rate achieved by varying p for the scenario: $P_1 = 5$, $P_2 = 50$, $\eta = 3$ dB, $n_t = n_r = 4$, $t_1 = 0.4$, $t_2 = 0.3$. Even in this scenario, which was thought to be a bad case in terms of sub-optimality, the sum-rate is not far from the sum-capacity of the centralized MAC. For the same scenario, we have plotted in Fig. 4 the achievable rate region and compared it to that obtained with SUD. We observe that in large MIMO MAC channels, the capacity region comprises a full cooperation segment (approximately) just like SISO MAC channels. The coordination signal allows one to move along an almost straight line, corresponding to a relatively large range of rates.

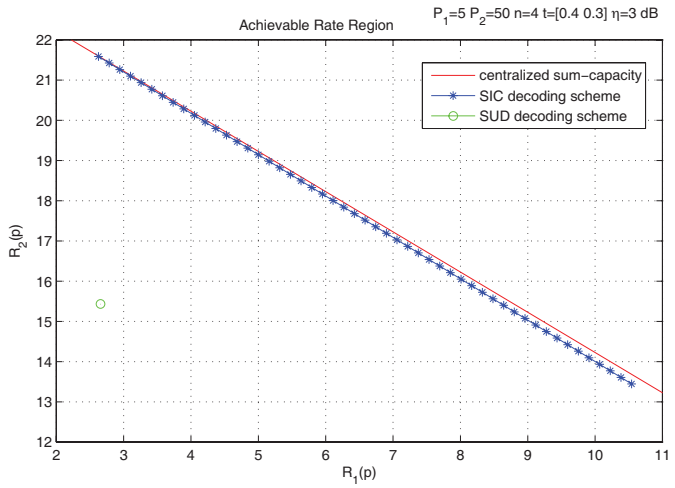

Fig. 4. Spatial PA game. Achievable rate region for $P_1 = 5$, $P_2 = 50$, $n_r = n_t = 4$, $\eta = 3$ dB, $t_1 = 0.4$, $t_2 = 0.3$. By varying p allows to move along a segment close to the centralized sum-capacity, similar to the SISO MAC channels.

VI. Conclusion

We have provided complete proofs for the existence and uniqueness of an NE in fast fading MIMO MACs with CSIR and CDIT where the transmission rate is chosen as user utility. By exploiting random matrix theory, we have also provided the corresponding optimum selfish PA policies. We have seen that the BS can, through a single parameter (i.e., $p \in [0, 1]$, which represents the distribution of the coordination signal), force the system to operate at many different points that correspond to a relatively large range of achievable transmission rate pairs. We know, from [1], [9] that for Gaussian MACs with single antenna terminals, this set of rate pairs corresponds to the full cooperation segment of the centralized MAC. Said otherwise a *decentralized* Gaussian SISO MAC with

coordination achieves the same rate pairs as a MAC with full cooperation or *virtual MIMO system*. The goal here was to know to what extent this key result is valid for fading MAC with multi-antenna terminals. It turns out this is almost true in the MIMO setting. In the cases of interest, where the power is optimally allocated either over space or time, the performance gap is relatively small even though the proposed coordination mechanism was a priori sub-optimal since it does take into account the channel realizations (known to the receiver). Interestingly in large MIMO MACs, the capacity region comprises a full cooperation segment just like SISO MACs. The coordination signal precisely allows one to move along the corresponding (almost) straight line. This shows the relevance of large systems in decentralized networks since they allow to determine the capacity region of certain systems whereas it is unknown in the finite setting. Furthermore, they induce an averaging effect, which makes the users' behavior predictable. Indeed, in large MIMO MACs the knowledge of the CSIT does not improve the performance w.r.t. the case with CDIT. To conclude we review some extensions of this work which we have suggested throughout it. It would be interesting to study the case of the decentralized space-time PA, which, in particular, would require the generalization of Lemma 1 to arbitrary positive matrices and exploitation of some results in [21]. A second useful extension would be to evaluate the impact of a non-perfect SIC on the PA problem. At last, we will mention that it would be useful to evaluate analytically or bounding the price of anarchy of the NE, which would require to find a bounding technique different from that used for non-atomic games [34], [35], [36].

Appendix A

We want to prove that the diagonally strict concavity condition is met for the time PA problem i.e., for all $(\alpha_1',\alpha_1'') \in \left(\mathcal{A}_1^{\mathrm{TPA}}\right)^2$ and $(\alpha_2',\alpha_2'') \in \left(\mathcal{A}_2^{\mathrm{TPA}}\right)^2$ such that either $\alpha_1' \neq \alpha_1''$ or $\alpha_2' \neq \alpha_2''$ we want to prove that:

$$\begin{aligned}\mathcal{C} &= (\alpha_1''-\alpha_1')\left[\tfrac{\partial R_1}{\partial \alpha_1}(\alpha_1',\alpha_2') - \tfrac{\partial R_1}{\partial \alpha_1}(\alpha_1'',\alpha_2'')\right] \\ &\quad +(\alpha_2''-\alpha_2')\left[\tfrac{\partial R_2}{\partial \alpha_2}(\alpha_1',\alpha_2') - \tfrac{\partial R_2}{\partial \alpha_2}(\alpha_1'',\alpha_2'')\right] > 0.\end{aligned} \tag{20}$$

We can write $\mathcal{C} = p\mathcal{T}^{(1)} + \overline{p}\mathcal{T}^{(2)}$ where for all $s \in \{1,2\}$:

$$\begin{aligned}\mathcal{T}^{(s)} &= (\alpha_1''-\alpha_1')\left[\tfrac{\partial R_1^{(s)}}{\partial \alpha_1}(\alpha_1',\alpha_2') - \tfrac{\partial R_1^{(s)}}{\partial \alpha_1}(\alpha_1'',\alpha_2'')\right] \\ &\quad +(\alpha_2''-\alpha_2')\left[\tfrac{\partial R_2^{(s)}}{\partial \alpha_2}(\alpha_1',\alpha_2') - \tfrac{\partial R_2^{(s)}}{\partial \alpha_2}(\alpha_1'',\alpha_2'')\right]\end{aligned} \tag{21}$$

By expanding $\mathcal{T}^{(1)}$ we have

$$\begin{aligned}&\mathcal{T}^{(1)} \\ &= (\alpha_1''-\alpha_1')\mathbb{E}\mathrm{Tr}\left\{[(\mathbf{I}+\rho_1\alpha_1'\mathbf{H}_1\mathbf{H}_1^H)^{-1} \right. \\ &\quad -(\mathbf{I}+\rho_1\alpha_1'\mathbf{H}_1\mathbf{H}_1^H)^{-1}]\rho_1\mathbf{H}_1\mathbf{H}_1^H\} \\ &\quad +(\alpha_2''-\alpha_2')\mathbb{E}\mathrm{Tr}\left\{(\mathbf{I}+\rho_1\alpha_1'\mathbf{H}_1\mathbf{H}_1^H \right. \\ &\quad +\tfrac{1-\overline{p}\alpha_2'}{p}\rho_2\mathbf{H}_2\mathbf{H}_2^H)^{-1}\rho_2\tfrac{-\overline{p}}{p}\mathbf{H}_2\mathbf{H}_2^H \\ &\quad \left. -(\mathbf{I}+\rho_1\alpha_1''\mathbf{H}_1\mathbf{H}_1^H+\tfrac{1-\overline{p}\alpha_2''}{p}\rho_2\mathbf{H}_2\mathbf{H}_2^H)^{-1}\rho_2\tfrac{-\overline{p}}{p}\mathbf{H}_2\mathbf{H}_2^H\right\} \\ &= \mathbb{E}\mathrm{Tr}\left\{(\mathbf{A}''-\mathbf{A}')[(\mathbf{I}+\mathbf{A}')^{-1}-(\mathbf{I}+\mathbf{A}'')^{-1}]\right. \\ &\quad \left.+(\mathbf{B}''-\mathbf{B}')[(\mathbf{I}+\mathbf{B}'+\mathbf{A}')^{-1}-(\mathbf{I}+\mathbf{B}''+\mathbf{A}'')^{-1}]\right\},\end{aligned} \tag{22}$$

where $\mathbf{A}' = \rho_1\alpha_1'\mathbf{H}_1\mathbf{H}_1^H$, $\mathbf{A}'' = \rho_1\alpha_1''\mathbf{H}_1\mathbf{H}_1^H$, $\mathbf{B}' = \rho_2\frac{1-\overline{p}\alpha_2'}{p}\mathbf{H}_2\mathbf{H}_2^H$, $\mathbf{B}'' = \rho_2\frac{1-\overline{p}\alpha_2''}{p}\mathbf{H}_2\mathbf{H}_2^H$. We observe that the matrices $\mathbf{A}'$, $\mathbf{A}''$, $\mathbf{B}'$ and $\mathbf{B}''$ verify the assumptions of Lemma 1. First, they are Hermitian and non-negative. Second, as they write as $\mathbf{A}' = a'\mathbf{H}_1\mathbf{H}_1^H$, $\mathbf{A}'' = a''\mathbf{H}_1\mathbf{H}_1^H$, $\mathbf{B}' = b'\mathbf{H}_2\mathbf{H}_2^H$ and $\mathbf{B}'' = b''\mathbf{H}_2\mathbf{H}_2^H$, we also see that the matrix order $\succeq$ is total for each of the pairs of matrices $(\mathbf{A}',\mathbf{A}'')$ and $(\mathbf{B}',\mathbf{B}'')$. This directly follows from the fact that the scalar order $\geq$ is total, which implies that either $a'' \geq a'$ or $a'' \leq a'$ and either $b'' \geq b'$ or $b'' \leq b'$. By considering the particular structure of the four matrices and applying Lemma 1, it is straightforward to see that the term $\mathcal{T}^{(1)}$ is strictly positive, $\mathcal{T}^{(1)} > 0$. In a similar way we can prove that $\mathcal{T}^{(2)} > 0$ and thus the diagonally strict concavity condition is met: $\mathcal{C} > 0$.

Appendix B

Proving Lemma 1 amounts to showing that

$$\begin{aligned}\mathcal{T} &= \mathrm{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{B}^{-1}-\mathbf{A}^{-1}) \right. \\ &\quad \left.+(\mathbf{C}-\mathbf{D})[(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}]\right\} \\ &> 0\end{aligned}$$

where the matrices $\mathbf{A} = \mathbf{I}+\mathbf{A}''$, $\mathbf{B} = \mathbf{I}+\mathbf{A}'$, $\mathbf{C} = \mathbf{B}''$ and $\mathbf{D} = \mathbf{B}'$ have been introduced for more clarity. Since the matrix order $\succeq$ is total for $\mathbf{A}$ and $\mathbf{B}$, and $\mathbf{C}$ and $\mathbf{D}$ it suffices to prove that $\mathcal{T} > 0$ for the four following cases: (1) $\mathbf{A} \succeq \mathbf{B}$ and $\mathbf{C} \succeq \mathbf{D}$; (2) $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{C} \preceq \mathbf{D}$; (3) $\mathbf{A} \succeq \mathbf{B}$ and $\mathbf{C} \preceq \mathbf{D}$; (4) $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{C} \succeq \mathbf{D}$.

Case (1): $\mathbf{A} \succeq \mathbf{B}$ and $\mathbf{C} \succeq \mathbf{D}$. To prove the desired result in this case we use the following lemma.

Lemma 2: If $\mathbf{M}$ is a Hermitian and non-negative ($\mathbf{M} = \mathbf{M}^H \succeq \mathbf{0}$) and $\mathbf{N}$ is non-negative ($\mathbf{N} \succeq 0$) but not necessarily Hermitian, then $\mathrm{Tr}(\mathbf{MN}) \geq 0$.

Proof: We write $\mathrm{Tr}(\mathbf{MN}) = \mathrm{Tr}(\mathbf{M}^{1/2}\mathbf{N}\mathbf{M}^{1/2}) \geq 0$. We have used the fact that $\mathbf{M}$ is a Hermitian non-negative matrix to write $\mathbf{M} = \mathbf{M}^{1/2}\mathbf{M}^{1/2}$. Knowing that $\mathbf{N}$ is a non-negative matrix one can easily check that $\mathbf{M}^{1/2}\mathbf{N}\mathbf{M}^{1/2}$ is also a non-negative matrix and thus the trace (sum of the non-negative eigenvalues) is non-negative. ■

The quantity $\mathcal{T}$ writes as $\mathcal{T} = \mathrm{Tr}(\mathbf{M}_1\mathbf{N}_1)+\mathrm{Tr}(\mathbf{M}_2\mathbf{N}_2)$ where $\mathbf{M}_1 = \mathbf{A}-\mathbf{B}$, $\mathbf{N}_1 = \mathbf{B}^{-1}-\mathbf{A}^{-1}$, $\mathbf{M}_2 = \mathbf{C}-\mathbf{D}$ and $\mathbf{N}_2 = (\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}$. Clearly these four matrices are Hermitian. Since by assumption $\mathbf{M}_1 \succeq 0$ and $\mathbf{M}_2 \succeq 0$ we only need to verify that $\mathbf{N}_1 \succeq 0$ and $\mathbf{N}_2 \succeq 0$ to be able to apply Lemma 2 to $\mathcal{T}$. The matrix $\mathbf{N}_1$ is non-negative because for any pair of invertible matrices $(\mathbf{X},\mathbf{Y})$: $\mathbf{X} \succeq \mathbf{Y} \Leftrightarrow \mathbf{Y}^{-1} \succeq \mathbf{X}^{-1}$ (see e.g., [33]). The same result applies to $\mathbf{N}_2$ since by assumption $\mathbf{A}+\mathbf{C} \succeq \mathbf{B}+\mathbf{D}$. Using lemma 2 concludes the proof.

Case (3): $\mathbf{A} \succeq \mathbf{B}$ and $\mathbf{C} \preceq \mathbf{D}$. To treat this case we first prove the following auxiliary Lemma.

Lemma 3: Let $\mathbf{X}$ and $\mathbf{Y}$ be two distinct, Hermitian and positive matrices of size n: $\mathbf{X} = \mathbf{X}^H \succ 0$, $\mathbf{Y} = \mathbf{Y}^H \succ 0$ and $\mathbf{X} \neq \mathbf{Y}$. Then $\mathrm{Tr}[(\mathbf{X}-\mathbf{Y})(\mathbf{Y}^{-1}-\mathbf{X}^{-1})] \geq 0$.

Proof: It is easy to see that $\mathrm{Tr}[(\mathbf{X}-\mathbf{Y})(\mathbf{Y}^{-1}-\mathbf{X}^{-1})] = \mathrm{Tr}[\mathbf{Z}+\mathbf{Z}^{-1}-2\mathbf{I}]$, with the Hermitian and positive matrix $\mathbf{Z} \triangleq \mathbf{X}^{\frac{1}{2}}\mathbf{Y}^{-1}\mathbf{X}^{\frac{1}{2}}$ and thus we further have $\mathrm{Tr}[(\mathbf{X}-\mathbf{Y})(\mathbf{Y}^{-1}-\mathbf{X}^{-1})] = \sum_{i=1}^{n}\frac{(\lambda_Z(i)-1)^2}{\lambda_Z(i)} \geq 0$ where the matrix

$\mathbf{\Lambda}_Z = \mathrm{Diag}(\lambda_Z(1), ..., \lambda_Z(n))$ corresponds to the spectral decomposition of . ■

By applying this lemma to $\mathcal{T}$ we have that:

$$\begin{aligned}\mathcal{T} &= \mathrm{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{B}^{-1}-\mathbf{A}^{-1})\right.\\ &\quad +[(\mathbf{C}+\mathbf{A})-(\mathbf{B}+\mathbf{D})][(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}]\\ &\quad \left.-(\mathbf{A}-\mathbf{B})[(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}]\right\}\\ &\geq \mathrm{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{B}^{-1}-\mathbf{A}^{-1})\right.\\ &\quad \left.-(\mathbf{A}-\mathbf{B})[(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{C})^{-1}]\right\}.\end{aligned} \tag{23}$$

We know that $\mathbf{C} \preceq \mathbf{D}$ then $\mathbf{C}+\mathbf{A} \preceq \mathbf{D}+\mathbf{A}$ and also that $(\mathbf{C}+\mathbf{A})^{-1} \succeq (\mathbf{D}+\mathbf{A})^{-1}$. Using the fact that $\mathbf{A} \succeq \mathbf{B}$ and also Lemma 2 we have that $\mathrm{Tr}[(\mathbf{A}-\mathbf{B})(\mathbf{C}+\mathbf{A})^{-1}] \geq \mathrm{Tr}[(\mathbf{A}-\mathbf{B})(\mathbf{D}+\mathbf{A})^{-1}]$ and the trace becomes lower bounded as $\mathcal{T} \geq \mathrm{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{B}^{-1}-\mathbf{A}^{-1})-(\mathbf{A}-\mathbf{B})[(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{D})^{-1}]\right\}$. Now, we are going to prove that this lower bound, say $\mathcal{T}_{LB}$, is positive:

$$\begin{aligned}&\mathcal{T}_{LB}\\ &= \mathrm{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{B}^{-1}-\mathbf{A}^{-1})\right.\\ &\quad \left.-[(\mathbf{A}+\mathbf{D})-(\mathbf{B}+\mathbf{D})]\left[(\mathbf{B}+\mathbf{D})^{-1}-(\mathbf{A}+\mathbf{D})^{-1}\right]\right\}\\ &= \mathrm{Tr}\left\{(\widetilde{\mathbf{A}}-\widetilde{\mathbf{B}})(\widetilde{\mathbf{B}}^{-1}-\widetilde{\mathbf{A}}^{-1})\right.\\ &\quad \left.-\left[(\widetilde{\mathbf{A}}+\mathbf{I})-(\widetilde{\mathbf{B}}+\mathbf{I})\right]\left[(\widetilde{\mathbf{B}}+\mathbf{I})^{-1}-(\widetilde{\mathbf{A}}+\mathbf{I})^{-1}\right]\right\}\end{aligned} \tag{24}$$

where we have made the following change of variables: $\mathbf{A} = \mathbf{D}^{1/2}\widetilde{\mathbf{A}}\mathbf{D}^{1/2}$, $\mathbf{B} = \mathbf{D}^{1/2}\widetilde{\mathbf{B}}\mathbf{D}^{1/2}$ such that $\widetilde{\mathbf{A}} = \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2} = \widetilde{\mathbf{A}}^H \succ 0$ and $\widetilde{\mathbf{B}} = \mathbf{D}^{-1/2}\mathbf{B}\mathbf{D}^{-1/2} = \widetilde{\mathbf{B}}^H \succ 0$. By applying the Woodbury formula $(\widetilde{\mathbf{A}}+\mathbf{I})^{-1} = \widetilde{\mathbf{A}}^{-1}-\widetilde{\mathbf{A}}^{-1}(\widetilde{\mathbf{A}}+\mathbf{I})^{-1}$ and $(\widetilde{\mathbf{B}}+\mathbf{I})^{-1} = \widetilde{\mathbf{B}}^{-1}-\widetilde{\mathbf{B}}^{-1}(\widetilde{\mathbf{B}}+\mathbf{I})^{-1}$, the lower bound $\mathcal{T}_{LB}^{(1)}$ rewrites as:

$$\begin{aligned}\mathcal{T}_{LB} &= \mathrm{Tr}\left\{(\widetilde{\mathbf{A}}-\widetilde{\mathbf{B}})\left[\widetilde{\mathbf{B}}^{-1}-\widetilde{\mathbf{A}}^{-1}-\widetilde{\mathbf{B}}^{-1}+\widetilde{\mathbf{B}}^{-1}(\widetilde{\mathbf{B}}+\mathbf{I})^{-1}\right.\right.\\ &\quad \left.\left.+\widetilde{\mathbf{A}}^{-1}-\widetilde{\mathbf{A}}^{-1}(\widetilde{\mathbf{A}}+\mathbf{I})^{-1}\right]\right\}\\ &= \mathrm{Tr}\left\{\widetilde{\mathbf{A}}\widetilde{\mathbf{B}}^{-1}(\widetilde{\mathbf{B}}+\mathbf{I})^{-1}+\widetilde{\mathbf{B}}\widetilde{\mathbf{A}}^{-1}(\widetilde{\mathbf{A}}+\mathbf{I})^{-1}\right.\\ &\quad \left.-(\widetilde{\mathbf{A}}+\mathbf{I})^{-1}-(\widetilde{\mathbf{B}}+\mathbf{I})^{-1}\right\}.\end{aligned} \tag{25}$$

Let us denote the ordered eigenvalues of the two matrices $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ as $\lambda_{\widetilde{A}}(1) \leq \lambda_{\widetilde{A}}(2) \leq \ldots \leq \lambda_{\widetilde{A}}(n)$ and $\lambda_{\widetilde{B}}(1) \leq \lambda_{\widetilde{B}}(2) \leq \ldots \leq \lambda_{\widetilde{B}}(n)$. From [37] we know that for two matrices $\mathbf{X}$ and $\mathbf{Y}$ of size n, $\mathrm{Tr}(\mathbf{XY}) \geq \sum_{i=1}^n \lambda_X(i)\lambda_Y(n-i+1)$, which implies directly that $\mathrm{Tr}(\mathbf{XY}^{-1}) \geq \sum_{i=1}^n \frac{\lambda_X(i)}{\lambda_Y(i)}$, where $\lambda_X(i)$ and $\lambda_Y(i)$ are the ordered eigenvalues (in the previously specified order) of the corresponding matrices. Applying this result we find that

$$\begin{cases}\mathrm{Tr}\left[\widetilde{\mathbf{A}}\widetilde{\mathbf{B}}^{-1}(\mathbf{I}+\widetilde{\mathbf{B}})^{-1}\right] \geq \displaystyle\sum_{i=1}^n \frac{\lambda_{\widetilde{A}}(i)}{\lambda_{\widetilde{B}}(i)(1+\lambda_{\widetilde{B}}(i))},\\ \mathrm{Tr}\left[\widetilde{\mathbf{B}}\widetilde{\mathbf{A}}^{-1}(\mathbf{I}+\widetilde{\mathbf{A}})^{-1}\right] \geq \displaystyle\sum_{i=1}^n \frac{\lambda_{\widetilde{B}}(i)}{\lambda_{\widetilde{A}}(i)(1+\lambda_{\widetilde{A}}(i))},\end{cases} \tag{26}$$

and finally obtain that:

$$\mathcal{T}_{LB} \geq \sum_{i=1}^n \frac{[\lambda_{\widetilde{A}}(i)-\lambda_{\widetilde{B}}(i)]^2[1+\lambda_{\widetilde{A}}(i)+\lambda_{\widetilde{B}}(i)]}{\lambda_{\widetilde{A}}(i)\lambda_{\widetilde{B}}(i)[1+\lambda_{\widetilde{A}}(i)][1+\lambda_{\widetilde{B}}(i)]} \geq 0. \tag{27}$$

To conclude the global proof one can easily check that Case (2) (resp. Case (4)) can be readily proved from the proof of Case (1) (resp. Case (3)) by interchanging the role of $\mathbf{A}$ and $\mathbf{B}$ and $\mathbf{C}$ and $\mathbf{D}$.

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Power allocation games in wireless networks of multi-antenna terminals

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Abstract We consider wireless networks that can be modeled by multiple access channels in which all the terminals are equipped with multiple antennas. The propagation model used to account for the effects of transmit and receive antenna correlations is the unitary-invariant-unitary model, which is one of the most general models available in the literature. In this context, we introduce and analyze two resource allocation games. In both games, the mobile stations selfishly choose their power allocation policies in order to maximize their individual uplink transmission rates; in particular they can ignore some specified centralized policies. In the first game considered, the base station implements successive interference cancellation (SIC) and each mobile station chooses his best space-time power allocation scheme; here, a coordination mechanism is used to indicate to the users the order in which the receiver applies SIC. In the second framework, the base station is assumed to implement single-user decoding. For these two games a thorough analysis of the Nash equilibrium is provided: the existence and uniqueness issues are addressed; the corresponding power allocation policies are determined by exploiting random matrix theory; the sum-rate efficiency of the equilibrium is studied analytically in the low and high signal-to-noise ratio regimes and by simulations in more typical scenarios. Simulations show that, in particular, the sum-rate efficiency is high for the type of systems investigated and the performance loss due to the use of the proposed suboptimum coordination mechanism is very small.

Keywords MIMO · MAC · Non-cooperative games · Nash equilibrium · Power allocation · Price of anarchy · Random matrix theory

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1 Introduction

In this paper, we consider the uplink of a decentralized network of several mobile stations (MS) and one base station (BS). This type of network is commonly referred to as the decentralized multiple access channel (MAC). The network is said to be decentralized in the sense that each user can freely choose his power allocation (PA) policy in order to selfishly maximize a certain individual performance criterion, which is called utility or payoff. This means that, even if the BS broadcasts some specified policies, every user is free to ignore the policy intended for him if the latter does not maximize his performance criterion.

To the best of the authors' knowledge, the problem of decentralized PA in wireless networks has been properly formalized for the first time in [1, 2]. Interestingly, this problem can be formulated quite naturally as a non-cooperative

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game with different performance criteria (utilities) such as the carrier-to-interference ratio [3], aggregate throughput [4] or energy efficiency [5, 6]. In this paper, we assume that the users want to maximize information-theoretic utilities and more precisely their Shannon transmission rates. Indeed, the point of view adopted here is close to the one proposed by the authors of [7] for DSL (digital subscriber lines) systems, which are modeled as a parallel interference channel; [8] for the single input single output (SISO) and single input multiple output (SIMO) fast fading MACs with global CSIR and global CSIT (Channel State Information at the Receiver/Transmitters); [9] for MIMO (Multiple Input Multiple Output) MACs with global CSIR, channel distribution information at the transmitters (global CDIT) and single-user decoding (SUD) at the receivers; [10, 11] for Gaussian MIMO interference channels with global CSIR and local CSIT and, by definition of the conventional interference channel [12], SUD at the receivers. Note that reference [13] where the authors considered Gaussian MIMO MACs with neither CSIT nor CDIT differs from our approach and that of [7–11] because in [13] the MIMO MAC is seen as a two-player zero-sum game where the first player is the group of transmitters and the second player is the set of MIMO sub-channels. The closest works to the work presented here are [9] and [14]. Although this paper is in part based on these works, it still provides significant contributions w.r.t. to them, as explained below.

In [9], the authors consider MIMO multiple access channels and assume SUD at the BS; the authors formulate the PA problem into a team game in which each user chooses his PA to maximize the network sum-rate. In [14], the same type of decentralized networks is considered but SIC is assumed at the BS. As each user needs to know his decoding rank in order to adapt his PA policy to maximize his individual transmission rate, a coordination mechanism has to be introduced: the coordination signal precisely indicates to all the users the decoding order used by the receiver. The present paper differs from these two contributions on at least four important technical points: (i) when SUD is assumed, the PA game is not formulated as a team game but as a non-cooperative one; (ii) we exploit several proof techniques that are different from [9]; (iii) while [9] and [14] assume a Kronecker propagation model with *common* receive correlation we assume here a more general model, the unitary-invariant-unitary (UIU) propagation model introduced by [21], for which the users can have different receive antenna correlation profiles. This is useful in practice since, for instance, it allows one to study propagation scenarios where some users can be in line of sight with the BS (the receive antenna are strongly correlated) whereas other users can be surrounded by many obstacles, which can strongly decorrelate the receive antennas for these users; (iv) while the authors of [14] restricted their attention to either a purely spatial PA problem or a purely temporal PA problem, we tackle here the general space-time PA problem.

In this context, our main objective is to study the equilibrium of two power allocation games associated with the two types of decoding schemes aforementioned (namely SIC and SUD). The motivation for this is that the existence of an equilibrium allows network designers to predict, with a certain degree of stability, the effective operating state(s) of the network. Clearly, in our context, uniqueness is a desirable feature of the equilibrium. As it will be seen, it is possible to prove the existence in both games under investigation. Uniqueness is proven in the case of SUD while it is conjectured for the case of SIC. In order to establish the corresponding results, the paper is structured as follows. After presenting the general system model in Sect. 2, we analyze in detail the space-time PA game when SIC and a corresponding coordination mechanism are assumed (Sect. 3). For this game, the existence and uniqueness of the NE are proven and the equilibrium is determined by exploiting random matrix theory when the numbers of antennas are sufficiently large. Its sum-rate efficiency is also analyzed. In Sect. 4, we analyze the case of SUD since this decoding scheme, although suboptimal in terms of performance (even in the case of a network with single-antenna terminals), has some features that can be found desirable in some contexts: the receiver complexity is low, there is no need for a coordination signal, there is no propagation error since the data flows are decoded in parallel and not successively and also it is intrinsically fair. To analyze the case of the SUD-based PA game, we will follow the same steps as in Sect. 3 and we will see that, the equilibrium analysis can be deduced, to a large extent, from the SIC case. Numerical results are provided in Sect. 5 to illustrate our theoretical analysis and to better assess the sum-rate efficiency of the considered games. Section 6 corresponds to the conclusion.

2 System model

We assume a MAC with arbitrary number of users, $K \geq 2$. Regarding the original definition of the MAC by [15] and [16], the system under consideration has two common features: all transmitters send at once and at different rates over the entire bandwidth, and the transmitters are using good codes in the sense of the Shannon rate. Our system differs from [15, 16] in the sense that multiple antennas are considered at the terminal nodes, channels vary over time and the BS does not dictate the PA policies to the MSs. Also, we assume the existence of coordination signal which is perfectly known to all the terminals. If the coordination signal is generated by the BS itself, this induces a certain cost in terms of downlink signaling but the distribution of the coordination signal can then be optimized. On the other hand, if the coordination signal comes from an external source, e.g., an FM

transmitter, the MSs can acquire their coordination signal for free in terms of downlink signaling. However this generally involves a certain sub-optimality in terms of uplink rate. In both cases, the coordination signal will be represented by a random variable denoted by $S \in \mathcal{S}$. Since we study the K-user MAC, $\mathcal{S} = \{0, 1, \ldots, K!\}$ is a $K!+1$-element alphabet. When the realization is in $\{1, \ldots, K!\}$, the BS applies SIC with a certain decoding order (game 1). When $S = 0$ the BS always applies SUD (game 2), where all users are decoded simultaneously (no interference cancellation). In a real wireless system the frequency at which the realizations would be drawn would be roughly proportional to the reciprocal of the channel coherence time (i.e., $1/T_{\text{coh}}$). Note that the proposed coordination mechanism is suboptimal because it does not depend on the realizations of the channel matrices. We will see that the corresponding performance loss is in fact very small.

We will further consider that each mobile station is equipped with n_t antennas whereas the base station has n_r antennas (thus we assume the same number of transmitting antennas for all the users). In our analysis, the flat fading channel matrices of the different links vary from symbol vector (or space-time codeword) to symbol vector. We assume that the receiver knows all the channel matrices (CSIR) whereas each transmitter has only access to the statistics of the different channels (CDIT). The equivalent baseband signal received by the base station can be written as:

$$\underline{Y}^{(s)}(\tau) = \sum_{k=1}^{K} \boldsymbol{H}_k(\tau)\underline{X}_k^{(s)}(\tau) + \underline{Z}^{(s)}(\tau), \tag{1}$$

where $\underline{X}_k^{(s)}(\tau)$ is the n_t-dimensional column vector of symbols transmitted by user k at time τ for the realization $s \in \mathcal{S}$ of the coordination signal, $\mathbf{H}_k(\tau) \in \mathbb{C}^{n_r \times n_t}$ is the channel matrix (stationary and ergodic process) of user k and $\underline{Z}^{(s)}(\tau)$ is a n_r-dimensional complex white Gaussian noise distributed as $\mathcal{N}(\underline{0}, \sigma^2 \mathbf{I}_{n_r})$. For the sake of clarity we will omit the time index τ from our notations.

In order to take into account the antenna correlation effects at the transmitters and receiver, we will assume the different channel matrices to be structured according to the unitary-independent-unitary model introduced in [21]:

$$\forall k \in \{1, \ldots, K\}, \quad \mathbf{H}_k = \mathbf{V}_k \tilde{\mathbf{H}}_k \mathbf{W}_k, \tag{2}$$

where $\mathbf{V}_k$ and $\mathbf{W}_k$ are deterministic unitary matrices that allow one to take into consideration the correlation effects at the receiver and transmitter. Also $\tilde{\mathbf{H}}_k$ is an $n_r \times n_t$ matrix whose entries are zero-mean independent complex Gaussian random variables with an arbitrary profile of variances, such that $\mathbb{E}|\tilde{H}_k(i,j)|^2 = \frac{\sigma_k(i,j)}{n_t}$. The Kronecker propagation model for which the channel transfer matrices factorizes as $\mathbf{H}_k = \mathbf{R}_k^{1/2} \tilde{\mathbf{\Theta}}_k \mathbf{T}_k^{1/2}$ is a special case of the UIU model where the profile of variances is separable i.e., $\mathbb{E}|\tilde{H}_k(i,j)|^2 = \frac{d_k^{(\mathrm{R})}(i) d_k^{(\mathrm{T})}(j)}{n_t}$, with for each k: $\mathbf{\Theta}_k$ is a random matrix with zero-mean i.i.d. entries, $\mathbf{T}_k$ is the transmit antenna correlation matrix, $\mathbf{R}_k$ is the receive antenna correlation matrix, $\{d_k^{(\mathrm{T})}(j)\}_{j \in \{1,\ldots,n_t\}}$ and $\{d_k^{(\mathrm{R})}(i)\}_{i \in \{1,\ldots,n_r\}}$ are their associated eigenvalues. In this paper we will consider that $\mathbf{V}_k = \mathbf{V}$ for all users. The reason for assuming this will be made clearer a little further. In spite of this simplification, we will still be able to deal with some useful scenarios where the users see different propagation conditions in terms of receive antenna correlation.

3 Successive interference cancellation

When SIC is assumed at the BS, the strategy of user $k \in \{1, 2, \ldots, K\}$, consists in choosing the best vector of precoding matrices $\mathbf{Q}_k = (\mathbf{Q}_k^{(1)}, \mathbf{Q}_k^{(2)}, \ldots, \mathbf{Q}_k^{(K!)})$ where $\mathbf{Q}_k^{(s)} = \mathbb{E}[\underline{X}_k^{(s)} \underline{X}_k^{(s),H}]$, for $s \in \mathcal{S}$, in the sense of his utility function. For clarity sake, we will introduce another notation which will be used in the remaining of this section to replace the realization s of the coordination signal. We denote by $\mathcal{P}_K$ the set of all possible permutations of K elements, such that $\pi \in \mathcal{P}_k$ denotes a certain decoding order for the K users and $\pi(k)$ denotes the rank of user $k \in \mathcal{K}$ and $\pi^{-1} \in \mathcal{P}_K$ denotes the inverse permutation (i.e. $\pi^{-1}(\pi(k)) = k$) such that $\pi^{-1}(r)$ denotes the index of the user that is decoded with rank $r \in \mathcal{K}$. We denote by $p_\pi \in [0, 1]$ the probability that the receiver implements the decoding order $\pi \in \mathcal{P}_K$, which means that $\sum_{\pi \in \mathcal{P}_K} p_\pi = 1$. At last note that there is a one-to-one mapping between the set of realizations of the coordination signal $\mathcal{S}$ and the set of permutations $\mathcal{P}_K$, i.e. $\xi : \mathcal{S} \to \mathcal{P}_k$ such that $\xi(\cdot)$ is a bijective function. This is the reason why the index s can be replaced with the index π without introducing any ambiguity or loss of generality. The vector of precoding matrices can be denoted by $\mathbf{Q} = (\mathbf{Q}_k^{(\pi)})_{\pi \in \mathcal{P}_K}$ and the utility function can be written as:

$$u_k^{\mathrm{SIC}}(\mathbf{Q}_k, \mathbf{Q}_{-k}) = \sum_{\pi \in \mathcal{P}_K} p_\pi R_k^{(\pi)}(\mathbf{Q}_k^{(\pi)}, \mathbf{Q}_{-k}^{(\pi)}), \tag{3}$$

where

$$\begin{aligned} &R_k^{(\pi)}(\mathbf{Q}_k^{(\pi)}, \mathbf{Q}_{-k}^{(\pi)}) \\ &\quad = \mathbb{E}\log_2 \left| \mathbf{I} + \rho \mathbf{H}_k \mathbf{Q}_k^{(\pi)} \mathbf{H}_k^H + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \mathbf{H}_\ell \mathbf{Q}_\ell^{(\pi)} \mathbf{H}_\ell^H \right| \\ &\quad\quad - \mathbb{E}\log_2 \left| \mathbf{I} + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \mathbf{H}_\ell \mathbf{Q}_\ell^{(\pi)} \mathbf{H}_\ell^H \right| \end{aligned} \tag{4}$$

with $\rho = \frac{1}{\sigma^2}$ and $\mathcal{K}_k^{(\pi)} = \{\ell \in \mathcal{K} | \pi(\ell) \geq \pi(k)\}$ represents, for a given decoding order π, the subset of users that will be decoded after user k. Also, we use the standard notation $-k$, which stands for the other players than k. An important point to mention here is the power constraint under which the utilities are maximized. Indeed for user $k \in \{1, \ldots, K\}$, the strategy set is defined as follows:

$$\mathcal{A}_k^{\mathrm{SIC}} = \Bigg\{ \mathbf{Q}_k = (\mathbf{Q}_k^{(\pi)})_{\pi \in \mathcal{P}_K} \mid \forall \pi \in \mathcal{P}_K, \mathbf{Q}_k^{(\pi)} \succeq 0, \sum_{\pi \in \mathcal{P}_K} p_\pi \mathrm{Tr}(\mathbf{Q}_k^{(\pi)}) \leq n_t \overline{P}_k \Bigg\}. \tag{5}$$

In order to tackle the existence and uniqueness issues for Nash equilibria in the general space-time PA game, we exploit and extend the results from Rosen [17], which we will briefly state here below in order to make this paper sufficiently self-contained.

Theorem 1 [17] *Let $\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k \in \mathcal{K}}, \{u_k\}_{k \in \mathcal{K}})$ be a game where $\mathcal{K} = \{1, \ldots, K\}$ is the set of players, $\mathcal{A}_1, \ldots, \mathcal{A}_K$ the corresponding sets of strategies and $u_1, \ldots, u_k$ the utilities of the different players. If the following three conditions are satisfied*: (i) *each u_k is continuous in the all the strategies $\underline{a}_j \in \mathcal{A}_j, \forall j \in \mathcal{K}$*; (ii) *each u_k is concave in $\underline{a}_k \in \mathcal{A}_k$*; (iii) *$\mathcal{A}_1, \ldots, \mathcal{A}_K$ are compact and convex sets*; *then $\mathcal{G}$ has at least one NE.*

Theorem 2 [17] *Consider the K-player concave game of Theorem 1. If the following (diagonally strict concavity) condition is met: for all $k \in \mathcal{K}$ and for all $(\underline{a}'_k, \underline{a}''_k) \in \mathcal{A}_k^2$ such that there exists at least one index $j \in \mathcal{K}$ for which $\underline{a}'_j \neq \underline{a}''_j$, $\sum_{k=1}^K (\underline{a}''_k - \underline{a}'_k)^T [\nabla_{\underline{a}_k} u_k(\underline{a}'_k, \underline{a}'_{-k}) - \nabla_{\underline{a}_k} u_k(\underline{a}''_k, \underline{a}''_{-k})] > 0$; then the uniqueness of the NE is insured.*

In the space-time power allocation game under investigation, the obtained results are stated in the following theorem.

Theorem 3 (Existence of an NE) *The joint space-time power allocation game described by: the set of players $k \in \{1, 2\}$; the sets of actions $\mathcal{A}_k^{\mathrm{SIC}}$ and the utility functions $u_k^{\mathrm{SIC}}(\mathbf{Q}_k, \mathbf{Q}_{-k})$ given in (3), has a Nash equilibrium.*

Proof It is quite easy to prove that the strategy sets $\mathcal{A}_k^{\mathrm{SIC}}$ are convex and compact sets and that the utility functions $u_k^{\mathrm{SIC}}(\mathbf{Q}_k, \mathbf{Q}_{-k})$ are concave w.r.t. $\mathbf{Q}_k$ and continuous w.r.t. to $(\mathbf{Q}_k, \mathbf{Q}_{-k})$ and by Theorem 1 at least one Nash equilibrium exists. For more details, the reader is referred to Appendix A. □

Theorem 4 (Sufficient condition for uniqueness) *If the following condition is met*

$$\sum_{\pi \in \mathcal{P}_K} \sum_{k=1}^K \mathrm{Tr}\{(\mathbf{Q}_k^{(\pi)''} - \mathbf{Q}_k^{(\pi)'})(\nabla_{Q_k^{(\pi)}} u_k^{\mathrm{SIC}}(\mathbf{Q}'_k, \mathbf{Q}'_{-k}) - \nabla_{Q_k^{(\pi)}} u_k^{\mathrm{SIC}}(\mathbf{Q}''_k, \mathbf{Q}''_{-k}))\} > 0 \tag{6}$$

for all $\mathbf{Q}'_k = (\mathbf{Q}_k^{(\pi)'})_{\pi \in \mathcal{P}_K}, \mathbf{Q}''_k = (\mathbf{Q}_k^{(\pi)''})_{\pi \in \mathcal{P}_K} \in \mathcal{A}_k^{\mathrm{SIC}}$ such that $(\mathbf{Q}'_1, \ldots, \mathbf{Q}'_K) \neq (\mathbf{Q}''_1, \ldots, \mathbf{Q}''_K)$, then the Nash equilibrium in the power allocation game of Theorem 3 is unique.

This theorem corresponds to the matrix generalization of the diagonally strict concavity (DSC) condition of [17] and is proven in Appendix B. To know whether this condition is verified or not in the MIMO MAC one needs to re-write it in a more exploitable manner. It can be checked that $\mathcal{C}$ expresses as $\mathcal{C} = \sum_{\pi \in \mathcal{P}_K} p_\pi \mathcal{T}_\pi$ where for each $\pi \in \mathcal{P}_K$, $\mathcal{T}_\pi$ is given by:

$$\begin{aligned}
\mathcal{T}_\pi &= \sum_{k=1}^K \mathrm{Tr}\{(\mathbf{Q}_k^{(\pi)''} - \mathbf{Q}_k^{(\pi)'})[\nabla_{Q_k^{(\pi)}} R_k^{(\pi)}(\mathbf{Q}_k^{(\pi)'}, \mathbf{Q}_{-k}^{(\pi)'}) \\
&\quad - \nabla_{Q_k^{(\pi)}} R_k^{(\pi)}(\mathbf{Q}_k^{(\pi)''}, \mathbf{Q}_{-k}^{(\pi)''})]\} \\
&= \mathbb{E} \sum_{r=1}^K \mathrm{Tr} \Bigg\{ \rho \mathbf{H}_{\pi^{-1}(r)} (\mathbf{Q}_{\pi^{-1}(r)}^{(\pi)''} - \mathbf{Q}_{\pi^{-1}(r)}^{(\pi)'}) \mathbf{H}_{\pi^{-1}(r)}^H \\
&\quad \times \Bigg[\Bigg(\mathbf{I} + \rho \mathbf{H}_{\pi^{-1}(r)} \mathbf{Q}_{\pi^{-1}(r)}^{(\pi)'} \mathbf{H}_{\pi^{-1}(r)}^H \\
&\quad + \rho \sum_{s=r+1}^K \mathbf{H}_{\pi^{-1}(s)} \mathbf{Q}_{\pi^{-1}(s)}^{(\pi)'} \mathbf{H}_{\pi^{-1}(s)}^H \Bigg)^{-1} \\
&\quad - \Bigg(\mathbf{I} + \rho \mathbf{H}_{\pi^{-1}(r)} \mathbf{Q}_{\pi^{-1}(r)}^{(\pi)''} \mathbf{H}_{\pi^{-1}(r)}^H \\
&\quad + \rho \sum_{s=r+1}^K \mathbf{H}_{\pi^{-1}(s)} \mathbf{Q}_{\pi^{-1}(s)}^{(\pi)''} \mathbf{H}_{\pi^{-1}(s)}^H \Bigg)^{-1} \Bigg] \Bigg\} \\
&= \mathbb{E} \sum_{r=1}^K \mathrm{Tr}(\mathbf{A}_r^{(\pi)''} - \mathbf{A}_r^{(\pi)'}) \Bigg[\Bigg(\mathbf{I} + \sum_{s=r}^K \mathbf{A}_r^{(\pi)'} \Bigg)^{-1} \\
&\quad - \Bigg(\mathbf{I} + \sum_{s=r}^K \mathbf{A}_r^{(\pi)''} \Bigg)^{-1} \Bigg],
\end{aligned} \tag{7}$$

where $\mathbf{A}_r^{(\pi)'} = \rho \mathbf{H}_{\pi^{-1}(r)} \mathbf{Q}_{\pi^{-1}(r)}^{(\pi)'} \mathbf{H}_{\pi^{-1}(r)}^H$, $\mathbf{A}_r^{(\pi)''} = \rho \mathbf{H}_{\pi^{-1}(r)} \times \mathbf{Q}_{\pi^{-1}(r)}^{(\pi)''} \mathbf{H}_{\pi^{-1}(r)}^H$ and the users have been ordered using their decoding rank rather than their index.

Theorem 5 (A sufficient condition for DSC) *If for any positive definite matrices* $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{A}_i \neq \mathbf{B}_i$, $i \in \{1, \ldots, K\}$ *we have that*

$$\sum_{i=1}^{K} \mathrm{Tr}\left\{ (\mathbf{A}_i - \mathbf{B}_i)\left[\left(\sum_{j=1}^{i} \mathbf{B}_j\right)^{-1} - \left(\sum_{j=1}^{i} \mathbf{A}_j\right)^{-1} \right] \right\} > 0, \tag{8}$$

then the DSC condition is met: $\mathcal{C} > 0$.

It turns out that the trace inequality (9) always holds for any K et for any positive matrices.

Lemma 1 Trace inequality *For any positive definite matrices* $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{A}_i \neq \mathbf{B}_i$, $i \in \{1, \ldots, K\}$ *we have that*

$$\sum_{i=1}^{K} \mathrm{Tr}\left\{ (\mathbf{A}_i - \mathbf{B}_i)\left[\left(\sum_{j=1}^{i} \mathbf{B}_j\right)^{-1} - \left(\sum_{j=1}^{i} \mathbf{A}_j\right)^{-1} \right] \right\} > 0. \tag{9}$$

The proof can be found in [27], for $K = 2$, and in [28] for arbitrary $K \geq 2$.

Determination of the Nash equilibrium. In order to find the optimal covariance matrices, we proceed in the same way as described in [9]. First we will focus on the optimal eigenvectors and then we will determine the optimal eigenvalues by approximating the utility functions under the large system assumption.

Theorem 6 (Optimal eigenvectors) *For all* $k \in \mathcal{K}$, $\mathbf{Q}_k \in \mathcal{A}_k^{\mathrm{SIC}}$ *there is no loss of optimality by imposing the structure* $\mathbf{Q}_k = (\mathbf{Q}_k{}^{(\pi)})_{\pi \in \mathcal{P}_K}$, $\mathbf{Q}_k^{(\pi)} = \mathbf{W}_k \mathbf{P}_k{}^{(\pi)} \mathbf{W}_k{}^H$, *in the sense that*:

$$\max_{\mathbf{Q}_k \in \mathcal{A}_k^{\mathrm{SIC}}} u_k^{\mathrm{SIC}}(\mathbf{Q}_k, \mathbf{Q}_{-k}) = \max_{\mathbf{Q}_k \in \mathcal{S}_k^{\mathrm{SIC}}} u_k^{\mathrm{SIC}}(\mathbf{Q}_k, \mathbf{Q}_{-k}),$$

where $\mathcal{S}_k^{\mathrm{SIC}} = \{\mathbf{Q}_k = (\mathbf{Q}_k{}^{(\pi)})_{\pi \in \mathcal{P}_k} \in \mathcal{A}_k^{\mathrm{SIC}} | \mathbf{Q}_k^{(\pi)} = \mathbf{W}_k \mathbf{P}_k^{(\pi)} \mathbf{W}_k^H\}$, $s \in \mathcal{S}$, *model from* (2) *and* $\mathbf{P}_k^{(s)} = \mathrm{Diag}(P_k^{(\pi)}(1), \ldots, P_k^{(\pi)}(n_t))$.

The detailed proof of this result is given in Appendix C. This result, although easy to obtain, it is instrumental in our context for two reasons. First, the search of the optimum precoding matrices boils down to the search of the eigenvalues of these matrices. Second, as the optimum eigenvectors are known, available results in random matrix theory can be exploited to find an accurate approximation of these eigenvalues. Indeed, the eigenvalues are not easy to find in the finite setting. They might be found using numerical techniques based on extensive search. Here, our approach consists in approximating the utilities in order to obtain expressions which are not only easier to interpret but also easier to be optimized w.r.t. the eigenvalues of the precoding matrices. The key idea is to approximate the different transmission rates by their large-system equivalent in the regime of large number of antennas. The corresponding approximates can be found to be accurate even for relatively small number of antennas (see e.g., [18, 19] for more details).

Since we have assumed $\mathbf{V}_k = \mathbf{V}$, we can exploit the results in [20, 21] for single-user MIMO channels, assuming the asymptotic regime in terms of the number of antennas: $n_r \to \infty$, $n_t \to \infty$, $\frac{n_r}{n_t} \to \beta$. The corresponding approximated utility for user k is:

$$\tilde{u}_k^{\mathrm{SIC}}(\{\mathbf{P}_k^{(\pi)}\}_{k \in \mathcal{K}, \pi \in \mathcal{P}_K}) = \sum_{\pi \in \mathcal{P}_K} p_\pi \tilde{R}_k^{(\pi)}(\mathbf{P}_k^{(\pi)}, \mathbf{P}_{-k}^{(\pi)}), \tag{10}$$

where

$$\begin{aligned}
&\tilde{R}_k^{(\pi)}(\mathbf{P}_k^{(\pi)}, \mathbf{P}_{-k}^{(\pi)}) \\
&= \frac{1}{n_r} \sum_{\ell \in \mathcal{K}_k^{(\pi)} \cup \{k\}} \sum_{j=1}^{n_t} \log_2(1 + (N_k^{(\pi)} + 1) \\
&\quad \times \rho P_\ell^{(\pi)}(j) \gamma_\ell^{(\pi)}(j)) \\
&\quad + \frac{1}{n_r} \sum_{i=1}^{n_r} \log_2\left(1 + \frac{1}{(N_k^{(\pi)} + 1) n_t} \right. \\
&\quad \left. \times \sum_{\ell \in \mathcal{K}_k^{(\pi)} \cup \{k\}} \sum_{j=1}^{n_t} \sigma_\ell(i, j) \delta_\ell^{(\pi)}(j)\right) \\
&\quad - \frac{1}{n_r} \sum_{\ell \in \mathcal{K}_k^{(\pi)} \cup \{k\}} \sum_{j=1}^{n_t} \gamma_\ell^{(\pi)}(j) \delta_\ell^{(\pi)}(j) \log_2 e \\
&\quad - \frac{1}{n_r} \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \sum_{j=1}^{n_t} \log_2(1 + N_k^{(\pi)} \rho P_\ell^{(\pi)}(j) \phi_\ell^{(\pi)}(j)) \\
&\quad - \frac{1}{n_r} \sum_{i=1}^{n_r} \log_2\left(1 + \frac{1}{N_k^{(\pi)} n_t} \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \sum_{j=1}^{n_t} \sigma_\ell(i, j) \psi_\ell^{(\pi)}(j)\right) \\
&\quad + \frac{1}{n_r} \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \sum_{j=1}^{n_r} \phi_\ell^{(\pi)}(j) \psi_\ell^{(\pi)}(j) \log_2 e,
\end{aligned} \tag{11}$$

where $N_k^{(\pi)} = |\mathcal{K}_k^{(\pi)}|$ and the parameters $\gamma_k^{(\pi)}(j)$ and $\delta_k^{(\pi)}(j)$ $\forall j \in \{1, \ldots, n_t\}$, $k \in \mathcal{K}$, $\pi \in \mathcal{P}_K$ are the solutions of:

$$\begin{cases}
\forall j \in \{1, \ldots, n_t\}, \ell \in \mathcal{K}_k^{(\pi)} \cup \{k\}: \\
\gamma_\ell^{(\pi)}(j) = \frac{1}{(N_k^{(\pi)}+1) n_t} \sum_{i=1}^{n_r} \frac{\sigma_\ell(i,j)}{1 + \frac{1}{(N_k^{(\pi)}+1) n_t} \sum_{r \in \mathcal{K}_k^{(\pi)} \cup \{k\}} \sum_{m=1}^{n_t} \sigma_r(i,m) \delta_r^{(\pi)}(m)} \\
\delta_\ell^{(\pi)}(j) = \frac{(N_k^{(\pi)}+1) \rho P_\ell^{(\pi)}(j)}{1 + (N_k^{(\pi)}+1) \rho P_\ell^{(\pi)}(j) \gamma_\ell^{(\pi)}(j)},
\end{cases} \tag{12}$$

and $\phi_\ell^{(\pi)}(j)$, $\psi_\ell^{(\pi)}(j)$, $\forall j \in \{1,\dots,n_t\}$ and $\pi \in \mathcal{P}_K$ are the unique solutions of the following system:

$$\begin{cases} \forall j \in \{1,\dots,n_t\}, \ell \in \mathcal{K}_k^{(\pi)}: \\ \phi_\ell^{(\pi)}(j) = \frac{1}{N_k^{(\pi)} n_t} \sum_{i=1}^{n_r} \frac{\sigma_\ell(i,j)}{1+\frac{1}{N_k^{(\pi)} n_t} \sum_{r\in\mathcal{K}_k^{(\pi)}} \sum_{m=1}^{n_t} \sigma_r(i,m)\psi_r^{(\pi)}(m)} \\ \psi_\ell^{(\pi)}(j) = \frac{N_k^{(\pi)} \rho P_\ell^{\pi)}(j)}{1+N_k^{(\pi)} \rho P_\ell^{(\pi)}(j)\phi_\ell^{(\pi)}(j)}. \end{cases} \tag{13}$$

The corresponding water-filling solution is:

$$P_k^{(\pi),\mathrm{NE}}(j) = \left[\frac{1}{\ln 2 n_r \lambda_k} - \frac{1}{N_k^{(\pi)} \rho \gamma_k^{(\pi)}(j)}\right]^+, \tag{14}$$

where $\lambda_k \geq 0$ is the Lagrangian multiplier tuned in order to meet the power constraint:

$$\sum_{\pi\in\mathcal{P}_K} \sum_{j=1}^{n_t} p_\pi \left[\frac{1}{\ln 2 n_r \lambda_k} - \frac{1}{N_k^{(\pi)} \rho \gamma_k^{(\pi)}(j)}\right]^+ = n_t \overline{P}_k.$$

Note that to solve the system of equations given above, we can use the same iterative power allocation algorithm as the one described in [9].

At this point, an important point has to be mentioned. The existence and uniqueness issues have be analyzed in the finite setting (exact game) whereas the determination of the NE is performed in the asymptotic regime (approximated game). It turns out that large system approximates of ergodic transmission rates have the same properties as their exact counterparts, as shown recently by [23], which therefore ensures the existence and uniqueness of the NE in the approximated game.

Nash Equilibrium efficiency. In order to measure the efficiency of the decentralized network w.r.t. its centralized counterpart we introduce the following quantity:

$$\mathrm{SRE} = \frac{R_{\mathrm{sum}}^{\mathrm{NE}}}{C_{\mathrm{sum}}} \leq 1, \tag{15}$$

where SRE stands for sum-rate efficiency; the quantity $R_{\mathrm{sum}}^{\mathrm{NE}}$ represents the sum-rate of the decentralized network at the Nash equilibrium, which is achieved for certain choices of coding and decoding strategies; the quantity C_{sum} corresponds to the sum-capacity of the centralized network, which is reached only if the optimum coding and decoding schemes are known. Note that this is the case for the MAC but not for other channels like the interference channel. Obviously, the efficiency measure we introduce here is strongly connected to the price of anarchy [24] (POA). The difference between SRE and POA is subtle. In our context, information theory provides us with fundamental physical limits on the social welfare (network sum-capacity) while in general no such upper bound is available. In our case, the sum-capacity is given by:

$$C_{\mathrm{sum}} = \max_{(\mathbf{\Omega}_1,\dots,\mathbf{\Omega}_K)\in\mathcal{A}^{(C)}} \mathbb{E}\log\left|\mathbf{I} + \rho\sum_{k=1}^{K} \mathbf{H}_k \mathbf{\Omega}_k \mathbf{H}_k^H\right|, \tag{16}$$

with

$$\mathcal{A}^{(C)} = \{(\mathbf{\Omega}_1,\dots,\mathbf{\Omega}_K) | \forall k \in \mathcal{K}, \mathbf{\Omega}_k \succeq 0, \\ \mathbf{\Omega}_k = \mathbf{\Omega}_k^H, \mathrm{Tr}(\mathbf{\Omega}_k) \leq n_t \overline{P}_k\}. \tag{17}$$

In general, it is not easy to find a closed-form expression of the SRE. This is why we will respectively analyze the SRE in the regimes of high and low signal-to-noise ratio (SNR), and for intermediate regimes simulations will complete our analysis. It turns out that the SRE tends to 1 in the two mentioned extreme regimes, which is the purpose of what follows.

In the *high SNR regime*, where $\rho \to \infty$, we observe from (12) that $\delta_\ell^{(\pi)}(j) \to \frac{1}{\gamma_\ell^{(\pi)}(j)}$. Under this condition, it is easy to check that by setting the derivatives of $\mathcal{L}_k$ w.r.t. $P_k^{(s)}(j)$ to zero, we obtain that the power allocation policy at the NE is the uniform power allocation $\mathbf{P}_k^{(\pi),\mathrm{NE}} = \overline{P}_k \mathbf{I}$, regardless the realization of the coordination signal S. Furthermore, in the high SNR regime, the sum-capacity is achieved by the uniform power allocation. Thus, we obtain that the gap between the NE achievable sum-rate and the sum-capacity is optimal, SRE = 1 for any distribution of S.

In the *low SNR regime*, where $\rho \to 0$, from (12) we obtain that $\delta_\ell^{(\pi)}(j) \to 0$ and that $\gamma_\ell^{(\pi)}(j) = \frac{1}{(N_k^{(\pi)}+1)n_t} \sum_{i=1}^{n_r} \sigma_\ell(i,j)$. By approximating $\ln(1+x) \approx x$ when $x \ll 1$, the power allocations policies at the NE are the solutions of the following linear programs:

$$\max_{\{P_k^{(\pi)}(j)\}_{1\leq j\leq n_t}} \sum_{j=1}^{n_t} \left\{ \sum_{p_\pi\in\mathcal{P}_K} \pi P_k^{(\pi)}(j) \sum_{i=1}^{n_r} \sigma_1(i,j) \right\} \\ \text{s.t.} \sum_{j=1}^{n_t} \sum_{\pi\in\mathcal{P}_K} P_k^{(\pi)}(j) \leq \overline{P}_k n_t, \tag{18}$$

given by:

$$\sum_{\pi\in\mathcal{P}_K} p_\pi P_k^{(\pi),\mathrm{NE}}(j) \\ = \begin{cases} n_t \overline{P}_k & \text{if } j = \arg\max_{1\leq m\leq n_t} \sum_{i=1}^{n_r} \sigma_k(i,m) \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

The optimal power allocation that achieves the sum-capacity is equal to the equilibrium power allocation, $\mathbf{P}_k^* = \sum_{\pi\in\mathcal{P}_K} p_\pi \mathbf{P}_k^{(\pi),\mathrm{NE}}(j)$ Thus, the achievable sum-rate at the NE is equal to the centralized upper bound and thus SRE = 1 for any distribution of S. In conclusion, when either the low

or high SNR regime is assumed, the sum-capacity of the fast fading MAC is achieved at the NE although a sub-optimum coordination mechanism is assumed and also regardless of the distribution of the coordination channel.

4 Single user decoding

In this section the coordination signal is deterministic (namely $\Pr[S = s] = \delta(s)$, δ being the Kronecker symbol) and therefore the amount of downlink signalling the BS needs in order to indicate to the MSs that it is using SUD can be made arbitrary small (by letting the frequency at which the realizations of the coordination signal are drawn tend to zero). In this framework, each user has to optimize only one precoding matrix. Indeed, the strategy of user $k \in \mathcal{K}$, consists in choosing the best precoding matrix $\mathbf{Q}_k^{(0)} = \mathbb{E}[\underline{X}_k^{(0)} \underline{X}_k^{(0)H}]$, in the sense of his utility function obtained with SUD:

$$u_k^{\mathrm{SUD}}(\mathbf{Q}_k^{(0)}, \mathbf{Q}_{-k}^{(0)}) = \mathbb{E}\log\left|\mathbf{I} + \rho\mathbf{H}_k\mathbf{Q}_k^{(0)}\mathbf{H}_k^H + \rho\sum_{\ell\neq k}\mathbf{H}_\ell\mathbf{Q}_\ell^{(0)}\mathbf{H}_\ell^H\right| - \mathbb{E}\log\left|\mathbf{I} + \rho\sum_{\ell\neq k}\mathbf{H}_\ell\mathbf{Q}_\ell^{(0)}\mathbf{H}_\ell^H\right|. \tag{20}$$

. The strategy set of user k becomes

$$\mathcal{A}_k^{\mathrm{SUD}} = \{\mathbf{Q}_k^{(0)} \succeq 0, \mathbf{Q}_k^{(0)} = \mathbf{Q}_k^{(0),H}, \mathrm{Tr}(\mathbf{Q}_k^{(0)}) \leq n_t\overline{P}_k\}. \tag{21}$$

It turns out that the equilibrium analysis in the game with SUD can be, to a large extent, deduced from the game with SIC. For this reason, we will not detail the corresponding proofs. The existence and uniqueness issues are given in the following theorem.

Theorem 7 (Existence and uniqueness of an NE) *The space power allocation game described by: the set of players $k \in \mathcal{K}$; the sets of actions $\mathcal{A}_k^{\mathrm{SUD}}$ and the payoff functions $u_k^{\mathrm{SUD}}(\mathbf{Q}_k^{(0)}, \mathbf{Q}_{-k}^{(0)})$ given in (20), has a unique Nash equilibrium.*

To prove the *existence* of a Nash equilibrium we also exploit Theorem 1 and the four necessary conditions on the utility functions and strategy sets can be verified using the same tools as described in Appendix A.

Uniqueness of the Nash equilibrium. Here we can specialize Theorem 4, which is the matrix extension of Theorem 2. When the strategies sets are not sets of pairs of matrices but only sets of matrices, the diagonally strict concavity condition in (6) can be written as follows. For all $\mathbf{Q}_k^{(0)'}, \mathbf{Q}_k^{(0)''} \in \mathcal{A}_k^{\mathrm{SUD}}$ such that $(\mathbf{Q}_1^{(0)'}, \ldots, \mathbf{Q}_K^{(0)'}) \neq (\mathbf{Q}_1^{(0)''}, \ldots, \mathbf{Q}_K^{(0)''})$:

$$\mathcal{C} = \sum_{k=1}^{K}\mathrm{Tr}\{(\mathbf{Q}_k^{(0)''} - \mathbf{Q}_k^{(0)'})[\nabla_{Q_k^{(0)}}u_1(\mathbf{Q}_k^{(0)'}, \mathbf{Q}_k^{(0)'}) - \nabla_{Q_k^{(0)}}u_1(\mathbf{Q}_k^{(0)''}, \mathbf{Q}_k^{(0)''})]\}. \tag{22}$$

Now we can evaluate $\mathcal{C}$ and obtain that:

$$\begin{aligned}\mathcal{C} &= \sum_{k=1}^{K}\mathrm{Tr}\left\{[\rho\mathbf{H}_k(\mathbf{Q}_k^{(0)'} - \mathbf{Q}_k^{(0)''})\mathbf{H}_k^H] \times\left[\left(\mathbf{I} + \rho\sum_{\ell=1}^{K}\mathbf{H}_\ell\mathbf{Q}_\ell^{(0)''}\mathbf{H}_\ell^H\right)^{-1} - \left(\mathbf{I} + \rho\sum_{\ell=1}^{K}\mathbf{H}_\ell\mathbf{Q}_\ell^{(0)'}\mathbf{H}_\ell^H\right)^{-1}\right]\right\} \\ &= \mathrm{Tr}\{(\mathbf{A}' - \mathbf{A}'')[(\mathbf{A}'')^{-1} - (\mathbf{A}')^{-1}]\},\end{aligned} \tag{23}$$

which is strictly positive for all $\mathbf{A}' \neq \mathbf{A}''$, $\mathbf{A}' \succ \mathbf{0}$, $\mathbf{A}'' \succ \mathbf{0}$ after [27] applied when $K = 1$. This result can be applied here since we have

$$\mathbf{A}' = \mathbf{I} + \rho\sum_{\ell=1}^{K}\mathbf{H}_\ell\mathbf{Q}_\ell^{(0)'}\mathbf{H}_\ell^H,$$

$$\mathbf{A}'' = \mathbf{I} + \rho\sum_{\ell=1}^{K}\mathbf{H}_\ell\mathbf{Q}_\ell^{(0)''}\mathbf{H}_\ell^H.$$

Determination of the Nash equilibrium. As for the optimal eigenvectors of the covariance matrices, we follow the same lines as in Appendix C. In this case also there is no loss of optimality by choosing the covariance matrices $\mathbf{Q}_k^{(0)} = \mathbf{W}_k\mathbf{P}_k^{(0)}\mathbf{W}_k^H$, where $\mathbf{W}_k$ is the same unitary matrix as in (2) and $\mathbf{P}_k$ is the diagonal matrix containing the eigenvalues of $\mathbf{Q}_k^{(0)}$.

