Contrôle d'équations aux dérivées partielles non linéaires dispersives

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Contrôle d'équations aux dérivées partielles non linéaires dispersives

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Résumé

Dans cette thèse, on étudie la contrôlabilité et la stabilisation de certaines équations aux dérivées partielles dispersives.

On s'intéresse d'abord au problème du contrôle interne. Grâce à des méthodes d'analyse microlocale et à l'utilisation des espaces de Bourgain, on prouve la stabilisation et la contrôlabilité en grand temps de l'équation de Schrödinger non linéaire, d'abord sur un intervalle, puis sur des variétés de dimension 3. Dans le cas d'un intervalle, on raisonne à la régularité L^2, permettant ainsi de traiter une non-linéarité focalisante ou défocalisante. On obtient aussi des résultats de régularité supplémentaire pour le contrôle. De plus, on prouve la contrôlabilité aux trajectoires, dont on déduit une deuxième preuve de la contrôlabilité globale.

On applique ensuite ces méthodes à l'équation de Korteweg-de Vries en données périodiques. Pour cette équation, on donne aussi un terme d'amortissement dépendant du temps permettant d'avoir un taux de décroissance exponentielle arbitraire.

On étudie aussi l'équation de Klein-Gordon sur des variétés compactes avec une non-linéarité critique. Sous des hypothèses légèrement plus fortes que la condition de contrôle géométrique, on prouve la stabilisation et la contrôlabilité en grand temps pour des données haute fréquence. La preuve nécessite la mise en œuvre d'une décomposition en profils sur des variétés pour laquelle des effets géométriques doivent être analysés.

Dans une dernière partie, on étudie le contrôle bilinéaire. Grâce à un effet régularisant, on établit la contrôlabilité locale de l'équation de Schrödinger sur un intervalle avec une preuve plus simple que dans la littérature existante, permettant ainsi d'atteindre les espaces optimaux et en temps arbitraire. La méthode est assez robuste pour être étendue à d'autres situations : les données radiales sur la boule, l'équation de Schrödinger non linéaire et des ondes non linéaire sur un intervalle.

Mots-clefs : Contrôle, Stabilisation, équations de Schrödinger et des ondes non linéaires, équation de Korteweg-de Vries, analyse microlocale.

Control of nonlinear dispersive partial differential equations

Abstract

In this thesis, we study the controllability and the stabilization of some dispersive partial differential equations

We are first interested in the internal control. Thanks to some microlocal analysis methods and the use of Bourgain spaces, we prove stabilization and control in large time for the non linear Schrödinger equation on an interval and then on some manifolds of dimension 3. In the case of an interval, we work at the L^2 regularity, which allows to deal with both focusing and defocusing nonlinearity. We also obtain additional results about the regularity of the control. Moreover, we prove the controllability near trajectories, from which we deduce a second proof of global controllability.

We then apply these methods to the Korteweg-de Vries equation on a periodic domain. For this equation, we also provide a time dependent damping term which enables an arbitrary exponential decay rate.

We also study the Klein Gordon equation with a critical nonlinearity on some compact manifolds. Under some assumptions slightly stronger than the geometric control condition, we prove the stabilization and controllability in large time for high frequency data. The proof requires the statement of a profile decomposition on manifolds for which some geometric effects have to be analysed.

In a last part, we study the bilinear control. Thanks to a regularizing effect, we establish the local controllability of the Schrödinger equation on an interval with a proof simpler than in the available litterature, allowing to reach the optimal spaces and in an arbitrary time. The method is robust enough to be extended to other situations: radial data on a ball, non linear Schrödinger equation and non linear wave equation on an interval.

Keywords : Control, Stabilization, non linear Schrödinger and wave equation, Korteweg-de Vries equation, Microlocal analysis.

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Table des matières

Chapitre 1

Introduction générale

Contents

Dans cette thèse, on s'intéresse à la contrôlabilité et à la stabilisation d'équations aux dérivées partielles dispersives non linéaires. Le problème de la contrôlabilité consiste à analyser si on peut amener la solution d'une EDP d'un état initial à un état final fixé à l'avance en agissant par l'intermédiaire d'un contrôle. Ce contrôle peut être de plusieurs types dont les plus communément étudiés sont :

- le contrôle interne, qui consiste à rajouter un terme source localisé en espace,
- le contrôle au bord, où l'on agit sur les conditions au bord de la solution,
- le contrôle bilinéaire, où l'on rajoute un terme de potentiel.

On parle de stabilisation lorsque l'équation est modifiée sous la forme d'un terme de retour ne dépendant que de la solution elle-même. Le système est alors en boucle fermée et on se pose la question de savoir s'il va converger vers un état stationnaire, le plus souvent la solution nulle. Le contrôle et la stabilisation sont très liés et leur étude utilise souvent des techniques proches. Dans notre cas, la stabilisation sera aussi souvent utilisée pour démontrer des théorèmes de contrôle pour de grandes données.

La première partie de cette thèse analysera le problème du contrôle et de la stabilisation interne pour les équations de Schrödinger non linéaires sur des variétés (chapitres 2 et 3)

$$i\partial_t u + \Delta u \pm |u|^2 u = \chi_\omega g,$$

pour l'équation de Korteweg de-Vries à donnéees périodiques (chapitre 4)

$$\partial_t u + \partial_x^3 u + u\partial_x u = \chi_\omega g$$

et des ondes non linéaire critique sur des variétés (chapitre 5)

$$\Box u + u + u^5 = \chi_\omega g.$$

Dans la dernière partie, correspondant au chapitre 6, relativement indépendante des précédentes quant aux méthodes de résolution, on s'intéressera au contrôle bilinéaire que l'on appliquera à l'équation de Schrödinger et des ondes linéaires et non linéaires.

1.1 Contrôle interne

La majorité des résultats de cette thèse concerne des équations non linéaires. Cependant, comme c'est aussi le cas pour le problème de Cauchy, l'étude des problèmes non linéaires nécessite une bonne compréhension des propriétés de l'équation linéaire associée. Bien souvent, il est utile de comparer le plus possible la solution du problème non linéaire à celle de son analogue linéaire.

On se propose donc de passer en revue les résultats linéaires de contrôlabilité sur quelques équations. Ensuite, à la section suivante, on décrira les techniques employées dans les problèmes non linéaires. On se limitera aux équations les plus classiques. On n'évoquera pas le problème des systèmes d'équations, bien que celui-ci fasse actuellement l'objet de nombreuses recherches. Cette présentation ne se veut pas exhaustive mais a pour but de faire le lien avec les problèmes non linéaires qui vont être abordés dans la thèse.

1.1.1 Revue de résultats connus pour le contrôle linéaire

1.1.1.1 Contrôlabilité et observabilité

Les problèmes de contrôle d'EDP linéaires peuvent souvent se mettre sous la forme

$$\left\{ \begin{array}{rcll} \dot{y} & = & \mathcal{A}y + Bg & t \in [0,T] \\ y(0) & = & y_0. & \end{array} \right. \tag{1.1}$$

où $\mathcal{A}$ est un opérateur non borné sur un espace de Hilbert H qui génère un semi-groupe fortement continu, B est l'opérateur de contrôle et g est le contrôle à choisir. Le contrôle interne correspond au cas où $B = 1_\omega$ est la multiplication par la fonction indicatrice d'un ouvert ω (pour les ondes, on met l'équation sous forme de système et $B = (0, 1_\omega)$. De plus, on est souvent amené par commodité à remplacer la fonction 1_ω par une fonction régulière $a(x) \approx 1_\omega$, de sorte que l'opérateur B soit borné sur H.

La question de la contrôlabilité est de savoir si étant donné un état initial y_0, et un état final y_1 fixés à l'avance, on peut trouver un contrôle $g \in L^2([0,T], H)$ tel que la solution de (1.1) satisfasse à $y(T) = y_1$. La méthode HUM de Jacques-Louis Lions [67] ramène la contrôlabilité à zéro (ou aux trajectoires), c'est à dire $y_1 = 0$, à l'obtention d'une inégalité d'observabilité pour le problème adjoint. Plus précisément, on considère le système

$$\left\{ \begin{array}{rcll} \dot{z} & = & -\mathcal{A}^* z & t \in [0,T] \\ z(T) & = & z_T & \end{array} \right. \tag{1.2}$$

et on est ramené à la preuve de l'inégalité d'observabilité

$$\|z(0)\|_H^2 \leq C \int_0^T \|B^* z(t)\|_H^2 \, dt.$$

Les équations que l'on va considérer dans cette thèse sont toutes réversibles (ou au moins l'équation adjointe aura des propriétés similaires) et on aura $\mathcal{A}^* = -\mathcal{A}$ et la contrôlabilité à zéro impliquera la contrôlabilité. Ce n'est pas le cas par exemple pour l'équation de la chaleur.

Notons aussi que dans le cas du contrôle interne, une inégalité d'observabilité mesure la proportion d'énergie de la solution qui va passer par la zone ω. Les deux principales questions que l'on se pose sont alors les conditions géométriques sur ω et le temps minimal nécessaire pour avoir observabilité et donc contrôle. C'est donc une manière d'analyser comment les solutions libres se propagent dans l'espace. D'autre part, lorsque l'inégalité d'observabilité est vérifiée, la donnée de l'observation d'une solution sur le sous-domaine ω permet de retrouver la condition initiale et donc la solution elle-même. L'observabilité est donc aussi une question pertinente pour elle-même.

Les résultats de contrôlabilité que l'on déduit alors reflètent le mode de propagation dans l'espace des équations considérées :

- les solutions de l'équation des ondes se propagent le long des rayons de l'optique géométrique, ce qui rend naturelle la condition de contrôle géométrique.
- les solutions de l'équation de Schrödinger se propagent à vitesse infinie selon les rayons de l'optique géométrique. Cependant, sur certains domaines bornés, les phénomènes de dispersion induits par cette propagation ne sont pas encore parfaitement compris.
- les solutions de l'équation de KdV se propagent à vitesse infinie dans une direction. Lorsqu'on les considère sur un domaine périodique, on a donc propagation instantanée dans tout l'espace.
- L'équation de la chaleur est parabolique, et les solutions diffusent immédiatement dans le milieu. Cela explique que l'on n'ait pas de conditions géométriques pour l'observation.

L'obtention d'inégalité d'observabilité est donc centrale pour les problèmes linéaires. Dans la partie de cette thèse sur le contrôle de l'équation de Schrödinger, on montrera des inégalités d'observabilités pour des équations de Schrödinger linéaires perturbées par un potentiel. On prouvera aussi des inégalités d'observabilités directement sur les systèmes non linéaires, notamment pour la stabilisation. Cette approche sera précisée à la section 1.1.2.

1.1.1.2 Quelques résultats sur des équations classiques

Dans la suite de cette section, on se propose de décrire les résultats connus pour les équations classiques. On les énoncera parfois dans le cas d'une variété compacte pour mieux faire le lien avec les résultats de la thèse. Cependant, des résultats analogues existent bien sûr pour des ouverts bornés de $\mathbb{R}^n$ autant pour le contrôle interne que pour le contrôle au bord.

Equation des ondes L'heuristique issue de la physique selon laquelle les solutions de l'équation des ondes suivent à haute fréquence les lois de l'optique géométrique s'exprime mathématiquement sous la forme de la propagation du front d'onde ou de la mesure de défaut microlocal. Ce sont ces méthodes microlocales qui permettent de montrer la contrôlabilité sous les hypothèses optimales de contrôle géométrique introduites par

Rauch-Taylor [76] et Bardos-Lebeau-Rauch [8]. La condition nécessaire et suffisante est alors la condition de contrôle géométrique.

Soit M une variété compacte. On dit qu'un couple (ω, T), où ω est un ouvert de M et $T > 0$, vérifie la **condition de contrôle géométrique** si toute géodésique voyageant à vitesse 1 passe par ω en un temps $0 < t < T$. On dit que ω vérifie la condition de contrôle géométrique s'il existe un temps $T > 0$ tel que (ω, T) vérifie cette condition.

Théorème 1 (Bardos-Lebeau-Rauch [8]). *Soit M une variété compacte et (ω, T) vérifiant la condition de contrôle géométrique. Alors, pour tout $(v_0, v_1) \in H^1 \times L^2$, il existe $g \in L^2([0,T], L^2)$ supporté dans $[0,T] \times \omega$ tel que l'unique solution de*

$$\left\{ \begin{array}{rcll} \partial_t^2 v - \Delta v &=& g & \text{dans } [0,T] \times M \\ (v, \partial_t v)(0) &=& (v_0, v_1) & \end{array} \right.$$

satisfasse $(v, \partial_t v)(T) = 0$.

Un résultat similaire peut aussi être obtenu pour le contrôle au bord avec quelques subtilités supplémentaires liées aux rayons diffractifs (voir aussi Burq-Gérard [22]). Ces résultats ont été initialement prouvés par l'analyse de la propagation du front d'onde. Cependant, des preuves plus simples utilisant les mesures de défaut microlocal (voir sous-section 1.1.2.4) de Patrick Gérard [49] et Luc Tartar [85] ont été mises aux point par la suite. Elles ont aussi permis de comprendre plus finement le taux de décroissance de l'équation amortie dans Lebeau [64] et d'améliorer la régularité nécessaire pour le bord dans Burq [20].

Dans le chapitre 5 de cette thèse, on utilisera ces mesures pour obtenir un résultat de stabilisation des ondes non linéaires critiques sur des variétés compactes. Cependant, dans le cas critique, les solutions ne sont pas proches de solutions linéaires et les théorèmes de propagation seront un peu différents, menant à des restrictions plus fortes sur la géométrie.

Equation de Schrödinger Le résultat le plus général pour le contrôle interne de l'équation de Schrödinger est celui de G. Lebeau [63].

Théorème 2 (Lebeau [63]). *Soit $T > 0$. Soit M une variété compacte et ω un ouvert vérifiant la condition de contrôle géométrique. Alors, pour tout $v_0 \in H^1(M)$, il existe $g \in L^2([0,T], H^1)$ supporté dans $[0,T] \times \omega$ tel que l'unique solution de*

$$\left\{ \begin{array}{rcll} i\partial_t v + \Delta v &=& g & \text{dans } [0,T] \times M \\ v(0) &=& v_0 & \end{array} \right.$$

satisfait $v(T) = 0$.

Le théorème n'est pas écrit ainsi dans l'article mais les méthodes introduites permettent d'en déduire ce résultat. On renvoie aussi à Dehman-Gérard-Lebeau [42] dans le cadre de variétés avec une preuve plus simple par les mesures de défaut.

Cependant, on sait que la condition de contrôle géométrique n'est pas nécessaire. En effet, pour le tore $\mathbb{T}^n$, n'importe quel ouvert ω non vide suffit pour avoir contrôlabilité.

Théorème 3 (Jaffard [56], Komornik [60], Burq-Zworski [28]). *Le théorème 2 est vrai sur le tore $\mathbb{T}^d$ avec uniquement l'hypothèse que ω est non vide.*

Ce résultat a été démontré avec différentes méthodes. La première preuve est celle de S. Jaffard [56] en dimension 2 étendue par Komornik [60] à des dimensions supérieures. Elle utilise des méthodes d'analyse harmonique sur les séries de Fourier lacunaires et semble donc très spécifique au cas du tore. La preuve de Burq-Zworski [28] combine un argument général sur le contrôle sur des variétés produits (qui donne d'ailleurs une preuve simple du contrôle sur une bande du tore) et la mesure de défaut microlocal. Elle s'étend aussi à des situations un peu différentes (le tore avec un "trou" avec ω un voisinage du "trou"). De plus, des résultats de contrôlabilité plus faibles peuvent être montrés dans certaines géométries où seules des trajectoires instables sont évitées, voir Burq [19].

Pour l'équation de Schrödinger non linéaire, on utilisera les méthodes microlocales (avec une preuve par les mesures de défaut) issues du théorème 2 pour obtenir la stabilisation pour de grandes données. On montrera aussi des théorèmes de contrôle local en utilisant les résultats du théorème 3 de façon perturbative.

Equation de KdV linéaire (ou équation d'Airy) L'équation de Korteweg-de Vries linéarisée est l'équation $\partial_t y + \partial_x y + \partial_{xxx} y = 0$. Son comportement peut sensiblement varier selon les conditions aux bord que l'on impose. Sur $\mathbb{R}$ ou avec des conditions périodiques, l'équation a des propriétés similaires à l'équation de Schrödinger.

Ainsi, dans le cas du contrôle interne sur $\mathbb{T}^1$, Russel et Zhang [82] ont prouvé la contrôlabilité en temps arbitraire et sans hypothèse sur ω.

Pour $a(x) \in C^\infty(\mathbb{T}^1)$ de moyenne 1, on introduit l'opérateur A définit par $Ag(x) = a(x)\left(g(x) - \int_{\mathbb{T}^1} a(y)g(y)dy\right)$. Celui-ci sert à conserver la moyenne des solutions, ce qui sera utile pour les applications au problème non linéaire. Les auteurs obtiennent alors le résultat suivant.

Théorème 4 (Russel-Zhang [82]). *Soit $T \geq 0$ et $s \geq 0$. Alors, pour tout v_0, $v_1 \in H^s(\mathbb{T}^1)$ avec $\int_{\mathbb{T}^1} v_0 = \int_{\mathbb{T}^1} v_1$, il existe un contrôle $g \in L^2([0,T], H^s(\mathbb{T}^1))$ tel que la solution $v \in C([0,T], H^s(\mathbb{T}^1))$ de l'équation*

$$\left\{ \begin{array}{rcl} \partial_t v + \partial_{xxx} v &=& Ag \\ v(0) &=& v_0 \end{array} \right.$$

vérifie $v(T) = v_1$.

La méthode utilisée par Russel-Zhang repose sur le théorème d'Ingham. Le chapitre 4 de la thèse donne une preuve alternative de ce résultat qui est suffisamment robuste pour être appliquée au problème non linéaire à grande donnée. Elle repose sur des théorèmes de propagation de la régularité et de la compacité, mettant en valeur la propagation instantanée des solutions de KdV.

Notons que dans le cas du contrôle au bord, la situation est plus compliquée. Par exemple, sur le segment $[0, L]$ avec conditions au bord $y(t, 0) = y(t, L) = 0$ et un contrôle $u(t) = \partial_x y(t, L)$, Rosier [77] a montré la contrôlabilité pour un ensemble de longueurs L

non critiques. Il utilise alors l'effet régularisant de cette équation (avec $u(t) = 0$) : une donnée initiale dans $L^2([0, L])$ induit une solution dans $L^2([0, T], H^1(]0, L[))$. Ceci n'est pas le cas avec des données périodiques, ce qui compliquera les choses lors de l'analyse du problème non linéaire. De plus, d'autres choix de contrôles au bord peuvent donner des comportements très différents, de type parabolique, voir O. Glass et S. Guerrero [53].

Equation de la chaleur On ne traitera pas de l'équation de la chaleur dans cette thèse. Cependant, les méthodes introduites pour l'équation de la chaleur sont aussi utilisées dans cette thèse pour prouver des résultats de prolongement unique. On cite donc par souci de complétude le résultat linéaire principal sur ce sujet. Dans le cas de l'équation de la chaleur, la dissipation immédiate de la solution permet d'obtenir le contrôle interne sans hypothèse sur le temps de contrôle et la localisation de l'ouvert.

Théorème 5 (Lebeau-Robbiano [66] et Fursikov-Imanuvilov [47])**.** *Soient Ω un domaine borné régulier de $\mathbb{R}^n$ et $\omega \subset\subset \Omega$ un ouvert non vide. Soient $T > 0$ et $v_0 \in L^2(\Omega)$. Alors, il existe un contrôle $g \in L^2(]0, T[\times\omega)$ tel que la solution de*

$$\left\{ \begin{array}{rcll} \partial_t v - \Delta v &=& 1_\omega g & \text{dans } [0, T] \times \Omega \\ v(0) &=& v_0 & \end{array} \right.$$

satisfait $v(T) = 0$.

Ce résultat a été prouvé indépendamment et de deux manières différentes.

Dans [66], Lebeau et Robbiano utilisent des inégalités de Carleman elliptiques pour construire le contrôle en une infinité d'étapes où l'on "tue" les basses fréquences, puis on utilise la forte dissipation de l'équation de la chaleur à haute fréquence. On renvoie au survey de G. Lebeau et J. Le Rousseau [62] pour une présentation pédagogique de ce résultat.

La méthode de Fursikov et Imanuvilov [47] consiste à prouver des inégalités de Carleman globales pour l'opérateur de la chaleur. Ces idées seront ensuite reprises par d'autres auteurs pour les appliquer à d'autres équations comme par exemple l'équation de Schrödinger. L'avantage des inégalités de Carleman est qu'elles sont très robustes et restent vraies lorsqu'on ajoute un potentiel.

Ainsi, dans l'appendice de la partie 3, on établit des inégalités de Carleman globales pour l'équation de Schrödinger sur une variété dans le but d'obtenir des résultats de prolongement unique. Cependant, contrairement à l'équation de la chaleur, des conditions géométriques sont nécessaires pour obtenir de telles inégalités. De même, ce sont aussi des inégalités de Carleman qui permettent d'obtenir des résultats de prolongement unique pour KdV.

1.1.1.3 Régularité du contrôle

Lorsqu'une inégalité d'observabilité est vérifiée, la méthode HUM permet de construire un opérateur de contrôle qui, pour chaque donnée initiale, fournit un contrôle ramenant la solution à zéro. Par exemple, pour le contrôle interne de l'équation des ondes sur une variété, la méthode HUM fournit un contrôle $L^2([0,T],L^2)$ si on veut contrôler des données dans $H^1 \times L^2$. Ce contrôle est celui qui minimise la norme $L^2([0,T],L^2)$. Bien sûr, on peut faire une autre construction basée sur une autre norme, par exemple contrôler des données $H^2 \times H^1$ avec un contrôle $L^2([0,T],H^1)$ mais ce sera un autre opérateur qui ne minimisera pas la même norme. Une question naturelle que l'on peut se poser est la régularité que l'on obtient si l'on applique l'opérateur HUM construit sur $H^1 \times L^2$ à une donnée plus régulière. Cette question a d'abord été abordée dans l'article de Dehman-Lebeau [43] en vue de l'appliquer à l'équation des ondes non linéaire. Ils prouvent cette propriété de préservation de la régularité de l'opérateur HUM pour les ondes sur un domaine, pourvu que l'on rajoute une fonction de troncature en temps. Pour le cas d'une variété sans bord, l'opérateur HUM est en fait un opérateur pseudo-différentiel elliptique.

Dans les chapitres 2 et 3, on applique cette idée à l'équation de Schrödinger. Prenons par exemple Schrödinger linéaire sur $\mathbb{T}^1$.

Si on note S l'opérateur HUM défini par

$$S\Phi_0 = i\int_0^T e^{-it\partial_x^2}a^2(x)e^{it\partial_x^2}\Phi_0\, dt,$$

alors, on prouve que S est inversible sur H^s, $s \geq 0$. Comme S est supposé inversible sur L^2, il s'agit alors seulement d'un problème de régularité. Pour le prouver, il suffit juste d'appliquer un opérateur de dérivée fractionnaire sur cette expression, de faire apparaître un commutateur avec $a(x)$ puis d'utiliser l'inversibilité sur L^2. On obtient même une estimée de l'inverse de S dans H^s (on note C_s si la constante dépend de $s \geq 0$)

$$\left\|S^{-1}\Psi_0\right\|_{H^s} \leq C\left\|\Psi_0\right\|_{H^s} + C_s\left\|\Psi_0\right\|_{H^{s-1}}. \tag{1.3}$$

Lorsqu'on va appliquer cette inégalité au problème non linéaire, on utilisera de façon cruciale que la constante C du terme principal est indépendante de s. On prouvera qu'un contrôle local construit par perturbation sur L^2 préserve aussi la régularité, et ce quels que soient s et la taille dans H^s.

Pour finir, on note que S. Ervedoza et E. Zuazua [45] ont par la suite généralisé cette préservation de la régularité du contrôle HUM à des opérateurs abstraits. Leur preuve est plus simple, notamment dans le cas où l'opérateur de contrôle n'est pas borné sur l'espace de résolution. Ce phénomène est donc très général.

Pour conclure cette sous-section sur le contrôle linéaire, on note que la comparaison du problème non linéaire à son analogue linéaire ne permet souvent d'obtenir que des résultats locaux, prouvés par un argument perturbatif. Cependant, il peut arriver que le linéarisé près d'un point d'équilibre ne soit pas contrôlable et qu'on obtienne tout de même la contrôlabilité locale par d'autres méthodes, par exemple

- la méthode du retour qui consiste à trouver une trajectoire qui part et qui revient à ce point d'équilibre et qui possède de meilleures propriétés pour le linéarisé (voir par exemple [36] pour l'équation d'Euler ou [10])
- l'utilisation des ordres supérieurs : cela consiste à regarder les ordres supérieurs du développement de Taylor du système de contrôle. Cette méthode s'est avérée particulièrement efficace dans de nombreux exemples où l'ensemble des données

contrôlables du linéarisé était de codimension finie (voir [39, 32] pour KdV ou [13] pour le contrôle bilinéaire de Schrödinger).

On renvoie au livre de Jean-Michel Coron [40] pour une présentation des méthodes et de nombreux exemples. Cependant, dans les situations abordées dans cette thèse, le linéarisé sera contrôlable et on ne fera pas appel à ces méthodes.

1.1.2 Principales méthodes utilisées

1.1.2.1 Le problème de Cauchy et les espaces fonctionnels

Les méthodes actuelles pour résoudre des équations dispersives non linéaires utilisent des espaces fonctionnels choisis pour améliorer les estimées sur les termes non linéaires et faire un point fixe avec le terme de Duhamel. Le terme de Duhamel sera du type (par exemple pour Schrödinger non linéaire)

$$F(u) = e^{it\Delta}u_0 \pm i \int_0^t e^{i(t-s)\Delta}|u|^2u(s)ds. \tag{1.4}$$

Il s'agit donc de trouver un point fixe de F dans les espaces fonctionnels adaptés. Or, on veut résoudre notre équation dans des espaces qui ne sont pas des algèbres (par exemple $C([0,T],L^2)$) et ne sont pas stables par l'application $u \mapsto |u|^2u$. On va donc chercher la solution dans des espaces plus restrictifs.

Dans cette thèse, on utilisera deux types d'espaces : les espaces de Strichartz et les espaces de Bourgain.

Les espaces de Strichartz Les espaces de Strichartz sont, par exemple, pour l'équation des ondes sur une variété de dimension 3 à données dans $H^1 \times L^2$, les espaces $L^p([0,T],L^q(M))$ où les couples (p,q) vérifient les conditions d'admissibilité suivantes (ces conditions sont imposées par les dilatations de $\mathbb{R}\times\mathbb{R}^3$ laissant l'équation invariante)

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{2}, \quad p > 2.$$

L'intérêt de ces espaces est que les solutions de l'équation linéaire y appartiennent : toute solution de

$$\left\{ \begin{array}{rcl} \Box v & = & f \quad \text{on} \quad [-T,T]\times M \\ (v(0),\partial_t v(0)) & = & (u_0,u_1). \end{array} \right.$$

vérifie

$$\|v\|_{L^p([-T,T],L^q(M))} \leq C(\|(u_0,u_1)\|_{H^1\times L^2} + \|f\|_{L^1([-T,T],L^2)}). \tag{1.5}$$

On a ainsi gagné de l'intégrabilité. La solution est par exemple dans L^8L^8 alors qu'une fonction H^1 n'appartient en général pas à L^8 en dimension 3. Cette propriété est presque "miraculeuse" : sachant que l'équation est réversible, cette propriété ne peut être vraie pour un temps fixé mais seulement presque partout. Cela reflète les propriétés dispersives de l'équation des ondes : une donnée initiale très concentrée va avoir tendance à s'étaler, malgré la conservation de l'énergie.

L'utilisation des espaces de Strichartz consiste à faire un point fixe dans un espace X_T qui sera l'intersection de l'espace d'énergie (de type $C([0,T],H^s)$ pour Schrödinger ou $C([0,T],H^s \times H^{s-1})$ pour les ondes) et d'un espace de Strichartz. La non-linéarité sera estimée grâce au fait que u appartient à un espace de Strichartz et l'estimée (1.5) permet de montrer que l'on reproduit bien l'espace X_T. Prenons comme exemple les ondes sous-critiques en dimension 3. Si u est dans $X_T = C([0,T],H^1) \cap L^8([0,T],L^8) \times C([0,T],L^2)$, la solution $v = F(u)$ de

$$\left\{ \begin{array}{rcl} \Box v & = & -|u|^3 u \\ (v(0), \partial_t v(0)) & = & (u_0, u_1) \end{array} \right.$$

vérifie

$$\begin{aligned} \|v\|_{L^8([0,T],L^8(M))} &\leq C(\|(u_0,u_1)\|_{H^1\times L^2} + \left\| |u|^3 u \right\|_{L^1([0,T],L^2)}) \\ &\leq C(\|(u_0,u_1)\|_{H^1\times L^2} + T^{1/2} \|u\|_{L^8([0,T],L^8)})^4 \end{aligned}$$

En faisant le même type d'estimations avec une inégalité d'énergie, on obtient que F reproduit une certaine boule de X_T pour T assez petit, la petitesse ne dépendant que de la norme des données initiales. En montrant de même que F est contractante, on obtient un point fixe de F et donc une solution.

L'exposant critique $|u|^4 u$, traité dans le chapitre 5, correspond exactement à l'exposant pour lequel le même raisonnement amène à une inégalité similaire mais sans puissance de T devant le terme non linéaire. On obtient alors une solution pour des petites données. Pour de grandes données, on écrit $u = w + r$ avec w solution de l'équation libre avec donnée initiales (u_0, u_1). On veut alors résoudre

$$\left\{ \begin{array}{rcl} \Box r & = & -|w+r|^4 (w+r) \\ (r(0), \partial_t r(0)) & = & (0,0). \end{array} \right.$$

Le même type d'estimations amène à obtenir un point fixe dans X_T intersection de l'espace d'énergie et l'espace de Strichartz $L^5 L^{10}$ à la condition cette fois que T soit assez petit pour que $\|w\|_{L^5([0,T],L^{10})}$ le soit aussi. On a donc une solution locale, mais cette fois le temps d'existence ne dépend pas que de l'énergie des conditions initiales mais aussi de leur "forme". C'est pour cette raison que l'existence globale nécessite des arguments supplémentaires (inégalités de Morawetz), même dans le cas défocalisant.

Des inégalités de Strichartz ont été établies pour l'équation des ondes sur $\mathbb{R}^d$ par Ginibre-Velo [52], sur une variété par Kapitanski [57], sur un ouvert à bord par Burq-Lebeau-Planchon [27] et Blair-Smith-Sogge [16] (avec des restrictions sur les exposants). Pour l'équation de Schrödinger, on dispose des inégalités de Strichartz sur $\mathbb{R}^d$ prouvées par Ginibre-Velo [51] et Keel-Tao [58]. Mais des pertes peuvent apparaître dès qu'on s'intéresse à des domaines bornés comme des variétés compactes dans Burq-Gérard-Tzvetkov [25] ou des domaines à bord comme dans Anton [5] et Blair-Smith-Sogge [15].

Les espaces de Bourgain L'utilisation des espaces de Bourgain consiste à inclure dans l'espace X_T de résolution la propriété "être proche d'une solution linéaire". Les espaces de Bourgain pour Schrödinger en dimension 1, par exemple, mesurent ainsi une régularité en "opérateur de Schrödinger".

$$\begin{aligned}\|u\|_{X^{s,b}}^2 &= \sum_k \int_{\mathbb{R}} \langle k\rangle^{2s} \left\langle \tau + k^2\right\rangle^{2b} \left|\widehat{\widehat{u}}(\tau,k)\right|^2 d\tau \\ &= \left\|u^{\#}\right\|^2_{H^b(\mathbb{R},H^s(\mathbb{T}^1))}\end{aligned}$$

où l'indice τ (resp. k) est celui de la transformée de Fourier en temps (resp. espace) et $u^{\#}(t) = e^{-it\Delta}u(t)$. L'espace $X_T^{s,b}$ est alors l'espace des restrictions à $[0,T]$. L'indice s mesure une régularité de type Sobolev en espace, alors que l'indice b mesure combien les fréquences en espace-temps de la fonction sont proches de la variété caractéristique $\tau = -\xi^2$ des solutions de l'équation de Schrödinger libre. C'est une sorte de régularité en "opérateur de Schrödinger". Ils sont en ce sens plus précis que les espaces de Strichartz puisqu'ils gardent toutes les propriétés dispersives de l'équation libre.

Les estimées multilinéaires sont en général plus difficiles à démontrer mais donnent des résultats plus précis que les inégalités de Strichartz, spécialement sur des domaines bornés. Par exemple, pour NLS cubique sur S^2, N. Burq P. Gérard et N.Tzvetkov [25] ont démontré des inégalités de Strichartz optimales qui permettent de résoudre l'équation à la régularité $H^{1/2}$. Cependant, dans les articles [23, 24], ces mêmes auteurs ont démontré par les espaces de Bourgain que la régularité minimale pour que le flot soit bien posé était $H^{1/4-}$.

Cette fois, pour résoudre à la régularité H^s, on résout dans l'espace $X_T^{s,b}$ avec un bon choix de b. Lorsque c'est possible, on choisit $b > 1/2$ de sorte que l'on ait $X_T^{s,b} \subset C([0,T],H^s)$ par l'injection de Sobolev. Le point fixe se fera avec les estimations suivantes, pour des exposants $0 < b' < 1/2 < b$, $b \leq -b' + 1$ appropriés, et $\psi \in C_0^\infty(\mathbb{R})$

$$\begin{aligned}\left\|\psi(t)e^{it\Delta}u_0\right\|_{X^{s,b}} &\leq C\left\|u_0\right\|_{H^s} \\ \left\|\psi(t/T)\int_0^t e^{i(t-\tau)\Delta}N(\tau)d\tau\right\|_{X^{s,b}} &\leq CT^{1-b-b'}\left\|N\right\|_{X^{s,-b'}} . \qquad (1.6)\end{aligned}$$

La première estimation est une conséquence directe de la deuxième définition de la norme $X^{s,b}$ et la deuxième exprime le fait que l'intégration "fait gagner" une dérivée dans les exposants de Sobolev b. Pour conclure un point fixe avec une formule de Duhamel de type (1.4), la partie la plus difficile consiste alors à prouver des inégalité multilinéaires de type

$$\left\||u|^2u\right\|_{X_T^{s,-b'}} \leq C\left\|u\right\|^3_{X_T^{s,b}} .$$

De telles inégalités ont été prouvées par Bourgain [17, 18] pour les tores, et Burq-Gérard-Tzvetkov [23, 26] pour certaines variétés, telles que S^2, S^3 et $S^2 \times S^1$.

La résolution de KdV en domaine périodique utilisera aussi ces espaces. Cependant, on ne pourra pas choisir $b > 1/2$ mais on sera contraint de prendre $b = 1/2$. En effet, une inégalité du type $\|\partial_x(uv)\|_{X_T^{0,b-1}} \lesssim \|u\|_{X_T^{0,b}}\|v\|_{X_T^{0,b}}$ ne peut être vraie. On le voit en prenant $u = e^{in^3t}e^{inx}$ et $v = e^{ix}$, de sorte que $e^{t\partial_x^3}[\partial_x(uv)] = i(n+1)e^{i(n^3-(n+1)^3)t}e^{i(n+1)x}$ et $\|\partial_x(uv)\|_{X_T^{0,b-1}} = \left\|e^{t\partial_x^3}[\partial_x(uv)]\right\|_{H_T^{b-1}(L^2)} \approx n^{1+2(b-1)}$. Or, comme u et v ont des normes $X_T^{0,b}$ de l'ordre de 1, cela impose $b \leq 1/2$.

Hélas, (1.6) n'est plus vraie pour $b \leq 1/2$. Bourgain introduit alors un deuxième espace $Y^{s,b}$

$$||u||^2_{Y^{s,b}} \quad := \quad \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}} \langle k \rangle^s \left\langle \tau - k^3 \right\rangle^b \left| \widehat{\widehat{u}}(k,\tau) \right| d\tau \right)^2 .$$

On travaille alors dans les espaces $Z_T^{s,b} = X_T^{s,b} \cap Y_T^{s,b-1/2}$. On fait alors le point fixe dans l'espace $Z^{0,1/2}$ et on utilise l'estimée bilinéaire, pour $s \geq 0$, et u d'intégrale nulle

$$\left\| \partial_x(u^2) \right\|_{Z_T^{s,-1/2}} \leq \|u\|^2_{X_T^{s,1/2}} . \tag{1.7}$$

Notons que le deuxième espace $Y_T^{s,0}$ permet d'obtenir une solution qui est bien dans $C([0,T], H^s)$.

Précisons aussi qu'en prenant $u = e^{in^3 t} e^{inx}$ et $v = 1$, on voit que l'inégalité bilinéaire associée à (1.7) n'est plus vraie si l'on n'impose plus une moyenne nulle. Ceci n'est pas un problème pour l'équation libre qui conserve le volume : un changement de variable simple permet, quitte à changer légèrement l'équation, de trouver des solutions avec des moyennes non nulles. C'est pour cette raison que l'on choisira un terme de contrôle et de stabilisation qui conserve le volume des solutions. Le théorème de contrôle que l'on trouvera sera alors entre données de même volume.

1.1.2.2 La stabilisation pour avoir le contrôle

Dans le cas d'une équation linéaire, la contrôlabilité pour des données de norme petite entraîne la contrôlabilité pour toutes les données de cet espace. Ce n'est évidemment pas le cas pour des équations non linéaires. Si l'on dispose d'un bon théorème d'existence, il est en général facile avec un théorème de point fixe de déduire un contrôle pour des petites données à partir d'un théorème de contrôle linéaire. La difficulté principale, une fois que l'on sait que l'équation est bien posée, est essentiellement pour les grandes données.

La stratégie classique que l'on va employer consiste à trouver un bon terme de stabilisation qui va ramener notre système près de 0. Pendant ce laps de temps, on prend comme terme de contrôle le terme de stabilisation donné par l'équation stabilisée. Par unicité, la solution avec ce contrôle est la même que la solution stabilisée. On a donc trouvé un contrôle qui ramène notre système près de 0. En combinant cette construction avec un théorème local de contrôle à zéro, on obtient un théorème de contrôlabilité à zéro pour des données de taille arbitraire. Si, en plus, ce qui sera toujours le cas par la suite, l'équation considérée est réversible, on obtient un théorème de contrôle global en faisant le même raisonnement à partir de la donnée finale à contrôler et en inversant le temps. Cette stratégie est illustrée par la figure 1.1 où le terme "énergie" signifie la norme adaptée à l'équation considérée, et où u_0 et u_1 sont les données initiale et finale que l'on veut contrôler.

La difficulté principale dans ce schéma est donc de trouver un bon terme de stabilisation. En général, on arrive à montrer la stabilisation exponentielle grâce à une inégalité d'observabilité qui montre que pendant un temps T, on dissipe au moins une certaine proportion de l'énergie.

Dans le chapitre 3, on donnera aussi une méthode alternative. On cherche toujours à trouver des contrôles qui vont faire tendre la solution vers 0, mais cette fois, on le fait par des contrôles successifs près de trajectoires libres. On se donne un ε tel que, pour toute trajectoire libre menant un $\tilde{u}_0$ à un $\tilde{u}_1$, on puisse contrôler $\tilde{u}_0$ vers tout état u_f tel que $\|u_f - \tilde{u}_1\|_E \leq \varepsilon$, où E désigne une "énergie". Comme chaque trajectoire libre conserve l'énergie, on peut choisir u_f tel que $\|u_f\|_E \leq \|\tilde{u}_0\|_E - \varepsilon$, de sorte que l'énergie diminue à chaque étape. On obtient ainsi le contrôle vers 0. Par réversibilité, on peut faire de même à partir de l'état final à atteindre. Ceci est illustré à la figure 1.2. On a ici un peu simplifié le propos par le fait que pour NLS, l'énergie non linéaire qui est conservée n'est pas exactement la norme H^1 dans laquelle on fait des estimations. Dans ce schéma de preuve, la difficulté est donc de prouver un résultat de contrôle local près de toutes les trajectoires libres. De plus, pour que cette stratégie fonctionne, il faut un ε qui soit uniforme pour toutes les trajectoires dans une boule de H^1, donc à régularité relativement faible. L'inconvénient de ces deux méthodes est que l'on obtient

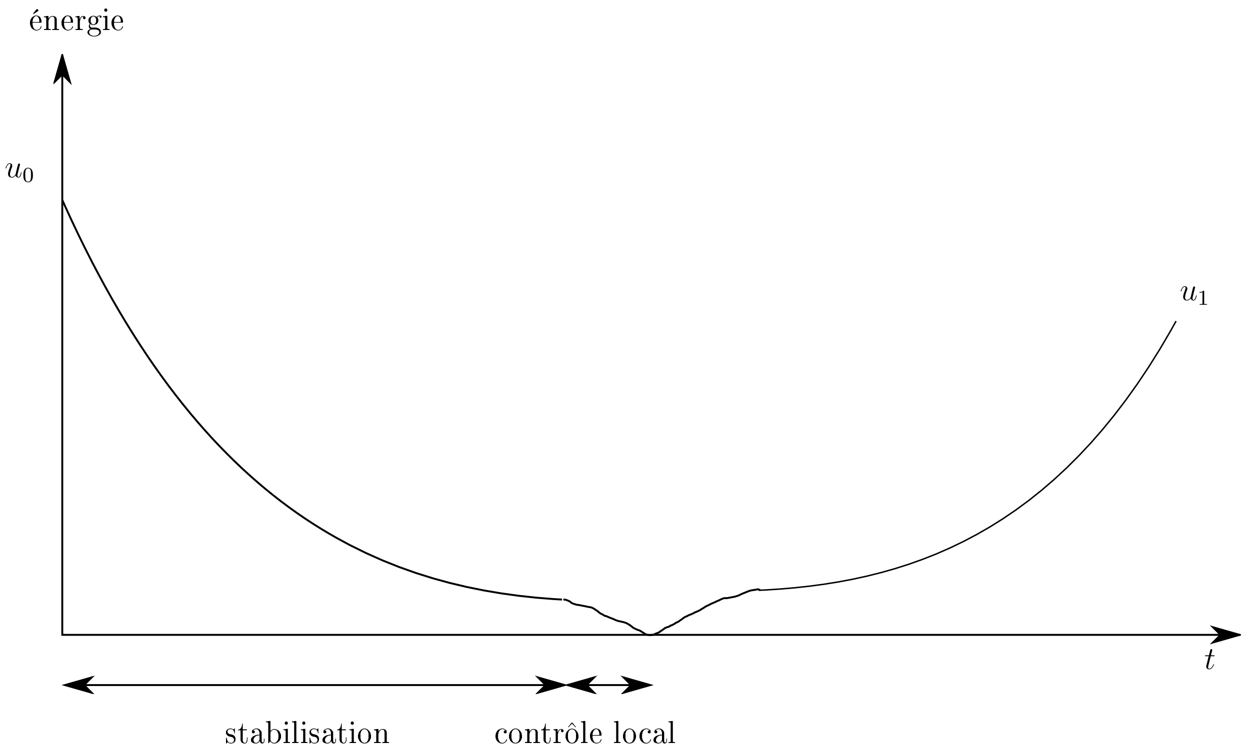

FIGURE 1.1 – Stratégie globale par stabilisation

un théorème de contrôle en temps long, qui dépend de la taille de la donnée. Ceci est en contraste avec les théorèmes linéaires où le temps nécessaire au contrôle ne dépend que de l'équation et pas de la taille des données. Ceci est encore plus frappant par exemple pour l'équation de Schrödinger où le contrôle linéaire se fait en temps arbitraire alors qu'on a besoin ici d'un grand temps pour le problème non linéaire. La nécessité de ce temps long est un problème ouvert.

1.1.2.3 La méthode de compacité-unicité

La méthode de compacité-unicité est une méthode maintenant classique en théorie du contrôle pour établir des inégalités d'observabilités dans les cas linéaires ou non linéaires.

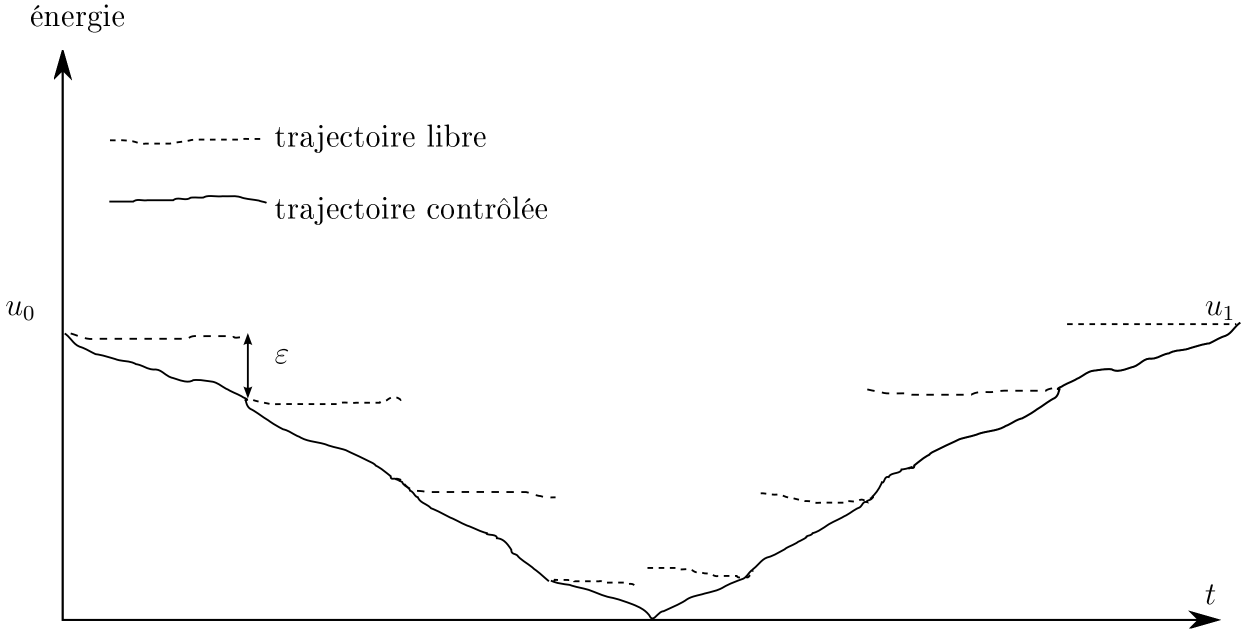

FIGURE 1.2 – Stratégie globale par contrôles successifs

Dans cette thèse, on utilisera cette méthode à de nombreuses reprises et principalement pour atteindre deux objectifs :

- montrer des inégalités d'observabilité pour des équations linéaires et en déduire un résultat de contrôlabilité par la méthode HUM. Ces équations linéaires seront en fait les linéarisées de l'équation non linéaire près d'une trajectoire. Le but est alors d'obtenir un contrôle local près d'une trajectoire.
- montrer une inégalité d'observabilité sur une équation non linéaire amortie. Une conséquence directe est alors la décroissance exponentielle de la solution.

On se propose dans cette sous-partie de décrire rapidement cette méthode sur le cas modèle de l'équation de Schrödinger non linéaire amortie sur $\mathbb{T}^1$:

$$i\partial_t u + \partial_x^2 u + ia^2(x)u = \pm|u|^2 u.$$

Pour établir la décroissance exponentielle en norme L^2, on doit prouver, pour des solutions de norme bornée par une constante, l'inégalité :

$$\|u_0\|_{L^2}^2 \leq C \int_0^T \|au(t)\|_{L^2}^2 \, dt.$$

On raisonne par contradiction. Soit u_n une suite de solutions contredisant l'observabilité :

$$\int_0^T \|au_n(t)\|_{L^2}^2 \, dt < \frac{1}{n} \|u_{n,0}\|_{L^2}^2 . \tag{1.8}$$

Supposons d'abord que $\|u_{n,0}\|_{L^2}$ ne converge pas vers zéro, le cas contraire étant plus simple puisque la solution est asymptotiquement linéaire. Comme la suite u_n est bornée, on peut extraire une sous-suite qui converge faiblement vers une fonction u. Si on appelle ω un ouvert où $a(x) \geq \eta > 0$, (1.8) établit la convergence forte vers zéro de u_n dans $L^2([0,T], L^2(\omega))$. Par un argument de propagation de la compacité (voir sous-section 1.1.2.4), on établit que u_n converge en fait fortement vers u. Ceci peut être prouvé par

une mesure de défaut microlocal ou un argument élémentaire (mais inspiré de l'analyse microlocale) en dimension 1. C'est le premier argument de "propagation de l'information". On en déduit aussi que u est solution de l'équation non linéaire.
Mais (1.8) implique $u \equiv 0$ sur ω. En particulier, u est C^∞ sur ω. On utilise alors un argument de propagation de la régularité pour prouver que la limite faible u est régulière. On conclut par un argument de prolongement unique (souvent prouvé par des inégalités de Carleman) que $u \equiv 0$, ce qui fournit une contradiction puisque cela implique que u_n converge fortement vers zéro dans L^2.

