Adaptive and anisotropic finite element approximation: Theory and algorithms

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Motivation: anisotropic phenomena

The solutions of many PDE’s exhibit a strongly anisotropic behavior.

- **Boundary layers** in fluid simulation.
- **Spikes and edges** of metallic objects in electromagnetism.
- **Shockwaves** in transport equations.

Figure: Fluid simulation around a supersonic plane (F. Alauzet).
Mesh optimization

A general objective: reduce at best the trade-off between accuracy and numerical complexity.

- **Accuracy**: for example, the error between the solution and its approximation in some given norm.

- **Complexity**: typically tied to the cardinality of the mesh.
An appetizer: given a function $f : \Omega \rightarrow \mathbb{R}$ and an integer $N$, construct a triangulation $\mathcal{T}$ of $\Omega$ which minimizes
\[
\| \nabla (f - I_{\mathcal{T}} f) \|_{L^2(\Omega)},
\]
over all triangulations such that $\#(\mathcal{T}) \leq N$, with $I_{\mathcal{T}}$ the piecewise linear interpolant.

In the numerical examples $N = 500$ and
\[
f(x, y) := \tanh(10(\sin(5y) - 2x)) + x^2y + y^3.
\]
An appetizer: given a function $f : \Omega \to \mathbb{R}$ and an integer $N$, construct a triangulation $\mathcal{T}$ of $\Omega$ which minimizes
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In the numerical examples $N = 500$ and
\[ f(x, y) := \tanh(10(\sin(5y) - 2x)) + x^2 y + y^3. \]

Figure: Sharp transition along the curve $\sin(5y) = 2x$, of width $1/10$. 
A classical result

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, let $f \in H^2(\Omega)$ and let $\mathcal{T}$ be a triangulation.

Ciarlet-Raviart

On each $T \in \mathcal{T}$, the local error satisfies

$$\|\nabla(f - I_T f)\|_{L^2(T)} \leq C_0 \frac{h_T^2}{r_T} \|d^2 f\|_{L^2(T)},$$

where $h_T := \text{diam}(T)$ and $r_T$ is the radius of the largest disc inscribed in $T$, and $C_0$ is an absolute constant.

Consequence: with $h = \max_{T \in \mathcal{T}} h_T$

$$\|\nabla(f - I_T f)\|_{L^2(\Omega)} \leq C(\mathcal{T}) h \|d^2 f\|_{L^2(\Omega)},$$

with $C(\mathcal{T}) = C_0 \max_{T \in \mathcal{T}} \frac{h_T}{r_T}$ that remains bounded for isotropic triangulations.
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In terms of $N = \#(\mathcal{T})$, this gives

$$\sqrt{N} \| \nabla (f - I_T f) \|_{L^2(\Omega)} \leq C'(\mathcal{T}) \| d^2 f \|_{L^2(\Omega)},$$

where $C'(\mathcal{T}) = C_0 \sqrt{|\Omega|} \frac{\max_{T \in \mathcal{T}} h_T^2 / r_T}{\min_{T \in \mathcal{T}} \sqrt{|T|}}$, that remains bounded for uniform triangulations:

$$h \sim h_T \sim r_T \sim \sqrt{|T|} \Rightarrow h \sim N^{-1/2}.$$
The parameters of a triangle.

Position

Area

Aspect ratio and orientation

Angles

Parameters
- Position
- Area
- Aspect ratio and orientation
- Angles

Questions raised
- Equivalence of meshes and metrics
- Arbitrary degree
- Smoothness classes
- Hierarchical triangulations

Conclusion

Uniform triangulation

Isotropic triangulation

Anisotropic triangulation

Optimal anisotropic triangulation
The parameters of a triangle.

- **Position**
- **Area**
- **Aspect ratio and orientation**
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**Conclusion**
- Uniform triangulation
- Isotropic triangulation
- Anisotropic triangulation
- Optimal anisotropic triangulation
Isotropic meshes: the triangle seen as a disk.