Here also we further exploit the asymptotic results for the MIMO channel given in [20, 21]. The approximated utility for user k is:

$$\begin{aligned}\tilde{u}_k^{\mathrm{SUD}}(\mathbf{P}_k^{(0)}, \mathbf{P}_{-k}^{(0)}) &= \frac{1}{n_r}\sum_{k=1}^{K}\sum_{j=1}^{n_t}\log_2(1 + K\rho P_k^{(0)}(j)\gamma_k(j)) \\ &\quad + \frac{1}{n_r}\sum_{i=1}^{n_r}\log_2\left(1 + \frac{1}{Kn_t}\sum_{k=1}^{K}\sum_{j=1}^{n_t}\sigma_k(i,j)\delta_k(j)\right) \\ &\quad - \frac{1}{n_r}\sum_{k=1}^{K}\sum_{j=1}^{n_t}\gamma_k(j)\delta_k(j)\log_2 e\end{aligned}$$

$$-\frac{1}{n_r}\sum_{\ell\neq k}\sum_{j=1}^{n_t}\log_2(1+(K-1)\rho P_\ell^{(0)}(j)\phi_\ell(j))$$
$$-\frac{1}{n_r}\sum_{i=1}^{n_r}\log_2\left(1+\frac{1}{(K-1)n_t}\sum_{\ell\neq k}\sum_{j=1}^{n_t}\sigma_\ell(i,j)\psi_\ell(j)\right)$$
$$+\frac{1}{n_r}\sum_{\ell\neq k}\sum_{j=1}^{n_r}\phi_\ell(j)\psi_\ell(j)\log_2 e, \tag{24}$$

where the parameters $\gamma_k(j)$ and $\delta_k(j)$ $\forall j\in\{1,\ldots,n_t\}$, $k\in\{1,2\}$ are solution of:

$$\begin{cases}\forall j\in\{1,\ldots,n_t\}, k\in\mathcal{K}:\\ \gamma_k(j)=\frac{1}{Kn_t}\sum_{i=1}^{n_r}\frac{\sigma_k(i,j)}{1+\frac{1}{Kn_t}\sum_{\ell=1}^{K}\sum_{m=1}^{n_t}\sigma_\ell(i,m)\delta_\ell(m)}\\ \delta_k(j)=\frac{K\rho P_k^{(0)}(j)}{1+K\rho P_k^{(0)}(j)\gamma_k(j)}\end{cases} \tag{25}$$

and $\phi_\ell(j)$, $\psi_\ell(j)$, $\forall j\in\{1,\ldots,n_t\}$ are the unique solutions of the following system:

$$\begin{cases}\forall j\in\{1,\ldots,n_t\}, \ell\in\mathcal{K}\setminus\{k\}:\\ \phi_\ell(j)=\frac{1}{(K-1)n_t}\sum_{i=1}^{n_r}\frac{\sigma_\ell(i,j)}{1+\frac{1}{(K-1)n_t}\sum_{r\neq k}\sum_{m=1}^{n_t}\sigma_r(i,m)\psi_r(m)}\\ \psi_\ell(j)=\frac{(K-1)\rho P_\ell^{(0)}(j)}{1+(K-1)\rho P_\ell^{(0)}(j)\phi_\ell(j)}.\end{cases} \tag{26}$$

The corresponding water-filling solution is:

$$P_k^{(0),\mathrm{NE}}(j)=\left[\frac{1}{\ln 2n_r\lambda_k}-\frac{1}{K\rho\gamma_k(j)}\right]^+, \tag{27}$$

where $\lambda_k\geq 0$ is the Lagrangian multiplier tuned in order to meet the power constraint:

$$\sum_{j=1}^{n_t}\left[\frac{1}{\ln 2n_r\lambda_k}-\frac{1}{K\rho\gamma_k(j)}\right]^+=n_t\overline{P}_k.$$

In what the efficiency of the NE point is concerned, we already know that the SUD decoding technique is sub-optimal in the centralized case (SUD does allow the network to operate at an arbitrary point of the centralized MAC capacity region) and it is impossible to reach the sum-capacity C_{sum} even if the high and low SNR regime are assumed.

5 Simulation results

In what follows, we assume the regime of large numbers of antennas. From [9, 20, 21], we know that the approximates of the ergodic achievable rates in the asymptotic regime are accurate even for relatively small number of antennas. For the channel matrices, we assume the Kronecker model $\mathbf{H}_k=\mathbf{R}_k^{1/2}\Theta_k\mathbf{T}_k^{1/2}$ mentioned in Sect. 2, where the receive

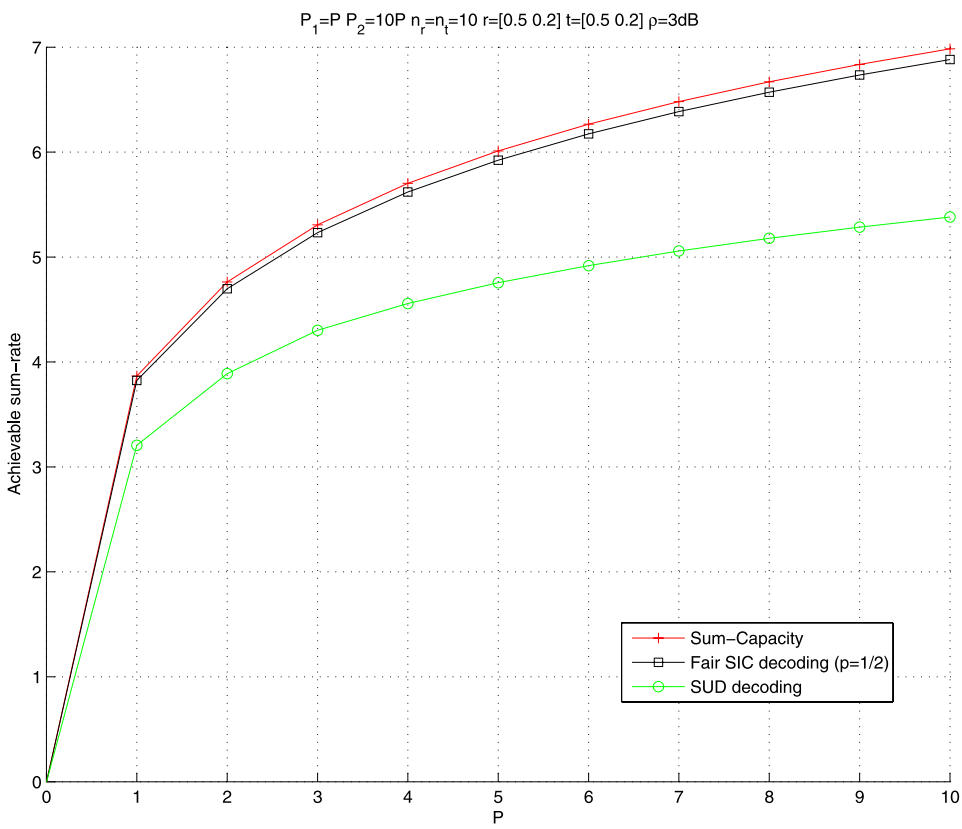

Fig. 1 Fair SIC (joint space-time power allocation) vs. SUD decoding. Achievable network sum-rate versus the available transmit power P for $p=\frac{1}{2}$, $n_r=n_t=10$, $r=[0.5,0.2]$, $t=[0.5,0.2]$, $\rho=3$ dB. The fair SIC performs much closer to the sum-capacity upper bound than SUD

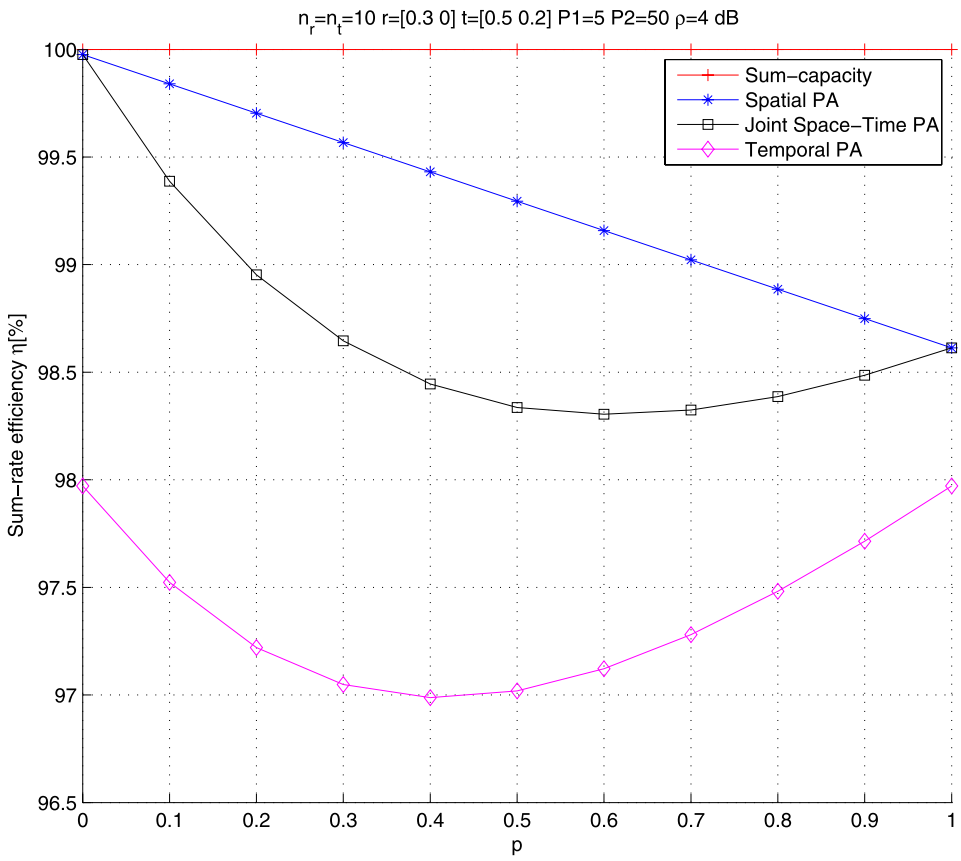

Fig. 2 SIC decoding, comparison between the joint space-time PA and the two special cases: the space PA and temporal PA. Sum-rate efficiency versus the distribution of the coordination signal $p \in [0, 1]$ for $n_r = n_t = 10$, $r = [0.3, 0]$, $t = [0.5, 0.2]$, $\rho = 4$ dB, $P_1 = 5$, $P_2 = 50$. The spatial PA outperforms the joint space-time PA (Braess paradox)

and transmit correlation matrices $\mathbf{R}_k$, $\mathbf{T}_k$ follow an exponential profile characterized by the correlation coefficients (see e.g., [25, 26]) $r = [r_1, r_2]$ and $t = [t_1, t_2]$ such that $\mathbf{R}_k(i, j) = r_k^{|i-j|}$, $\mathbf{T}_k(i, j) = t_k^{|i-j|}$. By assuming that the receive antenna is a uniform linear array (ULA) and knowing that, when the dimensions of Toeplitz matrices increase they can be approximated by circular matrices we obtain that all the receive correlation matrices $\mathbf{R}_k$ can be diagonalized in the same vector basis (i.e., the Fourier basis). Thus the considered model is included in the UIU model that we studied where $\mathbf{V}_k = \mathbf{V}$.

Fair SIC decoding versus SUD decoding. First we compare the results of the general space-time PA game considered in Sect. 3, where SIC decoding is used at the receiver, and the game described in Sect. 4, where SUD decoding is used. Figure 1 depicts the achievable sum-rate at the equilibrium as a function of the transmit power $P_1 = P_2 = P$, for the scenario $n_r = n_t = 10$, $r = [0.5, 0.2]$, $t = [0.5, 0.2]$, $\rho = 3$ dB. In order to have a fair comparison we assume that $p = \frac{1}{2}$ (on average each user is decoded second half of the time when SIC is assumed). We observe that, even in this scenario, which was thought to be a bad one in terms of suboptimality, the sum-rate obtained with the first game is very close to the sum-capacity upper bound. Also, the sum-rate reached when the BS uses SUD is clearly much lower than the sum-rate obtained by using SIC.

SIC decoding, comparison between the joint space-time PA and the special cases of spatial PA and temporal PA. Now we want to compare the results of the general space-time PA with the two particular cases that were studied in [14]: the spatial PA, where the users are forced to allocate their power uniformly over time (regardless of their decoding rank) but are free to allocate their power over the transmit antennas; the temporal PA, where the users are forced to allocate their power uniformly over their antennas but they can adjust their power as a function of the decoding rank at the receiver. Figure 2 represents the sum-rate efficiency as a function of the coordination signal distribution parameter $p \in [0, 1]$ when $n_r = n_t = 10$, $r = [0.3, 0]$, $t = [0.5, 0.2]$, $\rho = 4$ dB, $P_1 = 5$, $P_2 = 50$. We observe that the three types of power allocation policies perform very close to the upper bound. What is most interesting is the fact that the performance of the network at the equilibrium is better by using a purely spatial PA instead of the most general space-time PA. This has been confirmed by many other simulations and illustrates a Braess paradox: although the sets of strategies for the space-time case include those of the purely spatial case, the performance obtained at the NE are not better in the space-time case.

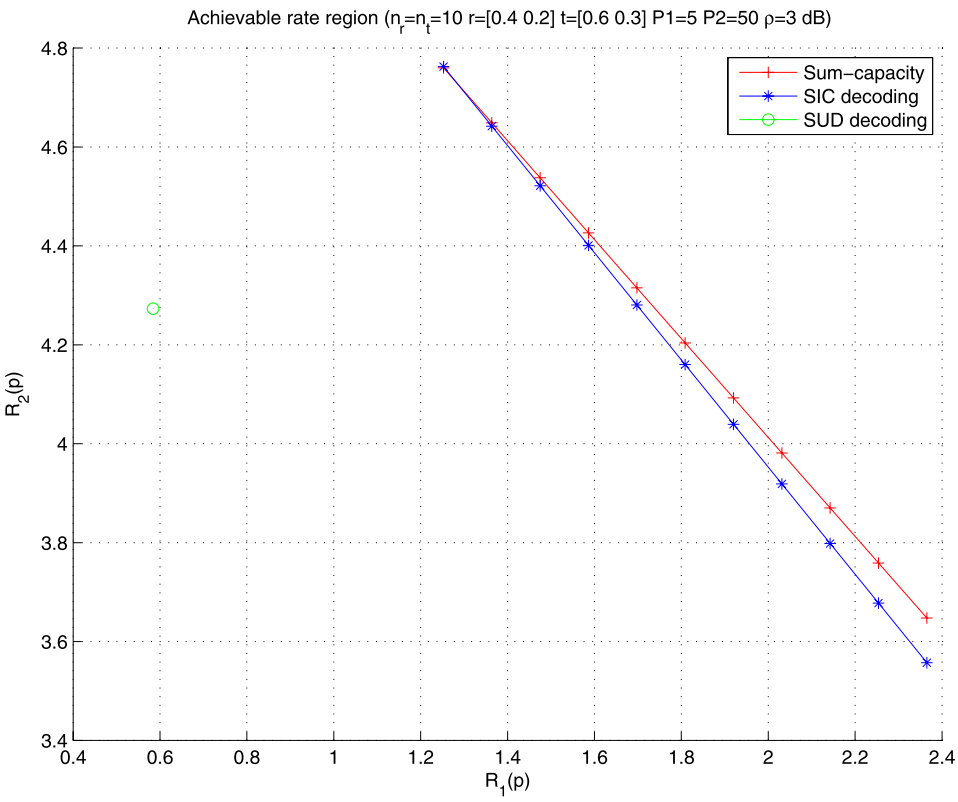

Fig. 3 SIC decoding, space PA. The achievable rate region at the NE versus the distribution of the coordination signal $p \in [0, 1]$ for $n_r = n_t = 10$, $r = [0.4, 0.2]$, $t = [0.6, 0.3]$, $\rho = 3$ dB, $P_1 = 5$, $P_2 = 50$. Varying p allows to move along a segment close to the sum-capacity, similar to SISO MAC

SIC decoding, spatial PA, achievable rate region. In Fig. 3, we observe that the rate region achieved at the NE of the space PA as a function of the distribution of the coordination signal p for the scenario $n_r = n_t = 10$, $r = [0.4, 0.2]$, $t = [0.6, 0.3]$, $\rho = 3$ dB, $P_1 = 5$, $P_2 = 50$. It is quite remarkable that in large MIMO MACs, the capacity region comprises a full cooperation segment just like the SISO MACs. The coordination signal precisely allows one to move along the corresponding line. This shows the relevance of large systems in decentralized networks since they allow to determine the capacity region of certain systems whereas it is unknown in the finite setting. Furthermore, they induce an averaging effect, which makes the users' behavior predictable.

6 Conclusions

Interestingly, the existence and uniqueness of the Nash equilibrium can be proven in multiple access channels with multi-antenna terminals for a general propagation channel model (namely the unitary-invariant-unitary model) and the most general case of space-time power allocation schemes. In particular, the uniqueness proof requires a matrix generalization of the second theorem of Rosen [17] and proving a trace inequality [28]. For all the types of power allocation policies (purely temporal PA, purely spatial PA, space-time PA), the sum-rate efficiency of the decentralized network is close to one when SIC is assumed and the network is coordinated by the proposed suboptimum coordination mechanism. Quite surprisingly, the space-time power allocation performs a little worse than its purely spatial counterpart, which puts in evidence a Braess paradox in the types of wireless networks under consideration. One of the interesting extensions of this work would be to analyze the impact of a non-perfect SIC on the PA problem. Indeed, the effect of propagation errors could then be assessed (which does not exist with SUD).

Appendix A

A.1 Concavity of the utility functions u_k^{SIC}

Let us focus on user $k \in \mathcal{K}$. We want to prove that $u_k^{\mathrm{SIC}}(\mathbf{Q}_k, \mathbf{Q}_{-k})$ is concave w.r.t. $\mathbf{Q}_k \in \mathcal{A}_1^{\mathrm{SIC}}$. We observe that the term $R_k^{(\pi)}(\mathbf{Q}_k^{(\pi)}, \mathbf{Q}_{-k}^{(\pi)})$ in (3) depends only on $\mathbf{Q}_k^{(\pi)}$ and $\mathbf{Q}_{-k}^{(\pi)}$ and not on the covariance matrices $\mathbf{Q}_k^{(\tau)}$, $\mathbf{Q}_{-k}^{(\tau)}$ for any other possible decoding rule $\tau \in \mathcal{P}_K \setminus \{\pi\}$. Thus, in order to prove that $u_k^{\mathrm{SIC}}(\mathbf{Q}_k, \mathbf{Q}_k)$ is strictly concave w.r.t. to $\mathbf{Q}_k = (\mathbf{Q}_k^{(\pi)})_{\pi \in \mathcal{P}_K}$, it suffices to prove that $R_k^{(\pi)}(\mathbf{Q}_k^{(\pi)}, \mathbf{Q}_{-k}^{(\pi)})$ is concave w.r.t. $\mathbf{Q}_k^{(\pi)}$ for all $\pi \in \mathcal{P}_K$.

To this end, we study the concavity of the function $f(\lambda) = R_k^{(\pi)}(\lambda \mathbf{Q}_k^{(\pi)'} + (1-\lambda)\mathbf{Q}_k^{(\pi)''})$ over the interval $[0, 1]$

for any pair of matrices $(\mathbf{Q}_k^{(\pi)'}, \mathbf{Q}_k^{(\pi)''})$. The second derivative of f is equal to:

$$\begin{aligned}\frac{\partial^2 f}{\partial \lambda^2}(\lambda) = -\mathbb{E}\mathrm{Tr}\Bigg[& \rho^2 \mathbf{H}_k^H \Bigg(\mathbf{I} + \rho \mathbf{H}_k \mathbf{Q}_k^{(\pi)''} \mathbf{H}_k^H \\ & + \rho\lambda \mathbf{H}_k \Delta\mathbf{Q}_k^{(\pi)} \mathbf{H}_k^H \\ & + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \mathbf{H}_\ell \mathbf{Q}_\ell^{(\pi)} \mathbf{H}_\ell^H \Bigg)^{-1} \mathbf{H}_k \Delta\mathbf{Q}_k^{(\pi)} \\ & \times \mathbf{H}_k^H \Bigg(\mathbf{I} + \rho \mathbf{H}_k \mathbf{Q}_k^{(\pi)''} \mathbf{H}_k^H \\ & + \rho\lambda \mathbf{H}_k \Delta\mathbf{Q}_k^{(\pi)} \mathbf{H}_k^H \\ & + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \mathbf{H}_\ell \mathbf{Q}_\ell^{(\pi)} \mathbf{H}_\ell^H \Bigg)^{-1} \mathbf{H}_k \Delta\mathbf{Q}_k^{(\pi)} \Bigg] \\ = -\mathbb{E}\mathrm{Tr}[& \mathbf{A}\Delta\mathbf{Q}_k^{(\pi)} \mathbf{A} \Delta\mathbf{Q}_k^{(\pi)}],\end{aligned}$$

with

$$\begin{aligned}\mathbf{A} = \rho^2 \mathbf{H}_k^H \Bigg(& \mathbf{I} + \rho \mathbf{H}_k \mathbf{Q}_k^{(\pi)''} \mathbf{H}_k^H \\ & + \rho\lambda \mathbf{H}_k \Delta\mathbf{Q}_k^{(\pi)} \mathbf{H}_k^H + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \mathbf{H}_\ell \mathbf{Q}_\ell^{(\pi)} \mathbf{H}_\ell^H \Bigg)^{-1} \mathbf{H}_k,\end{aligned}$$

which can be proven to be a Hermitian positive definite matrix, $\Delta\mathbf{Q}_k^{(\pi)} = \mathbf{Q}_k^{(\pi)'} - \mathbf{Q}_k^{(\pi)''}$ also a Hermitian matrix, and $\rho = \frac{1}{\sigma^2}$.

$$\begin{aligned}\frac{\partial^2 f}{\partial \lambda^2}(\lambda) &= -\mathbb{E}\mathrm{Tr}[\mathbf{A}^{1/2}\Delta\mathbf{Q}_k^{(\pi)}\mathbf{A}^{1/2}\mathbf{A}^{1/2}\Delta\mathbf{Q}_k^{(\pi)}\mathbf{A}^{1/2}] \\ &= -\mathbb{E}\mathrm{Tr}[\mathbf{B}\mathbf{B}^H] < 0,\end{aligned}$$

with $\mathbf{B} = \mathbf{A}^{1/2}\Delta\mathbf{Q}_k^{(\pi)}\mathbf{A}^{1/2}$.

A.2 Continuity of the utility functions u_k^{SIC}

Considering the Leibniz formula, the determinant of a matrix can be expressed as a weighted sum of products of its entries. Knowing that the product and the sum of continuous functions are continuous, we conclude that the determinant function is continuous. Also, it is well known that the logarithmic function is a continuous function. Thus, for any $\pi \in \mathcal{P}_K$, the function $R_k^{(\pi)}(\mathbf{Q}_k^{(\pi)}, \mathbf{Q}_{-k}^{(\pi)})$ is nothing else but the composition of two continuous functions which is also continuous w.r.t. $(\mathbf{Q}_k^{(\pi)}, \mathbf{Q}_{-k}^{(\pi)})$. This suffices to prove that $u_k^{\mathrm{SIC}}(\mathbf{Q}_k, \mathbf{Q}_{-k})$ is continuous w.r.t. $(\mathbf{Q}_k, \mathbf{Q}_{-k})$.

A.3 Convexity of the strategy sets $\mathcal{A}_k^{\mathrm{SIC}}$

In order to prove that the set $\mathcal{A}_k^{\mathrm{SIC}}$ is convex, we need to verify that, for any two matrices $(\mathbf{Q}_k', \mathbf{Q}_k'') \in \mathcal{A}_k^{\mathrm{SIC}} \times \mathcal{A}_k^{\mathrm{SIC}}$, we have:

$$\alpha\mathbf{Q}_1' + (1-\alpha)\mathbf{Q}_1'' \in \mathcal{A}_k^{\mathrm{SIC}},$$

for all $\alpha \geq 0$.

For any $\mathbf{Q}_k', \mathbf{Q}_k'' \in \mathcal{A}_k^{(\mathrm{SIC})}$, the matrices $\mathbf{Q}_k^{(\pi)}$ are Hermitian which implies that $\alpha\mathbf{Q}_k^{(\pi)'} + (1-\alpha)\mathbf{Q}_k^{(\pi)''}$ are also Hermitian matrices, for all $\pi \in \mathcal{P}_K$.

Furthermore, for any $\mathbf{Q}_k', \mathbf{Q}_k'' \in \mathcal{A}_k^{\mathrm{SIC}}$, we have that $\mathbf{Q}_k^{(\pi)'}$, $\mathbf{Q}_k^{(\pi)''}$ are non-negative matrices which implies that $\alpha\mathbf{Q}_k^{(\pi)'} + (1-\alpha)\mathbf{Q}_k^{(\pi)''}$ are also non-negative matrices, for all $\pi \in \mathcal{P}_K$.

Finally, knowing that the trace is a linear application we have that:

$$\begin{aligned}&\sum_{\pi \in \mathcal{P}_k} p_\pi \mathrm{Tr}(\alpha\mathbf{Q}_k^{(\pi)'} + (1-\alpha)\mathbf{Q}_k^{(\pi)''}) \\ &\quad = \alpha \sum_{\pi \in \mathcal{P}_k} p_\pi \mathrm{Tr}(\mathbf{Q}_k^{(\pi)'}) + (1-\alpha) \sum_{\pi \in \mathcal{P}_k} p_\pi \mathrm{Tr}(\mathbf{Q}_k^{(\pi)'}) \\ &\quad \leq \alpha n_t \overline{P}_k + (1-\alpha) n_t \overline{P}_k \\ &\quad = n_t \overline{P}_k.\end{aligned}$$

Thus $\alpha\mathbf{Q}_k' + (1-\alpha)\mathbf{Q}_k'' \in \mathcal{A}_k^{\mathrm{SIC}}$ and the set is convex.

A.4 Compactness of the strategy sets $\mathcal{A}_k^{\mathrm{SIC}}$

To prove that the strategy sets are compact sets we use the fact that, in finite dimension spaces, a closed and bounded set is compact.

First let us prove that $\mathcal{A}_k^{\mathrm{SIC}}$ is a closed set. We define the function $g : \mathcal{A}_k^{\mathrm{SIC}} \longrightarrow [0, n_t \overline{P}_k]$, with

$$f(\mathbf{Q}_k) = \sum_{\pi \in \mathcal{P}_K} p_\pi \mathrm{Tr}(\mathbf{Q}_k^{(\pi)}).$$

We see that $g(\cdot)$ is a continuous function and that its image is a compact and thus closed set. Knowing that the continuous inverse image of a closed set is closed, we conclude that $\mathcal{A}_k^{\mathrm{SIC}}$ is closed.

Now we want to prove that the set $\mathcal{A}_k^{\mathrm{SIC}}$ is a bounded set. We associate to the tuple of matrices $(\mathbf{Q}_k^{(\pi)})_{\pi \in \mathcal{P}_K}$ the following norm $\|\mathbf{Q}_k\| = \sqrt{\sum_{\pi \in \mathcal{P}_K} \|\mathbf{Q}_k^{(\pi)}\|_2^2}$ where $\|.\|_2$ is the spectral norm of a matrix.

$$\|\mathbf{Q}_k^{(\pi)}\|_2 = \sqrt{\max\{\lambda_{\mathbf{Q}_k^{(\pi)H}\mathbf{Q}_k^{(\pi)}}(i)\}_{i=1}^n}.$$

Since for all $\mathbf{Q}_k \in \mathcal{A}_k^{\text{SIC}}$, $\mathbf{Q}_k^{(\pi)}$ is a non-negative, Hermitian matrix we have that:

$$\max\{\lambda_{\mathbf{Q}_k^{(\pi)}}(i)\}_{i=1}^n \leq Tr(\mathbf{Q}_k^{(\pi)}) \leq \infty,$$

and thus:

$$\|\mathbf{Q}_k^{(\pi)}\|_2 = \sqrt{\max\{\lambda_{\mathbf{Q}_k^{(\pi)2}}(i)\}_{i=1}^n}$$
$$= \sqrt{\max\{\lambda_{\mathbf{Q}_k^{(\pi)}}(i)^2\}_{i=1}^n} \leq \infty.$$

In conclusion the associated norm $\|\mathbf{Q}_k\| \leq \infty$.

Appendix B

We suppose that there exist two different equilibrium strategy profiles: $(\widetilde{\mathbf{Q}}_k, \widetilde{\mathbf{Q}}_{-k}) \in \mathcal{A}_k^{\text{SIC}} \times \mathcal{A}_{-k}^{\text{SIC}}$ and $(\widehat{\mathbf{Q}}_k, \widehat{\mathbf{Q}}_{-k}) \in \mathcal{A}_k^{\text{SIC}} \times \mathcal{A}_{-k}^{\text{SIC}}$, such that $(\widetilde{\mathbf{Q}}_k, \widetilde{\mathbf{Q}}_{-k}) \neq (\widehat{\mathbf{Q}}_k, \widehat{\mathbf{Q}}_{-k})$. Then the condition given in the theorem, $\mathcal{C} > 0$ is met for the particular choice of $(\mathbf{Q}'_k, \mathbf{Q}'_{-k}) = (\widetilde{\mathbf{Q}}_k, \widetilde{\mathbf{Q}}_{-k})$ and $(\mathbf{Q}''_k, \mathbf{Q}''_{-k}) = (\widehat{\mathbf{Q}}_k, \widehat{\mathbf{Q}}_{-k})$.

By the definition of the Nash Equilibrium, the strategies $\widetilde{\mathbf{Q}}_k$, $k \in \mathcal{K}$, are the solutions of the following maximization problems:

$$\max_{\mathbf{Q}_k \in \mathcal{A}_k^{\text{SIC}}} u_k(\mathbf{Q}_k, \widetilde{\mathbf{Q}}_{-k}).$$

Thus, $\widetilde{\mathbf{Q}}_k$ satisfy the following Kuhn-Tucker optimality conditions:

(1) $\widetilde{\mathbf{Q}}_k \in \mathcal{A}_k^{\text{SIC}}$, which means that:

$$\begin{cases} \widetilde{\mathbf{Q}}_k^{(\pi)} = (\widetilde{\mathbf{Q}}_k^{(\pi)})^H \succeq 0, \quad \forall \pi \in \mathcal{P}_K \\ \sum_{\pi \in \mathcal{P}_K} p_\pi \operatorname{Tr}(\widetilde{\mathbf{Q}}_k^{(\pi)}) \leq n_t \overline{P}_k, \end{cases}$$

(2) There exist $\widetilde{\lambda}_k \geq 0$, and the following Hermitian non-negative matrices of rank 1, $\widetilde{\Phi}_k^{(\pi)}$, for all $\pi \in \mathcal{P}_K$, such that:

$$\begin{cases} \widetilde{\lambda}_k [\sum_{\pi \in \mathcal{P}_K} p_\pi \operatorname{Tr}(\widetilde{\mathbf{Q}}_k^{(\pi)}) - n_t \overline{P}_k] = 0 \\ \operatorname{Tr}(\widetilde{\Phi}_k^{(\pi)} \widetilde{\mathbf{Q}}_k^{(\pi)}) = 0, \quad \forall \pi \in \mathcal{P}_K, \end{cases}$$

(3)

$$\begin{cases} \forall \pi \in \mathcal{P}_K : \\ \nabla_{\mathbf{Q}_k^{(\pi)}} u_k(\widetilde{\mathbf{Q}}_k, \widetilde{\mathbf{Q}}_{-k}) = p_\pi \widetilde{\lambda}_k \mathbf{I} - \widetilde{\Phi}_k^{(\pi)}. \end{cases}$$

Having assumed that $(\widehat{\mathbf{Q}}_k, \widehat{\mathbf{Q}}_{-k})$ is also a Nash Equilibrium, $\widehat{\mathbf{Q}}_k$, with $k \in \mathcal{K}$ are the solution of:

$$\max_{\mathbf{Q}_k \in \mathcal{A}_k^{\text{SIC}}} u_k(\mathbf{Q}_k, \widehat{\mathbf{Q}}_{-k}),$$

and thus $\widehat{\mathbf{Q}}_k$ satisfy the following Kuhn-Tucker optimality conditions:

(4) $\widehat{\mathbf{Q}}_k \in \mathcal{A}_k^{\text{SIC}}$, which means that:

$$\begin{cases} \widehat{\mathbf{Q}}_k^{(\pi)} = (\widehat{\mathbf{Q}}_k^{(\pi)})^H \succeq 0, \quad \forall \pi \in \mathcal{P}_K \\ \sum_{\pi \in \mathcal{P}_K} p_\pi \operatorname{Tr}(\widehat{\mathbf{Q}}_k^{(\pi)}) \leq n_t \overline{P}_k, \end{cases}$$

(5) There exist $\widehat{\lambda}_k \geq 0$, $k \in \mathcal{K}$ and the following non-negative, Hermitian matrices of rank 1, $\widehat{\Phi}_k^{(\pi)}$, for all $\pi \in \mathcal{P}_K$ such that:

$$\begin{cases} \widehat{\lambda}_k [\sum_{\pi \in \mathcal{P}_K} p_\pi \operatorname{Tr}(\widehat{\mathbf{Q}}_k^{(\pi)}) - n_t \overline{P}_k] = 0 \\ \operatorname{Tr}(\widehat{\Phi}_k^{(\pi)} \widehat{\mathbf{Q}}_k^{(\pi)}) = 0, \quad \forall \pi \in \mathcal{P}_K, \end{cases}$$

(6)

$$\begin{cases} \forall \pi \in \mathcal{P}_K : \\ \nabla_{\mathbf{Q}_k^{(\pi)}} u_k(\widehat{\mathbf{Q}}_k, \widehat{\mathbf{Q}}_{-k}) = p_\pi \widehat{\lambda}_k \mathbf{I} - \widehat{\Phi}_k^{(\pi)}. \end{cases}$$

Using the third and the sixth optimality conditions, the condition given in (6) becomes:

$$\begin{aligned} \mathcal{C} = \sum_{\pi \in \mathcal{P}_K} \sum_{k=1}^K \{ & p_\pi \tilde{\lambda}_k \operatorname{Tr}(\widehat{\mathbf{Q}}_k^{(\pi)}) + p_\pi \hat{\lambda}_k \operatorname{Tr}(\widetilde{\mathbf{Q}}_k^{(\pi)}) \\ & - p_\pi \tilde{\lambda}_k \operatorname{Tr}(\widetilde{\mathbf{Q}}_k^{(\pi)}) - p_\pi \hat{\lambda}_k \operatorname{Tr}(\widehat{\mathbf{Q}}_k^{(\pi)}) \\ & - \operatorname{Tr}(\widehat{\mathbf{Q}}_k^{(\pi)} \widetilde{\Phi}_k^{(\pi)}) - \operatorname{Tr}(\widetilde{\mathbf{Q}}_k^{(\pi)} \widehat{\Phi}_k^{(\pi)}) \\ & + \operatorname{Tr}(\widetilde{\mathbf{Q}}_k^{(\pi)} \widetilde{\Phi}_k^{(\pi)}) + \operatorname{Tr}(\widehat{\mathbf{Q}}_k^{(\pi)} \widehat{\Phi}_k^{(\pi)}) \} \\ \leq \sum_{k=1}^K \Bigg\{ & \tilde{\lambda}_k \Bigg[\sum_{\pi \in \mathcal{P}_K} p_\pi \operatorname{Tr}(\widehat{\mathbf{Q}}_k^{(\pi)}) - n_t \overline{P}_k \Bigg] \\ & + \hat{\lambda}_k \Bigg[\sum_{\pi \in \mathcal{P}_K} p_\pi \operatorname{Tr}(\widetilde{\mathbf{Q}}_k^{(\pi)}) - n_t \overline{P}_k \Bigg] \Bigg\} \\ \leq 0. \end{aligned}$$

From the other four K-T conditions, we obtain that all the terms on the right are negative and thus $\mathcal{C} \leq 0$. But this contradicts the diagonally strict concavity condition and so the Nash Equilibrium is unique.

Appendix C

We want to prove that there is no optimality loss when restricting the search for the optimal covariance matrices to $\mathbf{Q}_k \in \mathcal{A}_k^{\text{SIC}}$ such that $\mathbf{Q}_k^{(\pi)} = \mathbf{W}_k \mathbf{P}_k^{(\pi)} \mathbf{W}_k^H$, for all $\pi \in \mathcal{P}_K$.

Let us consider user $k \in \mathcal{K}$. We have that:

$$\begin{aligned}
&\arg \max_{\mathbf{Q}_k \in \mathcal{A}_k^{\mathrm{SIC}}} u_k(\mathbf{Q}_k, \mathbf{Q}_{-k}) \\
&= \arg \max_{\mathbf{Q}_k \in \mathcal{A}_k^{\mathrm{SIC}}} \left\{ \sum_{\pi \in \mathcal{P}_K} p_\pi \mathbb{E} \log_2 \left| \mathbf{I} + \rho \mathbf{H}_k \mathbf{Q}_k^{(\pi)} \mathbf{H}_k^H \right. \right. \\
&\quad \left. \left. + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \mathbf{H}_\ell \mathbf{Q}_\ell^{(\pi)} \mathbf{H}_\ell^H \right| \right\} \\
&= \arg \max_{\mathbf{Q}_k \in \mathcal{A}_k^{\mathrm{SIC}}} \left\{ \sum_{\pi \in \mathcal{P}_K} p_\pi \mathbb{E} \right. \\
&\quad \times \log_2 \left| \mathbf{I} + \rho \mathbf{V} \tilde{\mathbf{H}}_k \mathbf{W}_k^H \mathbf{Q}_k^{(\pi)} \mathbf{W}_k \tilde{\mathbf{H}}_k^H \mathbf{V}^H \right. \\
&\quad \left. \left. + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \mathbf{V} \tilde{\mathbf{H}}_\ell \mathbf{W}_\ell^H \mathbf{Q}_\ell^{(\pi)} \mathbf{W}_\ell \tilde{\mathbf{H}}_\ell^H \mathbf{V}^H \right| \right\} \\
&= \arg \max_{\mathbf{Q}_k \in \mathcal{A}_k^{\mathrm{SIC}}} \left\{ \sum_{\pi \in \mathcal{P}_K} p_\pi \mathbb{E} \right. \\
&\quad \times \log_2 \left| \mathbf{I} + \rho \tilde{\mathbf{H}}_k \mathbf{W}_k^H \mathbf{Q}_k^{(\pi)} \mathbf{W}_k \tilde{\mathbf{H}}_k^H \right. \\
&\quad \left. \left. + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \tilde{\mathbf{H}}_\ell \mathbf{W}_\ell^H \mathbf{Q}_\ell^{(\pi)} \mathbf{W}_\ell \tilde{\mathbf{H}}_\ell^H \right| \right\} \\
&= \arg \max_{\mathbf{Q}_k \in \mathcal{A}_k^{\mathrm{SIC}}} \left\{ \sum_{\pi \in \mathcal{K}_k^{(\pi)}} \mathbb{E} \log_2 \left| \mathbf{I} + \rho \tilde{\mathbf{H}}_k \mathbf{X}_k^{(\pi)} \tilde{\mathbf{H}}_k^H \right. \right. \\
&\quad \left. \left. + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \tilde{\mathbf{H}}_\ell \mathbf{W}_\ell^H \mathbf{Q}_\ell^{(\pi)} \mathbf{W}_\ell \tilde{\mathbf{H}}_\ell^H \right| \right\}, \qquad (28)
\end{aligned}$$

where we denoted with $\mathbf{X}_k^{(\pi)} \triangleq \mathbf{W}_k^H \mathbf{Q}_k^{(\pi)} \mathbf{W}_k$. Knowing that the utility function is concave w.r.t. the new defined matrices $\mathbf{X}_k^{(\pi)}$, and the channel matrix $\mathbf{H}_k$ has independent entries, we can directly apply the results given in [22] to prove that annulling the non-diagonal entries of $\mathbf{X}_k^{(\pi)}$ can only increase the values of the functions

$$\mathbb{E} \log_2 \left| \mathbf{I} + \rho \tilde{\mathbf{H}}_k \mathbf{X}_k^{(\pi)} \tilde{\mathbf{H}}_k^H + \rho \sum_{\ell \in \mathcal{K}_k^{(\pi)}} \tilde{\mathbf{H}}_\ell \mathbf{W}_\ell^H \mathbf{Q}_\ell^{(\pi)} \mathbf{W}_\ell \tilde{\mathbf{H}}_\ell^H \right|.$$

In conclusion the optimal matrices $\mathbf{X}_k^{(\pi)}$ are diagonal, that we will denote with $\mathbf{P}_k^{(\pi)}$. The spectral decomposition of the optimal covariance matrices are: $\mathbf{Q}_k^{(\pi)} = \mathbf{W}_k \mathbf{P}_k^{(\pi)} \mathbf{W}_k^H$.

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A.3 Erratum

Successive Interference Cancelation

The proof of the uniqueness of the Nash equilibrium in Section 3 [27] (see Appendix A.2), on page 3 (SIC) should be read as follows:

"This theorem corresponds to the matrix generalization of the diagonally strict concavity (DSC) condition of [17] and is proven in Appendix B. To know whether this condition is verified or not in the MIMO MAC one needs to re-write it in a more exploitable manner. It can be checked that $\mathcal{C}$ expresses as $\mathcal{C} = \sum_{\pi \in \mathcal{P}_K} p_\pi \mathcal{T}_\pi$ where for each $\pi \in \mathcal{P}_K$, $\mathcal{T}_\pi$ is given by:

$$\begin{aligned}
\mathcal{T}_\pi &= \sum_{k=1}^{K} \mathrm{Tr}\left\{\left(\mathbf{Q}_k^{(\pi)''} - \mathbf{Q}_k^{(\pi)'}\right)\left[\nabla_{Q_k^{(\pi)}} R_k^{(\pi)}(\mathbf{Q}_k^{(\pi)'}, \mathbf{Q}_{-k}^{(\pi)'}) - \nabla_{Q_k^{(\pi)}} R_k^{(\pi)}(\mathbf{Q}_k^{(\pi)''}, \mathbf{Q}_{-k}^{(\pi)''})\right]\right\} \\
&= \mathbb{E}\sum_{r=1}^{K} \mathrm{Tr}\left\{\rho \mathbf{H}_{\pi^{-1}(r)}(\mathbf{Q}_{\pi^{-1}(r)}^{(\pi)''} - \mathbf{Q}_{\pi^{-1}(r)}^{(\pi)'})\mathbf{H}_{\pi^{-1}(r)}^H \right. \\
&\quad \left[\left(\mathbf{I} + \rho \mathbf{H}_{\pi^{-1}(r)}\mathbf{Q}_{\pi^{-1}(r)}^{(\pi)'}\mathbf{H}_{\pi^{-1}(r)}^H + \rho \sum_{s=r+1}^{K} \mathbf{H}_{\pi^{-1}(s)}\mathbf{Q}_{\pi^{-1}(s)}^{(\pi)'}\mathbf{H}_{\pi^{-1}(s)}^H\right)^{-1} - \right. \\
&\quad \left.\left.\left(\mathbf{I} + \rho \mathbf{H}_{\pi^{-1}(r)}\mathbf{Q}_{\pi^{-1}(r)}^{(\pi)''}\mathbf{H}_{\pi^{-1}(r)}^H + \rho \sum_{s=r+1}^{K} \mathbf{H}_{\pi^{-1}(s)}\mathbf{Q}_{\pi^{-1}(s)}^{(\pi)''}\mathbf{H}_{\pi^{-1}(s)}^H\right)^{-1}\right]\right\} \\
&= \mathbb{E}\sum_{r=1}^{K} \mathrm{Tr}\left(\mathbf{A}_r^{(\pi)''} - \mathbf{A}_r^{(\pi)'}\right)\left[\left(\mathbf{I} + \sum_{s=r}^{K} \mathbf{A}_s^{(\pi)'}\right)^{-1} - \left(\mathbf{I} + \sum_{s=r}^{K} \mathbf{A}_s^{(\pi)''}\right)^{-1}\right] \\
&\triangleq \mathbb{E}\left[F_\pi(\mathbf{H})\right]
\end{aligned} \tag{A.1}$$

where $\mathbf{A}_r^{(\pi)'} = \rho \mathbf{H}_{\pi^{-1}(r)}\mathbf{Q}_{\pi^{-1}(r)}^{(\pi)'}\mathbf{H}_{\pi^{-1}(r)}^H$, $\mathbf{A}_r^{(\pi)''} = \rho \mathbf{H}_{\pi^{-1}(r)}\mathbf{Q}_{\pi^{-1}(r)}^{(\pi)''}\mathbf{H}_{\pi^{-1}(r)}^H$ and the users have been ordered using their decoding rank rather than their index. Notice that since the expectation operator is linear we can switch between the trace and expectation.

Let us denote by $\mathbf{H} = [\mathbf{H}_1, \ldots, \mathbf{H}_K]$, $\mathbf{Q}' = (\mathbf{Q}_k')_{k\in\mathcal{K}}$, , $\mathbf{Q}'' = (\mathbf{Q}_k'')_{k\in\mathcal{K}}$, $\mathbf{Q}^{(\pi)'} = \left(\mathbf{Q}_k^{(\pi)'}\right)_{k\in\mathcal{K}}$, , $\mathbf{Q}^{(\pi)''} = \left(\mathbf{Q}_k^{(\pi)''}\right)_{k\in\mathcal{K}}$, $\mathbf{A}' = \left(\mathbf{A}_k^{(\pi)'}\right)_{\pi\in\mathcal{P}_K, k\in\mathcal{K}}$, $\mathbf{A}'' = \left(\mathbf{A}_k^{(\pi)''}\right)_{\pi\in\mathcal{P}_K, k\in\mathcal{K}}$.

In order to prove that the DSC condition holds we have to prove that for all $\mathbf{Q}' \neq \mathbf{Q}''$ we have $\mathcal{C} > 0$.

Let us give a very useful result.

Lemma 1 For any positive definite matrices $\mathbf{A}_1$, $\mathbf{B}_1$, and any positive semi-definite matrices $\mathbf{A}_i$, $\mathbf{B}_i$, $i \in \{2, \ldots, K\}$, we have that

$$\sum_{i=1}^{K} \mathrm{Tr}\left\{(\mathbf{A}_i - \mathbf{B}_i)\left[\left(\sum_{j=1}^{i} \mathbf{B}_j\right)^{-1} - \left(\sum_{j=1}^{i} \mathbf{A}_j\right)^{-1}\right]\right\} \geq 0 \tag{A.2}$$

where the equality holds if and only if $\mathbf{A}_j = \mathbf{B}_j$ *for all* $j \in \{1, \ldots, K\}$

The proof can be found in [42], for $K = 2$, and in [43] for arbitrary $K \geq 2$. Using this result, we can prove that for any channel realization, any $\mathbf{Q}', \mathbf{Q}''$ and any $\pi \in \mathcal{P}_K$:

$$F_\pi(\mathbf{H}) = \mathrm{Tr}\left(\mathbf{A}_r^{(\pi)''} - \mathbf{A}_r^{(\pi)'}\right)\left[\left(\mathbf{I} + \sum_{s=r}^{K} \mathbf{A}_s^{(\pi)'}\right)^{-1} - \left(\mathbf{I} + \sum_{s=r}^{K} \mathbf{A}_s^{(\pi)''}\right)^{-1}\right] \geq 0 \quad \text{(A.3)}$$

implying that $\mathcal{T}_\pi \geq 0$ and that $\mathcal{C} \geq 0$. Let us consider now two arbitrary covariance matrices such that $\mathbf{Q}' \neq \mathbf{Q}''$. This means that there is at least one decoding order $\vartheta \in \mathcal{P}_K$ such that $\mathbf{Q}^{(\vartheta)'} \neq \mathbf{Q}^{(\vartheta)''}$. We will prove that $\mathcal{T}_\vartheta > 0$ which will imply the desired result $\mathcal{C} > 0$.

Remark: Assuming that $\mathrm{rank}(\mathbf{H}_k^H \mathbf{H}_k) = n_t$, for all $k \in \mathcal{K}$, and $n_t^2 \leq n_r^2 + 1$, then $\mathbf{Q}' \neq \mathbf{Q}''$ implies that $\mathbf{A}' \neq \mathbf{A}''$. This means that for any channel realization we have $F_\vartheta(\mathbf{H}) > 0$ which implies directly $\mathcal{T}_\vartheta > 0$ and $\mathcal{C} > 0$.

For the general proof, let us define the following sets:

$$\begin{aligned} \mathcal{A}_H(\mathbf{Q}^{(\vartheta)'}, \mathbf{Q}^{(\vartheta)''}) &= \left\{\mathbf{H} \in \mathcal{D}_H | \forall k \in \mathcal{K} : \mathbf{H}_k(\mathbf{Q}_k{}^{(\vartheta)'} - \mathbf{Q}_k{}^{(\vartheta)''})\mathbf{H}_k^H = 0\right\} \\ \tilde{\mathcal{A}}_H(\mathbf{Q}^{(\vartheta)'}, \mathbf{Q}^{(\vartheta)''}) &= \left\{\mathbf{H} \in \mathcal{D}_H | \exists k \in \mathcal{K} : \mathbf{H}_k(\mathbf{Q}_k^{(\vartheta)'} - \mathbf{Q}_k^{(\vartheta)''})\mathbf{H}_k^H \neq 0\right\} \end{aligned} \quad \text{(A.4)}$$

We know that:

$$\begin{aligned} \mathcal{T}_\vartheta &= \mathbb{E}[F_\vartheta(\mathbf{H})] \\ &= \int_{\mathcal{D}_H} F_\vartheta(\mathbf{H}) L(\mathbf{H}) \mathrm{d}\mathbf{H} \\ &= \int_{\mathcal{A}_H(\mathbf{Q}^{(\vartheta)'}, \mathbf{Q}^{(\vartheta)''})} F_\vartheta(\mathbf{H}) L(\mathbf{H}) \mathrm{d}\mathbf{H} + \int_{\tilde{\mathcal{A}}_H(\mathbf{Q}^{(\vartheta)'}, \mathbf{Q}^{(\vartheta)''})} F_\vartheta(\mathbf{H}) L(\mathbf{H}) \mathrm{d}\mathbf{H} \\ &= \int_{\tilde{\mathcal{A}}_H(\mathbf{Q}^{(\vartheta)'}, \mathbf{Q}^{(\vartheta)''})} F_\vartheta(\mathbf{H}) L(\mathbf{H}) \mathrm{d}\mathbf{H} \end{aligned} \quad \text{(A.5)}$$

where $L(\mathbf{H}) > 0$ stands for the p.d.f. of $\mathbf{H} \in \mathcal{D}_H \equiv \mathbb{C}^{n_r \times K n_t}$. The second equality follows since $\mathcal{D}_H = \mathcal{A}_H(\mathbf{Q}^{(\vartheta)'}, \mathbf{Q}^{(\vartheta)''}) \cup \tilde{\mathcal{A}}_H(\mathbf{Q}^{(\vartheta)'}, \mathbf{Q}^{(\vartheta)''})$. The third equality follows since for all $\mathbf{H} \in \mathcal{A}_H(\mathbf{Q}^{(\vartheta)'}, \mathbf{Q}^{(\vartheta)''})$ we have that $F_\vartheta(\mathbf{H}) = 0$ from Lemma A.2. We know that for all $\mathbf{H} \in \tilde{\mathcal{A}}_H(\mathbf{Q}^{(\vartheta)'}, \mathbf{Q}^{(\vartheta)''})$ we have that $F_\vartheta(\mathbf{H}) > 0$. It suffices to prove that $\tilde{\mathcal{A}}_H(\mathbf{Q}^{(\vartheta)'}, \mathbf{Q}^{(\vartheta)''})$ is a subset of non-zero Lebesgue measure to imply that $\mathcal{T}_\vartheta > 0$ and thus that $\mathcal{C} > 0$. It is easy to see that the complementary set $\mathcal{A}_H(\mathbf{Q}^{(\tau)'}, \mathbf{Q}^{(\tau)''})$ has a zero Lebesgue measure. Following Sard's theorem the image of the set of critical points of a smooth function from one Euclidean space or manifold to another has Lebesgue measure zero, i.e. they form a null set. Thus, we have the desired result $\mathcal{C} > 0$."

Single User Decoding

The proof of uniqueness of the Nash equilibrium in Section 4 [27] (see Appendix A.2) on page 12 should be read as follows:

"*Uniqueness of the Nash equilibrium.* Here we can specialize Theorem 4, which is the matrix extension of Theorem 2. When the strategies sets are not sets of pairs of matrices

but only sets of matrices, the diagonally strict concavity condition in (6) can be written as follows. For all $\mathbf{Q}_k^{(0)'}, \mathbf{Q}_k^{(0)''} \in \mathcal{A}_k^{\text{SUD}}$ such that $(\mathbf{Q}_1^{(0)'}, \ldots, \mathbf{Q}_K^{(0)'}) \neq (\mathbf{Q}_1^{(0)''}, \ldots, \mathbf{Q}_K^{(0)''})$:

$$\mathcal{C} = \sum_{k=1}^{K} \text{Tr}\left\{(\mathbf{Q}_k^{(0)''} - \mathbf{Q}_k^{(0)'})\left[\nabla_{Q_k^{(0)}} u_1(\mathbf{Q}_k^{(0)'}, \mathbf{Q}_k^{(0)'}) - \nabla_{Q_k^{(0)}} u_1(\mathbf{Q}_k^{(0)''}, \mathbf{Q}_k^{(0)''})\right]\right\}. \tag{A.6}$$

Now we can evaluate $\mathcal{C}$ and obtain that:

$$\begin{aligned}
\mathcal{C} &= \mathbb{E}\sum_{k=1}^{K} \text{Tr}\left\{\left[\rho \mathbf{H}_k(\mathbf{Q}_k^{(0)'} - \mathbf{Q}_k^{(0)''})\mathbf{H}_k^H\right]\left[\left(\mathbf{I} + \rho\sum_{\ell=1}^{K}\mathbf{H}_\ell \mathbf{Q}_\ell^{(0)''}\mathbf{H}_\ell^H\right)^{-1}\right.\right. \\
&\quad \left.\left. - \left(\mathbf{I} + \rho\sum_{\ell=1}^{K}\mathbf{H}_\ell \mathbf{Q}_\ell^{(0)'}\mathbf{H}_\ell^H\right)^{-1}\right]\right\} \\
&= \mathbb{E}\text{Tr}\{(\mathbf{B}' - \mathbf{B}'')[(\mathbf{B}'')^{-1} - (\mathbf{B}')^{-1}]\}, \\
&= \mathbb{E}[F_0(\mathbf{H})]
\end{aligned} \tag{A.7}$$

which is positive for any $\mathbf{B}' = \mathbf{I} + \sum_{\ell=1}^{K}\mathbf{H}_\ell \mathbf{Q}^{(0)'}\mathbf{H}_\ell^H$, $\mathbf{B}'' = \mathbf{I} + \sum_{\ell=1}^{K}\mathbf{H}_\ell \mathbf{Q}^{(0)''}\mathbf{H}_\ell^H$ from (A.2) for $K = 2$. We need to prove that for any $(\mathbf{Q}_1^{(0)'}, \ldots, \mathbf{Q}_K^{(0)'}) \neq (\mathbf{Q}_1^{(0)''}, \ldots, \mathbf{Q}_K^{(0)''})$ we have $\mathcal{C} > 0$.

Remark: Assuming that $\text{rank}(\mathbf{H}^H\mathbf{H}) = Kn_t$ and $Kn_t^2 \leq n_r^2 + K$, then $\mathbf{Q}' \neq \mathbf{Q}''$ implies that $\mathbf{B}' \neq \mathbf{B}''$. This means that for any channel realization we have $F_0(\mathbf{H}) > 0$ which implies directly that $\mathcal{C} > 0$.

For the general proof, we define the following sets:

$$\begin{aligned}
\mathcal{B}_H(\mathbf{Q}^{(0)'}, \mathbf{Q}^{(0)''}) &= \left\{\mathbf{H} \in \mathcal{D}_H | \textstyle\sum_{k=1}^{K} \mathbf{H}_k(\mathbf{Q}_k{}^{(0)'} - \mathbf{Q}_k{}^{(0)''})\mathbf{H}_k^H = 0\right\} \\
\tilde{\mathcal{B}}_H(\mathbf{Q}^{(0)'}, \mathbf{Q}^{(0)''}) &= \left\{\mathbf{H} \in \mathcal{D}_H | \textstyle\sum_{k=1}^{K} \mathbf{H}_k(\mathbf{Q}_k{}^{(0)'} - \mathbf{Q}_k{}^{(0)''})\mathbf{H}_k^H \neq 0\right\}
\end{aligned} \tag{A.8}$$

We know that:

$$\begin{aligned}
\mathcal{C} &= \mathbb{E}[F_0(\mathbf{H})] \\
&= \int_{\mathcal{D}_H} F_0(\mathbf{H})L(\mathbf{H})\text{d}\mathbf{H} \\
&= \int_{\mathcal{B}_H(\mathbf{Q}^{(0)'}, \mathbf{Q}^{(0)''})} F_0(\mathbf{H})L(\mathbf{H})\text{d}\mathbf{H} + \int_{\tilde{\mathcal{B}}_H(\mathbf{Q}^{(0)'}, \mathbf{Q}^{(0)''})} F_0(\mathbf{H})L(\mathbf{H})\text{d}\mathbf{H} \\
&= \int_{\tilde{\mathcal{B}}_H(\mathbf{Q}^{(0)'}, \mathbf{Q}^{(0)''})} F_0(\mathbf{H})L(\mathbf{H})\text{d}\mathbf{H}
\end{aligned} \tag{A.9}$$

The second equality follows since $\mathcal{D}_H = \mathcal{B}_H(\mathbf{Q}^{(0)'}, \mathbf{Q}^{(0)''}) \cup \tilde{\mathcal{B}}_H(\mathbf{Q}^{(0)'}, \mathbf{Q}^{(0)''})$. The third equality follows because $F_0(\mathbf{H}) = 0$ for all $\mathbf{H} \in \mathcal{B}_H(\mathbf{Q}^{(0)'}, \mathbf{Q}^{(0)''})$ from Lemma 1. We also know that $F_0(\mathbf{H}) > 0$ for all $\mathbf{H} \in \tilde{\mathcal{B}}_H(\mathbf{Q}^{(0)'}, \mathbf{Q}^{(0)''})$. It suffices to prove that $\tilde{\mathcal{B}}_H(\mathbf{Q}^{(0)'}, \mathbf{Q}^{(0)''})$ is a subset of non-zero Lebesgue measure to imply that $\mathcal{C} > 0$. Here

as well, is easy to see that the complementary set $\mathcal{B}_H(\mathbf{Q}^{(\tau)'}, \mathbf{Q}^{(\tau)''})$ has a zero Lebesgue measure (similarly to the proof for the SIC decoding technique by using Sard's theorem).
”

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A GENERALIZATION OF A TRACE INEQUALITY FOR POSITIVE DEFINITE MATRICES.

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ABSTRACT. In this note, we provide the generalization of the trace inequality derived in [1]. More precisely, we prove that $\mathrm{Tr}\left\{\sum_{k=1}^{K}(\mathbf{A}_k-\mathbf{B}_k)\left[\left(\sum_{\ell=1}^{k}\mathbf{B}_\ell\right)^{-1}-\left(\sum_{\ell=1}^{k}\mathbf{A}_\ell\right)^{-1}\right]\right\}\geq 0$, for arbitrary $K\geq 1$ where $\mathrm{Tr}(\cdot)$ denotes the matrix trace operator, $\mathbf{A}_1$, $\mathbf{B}_1$ are any positive definite matrices and $\mathbf{A}_k$, $\mathbf{B}_k$, for all $k\in\{2,\ldots,K\}$, are any positive semidefinite matrices.

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1. INTRODUCTION

Trace inequalities are useful in many areas such as the multiple input multiple output (MIMO) systems in control theory and communications. The trace inequality derived in this paper is used to prove the sufficient condition that guarantees the uniqueness of a Nash equilibrium in certain MIMO communications games with $K \geq 1$ players (see [2] for details). To be more specific, the diagonally strict concavity condition of Rosen [3] is proven to be satisfied in the scenario of [2]. The trace inequality under discussion has already been proven in two special cases: in [4] for $K = 1$ and in [1] for $K = 2$. In what follows, we will provide the proof for the general case where $K \geq 1$ is arbitrary.

Theorem 1.1. *Let K be a strictly positive integer, $\mathbf{A}_1$, $\mathbf{B}_1$ be any positive definite matrices and $\forall k \in \{2, \ldots, K\}$, $\mathbf{A}_k$, $\mathbf{B}_k$ be any positive semidefinite matrices. Then*

$$\mathcal{T}_K \triangleq \mathrm{Tr}\left\{\sum_{k=1}^{K}(\mathbf{A}_k - \mathbf{B}_k)\left[\left(\sum_{\ell=1}^{k}\mathbf{B}_\ell\right)^{-1} - \left(\sum_{\ell=1}^{k}\mathbf{A}_\ell\right)^{-1}\right]\right\} \geq 0, \tag{1.1}$$

where $\mathrm{Tr}(\cdot)$ denotes the matrix trace operator.

2. AUXILIARY RESULTS

In order to prove Theorem 1.1, we will use the following auxiliary results.

Lemma 2.1. [1] *Let $\mathbf{A}$, $\mathbf{B}$ be two positive definite matrices and $\mathbf{C}$, $\mathbf{D}$ be two positive semidefinite matrices and $\mathbf{X}$ a Hermitian matrix. Then*

$$\mathrm{Tr}\left\{\mathbf{X}\mathbf{A}^{-1}\mathbf{X}\mathbf{B}^{-1}\right\} - \mathrm{Tr}\left\{\mathbf{X}(\mathbf{A}+\mathbf{C})^{-1}\mathbf{X}(\mathbf{B}+\mathbf{D})^{-1}\right\} \geq 0. \tag{2.1}$$

The proof can be found in [1].

Lemma 2.2. *Let $\mathbf{A}$, $\mathbf{B}$ be two positive definite matrices, $\mathbf{C}$, $\mathbf{D}$, two positive semi-definite matrices. Then*

$$\begin{aligned} \mathrm{Tr}\left\{(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}\right\} &= \\ \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}\right\} &\in \mathbb{R}. \end{aligned} \tag{2.2}$$

Proof. To prove this result, let us define $\mathcal{E}$ as follows:

$$\mathcal{E} = \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})\left[(\mathbf{B}+\mathbf{D})^{-1} - (\mathbf{A}+\mathbf{C})^{-1}\right]\right\} \tag{2.3}$$

We observe that $\mathcal{E}$ can be written in two different ways:

$$\begin{aligned} \mathcal{E} &= \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}[\mathbf{A}+\mathbf{C}-\mathbf{B}-\mathbf{D}](\mathbf{A}+\mathbf{C})^{-1}\right\} \\ &= \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}\right\} + \\ &\quad \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}\right\}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \mathcal{E} &= \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}[\mathbf{A}+\mathbf{C}-\mathbf{B}-\mathbf{D}](\mathbf{B}+\mathbf{D})^{-1}\right\} \\ &= \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}\right\} + \\ &\quad \mathrm{Tr}\left\{(\mathbf{C}-\mathbf{D})(\mathbf{A}+\mathbf{C})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{D})^{-1}\right\}. \end{aligned} \tag{2.5}$$

Using this fact and the commutative property of the trace of a matrix product, the desired result follows directly. The only thing left to be proven is that $\mathcal{E}$ is real. To this end, if we denote by

$\mathbf{M} = (\mathbf{C}-\mathbf{D})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{C})^{-1}$, we observe that $\mathbf{M}^H = (\mathbf{A}+\mathbf{C})^{-1}(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{C}-\mathbf{D})$. Therefore, we obtain that $\mathrm{Tr}(\mathbf{M}^H) = \mathrm{Tr}(\mathbf{M})$ and also $\mathrm{Tr}(\mathbf{M}) \in \mathbb{R}$. ∎

3. PROOF OF THEOREM 1.1

Define, for all $k \geq 1$, $\mathbf{X}_k = \sum_{i=1}^{k} \mathbf{A}_i$ and $\mathbf{Y}_k = \sum_{i=1}^{k} \mathbf{B}_i$. Notice that $\mathbf{X}_k$ and $\mathbf{Y}_k$ are positive definite matrices. We observe that $\mathcal{T}_K$ can be re-written recursively as follows:

$$\begin{cases} \mathcal{T}_1 &= \mathrm{Tr}\left\{(\mathbf{A}_1-\mathbf{B}_1)\mathbf{Y}_1^{-1}(\mathbf{A}_1-\mathbf{B}_1)\mathbf{X}_1^{-1}\right\} \\ \mathcal{T}_K &= \mathcal{T}_{K-1} + \mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\} + \\ & \quad \mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\} \end{cases} \tag{3.1}$$

We proceed in two steps. First, we find a lower bound for $\mathcal{T}_K$ and then we prove that this bound is positive.