Pour résumer, la méthode ramène principalement à prouver pour des équations non linéaires :

- un théorème de propagation de la compacité
- un théorème de propagation de la régularité
- un théorème de prolongement unique.

Les deux premières étapes étudient le comportement haute fréquence alors que la dernière est essentiellement un problème basse fréquence. Par exemple, dans le chapitre 5 sur l'équation des ondes critique, on ne parvient pas à accomplir cette troisième étape et on obtient alors seulement un théorème de stabilisation à haute fréquence.

Dans le problème basse fréquence du prolongement unique, la non linéarité joue un rôle important. On utilise alors des inégalités de Carleman que l'on décrira brièvement à la sous-section 1.1.2.6.

1.1.2.4 Théorèmes de propagation et mesure de défaut microlocal

Comme on l'a vu dans la sous-section précédente, le raisonnement par compacité-unicité nécessite de prouver un théorème de propagation de la compacité. Notons que dans certains cas qui ne seront pas abordés ici, cette étape est directement prouvée par un effet régularisant de l'équation. C'est par exemple ce qui se passe pour KdV avec certaines conditions au bord pour lesquelles une donnée L^2 produit une solution $L^2([0,T], H^1)$ (voir [77] et [73] par exemple). Dans toutes les équations étudiées ici, il n'y aura pas de tel effet régularisant. Cette compacité ne peut alors provenir que d'arguments de propagation. On prouvera donc cette propagation par des arguments microlocaux et des mesures de défaut microlocal. En dimension 1, on pourra faire un raisonnement élémentaire mais dont les idées resteront les mêmes.

Dans des géométries plus compliquées, l'outil fondamental sera les mesures de défaut microlocal de Patrick Gérard [49] et Luc Tartar [85]. Elles mesurent le défaut de convergence forte (locale) d'une suite u_n, faiblement convergente vers 0 dans $L^2(\mathbb{R}^d)$. Ainsi, pour une telle suite, $|u_n|^2$ est une suite bornée de L^1 et on peut donc en extraire une sous-suite qui converge au sens faible $*$ vers une mesure ν sur $\mathbb{R}^d$. Cette mesure est bien adaptée pour analyser certains défauts de convergences comme les phénomènes de concentration (par exemple $u_n = n^{d/2}\psi(nx)$). Cependant, elle ne permet pas de bien comprendre les phénomènes d'oscillation (par exemple $u_n = \psi(x)e^{inx\cdot\xi_0}$ où $\xi_0 \in \mathbb{R}^d$ est un vecteur fixe), qui sont pourtant très importants dans les équations des ondes ou de Schrödinger.

La mesure de défaut microlocal est une extension au niveau microlocal de ces mesures. C'est une mesure sur le fibré cotangent en sphères $S^*\mathbb{R}^d \approx \mathbb{R}^d_x \times S^{d-1}_\xi$: la variable x est le point de l'espace et la variable ξ va mesurer la direction d'oscillation. Plus précisément, on peut prouver qu'il existe une sous-suite et une mesure positive μ sur $S^*\mathbb{R}^d$ telles que pour tout A opérateur pseudo-différentiel, à noyau à support compact, polyhomogène d'ordre 0, on a

$$(Au_n, u_n)_{L^2} \underset{n\to\infty}{\longrightarrow} \int_{S^*\mathbb{R}^d} a(x,\xi) d\mu(x,\xi),$$

où a est le symbole principal de A. Ainsi, notre suite d'oscillations $u_n = \psi(x)e^{inx\cdot\xi_0}$ admet pour mesure microlocale $|\psi(x)|^2 dx \otimes \delta_{\xi=\xi_0/|\xi_0|}$, exprimant le fait qu'elle oscille beaucoup dans la direction ξ_0. Le lien avec la mesure précédente ν peut alors être fait en prenant comme opérateur test une fonction $\varphi(x) \in C_0^\infty$. On obtient alors que si la mesure d'une suite est nulle sur $S^*\omega$ où ω est un ouvert de $\mathbb{R}^d$, alors $\varphi(x)u_n$ converge fortement vers 0 dans L^2 pour $\varphi(x) \in C_0^\infty(\omega)$.

De plus, ces mesures se comportent bien vis-à-vis du changement de variable et peuvent ainsi être définies sur une variété. L'espace L^2 peut aussi être remplacé par un espace de Sobolev H^s.

On pourra démontrer des théorèmes de propagation de ces mesures pour des solutions d'équations, menant à des théorèmes de propagation de la compacité. Par exemple, pour des solutions de l'équation des ondes, on peut considérer la mesure en espace-temps et montrer qu'elle se propage selon le flot bicaractéristique. On a

$$H_p \mu = 0$$

où H_p est le champ hamiltonien associé au symbole de l'opérateur des ondes $p = \tau^2 - |\xi|_x^2$ où τ est la variable duale du temps t et $|\xi|_x^2$ est la norme induite sur T^*M par la structure Riemannienne. Ceci exprime la propagation de l'information le long des géodésiques.

Cependant, pour l'équation de Schrödinger, le temps et l'espace jouent des rôles très distincts et on doit considérer des opérateurs tangentiels qui ne contiennent pas de "dérivée" en temps (c'est à dire des opérateurs qui ne dépendent pas de la variable duale τ du temps). Cette idée a été introduite par Bardos-Masrour dans [9] et conduit à des théorèmes de propagation à "vitesse infinie". Par exemple, citons un théorème issu de [42] :

Théorème 6 ([42])**.** *Soit M une variété compacte et $T > 0$ arbitraire. Soit u_n faiblement convergente vers 0 dans $L^\infty([0,T], H^1(M))$ et vérifiant*

$$\left\{ \begin{array}{rcl} i\partial_t u_n + \Delta u_n & \underset{n\to\infty}{\longrightarrow} & 0 \text{ dans } L^2([0,T], H^1(M)) \\ u_n & \underset{n\to\infty}{\longrightarrow} & 0 \text{ dans } L^2([0,T], H^1(\omega)) \end{array} \right.$$

où ω satisfait la condition de contrôle géométrique. Alors u_n converge fortement vers 0 dans $L^\infty([0,T], H^1(M))$.

Dans cette thèse, on sera amené à montrer plusieurs variantes de ce théorème, notamment dans les espaces de Bourgain. On le montrera aussi de façon élémentaire pour Schrödinger et KdV en dimension 1.

Pour prouver un tel théorème de propagation, on prouve que μ vérifie l'équation de transport $H_p \mu = 0$ où cette fois p est le symbole du Laplacien en espace. La mesure se propage le long des géodésiques, à temps t fixe, donc de façon immédiate. Notons que contrairement à l'équation des ondes, il n'y a en général pas de lien entre la trace de cette mesure au temps t, si elle existe, avec la mesure associée à $u_n(t)$ (voir Macía [69] pour des exemples où ce n'est pas le cas, dans un cadre semi-classique).

Notons aussi qu'il existe des variantes des mesures pour des suites u_n à valeur dans un espace de Hilbert. Nous n'utiliserons pas ces raffinements. Cependant, dans le chapitre 5, nous utiliserons des mesures jointes : pour deux suites u_n, $\widetilde{u}_n$, on cherche le comportement de $(Au_n, \widetilde{u}_n)_{L^2}$, c'est à dire le défaut d'othogonalité des deux suites. On a bien dans ce cas l'existence de mesure de défaut microlocal. Cependant, pour les définir, on doit considérer la suite $U_n = (u_n, \widetilde{u}_n)$ à valeur dans $\mathbb{C}^2$ et la limite est une matrice 2×2 hermitienne positive à valeur mesure. Les termes diagonaux sont les mesures respectives de u_n et $\widetilde{u}_n$ alors que les termes croisés sont conjugués et sont bien la mesure de défaut d'orthogonalité cherchée. Cette mesure est alors complexe et n'a plus de raison d'être positive. Elle vérifie en tout cas les même propriétés de propagation que les mesures de solutions de l'équation des ondes.

On renvoie à [21] pour une présentation plus détaillée sur les mesures.

1.1.2.5 Linéarisabilité et décomposition en profils

Dans l'étape de propagation de la compacité de la méthode de compacité-unicité, on a une suite faiblement convergente et on veut prouver sa convergence forte. Dans le but d'appliquer les théorèmes de la section précédente, on peut alors utiliser le concept de linéarisabilité, introduit par Patrick Gérard [50]. Il consiste, pour une équation non linéaire, à se demander si une suite de données initiales faiblement convergente produit des solutions non linéaires proches des solutions linéaires associées.

De façon un peu caricaturale, on peut décrire la situation de la façon suivante.

Pour le problème haute fréquence, lorsqu'on est face à une équation sous-critique (voir une description à la sous-section 1.1.5), le problème non linéaire est proche du problème linéaire. On est alors dans le cas "linéarisable" décrit par Patrick Gérard [50] : pour une suite de données initiales (φ_n, ψ_n) faiblement convergente vers 0 (dans l'espace d'énergie $\mathcal{E} = \dot{H}^1 \times L^2$) et à support compact fixe, les solutions de l'équation non linéaire $\Box u_n + |u|^{p-1}u$ $(p < 5)$ seront asymptotiquement proches des solutions linéaires de $\Box v_n = 0$ avec mêmes données initiales. On aura $\|u_n - v_n\|_{L^\infty([0,T],\mathcal{E})} \underset{n\to\infty}{\longrightarrow} 0$. Dans ce cas, on peut utiliser les résultats linéaires connus et décrits à la section 1.1.2.4 et 1.1.1. En pratique, cela consiste à observer que dans l'estimation du terme non linéaire, on peut faire des estimations "douces" (c'est-à-dire un produit de normes dont certaines sont moins fortes) dans lesquelles apparaîtra un terme qui sera compact par rapport à la topologie "ambiante". Cette analyse sera utilisée dans le chapitre 3 pour l'équation de Schrödinger non linéaire en dimension 3. L'inégalité douce qui permet d'avoir la linéarisabilité est, pour un petit $\varepsilon > 0$,

$$\left\||u|^2 u\right\|_{X^{1,-b'}} \leq \|u\|^2_{X^{1-\varepsilon,b'}} \|u\|_{X^{1,b'}} .$$

La compacité est alors obtenue grâce au terme $X^{1-\varepsilon,b'}$.

En revanche, dans le cas critique, des grandes données initiales haute fréquence peuvent créer des comportements non linéaires. C'est ce qui se passe par exemple pour l'équation des ondes quintique en dimension 3. Heureusement, on est capable de décrire précisément quel est le défaut de linéarisabilité d'un suite faiblement convergente. C'est la décomposition en profils de Bahouri-Gérard [6] sur $\mathbb{R}^3$. Le défaut de linéarisabilité provient des suites de solutions concentrantes, c'est à dire de la forme

$$p_n = \frac{1}{\sqrt{h_n}} p\left(\frac{t-t_n}{h_n}, \frac{x-x_n}{h_n}\right). \tag{1.9}$$

avec $h_n \underset{n\to\infty}{\longrightarrow} 0$. On les appellera ondes de concentration. Le résultat principal de [6] peut être résumé ainsi.

Décomposition en profils. Toute suite u_n bornée de solutions , localisées en espace, de l'équation non linéaire peut s'écrire, à extraction près, comme la somme de :
- la limite faible u de u_n qui est solution de l'équation non linéaire,
- une suite de solutions linéaires,
- une somme infinie d'ondes de concentration,
- un reste arbitrairement petit.

Dans le chapitre 5 de cette thèse on étendra ce résultat au cadre d'une variété compacte. Comme on ne dispose pas de dilatation dans ce cas, on doit alors redéfinir ce qu'est une onde de concentration et faire attention aux effets géométriques, notamment à la reconcentration.

Notons que parfois, la régularité limite (au sens de la régularité minimale pour laquelle le flot non linéaire est bien posé) n'est pas nécessairement celle dictée par le scaling, ceci pour des raisons variées : géométriques (voir [24], [87]), perturbation du mode 0 (voir [71]), etc... Par exemple, pour le cas du contrôle de l'équation de Schrödinger non linéaire sur $\mathbb{T}^1$ abordé au chapitre 2, la régularité critique provenant du scaling est $H^{-\frac{1}{2}}$ mais le flot n'est Lipschitz sur aucun $H^{-\varepsilon}$, $\varepsilon > 0$. En ce sens, la régularité L^2 peut être vue comme critique et la propriété de linéarisabilité à ce niveau est fausse (voir L. Molinet [71] pour une description des limites de suites faiblement convergentes). Dans ce cas, on concluera sans utiliser la linéarisabilité, grâce à de bonnes propriétés de propagation de la compacité.

1.1.2.6 Prolongement unique et inégalités de Carleman

Comme on l'a précisé précédemment, la troisième étape de la méthode de compacité-unicité, l'étape basse fréquence, consiste à prouver un théorème de prolongement unique. On appelle prolongement unique un théorème du type (on prend l'exemple de Schrödinger) :

Prolongement unique L'unique solution de

$$\left\{\begin{array}{rcl} i\partial_t u + \Delta u &=& a(t,x)u + b(t,x)\overline{u} \\ u &\equiv& 0 \text{ sur } \omega \end{array}\right.$$

est la solution $u \equiv 0$.

Bien sûr, dans chaque cas, il faut préciser la régularité des solutions u et des potentiels a et b. Cependant, dans les cas étudiés, on aura effectué auparavant l'étape de propagation de la régularité, de sorte que ceux-ci auront la régularité "suffisante".

Dans certains cas favorables, on peut utiliser le caractère analytique des coefficients en une ou toutes les variables. Ce ne sera pas le cas dans nos exemples où les potentiels dépendant de l'espace et du temps sont obtenus à partir d'une équation non linéaire. Pour prouver ce genre de théorème dans ces situations, on utilise des inégalités de Carleman. Ce sont des inégalités à poids où on fait intervenir un grand paramètre s qui va accentuer l'influence de ce poids $\varphi(x)$. Les inégalités prennent alors la forme pour des fonctions à support compact u (les puissances de s sont celles pour l'opérateur de Schrödinger P)

$$\left\|e^{-s\varphi}Pv\right\|_{L^2}^2 \geq Cs\left\|e^{-s\varphi}\nabla v\right\|_{L^2}^2 + Cs^3\left\|e^{-s\varphi}v\right\|_{L^2}^2.$$

Esquissons l'idée générale pour montrer cette inégalité. Si P est notre opérateur différentiel linéaire, on va conjuguer P par l'opérateur de multiplication $e^{-s\varphi}$ pour donner l'opérateur différentiel $P_\varphi = e^{-s\varphi}Pe^{s\varphi}$ (dont les coefficients seront des produits de puissances de s et de dérivées de φ). En posant $v = e^{s\varphi}u$ et après absorption de certains termes si s est grand, l'inégalité revient à prouver

$$\left\|P_\varphi u\right\|_{L^2}^2 \geq Cs\left\|\nabla u\right\|_{L^2}^2 + Cs^3\left\|u\right\|_{L^2}^2.$$

On décompose $P_\varphi = P_\varphi^r + P_\varphi^i$ en ses parties autoadjointe et antiautoadjointe avec $P_\varphi^r = (P_\varphi + P_\varphi^*)/2$ et $P_\varphi^i = (P_\varphi - P_\varphi^*)/2$. Les exposants r et i de la notation choisie soulignent le fait que les symboles principaux de P_φ^r et P_φ^i sont reliés aux parties réelle et imaginaire du symbole principal de P_φ. On a alors

$$\left\|P_\varphi u\right\|_{L^2}^2 = \left\|P_\varphi^r u\right\|_{L^2}^2 + \left\|P_\varphi^i u\right\|_{L^2}^2 + ([P_\varphi^r, P_\varphi^i]u, u)_{L^2}.$$

On va alors surtout utiliser ce terme de commutateur et montrer qu'il est positif au sens des opérateurs autoadjoints pour s grand. Le but est de démontrer que les termes dominants (en dérivabilité et en puissance de s) sont positifs. Ceci peut être fait en calculant explicitement le terme de commutateur par des intégrations par parties (licites puisque u est à support compact). On extrait le terme dominant (par exemple dans notre cas avec une puissance 3 en s pour les termes d'ordre 0 et une puissance 1 de s pour ceux d'ordre 1). Cela nous donne alors une condition sur φ (dite pseudo-convexité par rapport à l'opérateur P) pour que ce terme soit strictement positif et permette d'absorber les termes de degré plus faible en prenant s assez grand. En pratique, on cherche à le mettre sous une forme quadratique positive en u et ses dérivées. C'est ce qu'on fera dans cette thèse, où les calculs explicites seront menés pour obtenir des inégalités de Carleman pour Schrödinger à coefficients variables.

Une autre manière de procéder est de voir ce problème comme un problème semi-classique avec petit paramètre $h = 1/s$. Cette façon de prouver des inégalités de Carleman peut sembler plus complexe, mais elle a l'intérêt de permettre une meilleure interprétation des conditions que l'on obtient sur φ.

Par commodité d'écriture, on pose $P_\varphi^i = i\widetilde{P}_\varphi^i$ de sorte que $\widetilde{P}_\varphi^i$ soit autoadjoint à symbole réel. On obtient

$$\left\|P_\varphi u\right\|_{L^2}^2 = \Big(((P_\varphi^r)^2 + (\widetilde{P}_\varphi^i)^2 + i[P_\varphi^r, \widetilde{P}_\varphi^i])u, u\Big)_{L^2}.$$

Or, l'opérateur du membre de droite est un opérateur semi-classique de symbole $p_r^2 + \tilde{p}_i^2 + h\{p_r, p_i\}$. Il reste juste maintenant à trouver des bonnes conditions sur φ pour que ce symbole soit positif pour h assez petit et permette l'utilisation de l'inégalité de Gårding. On renvoie à Le Rousseau-Lebeau [62] ou Tataru [86] pour une présentation de ces idées.

Quelle que soit la méthode (ce sont en fait mathématiquement les mêmes, seule la mise en œuvre change), les conditions pour φ que l'on obtient dépendent de l'opérateur P que l'on considère et de la géométrie de son symbole. De plus, dans le cas de Schrödinger par exemple, l'anisotropie de l'opérateur complique un peu le raisonnement. Dans ce cas précis, on prend une fonction poids

$$\varphi(t,x) = \frac{e^{\lambda C} - e^{\lambda\psi(x)}}{(T-t)(T+t)}.$$

Une condition pour ψ est alors $\nabla\psi_x \neq 0$ et

$$Hess(\psi)_x(\xi,\xi) + |\nabla\psi_x \cdot \xi| > 0$$

pour tout $\xi \in T_xM$ et x dans le support de u.

Notons que si les conditions sur ψ sont valables partout sauf sur un ouvert ω, on obtient l'inégalité de Carleman avec un terme d'observation en plus, c'est-à-dire un terme supporté dans ω dans le membre de gauche. Cela permet de trouver des ψ effectifs qui remplissent les conditions de pseudo-convexité. On obtient alors des inégalités du type, en notant $\theta(t,x) = \frac{e^{\lambda\psi(x)}}{(T-t)(T+t)} > 0$

$$\iint \left[s^3\lambda^4\theta^3|v|^2 + s\lambda\theta\,|\nabla v|^2\right] e^{-2s\varphi} \leq C \iint_\omega \left[s^3\lambda^4\theta^3|v|^2 + s\lambda\theta\,|\nabla v|^2\right] e^{-2s\varphi}.$$

pour v solution à support compact de $Pv = 0$. En faisant tendre s vers l'infini, on montre que v est nulle sur des domaines où ψ prend des valeurs strictement plus grandes que sur ω.

Notons qu'une compréhension superficielle de cette méthode peut donner l'impression que l'on peut montrer que toute solution est nulle. Cependant, on ne pourra jamais appliquer ce théorème directement de façon globale sur une variété compacte puisque ω doit alors contenir les points où le gradient de ψ s'annule donc au moins son maximum et son minimum. On l'appliquera alors sur des ouverts particuliers de cette variété et on utilisera de façon cruciale que la solution est à support compact.

Précisons que d'autres versions proches des inégalités de Carleman que l'on prouve dans cette thèse existaient déjà [55, 61, 88] (dont certaines qui m'ont été signalées par le referee d'un des articles) mais pas dans la situation exacte qui permettait de conclure facilement. Cependant, un autre intérêt de cette démonstration est qu'elle étend à des métriques variables les résultats de Mercado-Rosier-Osses [70] qui prouvent des inégalités de Carleman affaiblies avec des poids faiblement pseudo-convexes. Ces inégalités permettent d'obtenir des résultats de prolongement unique dans des cas où la condition de contrôle géométrique n'est pas vérifiée (par exemple une bande sur un tore) et pourraient se révéler utiles dans d'autres situations.

Pour le moment, on ne sait pas encore très bien quelles sont les conditions optimales globales pour le prolongement unique. Les conditions de pseudo-convexité semblent donner des conditions presque optimales pour le prolongement unique local, mais on ne sait pas vraiment en déduire des critères globaux généraux. Une question naturelle est par exemple de savoir si la condition de contrôle géométrique implique la propriété de prolongement unique pour l'équation de Schrödinger.

Après cette introduction générale, on se propose maintenant de détailler un peu plus les résultats de cette thèse. On a naturellement divisé la présentation selon l'équation étudiée.

1.1.3 Contrôle de l'équation de Schrödinger non linéaire

Les chapitres 2 et 3 concernent le contrôle interne de l'équation de Schrödinger non linéaire sur des variétés compactes. Nous nous intéresserons principalement à l'équation de Schrödinger non linéaire cubique

$$i\partial_t u + \Delta u = \pm |u|^2 u.$$

Rappelons que cette équation possède deux énergies formellement conservées :

$$\begin{aligned} \text{la masse} \quad \|u\|_{L^2}^2 &= \int_M |u|^2 dx \\ \text{l'énergie non linéaire} \quad E(u) &= \frac{1}{2}\int_M |\nabla u|^2 dx \pm \frac{1}{4}\int_M |u|^4 dx. \end{aligned}$$

Ces deux énergies permettent de prouver des résultats d'existence globale. L'équation sur $\mathbb{R}^d$ préserve aussi l'impulsion, mais cela n'a pas d'équivalent (à ma connaissance) sur une variété.

Le chapitre 2 de cette thèse étudie cette équation sur un intervalle et le chapitre 3 sur des variétés compactes de dimension 3. Les résultats sur la dimension 1 forment une partie à part puisqu'ils sont plus forts que ce qu'on obtient en dimension supérieure, notamment quant à la régularité et par le fait qu'on aborde la non-linéarité focalisante et défocalisante. Les résultats du chapitre 3 peuvent aussi s'appliquer aux dimensions inférieures mais apportent moins de nouveauté dans ces cas.

1.1.3.1 Résultats antérieurs

Les premiers résultats de contrôlabilité pour l'équation de Schrödinger non linéaire sont des résultats locaux, c'est-à-dire pour des données petites, souvent prouvés par perturbation de résultats linéaires. Citons par exemple Rosier-Zhang [79] pour des intervalles puis des rectangles [81].

Le premier résultat de contrôlabilité de grandes données pour l'équation de Schrödinger non linéaire apparaît dans l'article de Dehman-Gérard-Lebeau [42]. La preuve se fait par stabilisation et suit le schéma indiqué aux sections 1.1.2.2 et 1.1.2.3. Ils prouvent un théorème de contrôlabilité globale en grand temps pour des variétés compactes de dimension 2 sous deux hypothèses principales

- l'hypothèse de contrôle géométrique comme celle décrite pour le problème linéaire (voir sous section 1.1.1)

– le prolongement unique pour une équation de Schrödinger avec potentiel comme décrit dans la sous-section 1.1.2.6.

Le théorème principal qu'ils obtiennent est le suivant :

Théorème 7 (Dehman-Gérard-Lebeau [42]). *Soit P un polynôme réel vérifiant $P(0) = 0$ et $P'(r) \underset{r\to+\infty}{\longrightarrow} +\infty$. Soit M une variété compacte de dimension 2 et $\omega \subset M$ un ouvert vérifiant les conditions de contrôle géométrique et de prolongement unique décrits précédemment.*

Alors, pour tout $R_0 > 0$, il existe $T > 0$ tel que pour tout u_0, u_1 satisfaisant $\|u_0\|_{H^1} \leq R_0$ et $\|u_1\|_{H^1} \leq R_0$, il existe un contrôle $g \in C([0,T], H^1)$ supporté dans $[0,T] \times \omega$ tel que l'unique solution dans $C([0,T], H^1)$ de

$$\left\{ \begin{array}{rcl} i\partial_t u + \Delta u - P'(|u|^2)u & = & g \\ u(0) = u_0 & & \end{array} \right.$$

satisfasse $u(T) = u_1$.

Les auteurs utilisent les estimées de Strichartz sur des variétés compactes prouvées par N. Burq, P. Gérard et N. Tzvetkov [25]. Cela leur permet de raisonner dans les espaces de Sobolev, notamment pour les théorèmes de propagation, ce que l'on ne pourra pas faire dans les exemples traités dans cette thèse. De plus, on est ici dans le cas "linéarisable", c'est-à-dire qu'à haute fréquence, les solutions sont proches des solutions linéaires avec mêmes données initiales.

Les auteurs suivent le schéma de preuve décrit dans la sous-section 1.1.2.2 en utilisant l'équation amortie suivante :

$$i\partial_t u + \Delta u - a(x)(1-\Delta)^{-1}a(x)\partial_t u = P'(|u|^2)u.$$

Ce terme de stabilisation permet d'avoir décroissance de l'énergie $E(u) = \int_M |\nabla u|^2 + P(|u|^2)$. Le terme le plus naturel pour faire décroître cette énergie aurait plutôt été $a(x)^2\partial_t u$, mais ce choix différent permet de considérer le terme de stabilisation comme un terme source dans H^1 (si u est dans H^1, on aura seulement $\partial_t u \in H^{-1}$ par l'équation).

Dans le chapitre 3, on utilisera ce même terme de stabilisation pour prouver le même type de résultats sur des variétés de dimension 3. Cependant, dans le chapitre 2, on utilisera une autre équation faisant décroître l'énergie L^2.

1.1.3.2 Résultats principaux de la thèse

Commençons d'abord par décrire les résultats concernant la dimension 1.

Résultats en dimension 1 On étudiera le contrôle de l'équation de Schrödinger sur un intervalle avec conditions périodiques, de Dirichlet ou Neumann.

Tout d'abord, une remarque simple s'impose : on peut déduire facilement des résultats de contrôle avec conditions de Dirichlet ou Neumann à partir de résultats sur le tore $\mathbb{T}^1$, correspondant à des données périodiques sur $\mathbb{R}$. En effet, on peut identifier le domaine du Laplacien de Dirichlet $D(-\Delta_D)$ (resp $D(-\Delta_N)$) sur l'intervalle $[0,\pi]$ avec le sous-espace fermé de $H^2(\mathbb{R}/2\pi\mathbb{Z})$ formé des solutions impaires (resp paires). On doit juste vérifier le long de la preuve que le contrôle que l'on construit sur $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$ reste impair (resp pair) si u_0 l'est. La préservation de la régularité prendra alors la forme : si $u_0 \in D(-\Delta_D^s)$, alors le contrôle g est dans $C([0,T], D(-\Delta_D^s))$ (de même pour Δ_N). On ne précisera plus cela par la suite et on raisonnera uniquement sur $\mathbb{T}^1$. Notons que cet artifice fonctionne parce que l'opérateur de Schrödinger ne privilégie pas de direction. Il ne fonctionne pas pour KdV où il y a un sens de propagation privilégié.

La géométrie est donc ici plus simple que dans [42]. Cependant, on cherche à prouver un résultat de contrôlabilité au niveau de régularité L^2. Outre l'intérêt intrinsèque d'abaisser le plus possible la régularité, on gagne surtout le fait de pouvoir utiliser la norme L^2 qui est conservée par le flot de NLS. Ceci permet ainsi d'obtenir un résultat valable pour une équation focalisante ou défocalisante, c'est-à-dire quel que soit le signe devant la non-linéarité. Un autre avantage de raisonner à la régularité L^2 est que l'équation amortie associée est plus naturelle (ici λ est un réel quelconque) :

$$\left\{ \begin{array}{rcl} i\partial_t u + \partial_x^2 u + ia(x)u &=& \lambda |u|^2 u \quad \text{sur} \quad [0, +\infty[\times \mathbb{T}^1 \\ u(0) &=& u_0 \in L^2(\mathbb{T}^1). \end{array} \right. \tag{1.10}$$

Cependant, pour raisonner à ce faible niveau de régularité, les espaces de Strichartz utilisés dans [42] ne sont plus suffisants et on utilise les espaces de Bourgain. Ceci occasionne des difficultés techniques supplémentaires. Il faut ainsi démontrer des théorèmes de propagation adaptés à ces espaces. De plus, ces espaces sont plus difficiles à manier et, par exemple, ne sont pas stables par la multiplication par une fonction C^∞ (ceci exprime le fait que cette multiplication commute mal avec l'opérateur de Schrödinger).

Une autre difficulté est que les suites de solutions de l'équation de Schrödinger non linéaire ne sont pas linéarisables (au sens de Patrick Gérard comme on en a discuté à la section 1.1.2.3) sur L^2. Par exemple, les solutions explicites de données initiales e^{inx} : $e^{-i(n^2 \mp 1)t + inx}$ ne sont pas asymptotiquement proches dans L^2 de la solution linéaire $e^{in^2 t + inx}$. On raisonne alors de façon différente. Bien que les solutions haute fréquence de l'équation non linéaire ne soient pas proches en norme L^2 des mêmes solutions de l'équation linéaire, elles conservent leurs propriétés de propagation de la compacité et de la régularité. Ces propriétés de propagation s'expriment par le fait que le second membre de l'équation de Schrödinger est autorisé à converger vers 0 dans un espace plus faible.

Le résultat principal que l'on obtient est le suivant :

Théorème 8. *Soit $1/2 < b < 5/8$. Pour tout ouvert non vide $\omega \subset \mathbb{T}^1$ et $R_0 > 0$, il existe $T > 0$ et $C > 0$ tels que pour tous u_0 et u_1 dans $L^2(\mathbb{T}^1)$ avec*

$$\|u_0\|_{L^2} \leq R_0 \quad \text{et} \quad \|u_1\|_{L^2} \leq R_0$$

il existe un contrôle $g \in C([0,T], L^2)$ avec $\|g\|_{L^\infty([0,T],L^2)} \leq C$ supporté dans $[0,T] \times \omega$, tel que l'unique solution u dans $X_T^{0,b}$ du problème de Cauchy

$$\left\{ \begin{array}{rcl} i\partial_t u + \partial_x^2 u &=& \lambda |u|^2 u + g \quad \text{sur} \quad [0,T] \times \mathbb{T}^1 \\ u(0) &=& u_0 \in L^2(\mathbb{T}^1) \end{array} \right.$$

satisfasse à $u(T) = u_1$.
De plus, si u_0, $u_1 \in H^s$, avec $s \geq 0$, on peut imposer $g \in C([0,T], H^s)$.

La deuxième partie du théorème est un résultat de régularité du contrôle. Par exemple, si $u_0, u_1 \in C^\infty(\mathbb{T}^1)$, on obtient un contrôle et donc une solution dans $C^\infty([0,T] \times M)$. La difficulté pour prouver cette régularité supplémentaire du contrôle si $u_i \in H^s$ est surtout contenue dans le fait que cette régularité reste vraie quel que soit $s > 0$ et sans hypothèse sur la taille de u_0, u_1 dans H^s.

Comme décrit dans la sous-section 1.1.2.2, on prouve ce résultat en deux étapes : stabilisation et contrôle local. Le théorème de stabilisation est le suivant :

Théorème 9. *On suppose $a(x)^2 > \eta > 0$ sur un ouvert non vide. Alors, pour tout $R_0 > 0$, il existe $C > 0$ et $\gamma > 0$ tels que l'inégalité*

$$\|u(t)\|_{L^2} \leq Ce^{-\gamma t} \|u_0\|_{L^2} \quad t > 0$$

soit vérifiée pour toute solution u du système (1.10) avec donnée initiale u_0 telle que $\|u_0\|_{L^2} \leq R_0$.

On le prouve en suivant la méthode de compacité-unicité. Le prolongement de la régularité et de la compacité adapté aux espaces de Bourgain est alors montré de façon élémentaire mais en gardant toujours le même principe issu de l'analyse microlocale de faire apparaître un commutateur. Précisons aussi que comme l'a remarqué Luc Molinet [71], les solutions correspondant à une suite de données initiales faiblement convergente dans L^2 ne convergent (même faiblement) pas forcément vers une solution de l'équation initiale. On ne peut affirmer que la limite faible est solution de NLS qu'au dernier moment, après avoir montré la convergence forte. C'est alors seulement qu'on peut en déduire que la limite est nulle par prolongement unique.

Le théorème de contrôle local est le suivant :

Théorème 10. *Soit ω un ouvert non vide de $\mathbb{T}^1$ et $T > 0$. Alors, il existe $\varepsilon > 0$ et $\eta > 0$ tels que pour tout $u_0 \in L^2$ avec $\|u_0\|_{L^2} < \varepsilon$, il existe $g \in C([0,T], L^2)$, avec $\|g\|_{L^\infty([0,T],L^2)} \leq \eta$, à support dans $]0,T[\times\omega$ tel que l'unique solution u dans $X_T^{0,b}$ de*

$$\begin{cases} i\partial_t u + \partial_x^2 u &= \lambda |u|^2 u + g \\ u(x,0) &= u_0(x) \end{cases} \tag{1.11}$$

satisfasse $u(T) = 0$.
De plus, si $u_0 \in H^s$, avec $s \geq 0$, éventuellement avec une grande norme H^s, on peut imposer $g \in C([0,T], H^s)$.

Pour la régularité supplémentaire, on ne se préoccupe jamais de la taille H^s de la solution et le théorème de contrôlabilité locale reste vrai dans H^s, $s \geq 0$, avec seulement une hypothèse de taille L^2. Ce théorème est prouvé par perturbation du contrôle linéaire. On fait le point fixe dans L^2. On veut ensuite montrer que l'opérateur non linéaire de contrôle local ainsi construit préserve la régularité. Les deux ingrédients essentiels sont alors :

- l'opérateur HUM linéaire préserve la régularité avec des estimations "presque" uniformes en s (voir estimation (1.3)), comme précisé à la sous-section 1.1.1.3
- on a des estimations "douces".

$$\left\| |u|^2 u \right\|_{X_T^{s,-3/8}} \leq C \|u\|^2_{X_T^{0,3/8}} \|u\|_{X_T^{s,3/8}} + C_s \|u\|_{X_T^{s-1,3/8}} \|u\|_{X_T^{1,3/8}} \|u\|_{X_T^{0,3/8}}.$$

La subtilité de cette estimation est que le terme qui apparaît devant la norme de grande régularité s est : une constante C qui ne dépend pas de s et une norme dans $X_T^{0,3/8}$ donc de l'ordre de régularité de L^2. Une fois que la norme L^2 est bornée, le comportement des normes H^s est donc presque linéaire. Lors de l'estimation de point fixe, on aura alors seulement besoin d'une petitesse dans L^2 uniforme en s. Les termes faisant intervenir des régularités plus faibles n'ont alors pas besoin d'être petits.

Résultats en dimension 3 Les résultats de cette partie sont d'abord une généralisation des résultats de contrôlabilité globale en grand temps de [42] à la dimension 3. Cependant, on le démontre de deux manières différentes décrites dans la sous-section 1.1.2.2 : par stabilisation puis contrôle local et par contrôle près de trajectoires. La première méthode est donc la même que dans [42] alors que la deuxième n'avait, à ma connaissance, pas été utilisée pour ce genre de problèmes.

Pour obtenir ce résultat en dimension 3 avec l'équation cubique, on est contraint d'utiliser les espaces de Bourgain. En effet, les estimations de Strichartz de Burq-Gérard-Tzvetkov [25] ne permettent alors d'avoir un flot uniformément bien posé que sur H^s avec $s > 1$, manquant ainsi l'espace d'énergie. Les auteurs parviennent toutefois à prouver existence et unicité sur H^1, mais pas avec un flot uniformément continu, ce qui semble crucial pour prouver un problème de contrôlabilité. Cependant, les espaces de Bourgain permettent d'obtenir sur certaines variétés de meilleurs résultats pour le problème de Cauchy. En fait, en se basant sur les espaces mis au point par Bourgain, Burq-Gérard-Tzvetkov [23] ont montré que l'existence d'un flot régulier sur H^s, $s > s_0$, était essentiellement équivalente à l'inégalité multilinéaire

$$\|u_1 u_2\|_{L^2([0,T]\times M)} \leq C \min(N_1, N_2)^{s_0} \|f_1\|_{L^2(M)} \|f_2\|_{L^2(M)}$$
$$u_j(t) = e^{it\Delta} f_j, \quad j = 1, 2$$

pour des f_j vérifiant $f_j = \mathbf{1}_{\sqrt{1-\Delta}\in[N_j,2N_j[}(f_j)$, c'est-à-dire oscillant à la fréquence N_j.

On sait que cette propriété est vérifiée pour les exemples suivants ($1/2+$ signifie tout $s_0 > 1/2$) :
- $\mathbb{T}^3$ avec $s_0 = 1/2+$, voir [17]
- le tore irrationnel $\mathbb{R}^3/(\theta_1\mathbb{Z}\times\theta_2\mathbb{Z}\times\theta_3\mathbb{Z})$ avec $\theta_i \in \mathbb{R}$, pour lequel une estimée $s_0 = 2/3+$ a récemment été obtenue dans [18]. Une preuve plus simple pour $s_0 = 3/4+$ peut aussi être trouvée dans le début de [18] et dans [29]
- S^3 avec $s_0 = 1/2+$, voir [26]
- $S^2 \times S^1$ avec $s_0 = 3/4+$, voir [26].

Dans tous ces exemples, on a $s_0 < 1$ et on peut alors obtenir un problème globalement bien posé pour l'équation défocalisante grâce à la conservation de l'énergie. Dans la suite, les résultats énoncés le seront avec certaines hypothèses sur la variété et sur l'ouvert de contrôle ω. Les hypothèses seront de deux sortes :

– sur la variété M. On demandera que le problème soit régulièrement bien posé sur H^1 et donc que $s_0 < 1$. On appellera cette hypothèse "**problème bien posé**". On aura aussi besoin d'estimations de commutateurs que l'on ne précisera pas dans cette introduction mais seulement dans les chapitres concernés.
– des hypothèses sur le support du contrôle ω. Ce sera surtout le contrôle géométrique et un théorème de prolongement unique. La régularité sur les potentiels

pourra dépendre du cas considéré. On renvoie alors à l'énoncé précis dans les articles pour plus de précisions.

Ces propriétés seront vérifiées dans les exemples suivants :

- $\mathbb{T}^3$ avec $\omega = \{x \in \mathbb{R}^3/(\theta_1\mathbb{Z} \times \theta_2\mathbb{Z} \times \theta_3\mathbb{Z}) \,|\exists i \in \{1,2,3\}, x_i \in]-\varepsilon, \varepsilon[+\theta_i\mathbb{Z}\}$ (c'est-à-dire un voisinage de chaque face du "cube", volume fondamental de $\mathbb{T}^3$) avec $\theta_i \in \mathbb{R}$. De plus, on peut facilement étendre ce résultat à un "cube" avec des conditions de Dirichlet ou Neumann comme on l'a fait en dimension 1.
- S^3 où ω est un voisinage de $\{x_4 = 0\}$ dans $S^3 \subset \mathbb{R}^4$.
- $S^2 \times S^1$ avec $\omega = (\omega_1 \times S^1) \cup (S^2 \times]0, \varepsilon[)$ où ω_1 est un voisinage de l'équateur de S^2.

On obtient alors un résultat de contrôle global en grand temps.

Théorème 11. *Pour tout (ω, M), tels que les hypothèses de problème bien posé, de contrôle géométrique et de prolongement unique soient vérifiées, et $R_0 > 0$, il existe $T > 0$ et $C > 0$ tels que pour tous u_0 et u_1 dans $H^1(M)$ avec*

$$\|u_0\|_{H^1(M)} \leq R_0 \quad \text{et} \quad \|u_1\|_{H^1(M)} \leq R_0$$

il existe un contrôle $g \in C([0,T], H^1)$ avec $\|g\|_{L^\infty([0,T],H^1)} \leq C$ supporté dans $[0,T] \times \overline{\omega}$, tel que l'unique solution u dans $X_T^{1,b}$ du problème de Cauchy

$$\left\{ \begin{array}{rcll} i\partial_t u + \Delta u &=& |u|^2 u + g & \text{on} \quad [0,T] \times M \\ u(0) &=& u_0 \in H^1(M) & \end{array} \right.$$

satisfasse à $u(T) = u_1$.

Sous les mêmes hypothèses, on prouve pour cela un théorème de stabilisation similaire à ce qu'on obtient en dimension 1, mais pour l'équation suivante (la même que dans [42]) avec un $a(x)$ supporté dans ω),

$$i\partial_t u + \Delta u - a(x)(1-\Delta)^{-1} a(x)\partial_t u = (1 + |u|^2)u. \tag{1.12}$$

La deuxième méthode de preuve de contrôlabilité globale utilise le contrôle près d'une trajectoire. Celui-ci prend la forme suivante

Théorème 12. *Soit $T > 0$, M et ω tels que les hypothèses de "problème bien posé", de contrôle géométrique et de prolongement unique soient vérifiées. Soit $1 \geq s > s_0$ et $w \in X_T^{1,b}$ une solution de*

$$\left\{ \begin{array}{rcl} i\partial_t w + \Delta w \pm |w|^2 w &=& g \\ w(x,0) &=& w_0(x) \end{array} \right. \tag{1.13}$$

avec $g \in C([0,T], H^1)$ supporté dans $[0,T] \times \overline{\omega}$.

Alors, il existe $\varepsilon > 0$, tels que pour tout $u_0 \in H^s$ avec $\|u_0 - w_0\|_{H^s} < \varepsilon$, il existe $g_1 \in C([0,T], H^s)$ supporté dans $[0,T] \times \overline{\omega}$ tel que l'unique solution u in $X_T^{s,b}$ de (1.13) avec $u(0) = u_0$ et g remplacé par g_1 satisfasse à $u(T) = w(T)$.

De plus, pour tout $u_0 \in H^1$ avec $\|u_0 - w_0\|_{H^s} < \varepsilon$, on a la même conclusion avec $g_1 \in C([0,T], H^1)$.

On peut ainsi déduire plusieurs corollaires de ce théorème :

- Cela donne une seconde preuve du théorème de contrôlabilité global. Il faut pour cela suivre la deuxième méthode décrite dans la sous-section 1.1.2.2. Notons qu'il est pour cela indispensable que le ε ne dépende que de $\|w\|_{X_T^{1,b}}$ et pas de la solution w elle même.
- Pour tout temps T et avec les hypothèses du théorème, l'ensemble des données contrôlables à partir d'un état u_0 est un ouvert. Evidemment, pour un certain R_0, le théorème de contrôlabilité globale prouve qu'à partir d'un certain temps T on peut contrôler toute la boule de rayon R_0. Cependant, la contrôlabilité aux trajectoires montre que si le temps est trop petit, la seule obstruction à la contrôlabilité est la taille des données.
- La deuxième partie du théorème sur la régularité implique immédiatement la contrôlabilité "haute fréquence" près de la trajectoire. En effet, une fois que l'on a fixé une borne R_0 dans H^1 pour les données initiale u_0 et finale u_1, et pour la trajectoire w, on peut trouver un N et ε tels que la condition $\sum_{|k|\leq N} \left|\widehat{u_0}(k) - \widehat{w_0}(k)\right|^2 \leq \varepsilon$ (le $\widehat{}$ désigne la composante selon la k-ième fonction propre), implique que u_0 est assez proche de w_0 dans H^s, $s < 1$ (de même pour u_1 et w_T). Cela suffit alors à trouver un contrôle H^1 amenant u_0 à u_1. Cela constitue une généralisation au contrôle près d'une trajectoire de ce qui avait été prouvé par B. Dehman et G. Lebeau [43] pour l'équation des ondes et la trajectoire nulle. Notons que ce phénomène n'est possible que parce que l'équation est sous-critique. Il avait été prouvé dans [43] par une analyse en fréquences.

Le théorème de contrôle près d'une trajectoire est obtenu de façon perturbative. La partie difficile est alors de démontrer la contrôlabilité d'un opérateur linéarisé près d'une trajectoire w :

$$i\partial_t r + \Delta r = 2|w|^2 r + w^2 \overline{r} = \widetilde{g}. \tag{1.14}$$

Par la méthode HUM, on est ramené à prouver une inégalité d'observabilité pour l'équation duale que l'on résout donc dans des espaces de Sobolev à régularité proche de -1. Ensuite, on utilise la méthode de compacité-unicité. Celle-ci est rendue problématique par la faible régularité des potentiels $|w|^2$ et w^2. On y parvient tout de même grâce aux propriétés fortes de propagation de la compacité et de la régularité.

La deuxième partie du théorème, concernant la régularité du contrôle, se prouve comme en dimension 1 en prouvant que le contrôle HUM associé à (1.14) préserve lui aussi la régularité (comme décrit en sous-section 1.1.1.3). Si w était régulière, il suffirait d'appliquer un opérateur de dérivation $(-\Delta)^{\varepsilon/2}$ et d'appliquer des inégalités de commutateurs. C'est aussi ce que l'on fait mais on doit prouver ces inégalités pour des w dans $X_T^{1,b}$. Typiquement, on doit prouver que l'opérateur $[(-\Delta)^{\varepsilon/2}, |w|^2]$ agit de $X^{s,b}$ dans $X^{s,b}$. Il faut alors refaire des estimations proches de celles faites dans les preuves de Bourgain [17] et Burq-Gérard-Tzverkov [23, 26] sur les produits dans les espaces $X^{s,b}$. Ces preuves sont assez techniques et spécifiques aux variétés considérées.

Dans le cas où on veut faire un contrôle près de 0, il suffit d'appliquer les résultats de contrôle linéaire que l'on connaît. Par exemple, un contrôle près de 0 sur le tore ne nécessite pas de condition géométrique.

Théorème 13. *Si* (M, ω) *est soit :*
-($\mathbb{T}^3$,un ouvert arbitraire)
-($S^2 \times S^1$, $\omega_1 \times S^1$) où ω_1 est un voisinage de l'équateur S^2

-($S^2 \times S^1$,$S^2 \times]0, \varepsilon[$)
Alors, on a contrôle local près de 0 *et la préservation de la régularité comme au théorème 12.*

Pour le contrôle global, on ne sait pas si les hypothèses géométriques que l'on a faites sont vraiment nécessaires. Il y a cependant un cas où on sait que le contrôle géométrique est au moins nécessaire pour le théorème de stabilisation.

Proposition 1. *Soit* Γ *une géodésique de* S^3 *telle que* $Supp(a) \cap \Gamma = \emptyset$. *Alors, on n'a pas stabilisation exponentielle pour l'équation (1.12).*

La proposition 1 repose sur l'utilisation des modes de concentrations sur l'équateur $c_n(x_1 + ix_2)^n$ et la linéarisabilité à haute fréquence.

Pour finir, le prolongement unique est démontré dans tous les exemples considérés. On a recours pour cela à des inégalités de Carleman adaptées à l'équation de Schrödinger.

1.1.3.3 Problèmes ouverts et perspectives

De nombreux problèmes restent ouverts pour le contrôle de l'équation de Schrödinger.