Theorem (Adaptive approximation: DeVore-Yu)

For any \( f \in W^{2,1}(\Omega), \Omega = ]0,1[^2 \), there exists a sequence \((\mathcal{T}_N)_{N \geq 2}\) of (isotropic) triangulations of \( \Omega \), \( \#(\mathcal{T}_N) \leq N \), such that

\[
\sqrt{N} \| \nabla (f - I_{\mathcal{T}_N} f) \|_{L^2(\Omega)} \leq C \| M(d^2 f) \|_{L^1(\Omega)}
\]

\( M(g) \) : Hardy-Littlewood maximal function of \( g \).
Key principle: error equidistribution

Such sequences of triangulations may be obtained by a hierarchical refinement algorithm, starting from a coarse mesh.

- Refine the triangle with largest local error
  \[ \| \nabla (f - I_T f) \|_{L^2(T)}. \]
- Propagate the refinement to preserve conformity.
- Iterate until prescribed number of triangles is met.
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Such sequences of triangulations may be obtained by a hierarchical refinement algorithm, starting from a coarse mesh.

- Refine the triangle with largest local error $\|\nabla (f - I_T f)\|_{L^2(T)}$.
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Aspect ratio: the triangle seen as an ellipse.

The ellipse of minimal area containing a triangle $T$ is defined by

$$(z - z_T)^T \mathcal{H}_T (z - z_T) \leq 1,$$

where $\mathcal{H}_T$ is a symmetric positive definite matrix and $z_T$ is the barycenter of $T$. 

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- Position
- Area
- Aspect ratio and orientation
- Angles

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Conclusion
Anisotropic mesh generation

Given a metric $H : \Omega \rightarrow S_2^+$ produce a triangulation $\mathcal{T}$ such that: for any $T \in \mathcal{T}$ and any $z \in T$,

$$H(z) \simeq \mathcal{H}_T$$

Figure: A metric and an adapted triangulation (credit: J. Schoen)

Theoretical results by Boissonat & al, Shewchuk & al.
\[ \pi = ax^2 + 2bxy + cy^2 : \text{homogeneous quadratic polynomial.} \]

\[
L_G(\pi) := \inf_{\det H=1} \sup_{\mathcal{H}_T=H} \| \nabla (\pi - I_T \pi) \|_{L^2(T)}.
\]

(Near) Minimizing matrix \( H \)

**Figure:** Level lines of \( \pi \) (red, dashed), ellipse (blue, thick) associated to (near) optimal \( H \) which is proportional to the absolute value of the matrix associated to \( \pi \).

**Explicit equivalent of** \( L_G \)

\[
L_G(\pi) \simeq \sqrt{\| \pi \|^4} \sqrt{|\det \pi|},
\]

where \( \| \pi \| \) and \( \det \pi \) are the norm and determinant of the symmetric matrix associated to \( \pi \).
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Local model:
\[
f(x_0 + x, y_0 + y) = \alpha + (\beta x + \gamma y) + (ax^2 + 2bxy + cy^2) + O(|x|^3 + |y|^3)
\]
\[
\pi = \frac{1}{2} d^2f(x_0, y_0)
\]

**Theorem**

For any bounded polygonal domain \(\Omega\) and any \(f \in C^2(\Omega)\) there exists a sequence \((\mathcal{T}_N)_{N \geq N_0}\) of triangulations of \(\Omega\), \(\#(\mathcal{T}_N) \leq N\), such that

\[
\limsup_{N \to \infty} \sqrt{N} \| \nabla (f - I_{\mathcal{T}_N} f) \|_{L^2(\Omega)} \leq C \| L_G(d^2 f) \|_{L^1(\Omega)}
\]
An unusual estimate

\[ \lim_{N \to \infty} \sqrt{N} \| \nabla (f - I_{\mathcal{T}_N} f) \|_{L^2(\Omega)} \leq C \| L_G(d^2 f) \|_{L^1(\Omega)} \]

- The quantity \( L_G(d^2 f(z)) \simeq \sqrt{\| d^2 f(z) \|} \cdot 4 \sqrt{\det(d^2 f(z))} \)
depends nonlinearly on \( f \).
- Defining \( A(f) := \| L_G(d^2 f) \|_{L^1} \) we generally do not have \( A(f + g) \leq C(A(f) + A(g)) \).
- The estimate holds asymptotically as \( N \to \infty \).