We start by proving that, for all $K \geq 1$:

$$\mathcal{T}_K \geq \frac{1}{2}\sum_{i=1}^{K}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\} + \frac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_K-\mathbf{Y}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_K-\mathbf{Y}_K)\mathbf{X}_K^{-1}\right\} \tag{3.2}$$

To this end we proceed by induction on K. For all $K \geq 1$, define the proposition:

$$\mathcal{P}_K : \mathcal{T}_K \geq \frac{1}{2}\sum_{i=1}^{K}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\} + \frac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_K-\mathbf{Y}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_K-\mathbf{Y}_K)\mathbf{X}_K^{-1}\right\}. \tag{3.3}$$

It is easy to check that, for $K = 1$, $\mathcal{P}_1$ is true:

$$\begin{aligned} \mathcal{T}_1 &= \mathrm{Tr}\left\{(\mathbf{A}_1-\mathbf{B}_1)\mathbf{Y}_1^{-1}(\mathbf{A}_1-\mathbf{B}_1)\mathbf{X}_1^{-1}\right\} \\ &= \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{A}_1-\mathbf{B}_1)\mathbf{Y}_1^{-1}(\mathbf{A}_1-\mathbf{B}_1)\mathbf{X}_1^{-1}\right\} + \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_1-\mathbf{Y}_1)\mathbf{Y}_1^{-1}(\mathbf{X}_1-\mathbf{Y}_1)\mathbf{X}_1^{-1}\right\}. \end{aligned} \tag{3.4}$$

Now, let us assume that $\mathcal{P}_{K-1}$ is true and prove that $\mathcal{P}_K$ is also true. We have that:

$$\begin{aligned} \mathcal{T}_{K-1} &\geq \tfrac{1}{2}\sum_{i=1}^{K-1}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\} + \\ & \quad \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_{K-1}^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_{K-1}^{-1}\right\}. \end{aligned} \tag{3.5}$$

From (3.5) and the recursive formula (3.1), we further obtain:

(3.6)

$$
\begin{aligned}
\mathcal{T}_K \quad \geq \quad & \tfrac{1}{2}\sum_{i=1}^{K-1}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_{K-1}^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_{K-1}^{-1}\right\}+ \\
& \mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\}+\mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\} \\
\overset{(a)}{\geq} \quad & \tfrac{1}{2}\sum_{i=1}^{K-1}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\}+ \\
& \mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\}+\mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\} \\
\overset{(b)}{=} \quad & \tfrac{1}{2}\sum_{i=1}^{K-1}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\}+ \\
& \mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\}+\tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\} \\
= \quad & \tfrac{1}{2}\sum_{i=1}^{K}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\}+\tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{A}_K-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{X}_K^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}-\mathbf{Y}_{K-1})\mathbf{Y}_K^{-1}(\mathbf{A}_K-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\} \\
= \quad & \tfrac{1}{2}\sum_{i=1}^{K}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\}+ \\
& \tfrac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_{K-1}+\mathbf{A}_K-\mathbf{Y}_{K-1}-\mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1}+\mathbf{A}_K-\mathbf{Y}_{K-1}-\mathbf{B}_K)\mathbf{X}_K^{-1}\right\} \\
\overset{(c)}{=} \quad & \tfrac{1}{2}\sum_{i=1}^{K}\mathrm{Tr}\left\{(\mathbf{A}_i-\mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i-\mathbf{B}_i)\mathbf{X}_i^{-1}\right\}+\frac{1}{2}\mathrm{Tr}\left\{(\mathbf{X}_K-\mathbf{Y}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_K-\mathbf{Y}_K)\mathbf{X}_K^{-1}\right\}.
\end{aligned}
$$

The inequality (a) follows by applying Lemma 2.1 to the second term on the right and also by considering that $\mathbf{X}_K=\mathbf{X}_{K-1}+\mathbf{A}_K$ and $\mathbf{Y}_K=\mathbf{Y}_{K-1}+\mathbf{B}_K$. The equality (b) follows from Lemma 2.2.

The last step of the proof of Theorem 1.1 reduces to showing that the term on the right side of the equality (c) is positive. This can be easily checked by observing that all the terms of the form $\mathrm{Tr}\{\mathbf{X}\mathbf{B}^{-1}\mathbf{X}\mathbf{A}^{-1}\}$, with $\mathbf{X}$ a Hermitian matrix, $\mathbf{A}$ and $\mathbf{B}$ two positive definite matrices, can be re-written as $\mathrm{Tr}(\mathbf{N}\mathbf{N}^H) \geq 0$ with $\mathbf{N} = \mathbf{A}^{-1/2}\mathbf{X}\mathbf{B}^{-1/2}$. Thus, for all $K \geq 1$, $\mathcal{T}_K \geq 0$ and the desired result has been proven.

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A.5 ACM/ICST-VALUETOOLS-2009

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On the Base Station Selection and Base Station Sharing in Self-Configuring Networks

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ABSTRACT

We model the interaction of several radio devices aiming to obtain wireless connectivity by using a set of base stations (BS) as a non-cooperative game. Each radio device aims to maximize its own spectral efficiency (SE) in two different scenarios: First, we let each player to use a unique BS (BS selection) and second, we let them to simultaneously use several BSs (BS Sharing). In both cases, we show that the resulting game is an exact potential game. We found that the BS selection game posses multiple Nash equilibria (NE) while the BS sharing game posses a unique one. We provide fully decentralized algorithms which always converge to a NE in both games. We analyze the price of anarchy and the price of stability for the case of BS selection. Finally, we observed that depending on the number of transmitters, the BS selection technique might provide a better global performance (network spectral efficiency) than BS sharing, which suggests the existence of a Braess type paradox.

General Terms

Game Theory, Potential Games, Base Station Selection, Base Station Sharing, Self-Configuring Networks, Braess Paradox.

1. INTRODUCTION

In this paper, we consider the case where several radio devices aim to obtain wireless connectivity by using several base stations (BS). Here, each device must strategically determine the set of BSs to use, as well as the corresponding power level allocated to each BS to maximize its own spectral efficiency in bps/Hz. In this context, we consider two different scenarios. First, we let each player to use a unique BS (BS selection) and second, we let them to simultaneously use several BSs (BS Sharing).

Note that if only one BS is considered, our model simplifies to a multiple access channel (MAC). Here, when all transmitters access the BS using the same carrier, each device uses its maximum transmit power. When, the BS is accessible through out several carriers, each transmitter uses a water-filling power allocation (PA) considering the observed multiple access interference as background noise at each carrier [29]. In the first case, such solution is Pareto optimal, if and only if the sum of the achieved Shannon rates lies in the convex hulk of the capacity region of the corresponding MAC [10]. Generally, this condition may require certain coordination between the transmitters, which can be achieved by using pricing methods [1]. Conversely, in the second case, the solution is always Pareto optimal [29]. In a more general context, when there exists several BSs and regardless of the performance metric, the model remains being a subject of intensive research [3], [4], [12], [11], [24], [28].

Up to the knowledge of the authors, the state of art of the BS sharing and BS selection scenarios is described by the following contributions: [18], [3], [4], [24]. In [18], the BS selection problem is investigated by considering that each node is characterized by a fixed single user spectral efficiency. Here, the authors showed that based on the scheme of exponential learning, players converge to an evolutionarily stable equilibrium. Additionally, the authors showed that the price of anarchy of such a game is unaffected by disparities in the nodes' characteristics. In [24], the authors studied the BS selection scenario assuming that the transmitters aim to minimize their transmit power level required to achieve a target signal to interference plus noise ratio (SINR). Here, the interaction between the radio devices is modeled as an atomic and non-atomic potential game [19] to study the existence, uniqueness and efficiency of the NE. Other contributions using potential games for radio resource allocation are [23, 27,

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16, 8, 9]. In [4], the non-atomic extension of the BS selection game and the atomic extension of the BS sharing game are investigated. Therein, the performance metric is the Shannon rate and channel realizations are considered identical for all transmitters. Regardless of the possibly unrealistic assumption, the authors of [4] identified the existence of at least one NE in the non-atomic BS selection game and the existence of a unique equilibrium in the atomic BS sharing game. In [3], the authors study the BS selection and sharing scenarios when the number of receivers is equal to the number of transmitters and the performance metric is their overall SINR, i.e., the sum of the SINRs obtained at each BS. In this context, it is showed that when all players observed the same channel realization (as in [4]), restricting each player to choose only one BS produces a socially optimal NE. Conversely, when the players are left free to share their powers among several BSs, their utilities are strongly decreased. This effect is known as the Braess Paradox in the frame of congestion games [6].

In this paper, we tackle the BS selection and BS Sharing scenarios by modeling them as potential games. Contrary to previous works [24], we consider as performance metric the spectral efficiency of each player and we let the channel realization for each transmitter to be independently drawn from a given probability distribution. In the former case, we study both the atomic and non-atomic extensions of the game. In the atomic case, we show the existence of multiple NE and we use the best response dynamics to provide fully distributed algorithms to achieve a NE. This algorithm is proved to converge independently of the channel realization and the bandwidth allocated to each BS. We measure the price of anarchy of this solution and we observed that the performance of the fully decentralized solution (self-configured network) is close to that one obtained when there exists a central controller (optimally configured networks). In the non-atomic extension, we provide the optimal fractions of transmitter which must join each BS depending on their available bandwidths. Regarding, the BS sharing game, we show the existence of a unique NE. As in the previous case, we provide a fully decentralized algorithm which allows achieving a NE with probability one. Finally, we compare both scenarios and identify that BS selection performs better than BS sharing when there exists almost the same number or more transmitters than BSs. As identified in [3], this observation constitutes a Braess type paradox, which implies that increasing the space of strategies of each player, i.e., the number of BSs each player can use, ends up degenerating the global performance of the network.

2. SYSTEM MODEL

Notation: In the sequel, matrices, vectors and scalars are respectively denoted by boldface upper case symbols, boldface lower case symbols, and italic lower case symbols. The transpose and Hermitian transpose of a vector $\boldsymbol{x}$ (matrix $\boldsymbol{X}$) is denoted by $\boldsymbol{x}^T$ and $\boldsymbol{x}^H$ (resp. $\boldsymbol{X}^T$ and $\boldsymbol{X}^H$). The sets of natural and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. Finite sets of natural numbers are denoted by calligraphic upper case letters. Given two sets denoted by $\mathcal{A}$ and $\mathcal{B}$, their Cartesian product is denoted by $\mathcal{A}\times\mathcal{B}$. The cardinality of set $\mathcal{A}$ is denoted by $|\mathcal{A}|$. The S-dimensional vectors $\boldsymbol{e}_s$, for all $s \in \{1,\dots,S\}$ and $S \in \mathbb{N}$, denotes a vector with zeros in all its entries except the s-th entry which is unitary. The operator $[x]^+$, with $x \in \mathbb{R}$, represents the operation $\max(0,x)$.

Consider a set $\mathcal{K} = \{1,\dots,K\}$ of transmitters and a set of $\mathcal{S} = \{1,\dots,S\}$ receivers, e.g., base stations (BS) or access points (AP). Each transmitter can access the network by using a (non-empty) set of BSs. Each BS operates in a specific frequency band and we neglect any type of interference due to the adjacent bands (adjacent channel interference). We denote the bandwidth associated with BS $s \in \mathcal{S}$ by B_s and the total network bandwidth by $B = \sum_{s=1}^{S} B_s$. Each transmitter sends private messages only to its corresponding BSs and it does not exist any kind of information exchange between transmitters neither before nor during the whole transmission. Both transmitters and BSs are equipped with single antennas. Transmitter $k \in \mathcal{K}$ is able to simultaneously transmit to all the BSs subject to a power constraint,

$$\forall k \in \mathcal{K}, \quad \sum_{s=1}^{S} p_{k,s} \leqslant p_{k,\max}, \tag{1}$$

where $p_{k,s}$ and $p_{k,\max}$ respectively denote the transmit power dedicated to BS s by transmitter k and its maximum total transmit power. Without any loss of generality, we assume that all transmitters are limited by the same maximum transmit power level, i.e., $\forall k \in \mathcal{K}$, $p_{k,\max} = p_{\max}$.

For all $(k,s) \in \mathcal{K}\times\mathcal{S}$, we denote the channel coefficients between transmitter k and BS s by $h_{k,s}$. Each channel coefficient $h_{k,s}$ is a realization of a circularly symmetric complex Gaussian random variable h with zero mean and unit variance. We consider a slow fading channel, so that all channel realizations remain constant during the transmission time. The baseband received signals sampled at symbol rate at BS s, denoted by y_s, can be written as a vector $\boldsymbol{y} = (y_1,\dots,y_S)^T$, such that

$$\forall s \in \mathcal{S}, \quad y_s = \boldsymbol{h}_s \boldsymbol{x}_s^T + \boldsymbol{w}. \tag{2}$$

Here, for all $(k,s) \in \mathcal{K}\times\mathcal{S}$, the K-dimensional vector $\boldsymbol{h}_s = (h_{1,s},\dots,h_{K,s})$. The K-dimensional vector $\boldsymbol{x}_s = (x_{1,s},\dots,x_{K,s})$, and $x_{k,s}$ represents the symbol sent by transmitter k to BS s. The power allocation vector of transmitter k is the vector $(p_{k,1},\dots,p_{k,S})$, and $p_{k,s} = \mathbb{E}\left[x_{k,s}x_{k,s}^*\right]$ represents the power transmitted toward BS s by player k. The S-dimensional vector $\boldsymbol{w} = (w_1,\dots,w_S)$, with $w_s \sim \mathcal{N}\left(0,\sigma_s^2\right)$ represents the noise at the receivers. Here, $\sigma_s^2 = N_0 B_s$, where N_0 denotes the noise spectral density.

The SINR of transmitter k at BS s is denoted by $\gamma_{k,s}$ and $\forall (k,s) \in \mathcal{K}\times\mathcal{S}$,

$$\gamma_{k,s} = \frac{p_{k,s} g_{k,s}}{\zeta_{k,s}}, \tag{3}$$

where, $\zeta_{k,s} = \sigma_s^2 + \sum_{j\in\mathcal{K}\setminus k} p_{j,s} g_{j,s}$ represents the noise plus multiple access interference (MAI) undergone by player k at BS s and $g_{k,s} = |h_{k,s}|^2$ represents the channel gains. We denote by $\mathcal{K}_s$ the set of transmitters using the BS s. Then, we define two different scenarios depending on the conditions over the sets $\mathcal{K}_s$ for all $s \in \mathcal{S}$. In the first scenario, named BS selection, each transmitter uses a unique BS. Thus, for all $s \in \mathcal{S}$, the sets $\mathcal{K}_s$ such that $|\mathcal{K}_s| > 0$ form a partition of the set $\mathcal{K}$, i.e., $\forall (j,k) \in \mathcal{S}^2$ and $j \neq k$, $\mathcal{K}_j \cap \mathcal{K}_k = \emptyset$

and $\mathcal{K}_1 \cup \ldots \cup \mathcal{K}_S = \mathcal{K}$. In the second scenario, named BS Selection, a given transmitter is allowed to simultaneously use several BSs. Thus, for all $s \in \mathcal{S}$, the sets $\mathcal{K}_s$ form a cover of the set $\mathcal{K}$, i.e., $\forall s \in \mathcal{S}, \quad \mathcal{K}_s \subseteq \mathcal{K}$. In the following two sections, we study both scenarios. Later, we compare their performance by simulation results.

3. BASE STATION SELECTION GAMES

Assume that each transmitter can be modeled as a rational selfish player and that such an assumption is common knowledge among all players. Then, the BS selection scenario can be modeled by a non-cooperative game $\mathcal{G}_1$ described by the tuple $(\mathcal{K}, (\mathcal{P}_k)_{k\in\mathcal{K}}, (u_k)_{k\in\mathcal{K}})$. Here, the set of transmitters $\mathcal{K}$ is the set of players. The strategy of a given player $k \in \mathcal{K}$ is its PA scheme, i.e., the S-dimensional PA vector $\boldsymbol{p}_k = (p_{k,1}, \ldots, p_{k,S}) \in \mathcal{P}_k$, where $\mathcal{P}_k$ is the set of all actions of player k (strategy set). Since each player only transmits to a unique BS, its strategy set is defined as a finite set $\mathcal{P}_k$,

$$\begin{aligned}\mathcal{P}_k \quad = \quad & \{p_k\, \boldsymbol{e}_s : p_k \in [0, p_{\max}]\,,\ \forall s \in \mathcal{S},\ \boldsymbol{e}_s = (e_{s,1}, \ldots, e_{s,S}) \\ & \text{and } \ \forall r \in \mathcal{S} \setminus s, e_{s,r} = 0, \text{ and } e_{s,s} = 1\}\,. \end{aligned} \tag{4}$$

Then, a strategy profile of the game is a super vector

$$\boldsymbol{p} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_K) \in \mathcal{P},$$

where $\mathcal{P}$ is a finite set obtained from the Cartesian product of the strategy sets $\mathcal{P}_k$, for all $k \in \mathcal{K}$, i.e., $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_K$. Let us denote by $\boldsymbol{p}_{-k}$ any vector in the finite set $\mathcal{P}_{-k} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_{k-1} \times \mathcal{P}_{k+1} \times \ldots \times \mathcal{P}_K$. For a given $k \in \mathcal{K}$, the vector denoted by $\boldsymbol{p}_{-k}$ represents the strategies adopted by all the players other than player k. The utility function for player k, is defined as $u_k : \ \mathcal{P} \to \mathbb{R}_+$ and measures the satisfaction of player k with respect to its chosen strategy [21]. In this study, we define the utility function for all players as their spectral efficiency, i.e., the ratio between their Shannon transmission rate and the available bandwidth B:

$$u_k(\boldsymbol{p}_k, \boldsymbol{p}_{-k},) = \sum_{s\in\mathcal{S}} \frac{B_s}{B} \log_2 (1 + \gamma_{k,s})\,, \tag{5}$$

where $\gamma_{k,s}$ is given by (Eq. 3) and $\boldsymbol{p} \in \mathcal{P}$.

In the sequel, we consider a finite number of transmitters (players) such that each player is concerned with the strategy (BS selection and transmit power allocation) adopted by all the other players due to mutual interference. We name this model: atomic BS selection game. In the second part, we consider a high number of players such that each of them is indifferent to the strategy adopted by every single player. In this case, each player is rather concerned with the fraction of players adopting the same strategy. We name this model non-atomic BS selection games.

3.1 Atomic BS Selection Games

In the atomic extension of the BS selection game $\mathcal{G}_1$, our interest is to find a strategy profile $\boldsymbol{p}^* \in \mathcal{P}$ such that no player is interested in changing its own strategy. Once the network configuration $\boldsymbol{p}^*$ is reached, any unilateral deviation of a given player decreases its own utility. A network configuration $\boldsymbol{p}^*$ is known as a Nash equilibrium [20].

Definition 1 (Nash Equilibrium). In the strategic game $\mathcal{G}_1$, a strategy profile $\boldsymbol{p} \in \mathcal{P}$ is an NE if it satisfies, for all $k \in \mathcal{K}$ and for all $\boldsymbol{p}'_k \in \mathcal{P}_k$, that

$$u_k(\boldsymbol{p}_k, \boldsymbol{p}_{-k}) \geqslant u_k(\boldsymbol{p}'_k, \boldsymbol{p}_{-k}). \tag{6}$$

In the following, we analyze the existence, multiplicity and determination of such strategy profiles.

3.1.1 Existence of at least one NE

Our first step toward identifying the strategy profiles leading to a NE is to prove that there exists at least one NE for any specific number of transmitters and BSs regardless of the channel realizations. There exist several methodologies for proving this [15]. In our case, we first show that the game $\mathcal{G}_1$ is a potential game (PG).

Definition 2 (Exact Potential Game). *Any strategic game $\mathcal{G}$ defined by the tuple $\left(\mathcal{K}, (\mathcal{P}_k)_{k\in\mathcal{K}}, (u_k)_{k\in\mathcal{K}}\right)$ is an exact potential game (PG) if there exists a function $\phi(\boldsymbol{p})$ for all $\boldsymbol{p} \in \mathcal{P}$ such that for all players $k \in \mathcal{K}$ and for all $\boldsymbol{p}'_k \in \mathcal{P}_k$, it holds that*

$$u_k(\boldsymbol{p}_k, \boldsymbol{p}_{-k}) - u_k(\boldsymbol{p}'_k, \boldsymbol{p}_{-k}) \quad = \quad \phi(\boldsymbol{p}) - \phi(\boldsymbol{p}'), \tag{7}$$

where $\boldsymbol{p}' = \left(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{k-1}, \boldsymbol{p}'_k, \boldsymbol{p}_{k+1}, \ldots, \boldsymbol{p}_K\right)$.

Def. 2 together with Eq. (5) allow us to write the following proposition:

Proposition 3. *The strategic game $\mathcal{G}_1$ is an exact potential game with potential function*

$$\phi(\boldsymbol{p}) = \sum_{s\in\mathcal{S}} \frac{B_s}{B} \log_2 \left(\sigma_s^2 + \sum_{k=1}^{K} p_{k,s} g_{k,s}\right). \tag{8}$$

Since the BS selection game $\mathcal{G}_1$ is a PG (Prop. 3), the following proposition (Prop. 4) is an immediate consequence of *Corollary 2.2* in [19].

Proposition 4 (Existence of the NE). *The BS selection game $\mathcal{G}_1$ always has at least one NE in pure strategies.*

3.1.2 Multiplicity of the NE

Once we have ensured the existence of at least one NE, we determine whether there exists a unique NE or several NE. As a first step, we show that rational players always transmit at the maximum power level $p_{\max}$:

Proposition 5. *In the BS selection game $\mathcal{G}_1$, all players will always transmit at the maximum power independently of the channel chosen to transmit.*

Proof. The utility function of player $k \in \mathcal{K}$ transmitting to a given BS $s \in \mathcal{S}$ is $u_k(p_k \boldsymbol{e}_s, \boldsymbol{p}_{-k}) = \log_2 \left(1 + \frac{p_k g_{k,s}}{\zeta_{k,s}}\right)$. Then, since the logarithmic function is an increasing function, we have that $\forall (k,s) \in \mathcal{K} \times \mathcal{S}$, and $\forall p_k \in [0, p_{\max}]$, $u_k(p_k \boldsymbol{e}_s, \boldsymbol{p}_{-k}) = \log_2(1 + \frac{p_k g_{k,s}}{\zeta_{k,s}}) \leqslant u_k(p_{\max} \boldsymbol{e}_s, \boldsymbol{p}_{-k}) = \log_2(1 + \frac{p_{\max} g_{k,s}}{\zeta_{k,s}})$. Hence, rational players will always use their maximum transmit power level. □

Prop. 5 shows that the strategy set in (4) can be re-defined as follows

$$\begin{aligned}\mathcal{P}_k \quad = \quad & \{p_{\max}\, \boldsymbol{e}_s : \ \forall s \in \mathcal{S},\ \boldsymbol{e}_s = (e_{s,1}, \ldots, e_{s,S}) \\ & \text{and } \ \forall r \in \mathcal{S} \setminus s, e_{s,r} = 0, \text{ and } e_{s,s} = 1\}\,. \end{aligned} \tag{9}$$

The re-definition of the strategy sets $\mathcal{P}_k$ in Eq. (4) allows us to study the multiplicity of the NE by using basic elements of graph theory. First, let us index the elements of the strategy set $\mathcal{P}$ by using the set $\mathcal{I} = \{1, \ldots, S^K\}$ such that they are ordered following the index $i \in \mathcal{I}$. Denote by $\boldsymbol{p}^{(i)}$ the i-th element of the strategy set $\mathcal{P}$. Let us write each vector $\boldsymbol{p}^{(i)}$ with $i \in \mathcal{I}$, as a vector $\boldsymbol{p}^{(i)} = \left(\boldsymbol{p}_1^{(i)}, \ldots, \boldsymbol{p}_K^{(i)}\right)$, where for all $j \in \mathcal{K}$, $\boldsymbol{p}_j^{(i)} \in \mathcal{P}_j$. Second, consider that each of the strategy profiles $\boldsymbol{p}^{(i)}$ with $i \in \mathcal{I}$ is represented by a vertex v_i in a given non-directed graph G. Each vertex is adjacent to the $K(S-1)$ vertices representing the strategy profiles obtained by letting only one player to change its own strategy. Let us denote by $\mathcal{V}_i$ the set of indices of the adjacent vertices of vertex v_i. More precisely, the graph G can be defined by the tuple $G = (\mathcal{V}, \boldsymbol{A})$, where the set $\mathcal{V} = \{v_1, \ldots, v_{S^K}\}$ contains the S^K possible strategy profiles of the game and $\boldsymbol{A}$ is a symmetric matrix (adjacency matrix of G) with dimensions $S^K \times S^K$ and entries defined as follows $\forall (i,j) \in \mathcal{I}^2$ and $i \neq j$,

$$a_{i,j} = a_{j,i} = \begin{cases} 1 & \text{if } i \in \mathcal{V}_j \\ 0 & \text{otherwise} \end{cases}, \tag{10}$$

and $a_{i,i} = 0$ for all $i \in \mathcal{I}$. In the non-directed graph G, we define the distance between vertices v_i and v_j, for all $(i,j) \in \mathcal{I}^2$ as the length of the shortest path between v_i and v_j. Considering the structure of G, a more precise definition can be formulated for the shortest path,

Definition 6. *[Shortest Path] The distance (shortest path) between vertices v_i and v_j, with $(i,j) \in \mathcal{I}^2$ in a given non-directed graph $G = (\mathcal{V}, A)$, denoted by $d_{i,j}(G) \in \mathbb{N}$ is:*

$$d_{i,j}(G) = d_{j,i}(G) = \sum_{k=1}^{K} \mathbb{1}_{\left\{\boldsymbol{p}_k^{(i)} \neq \boldsymbol{p}_k^{(j)}\right\}}. \tag{11}$$

Note that the non-directed graph G satisfies the property: $\forall (i,j) \in \mathcal{I}^2$, with $i \neq j$, $\quad 1 \leqslant d_{i,j}(G) \leqslant K$. Thus, for a specific number S of BSs and K transmitters, the maximum number of NE which can be observed is obtained as follows:

Proposition 7. *In a given BS selection game $\mathcal{G}_1$ where the condition*

$$\forall (i,j) \in \mathcal{I}, \text{ with } i \neq j, \quad \phi\left(\boldsymbol{p}^{(i)}\right) \neq \phi\left(\boldsymbol{p}^{(j)}\right) \tag{12}$$

always holds, the maximum number of NE which can be observed is S^{K-1}.

Proof. Assume that a given strategy profile $\boldsymbol{p}^{(i)}$ (vertex v_i) with $i \in \mathcal{I}$ is a NE (Prop. 4). Then, given the condition (12) it follows that none of the vertices in the set $\mathcal{V}_i$ is a NE. Hence, two NE vertices must be separated by a minimum distance two in the non-directed graph $G = (\mathcal{V}, \boldsymbol{A})$. Thus, we obtain the maximum number of NE by calculating the maximum number of vertices mutually separated by minimum distance two in G. Given any two vertices v_i and v_j, for all $(i,j) \in \mathcal{I}^2$ with $i \neq j$ we have that $d_{i,j}(G) \geqslant 1$. Then, the vertex v_i and all the vertices v_j such that $j \in \mathcal{J}_{i,k} = \left\{n \in \mathcal{I} \setminus \{i\} : \boldsymbol{p}_k^{(n)} \neq \boldsymbol{p}_k^{(i)}\right\}$, for any $k \in \mathcal{K}$, are separated by minimum distance $d_{i,j}(G) \geqslant 2$. Then, for any $(i,k) \in \mathcal{I} \times \mathcal{K}$, the set $\mathcal{J}_{i,k}$ has cardinality $|\mathcal{J}_{i,k}| = S^{K-1} - 1$. Then, the total number of points mutually separated by minimum distance 2 (including the reference vertex v_i) is $|\mathcal{J}_{i,k}| + 1 = S^{K-1}$, which completes the proof. $\square$

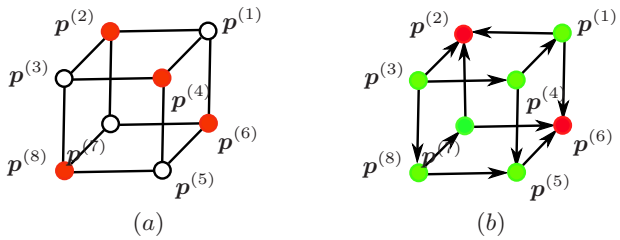

Figure 1: (a) Non-oriented graph and (b) oriented graph representing the BS Selection game with $K = 3$, $S = 2$, under the condition $\phi(\boldsymbol{p}^{(2)}) > \phi(\boldsymbol{p}^{(6)}) > \phi(\boldsymbol{p}^{(1)}) > \phi(\boldsymbol{p}^{(5)}) > \phi(\boldsymbol{p}^{(4)}) > \phi(\boldsymbol{p}^{(7)}) > \phi(\boldsymbol{p}^{(8)}) > \phi(\boldsymbol{p}^{(3)})$. Total number of vertices: $S^K = 8$, number of neighbors per vertex: $K(S-1) = 3$. Maximum Number of NE: $S^{K-1} = 4$. Number of NE: 2 (red vertices in (b)).

3.1.3 Determination of the NE

To evaluate the number of NE of the game $\mathcal{G}_1$ for a specific set of channel gains, we use an oriented graph $\hat{G} = \left(\mathcal{V}, \hat{\boldsymbol{A}}\right)$, where the adjacency matrix $\hat{\boldsymbol{A}}$ is a non-symmetric square matrix whose entries are $\forall (i,j) \in \mathcal{I}^2$ and $i \neq j$,

$$\hat{a}_{i,j} = \begin{cases} 1 & \text{if} \quad i \in \mathcal{V}_j \text{ and } \phi\left(\boldsymbol{p}^{(j)}\right) > \phi\left(\boldsymbol{p}^{(i)}\right) \\ 0 & \text{otherwise} \end{cases}, \tag{13}$$

and $a_{i,i} = 0$ for all $i \in \mathcal{I}$.

In the graph $\hat{G}$, we say that a given vertex v_i is adjacent to vertex v_j, if and only if $\phi(\boldsymbol{p}_i) > \phi(\boldsymbol{p}_j)$ and $d_{ij}(G) = 1$. Note that the condition for adjacency in $\hat{G}$ represents the rationality assumption of players: A player changes its strategy if the new strategy brings a higher utility function, i.e., increases the potential function. In Fig. 1, we show an example of the non-directed G and oriented $\hat{G}$ graphs for the case where $K = 3$ and $S = 2$.

From the definition of the matrix $\hat{\boldsymbol{A}}$, we have that a necessary and sufficient condition for a vertex v_i to represent a NE strategy profile is to have a null out-degree: $\deg^+(v_i) = 0$ (sink vertex), in the oriented graph $\hat{G}$. Hence, obtaining the number of NE in the game $\mathcal{G}_1$ boils down to identifying all the sinks in the oriented graph $\hat{G}$. For doing so, it suffices to identify the indices of the rows of matrix $\hat{A}$ containing only zeros. If the i-th row of matrix $\hat{A}$ is a null vector, then the strategy profile $\boldsymbol{p}^{(i)}$ is a NE. However, building the matrix $\hat{A}$ requires complete CSI, since it is necessary to determine whether $\phi(\boldsymbol{p}^{(i)}) > \phi(\boldsymbol{p}^{(j)})$, $\phi(\boldsymbol{p}^{(i)}) = \phi(\boldsymbol{p}^{(j)})$ or $\phi(\boldsymbol{p}^{(i)}) < \phi(\boldsymbol{p}^{(j)})$ for all $i \in \mathcal{I}$ and $j \in \mathcal{V}_i$.

To determine a strategy profile leading to a NE, in a distributed fashion with a less restrictive CSI at each radio device, we introduce the following definition:

Definition 8 (Random Walks). *A walk through an oriented graph $\hat{G}$ is an ordered list of vertices $v_{i_1}, \ldots, v_{i_N}$ such that vertex $v_{i_{n+1}}$ is adjacent to vertex v_{i_n}, with $i_n \in \mathcal{I}$ for all $n \in \{1, \ldots, N\}$, and $N \leqslant S^K$. We say that a walk*

is random if given a vertex v_{i_n}, the vertex $v_{i_{n+1}}$ is chosen randomly from the set $\mathcal{V}_{i_n}$.

From Def. 8, we have the following result:

PROPOSITION 9. *Any random walk in the oriented graph $\hat{G}$ ends in a vertex representing a NE.*

PROOF. Each step of the walk, i.e. the transition from vertex v_{i_n} to $v_{i_{n+1}}$, can be interpreted as changing from one strategy profile $\boldsymbol{p}^{(i_n)}$, $i_n \in \mathcal{I}$ to another strategy profile $\boldsymbol{p}^{(i_{n+1})}$, $i_{n+1} \in \mathcal{V}_{i_n}$ such that $\phi(\boldsymbol{p}^{(i_n)}) < \phi(\boldsymbol{p}^{(i_{n+1})})$. Since there exists a finite number of vertices in the graph, it turns out that any sequence $\phi(\boldsymbol{p}^{(i_1)}) < \phi(\boldsymbol{p}^{(i_2)}) < \ldots < \phi(\boldsymbol{p}^{(i_N)})$, with $i_n \in \mathcal{I}$ for all $n \in \{1, \ldots, N\}$, is finite, i.e., $N \leqslant S^K$. Moreover, the walk is ended if and only if the vertex v_N does not have any adjacent node, i.e., vertex v_N is a sink vertex. From the definition of the adjacency matrix $\hat{A}$ in Eq. (13) it follows that any sink vertex represents a NE (Def. 1). Thus, any random walk in the oriented graph $\hat{G}$ ends in a NE. This completes the proof. □

In practical terms, to perform a walk through the oriented graph $\hat{G}$ implies imposing certain rules on each transmitter of a given self-configuring network: (a) A given player changes its strategy if and only if the potential function can be strictly increased. (b) Two or more players do not change their strategy simultaneously. (c) All players have the same chances to update their strategies. The first condition derives from the fact that each player aims to maximize its own utility function. The second condition is to ensure that each step in the random walk is equivalent to going from a given vertex to one of its adjacent vertices. The third condition is to ensure a random walk, i.e., to ensure that each step is done with the same probability to any of the adjacent vertices. The last two conditions might require certain synchronization system among the transmitters.

Algorithm 1 Base Station Selection Algorithm

Require: $\forall k \in \mathcal{K}$,
  MAI Vector: $\boldsymbol{\zeta}_k(0) = \left(\boldsymbol{\zeta}_{k,1}(0), \ldots, \boldsymbol{\zeta}_{k,S}(0)\right)$
  Channel Realizations: $\boldsymbol{g}_k = (g_{k,1}, \ldots, g_{k,S})$, $\forall k \in \mathcal{K}$
  $t \leftarrow 0$.
  repeat
    $t \leftarrow t + 1$
    for $k = 1$ to K **do**
      $s \leftarrow \underset{i \in \mathcal{S}}{\operatorname{argmax}} \ \log_2 \left(p_{\max} g_{k,s} + \xi_{k,s}(t-1)\right)$
      $\boldsymbol{p}_k(t) \leftarrow p_{\max} \boldsymbol{e}_s$
      $\boldsymbol{\zeta}_k(t) \leftarrow \boldsymbol{\zeta}_k(t-1) + \left(\boldsymbol{p}_k(t) - \boldsymbol{p}_k(t-1)\right) \boldsymbol{g}_k^T$
  until $\boldsymbol{p}(t) = \boldsymbol{p}(t-1)$

Note that if the algorithm is implemented in a distributed way, each player $k \in \mathcal{K}$ requires the knowledge of two parameters. First, the MAI level at each BS, i.e., the vector $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_S)$, where $\zeta_s = \sigma_s^2 + \sum_{k \in \mathcal{K}} p_{k,s} g_{k,s}$ and which is common to all the players. Second, the channel realization with respect to each BS, i.e., the vector $\boldsymbol{g}_k = (g_{k,1}, \ldots, g_{k,S})$. Each element of the vector $\boldsymbol{\zeta}$ is obtained by feedback from the corresponding BS at a frequency higher than the reciprocal of the channel coherence time. Each element of the vector $\boldsymbol{g}_k$ must be estimated by transmitter k using channel estimation techniques, e.g., combining channel reciprocity and training sequences in a two-way link. An important remark regarding the proposed algorithm is that the NE where a given walk ends, mainly depends on the starting vertex in the graph $\hat{G}$ and the order we let each player to update its strategy. Thus, if each player is randomly chosen for updating at a given point of time, it is not possible to predict the NE where a walk ends. Hence, this might lead us to the situation where the convergence point is a non-optimal NE regarding a global metric, e.g., the network spectral efficiency. We analyze the optimality issues in Sec. 3.3. In Fig. 2 we show a walk through the directed graph $\hat{G}$ of a given BS selection game with $K = 5$ and $S = 2$ and a given set of channel realizations. The potential obtained at each possible strategy profile $\boldsymbol{p}^{(i)}$, i.e., $\phi\left(\boldsymbol{p}^{(i)}\right)$ with $i \in \mathcal{I}$ is plotted in Fig. 3 as a function of their index i. In Fig. 2, it can be seen how different walks end in different NE.

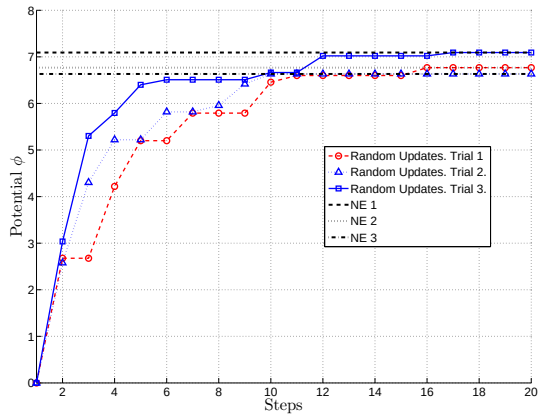

Figure 2: Evolution of the potential at each update of the players using the BS Selection Alg. 1. At each step one player is randomly chosen to update its strategy. All the sequences are obtained using the same set of channel realizations $h_{k,s}$, $\forall (k,s) \in \mathcal{K} \times S$. Number of players $K = 5$, Number of BSs $S = 3$, $\frac{B_1}{B} = 0.14$, $\frac{B_2}{B} = 0.40$, and $\frac{B_3}{B} = 0.46$. $SNR = 10 \log_{10}\left(\frac{p_{\max}}{N_0 B}\right) = 10$ dB.

3.2 Non-Atomic Base Station Selection Games

In the non-atomic BS selection game, we consider that there exists a large number of players, such that players are indifferent to the strategy adopted by any single player. Each player is rather concerned with simultaneous deviations of fractions of the total number of players. Let us denote by x_s the fraction of players transmitting to BS s, and assume that

$$\begin{aligned} \forall s \in \mathcal{S}_k, \quad x_s &= \tfrac{|\mathcal{K}_s|}{K} \\ \sum_{i=1}^{S} x_i &= 1. \end{aligned} \tag{14}$$

We denote the ratio between the available total bandwidth B and the total number of transmitters K by $\alpha = \frac{B}{K}$. Thus,

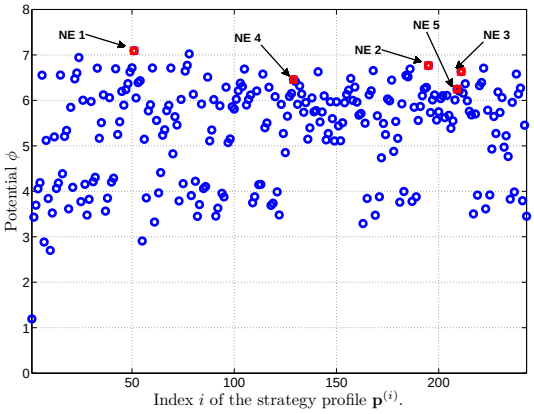

Figure 3: Potential associated with each strategy profile $\boldsymbol{p}^{(i)}$ as a function of $i \in \mathcal{I}$. The set of channel realizations $h_{k,s}$, $\forall (k,s) \in \mathcal{K} \times S$ is identical to that one used in Fig. 2. Nash Equilibria are pointed by arrows. Number of players $K = 5$, Number of BSs $S = 3$, $\frac{B_1}{B} = 0.14$, $\frac{B_2}{B} = 0.40$, and $\frac{B_3}{B} = 0.46$. $SNR = 10\log_{10}\left(\frac{p_{\max}}{N_0 B}\right) = 10$ dB.

the ratio between the available bandwidth at BS s and K, denoted by $\alpha_s = \frac{B_s}{K}$, satisfies $\sum_{s=1}^{S} \alpha_s = \alpha$. Using, these notations, the potential function ϕ can be written as follows

$$\begin{aligned}
\phi(\boldsymbol{p}) &= \sum_{s\in\mathcal{S}} \frac{B_s}{B} \log_2\left(\sigma_s^2 + \sum_{k=1}^{K} p_{k,s} g_{k,s}\right) \\
&= \sum_{s\in\mathcal{S}} \frac{B_s}{B} \log_2\left(\sigma_s^2 + p_{\max} \sum_{k\in\mathcal{K}_s} g_{k,s}\right) \\
&= \sum_{s\in\mathcal{S}} \frac{B_s}{B} \log_2\left(K\, N_0\, \alpha_s + p_{\max} \underbrace{\frac{|\mathcal{K}_s|}{K}}_{x_s} \frac{K}{|\mathcal{K}_s|} \sum_{k\in\mathcal{K}_s} g_{k,s}\right) \\
&= \sum_{s\in\mathcal{S}} \frac{B_s}{B} \log_2\left(N_0\, \alpha_s + (x_s p_{\max}) \frac{1}{|\mathcal{K}_s|} \sum_{k\in\mathcal{K}_s} g_{k,s}\right) \\
&\quad + \sum_{s\in\mathcal{S}} \frac{B_s}{B} \log_2(K) \qquad (15)
\end{aligned}$$

Note that the term $\sum_{s\in\mathcal{S}} \frac{B_s}{B} \log_2(K)$ does not depend on the strategy of the players. Thus, following *Lemma 2.7* in [19], the function

$$\tilde{\phi}(\boldsymbol{p}) = \sum_{s\in\mathcal{S}} \frac{B_s}{B} \log_2\left(N_0\, \alpha_s + (x_s p_{\max}) \frac{1}{|\mathcal{K}_s|} \sum_{k\in\mathcal{K}_s} g_{k,s}\right) \qquad (16)$$

can be considered as another exact potential function of the BS selection game $\mathcal{G}_1$. Now, we assume that the number of players grows toward infinity at the same rate that the bandwidth available at each BS, i.e.,

- $B \longrightarrow \infty$ and $K \longrightarrow \infty$,
- $\lim_{B,K\to\infty} \frac{B}{K} = \alpha < \infty$, and
- $\forall s \in \mathcal{S}$, $\lim_{B_s,K\to\infty} \frac{B_s}{K} = \alpha_s < \infty$.

From a practical point of view, when the number of transmitters grows toward infinity while the total bandwidth or number of BSs remain constant, the MAI becomes a dominant parameter and thus, independently of the strategy adopted by each player, their own utility function tends to zero. Thus, no quality of service can be guaranteed, for instance, in terms of minimum transmission rates. For avoiding such a situation, we have considered that the number of players grows to infinity at the same rate as the total bandwidth. This ensures that the utility function of each player does not tend to zero when the load (number of transmitters per BS) of the network is increased. Under these conditions, it holds that for all $s \in \mathcal{S}$, $|\mathcal{K}_s| \to \infty$, and thus:

$$\frac{1}{|\mathcal{K}_s|} \sum_{k\in\mathcal{K}_s} g_{k,s} \to \int_{-\infty}^{\infty} \lambda \mathrm{d}F_g(\lambda) = \Omega, \qquad (17)$$

where F_g is the cumulative probability function associated with the probability density function f_g of the random variable g (channel gains) described in Sec. 2: $\mathrm{d}F_g(\lambda) = f(\lambda)\mathrm{d}\lambda$.

This result allows us to write the function ϕ as a function of the fractions $x_1, \ldots, x_S$,

$$\tilde{\phi}(x_1, \ldots, x_S) = \sum_{s\in\mathcal{S}} \frac{\alpha_s}{\alpha} \log_2\left(N_0\, \alpha_s + x_s p_{\max}\Omega\right), \qquad (18)$$

and thus, finding a set of fractions such that no player is interested to modify, i.e. a NE in the non-atomic extension of the game $\mathcal{G}_1$ boils down to solve the following optimization problem (OP) [26],

$$\begin{cases} \max\limits_{x_1,\ldots,x_S\in\mathrm{R}_+} & \sum\limits_{s\in\mathcal{S}} \frac{\alpha_s}{\alpha} \log_2\left(N_0\, \alpha_s + x_s p_{\max}\Omega\right), \\ \text{s.t.} & \sum\limits_{i=1}^{S} x_i = 1 \;\text{ and }\; \forall i \in \mathcal{S},\; x_i \geqslant 0, \end{cases} \qquad (19)$$

which has a unique solution of the form

$$\forall s \in \mathcal{S}, \quad x_s = \frac{B_s}{B}. \qquad (20)$$

In Fig. 4, we show the fractions x_s, with $s \in \mathcal{S}$, obtained by Monte-Carlo simulations and using Eq. (20) for a network with $S = 6$ BSs and $K = 100$ transmitters. Therein, it becomes evident that Eq. (20) is a precise estimation of the outcome of the non-atomic BS selection game. Note that if all the BSs are allocated with the same bandwidth $B_s = \frac{B}{S}$ $\forall s \in \mathcal{S}$, the fraction of players at each BS is identical, i.e., $\forall s \in \mathcal{S}$, $x_s = \frac{1}{S}$. This result is a generalization of the one in [4], where similar fractions were obtained for the case where each BS is allocated with the same bandwidth and players observe the same channel gains, i.e., $\forall (k,s) \in \mathcal{K} \times \mathcal{S}$, $g_{k,s} = 1$.

3.3 Efficiency of the Nash Equilibria

Here, we evaluate the performance of the network when a completely decentralized stable configuration is achieved (Nash equilibrium) and the performance when there exists a central controller that dictates a configuration which maximizes a given global metric. In this study, we consider as global metric, the sum of the utilities of each player, i.e., the network spectral efficiency. To carry out such a study

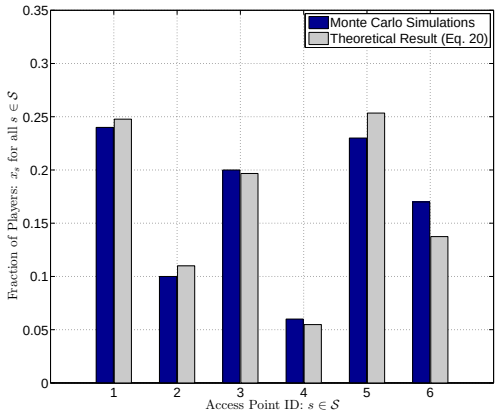

Figure 4: Fraction of players transmitting to BS s, with $s \in \mathcal{S}$, calculated using Monte-Carlo simulations and using Eq. (20) for a network with $S = 6$ BSs and $K = 100$ transmitters, with $\boldsymbol{\alpha} = (\alpha_s)_{\forall s \in \mathcal{S}} = (0.25, 0.11, 0.20, 0.05, 0.25, 0.14)$. $SNR = 10\log_{10}\left(\frac{p_{\max}}{N_0 B}\right) = 10$ dB.

we consider two metrics known in the game theory jargon as price of anarchy and price of stability [14]. We formally define such parameters as follows:

Definition 10. *In the strategic game $\left(\mathcal{K}, (\mathcal{P}_k)_{k\in\mathcal{K}}, (u_k)_{k\in\mathcal{K}}\right)$, denote the set of NE strategy profiles by $\mathcal{P}^* \subseteq \mathcal{P}$. Then, the price of anarchy (PoA) and the price of stability (PoS) are the ratios,*

$$\text{PoA} = \frac{\max\limits_{\boldsymbol{p}\in\mathcal{P}} \sum\limits_{k=1}^{K} u_k(\boldsymbol{p})}{\min\limits_{\boldsymbol{p}\in\mathcal{P}^*} \sum\limits_{k=1}^{K} u_k(\boldsymbol{p})}, \tag{21}$$

and

$$\text{PoS} = \frac{\max\limits_{\boldsymbol{p}\in\mathcal{P}} \sum\limits_{k=1}^{K} u_k(\boldsymbol{p})}{\max\limits_{\boldsymbol{p}\in\mathcal{P}^*} \sum\limits_{k=1}^{K} u_k(\boldsymbol{p})}, \tag{22}$$

respectively.

The discrete nature of the strategy set of the players makes obtaining a closed-form expression for both the PoA and PoS, a very difficult task. Thus, we evaluate both PoA and PoS using Monte-Carlo simulations. For instance, in Fig 5, we respectively plot the PoA and PoS for a network with $S \in \{2, 3\}$ BSs, and $K \in \{1, \ldots, 9\}$ players. Therein, we observe that the loss due to decentralization is minimum since $PoA \longrightarrow 1$, when the number of players increase. Similarly to the PoA, the PoS is also close to unity but different from the PoA. This implies that effectively, the case where several NE are observed, takes place often. As a concluding remark regarding the efficiency of NE, we state that the self-configuring nature of the network does not imply a significant loss of optimality, i.e., if the network were centralized by enforcing signaling protocols between all transmitters and the different BSs, the gain in network spectral efficiency will not justify the increment of signaling traffic due to the feedback of the optimal strategies.

3.4 Equilibrium in Mixed Strategies

For any player $k \in \mathcal{K}$, let the vector $\boldsymbol{q}_k = (q_{k,1}, \ldots, q_{k,S})$ represent a discrete probability distribution over the set of pure strategies $\mathcal{P}_k$. Here, $q_{k,s}$ represents the probability of player k transmitting to BS s. The mixed-strategy space of player k is the standard simplex $\mathcal{Q}_k$:

$$\begin{aligned}\mathcal{Q}_k \quad = \quad & \{(q_{k,1}, \ldots, q_{k,s}) \in \mathbb{R}^S : \sum_{s=1}^{S} q_{k,s} = 1, \\ & \text{and } \forall s \in \mathcal{S},\ q_{k,s} \geqslant 0\},\end{aligned} \tag{23}$$

and the space of mixed-strategies is $\mathcal{Q} = \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_K$. Let $\boldsymbol{s} = (s_1, \ldots, s_K)$ be a vector in the discrete set $\mathcal{S}^K$. Let us also index each element of the set $\mathcal{S}^K$ with the set $\{1, \ldots, S^K\}$ such that elements are ordered following the index $i \in \{1, \ldots, S^K\}$. Denote by $\boldsymbol{s}^{(i)} = \left(s_1^{(i)}, \ldots, s_K^{(i)}\right)$, the i-th element of such a set $\mathcal{S}^K$. Denote by $\boldsymbol{p}_{-k}^{(i)}$, the vector

$$\boldsymbol{p}_{-k}^{(i)} = \left(\boldsymbol{p}_1^{(i)}, \ldots, \boldsymbol{p}_{k-1}^{(i)}, \boldsymbol{p}_{k+1}^{(i)}, \ldots, \boldsymbol{p}_K^{(i)}\right),$$

where for all $j \in \mathcal{K}$, $\boldsymbol{p}_j^{(i)} = p_{\max}\boldsymbol{e}_{s_j^{(i)}} \in \mathcal{P}_j$. In the mixed-strategy extension of the BS selection game $\mathcal{G}_1$, the utility function $U_k(\boldsymbol{q})$, with $\boldsymbol{q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_K) \in \mathcal{Q}$, is defined as the expected value of the corresponding pure strategy utilities with respect to the probability distributions $\boldsymbol{q}_k$ for all $k \in \mathcal{K}$, i.e.

$$U_k(\boldsymbol{q}_k, \boldsymbol{q}_{-k}) = \sum_{i=1}^{S^K} \prod_{k=1}^{K} q_{k,s_k^{(i)}} u_k(\boldsymbol{p}_k^{(i)}, \boldsymbol{p}_{-k}^{(i)}). \tag{24}$$

Following *Lemma 2.10* in [19], and since, the game $\mathcal{G}_1$ is an exact PG (Prop. 3), we claim the existence of a potential function in the mixed-strategy extension of $\mathcal{G}_1$. We denote such a potential by $\bar{\phi}(\boldsymbol{q})$,

$$\bar{\phi}(\boldsymbol{q}) = \sum_{i=1}^{S^K} \prod_{k=1}^{K} q_{k,s_k^{(i)}} \phi(\boldsymbol{p}^{(i)}), \tag{25}$$

where $\boldsymbol{q} \in \mathcal{Q}$. From [20], we know that there always exists a NE in mixed strategies for the game $\mathcal{G}_1$. Thus the following OP must have at least one solution,

$$\begin{cases} \max\limits_{(q_{k,s})_{\forall(k,s)\in\mathcal{K}\times\mathcal{S}}} & \sum\limits_{i=1}^{S^K} \prod\limits_{k=1}^{K} q_{k,s_k^{(i)}} \phi(\boldsymbol{p}^{(i)}) \\ \text{s.t.} & \forall k \in \mathcal{K}, \quad \sum\limits_{s=1}^{S} q_{s,k} = 1, \\ & \forall (k,s) \in \mathcal{K}\times\mathcal{S}, \quad q_{s,k} \geqslant 0. \end{cases} \tag{26}$$

However, the solution to the OP (26) might not be necessarily a fully mixed strategy, i.e., a vector $\boldsymbol{q}_k$ with more than one entry different from zero. Indeed, depending on the channel realizations, it is possible that no NE in fully mixed strategies is observed, i.e.,

$$\exists (k,s) \in \mathcal{K}\times\mathcal{S} :\ \forall \boldsymbol{q} \in \mathcal{Q}, \quad U_k(\boldsymbol{e}_s, \boldsymbol{q}_{-k}) > U_k(\boldsymbol{q}_k, \boldsymbol{q}_{-k}). \tag{27}$$

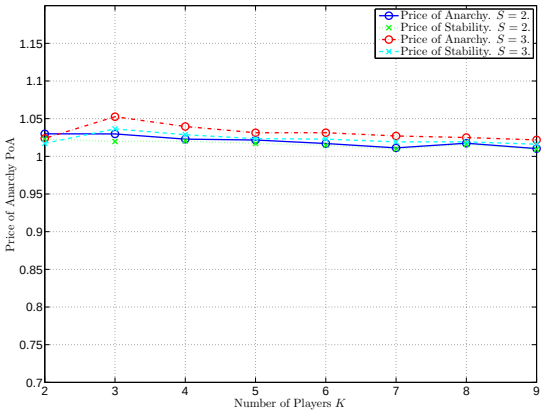

Figure 5: Price of anarchy (Def. 10) and price of stability (Def. 10)as a function of the number of players K for the case of two and three BSs, $S = 2$ and $S = 3$. $SNR = 10\log_{10}\left(\frac{p_{\max}}{N_0 B}\right) = 10$ dB.

For instance, consider the case when there exist two players and two BSs, i.e., $K = 2$, and $S = 2$. Then, we obtain that if there exists a pair $(k, s) \in \mathcal{K} \times \mathcal{S}$, such that

$$\frac{g_{k,-s}}{g_{k,s}} \leqslant \frac{\sigma_{-s}^2}{\sigma_s^2 + p_{\max} g_{-k,s}}, \tag{28}$$

the condition (27) always holds and thus, it does not exist a NE in fully mixed strategies. Here, we denote by $-s$ and $-k$ the element other than s and k, in the binary sets $\mathcal{S}$ and $\mathcal{K}$, respectively. Note that the non-existence of a NE in fully mixed strategies does not mean that it does not exist a NE in mixed strategies [20]. The existence of at least one NE in pure strategies has been proved (Prop. 4) and a pure strategy NE is also a (degenerated) mixed strategy NE.

There exists several algorithms to iteratively solve the OP (26). Those algorithms are known in the domain of machine learning theory as linear reward inaction and linear reward penalty [25]. Several applications of those algorithms are presented in [27, 22, 17]. Contrary to the algorithms presented in this paper, linear reward inaction and penalty algorithms requires to set up some parameters to refine the convergence speed and the accuracy of the obtained probability distributions [25]. These parameters depend on the channel realizations, which means that at each coherence time such parameters must be re-adjusted by training.

4. BASE STATION SHARING GAMES

In this section, we consider the case where each player can be associated with several BSs. Here, each player not only selects its set of BSs but also the specific power level to transmit to each of its BSs. We define this interaction as a strategic game denoted by $\mathcal{G}_2 = (\mathcal{K}, (\mathcal{P}_k)_{k\in\mathcal{K}}, (u_k)_{k\in\mathcal{K}})$, where the set $\mathcal{K}$ remains being the indices of each player as in the previous section, $\mathcal{P}$ represents the space of strategies, where $\mathcal{P} = \mathcal{P}_1 \times \ldots \mathcal{P}_K$ and for all $k \in \mathcal{K}$

$$\mathcal{P}_k = \left\{(p_{k,1}, \ldots, p_{k,S}) \in \mathbb{R}^S : \forall s \in \mathcal{S},\ p_{k,s} \geqslant 0, \text{ and } \sum_{s\in\mathcal{S}} p_{k,s} \leqslant p_{\max}\right\}.$$

The utility function remains being the spectral efficiency of each player as defined by Eq. (5).

4.1 Existence and Uniqueness of the NE

To study the NE of the BS Sharing game $\mathcal{G}_2$, we first introduce the following proposition:

PROPOSITION 11. *The BS sharing game $\mathcal{G}_2$ is an exact potential game with potential function $\phi(\boldsymbol{p})$ given by Eq. (8) for all $\boldsymbol{p} \in \mathcal{P}$.*

Prop. 11 leads us to the following result:

PROPOSITION 12. *In the strategic game $\mathcal{G}_2$ the strategy profile $\boldsymbol{p}^* = (\boldsymbol{p}_1^*, \ldots, \boldsymbol{p}_K^*)$, with $\boldsymbol{p}_k^* = (p_{k,1}^*, \ldots, p_{k,S}^*)$, where for all $(k, s) \in \mathcal{K} \times \mathcal{S}$,*

$$p_{k,s}^* = \left[\frac{B_s}{B\,\beta_k} - \frac{\zeta_{k,s}}{g_{k,s}}\right]^+, \tag{29}$$

is the unique NE of the game. The constant β_k for each player k is set to satisfy the condition $\sum_{s=1}^{S} p_{k,s} = p_{\max}$ and $\zeta_{k,s}$ represents the noise plus MAI overcome by player k at BS s.

PROOF. To prove the existence of at least one NE, we use the fact that the BS sharing game $\mathcal{G}_2$ is a PG (Prop. 11). Then, following *Corollary 2.2* in [19], the existence of at least one NE is ensured. Thus, proving the uniqueness of the NE ends up being equivalent to prove that the OP:

$$\max_{\boldsymbol{p}\in\mathcal{P}} \sum_{s\in\mathcal{S}} \frac{B_s}{B} \log_2\left(\sigma_s^2 + \sum_{j\in\mathcal{S}_s} p_{j,s} g_{j,s}\right) \tag{30}$$

posses a unique solution. Indeed, since the potential function ϕ is strictly concave on $\mathcal{P}$ and $\mathcal{P}$ is a simplex, and thus convex, the Karush-Khun-Tucker (KKT) conditions are necessary and sufficient conditions of optimality. Hence, we write:

$$\begin{cases} \forall (k,s) \in \mathcal{K}\times\mathcal{S}, & \frac{B_s}{B}\left(\frac{g_{k,s}}{p_{k,s} g_{k,s} + \zeta_{k,s}}\right) - \beta_k + \nu_k = 0 \\ \forall k \in \mathcal{K}, & \beta_k\left(\sum_{s=1}^{S} p_{k,s} - p_{\max}\right) = 0 \\ \forall (k,s) \in \mathcal{K}\times\mathcal{S}, & \nu_k p_{k,s} = 0, \end{cases} \tag{31}$$

The solution to the system of equations (31) is known to be unique and achieved by using the water-filling algorithm [10]. Such a solution is given by expression Eq. (29) where β_k is uniquely determined to satisfy the condition $\sum_{s=1}^{S} p_{k,s} = p_{\max}$, for all $k \in \mathcal{K}$. This ends up the proof. □

4.2 Determination of the NE

The NE of the BS sharing game $\mathcal{G}_2$ is fully determined by Eq. (29). Here, we study a decentralized algorithm such that the NE in Prop. 12 can be achieved by players in

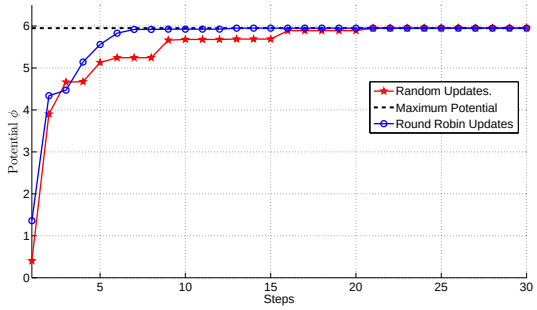

Figure 6: Evolution of the potential at each update of the players using the BS Sharing Alg. 2. Number of players $K = 6$, Number of BSs $S = 3$, $\frac{B_1}{B} = 0.75$, $\frac{B_2}{B} = 0.21$, and $\frac{B_3}{B} = 0.04$. $SNR = 10\log_{10}\left(\frac{p_{\max}}{N_0 B}\right) = 10$ dB.

a decentralized fashion. First, consider that the strategy space $\mathcal{P}$ is obtained from the Cartesian product of the closed convex sets $\mathcal{P}_k$, for all $k \in \mathcal{K}$. Second, note that the solution to the OP

$$\max_{x \in \mathcal{P}_k} \phi\left(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{k-1}, \boldsymbol{x}, \boldsymbol{p}_{k+1}, \ldots, \boldsymbol{p}_K\right) \tag{32}$$

for all $k \in \mathcal{K}$ and for all $\boldsymbol{p}_{-k} \in \mathcal{P}_{-k}$ is unique and can be determined by using the water-filling algorithm [10]. Thus, the OP (30) can be solved iteratively by using a non-linear Gauss-Seidel method. Denote by $\boldsymbol{p}_k(t)$ the solution to $\boldsymbol{p}_k$ at iteration t, where $t \in \mathbb{N}$ and $\boldsymbol{p}_k(t+1)$ is given by

$$\operatorname{argmax} \phi\left(\boldsymbol{p}_1(t+1), ..., \boldsymbol{p}_{k-1}(t+1), \boldsymbol{p}_k(t), \boldsymbol{p}_{k+1}(t), ..., \boldsymbol{p}_K(t)\right).$$

Then, following Prop. 2.7.1 in [5], the convergence of the sequence $\{p_k(t)\}_{t \in \{1,\ldots,N\}}$ for $N >> 0$ is ensured. Based on this result, we introduce the algorithm Alg. 2 for the BS sharing game:

Algorithm 2 Base Station Sharing Algorithm

Require: $\forall k \in \mathcal{K}$,
  MAI Vector: $\boldsymbol{\zeta}_k(0) = \left(\boldsymbol{\zeta}_{k,1}(0), \ldots, \boldsymbol{\zeta}_{k,S}(0)\right)$
  Channel Realizations: $\boldsymbol{g}_k = (g_{k,1}, \ldots, g_{k,S})$, $\forall k \in \mathcal{K}$
  $t \leftarrow 0$.
  repeat
    $t \leftarrow t + 1$
    for $k = 1$ to K **do**
      $\boldsymbol{p}_k(t) \leftarrow \underset{\boldsymbol{p}_k \in \mathcal{P}_k}{\operatorname{argmax}} \sum_{s \in \mathcal{S}} \frac{B_s}{B} \log_2\left(p_{k,s} g_{k,s} + \xi_{k,s}(t-1)\right)$
      $\boldsymbol{\zeta}_k(t) \leftarrow \boldsymbol{\zeta}_k(t-1) + \left(\boldsymbol{p}_k(t) - \boldsymbol{p}_k(t-1)\right) \boldsymbol{g}_k^T$
  until $\boldsymbol{p}(t) = \boldsymbol{p}(t-1)$

In Fig. 6 we show the convergence to the maximum of the potential function ϕ using Alg. 2 for the case of a network with $K = 6$ transmitters and $S = 3$ BSs. Therein, we show both a round Robin and random updates. In both cases, the convergence is achieved in very few iterations.

5. PERFORMANCE ANALYSIS

In this section, we use algorithms Alg. 1 and Alg. 2 to compare the global performance of the network when BS selection and BS sharing are used. We choose the network spectral efficiency as the global performance metric, i.e., the sum of the utilities of all players. In fig. 7, we plot the network spectral efficiency for a network with $S \in \{2, 4, 8\}$ BSs and $K \in \{2, \ldots, 60\}$ transmitters assuming an SNR of 10 dB for each player. We observe that when $K < S$ the BS sharing technique performs better than BS selection. However, when $K \geqslant S$, the performance of the BS selection is strongly superior to BS sharing. For a large number of transmitters both techniques perform similarly.