Tout d'abord, pour l'équation de Schrödinger linéaire, la détermination de la condition géométrique nécessaire et suffisante pour avoir contrôlabilité est un problème largement ouvert. On sait que le contrôle géométrique est suffisant mais pas nécessaire. Il est cependant nécessaire si les trajectoires sont stables. Les modes de concentrations de Ralston [75] (ainsi que les modes explicites que l'on a utilisés sur la sphère) montrent que l'ouvert doit nécessairement intersecter toutes les trajectoires périodiques stables. Il s'agirait de comprendre quelles sont les trajectoires sur lesquelles des solutions hautes fréquences sont susceptibles de s'accumuler. Ce problème est relativement proche de la détermination des mesures semi-classiques des suites de fonctions propres du Laplacien. Notons que des idées nouvelles sont récemment apparues dans ce domaine et pourraient servir pour le problème du contrôle : utilisation de mesures 2-microlocales par F. Macía [68] et Anantharaman-Macía [4] ou l'utilisation de propriété dynamiques globales comme l'ergodicité par N. Anantharaman [3].

Le problème non linéaire ajoute de nouvelles questions. Ainsi, pour le contrôle de grandes données, les techniques employées dans cette thèse ne permettent d'avoir contrôlabilité que si l'on a contrôle géométrique. Pour le tore, on sait prouver le contrôle linéaire pour un ouvert ω arbitraire mais on ne sait pas pour l'instant en déduire un résultat non linéaire pour des grandes données. Une première étape pourrait être de prouver la contrôlabilité à partir d'une bande du tore, sachant que les inégalités de Carleman faibles que l'on obtient (ainsi que celle de [70]) permettent d'obtenir un théorème de prolongement unique pour cette situation.

Un autre question que l'on peut se poser est celle du prolongement unique. En dimension 3, le théorème que l'on prouve (et c'est aussi le cas dans [42]) est sous l'hypothèse de prolongement unique. On parvient à prouver ce prolongement unique sur des exemples non triviaux. Cependant, il serait bien sûr très intéressant d'avoir une condition géométrique plus intrinsèque pour cette propriété. Par exemple, est-ce que la condition de contrôle géométrique implique le prolongement unique ?

Une autre question naturelle est de prouver le même type de résultats pour des ouverts à bord. Des inégalités de Strichartz pour l'équation de Schrödinger ont ainsi été prouvées par R. Anton [5] et M. Blair, H. Smith et C. Sogge [15]. Les problèmes sont alors de deux sortes et de difficulté croissante : le problème du contrôle interne et celui du contrôle au bord. Pour le premier, il faudrait donc prouver des théorèmes de propagation adaptés. Pour le deuxième, l'existence de solutions au problème de Cauchy avec données au bord non homogènes serait déjà une première étape.

Se pose enfin la question du temps de contrôle. Dans tous les théorèmes de contrôlabilité globale que l'on a démontrés, le temps de contrôle dépend de la donnée. Ce long temps est-il nécessaire, alors que l'équation de Schrödinger se propage à vitesse infinie ? Si oui, de quelle manière dépend-il de la taille de la donnée ? Notons que pour certaines équations paraboliques non linéaires, il a été montré qu'il ne pouvait pas y avoir contrôlabilité en temps arbitraire, voir [46].

1.1.4 Contrôle de l'équation de Korteweg-de Vries

Le chapitre 4, écrit en collaboration avec Lionel Rosier et Bing-Yu Zhang considère l'équation de Korteweg-de Vries

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0$$

en domaine périodique. Cette équation présente une infinité de quantités conservées mais nous n'en utiliserons que deux :

$$\begin{array}{rl} \text{le volume} & \int_{\mathbb{T}^1} u(t,x)\ dx \\ \text{l'énergie} & \int_{\mathbb{T}^1} u(t,x)^2\ dx. \end{array}$$

1.1.4.1 Résultats antérieurs

Pour notre problème précis de contrôle interne en données périodiques, les résultats antérieurs sont des résultats locaux dus à D. Russel et B-Y. Zhang [83]. En utilisant le théorème de contrôlabilité et de stabilisation de l'équation linéarisée de [82], ils établissent la contrôlabilité et la stabilisation interne locale de KdV en domaine périodique. Ils utilisent pour cela, et comme on va le faire dans le chapitre 4, les espaces de Bourgain adaptés à KdV. L'argument est donc perturbatif.

Un résultat global a été prouvé par A. Pazoto [73] (poursuivant des résultats de G. Perla-Menzala, C.F. Vasconcellos et E. Zuazua [74]) pour la stabilisation interne de KdV sur un intervalle avec des conditions au bord $u(t,0) = u(t,L) = \partial_x u(t,L) = 0$. Il utilise la méthode de compacité-unicité et est en cela aidé par le fait que les conditions au bord induisent un effet régularisant. Pour montrer la régularité de la limite faible (argument de propagation de la régularité), il montre une inégalité d'observabilité faible pour l'équation linéarisée. Celle-ci est établie par la méthode des multiplicateurs, toujours en s'aidant de la régularité supplémentaire des solutions.

Notons que le contrôle au bord de KdV a été intensément étudié ces dernières années.

Il s'agit du problème de contrôle suivant

$$\left\{\begin{array}{rcl} \partial_t u + \partial_x^3 u + \partial_x u + u\partial_x u &=& 0 \quad x \in]0, L[, \ t \in]0, T[\\ u(t,0) = h_1(t); u(t,L) = h_2(t); \partial_x u(t,L) &=& h_3(t) \quad t \in]0, T[\end{array}\right.$$

Pour le contrôle de la dérivée à droite avec $h_1 = h_2 = 0$, L. Rosier [77] a montré que le linéarisé était contrôlable pour des longueurs L n'appartenant pas à un ensemble de valeurs critiques. Il en déduit alors la contrôlabilité locale en dehors de ces longueurs. Pour le linéarisé, il utilise la méthode de compacité-unicité pour en déduire l'observabilité. La compacité est assurée par un argument de gain de régularité (une solution de donnée $u_0 \in L^2$ donne une solution $L^2([0,T], H^1)$ et la méthode des multiplicateurs. Il faut alors vérifier l'unicité qui n'est justement pas vérifiée pour certaines longueurs qu'il détermine explicitement. Le défaut de contrôlabilité n'est en ce sens pas très fort puisqu'il ne manque qu'un nombre fini de directions. C'est cette observation qui a permis la preuve de la contrôlabilité locale même pour les longueurs critiques grâce à des développements à des ordres supérieurs, voir les articles de J-M. Coron et E. Crépeau [39], E. Cerpa [30] E. Cerpa et E. Crépeau [31].

Citons aussi pour le contrôle à gauche, les résultats locaux de L. Rosier [78] et O. Glass et S. Guerrero [53] reposant sur des inégalités de Carleman. On obtient aussi des résultat globaux si on autorise plus de contrôles, voir M. Chapouly [34]. On renvoie au survey de L. Rosier et B.Y. Zhang [80] pour une présentation des résultats connus sur le contrôle de KdV.

1.1.4.2 Résultats principaux de la thèse

Dans un premier temps, les résultats correspondant à ce chapitre s'appliquent principalement à étendre les résultats globaux que l'on a prouvés sur l'équation de Schödinger non linéaire sur $\mathbb{T}^1$ à KdV sur un domaine périodique. Dans un deuxième temps, on cherchera un terme d'amortissement permettant d'obtenir un taux de décroissance arbitraire.

Comme pour l'équation de Schrödinger non linéaire en dimension 1, on va utiliser les espaces de Bourgain qui permettent d'avoir un problème bien posé sur L^2.

Le système de contrôle qui nous intéresse est donc le suivant

$$\partial_t u + u\partial_x u + \partial_x^3 u = Ag, \quad x \in \mathbb{T}, \ t \in \mathbb{R} \tag{1.15}$$

où A est l'opérateur $Ag(x) = a(x)\left(g(x) - \int_{\mathbb{T}^1} a(y)g(y)dy\right)$ déjà décrit pour la partie linéaire, $a(x) \in C^\infty(\mathbb{T}^1)$ étant de moyenne $[a] = \frac{1}{2\pi}\int_{\mathbb{T}} a(x)dx = 1$. Cet opérateur permet de conserver le volume des solutions, ce qui est à la fois cohérent physiquement et indispensable pour l'utilisation des espaces de Bourgain.

Le théorème principal que l'on obtient est alors un théorème de contrôle en grand temps dans tous les espaces H^s.

Théorème 14. *Soit $s \geq 0$, $R_0 > 0$, et $\mu \in \mathbb{R}$. Il existe un temps $T > 0$ tel que si $u_0, u_1 \in H^s(\mathbb{T})$ de même moyenne $[u_0] = [u_1] = \mu$ sont tels que*

$$\|u_0\|_s \leq R_0, \qquad \|u_1\|_s \leq R_0,$$

alors, on peut trouver un contrôle $g \in L^2(0,T;H^s(\mathbb{T}))$ tel que le système (1.15) admette une solution $u \in C([0,T];H^s(\mathbb{T}))$ satisfaisant

$$u(x,0) = u_0(x), \qquad u(x,T) = u_1(x).$$

On procède par stabilisation puis contrôle local. Le système amorti est le suivant :

$$\partial_t u + u\partial_x u + \partial_x^3 u = -AA^* u, \quad x \in \mathbb{T},\ t \in \mathbb{R}. \tag{1.16}$$

Théorème 15. *Soient $s \geq 0$ et $\mu \in \mathbb{R}$. Il existe une constante $\kappa > 0$ telle que pour tout $u_0 \in H^s(\mathbb{T}^1)$ avec $[u_0] = \mu$, la solution correspondante u du système (1.16) satisfasse à*

$$\|u(\cdot,t) - [u_0]\|_s \leq \alpha_{s,\mu}(\|u_0 - [u_0]\|_0)e^{-\kappa t}\|u_0 - [u_0]\|_s \qquad t \geq 0,$$

où $\alpha_{s,\mu} : \mathbb{R}^+ \to \mathbb{R}^+$ est une fonction croissante.

La preuve de ce théorème, pour $s = 0$, repose aussi sur la méthode de compacité-unicité. Pour la propagation de la compacité, il a donc fallu établir des théorèmes de propagation de la compacité et de la régularité adaptés à l'équation de KdV et dans les espaces de Bourgain, ce qui, à notre connaissance, n'existait pas dans la littérature. On prouve par exemple le théorème suivant :

Théorème 16. *Soit $u \in X_T^{0,1/2}$ solution de $\partial_t u + \partial_x^3 u + u\partial_x u = 0$ telle que $u \in C^\infty(]0,T[\times\omega)$ où $\omega \subset \mathbb{T}^1$ est un ouvert non vide.*

Alors, $u \in C^\infty(]0,T[\times\mathbb{T}^1)$.

Bien que le terme de stabilisation soit naturellement construit pour faire décroitre la norme L^2, on obtient aussi une décroissance exponentielle dans toutes les normes H^s, $s \geq 0$. Pour cela, on prouve d'abord que l'on a bien décroissance exponentielle des normes H^s pour l'équation linéaire. Or, on sait déjà que l'on a décroissance exponentielle dans L^2. On peut alors faire un argument perturbatif une fois que la norme L^2 de la solution est suffisamment petite. On obtient alors la décroissance exponentielle en comparant la décroissance forte induite par la partie linéaire de l'équation avec la petitesse du terme non linéaire que l'on va absorber.

Cependant, dans le terme de stabilisation "naturel" que l'on choisit pour le résultat précédent, le taux de décroissance n'est a priori pas connu. On voudrait pouvoir imposer ce taux de décroissance. Pour des équations linéaires, Slemrod [84] a mis au point une méthode permettant d'obtenir un stabilisateur produisant un taux de décroissance choisi. Ce terme de stabilisation est construit à partir d'une variante de l'opérateur HUM. En l'utilisant sur notre équation non linéaire, on obtient alors par un argument perturbatif un taux de décroissance exponentielle arbitraire pour des petites données.

Théorème 17. *Soient $\lambda > 0$, $s \geq 0$, et $\mu \in \mathbb{R}$. Alors, il existe $\delta > 0$, $C > 0$ et un opérateur linéaire Q_λ borné de $H^s(\mathbb{T}^1)$ dans $H^s(\mathbb{T}^1)$ tel que pour tout $u_0 \in H^s$ avec $\|u_0\|_s \leq \delta$ et $[u_0] = \mu$, la solution du système*

$$\partial_t u + u\partial_x u + \partial_x^3 u = -AQ_\lambda u, \quad u(x,0) = u_0(x),\ x \in \mathbb{T}^1$$

vérifie

$$\|u(\cdot,t) - [u_0]\|_s \leq Ce^{-\lambda t}\|u_0 - [u_0]\|_s \qquad t \geq 0.$$

Ce stabilisateur ne semble pas fonctionner pour de grandes données (il ne fait a priori même pas décroître la norme). Pour avoir ce type de résultat pour de grandes données, on est amené à utiliser un terme de stabilisation dépendant du temps et combinant les deux résultats de stabilisation précédents. On peut alors imposer un taux de décroissance exponentielle arbitraire pour l'équation non linéaire.

Théorème 18. *Soient $\lambda > 0$, $s \geq 0$, et $\mu \in \mathbb{R}$. Il existe une application régulière $\widetilde{Q}_\lambda$ de $H^s(\mathbb{T}^1) \times \mathbb{R}$ vers $H^s(\mathbb{T}^1)$ périodique par rapport à la seconde variable ($t \in \mathbb{R}$), et telle que pour tout $u_0 \in H^s(\mathbb{T}^1)$ avec $[u_0] = \mu$, la solution u du système*

$$\partial_t u + u\partial_x u + \partial_x^3 u = -A\widetilde{Q}_\lambda(u,t), \qquad u(\cdot,0) = u_0$$

satisfasse

$$||u(\cdot,t) - [u_0]||_s \leq \alpha_{s,\lambda,\mu}(||u_0 - [u_0]||_s)e^{-\lambda t}||u_0 - [u_0]||_s \qquad t \geq 0,$$

où $\alpha_{s,\lambda,\mu} : \mathbb{R}^+ \to \mathbb{R}^+$ est une fonction croissante.

On renvoie au chapitre 4 pour l'expression explicite de ce stabilisateur. On peut tout de même décrire grossièrement son action. Dans un premier temps, lorsque $\|u\|_{H^s}$ est grande, on a seulement le stabilisateur habituel AA^* qui permet de s'assurer de la décroissance tant que la solution est grande. Ensuite, après une période de transition, on rentre dans une phase où le stabilisateur va osciller de façon périodique en temps avec une période fixée T. Durant chaque période, on a trois étapes :

- un moment où le stabilisateur AQ_λ menant vers une décroissance en $e^{-\lambda t}$,
- une courte période de transition où il peut y avoir des pertes,
- un moment où le stabilisateur AG^* est actif, menant à une décroissance en $e^{-\kappa t}$.

On pourrait se dire qu'il est plus simple d'imposer l'amortissement AA^* quand la solution est grande et AQ_λ quand elle est petite. Le problème est qu'il y a alors une discontinuité du stabilisateur, problématique pour la définition de la solution. Si on essaie alors de faire une transition entre les deux amortissements, ne dépendant que de la norme de la solution par exemple, on ne peut pas s'assurer que la solution ne va pas stagner dans un état stable nouveau créé par cette transition.

1.1.4.3 Problèmes ouverts et perspectives

Parmi les questions directement liées aux théorèmes que l'on a prouvés, les plus directes sont :

- dans le théorème de contrôlabilité globale que l'on trouve, le temps de contrôlabilité dans H^s dépend de la norme H^s de la solution. Peut-on faire ce que l'on a fait pour Schrödinger pour ne faire dépendre ce temps que de la norme L^2 ? Bien évidemment, il y a toujours la question générale de se demander s'il y a vraiment un temps minimal de contrôle.
- Existe-t-il un terme d'amortissement indépendant du temps permettant d'obtenir un taux de décroissance arbitraire ?

Il y a aussi beaucoup de questions ouvertes pour le contrôle au bord avec des grandes données pour lequel peu de résultats existent si l'on ne s'autorise pas tous les contrôles. Par exemple, pour le contrôle au bord avec la donnée de Neumann à droite, il n'y a pour l'instant que des résultats locaux, qui ont déjà demandé des méthodes astucieuses.

1.1.5 Contrôle de l'équation des ondes non linéaire critique

Dans cette sous-section, correspondant au chapitre 5, on s'intéresse au contrôle et à la stabilisation de l'équation de Klein-Gordon critique. Avant de commencer à décrire les résultats, expliquons l'utilisation du terme "critique" qui signifie ici critique pour la norme d'énergie (voir aussi la fin de la partie sur Strichartz de la sous-section 1.1.2.1). Considérons l'équation des ondes non linéaire quintique sur $\mathbb{R}^3$, critique pour l'énergie :

$$\partial_t^2 u - \Delta u + u^5 = 0.$$

Cette équation admet pour quantité formellement conservée l'énergie non linéaire :

$$E(t) = \frac{1}{2}\left(\int_M |\partial_t u|^2 + \int_M |\nabla u|^2\right) + \frac{1}{6}\int_M |u|^6 .$$

L'équation est invariante par le scaling $u \to u_\lambda = \frac{1}{\sqrt{\lambda}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)$ pour $\lambda > 0$. C'est-à-dire que si u est solution sur un intervalle $[0,T]$, alors u_λ le sera aussi sur un intervalle $[0, \lambda T]$. On aura alors $\|(u_\lambda, \partial_t u_\lambda)(0)\|_{\dot{H}^s \times \dot{H}^{s-1}} = \lambda^{1-s} \|(u, \partial_t u)(0)\|_{\dot{H}^s \times \dot{H}^{s-1}}$. Cette transformation laisse donc invariante la norme d'énergie correspondant à $s = 1$. Faisons maintenant tendre λ vers $+\infty$. On distingue alors trois cas :

- Si $s > 1$, le cas sous-critique : le scaling a tranformé une solution sur $[0,T]$ en une solution sur $[0, \lambda T]$ de norme $\dot{H}^s \times \dot{H}^{s-1}$ petite. Cela semble compatible avec l'existence d'un flot régulier local : existence en grand temps pour des petites données et en petit temps pour des grandes données. En effet, on arrive bien dans ce cas à avoir un flot local régulier.
- Si $s = 1$, le cas critique : le scaling laisse la norme d'énergie invariante et transforme une solution sur $[0,T]$ en une solution en grand temps. On sent alors que cela va être problématique et qu'il va être difficile d'avoir une solution en un temps ne dépendant que de la taille en énergie de la donnée. En effet, on a vu à la sous-section 1.1.2.1 qu'on a un temps d'existence local qui dépend de la "forme" de la donnée initiale. On va avoir besoin d'arguments plus forts que la conservation de l'énergie pour avoir l'existence globale (les inégalités de Morawetz).
- Si $s < 1$, le cas sur-critique : le scaling transforme une solution sur $[0,T]$ en une solution sur $[0, \lambda T]$ de norme petite. L'existence d'un flot régulier pendant un temps T pour des données bornées dans $\dot{H}^s \times \dot{H}^{s-1}$ impliquerait alors l'existence d'un flot en grand temps pour des grandes données. On sent alors bien que cela sera difficile à obtenir si la non-linéarité n'a pas de propriétés particulières. En effet, on sait que le flot est mal posé dans les cas surcritiques (voir Lebeau [65] et Christ-Colliander-Tao [35]).

Le cas de l'équation quintique est ainsi intéressant mathématiquement puisque c'est celui où la régularité critique correspond à l'espace d'énergie où on dispose d'une quantité conservée positive. Dans cette thèse, on étudiera cette équation sur des variétés compactes. Les dilatations évoquées sur $\mathbb{R}^3$ n'ont alors plus de sens. Cependant, on verra que localement près d'un point de la variété et en temps court, le comportement non linéaire reste proche de celui de $\mathbb{R}^3$, notamment pour des dilatations qui ont tendance à concentrer ($\lambda \to 0$ dans la description précédente). Le fait que l'équation soit critique pour l'énergie produit alors des phénomènes mathématiques nouveaux.

1.1.5.1 Résultats antérieurs

Revenons maintenant au problème du contrôle. Comme indiqué dans la sous-section 1.1.1.2, le problème du contrôle et de la stabilisation de l'équation des ondes linéaire est bien compris depuis les travaux de Taylor-Rauch [76] et Bardos-Lebeau-Rauch [8] et l'introduction de la condition de contrôle géométrique.

Les premiers résultats sur l'équation non linéaire sont des résultats perturbatifs et s'appliquent à des non-linéarités légèrement sur-linéaires, ou alors supposent des petites données, voir Zuazua [89].

Pour les cas sous critiques à grandes données, le résultat le plus avancé est celui de Dehman-Lebeau-Zuazua [44]. Il prouve la contrôlabilité en grand temps pour des non linéarités sous-critiques et défocalisantes. Le problème est posé sur $\mathbb{R}^3$ avec un contrôle en dehors d'une boule mais la méthode peut s'étendre à d'autres géométries (avec un théorème de prolongement unique adapté). Les auteurs parviennent à mettre en œuvre la méthode de compacité-unicité. Deux éléments sont alors cruciaux dans l'argument par contradiction :
- le résultat de linéarisabilité de P. Gérard [50] permet à haute fréquence de se ramener à une suite de solutions de l'équation linéaire pour laquelle on peut prouver la propagation de la compacité par les mesures.
- ils prouvent que la limite faible u, a priori dans l'espace d'énergie, est plus régulière que ce à quoi on s'attend. Ils prouvent que la non-linéarité $f(u)$ est mieux que $L^1([0,T],L^2)$, elle est en fait $L^1([0,T],H^\varepsilon)$. L'argument est général et n'utilise que le fait que u est dans l'espace d'énergie et les espaces de Strichartz. Ensuite, cette régularité supplémentaire du second membre permet de prouver la propagation de la régularité (l'argument est microlocal cette fois) et donne $u \in C([0,T],H^{1+\varepsilon})$. Il reste alors à itérer le processus pour avoir assez de régularité.

Cependant, comme l'indiquent les auteurs, ces deux étapes ne s'étendent pas à des non-linéarités critiques.

Toujours dans le cas sous-critique, un autre résultat intéressant est celui de Dehman-Lebeau [43] sur le contrôle interne de l'équation des ondes non linéaire sous-critique sur des domaines bornés. Grâce à une analyse fine de l'opérateur HUM linéaire, les auteurs établissent la contrôlabilité de données haute-fréquence et éventuellement grandes en énergie. Le contrôle se fait alors en temps uniforme, celui de l'équation linéaire. Le temps ne dépend donc plus de la taille des données. Notons que le chapitre 3, déjà évoqué, contient une extension de ces résultats à l'équation de Schrödinger.

Pour l'équation critique sur $\mathbb{R}^3$, l'article de Dehman-Gérard [41] fournit une solution pour combler les arguments de [44] ne fonctionnant pas pour l'équation critique. Le premier argument qui tombe en défaut est la linéarisabilité. Les solutions non linéaires ayant pour données initiales une suite faiblement convergente vers 0 ne sont plus asymptotiquement proches des solutions linéaires associées. Or, H. Bahouri et P. Gérard [6] ont décrit très précisément ce défaut de linéarisabilité par une décomposition en profils, voir section 1.1.2.5. B. Dehman et P. Gérard prouvent alors que dans ce cas précis, les profils sont nuls et la linéarisabilité est vraie. Ce premier argument a permis aux auteurs de prouver la propagation de la compacité. Ensuite, ils ne cherchent pas à prouver la propagation de la régularité. Ils prouvent le prolongement unique directement en utilisant l'existence de l'opérateur de scattering. Dans le chapitre 5, on suivra en partie la première étape de ce raisonnement. On sera amené à prouver l'existence d'une décomposition en profils sur une variété, mais avec des difficultés supplémentaires dues à l'absence des dilatations. En revanche, la deuxième étape ne peut pas fonctionner puisqu'on n'a pas d'opérateur de scattering sur une variété compacte.

Toujours pour l'équation critique, on peut aussi citer l'article récent de L. Aloui, S. Ibrahim et K. Nakanishi [2] pour $\mathbb{R}^d$. Ils prouvent la stabilisation globale pour un amortissement en dehors d'une boule pour une non-linéarité quelconque, pourvu que la solution soit globale. Leur preuve utilise des inégalités de type Morawetz. Cependant, cette méthode ne semble pas s'appliquer à des géométries plus compliquées.

1.1.5.2 Résultats principaux de la thèse

Dans le chapitre 5, on prouve un résultat de contrôlabilité haute fréquence pour l'équation de Klein-Gordon non linéaire critique sur des variétés compactes de dimension 3. On remplace l'équation des ondes par l'équation de Klein-Gordon car la stabilisation exponentielle n'est plus vraie pour les ondes. Cette obstruction n'est pas très profonde et est principalement due au fait que les constantes ne sont pas amorties pour l'équation linéaire. Afin d'énoncer les résultats, on va introduire quelques définitions.

On dit que $(x_1, x_2, t) \in M \times M \times \mathbb{R}_+^*$ est un couple de foyers à distance t si l'ensemble

$$F_{x_1,x_2,t} := \left\{ \xi \in S^*_{x_1}M \middle| \, exp_{x_1} t\xi = x_2 \right\}$$

des directions des géodésiques partant de x_1 et arrivant à x_2 en un temps t, a une mesure de surface positive.

On note T_{focus} l'infimum des $t \in \mathbb{R}_+^*$ tels qu'il existe un couple de foyers à distance t.

Si M est compacte, on a nécessairement $T_{focus} > 0$.

La condition que l'on imposera sera légèrement plus forte que la condition de contrôle géométrique :
Contrôle géométrique avant refocalisation On impose que (ω, T_0) vérifie la condition de contrôle géométrique avec $T_0 < T_{focus}$.

Par exemple, pour $\mathbb{T}^3$, il n'y a pas de refocalisation et on a donc la condition de contrôle géométrique habituelle. Cependant, cette hypothèse est plus forte pour la sphère S^3. Par exemple, un voisinage d'un demi-équateur est suffisant pour la condition de contrôle géométrique alors que notre hypothèse demande l'équateur entier.

Le résultat principal que l'on obtient est un résultat de contrôle haute-fréquence en temps long.

Théorème 19. *Soit $R_0 > 0$ et ω satisfaisant la condition de contrôle géométrique avant refocalisation. Alors, il existe $T > 0$ et $\delta > 0$ tels que pour tout (u_0, u_1) et $(\tilde{u}_0, \tilde{u}_1)$ dans $\mathcal{E} = H^1 \times L^2$, avec*

$$\begin{aligned} \|(u_0, u_1)\|_{\mathcal{E}} &\leq R_0; & \|(\tilde{u}_0, \tilde{u}_1)\|_{\mathcal{E}} &\leq R_0 \\ \|(u_0, u_1)\|_{L^2 \times H^{-1}} &\leq \delta; & \|(\tilde{u}_0, \tilde{u}_1)\|_{L^2 \times H^{-1}} &\leq \delta \end{aligned}$$

il existe $g \in L^\infty([0,T], L^2)$ *supporté dans* $[0,T] \times \omega$ *tel que l'unique solution forte de*

$$\begin{cases} \Box u + u + |u|^4 u &= g \quad \text{sur} \quad [0,T] \times M \\ (u(0), \partial_t u(0)) &= (u_0, u_1). \end{cases}$$

satisfasse $(u(T), \partial_t u(T)) = (\tilde{u}_0, \tilde{u}_1)$.

L'hypothèse de taille est alors du type de celle qu'on trouve dans Dehman-Lebeau [43] dans le cas sous-critique. Cependant, ils obtiennent un contrôle en temps uniforme, celui du contrôle géométrique, alors qu'on n'a ici qu'un contrôle en temps long. En ce sens, la conclusion du théorème est plus proche de Dehman-Lebeau-Zuazua, c'est-à-dire un temps de contrôle dépendant de la taille des données.

On obtient ce résultat, comme dans les exemples précédents, à partir d'un théorème de stabilisation et d'un théorème de contrôle local. Le théorème de contrôle local se prouve assez simplement : dans les cas critiques, il est usuel que les petites données ne posent pas trop de problème. La partie principale de la preuve est donc la stabilisation à haute fréquence.

Théorème 20. *Soit* $R_0 > 0$*,* ω *satisfaisant l'hypothèse de contrôle géométrique avant refocalisation et* $a \in C^\infty(M)$ *avec* $a(x) > \eta > 0$ *pour* $x \in \omega$*. Alors, il existe* $C, \gamma > 0$ *et* $\delta > 0$ *tels que pour tout* (u_0, u_1) *dans* $\mathcal{E}$*, avec*

$$\|(u_0, u_1)\|_{\mathcal{E}} \leq R_0; \qquad \|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq \delta;$$

l'unique solution forte de

$$\begin{cases} \Box u + u + |u|^4 u + a(x)^2 \partial_t u &= 0 \quad \text{on} \quad [0,T] \times M \\ (u(0), \partial_t u(0)) &= (u_0, u_1). \end{cases} \tag{1.17}$$

vérifie $E(u)(t) \leq Ce^{-\gamma t} E(u)(0)$.

Pour prouver ce théorème, il faut encore prouver une inégalité d'observabilité et on utilise la méthode de compacité-unicité. Le fait de supposer en plus que les données initiales sont petites dans $L^2 \times H^{-1}$ est en fait l'ingrédient qui va nous permettre de ne pas avoir à démontrer que la limite est nulle. On ne montre ainsi qu'une inégalité d'observabilité faible (avec un second membre dans la norme $L^2 \times H^{-1}$ en plus de l'observation) et on est ramené au problème de la compacité. Comme on l'a précisé auparavant, la linéarisabilité (voir section 1.1.2.5) est fausse en toute généralité et on est amené à établir une décomposition en profils de type [6] sur une variété.

Cependant, comme les dilatations n'ont pas de vrai sens sur une variété, on définit une onde de concentration de la manière suivante. On a besoin de plusieurs paramètres :
- la vitesse de concentration : une suite $(h_n)_{n\in\mathbb{N}}$ de nombres strictement positifs tendant vers 0,
- les cœurs de concentration en espace : c'est une suite $(x_n)_{n\in\mathbb{N}}$ de M convergeant vers x_∞, qui va être le lieu de concentration,
- le profil de concentration : c'est un couple de fonctions $(\varphi, \psi) \in (\dot{H}^1 \times L^2)(T_{x_\infty} M)$.

On appelle alors **données de concentration** associées à $[(\varphi, \psi), \underline{h}, \underline{x}]$ la classe d'équivalence, modulo la convergence dans l'espace d'énergie, des suites ayant pour expression en coordonnées locales

$$h_n^{-\frac{1}{2}} \Psi_U(x) \left(\varphi, \frac{1}{h_n}\psi\right) \left(\frac{x - x_n}{h_n}\right) + o(1)_{\mathcal{E}} \tag{1.18}$$

dans un ouvert de carte $U_M \approx U \subset \mathbb{R}^d$ contenant x_∞ et pour une fonction troncature $\Psi_U \in C_0^\infty(U)$ telle que $\Psi_U(x) = 1$ au voisinage de x_∞. (Ici, on a identifié x_n, x_∞ avec son image dans U).

On va maintenant rajouter le paramètre de cœur de concentration en temps : c'est une suite t_n de $[-T, T]$ convergeant vers t_∞ qui va être le temps de concentration.

On appelle alors **onde de concentration linéaire (amortie)** associée à $[(\varphi, \psi), \underline{h}, \underline{x}, \underline{t}]$ une suite de solutions v_n de l'équation des ondes $\Box v_n + v_n = 0$ (ou l'équation $\Box v_n + v_n + a(x)\partial_t v_n = 0$ dans le cas amorti) avec données initiales au temps t_n les données de concentration associées à $[(\varphi, \psi), \underline{h}, \underline{x}]$.

Etant donné une onde de concentration linéaire (amortie), on lui associe son **onde de concentration non linéaire (amortie)** vérifiant les mêmes données de Cauchy au temps 0 :

$$\begin{cases} \Box u_n + u_n + a(x)\partial_t u_n + |u_n|^4 u_n &= 0 \\ (u_n, \partial_t u_n)_{|t=0} &= (v_n, \partial_t v_n)_{|t=0}. \end{cases}$$

Il aurait pu paraître plus logique de la définir comme la solution ayant les mêmes données au temps t_n, c'est-à-dire la donnée de concentration, mais cela sera plus pratique ainsi en vue de l'application que l'on va en faire. Le comportement de ces ondes de concentration a été étudié par S. Ibrahim [54]. Avec ces définitions, on peut établir le théorème de décomposition en profils sur des variétés.

Théorème 21. *Soit $(v_n)_{n\in\mathbb{N}}$ une suite de solutions de l'équation de Klein-Gordon amortie*

$$\begin{cases} \Box v_n + v_n + a(x)\partial_t v_n &= 0 \\ (v_n, \partial_t v_n)_{t=0} &= (\varphi_n, \psi_n) \end{cases}$$

avec (φ_n, ψ_n) borné dans $\mathcal{E}$. Alors, quitte à extraire, il existe une suite d'ondes de concentration linéaires amorties $(\underline{p}^{(j)})$, associées aux données de concentration $[(\varphi^{(j)}, \psi^{(j)}), \underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)}]$, telles que pour tout $l \in \mathbb{N}^*$,

$$v_n(t,x) = v(t,x) + \sum_{j=1}^{l} p_n^{(j)}(t,x) + w_n^{(l)}(t,x),$$

$$\forall T > 0, \quad \overline{\lim_{n\to\infty}} \left\| w_n^{(l)} \right\|_{L^\infty([-T,T],L^6(M)) \cap L^5([-T,T],L^{10})} \xrightarrow[l\to\infty]{} 0$$

$$\|(v_n, \partial_t v_n)\|_{\mathcal{E}}^2 = \sum_{j=1}^{l} \left\|(p_n^{(j)}, \partial_t p_n^{(j)})\right\|_{\mathcal{E}}^2 + \left\|(w_n^{(l)}), \partial_t w_n^{(l)})\right\|_{\mathcal{E}}^2 + o(1), \text{ quand } n \to \infty,$$

où $o(1)$ est uniforme pour $t \in [-T, T]$.

Le flot non linéaire suit alors cette décomposition, à une erreur petite près dans la norme énergie-Strichartz :

$$|||u|||_I = \|u\|_{L^\infty(I,H^1(M))} + \|\partial_t u\|_{L^\infty(I,L^2(M))} + \|u\|_{L^5(I,L^{10}(M))}.$$

Théorème 22. *Soit $T < T_{focus}/2$. Soit $(u_n)_{n\in\mathbb{N}}$ une suite de solutions de l'équation de Klein-Gordon non linéaire amortie (1.17) avec données initiales, au temps 0, (φ_n, ψ_n)*

bornées dans $\mathcal{E}$. Si on dénote $p_n^{(j)}$ (resp. v la limite faible) les ondes de concentration linéaires amorties données par le Théorème 21 et $q_n^{(j)}$ les ondes de concentration non linéaires associées (resp. u la solution de l'équation non linéaire associée avec $(u, \partial_t u)_{t=0} = (v, \partial_t v)_{t=0}$). Alors, on a, à extraction près,

$$u_n(t,x) = u + \sum_{j=1}^{l} q_n^{(j)}(t,x) + w_n^{(l)}(t,x) + r_n^{(l)} \tag{1.19}$$

$$\overline{\lim_{n\to\infty}} \left|\left|\left| r_n^{(l)} \right|\right|\right|_{[-T,T]} \underset{l\to\infty}{\longrightarrow} 0 \tag{1.20}$$

où $w_n^{(l)}$ est le même que dans le Théorème 21.

Le même théorème reste vrai si M est la sphère S^3 et $a \equiv 0$ (équation non amortie) sans hypothèse sur le temps T.

Le résultat plus précis que l'on obtient sur la sphère S^3 ne sera pas utilisé pour notre preuve de la contrôlabilité. Cependant, il donne un exemple d'un cas où on sait ce qu'il se passe après la refocalisation (étudié dans [54]), ce qui permet d'avoir une décomposition en grand temps.

Esquisse de preuve de la stabilisation Grâce à notre hypothèse de petitesse dans la norme $L^2 \times H^{-1}$, on est ramené à prouver une inégalité d'observabilité faible pour les solutions de l'équation non linéaire amortie. On veut prouver, pour des données bornées en énergie par une constante, l'inégalité :

$$E(u)(0) \leq C \left(\iint_{[0,T]\times M} |a(x)\partial_t u|^2 \; dtdx + \|(u_0,u_1)\|_{L^2\times H^{-1}} E(u)(0) \right).$$

La preuve se fait toujours par compacité-unicité, l'unicité étant fournie par le deuxième terme de l'inégalité. On prouve la compacité en deux étapes : dans un premier temps, on prouve la linéarisabilité et une fois cela prouvé, on sait que la solution est proche d'une solution linéaire. On peut alors appliquer l'argument linéaire classique, sachant que la condition de contrôle géométrique est vérifiée.

La principale étape est donc la preuve de la linéarisabilité. Une fois la décomposition en profils effectuée (Théorème 22), cela consiste à prouver que tous les profils sont nuls et qu'il ne reste dans la décomposition que la solution de l'équation linéaire w_n et un petit reste. L'argument est alors le suivant : on prouve que dans la décomposition en profils non linéaires, les énergies de chaque profil se comportent de façon indépendante. On obtient ainsi que la mesure énergie limite de la suite u_n est la somme des énergies limites de chaque élément de la décomposition (les profils et la partie linéaire). On déduit alors de l'observabilité que la mesure énergie limite de chaque profil est nulle sur ω. L'hypothèse de contrôle géométrique avant refocalisation et la description des profils faite par S. Ibrahim [54] vont alors permettre de trouver, pour chaque profil, un intervalle de temps de longueur $L < T_{focus}$ pendant lequel le profil ne se concentre pas et tel que (ω, L) vérifie la condition de contrôle géométrique habituelle. Or, pendant les temps où il n'y a pas de concentration, un profil non linéaire se comporte comme la solution d'une équation d'onde linéaire. On peut donc appliquer les arguments linéaires habituels sur ce profil particulier et montrer grâce à la condition de contrôle géométrique vérifiée par (ω, L) que ce profil est nul.

Esquisse de preuve de la décomposition en profils La démonstration étant relativement longue et technique, on essaie ici d'en donner seulement un aperçu, quitte à faire quelques simplifications. La preuve suit la démonstration de [6] et [48]. Cependant, deux obstacles viennent s'ajouter : le terme d'amortissement et le fait que l'on soit sur une variété. Le terme d'amortissement pose plutôt problème dans la partie linéaire où la conservation de l'énergie aidait beaucoup. La géométrie de la variété pose surtout problème pour la décomposition non linéaire et à cause des possibles reconcentrations.

Commençons par la décomposition linéaire, Théorème 21. Celle-ci se fait en deux étapes.

La première consiste en la décomposition de v_n selon ses composantes h_n oscillante où h_n sont des suites convergeant vers 0. Grâce à une méthode d'exhaustion, on décompose v_n en une somme de solutions linéaires qui oscillent toutes à des fréquences orthogonales entre elles, dans un sens à préciser. On prouve alors que le reste est petit dans une norme de Besov plus faible que la norme d'énergie mais qui permettra tout de même d'avoir une petitesse dans $L^\infty L^6$ grâce à une inégalité de Sobolev précisée.

Ensuite, pour chaque composante oscillant à une fréquence h_n, on va chercher les points de concentration. On va encore utiliser une méthode d'exhaustion qui va en quelque sorte "traquer" les possibles points de concentration à la vitesse h_n de l'énergie. Chaque point de concentration produit un profil que l'on extrait. Il faut ensuite prouver que le reste est de plus en plus petit dans $L^\infty L^6$. On déduit alors la petitesse dans $L^5 L^{10}$ par interpolation avec une inégalité de Strichartz.

Il reste ensuite à recombiner toutes les décompositions obtenues.

Pour la décomposition non linéaire, Théorème 22, on veut montrer que les ondes non linéaires associées à la décomposition linéaire précédente n'interagissent pas entre elles. Il faut pour cela, dans l'étape de décomposition linéaire, avoir prouvé que, pour chaque profil, les points de concentrations et les suites x_n qui convergent vers eux sont orthogonaux, dans un sens à préciser. Ensuite, il faut prouver que deux profils non linéaires ayant des points de concentration orthogonaux n'interagissent pas. On a alors ici besoin de façon cruciale de la description des ondes de concentration sur des variétés effectuée par S. Ibrahim [54]. Celui-ci prouve que si on suppose $T < T_{focus}$, alors les ondes de concentrations ne vont pas se reconcentrer. Il donne alors leur comportement précis. Près de l'unique point et temps de concentration, l'onde de concentration non linéaire sera proche d'une onde de concentration non linéaire sur $\mathbb{R}^3$, alors que dès que l'on est loin du temps et du point de concentration, la solution est proche d'une solution linéaire. Grâce à cette description précise, on prouve que les ondes de concentrations évoluent indépendamment. Or, le terme w_n^l étant petit dans $L^5 L^{10}$, il va être linéarisable et évoluera linéairement. On obtient finalement que le reste $r_n^{(l)}$ dans la décomposition est petit pour l grand.

1.1.5.3 Problèmes ouverts et perspectives

Le théorème que l'on a prouvé est un théorème haute fréquence et nécessite des données petites dans une norme plus basse que la norme d'énergie. Cependant, la même méthode pourrait être appliquée pour obtenir un théorème de contrôlabilité globale en grand temps et achever la stratégie de compacité-unicité. Il resterait seulement à prouver un théorème de prolongement unique de la forme :

$u \equiv 0$ est l'unique solution forte dans l'espace d'énergie de

$$\begin{cases} \Box u + u + |u|^4 u &= 0 \quad \text{sur} \quad [0,T] \times M \\ \partial_t u &= 0 \quad \text{sur} \quad [0,T] \times \omega. \end{cases}$$

Des théorèmes de prolongements uniques permettent d'atteindre les non-linéarités critiques (voir par exemple [59]) mais avec la condition $u_{|\omega} \equiv 0$ au lieu de $\partial_t u_{|\omega} \equiv 0$. La preuve d'un tel théorème de prolongement unique serait donc d'un grand intérêt.

L'autre question qui demeure est évidemment de se demander si l'hypothèse géométrique supplémentaire que l'on a imposée sur ω est vraiment nécessaire. Pour cela, il faudrait plusieurs avancées. D'abord, il faudrait comprendre ce phénomène de refocalisation pour des variétés générales. Pour la sphère, on comprend bien ce qui se passe : les solutions de données qui se focalisent au pôle sud se refocalisent au pôle nord avec le même profil qu'au départ (ou obtenu par une transformation linéaire simple). Or, pour d'autres variétés, les phénomènes de refocalisation peuvent être bien plus complexes. Un faisceau de géodésiques partant d'un même point peut se refocaliser partiellement en un point pour un temps alors qu'une autre partie se refocalisera ailleurs et en un autre temps. Sur ces géométries, la compréhension des solutions de l'équation des ondes linéaires associées à des profils reste très partielle.

Même pour la sphère où on comprend bien le comportement des profils linéaires, on ne saurait pas conclure sur la nécessité de notre hypothèse géométrique supplémentaire. En effet, le comportement avant et après focalisation est très lié à l'opérateur non linéaire de scattering sur $\mathbb{R}^3$ (on renvoie à la remarque 5.0.1 du chapitre 5 pour une discussion à ce sujet).

1.2 Contrôle bilinéaire

Le contrôle bilinéaire est un domaine qui a récemment connu un fort intérêt, notamment pour les nombreuses applications qu'il a dans des problèmes physiques concrets. Ce sont des situations qui arrivent typiquement lorsqu'on fait agir un laser sur un système quantique. Le système modèle qui nous intéressera (même si on traitera d'autres cas par la suite) est alors l'équation de Schrödinger

$$i\partial_t \psi(t,x) + \partial_x^2 \psi(t,x) = -u(t)\mu(x)\psi(t,x). \tag{2.21}$$

à laquelle on peut par exemple ajouter les conditions de Dirichlet au bord. On veut agir sur notre système par un terme de potentiel $u(t)\mu(x)\psi(t,x)$ qui a donc un profil fixe $\mu(x)$ dont on ne peut modifier que l'amplitude $u(t)$. On a ainsi beaucoup moins de marge de manœuvre que dans le contrôle interne où, pour tout temps t, on avait le choix d'une fonction de contrôle $g(t,x)$ définie sur un ouvert ω. Ici, pour chaque temps t, on n'a que le choix d'un réel $u(t)$. Remarquons que c'est aussi la situation dans laquelle on se trouve pour le contrôle au bord en dimension 1.

Pour des raisons physiques, on se limitera à des potentiels $\mu(x)$ et des contrôles $u(t)$ à valeurs réelles de sorte qu'il y ait conservation de la norme L^2. On se limitera donc à un contrôle dans la sphère unité $\mathcal{S}$ de L^2.

1.2.0.4 Résultats antérieurs et méthode générale

Un des premiers résultats significatifs concernant le contrôle bilinéaire n'est à vrai dire pas très encourageant, c'est le résultat négatif de Ball-Marsden-Slemrod [7].

Théorème 23. *Soit X un espace de Hilbert de dimension infinie. Soit A le générateur d'un semi-groupe continu d'opérateurs bornés sur X et B un opérateur linéaire borné sur X. Soit $x_0 \in X$. Si, pour $u \in L^1_{loc}([0,\infty[,\mathbb{R})$, $x(t,u,x_0)$ désigne la solution de*

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & Ax + u(t)Bx \\ x(0) & = & x_0 \end{array}\right.$$

alors l'ensemble $S(x_0)$ des points atteignables à partir de x_0,

$$S(x_0) := \{x(t,u,x_0) : t \geq 0, r > 1, u \in L^r_{loc}([0,\infty[,\mathbb{R})\}$$

est contenu dans une réunion dénombrable de sous ensembles compacts de X. En particulier, il admet un complémentaire dense dans X.

Pour un système conservatif comme Schrödinger avec μ réel, l'ensemble atteignable est de complémentaire dense dans $\mathcal{S}$.

La preuve (pour T fixé et u borné) consiste à remarquer que si u_n converge faiblement vers u dans $L^1([0,T],\mathbb{R})$, alors $x_n = x(T,u_n,x_0)$ converge fortement vers $x = x(T,u,x_0)$ dans X. Pour le vérifier, on écrit pour $r_n = x_n - x$

$$r_n(t) = \int_0^t (u_n(s) - u(s))e^{(t-s)A}Bx(s)ds + \int_0^t u_n(s)e^{(t-s)A}Br_n(s)ds.$$

Le premier terme tend fortement vers 0 dans X par convergence faible de u_n et on conclut la convergence forte par le lemme de Gronwall.

On verra à la Remarque 1 ce qui fait échouer cette preuve dans le cas qui nous intéresse et permet alors la contrôlabilité.

Pour le contrôle bilinéaire de l'équation de Schrödinger, A sera l'opérateur $i\partial_x^2$ de Dirichlet et B l'opérateur de multiplication par un potentiel $i\mu(x)$. On constate à ce stade que si la multiplication par μ agit sur l'espace dans lequel on raisonne, on ne peut pas avoir contrôlabilité exacte dans cet espace. Pourtant, ce résultat négatif n'est pas sans espoir.

Une première approche consiste à se dire que le résultat précédent n'empêche pas la contrôlabilité approchée. Ainsi, de nombreux résultats récents vont dans cette direction, parfois avec des techniques mathématiques très différentes. Citons par exemple le résultat de V. Nersesyan [72] qui utilise un terme de stabilisation vers l'état fondamental et celui de T. Chambrion, P. Mason, M. Sigalotti et U. Boscain [33] qui utilise les propriétés de contrôlabilité des approximations de Galerkin du système. Bien que ceci soit d'un grand intérêt pratique et amène à de nombreux problèmes très intéressants, nous ne suivrons pas cette approche dans la suite de la thèse.

Une deuxième approche, si on insiste pour avoir un contrôle exact, consiste à se dire que le résultat négatif de Ball-Marsden-Slemrod pourraît être dû à un mauvais choix d'espace fonctionnnel. Le théorème affirme que, si B est borné sur X, l'ensemble atteignable, en un temps T et pour des contrôles bornés, est compact dans X. Mais rien n'empêche à cet ensemble de contenir par exemple une boule dans une norme plus fine (intersectée avec la sphère unité de X). Cependant, pour ne pas être dans le cadre du théorème 23, l'opérateur B doit être "irrégulier" pour cette topologie. Par exemple, si on considère un potentiel μ régulier mais avec des mauvaises conditions au bord, cet opérateur agit sur l'espace $H^2_{(0)}$ (domaine du Laplacien de Dirichlet Δ_D) mais pas sur l'espace $H^3_{(0)} = D((-\Delta_D)^{3/2})$. On pourrait donc espérer avoir contrôlabilité dans $H^3_{(0)}$, dont la boule unité est bien compacte dans $H^2_{(0)}$. Cependant, la difficulté est que comme le potentiel μ ne doit pas être borné sur $H^3_{(0)}$, on se dit que l'on va avoir du mal à avoir un problème de Cauchy bien posé. On est donc coincé entre deux problèmes : si le potentiel est trop régulier, le théorème de Ball-Marsden-Slemrod empêchera la contrôlabilité, s'il n'est pas assez régulier, le problème de Cauchy sera mal posé.