An earlier estimate of this type was known for the \( L^p \)-norm.

Theorem (Chen, Sun, Xu; Babenko)

If \( f \in C^2(\Omega) \) and \( 1 \leq p \leq \infty \), then there exists a sequence \( (\mathcal{T}_N)_{N \geq N_0} \) of triangulations of \( \Omega \), \( \#(\mathcal{T}_N) \leq N \), such that

\[ \lim_{N \to \infty} N \| f - I_{\mathcal{T}_N} f \|_{L^p(\Omega)} \leq C \left\| \sqrt{\det \, d^2 f} \right\|_{L^\tau(\Omega)}, \quad \frac{1}{\tau} := 1 + \frac{1}{p} \]
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**Theorem (Chen, Sun, Xu; Babenko)**

If \( f \in C^2(\overline{\Omega}) \) and \( 1 \leq p \leq \infty \), then there exists a sequence \( (T_N)_{N \geq N_0} \) of triangulations of \( \Omega \), \( \#(T_N) \leq N \), such that

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- The quantity \( L_G (d^2 f(z)) \) \( \simeq \sqrt{\| d^2 f(z) \|} \left( 4 \sqrt{| \det( d^2 f(z) ) |} \right) \)
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  \[ A(f + g) \leq C (A(f) + A(g)). \]
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\]
Angles: the triangle seen as a triangle (!)

\[ \pi = ax^2 + 2bxy + cy^2 : \] homogeneous quadratic polynomial.

\[
L_A(\pi) := \inf_{|T|=1} \| \nabla (\pi - \mathbf{I}_T \pi) \|_{L^2(T)}.
\]

Figure: Interpolation of a parabola on a acute or obtuse mesh.
**Angles: the triangle seen as a triangle (!)**

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(Near) Minimizing triangle

\[ L_G \Rightarrow \frac{h_T}{r_T} \approx \frac{1}{\sqrt{\epsilon}} \]

\[ L_A \Rightarrow \frac{h_T}{r_T} \approx \frac{1}{\epsilon} \]

\[ \pi = x^2 \epsilon + y^2 \]

**Figure:** The minimizing triangle for \( L_A \) has acute angles and is more anisotropic than the minimizing ellipse for \( L_G \).

Explicit equivalent of \( L_A \)

\[ L_A(\pi) \approx \sqrt{\det \pi}. \]
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Explicit equivalent of \( L_A \)

\[ L_A(\pi) \sim \sqrt{\det \pi}. \]
Theorem

For any bounded polygonal domain $\Omega$ and any $f \in C^2(\mathring{\Omega})$ there exists a sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations of $\Omega$, $\#(\mathcal{T}_N) \leq N$, such that

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Furthermore for any admissible sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations, $\#(\mathcal{T}_N) \leq N$, one has

$$\liminf_{N \to \infty} \sqrt{N} \| \nabla (f - I_{\mathcal{T}_N} f) \|_{L^2(\Omega)} \geq \left\| L_A \left( \frac{d^2 f}{2} \right) \right\|_{L^1(\Omega)}.$$ 

Admissibility:

$$\sup_{N \geq N_0} \left( N^{\frac{1}{2}} \sup_{T \in \mathcal{T}_N} \text{diam}(T) \right) < \infty.$$
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Admissibility: 

$$\sup_{N \geq N_0} \left( N^\frac{1}{2} \sup_{T \in T_N} \text{diam}(T) \right) < \infty.$$
Guideline of the upper estimate (heuristic)

The asymptotically optimal sequence is built using a two-scale local patching strategy. (Not suited for applications)