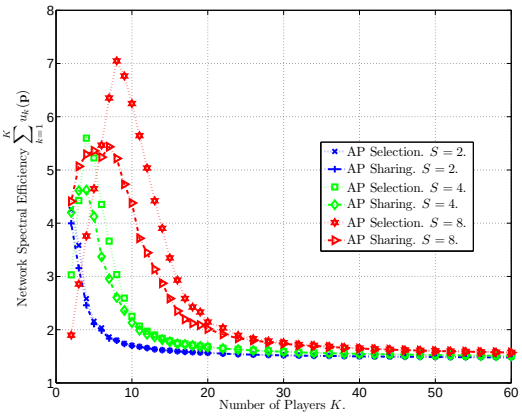

Figure 7: Sum spectral efficiency for the BS sharing and BS Selection algorithms as a function of the number of players present in the network. Players operate at a transmit SNR $10\log_{10}\left(\frac{p_{\max}}{N_0 B}\right) = 10$ dB and the fractions $\frac{B_s}{B} = \frac{1}{S}$, $\forall s \in \mathcal{S}$. $SNR = 10\log_{10}\left(\frac{p_{\max}}{N_0 B}\right) = 10$ dB.

Note that the strategy space of each player is bigger in the BS sharing scenario. Thus, one can think that a better performance is always obtained by using BS sharing than using BS selection. Paradoxically, we have found that on the contrary, for nearly fully and fully loaded networks i.e., $K \simeq S$ and $K > S$, increasing the space of strategies of each player produces a global loss of performance. A similar paradox is observed in congestion games where adding extra capacity to the network ends up reducing the overall performance [6]. A similar paradox to the one presented in this work is also observed in [3, 7, 13, 2].

6. CONCLUDING REMARKS

We have investigated the BS selection and BS sharing scenarios in the context of self-configuring networks using a non-cooperative model focusing on the spectral efficiency of each transmitter. We have proved the existence of at least one NE in both cases. In the BS sharing game a unique NE is observed, whereas BS selection games might exhibit several. We have provided fully decentralized algorithms such that players can calculate their NE strategy based on local information and the MAI observed at each BS. We have observed that no significant gain would be achieved by introducing a central controller in the case of BS selection. The self-configured network performs almost identical to the

optimally configured network.

Finally, we have identified that depending on the number of transmitters BS selection might perform better than BS sharing. This result implies a Braess type paradox, where increasing the strategy space of each players produces a degeneration of the global network spectral efficiency.

7. ACKNOWLEDGMENTS

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Power Allocation Games in Interference Relay Channels: Existence Analysis of Nash Equilibria

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***Abstract*—We consider a network composed of two interfering point-to-point links where the two transmitters can exploit one common relay node to improve their individual transmission rate. Communications are assumed to be multi-band and transmitters are assumed to selfishly allocate their resources to optimize their individual transmission rate. The main objective of this paper is to show that this conflicting situation (modeled by a non-cooperative game) has some stable outcomes, namely Nash equilibria. This result is proved for three different types of relaying protocols: decode-and-forward, estimate-and-forward, and amplify-and-forward. We provide additional results on the problems of uniqueness, efficiency of the equilibrium, and convergence of a best-response based dynamics to the equilibrium. These issues are analyzed in a special case of the amplify-and-forward protocol and illustrated by simulations in general.**

***Index Terms*—Cognitive radio, game theory, interference channel, interference relay channel, Nash equilibrium, power allocation games, relay channel.**

I. Introduction

A possible way to improve the performance in terms of range, transmission rate or quality of a network composed of mutual interfering independent source-destination links, is to add some relaying nodes in the network. This approach can be relevant in both wired and wireless networks. For example, it can be desirable and even necessary to improve the performance of the (wired) link between the digital subscriber line (DSL) access multiplexors (or central office) and customers' facilities and/or the (wireless) links between some access points and their respective receivers (personal computers, laptops, etc). The mentioned scenarios give a strong motivation for studying the following system composed of two transmitters communicating with their respective receivers and which can use a relay node. The channel model used to analyze this type of network has been called the interference relay channel (IRC) in [3][4] where the authors introduce a channel with two transmitters, two receivers, and one relay, all of them operating in the same frequency band. The main contribution of [3][4] is to derive achievable transmission rate regions for Gaussian IRCs assuming that the relay is implementing the decode-and-forward protocol (DF) and dirty paper coding.

In this paper, we consider multi-band interference relay channels and three different types of protocols at the relay, namely DF, estimate-and-forward (EF), and amplify-and-forward (AF). One of our main objectives is to study the corresponding power allocation (PA) problems at the transmitters. To this end, we proceed in two main steps. First, we provide achievable transmission rates for single-band Gaussian IRCs when DF, EF, and AF are respectively assumed. Second, we use these results to analyze the properties of the transmission rates for the multi-band case. In the multi-band case, we assume that the transmitters are decision makers that can freely choose their own resource allocation policies while selfishly maximizing their transmission rates. This resource allocation problem can be modeled as a static non-cooperative game. The closest works concerning the game-theoretic approach we adopt here seem to be [5][6][7][8] and [9][10][11]. In [5][6], the authors study the frequency selective and the parallel interference channels and provide sufficient conditions on the channel gains that ensure the existence and uniqueness of the Nash equilibrium (NE) and convergence of iterative water-filling algorithms. These conditions have been further refined in [7]. In [9], a traffic game in parallel relay networks is considered where each source chooses its power allocation policy to minimize a certain cost function. The price of anarchy [12] is analyzed in such a scenario. In [10], a quite similar analysis is conducted for multi-hop networks. In [11], the authors consider a special case of the Gaussian IRC where there are no direct links between the sources and destinations and there are two dedicated relays (one for each source-destination pair) implementing DF. The power allocation game consists in sharing the user's power between the source and relay transmission. The existence, uniqueness of, and convergence to a NE issues are addressed. In the present paper however, we mainly focus on the existence issue of an NE in the games under study, which is already a non-trivial problem. The uniqueness, efficiency, and the design of convergent distributed power allocation algorithms are studied only in a special case and the generalization is left as very useful extension of the present paper.

This paper is structured as follows. Sec. II describes the system model and assumptions in multi-band IRCs. Sec. III provides achievable transmission rates for single-band IRCs. These rates are exploited further in multi-band IRCs (as users' utility functions) analyzed in Sec. IV where the existence issue of NE in the non-cooperative power allocation game is studied. Three relaying protocols are considered: DF, EF, and AF. Sec. IV provides additional results on uniqueness of NE and convergence to NE for the AF protocol. Sec. V illustrates simulations highlighting the importance of optimally locating the relay and the efficiency of the possible NE. We conclude

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with summarizing remarks and possible extensions in Sec. VI.

II. System Model

The system under investigation is represented in Fig. 1. It is composed of two source nodes $\mathcal{S}_1$, $\mathcal{S}_2$ (also called transmitters), transmitting their private messages to their respective destination nodes $\mathcal{D}_1$, $\mathcal{D}_2$ (also called receivers). To this end, each source can exploit Q non-overlapping frequency bands (the notation (q) will be used to refer to band $q \in \{1, \ldots, Q\}$) which are assumed to be of unit bandwidth. The signals transmitted by $\mathcal{S}_1$ and $\mathcal{S}_2$ in band (q), denoted by $X_1^{(q)}$ and $X_2^{(q)}$, respectively, are assumed to be independent and power constrained:

$$\forall i \in \{1,2\},\ \sum_{q=1}^{Q} \mathbb{E}|X_i^{(q)}|^2 \leq P_i. \tag{1}$$

For $i \in \{1,2\}$, we denote by $\theta_i^{(q)}$ the fraction of power that is used by $\mathcal{S}_i$ for transmitting in band (q) that is, $\mathbb{E}|X_i^{(q)}|^2 = \theta_i^{(q)} P_i$. Additionally, we assume that there exists a multi-band relay $\mathcal{R}$. With these notations, the signals received by $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{R}$ in band (q) express as:

$$\begin{cases} Y_1^{(q)} &= h_{11}^{(q)} X_1^{(q)} + h_{21}^{(q)} X_2^{(q)} + h_{r1}^{(q)} X_r^{(q)} + Z_1^{(q)} \\ Y_2^{(q)} &= h_{12}^{(q)} X_1^{(q)} + h_{22}^{(q)} X_2^{(q)} + h_{r2}^{(q)} X_r^{(q)} + Z_2^{(q)} \\ Y_r^{(q)} &= h_{1r}^{(q)} X_1^{(q)} + h_{2r}^{(q)} X_2^{(q)} + Z_r^{(q)} \end{cases} \tag{2}$$

where $Z_i^{(q)} \sim \mathcal{N}(0, N_i^{(q)})$, $i \in \{1,2,r\}$, represents the Gaussian complex noise on band (q) and, for all $(i,j) \in \{1,2\}^2$, $h_{ij}^{(q)}$ is the channel gain between $\mathcal{S}_i$ and $\mathcal{D}_j$ and $h_{ir}^{(q)}$ is the channel gain between $\mathcal{S}_i$ and $\mathcal{R}$ in band (q). The channel gains are considered to be static. In wireless networks, this would amount, for instance, to considering a realistic situation where only large scale propagation effects can be taken into account by the transmitters to optimize their rates. The proposed approach can be applied to other types of channel models. Concerning channel state information (CSI), we will always assume coherent communications for each transmitter-receiver pair $(\mathcal{S}_i, \mathcal{D}_i)$ whereas, at the transmitters, the information assumptions will be context dependent. The single-user decoding (SUD) will always be assumed at $\mathcal{D}_1$ and $\mathcal{D}_2$. This is a realistic assumption in a framework where devices communicate in an a priori uncoordinated manner. At the relay, the implemented reception scheme will depend on the protocol assumed. The expressions of the signals transmitted by the relay, $X_r^{(q)}, q \in \{1, ..., Q\}$, depend on the relay protocol assumed and will therefore also be explained in the corresponding sections. So far, we have not mentioned any power constraint on the signals $X_r^{(q)}$. Note that the signal model (2) is sufficiently general for addressing two important scenarios. If one imposes an overall power constraint $\sum_{q=1}^{Q} \mathbb{E}|X_r^{(q)}|^2 \leq P_r$, multi-carrier IRCs with a single relay can be studied. On the other hand, if one imposes $\mathbb{E}|X_r^{(q)}|^2 \leq P_r^{(q)}, q \in \{1, ..., Q\}$, multi-band IRCs where a relay is available on each band (the relays are not necessarily co-located) can be studied. In this paper, for simplicity reasons and as a first step towards solving the general problem (where both source and relaying nodes optimize their PA policies) we will assume that the relay implements a fixed power allocation policy between the Q available bands ($\mathbb{E}|X_r^{(q)}|^2 = P_r^{(q)}, q \in \{1, ..., Q\}$).

To conclude this section, we will mention and justify one additional assumption. As in [4][3][13], the relay will be assumed to operate in the full-duplex mode. Mathematically, it is known from [14] that the achievability proofs for the full-duplex case can be almost directly applied the half-duplex case. But this is not our main motivation. Our main motivation is that, in some communication scenarios, the full-duplex assumption is realistic (see e.g., [1] where the transmit and receive radio-frequency parts are not co-located) and even more suited. In the scenario of DSL systems mentioned in Sec. I, the relay is connected to the source and destination through wired links. This allows the implementation of full-duplex repeaters, amplifiers, or digital relays. The same comment can be applied to optical communications.

Notational conventions

The capacity function for complex signals is denoted by $C(x) \triangleq \log_2(1+x)$; for all $a \in [0,1]$, the quantity $\overline{a}$ stands for $\overline{a} = 1 - a$; the notation $-i$ means that $-i = 1$ if $i = 2$ and $-i = 2$ if $i = 1$; for all complex numbers $c \in \mathbb{C}$, c^*, $|c|$, $\mathcal{R}e(c)$ and $\mathcal{I}m(c)$ denote the complex conjugate, modulus and the real and imaginary parts respectively.

III. Achievable transmission rates for single-band IRCs

This section provides preliminary results regarding the achievable rate regions for the IRCs assuming DF, EF, and AF protocols. They are necessary to express transmission rates in the multi-band case. Thus, we do not aim at improving available rate regions for IRCs as in [13] and related works [15][16][17]. In the latter references, the authors consider some special cases of the discrete IRC and derive rate regions based on the DF protocol and different coding-decoding schemes. In what follows, we make some suboptimal choices for the used coding-decoding schemes and relaying protocols which are motivated by a decentralized framework where each destination does not know the codebook used by the other destination. This approach, facilitates the deployment of relays since the receivers do not need to be modified. In particular, this explains why we do not exploit techniques like rate-splitting or successive interference cancellation. As we assume single-band IRCs, we have that $Q = 1$. For the sake of clarity, we omit the superscript (1) from the different quantities used e.g., $X_i^{(1)}$ becomes in this section X_i.

A. Transmission rates for the DF protocol

One of the purposes of this section is to state a corollary from [3]. Indeed, the given result corresponds to the special case of the rate region derived in [3] where each source sends to its respective destination a private message only (and not both public and private messages as in [3]). The reason for providing this region here is threefold: it is necessary for the multi-band case, it is used in the simulation part to establish a comparison between the different relaying protocols under consideration in this paper, and it makes the paper sufficiently

self-contained. The principle of the DF protocol is detailed in [14] and we give here only the main idea behind it. Consider a Gaussian relay channel where the source-relay link has a better quality than the source-destination link. From each message intended for the destination, the source builds a coarse and a fine message. With these two messages, the source superposes two codewords. The rates associated with these codewords (or messages) are such that the relay can reliably decode both of them while the destination can only decode the coarse message. After decoding this message, the destination can subtract the corresponding signal and try to decode the fine message. To help the destination to do so, the relay cooperates with the source by sending some information about the fine message. Mathematically, this translates as follows. The signal transmitted by $\mathcal{S}_i$ is structured as $X_i = X_{i0} + \sqrt{\frac{\tau_i}{\nu_i}\frac{P_i}{P_r}}X_{ri}$. The signals X_{i0} and X_{ri} are independent and correspond to the coarse and fine messages respectively; the parameter ν_i represents the fraction of transmit power the relay allocates to user i, hence we have $\nu_1+\nu_2 \leq 1$; the parameter τ_i represents the fraction of transmit power $\mathcal{S}_i$ allocates to the cooperation signal (conveying the fine message). Therefore, we have the following result.

Corollary 3.1 ([3]): When DF is assumed, the region in (3) is achievable; for $i \in \{1,2\}$, where $j=-i$, $(\nu_1,\nu_2) \in [0,1]^2$ s.t. $\nu_1+\nu_2 \leq 1$ and $(\tau_1,\tau_2) \in [0,1]^2$, $\tau_1+\tau_2 \leq 1$.

In a context of decentralized networks, each source $\mathcal{S}_i$ has to optimize the parameter τ_i in order to maximize its transmission rate R_i. In the rate region above, one can observe that this choice is not independent of the choice of the other source. Therefore, each source finds its optimal strategy by optimizing its rate w.r.t. $\tau_i^*(\tau_j)$. In order to do that, each source has to make some assumptions on the value τ_j used by the other source. This is precisely a non-cooperative game where each player makes some assumptions on the other player's behavior and maximizes its own utility. Interestingly, we see that, even in the single-band case, the DF protocol introduces a power allocation game through the parameter τ_i representing the cooperation degree between the source $\mathcal{S}_i$ and relay. In this paper, for obvious reasons of space, we will restrict our attention to the case where the cooperation degrees are fixed. In other words, in the multi-band scenario, the transmitter strategy will consist in choosing only the power allocation policy over the available bands. For more details on the game induced by the cooperation degrees the reader is referred to [2].

B. Transmission rates for the EF protocol

Here, we consider a second main class of relaying protocols, namely the estimate-and-forward protocol. A well-known property of the EF protocol for the relay channel [14] is that it always improves the performance of the receiver w.r.t. the case without relay (in contrast with DF protocols which can degrade the performance of the point-to-point link). The principle of the EF protocol for the standard relay channel is that the relay sends an approximated version of its observation signal to the receiver. More precisely, from an information-theoretic point of view [14], the relay compresses its observation in the Wyner-Ziv manner [18], i.e., knowing that the destination also receives a direct signal from the source that is correlated with the signal to be compressed. The compression rate is precisely tuned by taking into account this correlation degree and the quality of the relay-destination link. In our setup, we have two different receivers. The relay can either create a single quantized version of its observation, common to both receivers, or two quantized versions, one adapted for each destination. We have chosen the second type of quantization which we call the "bi-level compression EF". We note the work by [19] where the authors consider a different channel, namely a separated two-way relay channel, and exploit a similar idea, namely using two quantization levels at the relay. In the scheme used here, each receiver decodes independently its own message, which is less demanding than a joint decoding scheme in terms of information assumptions. As we have already mentioned, the relay implements the Wyner-Ziv compression and superposition coding similarly to a broadcast channel. The difference with the broadcast channel is that each destination also receives the two direct signals from the source nodes. The rate region which can be obtained by using such a coding scheme is given by the following theorem proved in Appendix A.

Theorem 3.2: For the Gaussian IRC with private messages and bi-level compression EF protocol, any rate pair (R_1,R_2) is achievable where

1) *if* $C\left(\frac{|h_{r1}|^2\nu_2 P_r}{|h_{11}|^2P_1+|h_{21}|^2P_2+|h_{r1}|^2\nu_1P_r+N_1}\right) \geq C\left(\frac{|h_{r2}|^2\nu_2 P_r}{|h_{22}|^2P_2+|h_{12}|^2P_1+|h_{r2}|^2\nu_1P_r+N_2}\right)$, *we have the rates in (4), (5) subject to the constraints*

$$\begin{cases} N_{wz}^{(1)} & \geq \frac{\left(|h_{11}|^2P_1+|h_{21}|^2P_2+N_1\right)A-A_1^2}{|h_{r1}|^2\nu_1P_r} \\ N_{wz}^{(2)} & \geq \frac{\left(|h_{22}|^2P_2+|h_{12}|^2P_1+|h_{r2}|^2\nu_1P_r+N_2\right)A-A_2^2}{|h_{r2}|^2\nu_2P_r}, \end{cases}$$

2) *else, if* $C\left(\frac{|h_{r2}|^2\nu_1 P_r}{|h_{22}|^2P_2+|h_{12}|^2P_1+|h_{r2}|^2\nu_2P_r+N_2}\right) \geq C\left(\frac{|h_{r1}|^2\nu_1 P_r}{|h_{11}|^2P_1+|h_{21}|^2P_1+|h_{r1}|^2\nu_2P_r+N_1}\right)$, *we have the rates in (6), (7) subject to the constraints*

$$\begin{cases} N_{wz}^{(1)} & \geq \frac{\left(|h_{11}|^2P_1+|h_{21}|^2P_2+|h_{r1}|^2\nu_2P_r+N_1\right)A-A_1^2}{|h_{r1}|^2\nu_1P_r} \\ N_{wz}^{(2)} & \geq \frac{\left(|h_{22}|^2P_2+|h_{12}|^2P_1+N_2\right)A-A_2^2}{|h_{r2}|^2\nu_2P_r}, \end{cases}$$

3) *else the rates are given by (8), (9) subject to the constraints*

$$\begin{cases} N_{wz}^{(1)} & \geq \frac{\left(|h_{11}|^2P_1+|h_{21}|^2P_2+|h_{r1}|^2\nu_2P_r+N_1\right)A-A_1^2}{|h_{r1}|^2\nu_1P_r} \\ N_{wz}^{(2)} & \geq \frac{\left(|h_{22}|^2P_2+|h_{12}|^2P_1+|h_{r2}|^2\nu_1P_r+N_2\right)A-A_2^2}{|h_{r2}|^2\nu_2P_r}, \end{cases}$$

with $N_{wz}^{(i)}$ representing the quantization noise corresponding to receiver i, $(\nu_1,\nu_2) \in [0,1]^2, \nu_1+\nu_2 \leq 1$, the relay PA, $A = |h_{1r}|^2P_1+|h_{2r}|^2P_2+N_r$, $A_1 = 2\mathcal{R}e(h_{11}h_{1r}^*)P_1+2\mathcal{R}e(h_{21}h_{2r}^*)P_2$ and $A_2 = 2\mathcal{R}e(h_{12}h_{1r}^*)P_1+2\mathcal{R}e(h_{22}h_{2r}^*)P_2$.

The three scenarios emphasized in this theorem correspond to the following situations: 1) $\mathcal{D}_1$ has the better link (in the sense of the theorem) and can decode both the relay message intended for $\mathcal{D}_2$ and its own message; 2) this scenario is the dual of scenario 1); 3) in this latter scenario, each destination node sees the cooperation signal intended for the other destination node as interference.

$$R_i \leq \min\left\{ C\left(\frac{|h_{ir}|^2\overline{\tau_i}P_i}{|h_{jr}|^2\overline{\tau_j}P_j+N_r}\right), C\left(\frac{|h_{ii}|^2P_i+|h_{ri}|^2\nu_iP_r+2\mathcal{R}e(h_{ii}h_{ri}^*)\sqrt{\tau_iP_i\nu_i\overline{P_r}}}{|h_{ji}|^2P_j+|h_{ri}|^2\nu_jP_r+2\mathcal{R}e(h_{ji}h_{ri}^*)\sqrt{\tau_jP_j\nu_j\overline{P_r}}+N_i}\right)\right\} \quad (3)$$

$$R_1 \leq C\left(\frac{|h_{11}|^2P_1}{N_1+\frac{|h_{21}|^2P_2\left(N_r+N_{wz}^{(1)}\right)}{|h_{2r}|^2P_2+N_r+N_{wz}^{(1)}}}+\frac{|h_{1r}|^2P_1}{N_r+N_{wz}^{(1)}+\frac{|h_{2r}|^2P_2N_1}{|h_{21}|^2P_2+N_1}}\right), \quad (4)$$

$$R_2 \leq C\left(\frac{|h_{22}|^2P_2}{N_2+|h_{r2}|^2\nu_1P_r+\frac{|h_{12}|^2P_1\left(N_r+N_{wz}^{(2)}\right)}{|h_{1r}|^2P_1+N_r+N_{wz}^{(2)}}}+\frac{|h_{2r}|^2P_2}{N_r+N_{wz}^{(2)}+\frac{|h_{1r}|^2P_1(|h_{r2}|^2\nu_1P_r+N_2)}{|h_{12}|^2P_1+|h_{r2}|^2\nu_1P_r+N_2}}\right), \quad (5)$$

$$R_1 \leq C\left(\frac{|h_{11}|^2P_1}{N_1+|h_{r1}|^2\nu_2P_r+\frac{|h_{21}|^2P_2\left(N_r+N_{wz}^{(1)}\right)}{|h_{2r}|^2P_2+N_r+N_{wz}^{(1)}}}+\frac{|h_{1r}|^2P_1}{N_r+N_{wz}^{(1)}+\frac{|h_{2r}|^2P_2(|h_{r1}|^2\nu_2P_r+N_1)}{|h_{21}|^2P_2+|h_{r1}|^2\nu_2P_r+N_1}}\right), \quad (6)$$

$$R_2 \leq C\left(\frac{|h_{22}|^2P_2}{N_2+\frac{|h_{12}|^2P_1\left(N_r+N_{wz}^{(2)}\right)}{|h_{1r}|^2P_1+N_r+N_{wz}^{(2)}}}+\frac{|h_{2r}|^2P_2}{N_r+N_{wz}^{(2)}+\frac{|h_{1r}|^2P_1N_2}{|h_{12}|^2P_1+N_2}}\right), \quad (7)$$

$$R_1 \leq C\left(\frac{|h_{11}|^2P_1}{N_1+|h_{r1}|^2\nu_2P_r+\frac{|h_{21}|^2P_2\left(N_r+N_{wz}^{(1)}\right)}{|h_{2r}|^2P_2+N_r+N_{wz}^{(1)}}}+\frac{|h_{1r}|^2P_1}{N_r+N_{wz}^{(1)}+\frac{|h_{2r}|^2P_2(|h_{r1}|^2\nu_2P_r+N_1)}{|h_{21}|^2P_2+|h_{r1}|^2\nu_2P_r+N_1}}\right), \quad (8)$$

$$R_2 \leq C\left(\frac{|h_{22}|^2P_2}{N_2+|h_{r2}|^2\nu_1P_r+\frac{|h_{12}|^2P_1\left(N_r+N_{wz}^{(2)}\right)}{|h_{1r}|^2P_1+N_r+N_{wz}^{(2)}}}+\frac{|h_{2r}|^2P_2}{N_r+N_{wz}^{(2)}+\frac{|h_{1r}|^2P_1(|h_{r2}|^2\nu_1P_r+N_2)}{|h_{12}|^2P_1+|h_{r2}|^2\nu_1P_r+N_2}}\right), \quad (9)$$

C. Transmission rates for the AF protocol

In this section, the relay is assumed to implement an analog amplifier which does not introduce any delay on the relayed signal. The main features of AF-type protocols are well-known by now (e.g., such relays are generally cheap, involve low complexity relay transceivers, and generally induce negligible processing delays in contrast with DF and EF-type relaying protocols). The relay merely sends $X_r = a_rY_r$ where a_r corresponds to the relay amplification factor/gain. We call the corresponding protocol the zero-delay scalar amplify-and-forward (ZDSAF). The type of assumptions we make here fits well to the setting of DSL or optical communication networks. In wireless networks, the assumed protocol can be seen as an approximation of a scenario with a relay equipped with a power amplifier only. The following theorem provides a region of transmission rates that can be achieved when the transmitters send private messages to their respective receivers, the relay implements the ZDSAF protocol and the receivers implement single-user decoding. The considered framework is attractive in the sense that an AF-based relay can be added to the network without changing the receivers.

Theorem 3.3 (Transmission rate region for the IRC with ZDSAF): Let R_i, $i \in \{1,2\}$, be the transmission rate for the source node $\mathcal{S}_i$. When ZDSAF is assumed the following region is achievable:

$$\forall i \in \{1,2\}, R_i^{\text{AF}} \leq C\left(\frac{|a_r\ h_{ir}h_{ri}+h_{ii}|^2\rho_i}{|a_r\ h_{jr}h_{ri}+h_{ji}|^2\rho_j\frac{N_j}{N_i}+a_r^2|h_{ri}|^2\frac{N_r}{N_i}+1}\right) \quad (10)$$

where $\rho_i = \frac{P_i}{N_i}$, $j = -i$, and a_r is the relay amplification gain.

The proof of this result is standard [20] and will therefore be omitted. Only two points are worth being mentioned. First, the proposed region is achieved by using Gaussian codebooks. Second, the choice of the value of the amplification gain a_r is not always straightforward. In the vast majority of the papers available in the literature, a_r is chosen in order to saturate the power constraint at the relay ($\mathbb{E}|X_r|^2 = P_r$) that is: $a_r = \overline{a}_r \triangleq \sqrt{\frac{P_r}{\mathbb{E}|Y_r|^2}} = \sqrt{\frac{P_r}{|h_{1r}|^2P_1+|h_{2r}|^2P_2+N_r}}$. However, as mentioned in some works [21][22][23][24], this choice can be sub-optimal in the sense of certain performance criteria. The intuitive reason for this is that the AF protocol not only amplifies the useful signal but also the received noise. This

effect can be negligible in certain scenarios for the standard relay channel but can be significant for the IRC. Indeed, even if the noise at the relay is negligible, the interference term for user i (i.e., the term $h_{jr}X_j$, $j=-i$) can be influential. In order to assess the importance of choosing amplification factor a_r adequately (i.e., to maximize the transmission rate of a given user or the network sum-rate) we derive its best value. The proposed derivation differs from [21][23] because, here, we consider a different system (an IRC instead of a relay channel with no direct link), a specific relaying function (linear relaying functions instead of arbitrary functions) and a different performance metric (individual transmission rate and sum-rate instead of raw bit error rate [21] and mutual information [23]). Our problem is also different from [24] since we do not consider the optimal clipping threshold in the sense of the end-to-end distortion for frequency division relay channels. At last, the main difference with [22] is that, for the relay channel, the authors discuss the choice of the optimal amplification gain in terms of transmission rate for a vector AF protocol having a delay of at least one symbol duration; here we focus on a scalar AF protocol with no delay and a different system namely the IRC. In this setup, we have found an analytical expression for the best a_r in the sense of $R_i(a_r)$ for a given user $i\in\{1,2\}$. We have also noticed that the a_r maximizing the network sum-rate has to be computed numerically in general. The corresponding analytical result is stated in the following theorem.

Theorem 3.4: [Optimal amplification gain for the ZDSAF in the IRC] *The transmission rate of user i, $R_i(a_r)$, as a function of $a_r\in[0,\overline{a}_r]$ can have several critical points which are the real solutions, denoted by $c_{r,i}^{(1)}$ and $c_{r,i}^{(2)}$, to the following second degree equation:*

$$\begin{aligned}&a_r^2\left[|m_i|^2\mathrm{Re}(p_iq_i^*)-(|p_i|^2+s_i)\mathrm{Re}(m_in_i^*)\right]+\\&a_r\left[|m_i|^2(|q_i|^2+1)-|n_i|^2(|p_i|^2+s_i)\right]+\\&(|q_i|^2+1)\mathrm{Re}(m_in_i^*)-|n_i|^2\mathrm{Re}(p_iq_i^*)=0\end{aligned}\tag{11}$$

where $m_i=h_{ir}h_{ri}\sqrt{\rho_i}$, $n_i=h_{ii}\sqrt{\rho_i}$, $p_i=h_{jr}h_{ri}\sqrt{\rho_j}$, $q_i=h_{ji}\sqrt{\rho_j}$, $s_i=|h_{ri}|^2$, $i\in\{1,2\}$ and $j=-i$. Thus, depending on the channel parameters, the optimal amplification gain $a_r^*=\arg\max_{a_r\in[0,\overline{a}_r]}R_i(a_r)$ takes one value in the set $a_r^*\in\{0,\overline{a}_r,c_{r,i}^{(1)},c_{r,i}^{(2)}\}$. If, additionally, the channel gains are reals then the two critical points write as: $c_{r,i}^{(1)}=-\frac{n_i}{m_i}$ and $c_{r,i}^{(2)}=-\frac{m_iq_i^2+m_i-p_iq_in_i}{m_iq_ip_i-p_i^2n_i-n_is_i}$.

The proof of this result is provided in Appendix B. Of course, in practice, if the receive signal-to-noise plus interference ratio (viewed from a given user) at the relay is low, choosing the amplification factor a_r adequately does not solve the problem. It is well known that in real systems, a more efficient way to combat noise is to implement error correcting codes. This is one of the reasons why DF is also an important relaying protocol, especially for digital relay transceivers for which AF cannot be implemented in its standard form (see e.g., [24] for more details).

D. Time-Sharing

In terms of achievable Shannon rates, distributed channels differ from their centralized counterpart. Indeed, rate regions are not necessarily convex since the time-sharing argument can be invalid (if no synchronization is possible). Similarly, depending on the channel gains, the achievable rate for a given transmitter can be non-concave with respect to its power allocation policy. This happens if the transmitters cannot be coordinated (distributed channels).

Assuming that the users can be coordinated, we discuss here a standard time-sharing procedure similarly to the approach in [25] for the frequency-division relay channel. More specifically, we assume that user 1 decides to transmit only during a fraction α_1 of the time using the power $\frac{P_1}{\alpha_1}$ and user 2 transmits only with a fraction α_2 percent of the time using the power $\frac{P_2}{\alpha_2}$.

The achievable rate-region with coordinated time-sharing, irrespective of the relay protocol, is:

$$\forall i\in\{1,2\},R_i^{\mathrm{TS}}\leq\alpha_i\overline{\beta_j}R_i\left(\frac{P_i}{\alpha_i},0\right)+\alpha_i\beta_jR_i\left(\frac{P_i}{\alpha_i},\frac{P_j}{\alpha_j}\right),\tag{12}$$

where $j=-i$, $(\alpha_i,\alpha_j)^2\in[0,1]^2$, $(\beta_i,\beta_j)^2\in[0,1]^2$ such that $\beta_1\alpha_2=\beta_2\alpha_1$. The rate $R_i\left(\frac{P_i}{\alpha_i},0\right)$ represents the achievable rate of user i (depends on the relay protocol and was provided in the previous subsections) when user j doesn't transmit and user i transmits with power $\frac{P_i}{\alpha_i}$, $R_i\left(\frac{P_i}{\alpha_i},\frac{P_j}{\alpha_j}\right)$ is the achievable rate when user i transmits with power $\frac{P_i}{\alpha_i}$ and user j transmits with power $\frac{P_j}{\alpha_j}$. In order to achieve this rate region, the users have to be coordinated. This means that they have to know at each instant if the other user is transmitting or not. User i also has to know the parameters α_i and α_j. The parameter $\beta_j\in[0,1]$ represents the fraction of time when user j interferes with user i. Considering the time when both users transmit with non-zero power, we obtain the condition: $\beta_1\alpha_2=\beta_2\alpha_1$.

IV. Power allocation games in multi-band IRCs and Nash equilibrium analysis

In the previous section, we have considered the system model presented in Sec. II for $Q=1$. Here, we consider multi-band IRCs for which $Q\geq 2$. As communications interfere on each band, choosing the power allocation policy at a given transmitter is not a simple optimization problem. Indeed, this choice depends on what the other transmitter does. Each transmitter is assumed to optimize its transmission rate in a selfish manner by allocating its transmit power P_i between Q sub-channels and knowing that the other transmitters want to do the same. This interaction can be modeled as a strategic form non-cooperative game, $\mathcal{G}=(\mathcal{K},(\mathcal{A}_i)_{i\in\mathcal{K}},(u_i)_{i\in\mathcal{K}})$, where: (i) the *players* of the game are the two information sources or transmitters and $\mathcal{K}=\{1,2\}$ is used to refer to the set of players; (ii) the strategy of transmitter i consists in choosing $\underline{\theta}_i=(\theta_i^{(1)},\ldots,\theta_i^{(Q)})$ in its *strategy set* $\mathcal{A}_i=\left\{\underline{\theta}_i\in[0,1]^Q\,\middle|\,\sum_{q=1}^{Q}\theta_i^{(q)}\leq 1\right\}$ where $\theta_i^{(q)}$ represents the fraction of power used in band (q); (iii) $u_i(\cdot)$ is the *utility* function of user $i\in\{1,2\}$ or its achievable rate depending on the relaying protocol. From now on, we will call *state*

of the network the (concatenated) vector of power fractions that the transmitters allocate to the IRCs i.e., $\underline{\theta} = (\underline{\theta}_1, \underline{\theta}_2)$. An important issue is to determine whether there exist some outcomes to this conflicting situation. A natural solution concept in non-cooperative games is the Nash equilibrium [26]. In distributed networks, the existence of a stable operating state of the system is a desirable feature. In this respect, the NE is a stable state from which the users do not have any incentive to unilaterally deviate (otherwise they would lose in terms of utility). The mathematical definition is the following.

Definition 4.1: [Nash equilibrium] *The state* $(\underline{\theta}_i^*, \underline{\theta}_{-i}^*)$ *is a pure NE of the strategic form game* $\mathcal{G}$ *if* $\forall i \in \mathcal{K}, \forall \underline{\theta}_i' \in \mathcal{A}_i,\ u_i(\underline{\theta}_i^*, \underline{\theta}_{-i}^*) \geq u_i(\underline{\theta}_i', \underline{\theta}_{-i}^*)$.

In this section, we mainly focus on the problem of existence of such a solution, which is the first step towards equilibria characterization in IRCs. The problems of equilibrium uniqueness, selection, convergence, and efficiency are therefore left as natural extensions of the work reported here.

A. Equilibrium existence analysis for the DF protocol

As explained in Sec. III-A the signals transmitted by $\mathcal{S}_1$ and $\mathcal{S}_2$ in band (q) have the following form: $X_i^{(q)} = X_{i,0}^{(q)} + \sqrt{\frac{\tau_i^{(q)}}{\nu_i^{(q)}} \frac{\theta_i^{(q)} P_i}{P_r^{(q)}}} X_{r,i}^{(q)}$ where the signals $X_{i,0}^{(q)}$ and $X_{r,i}^{(q)}$ are Gaussian and independent. At the relay $\mathcal{R}$, the transmitted signal writes as: $X_r^{(q)} = X_{r,1}^{(q)} + X_{r,2}^{(q)}$. For a given allocation policy $\underline{\theta}_i = \left(\theta_i^{(1)}, ..., \theta_i^{(Q)}\right)$, the source-destination pair $(\mathcal{S}_i, \mathcal{D}_i)$ achieves the transmission rate $\sum_{q=1}^{Q} R_i^{(q),\mathrm{DF}}$ where

$$\left\{ \begin{array}{lcl} R_1^{(q),\mathrm{DF}} & = & \min \left\{ R_{1,1}^{(q),\mathrm{DF}}, R_{1,2}^{(q),\mathrm{DF}} \right\} \\ R_2^{(q),\mathrm{DF}} & = & \min \left\{ R_{2,1}^{(q),\mathrm{DF}}, R_{2,2}^{(q),\mathrm{DF}} \right\} \end{array} \right. , \tag{13}$$

and $R_{1,1}^{(q),\mathrm{DF}}$, $R_{1,2}^{(q),\mathrm{DF}}$,$R_{2,1}^{(q),\mathrm{DF}}$,$R_{2,2}^{(q),\mathrm{DF}}$ are given in (14) and $(\nu^{(q)}, \tau_1^{(q)}, \tau_2^{(q)})$ is a given triple of parameters in $[0,1]^3$, $\tau_1^{(q)} + \tau_2^{(q)} \leq 1$.

The achievable transmission rate of user i is given by:

$$u_i^{\mathrm{DF}}(\underline{\theta}_i, \underline{\theta}_{-i}) = \sum_{q=1}^{Q} R_i^{(q),\mathrm{DF}}(\theta_i^{(q)}, \theta_{-i}^{(q)}). \tag{15}$$

We suppose that the game is played once (one-shot or static game), the users are rational (each selfish player does what is best for itself), rationality is common knowledge, and the game is with complete information that is, every player knows the triplet $\mathcal{G}^{\mathrm{DF}} = (\mathcal{K}, (\mathcal{A}_i)_{i\in\mathcal{K}}, (u_i^{\mathrm{DF}})_{i\in\mathcal{K}})$. Although this setup might seem to be demanding in terms of CSI at the source nodes, it turns out that the equilibria predicted in such a framework can be effectively observed in more realistic frameworks where one player observes the strategy played by the other player and reacts accordingly by maximizing his utility, the other player observes this and updates its strategy and so on. We will come back to this later on. The existence theorem for the DF protocol is given hereunder.

Theorem 4.2: [Existence of an NE for the DF protocol] *If the channel gains satisfy the condition* $\mathcal{R}e(h_{ii}^{(q)} h_{ri}^{(q)*}) \geq 0$, *for all* $i \in \{1,2\}$ *and* $q \in \{1, \ldots, Q\}$ *the game defined by* $\mathcal{G}^{\mathrm{DF}} = (\mathcal{K}, (\mathcal{A}_i)_{i\in\mathcal{K}}, (u_i^{\mathrm{DF}}(\underline{\theta}_i, \underline{\theta}_{-i}))_{i\in\mathcal{K}})$ *with* $\mathcal{K} = \{1,2\}$ *and* $\mathcal{A}_i = \left\{ \underline{\theta}_i \in [0,1]^Q \left| \sum_{q=1}^{Q} \theta_i^{(q)} \leq 1 \right. \right\}$, *has always at least one pure NE.*

Proof: The proof is based on Theorem 1 of [27]. The latter theorem states that in a game with a finite number of players, if for every player 1) the strategy set is convex and compact, 2) its utility is continuous in the vector of strategies and 3) concave in its own strategy, then the existence of at least one pure NE is guaranteed. In our setup checking that conditions 1) and 2) are met is straightforward. The condition $\mathcal{R}e(h_{ii}^{(q)} h_{ri}^{(q)*}) \geq 0$ is a sufficient condition that ensures the concavity of $R_{i,2}^{\mathrm{DF}}$ w.r.t. $\theta_i^{(q)}$. The intuition behind this condition is that the superposition of the two signals carrying useful information for user i (i.e., signal from $\mathcal{S}_i$ and $\mathcal{R}$) has to be constructive w.r.t. the amplitude of the resulting signal. As the sum of concave functions is a concave function, the *min* of two concave functions is a concave function (see [28] for more details on operations preserving concavity), and $R_{i,j}^{(q)}$ is a concave function of $\underline{\theta}_i$, it follows that 3) is also met, which concludes the proof. ■

Theorem indicates, in particular, that for the pathloss model where the channel gains are positive real scalars (i.e., $h_{ij} > 0$, $(i,j) \in \{1,2,r\}^2$) there always exists an equilibrium. As a consequence, if some relays are added in the network, the transmitters will adapt their PA policies accordingly and, whatever the locations of the relays, an equilibrium will be observed. This is a nice property for the system under investigation. As the PA game with DF is concave it is tempting to try to verify whether the sufficient condition for uniqueness of [27] is met here. It turns out that the diagonally strict concavity condition of [27] is not trivial to be checked. Additionally, it is possible that the game has several equilibria as it is proven to be the case for the AF protocol.

B. Equilibrium existence analysis for the EF protocol

In this section, we make the same assumptions as in Sec. IV-A concerning the reception schemes and PA policies at the relays: we assume that each node $\mathcal{R}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ implements single-user decoding and the PA policy at each relay i.e., $\underline{\nu} = \left(\nu^{(1)}, ..., \nu^{(Q)}\right)$ is fixed. Each relay now implements the EF protocol. Under this assumption, the utility for user $i \in \{1,2\}$ can be expressed as follows

$$u_i^{\mathrm{EF}}(\underline{\theta}_i, \underline{\theta}_{-i}) = \sum_{q=1}^{Q} R_i^{(q),\mathrm{EF}} \tag{16}$$

where $R_i^{(q),\mathrm{EF}}$ are given in (17) $\nu^{(q)} \in [0,1]$, $A^{(q)} = |h_{1r}^{(q)}|^2 \theta_1^{(q)} P_1 + |h_{2r}^{(q)}|^2 \theta_2^{(q)} P_2 + N_r^{(q)}$, $A_1^{(q)} = h_{11}^{(q)} h_{1r}^{(q),*} \theta_1^{(q)} P_1 + h_{21}^{(q)} h_{2r}^{(q),*} \theta_2^{(q)} P_2$ and $A_2^{(q)} = h_{12}^{(q)} h_{1r}^{(q),*} \theta_1^{(q)} P_1 + h_{22}^{(q)} h_{2r}^{(q),*} \theta_2^{(q)} P_2$. What is interesting with this EF protocol is that, here again, one can prove that the utility is concave for every user. This is the purpose of the following theorem.

Theorem 4.3: [Existence of an NE for the bi-level compression EF protocol] *The game defined by* $\mathcal{G}^{\mathrm{EF}} =$

$$\left\{\begin{array}{lcl} R_{1,1}^{(q),\mathrm{DF}} & = & C\left(\frac{\left|h_{1r}^{(q)}\right|^2\overline{\tau_1^{(q)}}\theta_1^{(q)}P_1}{\left|h_{2r}^{(q)}\right|^2\overline{\tau_2^{(q)}}\theta_2^{(q)}P_2+N_r^{(q)}}\right) \\ R_{2,1}^{(q),\mathrm{DF}} & = & C\left(\frac{\left|h_{2r}^{(q)}\right|^2\overline{\tau_2^{(q)}}\theta_2^{(q)}P_2}{\left|h_{1r}^{(q)}\right|^2\overline{\tau_1^{(q)}}\theta_1^{(q)}P_1+N_r^{(q)}}\right) \\ R_{1,2}^{(q),\mathrm{DF}} & = & C\left(\frac{\left|h_{11}^{(q)}\right|^2\theta_1^{(q)}P_1+\left|h_{r1}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}+2\mathrm{Re}\left(h_{11}^{(q)}h_{r1}^{(q),*}\right)\sqrt{\tau_1^{(q)}\theta_1^{(q)}P_1\nu^{(q)}P_r^{(q)}}}{\left|h_{21}^{(q)}\right|^2\theta_2^{(q)}P_2+\left|h_{r1}^{(q)}\right|^2\overline{\nu^{(q)}}P_r^{(q)}+2\mathrm{Re}\left(h_{21}^{(q)}h_{r1}^{(q),*}\right)\sqrt{\tau_2^{(q)}\theta_2^{(q)}P_2\overline{\nu^{(q)}}P_r^{(q)}}+N_1^{(q)}}\right) \\ R_{2,2}^{(q),\mathrm{DF}} & = & C\left(\frac{\left|h_{22}^{(q)}\right|^2\theta_2^{(q)}P_2+\left|h_{r2}^{(q)}\right|^2\overline{\nu^{(q)}}P_r^{(q)}+2\mathrm{Re}\left(h_{22}^{(q)}h_{r2}^{(q),*}\right)\sqrt{\tau_2^{(q)}\theta_2^{(q)}P_2\overline{\nu^{(q)}}P_r^{(q)}}}{\left|h_{12}^{(q)}\right|^2\theta_1^{(q)}P_1+\left|h_{r2}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}+2\mathrm{Re}\left(h_{12}^{(q)}h_{r2}^{(q),*}\right)\sqrt{\tau_1^{(q)}\theta_1^{(q)}P_1\nu^{(q)}P_r^{(q)}}+N_2^{(q)}}\right), \end{array}\right. \tag{14}$$

$$\left\{\begin{array}{lcl} R_1^{(q),\mathrm{EF}} & = & C\left(\frac{\left(\left|h_{2r}^{(q)}\right|^2\theta_2^{(q)}P_2+N_r^{(q)}+N_{wz,1}^{(q)}\right)\left|h_{11}^{(q)}\right|^2\theta_1^{(q)}P_1+\left(\left|h_{21}^{(q)}\right|^2\theta_2^{(q)}P_2+\left|h_{r1}^{(q)}\right|^2\overline{\nu^{(q)}}P_r^{(q)}+N_1^{(q)}\right)\left|h_{1r}^{(q)}\right|^2\theta_1^{(q)}P_1}{\left(N_r^{(q)}+N_{wz,1}^{(q)}\right)\left(|h_{21}^{(q)}|^2\theta_2^{(q)}P_2+|h_{r1}^{(q)}|^2\overline{\nu^{(q)}}P_r^{(q)}+N_1^{(q)}\right)+|h_{2r}^{(q)}|^2\theta_2^{(q)}P_2\left(|h_{r1}^{(q)}|^2\overline{\nu^{(q)}}P_r^{(q)}+N_1^{(q)}\right)}\right) \\ R_2^{(q),\mathrm{EF}} & = & C\left(\frac{\left(|h_{1r}|^2\theta_1^{(q)}P_1+N_r^{(q)}+N_{wz,2}^{(q)}\right)\left|h_{22}^{(q)}\right|^2\theta_2^{(q)}P_2+\left(\left|h_{12}^{(q)}\right|^2\theta_1^{(q)}P_1+\left|h_{r2}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}+N_2^{(q)}\right)\left|h_{2r}^{(q)}\right|^2\theta_2^{(q)}P_2}{\left(N_r^{(q)}+N_{wz,2}^{(q)}\right)\left(\left|h_{12}^{(q)}\right|^2\theta_1^{(q)}P_1+\left|h_{r2}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}+N_2^{(q)}\right)+\left|h_{1r}^{(q)}\right|^2\theta_1^{(q)}P_1\left(\left|h_{r2}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}+N_2^{(q)}\right)}\right) \end{array}\right., \tag{17}$$

$$\left\{\begin{array}{lcl} N_{wz,1}^{(q)} & = & \frac{\left(\left|h_{11}^{(q)}\right|^2\theta_1^{(q)}P_1+\left|h_{21}^{(q)}\right|^2\theta_2^{(q)}P_2+\left|h_{r1}^{(q)}\right|^2\overline{\nu^{(q)}}P_r^{(q)}+N_1^{(q)}\right)A^{(q)}-\left|A_1^{(q)}\right|^2}{\left|h_{r1}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}} \\ N_{wz,2}^{(q)} & = & \frac{\left(\left|h_{22}^{(q)}\right|^2\theta_2^{(q)}P_2+\left|h_{12}^{(q)}\right|^2\theta_1^{(q)}P_1+\left|h_{r2}^{(q)}\right|^2\nu^{(q)}P_r^{(q)}+N_2^{(q)}\right)A^{(q)}-\left|A_2^{(q)}\right|^2}{\left|h_{r2}^{(q)}\right|^2\overline{\nu^{(q)}}P_r^{(q)}} \end{array}\right. \tag{18}$$

$(\mathcal{K},(\mathcal{A}_i)_{i\in\mathcal{K}},(u_i^{\mathrm{EF}}(\underline{\theta}_i,\underline{\theta}_{-i}))_{i\in\mathcal{K}})$ *with* $\mathcal{K}=\{1,2\}$ *and* $\mathcal{A}_i=\left\{\underline{\theta}_i\in[0,1]^Q\,\middle|\,\sum_{q=1}^{Q}\theta_i^{(q)}\leq 1\right\}$, *has always at least one pure NE.*

The proof is similar to the proof of Theorem IV-A. To be able to apply Theorem 1 of Rosen [27], we have to prove that the utility u_i^{EF} is concave w.r.t. $\underline{\theta}_i$. The problem is less simple than for DF because the compression noise $N_{wz,i}^{(q)}$ which appears in the denominator of the capacity function in Eq. (17) depends on the strategy $\underline{\theta}_i$ of transmitter i. It turns out that it is still possible to prove the desired result as shown in Appendix C.

C. Equilibrium analysis for the AF protocol

Here, we assume that the relay implements the ZDSAF protocol, which has already been described in Sec. III-C. One of the nice features of the (analog) ZDSAF protocol is that relays are very easy to be deployed since they can be used without any change on the existing (non-cooperative) communication system. The amplification factor/gain for the relay on band (q) will be denoted by $a_r^{(q)}$. Here, we consider the most common choice for the amplification factor that it, the one that exploits all the transmit power available on each band. The achievable transmission rate is given by

$$u_i^{\mathrm{AF}}(\underline{\theta}_i,\underline{\theta}_{-i})=\sum_{q=1}^{Q}R_i^{(q),\mathrm{AF}}(\theta_i^{(q)},\theta_{-i}^{(q)}) \tag{19}$$

where $R_i^{(q),\mathrm{AF}}$ is the rate user i obtains by using band (q) when the ZDSAF protocol is used by the relay $\mathcal{R}$. After Sec. III-C the latter quantity is:

$$R_i^{(q),\mathrm{AF}}=C\left(\frac{\left|a_r^{(q)}\ h_{ir}^{(q)}h_{ri}^{(q)}+h_{ii}^{(q)}\right|^2\theta_i^{(q)}\rho_i}{\left|a_r^{(q)}\ h_{jr}^{(q)}h_{ri}^{(q)}+h_{ji}^{(q)}\right|^2\rho_j\theta_j^{(q)}\frac{N_j^{(q)}}{N_i^{(q)}}+\left(a_r^{(q)}\right)^2\left|h_{ri}^{(q)}\right|^2\frac{N_r^{(q)}}{N_i^{(q)}}+1}\right) \tag{20}$$

where $a_r^{(q)}=\tilde{a}_r^{(q)}(\theta_1^{(q)},\theta_2^{(q)})\triangleq\sqrt{\frac{P_r}{\left|h_{1r}^{(q)}\right|^2P_1+\left|h_{2r}^{(q)}\right|^2P_2+N_r}}$ and $\rho_i^{(q)}=\frac{P_i}{N_i^{(q)}}$. Without loss of generality and for the sake of clarity we will assume in Sec. IV-C that $\forall(i,q)\in\{1,2,r\}\times\{1,\ldots,Q\}$, $N_i^{(q)}=N$, $P_r^{(q)}=P_r$ and we introduce the quantities $\rho_i=\frac{P_i}{N}$. In this setup the following existence theorem can be proven.

Theorem 4.4: [Existence of an NE for ZDSAF] *If any of the following conditions are met: i)* $\left|a_r^{(q)}h_{ir}^{(q)}h_{ri}^{(q)}\right|\gg\left|h_{ii}^{(q)}\right|$ *and* $\left|a_r^{(q)}h_{jr}^{(q)}h_{ri}^{(q)}\right|\gg\left|h_{ji}^{(q)}\right|$ *(negligible direct links), ii)* $\left|h_{ii}^{(q)}\right|\gg\left|a_r^{(q)}h_{ir}^{(q)}h_{ri}^{(q)}\right|$ *and* $\left|h_{ji}^{(q)}\right|\gg\left|a_r^{(q)}h_{jr}^{(q)}\right|\min\left\{1,\left|h_{ri}^{(q)}\right|\right\}$ *(negligible relay links), iii)* $a_r^{(q)}=A_r^{(q)}\in[0,\tilde{a}_r^{(q)}(1,1)]$ *(constant amplification gain), there exists at least one pure NE in the PA game* $\mathcal{G}^{\mathrm{AF}}$.

The proof is similar to the proof of Theorem IV-A. The sufficient conditions ensure the concavity of the function $R_i^{(q),\mathrm{AF}}$ w.r.t. $\theta_i^{(q)}$. For the first case i) where the direct links between the sources and destinations are negligible (e.g., in the wired DSL setting these links are missing and the transmission is only possible using the relay nodes), the achievable rates are given $\forall i\in\{1,2\}$, in (21) and it can be proven that $R_i^{(q),\mathrm{AF}}$ is concave w.r.t. $\theta_i^{(q)}$. The other two cases are easier to prove

$$R_i^{(q),\mathrm{AF}} = C\left(\frac{\left|h_{ir}^{(q)}h_{ri}^{(q)}\right|^2\theta_i^{(q)}\rho_i\rho_r\frac{N_r}{N_i}}{\left|h_{ri}^{(q)}\right|^2\theta_i^{(q)}\rho_i+\left(\left|h_{rj}^{(q)}\right|^2\theta_j^{(q)}\rho_j\frac{N_j}{N_i}+\frac{N_r}{N_i}\right)\left(\left|h_{ri}^{(q)}\right|^2\rho_r\frac{N_r}{N_i}+1\right)}\right) \tag{21}$$

since the amplification gain is either constant or not taken into account and the rate $R_i^{(q),\mathrm{AF}}$ is a composition of a logarithmic function and a linear function of $\theta_i^{(q)}$ and thus concave.

The determination of NE and the convergence issue to one of the NE are far from being trivial in this case. For example, potential games [29] and supermodular games [30] are known to have attractive convergence properties. It can be checked that, the PA game under investigation is neither a potential nor a supermodular game in general. To be more precise, it is a potential game for a set of channel gains which corresponds to a scenario with probability zero (e.g., the parallel multiple access channel). The authors of [31] studied supermodular games for the interference channel with $K = 2$, $Q = 3$, assuming that only one band is shared by the users (IC) while the other bands are private (one interference-free band for each user). Therefore, each user allocates its power between two bands. Their strategies are designed such that the game has strategic complementarities. However, as stated in [31], this design trick does not work for more than two players or if the users can access more than two frequency bands. In conclusion, general convergence results seem to require more advanced tools and further investigations.

Special case study

As we have just mentioned, the uniqueness/convergence/efficiency analysis of NE for the DF and EF protocols requires a separate work to be treated properly. However, it is possible to obtain relatively easy some interesting results in a special case of the AF protocol. The reason for analyzing this special case is threefold: a) it corresponds to a possible scenario in wired communication networks; b) it allows us to introduce some game-theoretic concepts that can be used to treat more general cases and possibly the DF and EF protocols; c) it allows us to have more insights on the problem with a more general choice for $a_r^{(q)}$. The special case under investigation is as follows: $Q = 2$ and $\forall q \in \{1,2\}$, $a_r^{(q)} = A_r^{(q)} \in [0, \tilde{a}_r(1,1)]$ are constant w.r.t. $\underline{\theta}$. We observe that the strategy set of user i are scalar spaces $\theta_i \in [0,1]$ because we can consider $\theta_i^{(1)} = \theta_i$ and $\theta_i^{(2)} = \overline{\theta_i}$. For the sake of clarity, we denote by $h_{ij} = h_{ij}^{(1)}$ and $g_{ij} = h_{ij}^{(2)}$. Note that the case $a_r^{(q)} = A_r^{(q)}$ can also be seen as an interference channel for which there is an additional degree of freedom on each band. The choice $Q = 2$ is totally relevant in scenarios where the spectrum is divided in two bands, one shared band where communications interfere and one protected band where they do not (see e.g., [32]). The choice $a_r^{(q)} = const.$ has the advantage of being mathematically simple and allows us to initialize the uniqueness/convergence analysis. Moreover, it corresponds to a suitable model for an analog repeater in the linear regime in wired networks or, more generally, to a power amplifier for which neither automatic gain control is available nor received power estimation mechanism. By making these two assumptions, it is possible to determine exactly the number of Nash equilibria through the notion of best response (BR) functions. The BR of player i to player j is defined by $\mathrm{BR}_i(\theta_j) = \arg\max_{\theta_i} u_i(\theta_i, \theta_j)$. In general, it is a correspondence but in our case it is just a function. The equilibrium points are the intersection points of the BRs of the two players. In this case, using the Lagrangian functions to impose the power constraint, it can be checked that:

$$\mathrm{BR}_i(\theta_j) = \left| \begin{array}{ll} F_i(\theta_j) & \text{if } 0 < F_i(\theta_j) < 1 \\ 1 & \text{if } F_i(\theta_j) \geq 1 \\ 0 & \text{otherwise} \end{array} \right. \tag{22}$$

where $j = -i$, $F_i(\theta_j) \triangleq -\frac{c_{ij}}{c_{ii}}\theta_j + \frac{d_i}{c_{ii}}$ is an affine function of θ_j for $(i,j) \in \{(1,2),(2,1)\}$, $c_{ii} = 2|A_r^{(1)}h_{ri}h_{ir} + h_{ii}|^2|A_r^{(2)}g_{ri}g_{ir} + g_{ii}|^2\rho_i$; $c_{ij} = |A_r^{(1)}h_{ri}h_{ir} + h_{ii}|^2|A_r^{(2)}g_{ri}g_{jr} + g_{ji}|^2\rho_j + |A_r^{(1)}h_{ri}h_{jr} + h_{ji}|^2|A_r^{(2)}g_{ri}g_{ir} + g_{ii}|^2\rho_j$; $d_i = |A_r^{(1)}h_{ri}h_{ir} + h_{ii}|^2[|A_r^{(2)}g_{ri}g_{ir} + g_{ii}|^2\rho_i + |A_r^{(2)}g_{ri}g_{jr} + g_{ji}|^2\rho_j + A_r^{(2)}|g_{ri}|^2 + 1] - |A_r^{(2)}g_{ri}g_{ir} + g_{ii}|^2(A_r^{(1)}|h_{ri}|^2 + 1)$. By studying the intersection points between BR_1 and BR_2, one can prove the following theorem (the proof is provided in Appendix D).

Theorem 4.5 (Number of Nash equilibria for ZDSAF): For the game $\mathcal{G}^{\mathrm{AF}}$ with fixed amplification gains at the relays, (i.e., $\frac{\partial a_r}{\partial \theta_i^{(q)}} = 0$), there can be a unique NE, two NE, three NE or an infinite number of NE, depending on the channel parameters (i.e., h_{ij}, g_{ij}, ρ_i, $A_r^{(q)}$, $(i,j) \in \{1,2,r\}^2$, $q \in \{1,2\}$.

Notice that, if $A_r = 0$, we obtain the complete characterization of the NE set for the two-users two-channels parallel interference channel. In the proof in Appendix D, we give explicit expressions of the possible NE in function of the system parameters (i.e., the amplification gain A_r and the channel gains). If the channel gains are the realizations of continuous random variables, it is easy to prove that the probability of observing the necessary conditions on the channel gains for having two NEs or an infinite number of NEs is zero. Said otherwise, considering the pathloss model and arbitrary nodes positioning, there will be, with probability one, either one or three NE, depending on the channel gains. When the channel gains are such that the NE is unique, the unique NE can be shown to be:

$$\underline{\theta}^{\mathrm{NE}} = \underline{\theta}^* = \left(\frac{c_{22}d_1 - c_{12}d_2}{c_{11}c_{22} - c_{12}c_{21}}, \frac{c_{11}d_2 - c_{21}d_1}{c_{11}c_{22} - c_{12}c_{21}}\right). \tag{23}$$

When there are three NE, it seems a priori impossible to predict the NE that will be effectively observed in the one-shot game. In fact, in practice, in a context of adaptive/cognitive transmitters (note that what can be adapted is also the PA policy chosen by the designer/owner of the transmitter), it is possible to predict the equilibrium of the network. First,

in general, there is no reason why the sources should start transmitting at the same time. Thus, one transmitter, say i, will be alone and using a certain PA policy. The transmitter coming after, namely $\mathcal{S}_{-i}$, will sense/measure/probe its environment and play its BR to what it observes. As a consequence, user i will move to a new policy, maximizing its utility to what transmitter $-i$ has played and so on. The key question is: does this procedure converge? This procedure is guaranteed to converge to one of the NE and a detailed discussion about the asymptotic stability of the NE can be found in Appendix D. The arguments for proving this have been used for the first time in [33] where the "Cournot duopoly" was introduced. In [33], the BRs of each player is purely affine, which leads in this case to a unique equilibrium. The corresponding iterative procedure is called the Cournot tâtonnement process in [34]. In the case with three NE, the effectively observed NE can be predicted by knowing the initial network state that is, the PA policy played by the first transmitting player (see Sec. V). To implement such an iterative procedure, it can be checked [1] that the transmitters need to know less network parameters than in the original game where the amplification factor saturates the constraint. In fact, the needed parameters can be acquired by realistic sensing/probing techniques or feedback mechanisms based on standard estimation procedures. As a comment, note that in the (modern) literature of decentralized or distributed communications networks where the optimal PA policy of a transmitter is to water-fill, the mentioned iterative procedure is called iterative water-filling.

D. Equilibrium analysis for the Time-Sharing scheme

In the previous subsections, we have given sufficient conditions that ensure the existence of the Nash equilibrium. Our approach is based on the concave games studied in [27] and consists in finding the sufficient conditions that ensure the concavity of the transmission achievable rates. We have seen that, assuming ZDSAF or DF relaying protocols, the achievable rates are not necessarily concave.

Assuming that the transmitters can be coordinated, and by using the time-sharing scheme similarly to Subsec. III-D, the achievable transmission rate of user i is given by:

$$u_i^{\mathrm{TS}}(\underline{\theta}_i, \underline{\theta}_{-i}) = \sum_{q=1}^{Q} R_i^{(q),\mathrm{TS}}(\theta_i^{(q)}, \theta_{-i}^{(q)}) \tag{24}$$

where $R_i^{(q),\mathrm{TS}}$ is the rate user i obtains by using band (q) and time-sharing technique. After Sec. III-D the latter quantity is:

$$\begin{aligned} \forall i \in \{1,2\}, R_i^{(q),\mathrm{TS}} \quad = \quad & \alpha_i^{(q)} \overline{\beta_j}^{(q)} R_i^{(q)} \left(\frac{\theta_i^{(q)} P_i}{\alpha_i}, 0 \right) + \\ & \alpha_i^{(q)} \beta_j^{(q)} R_i^{(q)} \left(\frac{\theta_i^{(q)} P_i}{\alpha_i}, \frac{\theta_j^{(q)} P_j}{\alpha_j} \right) \end{aligned} , \tag{25}$$

where $j = -i$, $(\alpha_i^{(q)}, \alpha_j^{(q)}) \in [0,1]^2$, $(\beta_i^{(q)}, \beta_j^{(q)}) \in [0,1]^2$ such that $\beta_1^{(q)} \alpha_2^{(q)} = \beta_2^{(q)} \alpha_1^{(q)}$. These parameters are fixed and chosen such that the achievable rates are maximized. The rates $R_i^{(q)} \left(\frac{\theta_i^{(q)} P_i}{\alpha_i}, 0 \right)$, $R_i^{(q)} \left(\frac{\theta_i^{(q)} P_i}{\alpha_i}, \frac{\theta_j^{(q)} P_j}{\alpha_j} \right)$ represent the achievable rates in band (q) when time-sharing is used. These rates depend on the relaying protocol and are given by Eq. (13) for DF and by Eq. (20) for ZDSAF. Notice that, when EF is assumed, the rates are always concave irrespective of the channel gains and time-sharing techniques do not change the achievable rate-region.

Theorem 4.6: [Existence of an NE for TS] *There always exists at least one pure NE in the PA game $\mathcal{G}^{\mathrm{TS}}$, regardless of the used relaying scheme and the values of the channel gains.*

If the users are coordinated (i.e., each user is aware of the moments where the other user is transmitting or not) then their achievable rates $R_i^{(q),\mathrm{TS}}$ are always concave w.r.t. $\theta_i^{(q)}$. This implies directly that [27], irrespective of the relaying technique and of the channel gains, the existence of an NE will be guaranteed.

In the particular case where either $P_r^{(q)} = 0$ or $h_{ir}^{(q)} = 0$, for all $q \in \{1, \ldots, Q\}$ and $i \in \{1, 2\}$, the parallel IRC reduces to the parallel interference channel [6]. The time-sharing scheme is useless since the achievable rates are already concave and $\alpha_i = 1$, $\beta_i = 1$ are optimal. Therefore, Theorem 4.6 guarantees the existence of the NE in this case and is consistent with the known results in [6].