La première solution à ce premier problème de régularité a été résolue par Karine Beauchard [10] et a consisté à utiliser le théorème de Nash-Moser (voir par exemple le livre de Serge Alinhac et Patrick Gérard [1] pour une présentation pédagogique). Ce théorème permet d'obtenir un théorème d'inversion locale lorsqu'un opérateur présente des pertes de dérivées par rapport à son linéarisé en un point. Cependant, il nécessite de très longs calculs et ne fournit pas les régularités optimales. Précisons à ce stade que le principal apport de cette thèse pour ce problème (écrit en collaboration avec K. Beauchard) consiste à se passer de ce théorème grâce un effet régularisant : on va prouver que le problème de Cauchy est bien posé sur $H^3_{(0)}$ avec un choix de μ tel que celui-ci ne soit pas borné sur $H^3_{(0)}$.

Précisons à présent un peu plus les résultats et les méthodes qui avaient été employées. φ_1 désigne ici la première fonction propre du laplacien de Dirichlet sur $]-1/2,1/2[$.

Théorème 24 (K. Beauchard [10]). *Soit ϕ_1, $\phi_0 \in \mathbb{R}$. Il existe $T>0$ et $\eta>0$ tels que pour tout ψ_0, ψ_f dans $\mathcal{S}\cap H^7_{(0)}(]-1/2,1/2[)$ satisfaisant*

$$\left\|\psi_0 - e^{i\phi_0}\varphi_1\right\|_{H^7} < \eta, \quad \left\|\psi_f - e^{i\phi_1}\varphi_1\right\|_{H^7} < \eta,$$

il existe $u \in H^1_0(]0,T[,\mathbb{R})$ telle que la solution de

$$\left\{\begin{array}{rcl} i\partial_t\psi(t,x) &=& -\frac{1}{2}\partial_x^2\psi(t,x) - u(t)x\psi(t,x) \\ \psi(t,-1/2) &=& \psi(t,1/2)=0 \\ \psi(0,x) &=& \psi_0(x) \end{array}\right.$$

satisfasse $\psi(T)=\psi_f$.

Décrivons d'abord la méthode générale de preuve. On se placera sur un intervalle borné avec des conditions de Dirichlet. On notera φ_k, λ_k, $k\in\mathbb{N}^*$ les fonctions et valeurs propres de l'opérateur $-\partial_x^2$.

Observons d'abord que la fonction $\Theta_T : u \mapsto \psi(T)$ où ψ est la solution de (2.21) avec condition initiale φ_1, est non linéaire. Donc, même si les équations que l'on considère sont linéaires, le problème de contrôle ne l'est pas. La méthode consiste à regarder le linéarisé près de la trajectoire $\psi_1(t,x) = e^{-i\lambda_1 t}\varphi_1$, correspondant à $u\equiv 0$. Notons que comme l'équation conserve la norme L^2, l'opérateur linéarisé a nécessairement son image contenue dans le plan tangent de la sphère unité $\mathcal{S}$ de L^2 au point $\psi_1(T)$ et ne peut pas être surjectif. Or, comme on ne cherche à contrôler que des données dans $\mathcal{S}$, on n'aura besoin que de la surjectivité de la projection sur le plan tangent. On ne précisera plus ce détail par la suite.

Le linéarisé près de la solution ψ_1 s'écrit $d\Theta_T(0) \cdot v = \Psi(T)$ où Ψ est solution de

$$\begin{cases} i\frac{\partial \Psi}{\partial t} = -\partial_x^2 \Psi - v(t)\mu(x)\psi_1, & x \in (0,1), t \in (0,T), \\ \Psi(t,0) = \Psi(t,1) = 0, \\ \Psi(0,x) = 0, \end{cases} \tag{2.22}$$

de sorte que l'on ait

$$\Psi(T) = \sum_{k=1}^{\infty} i\langle \mu\varphi_1, \varphi_k\rangle \left(\int_0^T v(t) e^{i(\lambda_k - \lambda_1)t} dt \right) e^{-i\lambda_k T} \varphi_k.$$

La surjectivité (modulo la projection sur le plan tangent) de cet opérateur se ramène alors à un problème de moments. Etant donné une suite $d_{k-1}(\Psi_f) := \frac{\langle \Psi_f, \varphi_k\rangle e^{i\lambda_k T}}{i\langle \mu\varphi_1, \varphi_k\rangle}$ et $\omega_{k-1} := \lambda_k - \lambda_1$, existe-t-il une fonction $v(t)$ telle que $\int_0^T v(t) e^{i\omega_k} = d_k$? Sous des hypothèses de gap sur les λ_i, un théorème d'Ingham fournit alors une solution à ce problème. On obtient typiquement

Lemme 1. *Soit $T > 0$ et $(\omega_k)_{k\in\mathbb{N}}$ une suite strictement croissante de $[0, +\infty)$ telle que $\omega_0 = 0$ et*

$$\omega_{k+1} - \omega_k \to +\infty \text{ quand } k \to +\infty.$$

Alors il existe une application continue

$$\begin{array}{rcl} L: \ \mathbb{R} \times l^2(\mathbb{N}^*, \mathbb{C}) & \to & L^2((0,T), \mathbb{R}) \\ d = (d_0, \tilde{d}) & \mapsto & L(d) \end{array}$$

telle que, pour tout $d = (d_k)_{k\in\mathbb{N}} \in \mathbb{R} \times l^2(\mathbb{N}^*, \mathbb{C})$, la fonction $v := L(d)$ vérifie

$$\int_0^T v(t) e^{i\omega_k t} dt = d_k, \forall k \in \mathbb{N}.$$

Donc, si on veut que le linéarisé soit surjectif de $L^2([0,T])$ dans $H^3_{(0)}$, il faut avoir $d(\Psi_f) \in l^2$ dès que $\Psi_f \in H^3_{(0)}$ et donc

$$\frac{C_1}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k\rangle| \leqslant \frac{C_2}{k^3}, \forall k \in \mathbb{N}^*. \tag{2.23}$$

Cependant, dans les cas où cette condition est vérifiée, la régularité de μ ne sera pas suffisante pour que la multiplication par μ agisse sur l'espace de résolution (par exemple, l'inégalité empêche que l'on ait $\mu \in H^3_{(0)}$).

La preuve consiste alors à appliquer un théorème d'inversion locale pour montrer que le contrôle non linéaire Θ_T est localement surjectif. Dans [10], pour compenser le manque de régularité apparent du système non linéaire par rapport au linéarisé, l'inversion locale est prouvée par une variante du théorème de Nash-Moser. Celui-ci nécessite la construction d'inverses du linéarisé dans différents espaces et en différents points. La preuve est très complexe et donne un contrôle dans des espaces non optimaux. La méthode décrite dans cette thèse à la sous-section suivante, permet d'éviter ces problèmes.

Pour l'instant, on a décrit la méthode de preuve quand l'hypothèse (2.23) est vérifiée. Le chapitre 6 restera dans ce cadre. Cependant, elle n'est pas vérifiée dans le théorème 24 avec $\mu(x) = x$. Par souci de complétude, décrivons rapidement les méthodes qui permettent de surmonter ce problème. On réfère au livre de Jean-Michel Coron [38] pour une présentation générale de ces techniques ainsi que pour l'application précise au problème du contrôle bilinéaire.

Dans [10], K. Beauchard utilise la méthode du retour introduite par J-M. Coron. Elle consiste à trouver une meilleure trajectoire auprès de laquelle le linéarisé sera contrôlable. Il s'agit alors de la trajectoire $e^{-i\lambda_{1,\gamma}t}\varphi_{1,\gamma}$ (avec $u \equiv \gamma$) où $\varphi_{1,\gamma}$ est la première valeur propre de l'opérateur $-\partial_x^2 - \gamma x$. Une inégalité du type (2.23) est alors vérifiée pour $\gamma > 0$ assez petit. Le théorème de contrôlabilité local donne alors des voisinages V_0, V_T de $\varphi_{1,\gamma}$ et $e^{-i\lambda_{1,\gamma}T}\varphi_{1,\gamma}$ pour lesquels on sait amener toute donnée de V_0 à une donnée de V_T. Ensuite, l'auteur trouve, par des transformations quasi-statiques, des contrôles qui amènent un voisinage de $\varphi_1 e^{i\phi_0}$ vers V_0 et d'un sous ensemble de V_T vers un voisinage de $\varphi_1 e^{i\phi_1}$. Finalement, on peut donc aller d'un voisinage de $\varphi_1 e^{i\phi_0}$ à un voisinage de $\varphi_1 e^{i\phi_1}$. Notons que l'on a alors un contrôle local en temps grand malgré la vitesse infinie de propagation. Il a été montré par J-M. Coron [37] qu'un temps minimal est nécessaire si on cherche en plus à contrôler la position et la vitesse du puit de potentiel.

On peut aussi mentionner le résultat presque global de K. Beauchard et J-M. Coron [13] qui est démontré grâce à des résultats locaux près de trajectoires périodiques. Il prouve que l'on peut passer d'un voisinage d'une fonction propre à un autre.

On donne un aperçu de la technique pour passer du mode propre φ_1 au mode φ_2. On note $\psi_1(t) = e^{-i\lambda_1 t}\varphi_1$ (idem pour $\psi_2(t)$) la trajectoire libre avec $u \equiv 0$. Pour $\theta \in [0, 1]$, on remarque que les fonctions $f_\theta(t) = \sqrt{1-\theta}\psi_1(t) + \sqrt{\theta}\psi_2(t)$ sont des solutions libres périodiques ($u \equiv 0$) dans la boule unité de L^2. Cela constitue donc un chemin reliant φ_1 à φ_2 avec des solutions libres. Les auteurs parviennent alors à montrer la contrôlabilité locale près de chaque trajectoire. Par compacité, cela permet alors de passer de φ_1 à φ_2 par des contrôles successifs.

Notons aussi que dans la preuve de [13], lorsque les auteurs prouvent le contrôle local près d'une trajectoire f_θ, $\theta \in]0, 1[$, le linéarisé n'est pas contrôlable : l'image est de codimension finie et ils doivent utiliser un développement à l'ordre 2 pour récupérer les dimensions manquantes.

On peut ici faire le lien avec le chapitre 3 de cette thèse sur le contrôle interne de NLS. On obtient aussi un résultat de contrôlabilité globale à partir de résultats de contrôlabilité près de trajectoires. Ainsi, à chaque étape, on fait un contrôle près d'une trajectoire qui va nous faire baisser l'énergie. Dans la méthode de Beauchard-Coron, le paramètre θ joue alors le rôle de l'énergie que l'on utilisait pour NLS.

Pour d'autres équations, le linéarisé n'est contrôlable que pour un temps suffisamment grand. Cela arrive typiquement lorsque le gap spectral nécessaire à l'inégalité d'Ingham est asymptotiquement minoré par une constante non nulle. C'est par exemple le cas dans l'article [11] pour l'équation des ondes. De plus, K. Beauchard prouve que sur un intervalle de longueur 1 et avec un μ vérifiant les conditions appropriées, on a contrôlabilité locale pour $T > 2$ et cette propriété est fausse pour $T \leq 2$. Pour $T < 2$, l'ensemble atteignable est localement une sous variété non plate (et de codimension infinie). La non-contrôlabilité n'est donc ici pas due à un mauvais choix d'espace fonctionnel.

Pour conclure cette section, on peut préciser que dans tous ces résultats, des techniques non linéaires sont venues se greffer à l'argument de perturbation reposant sur le théorème de Nash-Moser. L'amélioration que nous avons apportée repose surtout sur la simplification de l'étape de perturbation du linéaire. Il est très probable que le gain de régularité que l'on obtient dans le chapitre 6 s'étende à ces problèmes où le linéarisé n'est pas contrôlable, en particulier ceux décrits précédemment.

1.2.0.5 Résultats principaux de la thèse

Comme précisé dans la sous-section précédente, le résultat principal de cette thèse pour le contrôle bilinéaire, contenu dans l'article [14] écrit avec Karine Beauchard, est la découverte d'un effet régularisant qui permet d'éviter l'utilisation du théorème de Nash-Moser. Pour être plus précis, on prouve que le problème de Cauchy est bien posé avec un flot C^1 sur $H^3_{(0)}$ avec un $\mu \in H^3$, avec éventuellement des conditions au bord non homogènes. Ainsi, on peut trouver des $\mu \in H^3$ (donc on aura un flot C^1) qui vérifient les conditions (2.23) (donc le linéarisé sera inversible de $L^2([0,T])$ dans $H^3_{(0)}$). La stratégie est alors simplement d'appliquer le théorème d'inversion locale classique.

La multiplication par μ n'est alors plus bornée sur $H^3_{(0)}$, donc ce n'est pas en contradiction avec le théorème de Ball-Marsden-Slemrod. Notons que si on considère la régularité $H^2_{(0)}$, la multiplication par μ est bornée et ce résultat négatif s'applique. $H^3_{(0)}$ est donc la régularité optimale.

Pour Schrödinger, on obtient alors des résultats dans les espaces optimaux et sans restriction sur le temps. De plus, la méthode est suffisamment robuste pour qu'on l'applique à d'autres situations y compris non linéaires : cas de données radiales sur une boule 3D, équation de Schrödinger non linéaire sur un intervalle et équation des ondes non linéaire sur un intervalle.

Dans chaque cas, on montre que la version adaptée au problème de l'hypothèse (2.23) est générique, et on présente des potentiels μ explicites et simples qui la vérifient.

Equation de Schrödinger

$$\left\{ \begin{array}{l} i\frac{\partial\psi}{\partial t}(t,x) = -\frac{\partial^2\psi}{\partial x^2}(t,x) - u(t)\mu(x)\psi(t,x), x \in (0,1), t \in (0,T), \\ \psi(t,0) = \psi(t,1) = 0, \end{array} \right. \tag{2.24}$$

On note $A = -\partial_x^2$ le Laplacien de Dirichlet et $H^k_{(0)} = D(A^{k/2})$. Le gain de régularité dont on a parlé à la sous-section précédente est contenu dans le lemme suivant.

Lemme 2. *Soit $T > 0$ et $f \in L^2((0,T), H^3 \cap H^1_0)$. Alors, la fonction $G : t \mapsto \int_0^t e^{iAs} f(s)ds$ appartient à $C^0([0,T], H^3_{(0)})$, et de plus*

$$\|G\|_{L^\infty((0,T),H^3_{(0)})} \leqslant c_1(T)\|f\|_{L^2((0,T),H^3\cap H^1_0)} \tag{2.25}$$

où les constantes $c_1(T)$ sont bornées uniformément pour T borné.

L'intérêt de ce lemme réside dans le fait que l'on n'a pas fait d'hypothèse sur les conditions au bord de f. La preuve, très élémentaire, consiste à faire des intégrations par parties nécessaires pour avoir deux termes : la dérivée troisième et les termes de bord. L'estimation des termes de bord utilisera alors une sorte de moyennisation due aux oscillations.

Notons que l'on peut obtenir la même version du théorème au niveau H^1 et obtenir la régularité H^1_0. On a ici décalé de deux dérivées pour le besoin des applications.

Le théorème pour le contrôle linéaire est alors le suivant. C'est un contrôle local près de la trajectoire $\psi_1 = e^{-i\lambda_1 t}\varphi_1$ correspondant à l'état fondamental.

Théorème 25. *Soit $T > 0$ et $\mu \in H^3((0,1),\mathbb{R})$ tel que*

$$\exists c > 0 \text{ tel que } \frac{c}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k\rangle|, \forall k \in \mathbb{N}^*. \tag{2.26}$$

Il existe $\delta > 0$ et une application C^1

$$\Gamma : \mathcal{V}_T \rightarrow L^2((0,T),\mathbb{R})$$

où

$$\mathcal{V}_T := \{\psi_f \in \mathcal{S} \cap H^3_{(0)}((0,1),\mathbb{C}); \|\psi_f - \psi_1(T)\|_{H^3} < \delta\},$$

telle que, $\Gamma(\psi_1(T)) = 0$ et pour tout $\psi_f \in \mathcal{V}_T$, la solution de (2.24) avec condition initiale

$$\psi(0) = \varphi_1 \tag{2.27}$$

et avec le contrôle $u = \Gamma(\psi_f)$ satisfasse à $\psi(T) = \psi_f$.

Il y avait un théorème similaire dans [10] mais avec une régularité et un temps non optimaux, et surtout une preuve bien plus complexe.

Remarque 1. *Il est intéressant de se demander ce qui ne fonctionne plus dans la preuve de Ball-Marsden-Slemrod dans notre cas précis. Si on suit cette preuve, on doit prouver que si $u_n \rightharpoonup u$ dans $L^2([0,T])$, alors $\int_0^T e^{-i(t-\tau)A}\mu(u_n - u)w\, d\tau$ converge vers 0 dans $H^3_{(0)}$ (on garde les notations introduites juste après l'énoncé du Théorème 23). Pour simplifier, on prend $p = 0$ et $w_0 = \varphi_1$ de sorte que $x = e^{-i\lambda_1 t}\varphi_1$. On prend alors $u_n = e^{i(-\lambda_n+\lambda_1)t}$ et $T = 2/\pi$ de sorte que*

$$\int_0^{2/\pi} e^{-iA(2/\pi-\tau)}\mu(u_n - u)w\, d\tau = 2/\pi e^{-2\lambda_n/\pi} \langle \mu\varphi_1, \varphi_n\rangle \varphi_n$$

qui ne converge pas vers 0 dans $H^3_{(0)}$ vu l'hypothèse (2.23) faite sur μ.

En utilisant la description explicite (et le fait que le comportement est moralement unidimensionnel), on obtient un résultat similaire pour des données radiales de la boule en dimension 3. Le raisonnement est ici simplifié par la connaissance explicite des valeurs propres du Laplacien. Il est très probable qu'un résultat semblable puisse être obtenu dans d'autres dimensions $n \leq 5$ (pour que H^3 soit une algèbre) en utilisant une asymptotique des zéros des fonctions de Bessel.

De plus, la meilleure compréhension du problème de Cauchy nous permet d'obtenir un théorème avec des contrôles plus réguliers si les données le sont.

Théorème 26. *Soit $T > 0$ et $\mu \in H^5((0,1),\mathbb{R})$ tels que (2.26) soit vérifiée. Il existe $\delta > 0$ et une application C^1*

$$\begin{array}{rccl} \Gamma: & \mathcal{V}_T & \to & H^1_0((0,T),\mathbb{R}) \\ & \psi_f & \mapsto & \Gamma(\psi_f) \end{array}$$

où

$$\mathcal{V}_T := \{\psi_f \in \mathcal{S} \cap H^5_{(0)}((0,1),\mathbb{C}); \|\psi_f - \psi_1(T)\|_{H^5} < \delta\},$$

tels que $\Gamma(\psi_1(T)) = 0$ et, pour tout $\psi_f \in \mathcal{V}_T$, la solution de (2.24), (2.27) avec contrôle $u = \Gamma(\psi_f)$ satisfasse à $\psi(T) = \psi_f$.

En fait, dans la preuve de ce théorème, on prouve que la solution vérifie $A\psi(t) - u(t)\mu\psi(t) \in C^0([0,T], H^3_{(0)})$. $\psi(t)$ est donc bien dans $H^5([0,1])$ pour tout temps mais pas dans $H^5_{(0)}(]0,1[)$. Les valeurs au bord de la solution peuvent d'ailleurs être calculées en fonction de $u(t)$ et des valeurs au bord de μ.

Bien sûr, on peut poursuivre la stratégie aux ordres supérieurs pour prouver la contrôlabilité locale autour de l'état fondamental
- dans $H^7_{(0)}(0,1)$ avec contrôle dans $H^2_0((0,T),\mathbb{R})$,
- dans $H^9_{(0)}(0,1)$ avec contrôle dans $H^3_0((0,T),\mathbb{R})$, etc.

Equation de Schrödinger non linéaire La méthode s'applique aussi à des équations non linéaires. En effet, comme on travaille avec des grandes régularités pour lesquelles H^s est une algèbre, le terme non linéaire pose peu de problèmes tant qu'on ne fait que des raisonnements locaux près de trajectoires bien définies. Commençons par l'équation de Schrödinger non linéaire.

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = -\frac{\partial^2\psi}{\partial x^2}(t,x) + |\psi|^2\psi(t,x) - u(t)\mu(x)\psi(t,x), x \in (0,1), t \in (0,T), \\ \frac{\partial\psi}{\partial x}(t,0) = \frac{\partial\psi}{\partial x}(t,1) = 0. \end{cases} \tag{2.28}$$

Ici, le choix de conditions au bord de Neumann nous permet d'avoir sous la main une trajectoire simple ($\psi_{ref}(t,x) := e^{-it}, u_{ref}(t) = 0$). On aurait aussi pu choisir le signe "$-$" devant la non linéarité, correspondant au cas focalisant. Cela aurait juste changé un peu le raisonnement sur le linéarisé puisqu'il peut arriver des cas où certaines fréquences n'oscillent plus mais sont dissipées.

On obtient alors le théorème de contrôle local suivant.

Théorème 27. *Soit $T > 0$ et $\mu \in H^2(0,1)$ tels que*

$$\exists c > 0 \text{ tel que } \left|\int_0^1 \mu(x)\cos(k\pi x)dx\right| \geqslant \frac{c}{\max\{1,k\}^2}, \forall k \in \mathbb{N}. \tag{2.29}$$

Il existe $\eta > 0$ et une application C^1

$$\Gamma : \mathcal{V}_T \to L^2((0,T),\mathbb{R})$$

où

$$\mathcal{V}_T := \{\psi_f \in \mathcal{S} \cap H^2(0,1); \psi_f'(0) = \psi_f'(1) = 0 \text{ et } \|\psi_f - e^{-iT}\|_{H^2} < \eta\}$$

telle que, pour tout $\psi_f \in \mathcal{V}_T$, la solution de (2.28) avec condition initiale

$$\psi(0,x) = 1, \forall x \in (0,1) \tag{2.30}$$

et contrôle $u := \Gamma(\psi_f)$ est définie sur $[0,T]$ satisfait $\psi(T) = \psi_f$.

Equation des ondes non linéaire L'équivalent du lemme 2 adapté à l'équation des ondes est aussi vérifié. Cela permet alors de prouver un théorème de contrôle pour l'équation des ondes non linéaires. On choisit une non-linéarité telle que la fonction constante égale à 1 soit solution et telle que le linéarisé soit l'équation des ondes libre. On obtient alors le contrôle local sous des hypothèse sur μ de type (2.29). Comme dans l'article de K. Beauchard [11] pour l'équation linéaire, un temps plus grand que 2 est nécessaire pour le contrôle sur un intervalle de longueur 1.

1.2.0.6 Questions, problèmes ouverts et perspectives

La première question que l'on peut se poser est la généralisation de ces théorèmes à des dimensions supérieures. Or, les situations que l'on a présentées ont toutes en commun l'existence d'un gap spectral qui n'est pas garanti en dimensions supérieures. L'article de K. Beauchard, Y. Chitour, D. Khateb et R. Long [12] analyse le linéarisé de l'équation en dimensions 2 et 3 et explore la notion de contrôlabilité spectrale. Les résultats en dimension 3 semblent trop négatifs pour espérer avoir un contrôle sur le système non linéaire. Cependant, en dimension 2, la formule de Weyl peut faire espérer un comportement du type de celui qu'on a décrit pour l'équation des ondes avec l'existence d'un temps minimal. Le fait de comprendre la structure des données atteignables constitue en tout cas une question intéressante (mais sans doute difficile).

Pour l'équation des ondes (linéaire on non linéaire) sur un intervalle de longueur 1, on obtient donc un résultat de contrôle où le temps minimal est 2. Les résultats de K. Beauchard [11] montrent que, au moins si on se limite à un contrôle local, ce temps est optimal. L'interprétation de ce résultat n'est pas si évidente. On peut, dans un premier temps, se dire qu'il est naturel qu'il y ait un temps minimal de contrôle pour l'équation des ondes à cause de la vitesse finie de propagation. C'est d'ailleurs le cas pour le contrôle au bord de l'équation des ondes. Cependant, dans le contrôle bilinéaire, le contrôle apparaît distribué sur tout l'intervalle. D'autre part, pour le contrôle interne des ondes distribué sur tout l'espace, on n'a pas de temps minimal. Une première explication que l'on pourrait donner est que d'après le résultat de Ball-Marsden-Slemrod [7] discuté précédemment, on ne peut avoir contrôle que si le potentiel est dans un certain sens singulier. Dans notre cas, la singularité provient alors des conditions au bord non nulles du potentiel qui permettent d'avoir la condition (2.29). Heuristiquement, on a donc envie de dire que, en fait, on contrôle "par le bord". Seule la singularité au bord que l'on introduit par notre potentiel permet d'atteindre des données hors d'un compact de l'espace où on contrôle.

Le temps 2 est exactement le temps de contrôle géométrique pour un contrôle au bord à partir d'un seul côté. Notre situation semble donc proche du contrôle au bord des ondes à partir d'un seul côté et fournirait une explication pour le temps minimal 2. Cette explication parfaitement heuristique n'a absolument aucun fondement mathématique et ne constitue qu'une tentative d'interprétation de ce temps 2. Il serait intéressant de trouver une vraie justification géométrique de ce temps 2, par exemple par la propagation du front d'onde.

Remarque 2. *Dans le but d'homogénéiser les notations de l'introduction, nous avons changé quelques notations par rapport aux chapitres correspondant aux articles de la thèse. Voici les changements notables :*

- *dans le chapitre 4, on écrira* $X_{b,s}$ *au lieu de* $X^{s,b}$ *avec un ordre inversé des indices.*
- *dans le chapitre 4, la fonction de troncature ne sera pas* $a(x)$ *mais* $g(x)$ *et l'opérateur pour le contrôle ne sera pas* A *mais* G*.*

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Chapitre 2

Contrôlabilité et stabilisation globale pour l'équation de Schrödinger non linéaire sur un intervalle

Contents

Ce chapitre est la reprise d'un article publié dans le journal ESAIM : Control, Optimisation and Calculus of Variations [15].

Introduction

In this article, we study the stabilization and exact controllability for the periodic one-dimensional nonlinear Schrödinger equation (NLS).

$$\left\{ \begin{array}{rcl} i\partial_t u + \partial_x^2 u & = & \lambda |u|^2 u \quad \text{on} \quad [0,+\infty[\times \mathbb{T}^1 \\ u(0) & = & u_0 \in L^2(\mathbb{T}^1) \end{array} \right. \tag{0.1}$$

with $\lambda \in \mathbb{R}$.

The well posedness in such a low regularity was proved by J. Bourgain [3]. The proof uses the so called Bourgain spaces $X^{s,b}$ to get local well posedness and the conservation of the L^2 mass for global existence.

The aim of this article is to prove exact internal controllability of system (0.1) in large time for a control supported in any small open subset of $\mathbb{T}^1$. We also extend these results to $]0,\pi[$ with Dirichlet or Neumann boundary conditions. The strategy follows the one of B. Dehman, P. Gérard and G. Lebeau [7] where exact controllability in H^1 is proved for defocusing NLS on compact surfaces. Our result differs from this one because we obtain a control at a lower regularity. This allows to consider the focusing and defocusing equation and to use a different stabilization term, which seems more natural. Moreover, if the Cauchy data are smoother, that is H^s with $s \geq 0$, the control we build on L^2 keeps that regularity, without any assumption on the size in H^s. Yet, in this low regularity, Strichartz inequality of [5] does not provide uniform well posedness, and this forces us to use $X^{s,b}$ spaces. These spaces are also used in the prior and independent paper of L. Rosier and B. Y. Zhang [21] where they obtain results of local controllability near 0 for the same problem.

The strategy is first to prove stabilization and to combine it with local exact controllability near 0 to get null controllability. Then, we remark that the equation obtained by reversing time fulfills exactly the same properties and this allows to establish exact controllability.
Let $a = a(x) \in L^\infty(\mathbb{T}^1)$ real valued, the stabilization system we consider is

$$\left\{ \begin{array}{rcl} i\partial_t u + \partial_x^2 u + ia^2 u &=& \lambda |u|^2 u \quad \text{on} \quad [0,T] \times \mathbb{T}^1 \\ u(0) &=& u_0 \in L^2(\mathbb{T}^1). \end{array} \right. \tag{0.2}$$

The well posedness of this system will be proved in Section 2.2 and we can check that it satisfies the mass decay.

$$\|u(t)\|_{L^2}^2 - \|u(0)\|_{L^2}^2 = -2 \int_0^t \|au(\tau)\|_{L^2}^2 \, d\tau. \tag{0.3}$$

Our theorem states that we have an exponential decay.

Theorem 2.0.1. *Assume that $a(x)^2 > \eta > 0$ on some nonempty open set. Then, for every $R_0 > 0$, there exist $C > 0$ and $\gamma > 0$ such that inequality*

$$\|u(t)\|_{L^2} \leq Ce^{-\gamma t} \|u_0\|_{L^2} \quad t > 0$$

holds for every solution u of system (0.2) with initial data u_0 such that $\|u_0\|_{L^2} \leq R_0$.

Then, as a consequence of stabilization and local controllability near 0 established in Section 2.3, we obtain the following result.

Theorem 2.0.2. *Let $1/2 < b < 5/8$. For any nonempty open set $\omega \subset \mathbb{T}^1$ and $R_0 > 0$, there exist $T > 0$ and $C > 0$ such that for every u_0 and u_1 in $L^2(\mathbb{T}^1)$ with*

$$\|u_0\|_{L^2} \leq R_0 \quad \text{and} \quad \|u_1\|_{L^2} \leq R_0$$

there exists a control $g \in C([0,T], L^2)$ with $\|g\|_{L^\infty([0,T],L^2)} \leq C$ supported in $[0,T] \times \omega$, such that the unique solution u in $X_T^{0,b}$ to the Cauchy problem

$$\left\{ \begin{array}{rcl} i\partial_t u + \partial_x^2 u &=& \lambda |u|^2 u + g \quad \text{on} \quad [0,T] \times \mathbb{T}^1 \\ u(0) &=& u_0 \in L^2(\mathbb{T}^1) \end{array} \right. \tag{0.4}$$

satisfies $u(T) = u_1$.
Moreover, if u_0 and $u_1 \in H^s$, with $s \geq 0$, one can impose $g \in C([0,T], H^s)$.

We deduce the same results on $L^2(]0,\pi[)$ with the Dirichlet (respectively Neumann) Laplacian. To accomplish this, we use the identification of $D(-\Delta_D)$ (resp. $D(-\Delta_N)$) with the closed subspace of $H^2(\mathbb{R}/2\pi\mathbb{Z})$ of odd (resp. even) functions. We only have to check along the proof that the control we build on $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$ remains odd (resp. even) if u_0 is so. The propagation of regularity for the control takes the form : if $u_0 \in D(-\Delta_D^s)$, then one can choose $g \in C([0,T], D(-\Delta_D^s))$ (and similarly for Δ_N).

The continuity in time for g is obtained with time cutoff at each stage : the stabilization term is brought to 0 and the local control we build is identically zero at initial and final time. For example, if u_0 and u_1 are assumed in C^∞, it allows to impose u and g in $C^\infty([0,T] \times \mathbb{T}^1)$.

The independence of C, γ and the time of control T on the bound R_0 are an open problem. Yet, it is an interesting fact that even if we want a control in H^s, the time of controllability only depends on the size of the data in L^2. However, it is unknown whether there is really a minimal time of controllability. This is in strong contrast with the linear case where exact controllability occurs in arbitrary small time and the conditions are only geometric for the open set ω. For example, exact controllability is known to be true when Geometric Control Condition is realized, see G. Lebeau [16], but also for any open set ω of $\mathbb{T}^n$, see S. Jaffard [13] and V. Komornik [14]. N. Burq and M. Zworski [6] also proved the equivalence with a resolvent estimate. Moreover, some recent studies have analysed the explosion of the control cost when T tends to 0 : K.- D. Phung [20] by reducing to the heat or wave equation, L. Miller [18] with resolvent estimates, G. Tenenbaum and M. Tucsnak [23] with number theoretic arguments.

Let us now describe briefly the main arguments of the proof of Theorem 2.0.1 and 2.0.2. First, the functional spaces used are the Bourgain spaces which are especially suited for solving dispersive equations. In our problem, we use some multilinear estimates in $X^{s,b}$ (see the definition in Section 2.1). The first step is the following estimate for $b \geq 3/8$, uniformly for $T \leq 1$

$$\|u\|_{L^4([0,T]\times\mathbb{T}^1)} \leq C \|u\|_{X_T^{0,b}}. \tag{0.5}$$

This was first proved by J. Bourgain in [3]. A simpler proof, due to N. Tzvetkov, can be found in the book of T. Tao [22] p 104. This allows to prove multilinear estimates in $X^{s,b}$, as follows.

Lemma 2.0.1. *For every $s \geq 0$, $b, b' \geq 3/8$, there exists C_s independent on $T \leq 1$ such that for u and $\tilde{u} \in X_T^{s,b}$, we have*

$$\begin{aligned} \left\||u|^2 u\right\|_{X_T^{s,-b'}} &\leq C \|u\|_{X_T^{0,b}}^2 \|u\|_{X_T^{s,b}} &(0.6)\\ \left\||u|^2 u - |\tilde{u}|^2\tilde{u}\right\|_{X_T^{s,-b'}} &\leq C \left(\|u\|_{X_T^{s,b}}^2 + \|\tilde{u}\|_{X_T^{s,b}}^2\right) \|u-\tilde{u}\|_{X_T^{s,b}}. &(0.7)\end{aligned}$$

This type of multilinear estimates was introduced in [3], but we refer to [4] p 107 where the estimates we need are stated during the proof of Theorem 2.1 chapter V. In the Appendix, we recall the proof and precise some dependence in s of the estimates.

We prove the control near 0 by a perturbative argument near the one of E. Zuazua [24]. We use the fixed point theorem of Picard to deduce our result from the linear control. The propagation of H^s regularity from the state to the control is obtained using this property for the linear control and a local linear behavior. The idea comes from the work of B. Dehman and G. Lebeau [8] about the wave equation where only some smallness on a finite number of harmonics is required. A notable fact in our case is that no assumption of smallness is made on the H^s norm. We only need the L^2 norm to be small. Yet, to obtain a bound independent on s, we have to make some estimates with constants independent on s. This will only be possible up to weaker terms, but this will be enough to conclude.

The proof of stabilization is more intricate. In a contradiction argument, following B. Dehman, G. Lebeau, E. Zuazua [9] and [7], we are led to prove the strong convergence to zero in $X_T^{0,b}$ of some weakly convergent sequence (u_n) of solutions to damped NLS. In [7], the authors use some linearisability property of NLS in H^1. Yet, this is false in the L^2 case. Moreover, as it was seen by L. Molinet in [19], a weak limit u of solutions of NLS is in general not necessarily solution of the same equation. So, we have to proceed a little differently.

We first establish the strong convergence by some propagation of compactness. For a sequence (u_n) weakly convergent to 0 in $X_T^{0,b}$ satisfying

$$\begin{cases} i\partial_t u_n + \partial_x^2 u_n \to 0 \quad \text{in} \quad X_T^{-1+b,-b} \\ u_n \to 0 \quad \text{in} \quad L^2([0,T]\times\omega), \end{cases}$$

we prove that $u_n \to 0$ in $L^2_{loc}([0,T]\times\mathbb{T}^1)$. As the geometric control assumption is fulfilled, the propagation of compactness could be proved using microlocal defect measure introduced by P. Gérard [10], adapting to $X^{s,b}$ spaces the argument of [7] inspired by C. Bardos and T. Masrour [1]. In dimension 1, the microlocal analysis is much simpler and we have chosen, for the convenience of the reader, to prove it with elementary arguments (even if the ideas are the same).

Once we know that the convergence is strong, we infer that the limit u is solution to NLS. We use a classical unique continuation theorem to infer that it is 0.

Proposition 2.0.1. *For every $T > 0$ and ω any nonempty open set of $\mathbb{T}^1$, the only solution in $C^\infty([0,T]\times\mathbb{T}^1)$ to the system*

$$\begin{cases} i\partial_t u + \partial_x^2 u = b(t,x)u \text{ on } [0,T]\times\mathbb{T}^1 \\ u = 0 \text{ on } [0,T]\times\omega \end{cases}$$

where $b(t,x)\in C^\infty([0,T]\times\mathbb{T}^1)$ is the trivial one $u\equiv 0$.

This was proved by Isakov [12] (see Corollary 6.1) using Carleman estimates.

Yet, the weak limit a priori belongs to $X_T^{0,b}$. Therefore, to apply Proposition 2.0.1, we need u smooth enough. We prove that a solution of NLS with $u\in C^\infty([0,T]\times\omega)$ is actually smooth. The proof is an adaptation to the $X^{s,b}$ spaces of propagation results of microlocal regularity coming from [7]. Again, we present it in such a way that no knowledge of microlocal analysis is necessary, even if the ideas deeply come from this theory.

Notation Denote D^r the operator defined on $\mathcal{D}'(\mathbb{T}^1)$ by

$$\begin{aligned} \widehat{D^r u}(n) &= \operatorname{sgn}(n)|n|^r\widehat{u}(n) \qquad \text{if} \quad n\neq 0 \\ &= \widehat{u}(0) \qquad\qquad\qquad \text{if} \quad n=0. \end{aligned} \tag{0.8}$$

In this article, b and b' will be two constants, fixed for the rest of the article, such that $1 > b + b'$, $b > 1/2 > b'$, and estimates (0.6) and (0.7) hold, see Lemma 2.1.3 below for the justification of these assumptions. Actually, we can check that for any $1/2 < b < 5/8$, we can find a suitable constant b' with the needed properties.

C will denote any absolute constant whose value could change along the article. It could actually depend on s. Yet, when the dependence on s will be needed, this will be announced and we will denote C if it is independent on s and C_s otherwise.

2.1 Some properties of $X^{s,b}$ spaces

We equip the Sobolev space $H^s(\mathbb{T}^1)$ with the norm

$$\|u\|_{H^s}^2 = \|D^s u\|_{L^2}^2 = |\widehat{u}(0)|^2 + \sum_{k\neq 0} |k|^{2s} |\widehat{u}(k)|^2 .$$

The Bourgain space $X^{s,b}$ is equipped with the norm

$$\begin{aligned}\|u\|_{X^{s,b}}^2 &= \|\widehat{u}(.,0)\|_{H^b(\mathbb{R})}^2 + \sum_k \int_{\mathbb{R}} |k|^{2s} \left\langle \tau + k^2 \right\rangle^{2b} \left|\widehat{\widehat{u}}(\tau,k)\right|^2 d\tau \\ &= \left\|u^{\#}\right\|_{H^b(\mathbb{R},H^s(\mathbb{T}^1))}^2\end{aligned}$$

where $\langle . \rangle = \sqrt{1+|.|^2}$, $u = u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{T}^1$, and $u^{\#}(t) = e^{-it\partial_x^2}u(t)$. $\widehat{\widehat{u}}(\tau,k)$ denotes the Fourier transform of u with respect to the time variable (indice τ) and space variable (indice k). $\widehat{u}(t,k)$ denotes the Fourier transform in space variable.
$X_T^{s,b}$ is the associated restriction space, with the norm

$$\|u\|_{X_T^{s,b}} = \inf \left\{ \|\tilde{u}\|_{X^{s,b}} \left| \tilde{u} = u \text{ on } [0,T]\times\mathbb{T}^1 \right. \right\} .$$

Let us study the stability of the $X^{s,b}$ spaces with respect to some particular operations.

Lemma 2.1.1. *Let $\psi \in C_0^\infty(\mathbb{R})$ and $u \in X^{s,b}$ then $\psi(t)u \in X^{s,b}$.
If $u \in X_T^{s,b}$ then we have $\psi(t)u \in X_T^{s,b}$.*

Démonstration. We write

$$\|\psi u\|_{X^{s,b}} = \left\|e^{-it\partial_x^2}\psi(t)u\right\|_{H^b(H^s)} = \left\|\psi u^{\#}\right\|_{H^b(H^s)} \leq C \left\|u^{\#}\right\|_{H^b(H^s)} \leq C \|u\|_{X^{s,b}} .$$

We get the second result by applying the first one on any extension of u and taking the infimum. □

We easily get that D^r (using notation (0.8)) maps any $X^{s,b}$ into $X^{s-r,b}$. In the case of multiplication by $C^\infty(\mathbb{T}^1)$ function, we have to deal with a loss in $X^{s,b}$ regularity compared to what we could expect. Some regularity in the index b is lost, due to the fact that multiplication does not keep the structure in time of the harmonics. This loss is unavoidable : take $u_n = \psi(t)e^{inx}e^{in^2t}$ (where $\psi \in C_0^\infty(\mathbb{R})$ equal to 1 on $[-1,1]$) which is uniformly bounded in $X^{0,b}$ for every $b \geq 0$. Yet, if we consider the operator of multiplication by e^{ix}, we get $\|e^{ix}u_n\|_{X^{0,b}} \approx n^b$. We can prove that our example is the worst one.

Lemma 2.1.2. *Let $-1 \leq b \leq 1$, $s \in \mathbb{R}$ and $\varphi \in C^\infty(\mathbb{T}^1)$. Then, if $u \in X^{s,b}$ we have $\varphi(x)u \in X^{s-|b|,b}$.*
Similarly, multiplication by φ maps $X_T^{s,b}$ into $X_T^{s-|b|,b}$.

Démonstration. We first deal with the two cases $b=0$ and $b=1$ and we will conclude by interpolation and duality.
For $b=0$, $X^{s,0} = L^2(\mathbb{R}, H^s)$ and the result is obvious.
For $b=1$, we have $u \in X^{s,1}$ if and only if

$$u \in L^2(\mathbb{R}, H^s) \text{ and } i\partial_t u + \partial_x^2 u \in L^2(\mathbb{R}, H^s)$$

with the norm

$$\|u\|^2_{X^{s,1}} = \|u\|^2_{L^2(\mathbb{R},H^s)} + \left\|i\partial_t u + \partial_x^2 u\right\|^2_{L^2(\mathbb{R},H^s)}.$$

Then, we have

$$\begin{aligned}
\|\varphi(x)u\|^2_{X^{s-1,1}} &= \|\varphi u\|^2_{L^2(\mathbb{R},H^{s-1})} + \left\|i\partial_t(\varphi u) + \partial_x^2(\varphi u)\right\|^2_{L^2(\mathbb{R},H^{s-1})} \\
&\leq C\left(\|u\|^2_{L^2(\mathbb{R},H^{s-1})} + \left\|\varphi\left(i\partial_t u + \partial_x^2 u\right)\right\|^2_{L^2(\mathbb{R},H^{s-1})}\right. \\
&\quad \left. + \left\|\left[\varphi, \partial_x^2\right] u\right\|^2_{L^2(\mathbb{R},H^{s-1})}\right) \\
&\leq C\left(\|u\|^2_{L^2(\mathbb{R},H^{s-1})} + \left\|i\partial_t u + \partial_x^2 u\right\|^2_{L^2(\mathbb{R},H^{s-1})} + \|u\|^2_{L^2(\mathbb{R},H^s)}\right) \\
&\leq C\,\|u\|^2_{X^{s,1}}.
\end{aligned}$$

Here, we have used that $[\varphi, \partial_x^2] = -2(\partial_x\varphi)\partial_x - (\partial_x^2\varphi)$ is a differential operator of order 1. To conclude, we prove that $X^{s,b}$ spaces are in interpolation. For that, we consider $X^{s,b}$ as a weighted $L^2(\mathbb{R}\times\mathbb{Z}, \mu\otimes\delta)$ spaces, where μ is the Lebesgue measure on $\mathbb{R}$ and δ is the discret measure on $\mathbb{Z}$. Using the Fourier transform, we can interpret $X^{s,b}$ as the weighted L^2 space

$$L^2\left(\mathbb{R}\times\mathbb{Z}, w_{s,b}(\tau,k)\mu\otimes\delta\right)$$

where $w_{s,b}(\tau,k) = |k|_\wr^{2s}\left\langle\tau+k^2\right\rangle^{2b}$. Here, we denote

$$|k|_\wr = |k| \text{ if } k \neq 0 \text{ and } 1 \text{ otherwise.} \tag{1.9}$$

Then, we use the complex interpolation theorem of Stein-Weiss for weighted L^p spaces (see [2] p 114) : for $0<\theta<1$

$$\left(X^{s,0}, X^{s',1}\right)_{[\theta]} \approx L^2\left(\mathbb{R}\times\mathbb{Z}, |k|_\wr^{2s(1-\theta)+2s'\theta}\left\langle\tau+k^2\right\rangle^{2\theta}\mu\otimes\delta\right) \approx X^{s(1-\theta)+s'\theta,\theta}.$$

Since φ maps $X^{s,0}$ into $X^{s,0}$ and $X^{s,1}$ into $X^{s-1,1}$, we conclude that for $0 \leq b \leq 1$, φ maps $X^{s,b} = (X^{s,0}, X^{s,1})_{[b]}$ into $(X^{s,0}, X^{s-1,1})_{[b]} = X^{s-b,b}$ which yields the b loss of regularity as announced.
Then, by duality, this also implies that for $0 \leq b \leq 1$, $\varphi(x)$ maps $X^{-s+b,-b}$ into $X^{-s,-b}$. As there is no assumption on $s \in \mathbb{R}$, we also have the result for $-1 \leq b \leq 0$ with a loss $-b = |b|$.
To get the same result for the restriction spaces $X_T^{s,b}$, we write the estimate for an extension $\tilde{u}$ of u, which yields

$$\|\varphi u\|_{X_T^{s-|b|,b}} \leq \|\varphi\tilde{u}\|_{X^{s-|b|,b}} \leq C\,\|\tilde{u}\|_{X^{s,b}}.$$

Taking the infimum on all the $\tilde{u}$, we get the claimed result. □

We will also use (see [11] or [3])

Lemma 2.1.3. *Let $\Psi \in C_0^\infty(\mathbb{R})$ and (b, b') satisfying*

$$0 < b' < \frac{1}{2} < b, \qquad b + b' \leq 1.$$

If we note $F(t) = \Psi\left(\frac{t}{T}\right) \int_0^t f(t')dt'$, we have for $T \leq 1$

$$\|F\|_{H^b} \leq C T^{1-b-b'} \|f\|_{H^{-b'}}.$$

In the futur aim of using a boot-strap argument, we will need some continuity in T of the $X_T^{s,b}$ norm of a fixed function :

Lemma 2.1.4. *Let $0 < b < 1$ and u in $X^{s,b}$ then the function*

$$\left\{ \begin{array}{cccc} f & : &]0,T] & \longrightarrow & \mathbb{R} \\ & & t & \longmapsto & \|u\|_{X_t^{s,b}} \end{array} \right.$$

is continuous. Moreover, if $b > 1/2$, there exists C_b such that

$$\lim_{t \to 0} f(t) \leq C_b \|u(0)\|_{H^s}.$$

Démonstration. By reasoning on each component on the basis, we are led to prove the result in $H^b(\mathbb{R})$. The most difficult case is the limit near 0. It suffices to prove that if $u \in H^b(\mathbb{R})$, with $b > 1/2$, satisfies $u(0) = 0$, and $\Psi \in C_0^\infty(\mathbb{R})$ with $\Psi(0) = 1$, then

$$\Psi\left(\frac{t}{T}\right) u \underset{T \to 0}{\longrightarrow} 0 \quad \text{in} \quad H^b.$$

Such a function u can be written $\int_0^t f$ with $f \in H^{b-1}$. Then, Lemma 2.1.3 gives the result we want if $u \in H^{b+\varepsilon}$. Nevertheless, if we only have $u \in H^b$, $\Psi(\frac{t}{T})u$ is uniformly bounded. We conclude by a density argument. □

The following lemma will be useful to control solutions on large intervals that will be obtained by piecing together solutions on smaller ones. We state it without proof.