- Initial triangulation of the domain.
- The interior of each cell is tiled with a triangle "optimally adapted" in size and shape to the Taylor development of $f$.
- Additional triangles at the interfaces ensure conformity.
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Introduction : Parameters of a triangle

Questions raised in the thesis

Equivalence of meshes and metrics
Finite elements of arbitrary degree
Anisotropic smoothness classes
Hierarchical anisotropic triangulations

Conclusion and perspectives
Parameters
Position
Area
Aspect ratio and orientation
Angles

Questions raised
Equivalence of meshes and metrics
Arbitrary degree
Smoothness classes
Hierarchical triangulations

Conclusion
Metrics and triangulations on $\mathbb{R}^2$

Definition (Equivalence triangulation/metric)

A (conforming) triangulation $\mathcal{T}$ of $\mathbb{R}^2$ is $C$-equivalent to a metric $H \in C^0(\mathbb{R}^2, S_2^+)$ if for all $T \in \mathcal{T}$ and $z \in T$ one has

$$C^{-1}H(z) \leq H_T \leq CH(z).$$

Figure: A metric and an equivalent triangulation, Credit: J. Schoen
Metrics and triangulations on $\mathbb{R}^2$

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$$C^{-1}H(z) \leq \mathcal{H}_T \leq CH(z).$$

Definition (Equivalence collection of triangulations/collection of metrics)

A collection $\mathcal{T}$ of triangulations of $\mathbb{R}^2$ is equivalent to a collection $\mathcal{H} \subset C^0(\mathbb{R}^2, S^+_2)$ of metrics if there exists $C$ such that

- $\forall T \in \mathcal{T}, \exists H \in \mathcal{H}$, such that $T$ and $H$ are $C$-equivalent.
- $\forall H \in \mathcal{H}, \exists T \in \mathcal{T}$, such that $T$ and $H$ are $C$-equivalent.
Isotropic triangulations

Theorem (reformulation of earlier work)

The collection $\mathcal{T}$ of all triangulations $T$ satisfying for each $T \in \mathcal{T}$

$$\text{diam}(T)^2 \leq 4|T|$$

is equivalent to the collection $\mathcal{H}$ of metrics $H$ of the form

$$H(z) = \frac{\text{Id}}{s(z)^2} \quad \text{where} \quad |s(z) - s(z')| \leq |z - z'|$$

Triangulations produced by FreeFem
From isotropic to anisotropic metrics

Isotropic “Lipschitz” metrics

\[ H(z) = s(z)^{-2} \text{Id}. \] Two equivalent properties:

- (d) \( \forall z, z' \in \mathbb{R}^2, |s(z) - s(z')| \leq |z - z'| \)

- (r) \( \forall z, z' \in \mathbb{R}^2, \left| \ln \left( \frac{s(z')}{s(z)} \right) \right| \leq d_H(z, z') \)

where \( d_H \) denotes the Riemannian distance

\[
d_H(z, z') := \inf_{\gamma: \gamma(0)=z, \gamma(1)=z'} \int_0^1 \sqrt{\gamma'(t)^T H(\gamma(t)) \gamma'(t)} \, dt.
\]

Anisotropic “Lipschitz” metrics

\[ H(z) = S(z)^{-2}. \] Two natural (but non-equivalent) generalizations:

- (D) \( \forall z, z' \in \mathbb{R}^2, \|S(z) - S(z')\| \leq |z - z'| \)

- (R) \( \forall z, z' \in \mathbb{R}^2, \frac{1}{2} \left\| \ln \left( S(z)^{-1} S(z')^2 S(z)^{-1} \right) \right\| \leq d_H(z, z') \)
From isotropic to anisotropic metrics

Isotropic “Lipschitz” metrics

\[ H(z) = s(z)^{-2} \text{Id}. \] Two equivalent properties:

- (d) \( \forall z, z' \in \mathbb{R}^2, \quad |s(z) - s(z')| \leq |z - z'| \)