V. Simulation results

Single-band IRCs: AF vs DF vs EF. Here, we assume $Q = 1$ and a path loss exponent of 2 that is, $|h_{ij}| = \left(\frac{d_{ij}}{d_0}\right)^{-\frac{\gamma}{2}}$ for $(i,j) \in \{1,2,r\}^2$ where $d_0 = 5$ m is a reference distance and $\gamma = 2$ is the path loss exponent. The nodes $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{D}_1$, $\mathcal{D}_2$ are assumed to be located in a plane. The positions of the nodes will be indicated on each figure and are characterized by the distance between them which are chosen as follows: $d'_{11} = 11.5$ m, $d'_{22} = 10$ m, $d'_{12} = 11$ m and $d'_{21} = 14$ m. As for the relay, to avoid any divergence for the path loss in $d_{ij} = 0$, we assume that it is located a hight $\epsilon = 0.1$ m from this plane i.e., the relay location is given by the (x_r, y_r, z_r) where z_r is fixed and equals 0.1 m; thus $d_{ij} = \sqrt{d'^2_{ij} + \epsilon^2}$ for $i = r$ or $j = r$ and $i \neq j$. The noise levels at the receiver nodes are assumed to be normalized ($N_1 = N_2 = N_r = 1$). In terms of transmit power we analyze two cases: a symmetric case where $P_1 = P_2 = 10$ (normalized power) and an asymmetric one where $P_1 = 3$ and $P_2 = 10$. The relay transmit power is fixed: $P_r = 10$. For the symmetric scenario, Fig. 2 represents the regions of the plane $\left(\frac{x_r}{d_0}, \frac{y_r}{d_0}\right) \in [-4, +4] \times [-3, +4]$ (corresponding to the possible relay positions) where a certain protocol performs better than the two others in terms of system sum-rate. These regions are in agreement with what is generally observed for the standard relay channel. This type of information is useful, for example, when the relay has to be located in specific places because of different practical constraints and one has to choose the best protocol. Fig. 3 allows one to better quantify the differences in terms of sum-rate between the AF, DF and bi-level EF protocols since it represents the sum-rate versus x_r for a given $y_r = 0.5 d_0$. The discontinuity observed stems from the fact that for the bi-level EF protocol there is a frontier delineating the scenarios where one receiver is better than the other and can therefore suppress the interference of the relay (as explained in Sec. III-B).

Number of Nash equilibria for the AF protocol. First, we show that in the PA game with ZDSAF, one can have three possible Nash equilibria. For a given typical scenario composed of an IC in parallel with an IRC ($Q = 2$) and $\rho_1 = 1$, $\rho_2 = 3$, $\rho_r = 2$ and the channel gains $(g_{11}, g_{12}, g_{21}, g_{22}) = (2.76, 5.64, -3.55, -1.61)$, $(h_{11}, h_{12}, h_{21}, h_{22}) = (14.15, 3.4, 0, 1.38)$ and $(h_{1r}, h_{2r}, h_{r1}, h_{r2}) = (-3.1, 2.22, -3.12, 1.16)$, we plot the best response functions in Fig. 4. We see that there are three intersection points and therefore three Nash equilibria. As explained in Sec. IV-C, the effectively observed NE in a one-shot game is not predictable without any additional assumptions. However, the Cournot tatônnement procedure converges towards a given NE which can be predicted from the sole knowledge of the starting point of the game, namely θ_1^0 or θ_2^0.

Stackelberg formulation. We have mentioned that a strong motivation for studying IRCs is to be able to introduce relays in a network with non-coordinated and interfering pairs of terminals. For example, relays could be introduced by an operator aiming at improving the performance of the communications of his customers. In such a scenario, the operator acts as a player and more precisely as a game leader in the sense of [35]. In [35], the author introduced what is called nowadays a Stackelberg game. This type of hierarchical games comprises one leader which plays in the first step of the game and several players (the followers) which observe the leader's strategy and choose their actions accordingly. In our context, the game leader is the operator/engineer /relay who chooses the parameters of the relays. The followers are the adaptive/cognitive transmitters that adapt their PA policy to what they observe. In the preceding sections we have mentioned some of these parameters: the location of each relay; in the case of AF, the amplification gain of each relay; in the case of DF and EF, the power allocation policy between the two cooperative signals at each relay i.e., the parameter $\nu^{(q)}$. Therefore, the relay can be thought of as a player who maximizes its own utility. This utility can be either the individual utility of a given transmitter (picture one WiFi subscriber wanting to increase his downlink throughput by locating his cellular phone somewhere in his apartment while his neighbor can also exploit the same spectral resources) or the network sum-rate (in the case of an operator). In the latter case, the operator possesses some degrees of freedom to improve the efficiency of the equilibrium. In the remaining part of this section, we focus on the Stackelberg formulation where the strategy of the leader is respectively the relay amplification factor, position and power allocation between the cooperative signals. The system considered is composed of an IRC in parallel with an interference channel (IC) [36]. All the simulations provided are obtained by applying the Cournot tatônnement procedure. The simulation setup is as follows. The source and destination nodes are located in fixed locations in the region $[-L, L]^2$ of a plane, with $L = 10$ m, such that the relative distances between the nodes are: $d_{11} = 6.52$m, $d_{12} = 8.32$m, $d_{21} = 6.64$m, $d_{22} = 6.73$m. We assume a path loss model for the channel gains $|g_{ij}|$, $|h_{ij}|$. For the path loss model we take $|h_{ij}| = \left(\frac{d_{ij}}{d_0}\right)^{-\frac{\gamma^{(1)}}{2}}$ and $|g_{ij}| = \left(\frac{d_{ij}}{d_0}\right)^{-\frac{\gamma^{(2)}}{2}}$ for $(i, j) \in \{1, 2, r\}^2$ where $d_0 = 1$ m is a reference distance. The relay is at $\epsilon = 0.5$ m from the plane. We will also assume that $N_i^{(1)} = N_i^{(2)} = N_i$, $i \in \{1, 2\}$, $N_r^{(1)}$ will be denoted by N_r and also $P_r^{(1)} = P_r$, $A_r^{(1)} = A_r$, $a_r^{(1)} = a_r$, $\tilde{a}_r^{(1)} = a_r$, $\nu^{(1)} = \nu$.

Optimal relay amplification gain for the AF protocol. First we consider the ZDSAF relaying scheme assuming a fixed amplification gain $a_r = A_r$ (Sec. IV-C). We want to analyze the influence of the value of the amplification factor, $A_r \in [0, \tilde{a}_r(1, 1)]$, on the achievable network sum-rate at the NE. This is what Fig. 5 shows for the following scenario: $\epsilon = 0.5$ m, $P_1 = 20$ dBm, $P_2 = 23$ dBm, $P_r = 22$ dBm, $N_1 = 10$ dBm, $N_2 = 9$ dBm, $N_r = 7$ dBm, $\gamma^{(1)} = \gamma^{(2)} = 2$. We observe that the optimal value is $A_r^* = 0.05$ and is not equal to the one saturating the relay power constraint $\tilde{a}_r(1, 1) = 0.17$. This result illustrates for the sum-rate what we have proved analytically for the individual rate of a given user (see Sec. III-C). Note that the gap between the optimal choice for a_r and the choice saturating the power constraint is not that large and in fact other simulation results have shown is is generally of this order and even smaller typically.

Optimal relay location for the AF protocol. Now, we consider the ZDSAF when the full power regime is assumed at the relay, $a_r = \tilde{a}_r(\theta_1, \theta_2)$ (Sec. IV-C) and study the relay location problem. Fig. 6 represents the achievable network sum-rate as a function of the relay position $(x_{\mathcal{R}}, y_{\mathcal{R}}) \in [-L, L]^2$ for the scenario: $P_1 = 20$ dBm, $P_2 = 17$ dBm, $P_r = 22$ dBm, $N_1 = 10$ dBm, $N_2 = 9$ dBm, $N_r = 7$ dBm, $\gamma^{(1)} = 2.5$ and $\gamma^{(2)} = 2$. We observe that there are two local maximum that actually correspond to the points that maximize the individual achievable rates. Many simulation results have confirmed that, when the source nodes are sufficiently far away from each other, maximizing the individual rate of either user at the NE amounts to locating the relay on one of the the segment between $\mathcal{S}_i$ and $\mathcal{D}_i$. This interesting and quite generic observation can be explained as follows. For this purpose, consider Fig. 6 which is a temperature image representing the values of θ_1 and θ_2 for different relay positions in $[-L, L]^2$. The region where $(\theta_1, \theta_2) = (1, 0)$ (resp. $(\theta_1, \theta_2) = (0, 1)$) is the region around $\mathcal{S}_1$ (resp. $\mathcal{S}_2$). We see that the intersection between these regions corresponds to a small area. This quite general observation shows that the selfish behavior of the transmitters leads to self-regulating the interference in the network. Said otherwise, a selfish transmitter will not use at all a far away relay but leaves it to the other transmitter. Thus, when one transmitter uses the relay, it is often alone and sees no interference. In these conditions, by considering the path loss effects it can be proved that the optimal relay position is on the segment between the considered source and destination nodes. This also explains why the position that maximizes the network sum-rate lies also on one of the segments from $\mathcal{S}_i$ to $\mathcal{D}_i$.

Optimal relay power allocation at the relay for DF and EF. For the DF protocol, we fix the cooperation degrees $\tau_1 = 0$ and $\tau_2 = 0$. In Fig. 7, we plot the achievable sum-rate at the equilibrium as a function of the relay power allocation policy is $\nu \in [0, 1]$ (with the convention $\nu = \nu^{(1)}$) for the scenario:

$x_{\mathcal{R}} = 0$ m, $y_{\mathcal{R}} = 0$ m, $P_1 = 22$ dBm, $P_2 = 17$ dBm, $P_r = 23$ dBm, $N_1 = 7$ dBm, $N_2 = 9$ dBm, $N_r = 0$ dBm, $\gamma^{(1)} = 2.5$ and $\gamma^{(2)} = 2$. We observe that, for both protocols, the optimal power allocation $\nu^* = 1$, meaning that the relay allocates all its available power to the better receiver, $\mathcal{D}_1$. In this case, the relay is in very good conditions and can therefore reliably decode the source messages. This explains why DF outperforms EF which is in agreement with the observations we have made in Sec. III. We have observed that, in general, the network sum-rate is not concave w.r.t. $\nu \in [0, 1]$ and that the optimal power allocation lies on the borders $\nu^* \in \{0, 1\}$ for both relaying protocols. In Fig. 7, we also see that the fair PA policy that is, $\nu = \frac{1}{2}$ can lead to a relatively significant performance loss.

VI. Conclusion

The complete study of PA games in IRCs is a wide problem and we do not claim to fully characterize it here. One of the main objectives in this paper has been to know whether there exist some stable outcomes to the conflicting situation where two transmitters selfishly allocate their power between different sub-channels in multi-band interference relay channels in order to maximize their individual transmission rate. Our approach has been to consider transmission rates achievable in a decentralized framework where relays can be deployed with minor or even with no changes for the already existing receivers. For the three types of protocols considered, we have proved that the utility of the transmitters is a concave function of the individual strategy, which ensures the existence of Nash equilibria in the power allocation game after Rosen [27]. In a special case of the AF protocol, we have fully characterized the number of NE and the convergence problem of Cournot-type or iterative water-filling procedures to an NE. Although we have limited the scope of the paper, we have seen that studying IRCs deeply requires further investigations. Many interesting questions which can be considered as natural extensions of this work have arisen. Considering more efficient coding-decoding schemes and relaying protocols such as those of [15] and related works, is it possible to prove that the utilities are still concave functions? For these schemes and those considered in this paper, it is also important to fully determine the number of Nash equilibria and derive convergent iterative distributed power allocation algorithms. We have also seen that several power allocation games come into play and need to be studied when considering DF, EF and AF-type protocols: for allocating transmit power between the different bands at the sources, for choosing the cooperation degree at the sources, for allocating the power between the cooperation signals at the relay, for allocating the transmit power over time. Furthermore, a new agent can come into play (the relay) and several Stackelberg formulations can be used to improve the efficiency of the equilibria.

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Appendix A
Proof of Theorem 3.2 (achievable transmission rates for IRCs with the EF protocol)

In order to prove that the transmission rate region of Theorem 3.2 is achievable for Gaussian IRCs, we use a quite common approach [20] for proving coding theorems: we first prove that it is achievable for discrete input discrete output channels and obtain the Gaussian case from standard quantization and continuity arguments [20], and a proper choice of coding auxiliary variables.

Definitions and notations

We denote by $A_\epsilon^{(n)}(X)$ the weakly ϵ-typical set for the random variable X. If X is a discrete variable, $X \in \mathcal{X}$, then $\|\mathcal{X}\|$ denotes the cardinality of the finite set $\mathcal{X}$. We use x^n to indicate the vector $(x_1, x_2, \ldots, x_n)$.

Definition A.1: A two-user discrete memoryless interference relay channel (DMIRC) without feedback consists of three input alphabets $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_r$, and three output alphabets $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $\mathcal{Y}_r$, and a probability transition function that satisfies $p(y_1^n, y_2^n, y_r^n \mid x_1^n, x_2^n, x_r^n) = \prod_{k=1}^{n} p(y_{1,k}, y_{2,k}, y_{r,k} \mid x_{1,k}, x_{2,k}, x_{r,k})$ for some $n \in \mathbb{N}^*$.

Definition A.2: A $\left(2^{nR_1}, 2^{nR_2}, n\right)$-code for the DMIRC with private messages consists of two sets of integers $\mathcal{W}_1 = \left\{1, ..., 2^{nR_1}\right\}$ and $\mathcal{W}_2 = \left\{1, ..., 2^{nR_2}\right\}$, two encoders: $f_i : \mathcal{W}_i \to \mathcal{X}_i^n$,, a set of relay functions $\{f_{r,k}\}_{k=1}^n$ such that $x_{r,k} = f_{r,k}(y_{r,1}, y_{r,2}, ..., y_{r,k-1})$, $1 \le k \le n$ and two decoding functions $g_i : \mathcal{Y}_i^n \to \mathcal{W}_i$, $i \in \{1, 2\}$. The source node $\mathcal{S}_i$ intends to transmit W_i, the private message, to the receiver node $\mathcal{D}_i$.

Definition A.3: The average probability of error is defined as the probability that the decoded message pair differs from the transmitted message pair; that is, $P_e^{(n)} = \Pr[g_1(Y_1^n) \neq W_1 \text{ or } g_2(Y_2^n) \neq W_2 \mid (W_1, W_2)]$, where (W_1, W_2) is assumed to be uniformly distributed over $\mathcal{W}_1 \times \mathcal{W}_2$. We also define the the average probability of error for each receiver as $P_{ei}^{(n)} = \Pr[g_i(Y_i^n) \neq W_i \mid W_i]$. We have $0 \le \max\left\{P_{e1}^{(n)}, P_{e2}^{(n)}\right\} \le P_e^{(n)} \le P_{e1}^{(n)} + P_{e2}^{(n)}$. Hence $P_e^{(n)} \to 0$ implies that both $P_{e1}^{(n)} \to 0$ and $P_{e2}^{(n)} \to 0$, and conversely.

Definition A.4: A rate pair (R_1, R_2) is said to be achievable for the IRC if there exists a sequence of $\left(2^{nR_1}, 2^{nR_2}, n\right)$ codes with $P_e^{(n)} \to 0$ as $n \to \infty$.

Overview of coding strategy

At the end of the block k, the relay constructs two estimations $\hat{y}_{r1}^n(k)$ and $\hat{y}_{r2}^n(k)$ of its observation $y_r^n(i)$ that intends to transmit to the receivers $\mathcal{D}_1$ and $\mathcal{D}_2$ to help them resolve the uncertainty on $w_{1,k}$ and $w_{2,k}$ respectively at the end of the block $k+1$.

Details of the coding strategy

Codebook generation

- i Generate 2^{nR_i} i.i.d. codewords $x_i^n(w_i) \sim \prod_{k=1}^n p(x_{i,k})$, where $w_i \in \left\{1, \ldots, 2^{nR_i}\right\}$, $i \in \{1, 2\}$.
- ii Generate $2^{nR_0^{(1)}}$ i.i.d. codewords $u_1^n \sim \prod_{k=1}^n p(u_{1,k})$. Label these $u_1^n(s_1)$,

$s_1 \in \left\{1, \ldots, 2^{nR_0^{(1)}}\right\}$.

iii Generate $2^{nR_0^{(2)}}$ i.i.d. codewords $u_2^n \sim \prod_{k=1}^n p\left(u_{2,k}\right)$. Label these $u_2^n(s_2)$, $s_1 \in \left\{1, \ldots, 2^{nR_0^{(2)}}\right\}$.

iv For each pair $(u_1^n(s_1), u_2^n(s_2))$, choose a sequence x_r^n where $x_r^n \sim p\left(x_r^n | u_1^n(s_1), u_2^n(s_2)\right) = \prod_{k=1}^n p\left(x_{r,k} | u_{1,k}(s_1), u_{2,k}(s_2)\right)$.

v For each $u_1^n(s_1)$, generate $2^{n\hat{R}_1}$ conditionally i.i.d. codewords $\hat{y}_{r1}^n \sim \prod_{k=1}^n p\left(\hat{y}_{r1k} | u_{1,k}(s_1)\right)$ and label them $\hat{y}_{r1}^n(z_1|s_1)$, $z_1 \in \left\{1, \ldots, 2^{n\hat{R}_1}\right\}$. For each pair $(u_1, \hat{y}_{r1}) \in \mathcal{U}_1 \times \hat{\mathcal{Y}}_{r1}$, the conditional probability $p(\hat{y}_{r1}|u_1)$ is defined as $p(\hat{y}_{r1}|u_1) = \sum_{x_1,x_2,y_1,y_2,y_r} p\left(x_1\right) p\left(x_2\right) p\left(y_1, y_2, y_r | x_1, x_2, x_r\right) p\left(\hat{y}_{r1} | y_r, u_1\right)$.

vi For each $u_2^n(s_2)$, generate $2^{n\hat{R}_2}$ conditionally i.i.d. codewords $\hat{y}_{r2}^n \sim \prod_{k=1}^n p\left(\hat{y}_{r2k} | u_{2,k}(s_2)\right)$ and label them $\hat{y}_{r2}^n(z_2|s_2)$, $z_2 \in \left\{1, 2^{n\hat{R}_2}\right\}$. For each triplet $(u_2, \hat{y}_{r1}) \in \mathcal{U}_2 \times \hat{\mathcal{Y}}_{r1}$, the conditional probability $p(\hat{y}_{r2}|u_2)$ is defined as $p(\hat{y}_{r2}|u_2) = \sum_{x_1,x_2,y_1,y_2,y_r} p\left(x_1\right) p\left(x_2\right) p\left(y_1, y_2, y_r | x_1, x_2\right) p\left(\hat{y}_{r2} | y_r, u_2\right)$.

vii Randomly partition the message set $\left\{1, 2, \ldots, 2^{n\hat{R}_1}\right\}$ into $2^{nR_0^{(1)}}$ sets $\left\{S_1^{(1)}, S_2^{(1)}, \ldots, S_{2^{nR_0^{(1)}}}^{(1)}\right\}$ by independently and uniformly assigning each message in $\left\{1, \ldots, 2^{n\hat{R}_1}\right\}$ to an index in $\left\{1, \ldots, 2^{nR_0^{(1)}}\right\}$.

viii Also, randomly partition the message set $\left\{1, 2, \ldots, 2^{n\hat{R}_2}\right\}$ into $2^{nR_0^{(2)}}$ sets $\left\{S_1^{(2)}, S_2^{(2)}, \ldots, S_{2^{nR_0^{(2)}}}^{(2)}\right\}$ by independently and uniformly assigning each message in $\left\{1, \ldots, 2^{n\hat{R}_2}\right\}$ to an index in $\left\{1, \ldots, 2^{nR_0^{(2)}}\right\}$.

Encoding procedure Let $w_{1,k}$ and $w_{2,k}$ be the messages to be send in block k. $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively transmit the codewords $x_1^n(w_{1,k})$ and $x_2^n(w_{2,k})$. We assume that $(u_1^n(s_{1,k-1}), \hat{y}_{r1}^n(z_{1,k-1}|s_{1,k-1}), y_r^n(k-1)) \in A_\epsilon^{(n)}$ and $z_{1,k-1} \in S_{s_{1,k}}^{(1)}$ and also that $(u_2^n(s_{2,k-1}), \hat{y}_{r2}^n(z_{2,k-1}|s_{2,k-1}), y_r^n(k-1)) \in A_\epsilon^{(n)}$ with $z_{2,k-1} \in S_{s_{2,k}}^{(2)}$. Then the relay transmits the codeword $x_r^n(s_{1,k}, s_{2,k})$.

Decoding procedure In what follows, we will only detail the decoding procedure at the receiver node $\mathcal{D}_1$ (at $\mathcal{D}_2$ the decoding is analogous). At the end of block k:

i The receiver node $\mathcal{D}_1$ estimates $\hat{s}_{1,k} = s_1$ if and only if there exists a unique sequence $u_1^n(s_1)$ that is jointly typical with $y_1^n(k)$. We have $s_1 = s_{1,k}$ with arbitrarily low probability of error if n is sufficiently large and $R_0^{(1)} < I(U_1; Y_1)$.

ii Next, the receiver node $\mathcal{D}_1$ constructs a set $\mathcal{L}_1\left(y_1^n\left(k-1\right)\right)$ of indexes z_1 such that $\left(u_1^n\left(\hat{s}_{1,k-1}\right), \hat{y}_{r1}^n\left(z_1|\hat{s}_{1,k-1}\right), y_1^n\left(k-1\right)\right) \in A_\epsilon^{(n)}$. $\mathcal{D}_1$ estimates $\hat{z}_{1,k-1}$ by doing the intersection of sets $\mathcal{L}_1\left(y_1^n\left(k-1\right)\right)$ and $S_{\hat{s}_{1,k}}^{(1)}$. Similarly to [14, Theorem 6] and using [14, Lemma 3], one can show that $\hat{z}_{1,k-1} = z_{1,k-1}$ with arbitrarily low probability of error if n is sufficiently large and $\hat{R}_1 < I(\hat{Y}_{r1}; Y_1|U_1) + R_0^{(1)}$.

iii Using $\hat{y}_{r1}^n(\hat{z}_{1,k-1}|\hat{s}_{1,k-1})$ and $y_1^n(k-1)$, the receiver node $\mathcal{D}_1$ finally estimates the message $\hat{w}_{1,k-1} = w_1$ if and only if there exists a unique codeword $x_1^n(w_1)$ such that $(x_1^n(w_1), u_1^n(\hat{s}_{1,k-1}), y_1^n(i-1), \hat{y}_{r1}^n(\hat{z}_{1,k-1}|\hat{s}_{1,k-1})) \in A_\epsilon^{(n)}$. One can show that $w_1 = w_{1,k-1}$ with arbitrarily low probability of error if n is sufficiently large and

$$R_1 \quad < \quad I\left(X_1; Y_1, \hat{Y}_{r1} \mid U_1\right). \tag{26}$$

iv At the end of the block k, the relay looks for the suitable estimation of its observation that it intends to transmit to the receiver node $\mathcal{D}_1$ by estimating $\hat{z}_{1,k}$. It estimates $\hat{z}_{1,k} = z_1$ if there exists a sequence $\hat{y}_r^n(z_1|s_{1,k})$ such that $(u_1^n(s_{1,k}), \hat{y}_{r1}^n(z_1|s_{1,k}), y_r^n(k)) \in A_\epsilon^{(n)}$. There exists a such sequence if n is sufficiently large and $\hat{R}_1 > I(\hat{Y}_{r1}; Y_r|U_1)$.

From i, ii, iii we further obtain

$$I(\hat{Y}_{r1}; Y_r|U_1, Y_1) \quad < \quad I(U_1; Y_1). \tag{27}$$

The achievability proof for the second receiver node follows in a similar manner. Therefore, we have completed the proof.

From the discrete case to the Gaussian case

As mentioned in the beginning of this section, obtaining achievable transmission rates for Gaussian IRCs from those for discrete IRCs is an easy task. Indeed, the latter consists in using Gaussian codebooks everywhere and choosing the coding auxiliary variables properly namely choosing U_1, U_2, $\hat{Y}_{r,1}$, and $\hat{Y}_{r,2}$. The coding auxiliary variables U_1 and U_2 are chosen to be independent and distributed as $U_1 \sim \mathcal{N}(0, \nu_1 P_r)$ and $U_2 \sim \mathcal{N}(0, \nu_2 P_r)$. The corresponding codewords u_1^n and u_2^n convey the messages resulting from the compression of Y_r. The auxiliary variables $\hat{Y}_{r,1}$, $\hat{Y}_{r,2}$ write as $\hat{Y}_{r,1} = Y_r + Z_{wz}^{(1)}$ and $\hat{Y}_{r,2} = Y_r + Z_{wz}^{(2)}$ where the compression noises $Z_{wz}^{(1)} \sim \mathcal{N}(0, N_{wz}^{(1)})$ and $Z_{wz}^{(2)} \sim \mathcal{N}(0, N_{wz}^{(2)})$ are independent. At last, the relay transmits the signal $X_r = U_1 + U_2$ as in the case of a broadcast channel except that, here, each destination also receives two direct signals from the source nodes. By making these choices of random variables we obtain the desired rate region.

Appendix B
Proof of Theorem 3.4 (optimal amplification gain for ZDSAF in IRCs)

Using the notations given in Theorem 3.4 and also the signal-to-noise plus interference ratio in the capacity function of Eq. (10) the rate R_i can be written as:

$$R_i(a_r) = C\left(\frac{|m_i a_r + n_i|^2}{|p_i a_r + q_i|^2 + s_i a_r^2 + 1}\right).$$

$$R_i^{'}(a_r) = \frac{a_r^2\left[|m_i|^2\mathrm{Re}(p_iq_i^*) - (|p_i|^2+s_i)\mathrm{Re}(m_in_i^*)\right] + a_r\left[|m_i|^2(|q_i|^2+1) - |n_i|^2(|p_i|^2+s_i)\right] + (|q_i|^2+1)\mathrm{Re}(m_in_i^*) - |n_i|^2\mathrm{Re}(p_iq_i^*)}{[|p_ia_r+q_i|^2+s_ia_r^2+1][|m_ia_r+n_i|^2+|p_ia_r+q_i|^2+s_ia_r^2+1]}. \tag{28}$$

We observe that $R_i(0) = C\left(\frac{|n_i|^2}{|q_i|^2+1}\right)$ and that we have an horizontal asymptote $R_{i,\infty} \triangleq \lim_{a_r\to\infty} R_1(a_r) = C\left(\frac{|m_i|^2}{|p_i|^2+s_i}\right)$. Also the first derivative w.r.t. a_r isa_r is given in (28)

The explicit solution, a_r^* depends on the channel parameters and is given here below. We denote by Δ the discriminant of the nominator in the previous equation. If $\Delta < 0$, then in function of the sign of $|m_i|^2\mathrm{Re}(p_iq_i^*) - (|p_i|^2+s_i)\mathrm{Re}(m_in_i^*)$, the function $R_i(a_r)$ is either decreasing and $a_r^* = 0$ or increasing and $a_r^* = \overline{a}_r$. Let us now focus on the case where $\Delta \geq 0$.

1) If $|m_i|^2\mathrm{Re}(p_iq_i^*) - (|p_i|^2+s_i)\mathrm{Re}(m_in_i^*) \geq 0$ then
   a) if $c_{r,i}^{(1)} \leq 0$ and $c_{r,i}^{(2)} \leq 0$ then $a_r^* = \overline{a}_r$;
   b) if $c_{r,i}^{(1)} > 0$ and $c_{r,i}^{(2)} \leq 0$ then
      i) if $\overline{a}_r \geq c_{r,i}^{(1)}$ then $a_r^* = 0$;
      ii) if $\overline{a}_r < c_{r,i}^{(1)}$ then
         - if $R_i(0) \geq R_i(\overline{a}_r)$ then $a_r^* = 0$ else $a_r^* = \overline{a}_r$;
   c) if $c_{r,i}^{(1)} \leq 0$ and $c_{r,i}^{(2)} > 0$ then the analysis is similar to the previous case and $a_r^* \in \{0, \overline{a}_r\}$ depending on $a_r^{(2)}$ this time;
   d) if $c_{r,i}^{(1)} > 0$ and $c_{r,i}^{(2)} > 0$
      i) if $c_{r,i}^{(1)} < c_{r,i}^{(2)}$
         A) if $\overline{a}_r \leq c_{r,i}^{(1)}$ then $a_r^* = \overline{a}_r$;
         B) if $c_{r,i}^{(1)} < \overline{a}_r \leq c_{r,i}^{(2)}$ then $a_r^* = c_{r,i}^{(1)}$;
         C) if $\overline{a}_r > c_{r,i}^{(2)}$ then
            - if $R_i(c_{r,i}^{(1)}) \geq R_1(\overline{a}_r)$ then $a_r^* = c_{r,i}^{(1)}$ else $a_r^* = \overline{a}_r$;
      ii) if $c_{r,i}^{(1)} > c_{r,i}^{(2)}$ then the analysis is similar to the previous case, exchanging the roles of $c_{r,i}^{(1)}$ and $c_{r,i}^{(2)}$;
      iii) if $c_{r,i}^{(1)} = c_{r,i}^{(2)}$ then $a_r^* = \overline{a}_r$.
2) If $|m_i|^2\mathrm{Re}(p_iq_i^*) - (|p_i|^2+s_i)\mathrm{Re}(m_in_i^*) < 0$ then the analysis follows in the same lines and $a_r^* \in \{0, \overline{a_r}, c_{r,i}^{(1)}, c_{r,i}^{(2)}\}$.

Appendix C
Proof of Theorem 4.3 (Existence of an NE for the bi-level compression EF protocol)

We want to prove that for each user $R_i^{(q)}$ is concave w.r.t. $\theta_i^{(q)}$. Consider w.l.o.g. the case of user 1. The general case of complex channel gains is considered. We analyze the second derivative of $R_1^{(q)}$ given in Eq. (17). For the sake of clarity we denote by $\tilde{N}_1^{(q)} = |h_{r1}|^2\overline{\nu^{(q)}}P_r^{(q)} + N_1^{(q)}$, $\Gamma_0 = |h_{r1}|^2\nu^{(q)}P_r^{(q)}$ and $\Gamma_1 = |h_{21}|^2\theta_2^{(q)}P_2 + \tilde{N}_1^{(q)}$. After some manipulations we obtain the following relation: $\frac{d^2R_1^{(q)}}{d(\theta_1^{(q)})^2} = M_1 - M_2$ with $M_k = \frac{NM_k}{DM_k}$, $k \in \{1,2\}$ where (for the sake of clarity we have denoted $h_{ij}^{(q)}$ by h_{ij}):NM_1, NM_2, DM_1, DM_2 are defined by (29),(30).

We observe that the terms $\Lambda_k \geq 0$, $k \in \{2,\ldots,7\}$. Also we can easily see from Eq. (29) that $M_2 \geq 0$, $DM_1 \geq 0$. Thus if we prove that $NM_1 \leq 0$ the concavity of $R_1^{(q)}$ will be guaranteed. In this purpose we plug the expressions of Λ_5, Λ_6, Λ_7, Λ_8 into Eq. (29) and obtain that $NM_1 = \frac{NNM_1}{DNM_1}$ with NNM_1, DNM_1 given in (31), (32).

Therefore we obtain the desired result $NM_1 \leq 0$ and thus $M_1 \geq 0$, which implies that $\frac{d^2R_1^{(q)}}{d\left(\theta_1^{(q)}\right)^2} \leq 0$, whatever the channel parameters.

Appendix D
Proof of Theorem 4.5 (Number of Nash equilibria for ZDSAF)

Before discussing these situations in detail, let us first observe that the two functions $F_i(\theta_j)$ are decreasing w.r.t. θ_j and also $F_i(0) = \frac{d_i}{c_{ii}}$, $F_i(\theta_j^o) = 0$ where $\theta_j^o = \frac{d_i}{c_{ij}}$.

In this section, we will investigate the NE of the game and also their asymptotical stability of each NE point. A sufficient and necessary condition that guarantees the asymptotic stability of a certain NE point is related to the relative slopes of the best-response functions and is given by [37] [38]:

$$\left|\frac{\mathrm{dBR}_1}{\mathrm{d}\theta_2}\frac{\mathrm{dBR}_2}{\mathrm{d}\theta_1}\right| < 1 \tag{33}$$

in an open neighborhood of the NE point. We denote by $\mathcal{V}(\theta_1,\theta_2)$ an open neighborhood of $(\theta_1,\theta_2) \in [0,1]^2$.

1) If $d_1 \leq 0$ and $d_2 \leq 0$, then the BR are constants $\mathrm{BR}_i(\theta_j) = 0$ and thus the NE is unique $(\theta_1^{\mathrm{NE}}, \theta_2^{\mathrm{NE}}) = (0,0)$, for all $c_{ii} \geq 0$, $c_{ji} \geq 0$. The condition (33) is met since $\left|\frac{\mathrm{dBR}_1}{\mathrm{d}\theta_2}\frac{\mathrm{dBR}_2}{\mathrm{d}\theta_1}\right| = 0$ for $(\theta_1,\theta_2) \in \mathcal{V}(0,0)$ and thus the NE is asymptotically stable.
2) If $d_1 \leq 0$ and $d_2 > 0$, then it can be checked that the NE is unique, for all $c_{ii} \geq 0$, $c_{ji} \geq 0$: $\theta_1^{\mathrm{NE}} = 0$ and
$$\theta_2^{\mathrm{NE}} = \begin{cases} \frac{d_2}{c_{22}} & \text{, if} \quad d_2 < c_{22}, \\ 1 & \text{, otherwise.} \end{cases}$$
It can be checked that $\left|\frac{\mathrm{dBR}_1}{\mathrm{d}\theta_2}\frac{\mathrm{dBR}_2}{\mathrm{d}\theta_1}\right| = 0$ for $(\theta_1,\theta_2) \in \mathcal{V}(\theta_1^{\mathrm{NE}}, \theta_2^{\mathrm{NE}})$ and the NE is asymptotically stable.
3) If $d_1 > 0$ and $d_2 \leq 0$, then, similarly to the previous item, we have a unique NE, for all $c_{ii} \geq 0$, $c_{ji} \geq 0$: $\theta_2^{\mathrm{NE}} = 0$ and
$$\theta_1^{\mathrm{NE}} = \begin{cases} \frac{d_1}{c_{11}} & \text{, if} \quad d_1 < c_{11}, \\ 1 & \text{, otherwise.} \end{cases}$$
Here as well we have $\left|\frac{\mathrm{dBR}_1}{\mathrm{d}\theta_2}\frac{\mathrm{dBR}_2}{\mathrm{d}\theta_1}\right| = 0$ for $(\theta_1,\theta_2) \in \mathcal{V}(\theta_1^{\mathrm{NE}}, \theta_2^{\mathrm{NE}})$ and the NE is asymptotically stable.
4) If $d_1 > 0$ and $d_2 > 0$, we have to take into consideration the parameters $c_{ii} \geq 0$, $c_{ji} \geq 0$.

$$\begin{cases} NM_1 &= 2\frac{\left(|h_{11}|^2 {P_1}^2 |h_{1r}|^2 - {\Lambda_2}^2 {P_1}^2\right)|h_{11}|^2 \theta_1^{(q)} P_1}{\Gamma_0 \Lambda_5} + 2\,\frac{\Lambda_8\,|h_{11}|^2 P_1}{\Gamma_0\,\Lambda_5} - \\ & 2\left(\frac{\Lambda_8\,|h_{11}|^2 \theta_1^{(q)} P_1}{\Gamma_0} + \Lambda_6\,|h_{11}|^2 P_1 + |h_{1r}|^2 \Gamma_1 P_1\right)\frac{\Lambda_8\,\Gamma_1}{{\Lambda_5}^2 \Gamma_0} \\ & +2\,\frac{\Lambda_7\,{\Lambda_8}^2 {\Gamma_1}^2}{{\Gamma_0}^2 {\Lambda_5}^3} - 2\,\frac{\Lambda_7\left(|h_{11}|^2 {P_1}^2 |h_{1r}|^2 - {\Lambda_2}^2 {P_1}^2\right)\Gamma_1}{{\Lambda_5}^2 \Gamma_0}, \\ NM_2 &= \left[\left(\frac{\Lambda_8\,|h_{11}|^2 \theta_1^{(q)} P_1}{\Gamma_0} + \Lambda_6\,|h_{11}|^2 P_1 + |h_{1r}|^2 \Gamma_1 P_1\right)\frac{1}{\Lambda_5} - \frac{\Lambda_7\,\Lambda_8\,\Gamma_1}{{\Lambda_5}^2 \Gamma_0}\right]^2, \\ DM_1 &= 1 + \frac{\Lambda_7}{\Lambda_5}, \\ DM_2 &= DM_1^2, \end{cases} \tag{29}$$

$$\begin{cases} \Lambda_1 &= 2\mathcal{R}e(h_{11} h_{1r}^* h_{21}^* h_{2r}), \\ \Lambda_2 &= |h_{11} h_{1r}^*|, \\ \Lambda_3 &= A^{(q)} \\ \Lambda_4 &= |h_{11}|^2 \theta_1^{(q)} P_1 + \Gamma_1, \\ \Lambda_5 &= \left(N_r^{(q)} + N_{wz,1}^{(q)}\right)\Gamma_1 + |h_{2r}|^2 \theta_2^{(q)} P_2 \tilde{N}_1^{(q)}, \\ \Lambda_6 &= |h_{2r}|^2 \theta_2^{(q)} P_2 + N_r^{(q)} + N_{wz,1}^{(q)}, \\ \Lambda_7 &= \Lambda_6\,|h_{11}|^2 \theta_1^{(q)} P_1 + \Gamma_1\,|h_{1r}|^2 \theta_1^{(q)} P_1, \\ \Lambda_8 &= |h_{11}|^2 P_1 \Lambda_3 + |h_{1r}|^2 P_1 \Lambda_4 - 2\Lambda_2^2 \theta_1^{(q)} P_1^2 - \Lambda_1 \theta_2^{(q)} P_1 P_2. \end{cases} \tag{30}$$

$$\begin{aligned} NNM_1 &= 2P_1^2 \Gamma_0 \left[P_2^2 \left(\theta_2^{(q)}\right)^2 |h_{21}|^4 |h_{1r}|^2 + P_2^2 \left(\theta_2^{(q)}\right)^2 |h_{21}|^2 |h_{11}|^2 |h_{2r}|^2 + \right. \\ & \left. \left(\tilde{N}_1^{(q)}\right)^2 |h_{1r}|^2 + 2\,P_2\,\tilde{N}_1^{(q)} \theta_2^{(q)}\,|h_{21}|^2 |h_{1r}|^2\right] \\ & \left(|h_{21}|^2 \theta_2^{(q)}\,P_2 + \Gamma_0 + \tilde{N}_1^{(q)}\right)\left(\theta_2^{(q)}\,P_2\,N_r^{(q)}\,|h_{21}|^2 + |h_{2r}|^2 \theta_2^{(q)}\,P_2\,\tilde{N}_1^{(q)} + N_r^{(q)}\,\tilde{N}_1^{(q)}\right) \\ & \left(|h_{1r} h_{21} - h_{11} h_{2r}|^2 \theta_2^{(q)}\,P_2 + |h_{1r}|^2 \tilde{N}_1^{(q)} + |h_{11}|^2 N_r^{(q)}\right) \\ &\geq 0 \end{aligned} \tag{31}$$

$$\begin{aligned} DNM_1 &= -|h_{11} h_{2r} - h_{21} h_{1r}|^2 \theta_1^{(q)} P_1 \theta_2^{(q)} P_2 \tilde{N}_1^{(q)} - |h_{21}|^2 |h_{1r}|^2 \theta_1^{(q)} P_1 \theta_2^{(q)} P_2 \tilde{N}_1^{(q)} - |h_{11}|^2 \theta_1^{(q)} P_1 N_r^{(q)} \tilde{N}_1^{(q)} - \\ & |h_{21}|^2 |h_{2r}|^2 \left(\theta_2^{(q)}\right)^2 P_2^2 \tilde{N}_1^{(q)} - 2\,|h_{21}|^2 \theta_2^{(q)} P_2 N_r^{(q)} \tilde{N}_1^{(q)} - |h_{21}|^4 \left(\theta_2^{(q)}\right)^2 P_2^2 N_r^{(q)} - \\ & |h_{21} h_{1r} - h_{11} h_{2r}|^2 |h_{21}|^2 \theta_1^{(q)} P_1 \left(\theta_2^{(q)}\right)^2 P_2^2 - |h_{1r}|^2 \theta_1^{(q)} P_1 \left(\tilde{N}_1^{(q)}\right)^2 - |h_{2r}|^2 \theta_2^{(q)} P_2 \left(\tilde{N}_1^{(q)}\right)^2 - \\ & \left(\tilde{N}_1^{(q)}\right)^2 N_r^{(q)} - |h_{21}|^2 \theta_2^{(q)} P_2 N_r^{(q)} \Gamma_0 - N_r^{(q)} \Gamma_0 \tilde{N}_1^{(q)} - |h_{2r}|^2 \theta_2^{(q)} P_2 \tilde{N}_1^{(q)} \Gamma_0 - \\ & |h_{11}|^2 |h_{21}|^2 \theta_1^{(q)} P_1 \theta_2^{(q)} P_2 N_r^{(q)} \\ &\leq 0 \end{aligned} \tag{32}$$

a) If $F_1(1) \geq 1$ and $F_2(1) \geq 1$, then we have $d_1 \geq c_{12} + c_{11}$ and $d_2 \geq c_{21} + c_{22}$. In this case the BR are constants i.e., $\mathrm{BR}_i(\theta_j) = 1$ and thus the NE is unique $(\theta_1^{\mathrm{NE}}, \theta_2^{\mathrm{NE}}) = (1, 1)$. We have $\left|\frac{\mathrm{dBR}_1}{\mathrm{d}\theta_2}\frac{\mathrm{dBR}_2}{\mathrm{d}\theta_1}\right| = 0$ for $(\theta_1, \theta_2) \in \mathcal{V}(1, 1)$ and the NE is asymptotically stable.

b) If $F_1(1) \geq 1$ and $F_2(1) < 1$, then we have $d_1 \geq c_{12} + c_{11}$ and $d_2 < c_{21} + c_{22}$. Here also the NE is unique and $\theta_1^{\mathrm{NE}} = 1$ and

$$\theta_2^{\mathrm{NE}} = \begin{cases} \frac{d_2 - c_{21}}{\alpha_{22}} & \text{, if} \quad d_2 > c_{22}, \\ 0 & \text{, otherwise.} \end{cases}$$

Similarly, we have $\left|\frac{\mathrm{dBR}_1}{\mathrm{d}\theta_2}\frac{\mathrm{dBR}_2}{\mathrm{d}\theta_1}\right| = 0$ for $(\theta_1, \theta_2) \in \mathcal{V}(\theta_1^{\mathrm{NE}}, \theta_2^{\mathrm{NE}})$ and the NE is asymptotically stable.

c) If $F_1(1) < 1$ and $F_2(1) \geq 1$, then we have $d_1 < c_{12} + c_{11}$ and $d_2 \geq c_{21} + c_{22}$. Here also the NE is unique and $\theta_2^{\mathrm{NE}} = 1$ and

$$\theta_1^{\mathrm{NE}} = \begin{cases} \frac{d_1 - c_{12}}{c_{11}} & \text{, if} \quad d_1 > c_{11}, \\ 0 & \text{, otherwise.} \end{cases}$$

Here as well we have $\left|\frac{\mathrm{dBR}_1}{\mathrm{d}\theta_2}\frac{\mathrm{dBR}_2}{\mathrm{d}\theta_1}\right| = 0$ for $(\theta_1, \theta_2) \in \mathcal{V}(\theta_1^{\mathrm{NE}}, \theta_2^{\mathrm{NE}})$ and the NE is asymptotically stable.

d) If $F_1(1) < 1$ and $F_2(1) < 1$, then we'll have $d_1 < c_{12} + c_{11}$ and $d_2 < c_{21} + c_{22}$. This case is the most demanding one and will be treated in detail separately.

At this point an important observation is in order. The discussed scenarios, for which we have determined the unique NE, have a simple geometric interpretation. If the intersection point (θ_1^*, θ_2^*) is such that either $\theta_1^* \in \mathbb{R} \setminus [0, 1]$ or $\theta_2^* \in \mathbb{R} \setminus [0, 1]$ then the NE is unique and differs from this point $((\theta_1^{\mathrm{NE}}, \theta_2^{\mathrm{NE}}) \neq (\theta_1^*, \theta_2^*))$. The case 4.(d) corresponds

to the case where the intersection point $(\theta_1^*, \theta_2^*) \in [0,1]^2$ is an NE point. Now we are interested in finding whether this intersection point is the unique NE or there are more than one NE. If $0 < d_1 < c_{11} + c_{12}$ and $0 < d_2 < c_{22} + c_{21}$ we have the following situations:

1) If $c_{11}c_{22} = c_{21}c_{12}$, then the curves described by $\theta_i = F_i(\theta_j)$ are parallel.
   a) If $d_1 = d_2$, then the curves are superposed. In this special case we have an infinity of NE that can be characterized by $(\theta_1^{\rm NE}, \theta_2^{\rm NE}) \in \mathcal{T}$ where:
   $$\mathcal{T} = \left\{(\theta_1, \theta_2) \in [0,1]^2 \,\middle|\, \theta_1 = F_1(\theta_2^{\rm NE})\right\}.$$
   In this case we have an infinity of NE such that $\left|\frac{{\rm dBR}_1}{{\rm d}\theta_2}\frac{{\rm dBR}_2}{{\rm d}\theta_1}\right| = 1$ for $(\theta_1, \theta_2) \in \mathcal{V}(\theta_1^{\rm NE}, \theta_2^{\rm NE})$ and the NEs are not stable states. This can be easily understood since a small deviation from a certain NE drives the users to a new NE point. Thus, the users don't return to the initial state.
   b) If $d_1 \neq d_2$, then the two lines are only parallel. In this case it can be checked that the NE is unique and also asymptotically stable since again $\left|\frac{{\rm dBR}_1}{{\rm d}\theta_2}\frac{{\rm dBR}_2}{{\rm d}\theta_1}\right| = 0$ for $(\theta_1, \theta_2) \in \mathcal{V}(\theta_1^{\rm NE}, \theta_2^{\rm NE})$. In order to explicit the exact relation of the NE, one has to consider all scenarios in function of the sign of the following four relations $F_i(0) - 1$ and $\theta_j^o - 1$, $i \in \{1,2\}$. We will explicit only one of them. Let us assume that $F_i(0) - 1 < 0$ and $\theta_j^o < 0$ which means that $d_1 < \min\{c_{12}, \frac{c_{12}c_{21}}{c_{22}}\}$ and $d_2 < \min\{c_{21}, c_{22}\}$. Here we have two sub-cases:
      - If $\frac{d_1}{c_{12}} < \frac{d_2}{c_{22}}$, then the NE is characterized by $\theta_1^{\rm NE} = 0$ and $\theta_2^{\rm NE} = \frac{d_2}{c_{22}}$.
      - If $\frac{d_1}{c_{12}} > \frac{d_2}{c_{22}}$, then the NE is characterized by $\theta_1^{\rm NE} = \frac{d_1 c_{22}}{c_{12}c_{21}}$ and $\theta_2^{\rm NE} = 0$.
2) Consider $c_{11}c_{22} \neq c_{21}c_{12}$. Here we have to consider all cases in function of the sign of the four relations $F_i(0) - 1$ and $\theta_j^o - 1$, $i \in \{1,2\}$. We will focus on only one of them. Let us assume that $F_i(0) - 1 < 0$ and $\theta_j^o - 1 < 0$ and thus $d_1 < \min\{c_{12}, c_{11}\}$ and $d_2 < \min\{c_{21}, c_{22}\}$. Here we have four sub-cases:
   - If $\frac{d_2}{c_{22}} < \frac{d_1}{c_{12}}$ and $\frac{d_1}{c_{11}} > \frac{d_2}{c_{21}}$, then the NE is unique: $\theta_1^{\rm NE} = \theta_1^*$ and $\theta_2^{\rm NE} = \theta_2^*$. Also we have that $\left|\frac{{\rm dBR}_1}{{\rm d}\theta_2}\frac{{\rm dBR}_2}{{\rm d}\theta_1}\right| < 1$ for $(\theta_1, \theta_2) \in \mathcal{V}(\theta_1^*, \theta_2^*)$ and the NE is asymptotically stable.
   - If $\frac{d_2}{c_{22}} > \frac{d_1}{c_{12}}$ and $\frac{d_1}{c_{11}} < \frac{d_2}{c_{21}}$, then there are three different NE: $(\theta_1^{\rm NE}, \theta_2^{\rm NE}) \in \{(\theta_1^*, \theta_2^*), (0, \frac{d_2}{c_{22}}), (\frac{d_1}{c_{11}}, 0)\}$, the intersection point and two other NE's on the border. The intersection point is unstable since $\left|\frac{{\rm dBR}_1}{{\rm d}\theta_2}\frac{{\rm dBR}_2}{{\rm d}\theta_1}\right| > 1$ for $(\theta_1, \theta_2) \in \mathcal{V}(\theta_1^*, \theta_2^*)$ but the other two NE's are asymptotically stable since $\left|\frac{{\rm dBR}_1}{{\rm d}\theta_2}\frac{{\rm dBR}_2}{{\rm d}\theta_1}\right| = 0$ for $(\theta_1, \theta_2) \in \mathcal{V}(0, \frac{d_2}{c_{22}})$ and $(\theta_1, \theta_2) \in \mathcal{V}(\frac{d_1}{c_{11}}, 0)$.
   - If $\frac{d_2}{c_{22}} = \frac{d_1}{c_{12}}$ and $\frac{d_1}{c_{11}} < \frac{d_2}{c_{21}}$, then there are only two different NE: $(\theta_1^{\rm NE}, \theta_2^{\rm NE}) \in \{(0, \frac{d_2}{c_{22}}), (\frac{d_1}{c_{11}}, 0)\}$. In this case both of NEs are on the border, one of which represents the intersection point of the BR's. It turns out that the intersection point is not a stable NE because $\left|\frac{{\rm dBR}_1}{{\rm d}\theta_2}\frac{{\rm dBR}_2}{{\rm d}\theta_1}\right| > 1$ for $(\theta_1, \theta_2) \in \mathcal{V}(0, \frac{d_2}{c_{22}})$. However, the other NE is asymptotically stable since $\left|\frac{{\rm dBR}_1}{{\rm d}\theta_2}\frac{{\rm dBR}_2}{{\rm d}\theta_1}\right| = 0$ for $(\theta_1, \theta_2) \in \mathcal{V}(\frac{d_1}{c_{11}}, 0)$.
   - If $\frac{d_2}{c_{22}} > \frac{d_1}{c_{12}}$ and $\frac{d_1}{c_{11}} = \frac{d_2}{c_{21}}$, then there are two NE: $(\theta_1^{\rm NE}, \theta_2^{\rm NE}) \in \{(\frac{d_1}{c_{11}}, 0), (0, \frac{d_2}{c_{22}})\}$. Here the analysis of the stability of the two NE's is similar to the previous case.

In conclusion, the number of NE states depends on the geometrical properties of the best-response functions. Three different cases can be identified: 1) when the lines $\theta_i = F_i(\theta_j)$ are superposed the game has an infinity of NE which are not stable; 2) when the lines have a unique intersection point that lies outside of the borders $[0,1] \times [0,1]$, the NE is unique and asymptotically stable; 3) when the lines have a unique intersection point (θ_1^*, θ_2^*) that lies inside $[0,1] \times [0,1]$, there can be one, two or three different NE among which one is identical to this intersection point. In the case where the the NE is unique, it is also asymptotically stable. When the game has two or three NE, the intersection point (θ_1^*, θ_2^*) is an unstable equilibrium while the other/others are asymptotically stable. The best-response algorithm converges to one of the NE points depending on the initial state of the system.

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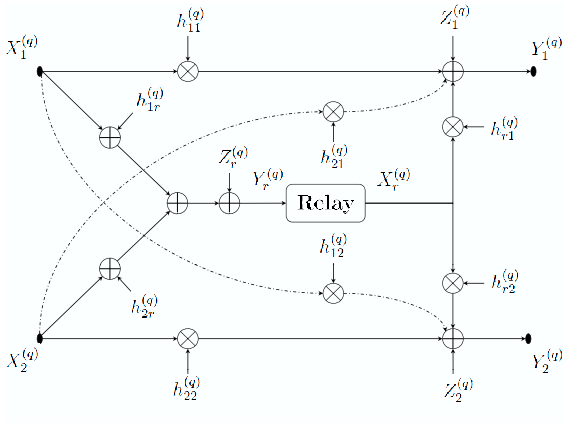


Fig. 1. System model: a Q-band interference channel with a relay; q is the band index and $q \in \{1, ..., Q\}$.

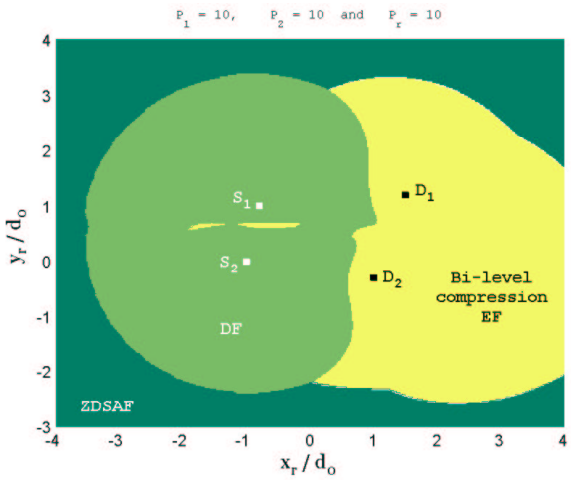


Fig. 2. For different relay positions in the plane $\left(\frac{x_r}{d_0}, \frac{y_r}{d_0}\right) \in [-4, +4] \times [-3, +3]$, the figure indicates the regions where one relaying protocol (AF, DF or bi-level EF) dominates the two others in terms of network sum-rate.

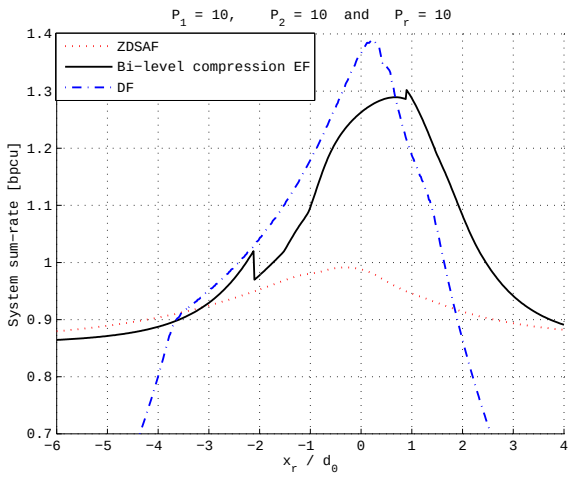


Fig. 3. Achievable system sum-rate versus x_r (abscissa for the relay position) for a fixed y_r ($y_r = 0.5d_0$), with AF, DF and bi-level EF.

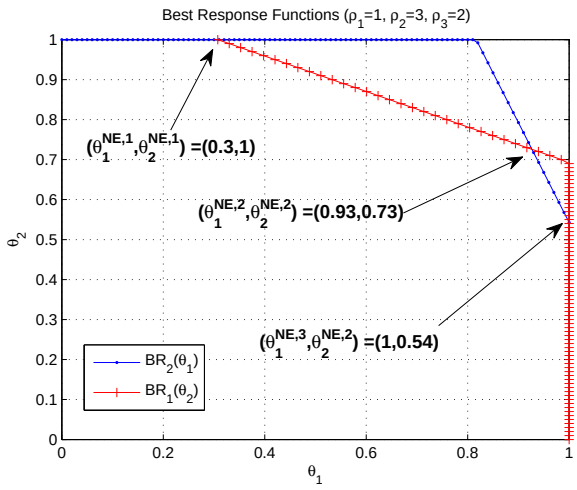


Fig. 4. Best replies for a system composed of an IC in band (1) and IRC in band (2) when the ZDSAF protocol is assumed (fixed amplification factor). The number of equilibria is generally three as indicated the figure.

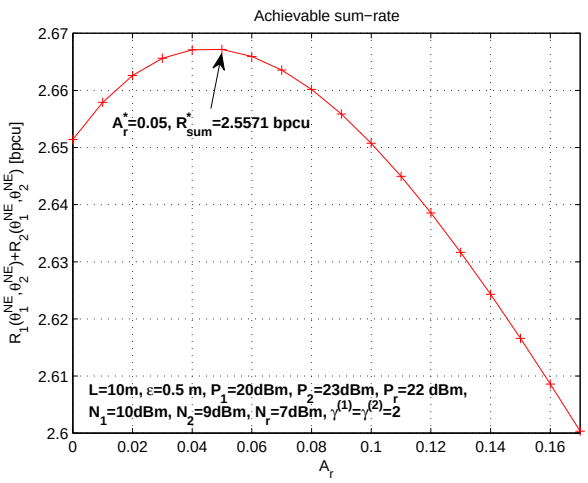
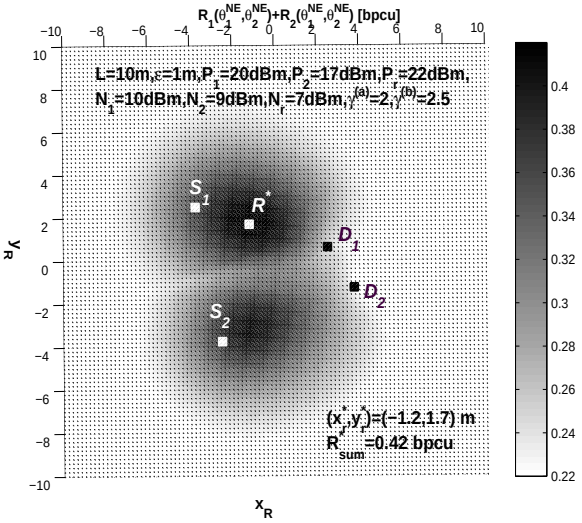
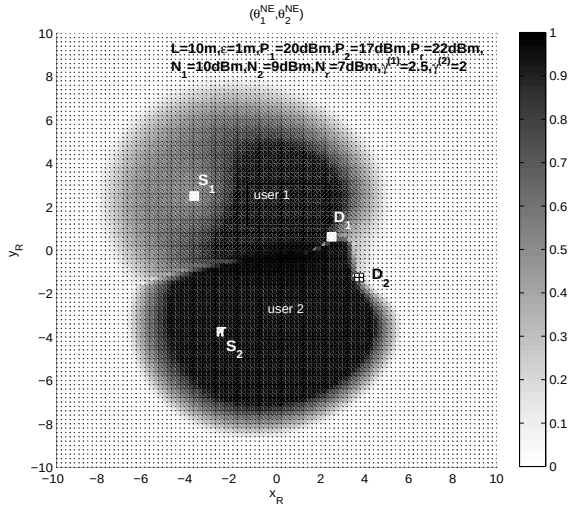


Fig. 5. ZDSAF relaying protocol with fixed amplification gain. Achievable network sum-rate at the NE as a function of $A_r \in [0, \tilde{a}_r]$ for $L = 10$m, $\epsilon = 0.5$m, $P_1 = 20$dBm, $P_2 = 23$dBm, $P_r = 22$dBm, $N_1 = 10$dBm, $N_2 = 9$dBm, $N_r = 7$dBm, $\gamma^{(1)} = \gamma^{(2)} = 2$. The optimal amplification gain $A_r^* = 0.05 \leq \tilde{a}_r(1,1) = 0.17$ meaning that saturating the relay power constraint is suboptimal.



(a)



(b)

Fig. 6. ZDSAF relaying protocol, full power regime. $L = 10$m, $\epsilon = 1$m, $P_1 = 20$dBm, $P_2 = 17$dBm, $P_r = 22$dBm, $N_1 = 10$dBm, $N_2 = 9$dBm, $N_r = 7$dBm, $\gamma^{(1)} = 2.5$ and $\gamma^{(2)} = 2$. (a) Achievable network sum-rate at the NE as a function of $(x_{\mathcal{R}}, y_{\mathcal{R}}) \in [-L, L]^2$ (the optimal relay position $(x^*_{\mathcal{R}}, y^*_{\mathcal{R}}) = (-1.2, 1.7)$ lies on the segment between $\mathcal{S}_1$ and $\mathcal{D}_1$). (b) Power allocation policies at the NE $(\theta_1^{NE}, \theta_2^{NE})$ as a function of $(x_{\mathcal{R}}, y_{\mathcal{R}}) \in [-L, L]^2$ (the regions where the uses allocate their power to IRC are almost non overlapping.).

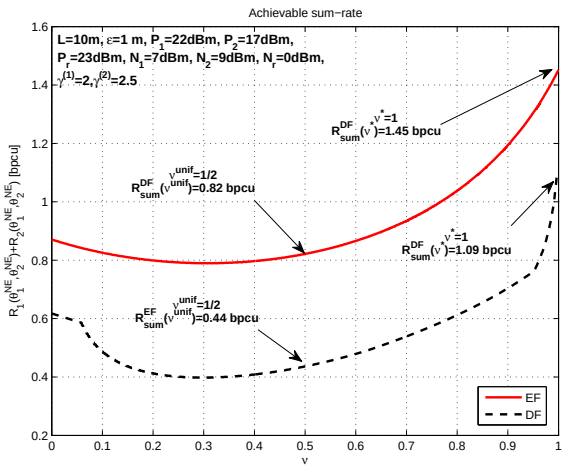


Fig. 7. EF vs. DF relaying protocol. Achievable network sum-rate at the NE as a function of $\nu \in [0, 1]$ for $L = 10\text{m}$, $\epsilon = 1\text{m}$, $P_1 = 22\text{dBm}$, $P_2 = 17\text{dBm}$, $P_r = 23\text{dBm}$, $N_1 = 7\text{dBm}$, $N_2 = 9\text{dBm}$, $N_r = 0\text{dBm}$, $\gamma^{(1)} = 2.5$ and $\gamma^{(2)} = 2$. The optimal relay PA $\nu^* = 1$ is in favor of the better user and outperforms the uniform relay PA $\nu = 0.5$ for both EF and DF.

Appendix B

Publications on Energy-Efficient Communications

B.1 IEEE-PIMRC-2010

E. V. Belmega, S. Lasaulce, and M. Debbah, "A survey on energy-efficient communications", *IEEE Intl. Symp. on Personal, Indoor and Mobile Radio Communications (PIMRC 2010)*, Istanbul, Turkey, Sep. 2010.

A Survey on Energy-Efficient Communications

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***Abstract*—In this paper, we review the literature on physical layer energy-efficient communications. The most relevant and recent works are mainly centered around two frameworks: the pragmatic and the information theoretical approaches. Both of them aim at finding the best transmit and/or receive policies which maximize the number of bits that can be reliably conveyed over the channel per unit of energy consumed. Taking into account both approaches, the analysis starts with the single user SISO (single-input single-output) channel, and is then extended to the MIMO (multiple-input multiple-output) and multi-user scenarios.**

I. Introduction

During the past decade, energy consumption has become an increasingly important issue in wireless networks. For instance, in the current cellular networks, the mobile terminals are equipped with relatively large screens, required to offer more and more functionalities and they need to operate at higher transmission rates for a longer period of time. At the fixed infrastructure level of these networks, the number of base stations has increased dramatically implying important energy costs. According to [2], these costs are expected to be multiplied by a factor of six within the decade 2002–2012. However, significant progress has been made in the art of designing wireless transmitters and receivers. This includes antennas and electronic circuits technology, signal processing algorithms, channel coding techniques and network protocols. The arising question is: Will technological progress be fast enough to control and decrease the energy consumption at the terminal and the network infrastructure sides? Answering such a question is a difficult task and only partial answers can be provided. For this purpose, different communication and information theoretical tools will be used. An important tool and one of the technological breakthroughs in communications is the MIMO concept (i.e., systems composed of multiple antenna terminals) [3][4][5]. It is well known that, for a point-to-point communication, using multiple antenna terminals in full diversity mode (i.e., all the transmmit antennas are used to send the same information over the channel) allows one to decrease the transmit power while ensuring a fixed quality of transmission (e.g., the bit error rate).

In this paper, we overview the literature on energy-efficient communications w.r.t. the number of bits that can be reliably conveyed over the channel per unit of energy consumed. The research on this topic has been focused on two main approaches: a pragmatic approach based on practical modulations, coding-decoding schemes, electronics and an information theoretical approach. In Tab. I, we have summarized the general assumptions for both approaches. The systems under investigation consist either of single or multiple antenna terminals. The multi-carrier scenario is a special case of MIMO channel which can be solved in closed-form in the pragmatic approach and, thus, will be considered separately. Regarding the channel coherence time, in the pragmatic approach, the quasi-static channel is considered assuming perfect channel state information at the transmitters (CSIT). The transmitter can adjust its power as a function of the channel state. In the second scenario three types of channels are considered: a) the static channel with perfect CSIT; b) the fast fading channel; c) the slow fading channel. For b) and c) only the statistics of the channel are required at the transmitter. In all scenarios, perfect channel state information is needed at the decoder. The main focus of this paper is the energy-efficiency power allocation (PA) problem although different degrees of freedom are also briefly reviewed. In most of the dedicated literature, only the transmit power at the output of the RF circuits (or the transmit power for reliable data) is considered. Even if this assumption may not be realistic, it allows one to characterize the upper bound on the maximum performance that can be achieved in practice. However, we will also review some works that have taken into account the consumed circuitry energy which may have a critical impact on the system energy-efficiency. Furthermore, only the single-user setting is investigated in the information theoretical approach, whereas for the pragmatic approach the multi-user scenario is also considered.