Lemma 2.1.5. *Let $0 < b < 1$. If $\bigcup]a_k, b_k[$ is a finite covering of $[0, 1]$, then there exists a constant C depending only of the covering such that for every $u \in X^{s,b}$*

$$\|u\|_{X_{[0,1]}^{s,b}} \leq C \sum_k \|u\|_{X_{[a_k,b_k]}^{s,b}}.$$

Finally, we have the following Rellich type lemma

Lemma 2.1.6. *For every $\delta > 0$, $\eta > 0$, s, $b \in \mathbb{R}$ and $T > 0$, we have*

$$X_T^{s+\eta, b+\delta} \subset X_T^{s,b}$$

with compact imbedding.

Démonstration. Using space Fourier transform and working with $u_n^\# = e^{-it\partial_x^2} u_n$, we are led to prove the compact embedding $H^{b+\delta}([0,T], l^2_{\langle k \rangle^{2(s+\eta)}}) \subset H^b([0,T], l^2_{\langle k \rangle^{2s}})$, where $l^2_{\langle k \rangle^{2s}}$ is the discret l^2 space with the weight $\langle k \rangle^{2s}$. This is an easy adaptation of Rellich's theorem. □

2.2 Existence of a solution to NLS with source and damping term

Theorem 2.2.1. *Let $T > 0$, $s \geq 0$, $\lambda \in \mathbb{R}$ and $a \in C^\infty(\mathbb{T}^1)$, $\varphi \in C_0^\infty(\mathbb{R})$ taking real values.*
For every $g \in L^2([-T,T], H^s)$ and $u_0 \in H^s$, there exists a unique solution u in $X_T^{s,b}$ to

$$\left\{ \begin{array}{rcl} i\partial_t u + \partial_x^2 u + i\varphi(t)^2 a(x)^2 u & = & \lambda |u|^2 u + g \text{ on } [-T,T] \times \mathbb{T}^1 \\ u(0) & = & u_0 \in H^s. \end{array} \right. \tag{2.10}$$

Moreover the flow map

$$\begin{array}{rcrcl} F & : & H^s(\mathbb{T}^1) \times L^2([-T,T], H^s(\mathbb{T}^1)) & \to & X_{[-T,T]}^{s,b} \\ & & (u_0, g) & \mapsto & u \end{array}$$

is Lipschitz on every bounded subset.
The same results occur for $s = 0$ with the weaker assumption $a \in L^\infty(\mathbb{T}^1)$.

Démonstration. It is strongly inspired by Bourgain's one (see [3], [4] and [11]). First, we notice that if $g \in L^2([-T,T], H^s)$, it also belongs to $X_T^{s,-b'}$ as $b' \geq 0$. We restrict ourself to positive times. The solution on $[-T,0]$ is obtained similarly. The distinction on the case $s = 0$ and $s > 0$ for the regularity assumption on a will appear along the proof with the following statement : with the assumptions of the Theorem, multiplication by a maps $X^{s,0} = L^2([0,T], H^s)$ into itself.
We consider the functional

$$\Phi(u)(t) = e^{it\partial_x^2} u_0 - i \int_0^t e^{i(t-\tau)\partial_x^2} \left[-ia^2\varphi^2 u + \lambda |u|^2 u + g \right](\tau) d\tau.$$

We will apply a fixed point argument on the Banach space $X_T^{s,b}$. Let $\psi \in C_0^\infty(\mathbb{R})$ be equal to 1 on $[-1,1]$. Then by construction, (see [11]) :

$$\left\| \psi(t) e^{it\partial_x^2} u_0 \right\|_{X^{s,b}} = \|\psi\|_{H^b(\mathbb{R})} \|u_0\|_{H^s}.$$

Therefore, for $T \leq 1$ we have

$$\left\| e^{it\partial_x^2} u_0 \right\|_{X_T^{s,b}} \leq C \|u_0\|_{H^s}.$$

The one dimensional estimate of Lemma 2.1.3 implies

$$\left\| \psi(t/T) \int_0^t e^{i(t-\tau)\partial_x^2} F(\tau) d\tau \right\|_{X^{s,b}} \leq C T^{1-b-b'} \|F\|_{X^{s,-b'}}$$

and then

$$\begin{array}{rl} & \left\| \displaystyle\int_0^t e^{i(t-\tau)\partial_x^2} \left[-ia^2\varphi^2 u + \lambda |u|^2 u + g \right](\tau) d\tau \right\|_{X_T^{s,b}} \\ \leq & C T^{1-b-b'} \left\| -ia^2\varphi^2 u + \lambda |u|^2 u + g \right\|_{X_T^{s,-b'}} \\ \leq & C T^{1-b-b'} \left[\left\| \varphi^2 a^2 u \right\|_{X_T^{s,0}} + \left\| |u|^2 u \right\|_{X_T^{s,-b'}} + \|g\|_{X_T^{s,-b'}} \right] \\ \leq & C T^{1-b-b'} \|u\|_{X_T^{s,b}} \left(1 + \|u\|_{X_T^{0,b}}^2 \right) + C T^{1-b-b'} \|g\|_{X_T^{s,-b'}}. \end{array} \tag{2.11}$$

Thus

$$\|\Phi(u)\|_{X_T^{s,b}} \leq C\|u_0\|_{H^s} + C\|g\|_{X_T^{s,-b'}} + CT^{1-b-b'}\|u\|_{X_T^{s,b}}\left(1+\|u\|^2_{X_T^{0,b}}\right) \tag{2.12}$$

and similarly,

$$\|\Phi(u)-\Phi(\tilde{u})\|_{X_T^{s,b}} \leq CT^{1-b-b'}\|u-\tilde{u}\|_{X_T^{s,b}}\left(1+\|u\|^2_{X_T^{s,b}}+\|\tilde{u}\|^2_{X_T^{s,b}}\right). \tag{2.13}$$

These estimates imply that if T is chosen small enough Φ is a contraction on a suitable ball of $X_T^{s,b}$.
Moreover, we have uniqueness in the class $X_T^{s,b}$ for the Duhamel equation. To get the uniqueness in $X_T^{s,b}$ for the Schrödinger equation itself, we prove that every solution u in $X_T^{s,b}$ of equation (2.10) in the distributional sense is also solution of the integral equation. Let us put

$$w(t) = e^{it\partial_x^2}u_0 - i\int_0^t e^{i(t-\tau)\partial_x^2}\left[-i\varphi^2a^2u+\lambda|u|^2u+g\right](\tau)d\tau.$$

As $u \in X_T^{s,b}$, we have $|u|^2u \in X_T^{s,-b'}$ and since $b' < 1/2$, we infer

$$\begin{aligned}\partial_t\left[\int_0^t e^{-i\tau\partial_x^2}\left[-ia^2\varphi^2u+\lambda|u|^2u+g\right](\tau)d\tau\right]\\ = e^{-it\partial_x^2}\left[-ia^2\varphi^2u+\lambda|u|^2u+g\right](t)\end{aligned}$$

in the distributional sense which implies that w is solution of

$$i\partial_t w + \partial_x^2 w + ia^2\varphi^2u = \lambda|u|^2u+g.$$

Then, $r = e^{-it\partial_x^2}(u-w)$ is solution of $\partial_t r = 0$ and $r(0)=0$. Hence, $r=0$ and u is solution of the integral equation. Actually, the above proof also gives that the solution u of the integral equation is also solution in the distributional sense.
We also prove propagation of regularity.
If $u_0 \in H^s$, with $s>0$, we have an existence time T for the solution in $X_T^{0,b}$ and another time $\tilde{T}$ for the existence in $X_{\tilde{T}}^{s,b}$. By uniqueness in $X_T^{0,b}$, the two solutions are the same on $[0,\tilde{T}]$. If we assume $\tilde{T} < T$, we have the explosion of $\|u(t,.)\|_{H^s}$ as t tends to $\tilde{T}$ whereas $\|u(t,.)\|_{L^2}$ remains bounded on this interval. Using local existence in L^2 and Lemma 2.1.5, we easily get that $\|u\|_{X_{\tilde{T}}^{0,b}}$ is finite. Then, using tame estimate (2.12) on a subinterval $[\widetilde{T}-\varepsilon,\widetilde{T}]$, with ε small enough such that $C\varepsilon^{1-b-b'}\left(1+\|u\|^2_{X^{0,b}_{[\widetilde{T}-\varepsilon,\widetilde{T}]}}\right) < 1/2$, we obtain

$$\|u\|_{X^{s,b}_{[\widetilde{T}-\varepsilon,\widetilde{T}]}} \leq C\|u(T-\varepsilon)\|_{H^s} + \|g\|_{X^{s,-b'}_{[\widetilde{T}-\varepsilon,\widetilde{T}]}}.$$

We conclude that $u \in X_{\tilde{T}}^{s,b}$, which contradicts the explosion of $\|u(t,.)\|_{H^s}$ near $\tilde{T}$. Therefore, the time of existence is the same for every $s \geq 0$.
Next, we use L^2 energy estimates to get global existence in $X_T^{0,b}$ and so in $X_T^{s,b}$. By multiplying equation (2.10) by $\overline{u}$, taking imaginary part and integrating, we get

$$\|u(t)\|^2_{L^2} - \|u(0)\|^2_{L^2} = -2\int_0^t\|a\varphi(\tau)u(\tau)\|^2_{L^2}d\tau + 2\Im\int_0^t\int_{\mathbb{T}^1} g\bar{u}$$

$$\begin{aligned}\|u(t)\|_{L^2}^2 &\leq \|u(0)\|_{L^2}^2 + C\int_0^t \|u(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \|u(\tau)\|_{L^2} \|g(\tau)\|_{L^2} \, d\tau \\ &\leq \|u(0)\|_{L^2}^2 + C\int_0^t \|u(\tau)\|_{L^2}^2 \, d\tau + C \|g\|_{L^2([-T,T],L^2)}^2 .\end{aligned}$$

Then, by Gronwall inequality, we have

$$\|u(t)\|_{L^2}^2 \leq C\left(\|u(0)\|_{L^2}^2 + \|g\|_{L^2([-T,T],L^2)}^2\right) e^{C|t|}. \tag{2.14}$$

This ensures that the L^2 norm remains bounded and the solution u is global in time. For the continuity of the flow, we use a slight modification of estimate (2.13) for two solutions u and $\tilde{u}$

$$\begin{aligned}\|u-\tilde{u}\|_{X_T^{s,b}} &\leq C\|u(0)-\tilde{u}(0)\|_{H^s} + C\|g-\tilde{g}\|_{X_T^{s,-b'}} \\ &+CT^{1-b-b'}\|u-\tilde{u}\|_{X_T^{s,b}}\left(1+\|u\|_{X_T^{s,b}}^2+\|\tilde{u}\|_{X_T^{s,b}}^2\right).\end{aligned}$$

Then, for T small enough (depending on the size of u_0, $\widetilde{u_0}$, g and $\tilde{g}$), we get

$$\|u-\tilde{u}\|_{X_T^{s,b}} \leq C\|u(0)-\tilde{u}(0)\|_{H^s} + C\|g-\tilde{g}\|_{X_T^{s,-b'}} .$$

Then, we just have to piece solutions together on small intervals. Using the control of the $X_T^{s,b}$ norm on $L^\infty([0,T],H^s)$ and Lemma 2.1.5, we get that F is Lipschitz on bounded sets for arbitrary T. □

After this point and until the end of the proof of local controllability, we will express the dependence on s of the constants by writing them C_s or $C(.)$ if some other dependence is considered. b, b', λ, a and φ being fixed, we will not write the dependence of constants in these variables.
The following Propositions establish a linear behavior on bounded sets of L^2.

Proposition 2.2.1. *For every $T>0$, $\eta>0$ and $s\geq 0$, there exists $C(T,\eta,s)$ such that for every $u\in X_T^{s,b}$ solution of (2.10) with $\|u_0\|_{L^2}+\|g\|_{L^2([0,T],L^2)}<\eta$, we have the following estimate*

$$\|u\|_{X_T^{s,b}} \leq C(T,\eta,s)\left(\|u_0\|_{H^s}+\|g\|_{L^2([0,T],H^s)}\right).$$

Démonstration. Assume $T\leq 1$. Using (2.12), we obtain that u satisfies

$$\|u\|_{X_T^{s,b}} \leq C\left(\|u_0\|_{H^s}+\|g\|_{L^2([0,T],H^s)}\right) + C_sT^{1-b-b'}\|u\|_{X_T^{s,b}}\left(1+\|u\|_{X_T^{0,b}}^2\right).$$

With T such that $C_sT^{1-b-b'}<1/2$, it yields

$$\|u\|_{X_T^{s,b}} \leq C\left(\|u_0\|_{H^s}+\|g\|_{L^2([0,T],H^s)}\right) + C_sT^{1-b-b'}\|u\|_{X_T^{s,b}}\|u\|_{X_T^{0,b}}^2 .$$

First we use it with $s=0$. As we have proved in Lemma 2.1.4 the continuity with respect to T of $\|u\|_{X_T^{0,b}}$ we are in position to apply a boot-strap argument : for $T^{1-b-b'} < \frac{1}{2C_0\left(\|u_0\|_{L^2}+\|g\|_{L^2([0,T],L^2)}\right)^2}$, we obtain :

$$\|u\|_{X_T^{0,b}} \leq C\left(\|u_0\|_{L^2}+\|g\|_{L^2([0,T],L^2)}\right). \tag{2.15}$$

The mass estimate (2.14) gives $\|u(t)\|_{L^2} \leq C\eta e^{C|t|} \leq C(\eta)$. Then, we have a constant $\varepsilon(\eta)$ such that (2.15) holds for every interval of length smaller than $\varepsilon(\eta)$. Repeating the argument on every small interval, using that $X_T^{0,b}$ controls $L^\infty(L^2)$ and matching solutions with Lemma 2.1.5, we get the same result for some large interval $[0,T]$, with $T \leq 1$, with a constant C dependent on η. It expresses a local linear behavior.
Then, returning to the case $s > 0$ and $C_s T^{1-b-b'} < 1/2$, we have the estimate

$$C_s T^{1-b-b'} \|u\|^2_{X_T^{0,b}} \leq C_s T^{1-b-b'} C(\eta)^2 \eta^2.$$

Then, for $T \leq \varepsilon(s,\eta)$, this can be bounded by $1/2$ and we have

$$\|u\|_{X_T^{s,b}} \leq C\left(\|u_0\|_{H^s} + \|g\|_{L^2([0,T],H^s)}\right). \tag{2.16}$$

Again, piecing solutions together, we get the same result for large $T \leq 1$ with C depending on s and η. The assumption $T \leq 1$ is removed similarly with a final constant $C(s,\eta,T)$. □

A notable consequence of this result is that NLS has a linear behavior in any H^s on any bounded set of L^2.

Yet, in the last estimate, the constants strongly depend on s. We will use the more precise estimates of the Appendix to eliminate this dependence in s, up to some weaker terms.

Proposition 2.2.2. *For every $T > 0$, $\eta > 0$, there exists $C(T,\eta)$ such that for every $s \geq 1$, we can find $C(T,\eta,s)$ such that for every $u \in X_T^{s,b}$ solution of (2.10) with $\|u_0\|_{L^2} + \|g\|_{L^2([0,T],L^2)} < \eta$, we have*

$$\begin{aligned}\|u\|_{X_T^{s,b}} &\leq C(\eta,T)\left(\|u_0\|_{H^s} + \|g\|_{L^2([0,T],H^s)}\right) \\ &\quad + C(s,\eta,T)\|u\|_{X_T^{s-1,b}}\|u\|_{X_T^{1,b}}\|u\|_{X_T^{0,b}} + C(s,\eta,T)\|u\|_{X_T^{s-1,b}}. \end{aligned} \tag{2.17}$$

Démonstration. First, we assume $T \leq 1$. Lemma 2.1.3 gives a constant C independant on s such that

$$\begin{aligned}\|u\|_{X_T^{s,b}} &\leq C\left(\|u_0\|_{H^s} + \|g\|_{L^2([0,T],H^s)}\right) \\ &\quad + CT^{1-b-b'}\left(\left\|a^2\varphi^2 u\right\|_{L^2([0,T],H^s)} + \left\||u|^2 u\right\|_{X_T^{s,-b'}}\right).\end{aligned}$$

Estimate (1.46) of Proposition 2.A.1 and Corollary 2.A.1 of the Appendix gives some constant C and C_s such that

$$\begin{aligned}\|u\|_{X_T^{s,b}} &\leq C\left(\|u_0\|_{H^s} + \|g\|_{L^2([0,T],H^s)}\right) \\ &\quad + T^{1-b-b'}\left(C\|u\|_{X_T^{s,b}} + C_s\|u\|_{X_T^{s-1,b}}\right) \\ &\quad + T^{1-b-b'}\left(C\|u\|^2_{X_T^{0,b}}\|u\|_{X_T^{s,b}} + C_s\|u\|_{X_T^{s-1,b}}\|u\|_{X_T^{1,b}}\|u\|_{X_T^{0,b}}\right).\end{aligned}$$

From the previous Proposition, we have

$$\|u\|_{X_T^{0,b}} \leq C(\eta,T)\left(\|u_0\|_{L^2} + \|g\|_{L^2([0,T],L^2)}\right) \leq C(\eta,T)\eta.$$

Actually, $C(\eta, T)$ can be bounded by $C(\eta) = C(\eta, 1)$ if $T \leq 1$.
Again, for T small enough (depending only on η and not on s), we have

$$\begin{aligned} \|u\|_{X_T^{s,b}} &\leq C\left(\|u_0\|_{H^s} + \|g\|_{L^2([0,T],H^s)}\right) \\ &\quad + C_s \|u\|_{X_T^{s-1,b}} \|u\|_{X_T^{1,b}} \|u\|_{X_T^{0,b}} + C_s \|u\|_{X_T^{s-1,b}}. \end{aligned}$$

Then, piecing solutions together, we finally obtain the result on a large interval $[0, T]$. $\square$

Remark 2.2.1. *If $g = 0$, the solution $u \in X_T^{0,b}$ of (2.10) actually satisfies*

$$\|u(t)\|_{L^2}^2 - \|u(0)\|_{L^2}^2 = -2 \int_0^t \|a\varphi(\tau)u(\tau)\|_{L^2}^2 \, d\tau.$$

Remark 2.2.2. *If a is even and $u \in X_T^{0,b}$ solution of (2.10) with source term g, then $\pm u(t, -x)$ is solution with source term $\pm g(t, -x)$. As a conclusion, by uniqueness in $X_T^{0,b}$, we infer that if u_0 and g are odd (resp. even), then u is also odd (resp. even). This gives an existence and uniqueness theorem for Dirichlet and Neumann conditions if $a \in C_0^\infty(]0, \pi[)$ (by identification it will become $a \in C^\infty(\mathbb{T}^1)$ even).*

2.3 Controllability near 0

We know (see [7], [16] or [17]) that any nonempty open set ω satisfies an observability estimate in L^2 in arbitrary small time $T > 0$. Namely, for any $a(x) \in C^\infty(\mathbb{T}^1)$ and $\varphi(t) \in C_0^\infty(]0, T[)$ real valued such that $a \equiv 1$ on ω and $\varphi \equiv 1$ on $[T/3, 2T/3]$ (we add the cutoff in time to impose that the control g is zero at 0 and T), there exists $C > 0$ such that

$$\|\Psi_0\|_{L^2}^2 \leq C \int_0^T \left\| a(x)\varphi(t)e^{it\partial_x^2}\Psi_0 \right\|_{L^2}^2 \, dt \tag{3.18}$$

for every $\Psi_0 \in L^2$.
As a consequence, using the HUM method of J-L. Lions, this implies exact controllability in L^2 for the linear equation. More precisely, we can follow [7] to construct an isomorphism of control S from L^2 to L^2. For every data Ψ_0 in L^2, there exists $\Phi_0 = S^{-1}\Psi_0$, $\Psi_0 = S\Phi_0$ such that if Φ is solution of the dual equation

$$\left\{ \begin{array}{rcl} i\partial_t\Phi + \partial_x^2\Phi &=& 0 \\ \Phi(x, 0) &=& \Phi_0(x) \end{array} \right. \tag{3.19}$$

and Ψ solution of

$$\left\{ \begin{array}{rcl} i\partial_t\Psi + \partial_x^2\Psi &=& a^2(x)\varphi^2(t)\Phi \\ \Psi(T) &=& 0 \end{array} \right. \tag{3.20}$$

we have $\Psi(0) = \Psi_0$.

Lemma 2.3.1. *S is an isomorphism of H^s for every $s \geq 0$.*

Démonstration. We easily see that S maps H^s into itself. So we just have to prove that $S\Phi_0 \in H^s$ implies $\Phi_0 \in H^s$, *i.e.* $D^s\Phi_0 \in L^2$ (with notation (0.8) of the end of the Introduction). We use the formula

$$S\Phi_0 = i\int_0^T e^{-it\partial_x^2}\varphi^2 a^2 e^{it\partial_x^2}\Phi_0\, dt.$$

Then, using that S^{-1} is continuous from L^2 into itself and Lemma 2.A.1 of the Appendix, we get

$$\begin{aligned}
\|D^s\Phi_0\|_{L^2} &\leq C\,\|SD^s\Phi_0\|_{L^2} \leq C\left\|\int_0^T e^{-it\partial_x^2}a^2\varphi^2 e^{it\partial_x^2}D^s\Phi_0\right\|_{L^2}\\
&\leq C\left\|D^s\int_0^T e^{-it\partial_x^2}a^2\varphi^2 e^{it\partial_x^2}\Phi_0\right\|_{L^2}\\
&\quad +C\left\|\int_0^T e^{-it\partial_x^2}\left[a^2, D^s\right]\varphi^2 e^{it\partial_x^2}\Phi_0\right\|_{L^2}\\
&\leq C\,\|S\Phi_0\|_{H^s} + C_s\,\|\Phi_0\|_{H^{s-1}}\,.
\end{aligned}$$

This yields the desired result for $s \in [0,1]$. We obtain it for every $s \geq 0$ by iteration. Moreover, if we track the dependence of each constant, especially their dependence in s, we get for $s \geq 1$

$$\left\|S^{-1}\Psi_0\right\|_{H^s} \leq C(a,\varphi,T)\,\|\Psi_0\|_{H^s} + C(a,\varphi,s,T)\,\|\Psi_0\|_{H^{s-1}}\,. \tag{3.21}$$

□

Theorem 2.3.1. *Let ω be any nonempty open subset of $\mathbb{T}^1$ and $T > 0$. Then there exist $\varepsilon > 0$ and $\eta > 0$ such that for every $u_0 \in L^2$ with $\|u_0\|_{L^2} < \varepsilon$, there exists $g \in C([0,T], L^2)$, with $\|g\|_{L^\infty([0,T],L^2)} \leq \eta$, compactly supported in $]0,T[\times\omega$ such that the unique solution u in $X_T^{0,b}$ of*

$$\left\{\begin{array}{rcl} i\partial_t u + \partial_x^2 u &=& \lambda|u|^2u + g\\ u(x,0) &=& u_0(x)\end{array}\right. \tag{3.22}$$

satisfies $u(T) = 0$.
Moreover, if $u_0 \in H^s$, with $s \geq 0$, eventually with a large H^s norm, we can impose $g \in C([0,T], H^s)$.

Démonstration. We first choose $a(x) \in C_0^\infty(\omega)$ and $\varphi(t) \in C_0^\infty(]0,T[)$ different from zero, so that, observability estimate (3.18) occurs. We seek g under the form $\varphi^2(t)a^2(x)\Phi$ where Φ is solution of system (3.19), as in linear control theory. The purpose is then to choose the adequate Φ_0 and the system is completely determined.
Actually, we consider the two systems

$$\left\{\begin{array}{rcl} i\partial_t \Phi + \partial_x^2 \Phi &=& 0\\ \Phi(x,0) &=& \Phi_0(x)\end{array}\right. \tag{3.23}$$

and

$$\left\{\begin{array}{rcl} i\partial_t u + \partial_x^2 u &=& \lambda|u|^2u + a^2\varphi^2\Phi\\ u(x,T) &=& 0.\end{array}\right. \tag{3.24}$$

Let us define the operator

$$\begin{array}{rccl} L: & L^2(\mathbb{T}^1) & \to & L^2(\mathbb{T}^1) \\ & \Phi_0 & \mapsto & L\Phi_0 = u_0 = u(0). \end{array}$$

We split $u = v + \Psi$ with Ψ solution of

$$\left\{ \begin{array}{rcl} i\partial_t \Psi + \partial_x^2 \Psi & = & a^2(x)\varphi^2(t)\Phi \\ \Psi(T) & = & 0. \end{array} \right. \tag{3.25}$$

This corresponds to the linear control, and therefore, $\Psi(0) = S\Phi_0$. As for function v, it is solution of

$$\left\{ \begin{array}{rcl} i\partial_t v + \partial_x^2 v & = & \lambda |u|^2 u \\ v(T) & = & 0. \end{array} \right. \tag{3.26}$$

Then, u, v, Ψ belong to $X_T^{0,b}$ and $u(0) = v(0) + \Psi(0)$, which we can write

$$L\Phi_0 = K\Phi_0 + S\Phi_0$$

where $K\Phi_0 = v(0)$.
$L\Phi_0 = u_0$ is equivalent to $\Phi_0 = -S^{-1}K\Phi_0 + S^{-1}u_0$. Defining the operator $B : L^2 \to L^2$ by

$$B\Phi_0 = -S^{-1}K\Phi_0 + S^{-1}u_0,$$

the problem $L\Phi_0 = u_0$ is now to find a fixed point of B. We will prove that if $\|u_0\|_{L^2}$ is small enough, B is a contraction (for the L^2 norm) and reproduces the closed set $F = B_{L^2}(0,\eta) \bigcap \left(\bigcap_{i=1}^{\lfloor s \rfloor -1} B_{H^i}(0, R_i) \right) \bigcap B_{H^s}(0, R_s)$ for η small enough and for some large R_i.
We may assume $T < 1$, and fix it (actually the norm of S^{-1} as an operator acting on L^2 or H^s depends on T and even explode when T tends to 0, see [20], [18] and [23]). In the rest of the proof, as we want a bound for η independent on s, we will denote C any constant depending only on a, φ, b, b' and T that are fixed. We will write C_s if a dependence on s is allowed.
Since S is an isomorphism of H^s, we have

$$\|B\Phi_0\|_{H^s} \leq C_s \left(\|K\Phi_0\|_{H^s} + \|u_0\|_{H^s} \right). \tag{3.27}$$

So, we are led to estimate $\|K\Phi_0\|_{H^s} = \|v(0)\|_{H^s}$.
Then, if we apply to equation (3.26) the same $X_T^{s,b}$ estimates (Lemma 2.1.3 and estimate (0.6) of Lemma 2.0.1) we used in the existence Theorem 2.2.1, we get

$$\begin{array}{rcl} \|v(0)\|_{H^s} & \leq & C \|v\|_{X_T^{s,b}} \\ & \leq & CT^{1-b-b'} \left\| |u|^2 u \right\|_{X_T^{s,-b'}} \\ & \leq & C \left\| |u|^2 u \right\|_{X_T^{s,-b'}} \\ & \leq & C_s \|u\|_{X_T^{0,b}}^2 \|u\|_{X_T^{s,b}}. \end{array} \tag{3.28}$$

Let us first consider the L^2 norm and use the local linear behavior of u (see Proposition 2.2.1). We obtain that for $\|\varphi^2 a^2 \Phi\|_{L^2([0,T],L^2)} \leq C \|\Phi_0\|_{L^2} < C\eta < 1$, we have

$$\|u\|_{X_T^{0,b}} \leq C \|\Phi_0\|_{L^2}.$$

Finally, applying (3.27) and (3.28) with $s=0$, this yields

$$\|B\Phi_0\|_{L^2} \leq C\left(\|\Phi_0\|_{L^2}^3 + \|u_0\|_{L^2}\right).$$

Choosing η small enough and $\|u_0\|_{L^2} \leq \eta/2C$, we obtain $\|B\Phi_0\|_{L^2} \leq \eta$ and B reproduces the ball B_η of L^2.
For the H^s norm, we distinguish two cases : $s \leq 1$ and $s > 1$.

For $s \leq 1$, we return to (3.28) with the new estimate in $X_T^{0,b}$.

$$\|v(0)\|_{H^s} \leq C_s\eta^2 \|u\|_{X_T^{s,b}}$$

$$\|B\Phi_0\|_{H^s} \leq C_s\left(\eta^2 \|u\|_{X_T^{s,b}} + \|u_0\|_{H^s}\right)$$

Then, using Proposition 2.2.1 we have a linear behavior in H^s norm when we have only a bounded L^2 norm. More precisely, for $\|\varphi^2 a^2 \Phi\|_{L^2([0,T],L^2)} \leq C\|\Phi_0\|_{L^2} < C\eta < 1$ we get

$$\|u\|_{X^{s,b}} \leq C_s \|\Phi_0\|_{H^s} \tag{3.29}$$

and

$$\|B\Phi_0\|_{H^s} \leq C_s\left(\eta^2 \|\Phi_0\|_{H^s} + \|u_0\|_{H^s}\right).$$

Then, for $C_s\eta^2 < 1/2$, B reproduces any ball in H^s of radius greater than $2C_s\|u_0\|_{H^s}$.

As a conclusion, we have proved that if $\eta < \tilde{C}_s$, $\|u_0\|_{L^2} \leq C(\eta)$ and $R \geq C(\|u_0\|_{H^s})$, then B reproduces F. Moreover, we can check that all the estimates are uniform for $s \leq 1$ and so the bound on η is uniform.

If $s > 1$, we choose the R_i by induction. R_1 is chosen as for the case $s \leq 1$ so that B reproduces $B_{H^1}(0,R_1)$. The crucial point will be to make some asumptions of smallness on η that will be independent on i and s. This will be possible using some estimates uniform in s, up to some smoother terms (that could be very large). First, we use estimate (3.21) about S^{-1}.

$$\|B\Phi_0\|_{H^i} \leq C\|K\Phi_0\|_{H^i} + C_i\|K\Phi_0\|_{H^{i-1}} + C_i\|u_0\|_{H^i}$$

The same analysis we made for the case $s \leq 1$ yields

$$\|K\Phi_0\|_{H^{i-1}} \leq C_{i-1}\eta^2\|\Phi_0\|_{H^{i-1}} \leq C_{i-1}\eta^2 R_{i-1}.$$

Then, using the more precise multilinear estimate (1.46) of Proposition 2.A.1 of the Appendix, we get

$$\begin{aligned}
\|v(0)\|_{H^i} &\leq C\left\||u|^2 u\right\|_{X_T^{i,-b'}} \\
&\leq C\|u\|_{X_T^{0,b}}^2 \|u\|_{X_T^{i,b}} + C_i\|u\|_{X_T^{i-1,b}}\|u\|_{X_T^{1,b}}\|u\|_{X_T^{0,b}}.
\end{aligned}$$

For the term with maximal derivative, we use the refinement (2.17) of Proposition 2.2.2 and Corollary 2.A.1 of the Appendix

$$\begin{aligned}
\|u\|_{X_T^{i,b}} &\leq C\left\|\varphi^2 a^2\Phi\right\|_{L^2([0,T],H^i)} + C_i\|u\|_{X_T^{i-1,b}}\|u\|_{X_T^{1,b}}\|u\|_{X_T^{0,b}} + C_i\|u\|_{X_T^{i-1,b}} \\
&\leq C\|\Phi_0\|_{H^i} + C_i\|\Phi_0\|_{H^{i-1}} + C_i\|u\|_{X_T^{i-1,b}}\|u\|_{X_T^{1,b}}\|u\|_{X_T^{0,b}} \\
&\quad + C_i\|u\|_{X_T^{i-1,b}}.
\end{aligned}$$

For the terms with lower derivative, we only need estimate (3.29), which yields

$$\begin{aligned}\|v(0)\|_{H^i} &\leq C\eta^2 \|u\|_{X_T^{i,b}} + C_i R_{i-1} R_1 \eta \\ &\leq C\eta^2 \|\Phi_0\|_{H^i} + C\eta^2 \left(C_i R_{i-1} + C_i R_{i-1} R_1 \eta\right) + C_i R_{i-1} R_1 \eta.\end{aligned}$$

Finally, we obtain

$$\|B\Phi_0\|_{H^i} \leq C\eta^2 \|\Phi_0\|_{H^i} + C(i, \eta, R_1, R_{i-1}, \|u_0\|_{H^i}).$$

If we choose $C\eta^2 < 1/2$ independant on s and $R_i = 2C(i, \eta, R_1, R_{i-1}, \|u_0\|_{H^i})$, we obtain that B reproduces $B_{H^i}(0, R_i)$. The same arguments work for $B_{H^s}(0, R_s)$ if $s \geq 1$.

Let us prove that B is contracting for L^2 norm. For that, we examine the systems

$$\begin{cases} i\partial_t(u-\tilde{u}) + \partial_x^2(u-\tilde{u}) = \lambda(|u|^2 u - |\tilde{u}|^2\tilde{u}) + a^2\varphi^2(\Phi - \widetilde{\Phi}) \\ (u-\tilde{u})(T) = 0 \end{cases} \tag{3.30}$$

$$\begin{cases} i\partial_t(v-\tilde{v}) + \partial_x^2(v-\tilde{v}) = \lambda(|u|^2 u - |\tilde{u}|^2\tilde{u}) \\ (v-\tilde{v})(T) = 0. \end{cases}$$

We obtain

$$\begin{aligned}\left\|B\Phi_0 - B\widetilde{\Phi}_0\right\|_{L^2} &\leq C\|(v-\tilde{v})(0)\|_{L^2} \\ &\leq CT^{1-b-b'} \left\||u|^2u - |\tilde{u}|^2\tilde{u}\right\|_{X_T^{0,-b'}} \\ &\leq C\left(\|u\|_{X_T^{0,b}}^2 + \|\tilde{u}\|_{X_T^{0,b}}^2\right)\|u-\tilde{u}\|_{X_T^{0,b}} \\ &\leq C\eta^2 \|u-\tilde{u}\|_{X_T^{0,b}}. \end{aligned} \tag{3.31}$$

Considering equation (3.30), we deduce

$$\begin{aligned}\|u-\tilde{u}\|_{X_T^{0,b}} &\leq CT^{1-b-b'} \left\||u|^2u - |\tilde{u}|^2\tilde{u}\right\|_{X_T^{0,-b'}} + C\left\|\varphi^2 a^2(\Phi - \widetilde{\Phi})\right\|_{L^2([0,T],L^2)} \\ &\leq \left(\|u\|_{X_T^{0,b}}^2 + \|\tilde{u}\|_{X_T^{0,b}}^2\right)\|u-\tilde{u}\|_{X_T^{0,b}} + C\left\|\Phi_0 - \widetilde{\Phi_0}\right\|_{L^2} \\ &\leq C\eta^2\|u-\tilde{u}\|_{X_T^{0,b}} + C\left\|\Phi_0 - \widetilde{\Phi_0}\right\|_{L^2}.\end{aligned}$$

If η is taken small enough (independent on s) it yields

$$\|u-\tilde{u}\|_{X_T^{0,b}} \leq C\left\|\Phi_0 - \widetilde{\Phi_0}\right\|_{L^2}. \tag{3.32}$$

Combining (3.32) with (3.31) we finally get

$$\left\|B\Phi_0 - B\widetilde{\Phi}\right\|_{L^2} \leq C\eta^2\left\|\Phi_0 - \widetilde{\Phi_0}\right\|_{L^2}.$$

Therefore, for η small enough (independent on s), B is a contraction of a closed set F of L^2 and has a fixed point that by construction belongs to H^s. This completes the proof of Theorem 2.3.1. □

Remark 2.3.1. *To get control for Dirichlet or Neumann condition, we have to check that if u_0 is odd (resp even), then the control we built is so. Suppose that $a \in C^\infty(\mathbb{T}^1)$ is even on $\mathbb{T}^1$ and u_0 is odd (resp even). Then $\check{u}(x) = -u(-x)$ is solution of (3.24) with Φ replaced by $\check{\Phi}(x) = -\Phi(-x)$. We have $u_0 = \check{u_0} = L\check{\Phi}_0$ and therefore, $B\check{\Phi}_0 = \check{\Phi}_0$. Since $\check{\Phi}_0$ has the same norm as Φ_0 and by uniqueness of the fixed point in the closed set F, we obtain $\check{\Phi}_0 = \Phi_0$ and Φ_0 is odd. Therefore, the control $a^2\varphi^2\Phi$ and u are odd. The same argument works similarly for u_0 even.*

2.4 Propagation of compactness

In this section, we adapt some theorems of Dehman-Gérard-Lebeau [7] in the case of $X^{s,b}$ spaces.

Theorem 2.4.1. *Let u_n be a sequence of solutions of*

$$i\partial_t u_n + \partial_x^2 u_n = f_n$$

such that for some $0 \leq b \leq 1$, we have

$$\|u_n\|_{X_T^{0,b}} \leq C, \quad \|u_n\|_{X_T^{-1+b,-b}} \to 0 \quad \textit{and} \quad \|f_n\|_{X_T^{-1+b,-b}} \to 0.$$

Moreover, we assume that there is a nonempty open set ω such that $u_n \to 0$ strongly in $L^2([0,T], L^2(\omega))$.
Then $u_n \to 0$ strongly in $L^2_{loc}([0,T], L^2(\mathbb{T}^1))$.

Démonstration. Let $\varphi \in C^\infty(\mathbb{T}^1)$ and $\Psi \in C_0^\infty(]0,T[)$ taking real values, that will be chosen later. Set $Bu = \varphi(x)D^{-1}$ and $A = \Psi(t)B$ where D^{-1} is the operator defined at the end of the Introduction in (0.8). We have $A^* = \Psi(t)D^{-1}\varphi(x)$.
Denote L the Schrödinger operator $L = i\partial_t + \partial_x^2$. For $\varepsilon > 0$, we denote $A_\varepsilon = Ae^{\varepsilon\partial_x^2} = \Psi(t)B_\varepsilon$ for the regularization. We write by a classical way

$$\begin{aligned} \alpha_{n,\varepsilon} &= (Lu_n, A_\varepsilon^* u_n)_{L^2(]0,T[\times\mathbb{T}^1)} - (A_\varepsilon u_n, Lu_n,)_{L^2(]0,T[\times\mathbb{T}^1)} \\ &= ([A_\varepsilon, \partial_x^2]u_n, u_n) - i(\Psi'(t)B_\varepsilon u_n, u_n). \end{aligned}$$

But we have also

$$\alpha_{n,\varepsilon} = (f_n, A_\varepsilon^* u_n)_{L^2(]0,T[\times\mathbb{T}^1)} - (A_\varepsilon u_n, f_n)_{L^2(]0,T[\times\mathbb{T}^1)}.$$

Using Lemma 2.1.2, we obtain

$$\begin{aligned} \left|(f_n, A_\varepsilon^* u_n)_{L^2(]0,T[\times\mathbb{T}^1)}\right| &\leq \|f_n\|_{X_T^{-1+b,-b}} \|A_\varepsilon^* u_n\|_{X_T^{1-b,b}} \\ &\leq \|f_n\|_{X_T^{-1+b,-b}} \|u_n\|_{X_T^{0,b}}. \end{aligned} \tag{4.33}$$

Then, $\sup_\varepsilon \left|(f_n, A_\varepsilon^* u_n)_{L^2(]0,T[\times M)}\right| \to 0$ when $n \to \infty$. The same estimate for the other terms gives $\sup_\varepsilon \alpha_{n,\varepsilon} \to 0$ and likewise for the term $(\Psi'(t)B_\varepsilon u_n, u_n)$.
Finally, taking the supremum on ε tending to 0, we get

$$([A, \partial_x^2]u_n, u_n)_{L^2(]0,T[\times\mathbb{T}^1)} \to 0 \text{ when } n \to \infty.$$

Then, as D^{-1} commutes with ∂_x^2, we have

$$[A, \partial_x^2] = -2\Psi(t)(\partial_x\varphi)\partial_x D^{-1} - \Psi(t)(\partial_x^2\varphi)D^{-1}.$$

Making the same estimates as in (4.33), we get

$$(\Psi(t)(\partial_x^2\varphi)D^{-1}u_n, u_n)_{L^2(]0,T[\times\mathbb{T}^1)} \to 0.$$

Moreover, $-i\partial_x D^{-1}$ is actually the orthogonal projection on the subspace of functions with $\widehat{u}(0) = 0$. Using weak convergence, we easily obtain that $\widehat{u_n}(0)(t)$ tends to 0 in $L^2([0,T])$ and so,

$$(\Psi(t)(\partial_x\varphi)\widehat{u_n}(0)(t), u_n)_{L^2(]0,T[\times\mathbb{T}^1)} \to 0.$$

Our final result is that for any $\varphi \in C^\infty(\mathbb{T}^1)$ and $\Psi \in C_0^\infty(]0,T[)$

$$(\Psi(t)(\partial_x\varphi)u_n, u_n)_{L^2(]0,T[\times\mathbb{T}^1)} \longrightarrow 0.$$

Now, we remark that the functions that can be written $\partial_x\varphi$ are actually all the functions ϕ that fulfill $\int_{\mathbb{T}^1}\phi = 0$. For example, for any $\chi \in C_0^\infty(\omega)$ and any $x_0 \in \mathbb{T}^1$, $\phi(x) = \chi(x) - \chi(x-x_0)$ can be written $\phi = \partial_x\varphi$.
The strong convergence in $L^2([0,T], L^2(\omega))$ implies

$$(\Psi(t)\chi u_n, u_n)_{L^2(]0,T[\times\mathbb{T}^1)} \longrightarrow 0.$$

Then for any $x_0 \in \mathbb{T}^1$

$$(\Psi(t)\chi(.-x_0)u_n, u_n)_{L^2(]0,T[\times\mathbb{T}^1)} \longrightarrow 0.$$

We close the proof by constructing a partition of unity of $\mathbb{T}^1$ with some functions $\chi_i(.-x_0^i)$ with $\chi_i \in C_0^\infty(\omega)$ and $x_0^i \in \mathbb{T}^1$. □

2.5 Propagation of regularity

We write Proposition 13 of [7] with some $X^{s,b}$ assumptions on the second term of the equation.

Theorem 2.5.1. *Let $T>0$, $0 \leq b < 1$ and $u \in X_T^{r,b}$, $r \in \mathbb{R}$ solution of*

$$i\partial_t u + \partial_x^2 u = f \in X_T^{r,-b}.$$

Moreover, we assume that there exists a nonempty open set ω such that $u \in L^2_{loc}(]0,T[, H^{r+\rho}(\omega))$ for some $\rho \leq \frac{1-b}{2}$.
Then $u \in L^2_{loc}(]0,T[, H^{r+\rho}(\mathbb{T}^1))$.

Démonstration. We first regularize : $u_n = e^{\frac{1}{n}\partial_x^2}u = \Xi_n u$ and $f_n = \Xi_n f$ with $\|u_n\|_{X_T^{r,b}} \leq C$ and $\|f_n\|_{X_T^{r,-b}} \leq C$. Set $s = r+\rho$.
We will make a proof near the one we did for propagation of compactness.
Let $\varphi \in C^\infty(\mathbb{T}^1)$ and $\Psi \in C_0^\infty(]0,T[)$ taking real values. Set $Bu = D^{2s-1}\varphi(x)$ and $A = \Psi(t)B$ (with notation (0.8) of the Introduction). If $L = i\partial_t + \partial_x^2$, we write

$$\begin{aligned} & (Lu_n, A^*u_n)_{L^2(]0,T[\times\mathbb{T}^1)} - (Au_n, Lu_n,)_{L^2(]0,T[\times\mathbb{T}^1)} \\ = \; & ([A,\partial_x^2]u_n, u_n)_{L^2(]0,T[\times\mathbb{T}^1)} - i(\Psi'(t)Bu_n, u_n) \end{aligned}$$

$$\begin{aligned} |(Au_n, f_n)_{L^2(]0,T[\times\mathbb{T}^1)}| &\leq \|Au_n\|_{X_T^{-r,b}}\|f_n\|_{X_T^{r,-b}} \\ &\leq \|u_n\|_{X_T^{r+2\rho-1+b,b}}\|f_n\|_{X_T^{r,-b}}. \end{aligned}$$

As we have chosen $\rho \leq \frac{1-b}{2}$, we have $r + 2\rho - 1 + b \leq r$. Therefore, we obtain

$$|(Au_n, f_n)_{L^2(]0,T[\times\mathbb{T}^1)}| \leq C\|u_n\|_{X_T^{r,b}}\|f_n\|_{X_T^{r,-b}} \leq C.$$

The same estimates for the other terms imply that $([A,\partial_x^2]u_n,u_n)_{L^2(]0,T[\times\mathbb{T}^1)}$ is uniformly bounded. Yet, we have

$$[A,\partial_x^2] = -2\Psi(t)D^{2s-1}(\partial_x\varphi)\partial_x - \Psi(t)D^{2s-1}(\partial_x^2\varphi)$$

while

$$\left|(\Psi(t)D^{2s-1}(\partial_x^2\varphi)u_n,u_n)_{L^2(]0,T[\times\mathbb{T}^1)}\right| \le C\|u_n\|_{X_T^{r,b}}\|u_n\|_{X_T^{r,-b}} \le C.$$

Finally we can control

$$|(\Psi(t)D^{2s-1}(\partial_x\varphi)\partial_x u_n,u_n)| \le C. \tag{5.34}$$

If $f\in C_0^\infty(\omega)$ then

$$\begin{aligned}
&(\Psi(t)D^{2s-1}f^2\partial_x u_n,u_n)\\
=\ &(\Psi(t)D^{s-1}f\partial_x u_n, fD^s u_n) + (\Psi(t)[D^{s-1},f]f\partial_x u_n, D^s u_n)\\
=\ &(\Psi(t)D^{s-1}f\partial_x u_n, D^s f u_n) + (\Psi(t)D^{s-1}f\partial_x u_n,[D^s,f]u_n)\\
&+(\Psi(t)[D^{s-1},f]f\partial_x u_n, D^s u_n).
\end{aligned}$$

Our asumption gives $fu\in L^2_{loc}([0,T],H^s)$ and $f\partial_x u\in L^2_{loc}([0,T],H^{s-1})$. Therefore, $fu_n=\Xi_n fu+[f,\Xi_n]u$ is uniformly bounded in $L^2_{loc}([0,T],H^s)$ thanks to Lemma 2.A.2 of Appendix and $s\le r+1$. Making the same reasoning for $f\partial_x u_n$, we obtain

$$|(\Psi(t)D^{s-1}f\partial_x u_n, D^s f u_n)| \le C.$$

Lemma 2.A.1 of the Appendix and $u\in L^2([0,T],H^r)$ yields (and likewise for the other term of commutator)

$$\begin{aligned}
\left|(\Psi(t)D^{s-1}f\partial_x u_n,[D^s,f]u_n)\right| &\le \left\|D^{r-1}f\partial_x u_n\right\|_{L^2(L^2)}\left\|D^\rho[D^s,f]u_n\right\|_{L^2(L^2)}\\
&\le \|u_n\|_{L^2(H^r)}\|u_n\|_{L^2(H^{s-1+\rho})} \le C.
\end{aligned}$$

And finally,

$$\left|(\Psi(t)D^{2s-1}f^2\partial_x u_n,u_n)\right| \le C.$$

Then, writing $\partial_x\varphi=f^2(x)-f^2(x-x_0)$ and using (5.34), we obtain

$$\left|(\Psi(t)D^{2s-1}f^2(.-x_0)\partial_x u_n,u_n)\right| \le C.$$

Finishing the proof as in Theorem 2.4.1 with a partition of unity, we obtain

$$|(\Psi(t)D^{2s-1}\partial_x u,u)| \le C$$

$$\int_0^T \sum_{k\neq 0}\Psi(t)|k|^{2s}|\widehat{u}(k,t)|^2 \quad dt \le C$$

which achieves the proof. □

Corollary 2.5.1. *Here $b>1/2$ and ω is any nonempty open set of $\mathbb{T}^1$. Let $u\in X_T^{0,b}$ solution of*

$$\begin{cases} i\partial_t u+\partial_x^2 u=\lambda|u|^2u \text{ on } [0,T]\times\mathbb{T}^1\\ u\in C^\infty(]0,T[\times\omega). \end{cases}$$

Then $u\in C^\infty(]0,T[\times\mathbb{T}^1)$.