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where \( d_H \) denotes the Riemannian distance

\[
d_H(z, z') := \inf_{\gamma: \gamma(0) = z, \gamma(1) = z'} \int_0^1 \sqrt{\gamma'(t)^T H(\gamma(t)) \gamma'(t)} \, dt.
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Graded Triangulations

Definition

A triangulation $\mathcal{T}$ of $\mathbb{R}^2$ is $K$-graded if for all $T, T' \in \mathcal{T}$,

\[ T \text{ intersects } T' \implies K^{-1} \mathcal{H}_T \leq \mathcal{H}_{T'} \leq K \mathcal{H}_T. \]

Non Graded

Graded

Theorem

For any $K \geq K_0$ the collection $\mathcal{T}$ of $K$-graded triangulations is equivalent to the collection $\mathbb{H}$ of metrics satisfying $(R)$.

Key ingredient: mesh generation results by Labelle, Shewchuk.
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Heuristic of the construction of $\mathcal{T}$ from $H$

Construct a collection $\mathcal{V} \subset \mathbb{R}^2$ of sites which satisfies:

- **covering** For all $z \in \mathbb{R}^2$, $d_H(z, \mathcal{V}) := \min_{v \in \mathcal{V}} d(z, v) \leq 1$.
- **separation** For all $v \neq w \in \mathcal{V}$, $d_H(v, w) \geq 1$. (or $\geq \delta_0 > 0$).

Connect sites when Anisotropic Voronoi regions intersect.

- Euclidean case $\text{Vor}(v) := \{z : |z - v| = \min_{w \in \mathcal{V}} |z - w|\}$.
- Peyrè, & al $\text{Vor}(v) := \{z : d_H(z, v) = \min_{w \in \mathcal{V}} d_H(z, w)\}$.
- Shewchuk, & al Define $\|v\|_M := \sqrt{v^T M v}$

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where $\text{Vor}(v) := \{z; \|z - v\|_{H(v)} = \min_{w \in \mathcal{V}} \|z - w\|_{H(w)}\}$. 

Parameters
- Position
- Area
- Aspect ratio and orientation
- Angles

Questions raised
- Equivalence of meshes and metrics
- Arbitrary degree
- Smoothness classes
- Hierarchical triangulations

Conclusion
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QuasiAcute triangulations

Definition

A triangulation $\mathcal{T}$ is $K$-QuasiAcute if

- $\mathcal{T}$ is $K$-graded.
- There exists a $K$-refinement $\mathcal{T}'$ of $\mathcal{T}$ such that any angle $\theta$ of any $T \in \mathcal{T}'$ satisfies

$$\theta \leq \pi - \frac{1}{K}.$$

$\mathcal{T}$ : $K$-QuasiAcute

$\mathcal{T}'$ : $K$-refinement of $\mathcal{T}$. 
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Theorem

For all $K \geq K_0$ the collection $\mathbb{T}$ of $K$-QuasiAcute triangulations is equivalent to the collection $\mathbb{H}$ of metrics satisfying simultaneously (R) and (D).
A comparison: how to capture a curvilinear discontinuity.

Objective: layer of width $\delta$ of triangles covering a smooth curve, using an Isotropic, QuasiAcute or Graded triangulation.

- **Isotropic**
  \[ \#(T) \simeq \delta^{-1} \]

- **QuasiAcute**
  \[ \#(T) \simeq \delta^{-\frac{1}{2}} \ln \delta \]

- **Graded**
  \[ \#(T) \simeq \delta^{-\frac{1}{2}} \]

- **No restriction**
  \[ \#(T) \simeq \delta^{-\frac{1}{2}} \]
Finite elements of arbitrary degree $m - 1$


In this talk we only consider the $W^{1,2}$ semi-norm and $L^2$ norm.
For all $\pi \in H_m$ (homogeneous polynomials of degree $m$)

$$L_A(\pi) := \inf_{|T|=1} \| \nabla (\pi - I_{T}^{m-1} \pi) \|_{L^2(T)}.$$ 

$I_{T}^{m-1}$ : Lagrange interpolant of degree $m - 1$.