TABLE I
System model and assumptions for the two energy-efficient approaches

	Pragmatic approach	Information theoretical approach
Dimensionality	SISO	SISO
	Multi-carrier	-
	MIMO	MIMO
Number of users	Single-user	Single-user
	Multi-user	-
Coherence time	Quasi-static, CSIT	Static channels, CSIT
		Fast fading, CDIT
		Slow fading, CDIT
Consumed power	RF signal power	RF signal power
	RF signal plus circuitry power	-

A. Notations

We define hereafter some general notations and acronyms that will be used throughout the paper. Let R denote the

transmit rate, γ the received SNR for the single user case or SINR for the multi-user case, $p \in (0, \overline{P}]$ denote the transmit power which is constrained by $\overline{P}$, h the channel gain, σ^2 the noise variance (the noise is assumed Gaussian). For the MIMO system we denote by n_t, n_r the number of available antennas at the transmitter and receiver, $\mathbf{H}$ the $n_r \times n_t$ channel matrix, $\underline{h}_j$ the $j-th$ column of $\mathbf{H}$, the input covariance matrix is $\mathbf{Q} = \mathbf{U}\text{diag}(p_1, \ldots, p_{n_t})\mathbf{U}^H$ where $\mathbf{U}$ is a unitary matrix, and $\underline{p} = (p_1, \ldots, p_{n_t})$ is the vector of the corresponding eigenvalues. The average power constraint is $\text{Tr}(\mathbf{Q}) = \sum_{j=1}^{n_t} p_j \leq \overline{P}$. The noise correlation matrix is $\Sigma_z = \sigma^2 \mathbf{I}$, unless otherwise specified.

Acronym	Definition
SISO	single-input single-output
MIMO	multiple-input multiple-output
CSIT	channel state information at the transmitter
CDIT	channel distribution information at the transmitter
PA	power allocation
RF	radio frequency
SNR	signal-to-noise ratio
SINR	signal-to-interference plus noise ratio
CDMA	code division multiple access
BER	bit error rate
FSK	frequency shift keying
bpcu	bits per channel use
NE	Nash equilibrium
OFDMA	orthogonal frequency-division multiple access
STBC	space-time block coding
AWGN	additive white Gaussian noise
UPA	uniform power allocation

B. A Generic Efficiency Function

The efficiency of a system can be defined in general as the ratio between what the system delivers to what it consumes. For example, we can define the efficiency function as:

$$E(x) = \frac{f(x)}{g(x)}, \tag{1}$$

where $x \in [0, \overline{X}]$ denotes the resource constrained by $\overline{X}$, $f(\cdot)$ the benefit function such that $f(0) = 0$ and $g(\cdot)$ is the cost of the resource. We assume also that $g(0) = 0$, which means that the cost in standby mode (no transmission) is zero. The problem of *efficient resource allocation* is to find the optimal x^* maximizing $E(x)$. Assuming a linear cost, $g(x) = \lambda x$ where $\lambda > 0$ represents the unit cost, then it is sufficient to study the function:

$$\tilde{E}(x) = \frac{f(x)}{x} \tag{2}$$

Depending on the shape of $f(x)$, two types of energy-efficiency functions can be distinguished:

Type I: $f(x)$ is an increasing S-shaped function. In [6], the authors show that, under this hypothesis, the efficiency $\tilde{E}(x)$ is quasi-concave w.r.t. x. The optimal solution is unique and non-trivial $x^* > 0$ and is given by $x^* = \min\{\overline{X}, \tilde{x}\}$ where $\tilde{x}$ is the solution of the equation:

$$xf'(x) - f(x) = 0. \tag{3}$$

The solution $\tilde{x}$ has a neat geometrical interpretation. It is the intersection point between the curve $y = f(x)$ and the tangent that passes through the origin $(0, 0)$. For example, if $f(x) = e^{-a_3/x}$ with $a_3 > 0$, the optimal solution is $x^* = \min\{\overline{X}, a_3\}$.

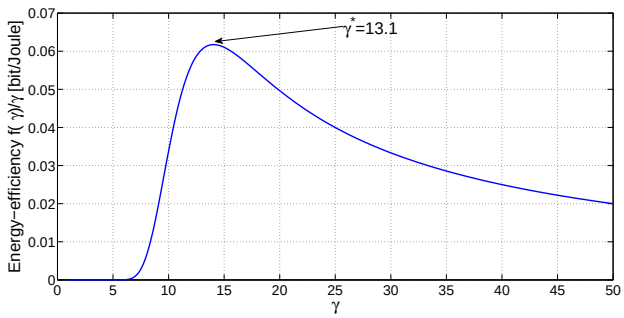

Fig. 1. Type I energy-efficiency vs. received SNR γ.

Type II: $f(x)$ is an increasing concave function. In this case, the optimal solution is trivial $x^* \to 0$. For example, for a logarithmic benefit function, $f(x) = a_1 \log(1 + a_2 x)$ with $a_1 > 0$ and $a_2 > 0$ it can be shown that the energy-efficiency function is convex and decreasing w.r.t. x. Thus, the optimal solution is trivial $x^* \to 0$. Intuitively speaking, if increasing the resource consumption results in a marginal increase of benefit, then the most efficient solution is not to consume the resource at all.

II. Pragmatic Approach

We will first study the pragmatic approach, starting with the simplest case of single antenna systems.

A. SISO

In [7][8], the authors study the uplink of a K-user CDMA Gaussian channel. A non-cooperative power control game is formulated where the transmitters tune their powers in order to maximize their individual performance in terms of energy-efficiency. The chosen performance metric for the single user case is defined as:

$$G(p, R) = \frac{LRf(\gamma)}{Mp}, \tag{4}$$

where L represents the information bits, M the packet size ($M > L$ after the channel coding). Also, $f(\gamma) = (1 - \text{BER})^M$ represents the probability of correct packet reception and BER denotes the bit error rate. In general, $f(\gamma)$ is an S-shaped function. The energy-efficiency is a Type I function and a non-trivial solution $\gamma^* > 0$ exists for the optimization of $\frac{f(\gamma)}{\gamma}$. This is illustrated in Fig. 1 for $R = 1$ bpcu (bits per channel use), $M = L = 80$ and a non-coherent FSK modulation [8]. The optimal transmit policy corresponds to the power achieving the optimal SNR γ^* while satisfying the power constraint. This result is shown to extend to the multi-user scenario where, at the Nash equilibrium (NE) state (see e.g., [9][10]), the optimal transmit policy for any user is the minimal power that allows it to achieve the optimal SINR equal to γ^* (independently of the user identity).

In [11], the authors showed that the performance obtained at the NE is inefficient. In order to obtain a Pareto improvement of the non-cooperative power control game, different methods have been proposed such as: pricing techniques [11], hierarchy among users with either successive interference cancellation

at the receiver or using the Stackelberg formulation [12] [13], repeated games framework [14].

Several extensions of [8] have been proposed by considering: The influence of other supplementary degrees of freedom on the system energy-efficiency, such as the transmit constellation size [15], transmission rate [16], [20], the coefficients of the receive filter [18], [22], [21]; multi-hop systems and introducing the circuitry consumed power [22], [23]; non-linear receivers [17]. For more details the reader is referred also to [19].

In [24], the non-cooperative power control game is studied in a frequency-selective environment for the uplink of an impulse-radio ultrawideband system. In this case, the problem is more challenging than single path because of the self-interference in addition to multiple access interference and every user achieves a different SINR at the output of its Rake receiver. The authors of [23] study the energy-efficiency non-cooperative power control game in large networks. The nodes are assumed to form clusters to send the local signal at distant receivers. In this scenario, the NE is characterized assuming that the players are the clusters that choose their average transmit power to maximize the energy-efficiency.

B. Multi-carrier

The authors in [25] have extended the analysis in [8] to the study of the PA problem in multi-carrier CDMA systems. The transmitter can send independent data flows over a number of $D \geq 2$ orthogonal carriers. The energy-efficiency utility writes as:

$$G(\underline{p}, R) = \frac{RL}{M} \frac{\sum_{d=1}^{D} f(\gamma_d)}{\sum_{d=1}^{D} p_d}, \tag{5}$$

where $\underline{p} = (p_1, \ldots, p_D)$, $p_d \geq 0$, represents the power allocated to the d-th carrier and γ_d is the receive SNR on the d-th carrier. The authors prove that the optimal PA policy is to use only the best carrier (w.r.t. the channel gain) and to transmit over this carrier with a power that achieves an SNR equal to γ^*. The result is extended to the multi-user case.

A different energy-efficiency function has been studied in [28][29] for multi-carrier frequency-selective OFDMA channels. This function is defined by the ratio of the throughput and the total power (transmit plus circuitry consumption). The throughput is the transmission rate depending on the SNR gap factor.

C. MIMO

The multi-carrier case can be seen as a particular MIMO channel where $n_r = n_t = D$ and $\mathbf{H}$ is a diagonal matrix. Now we will focus on the general MIMO case. The major difficulty in extending this pragmatic approach to the general MIMO case is that the output SNR will be strongly related to the encoding-decoding schemes implemented.

In [18], the authors study the SIMO (single-input multiple-output) case where the receiver is equipped with several antennas. The users tune the MMSE receiver coefficients (in this case matrices instead of vectors) and their transmit powers. A large system comparison between the MMSE filter, the matched filter and the decorrelator is also provided. In this case, since the transmitter is equipped with one antenna the problem remains essentially a power control problem. In [27], the framework in [26], is extended to multiuser MIMO wireless systems where each terminal can tune its transmit power, beamforming vector and receiver in order to maximize its own utility. Hence, the transmit covariance matrix is restricted to be a unit rank matrix.

In [30], the authors studied the two extreme cases w.r.t. the tradeoff between the diversity and multiplexing gains brought by MIMO systems: (a) the full multiplexing mode, where the transmitter sends independent data flows over its antennas; (b) the full diversity mode, where the transmitter sends the same information over its antennas.

In case (a), the transmit covariance matrix is diagonal $\mathbf{Q} = \mathrm{diag}(\underline{p})$ and the efficiency function has the same expression as (5) by replacing D with n_t. Here, γ_i is the output SINR of the matched filter receiver for the i-th component of the transmitted signal: $\gamma_i = p_i \underline{h}_i^H \left(\mathbf{\Sigma_z} + \sum_{\mathbf{j} \neq \mathbf{i}} \mathbf{p_j} \underline{\mathbf{h}}_\mathbf{j} \underline{\mathbf{h}}_\mathbf{j}^\mathbf{H} \right)^{-1} \underline{h}_i$. The authors proved a similar result as in [25] for the single user case. When independent information is sent over the transmit antennas and assuming a matched filter receiver, the optimal PA policy is beamforming in the direction that requires the minimal power to achieve the target SINR γ^*:

$$p_i^* = \left| \begin{array}{ll} \min\left\{ \overline{P}, \frac{\gamma^*}{\underline{h}_k^H \mathbf{\Sigma_z^{-1}} \underline{\mathbf{h}}_\mathbf{k}} \right\}, & \text{if } i = k, \\ 0, & \text{otherwise}, \end{array} \right. \tag{6}$$

where $k = \arg\max_{j \in \{1, \ldots, n_t\}} \underline{h}_j^H \mathbf{\Sigma_z^{-1}} \underline{\mathbf{h}}_\mathbf{j}$ is the index of the best channel and Σ_z is a general positive definite noise covariance matrix. This result was extended to any linear receiver [30].

In case (b), the transmit covariance matrix is a unit rank matrix $\mathbf{Q} = \underline{v}\underline{v}^H$ where $v_i = \sqrt{p_i}$ for all $i \in \{1, \ldots, n_t\}$. The received SNR at the output of the matched filter (or the MRC receiver) is:

$$\gamma_{\mathrm{MRC}} = \sum_{i=1}^{n_t} \sum_{j=1}^{n_t} \sqrt{p_i} \sqrt{p_j} \underline{h}_i^H \mathbf{\Sigma_z^{-1}} \underline{\mathbf{h}}_\mathbf{j}. \tag{7}$$

The energy-efficiency function to be maximized is $G(\underline{p}, R) = \frac{RL}{M} \frac{f(\gamma_{\mathrm{MRC}})}{\sum_{i=1}^{n_t} p_i}$, under the power constraint $\sum_{i=1}^{n_t} p_i \leq \overline{P}$. The problem is more difficult here and a closed-form solution can be obtained only for a particular case where $n_r = n_t = n$, $\mathbf{H} = \mathrm{diag}(h_{11}, \ldots, h_{nn})$ and $\mathbf{\Sigma_z} = \mathrm{diag}(\sigma_\mathbf{1}^\mathbf{2}, \ldots, \sigma_\mathbf{n}^\mathbf{2})$. Notice that this is the dual case of the one studied in [25] (where the transmitter sends independent information over the parallel sub-channels). The optimal solution corresponds to choosing only the link with the best output SNR (i.e., $k = \arg\max_{j \in \{1, \ldots, n\}} \frac{|h_{jj}|^2}{\sigma_j^2}$) and to transmit with a power that achieves γ^*. Notice that the same solution was obtained in [25].

There are several works that have studied the energy-efficiency in MIMO channels assuming space-time codes. In [31], the authors evaluate the improvement obtained by using multiple antenna terminals and implementing Alamouti diversity schemes. Assuming a fixed transmission rate and the BPSK input modulation, the MIMO system outperforms the SISO in terms of energy-efficiency if only the transmit power consumption is taken into account. When the circuitry energy consumption is also taken into account, this conclusion is no longer true. However, if the input constellation size can be optimized, the MIMO system can outperform the SISO system, in spite of the higer circuitry energy consumption. The authors consider also the scenario where the nodes of the network are single antenna terminals that can cooperate among each-other to form a virtual MIMO system. It turns out that, applying MIMO coding/decoding techniques reduces both, the total consumed energy and the total delay, even if the costs of the local exchange information among the nodes is accounted for. The STBC cooperative transmission is also addressed in [32] for sensor networks. The authors propose the low-energy adaptive clustering hierarcy (LEACH) framework to improve the energy-efficiency. In [35][36], the multi-level clustering techniques allowing far-off nodes to communicate to the base station are investigated. Other energy-efficient scheduling mechanisms are reviewed in [33]. In [34], the authors derive an adaptive MIMO approach where the transmitter adapts its modulation and rate and chooses either space-division multiplexing, space-time coding or single-antenna transmission. The authors show that this adaptive technique can improve the energy-efficiency up to 30% compared to non-adaptive systems.

III. Information theoretical approach

We will now overview the information theoretical approach. One of the first papers addressing energy-efficient communications from this point of view is [37] where the author determines the capacity per unit cost for various versions of the photon counting channel. In [38], the author studies the discrete memoryless channel where a cost $b[\cdot]$ is assigned to each symbol of the input alphabet. The maximum number of bits that can be transmitted reliably through the channel per unit cost is characterized as follows. Two different scenarios were considered depending on whether the input alphabet, $\mathcal{X}$, contains or not a zero cost symbol: $b[x_0] = 0$ (e.g., the silence conveys information).

Assuming that there is no zero cost symbol, the capacity per unit cost is:

$$\tilde{C} = \sup_{\beta>0} \frac{C(\beta)}{\beta} = \sup_{\beta>0} \frac{\sup_{X,\mathbb{E}[b[X]]\leq\beta} I(X;Y)}{\beta} \tag{8}$$

where $C(\beta) = \sup_{q(X),\mathbb{E}[b[X]]\leq\beta} I(X;Y)$ represents the capacity of an input-constrained memoryless stationary channel.

If the input alphabet contains a zero cost symbol, $b[x_0] = 0$, the capacity per unit-cost per unit cost is:

$$\tilde{C} = \sup_{x\in\mathcal{X}\setminus\{x_0\}} \frac{D(q_{Y|X=x}\|q_{Y|X=x_0})}{b[x]}. \tag{9}$$

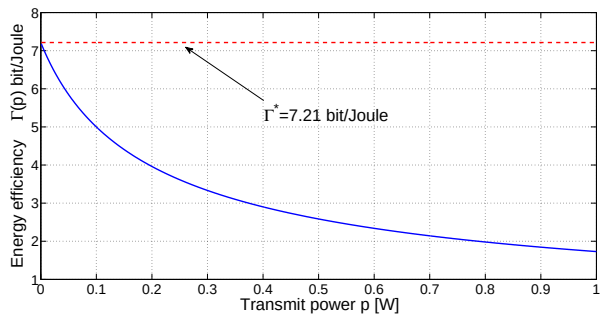

Fig. 2. Type II energy-efficiency vs. transmit power p.

Notice that the capacity per unit cost is easier to compute since the optimization is not done over the input probability ditributions $q(x)$ but over the symbols of the input alphabet. Furthermore, the divergence between two distributions, $D(\cdot\|\cdot)$, is easier to compute than the mutual information, $I(\cdot;\cdot)$ [43].

In [44], the authors consider the discrete memoryless channel with binary inputs where the 0 is a zero cost symbol. As opposed to [38] where the cost constraint is imposed on each symbol, in [44] the codeword is cost-constrained. In this case, it is not possible to guarantee an asymptotically small error probability and, thus, the Shannon capacity is zero. The capacity per unit energy is defined as the maximum rate, in bits per unit of energy consumed, that can be transmitted over the channel such that the maximum likelihood random coding error exponent is positive. Based on this notion, the authors of [45] define the capacity under a similar finite energy constraint as the maximum total number of bits that can be transmitted with a positive error exponent. Then they analyse the connections between this notion and the capacity per unit energy in [44]. In [46], the authors apply the results in [38], [44] to the wide-sense stationary and uncorrelated scattering (WSSUS) channel.

In the remaining part of the paper, we consider the continuous channels and assume that the input alphabet does not contain zero cost symbols unless otherwise specified.

A. SISO

We start with the static SISO AWGN channel. Following [38], the achievable rate per unit cost is:

$$\Gamma(p) = \frac{1}{2p}\log_2\left(1+\frac{p|h|^2}{\sigma^2}\right) \tag{10}$$

Notice that $\Gamma(p)$ is a Type II efficiency function. This can also be seen in Fig. 2 where we plot $\Gamma(p)$ for the scenario where $\rho = 10$ dB, $h = 1$. In this case, the capacity per unit cost is achieved when $p^* \to 0$ and given by $\Gamma^* \to \frac{1}{2}\frac{|h|^2}{\sigma^2}\log_2 \mathrm{e}$. Therefore, in order to be energy-efficient in the sense of the capacity per unit cost, the transmitter has to send information with very low power which implies low data rates. This solution may be realistic in sensor networks but is not acceptable in most common scenarios where minimum communication rates are required. For fast fading channels a similar result is proved in [41].

The case of slow fading channels is considered in [30] [41]. In this case, the Shannon achievable rate is equal to zero. Thus, a different information theoretical energy-efficiency function is proposed:

$$\Gamma(p,R) = \frac{R[1-\mathrm{P_{out}}(p,R)]}{p}, \tag{11}$$

where $\mathrm{P_{out}}(p,R) = \Pr\left[\log_2\left(1+\frac{p|h|^2}{\sigma^2}\right) < R\right]$ is the outage probability. The numerator, $R[1-\mathrm{P_{out}}(p,R)]$, can be seen as the long-term expected throughput. Assuming Rayleigh fading, the closed-form expression of the outage probability is given by $\mathrm{P_{out}}(p,R) = 1-\exp\{-\frac{\sigma^2(e^R-1)}{p}\}$. In this case, $\Gamma(p,R)$ is a Type I energy-efficiency function and a non-trivial solution exists and is given by: $p^* = \min\{\sigma^2(e^R-1), \overline{P}\}$. We observe that this result is very similar to the one obtained in Sec. II where the pragmatic energy-efficiency function is considered. This can be explained by the fact that, as opposed to the static and fast fading cases, in slow fading channels, there are outage events (i.e., non-zero error probablility) which imply the existence of an non trivial tradeoff between the throughput and power consumption.

A very similar notion with the capacity per unit cost is the minimum energy-per-bit. This notion is defined in [39] for the discrete-time AWGN relay channel. By considering the relay power equal to zero the minimum energy-per-bit becomes:

$$\varepsilon_b = \lim_{p\to 0}\frac{2p}{\log_2\left(1+\frac{p|h|^2}{\sigma^2}\right)} = \frac{2\sigma^2}{|h|^2\log_2 \mathrm{e}}$$

We obeserve that the minimum energy-per-bit is the inverse of the capacity per unit cost. In [40], the authors study the AWGN relay channel in the presence of circularly symmetric fast fading. They consider different relaying protocols and provide lower bounds on the minimum energy-per-bit.

B. MIMO

In [41], the authors investigated the case of MIMO channels assuming that the channel matrix $\mathbf{H}$ is a $n_r \times n_t$ random matrix with i.i.d. standard Gaussian entries. It turns out that for static and fast fading channels, the optimal energy-efficient solution is similar to the SISO case. More precisely, the optimal covariance matrix maximizing the achievable rate per unit cost goes to zero $\mathbf{Q}^* \to 0$. The capacity per unit cost for the static channel is $\Gamma^* \to \frac{1}{\ln 2}\frac{\mathrm{Tr}(\mathbf{HH}^H)}{\sigma^2}$. The result is extended to fast fading channels.

For the slow fading MIMO channel the problem is much more difficult. In contrast to the static and fast fading cases, the results obtained for the single-antenna case are not necessarily extendable to MIMO channels. In this case, even the optimal solution that minimizes the outage probability is still an open issue. This is due to the fact that the mutual information is a random variable that has an intractable probability distribution, and no closed-form expressions are available for the outage probability. Telatar conjectured in [5] that the optimal transmit policy is to spread all the available power, $\overline{P}$, uniformly over a subset of ℓ antennas where $\ell = \ell(R,\sigma^2)$ is a function

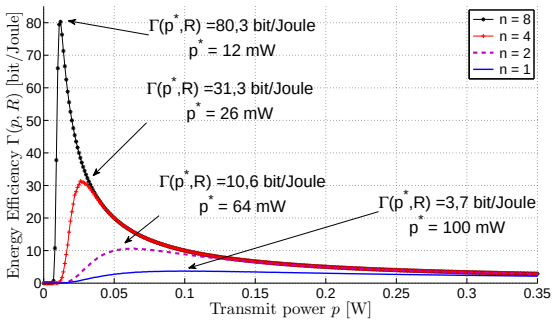

Fig. 3. Energy-efficiency vs. transmit power.

of the system parameters. This famous conjecture has been proved for the particular cases: $n_r = 1, n_t = 2$ in [47] and $n_r = 1, n_t \geq 2$ in [48]. Relying on [48], the authors of [49] have found the optimal covariance matrix maximizing the energy-efficiency for the case where $n_r = 1, n_t \geq 2$. They also conjectured the solution for the general MIMO case. It turns out that the cojectured solution has the exact same stucture as the one minimizing the outage probability. The difference is that, when optimizing the energy-efficiency function, it is not always optimal to use all the available power.

A particular case of interest is the case of UPA transmit policy where $\mathbf{Q} = \frac{p}{n_t}\mathbf{I}$. In [49], the authors conjecture that the energy-efficiency is a quasi-concave function w.r.t. p and that a non-trivial solution exists $p^* > 0$. This is illustrated numerically in Fig. 3 for the scenario: $n_r = n_t = n \in \{1,2,4,8\}$, $\rho = 10$ dB, $R = 1$ bpcu. We observe that the optimal energy-efficiency value is increasing with the system size and, thus, having several transmit antennas improves the energy-efficiency of the system.

The quasi-concavity property w.r.t. the transmit power p is important for example in the multi-user scenario. It allows one to prove the existence of NE states for non-cooperative energy-efficient games (see [9]).

In [42], the authors study the tradeoff between the minimum energy-per-bit and the spectral efficiency for wide-band MIMO channels assuming that the input alphabet contains a zero cost symbol and the UPA transmit policy.

IV. Conclusions

In this paper, we overviewed the literature on energy-efficient communications. The current research is focused on maximizing the number of bits per Joule that can be reliably conveyed through the channel. From an information theoretical point of view, the optimal transmit power allocation policy is trivial for the static and fast fading channels. When slow fading is assumed, a non-trivial solution exists and using mutiple-antennas terminals improves the system energy-efficiency. However, these conclusions do not hold necessarily in practical scenarios where the circuitry energy is also considered.

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Energy-Efficient Precoding for Multiple-Antenna Terminals

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***Abstract*—The problem of energy-efficient precoding is investigated when the terminals in the system are equipped with multiple antennas. Considering static and fast-fading multiple-input multiple-output (MIMO) channels, the energy-efficiency is defined as the transmission rate to power ratio and shown to be maximized at low transmit power. The most interesting case is the one of slow fading MIMO channels. For this type of channels, the optimal precoding scheme is generally not trivial. Furthermore, using all the available transmit power is not always optimal in the sense of energy-efficiency (which, in this case, corresponds to the communication-theoretic definition of the goodput-to-power (GPR) ratio). Finding the optimal precoding matrices is shown to be a new open problem and is solved in several special cases: 1. when there is only one receive antenna; 2. in the low or high signal-to-noise ratio regime; 3. when uniform power allocation and the regime of large numbers of antennas are assumed. A complete numerical analysis is provided to illustrate the derived results and stated conjectures. In particular, the impact of the number of antennas on the energy-efficiency is assessed and shown to be significant.**

***Index Terms*—Energy-efficiency, MIMO systems, outage probability, power allocation, precoding.**

I. Introduction

In many areas, like finance, economics or physics, a common way of assessing the performance of a system is to consider the ratio of what the system delivers to what it consumes. In communication theory, transmit power and transmission rate are respectively two common measures of the cost and benefit of a transmission. Therefore, the ratio transmission rate (say in bit/s) to transmit power (in J/s) appears to be a natural energy-efficiency measure of a communication system. An important question is then: what is the maximum amount of information (in bits) that can be conveyed per Joule consumed? As reported in [1], one of the first papers addressing this issue is [2] where the author determines the capacity per unit cost for various versions of the photon counting channel. As shown in [1], the normalized[1] capacity per unit cost for the well-known additive white Gaussian channel model $Y = X + Z$ is maximized for Gaussian inputs and is given by $\lim_{P\to 0} \frac{\log_2\left(1+\frac{P}{\sigma^2}\right)}{P} = \frac{1}{\sigma^2 \ln 2}$, where $\mathbb{E}|X|^2 = P$ and $Z \sim \mathbb{C}\mathcal{N}(0, \sigma^2)$. Here, the main message of communication theory to engineers is that energy-efficiency is maximized by operating at low transmit power and therefore at low transmission rates. However, this answer holds for static and single input single output (SISO) channels and it is legitimate to ask: what is the answer for multiple-input multiple-output (MIMO) channels? In fact, as shown in this paper, the case of slow fading MIMO channels is especially relevant to be considered. Roughly speaking, the main reason for this is that, in contrast to static and fast fading channels, in slow fading channels there are outage events which imply the existence of an optimum tradeoff between the number of successfully transmitted bits or blocks (called goodput in [3] and [4]) and power consumption. Intuitively, this can be explained by saying that increasing transmit power too much may result in a marginal increase in terms of quality or effective transmission rate.

First, let us consider SISO slow fading or quasi-static channels. The most relevant works related to the problem under investigation essentially fall into two classes corresponding to two different approaches. The first approach, which is the one adopted by Verdú in [1] and has already been mentioned, is an information-theoretic approach aiming at evaluating the capacity per unit cost or the minimum energy per bit (see e.g., [5], [6], [7], [8]). In [1], two different cases were investigated depending on whether the input alphabet contains or not a zero cost or free symbol. In this paper, only the case where the input alphabet does not contain a zero-cost symbol will be discussed (i.e., the silence at the transmitter side does not convey information). The second approach, introduced in [9] is more pragmatic than the previous one. In [9] and subsequent works [4], [10], the authors define the energy-efficiency of a SISO communication as $u(p) = \frac{Rf(\eta)}{p}$ where R is the effective transmission data rate in bits, η the signal-to-noise-plus-interference ratio (SINR) and f is a benefit function (e.g., the success probability of the transmission) which depends on the chosen coding and modulation schemes. To the authors' knowledge, in all works using this approach ([9], [4], [10], [11], [12], [13], etc.), the same (pragmatic) choice is made for f: $f(x) = (1 - e^{-\alpha x})^N$, where α is a constant and N the block length in symbols. Interestingly, the two mentioned approaches can be linked by making an appropriate choice for f. Indeed, if f is chosen to be the complementary of the outage probability, one obtains a counterpart of the capacity per unit cost for slow fading channels and gives an information-theoretic interpretation to the initial definition of [9]. To our knowledge, the resulting performance metric has not been considered so far in the literature. This specific metric, which we call goodput-to-power ratio (GPR), will be considered in this paper. Moreover, we consider MIMO channels where the transmitter and receiver are informed of the channel

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[1]In [1] the capacity per unit cost is in bit/s per Joule and not in bit/J, which amounts to normalize by a quantity in Hz.

distribution information (CDI) and channel state information (CSI) respectively. To conclude the discussion on the relevant literature, we note that some authors addressed the problem of energy-efficiency in MIMO communications but they did not consider the proposed energy-efficiency measure based on the outage probability. In this respect, the most relevant works seem to be [15], [16] and [17]. In [15], the authors adopt a pragmatic approach consisting in choosing a certain coding-modulation scheme in order to reach a given target data rate while minimizing the consumed energy. In [16], the authors study the tradeoff between the minimum energy-per-bit versus spectral efficiency for several MIMO channel models in the wide-band regime assuming a zero cost symbol in the input alphabet and unform power allocation over all the antennas. In [17], the authors consider a similar pragmatic approach to the one in [4], [10] and study a multi-user MIMO channel where the transmitters are constrained to using beamforming power allocation strategies.

This paper is structured as follows. In Sec. II, assumptions on the signal model are provided. In Sec. III, the proposed energy-efficiency measure is defined for static and fast-fading MIMO channels. As the case of slow fading channels is non-trivial, it will be discussed separately in Sec. IV. In Sec. IV, the problem of energy-efficient precoding is discussed for general MIMO slow fading channels and solved for the multiple input single output (MISO) case, whereas in Sec. V asymptotic regimes (in terms of the number of antennas and SNR) are assumed. In Sec. VI, simulations illustrating the derived results and stated conjectures are provided. Sec. VII provides concluding remarks and open issues.

II. General System Model

We consider a point-to-point communication with multiple antenna terminals. The signal at the receiver is modeled by:

$$\underline{y}(\tau) = \mathbf{H}(\tau)\underline{x}(\tau) + \underline{z}(\tau), \tag{1}$$

where $\mathbf{H}$ is the $n_r \times n_t$ channel transfer matrix and n_t (resp. n_r) the number of transmit (resp. receive) antennas. The entries of $\mathbf{H}$ are i.i.d. zero-mean unit-variance complex Gaussian random variables. The vector $\underline{x}$ is the n_t-dimensional column vector of transmitted symbols and $\underline{z}$ is an n_r-dimensional complex white Gaussian noise distributed as $\mathcal{N}(\underline{0}, \sigma^2\mathbf{I})$. In this paper, the problem of allocating the transmit power between the available transmit antennas is considered. We will denote by $\mathbf{Q} = \mathbb{E}[\underline{x}\underline{x}^H]$ the input covariance matrix (called the precoding matrix), which translates the chosen power allocation (PA) policy. The corresponding total power constraint is

$$\mathrm{Tr}(\mathbf{Q}) \leq \overline{P}. \tag{2}$$

At last, the time index τ will be removed for the sake of clarity. In fact, depending on the rate at which $\mathbf{H}$ varies with τ, three dominant classes of channel models can be distinguished:

1) the class of static channels;
2) the class of fast fading channels;
3) the class of slow fading channels.

The matrix $\mathbf{H}$ is assumed to be perfectly known at the receiver (coherent communication assumption) whereas only the statistics of $\mathbf{H}$ are available at the transmitter. The first two classes of channels are considered in Sec. III and the last one is treated in detail in Sec. IV and V.

III. Energy-efficient communications over static and fast fading MIMO channels

A. Case of static channels

Here the frequency at which the channel matrix varies is strictly zero that is, $\mathbf{H}$ is a constant matrix. In this particular context, both the transmitter and receiver are assumed to know this matrix. We are exactly in the same framework as [18]. Thus, for a given precoding scheme $\mathbf{Q}$, the transmitter can send reliably to the receiver $\log_2\left|\mathbf{I}_{n_r} + \rho\mathbf{H}\mathbf{Q}\mathbf{H}^H\right|$ bits per channel use (bpcu) with $\rho = \frac{1}{\sigma^2}$. Then, let us define the energy-efficiency of this communication by:

$$G_{\mathrm{static}}(\mathbf{Q}) = \frac{\log_2\left|\mathbf{I}_{n_r} + \rho\mathbf{H}\mathbf{Q}\mathbf{H}^H\right|}{\mathrm{Tr}(\mathbf{Q})}. \tag{3}$$

The energy-efficiency $G_{\mathrm{static}}(\mathbf{Q})$ corresponds to an achievable rate per unit cost for the MIMO channel as defined in [1]. Assuming that the cost of the transmitted symbol $\underline{x}$, denoted by $b[\underline{x}]$, is the consumed energy $b[\underline{x}] = \|\underline{x}\|^2 = \mathrm{Tr}(\underline{x}\underline{x}^H)$, the capacity per unit cost defined in [1] is: $\widetilde{C}_{\mathrm{slow}} \triangleq \sup\limits_{\underline{x},\mathbb{E}[b[\underline{x}]]\leq\overline{P}} \frac{I(\underline{x};\underline{y})}{\mathbb{E}[b[\underline{x}]]}$. The supremum is taken over the p.d.f. of $\underline{x}$ such that the average transmit power is limited $\mathbb{E}[b[\underline{x}]] \leq \overline{P}$.

It is easy to check that:

$$\begin{aligned}\widetilde{C}_{\mathrm{slow}} &= \sup_{\mathbf{Q},\mathrm{Tr}(Q)\leq\overline{P}} \frac{1}{\mathrm{Tr}(Q)} \sup_{\underline{x},\mathbb{E}(\underline{x}\underline{x}^H)=\mathbf{Q}} I(\underline{x};\underline{y}) \\ &= \sup_{\mathbf{Q},\mathrm{Tr}(\mathbf{Q})\leq\overline{P}} G_{\mathrm{static}}(\mathbf{Q}).\end{aligned} \tag{4}$$

The second equality follows from [18] where Telatar proved that the mutual information for the MIMO static channel is maximized using Gaussian random codes. In other words, finding the optimal precoding matrix which maximizes the energy-efficiency function corresponds to finding the capacity per unit cost of the MIMO channel where the cost of a symbol is the necessary power consumed to be transmitted. The question is then whether the strategy "transmit at low power" (and therefore at a low transmission rate) to maximize energy-efficiency, which is optimal for SISO channels, also applies to MIMO channels. The answer is given by the following proposition, which is proved in Appendix A.

Proposition 3.1 (Static MIMO channels): The energy-efficiency of a MIMO communication over a static channel, measured by G_{static}, is maximized when $\mathbf{Q} = \mathbf{0}$ and this maximum is

$$G^*_{\mathrm{static}} = \frac{1}{\ln 2}\frac{\mathrm{Tr}(\mathbf{H}\mathbf{H}^H)}{n_t\sigma^2}. \tag{5}$$

Therefore, we see that, for static MIMO channels, the energy-efficiency defined in Eq. (3) is maximized by transmitting at a very low power. This kind of scenario occurs for example, when deploying sensors in the ocean to measure a temperature field (which varies very slowly). In some applications however, the rate obtained by using such a scheme

can be not sufficient. In this case, considering the benefit to cost ratio can turn out to be irrelevant, meaning that other performance metrics have to be considered (e.g., minimize the transmit power under a rate constraint).

B. Case of fast fading channels

In this section, the frequency with which the channel matrix varies is the reciprocal of the symbol duration ($\underline{x}(\tau)$ being a symbol). This means that it can be different for each channel use. Therefore, the channel varies over a transmitted codeword (or packet) and, more precisely, each codeword sees as many channel realizations as the number of symbols per codeword. Because of the corresponding self-averaging effect, the following transmission rate (also called EMI for ergodic mutual information) can be achieved on each transmitted codeword by using the precoding strategy $\mathbf{Q}$:

$$R_{\text{fast}}(\mathbf{Q}) = \mathbb{E}_{\mathbf{H}}\left[\log_2\left|\mathbf{I}_{n_r} + \rho\mathbf{H}\mathbf{Q}\mathbf{H}^H\right|\right]. \tag{6}$$

Interestingly, $R_{\text{fast}}(\mathbf{Q})$ can be maximized w.r.t. $\mathbf{Q}$ by knowing only the statistics of $\mathbf{H}$ that is, $\mathbb{E}\left[\mathbf{H}\mathbf{H}^H\right]$, under the standard assumption that the entries of $\mathbf{H}$ are complex Gaussian random variables. In practice, this means that only the knowledge of the path loss, power-delay profile, antenna correlation profile, etc is required at the transmitter to maximize the transmission rate. At the receiver however, the instantaneous knowledge of $\mathbf{H}$ is required. In this framework, let us define energy-efficiency by:

$$G_{\text{fast}}(\mathbf{Q}) = \frac{\mathbb{E}_{\mathbf{H}}\left[\log_2\left|\mathbf{I}_{n_r} + \rho\mathbf{H}\mathbf{Q}\mathbf{H}^H\right|\right]}{\text{Tr}(\mathbf{Q})}. \tag{7}$$

By defining $\underline{g}_i$ as the i-th column of the matrix $\sqrt{\rho}\mathbf{H}\mathbf{U}$, $i \in \{1, \ldots, n_t\}$, $\mathbf{U}$ and $\{p_i\}_{i=1}^{n_t}$ an eigenvector matrix and the corresponding eigenvalues of $\mathbf{Q}$ respectively, and also by rewriting $G_{\text{fast}}(\mathbf{Q})$ as

$$G_{\text{fast}}(\mathbf{Q}) = \mathbb{E}_{\mathbf{H}}\left[\frac{\log_2\left|\mathbf{I}_{n_r} + \sum_{i=1}^{n_t} p_i \underline{g}_i \underline{g}_i^H\right|}{\sum_{i=1}^{n_t} p_i}\right], \tag{8}$$

it is possible to apply the proof of Prop. 3.1 for each realization of the channel matrix. This leads to the following result.

Proposition 3.2 (Fast fading MIMO channels): The energy-efficiency of a MIMO communication over a fast fading channel, measured by G_{fast}, is maximized when $\mathbf{Q} = \mathbf{0}$ and this maximum is

$$G^*_{\text{fast}} = \frac{1}{\ln 2}\frac{\text{Tr}(\mathbb{E}\left[\mathbf{H}\mathbf{H}^H\right])}{n_t\sigma^2}. \tag{9}$$

We see that, for fast fading MIMO channels, maximizing energy-efficiency also amounts to transmitting at low power. Interestingly, in slow fading MIMO channels, where outage events are unavoidable, we have found that the answer can be different. This is precisely what is shown in the remaining of this paper.

IV. Slow fading MIMO channels: from the general case to special cases

A. General MIMO channels

In this section and the remaining of this paper, the frequency with which the channel matrix varies is the reciprocal of the block/codeword/frame/packet/time-slot duration that is, the channel remains constant over a codeword and varies from block to block. As a consequence, when the channel matrix remains constant over a certain block duration much smaller than the channel coherence time, the averaging effect we have mentioned for fast fading MIMO channels does not occur here. Therefore, one has to communicate at rates smaller than the ergodic capacity (maximum of the EMI). The maximum EMI is therefore a rate upper bound for slow fading MIMO channels and only a fraction of it can be achieved (see [27] for more information about the famous diversity-multiplexing tradeoff). In fact, since the mutual information is a random variable, varying from block to block, it is not possible (in general) to guarantee at $100\ \%$ that it is above a certain threshold. A suited performance metric to study slow-fading channels [14] is the probability of an outage for a given transmission rate target R. This metric allows one to quantify the probability that the rate target R is not reached by using a good channel coding scheme and is defined as follows:

$$\text{P}_{\text{out}}(\mathbf{Q}, R) = \Pr\left[\log_2\left|\mathbf{I}_{n_r} + \rho\mathbf{H}\mathbf{Q}\mathbf{H}^H\right| < R\right]. \tag{10}$$

In terms of information assumptions, here again, it can be checked that only the second-order statistics of $\mathbf{H}$ are required to optimize the precoding matrix $\mathbf{Q}$ (and therefore the power allocation policy over its eigenvalues). In this framework, we propose to define the energy-efficiency as follows:

$$\Gamma(\mathbf{Q}, R) = \frac{R[1 - \text{P}_{\text{out}}(\mathbf{Q}, R)]}{\text{Tr}(\mathbf{Q})}. \tag{11}$$

In other words, the energy-efficiency or goodput-to-power ratio is defined as the ratio between the expected throughput (see [3],[20] for details) and the average consumed transmit power. The expected throughput can be seen as the average system throughput over many transmissions. In contrast with static and fast fading channels, energy-efficiency is not necessarily maximized at low transmit powers. This is what the following proposition indicates.

Proposition 4.1 (Slow fading MIMO channels): The goodput-to-power ratio $\Gamma(\mathbf{Q}, R)$ is maximized, in general, for $\mathbf{Q} \neq \mathbf{0}$.

The proof of this result is given in Appendix B. Now, a natural issue to be considered is the determination of the matrix (or matrices) maximizing the goodput-to-power ratio (GPR) in slow fading MIMO channels. It turns out that the corresponding optimization problem is not trivial. Indeed, even the outage probability minimization problem w.r.t. $\mathbf{Q}$ (which is a priori simpler) is still an open problem [18], [21], [22]. This is why we only provide here a conjecture on the solution maximizing the GPR.

Conjecture 4.2 (Optimal precoding matrices): There exists a power threshold $\overline{P}_0$ such that:

- *if* $\overline{P} \leq \overline{P}_0$ *then* $\mathbf{Q}^* \in \arg\min_{\mathbf{Q}} P_{\text{out}}(\mathbf{Q}, R) \Rightarrow \mathbf{Q}^* \in \arg\max_{\mathbf{Q}} \Gamma(\mathbf{Q}, R)$*;*
- *if* $\overline{P} > \overline{P}_0$ *then* $\Gamma(\mathbf{Q}, R)$ *has a unique maximum in* $\mathbf{Q}^* = \frac{p^*}{n_t}\mathbf{I}_{n_t}$ *where* $p^* \leq \overline{P}$*.*

This conjecture has been validated for all the special cases solved in this paper. One of the main messages of this conjecture is that, if the available transmit power is less than a threshold, maximizing the GPR is equivalent to minimizing the outage probability. If it is above the threshold, uniform power allocation is optimal and using all the available power is generally suboptimal in terms of energy-efficiency. Concerning the optimization problem associated with (11) several comments are in order. First, there is no loss of optimality by restricting the search for optimal precoding matrices to diagonal matrices: for any eigenvalue decomposition $\mathbf{Q} = \mathbf{U}\mathbf{D}\mathbf{U}^H$ with $\mathbf{U}$ unitary and $\mathbf{D} = \mathbf{Diag}(\underline{p})$ with $\underline{p} = (p_1, \ldots, p_{n_t})$, both the outage and trace are invariant w.r.t. the choice of $\mathbf{U}$ and the energy-efficiency can be written as:

$$\Gamma(\mathbf{D}, R) = \frac{R[1 - \mathrm{P}_{\text{out}}(\mathbf{D}, R)]}{\sum_{i=1}^{n_t} p_i}. \tag{12}$$

Second, the GPR is generally not concave w.r.t. $\mathbf{D}$. In Sec. IV-B, which is dedicated to MISO systems, a counter-example where it is not quasi-concave (and thus not concave) is provided.

Uniform Power Allocation policy

An interesting special case is the one of uniform power allocation (UPA): $\mathbf{D} = \frac{p}{n_t}\mathbf{I}_{n_t}$ where $p \in [0, \overline{P}]$ and $\Gamma_{\text{UPA}}(p, R) \triangleq \Gamma\left(\frac{p}{n_t}\mathbf{I}_{n_t}, R\right)$.

One of the reasons for studying this case is that the famous conjecture of Telatar given in [18]. This conjecture states that, depending on the channel parameters and target rate (i.e., σ^2, R), the power allocation (PA) policy minimizing the outage probability is to spread all the available power uniformly over a subset of $\ell^* \in \{1, \ldots, n_t\}$ antennas. If this can be proved, then it is straightforward to show that the covariance matrix $\mathbf{D}^*$ that maximizes the proposed energy-efficiency function is $\frac{p^*}{\ell^*}\mathbf{Diag}(\underline{e}_{\ell^*})$, where $\underline{e}_{\ell^*} \in \mathcal{S}_{\ell^*}$[2]. Thus, $\mathbf{D}^*$ has the same structure as the covariance matrix minimizing the outage probability except that using all the available power is not necessarily optimal, $p^* \in [0, \overline{P}]$. In conclusion, solving Conjecture 4.2 reduces to solving Telatar's conjecture and also the UPA case.

The main difficulty in studying the outage probability or/and the energy-efficiency function is the fact that the probability distribution function of the mutual information is generally intractable. In the literature, the outage probability is often studied by assuming a UPA policy over all the antennas and also using the Gaussian approximation of the p.d.f. of the mutual information. This approximation is valid in the asymptotic regime of large number of antennas. However, simulations show that it also quite accurate for reasonable small MIMO systems [23], [24].

[2]We denote by $\mathcal{S}_\ell = \{\underline{v} \in \{0,1\}^{n_t} | \sum_{i=1}^{n_t} v_i = \ell\}$ the set of n_t dimensional vectors containing ℓ ones and $n_t - \ell$ zeros, for all $\ell \in \{1, \ldots, n_t\}$.

	Is $\mathbf{D}^*$ known?	Is $\Gamma^{\text{UPA}}(p)$ quasi-concave?	Is p^* known?
General MIMO	Conjecture	Conjecture	Conjecture
MISO	Yes	Yes	Yes
1×1	Yes	Yes	Yes
Large MIMO	Conjecture	Yes	Yes
Low SNR	Yes	Yes	Yes
High SNR	Yes	Yes	Conjecture

TABLE I
SUMMARY OF PROVED RESULTS AND OPEN PROBLEMS

Under the UPA policy assumption, the GPR $\Gamma_{\text{UPA}}(p, R)$ is conjectured to be quasi-concave w.r.t. p. Quasi-concavity is not only useful to study the maximum of the GPR but is also an attractive property in some scenarios such as the distributed multiuser channels. For example, by considering MIMO multiple access channels with single-user decoding at the receiver, the corresponding distributed power allocation game where the transmitters' utility functions are their GPR is guaranteed to have a pure Nash equilibrium after Debreu-Fan-Glicksberg theorem [25].

Before stating the conjecture describing the behavior of the energy-efficiency function when the UPA policy is assumed, we study the limits when $p \to 0$ and $p \to +\infty$. First, let us prove that $\lim_{p \to 0} \Gamma_{\text{UPA}}(p, R) = 0$. Observe that $\lim_{p \to 0} P_{\text{out}}\left(\frac{p}{n_t}\mathbf{I}_{n_t}, R\right) = 1$ and thus the limit is not trivial to prove. The result can be proven by considering the equivalent $1 + \frac{\rho p}{n_t}\mathrm{Tr}(\mathbf{H}\mathbf{H}^H)$ of the determinant $\left|\mathbf{I}_{n_r} + \frac{\rho p}{n_t}\mathbf{H}\mathbf{H}^H\right|$ when $\sigma \to +\infty$. As the entries of the matrix $\mathbf{H}$ are i.i.d. complex Gaussian random variables, the quantity $\mathrm{Tr}(\mathbf{H}\mathbf{H}^H) = \sum_{i=1}^{n_t}\sum_{j=1}^{n_r}|h_{ij}|^2$ is a $2n_r n_t$ Chi-square distributed random variable. Thus $\Gamma_{\text{UPA}}(p, R)$ can be approximated by: $\widehat{\Gamma}_{\text{UPA}}(p, R) = R\exp\left(-\frac{d}{p}\right)\sum_{k=0}^{n_r n_t - 1}\frac{d^k}{k!}\frac{1}{p^{k+1}}$ with $d = n_t(2^R - 1)\sigma^2$. It is easy to see that this approximate tends to zero when $p \to 0$. Second, note that the limit $\lim_{p \to +\infty} \Gamma_{\text{UPA}}(p, R) = 0$. This is easier to check since $\lim_{p \to +\infty} P_{\text{out}}\left(\frac{p}{n_t}\mathbf{I}, R\right) = 0$.

Conjecture 4.3 (UPA and quasi-concavity of the GPR): Assume that $\mathbf{D} = \frac{p}{n_t}\mathbf{I}_{n_t}$*. Then* $\Gamma_{\text{UPA}}(p, R)$ *is quasi-concave w.r.t.* $p \in \left[0, \overline{P}\right]$*.*

Table IV-A distinguishes between what has been proven in this paper and the conjectures which remain to be proven.

B. MISO channels

In this section, the receiver is assumed to use a single antenna that is, $n_r = 1$, while the transmitter can have an arbitrary number of antennas, $n_t \geq 1$. The channel transfer matrix becomes a row vector $\underline{h} = (h_1, ..., h_{n_t})$. Without loss of optimality, the precoding matrix is assumed to be diagonal and is denoted by $\mathbf{D} = \mathbf{Diag}(\underline{p})$ with $\underline{p}^T = (p_1, ..., p_{n_t})$. Throughout this section, the rate target R and noise level

σ^2 are fixed and the auxiliary quantity c is defined by: $c = \sigma^2(2^R - 1)$. By exploiting the existing results on the outage probability minimization problem for MISO channels [22], the following proposition can be proved (Appendix C).

Proposition 4.4 (Optimum precoding matrices for MISO channels): For all $\ell \in \{1, ..., n_t - 1\}$, let c_ℓ be the unique solution of the equation (in x) $\Pr\left[\frac{1}{\ell+1}\sum_{i=1}^{\ell+1}|X_i|^2 \leq x\right] - \Pr\left[\frac{1}{\ell}\sum_{i=1}^{\ell}|X_i|^2 \leq x\right] = 0$ *where X_i are i.i.d. zero-mean Gaussian random variables with unit variance. By convention $c_0 = +\infty$, $c_{n_t} = 0$. Let ν_{n_t} be the unique solution of the equation (in y)* $\frac{y^{n_t}}{(n_t-1)!} - \sum_{i=0}^{n_t-1}\frac{y^i}{i!} = 0$. *Then the optimum precoding matrices have the following form:*

$$\mathbf{D}^* = \begin{cases} \frac{\overline{P}}{\ell}\mathbf{Diag}(\underline{e}_\ell) & \text{if } \overline{P} \in \left[\frac{c}{c_{\ell-1}}, \frac{c}{c_\ell}\right) \\ \min\left\{\frac{\sigma^2(2^R-1)}{\nu_{n_t}}, \frac{\overline{P}}{n_t}\right\}\mathbf{I} & \text{if } \overline{P} \geq \frac{c}{c_{n_t-1}} \end{cases} \tag{13}$$

where $c = \sigma^2(2^R - 1)$ and $\underline{e}_\ell \in \mathcal{S}_\ell$.

Similarly to the optimal precoding scheme for the outage probability minimization, the solution maximizing the GPR consists in allocating the available transmit power uniformly between only a subset $\ell \leq n_t$ antennas. As i.i.d entries are assumed for $\mathbf{H}$, the choice of these antennas does not matter. What matters is the number of antennas selected (denoted by ℓ), which depends on the available transmit power $\overline{P}$: the higher the transmit power, the higher the number of used antennas. The difference between the outage probability minimization and GPR maximization problems appears when the transmit power is greater than the threshold $\frac{c}{c_{n_t-1}}$. In this regime, saturating the power constraint is suboptimal for the GPR optimization. The corresponding sub-optimality becomes more and more severe as the noise level is low; simulations (Sec. VI) will help us to quantify this gap.

Unless otherwise specified, we will assume from now on that **UPA** is used at the transmitter. This assumption is, in particular, useful to study the regime where the available transmit power is sufficiently high (as conjectured in Proposition 4.1). Under this assumption, our goal is to prove that the GPR is quasi-concave w.r.t. $p \in [0, \overline{P}]$ with $\mathbf{D} = \frac{p}{n_t}\mathbf{I}_{n_t}$ and determine the (unique) solution p^* which maximizes the GPR. Note that the quasi-concavity property w.r.t. $\underline{p}$ is not always available for MISO systems (and thus is not always available for general MIMO channels). In Appendix D, a counter-example proving that in the case where $n_r = 1$ and $n_t = 2$ (two input single output channel, TISO) the energy-efficiency $\Gamma^{\text{TISO}}\left(\mathbf{Diag}(\underline{p}), R\right)$ is not quasi-concave w.r.t. $\underline{p} = (p_1, p_2)$ is provided.

Proposition 4.5 (UPA and quasi-concavity (MISO channels)): Assume the UPA, $\mathbf{Q} = \frac{p}{n_t}\mathbf{I}_{n_t}$, then $\Gamma(p, R)$ is quasi-concave w.r.t. $p \in \left[0, \overline{P}\right]$ and has a unique maximum point in $p^* = \min\left\{\frac{(2^R-1)n_t\sigma^2}{\nu_{n_t}}, \overline{P}\right\}$ where ν_{n_t} is the solution (w.r.t. y) of:

$$\frac{y^{n_t}}{(n_t-1)!} - \sum_{i=0}^{n_t-1}\frac{y^i}{i!} = 0. \tag{14}$$

Proof: Since the entries of $\underline{h}$ are complex Gaussian random variables, the sum $\sum_{k=1}^{n_t}|h_k|^2$ is a $2n_t-$ Chi-square distributed random variable, which implies that:

$$\begin{aligned} \Gamma^{\text{MISO}}(p, R) &= \frac{R\left\{1 - \Pr[\log_2\left(1 + \frac{\rho p}{n_t}\underline{h}^H\underline{h}\right) < R]\right\}}{p} \\ &= \frac{R\left\{1 - \Pr\left[\sum_{i=1}^{n_t}|h_i|^2 < \frac{d}{p}\right]\right\}}{p} \\ &= R \times \mathrm{e}^{-\frac{d}{p}}\sum_{i=0}^{n_t-1}\frac{d^i}{p^{i+1}}\frac{1}{i!}, \end{aligned} \tag{15}$$

with $d = cn_t = (2^R - 1)n_t\sigma^2$. The second order derivative of the goodput $R\left[\mathrm{e}^{-\frac{d}{p}}\sum_{i=0}^{n_t-1}\left(\frac{d}{p}\right)^i\frac{1}{i!}\right]$ w.r.t. p is $R\left[\frac{d^{n_t}}{p^{n_t+3}}\frac{1}{n_t!}\mathrm{e}^{-d/p}(d - (n_t+1)p)\right]$. Clearly, the goodput is a sigmoidal function and has a unique inflection point in $p_0 = \frac{d}{n_t+1}$. Therefore, the function $\Gamma^{\text{MISO}}(p, R)$ is quasi-concave [26] and has a unique maximum in $p^* = \min\left\{\frac{d}{\nu_{n_t}}, \overline{P}\right\}$ where ν_{n_t} is the root of the first order derivative of $\Gamma^{\text{MISO}}(p, R)$ that is, the solution of (14). ■

The **SIMO** case ($n_t = 1$, $n_r \geq 2$) follows directly since $|\mathbf{I} + \rho p\underline{h}\underline{h}^H| = 1 + \rho p\underline{h}^H\underline{h}$.

To conclude this section, we consider the most simple case of MISO channels namely the SISO case ($n_t = 1$, $n_r = 1$). We have readily that:

$$\Gamma^{\text{SISO}}(p, R) = \frac{e^{-\frac{c}{p}}}{p}. \tag{16}$$

To the authors' knowledge, in all the works using the energy-efficiency definition of [4] for SISO channels, the only choice of energy-efficiency function made is based on the empirical approximation of the block error rate which is $\frac{(1-e^{-x})^M}{x}$, M being the block length and x the operating SINR. Interestingly, the function given by (16) exhibits another possible choice. It can be checked that the function $e^{-\frac{c}{p}}$ is sigmoidal and therefore Γ^{SISO} is quasi-concave w.r.t. p [26]. The first order derivative of Γ^{SISO} is

$$\frac{\partial\Gamma^{\text{SISO}}}{\partial p} = R\frac{(c-p)e^{-\frac{c}{p}}}{p^3}. \tag{17}$$

The GPR is therefore maximized in a unique point which $p^* = c = \sigma^2(2^R - 1)$. To make the bridge between this solution and the one derived in [4] for the power control problem over multiple access channels, the optimal power level can be rewritten as:

$$p^* = \min\left\{\frac{\sigma^2}{\mathbb{E}|h|^2}(2^R - 1), \overline{P}\right\} \tag{18}$$

where $\mathbb{E}|h|^2 = 1$ in our case. In [4], instantaneous CSI knowledge at the transmitters is assumed while here only the statistics are assumed to be known at the transmitter. Therefore, the power control interpretation of (18) in a wireless scenario is that the power is adapted to the path loss (slow power control) and not to fast fading (fast power control).

V. Slow fading MIMO channels in asymptotic regimes

In this section, we first consider the GPR for the case where the size of the MIMO system is finite assuming the low/high SNR operating regime. Then, we consider the UPA policy and prove that Conjecture 4.3 claiming that $\Gamma_{\mathrm{UPA}}(p,R)$ is quasi-concave w.r.t. p (which has been proven for MISO, SIMO, and SISO channels) is also valid in the asymptotic regimes where either at least one dimension of the system (n_t, n_r) is large but the SNR is finite. Here again, the theory of large random matrices is successfully applied since it allows one to prove some results which are not available yet in the finite case (see e.g., [19], [28] for other successful examples).

A. Extreme SNR regimes

Here, all the channel parameters (n_t, n_r, and $\overline{P}$ in particular) are fixed. The low (resp. high) SNR regime is defined by $\sigma^2 \to +\infty$ (resp. $\sigma^2 \to 0$). In both cases, we will consider the GPR and the optimal power allocation problem.

1) Low SNR regime: Let us consider the general power allocation problem where $\mathbf{D} = \mathbf{Diag}(\underline{p})$ with $\underline{p} = (p_1, \ldots, p_{n_t})$. In [22], the authors extended the results obtained in the low and high SNR regimes for the MISO channel to the MIMO case. In the low SNR regime, the authors of [22] proved that the outage probability $P_{\mathrm{out}}(\mathbf{Diag}(\underline{p}), R)$ is a Schur-concave (see [29] for details) function w.r.t. $\underline{p}$. This implies directly that beamforming power allocation policy maximizes the outage probability. These results can be used (see Appendix E) to prove the following proposition:

Proposition 5.1 (Low SNR regime): When $\sigma^2 \to +\infty$, the energy-efficiency function $\Gamma(\mathbf{Diag}(\underline{p}), R)$ is Schur-concave w.r.t. $\underline{p}$ and maximized by a beamforming power allocation policy $\mathbf{D}^* = \overline{P}\mathbf{Diag}(\underline{e}_1)$.

2) High SNR regime: Now, let us consider the high SNR regime. It turns out that the UPA policy maximizes the energy-efficiency function. In this case also, the proof of the following proposition is based on the results in [22] (see Appendix E).

Proposition 5.2 (High SNR regime): When $\sigma^2 \to 0$, the energy-efficiency function $\Gamma(\mathbf{Diag}(\underline{p}), R)$ is Schur-convex w.r.t. $\underline{p}$ and maximized by an uniform power allocation policy $\mathbf{D}^* = \frac{p^*}{n_t}\mathbf{I}_{n_t}$ with $p^* \in (0, \overline{P}]$. Furthermore, the limit when $p \to 0$ such that $\frac{p}{\sigma^2} \to \xi$ is $\Gamma\left(\frac{p}{n_t}\mathbf{I}_{n_t}, R\right) \to +\infty$ which implies that $p^* \to 0$.

In other words, in the high SNR regime, the optimal structure of the covariance matrix is obtained by uniformly spreading the power over all the antennas, $\mathbf{D}^* = \frac{p^*}{n_t}\mathbf{I}_{n_t}$ the same structure which minimizes the outage probability in this case. Nevertheless, in contrast to the outage probability optimization problem, in order to be energy-efficient it is not optimal to use all the available power $\overline{P}$ but to transmit with zero power.

B. Large MIMO channels

The results we have obtained can be summarized in the following proposition.

Proposition 5.3 (Quasi-concavity for large MIMO systems): If the system operates in one of the following asymptotic regimes:

(a) $n_t < +\infty$ and $n_r \to +\infty$;
(b) $n_t \to +\infty$ and $n_r < +\infty$;
(c) $n_t \to +\infty$, $n_r \to +\infty$ with $\lim\limits_{n_i \to +\infty, i \in \{t,r\}} \frac{n_r}{n_t} = \beta < +\infty$,

then $\Gamma_{\mathrm{UPA}}(p,R)$ is quasi-concave w.r.t. $p \in [0, \overline{P}]$.

Proof: Here we prove each of the three statements made above and provide comments on each of them at the same time.

Regime (a): $n_t < +\infty$ and $n_r \to \infty$. The idea of the proof is to consider a large system equivalent of the function $\Gamma_{\mathrm{UPA}}(p,R)$. This equivalent is denoted by $\widehat{\Gamma}^{\mathrm{a}}_{\mathrm{UPA}}(p,R)$ and is based on the Gaussian approximation of the mutual information $\log_2\left|\mathbf{I} + \frac{\rho p}{n_t}\mathbf{H}\mathbf{H}^H\right|$ (see e.g., [30]). The goal is to prove that the numerator of $\widehat{\Gamma}^{\mathrm{a}}_{\mathrm{UPA}}(p,R)$ is a sigmoidal function w.r.t. p which implies that $\widehat{\Gamma}^{\mathrm{a}}_{\mathrm{UPA}}(p,R)$ is a quasi-concave function [26]. In the considered asymptotic regime, we know from [30] that:

$$\log_2\left|\mathbf{I} + \frac{\rho p}{n_t}\mathbf{H}\mathbf{H}^H\right| \to \mathcal{N}\left(n_t \log_2\left(1 + \frac{n_r}{n_t}\rho p\right), \frac{n_t}{n_r}\log_2(e)\right). \tag{19}$$

A large system equivalent of the numerator of $\Gamma_{\mathrm{UPA}}(p,R)$, which is denoted by $\widehat{N}_a(p,R)$, follows:

$$\widehat{N}_a(p,R) = RQ\left(\frac{R - n_t\log_2\left(1 + \frac{n_r}{n_t}\rho p\right)}{\sqrt{\frac{n_t}{n_r}\log_2(e)}}\right) \tag{20}$$

where $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{+\infty}\exp\left(-\frac{t^2}{2}\right)\mathrm{d}t$. Denote the argument of Q in (20) by α_a. The second order derivative of $\widehat{N}_a(p,R)$ w.r.t. p

$$\frac{\partial^2 \widehat{N}_a(p,R)}{\partial p^2} = \frac{1}{\sqrt{2\pi}}\left[\alpha_a(p)(\alpha_a'(p))^2 - \alpha_a''(p)\right]\exp\left(-\frac{\alpha_a(p)^2}{2}\right). \tag{21}$$

Therefore $\widehat{N}_a(p,R)$ has a unique inflection point

$$\tilde{p}_a = \frac{n_t}{n_r\rho}\left\{2^{\left[\frac{1}{n_t}\left(R - \frac{1}{n_t}\left(\frac{n_t\log_2(e)}{n_r}\right)^{3/2}\right)\right]} - 1\right\}. \tag{22}$$

Clearly, for each equivalent of $\Gamma_{\mathrm{UPA}}(p,R)$, the numerator has a unique inflection point and is sigmoidal, which concludes the proof. In fact, in the considered asymptotic regime we have a stronger result since $\lim\limits_{n_r \to +\infty}\tilde{p}_a = 0$, which implies that $\widehat{N}_a(p,R)$ is concave and therefore $\widehat{\Gamma}^{\mathrm{a}}_{\mathrm{UPA}}(p,R)$ is maximized in $p_a^* = 0$ as in the case of static MIMO channels. This translates the well-known channel hardening effect [30]. However, in contrast to the static case, the energy-efficiency becomes infinite here since $\Gamma_{\mathrm{UPA}}(p,R) \to \frac{1}{p}$ with $p_a^* \to 0$.