Démonstration. We have $|u|^2u \in X_T^{0,-b}$ by multilinear estimates.
By applying once Theorem 2.5.1, we get $u \in L^2_{loc}([0,T], H^{1+\frac{1-b}{2}})$. Then we can choose t_0 such that $u(t_0) \in H^{1+\frac{1-b}{2}}$. We can then solve in $X^{1+\frac{1-b}{2},b}$ our nonlinear Schrödinger equation with initial data $u(t_0)$. By uniqueness in $X_T^{0,b}$, we conclude that $u \in X_T^{1+\frac{1-b}{2},b}$.
By iteration of this process, we get that $u \in L^2(]0,T[,H^r)$ for every $r \in \mathbb{R}$ and $u \in C^\infty([0,T],\mathbb{T}^1)$. □

Corollary 2.5.2. *Let ω be any nonempty open set of $\mathbb{T}^1$ and $u \in X_T^{0,b}$ solution of*

$$\left\{\begin{array}{rcl} i\partial_t u + \partial_x^2 u &=& \lambda|u|^2u \text{ on } [0,T]\times\mathbb{T}^1 \\ u &=& 0 \text{ on }]0,T[\times\omega. \end{array}\right.$$

Then $u = 0$

Démonstration. Using Corollary 2.5.1, we infer that $u \in C^\infty(]0,T[\times\mathbb{T}^1)$.
Proposition 2.0.1 of unique continuation implies $u = 0$. □

Remark 2.5.1. *We have the same conclusion for $u \in X_T^{0,b}$ solution of*

$$\left\{\begin{array}{rcl} i\partial_t u + \partial_x^2 u &=& 0 \text{ on } [0,T]\times\mathbb{T}^1 \\ u &=& 0 \text{ on }]0,T[\times\omega \end{array}\right.$$

2.6 Stabilization

Theorem 2.0.1 is a direct consequence of the following Proposition.

Proposition 2.6.1. *Let $a \in L^\infty(\mathbb{T}^1)$ taking real values such that $a^2(x) > \eta$ on a nonempty open set ω of $\mathbb{T}^1$, for some constant $\eta > 0$.*
For every $T > 0$ and every $R_0 > 0$, there exists a constant $C > 0$ such that inequality

$$\|u(0)\|_{L^2}^2 \leq C \int_0^T \|au\|_{L^2}^2\, dt$$

holds for every solution $u \in X_T^{0,b}$ of the damped equation

$$\left\{\begin{array}{rcl} i\partial_t u + \partial_x^2 u + ia^2u &=& \lambda|u|^2u \text{ on } [0,T]\times\mathbb{T}^1 \\ u(0) &=& u_0 \in L^2 \end{array}\right. \tag{6.35}$$

and $\|u_0\|_{L^2} \leq R_0$.

Démonstration. We argue by contradiction, we suppose the existence of a sequence (u_n) of solutions of (6.35) such that

$$\|u_n(0)\|_{L^2} \leq R_0$$

and

$$\int_0^T \|au_n\|_{L^2}^2\, dt \leq \frac{1}{n}\|u_{0,n}\|_{L^2}^2. \tag{6.36}$$

Denote $\alpha_n = \|u_{0,n}\|_{L^2} \leq R_0$. Up to extraction, we can suppose that $\alpha_n \longrightarrow \alpha$.
We will distinguich two cases : $\alpha > 0$ and $\alpha = 0$.

First case : $\alpha_n \longrightarrow \alpha > 0$
By decreasing of the L^2 norm, (u_n) is bounded in $L^\infty([0,T], L^2)$ and therefore in $X_T^{0,b}$. Then, as $X_T^{0,b}$ is a separable Hilbert we can extract a subsequence such that $u_n \rightharpoonup u$ weakly in $X_T^{0,b}$ for some $u \in X_T^{0,b}$.
By compact embedding, as we have $b < 1$ and $-b < 0$, we can also extract a subsequence such that we have strong convergence in $X_T^{-1+b,-b}$.
At this stage, we have to be careful because as it was seen by L. Molinet in [19], the weak limit u is not necessarily solution to NLS. See Remark 2.6.1 below. Thus, $\lambda|u_n|^2 u_n$ is bounded in $X_T^{0,-b'}$. We can extract a subsequence such that it converges weakly in $X_T^{0,-b'}$ to some f and strongly in $X_T^{-1+b,-b}$ (here, we use $b > b'$).
Using (6.36) and passing to the limit in the equation verified by u_n, we get

$$\left\{ \begin{array}{rcl} i\partial_t u + \partial_x^2 u &=& f \text{ on } [0,T] \times \mathbb{T}^1 \\ u &=& 0 \text{ on } [0,T] \times \omega. \end{array} \right. \tag{6.37}$$

Denote $r_n = u_n - u$ and $f_n = -ia^2 u_n + \lambda|u_n|^2 u_n - f$, we have

$$i\partial_t r_n + \partial_x^2 r_n = f_n$$

Moreover, because of (6.36) we have

$$a(x)u_n \underset{L^2([0,T],L^2)}{\longrightarrow} 0.$$

and so, f_n converges strongly to 0 in $X_T^{-1+b,-b}$.
It also implies that $u_n \underset{L^2([0,T],L^2(\omega))}{\longrightarrow} 0$ and the same for r_n.
We are then in position to apply Theorem 2.4.1. We infer

$$r_n \underset{L^2_{loc}([0,T],L^2)}{\longrightarrow} 0.$$

Then, we can pick one $t_0 \in [0,T]$ such that $r_n(t_0)$ tends to 0 strongly in L^2 and therefore $u_n(t_0) \to u(t_0)$ in L^2. Denote v the solution of

$$\left\{ \begin{array}{rcl} i\partial_t v + \partial_x^2 v &=& \lambda|v|^2 v \text{ on } [0,T] \times \mathbb{T}^1 \\ v(t_0) &=& u(t_0). \end{array} \right. \tag{6.38}$$

The main problem is, at this point, we still do not know whether $u = v$.
Yet, we have seen in the existence Theorem 2.2.1 that the flow (even backward) is Lipschitz on bounded sets. Then, as we have $u_n(t_0) \to v(t_0)$ and $ia^2 u_n \to 0$ in $L^2([0,T], L^2)$, we get $u_n \to v$ in $X_T^{0,b}$. Therefore, $u = v$ and u is solution of (6.38). Corollary 2.5.2 implies $u = 0$.
In particular, we have $\|u_n(0)\|_{L^2} \to 0$ which is a contradiction to our hypothesis $\alpha > 0$.

Second case : $\alpha_n \longrightarrow 0$
Let us make the change of unknown $v_n = u_n/\alpha_n$. v_n is solution of the system

$$i\partial_t v_n + \partial_x^2 v_n + ia^2 v_n = \lambda \alpha_n^2 |v_n|^2 v_n$$

and

$$\int_0^T \|av_n\|_{L^2}^2 \, dt \leq \frac{1}{n}. \tag{6.39}$$

Thus, we have

$$\|v_n(0)\|_{L^2} = 1 \tag{6.40}$$

and v_n is bounded in $L^\infty([0,T],L^2)$ as the L^2 norm of u_n decreases.
By Duhamel formula and multilinear estimates, we obtain

$$\|v_n\|_{X_T^{0,b}} \leq C\,\|v_n(0)\|_{L^2} + CT^{1-b-b'}\left(\|v_n\|_{X_T^{0,b}} + \alpha_n^2\,\|v_n\|^3_{X_T^{0,b}}\right).$$

Then, if we take $CT^{1-b-b'} < 1/2$, independant of v_n, we have

$$\|v_n\|_{X_T^{0,b}} \leq C + C\alpha_n^2\,\|v_n\|^3_{X_T^{0,b}}.$$

Lemma 2.1.4 states that $\|v_n\|_{X_T^{0,b}}$ is continuous in T. Since it is bounded near $t=0$ and $\alpha_n \to 0$, a classical boot strap argument gives that v_n is bounded on $X_T^{0,b}$. Using Lemma 2.1.5, we conclude that it is bounded in $X_T^{0,b}$ even for large T. Therefore, $\alpha_n^2|v_n|^2 v_n$ tends to 0 in $X_T^{0,-b'}$ and so in $X_T^{-1+b,-b}$.
Then, we can extract a subsequence such that $v_n \rightharpoonup v$ in $X_T^{0,b}$ and strongly in $X_T^{-1+b,-b}$.
v is solution of

$$\left\{\begin{array}{rcl} i\partial_t v + \partial_x^2 v &=& 0 \text{ on } [0,T]\times\mathbb{T}^1 \\ v &=& 0 \text{ on }]0,T[\times\omega \end{array}\right. \tag{6.41}$$

which implies $v=0$ by Remark 2.5.1 of unique continuation.
Estimate (6.39) implies

$$ia^2 v_n \underset{L^2([0,T],L^2)}{\longrightarrow} 0$$

and so in $X_T^{-1+b,-b}$.
Then, we can apply Theorem 2.4.1 as in the first case, to get that v_n converges to 0 in $L^2_{loc}([0,T],L^2)$. Take t_0 such that $v_n(t_0)$ strongly converges to 0 in L^2 and solve with initial data $v_n(t_0)$, we obtain that v_n converges to 0 in $X_T^{0,b}$. This contradicts (6.40). □

Remark 2.6.1. *We could have used a variant of Theorem 1 of [19] to get directly that the weak limit can only be zero.*

2.A APPENDIX

In this Appendix, we recall some basic microlocal analysis estimates that can be easily proved in dimension 1, without using any general theory. We also give the proof of some multilinear Bourgain estimates.

Following notation (0.8) of the Introduction, we have

Lemma 2.A.1. *Let f denote the operator of multiplication by $f \in C^\infty(\mathbb{T}^1)$.*
Then, $[D^r, f]$ maps any H^s into H^{s-r+1}.

Démonstration. We denote $|\ |_\wr$ the modification (1.9) of $|\ |$ introduced in Lemma 2.1.2. We also write $\mathrm{sgn}(0) = 1$.

We have

$$\widehat{D^r(fu)}(n) = \mathrm{sgn}(n)\,|n|_\wr^r \sum_k \widehat{f}(n-k)\widehat{u}(k)$$
$$\widehat{(fD^r u)}(n) = \sum_k \widehat{f}(n-k)\mathrm{sgn}(k)\,|k|_\wr^r\,\widehat{u}(k).$$

And then

$$\begin{aligned}\widehat{[D_r,f]u}(n) &= \sum_k \widehat{f}(n-k)(\mathrm{sgn}(n)\,|n|_\wr^r - \mathrm{sgn}(k)\,|k|_\wr^r)\widehat{u}(k)\\ \left|\widehat{[D_r,f]u}(n)\right| &\le C\sum_k |\widehat{f}(n-k)||n-k|(|n|_\wr^{r-1} + |k|_\wr^{r-1})|\widehat{u}(k)|.\end{aligned}$$

Using $|n|_\wr^{2\rho} \le C\,|n-k|_\wr^{2|\rho|}\,|k|_\wr^{2\rho}$ for any $\rho \in \mathbb{R}$, we get

$$\begin{aligned}\|[D_r,f]u\|^2_{H^{s-r+1}} &\le \sum_n |n|_\wr^{2s}\left(\sum_k \left|\widehat{f}(n-k)(n-k)\right|\,|\widehat{u}(k)|\right)^2\\ &+ \sum_n\left(\sum_k |n-k|_\wr^{|s-r+1|}\,|k|_\wr^s\left|\widehat{f}(n-k)(n-k)\right|\,|\widehat{u}(k)|\right)^2\\ &\le \sum_n\left(\sum_k |n-k|_\wr^{|s|}\,|k|_\wr^s\left|\widehat{f}(n-k)(n-k)\right|\,|\widehat{u}(k)|\right)^2 \qquad (1.42)\\ &+ \sum_n\left(\sum_k |n-k|_\wr^{|s-r+1|}\,|k|_\wr^s\left|\widehat{f}(n-k)(n-k)\right|\,|\widehat{u}(k)|\right)^2 .(1.43)\end{aligned}$$

We estimate (1.42) using Cauchy-Schwarz inequality, and it is the same for (1.43).

$$\begin{aligned}(1.42) &\le \sum_n\left(\sum_k |n-k|_\wr^{|s|}\,|\widehat{f}(n-k)(n-k)|\right)\times\\ &\quad\left(\sum_k |n-k|_\wr^{|s|}\,|\widehat{f}(n-k)(n-k)|\,|k|_\wr^{2s}\,|\widehat{u}(k)|^2\right)\\ &\le \left(\sum_k |k|_\wr^{|s|}\,|k\widehat{f}(k)|\right)^2\left(\sum_k |k|_\wr^{2s}\,|\widehat{u}(k)|^2\right)\\ &\le C_f\,\|u\|^2_{H^s}.\end{aligned}$$

□

Corollary 2.A.1. *If $f \in C^\infty(\mathbb{T}^1)$, there exists some constant C such that for every $s \in \mathbb{R}$, there exists C_s such that the following estimate holds*

$$\|fu\|_{H^s} \le C\,\|u\|_{H^s} + C_s\,\|u\|_{H^{s-1}}.$$

Démonstration. We just write $D^s(fu) = fD^su + [D^s, f]u$. □

Lemma 2.A.2. *Let $f \in C^\infty(\mathbb{T}^1)$ and $\rho_\varepsilon = e^{\varepsilon^2\partial_x^2}$ with $0 \leq \varepsilon \leq 1$.
Then, $[\rho_\varepsilon, f]$ is uniformly bounded as an operator from H^s into H^{s+1}.*

Démonstration. It is exactly the same as for Lemma 2.A.1 using

$$\left|e^{-\varepsilon^2 n^2} - e^{-\varepsilon^2 k^2}\right| \leq C|n-k| \left(\langle n\rangle^{-1} + \langle k\rangle^{-1}\right)$$

because

$$\left|\partial_\xi\left(e^{-\varepsilon^2\xi^2}\right)\right| \leq C\langle\xi\rangle^{-1}.$$

□

We give the proof of multilinear Bourgain estimates. We also get some information about the dependence on s of the estimates.

Proposition 2.A.1. *For every $s \geq 0$, we have uniformly on $T \leq 1$*

$$\left\||u|^2u\right\|_{X_T^{s,-3/8}} \leq C3^s \|u\|^2_{X_T^{0,3/8}} \|u\|_{X_T^{s,3/8}} \tag{1.44}$$

$$\left\||u|^2u - |\tilde{u}|^2\tilde{u}\right\|_{X_T^{s,-3/8}} \leq C3^s\left(\|u\|^2_{X_T^{s,3/8}} + \|\tilde{u}\|^2_{X_T^{s,3/8}}\right)\|u-\tilde{u}\|_{X_T^{s,3/8}}. \tag{1.45}$$

Moreover, there exists $C > 0$ such that for every $s \geq 1$, we can find $C_s > 0$ such that for every $T \leq 1$

$$\begin{aligned}\left\||u|^2u\right\|_{X_T^{s,-3/8}} \leq\;& C\|u\|^2_{X_T^{0,3/8}}\|u\|_{X_T^{s,3/8}} \\ &+C_s\|u\|_{X_T^{s-1,3/8}}\|u\|_{X_T^{1,3/8}}\|u\|_{X_T^{0,3/8}}.\end{aligned} \tag{1.46}$$

Démonstration. We follow closely [4] p 107. For estimates (1.44) and (1.45), it is enough to prove

$$\begin{aligned}\|u_1\overline{u_2}u_3\|_{X^{s,-3/8}} \leq C(&\|u_1\|_{X^{s,3/8}}\|u_2\|_{X^{0,3/8}}\|u_3\|_{X^{0,3/8}} \\ +\|u_1\|_{X^{0,3/8}}\|u_2\|_{X^{s,3/8}}\|u_3\|_{X^{0,3/8}} &+ \|u_1\|_{X^{0,3/8}}\|u_2\|_{X^{0,3/8}}\|u_3\|_{X^{s,3/8}}).\end{aligned}$$

Denote $w = u_1\overline{u_2}u_3$. We argue by duality. Let $v \in X^{-s,3/8}$.
We write $\widehat{v}(\lambda,k)$ instead of $\widehat{\widehat{v}}(\lambda,k)$ the Fourier transform in time and space variable. $|\ |_\wr$ still denotes the modification (1.9) of $|\ |$ defined in the proof of Lemma 2.1.2.

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{T}^1} w\overline{v} &= \sum_k\int_\lambda \widehat{w}(\lambda,k)\overline{\widehat{v}(\lambda,k)} = \sum_k\int_\lambda |k|_\wr^s\,\widehat{w}(\lambda,k)\,|k|_\wr^{-s}\,\overline{\widehat{v}(\lambda,k)} \\ &= \sum_{k_1,k_2,k_3}\int_{\lambda_1,\lambda_2,\lambda_3} |k|_\wr^s\,\widehat{u_1}(\lambda_1,k_1)\overline{\widehat{u_2}(\lambda_2,k_2)}\widehat{u_3}(\lambda_3,k_3)\,|k|_\wr^{-s}\,\overline{\widehat{v}(\lambda,k)}\end{aligned} \tag{1.47}$$

where $k = k_1 - k_2 + k_3$ and $\lambda = \lambda_1 - \lambda_2 + \lambda_3$.
Observe that $|k|_\wr^s \leq 3^s\max(|k_1|_\wr^s, |k_2|_\wr^s, |k_3|_\wr^s)$. We assume $|k|_\wr^s \leq 3^s|k_1|_\wr^s$, and the other possibilities will produce the other terms of the right hand side of the estimate we want

(we do not write them any more, each inequality is true if we add the same term with u_2 and u_3).

$$(1.47) \leq 3^s \sum_{k_1,k_2,k_3} \int_{\lambda_1,\lambda_2,\lambda_3} |k_1|_\wr^s \, |\widehat{u_1}(\lambda_1,k_1)| \, |\widehat{u_2}(\lambda_2,k_2)| \, |\widehat{u_3}(\lambda_3,k_3)| \, |k|_\wr^{-s} \, |\widehat{v}(\lambda,k)|$$

Denote $u_1^\S$ the function with Fourier transform equal to $|\widehat{u_1}(\lambda,k)|$. Then, using dispersive estimate (0.5)

$$\begin{aligned}
(1.47) \quad &\leq \quad 3^s \int_{\mathbb{R}} \int_{\mathbb{T}^1} (D^s u_1^\S) \overline{u_2^\S} u_3^\S \overline{D^{-s} v^\S} \\
&\leq \quad C 3^s \left\| D^s u_1^\S \right\|_{L^4} \left\| u_2^\S \right\|_{L^4} \left\| u_3^\S \right\|_{L^4} \left\| D^{-s} v^\S \right\|_{L^4} \\
&\leq \quad C 3^s \left\| D^s u_1^\S \right\|_{X^{0,3/8}} \left\| u_2^\S \right\|_{X^{0,3/8}} \left\| u_3^\S \right\|_{X^{0,3/8}} \left\| D^{-s} v^\S \right\|_{X^{0,3/8}} \\
&\leq \quad C 3^s \left\| u_1 \right\|_{X^{s,3/8}} \left\| u_2 \right\|_{X^{0,3/8}} \left\| u_3 \right\|_{X^{0,3/8}} \left\| v \right\|_{X^{-s,3/8}} .
\end{aligned}$$

Estimate (1.46) is obtained similarly using the following inequality, if for example $|k_1| = \max(|k_1|, |k_2|, |k_3|)$,

$$|k_1 - k_2 + k_3|_\wr^s \leq |k_1|_\wr^s + C_s |k_1|_\wr^{s-1} \left(|k_2|_\wr + |k_3|_\wr \right) .$$

This is a consequence of the fundamental theorem of calculus applied to the function $(1 + x + y)^s$. $\square$

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Chapitre 3

Contrôle et stabilisation de l'équation de Schrödinger non linéaire sur des variétés compactes de dimension 3

Contents

Ce chapitre est la reprise d'un article publié dans le journal SIAM Journal on Mathematical Analysis [33].

3.1 Introduction

In this article, we study the internal stabilization and exact controllability for the defocusing nonlinear Schrödinger equation (NLS) on some compact manifolds of dimension 3.

$$\left\{\begin{array}{rcl} i\partial_t u+\Delta u &=& |u|^2 u \quad \text{on} \quad [0,+\infty[\times M \\ u(0) &=& u_0\in H^1(M). \end{array}\right. \tag{1.1}$$

where Δ is the Laplace-Beltrami operator on M. The solution displays two conserved energy : the L^2 energy $\|u\|_{L^2}$ and the nonlinear energy, or H^1 energy

$$E(t)=\frac{1}{2}\int_M |\nabla u|^2+\frac{1}{4}\int_M |u|^4 .$$

This equation arises in Nonlinear Optics where it is obtained as an asymptotic regime of the Maxwell equations in a nonlinear medium (see e.g. Sulem-Sulem [43]). In this context, the metric g can be interpreted as an inhomogeneity of the optical index. A more physically relevant situation could be to consider this equation on a domain. However, for the moment, this equation is not known to be globally well posed on an open set of dimension 3 (see [2], [6] for the 2D case and [1] for the radial solutions on a ball). A compact manifold makes a good framework to understand the effect of geometry.

For the study of controllability, some similar results where obtained in dimension 2 in the article of B. Dehman, P. Gérard and G. Lebeau [16] where exact controllability in H^1 is proved for defocusing NLS on compact surfaces. Yet, the proof is based on Strichartz estimates which provide uniform wellposedness in dimension 3, only in H^s for $s>1$. In [11], N. Burq, P. Gérard and N. Tzvetkov managed to prove global existence and uniqueness in H^1 but failed to prove uniform wellposedness, which appears of great importance in control problems. However, for certain specific manifolds, the strategy of $X^{s,b}$ spaces of J. Bourgain, extended to some other manifolds by Burq, Gérard and Tzvetkov, succeeded in proving uniform wellposedness in H^s for some lower regularities. So, up to our knowledge, this paper is the first one dealing with global controllability for cubic NLS in three dimensions.

For control results, the $X^{s,b}$ spaces have already been used in dimension 1 at L^2 regularity : first L. Rosier and B. Y. Zhang [42] obtained local results and independently, we proved global controllability in large time in [32]. We also quote the recent paper [41] about the control of NLS on rectangles, but still with local results. The $X^{s,b}$ spaces will also be our framework in this paper.

Under some specific assumptions that will be precised later, we prove global controllability in large time by two different ways, interesting for their own : by stabilization and control near 0 or by some successive controls near some trajectories. This will provide global controllability towards 0 and the general result will follow by reversing time. The first strategy is very classical in control theory and has been used unnumerable times (see for example Lee-Markus book [35] p 397 in finite dimension). The second strategy seems less classical, at least in this framework.

Our assumptions are fulfilled in the following cases ($\omega\subset M$ is the support of the control) :

- $\mathbb{T}^3$ with $\omega = \{x \in \mathbb{R}^3/(\theta_1\mathbb{Z} \times \theta_2\mathbb{Z} \times \theta_3\mathbb{Z}) \,|\exists i \in \{1,2,3\}, x_i \in]-\varepsilon, \varepsilon[+\theta_i\mathbb{Z}\}$ (that is a neighborhood of each face of the "cube", fundamental volume of $\mathbb{T}^3$) with $\theta_i \in \mathbb{R}$. Moreover, we can easily extend this result to a cuboid with Dirichlet or Neumann boundary conditions, see [32] or [42].
- S^3 with ω a neighborhood of $\{x_4 = 0\}$ in $S^3 \subset \mathbb{R}^4$.
- $S^2 \times S^1$ with $\omega = (\omega_1 \times S^1) \cup (S^2 \times]0, \varepsilon[)$ where ω_1 is a neighborhood of the equator of S^2.

Theorem 3.1.1. *For any open set $\omega \subset M$ satisfying Assumption 3.1.1, 3.1.2, 3.1.3 (see below) and $R_0 > 0$, there exist $T > 0$ and $C > 0$ such that for every u_0 and u_1 in $H^1(M)$ with*

$$\|u_0\|_{H^1(M)} \leq R_0 \quad \text{and} \quad \|u_1\|_{H^1(M)} \leq R_0$$

there exists a control $g \in C([0,T], H^1)$ with $\|g\|_{L^\infty([0,T],H^1)} \leq C$ supported in $[0,T] \times \overline{\omega}$, such that the unique solution u in $X_T^{1,b}$ of the Cauchy problem

$$\left\{ \begin{array}{rcll} i\partial_t u + \Delta u & = & |u|^2 u + g & \text{on} \quad [0,T] \times M \\ u(0) & = & u_0 \in H^1(M) & \end{array} \right. \tag{1.2}$$

satisfies $u(T) = u_1$.

In all the rest of the article, ω will be related to a cut-off function $a = a(x) \in C^\infty(M)$ (whose existence is guaranteed by Whitney Theorem), taking real values and such that

$$\omega = \{x \in M : a(x) \neq 0\}. \tag{1.3}$$

The stabilization system we consider is

$$\left\{ \begin{array}{rcll} i\partial_t u + \Delta u - a(x)(1-\Delta)^{-1}a(x)\partial_t u & = & (1+|u|^2)u & \text{on} \quad [0,T] \times M \\ u(0) & = & u_0 \in H^1(M). & \end{array} \right. \tag{1.4}$$

The link with the original equation can be made by the change of variable $w = e^{-it}u$. A more physically relevant damping term would be $ia(x)u$ as used in the one dimensional case in [32]. Yet, the damping term in (1.4) is especially fitted to the H^1 energy which is the regularity at which we solve the equation. The well posedness of system (1.4) will be proved in Section 3.3.1 and we can check that it satisfies the energy decay

$$E(u(t)) - E(u(0)) = -\int_0^t \left\|(1-\Delta)^{-1/2}a(x)\partial_t u\right\|_{L^2}^2. \tag{1.5}$$

Our theorem states that under some geometrical hypotheses, this yields an exponential decay.

Theorem 3.1.2. *Let M, ω satisfying Assumption 3.1.1, 3.1.2, 3.1.3. Let $a \in C^\infty(M)$, as in (1.3). There exists $\gamma > 0$ such that for every $R_0 > 0$, there is a constant $C > 0$ such that inequality*

$$\|u(t)\|_{H^1} \leq Ce^{-\gamma t} \|u_0\|_{H^1} \quad t > 0$$

holds for every solution u of system (1.4) with initial data u_0 such that $\|u_0\|_{H^1} \leq R_0$.

The independence of C and of the time of control T on the bound R_0 are an open problem. The fact that γ is independant on the size lies on the fact that it only describes the behavior near 0. However, it is unknown whether there is really a minimal time of controllability. This is in strong contrast with the linear case where exact controllability occurs in arbitrary small time and the conditions are only geometric for the open set ω. Moreover, some recent studies have analysed the explosion of the control cost when T tends to 0 : K.- D. Phung [39] by reducing to the heat or wave equation, L. Miller [37] with resolvent estimates, G. Tenenbaum and M. Tucsnak [44] with number theoretic arguments.

Let us now describe our assumptions. The first two deal with classical geometrical assumptions in control theory.

Assumption 3.1.1. *Geometric control : there exists $T_0 > 0$ such that every geodesic of M, travelling with speed 1 and issued at $t = 0$, enters the set ω in a time $t < T_0$.*

This condition is known to be sufficient for linear controllability, see G. Lebeau [34]. In Section 3.10, we prove that it is necessary on S^3 for the nonlinear stabilization. However, there are some geometrical situation (especially when there are some unstable geodesics) in which it is not necessary. For example, we have linear controllability for any open set ω of $\mathbb{T}^3$, see S. Jaffard [26] and V. Komornik [28] (see also [14]). This also holds for $M = S^2 \times S^1$ with $\omega = S^2 \times]0, \varepsilon[$ or $\omega = \omega_1 \times S^1$ where ω_1 is a neighborhood of the equator. In that cases, our method fails to prove global results and we can only prove local controllability by perturbation (see Theorem 3.1.4).

Assumption 3.1.2. *Unique continuation : For every $T > 0$, the only solution in $C^\infty([0,T] \times M)$ to the system*

$$\left\{ \begin{array}{c} i\partial_t u + \Delta u + b_1(t,x)u + b_2(t,x)\overline{u} = 0 \text{ on } [0,T] \times M \\ u = 0 \text{ on } [0,T] \times \omega \end{array} \right. \tag{1.6}$$

where $b_1(t,x)$ and $b_2(t,x) \in C^\infty([0,T] \times M)$ is the trivial one $u \equiv 0$.

We do not know if there exists a link between these two assumptions. In our three particular cases, this can be proved using Carleman estimates. There are some existing results about this, as the one of V. Isakov [25](for general anisotropic PDE's), L. Baudouin and J.P. Puel [4] (global Carleman estimates) or A. Mercado, A. Osses and L. Rosier [36] (in the special case of Schrödinger with flat metric but weaker geometrical assumptions). In the case of Riemannian manifold with boundary, some Carleman estimates were obtained by R. Triggiani and X. Xu [47] (see also an interesting discussion Section 10 about the existence of convex weights). Note also that these Carleman estimates can be used to treat directly some controllability problems, see for instance I. Lasiecka and R. Triggiani [29], I. Lasiecka, R. Triggiani and X. Zhang [31, 30] and R. Triggiani [46]. For the convenience of the reader, we have chosen to give a proof of Carleman estimates on a compact manifold. It is given in the Appendix, Section 3.B. They are quite similar to the one of [47] but simpler because without boundary terms. We also believe that these estimates are of independant interest if the weight is weakly convex (as in [36] but with a metric).

The last assumption is a technical assumption that ensures that the Cauchy problem is well posed in H^1. It yields a bilinear loss of $s_0 < 1$.

Assumption 3.1.3. *There exists $C > 0$ and $0 \leq s_0 < 1$ such that for any $f_1, f_2 \in L^2(M)$ satisfying*

$$f_j = \mathbf{1}_{\sqrt{1-\Delta}\in[N_j,2N_j[}(f_j), \quad j = 1,2.$$

one has the following bilinear estimates

$$\|u_1u_2\|_{L^2([0,T]\times M)} \leq C \min(N_1,N_2)^{s_0} \|f_1\|_{L^2(M)} \|f_2\|_{L^2(M)} \tag{1.7}$$
$$u_j(t) = e^{it\Delta} f_j, \quad j = 1,2.$$

It is known to be true in the following examples ($1/2+$ means any $s > 1/2$) :
- $\mathbb{T}^3$ with $s_0 = 1/2+$, see [7].
- the irrational torus $\mathbb{R}^3/(\theta_1\mathbb{Z}\times\theta_2\mathbb{Z}\times\theta_3\mathbb{Z})$ with $\theta_i \in \mathbb{R}$, for which an estimate with $s_0 = 2/3+$ has been recently obtained in [8]. An easier proof for $s_0 = 3/4+$ can also be found in the beginning of [8] and in [15].
- S^3 with $s_0 = 1/2+$, see [13].
- $S^2 \times S^1$ with $s_0 = 3/4+$, see [13].

It yields some trilinear estimates in Bourgain spaces (see the definition below). For the control near a trajectory, we still have some particular assumptions that are again fulfilled in the particular geometries described above. Our result reads as follow

Theorem 3.1.3. *Let $T > 0$ and M, ω such that Assumptions 3.1.1, 3.1.3, 3.1.4 and 3.1.5 are fulfilled (see below). Let $1 \geq s > s_0$ and $w \in X_T^{1,b}$ be a solution of*

$$\begin{cases} i\partial_t w + \Delta w \pm |w|^2 w &= g \\ w(x,0) &= w_0(x) \end{cases} \tag{1.8}$$

with $g \in C([0,T],H^1)$ supported in $[0,T]\times\overline{\omega}$.
Then, there exists $\varepsilon > 0$, such that for every $u_0 \in H^s$ with $\|u_0 - w_0\|_{H^s} < \varepsilon$, there exists $g_1 \in C([0,T],H^s)$ supported in $[0,T]\times\overline{\omega}$ such that the unique solution u in $X_T^{s,b}$ of (1.8) with $u(0) = u_0$ and g replaced by g_1 fulfills $u(T) = w(T)$.
Moreover, for any $u_0 \in H^1$ with $\|u_0 - w_0\|_{H^s} < \varepsilon$, the same conclusion holds with $g \in C([0,T],H^1)$.

An interesting fact is that the smallness assumption only concerns the H^s norm, even if we want a control in H^1. For example, as in [17], if we assume $\|u_0\|_{H^1} \leq R_0$, we can find $N \in \mathbb{N}$ large enough such that the smallness asumption only concerns the N first frequencies (see Corollary 3.9.2). Of course, this result remains true in lower dimension where it was only known for the trajectory $w = 0$ (see [16]).

Let us describe the new hypothesis. Assumption 3.1.4 is a unique continuation result at weaker regularity.

Assumption 3.1.4. *Unique continuation in H^1 : For every $T > 0$, the only solution in $C([0,T],H^1)$ to the system*

$$\begin{cases} i\partial_t u + \Delta u + b_1(t,x)u + b_2(t,x)\overline{u} = 0 \text{ on } [0,T]\times M \\ u = 0 \text{ on } [0,T]\times\omega \end{cases} \tag{1.9}$$

where $b_1(t,x)$ and $b_2(t,x) \in L^\infty([0,T],L^3)$ is the trivial one $u \equiv 0$.

We do not know if it is really stronger than Assumption 3.1.2 but for the moment, there are some example where we are able to prove Assumption 3.1.2 and not Assumption 3.1.4 using some weak Carleman estimates (see Appendix, Section 3.B). For instance, on $\mathbb{T}^3$, we are able to prove Assumption 3.1.2 for $\omega = \{x \in \mathbb{R}^3/\mathbb{Z}^3 \,|x_1 \in]0, \varepsilon[+\mathbb{Z}\}$ but not Assumption 3.1.4. Yet, for the moment, we do not manage to deduce a controllability result from this statement.

The other new assumption is technical and yields quadrilinear estimates for a commutator

Assumption 3.1.5. *There exists $0 \leq s_0 < 1$ so that for any $\varepsilon \in [0,1]$, we can find one constant $C > 0$ such that for any f_1, f_2, f_3, $f_4 \in L^2(M)$ satisfying*

$$f_j = \mathbf{1}_{\sqrt{1-\Delta}\in[N_j,2N_j[}(f_j), \quad j = 1,2,3,4$$

one has the following quadrilinear estimate

$$\begin{aligned} &\sup_{\tau\in\mathbb{R}} \left| \int_{\mathbb{R}} \int_M \chi(t) e^{it\tau} u_1 u_2 \left((-\Delta)^{\varepsilon/2} u_3 u_4 - u_3 (-\Delta)^{\varepsilon/2} u_4 \right) dxdt \right| \qquad (1.10)\\ &\leq C(N_1^\varepsilon + N_2^\varepsilon) \left(m(N_1,\cdots,N_4)\right)^{s_0} \|f_1\|_{L^2(M)} \|f_2\|_{L^2(M)} \|f_3\|_{L^2(M)} \|f_4\|_{L^2(M)} \\ &\qquad u_j(t) = e^{it\Delta} f_j, \quad j = 1,2,3,4 \end{aligned}$$

where $\chi \in C_0^\infty(\mathbb{R})$ is arbitrary and $m(N_1, \cdots, N_4)$ is the product of the smallest two numbers among N_1, N_2, N_3, N_4.
Moreover, the same result holds with u_i replaced by $\overline{u_i}$ for i in a subset of $\{1,2,3,4\}$.

For the three treated examples, we prove in Appendix, Section 3.A, that Assumption 3.1.5 holds true with the same s_0 as in Assumption 3.1.3. We believe that it is the case for any manifold, but we did not manage to prove it.

As explained before, there are some examples for which we know that geometric control assumption is not necessary. For instance, for any pair of manifolds M_1, M_2 and $\omega_1 \subset M_1$ such that ω_1 satisfies observability estimate, $\omega_1 \times M_2$ satisfies observability estimate for the linear Schrödinger equation. We can then use this remark and the work of S. Jaffard [26] and V. Komornik [28] for the linear equation on $\mathbb{T}^n$ to get some local nonlinear results . Since Theorem 3.1.3 is proved by a perturbative argument, we can also deduce controllability near 0 from these already known linear control results.

Theorem 3.1.4. *If $w \equiv 0$ and (M, ω) is either :*
-($\mathbb{T}^3$,any open set)
-($S^2 \times S^1$, $\omega_1 \times S^1$) where ω_1 is a neighborhood of the equator of S^2
-($S^2 \times S^1$,$S^2 \times]0, \varepsilon[$)
Then, the same conclusion as Theorem 3.1.3 is true.

Rosier and Zhang [41] simultaneoulsy obtained the same result for $\mathbb{T}^3$.

The proof of stabilization and of linear control with potential follows the same scheme as [16]. In a contradiction argument, we are led to prove the strong convergence to zero in $X_T^{s,b}$ of some weakly convergent sequence (u_n) solution to damped NLS or Schrödinger with potential. Since the equation is subcritical, we use some linearizability properties of NLS in H^1 (see the work of P. Gérard [22] for the wave equation).

We first establish the strong convergence by some propagation of compactness. We adapt the argument of [16] inspired by C. Bardos and T. Masrour [3]. We use microlocal defect measures introduced by P. Gérard [21]. For a sequence (u_n) weakly convergent to 0 in $X_T^{s,b}$ satisfying

$$\begin{cases} i\partial_t u_n + \Delta u_n \to 0 \quad \text{in} \quad X_T^{s-1+b,-b} \\ a(x)u_n \to 0 \quad \text{in} \quad L^2([0,T],H^s), \end{cases}$$

we prove that $u_n \to 0$ in $L^2_{loc}([0,T],H^s)$.

Once we know that the convergence is strong, we infer that the limit u is solution to NLS. We would like to use Assumption 3.1.2 or 3.1.4 of unique continuation to prove that it is 0. However, more regularity is needed to apply them. Again, we adapt the proof for $X^{s,b}$ spaces of propagation results of microlocal regularity coming from [16].

The rest of this article is organized as follows.
The first section states and recall some properties of the Bourgain spaces that will be used all along the paper. The second section proves well posedness of the nonlinear equation with source and damping term and its associated linearization near trajectories. In section 3, we prove that the equation is linearizable, namely that at high frequency, the nonlinear equation behaves as the linear one. Section 4 and 5 are devoted to the propagation of regularity and compactness along the bicharacteristics which will be the essential tools for the proofs of stabilization and controllability. The main results of the article are proved in the last sections. The stabilization result is proved in section 6. In section 7 we prove the controllability of the linear equation that are obtained by linearization of the nonlinear one. This permits to prove control near trajectories in section 8. In section 9, we prove that on S^3, our geometrical assumption is nearly optimal. In the appendix section A, we prove some commutator estimates used in the proof of the regularity result of the control constructed in section 7. The section B is used to prove the assumption of unique continuation in our specific geometries thanks to some Carleman estimates.

In this article, b' will be a constant such that estimates of Lemma 3.2.1 holds. Actually, each of the trilinear estimates (with different s) that will be done will yield one $b' < 1/2$ but remains true if we choose a greater one. So we take $b' < 1/2$ as the largest of these constants. This allows to choose one $b > 1/2$ with $1 > b + b'$.

In all the rest of the paper, C will denote any constant whose value could change along the article.

3.2 Some properties of $X^{s,b}$ spaces

Since M is compact, Δ has a compact resolvent and thus, the spectrum of Δ is discrete. We choose $e_k \in L^2(M)$, $k \in M$ an orthonormal basis of eigenfunctions of $-\Delta$, associated to eigenvalues λ_k. Denote P_k the orthogonal projector on e_k. We equip the Sobolev space $H^s(M)$ with the norm (with $\langle x\rangle = \sqrt{1+|x|^2}$),

$$\|u\|^2_{H^s(M)} = \sum_k \langle\lambda_k\rangle^s \|P_k u\|^2_{L^2(M)}.$$

The Bourgain space $X^{s,b}$ is equipped with the norm

$$\|u\|^2_{X^{s,b}} = \sum_k \langle\lambda_k\rangle^s \left\|\langle\tau+\lambda_k\rangle^b \widehat{P_k}(\tau)u\right\|^2_{L^2(\mathbb{R}_\tau\times M)} = \left\|u^\#\right\|^2_{H^b(\mathbb{R},H^s(M))}$$

where $u = u(t,x)$, $t\in\mathbb{R}$, $x\in M$, $u^\#(t) = e^{-it\Delta}u(t)$ and $\widehat{P_k u}(\tau)$ denotes the Fourier transform of $P_k u$ with respect to the time variable.

$X^{s,b}_T$ is the associated restriction space with the norm

$$\|u\|_{X^{s,b}_T} = \inf\left\{\|\tilde u\|_{X^{s,b}} \,|\tilde u = u \text{ on }]0,T[\times M\right\}$$

We also write $\|u\|_{X^{s,b}_I}$ if the infinimum is taken on functions $\tilde u$ equalling u on an interval I. The following properties of $X^{s,b}_T$ spaces are easily verified.

1. $X^{s,b}$ and $X^{s,b}_T$ are Hilbert spaces.
2. If $s_1 \le s_2$, $b_1 \le b_2$ we have $X^{s_2,b_2} \subset X^{s_1,b_1}$ with continuous embedding.
3. For every $s_1 < s_2$, $b_1 < b_2$ and $T > 0$, we have $X^{s_2,b_2}_T \subset X^{s_1,b_1}_T$ with compact imbedding.
4. For $0<\theta<1$, the complex interpolation space $\left(X^{s_1,b_1}, X^{s_2,b_2}\right)_{[\theta]}$ is $X^{(1-\theta)s_1+\theta s_2,(1-\theta)b_1+\theta b_2}$.

4. can be proved with the interpolation theorem of Stein-Weiss for weighted L^p spaces (see [5] p 114).

Then, we list some additional trilinear estimates that will be used all along the paper.

Lemma 3.2.1. *If Assumption 3.1.3 is fulfilled, for every $r \ge s > s_0$, there exist $0 < b' < 1/2$ and $C > 0$ such that for any u and $\tilde u \in X^{r,b'}$*

$$\left\||u|^2u\right\|_{X^{r,-b'}} \le C\|u\|^2_{X^{s,b'}}\|u\|_{X^{r,b'}} \tag{2.11}$$
$$\left\||u|^2\widetilde{u}\right\|_{X^{r,-b'}} \le C\|u\|_{X^{s,b'}}\|u\|_{X^{r,b'}}\|\widetilde{u}\|_{X^{r,b'}} \tag{2.12}$$
$$\left\||u|^2u - |\widetilde{u}|^2\widetilde{u}\right\|_{X^{s,-b'}} \le C\left(\|u\|^2_{X^{s,b'}} + \|\tilde u\|^2_{X^{s,b'}}\right)\|u-\tilde u\|_{X^{s,b'}}. \tag{2.13}$$

Moreover, the same estimates hold with $z_1\overline{z_2}z_3$ replaced by any $\mathbb{R}$-trilinear form on $\mathbb{C}$.

The proof of the previous lemma can be found in [9], [12] or [23]. Yet, in the Appendix, we prove some slightly different estimates, but the proof gives an idea of how Lemma 3.2.1 is established. We also give some variants that will be used in the linearized version of our equations.

Lemma 3.2.2. *If Assumption 3.1.3 is fulfilled, for every $-1 \leq s \leq 1$ and any $s_0 < r \leq 1$, there exist $0 < b' < 1/2$ and $C > 0$ such that for any $u \in X^{s,b'}$ and $a_1, a_2 \in X^{1,b'}$*

$$\|a_1\overline{a_2}u\|_{X^{s,-b'}} \leq C\|a_1\|_{X^{1,b'}}\|a_2\|_{X^{1,b'}}\|u\|_{X^{s,b'}} \tag{2.14}$$
$$\left\||a_1|^2u\right\|_{X^{s,-b'}} \leq C\|a_1\|_{X^{1,b'}}\|a_1\|_{X^{r,b'}}\|u\|_{X^{s,b'}} \tag{2.15}$$

Moreover, the same estimates hold with $z_1\overline{z_2}z_3$ replaced by any $\mathbb{R}$-trilinear form on $\mathbb{C}$.

Démonstration. We first prove (2.15). Estimate (2.12) of Lemma 3.2.1 implies that the operator of multiplication by $|a_1|^2$ maps $X^{1,b'}$ into $X^{1,-b'}$ with norm $\|a_1\|_{X^{1,b'}}\|a_1\|_{X^{r,b'}}$. By duality, it maps $X^{-1,b'}$ into $X^{-1,-b'}$ with the same norm. We get the same result for $-1 \leq s \leq 1$ by interpolation, which yields (2.15). For (2.14), we observe that estimate

$$\|a_1\overline{a_2}u\|_{X^{1,-b'}} \leq C\|a_1\|_{X^{1,b'}}\|a_2\|_{X^{1,b'}}\|u\|_{X^{1,b'}}$$

holds whatever the position of the conjugate operator and we conclude similarly. □

Let us study the stability of the $X^{s,b}$ spaces with respect to some particular operations.

Lemma 3.2.3. *Let $\varphi \in C_0^\infty(\mathbb{R})$ and $u \in X^{s,b}$ then $\varphi(t)u \in X^{s,b}$.*
If $u \in X_T^{s,b}$ then we have $\varphi(t)u \in X_T^{s,b}$.

Démonstration. We write

$$\|\varphi u\|_{X^{s,b}} = \left\|e^{-it\Delta}\varphi(t)u(t)\right\|_{H^b(\mathbb{R},H^s)} = \left\|(\varphi u)^\#\right\|_{H^b(\mathbb{R},H^s)} \leq C\left\|u^\#\right\|_{H^b(\mathbb{R},H^s)} \leq C\|u\|_{X^{s,b}}$$

We get the second result by applying the first one on any extension of u and taking the infinimum. □

In the case of pseudodifferential operators in the space variable, we have to deal with a loss in $X^{s,b}$ regularity compared to what we could expect. Some regularity in the index b is lost, due to the fact that a pseudodifferential operator does not keep the structure in time of the harmonics.
This loss is unavoidable as we can see, for simplicity on the torus $\mathbb{T}^1$: we take $u_n = \psi(t)e^{inx}e^{i|n^2|t}$ (where $\psi \in C_0^\infty$ equal to 1 on $[-1,1]$) which is uniformly bounded in $X^{0,b}$ for every $b \geq 0$. However, if we consider the operator B of order 0 of multiplication by e^{ix}, we get $\|e^{ix}u_n\|_{X^{0,b}} \approx n^b$. Yet, we do not have such loss for operator of the form $(-\Delta)^r$ which acts from any $X^{s,b}$ to $X^{s-2r,b}$. But if we do not make any further assumption on the pseudodifferential operator, we can show that our example is the worst one :

Lemma 3.2.4. *Let $-1 \leq b \leq 1$ and B be a pseudodifferential operator in the space variable of order ρ. For any $u \in X^{s,b}$ we have $Bu \in X^{s-\rho-|b|,b}$.*
Similarly, B maps $X_T^{s,b}$ into $X_T^{s-\rho-|b|,b}$.

Démonstration. We first deal with the two cases $b=0$ and $b=1$ and we will conclude by interpolation and duality.
For $b=0$, $X^{s,0}=L^2(\mathbb{R},H^s)$ and the result is obvious.
For $b=1$, we have $u\in X^{s,1}$ if and only if

$$u\in L^2(\mathbb{R},H^s) \text{ and } i\partial_t u+\Delta u\in L^2(\mathbb{R},H^s)$$

with the norm

$$\|u\|^2_{X^{s,1}}=\|u\|^2_{L^2(\mathbb{R},H^s)}+\|i\partial_t u+\Delta u\|^2_{L^2(\mathbb{R},H^s)}.$$

Then, we have

$$\begin{aligned}
\|Bu\|^2_{X^{s-\rho-1,1}} &= \|Bu\|^2_{L^2(\mathbb{R},H^{s-\rho-1})}+\|i\partial_t Bu+\Delta Bu\|^2_{L^2(\mathbb{R},H^{s-\rho-1})}\\
&\le C\Big(\|u\|^2_{L^2(\mathbb{R},H^{s-1})}+\|B\left(i\partial_t u+\Delta u\right)\|^2_{L^2(\mathbb{R},H^{s-\rho-1})}\\
&\quad +\|[B,\Delta]\,u\|^2_{L^2(\mathbb{R},H^{s-\rho-1})}\Big)\\
&\le C\Big(\|u\|^2_{L^2(\mathbb{R},H^{s-1})}+\|i\partial_t u+\Delta u\|^2_{L^2(\mathbb{R},H^{s-1})}+\|u\|^2_{L^2(\mathbb{R},H^s)}\Big)\\
&\le C\,\|u\|^2_{X^{s,1}}.
\end{aligned}$$

Hence, B maps $X^{s,0}$ into $X^{s-\rho,0}$ and $X^{s,1}$ into $X^{s-\rho-1,1}$. Then, we conclude by interpolation that B maps $X^{s,b}=\left(X^{s,0},X^{s,1}\right)_{[b]}$ into $\left(X^{s-\rho,0},X^{s-\rho-1,1}\right)_{[b]}=X^{s-\rho-b,b}$ which yields the b loss of regularity as announced.