**Theorem (Optimal asymptotic interpolation error)**

For any bounded polygonal domain $\Omega$, and any $f \in C^m(\overline{\Omega})$ there exists a sequence $(T_N)_{N \geq N_0}$ of triangulations of $\Omega$, $\#(T_N) \leq N$, such that

$$\lim \sup_{N \to \infty} N^{\frac{m-1}{2}} \| \nabla (f - I_{T_N}^{m-1} f) \|_{L^2(\Omega)} \leq \left\| L_A \left( \frac{d^m f}{m!} \right) \right\|_{L^2_m(\Omega)}.$$ 

Furthermore for any admissible sequence of triangulations $(T_N)_{N \geq N_0}$, $\#(T_N) \leq N$, one has

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Proposition (Explicit minimizing triangle, $m = 2$ or $3$)
Any acute triangle $T$, $|T| = 1$, such that $H_T$ is proportional to the following matrix is a (near) minimizer of the optimization defining $L_A(\pi)$.

- $m = 2$, $\pi = ax^2 + 2bxy + cy^2$, matrix: $\begin{pmatrix} a & b \\ b & c \end{pmatrix}^2$
- $m = 3$, $\pi = ax^3 + 3bx^2y + 3cxy^2 + dy^3$, matrix:

\[
M_A(\pi) := \sqrt{\left( \begin{pmatrix} a & b \\ b & c \end{pmatrix}^2 + \begin{pmatrix} b & c \\ c & d \end{pmatrix}^2 \right)}.
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Explicit minimizing triangle for $m > 3$: open problem.
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Explicit minimizing triangle for $m > 3$: open problem.
Optimal metric for the approximation of $f$

(heuristic)

Set

$$H(z) := \lambda (\det M(z))^{\frac{-1}{2m}} M(z),$$

where $\lambda > 0$ is a sufficiently large constant and

- $(m = 2)$, $M(z) \simeq [d^2 f(z)]^2$.
- $(m = 3)$, $M(z) \simeq M_A(d^3 f(z))$.

Mesh generation:

1. Produce a QuasiAcute triangulation $\mathcal{T}$ which is $C$-equivalent to $H$.
2. Interpolate $f$ on the refinement $\mathcal{T}'$ on which angles are uniformly bounded.

$\Rightarrow$ Optimal estimate up to a fixed multiplicative constant.
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Polynomials on $\mathcal{H}_m$: $Q(\pi) = \tilde{Q}(a_0, \ldots, a_m)$ if $\pi = a_0x^m + a_1x^{m-1}y + \cdots + a_my^m$.

**Proposition (Explicit equivalent of $L_A$)**

There exists a polynomial $Q$ on $\mathcal{H}_m$, of degree $r$, such that

$$L_A(\pi) \simeq |Q(\pi)|^{1/r}$$

uniformly.

- $m = 2$, $L_A(\pi) \simeq \sqrt{|\det \pi|}$.
- $m = 3$, $L_A(\pi) \simeq \sqrt{|\det M_A(\pi)|}$.
- $m \geq 4$ : explicit polynomials $Q$ are obtained using Hilbert’s theory of invariants.
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Optimizing only the aspect ratio:

\[ L_G(\pi) := \inf_{\det H=1} \sup_{\mathcal{H}_T=H} \|\nabla (\pi - I_T^{m-1} \pi)\|_2. \]

**Proposition (Explicit minimizing ellipse, \( m = 2 \) or \( 3 \))**

The matrix \( H \) such that \( \det H = 1 \) and which is proportional to the following is a (near) minimiser of the optimization problem defining \( L_G(\pi) \).

- \( m = 2 \), \( \pi = ax^2 + 2bxy + cy^2 \), matrix : \[
\begin{pmatrix}
  a & b \\
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\].