Regime (b): $n_t \to +\infty$ and $n_r < +\infty$. To prove the corresponding result the same reasoning as in (a) is applied. From [30] we know that:

$$\log_2\left|\mathbf{I} + \frac{\rho p}{n_t}\mathbf{H}\mathbf{H}^H\right| \to \mathcal{N}\left(n_r\log_2(1+\rho p), \left(\sqrt{\frac{n_r}{n_t}}\log_2(e)\frac{\rho p}{1+\rho p}\right)^2\right). \tag{23}$$

A large system equivalent of the numerator of $\Gamma_{\mathrm{UPA}}(p,R)$ is $\widehat{N}_b(p,R) = RQ\left(\alpha_b(p)\right)$ with

$$\alpha_b(p) = \sqrt{\frac{n_t}{n_r}}\log_2(e)\frac{1+\rho p}{\rho p}[R - n_r\log_2(1+\rho p)]. \quad (24)$$

The numerator function $\widehat{N}_b(p,R)$ can be checked to have a unique inflection point given by:

$$\tilde{p}_b = \sigma^2\left(2^{\frac{R}{n_r}} - 1\right) \quad (25)$$

and is sigmoidal, which concludes the proof. We see that the inflection point does not vanish this time (with n_t here) and therefore the function $\widehat{N}_b(p,R)$ is quasi-concave but not concave in general. From [26], we know that the optimal solution p_b^* represents the point where the tangent that passes through the origin intersects the S-shaped function $RQ\left(\alpha_b(p)\right)$. As n_t grows large, the function $Q\left(\alpha_b(p)\right)$ becomes a Heavyside step function since $\forall p \leq \tilde{p}_b$, $\lim_{n_t\to+\infty} Q\left(\alpha_b(p)\right) = 0$ and $\forall p \geq \tilde{p}_b$, $\lim_{n_t\to+\infty} Q\left(\alpha_b(p)\right) = 1$. This means that the optimal power p_b^* that maximizes the energy-efficiency approaches $\tilde{p}_b$ as n_t grows large, $p_b^* \to \sigma^2\left(2^{\frac{R}{n_r}} - 1\right)$. The optimal energy-efficiency tends to $\frac{\widehat{N}_b(p_b^*,R)}{p_b^*} \to \frac{1}{2\sigma^2\left(2^{\frac{R}{n_r}}-1\right)}$ when $n_t \to +\infty$.

Regime (c): $n_t \to +\infty$, $n_r \to \infty$. Here we always apply the same reasoning but exploit the results derived in [31]. From [31], we have that:

$$\log_2\left|\mathbf{I} + \frac{\rho p}{n_t}\mathbf{H}\mathbf{H}^H\right| \to \mathcal{N}\left(n_t\mu_I, \sigma_I^2\right) \quad (26)$$

where $\mu_I = \beta\log_2(1+\rho p(1-\gamma)) - \gamma + \log_2(1+\rho p(\beta-\gamma))$, $\sigma_I^2 = -\log_2\left(1-\frac{\gamma^2}{\beta}\right)$, $\gamma = \frac{1}{2}\left(1+\beta+\frac{1}{\rho p} - \sqrt{(1+\beta+\frac{1}{\rho p})^2 - 4\beta}\right)$. It can be checked that $(\alpha_c'(p))^2\alpha_c(p) - \alpha_c''(p) = 0$ has a unique solution where $\alpha_c(p) = \frac{R - n_t\mu_I(p)}{\sigma_I(p)}$. We obtain $\alpha_c'(p) = \frac{n_t\mu_I\sigma_I' - n_t\mu_I'\sigma_I - R\sigma_I'}{\sigma_I^2}$ and $\alpha_c''(p) = \frac{(n_t\mu_I\sigma_I'' - n_t\mu_I''\sigma_I - R\sigma_I'')\sigma_I^2 - 2\sigma_I\sigma_I'(n_t\mu_I\sigma_I' - n_t\mu_I'\sigma_I - R\sigma_I')}{\sigma_I^4}$. We observe that, in the equation $(\alpha_c'(p))^2\alpha_c(p) - \alpha_c''(p) = 0$, there are terms in n_t^3, n_t^2, n_t and constant terms w.r.t. n_t. When n_t becomes sufficiently large the first order terms can be neglected, which implies that the solution is given by $\mu_I(p) = 0$. It can be shown that $\mu_I(0) = 0$ and that μ_I is an increasing function w.r.t. p which implies that the unique solution is $\tilde{p}_c = 0$. Similarly to regime (a) we obtain the trivial solution $p_c^* = 0$.

■

VI. Numerical results

In this section, we present several simulations that illustrate our analytical results and verify the two conjectures stated. Since closed-form expressions of the outage probability are not available in general, Monte Carlo simulations will be implemented. The exception is the MISO channel for which the optimal energy-efficiency can be computed numerically (as we have seen in Sec. IV-B) without the need of Monte Carlo simulations.

UPA, the quasi-concavity property and the large MIMO channels.

Let us consider the case of UPA. In Fig. 1, we plot the GPR $\Gamma_{\mathrm{UPA}}(p,R)$ as a function of the transmit power $p \in [0,\overline{P}]$ W for an MIMO channel where $n_r = n_t = n$ with $n \in \{1,2,4,8\}$ and $\rho = 10$ dB, $R = 1$ bpcu, $\overline{P} = 1$ W. First, note that the energy-efficiency for UPA is a quasi-concave function w.r.t. p, illustrating Conjecture 4.3. Second, we observe that the optimal power p^* maximizing the energy-efficiency function is decreasing and approaching zero as the number of antennas increases and also that $\Gamma_{\mathrm{UPA}}(p^*,R)$ is increasing with n. In Fig. 2, this dependence of the optimal energy-efficiency and the number of antennas n is depicted explicitly for the same scenario. These observations are in accordance with the asymptotic analysis in subsection V-B for Regime (c).

Similar simulation results were obtained for the case where n_t is fixed and n_r is increasing, thus illustrating the asymptotic analysis in subsection V-B for Regime (a).

In Fig. 3, we plot the energy-efficiency $\Gamma_{\mathrm{UPA}}(p,R)$ as a function of the transmit power $p \in [0,\overline{P}]$ W for MIMO channel such that $n_r = 2$, $n_t \in \{1,2,4,8\}$ and $\rho = 10$ dB, $R = 1$ bpcu, $\overline{P} = 1$ W. The difference w.r.t. the previous case, is that the optimal power p^* does not go to zero when n_t increases. This figure illustrates the results obtained for Regime (b) in section V-B where the optimal power allocation $p_b^* \to \frac{2^{\frac{R}{n_r}}-1}{\rho} = 0.0414$ W and the optimal energy-efficiency $\Gamma_{UPA}^* \to \frac{\rho}{2(2^{\frac{R}{n_r}}-1)} = 12,07$ bit/Joule when $n_t \to +\infty$.

UPA and the finite MISO channel

In Fig. 4, we illustrate Proposition 4.4 for $n_t = 4$. We trace the cases where the transmitter uses an optimal UPA over only a subset of $\ell \in \{1,2,3,4\}$ antennas for $\rho = 10$ dB, $R = 3$ bpcu. We observe that: i) if $\overline{P} \leq \frac{c}{c_1}$ then the beamforming PA is the generally optimal structure with $\mathrm{D}^* = \overline{P}\ \mathbf{Diag}(\underline{e}_1)$; ii) if $\overline{P} \in \left[\frac{c}{c_1}\frac{c}{c_2}\right)$ then using UPA over three antennas is the generally optimal structure with $\mathrm{D}^* = \overline{P}/2\ \mathbf{Diag}(\underline{e}_2)$; iii) if $\overline{P} \in \left[\frac{c}{c_2}\frac{c}{c_3}\right)$ then using UPA over three antennas is generally optimal with $\mathrm{D}^* = \overline{P}/3\ \mathbf{Diag}(\underline{e}_3)$; iv) if $\overline{P} \geq \frac{c}{c_4}$ then the UPA over all the antennas is optimal with $\mathrm{D}^* = \frac{1}{4}\min\left\{\frac{4*c}{\nu_4}, \overline{P}\right\}\ \mathbf{I}_4$. The saturated regime illustrates the fact that it is not always optimal to use all the available power after a certain threshold.

UPA and the finite MIMO channel

Fig. 5 represents the success probability, $1 - P_{\mathrm{out}}(\mathbf{D},R)$, in function of the power constraint $\overline{P}$ for $n_t = n_r = 2$, $R = 1$ bpcu, $\rho = 3$ dB. Since the optimal PA that maximizes the success probability is unknown (unlike the MISO case) we use Monte-Carlo simulations and exhaustive search to compare the optimal PA with the UPA and the beamforming PA. We observe that the result is in accordance with Telatar's conjecture. There exists a threshold $\delta = 0.16$ W such that if $\overline{P} \leq \delta$, the beamforming PA is optimal and otherwise the UPA is optimal. Of course, using all the available power is always optimal when maximizing the success probability. The objective is to check whether Conjecture 4.2 is verified in this particular case. To this purpose, Fig. 6 represents the energy-

efficiency function for the same scenario. We observe that for the exact threshold $\delta = 0.16$ W, we obtain that if $\overline{P} \leq \delta$ the beamforming PA using all the available power is optimal. If $\overline{P} > \delta$ the UPA is optimal. Here, similarly to the MISO case, we observe a saturated regime which means that after a certain point it is not optimal w.r.t. energy-efficiency to use up all the available transmit power. In conclusion, our conjecture has been verified in this simulation.

Note that for the beamforming PA case we have explicit relations for both the outage probability and the energy-efficiency (it is easy to check that the MIMO with beamforming PA reduces to the SIMO case) and thus Monte-Carlo simulations have not been used.

VII. Conclusion

In this paper, we propose a definition of energy-efficiency metric which is the extension of the work in [1] to static MIMO channels. Furthermore, our definition bridges the gap between the notion of capacity per unit cost [1] and the empirical approach of [4] in the case of slow fading channels. In static and fast fading channels, the energy-efficiency is maximized at low transmit power and the corresponding rates are also small. On the the other hand, the case of slow fading channel is not trivial and exhibits several open problems. It is conjectured that solving the (still open) problem of outage minimization is sufficient to solve the problem of determining energy-efficient precoding schemes. This conjecture is validated by several special cases such as the MISO case and asymptotic cases. Many open problems are introduced by the proposed performance metric, here we just mention some of them:

- First of all, the conjecture of the optimal precoding schemes for general MIMO channels needs to be proven.
- The quasi-concavity of the goodput-to-power ratio when uniform power allocation is assumed remains to be proven in the finite setting.
- A more general channel model should be considered. We have considered i.i.d. channel matrices but considering non zero-mean matrices with arbitrary correlation profiles appears to be a challenging problem for the goodput-to-power ratio.
- The connection between the proposed metric and the diversity-multiplexing tradeoff at high SNR has not been explored.
- Only single-user channels have been considered. Clearly, multi-user MIMO channels such as multiple access or interference channels should be considered.
- The case of distributed multi-user channels become more and more important for applications (unlicensed bands, decentralized cellular networks, etc.). Only one result is mentioned in this paper: the existence of a pure Nash equilibrium in distributed MIMO multiple access channels assuming uniform power allocation transmit policy.

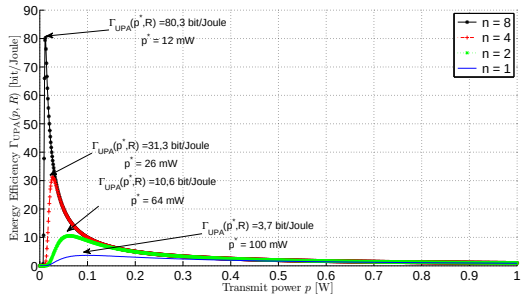

Fig. 1. Energy-efficiency (GPR) vs. transmit power $p \in [0, 1]$ W for MIMO channels where $n_r = n_t = n \in \{1, 2, 4, 8\}$, UPA $\mathbf{D} = \frac{p}{n_t}\mathbf{I}_{n_t}$, $\rho = 10$ dB, $R = 1$ bpcu. Observe that the energy-efficiency is a quasi-concave function w.r.t. p. The optimal point p^* is decreasing and $\Gamma_{\text{UPA}}(p^*, R)$ is increasing with n.

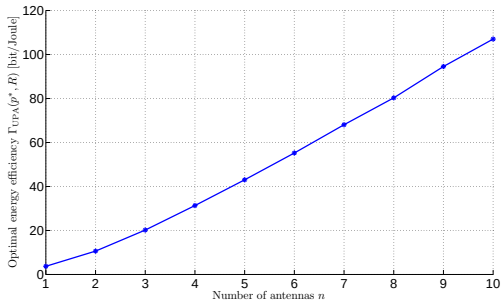

Fig. 2. Energy-efficiency vs. the number of antennas n for MIMO $n_r = n_t = n \in \{1, 2, 4, 8\}$, UPA, $\mathbf{D} = \frac{p}{n_t}\mathbf{I}_{n_t}$, $\rho = 10$ dB, $R = 1$ bpcu and $\overline{P} = 1$ W. Observe that $\Gamma_{\text{UPA}}(p^*, R)$ is increasing with n.

Appendix A
Proof of Proposition 3.1

As $\mathbf{Q}$ is a positive semi-definite Hermitian matrix, it can always be spectrally decomposed as $\mathbf{Q} = \mathbf{U}\mathbf{D}\mathbf{U}^H$ where $\mathbf{D} = \mathbf{Diag}(p_1, \ldots, p_{n_t})$ is a diagonal matrix representing a given PA policy and $\mathbf{U}$ a unitary matrix. Our goal is to prove that, for every $\mathbf{U}$, G_{static} is maximized when $\mathbf{D} = \mathbf{Diag}(0, 0, ..., 0)$.

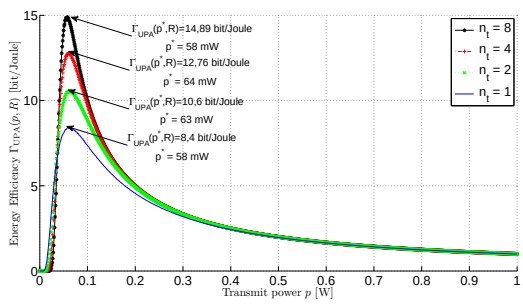

Fig. 3. Energy-efficiency vs. transmit power $p \in [0, 1]$ W for MIMO $n_r = 2$, $n_t \in \{1, 2, 4, 8\}$, UPA $\mathbf{D} = \frac{p}{n_t}\mathbf{I}_{n_t}$, $\rho = 10$ dB, $R = 1$ bpcu. Observe that the energy-efficiency is a quasi-concave function w.r.t. p. The optimal point p^* is not decreasing with n but almost constant.

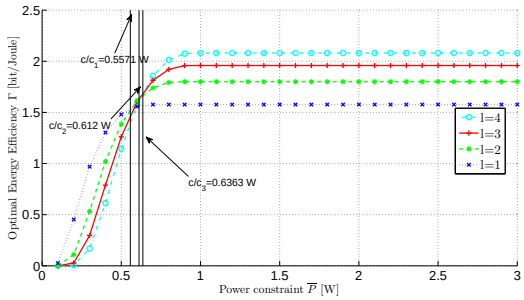

Fig. 4. Optimal energy-efficiency vs. constraint power for MISO $n_t = 4$, $n_r = 1$, UPA over a subset of $\ell \in \{1, 2, 3, 4\}$ antennas, $\rho = 10$ dB, $R = 3$ bpcu. We illustrate the results of Proposition 4.4. If $\overline{P} \leq \frac{c}{c_1}$ is low enough, the beamforming PA with full power is optimal. If $\overline{P} \geq \frac{c}{c_2}$ is high enough, the UPA is optimal but not with full power necessarily $\left(p^* = \min\{\frac{c}{\nu_4}, \overline{P}\}\right)$ which explains the saturated regime.

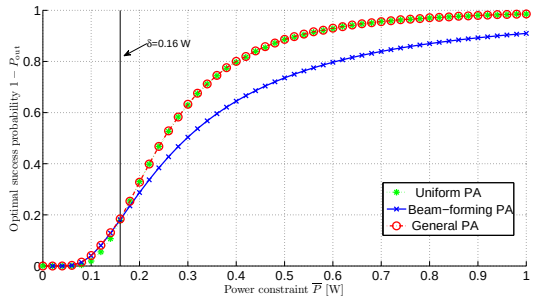

Fig. 5. Success probability vs. power constraint $\overline{P}$, comparison between beamforming PA, UPA and General PA for MIMO $n_t = n_r = 2$, $R = 1$ bpcu, $\rho = 3$ dB. We observe that Telatar's conjecture is validated. There is a threshold, $\delta = 0.16$ W, below which ($\overline{P} \leq \delta$) the beamforming PA is optimal and above it, UPA is optimal.

To this end we rewrite G_{static} as

$$G_{\text{static}}(\mathbf{U}\ \mathbf{Diag}(p_1, \ldots, p_{n_t})\ \mathbf{U}^H) = \frac{\log_2 \left| \mathbf{I}_{n_r} + \sum_{i=1}^{n_t} p_i \underline{g}_i \underline{g}_i^H \right|}{\sum_{i=1}^{n_t} p_i}, \tag{27}$$

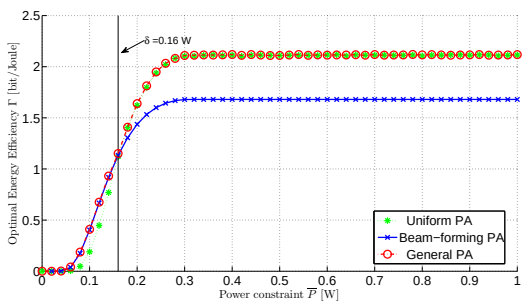

Fig. 6. Optimal energy-efficiency vs. power constraint $\overline{P}$, comparison between beamforming PA, UPA and General PA for MIMO $n_t = n_r = 2$, $R = 1$ bpcu, $\rho = 3$ dB. We observe that our Conjecture 4.2 is validated. For the exact same $\delta = 0.16$ W, we have that for $\overline{P} \leq \delta$ the beamforming PA structure optimal and above it, UPA structure is optimal.

where $\underline{g}_i$ represents the ith column of the $n_r \times n_t$ matrix $\mathbf{G} = \sqrt{\rho}\mathbf{HU}$ and proceed by induction on $n_t \geq 1$.

First, we introduce an auxiliary quantity (whose role will be made clear a little further)

$$\begin{aligned} E^{(n_t)}(p_1, \ldots, p_{n_t}) \triangleq\ & \text{Tr}\left(\mathbf{I}_{n_r} + \sum_{i=1}^{n_t} p_i \underline{g}_i \underline{g}_i^H\right)^{-1} \left(\sum_{i=1}^{n_t} p_i \underline{g}_i \underline{g}_i^H\right) \\ & - \log_2 \left| \mathbf{I}_{n_r} + \sum_{i=1}^{n_r} p_i \underline{g}_i \underline{g}_i^H \right|. \end{aligned} \tag{28}$$

and prove by induction that it is negative that is, $\forall (p_1, \ldots, p_{n_t}) \in \mathbb{R}_+^{n_t}$, $E^{(n_t)}(p_1, \ldots, p_{n_t}) \leq 0$.

For $n_t = 1$, we have $E^{(1)}(p_1) = \text{Tr}\left[(\mathbf{I}_{n_r} + p_1 \underline{g}_1 \underline{g}_1^H)^{-1} p_1 \underline{g}_1 \underline{g}_1^H\right] - \log_2 \left|\mathbf{I}_{n_r} + p_1 \underline{g}_1 \underline{g}_1^H\right|$. The first order derivative of $E^{(1)}(p_1)$ w.r.t. p_1 is:

$$\frac{\partial E^{(1)}}{\partial p_1} = -p_1 [\underline{g}_1^H (\mathbf{I}_{n_r} + p_1 \underline{g}_1 \underline{g}_1^H)^{-1} \underline{g}_1]^2 \leq 0 \tag{29}$$

and thus $E^{(1)}(p_1) \leq E^{(1)}(0) = 0$.

Now, we assume that $E^{(n_t-1)}(\underline{p}) \leq 0$ and want to prove that $E^{(n_t)}(\underline{p}, p_{n_t}) \leq 0$, where $\underline{p} = (p_1, \ldots, p_{n_t-1})$. It turns out that:

$$\frac{\partial E^{(n_t)}}{\partial p_{n_t}} = -\sum_{j=1}^{n_t} p_j \left| \underline{g}_j^H \left(\mathbf{I}_{n_r} + \sum_{i=1}^{n_t} p_i \underline{g}_i \underline{g}_i^H\right)^{-1} \underline{g}_{n_t} \right|^2 \leq 0, \tag{30}$$

and therefore $E^{(n_t)}(p_1, \ldots, p_{n_t-1}, p_{n_t}) \leq E^{(n_t)}(p_1, \ldots, p_{n_t-1}, 0) = E^{(n_t-1)}(p_1, \ldots, p_{n_t-1}) \leq 0$.

As a second step of the proof, we want to prove by induction on $n_t \geq 1$ that

$$\arg\max_{\underline{p}, p_{n_t}} G_{\text{static}}^{(n_t)}(\underline{p}, p_{n_t}) = \underline{0}. \tag{31}$$

For $n_t = 1$ we have $G_{\text{static}}^{(1)}(p_1) = \frac{\log_2 |\mathbf{I}_{n_r} + p_1 \underline{g}_1 \underline{g}_1^H|}{p_1} = \frac{\log_2(1 + p_1 \underline{g}_1^H \underline{g}_1)}{p_1}$ which reaches its maximum in $p_1 = 0$.

Now, we assume that $\arg\max_{\underline{p}} G_{\text{static}}^{(n_t-1)}(\underline{p}) = \underline{0}$ and want to prove that $\arg\max_{(\underline{p}, p_{n_t})} G_{\text{static}}^{(n_t)}(\underline{p}, p_{n_t}) = \underline{0}$.

Let $k = \arg\min_{i \in \{1, \ldots, n_t\}} \text{Tr}\left[\left(\mathbf{I}_{n_r} + \sum_{j=1}^{n_t} p_j \underline{g}_j \underline{g}_j^H\right)^{-1} \underline{g}_i \underline{g}_i^H\right]$.

By calculating the first order derivative of $G_{\text{static}}^{(n_t)}$ w.r.t. p_k one obtains that:

$$\frac{\partial G_{\text{static}}^{(n_t)}}{\partial p_k} = \frac{\mathcal{N}}{\left(\sum_{i=1}^{n_t} p_i\right)^2}, \tag{32}$$

with

$$\begin{aligned} \mathcal{N} =\ & \left(\sum_{i=1}^{n_t} p_i\right) \text{Tr}\left[\left(\mathbf{I}_{n_r} + \sum_{j=1}^{n_t} p_j \underline{g}_j \underline{g}_j^H\right)^{-1} \underline{g}_k \underline{g}_k^H\right] \\ & - \log_2 \left| \mathbf{I}_{n_r} + \sum_{i=1}^{n_t} p_i \underline{g}_i \underline{g}_i^H \right| \end{aligned} \tag{33}$$

and thus $\frac{\partial G_{\text{static}}^{(n_t)}}{\partial p_k} \leq \frac{E^{(n_t)}(p_1,\ldots,p_{n_t})}{\left(\sum_{i=1}^{n_t} p_i\right)^2} \leq 0$ and $p_k^* = 0$ for all $p_1,\ldots,p_{k-1},p_{k+1},\ldots,p_{n_t}$. We obtain that $F^{(n_t)}(p_1,\ldots,p_{k-1},0,p_{k+1},\ldots,p_{n_t})$ $= F^{(n_t-1)}(p_1,\ldots,p_{k-1},p_{k+1},\ldots,p_{n_t})$, which is maximized when $(p_1,\ldots,p_{k-1},p_{k+1},\ldots,p_{n_t}) = \underline{0}$ by assumption. We therefore have that $\mathbf{Q}^* = \mathbf{U0U}^H = \mathbf{0}$ is the solution that maximizes the function $G_{\text{static}}(\mathbf{Q})$. At last, to find the maximum reached by G_{static} one just needs to consider the the equivalent of the $\log_2\left|\mathbf{I}_{n_r} + \rho\mathbf{HQH}^H\right|$ around $\mathbf{Q} = \mathbf{0}$

$$\log_2\left|\mathbf{I}_{n_r} + \rho\mathbf{HQH}^H\right| \sim \frac{\rho}{n_t}\text{Tr}(\mathbf{HH}^H) \tag{34}$$

and takes $\mathbf{Q} = \frac{q}{n_t}\mathbf{I}_{n_t}$ with $q \to 0$.

Appendix B
Proof of Proposition 4.1

The proof has two parts. First, we start by proving that if the optimal solution is different than the uniform spatial power allocation $\mathbf{P}^* \neq \frac{p}{n_t}\mathbf{I}_{n_t}$ with $p \in \left[0,\overline{P}\right]$ then the solution is not trivial $\mathbf{P}^* \neq \mathbf{0}$. We proceed by reductio ad absurdum. We assume that the optimal solution is trivial $\mathbf{P}^* = \mathbf{0}$. This means that when fixing $(p_2,\ldots,p_{n_t}) = (0,\ldots,0)$ the optimal $p_1 \in [0,\overline{P}]$ that maximizes the energy-efficiency function is $p_1^* = 0$. The energy-efficiency function becomes:

$$\Gamma(\textbf{Diag}(p_1,0,\ldots,0),R) = R\frac{1-\Pr\left[\log_2(1+\rho p_1\|\underline{h}_1\|^2) < R\right]}{p_1} \tag{35}$$

where $\underline{h}_1$ represents the first column of the channel matrix $\mathbf{H}$. Knowing that the elements in $\underline{h}_1$ are i.i.d. $h_{1j} \sim \mathcal{CN}(0,1)$ for all $j \in \{1,\ldots,n_r\}$ we have that $|h_{1j}|^2 \sim \text{expon}(1)$. The random variable $\|\underline{h}_1\|^2 = \sum_{j=1}^{n_r}|h_{1j}|^2$ is the sum of n_r i.i.d. exponential random variables of parameter $\lambda = 1$ and thus follows an $2n_r$ chi-square distribution (or an n_r Erlang distribution) whose c.d.f. is known and given by $\varsigma(x) = 1 - \exp(-x)\sum_{k=0}^{n_r-1}\frac{x^k}{k!}$. We can explicitly calculate the outage probability and obtain the energy-efficiency function:

$$\Gamma(\textbf{Diag}(p_1,0,\ldots,0),R) = R\exp\left(-\frac{c}{p_1}\right)\sum_{k=0}^{n_r-1}\frac{c^k}{k!}\frac{1}{p_1^{k+1}} \tag{36}$$

where $c = \frac{2^R-1}{\rho} > 0$. It is easy to check that $\lim_{p_1\to 0}\Gamma(p_1,R) = 0$, $\lim_{p_1\to\infty}\Gamma(p_1,R) = 0$. By evaluating the first derivative w.r.t. p_1, it is easy to check that the maximum is achieved for $p_1^* = \frac{c}{\nu_{n_r}} \geq 0$ where ν_{n_r} is the unique positive solution of the following equation (in y):

$$\frac{1}{(n_r-1)!}y^{n_r} - \sum_{k=0}^{n_r-1}\frac{y^k}{k!} = 0. \tag{37}$$

Considering the power constraint the optimal transmission power is $p_1^* = \min\{\frac{2^R-1}{\nu_{n_r}\rho},\overline{P}\}$, which contradicts the hypothesis and thus if the optimal solution is different than the uniform spatial power allocation then the solution is not trivial $\mathbf{P}^* \neq \mathbf{0}$.

Appendix C
Proof Proposition 4.4

Let $\underline{p}^T = (p_1,...,p_{n_t})$ be the vector of powers allocated to the different antennas $i \in \{1,...,n_t\}$ and thus $\mathbf{D} = \textbf{Diag}(\underline{p})$. Define the two sets: $\mathcal{C}(x) = \left\{\underline{p} \geq 0, \sum_{i=1}^{n_t} p_i \leq x\right\}$ and $\Delta(x) = \left\{\underline{p} \geq 0, \sum_{i=1}^{n_t} p_i = x\right\}$. Using these notations, they key observation to be made is the following:

$$\begin{aligned}\sup_{\underline{p}\in\mathcal{C}(\overline{P})}\Gamma^{\text{MISO}}(\mathbf{D},R) &\overset{(a)}{=} R\sup_{\underline{p}\in\mathcal{C}(\overline{P})}\frac{1-P_{\text{out}}^{\text{MISO}}(\mathbf{D},R)}{\sum_{i=1}^{n_t}p_i}\\ &\overset{(b)}{=} R\sup_{x\in[0,\overline{P}]}\sup_{\underline{p}\in\Delta(x)}\frac{1-P_{\text{out}}^{\text{MISO}}(\mathbf{D},R)}{x}\\ &\overset{(c)}{=} R\sup_{x\in[0,\overline{P}]}\frac{g\left(\frac{c}{x}\right)}{x}\end{aligned} \tag{38}$$

where $P_{\text{out}}^{\text{MISO}} = \Pr\left[\log\left(1+\rho\sum_{i=1}^{n_t}p_i|h_i|^2\right) \leq R\right]$: (a) translates the definition of the GPR; (b) follows from the property $\sup\{A\cup B\} = \sup\{\sup\{A\},\sup\{B\}\}$ for two sets A and B, applied to our context; in (c) the function $g(z) = \left\{g_\ell(z),\ \text{if} z \in \left[\frac{c}{c_{\ell-1}},\frac{c}{c_\ell}\right)\right.$ is a piecewise continuous function where $g_\ell(z) = 1-\Pr\left[\frac{1}{\ell}\sum_{i=1}^{n_t}|h_i|^2 \leq z\right]$ for $z \in \left[\frac{c}{c_{\ell-1}},\frac{c}{c_\ell}\right)$ and $\ell \in \{1,\ldots,n_t\}$. The function $g(z)$ corresponds to the solution of the minimization problem of the outage probability [22].

Now, we study the function g_ℓ. By calculating the first order derivative of $\frac{1}{x}g_\ell\left(\frac{c}{x}\right)$ w.r.t. x we obtain:

$$\frac{\text{d}}{\text{d}x}\left\{\frac{1}{x}g_\ell\left(\frac{c}{x}\right)\right\} = \frac{\text{e}^{-\frac{\ell c}{x}}}{x^2}\left[\frac{1}{(\ell-1)!}\left(\frac{\ell c}{x}\right)^\ell - \sum_{j=0}^{\ell-1}\frac{1}{j!}\left(\frac{\ell c}{x}\right)^j\right]. \tag{39}$$

Thus the function $\frac{1}{x}g\left(\frac{c}{x}\right)$ is increasing for $x \in (0,x_\ell)$ and decreasing on $x \in (x_\ell,\infty)$. The maximum point is reached in $x_\ell = \frac{\ell c}{y_\ell}$ where y_ℓ is the unique positive solution of the equation $\phi_\ell(y) = 0$ where

$$\phi_\ell(y) = \frac{1}{(\ell-1)!}y^\ell - \sum_{i=0}^{\ell-1}\frac{1}{i!}y^i. \tag{40}$$

We have that $\phi(0) = -1 < 0$ and

$$\begin{aligned}\phi_\ell(\ell) &= \tfrac{1}{(\ell-1)!}\ell^\ell - \sum_{i=0}^{\ell-1}\frac{1}{i!}\ell^i\\ &= \sum_{i=0}^{\ell-1}\frac{\ell-i-1}{i!}\ell^i\\ &> 0.\end{aligned} \tag{41}$$

This implies that $y_\ell \leq \ell$ and thus $x_\ell \geq c$. Since $c_{n_t-1} \geq 1$ we also have $x_\ell \geq \frac{c}{c_{n_t-1}}$ for all $\ell \in \{1,\ldots,n_t-1\}$.

Therefore, all the functions $\frac{1}{x}g_\ell\left(\frac{c}{x}\right)$ are increasing on the intervals $\left(0,\frac{c}{c_{n_t-1}}\right)$. Moreover, on the interval $\left(\frac{c}{c_{n_t-1}},\infty\right)$,

they are increasing on $\left(\frac{c}{c_{n_t-1}}, x_\ell\right]$ and decreasing on $[x_\ell, \infty)$. Proposition 4.4 follows directly.

APPENDIX D
COUNTER-EXAMPLE, TISO

Consider the particular case where $n_t = 2$ and $n_r = 1$. From Proposition 4.4, it follows that for a power constraint $\overline{P} < \frac{c}{c_1}$ the beamforming power allocation policy maximizes the energy-efficiency and $\Gamma^{\text{TISO}}(\mathbf{Diag}(\overline{P}, 0), R) = \Gamma^{\text{TISO}}(\mathbf{Diag}(0, \overline{P}), R) > \Gamma^{\text{TISO}}\left(\mathbf{Diag}\left(\frac{\overline{P}}{2}, \frac{\overline{P}}{2}\right), R\right)$. The function $\Gamma^{\text{TISO}}(\mathbf{Diag}(p_1, p_2), R)$ with $(p_1, p_2) \in \mathcal{P}_2 \triangleq \{(p_1, p_2) \in \mathbb{R}_+^2 \mid p_1 + p_2 \leq \overline{P}\}$ denotes the energy-efficiency function. We want to prove that $\Gamma^{\text{TISO}}(\mathbf{Diag}(p_1, p_2), R)$ is not quasi-concave w.r.t. $(p_1, p_2) \in \mathcal{P}_2$. This amounts to finding a level $\gamma \geq 0$ such that the corresponding upper-level set $\mathcal{U}_\gamma = \left\{(p_1, p_2) \in \mathcal{P}_2 \mid \Gamma^{\text{TISO}}(\mathbf{Diag}(p_1, p_2), R) \geq \gamma\right\}$ is not a convex set (see [32] for a detailed analysis on quasi-concave functions). Consider an arbitrary $0 < q < \min\left\{\overline{P}, \frac{c}{c_1}\right\}$ such that $\Gamma^{\text{TISO}}(\mathbf{Diag}(q, 0), R) = \Gamma^{\text{TISO}}(\mathbf{Diag}(0, q), R) < \Gamma^{\text{TISO}}\left(\mathbf{Diag}\left(\frac{q}{2}, \frac{q}{2}\right), R\right)$. It turns out that all upper-level sets $\mathcal{U}_{\gamma_q}$ with $\gamma_q \triangleq \Gamma^{\text{TISO}}(\mathbf{Diag}(q, 0), R)$ are not convex sets. This follows directly from the fact that $(q, 0), (0, q) \in \mathcal{U}_{\gamma_q}$ but $\left(\frac{q}{2}, \frac{q}{2}\right) \notin \mathcal{U}_{\gamma_q}$ since $\Gamma^{\text{TISO}}\left(\mathbf{Diag}\left(\frac{q}{2}, \frac{q}{2}\right), R\right) < \gamma_q$.

APPENDIX E
EXTREME SNR CASES, GPR

In [22], the authors proved that in the low SNR regime the outage probability $P_{\text{out}}(\underline{p}, R)$ is Schur-concave w.r.t. $\underline{p}$. This means that for any vectors $\underline{p}$, $\underline{q}$ such that $\underline{p} \succ \underline{q}$ then $P_{\text{out}}(\underline{p}, R) \leq P_{\text{out}}(\underline{q}, R)$. The operator $\succ$ denotes the majorization operator which will be briefly described (see [29] for details). For any two vectors $\underline{p}, \underline{q} \in \mathbb{R}_+^{n_t}$, $\underline{p}$ majorizes $\underline{q}$ (denoted by $\underline{p} \succ \underline{q}$) if $\sum_{k=1}^{m} p_k \geq \sum_{k=1}^{m} q_k$, for all $m \in \{1, \ldots, n_t - 1\}$ and $\sum_{k=1}^{n_t} p_k = \sum_{k=1}^{n_t} q_k$. This operator induces only a partial ordering. The Schur-convexity and $\prec$ operator can be defined in an analogous way. Also, an important observation to be made is that the beamforming vector majorizes any other vector, whereas the uniform vector is majorized by any other vector (provided the sum of all elements of the vectors is equal). Otherwise stated, $x\underline{e}_1 \succ \underline{p} \succ \frac{x}{n_t}\underline{\mathbf{1}}$ for any vector $\underline{p}$ such that $\sum_{i=1}^{n_t} p_i = x$ and $\underline{\mathbf{1}} = (1, 1, \ldots, 1)$ and $\underline{e}_1 \in \mathcal{S}_1$.

It is straightforward to see that if $P_{\text{out}}(\mathbf{Diag}(\underline{p}), R)$ is Schur-concave w.r.t. $\underline{p}$ then $1 - P_{\text{out}}(\mathbf{Diag}(\underline{p}), R)$ is Schur-convex w.r.t. $\underline{p}$. Since the majorization operator implies the sum of all elements of the ordered vectors to be identical, $\Gamma(\mathbf{Diag}(\underline{p}), R) = \frac{1 - P_{\text{out}}(\mathbf{Diag}(\underline{p}), R)}{\sum_{i=1}^{n_t} p_i}$ will also be Schur-convex w.r.t. $\underline{p}$ and thus is maximized by a beamforming vector. Using the same notations as in Appendix C we obtain:

$$\begin{aligned}
&\sup_{\underline{p} \in \mathcal{C}(\overline{P})} \Gamma(\mathbf{Diag}(\underline{p}), R) \\
&= \sup_{x \in [0, \overline{P}]} \frac{1}{x} \sup_{\underline{p} \in \Delta(x)} [1 - P_{\text{out}}(\mathbf{Diag}(\underline{p}), R)] \\
&\overset{(a)}{=} \sup_{x \in [0, \overline{P}]} \frac{1}{x} [1 - \Pr[\log(1 + x\rho \underline{h}_1^H \underline{h}_1) \leq R], \\
&= \sup_{x \in [0, \overline{P}]} \frac{1}{x} \left\{1 - Pr\left[\frac{1}{n_r} \sum_{j=1}^{n_r} |h_{1j}|^2 \leq \frac{c}{n_r x}\right]\right\}, \\
&\overset{(b)}{=} \sup_{x \in [0, \overline{P}]} \frac{g_{n_r}\left(\frac{c}{n_r x}\right)}{x},
\end{aligned} \tag{42}$$

where (a) follows by considering beamforming power allocation policy on the first transmit antenna (with no generality loss) and replacing $\underline{p} = x\underline{e}_1$ with $\underline{e}_1 = (1, 0, \ldots, 0)$ and $\underline{h}_1$ denoting the first column of the channel matrix; in (c) we make use the definition in Appendix C for the function $\frac{1}{x} g_{n_r}\left(\frac{c}{n_r x}\right)$ which has a unique optimal point in $\min\left\{\frac{c}{y_{n_r}}, \overline{P}\right\}$, with y_{n_r} the unique solution of $\Phi_{n_r}(y) = 0$. Since $\sigma^2 \to 0$ then $c \to +\infty$ and thus the optimal power allocation is $\underline{p}^* = \overline{P}\underline{e}_1$.

Similarly, for the high SNR case we have:

$$\begin{aligned}
\sup_{\underline{p} \in \mathcal{C}(\overline{P})} \Gamma(\mathbf{Diag}(\underline{p}), R) &= \sup_{x \in [0, \overline{P}]} \frac{1}{x} \sup_{\underline{p} \in \Delta(x)} [1 - P_{\text{out}}(\mathbf{Diag}(\underline{p}), R)] \\
&= \sup_{x \in [0, \overline{P}]} \frac{1}{x} \left[1 - P_{\text{out}}\left(\mathbf{Diag}\left(\frac{x}{n_t}(1, \ldots, 1)\right), R\right)\right].
\end{aligned} \tag{43}$$

We have used the results in [22], where the UPA was proven to minimize the outage probability.

Let us now consider the limit of the energy-efficiency function when $p \to 0$, $\sigma^2 \to 0$ such that $\frac{p}{\sigma^2} \to \xi$ with ξ a positive finite constant. We obtain that $1 - P_{\text{out}}\left(\frac{x}{n_t}\mathbf{I}_{n_t}, R\right) \to \Pr\left[\left|\mathbf{I}_{n_r} + \frac{\xi}{n_t}\mathbf{H}\mathbf{H}^H\right|\right] > 0$ which implies directly that $\Gamma\left(\frac{x}{n_t}\mathbf{I}_{n_t}, R\right) \to +\infty$.

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Appendix C

Publication on Learning Algorithms

C.1 EURASIP-EUSIPCO-2010

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LEARNING DISTRIBUTED POWER ALLOCATION POLICIES IN MIMO CHANNELS

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ABSTRACT

In this paper[1], we study the discrete power allocation game for the fast fading multiple-input multiple-output multiple access channel. Each player or transmitter chooses its own transmit power policy from a certain finite set to optimize its individual transmission rate. First, we prove the existence of at least one pure strategy Nash equilibrium. Then, we investigate two learning algorithms that allow the players to converge to either one of the NE states or to the set of correlated equilibria. At last, we compare the performance of the considered discrete game with the continuous game in [7].

1. INTRODUCTION

Game theory appears to be a suitable framework to analyze self-optimizing wireless networks. The transmitters, based on their knowledge on the environment and cognitive capabilities, allocate their own resources to optimize their individual performance with very little or no intervention from a central authority.

Game theoretical tools have recently been used to study the power allocation problem in networks with multiple antenna terminals. In [1],[2],[3],[4],[5], the authors studies the MIMO slow fading interference channel, in [6] the MIMO cognitive radio channel, and in [7] the multiple access channel. The main drawback of these approaches is the fact that the action sets (or possible choices) of the transmitters are the convex cones of positive semi-definite matrices. In practice, this is an unrealistic assumption and discrete finite action sets should be considered. Another raising issue is related to the iterative water-filling type algorithms that converge to the games' Nash equilibria (NE) states. In order to apply these algorithms, the transmitters are assumed to be strictly rational players that perfectly know the structure of the game (at least their own payoff functions) and the strategies played by the others in the past.

An alternative way of explaining how the players may converge to an NE is the theory of learning [14]. Learning algorithms are long-run processes in which players, with very little knowledge and rationality constraints, try to optimize their benefits. In [8], the authors propose two stochastic learning algorithms that converge to the pure strategy NE and to mixed strategy NE of the energy efficiency game in a single-input single-output (SISO) interference channel. In [10], the multiple access point wireless network is investigated where a large number of users can learn the correlated equilibrium of the game. A similar scenario is studied in [12]. In [9], learning algorithms are proposed in a wireless network where users compete dynamically for the available spectrum. In [11], the authors study learning algorithms in cellular networks where the links are modeled as collision channels. An adaptive algorithm was proposed in [1] for the MIMO interference channel. The proposed algorithm allows the users to converge to a Stackelberg equilibrium by learning the ranks of their own covariance matrices that maximize the system sum-rate.

In this paper, we study the power allocation game in fast fading multiple-input multiple-output (MIMO) multiple access channels (MAC), similarly to [7]. We assume that the action sets of the transmitters are discrete finite sets and consist in uniformly spreading their powers over a subset of antennas. Assuming the single user decoding scheme at the receiver, we show that the proposed game is a potential one and the existence of a pure strategy Nash equilibrium (NE) follows directly. However, the uniqueness of the NE cannot be ensured in general and, thus, several iterative algorithms that converge to one of the NE states are studied. A best-response type algorithm is compared with a reinforcement learning algorithm in terms of system performance, required information, and cognitive capabilities of players. To improve the system performance, we consider a second learning algorithm based on regret matching that converges to the set of correlated equilibria (CE).

We begin our analysis by describing the system model in Sec. 2 and introducing some basic game theoretical concepts. Then, in Sec. 3, we analyze the Nash equilibria of the power allocation game. First, we review the setting of [7] in Subsec. 3.1 and then, study the discrete game in Subsec. 3.2. In Sec. 4, we study two learning algorithms: One that allows the users to converge to one of the NE (see Subsec. 4.1) and another that allows the users to converge to the set of CE (see Subsec. 4.2). We analyze the performance of the different scenarios via numerical simulations in Sec. 5 and conclude with several remarks in Sec. 6.

2. SYSTEM MODEL

We consider a multiple access channel (MAC) composed of an arbitrary number of mobile stations (MS) $K \geq 2$ and a single base station (BS). We further assume that each mobile station is equipped with n_t antennas whereas the base station has n_r antennas. We assume the fast fading model where the receiver has perfect knowledge of the channel matrices. The knowledge required at the transmitters depends on the different scenarios and will be defined accordingly. The equivalent

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baseband signal received at the base station is:

$$\underline{Y} = \sum_{k=1}^{K} \mathbf{H}_k \underline{X}_k + \underline{Z}, \tag{1}$$

where the time index has been ignored and $\underline{X}_k$ is the n_t-dimensional column vector of symbols transmitted by user k, $\mathbf{H}_k \in \mathbb{C}^{n_r \times n_t}$ is the channel matrix (stationary and ergodic process) of user k and $\underline{Z}$ is a n_r-dimensional complex white Gaussian noise distributed as $\mathcal{N}(\underline{0}, \sigma^2 \mathbf{I}_{n_r})$.

In order to take into account the antenna correlation effects at the transmitters and receiver, we will assume the different channel matrices to be structured according to the unitary-independent-unitary model introduced in [23], $\forall k \in \mathcal{K}$, $\mathbf{H}_k = \mathbf{V}_k \tilde{\mathbf{H}}_k \mathbf{W}_k$, where $\mathcal{K} = \{1, ..., K\}$, $\mathbf{V}_k$ and $\mathbf{W}_k$ are deterministic unitary matrices. Also $\tilde{\mathbf{H}}_k$ is an $n_r \times n_t$ matrix whose entries are zero-mean independent complex Gaussian random variables with an arbitrary profile of variances, such that $\mathbb{E}|\tilde{H}_k(i,j)|^2 = \frac{\sigma_k(i,j)}{n_t}$. Note that the Kronecker propagation model (where the channel matrices are of the form $\mathbf{H}_k = \mathbf{R}_k^{1/2} \tilde{\Theta}_k \mathbf{T}_k^{1/2}$) is a special case of the UIU model. The BS is assumed to use a simple single user decoding (SUD) technique. The achievable ergodic rate of user $k \in \mathcal{K}$ is given by:

$$u_k(\mathbf{Q}_k, \mathbf{Q}_{-k}) = \mathbb{E}[i_k(\mathbf{Q}_k, \mathbf{Q}_{-k})], \tag{2}$$

where $i_k(\mathbf{Q}_k, \mathbf{Q}_{-k})$ denotes the instantaneous mutual information

$$\begin{aligned} i_k(\mathbf{Q}_k, \mathbf{Q}_{-k}) \quad = \quad & \log_2 \left| \mathbf{I}_{n_r} + \rho \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H + \rho \sum_{\ell \neq k} \mathbf{H}_\ell \mathbf{Q}_\ell \mathbf{H}_\ell^H \right| - \\ & \log_2 \left| \mathbf{I}_{n_r} + \rho \sum_{\ell \neq k} \mathbf{H}_\ell \mathbf{Q}_\ell \mathbf{H}_\ell^H \right|. \end{aligned} \tag{3}$$

In this paper, we study the power allocation game where the players are autonomous non-cooperative devices that choose their power allocation policies, $\mathbf{Q}_k$, to maximize their own transmission rates, $u_k(\mathbf{Q}_k, \mathbf{Q}_{-k})$.

2.1 Non-Cooperative Game Framework

In what follows, we briefly define some basic game theoretical concepts (see e.g. [13] for details) and standard notations that will be used throughout the paper. A normal-form game is defined as the triplet $\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k \in \mathcal{K}}, \{u_k\}_{k \in \mathcal{K}})$ where $\mathcal{K}$ is the set of players (the K transmitters), $\mathcal{A}_k$ represents the set of actions (discrete or continuous) that player k can take (different power allocation policies), and $u_k : \mathcal{A} \to \mathbb{R}_+$ is the payoff function of user k that depends on his own choice but also the choices of the others (the ergodic achievable rate in (2)) where $\mathcal{A} = \times_{k \in \mathcal{K}} \mathcal{A}_k$ represents the overall action space. We denote by $\underline{a} \in \mathcal{A}$ a strategy profile and by a_{-k} the strategies of all the players except k.

The Nash equilibrium has been introduced in [15] and appears to be the natural solution in non-cooperative games. The mathematical definition of a pure-strategy NE is given by:

Definition 1 *A strategy profile $\underline{a}^* \in \mathcal{A}$ is a Nash equilibrium for the game $\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k \in \mathcal{K}}, \{u_k\}_{k \in \mathcal{K}})$ if for all $k \in \mathcal{K}$ and all $a_k \in \mathcal{A}_k$: $u_k(a_k^*, a_{-k}^*) \geq u_k(a_k, a_{-k}^*)$.*

This definition translates the fact that the NE is a stable state from which no user has any incentive to deviate unilaterally. A mixed strategy for user k is a probability distribution over its own action set $\mathcal{A}_k$. Let $\Delta(\mathcal{A}_k)$ denote the set of probability distributions over the set $\mathcal{A}_k$. The mixed NE is defined similarly to pure-strategy NE by replacing the pure strategies with the mixed strategies. The existence of NE has been proven in [15] for all discrete games. If the action spaces are discrete finite sets, then $\underline{p}_k \in \Delta(\mathcal{A}_k)$ denotes the probability vector such that $p_{k,j}$ represents the probability that user k chooses a certain action $a_k^{(j)} \in \mathcal{A}_k$ and $\sum_{a_k^{(j)} \in \mathcal{A}_k} p_{k,j} = 1$.

We also define the concept of correlated equilibrium [16] which can be viewed as the NE of a game where the players receive some private signaling or playing recommendation from a common referee or mediator. The mathematical definition is as follows:

Definition 2 *A joint probability distribution $\underline{q} \in \Delta(\mathcal{A})$ is a correlated equilibrium if for all $k \in \mathcal{K}$ and all $a_k^{(j)}, a_k^{(i)} \in \mathcal{A}_k$*

$$\sum_{\underline{a} \in \mathcal{A} : a_k = a_k^{(j)}} q_{\underline{a}} \left[u_k(a_k^{(j)}, a_{-k}) - u_k(a_k^{(i)}, a_{-k}) \right] \geq 0, \tag{4}$$

where $q_{\underline{a}}$ denotes the probability associated to the action profile $\underline{a} \in \mathcal{A}$.

At the CE, User k has no incentive in deviating from the mediator's recommandation to play $a_k^{(j)} \in \mathcal{A}_k$ knowing that all the other players follow as well the mediator's recommendation (a_{-k}). Notice that the set of mixed NE is included in the set of CE by considering independent p.d.f's. Similarly, the set of pure strategy NE is included in the set of mixed strategy NE by considering degenerate p.d.f.'s (i.e. $p_{k,j} \in \{0,1\}$) over the action sets of users.

3. NON-COOPERATIVE POWER ALLOCATION GAME

In this section, we analyse the NE of the power allocation game in fast fading MIMO MAC. First, we briefly review the case where the action sets of the users are continuous [7]. Then, we focus our attention on the practical case where the action sets of the users are discrete and finite. In this section, the players are assumed to be strictly rational transmit devices. Based on the available information, the transmitters choose the power allocation policy maximizing their own transmission rates. Furthermore, rationality is assumed to be common knowledge.

3.1 Compact and Convex Action Sets

We consider the same scenario as [7]. The transmitters are assumed to know only the statistics of the channels. The non-cooperative normal-form game is denoted by $\mathcal{G}_C = (\mathcal{K}, \{\mathcal{C}_k\}_{k \in \mathcal{K}}, \{u_k\}_{k \in \mathcal{K}})$. Each mobile station $k \in \mathcal{K}$ chooses its own input transmit covariance matrix $\mathbf{Q}_k \in \mathcal{C}_k$ to maximize its own achievable ergodic rate defined in (2). The action set of player $k \in \mathcal{K}$ is the convex cone of positive semi-definite matrices:
$\mathcal{C}_k = \left\{ \mathbf{Q}_k \in \mathbb{C}^{n_t \times n_t} | \mathbf{Q}_k \succeq 0, \mathrm{Tr}(\mathbf{Q}_k) \leq \overline{P}_k \right\}$. In [7], the authors proved the existence and uniqueness of NE using Theorems 1 and 2 in [17]. We provide here an alternative proof based on the notion of potential games [18].

Definition 3 *A normal form game* $\mathscr{G} = (\mathscr{K}, \{\mathscr{A}_k\}_{k\in\mathscr{K}}, \{u_k\}_{k\in\mathscr{K}})$ *is a* potential game *if there exists a potential function* $P : \mathscr{A} \to \mathbb{R}_+$ *such that, for all* $k \in \mathscr{K}$ *and every* $\underline{a}, \underline{b} \in \mathscr{A}$

$$u_k(a_k, \underline{a}_{-k}) - u_k(b_k, a_{-k}) = P(a_k, a_{-k}) - P(b_k, \underline{a}_{-k}). \quad (5)$$

Following [18], the local maxima of the potential function are the NE of the game. Thus, every potential game has at least one NE. For the game $\mathscr{G}_C$, the system achievable sum-rate:

$$R(\mathbf{Q}_1, \ldots, \mathbf{Q}_K) = \mathbb{E}\log_2 \left| \mathbf{I} + \rho \sum_{k=1}^{K} \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H \right|, \quad (6)$$

is a potential function. It can be checked that $R(\mathbf{Q})$ is strictly concave w.r.t. $(\mathbf{Q}_1, \ldots, \mathbf{Q}_K)$. Thus, it has a unique global maximizer which corresponds to the unique NE of the game. Furthermore, based on the finite improvement path (FIP) property [18], the iterative water-filling type algorithm in [7] converges to the unique NE. In [19], the author proves that for strict concave potential games, the CE is unique and consists in playing with one probability the unique pure NE. So the CE reduces to the unique NE of the game.

There are several drawbacks of this distributed power allocation framework: i) The action sets of users are assumed to be compact and convex sets (unrealistic in practical scenarios); ii) In order to implement the iterative water-filling algorithm, the transmitters need to know the global channel distribution information and to observe, at every iteration, the strategies chosen by the other players (very demanding in terms of information assumptions and signaling cost).

3.2 Finite Action Sets

Let us now consider the scenario where the action sets of users are discrete finite sets. The discrete game is very similar to $\mathscr{G}_C$ and is denoted by $\mathscr{G}_D = (\mathscr{K}, \{\mathscr{D}_k\}_{k\in\mathscr{K}}, \{u_k\}_{k\in\mathscr{K}})$. The action set of user k is a simple quantized version of $\mathscr{C}_k$:

$$\mathscr{D}_k = \left\{ \frac{\overline{P}_k}{\ell} \mathrm{Diag}(\underline{e}_\ell) \,\middle|\, \ell \in \{1, \ldots, n_t\}, \underline{e}_\ell \in \{0,1\}^{n_t}, \sum_{i=1}^{n_t} e_\ell(i) = \ell \right\}. \quad (7)$$

$\mathscr{D}_k$ represents the set of diagonal matrices that consists in allocating uniform power over only a subset of ℓ eigenmodes. Note that the discrete game $\mathscr{G}_D$ remains a potential game with the same potential function in (6). Thus, the existence of at least one pure NE is guaranteed. However, the uniqueness property of the NE is lost in general.

We consider hereunder two particular scenarios that illustrate the extreme cases where either all strategy profiles in $\mathscr{D} = \times_k \mathscr{D}_k$ are NE or where the NE is unique.

3.2.1 *Completely Correlated Antennas*

Let us assume the Kronecker model where the transmit antennas and receive antennas are completely correlated, i.e., for all k, $\mathbf{R}_k = \mathbf{J}_{n_r}$ and $\mathbf{T}_k = \mathbf{J}_{n_t}$. The matrix $\mathbf{J}_n$ is a $n \times n$ matrix with all entries equal to one. In this case, the potential function is constant and independent of the users' covariance matrices:

$$R(\mathbf{Q}_1, \ldots, \mathbf{Q}_K) = \mathbb{E}\log_2 \left| \mathbf{I}_{n_r} + \rho \overline{P} \sum_{k=1}^{K} \sum_{i=1}^{n_r} \sum_{j=1}^{n_t} |h_k(i,j)|^2 \mathbf{J}_{n_r} \right|. \quad (8)$$

This means that all the possible action profiles in $(\mathbf{Q}_1, \ldots, \mathbf{Q}_K) \in \mathscr{D}$ are potential maximizers and thus NE of $\mathscr{G}_D$.

3.2.2 *Independent Antennas*

Now, we consider the other extreme case where the antennas at the terminals are completely uncorrelated, i.e., for all k, $\mathbf{R}_k = \mathbf{I}_{n_r}$ and $\mathbf{T}_k = \mathbf{I}_{n_t}$. In other words, $\mathbf{H}_k$ is a random matrix with i.i.d. complex Gaussian entries. Let us recall that in the continuous setting derived in Subsec. 3.1, if $\mathbf{H}_k$ are i.i.d. matrices, then the NE policy for all users is spread their powers uniformly over all the antennas: $\forall k, \mathbf{Q}_k^{(\mathrm{UPA})} = \frac{\overline{P}_k}{n_t}\mathbf{I}_{n_t}$. In the continuous case, the potential function is strictly concave. Thus, for that any user k the strategy $\mathbf{Q}_k^{(\mathrm{UPA})}$ strictly dominates all the other strategies in $\mathscr{C}_k$. From the fact that $\mathscr{D}_k \subset \mathscr{C}_k$, the strategy $\mathbf{Q}_k^{(\mathrm{UPA})}$ strictly dominates all the other strategies in $\mathscr{D}_k$ also. In conclusion, the NE is unique and corresponds to the same solution as in the continuous game. Note that this is a very particular case and occurs only because the NE profile in the continuous case, $(\mathbf{Q}_1^{(\mathrm{UPA})}, \ldots, \mathbf{Q}_K^{(\mathrm{UPA})}) \in \mathscr{C} = \times_k \mathscr{C}_k$ happens to be also in the discrete set $\mathscr{D}$.

We see that, when quantizing the action sets of players, the uniqueness of the NE is no longer guaranteed. This raises an important issue when playing the one-shot game. There is a priori no explanation for users to expect the same equilibrium point. Because of this, their actions may not even correspond to an NE at all. A possible way to cope with this problem is to consider distributed iterative algorithms that converge to one of the NE points. Let us consider the iterative algorithm based on the best-response functions (similarly to [7]). Knowing that $\mathscr{G}_D$ is a potential game, by the FIP property, the users converge to one of the possible NE depending on the starting point. At each iteration, only one of the players updates his action by choosing its best action w.r.t. its own payoff. For exemple, at iteration t user k chooses $Q_k^{[t]} = \arg\max_{\mathbf{Q}_k \in \mathscr{D}_k} u_k\left(\mathbf{Q}_k, \mathbf{Q}_{-k}^{[t-1]}\right)$, while the other users don't do anything and $\mathbf{Q}_{-k}^{[t]} = \mathbf{Q}_{-k}^{[t-1]}$. Notice that user k is supposed to know the previous actions of the other players $\mathbf{Q}_k^{[t-1]}$. This involves a high amount of signaling between players. At the end of each iteration, the user that updated its choice needs to send it to all the other users. Furthermore, the users are assumed to be strictly rational and need to know the structure of the game and their own payoff in order to compute the best-response functions.

4. LEARNING ALGORITHMS

In this section, we discuss a different class of iterative algorithms that converge to the equilibrium points of the discrete game $\mathscr{G}_D$ described in Subsec. 3.2. As opposed to the best-response algorithm, the users are no longer rational devices but simple automata that know only their own action sets. They start at a completely naive state choosing randomly their action (following the uniform distribution over their own action sets for exemple). After the play, each users obtains a certain feedback from the nature (e.g., the realization of a random variable, the value of its own payoff). Based only on this value, each user applies a simple updating rule of its mixed strategy. It turns out that, in the long-run,

the updating rules converge to some desirable system states (NE, CE). Note that the rationality assumption is no longer needed. The transmitters don't even need to know the structure of the game or even that a game is played at all. The price to pay will be reflected in slower convergence time.

4.1 A Reinforcement Learning Algorithm

We consider a stochastic learning algorithm similar to [20]. Let us index the elements of $\mathscr{D}_K = \{\mathbf{D}_k^{(1)}, \ldots, \mathbf{D}_k^{(m_k)}\}$ with $m_k = \text{Card}(\mathscr{D}_k)$ (i.e., the cardinal of $\mathscr{D}_k$). At step $t > 0$ of the iterative process, User k randomly chooses a certain action $\mathbf{Q}_k^{[t]} \in \mathscr{D}_k$ based on the probability distribution $\underline{p}_k^{[t-1]}$ from the previous iteration. As a consequence, it obtains the realization of a random variable, which is, in our case, the normalized instantaneous mutual information $i_k^{[t]} = \frac{\tilde{i}_k\left(\mathbf{Q}_k^{[t]}, \mathbf{Q}_{-k}^{[t]}\right)}{I_{\max}} \in [0,1]$. Where $\tilde{i}_k(\cdot,\cdot)$ is a finite approximation of the mutual information $i_k(\cdot,\cdot)$ such that:

$$\tilde{i}_k(\cdot,\cdot) = \begin{cases} i_k(\cdot,\cdot)\text{ , if } i_k(\cdot,\cdot) \leq I_{\max} \\ I_{\max}\text{ , otherwise} \end{cases}, \tag{9}$$

where $I_{\max}$ is chosen such that the expectation of $\tilde{i}_k(\cdot,\cdot)$ approximates the expected mutual information and thus depends on the system's parameters (n_r, n_t, ρ). Based on this value, User k updates its own probability distribution as follows:

$$p_{k,j}^{[t]} = \begin{cases} p_{k,j}^{[t-1]} - b i_k^{[t]} p_{k,j}^{[t-1]}, & \text{if} \quad \mathbf{Q}_k^{[t]} \neq \mathbf{D}_k^{(j)}, \\ p_{k,j}^{[t-1]} + b i_k^{[t]} (1 - p_{k,j}^{[t-1]}), & \text{if} \quad \mathbf{Q}_k^{[t]} = \mathbf{D}_k^{(j)}, \end{cases} \tag{10}$$

where $0 < b < 1$ is a step size and $p_{k,j}^{[t]}$ represents the probability that user k choses $\mathbf{D}_k^{(j)}$ at iteration t. Using well known results in weak convergence of random processes [20], the sequence will converge, when $b \to 0$ to the solution of a deterministic ordinary differential equation (ODE). Similarly to [21], it can be checked that the potential function in (6) is a Lyapunov function for this ODE. This means that the stationary stable points of the ODE correspond to the maxima of the potential and, thus, to the pure strategy NE of $\mathscr{G}_D$. In conclusion, when $t \to +\infty$, the updating rule (10) converge to one of the pure strategy NE. This means that the users learn their own NE strategies knowing only the realization of their mutual information and using a simple updating rule.

4.2 Learning Correlated Equilibria

In general, the performance at the NE for discrete games depends on the quantized choice of the action sets of users. In order to improve the users' performance, we study a different learning algorithm which allows them to converge towards a correlated equilibrium.

We consider the modified regret matching algorithm introduced in [22] which allows the players to converge to the set of correlated equilibria. Each user needs only the knowledge of its own payoff values received over the time.

At iteration t, User k choses randomly an action $\mathbf{Q}_k^{[t]}$ following the distribution $\underline{p}_k^{[t-1]}$ and obtains the value of its payoff $u_k^{[t]} = u_k(\mathbf{Q}_k^{[t]}, \mathbf{Q}_{-k}^{[t]})$. Without loss of generality, assume $\mathbf{Q}_k^{[t-1]} = \mathbf{D}_k^{(j)}$. The play probabilities are updated as follows:

$$\begin{cases} p_{k,i}^{[t]} &= \left(1 - \frac{\delta}{t^\gamma}\right) \min\left\{\frac{1}{\mu} M_k^{[t-1]}(j,i), \frac{1}{m_k - 1}\right\} + \frac{\delta}{t^\gamma}\frac{1}{m_k}, \quad \text{for} \quad i \neq j, \\ p_{k,j}^{[t]} &= 1 - \sum_{i \neq j} p_{k,i}^{[t]}, \end{cases} \tag{11}$$

where $0 < \delta < 1$, $0 < \gamma < 1/4$, $\mu > 0$ a sufficiently large parameter that ensures the probabilities are well defined. We observe that User k needs to know not only $u_k^{[t]}$ but also all the past values of its payoff $\left\{u_k^{[\tau]}\right\}_{\tau < t}$. The basic idea is that if at time t a player plays action $\mathbf{D}_k^{(j)}$ then the probability that at time $t+1$ the player chooses a different action $\mathbf{D}_k^{(i)}$ is proportional to the regret for not having chosen action $\mathbf{D}_k^{(i)}$ instead of $\mathbf{D}_k^{(j)}$. The regret is measured as an approximation of the increase in average payoff (if any) resulting if User k had chosen action $\mathbf{D}_k^{(i)}$ in all the past when $\mathbf{D}_k^{(j)}$ was chosen and is denoted by $M_k^{[t]}(j,i)$:

$$M_k^{[t]}(j,i) = \left[\frac{1}{t} \sum_{\tau \leq t, \mathbf{Q}_k^{[\tau]} = \mathbf{D}_k^{(i)}} \frac{p_{k,j}^{[\tau]}}{p_{k,i}^{[\tau]}} u_k^{[\tau]} - \frac{1}{t} \sum_{\tau \leq t, \mathbf{Q}_k^{[\tau]} = \mathbf{D}_k^{(j)}} u_k^{[\tau]}\right]^+. \tag{12}$$

It turns out (see [22]) that the empirical distribution of play up to t denoted by $\underline{z}_t \in \Delta(\mathscr{D})$

$$z_t(\mathbf{Q}_1, \ldots, \mathbf{Q}_K) = \frac{1}{t}\text{Card}\{\tau \leq t : (\mathbf{Q}_1^{[\tau]}, \ldots, \mathbf{Q}_K^{[\tau]}) = (\mathbf{Q}_1, \ldots, \mathbf{Q}_K)\}, \tag{13}$$

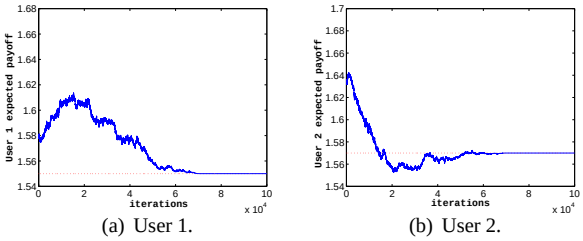
for all $(\mathbf{Q}_1, \ldots, \mathbf{Q}_K) \in \mathscr{D}$ converges almost surely as $t \to +\infty$ to the set of correlated equilibria.