By duality, this also implies that for $0\le b\le 1$, B^* maps $X^{-s+\rho+b,-b}$ into $X^{-s,-b}$. As there is no assumption on $s\in\mathbb{R}$, we also have the result for $-1\le b\le 0$ with a loss $-b=|b|$.
To get the same result for the restriction spaces $X^{s,b}_T$, we write the inequality for an extension $\tilde{u}$ of u, which yields

$$\|Bu\|_{X_T^{s-\rho-|b|,b}}\le\|B\tilde{u}\|_{X^{s-\rho-|b|,b}}\le C\,\|\tilde{u}\|_{X^{s,b}}.$$

Taking the infinimum on all the $\tilde{u}$, we get the claimed result. □

We will also use the following elementary estimate (see e.g. [24] or [7]).

Lemma 3.2.5. *Let (b,b') satisfying*

$$0<b'<\frac{1}{2}<b,\qquad b+b'\le 1. \tag{2.16}$$

If $f\in H^{-b'}(\mathbb{R})$ and we note $F(t)=\Psi\left(\frac{t}{T}\right)\int_0^t f(t')dt'$, we have for $T\le 1$

$$\|F\|_{H^b(\mathbb{R})}\le CT^{1-b-b'}\,\|f\|_{H^{-b'}(\mathbb{R})}.$$

In the futur aim of using a boot-strap argument, we will need some continuity in T of the $X^{s,b}_T$ norm of a fixed function :

Lemma 3.2.6. *Let $0<b<1$ and u in $X^{s,b}$ then the function*

$$\left\{\begin{array}{ccccc} f & : &]0,T] & \longrightarrow & \mathbb{R}\\ & & t & \longmapsto & \|u\|_{X_t^{s,b}}\end{array}\right.$$

is continuous. Moreover, if $b>1/2$, there exists C_b such that

$$\lim_{t\to 0} f(t)\le C_b\,\|u(0)\|_{H^s}.$$

Démonstration. By reasoning on each component on the basis, we are led to prove the result in $H^b(\mathbb{R})$. The most difficult case is the limit near 0. It suffices to prove that if $u \in H^b(\mathbb{R})$, with $b > 1/2$, satisfies $u(0) = 0$, and $\Psi \in C_0^\infty(\mathbb{R})$ with $\Psi(0) = 1$, then

$$\Psi\left(\frac{t}{T}\right) u \underset{T \to 0}{\longrightarrow} 0 \quad \text{in} \quad H^b.$$

Such a function u can be written $\int_0^t f$ with $f \in H^{b-1}$. Then, Lemma 3.2.5 gives the result we want if $u \in H^{b+\varepsilon}$. Nevertheless, if we only have $u \in H^b$, $\Psi(\frac{t}{T})u$ is uniformly bounded. We conclude by a density argument. □

The following lemma will be useful to control solutions on large intervals that will be obtained by piecing together solutions on smaller ones. We state it without proof.

Lemma 3.2.7. *Let $0 < b < 1$. If $\bigcup]a_k, b_k[$ is a finite covering of $[0,1]$, then there exists a constant C depending only of the covering such that for every $u \in X^{s,b}$*

$$\|u\|_{X^{s,b}_{[0,1]}} \leq C \sum_k \|u\|_{X^{s,b}_{[a_k,b_k]}}.$$

3.3 Existence of solution to NLS with source and damping term

3.3.1 Nonlinear equation

Let $a \in C^\infty(M)$ taking real values fixed.
We will prove the existence for defocusing non linearity of degree 3 : they will have the form $\alpha u + \beta |u|^2 u$, with $\alpha, \beta \geq 0$.

Proposition 3.3.1. *Let $T > 0$ and $s \geq 1$. Assume that M satisfies Assumption 3.1.3. Then, for every $g \in L^2([0,T], H^s)$ and $u_0 \in H^s$, there exists a unique solution u on $[0,T]$ in $X_T^{s,b}$ to the Cauchy problem*

$$\left\{ \begin{array}{rcl} i\partial_t u + \Delta u - \alpha u - \beta |u|^2 u &=& a(x)(1-\Delta)^{-1}a(x)\partial_t u + g \text{ on } [0,T] \times M \\ u(0) &=& u_0 \in H^s \end{array} \right. \tag{3.17}$$

Moreover the flow map

$$\begin{array}{rcl} F \;:\; H^s(M) \times L^2([0,T], H^s(M)) & \rightarrow & X_T^{s,b} \\ (u_0, g) & \mapsto & u \end{array}$$

is Lipschitz on every bounded subset.

Démonstration. It is strongly inspired by the one of Bourgain [7] and Dehman, Gérard, Lebeau [16] for the stabilization term. The proof is mainly based on estimates of Lemma 3.2.1.

First, we establish that the operator J defined by $Jv = (1 + ia(x)(1-\Delta)^{-1}a(x))v$ is an isomorphism of H^s and $X^{s,b}$ ($s \in \mathbb{R}$ and $-1 \leq b \leq 1$).

J is an isomorphism of L^2 because of its decomposition in identity plus an antiselfadjoint part $J = 1 + A$. It is then an isomorphism of H^s with $s \geq 0$ by ellipticity and for every $s \in \mathbb{R}$ by duality. Using Lemma 3.2.4, we infer that if $-1 \leq b \leq 1$, A maps $X^{s,b}$ into itself. Moreover, J^{-1} (considered for example acting on $L^2([0,T] \times M)$) is a pseudodifferential operator of order 0 and satisfies $J^{-1} = 1 - AJ^{-1}$. Then, using again Lemma 3.2.4, we get that AJ^{-1} maps $X^{s,b}$ into $X^{s-|b|+2,b}$ and J is an isomorphism of $X^{s,b}$.

In the sequel of the proof, v will denote Ju. Hence, we can write system (3.17) as

$$\left\{ \begin{array}{rcl} \partial_t v - i\Delta v - R_0 v + i\beta|u|^2 u & = & -ig \text{ on } [0,T] \times M \\ v & = & Ju \\ v(0) & = & v_0 = Ju_0 \in H^s \end{array} \right. \tag{3.18}$$

where $R_0 = -i\Delta A J^{-1} + i\alpha J^{-1}$ is a pseudo-differential operator of order 0.

First, we notice that if $g \in L^2([0,T], H^s)$, it also belongs to $X_T^{s,-b'}$ as $b' \geq 0$.
We consider the functional

$$\Phi(v)(t) = e^{it\Delta} v_0 + \int_0^t e^{i(t-\tau)\Delta} \left[R_0 v - i\beta \left|u\right|^2 u - ig \right](\tau) d\tau.$$

We will apply a fixed point argument on the Banach space $X_T^{s,b}$. Let $\psi \in C_0^\infty(\mathbb{R})$ be equal to 1 on $[-1,1]$. Then by construction, (see [24]) :

$$\left\| \psi(t) e^{it\Delta} v_0 \right\|_{X^{s,b}} = \left\| \psi \right\|_{H^b(\mathbb{R})} \left\| v_0 \right\|_{H^s}.$$

Thus, for $T \leq 1$ we have

$$\left\| e^{it\Delta} v_0 \right\|_{X_T^{s,b}} \leq C \left\| v_0 \right\|_{H^s} \leq C \left\| u_0 \right\|_{H^s}.$$

For $T \leq 1$, the one dimensional estimate of Lemma 3.2.5 implies

$$\left\| \psi(t/T) \int_0^t e^{i(t-\tau)\Delta} F(\tau) \right\|_{X^{s,b}} \leq C T^{1-b-b'} \left\| F \right\|_{X^{s-b'}}$$

and then

$$\begin{aligned} & \left\| \int_0^t e^{i(t-\tau)\Delta} \left[R_0 v - i\beta \left|u\right|^2 u - ig \right](\tau) d\tau \right\|_{X_T^{s,b}} \qquad (3.19) \\ & \leq C T^{1-b-b'} \left\| R_0 v - \beta i \left|u\right|^2 u - ig \right\|_{X_T^{s,-b'}} \\ & \leq C T^{1-b-b'} \left\| R_0 v \right\|_{X_T^{s,0}} + \left\| \left|u\right|^2 u \right\|_{X_T^{s,-b'}} + \left\| g \right\|_{X_T^{s,-b'}} \\ & \leq C T^{1-b-b'} \left\| v \right\|_{X_T^{s,b}} \left(1 + \left\| v \right\|_{X_T^{1,b}}^2 \right) + \left\| g \right\|_{X_T^{s,-b'}}. \qquad (3.20) \end{aligned}$$

Thus

$$\left\| \Phi(v) \right\|_{X_T^{s,b}} \leq C \left\| u_0 \right\|_{H^s} + \left\| g \right\|_{X_T^{s,-b'}} + C T^{1-b-b'} \left\| v \right\|_{X_T^{s,b}} \left(1 + \left\| v \right\|_{X_T^{1,b}}^2 \right) \tag{3.21}$$

and similarly,

$$\left\| \Phi(v) - \Phi(\tilde{v}) \right\|_{X_T^{s,b}} \leq C T^{1-b-b'} \left\| v - \tilde{v} \right\|_{X_T^{s,b}} \left(1 + \left\| v \right\|_{X_T^{s,b}}^2 + \left\| \tilde{v} \right\|_{X_T^{s,b}}^2 \right). \tag{3.22}$$

These estimates imply that if T is chosen small enough Φ is a contraction on a suitable ball of $X_T^{s,b}$. Moreover, we have uniqueness in the class $X_T^{s,b}$ for the Duhamel equation and therefore for the Schrödinger equation.

We also prove propagation of regularity.
If $u_0 \in H^s$, with $s > 1$, we have an existence time T for the solution in $X_T^{1,b}$ and another time $\tilde{T}$ for the existence in $X_{\tilde{T}}^{s,b}$. By uniqueness in $X_T^{1,b}$, the two solutions are the same on $[0, \tilde{T}]$. Assume $\tilde{T} < T$. Then, $\|u(t,.)\|_{H^s}$ explodes as t tends to $\tilde{T}$ whereas $\|u(t,.)\|_{H^1}$ remains bounded. Using local existence in H^1 and Lemma 3.2.7, we get that $\|u\|_{X_{\tilde{T}}^{1,b}}$ is finite. Applying tame estimate (3.21) on a subinterval $[T-\varepsilon, T]$, with ε small enough such that $C\varepsilon^{1-b-b'}\left(1 + \|v\|^2_{X_T^{1,b}}\right) < 1/2$, we obtain

$$\|v\|_{X_T^{s,b}} \leq C \|u(T-\varepsilon)\|_{H^s} + \|g\|_{X_T^{s,-b'}} .$$

Therefore, $u \in X_{\tilde{T}}^{s,b}$, and this contradicts the explosion of $\|u(t,.)\|_{H^s}$ near $\tilde{T}$.

Next, we use energy estimates to get global existence.
First, we will consider the energy :

$$E(t) = \frac{1}{2}\int_M |\nabla u|^2 + \frac{1}{2}\alpha \int_M |u|^2 + \beta\frac{1}{4}\int_M |u|^4 .$$

The energy is conserved if $g = 0$ and $a = 0$. It is nonincreasing if $g = 0$. In general, multiplying our equation by $\partial_t \bar{u}$, we have the relation :

$$\begin{aligned}
E(t) - E(0) &= -\int_0^t \left\|(1-\Delta)^{-1/2} a(x)\partial_t u\right\|^2_{L^2} - \Re\int_0^t \int_M g\overline{\partial_t u} \\
&= -\int_0^t \left\|(1-\Delta)^{-1/2} a(x)\partial_t u\right\|^2_{L^2} - \Re\int_0^t \int_M (J^{-1*}g)\overline{\partial_t v} \\
&= -\int_0^t \left\|(1-\Delta)^{-1/2} a(x)\partial_t u\right\|^2_{L^2} \\
&\quad -\Re\int_0^t \int_M (J^{-1*}g)\overline{i\Delta v + R_0 v - i\beta|u|^2 u - ig}.
\end{aligned}$$

If $0 \leq t \leq T$ (for this equation, there is not global existence in negative time) and $\beta > 0$, we get

$$\begin{aligned}
E(t) &\leq E(0)) + C\int_0^t \left\|\nabla(J^{-1*}g)\right\|_{L^2} \|\nabla u\|_{L^2} + \int_0^t \|g\|_{L^2}\|u\|_{L^2} \\
&\quad + \int_0^t \|g\|_{L^4}\|u\|^3_{L^4} + \|g\|^2_{L^2([0,T]\times M)} \\
&\leq E(0) + C\int_0^t \|g(\tau)\|_{H^1}\sqrt{E(\tau)} + C\int_0^t \|g(\tau)\|_{L^2}(E(\tau))^{1/4} \\
&\quad + C\int_0^t \|g(\tau)\|_{H^1}(E(\tau))^{3/4} + \|g\|^2_{L^2([0,T]\times M)} \\
&\leq E(0) + C\int_0^t \|g(\tau)\|_{H^1}\left[1 + (E(\tau))^{3/4}\right] + \|g\|^2_{L^2([0,T]\times M)} .
\end{aligned}$$

Therefore

$$\max_{0\le\tau\le t} E(\tau) \le E(0) + C(\left[1+\max_{0\le\tau\le t} E(\tau)^{3/4}\right] \|g\|_{L^1([0,T],H^1)} + \|g\|^2_{L^2([0,T]\times M)}.$$

So we have finally

$$E(t) \le C\left(1+E(0)^4 + \|g\|^8_{L^2([0,T]\times M)} + \|g\|^4_{L^1([0,T],H^1)}\right). \tag{3.23}$$

This implies that the energy is bounded if $g\in L^2([0,T],H^1)$ and yields global existence in $X^{1,b}_T$ for every $T>0$. The fact that the flow is locally Lipschitz follows from estimate (3.22). □

Remark 3.3.1. *If $g=0$, the solution of (3.17) satisfies the energy decay*

$$E(t)-E(0) = -\int_0^t \left\|(1-\Delta)^{-1/2}a(x)\partial_t u\right\|^2_{L^2}.$$

This is obtained for initial data in H^2 by multiplying the equation by $\partial_t\bar{u}$ and can be extended to initial data in H^1 by approximation.

Remark 3.3.2. *We have also proved that for any u_0, g with $\|u_0\|_{H^1}+\|g\|_{L^2([0,T],H^1)}\le A$, the solution u of (3.17) satisfies*

$$\|u\|_{X^{1,b}_T} \le C(T,A).$$

Remark 3.3.3. *If we look carefully at inequality (3.19), we see that we have for $0<\varepsilon<1-b-b'$*

$$\begin{aligned}
&\left\|\int_0^t e^{(t-\tau)\Delta}\left[R_0 v - i\,|u|^2\,u - iJg\right](\tau)d\tau\right\|_{X^{1,b+\varepsilon}}\\
&\le CT^{1-b-b'-\varepsilon}\left\|R_0 v - i\,|u|^2\,u - iJg\right\|_{X^{1,-b'}_T}\\
&\le CT^{1-b-b'-\varepsilon}\|v\|_{X^{1,b}_T}\left(1+\|v\|^2_{X^{1,b}_T}\right)+\|g\|_{L^2([0,T],H^1)}
\end{aligned} \tag{3.24}$$

and we can then conclude that u is bounded in $X^{1,b+\varepsilon}_T$.

Remark 3.3.4. *We notice that for a solution of the equation, the term of stabilization $a(x)(1-\Delta)^{-1}a(x)\partial_t u$ belongs to $L^\infty([0,T],H^1(M))$ as expected. Actually, for a solution, this term acts as an operator of order 0. This is more visible using the equation fulfilled by $v=Ju$.*

Then, in the aim of obtaining controllability near trajectories, we prove an appropriate existence theorem.

Proposition 3.3.2. *Suppose that Assumption 3.1.3 is fulfilled. Let $T>0$ and w solution in $X^{1,b}_T$ of*

$$\left\{\begin{array}{rcl} i\partial_t w+\Delta w &=& \pm|w|^2 w + g_1 \text{ on } [0,T]\times M\\ w(0) &=& w_0\in H^1\end{array}\right. \tag{3.25}$$

with $g_1\in L^2([0,T],H^1)$. Then, for any $s\in]s_0,1]$, there exists $\varepsilon>0$ such that for any $u_0\in H^s$ and $g\in L^2([0,T],H^s)$ with $\|u_0-w_0\|_{H^s}+\|g_1-g\|_{L^2([0,T],H^s)}\le\varepsilon$ there exists a

unique solution u in $X_T^{s,b}$ of equation (3.25).
Moreover for any $1 \geq r \geq s$ there exists $C = C(r, \|w\|_{X_T^{1,b}}, T) > 0$ such that, if $u_0 \in H^r$ and $g \in L^2([0,T], H^r)$, we have $u \in X_T^{r,b}$ and

$$\|u - w\|_{X_T^{r,b}} \leq C \left(\|u_0 - w_0\|_{H^r} + \|g_1 - g\|_{L^2([0,T],H^r)} \right). \tag{3.26}$$

Remark 3.3.5. *In the focusing case, the existence of w is not guaranteed for any w_0 g_1 and T, and the result we prove assumes this existence.*

Remark 3.3.6. *Here, we emphasize the fact that the asumption of smallness only concerns the H^s norm and not H^r. This is a consequence of the subcritical behavior.*

Démonstration. We want to linearize the equation. If $u = w + r$ and $g = g_1 + g_r$, then

$$\begin{aligned} |w+r|^2 (w+r) &= |w|^2 w + 2|w|^2 r + w^2 \bar{r} + 2|r|^2 w + r^2 \bar{w} + |r|^2 r \\ &= |w|^2 w + 2|w|^2 r + w^2 \bar{r} + F(w,r). \end{aligned}$$

We are looking for r solution of

$$\left\{ \begin{aligned} i\partial_t r + \Delta r &= 2|w|^2 r + w^2 \bar{r} + F(w,r) + g_r \\ r(x,0) &= r_0(x). \end{aligned} \right. \tag{3.27}$$

We make a proof similar to Proposition 3.3.1. We only write the necessary estimates. (2.11) and (2.12) yield

$$\begin{aligned} \|r\|_{X_T^{r,b}} \leq\ & C \left(\|r_0\|_{H^r} + \|g_r\|_{L^2([0,T],H^r)} \right) + CT^{1-b-b'} \|w\|^2_{X_T^{1,b}} \|r\|_{X_T^{r,b}} \\ & + CT^{1-b-b'} \left(\|w\|_{X_T^{1,b}} \|r\|_{X_T^{r,b}} \|r\|_{X_T^{s,b}} + \|r\|_{X_T^{r,b}} \|r\|^2_{X_T^{s,b}} \right). \end{aligned}$$

With T such that $CT^{1-b-b'} \|w\|^2_{X_T^{1,b}} < 1/2$, it yields

$$\begin{aligned} \|r\|_{X_T^{r,b}} \leq\ & C \left(\|r_0\|_{H^r} + \|g_r\|_{L^2([0,T],H^r)} \right) \\ & + CT^{1-b-b'} \left(\|w\|_{X_T^{1,b}} \|r\|_{X_T^{r,b}} \|r\|_{X_T^{s,b}} + \|r\|_{X_T^{r,b}} \|r\|^2_{X_T^{s,b}} \right). \end{aligned} \tag{3.28}$$

First, we apply this with $r = s$. As we have proved in Lemma 3.2.6 the continuity with respect to T of $\|r\|_{X_T^{s,b}}$ we are in position to apply a boot-strap argument : for $\|r_0\|_{H^s} + \|g_r\|_{L^2([0,T],H^s)}$ small enough (depending only on $\|w\|_{X_T^{1,b}}$), we obtain :

$$\|r\|_{X_T^{s,b}} \leq C \|r_0\|_{H^s} + \|g_r\|_{L^2([0,T],H^s)}.$$

Repeating the argument on every small interval, using that $\|r\|_{X_T^{s,b}}$ controls $L^\infty(H^s)$ and matching solutions with Lemma 3.2.7, we get the same result for every large interval, with a smaller constant ε, depending only on s, T and $\|w\|_{X_T^{1,b}}$.
Then, we return to the general case $r \geq s$ and $CT^{1-b-b'} \|w\|^2_{X_T^{1,b}} < 1/2$. For T small enough (depending only on r, ε and $\|w\|_{X_T^{1,b}}$), estimate (3.28) becomes

$$\|r\|_{X_T^{r,b}} \leq C \left(\|r_0\|_{H^r} + \|g_r\|_{L^2([0,T],H^r)} \right)$$

Again, we obtain the desired result by piecing solutions together. □

3.3.2 Linear equation with rough potential

The control near trajectories will be obtained by a perturbation of control of linear Schrödinger equation with rough potential. The equation considered are the linearization of nonlinear equations and its dual version. We establish here the necessary estimates.

Proposition 3.3.3. *Suppose Assumption 3.1.3. Let $T > 0$, $s \in [-1,1]$, $A > 0$ and $w \in X_T^{1,b}$ with $\|w\|_{X_T^{1,b}} \leq A$.*

For every $u_0 \in H^s$ and $g \in X_T^{s,-b'}$ there exists a unique solution u in $X_T^{s,b}$ of equation

$$\left\{ \begin{array}{rcl} i\partial_t u + \Delta u & = & \pm 2|w|^2 u \pm w^2 \overline{u} + g \text{ on } [0,T] \times M \\ u(0) & = & u_0 \in H^s. \end{array} \right. \tag{3.29}$$

Moreover there exists $C = C(s,A,T) > 0$ such that

$$\|u\|_{X_T^{s,b}} \leq C \left(\|u_0\|_{H^s} + \|g\|_{X^{s,-b'}} \right). \tag{3.30}$$

Démonstration. We make the same arguments as above using estimates of Lemma 3.2.2. □

3.4 Linearisation in H^1

The following result show that any sequence of solutions with Cauchy data weakly convergent to 0 asymptotically behave as solutions of the linear equation. These types of results were first introduced by P. Gérard in [22] for the wave equation and are typical of subcritical situations.

Proposition 3.4.1. *Suppose Assumption 3.1.3 is fulfilled. Let $(u_n) \in X_T^{1,b}$ be a sequence of solutions of*

$$\left\{ \begin{array}{rcl} i\partial_t u_n + \Delta u_n - u_n - |u_n|^2 u_n & = & a(x)(1-\Delta)^{-1} a(x) \partial_t u_n \text{ on } [0,T] \times M \\ u_n(0) & = & u_{n,0} \in H^1(M) \end{array} \right. \tag{4.31}$$

such that

$$u_{n,0} \underset{H^1(M)}{\rightharpoonup} 0.$$

Then

$$|u_n|^2 u_n \underset{X_T^{1,-b'}}{\longrightarrow} 0.$$

Démonstration. We prove that any subsequence (still denoted u_n) admits another subsequence converging to 0. The main point is the tame $X_T^{s,b}$ estimate of Lemma 3.2.1. For one $s_0 < s < 1$ we have

$$\left\| |u_n|^2 u_n \right\|_{X_T^{1,-b'}} \leq C \left\| u_n \right\|_{X_T^{s,b}}^2 \left\| u_n \right\|_{X_T^{1,b}} \tag{4.32}$$

First, using Remark 3.3.2, we conclude that u_n is bounded in $X_T^{1,b}$, and actually by Remark 3.3.3, u_n is bounded in $X_T^{1,b+\varepsilon}$ for some $\varepsilon > 0$. By compact embedding of $X_T^{1,b+\varepsilon}$ into $X_T^{s,b}$ we obtain that u_n admits a subsequence converging weakly in $X_T^{1,b}$ and strongly in $X_T^{s,b}$ to a function $u \in X_T^{s,b}$ with $u(0) = 0$. $u_n(0)$ strongly converges to 0 in H^s and by continuity of the nonlinear flow in H^s, u_n strongly converges to 0 in $X_T^{s,b}$. This yields the desired result thanks to (4.32). □

3.5 Propagation of compactness

In this section, we adapt some theorems of Dehman-Gérard-Lebeau [16] in the case of $X^{s,b}$ spaces. We recall that S^*M denotes the cosphere bundle of the Riemannian manifold M,

$$S^*M = \{(x,\xi) \in T^*M \quad : \quad |\xi|_x = 1\}$$

Proposition 3.5.1. *Let $r \in \mathbb{R}$. Let u_n be a sequence of solutions to*

$$i\partial_t u_n + \Delta u_n = f_n$$

such that for one $0 \le b \le 1$, we have

$$\|u_n\|_{X_T^{r,b}} \le C, \quad \|u_n\|_{X_T^{r-1+b,-b}} \to 0 \quad \text{and} \quad \|f_n\|_{X_T^{r-1+b,-b}} \to 0$$

Then, there exists a subsequence $(u_{n'})$ of (u_n) and a positive measure μ on $]0,T[\times S^*M$ such that for every tangential (that is without time derivative) pseudodifferential operator $A = A(t,x,D_x)$ of order $2r$ and of principal symbol $\sigma(A) = a_{2r}(t,x,\xi)$,

$$(A(t,x,D_x)u_{n'}, u_{n'})_{L^2(]0,T[\times M)} \to \int_{]0,T[\times S^*M} a_{2r}(t,x,\xi)\ d\mu(t,x,\xi)$$

Moreover, if G_s denotes the geodesic flow on S^*M, one has for every $s \in \mathbb{R}$,

$$G_s(\mu) = \mu.$$

Démonstration. Existence of the measure : it is based on Gårding inequality, see [21] for an introduction.
Propagation : Denote L the operator $L = i\partial_t + \Delta$. Let $\varphi = \varphi(t) \in C_0^\infty(]0,T[)$, $B(x,D_x)$ be a pseudodifferential operator of order $2r-1$, with principal symbol b_{2r-1}, $A(t,x,D_x) = \varphi(t)B(x,D_x)$. For $\varepsilon > 0$, we denote $A_\varepsilon = \varphi B_\varepsilon = Ae^{\varepsilon\Delta}$ for the regularization.

As $A_\varepsilon u_n$ and $A_\varepsilon^* u_n$ are C^∞, we can write
$(Lu_n, A_\varepsilon^* u_n)_{L^2(]0,T[\times M)} = (f_n, A_\varepsilon^* u_n)_{L^2(]0,T[\times M)}$ and
$(A_\varepsilon u_n, Lu_n)_{L^2(]0,T[\times M)} = (A_\varepsilon u_n, f_n)_{L^2(]0,T[\times M)}$. We write by a classical way

$$\begin{aligned} \alpha_{n,\varepsilon} &= (Lu_n, A_\varepsilon^* u_n)_{L^2(]0,T[\times M)} - (A_\varepsilon u_n, Lu_n,)_{L^2(]0,T[\times M)} \\ &= ([A_\varepsilon, \Delta]u_n, u_n) - i(\partial_t(A_\varepsilon)u_n, u_n). \end{aligned}$$

We will strongly use Lemma 3.2.3 and 3.2.4 without citing them.

$\partial_t(A_\varepsilon)$ is of order $2r-1$ uniformly in ε, then,

$$\begin{aligned} \sup_\varepsilon (\partial_t(A_\varepsilon)u_n, u_n)_{L^2(]0,T[\times M)} &\le C\|\partial_t(A_\varepsilon)u_n\|_{X_T^{-r+1-b,b}} \|u_n\|_{X_T^{r-1+b,-b}} \\ &\le C\|u_n\|_{X_T^{r,b}} \|u_n\|_{X_T^{r-1+b,-b}} \end{aligned}$$

which tends to 0 if $n \to \infty$.
But we have also

$$\alpha_{n,\varepsilon} = (f_n, A_\varepsilon^* u_n)_{L^2(]0,T[\times M)} - (A_\varepsilon u_n, f_n)_{L^2(]0,T[\times M)}$$

$$\begin{aligned}\left|(f_n, A_\varepsilon^* u_n)_{L^2(]0,T[\times M)}\right| &\leq \|f_n\|_{X_T^{r-1+b,-b}} \|A_\varepsilon^* u_n\|_{X_T^{-r+1-b,b}} \\ &\leq \|f_n\|_{X_T^{r-1+b,-b}} \|u_n\|_{X_T^{r,b}}.\end{aligned}$$

Then, $\sup_\varepsilon \left|(f_n, A_\varepsilon^* u_n)_{L^2(]0,T[\times M)}\right| \to 0$ when $n \to \infty$. The same estimate for the other terms gives $\sup_\varepsilon \alpha_{n,\varepsilon} \to 0$.

Finally, taking the supremum on ε tending to 0, we get

$$(\varphi[B,\Delta]u_n, u_n)_{L^2(]0,T[\times M)} \to 0 \text{ when } n \to \infty$$

which means, in terms of measure

$$\int_{]0,T[\times S^*M} \varphi(t) \left\{\sigma_2(\Delta), b_{2r-1}\right\} \; d\mu(t,x,\xi) = 0.$$

This is precisely the propagation along the geodesic flow. □

Corollary 3.5.1. *Let $r \in \mathbb{R}$. Assume that $\omega \subset M$ satisfies Assumption 3.1.1 and $a \in C^\infty(M)$, as in (1.3). Let u_n be a sequence bounded in $X_T^{r,b'}$ with $0 < b' < 1/2$, weakly convergent to 0 and satisfying*

$$\begin{cases} i\partial_t u_n + \Delta u_n \to 0 \text{ in } X_T^{r,-b'} \\ a(x)u_n \to 0 \text{ in } L^2([0,T],H^r). \end{cases} \tag{5.33}$$

Then, we have $u_n \to 0$ in $X^{r,1-b'}$.

Démonstration. Let (u_{n_k}) be any subsequence of (u_n). The asumption on b' and compact embedding allow us to apply Proposition 3.5.1. We can attach to (u_{n_k}) a microlocal defect measure in $L^2([0,T],H^r)$ that propagates along the geodesics with infinite speed. The second assumption of (5.33) gives $a(x)\mu = 0$. By Assumption 3.1.1, and the fact that a is elliptic on ω, we have $\mu = 0$ on $]0,T[\times S^*M$, i.e., $(u_{n'}) \to 0$ in $L^2([0,T],H^r)$, and $u_n \to u$ in $L^2([0,T],H^r)$.
Then, we can pick t_0 such that $u_n(t_0) \to 0$ in H^r.
Using Lemma 3.2.5 and asumptions on b', we get for $T \leq 1$

$$\left\|\int_0^t e^{i(t-\tau)\Delta} f_n(\tau) d\tau\right\|_{X_T^{r,1-b'}} \leq C \left\|f_n\right\|_{X_T^{r,-b'}}.$$

Using Duhamel formula, we conclude $u_n \to 0$ in $X_T^{r,1-b'}$.
Then, the hypothesis $T \leq 1$ is easily removed by piecing solutions together as in Lemma 3.2.7. □

3.6 Propagation of regularity

We write Proposition 13 of [16] with some $X^{s,b}$ asumptions on the second term of the equation.

Proposition 3.6.1. *Let $T > 0$, $0 \leq b < 1$ and $u \in X_T^{r,b}$, $r \in \mathbb{R}$ solution of*

$$i\partial_t u + \Delta u = f \in X_T^{r,-b}$$

Given $\gamma_0 = (x_0, \xi_0) \in T^*M \setminus 0$, we assume that there exists a zeroth order pseudo-differential operator $\chi(x, D_x)$, elliptic in γ_0 such that

$$\chi(x, D_x)u \in L^2_{loc}(]0, T[, H^{r+\rho})$$

for some $\rho \leq \frac{1-b}{2}$. Then, for every $\gamma_1 \in \Gamma_{\gamma_0}$, the geodesic ray starting at γ_0, there exists a pseudodifferential operator $\Psi(x, D_x)$, elliptic in γ_1 such that

$$\Psi(x, D_x)u \in L^2_{loc}(]0, T[, H^{r+\rho}).$$

Corollary 3.6.1. *With the notations of the Proposition, if an open set ω satisfies Assumption 3.1.1 and $a(x)u \in L^2_{loc}(]0, T[, H^{r+\rho})$, with $a \in C^\infty(M)$, as in (1.3), then $u \in L^2_{loc}(]0, T[, H^{r+\rho}(M)$.*

Proof : We first regularize : $u_n = e^{\frac{1}{n}\Delta}u$ with $\|u_n\|_{X_T^{r,b}} \leq C$. Set $s = r + \rho$
Let $B(x, D_x)$ be a pseudodifferential operator of order $2s - 1 = 2r + 2\rho - 1$, that will be chosen later and $A = A(t, x, D_x) = \varphi(t)B(x, D_x)$ where $\varphi \in C_0^\infty(]0, T[)$.
If $L = i\partial_t + \Delta$, we write

$$\begin{aligned} & (Lu_n, A^*u_n)_{L^2(]0,T[\times M)} - (Au_n, Lu_n,)_{L^2(]0,T[\times M)} \\ = \; & ([A, \Delta]u_n, u_n)_{L^2(]0,T[\times M)} - (i\varphi' Bu_n, u_n)_{L^2(]0,T[\times M)} \end{aligned}$$

$$\begin{aligned} |(Au_n, f_n)_{L^2(]0,T[\times M)}| &\leq \|Au_n\|_{X_T^{-r,b}} \|f_n\|_{X_T^{r,-b}} \\ &\leq \|u_n\|_{X_T^{r+2\rho-1+b,b}} \|f_n\|_{X_T^{r,-b}}. \end{aligned}$$

As we have chosen $\rho \leq \frac{1-b}{2}$, we have $r + 2\rho - 1 + b \leq r$ and so

$$|(Au_n, f_n)_{L^2(]0,T[\times M)}| \leq C\|u_n\|_{X_T^{r,b}} \|f_n\|_{X_T^{r,-b}} \leq C.$$

Similarly

$$|(\varphi' Bu_n, u_n)_{L^2(]0,T[\times M)}| \leq C\|u_n\|_{X_T^{r,b}} \|u_n\|_{X_T^{r,-b}} \leq C.$$

Then,

$$([A, \Delta]u_n, u_n)_{L^2(]0,T[\times M)} = \int_0^T \varphi(t)([B, \Delta]u_n(t), u_n(t))_{L^2(M)} dt$$

is uniformly bounded. Then, we select B by means of symplectic geometry. Take $\gamma_1 \in \Gamma_{\gamma_0}$, U and V two small conical neighborhoods, respectively of γ_1 and γ_0. For every symbol $c(x, \xi)$, of order s, supported in U, one can find a symbol $b(x, \xi)$ of order $2s - 1$ such that

$$\frac{1}{i}\{\sigma_2(\Delta), b(x, \xi)\} = |c(x, \xi)|^2 + r(x, \xi)$$

with $r(x, \xi)$ of order $2s$ and supported in V. We take B a pseudodifferential operator with principal symbol b so that $[B, \Delta]$ is a pseudodifferential operator of principal symbol $|c(x, \xi)|^2 + r(x, \xi)$. Then, if we choose $c(x, \xi)$ elliptic at γ_1, we conclude

$$\int_0^t \varphi(t) \|c(x, D_x)u_n(t, x)\|^2_{L^2} dt \leq C.$$

This ends the proof of Proposition 3.6.1.

Corollary 3.6.2. *Here dim $M \leq 3$ and $b > 1/2$. Let $u \in X_T^{1,b}$ solution of*

$$\left\{\begin{array}{rcl} i\partial_t u + \Delta u &=& |u|^2 u + u \text{ on } [0,T] \times M \\ \partial_t u &=& 0 \text{ on }]0,T[\times\omega \end{array}\right. \tag{6.34}$$

where ω satisfies Assumption 3.1.1.
Then $u \in C^\infty(]0,T[\times M)$.

Démonstration. We have $u \in L^\infty([0,T],H^1)$, and so in $L^\infty([0,T],L^6)$ by Sobolev embedding. Then, we infer that $|u|^2 u \in L^\infty([0,T],L^2(M))$.
On $]0,T[\times\omega$, we have

$$\Delta u = |u|^2 u + u.$$

Therefore, $\Delta u \in L^2([0,T],L^2(\omega))$ and $u \in L^2(]0,T[,H^2(\omega))$. Since $H^2(\omega)$ is an algebra, we can go on the same reasonning to conclude that $u \in C^\infty(]0,T[\times\omega)$.
By applying once Corollary 3.6.1, we get $u \in L^2_{loc}([0,T],H^{1+\frac{1-b}{2}})$. Then we can pick t_0 such that $u(t_0) \in H^{1+\frac{1-b}{2}}$. We can then solve in $X^{1+\frac{1-b}{2},b}$ our nonlinear Schrödinger equation with initial data $u(t_0)$. By uniqueness in $X_T^{1,b}$, we can conclude that $u \in X_T^{1+\frac{1-b}{2},b}$.
By iteration, we get that $u \in L^2(]0,T[,H^r)$ for every $r \in \mathbb{R}$ and $u \in C^\infty([0,T]\times M)$. □

Corollary 3.6.3. *If, in addition to Corollary 3.6.2, ω satisfies Assumption 3.1.2, then $u = 0$.*

Démonstration. Using Corollary 3.6.2, we infer that $u \in C^\infty(]0,T[\times M)$.
Taking time derivative of equation (6.34), $v = \partial_t u$ satisfies

$$\left\{\begin{array}{rcl} i\partial_t v + \Delta v + f_1\, v + f_2\, \bar{v} &=& 0 \\ v &=& 0 \text{ on }]0,T[\times\omega \end{array}\right. \tag{6.35}$$

for some $f_1,\ f_2 \in C^\infty(]0,T[\times M)$. Assumption 3.1.2 gives $v = \partial_t u = 0$. Multiplying (6.34) by $\bar{u}$ and integrating , we get

$$\int_M |\nabla u|^2 + \int_M |u|^4 + \int_M |u|^2 = 0$$

and so $u = 0$. □

Remark 3.6.1. *We have the same conclusion for $u \in X_T^{1,b}$ solution of*

$$\left\{\begin{array}{rcl} i\partial_t u + \Delta u &=& u \text{ on } [0,T] \times M \\ \partial_t u &=& 0 \text{ on }]0,T[\times\omega. \end{array}\right. \tag{6.36}$$

3.7 Stabilization

Theorem 3.1.2 is a consequence of the following Proposition

Proposition 3.7.1. *Let $a \in C^\infty(M)$, as in (1.3). Under Assumptions 3.1.1, 3.1.2 and 3.1.3, for every $T > 0$ and every $R_0 > 0$, there exists a constant $C > 0$ such that inequality*

$$E(0) \leq C \int_0^T \left\|(1-\Delta)^{-1/2} a(x)\partial_t u\right\|_{L^2}^2 dt$$

holds for every solution u of the damped equation

$$\left\{ \begin{array}{rcl} i\partial_t u + \Delta u - (1+|u|^2)u &=& a(x)(1-\Delta)^{-1}a(x)\partial_t u \text{ on } [0,T]\times M \\ u(0) &=& u_0 \in H^1 \end{array} \right. \tag{7.37}$$

and $\|u_0\|_{H^1} \leq R_0$.

Proof of Proposition 3.7.1 $\Rightarrow$ Theorem 3.1.2. For any $f \in H^1(M)$, Sobolev embeddings yield

$$\begin{aligned} E(f) &\leq C\left(\|f\|_{H^1}^2 + \|f\|_{H^1}^4\right) \\ \|f\|_{H^1} &\leq C\,(E(f))^{1/2}. \end{aligned}$$

As the energy is decreasing, if $\|u_0\|_{H^1} \leq R_0$, we can find another $\widetilde{R}_0$ such that $\|u(t)\|_{H^1} \leq \widetilde{R}_0$ for any $t > 0$. For this range of values, we have

$$C^{-1}(E(f))^{1/2} \leq \|f\|_{H^1} \leq C\,(E(f))^{1/2} \tag{7.38}$$

for one $C > 0$ depending on R_0.

We apply Proposition 3.7.1 with this bound and obtain $E(t) \leq Ce^{-\gamma(R_0)t}E(0)$. Then, for $t > t(R_0)$, we have $\|u(t)\|_{H^1} \leq 1$.

We take $\gamma(1)$ the decay rate corresponding to the bound 1. Therefore, for $t > t(R_0)$, we get $\|u(t)\|_{H^1} \leq Ce^{-\gamma(1)(t-t(R_0))}\|u(t(R_0))\|_{H^1}$. This yields a decay rate independant of R_0 as announced, while the coefficient C may strongly depend on R_0. □

Remark 3.7.1. *If we make the change of unknown $w = e^{-it}u$, w is solution of the new damped equation*

$$\left\{ \begin{array}{rcl} i\partial_t w + \Delta w - |w|^2 w &=& a(x)(1-\Delta)^{-1}a(x)(\partial_t w - iw) \text{ on } [0,T]\times M \\ w(0) &=& u_0 \in H^1. \end{array} \right.$$

This modification is necessary because there is not exponential decay for the damped equation (7.37) with $|u|^2u$ instead of $(1+|u|^2)u$. We check for example that for $a = 1$, the solution $u(t)$ with constant Cauchy data u_0 is

$$|u(t)|^2 = \frac{|u_0|^2}{1+|u_0|^2 t}.$$

This can be seen by working in polar coordinates $u(t) = \rho(t)e^{i\theta(t)}$ so that the solution satisfies $\dot{\rho} + i\rho\dot{\theta} = \frac{1}{i-1}\rho^3$ and $\frac{d}{dt}\left(\frac{1}{\rho^2}\right) = 1$ by taking real part. Moreover, it also proves that the solution is global in time only on $\mathbb{R}^+$ (this restriction remains with the non linearity $(1+|u|^2)u$).

Proof of Proposition 3.7.1. We argue by contradiction, we suppose the existence of a sequence (u_n) of solutions of (7.37) such that

$$\|u_n(0)\|_{H^1} \leq R_0$$

and

$$\int_0^T \left\|(1-\Delta)^{-1/2}a(x)\partial_t u_n\right\|_{L^2}^2 dt \leq \frac{1}{n}E(u_n(0)). \tag{7.39}$$

We note $\alpha_n = E(u_n(0))^{1/2}$. By the Sobolev embedding for the L^4 norm, we have $\alpha_n \leq C(R_0)$. So, up to extraction, we can suppose that $\alpha_n \longrightarrow \alpha$.
We will distinguich two cases : $\alpha > 0$ and $\alpha = 0$.

First case : $\alpha_n \longrightarrow \alpha > 0$
By decreasing of the energy, (u_n) is bounded in $L^\infty([0,T], H^1)$ and so in $X_T^{1,b}$. Then, as $X_T^{1,b}$ is a separable Hilbert we can extract a subsequence such that $u_n \rightharpoonup u$ weakly in $X_T^{1,b}$ ans strongly in $X_T^{s,b'}$ for one $u \in X_T^{1,b}$ and $s > s_0$. Therefore, $|u_n|^2u_n$ converges to $|u|^2u$ in $X_T^{s,-b'}$.

Using (7.39) and passing to the limit in the equation verified by u_n, we get

$$\begin{cases} i\partial_t u + \Delta u &= |u|^2u + u \text{ on } [0,T] \times M \\ \partial_t u &= 0 \text{ on }]0,T[\times\omega \end{cases}$$

Using Corollary 3.6.3, we infer $u = 0$.
Therefore, we have, up to new extraction, $u_n(0) \rightharpoonup 0$ in H^1. Using Proposition 3.4.1 of linearisation, we infer that $|u_n|^2u_n \to 0$ in $X_T^{1,-b'}$.
Moreover, by (7.39) we have

$$a(x)(1-\Delta)^{-1}a(x)\partial_t u_n \underset{L^2([0,T],H^1)}{\longrightarrow} 0$$

and the convergence is also in $X_T^{1,-b'}$.
Then, estimate (7.39) also implies $a(x)\partial_t u_n \underset{L^2([0,T],H^{-1})}{\longrightarrow} 0$.
Using equation (7.37), we obtain

$$a(x)\left[\Delta u_n - u_n - |u_n|^2u_n - a(x)(1-\Delta)^{-1}a(x)\partial_t u_n\right] \underset{L^2([0,T],H^{-1})}{\longrightarrow} 0$$

By Sobolev embedding, u_n tends to 0 in $L^\infty([0,T], L^p)$ for any $p < 6$. Therefore, $|u_n|^2u_n$ converges to 0 in $L^\infty([0,T], L^q)$ for $q < 2$ and so in $L^2([0,T], H^{-1})$. Thus, we get

$$a(x)(\Delta - 1)u_n \underset{L^2([0,T],H^{-1})}{\longrightarrow} 0.$$

Therefore, $(1-\Delta)^{1/2}a(x)u_n = (1-\Delta)^{-1/2}a(x)(1-\Delta)u_n + (1-\Delta)^{-1/2}[(1-\Delta), a(x)]u_n$ converges to 0 in $L^2([0,T], L^2)$.

In conclusion, we have

$$\begin{cases} u_n \rightharpoonup 0 \quad \text{in} \quad X_T^{1,b'} \\ a(x)u_n \to 0 \quad \text{in} \quad L^2([0,T], H^1) \\ i\partial_t u_n + \Delta u_n - u_n \longrightarrow 0 \quad \text{in} \quad X_T^{1,-b'}. \end{cases}$$

Thus, changing u_n into $e^{it}u_n$ and using that the multiplication by e^{it} is continuous on any $X_T^{s,b}$ (see Lemma 3.2.3), we are in position to apply Corollary 3.5.1. Hence, as we have $1-b' > 1/2$, it yields

$$u_n(0) \underset{H^1}{\longrightarrow} 0.$$

In particular, $E(u_n(0)) \to 0$ which is a contradiction to our hypothesis $\alpha > 0$.

Second case : $\alpha_n \longrightarrow 0$
Let us make the change of unknown $v_n = u_n/\alpha_n$. v_n is solution of the system

$$i\partial_t v_n + \Delta v_n - a(x)(1-\Delta)^{-1}a(x)\partial_t v_n = v_n + \alpha_n^2 |v_n|^2 v_n$$

and

$$\int_0^T \left\|(1-\Delta)^{-1/2}a(x)\partial_t v_n\right\|_{L^2}^2 dt \le \frac{1}{n} \tag{7.40}$$

For a constant depending on R_0, we still have (7.38). Therefore, we write

$$\|v_n(t)\|_{H^1} = \frac{\|u_n(t)\|_{H^1}}{E(u_n(0))^{1/2}} \le C\frac{E(u_n(t))^{1/2}}{E(u_n(0))^{1/2}} \le C$$

Therefore

$$\|v_n(0)\|_{H^1} = \frac{\|u_n(0)\|_{H^1}}{E(u_n(0))^{1/2}} \ge C > 0 \tag{7.41}$$

Thus, we have $\|v_n(0)\|_{H^1} \approx 1$ and v_n is bounded in $L^\infty([0,T],H^1)$.
By the same estimates we made in the proof of Proposition 3.3.1, we obtain

$$\|v_n\|_{X_T^{1,b}} \le C\|v_n(0)\|_{H^1} + CT^{1-b-b'}\left(\|v_n\|_{X_T^{1,b}} + \alpha_n^2\|v_n\|_{X_T^{1,b}}^3\right)$$

Then, if we take $CT^{1-b-b'} < 1/2$, independant of v_n, we have

$$\|v_n\|_{X_T^{1,b}} \le C(1+\alpha_n^2\|v_n\|_{X_T^{1,b}}^3).$$

By a boot strap argument, we conclude that, $\|v_n\|_{X_T^{1,b}}$ is uniformly bounded. Using Lemma 3.2.7, we conclude that it is bounded on $X_T^{1,b}$ for some large T and then, $\alpha_n^2|v_n|^2v_n$ tends to 0 in $X_T^{1,-b'}$.
Then, we can extract a subsequence such that $v_n \rightharpoonup v$ in $X_T^{1,b}$ where v is solution of

$$\left\{\begin{array}{rcl} i\partial_t v + \Delta v &=& v \text{ on } [0,T]\times M \\ \partial_t v &=& 0 \text{ on }]0,T[\times\omega \end{array}\right.$$

It implies $v=0$ by Remark 3.6.1.
Estimate (7.40) yields that $a(x)(1-\Delta)^{-1}a(x)\partial_t v_n$ converges to 0 in $L^2([0,T],H^1)$ and so in $X_T^{1,-b'}$.