- \( m = 3 \), \( \pi = ax^3 + 3bx^2y + 3cxy^2 + dy^3 \)

\[ M_G(\pi) := M_A(\pi) + \left( \frac{-\text{disc}(\pi)}{\|\pi\|} \right)^{\frac{1}{3}} \text{Id}, \]

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Anisotropic metric (heuristic)

Set

\[ H(z) := \lambda (\det M(z))^{\frac{1}{2m}} M(z) \]

where \( \lambda > 0 \) is a constant and

- \((m = 2), M(z) := \|[d^2 f(z)]\| \|[d^2 f(z)]\|.
- \((m = 3), M(z) := M_G(d^3 f(z)).\)

Interpolate \( f \) on a mesh \( \mathcal{T} \) which is \( \mathcal{C} \)-equivalent to \( H \)

\[ \Rightarrow \text{estimate in terms of } L_G. \]

Figure: Interpolation with anisotropic \( P_2 \) elements.
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Interpolate $f$ on a mesh $\mathcal{T}$ which is $C$-equivalent to $H$

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Numerical experiments : $\|\nabla(f - I_T^{m-1} f)\|_L^2$, with 500 triangles.

<table>
<thead>
<tr>
<th>Uniform</th>
<th>Isotropic</th>
<th>Based on $L_G$</th>
<th>Based on $L_A$</th>
</tr>
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<td>11</td>
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<tr>
<td>$P_2$</td>
<td>79</td>
<td>14</td>
<td>0.88</td>
</tr>
</tbody>
</table>
Anisotropic smoothness classes: from finite element approximation to image models

Figure: A cartoon function, and an adapted triangulation. Picture: Gabriel Peyré

Approximation of cartoon functions

If \( g = \sum_{1 \leq i \leq r} g_i \chi_{\Omega_i} \) where \( g_i \in C^2(\overline{\Omega}_i) \) and \( \partial \Omega_i \) is piecewise \( C^2 \), then there exists a sequence \((\mathcal{T}_N)_{N \geq N_0}\) of triangulations such that

\[
N \| g - I_{\mathcal{T}_N} g \|_{L^2(\Omega)} \leq C(g).
\]

On the other hand, we have for smooth functions:

Theorem (Chen, Sun Xu; Babenko)

If \( f \in C^2(\overline{\Omega}) \) and \((\mathcal{T}_N)_{N \geq N_0}\) is an optimally adapted sequence then

\[
\limsup_{N \to \infty} N \| f - I_{\mathcal{T}_N} f \|_{L^2(\Omega)} \leq C \left\| \sqrt{|\det d^2 f|} \right\|_{L^3(\Omega)}^2.
\]

How to connect these estimates?

Does \( \left\| \sqrt{|\det d^2 g|} \right\|_{L^3}^2 \) make sense if \( g \) is a cartoon function?
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For any $f \in C^2(\Omega)$

\[
J(f) := \left\| \sqrt{|\det d^2f|} \right\|_{L^{\frac{2}{3}}}
\]

If $g$ is a cartoon function with discontinuity set $E$ we define

\[
J(g) := \lim_{\delta \to 0} J(g \ast \varphi_\delta),
\]

where $\varphi_\delta := \delta^{-2} \varphi(\delta^{-1}\cdot)$ is a mollifier.

Proposition

\[
J(g)_{\frac{2}{3}} = \left\| \sqrt{|\det d^2g|} \right\|_{L^{\frac{2}{3}}(\Omega \setminus E)}^{\frac{2}{3}} + C(\varphi) \left\| [g] \sqrt{\kappa} \right\|_{L^{\frac{2}{3}}(E)}^{\frac{2}{3}}
\]

where $[g]$ is the jump of $g$, and $\kappa$ the curvature of $E$.