There are several differences with the learning algorithm we discussed in Subsec. 4.1. Here, the learning process is no longer stochastic and the feedback each user gets at iteration t is the value of the deterministic payoff $u_k^{[t]} = u_k(\cdot,\cdot)$ instead of $i_k(\cdot,\cdot)$. The consequence is that the convergence is faster but the nature has to feedback not only the instantaneous mutual information but the ergodic achievable rate. Also, the updating rule for User k at iteration t depends on the whole history of received payoff values $\left\{u_k^{[\tau]}\right\}_{\tau \leq t}$ and not only on the current iteration $u_k^{[t]}$.

5. SIMULATION RESULTS

In what follows, we evaluate the gap between the results obtained at the equilibrium point of $\mathscr{G}_C$ in Subsec. 3.1 and $\mathscr{G}_D$ in Subsec. 3.2. We also analyze the performance of the two learning algorithms. We consider the following scenario: Two users ($K = 2$), $n_r = n_t = 2$, the Kronecker channel model where the transmit and receive correlation follow the exponential profile (i.e. $\mathbf{R}_k(i,j) = r_k^{|i-j|}$ and $\mathbf{T}_k = t_k^{|i-j|}$) characterized by the coefficients $r_1 = 0.7$, $r_2 = 0.5$, $t_1 = 0.2$, $t_2 = 0.4$, and $\sigma^2 = 1$ W.

First, we consider the discrete game in Subsec. 3.2. In Fig. 1, we plot the expected payoff depending on the probability distribution over the action sets at every iteration for User 1 in Fig. 1(a) and for User 2 in Fig. 1(b) assuming $\overline{P}_1 = \overline{P}_2 = 5$ W. We assume here that the stochastic reinforcement algorithm in Subsec. 4.1 is applied by both users in



(a) User 1. (b) User 2.

Figure 1: Expected payoff vs. iteration number for $K = 2$ users.

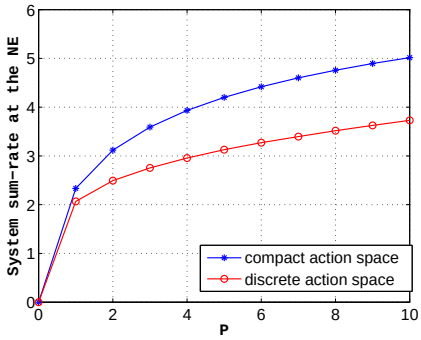


Figure 2: The achievable sum-rate at the NE. Compact action sets game vs. discrete action sets game. There is an optimality loss due to the quantization of the users' action sets.

order to learn their NE strategies. We observe that the users converge after approximately $8 \cdot 10^4$ iterations. By using a based response algorithm the convergence is almost instantaneous (only 2 or 3 iterations). However, the rationality assumption and perfect knowledge of the game structure for each player are required.

At last, we compare the performance of the overall system in terms of achievable sum-rate of the two games discussed in Sec. 3 as function of $P \in \{0, \ldots, 10\}$ W, assuming $\overline{P}_1 = \overline{P}_2 = P$. In Fig. 2, we plot the achievable sum-rate obtained at the NE with the iterative water-filling type algorithm proposed in [7] for $\mathscr{G}_C$. Also, we plot the achievable sum-rate obtained at the NE point of $\mathscr{G}_D$ to which the users applying the learning algorithm in Subsec. 4.1 converge. We observe that there is a performance loss due to the quantization of the action sets of users. The discrete action sets $\mathscr{D}_k$ can be further refined and the results of the algorithms improved. However this will result in a higher complexity and computational costs.

6. CONCLUSIONS

We study the discrete non-cooperative power allocation game in MIMO MAC systems. In the long-run, the transmitters can learn their optimal subset of active antennas. The players are not assumed to be rational but automata which apply simple updating rules on the p.d.f.'s over their possible power allocation policies. We evaluate the performance gap between the convergence NE state of the learning procedure and the NE of the analogous game with rational players and assuming compact and convex action sets.

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Appendix D

Résumé

D.1 Introduction

Le présent manuscrit est axé sur les problèmes d'allocation des ressources dans les réseaux sans fil MIMO. Ces problèmes émergent dans un contexte de communication dans lequel les réseaux intelligents sont capables de fonctionner avec un régime à haut rendement énergétique.

D.1.1 Contexte et Motivation

Bien que la théorie des jeux et la théorie de l'information ont été largement développées durant les dernières soixante années, à commencer par les contributions séminales de J. von Neumann et O. Morgenstern [1], J. Nash [2] et CE Shannon [3], ce n'est que récemment que les connexions et les interactions entre ces deux théories ont commencé être mises en évidence et exploitées à grande échelle. Toutefois, la première application des outils de la théorie des jeux aux communications fiables remonte à la thèse de doctorat de Mandelbrot [4], en 1952, et plus tard dans [5] et [6] où la communication entre un émetteur et le récepteur est modélisée par un jeu à deux joueurs et à somme nulle avec une fonction de paiement donnée par l'information mutuelle. L'émetteur joue contre une nature malveillante qui choisit la pire distribution du canal au sens de l'information mutuelle. Il s'avère que la solution de ce jeu non-coopératif est identique à la capacité maximin dans le pire des cas en supposant que l'émetteur n'a pas de connaissances sur les paramètres du canal (statistiques du bruit et des gains du canal).

Le récent regain d'intérêt dans l'application des outils de la théorie des jeux pour les communications est dû au développement des communications sans fil. Dans ce contexte, les multiples dispositifs, émetteurs et récepteurs, partagent le même environnement de communication. Ainsi, une compétition pour les ressources publiques (tels que les bandes de fréquences, les intervalles de temps, l'espace, la puissance d'émission ou l'énergie) apparaît naturellement. Ces ressources peuvent être, à priori, optimisées par une autorité centrale. Toutefois, l'approche centralisée présente plusieurs inconvénients: i) elle n'est pas réaliste dans un environnement partagé par de multiples prestataires de services ou opérateurs; ii) le problème d'optimisation conjoint par rapport à tous les paramètres du réseau est généralement un problème d'optimisation non-convexe et très complexe, impliquant des coûts de calcul élevés; iii) elle n'est pas échelonnable, c'est à dire, un léger changement dans la topologie du réseau peut conduire à un problème d'optimisation très complexe, voire intraitable, iv) elle implique un coût important en terme de signalisation, si le propriétaire du réseau doit envoyer les politiques optimales d'allocation de ressources à chaque utilisateur du réseau; v) la solution centralisée n'est pas nécessairement équitable par rapport à la qualité de service fournie à ses utilisateurs; vi) dans le contexte des utilisateurs autonomes et rationnels, lorsque la qualité de service d'un utilisateur n'est pas satisfaisante, l'utilisateur peut refuser la politique d'allocation centralisée, ce qui peut altérer le fonctionnement du réseau. Pour ces raisons, une solution distribuée peut être souhaitable, bien que la solution centralisée est généralement préférable du point de vue de la performance globale du réseau. Dans les environnements distribués, la compétition pour les ressources donne lieu à des situations interactives. La

théorie des jeux fournit un cadre mathématique approprié à l'étude des tels situations.

Dans ce contexte, l'un de nos principaux objectifs est d'étudier les jeux non-coopératifs d'allocation de ressources dans les réseaux de communication MIMO (avec plusieurs entrées et sorties) sans fil. La motivation du choix des canaux MIMO est double: i) les canaux MIMO peuvent être utilisés pour modéliser beaucoup de canaux de communication réalistes; ii) ces canaux offrent un cadre mathématique élégant (voir la notation compacte avec des matrices). Pour être plus précis, les agents ou joueurs sont les dispositifs autonomes d'émission. Ces dispositifs sont capables de détecter l'environnement et de décider leurs actions individuelles, à savoir, leur propre politique d'allocation de puissance. En supposant que ce sont des dispositifs rationnels et égoïstes, les actions choisies sont celles qui maximisent les benefices individuels, à savoir, leur débits de transmission de Shannon. Il y a beaucoup de raisons pour lesquelles ce type de paiement a été souvent considéré dans le littérature. Ici, nous ne citerons que les plus importantes. Tout d'abord, les taux de transmission de Shannon caractérisent les limites de performance d'un système de communication et nous permettent d'étudier des réseaux distribués où de bonnes méthodes de codage sont mises en œuvre. Deuxièmement, la relation directe entre le taux de transmission possible d'un utilisateur et son rapport signal-à interférence-plus-bruit (RSIB) nous permet d'optimiser des mesures de performances comme le RSIB ou les quantités relatives de même type (par exemple, le rapport porteuse-à-interférence). Troisièmement, du point de vue des mathématiques, les taux de Shannon ont de nombreuses propriétés intéressantes (par exemple, les propriétés de concavité), qui permettent d'effectuer des analyses complètes de performance. Par conséquent, elles fournissent des idées utiles et des concepts qui sont exploitables pour une conception pratique des réseaux décentralisés.

Toutefois, au cours de la dernière décennie, la consommation d'énergie est devenue un enjeu de plus en plus important dans les réseaux sans fil. Par exemple, dans les les réseaux cellulaires, les terminaux mobiles sont équipés d'écrans de taille relativement importante, exigés d'offrir des fonctionnalités de plus en plus complexes et aussi de fonctionner à des vitesses de transmission plus élevées pendant une plus longue durée. En outre, dans les réseaux de capteurs où le changement de batteries de dispositifs est très peu pratique ou, dans certains cas, voire impossible, la consommation d'énergie devient un enjeu crucial. Dans ces scénarios, l'optimisation des taux de Shannon sans tenir compte des coûts encourus n'est plus une métrique de performance convenable. Afin de tenir compte de la puissance consommée pour atteindre ces taux, une nouvelle mesure d'efficacité énergétique a été proposé dans la littérature [7] [8]. Cette métrique, que nous allons également étudier dans notre scénario de réseaux MIMO, mesure le nombre de bits qui peuvent être transmis d'une manière fiable à travers le canal par unité d'énergie consommée.

D'un point de vue pratique, le calcul des solutions des deux problèmes mentionnés ci-dessus, le problème d'allocation de puissance au sens du taux de Shannon ou de l'efficacité énergétique, implique souvent la mise en œuvre des algorithmes complexes au niveau de l'émetteur, une signalisation importante entre les émetteurs, l' hypothèse de la rationalité des émetteurs. Dans ce contexte, nous allons étudier les algorithmes d'apprentissage qui peuvent être utilisés pour modéliser la prise de décision adaptative

des dispositifs. Ce sont des algorithmes de faible complexité qui peuvent être mis en œuvre sans l'hypothèse de rationalité. Les dispositifs choisissent leur actions basées sur un simple feed-back de l'environnement (qui mesure la satisfaction de leur choix). Il s'avère que, dans le long terme, ils peuvent améliorer leurs performances tout en fonctionnant dans un environnement inconnu et converger vers des points de fonctionnement souhaitable des réseaux.

Le présent manuscrit est organisé en trois parties principales.

Nous commençons dans la section D.2 par l'analyse du problème d'allocation de puissance dans un cadre où de multiples utilisateurs cherchent à maximiser leurs propres débits de transmission (voir la section D.2). Ce problème est formulé comme un jeu non coopératif. L'existence et la multiplicité de la solution d'équilibre de Nash (EN) sera étudiée pour deux modèles de réseaux différents. Nous étudierons le canal à d'accès multiple (CAM) [9], [10], où plusieurs émetteurs envoient leurs messages vers un récepteur commun. Ensuite viendra l'étude du canal à interférence à relais (CIR) [11], [12], qui se compose d'un canal à interférence [13] (où plusieurs paires émetteur-récepteur coexistent dans le même environnement générant de l'interférence mutuelle) et plusieurs nœuds de relais qui peuvent être utilisés par les émetteurs afin d'améliorer la performance de leurs communications. Nous évaluerons également les performances obtenues aux points de fonctionnement EN via des simulations numériques et des algorithmes itératifs dits de *water-filling* ou de meilleure réponses.

Optimiser le débit atteignable de Shannon n'est pas toujours la meilleure politique, en particulier dans les réseaux où les terminaux sont équipés de batteries de capacité limitée. C'est pourquoi, dans la section D.3, nous étudierons une nouvelle métriques d'efficacité énergétique qui tient compte à la fois, du débit de transmission atteignable et de la puissance consommée pour atteindre cet débit. En raison des difficultés rencontrées, le problème d'allocation de puissance au sens de l'efficacité énergétique est étudiée uniquement pour le canal MIMO mono-utilisateur. Le scénario multi-utilisateur sera considéré comme une extension utile de ce travail. L'une des principales difficultés rencontrée réside dans le fait que l'optimisation de la probabilité de coupure pour les canaux MIMO à faible évanouissement est encore un problème ouvert [14].

Enfin, dans la section D.4, nous considérerons des algorithmes d'apprentissage. Ces algorithmes illustrent une autre façon pour les utilisateurs de converger vers certains points de fonctionnement souhaitable, comme l'équilibre de Nash des jeux non-coopératifs (Sec. D.2) et le point qui optimise la probabilité de coupure ou la métriques d'efficacité énergétique (Sec. D.3). Ces algorithmes peu complexes et adaptatifs ne nécessitent que peu de connaissance sur l'environnement et aucune hypothèse de rationalité.

Dans la section D.5, nous conclurons notre analyse par quelques remarques et questions ouvertes.

D.1.2 Élements de Théorie des Jeux Non-coopératifs

Nous passerons brièvement en revue ci-dessous quelques concepts de base de la théorie des jeux non-coopératifs qui seront utilisés tout au long de ce manuscrit. Par définition,

la théorie des jeux est le cadre mathématique dédié à l'étude des situations interactives entre des décideurs ou des agents autonomes. Nous allons considérer *l'hypothèse de la rationalité des joueurs* dans le sens où un joueur choisit sa meilleure stratégie pour maximiser son bénéfice [1]. La caractérisation mathématique d'un jeu sous forme stratégique est donné par la définition suivante.

Définition D.1.1 *Un jeu sous forme stratégique est un triplet* $\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{u_k\}_{k\in\mathcal{K}})$*, où* $\mathcal{K} = \{1, \ldots, K\}$ *représente* l'ensemble de joueurs*,* $\mathcal{A}_k$ *représente* l'ensemble des stratégies *ou des actions que le joueur* $k \in \mathcal{K}$ *peut prendre et* $u_k : \times_{\ell\in\mathcal{K}}\mathcal{A}_\ell \to \mathbb{R}$ *représente le bénéfice ou la fonction de paiement de l'utilisateur* k*, qui est une mesure de sa satisfaction.*

Dans le cas des jeux non-coopératifs, dans lequel les joueurs agissent de manière égoïste et indépendante, l'équilibre de Nash (EN), introduit en [2] représente un concept de solution du jeu. Le EN a été largement étudié dans les problèmes d'allocation des ressources parce que c'est un concept très important pour les concepteurs de réseaux. Il représente un point de fonctionnement qui est, à la fois, prévisible et robuste aux déviations unilatérales (ce qui est réaliste compte tenu du fait que les joueurs sont supposés être non-coopératifs et agir de manière isolée). Cela signifie qu'une fois que le système fonctionne dans cet état, aucun utilisateur n'a intérêt à changer de stratégie car il va perdre en terme de bénéfice. La définition mathématique du EN est comme suit:

Définition D.1.2 *Un profil de stratégies* $(\underline{a}_1^{\mathrm{NE}}, \ldots, \underline{a}_K^{\mathrm{NE}}) \in \times_{\ell\in\mathcal{K}}\mathcal{A}_\ell$ *est un équilibre de Nash si pour tout* $k \in \mathcal{K}$ *et pour tout* $\underline{a}'_k \in \mathcal{A}_k$ *nous avons* $u_k(\underline{a}_k^{\mathrm{NE}}, \underline{a}_{-k}^{\mathrm{NE}}) \geq u_k(\underline{a}'_k, \underline{a}_{-k}^{\mathrm{NE}})$*, où* $\underline{a}_{-k} = (\underline{a}_1, \ldots, \underline{a}_{k-1}, \underline{a}_{k+1}, \ldots, \underline{a}_K)$ *désigne l'ensemble des actions des autres joueurs.*

En fonction de la structure du jeu, des propriétés topologiques des ensembles de stratégies et des fonctions de paiement, les principales questions à résoudre sont les suivantes: i) l'existence d'au moins un EN; ii) la multiplicité des EN; iii) la conception d'algorithmes distribués qui permettent aux utilisateurs de converger vers un état EN en utilisant uniquement les connaissances locales de l'environnement; iv) de déterminer la performance du réseau dans les états d'équilibre. En ce qui concerne la conception d'algorithmes distribués, le EN possède une autre caractéristique séduisante. Comme nous le verrons dans la section D.4, il représente le résultat d'une simple adaptation itérative des actions, c'est à dire, des algorithmes d'apprentissage. Ce qui est remarquable, c'est que ces algorithmes itératifs nécessitent très peu de connaissances sur l'environnement. En particulier, ils ne nécessitent ni la connaissance de la structure du jeu ni même l'hypothèse de la rationalité des joueurs.

En général, la performance au EN n'est pas optimale par rapport à la performance globale du réseau, qui peut être mesurée par exemple par la somme de gains de chaque utilisateur $u(\underline{a}) = \sum_{k\in\mathcal{K}} u_k(\underline{a}_k, \underline{a}_{-k})$. En outre, ce n'est pas un état équitable à l'égard des performances individuelles des utilisateurs. Les points de fonctionnement qui nous intéressent sont donc les états dits Pareto-optimales. Un état du système est Pareto-optimale si aucun utilisateur ne préfère un état.

Définition D.1.3 *Soit $\underline{a}$ et $\underline{a}'$ deux profils de stratégies différents en $\times_{\ell\in\mathcal{K}}\mathcal{A}_\ell$. Ensuite, si*

$$\forall k \in \mathcal{K},\ \ u_k(\underline{a}_k, \underline{a}_{-k}) \geq u_k(\underline{a}'_k, \underline{a}'_{-k}), \tag{D.1}$$

avec l'inégalité stricte pour au moins un joueur, la stratégie $\underline{a}$ est Pareto-supérieure à $\underline{a}'$. S'il n'existe pas d'autre stratégies qui soient Pareto-supérieures à un profil de stratégies de $\underline{a}^{\mathrm{PO}}$, alors $\underline{a}^{\mathrm{PO}}$ est Pareto-optimale.

Toutefois, un état Pareto-optimale n'est pas nécessairement un état stable pour un environnement dans lequel multiples utilisateur non-coopératifs et égoïstes coexistent. En outre, la solution Pareto-optimale est plutôt une solution de type centralisé, car l'information globale sur les canaux du réseau est nécessaire pour calculer ces états.

Il existe différentes techniques qui peuvent être utilisées pour améliorer les performances du EN. En général, ces techniques impliquent l'intervention d'une autorité centralisée ou une sorte de coopération au niveau des utilisateurs. Il y a toujours un compromis entre les performances obtenues à l'état d'équilibre du réseau et le coût de signalisation que cela implique. Une étude plus détaillée des concepts théoriques des jeux non-coopératifs peut être trouvée dans la littérature spécialisée [15], [16]. Pour une analyse approfondie sur les méthodes d'analyse du EN dans les réseaux sans fil en général, le lecteur est renvoyé à [17] et [18].

Une classe importante de jeux sont les jeux de potentiel qui ont été introduits dans [19].

Définition D.1.4 *Un jeu sous forme stratégique $\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{u_k\}_{k\in\mathcal{K}})$ est un* jeu de potentiel exact *s'il existe une fonction de potentiel $V : \mathcal{A} \to \mathbb{R}_+$ de telle sorte que, pour tous les $k \in \mathcal{K}$ et tous les $\underline{a}, \underline{a}' \in \mathcal{A}$*

$$u_k(\underline{a}_k, \underline{a}_{-k}) - u_k(\underline{a}'_k, \underline{a}_{-k}) = V(\underline{a}_k, \underline{a}_{-k}) - V(\underline{a}'_k, \underline{a}_{-k}). \tag{D.2}$$

Cette définition traduit le fait que tous les utilisateurs ont les mêmes incitations à changer leurs actions, conduisant le système de l'état $\underline{a}$ à l'état $\underline{a}'$. Par exemple, les jeux de congestion ou les jeux de routage sont des exemples des jeux de potentiel. À la suite de [19], les maxima locaux de la fonction de potentiel sont des points de EN du jeu. Ainsi, le jeu de potentiel a au moins un EN en stratégies pures. En outre, dans les jeux finis, l'algorithme itératif basé sur les fonctions des meilleures réponse converge vers l'un des états EN (voir la propriété du chemin d'amélioration finie dans [19]) en fonction du point de depart.

Pour faire face aux questions de l'existence et l'unicité des équilibres de Nash, nous exploiterons souvent les propriétés des jeux concaves et les résultats de Rosen [20]. Ces résultats sont indiqués ci-dessous et sont valables pour le cas où les actions des joueurs sont des vecteurs.

Théorème D.1.5 *[20] Soit $\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{u_k\}_{k\in\mathcal{K}})$, un jeu sous forme stratégique. Si les trois conditions suivantes sont remplies: (i) chaque u_k est continue par rapport au profil de stratégies $\underline{a}_j \in \mathcal{A}_j, \forall j \in \mathcal{K}$; (ii) chaque u_k est concave en $\underline{a}_k \in \mathcal{A}_k$; (iii) $\mathcal{A}_1, ..., \mathcal{A}_K$ sont des ensembles compacts et convexes; alors $\mathcal{G}$ a au moins un EN.*

Théorème D.1.6 *[20] Considérons le jeu concave à K-joueurs du Théorème D.1.5. Si la condition suivante (la concavité diagonale stricte) est remplie: pour tous les $k \in \mathcal{K}$ et tous $(\underline{a}'_k, \underline{a}''_k) \in \mathcal{A}_k^2$ tel qu'il existe au moins un indice $j \in \mathcal{K}$ pour lequel $\underline{a}'_j \neq \underline{a}''_j$, $\sum_{k=1}^{K} (\underline{a}''_k - \underline{a}'_k)^T \left[\nabla_{\underline{a}_k} u_k(\underline{a}'_k, \underline{a}'_{-k}) - \nabla_{\underline{a}_k} u_k(\underline{a}''_k, \underline{a}''_{-k})\right] > 0$; alors l'unicité de la NE est assurée.*

Ces théorèmes se révélent particulièrement utiles dans les jeux non-coopératifs d'allocation de puissance au sens de l'efficacité du débit de transmission Shannon où les fonctions de paiement des utilisateurs sont généralement concaves par rapport à la puissance d'émission.

D'autres concepts de solution pour les jeux non coopératifs généralisent la notion de EN en stratégie pure: le EN en stratégie mixte et l'équilibre corrélé (EC). Une stratégie mixte pour l'utilisateur k est une distribution de probabilité sur l'ensemble des actions $\mathcal{A}_k$. Soit $\Delta(\mathcal{A}_k)$ désigne l'ensemble des distributions de probabilité sur l'ensemble $\mathcal{A}_k$. Le EN mixte est défini de façon similaire au EN en stratégies pures en remplaçant les stratégies pures avec les stratégies mixtes. L'existence du EN mixte a été prouvée dans [2] pour tous les jeux discrets. Si les espaces d'actions sont discrèts et finis, alors $\underline{p}_k \in \Delta(\mathcal{A}_k)$ désigne le vecteur de probabilité telle que $p_{k,j}$ représente la probabilité que l'utilisateur k choisit une certaine action $\underline{a}_k^{(j)} \in \mathcal{A}_k$ et $\sum_{\underline{a}_k^{(j)} \in \mathcal{A}_k} p_{k,j} = 1$.

Nous avons également défini le concept d'équilibre corrélé [21], qui peut être considéré comme le EN d'un jeu où les joueurs reçoivent certaines recommandations de la part d'un arbitre ou un d'un médiateur commun. La définition mathématique est la suivante:

Définition D.1.7 *Une distribution de probabilité conjointe $\underline{q} \in \Delta(\mathcal{A})$ est un équilibre corrélé si pour tout $k \in \mathcal{K}$ et tous les $\underline{a}_k^{(j)}, \underline{a}_k^{(i)} \in \mathcal{A}_k$*

$$\sum_{\underline{a} \in \mathcal{A} : \underline{a}_k = \underline{a}_k^{(j)}} q_{\underline{a}} \left[u_k(\underline{a}_k^{(j)}, \underline{a}_{-k}) - u_k(\underline{a}_k^{(i)}, \underline{a}_{-k})\right] \geq 0, \tag{D.3}$$

où $q_{\underline{a}}$ désigne la probabilité associée au profil d'action $\underline{a} \in \mathcal{A}$.

Lors du EC, l'utilisateur k a pas d'incitation à dévier de la recommandation du médiateur, $\underline{a}_k^{(j)} \in \mathcal{A}_k$, en sachant que tous les autres joueurs suivent ainsi la recommandation du médiateur $(\underline{a}_{-k})$. Notez que l'ensemble des EN mixtes est inclus dans l'ensemble des EC en considérant des distribution de probabilités indépendants. De même, l'ensemble des EN en stratégies pure est inclus dans l'ensemble des EN mixtes, en considérant des distribution de probabilités dégénérées (à savoir $p_{k,j} \in \{0, 1\}$) sur l'ensemble des stratégies pures.

D.1.3 Publications

Les travaux de recherche menés au cours des trois années de thèse ont conduit à plusieurs publications. Les articles sont classés ci-dessous en fonction des sujets connexes.

Jeux non-coopératifs d'allocation de puissance efficaces en terme de débit de transmission

Les contributions sur les jeux non-coopératifs d'allocation de puissance pour le ***canal à accès multiple MIMO à évanouissement rapide*** ont été publié dans quatre articles de journal, dont deux d'entre eux sont des revues de mathématique, et un papier conférence:

- E. V. Belmega, S. Lasaulce, M. Debbah, M. Jungers, and J. Dumont, "Power allocation games in wireless networks of multi-antenna terminals", *Springer Telecommunications Systems Journal*, DOI: 10.1007/s11235-010-9305-3, May 2010.

- E. V. Belmega, S. Lasaulce, and M. Debbah, "Power allocation games for MIMO multiple access channels with coordination", *IEEE Trans. on Wireless Communications*, vol. 8, no. 6, pp. 3182–3192, Jun. 2009.

- E. V. Belmega, M. Jungers, and S. Lasaulce, "A generalization of a trace inequality for positive definite matrices", *The Australian Journal of Mathematical Analysis and Applications (AJMAA)*, to appear, 2010.

- E. V. Belmega, S. Lasaulce, and M. Debbah, "A trace inequality for positive definite matrices", *Journal of Inequalities in Pure and Applied Mathematics (JIPAM)*, vol. 10, no. 1, pp. 1-4, 2009.

- E. V. Belmega, S. Lasaulce, and M. Debbah "Power Control in Distributed Multiple Access Channels with Coordination", *IEEE/ACM Proc. of the Intl. Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks and Workshops (WIOPT)*, Berlin, Germany, 1–8 Apr. 2008.

L'étude du problème de l'allocation de puissance pour le ***canal parallèle à accès multiples statique***, formulée comme un jeu de routage non-coopératif, a débouché sur deux papiers de conférence:

- S. Medina Perlaza, E. V. Belmega, S. Lasaulce, and M. Debbah, "On the base station selection and base station sharing in self-configuring networks", *International Conference on Performance Evaluation Methodologies and Tools (VALUETOOLS)*, Pisa, Italy, invited paper, Oct. 2009.

- E. V. Belmega and S. Lasaulce, "Information theoretic congestion games in heterogeneous wireless networks", *Game Theory for Analysis and Optimization of Computer Systems (GAMECOMP)*, Grenoble, France, invited talk, May 2008.

- E. V. Belmega, S. Lasaulce, and M. Debbah, "Decentralized handovers in cellular networks with cognitive terminals", *in the IEEE Proc. of the 3rd International Symposium on Communications, Control and Signal Processing (ISCCSP)*, St Julians, Malta, invited paper, 12–14 Mar. 2008.

Le jeu non-coopératif d'allocation de puissance pour le ***canal parallèle à interférence à relais*** a été étudié dans un papier journal, qui a été révisé, et quatre papiers de conférence:

- E. V. Belmega, B. Djeumou, and S. Lasaulce, "Power allocation games in interference relay channels: Existence analysis of Nash equilibria", *EURASIP Journal on Wireless Communications and Networking (JWCN)*, **accepted for publication**, Nov. 2010.
- E. V. Belmega, B. Djeumou, and S. Lasaulce "Resource allocation games in interference relay channels", *IEEE Intl. Conference on Game Theory for Networks (Gamenets)*, Istanbul, Turkey, invited paper, May 2009.
- E. V. Belmega, B. Djeumou, and S. Lasaulce "What happens when cognitive terminals compete for a relay node?", *IEEE Intl. Conference on Acoustics, Speech and Signal Processing (ICASSP)*, Taipei, Taiwan, 1–4 Apr. 2009.
- E. V. Belmega, B. Djeumou, and S. Lasaulce "Jeux d'allocation de puissance pour les canaux à interférence à relais", *GRETSI*, Dijon, France, Sep. 2009.
- B. Djeumou, E. V. Belmega, and S. Lasaulce, "Régions de taux atteignables pour le canal à interférence à relais", *GRETSI*, Dijon, France, Sep. 2009.

L'état de l'art par rapport à ce sujet sera publiée dans le chapitre du livre qui suit:

- E. V. Belmega, S. Lasaulce, and M. Debbah, "Shannon rate efficient power allocation games", *Game Theory for Wireless Communications and Networking*, Auerbach Publications, Taylor and Francis Group, CRC Press, **accepted for publication**, 2009.

Communication efficace en terme de consommation d'énergie

Le problème de l'allocation de puissance dans le sens de la maximisation de l'efficacité énergétique (i.e., le nombre de bits qui peut être transmis d'une manière fiable à travers le canal par unité d'énergie consommée) a été étudié dans le cadre du ***canal MIMO mono-utilisateur***. Les principaux résultats sont présentés dans un article de journal, et trois articles de conférence:

- E. V. Belmega, and S. Lasaulce, "Energy-efficient precoding for multiple-antenna terminals", *IEEE Trans. on Signal Processing*, **accepted for publication**, Sep. 2010.
- E. V. Belmega, S. Lasaulce, and M. Debbah, "A survey on energy-efficient communications", *IEEE Intl. Symp. on Personal, Indoor and Mobile Radio Communications (PIMRC 2010)*, Istanbul, Turkey, Sep. 2010.
- E. V. Belmega, and S. Lasaulce, "An information-theoretic look at MIMO energy-efficient communications", *International Conference on Performance Evaluation Methodologies and Tools (VALUETOOLS)*, Pisa, Italy, Oct. 2009.

- E. V. Belmega, S. Lasaulce, M. Debbah, and A. Hjørungnes "A new energy efficiency function for quasi-static MIMO channels", *International Wireless Communications and Mobile Computing Conference (IWCMC)*, Leipzig, Germany, invited paper, Jun. 2009.

Les algorithmes d'apprentissage dans les jeux d'allocation de ressources

L'étude des algorithmes d'apprentissage, qui permettent aux émetteurs de converger vers des états souhaitables du réseau avec peu de connaissances sur l'environnement de communication et aucune hypothèses de rationalité, a donné lieu à deux papiers de conférence:

- E. V. Belmega, H. Tembine, and S. Lasaulce, "Learning to precode in outage minimization games over MIMO interference channels", *IEEE Asilomar Conf. on Signals, Systems, and Computers*, Pacific Grove, CA, USA, invited paper, Nov. 2010.

- E. V. Belmega, S. Lasaulce, M. Debbah, and A. Hjørungnes, "Learning Distributed Power Allocation Policies in MIMO Channels", *European Signal Processing Conference (EUSIPCO)*, Aalborg, Denmark, invited paper, Aug. 2010.

Le présent manuscrit représente un guide sommaire des contributions principales publiées dans les articles précités. Les papiers les plus importants ont été ajouté dans les appendices. Ils seront utilisés comme références à des détails manquants d'analyse et de démonstrations mathématiques.

D'autres contributions qui ne seront pas abordées dans ce manuscrit ont été obtenues et/ou publiées dans deux articles de journal, un chapitre de livre et trois papiers de conférence:

- E. Altman, Y. Hayel, and E. V. Belmega, "Modelling competition between Hawks and Doves in a dynamic framework", *Special Issue on Evolutionary Games: Dynamic Games and Applications, Birkhauser-Springer*, **to be submitted**, Jan. 2011.

- E. V. Belmega, B. Djeumou, and S. Lasaulce, "Gaussian broadcast channels with an orthogonal and bidirectionnal cooperation link", *EURASIP J. on Wireless Communications and Networking (JWCN)*, pp.1–16, doi:10.1155/2008/341726, 2008.

- E. V. Belmega, S. Lasaulce, and M. Debbah, "Capacity of cooperative channels: three terminals case study", *Cooperative Wireless Communication, ISBN 142006469X*, Auerbach Publications, Taylor and Francis Group, CRC Press, Oct. 2008.

- P. Mertikopoulos, E. V. Belmega, A. L. Moustakas, and S. Lasaulce, "Dynamic Power Allocation Games in Parallel Multiple Access Channels", *Valuetools 2011, Paris, France*, **submitted, arXiv:1011.5610v1**, Nov. 2010.

- B. Djeumou, E. V. Belmega, and S. Lasaulce, "Recombinaison de signaux décodés et transférés pour le canal à relais à division fréquentielle", *Actes du GRETSI*, Troyes, France, 1–4 Sep. 2007.

- E. V. Belmega, B. Djeumou, and S. Lasaulce, "Performance analysis for the AF-based frequency division cooperative broadcast channel", *in the IEEE Proceedings of the Signal Processing Advances in Wireless Communications conference (SPAWC)*, Helsinki, Finland, 1–5 Jun. 2007.

D.2 Jeux Non-coopératifs d'Allocation de Puissance Efficaces en Terme de Débit de Transmission

Dans cette section, notre objectif est d'étudier des réseaux sans fil distribués ou autonomes. Dans de tels réseaux, les émetteurs sont en mesure de gérer leurs propres ressources avec peu ou, idéalement, aucune intervention de la part de l'autorité centrale. Les émetteurs sont supposés être rationnels, égoïstes et capables de choisir leur propre politique d'allocation de puissance pour maximiser la performance de leurs communications. La performance d'une communication est mesurée en termes de débit de transmission atteignable. L'interférence mutuelle crée par les émetteurs partageant le même environnement de communication induit une interaction entre les émetteurs. Cette interaction est étudiée en utilisant le cadre de la théorie des jeux non coopératifs. Sauf indication contraire, les composantes du jeu non-coopératif peuvent être identifiées comme suit. Les joueurs sont les émetteurs. Les fonctions de paiement sont les débits de transmission atteignables. Les actions des joueurs sont leurs stratégies de précodage.

Trois étapes peuvent être identifiées dans notre approche. Tout d'abord, nous étudions le jeu non-coopératif en un coup où les utilisateurs ont une connaissance parfaite de la structure du jeu. Nous étudions l'existence et la multiplicité de la solution d'équilibre de Nash. Plusieurs questions se posent. La connaissance parfaite de la structure du jeu au niveau des émetteurs est souvent une hypothèse irréaliste. En effet, en général, des expressions analytiques des points EN ne sont pas disponibles. En outre, si plusieurs états d'équilibre existent, il n'y a aucune raison de supposer que les émetteurs rationnels doivent prédire le même résultat du jeu. Dans cette situation, le réseau peut fonctionner dans un état qui ne correspond même pas à un état d'équilibre. Pour faire face à ces questions, une solution possible est de considérer des algorithmes itératifs. En conséquence, l'étape suivante de notre est d'étudier les algorithmes itératifs basés sur les meilleures réponses. Comme nous le verrons, ces algorithmes sont identiques aux algorithmes dits de *water-filling*. En plus, ils peuvent permettre aux utilisateurs de converger vers l'un des EN du jeu en un coup. Ces algorithmes sont distribués dans le sens qu'ils nécessitent moins d'informations sur la structure du jeu. Un autre inconvénient de la notion EN est le fait que c'est généralement un point de fonctionnement inefficace (par rapport à la performance globale du réseau, mais également par rapport aux performances individuelles des utilisateurs). Comme dernière étape, nous abordons aussi la formulation de Stackelberg [22] pour améliorer les performances du point EN. Cela

implique un certain coût en termes de signalisation centralisée de la part de l'autorité du système.

Les jeux non-coopératifs d'allocation de puissance efficaces en terme de débit de transmission sont étudiés pour deux modèles de base du réseau: les canaux à accès multiples (CAM) et les canaux à interférences à relais (CIR). Avant d'exposer nos contributions principales, nous allons discuter ci-dessous de plusieurs hypothèses importantes et de différences entre ces deux modèles de réseaux. En termes de temps de cohérence du canal, trois cas peuvent être distingués. Les gains de canal peuvent être: i) constants et déterministes, i.e., des liens statiques; ii) des variables aléatoires qui changent indépendamment à chaque utilisation du canal, i.e., liens à évanouissement rapide; iii) des variables aléatoires qui restent constantes pour toute la durée de la transmission, i.e. liens à faible évanouissement. Dans le troisième cas, les débits de Shannon atteignables sont strictement égaux à zéro [14] [23]. Ainsi, pour étudier le jeu d'allocation de puissance, il faut considérer une autre métrique pour mesurer la satisfaction des utilisateurs. Par exemple, comme nous le verrons dans la section D.3, on pourrait envisager une mesure de performance en fonction de la probabilité de coupure [24]. Cependant, ceci est hors de portée ici et ce cas ne sera pas considéré dans cette section. Pour le CAM, nous allons examiner brièvement le cas des liens statiques et ensuite nous allons nous concentrer sur le cas plus difficile de liens à évanouissement rapide. Du point de vue de la théorie de l'information, l'analyse du CIR est beaucoup plus difficile. C'est pourquoi, seul le cas des liens statiques sera pris en compte. Pour la même raison, le cas général des multiples dimensions (MIMO) ne sera étudié que pour le CAM. Pour le CIR, nous limiterons notre attention au cas des sous-canaux parallèles (ou orthogonaux). Une autre différence intrinsèque entre les deux modèles de réseaux est la technique de décodage. Pour le CAM, le récepteur doit décoder tous les messages des émetteurs et est donc censé connaître les alphabets utilisés par tous les émetteurs. Dans cette situation, deux techniques de décodage peuvent être utilisées: i) le décodage simultané des utilisateurs (DSU) (lorsque le récepteur décode le message d'un émetteur, les signaux des autres émetteurs sont considérés comme du bruit additif); ii) l'annulation d'interférence successive (AIS), à savoir, les message des émetteurs sont décodés séquentiellement (lors du décodage du message d'un émetteur, les messages décodés précédemment sont pris en compte pour réduire le niveau total de l'interférence reçue). Pour le CIR, chaque décodeur connaît que l'alphabet employé par son propre émetteur. Par conséquent, la technique de l'AIS, bien qu'attrayante en termes de taux de transmission, ne sera pas étudiée.

Dans ce qui quit, nous allons résumer nos contributions par rapport à ces sujets.

Pour *le CAM MIMO*, le détail des analyses et les preuves des résultats peuvent être trouvés en Appendice A.1, Appendice A.2, Appendice A.3, Appendice A.4 et Appendice A.5.

- Tout d'abord nous avons étendu les résultats de [20] au cas où les ensembles des actions des utilisateurs sont des ensembles de matrices au lieu des ensembles de vecteurs.
- À partir de cette base, nous avons étudié l'existence et l'unicité de l'état EN dans

le jeu en un coup. L'existence d'au moins un point de EN est garantie. Pour le cas des liens statiques, le EN n'est généralement pas unique et des conditions suffisantes sur les paramètres du canal assurant l'unicité ont été fournies. Pour le cas des liens à évanouissement rapide, l'EN est prouvé être unique.

- Pour le cas des liens à évanouissement rapide, déterminer le point EN n'est pas un problème trivial car les débits de transmission ergodiques n'ont pas d'expressions analytiques. Tout d'abord, nous avons déterminé les vecteurs propres optimaux des matrices de precodage. Ensuite, nous avons utilisé la théorie des matrices aléatoires pour approximer les taux ergodiques avec leurs équivalents déterministes. Enfin, nous proposons un algorithme itératif de type *water-filling* qui converge vers les valeurs propres optimales.

- Le jeu d'allocation de puissance a été analysé pour deux techniques de décodage différentes: DSU et AIS. Le décodage DSU est plus facile à mettre en œuvre, mais il s'avère inefficace par rapport au débit total du réseau. Pour le décodage AIS nous avons proposé un signal de coordination sous-optimal qui donne l'ordre de décodage au niveau du récepteur. Ce signal doit être connu au niveau du chaque émetteur et par conséquent, implique un certain coût de signalisation.

- Pour évaluer les performances du réseau au point de EN, nous avons introduit l'efficacité en terme de débit total du réseau. Cette mesure traduit l'écart entre le débit total atteignable au point du EN et la capacité-somme du CAM MIMO à évanouissement rapide. Dans les régimes extrêmes du RSB, en supposant que le décodeur utilise le décodage AIS, cet écart tend vers zéro. Pour RSB arbitraire, cette écart a été évaluée à travers les simulations numériques. Il s'avère que, en supposant que le décodage AIS, l'écart est très faible, même si le signal de coordination ne dépend pas de coefficients d'évanouissement. Comme nous l'avons prévu, décodage DSU est moins efficace. Un paradoxe intéressant du type Braess a été souligné. Si les utilisateurs sont obligés d'allouer leur puissances de manière uniforme dans le temps (et quel que soit le signal de la coordination) le débit total du réseau au point du EN sera plus grand que dans le cas général.

Pour *le CIR parallèle*, l'analyse complète se trouve dans l'Appendice A.6.

- Le jeu d'allocation de puissance a été étudié pour trois protocoles de transmission différents: Décoder-et-Transférer (DT), Estimer-et-Transférer (ET) et Amplifier-et-Transférer (AT).

- Des conditions suffisantes assurant l'existence d'au moins un EN ont été fournies pour DT et AT. En supposant que les nœuds de relais utilisent le protocole ET, l'existence du EN est toujours garantie.

- Basé sur une technique de partage du temps, l'existence du EN peut toujours être garantie indépendamment du protocole utilisé au niveau des relais. Toutefois, cela implique un certain niveau de coordination entre les émetteurs.

- L'analyse de la multiplicité des EN n'est pas triviale. Pour le cas particulier du protocole AT avec un gain d'amplification constant, l'analyse complète caractérisant le nombre de points EN en fonction des paramètres du canal a été fournie. En outre, sur la base du "duopole de Cournot", la convergence vers l'un des EN des algorithmes itératifs basés sur les meilleures réponses a été prouvée.

- Des simulations numériques ont été utilisées pour évaluer la performance du réseau au EN. Lorsque l'on compare les trois protocoles du relais en terme de débit total du réseau, des observations similaires à celles du canal à relais classique ont été faites: DT est optimal si le relais est situé à proximité des sources (très bon liens source-relais), tandis que ET est optimal si le relais se trouve à proximité des destinations (très bons liens relais-destination). Plusieurs formulations de Stackelberg, où le propriétaire du réseau choisit les paramètres du relais (i.e., la localisation spatiale, le gain d'amplification du AT, l'allocation de puissance pour les DT et ET) ont été évalués à l'aide de simulations numériques.

Plusieurs questions ouverts et extensions intéressantes sont indiqués ci-après:

- *Le CAM MIMO:* Une extension intéressante serait d'étudier le cas pour lequel les récepteurs ont une connaissance imparfaite des paramètres du canal. Une autre question intéressante et ouverte serait de déterminer les vecteurs propres optimaux et les valeurs propres des matrices de covariance lorsque la contrainte, $\mathbf{V}_k = \mathbf{V}$ pour tout $k \in \mathcal{K}$ est relâchée. En ce sens, les résultats récents de la théorie des matrices aléatoires en [59] peuvent être utilisés. Une question ouverte intéressante est de prouver mathématiquement la convergence des algorithmes itératifs proposés basés sur le *water-filling*.

- *Le CIR parallèle:* Une extension de ce travail serait d'examiner des protocoles de relayage et des techniques de codage-décodage plus efficaces tels que ceux de [60] et des ouvrages connexes. Il est également important de bien déterminer le nombre ou la topologie de l'ensemble des équilibres de Nash et d'étudier la convergence des algorithmes itératifs et distribués d'allocation de puissance. Nous avons également vu que des problèmes supplémentaires d'allocation de puissance entrent en jeu et doivent être pris en compte dans le problème général: le problème de l'allocation de puissance de transmission entre les différents sous-canaux au niveau des sources (AT, DT, ET), pour choisir le degré de coopération avec les relais au niveau des sources (DT), l'allocation de puissance entre les signaux de coopération au niveau des relais (ET et DT), la répartition la puissance d'émission en temps.

Dans cette section, nous avons considéré que les émetteurs allouent leurs puissances afin de maximiser leur taux de Shannon atteignables sans aucune considération à propose de la consommation d'énergie. La consommation d'énergie a un impact direct sur la durée de vie des batteries des dispositifs. Il existe des applications (e.g., les réseaux de capteurs) pour lesquels la durée de vie des batteries joue un rôle crucial et maximiser le débit de transmission n'est plus de première importance. Pour ce type d'applications,

une mesure de performance différente doit être considérée. Dans la section D.3, nous étudierons la métrique de l'efficacité énergétique mesurant le nombre de bits qui peuvent être transmis à travers le canal par unité d'énergie consommée.

D'autres questions importantes concernent les algorithmes itératifs de type *water-filling*. Tout d'abord, il s'agit de la rationalité des utilisateurs et la connaissance parfaite de leurs fonctions de paiement ou des meilleures réponse. En outre, ces algorithmes exige généralement la connaissance parfaite au niveau de l'émetteur de l'état du canal global (pour le cas statique) ou de son distribution (pour le cas évanouissement rapide). De plus, ces algorithmes nécessitent beaucoup de signalisation entre les utilisateurs. A chaque itération, l'utilisateur qui met à jour son choix, doit envoyer cette information à tous les autres utilisateurs. Toutes ces hypothèses peuvent être considérées comme étant irréalistes dans de nombreuses applications. Une solution possible est fournie par la théorie de l'apprentissage dans les jeux qui sera étudiée dans la section D.4.

D.3 Communications Efficaces en Terme de Consommation d'Énergie

Dans la section précédente, la performance de la communication a été mesurée en terme de débit de transmission atteignable. Le coût de la communication, à savoir, la puissance d'émission consommée pour atteindre les débits de transmission correspondants n'a pas été prise en compte. Dans les réseaux cellulaires ou les réseaux de capteurs, dans lesquels les terminaux mobiles ou les capteurs sont équipés de batteries ayant une capacité limitée, il est important d'optimiser la durée de vie de la batterie que d'optimisation des taux de transmission. Par conséquent, ce coût doit être pris en considération. Dans cette section, nous allons considérer une métrique de performance différente qui appartient à la théorie de l'information: l'efficacité énergétique. Cette mesure est définie comme le rapport entre le bénéfice de la transmission (par exemple, le nombre de bits transmis à travers le canal) et le coût de la transmission (i.e., la puissance d'émission).

La recherche sur ce thème a été axée sur deux grandes approches: une approche pragmatique basée sur des modulations pratiques, systèmes de codage-décodage, de l'électronique (voir [8], [61], [62], [63]), et une approche de la théorie d'information basée sur la capacité par unité de coût, notion introduite dans [7]. Une discussion détaillée et pertinente sur l'état de l'art des deux approches est présentée dans [64] (voir l'Appendice B.1). La plupart de cette recherche est centré sur des réseaux composés des dispositifs équipés d'une antenne unique. Il est bien connu que, pour une communication point à point, l'utilisation d'antennes multiples au niveau des terminaux [65] [66] [14] dans le mode de fonctionnement qui maximise la diversité (i.e., toutes les antennes d'émission sont utilisés pour envoyer les mêmes informations sur le canal) permet de diminuer la puissance de transmission tout en assurant une qualité de transmission constante (par exemple, le taux d'erreur binaire). Dans ce qui suit, nous nous concentrerons sur le problème de l'allocation de puissance d'un point de vue de la théorie de l'information dans les canaux MIMO comme dans la setion précédente. En outre, seule la puissance

d'émission à la sortie des circuits RF (ou la puissance de transmission fiable des données) est considérée. Même si cette hypothèse est irréaliste, elle nous permet de caractériser la limite supérieure des performances qui peuvent être atteintes dans la pratique.

Notre objectif initial était d'étudier le jeu non-coopératif d'allocation de puissqnce pour le canal MIMO à accès multiples, tel que décrit dans la section précédente. Dans notre cas, les joueurs, les émetteurs, choisissent leur meilleure matrice de covariance afin de maximiser leur fonction d'efficacité énergétique au lieu du débit de transmission Shannon. Toutefois, le problème s'est avéré être très difficile. Pour cette raison, nous nous sommes limités seulement à étude du cas particulier du canal MIMO mono-utilisateur (voir l'Appendice B.2). Notez que sous cette hypothèse, le jeu est réduit à un problème d'optimisation qui se révélera être généralement insolluble.

Ce problème est discuté en détail dans l'Appendice B.2. Pour les cas des liens statiques et liens à évanouissement rapide, la solution se révèle être triviale. Afin d'être économes en énergie, l'émetteur envoie des données avec une puissance de transmission très faible, ce qui implique aussi que les débits atteignables sont eux aussi très faibles. Pour les liens à faible évanouissement, le problème d'optimisation est plus difficile et la solution s'est avérée être non-triviale en général. Nos contributions sont les suivantes:

- Nous conjecturons la solution du problème général et nous donnons la solution dans des cas particuliers: dans les régimes extrêmes de RSB; dans les cas particuliers où l'un des dispositifs (le récepteur ou l'émetteur) est équipé d'une seule antenne (canaux MISO et SIMO).

- Un cas particulier intéressant est le cas de l'allocation uniforme de puissance. Dans ce cas, la fonction de l'efficacité énergétique est conjecturée être quasi-concave par rapport à la puissance d'émission. Une preuve rigoureuse est présentée en utilisant l'hypothèse de grands systèmes et la théorie des matrices aléatoires.

Pour le cas des liens statiques et à évanouissement rapide, la fonction de l'efficacité énergétique ne semble pas être une métrique de performance appropriée. La solution triviale qui maximise la fonction de l'efficacité énergétique peut être expliquée par le fait que la consommation d'énergie des circuits électriques n'est pas prise en compte. Une extension importante de ce travail serait de considérer la puissance des circuits. Dans ce cas, si l'émetteur ne transmet pas, le coût en terme de puissance consommée ne sera pas nul et la solution triviale ne sera plus optimale. De plus, avoir plusieurs antennes au niveau des dispositifs peut se révéler sous-optimal par rapport au cas avec une seule antenne. Une autre façon d'éviter la solution triviale consiste à considérer une contrainte de QoS minimale (par exemple, du débit de transmission minimal).

La solution n'est plus triviale pour le cas des liens à faible évanouissement, ce qui peut être expliqué par le fait que les bénéfices de la transmission sont fondamentalement différents. Pour les canaux statiques et à évanouissement rapide, la transmission est contrainte à être asymptotiquement fiable (avec probabilité d'erreur zéro). Cette contrainte se révèle à être trop rigide et une transmission économe en énergie n'est pas possible à une puissance d'émission non-nulle. Pour le canal à faible évanouissement, il s'avère qu'il y a un compromis optimal entre la probabilité de coupure et la débit

de transmission qui permet une communication efficace en terme d'énergie avec une puissance de transmission non-nulle.

De nombreux problèmes ouverts sont introduits par cette métriqu, ici nous allons citer quelques-uns:

- Tout d'abord, la conjecture sur la matrice de précodage optimale pour les canaux MIMO généraux doit être prouvée (voir Appendice B.2).
- La quasi-concavité de la fonction d'efficacité énergétique pour le cas de liens à faible évanouissement lorsque l'on suppose une allocation de puissance uniforme reste à être prouvée dans le régime fini des dimensions du système.
- Un modèle de canal plus général doit être envisagé. Nous avons considéré des matrices de canal avec des entrées i.i.d. standard Gaussiennes. Le modèle plus complexe des matrices aléatoires de moyenne non-nulle avec des profils de corrélation arbitraires apparaît comme un problème difficile à résoudre.
- Le lien entre la métrique proposée et le compromis entre la diversité et le multiplexage à fort RSB n'a pas été exploré.
- Seuls les canaux à un seul utilisateur ont été pris en considération. De toute évidence, les canaux MIMO à plusieurs utilisateurs comme les canaux à accès multiples ou les canaux à interférence doivent être étudiés. Le problème des canaux distribués à plusieurs utilisateurs et le jeu non-coopératif d'allocation de puissance associé est très intéressant. À cet égard, un seul résultat est mentionné ici: l'existence d'un équilibre de Nash pour les canaux à accès multiple en supposant l'allocation de puissance uniforme et le décodage simultané des utilisateurs.

La principale difficulté réside dans le fait que le problème d'optimisation de la probabilité de coupure est encore un problème ouvert. Nous avons vu que, pour résoudre le problème général, il suffit de démontrer la conjecture de Telatar et le cas d'attribution de puissance uniforme. Optimisation de la probabilité de coupure est un problème difficile, même à partir d'un point de vue numérique. C'est pourquoi, dans le chapitre suivant, nous verrons si l'utilisation simple d'apprentissage algorithmes, l'émetteur peut converger vers la solution qui maximiser la probabilité de coupure.

D.4 Les Algorithmes d'Apprentissage

Dans la section D.2, nous avons étudié les jeux non-coopératifs d'allocation de puissance où les émetteurs choisissent la politique d'allocation de puissance optimale pour maximiser leur taux atteignable. Des algorithmes itératifs ont été proposés pour calculer les points d'équilibre de Nash au niveau des émetteurs. Afin d'appliquer ces algorithmes, les émetteurs sont considérés comme des joueurs strictement rationnels qui connaissent parfaitement la structure du jeu et les stratégies jouées par les autres joueurs dans le passé. En outre, la rationalité des émetteurs est supposée être de notoriété publique.

Dans cette section, nous étudions une autre façon d'expliquer comment les joueurs peuvent converger vers un point d'équilibre du jeu en un coup. Cette alternative est offerte par la théorie de l'apprentissage dans les jeux [74]. Les algorithmes d'apprentissage sont des processus à long terme dans lesquels les hypothèses de connaissance et de rationalité des joueurs sont moins restrictives. Les utilisateurs optimisent leur paiement moyen en appliquant des mises à jour simples de la distribution définie sur l'espace des stratégies en fonction d'un retour d'information de la part de l'environment. Nous allons principalement étudier l'algorithme d'apprentissage par renforcement similaire à celui proposé en [75]. Dans ce cadre, les utilisateurs sont simplement des automates capables de choisir leurs actions dans un ensemble fini. Leurs choix sont basés sur les résultats obtenus par retour d'information de l'environnement. Ainsi, ils peuvent améliorer leurs performances au fil du temps tout en fonctionnant dans un environnement presque inconnu.

Deux scénarios différents seront considérés. Tout d'abord, nous étudions un jeu d'allocation de puissance semblable au CAM MIMO à évanouissement rapide où le récepteur applique la technique de décodage DSU (voir Sec. D.2). La différence réside dans les ensembles des actions de joueurs qui sont ici des ensembles discrets et finis. En raison de cette différence, l'analyse menée dans Sec. D.2 n'est plus valide. Par conséquent, nous devons d'abord analyser l'équilibre de Nash pour le jeu en un coup non-coopératif. Ensuite, nous verrons qu'en utilisant de simples règles d'adaptation, les joueurs convergent vers l'un des points du EN. Le deuxième scénario est le canal MIMO à faible évanouissement qui à été considéré dans la D.3 du point de vue des communications énergétiques. Nous allons voir que la matrice de covariance optimale qui minimize la probabilité de coupure (et qui est toujours un problème ouvert) peut être calculée en appliquant un algorithme d'apprentissage de renforcement similaire.

Dans l'Appendice C.1 le scénario du CAM MIMO à évanouissement rapide est étudié en détails. Nous avons observé que des algorithmes simples d'apprentissage par renforcement permettent aux émetteurs d'apprendre leur politique d'allocation. Au niveau du réseaux, les points de fonctionnement vers lesquels les émetteurs convergent sont des états désirables (e.g., le EN). Ces algorithmes ont plusieurs caractéristiques intéressantes. Ils sont des algorithmes de faible complexité et adaptatifs en temps. Les utilisateurs mettent à jour leurs choix d'actions basés sur un certain retour d'information de l'environnement qui leur permet d'améliorer leurs performances. La mise à jour ne nécessite aucune autre connaissance sur l'environnement (topologie du réseau, des informations sur les états du canal) ni l'hypothèse de la rationalité. Cependant tous ces avantages ont un coût en temps de convergence qui devient relativement large. En outre, ces algorithmes sont de nature stochastique et seule la convergence asymptotique en probabilité peut être assurée. Dans la pratique, cela se traduit par le fait qu'un choix très minutieux du pas de quantification doit être fait pour assurer une bonne performance des algorithmes. Nous avons vu qu'il existe un compromis entre la probabilité (ou bien la fréquence) de convergence et le temps nécessaire pour la convergence. Ces problèmes sont également causés par les propriétés inhérentes de l'apprentissage par renforcement et la dynamique du réplicateur. Les algorithmes d'apprentissage par renforcement permettent aux utilisateurs de converger vers les solutions de l'équation différentielle décrivant

la dynamique du réplicateur. Toutes les stratégies pures, même celles qui ne sont pas optimales, sont des points stationnaires de cette dynamique. C'est une des raisons pour laquelle les méthodes numériques ne parviennent pas à converger vers le points optimaux. Pour résoudre ces problèmes, d'autres techniques d'apprentissage fondées sur l'apprentissage du type Boltzman-Gibbs et autres [84] peuvent être étudiées. En outre, dans la littérature il y a des algorithmes stochastiques [81] [80] qui sont dédiés aux processus contraints sur des ensembles bornés. Ces algorithmes peuvent également être étudiés afin de relâcher quelques contraintes sur les matrices des canaux.

Nous avons aussi vu que les algorithmes d'apprentissage par renforcement nous permettent de calculer numériquement les solutions de problèmes ouverts tels que l'optimisation de la probabilité de coupure pour le canal MIMO à faible évanouissement. Cette analyse peut être étendue afin de trouver la matrice de covariance optimale qui maximise la fonction d'efficacité énergétique définie dans la section D.3. En outre, une extension intéressante est d'étudier le cas général du scénario avec plusieurs utilisateurs (e.g., le canal à accès multiple ou canal à interference). Une autre question intéressante est d'étudier les écarts entre le cas continu étudié dans les sections précédentes (i.e., section D.2 et section D.3 où les ensembles des actions des utilisateurs sont les cônes convexes de matrices positives semi-définies de trace finie) et le cas discret étudié dans ce chapitre. Notez que pour le canal standard de Rayleigh il n'y a pas d'écart d'optimalité entre les deux approches. Toutefois, pour les modèles de canaux généraux, le problème n'est plus trivial et une analyse mathématique en profondeur est requise.

D.5 Conclusions

Dans ce manuscrit, notre objectif principal a été d'étudier les réseaux sans fil dans lesquels les nœuds terminaux sont équipés de plusieurs antennes. Plusieurs thèmes d'actualité, tels que les réseaux intelligents auto-optimisants, les communications dites *green* ou vertes et algorithmes distribués ont été abordés plutôt d'un point de vue théorique. Dans ce but, nous avons utilisé une gamme diversifiée d'outils de la théorie des jeux, théorie de l'information, théorie des matrices aléatoires et théorie de l'apprentissage. Bien qu'il reste encore un grand écart à combler afin de rendre ces études réalistes, leur importance réside dans le fait qu'elles représentent les limites de performance atteignable en pratique.

Nous avons commencé notre analyse par l'étude du problème d'allocation de puissance dans les réseaux MIMO distribués. Les émetteurs sont censés être autonomes et capables de gérer leurs puissances afin d'optimiser leur taux de Shannon atteignables. Le cadre des jeux non-coopératifs a été utilisé pour étudier la solution de ce problème. Des algorithmes itératifs basés sur les meilleure réponses ont été mis en œuvre pour calculer les solutions de l'équilibre de Nash. Deux modèles différents ont été considérés: le canal à accès multiples MIMO et le canal à interférence à relais parallèle.

Le premier modèle se caractérise par le fait qu'une technique de décodage plus complexe que le décodage simultané des utilisateurs peut être mises en œuvre au niveau du récepteur: l'annulation d'interférence successive. Nous avons vu qu'en utilisant un

simple signal de coordination qui caractérise l'ordre de décodage au niveau du récepteur, le débit total du système à l'équilibre de Nash est assez proche de la solution du réseau centralisé. En outre, la distribution de ce signal public peut être manipulée de manière centralisée pour contrôler le point de fonctionnement du réseau. En supposant les liens à évanouissement rapide, la théorie des matrices aléatoires a été utilisé pour déterminer le point d'équilibre de Nash. Un cas particulier intéressant est le cas du CAM parallèle avec des liens statiques, qui a été étudié dans une perspective des jeux de routage. Plusieurs paradoxes de Braess ont également été mis en évidence.

Le second modèle, i.e. le canal à interférence à relais parallèle, est caractérisé par la présence des nœuds supplémentaires de relais. Ces nœuds peuvent être exploités par les émetteurs d'améliorer leurs performances de communication. Trois protocoles de relayage différents sont étudiés et leurs performances comparées par des simulations numériques: Amplifier-et-Transférer, Decoder-et-Transférer et Estimer-et-Transférer. Les paramètres des nœuds relais peuvent être manipulés par le propriétaire du système pour contrôler le point de fonctionnement du réseau. Plusieurs questions intéressantes se posent, l'étude du jeu général impliquant tous les degrés de liberté des émetteurs, la multiplicité des équilibres de Nash, les formulations Stackelberg.

Plusieurs enjeux majeurs apparaissent lorsque nous utilisons le cadre des jeux non-coopératifs et les algorithmes itératifs basés sur les meilleures réponses. Tout d'abord, les émetteurs sont supposés être des dispositifs strictement rationnels. Deuxièmement, les informations nécessaires au niveau des émetteurs par rapport à leurs propres fonctions de paiement et les paramètres des canaux ou leur statistiques sont souvent irréalistes. Troisièmement, les itérations impliquent beaucoup de signalisation entre les émetteurs car ils doivent rendre public leur choix d'action. Quatrièmement, prouver la convergence vers l'un des points d'équilibre est généralement un problème très difficile.

La théorie de l'apprentissage dans les jeux apparaît comme une solution candidate à toutes ces problèmes. Nous avons vu qu'avec l'utilisation de règles de mise à jour simples (e.g., l'apprentissage par renforcement), chaque utilisateur converge vers l'équilibre de Nash du jeu moyen en un coup. Dans ce cadre, les utilisateurs ne sont plus rationnels, mais des dispositifs automates. En outre, la seule connaissance de l'environnement nécessaire est un retour d'information (à savoir la valeur instantanée de la fonction du paiement) qui marque le choix des utilisateurs. Il s'avère que, sur la base de ce retour d'information, les utilisateurs peuvent s'adapter et apprendre dans le temps leurs stratégies optimales. Cependant, ces algorithmes sont de nature stochastique et seule une convergence probabiliste peut être garantie. En outre, la convergence dans la pratique implique un temps relativement long.

Un autre problème majeur est lié à la question de l'efficacité énergétique de la communication. Afin d'atteindre des débits de transmission élevés, la consommation d'énergie est également élevée. Dans les réseaux où la consommation d'énergie est une question critique, le débit de Shannon atteignable n'est plus une métrique de performance adaptée. C'est pourquoi nous avons également abordé le problème de l'optimisation d'une fonction d'efficacité énergétique. Cette métrique de performance traduit le nombre moyen de bits qui peuvent être transmis à travers le canal par unité d'énergie consommée. En raison des difficultés rencontrées, notre travail a été limité au canal MIMO mono-utilisateur.

Dans le cadre de la théorie l'information, si aucune erreur n'est tolérée en réception, une communication efficace en terme d'énergie n'est pas possible. Cependant, en supposant un débit de transmission fixe et qu'une probabilité d'erreur est tolérée, alors la communication efficace en termes d'énergie est possible avec une puissance de transmission strictement positive.

Pour le canal MIMO à faible évanouissement, le problème général qui consiste à trouver la matrice de covariance qui maximise la fonction d'efficacité énergétique est encore une question ouverte. Nous avons vu que ce problème revient à résoudre le cas particulier de l'allocation uniforme de puissance sur les antennes d'émission et de prouver que la conjecture Telatar de [14] est vraie. Cependant, trouver la matrice de covariance optimale qui minimise la probabilité de coupure se révèle être une question difficile. Comme nous l'avons vu, les algorithmes d'apprentissage fournissent des outils qui permettent de calculer la matrice de covariance optimale au niveau de l'émetteur. Une autre extension importante, est l'étude du scénario à plusieurs utilisateurs.

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