Compare with

\[
TV(g) = \| \nabla g \|_{L^1(\Omega \setminus E)} + \|[g]\|_{L^1(E)}.
\]
For any \( f \in C^2(\overline{\Omega}) \)

\[
J(f) := \left\| \sqrt{|\det d^2 f|} \right\|_{L^2_3}.
\]

If \( g \) is a cartoon function with discontinuity set \( E \) we define

\[
J(g) := \lim_{\delta \to 0} J(g * \varphi_\delta),
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Proposition

\[
J(g) \frac{2}{3}^3 = \left\| \sqrt{|\det d^2 g|} \right\|_{L^2_3(\Omega \setminus E)}^2 + C(\varphi) \left\| [g] \sqrt{\kappa} \right\|_{L^2_3(E)}^2
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$$TV(g) = \left\| \nabla g \right\|_{L^1(\Omega \setminus E)} + \left\| [g] \right\|_{L^1(E)}.$$
Piecewise constant functions

$$TV(g) = \int_{\Gamma} |[g]|$$

$$J(g)^{\frac{2}{3}} = \int_{\Gamma} |[g]|^{\frac{2}{3}} |\kappa|^{\frac{1}{3}}$$

Figure: $TV(g) \simeq J(g)$

Figure: $TV(g) \ll J(g)$
Hierarchical sequences of anisotropic triangulations


A. Cohen, J.-M. Mirebeau, *Greedy bisection generates optimally adapted triangulations*, accepted in Maths of Comp 2010
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Figure: An isotropic hierarchical refinement algorithm, used for PDEs
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Given a triangulation of a domain and a function $f$:

- Select the triangle on which the $L^2$ interpolation error is maximal $\|f - I_T f\|_{L^2(T)}$.

- Bisect it along one median, so as to minimize the resulting $L^1$ interpolation error.

- Repeat these steps until targeted number of triangles is met.

**Figure**: Algorithm proposed by: N. Dyn, F. Hecht, A. Cohen
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Proposition (Identification of the bisection)

The algorithm applied to \( f(x, y) = x^2 + y^2 \) chooses to cut the longest edge of the selected triangle.

Preserves isotropy.

For any triangle \( T \) with edges \(|a| \geq |b| \geq |c|\) we define

\[
s(T) := \frac{|b|^2 + |c|^2}{4|T|}.
\]

Restores isotropy.

If \( T_1, T_2 \) are obtained by bisecting the longest edge of \( T \) then

\[
\max\{s(T_1), s(T_2)\} \leq s(T).
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For any triangle $T$ with edges $|a| \geq |b| \geq |c|$ we define

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Proposition

One of the eight children \( (T_i)_{i=1}^8 \) of \( T \) obtained by bisecting recursively three times the longest edge satisfies

\[
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Theorem

The algorithm applied to any strongly convex function $f \in C^2(\bar{\Omega})$ produces a sequence $(T_N)_{N \geq N_0}$ of triangulations which satisfies the (optimal) estimate

$$\limsup_{N \to \infty} N \| f - I_{T_N} f \|_{L^2(\Omega)} \leq C \| \sqrt{\det d^2 f} \|_{L^\frac{2}{3}(\Omega)}$$

Figure: Algorithm applied to $f(x, y) = x^2 + 100y^2$
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The algorithm applied to any strongly convex function $f \in C^2(\bar{\Omega})$ produces a sequence $(\mathcal{I}_N)_{N \geq N_0}$ of triangulations which satisfies the (optimal) estimate

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Conclusion and perspectives
Conclusion:

► A result of algorithmic geometry for QuasiAcute triangulations.

► Sharp asymptotic estimates for $P_m$ interpolation error on optimal mesh, for $H^1$ but also $L^p$ and $W^{1,p}$ norms.

► Some quantities remain meaningful for cartoon functions. e.g. $J(f) = \| \sqrt{\det(d^2f)} \|_{L^2}^2$.

► Combining hierarchy and anisotropy is possible (without conformity).

Perspectives:

► Numerical applications to $P_2$ elements in PDEs.

► Realistic mesh generation algorithms for QuasiAcute triangulations.

► Extension to dimension $d > 2$.

► Non asymptotic error estimates.

Thank you for your attention.
Conclusion:

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