We finish the proof as in the first case to conclude the convergence of v_n to 0 in $X_T^{1,b}$. This contradicts (7.41). □

3.8 Controllability of the linear equation

3.8.1 Observability estimate

Proposition 3.8.1. *Assume that (M, ω) satisfies Assumptions 3.1.1, 3.1.3 and 3.1.4. Let $a \in C^\infty(M)$, as in (1.3), taking real values. Then, for every $-1 \leq s \leq 1$, $T > 0$ and $A > 0$, there exists C such that estimate*

$$\|u_0\|_{H^s}^2 \leq C \int_0^T \|au(t)\|_{H^s}^2 \, dt$$

holds for every solution $u(t,x) \in X_T^{s,b}$ of the system

$$\left\{ \begin{array}{rcl} i\partial_t u + \Delta u & = & \pm 2|w|^2 u \pm w^2 \bar{u} \quad \text{on} \quad [0,T] \times M \\ u(0) & = & u_0 \ \in H^s \end{array} \right. \tag{8.42}$$

with one w satisfying $\|w\|_{X_T^{1,b}} \leq A$.

Démonstration. We only treat the case with $2|w|^2 u + w^2 \bar{u}$. The others are similar. We argue by contradiction. Let $u_n \in X_T^{s,b}$ be a sequence of solution to (8.42) with some associated w_n such that

$$\|u_n(0)\|_{H^s} = 1, \quad \int_0^T \|au_n\|_{H^s}^2 \to 0 \tag{8.43}$$

and

$$\|w_n\|_{X_T^{1,b}} \leq A.$$

Proposition 3.3.3 of existence yields that u_n is bounded in $X_T^{s,b}$ and we can extract a subsequence such that u_n converges strongly in $X_T^{s-1+b,-b}$ to some $u \in X_T^{s,b}$ ($b < 1-b' < 1$). Then, using Lemma 3.2.2, we infer that $2|w_n|^2 u_n + w_n^2 \bar{u_n}$ is bounded in $X_T^{s,-b'}$. We can extract another subsequence such that it converges strongly in $X_T^{s-1+b,-b}$ (here we use $-b < -1/2 < -b'$) to some $\Psi \in X_T^{s,-b'}$.
Denoting $r_n = u_n - u$ and $f_n = 2|w_n|^2 u_n + w_n^2 \bar{u_n} - \Psi$, we can apply Proposition 3.5.1 of propagation of compactness. As ω satisfies geometric control and $au_n \to 0$ in $L^2([0,T], H^s)$, we obtain that $r_n \to 0$ in $L^2_{loc}([0,T], H^s)$.
r_n is also bounded in $X_T^{s,b}$ and we deduce, by interpolation, that r_n tends to 0 in $X_I^{s,b'}$ for every $I \subset\subset]0,T[$.

Now, we want to prove that $u \equiv 0$ using unique continuation. As w_n is bounded in $X_T^{1,b}$, we can extract a subsequence such that it converges weakly to some $w \in X_T^{1,b}$. We have to prove that u is solution of a linear Schrödinger equation with potential. But the fact that $|w_n|^2 u_n$ converges weakly to $|w|^2 u$ is not guaranteed and actually uses the fact that the regularity H^1 is subcritical (see the article of L. Molinet [38] where the limit of the product is not the expected one).
We decompose

$$\begin{array}{rcl} u_n|w_n|^2 - u|w|^2 & = & (u_n - u)|w_n|^2 + u\left[|w_n - w|^2 - w(\overline{w - w_n}) - \overline{w}(w - w_n)\right] \\ & = & I + II + III + IV. \end{array}$$

Term I converges strongly to 0 in $X_T^{s,-b'}$ because $u_n - u$ tends to 0 in $X_T^{s,b'}$ and w_n is bounded in $X_T^{1,b}$. For term II, we use tame estimate for ε such that $1-\varepsilon > s_0$

$$\left\| u|w_n - w|^2 \right\|_{X_T^{s,-b'}} \leq \|u\|_{X_T^{s,b}} \|w_n - w\|_{X_T^{1-\varepsilon,b'}} \|w_n - w\|_{X_T^{1,b'}}.$$

By compact embedding, $w_n - w$ converges, up to extraction, strongly to 0 in $X_T^{1-\varepsilon,b'}$ and Term II converges strongly in $X_T^{s,-b'}$. Terms III and IV converge weakly to 0 in $X_T^{-1,-b}$ and so in the distributional sense.

Finally, we conclude that the limit of $u_n|w_n|^2$ is $u|w|^2$. We obtain similarly that $w_n^2\overline{u_n}$ converges in the distributional sense to $w^2\bar{u}$. Therefore, u is solution of

$$\begin{cases} i\partial_t u + \Delta u = 2|w|^2 u + w^2 \bar{u} \\ u = 0 \text{ on } [0,T]\times\omega. \end{cases}$$

Using Corollary 3.6.1, we infer that $u \in L^2_{loc}([0,T], H^{s+\frac{1-b}{2}})$ and existence Proposition 3.3.3 yields that it actually belongs to $X_T^{s+\frac{1-b}{2},b}$. By iteration, we obtain that $u \in X_T^{1,b}$. Then, we can apply Assumption 3.1.4 and we have in fact $u = 0$.

We pick $t_0 \in [0,T]$ such that $u_n(t_0)$ converges strongly to 0 in H^s. Estimate (3.30) of existence Proposition 3.3.3 yields strong convergence to 0 of u_n in $X_T^{s,b}$. Therefore, $\|u_n(0)\|_{H^s}$ tends to 0, which contradicts (8.43). $\square$

3.8.2 Linear control

Proposition 3.8.2. *Assume that (M,ω) satisfies Assumptions 3.1.1, 3.1.3 and 3.1.4. Let $-1 \leq s \leq 1$, $T > 0$ and $w \in X_T^{1,b}$. For every $u_0 \in H^s(M)$ there exists a control $g \in C([0,T],H^s)$ supported in $[0,T]\times\overline{\omega}$, such that the unique solution u in $X_T^{s,b}$ of the Cauchy problem*

$$\begin{cases} i\partial_t u + \Delta u &=& \pm 2|w|^2 u \pm w^2\overline{u} + g \quad \text{on} \quad [0,T]\times M \\ u(0) &=& u_0 \in H^s(M) \end{cases} \tag{8.44}$$

satisfies $u(T) = 0$.

Démonstration. We only treat the case with $2|w|^2u + w^2\bar{u}$. Let $a(x) \in C^\infty(M)$ real valued, as in (1.3). We apply the HUM method of J.L. Lions. We consider the system

$$\begin{cases} i\partial_t u + \Delta u &=& 2|w|^2 u + w^2\overline{u} + g & g \in L^2([0,T],H^s) \quad u(T) = 0 \\ i\partial_t v + \Delta v &=& 2|w|^2 v - w^2\overline{v} & v(0) = v_0 \in H^{-s}. \end{cases}$$

These equations are well posed in $X_T^{s,b}$ and $X_T^{-s,b}$ thanks to Proposition 3.3.3. The equation verified by v is the dual as the one of u for the real duality (the equation is not $\mathbb{C}$ linear). Then, multiplying the first system by $i\overline{v}$, integrating and taking real part, we get (the computation is true for w, g and v_0 smooth, we extend it by approximation)

$$\Re(u_0, v_0)_{L^2} = \Re \int_0^T (ig, v)_{L^2} dt$$

where $(\cdot,\cdot)_{L^2}$ is the complex duality on $L^2(M)$. We define the continuous map $S : H^{-s} \to H^s$ by $Sv_0 = u_0$ with the choice

$$g = Av = -ia(x)(1-\Delta)^{-s}a(x).$$

This yields

$$\begin{aligned}\Re(Sv_0, v_0)_{L^2} &= \Re\int_0^T (a(x)(1-\Delta)^{-s}a(x)v, v) = \int_0^T \left\|(1-\Delta)^{-s/2}a(x)v\right\|_{L^2}^2 \\ &= \int_0^T \|a(x)v\|_{H^{-s}}^2 .\end{aligned}$$

Thus, S is self-adjoint and positive-definite thanks to observability estimate of Proposition 3.8.1. It therefore defines an isomorphism from H^{-s} into H^s. Moreover, we notice that the norms of S and S^{-1} are uniformly bounded as w is bounded in $X_T^{1,b}$. □

Proposition 3.8.3. *Assume $0 \leq s \leq 1$, $w = 0$ and (M, ω) is either :*
-($\mathbb{T}^3$,any open set)
-($S^2 \times S^1$,(a neighborhood of the equator)$\times S^1$)
-($S^2 \times S^1$,$S^2\times$ (any open set of S^1))
Then, the same conclusion as Proposition 3.8.2 holds.

Démonstration. By following the proof of Proposition 3.8.2, we are reduced to proving an observability estimate

$$\|u_0\|_{H^{-s}}^2 \leq C\int_0^T \left\|a(x)e^{it\Delta}u_0\right\|_{H^{-s}}^2 dt$$

These results are already known for $s = 0$:
-for $\mathbb{T}^3$, this was first proved by S. Jaffard [26] in dimension 2 and generalized to any dimension by V. Komornik [28].
-the others example are of the form $(M_1 \times M_2, \omega_1 \times M_2)$ were ω_1 satisfies observability estimate.

We can extend them to any s, with $0 \leq s \leq 1$ by writing $\|u_0\|_{H^{-s}} = \left\|(1-\Delta)^{-s/2}u_0\right\|_{L^2}$. We conclude using observablility estimate in L^2 and commutator estimates.
Actually, Proposition 3.8.4 of the next section proves that controllability in L^2 implies controllability in H^s, $0 \leq s \leq 1$, with the HUM operator constructed on L^2. This yields the observability estimate in H^{-s} and for that reason, we do not detail the previous argument. □

3.8.3 Regularity of the control

This section is strongly inspired by the work of B. Dehman and G. Lebeau [17]. It express the fact that the HUM operator constructed on a space H^s propagates some better regularity. We extend this result to the Schrödinger equation with some rough potentials.

Let $T > 0$, $s \in [-1, 1]$ and $w \in X_T^{1,b}$. As in the the proof of Proposition 3.8.2, we denote $S = S_{s,T,w,a} : H^{-s} \to H^s$ the HUM operator of control associated to the trajectory w by $S\Phi_0 = u_0$ where

$$\begin{cases} i\partial_t \Phi + \Delta\Phi &= 2|w|^2\Phi - w^2\overline{\Phi} \\ \Phi(x,0) &= \Phi_0(x) \in H^{-s} \end{cases}$$

and u solution of

$$\begin{cases} i\partial_t u + \Delta u &= 2|w|^2 u + w^2\overline{u} + A\Phi \\ u(T) &= 0 \end{cases}$$

where $A = -ia(x)(1-\Delta)^{-s}a(x)$.

Proposition 3.8.4. *Suppose Assumptions 3.1.3 and 3.1.5 are fulfilled. Let $0 \leq s_0 < s \leq 1$, $\varepsilon = 1 - s$ and $w \in X_T^{1,b}$. Denote $S = S_{s,T,w,a}$ the operator defined above. We assume that S is an isomorphism from H^{-s} into H^s. Then, S is also an isomorphism from $H^{-s+\varepsilon}$ into $H^{s+\varepsilon} = H^1$.*

Démonstration. First, we show that S maps $H^{-s+\varepsilon}$ into $H^{s+\varepsilon}$.

Let $\Phi_0 \in H^{-s+\varepsilon}$. By existence Proposition 3.3.3, we have $\Phi \in X_T^{-s+\varepsilon,b}$, then $A\Phi \in L^2([0,T], H^{s+\varepsilon})$ and existence Proposition 3.3.3 gives again $u \in X_T^{s+\varepsilon,b}$ and $u(0) = S\Phi_0 \in H^{s+\varepsilon}$.

To finish, we only have to prove that $S\Phi_0 = u_0 \in H^{s+\varepsilon}$ implies $\Phi_0 \in H^{-s+\varepsilon}$. As we already know that $\Phi_0 \in H^{-s}$, we need to prove that $(-\Delta)^{\varepsilon/2}\Phi_0 \in H^{-s}$. We use the fact that S is an isomorphism from H^{-s} into H^s. Denote $D^\varepsilon = (-\Delta)^{\varepsilon/2}$.

$$\begin{aligned} \|D^\varepsilon\Phi_0\|_{H^{-s}} \leq C\|SD^\varepsilon\Phi_0\|_{H^s} &\leq C\|SD^\varepsilon\Phi_0 - D^\varepsilon S\Phi_0\|_{H^s} + C\|D^\varepsilon S\Phi_0\|_{H^s} \\ &\leq C\|SD^\varepsilon\Phi_0 - D^\varepsilon u_0\|_{H^s} + C\|u_0\|_{H^{s+\varepsilon}} \end{aligned}$$

Let φ solution of

$$\begin{cases} i\partial_t \varphi + \Delta\varphi &= 2|w|^2\varphi - w^2\overline{\varphi} \\ \varphi(x,0) &= D^\varepsilon\Phi_0(x) \end{cases}$$

and v solution of

$$\begin{cases} i\partial_t v + \Delta v &= 2|w|^2 v + w^2\overline{v} + A\varphi \\ v(T) &= 0. \end{cases}$$

So that $v(0) = SD^\varepsilon\Phi_0$. We need to estimate $\|v(0) - D^\varepsilon u_0\|_{H^s}$. But $r = v - D^\varepsilon u$ is solution of

$$\begin{cases} i\partial_t r + \Delta r &= 2|w|^2 r + w^2\overline{r} - 2[D^\varepsilon, |w|^2]u - [D^\varepsilon, w^2]\overline{u} + A(\varphi - D^\varepsilon\Phi) - [D^\varepsilon, A]\Phi \\ r(T) &= 0. \end{cases}$$

Then, using Proposition 3.3.3 we obtain

$$\begin{aligned} \|r_0\|_{H^s} \leq C\|r\|_{X_T^{s,b}} &\leq C\Big(\big\|[D^\varepsilon, |w|^2]u\big\|_{X_T^{s,-b'}} + \big\|[D^\varepsilon, w^2]\big\|_{X_T^{s,-b'}} \\ &\quad + \|A(\varphi - D^\varepsilon\Phi)\|_{X_T^{s,-b'}} + \|[D^\varepsilon, A]\Phi\|_{X_T^{s,-b'}}\Big). \end{aligned}$$

Lemma 3.A.3 of the Appendix, Section 3.A gives us some estimates about the commutators. For the last term, we notice that $[D^\varepsilon, A]$ is a pseudodifferential operator of order $\varepsilon - 2s - 1 \leq -2s$.

$$\|r_0\|_{H^s} \leq C\left(\|w\|^2_{X_T^{s+\varepsilon,b'}} \|u\|_{X_T^{s,b'}} + \|A(\varphi - D^\varepsilon\Phi)\|_{X_T^{s,-b'}} + \|\Phi\|_{L^2([0,T],H^{-s})}\right).$$

We already know that $u \in X_T^{s,b'}$, $w \in X_T^{s+\varepsilon,b'}$ and $\Phi \in X_T^{-s,b}$. We only have to estimate $\|A(\varphi - D^\varepsilon\Phi)\|_{X_T^{s,-b'}} \leq C\|\varphi - D^\varepsilon\Phi\|_{L^2([0,T],H^{-s})}$. But $d = \varphi - D^\varepsilon\Phi$ is solution of

$$\begin{cases} i\partial_t d + \Delta d &= 2|w|^2 d - w^2\overline{d} - 2[D^\varepsilon, |w|^2]\Phi + [D^\varepsilon, w^2]\overline{\Phi} \\ d(x,0) &= 0. \end{cases}$$

Thus, using Proposition 3.3.3, we get

$$\|\varphi - D^\varepsilon\Phi\|_{L^2(H^{-s})} \leq C\|d\|_{X_T^{-s,b}} \leq C\left(\left\|[D^\varepsilon, |w|^2]\Phi\right\|_{X_T^{-s,-b'}} + \left\|[D^\varepsilon, w^2]\overline{\Phi}\right\|_{X_T^{-s,-b'}}\right).$$

The second part of Lemma 3.A.3 of the Appendix allows us to conclude. □

3.9 Control near a trajectory

Theorem 3.1.3 and 3.1.4 are consequences of the following Proposition.

Proposition 3.9.1. *Suppose Assumptions 3.1.3 and 3.1.5 are fulfilled. Let $T > 0$ and $w \in X_T^{1,b}$ a controlled trajectory, i.e. solution of*

$$i\partial_t w + \Delta w = \pm|w|^2 w + g_1 \text{ on } [0,T] \times M$$

with $g_1 \in L^2([0,T], H^1(M))$, supported in $\overline{\omega}$. Let $1 \geq s > s_0 \geq 0$. Assume that the HUM operator $S = S_{s,T,w,a}$, defined in Subsection 3.8.3, is an isomorphism from H^{-s} into H^s. There exists $\varepsilon > 0$ such that for every $u_0 \in H^s$ with $\|u_0 - w(0)\|_{H^s} < \varepsilon$, there exists $g \in C([0,T], H^s)$ supported in $[0,T] \times \overline{\omega}$ such that the unique solution u in $X_T^{s,b}$ of

$$\begin{cases} i\partial_t u + \Delta u &= \pm|u|^2 u + g \\ u(x,0) &= u_0(x) \end{cases} \tag{9.45}$$

fulfills $u(T) = w(T)$.
Moreover, we can find another $\varepsilon > 0$ depending only on T,s, ω and $\|w\|_{X_T^{1,b}}$ such that for any $u_0 \in H^1$ with $\|u_0 - w(0)\|_{H^s} < \varepsilon$, the same conclusion holds with $g \in C([0,T], H^1)$.

Démonstration. In the demonstration, we denote C some constants that could actually depend on T, $\|w\|_{X_T^{1,b}}$ and s. The final ε will have the same dependence. We make the proof for the defocusing case, but since there is no energy estimate, it is the same in the other situation.

We linearize the equation as in Proposition 3.3.2. If $u = w + r$, then r is solution of

$$\begin{cases} i\partial_t r + \Delta r &= 2|w|^2 r + w^2\bar{r} + F(w,r) + g - g_1 \\ r(x,0) &= r_0(x) \end{cases}$$

with $F(w,r) = 2|r|^2 w + r^2\overline{w} + |r|^2 r$. We seek g under the form $g_1 + A\Phi$ where Φ is solution of the dual linear equation and $A = -ia(x)(1-\Delta)^{-s}a(x)$, as in the linear control. The purpose is then to choose the adequat Φ_0 and the system is completely determined.
With $\|r_0\|_{H^s}$ small enough, we are looking for a control such that $r(T) = 0$.
More precisely, we consider the two systems

$$\left\{\begin{array}{rcl} i\partial_t\Phi + \Delta\Phi & = & 2|w|^2\Phi - w^2\overline{\Phi} \\ \Phi(x,0) & = & \Phi_0(x) \in H^{-s} \end{array}\right.$$

and

$$\left\{\begin{array}{rcl} i\partial_t r + \Delta r & = & 2|w|^2 r + w^2\overline{r} + F(w,r) + A\Phi \\ r(x,T) & = & 0. \end{array}\right.$$

Let us define the operator

$$\begin{array}{rrcl} L: & H^{-s}(M) & \rightarrow & H^s(M) \\ & \Phi_0 & \mapsto & L\Phi_0 = r(0). \end{array}$$

We split $r = v + \Psi$ with Ψ solution of

$$\left\{\begin{array}{rcl} i\partial_t\Psi + \Delta\Psi & = & 2|w|^2\Psi + w^2\overline{\Psi} + A\Phi \\ \Psi(T) & = & 0. \end{array}\right.$$

This corresponds to the linear control, and so $\Psi(0) = S\Phi_0$. v is solution of

$$\left\{\begin{array}{rcl} i\partial_t v + \Delta v & = & 2|w|^2 v + w^2\overline{v} + F(w,r) \\ v(T) & = & 0 \end{array}\right. \tag{9.46}$$

Then, r, v, Ψ belong to $X_T^{s,b}$ and $r(0) = v(0) + \Psi(0)$, which we can write

$$L\Phi_0 = K\Phi_0 + S\Phi_0$$

where $K\Phi_0 = v(0)$.
$L\Phi_0 = r_0$ is equivalent to $\Phi_0 = -S^{-1}K\Phi_0 + S^{-1}r_0$. Defining the operator $B : H^{-s} \rightarrow H^{-s}$ by

$$B\Phi_0 = -S^{-1}K\Phi_0 + S^{-1}r_0,$$

the problem $L\Phi_0 = r_0$ is now to find a fixed point of B near the origin of H^{-s}. We will prove that B is contracting on a small ball $B_{H^{-s}}(0,\eta)$ provided that $\|r_0\|_{H^s}$ is small enough.
We may assume $T < 1$, and fix it for the rest of the proof (actually the norm of S^{-1} depends on T and even explode when T tends to 0, see [37] and [44]).
We have

$$\|B\Phi_0\|_{H^{-s}} \leq C\left(\|K\Phi_0\|_{H^s} + \|r_0\|_{H^s}\right).$$

So, we are led to estimate $\|K\Phi_0\|_{H^s} = \|v(0)\|_{H^s}$.
If we apply to equation (9.46) the estimate of Proposition 3.3.3 we get

$$\begin{array}{rcl} \|v(0)\|_{H^s} & \leq & \|v\|_{X_T^{s,b}} \\ & \leq & C\|F(w,r)\|_{X_T^{s,-b'}} \\ & \leq & C\|w\|_{X_T^{1,b}}\|r\|^2_{X_T^{s,b}} + \|r\|^3_{X_T^{s,b}}. \end{array}$$

Then, we use the linear behavior near a trajectory of Proposition 3.3.2. We conclude that for
$\|A\Phi\|_{L^2([0,T],H^s)} \leq \|\Phi\|_{X_T^{-s,b}} \leq C\|\Phi_0\|_{H^{-s}} < C\eta$ (see Proposition 3.3.3) small enough, we have

$$\|r\|_{X_T^{s,b}} \leq C\|\Phi_0\|_{H^{-s}}.$$

This yields

$$\|B\Phi_0\|_{H^{-s}} \leq C\left(\|\Phi_0\|_{H^{-s}}^2 + \|\Phi_0\|_{H^{-s}}^3 + \|r_0\|_{H^s}\right).$$

Choosing η small enough and $\|r_0\|_{H^s} \leq \eta/2C$, we obtain $\|B\Phi_0\|_{H^{-s}} \leq \eta$ and B reproduces the ball $B_{H^{-s}}(0,\eta)$.

If $u_0 \in H^1$, we want one g in $C([0,T],H^1)$, that is $\Phi_0 \in H^{1-2s}$. We prove that B reproduces $B_{H^{-s}}(0,\eta) \cap B_{H^{1-2s}}(0,R)$ for R large enough.

Proposition 3.8.4 yields that S is an isomorphism from H^{1-2s} into H^1. Then, we have by the same arguments as above

$$\|B\Phi_0\|_{H^{1-2s}} \leq C\left(\|K\Phi_0\|_{H^1} + \|r_0\|_{H^1}\right)$$

$$\begin{aligned}
\|v(0)\|_{H^1} &\leq C\|v\|_{X_T^{1,b}} \\
&\leq C\|F(w,r)\|_{X_T^{1,-b'}} \\
&\leq C\|w\|_{X_T^{1,b}}\|r\|_{X_T^{s,b}}\|r\|_{X_T^{1,b}} + \|r\|_{X_T^{s,b}}^2\|r\|_{X_T^{1,b}}
\end{aligned}$$

and

$$\|r\|_{X_T^{1,b}} \leq C\|\Phi_0\|_{H^{1-2s}}.$$

Then,

$$\|B\Phi_0\|_{H^{1-2s}} \leq C\left(R\eta + R\eta^2 + \|r_0\|_{H^1}\right).$$

Choosing η such that $C(\eta+\eta^2) < 1/2$ (it is important to notice here that this bound does not depend on the size of r_0 in H^1) and R large enough, we obtain that B reproduces $B_{H^{-s}}(0,\eta) \cap B_{H^{1-2s}}(0,R)$.

Let us prove that it is contracting for the H^{-s} norm. For that, we examine the systems

$$\left\{\begin{array}{rcl}
i\partial_t(r-\tilde{r}) + \Delta(r-\tilde{r}) &=& 2|w|^2(r-\tilde{r}) + w^2\overline{(r-\tilde{r})} \\
&& +F(w,r) - F(w,\tilde{r}) + A(\Phi - \widetilde{\Phi}) \\
(r-\tilde{r})(T) &=& 0
\end{array}\right. \tag{9.47}$$

$$\left\{\begin{array}{rcl}
i\partial_t(v-\tilde{v}) + \Delta(v-\tilde{v}) &=& 2|w|^2(v-\tilde{v}) + w^2\overline{(v-\tilde{v})} + F(w,r) - F(w,\tilde{r}) \\
(v-\tilde{v})(T) &=& 0.
\end{array}\right.$$

We obtain

$$\begin{aligned}
\left\|B\Phi_0 - B\widetilde{\Phi}_0\right\|_{H^{-s}} &\leq C\|(v-\tilde{v})(0)\|_{H^s} \leq C\|F(w,r) - F(w,\tilde{r})\|_{X_T^{s,-b'}} \\
&\leq C\left(\|r\|_{X_T^{s,b}} + \|\tilde{r}\|_{X_T^{s,b}} + \|r\|_{X_T^{s,b}}^2 + \|\tilde{r}\|_{X_T^{s,b}}^2\right)\|r-\tilde{r}\|_{X_T^{s,b}} \\
&\leq C(\eta+\eta^2)\|r-\tilde{r}\|_{X_T^{s,b}} \leq C\eta\|r-\tilde{r}\|_{X_T^{s,b}}.
\end{aligned} \tag{9.48}$$

Considering equation (9.47), we deduce

$$\begin{aligned}\|r-\tilde{r}\|_{X_T^{s,b}} &\leq C\|F(w,r)-F(w,\tilde{r})\|_{X_T^{s,-b'}} + C\left\|A(\Phi-\widetilde{\Phi})\right\|_{L^2([0,T],H^s)}\\ &\leq C\eta\|r-\tilde{r}\|_{X_T^{s,b}} + C\left\|\Phi_0-\widetilde{\Phi}_0\right\|_{H^{-s}}.\end{aligned}$$

If η is taken small enough it yields

$$\|r-\tilde{r}\|_{X_T^{s,b}} \leq C\left\|\Phi_0-\widetilde{\Phi}_0\right\|_{H^{-s}}. \tag{9.49}$$

Combining (9.49) with (9.48) we finally get

$$\left\|B\Phi_0-B\widetilde{\Phi}_0\right\|_{H^{-s}} \leq C\eta\left\|\Phi_0-\widetilde{\Phi}_0\right\|_{H^{-s}}.$$

This yields that B is a contraction on a small ball $B_{H^{-s}}(0,\eta)$, which completes the proof of Proposition 3.9.1. □

Corollary 3.9.1. *Let $T>0$ and (M,ω) such that Assumptions 3.1.1, 3.1.3, 3.1.4 and 3.1.5 are fulfilled.*
Then, the set of reachable states is open in H^s for $s_0<s\leq 1$.

In the next corollary, $\widehat{f}(k)$ denotes the coordinates of a function f in the basis of eigenfunction of M.

Corollary 3.9.2. *Suppose the same assumptions as Proposition 3.9.1. Let $E_0>\|w_0\|_{H^1}$. Then, there exist N and ε such that for every u_0 and $u_1\in H^1$ with*

$$\|u_0\|_{H^1}\leq E_0, \qquad \|u_1\|_{H^1}\leq E_0 \tag{9.50}$$

$$\sum_{|k|\leq N}|\widehat{u_0}(k)-\widehat{w_0}(k)|^2\leq\varepsilon, \qquad \sum_{|k|\leq N}|\widehat{u_1}(k)-\widehat{w_T}(k)|^2\leq\varepsilon \tag{9.51}$$

we can find a control $g\in L^\infty([0,T],H^1)$ supported in $[0,T]\times\omega$ such that the unique solution of (9.45) with control g and $u(0)=u_0$ satisfies $u(T)=u_1$.

Démonstration. We build the control in two steps : the first brings the system from u_0 to $w(T/2)$ and the second from $w(T/2)$ to u_1. Actually, the second step is the same by reversing time and we only describe the first one.

Let $s_0<s<1$. We first check that the first part of the conclusion of Proposition 3.9.1 is true without Assumption 3.1.5. It gives one $\widetilde{\varepsilon}>0$ such that if $\|u_0-w_0\|_{H^s}\leq\widetilde{\varepsilon}$ we have a control to $w(T/2)$ in time $T/2$ with $g\in C([0,T/2],H^1)$. We only check that once E_0 is chosen, we can find N and ε such that assumptions (9.50) and (9.51) imply $\|u_0-w_0\|_{H^s}\leq\widetilde{\varepsilon}$. □

We also obtain a first proof of global controllability. The Assumptions we make are stronger than Theorem 3.1.1 that will be proved using stabilization. However, in the examples we treat, the Assumptions are fulfilled.

Corollary 3.9.3. *Theorem 3.1.1 is true under the stronger Assumptions 3.1.1, 3.1.3 and 3.1.4.*

Démonstration. We will make successives controls near some free nonlinear trajectory so that the energy decrease. The main argument is that the ε of Theorem 3.1.3 only depends on $\|w\|_{X_T^{1,b}}$ and if the trajectory is a free nonlinear trajectory, then the ε only depends on $\|w_0\|_{H^1}$. We just have to be careful that each new free trajectory fulfills $\|w\|_{X_T^{1,b}} \leq A$ for one fixed constant A.

Fix $T > 0$. There exist C_1 such that

$$\|f\|_{H^1} \leq C_1 \left(E(f) + \sqrt{E(f)}\right)^{1/2} \quad \forall f \in H^1(M).$$

Denote $A = C_1 \left(E(w_0) + \sqrt{E(w_0)}\right)$. There exists a constant such that $\|w_0\|_{H^1} \leq A$ implies $\|w\|_{X_T^{1,b}} \leq B$ for w solution of

$$\begin{cases} i\partial_t w + \Delta w &= |w|^2 w \text{ on } [0,T] \times M \\ w(0) &= w_0. \end{cases}$$

Let ε the constant so that Theorem 3.1.3 si true for any w with $\|w\|_{X_T^{1,b}} \leq B$. We choose the arrival point $u_T = (1 - \varepsilon/A) w_T$ such that

$$\|u_T - w_T\|_{H^1} = \varepsilon/A \, \|w_T\|_{H^1} \leq C_1 \left(E(w_T) + \sqrt{E(w_T)}\right) \varepsilon/A = \varepsilon.$$

We have a control g supported in $[0,T] \times \overline{\omega}$ such that the solution u of

$$\begin{cases} i\partial_t u + \Delta u &= |u|^2 u + g \text{ on } [0,T] \times M \\ u(0) &= w_0 \end{cases}$$

satisfies $u(T) = u_T$. If $1 - \varepsilon/A \in [0,1]$, we have

$$E(u_T) = \frac{1}{2} \int_M |(1-\varepsilon/A)\nabla w_T|^2 + \frac{1}{4}\int_M |(1-\varepsilon/A) w_T|^4 \leq (1-\varepsilon/A)^2 E(w_T).$$

Moreover, we still have

$$\|u_T\|_{H^1} \leq C_1 \left(E(u_T) + \sqrt{E(u_T)}\right)^{1/2} \leq A.$$

Then, we can reiterate this processus with the same ε. We construct a sequence of solutions $u_n \in X^{1,b}_{[nT,(n+1)T]}$ and of controls $g_n \in C([nT,(n+1)T], H^1)$ such that

$$\begin{cases} i\partial_t u_n + \Delta u_n &= |u_n|^2 u_n + g_n \text{ on } [nT,(n+1)T] \times M \\ u_n(nT) &= u_{n-1}(nT) \end{cases}$$

and

$$E(u_n(nT)) \leq (1-\varepsilon/A)^{2n} E(w_0) \leq C(1-\varepsilon/A)^{2n} \left(\|w_0\|_{H^1}^2 + \|w_0\|_{H^1}^4\right).$$

But, we have

$$\|u_n(nT)\|_{H^1}^2 \leq C_1 \left(E(u_n(nT)) + \sqrt{E(u_n(nT))}\right)^{1/2}.$$

Therefore, it can be made arbitrary small for large n. This allows to use local controllability near the trajectory 0. We obtain global controllability making the same proof in negative time. □

3.10 Necessity of geometric control assumption on S^3

In this section, we prove that on S^3, the geometric control is necessary for stabilization to occur. The argument uses some concentration of eigenfunctions. This concentration was also used by N. Burq, P. Gérard and N. Tzvetkov [10] to prove some ill-posedness results.

Proposition 3.10.1. *Let Γ be a geodesic of S^3 and $a \in C^\infty(S^3)$ such that $Supp(a) \cap \Gamma = \emptyset$. Then, for every $R_0 > 0$, C and $\gamma > 0$ there exist $T > 0$ and $u_0 \in H^1(S^3)$ with $\|u_0\|_{H^1} \leq R_0$ such that*

$$\|u(T)\|_{H^1} > Ce^{-\gamma T} \|u\|_{H^1}$$

for u solution of equation

$$\left\{ \begin{array}{rcl} i\partial_t u + \Delta u - (1+|u|^2)u & = & a(x)(1-\Delta)^{-1}a(x)\partial_t u \text{ on } [0,T] \times S^3 \\ u(0) & = & u_0 \in H^1. \end{array} \right. \tag{10.52}$$

Démonstration. Let T such that $Ce^{-\gamma T} \leq 1/2$.

By changes of coordinates, we can assume that $\Gamma = \{x_3 = x_4 = 0\}$. We will use the eigenfunctions $\Phi_n = c_n(x_1 + ix_2)^n$ that concentrates on the subset $\{x_3 = x_4 = 0\}$. c_n is chosen such that $\|\Phi_n\|_{H^1} = R_0$ and so $c_n \approx n^{1/2-1}$. We have $-\Delta \Phi_n = \lambda_n \Phi_n$ with $\lambda_n = n(n+2)$. Let u_n be the solution of (10.52) with $u_n(0) = \Phi_n$. Let $v_n = e^{i(\lambda_n - 1)t}\Phi_n$ be the solution of the linear equation

$$\left\{ \begin{array}{rcl} i\partial_t v_n + \Delta v_n - v_n & = & 0 \text{ on } [0,T] \times S^3 \\ v_n(0) & = & \Phi_n. \end{array} \right.$$

Then, $r_n = u_n - v_n$ is solution of

$$\left\{ \begin{array}{rcl} i\partial_t r_n + \Delta r_n - r_n & = & a(x)(1-\Delta)^{-1}a(x)\partial_t r_n + R_n \text{ on } [0,T] \times S^3 \\ r_n(0) & = & 0 \end{array} \right.$$

with $R_n = |u_n|^2 u_n + a(x)(1-\Delta)^{-1}a(x)\partial_t v_n$.

Proposition 3.4.1 of linearisation yields that $|u_n|^2 u_n \longrightarrow 0$ in $X_T^{1,-b'}$.
For the other term in R_n, we use the concentration of the Φ_n.

$$\begin{array}{rcl} \left\|a(x)(1-\Delta)^{-1}a(x)\partial_t v_n\right\|_{X_T^{1,-b'}} & \leq & \left\|a(x)(1-\Delta)^{-1}a(x)\partial_t v_n\right\|_{L^2([0,T],H^1)} \\ & \leq & \|a(x)\partial_t v_n\|_{L^2([0,T],H^{-1})} \leq (\lambda_n + 1)\|a(x)\Phi_n\|_{L^\infty(S^3)} \end{array}$$

Let $\delta > 0$, such that we have $x_3^2 + x_4^2 > \delta$ on Supp a. Hence, we have $|(x_1 + ix_2)|^2 = x_1^2 + x_2^2 = 1 - x_3^2 - x_4^2 < 1 - \delta$.

$$(\lambda_n + 1)\|a(x)\Phi_n\|_{L^\infty(S^3)} \leq C(\lambda_n + 1)c_n(1-\delta)^{n/2}$$

Since λ_n and c_n are at most polynomial in n, we deduce that R_n tends to 0 in $X_T^{1,-b'}$. By some arguments similar to the proof of the continuity of the flow map of Proposition 3.3.1,, we infer that r_n tends to 0 in $X_T^{1,b}$. Then, $\|u_n(T)\|_{H^1}$ tends to R_0 and for n large enough, we have $\|u_n(T)\|_{H^1} > R_0/2$. $\square$

With a similar proof, we could show the same result on $S^2\times S^1$ if $Supp(a)\cap(\Gamma\times S^1)=\emptyset$ for some geodesic Γ of S^2. Yet, it does not imply geometric control.

The construction of J. V. Ralston [40] proves that actually, a necessary condition for stabilization is that the support of $a(x)$ intersects any stable closed geodesic (see also the work of L. Thomann [45] where this concentration is used to prove ill-posedness). In the case of S^3, we use the geometric fact that every closed geodesic is stable.

3.A Some commutator estimates

This section is devoted to the proof of some commutator estimates used in Proposition 3.8.4. More precisely, we study the action of $[(-\Delta)^{\varepsilon/2}, a_1a_2]$ on $X^{s,b}$ where a_i are rough. We first give a simple proof for $\mathbb{T}^3$ (rational or not) and then a general one under Assumption 3.1.5. Then, we show that this assumption is fulfilled for S^3 and $S^2\times S^1$. We will need an elementary lemma.

Lemma 3.A.1. *If $0\leq\varepsilon\leq 1$, we have for any norm $||k|^{\varepsilon}-|k_3|^{\varepsilon}|\leq|k-k_3|^{\varepsilon}$.*

Démonstration. Using triangular inequality, we get $||k|-|k_3||^{\varepsilon}\leq|k-k_3|^{\varepsilon}$. Then, we are reduced to the case of $\mathbb{R}^{+*}$: we prove that for $z,\,t\in\mathbb{R}^{+*}$, we have $(z+t)^{\varepsilon}-z^{\varepsilon}\leq t^{\varepsilon}$, which is an easy consequence of Minkowsky inequality for $1\leq 1/\varepsilon\leq+\infty$. □

3.A.1 An easier proof for $\mathbb{T}^3$

Lemma 3.A.2. *Let $M=\mathbb{R}^3/(\theta_x\mathbb{Z}\times\theta_y\mathbb{Z}\times\theta_z\mathbb{Z})$ with $(\theta_x,\theta_y,\theta_z)\in\mathbb{R}^3$. Denote s_0 the constant taken from Assumption 3.1.3. Let $s>s_0$ and $0\leq\varepsilon\leq 1$.*
Then, there exists $b'<1/2$ such that $u_3\mapsto[\Delta^{\varepsilon/2},u_1u_2]u_3$ maps any $X^{s,b'}$ into $X^{s,-b'}$, where u_1u_2 denotes the operator of multiplication by u_1u_2 with $u_i\in X^{s+\varepsilon,b'}$ for $i\in\{1,2\}$. This function $[\Delta^{\varepsilon/2},u_1u_2]$ also maps $X^{-s,b'}$ into $X^{-s,-b'}$.
Moreover, the same result holds with u_i replaced by $\overline{u_i}$ for i in a subset of $\{1,2,3\}$.

Démonstration. We choose the norm $|k|=\sqrt{(\theta_xk_x)^2+(\theta_yk_y)^2+(\theta_zk_z)^2}$ so that

$$\widehat{-\Delta u}(k)=|k|^2\widehat{u}(k)$$

By duality, it is equivalent to prove

$$\int_{\mathbb{R}\times M}[(-\Delta)^{\varepsilon/2},u_1u_2]u\;\overline{v}\leq C\left\|u_1\right\|_{X^{s+\varepsilon,b'}}\left\|u_2\right\|_{X^{s+\varepsilon,b'}}\left\|u\right\|_{X^{s,b'}}\left\|v\right\|_{X^{-s,b'}}.$$

Using Parseval theorem and denoting $k=k_1+k_2+k_3$, $\tau=\tau_1+\tau_2+\tau_3$

$$\begin{aligned}&\int_{\mathbb{R}\times M}[(-\Delta)^{\varepsilon/2},u_1u_2]u\;\overline{v}\\&=\int_{\tau_1,\tau_2,\tau_3}\sum_{k_1,k_2,k_3}\widehat{u_1}(k_1,\tau_1)\widehat{u_2}(k_2,\tau_2)(|k|^{\varepsilon}-|k_3|^{\varepsilon})\widehat{u}(k_3,\tau_2)\widehat{\overline{v}}(k,\tau)\\&\leq\int_{\tau_1,\tau_2,\tau_3}\sum_{k_1,k_2,k_3}||k|^{\varepsilon}-|k_3|^{\varepsilon}|\left|\widehat{u_1}(k_1,\tau_1)\widehat{u_2}(k_2,\tau_2)\widehat{u}(k_3,\tau_3)\widehat{\overline{v}}(k,\tau)\right|.\end{aligned}$$

Lemma 3.A.1 and $k - k_3 = k_1 + k_2$ yields

$$\begin{aligned}&\left|\int_{\mathbb{R}\times M} [(-\Delta)^{\varepsilon/2}, u_1u_2]u\,\overline{v}\right| \\ &\leq C\int_{\tau_1,\tau_2,\tau_3}\sum_{k_1,k_2,k_3} (|k_1|^{\varepsilon}+|k_2|^{\varepsilon})\,|\widehat{u_1}(k_1,\tau_1)|\,|\widehat{u_2}(k_2,\tau_2)\,|\,|\,\widehat{u}(k_3,\tau_3)|\,|\widehat{v}(k,\tau)|\,.\end{aligned}$$

Denoting $u_1^{\S}$ the function with Fourier transform $|\widehat{u_1}(k_1,\tau_1)|$ we obtain.

$$\begin{aligned}\left|\int_{\mathbb{R}\times M} [(-\Delta)^{\varepsilon/2}, u_1u_2]u\,\overline{v}\right| &\leq C\int_{\mathbb{R}\times M}\left(\Delta^{\varepsilon/2}u_1^{\S}\right)u_2^{\S}u^{\S}\,\overline{v^{\S}} + \int_{\mathbb{R}\times M} u_1^{\S}\left(\Delta^{\varepsilon/2}u_2^{\S}\right)u^{\S}\,\overline{v^{\S}} \\ &\leq C\left\|u_1\right\|_{X^{s+\varepsilon,b'}}\left\|u_2\right\|_{X^{s+\varepsilon,b'}}\left\|u\right\|_{X^{s,b'}}\left\|v\right\|_{X^{-s,b'}}.\end{aligned}$$

Here, we have finished the proof using the trilinear Bourgain estimate because $s > s_0$. If we estimate this integral using the trilinear estimate at the negative level H^{-s}, we obtain the second result we announced. □

3.A.2 General proof under Assumption 3.1.5

Lemma 3.A.3. *Denote s_0 the constant taken from Assumption 3.1.5. Let $s > s_0$ and $0 \leq \varepsilon \leq 1$.*
Then, there exists $b' < 1/2$ such that $u_3 \mapsto [(-\Delta)^{\varepsilon/2}, u_1u_2]u_3$ maps any $X^{s,b'}$ into $X^{s,-b'}$, where u_1u_2 denotes the operator of multiplication by u_1u_2 with $u_i \in X^{s+\varepsilon,b'}$ for $i \in \{1,2\}$.
This function $[\Delta^{\varepsilon/2}, u_1u_2]$ also maps $X^{-s,b'}$ into $X^{-s,-b'}$.
Moreover, the same result holds with u_i replaced by $\overline{u_i}$ for i in a subset of $\{1,2,3\}$.

Démonstration. The proof follows the techniques of J. Bourgain and N. Burq, P. Gérard, N. Tzvetkov. Here, we were inspired more precisely by [23]. We recall the notations $u^{\#} = e^{-it\Delta}u(t)$, $u^N = \mathbf{1}_{\sqrt{1-\Delta}\in[N,2N[}u$ where N is a dyadic number and $\widehat{u}(\tau)$ is the Fourier transform of u with respect to the time variable. First, with some dyadic integers N_i fixed, we estimate the integral

$$\begin{aligned}I(N_1,..,N_4) &= \int_{\mathbb{R}\times M} u_1^{N_1}u_2^{N_2}\left[((-\Delta)^{\varepsilon/2}u_3^{N_3})\,\overline{u_4}^{N} - u_3^{N_3}\,(-\Delta)^{\varepsilon/2}\overline{u_4}^{N}\right]\,dtdx \\ &= \frac{1}{(2\pi)^4}\int_{\mathbb{R}_t\times M_x}\iiiint_{\mathbb{R}^4_{\tau_1,\tau_2,\tau_3,\tau_4}} e^{it(\tau_1+\tau_2+\tau_3-\tau)}e^{it\Delta}\widehat{u_1^{N_1\#}}(\tau_1)e^{it\Delta}\widehat{u_2^{N_2\#}}(\tau_2) \\ &\times\left[((-\Delta)^{\varepsilon/2}e^{it\Delta}\widehat{u_3^{N_3\#}}(\tau_3))\overline{e^{it\Delta}\widehat{u_4^{N\#}}(\tau)} - e^{it\Delta}\widehat{u_3^{N_3\#}}(\tau_3)(-\Delta)^{\varepsilon/2}\overline{e^{it\Delta}\widehat{u_4^{N_4\#}}(\tau)}\right]\end{aligned}$$

By nearly orthogonality in H^b and partition of unity, $u_j = \sum_{n\in\mathbb{Z}}\varphi(t-n/2)u_j(t)$, we are led to the special case where the u_j are supported in time in the interval $]0,1[$. Select $\chi \in C_0^{\infty}(\mathbb{R})$ such that $\chi = 1$ on $[0,1]$. Thus, estimates (1.10), applied with τ_j fixed, and Cauchy-Schwarz inequality in $(\tau_1,\tau_2,\tau_3,\tau_4)$ gives for any $b > 1/2$

$$\begin{aligned}|I(N_1,..,N_4)| &\leq C(N_1^{\varepsilon}+N_2^{\varepsilon})\,(m(N_1,\cdots,N_4))^{s_0}\prod_{j=1}^{4}\int_{\tau_j}\left\|\widehat{u_j^{N_j\#}}(\tau_j)\right\|_{L^2(M)} \\ &\leq C(N_1^{\varepsilon}+N_2^{\varepsilon})\,(m(N_1,\cdots,N_4))^{s_0}\prod_{j=1}^{4}\left\|u_j^{N_j}\right\|_{X^{0,b}(\mathbb{R}\times M)}. \qquad (1.53)\end{aligned}$$

This estimate is very satisfactory for the space regularity. Yet, for the regularity in time, it requires $b > 1/2$ which is too much for our purpose. We will interpolate with some crude estimates in space but better in time.
For the case where N_1 is large, we estimate $|I(N_1, \cdots, N)|$ using Sobolev embeddings $H^{1/4}(\mathbb{R}) \subset L^4(\mathbb{R})$:

$$|I(N_1, \cdots, N_4)| \leq C(N_3^\varepsilon + N_4^\varepsilon)\,(m(N_1, \cdots, N_4))^{3/2} \prod_{j=1}^{4} \left\| u_j^{N_j} \right\|_{X^{0,1/4}(\mathbb{R}\times M)} . \tag{1.54}$$

In another case where the frequency N_3 is large, we will use an argument near [18]. In that case, we can not afford a loss in the frequency N_3. We use the fact that $[u_1^{N_1} u_2^{N_2}, \Delta^{\varepsilon/2}]$ is a pseudodifferential operator of order less than 0 (if $\varepsilon \leq 1$). Then,

$$\begin{aligned} |I(N_1, \cdots, N_4)| &= \left| \int_{\mathbb{R}\times M} [u_1^{N_1} u_2^{N_2}, \Delta^{\varepsilon/2}] u_3^{N_3} \overline{u_4^{N_4}} \right| \\ &\leq C \int_{\mathbb{R}} \left\| [u_1(t)^{N_1} u_2(t)^{N_2}, \Delta^{\varepsilon/2}] \right\|_{L^2 \to L^2} \|u_3(t)\|_{L^2(M)} \|u_4(t)\|_{L^2(M)}\, dt \\ &\leq \int_{\mathbb{R}} \sum_{\alpha=0}^{m} \|\partial^\alpha u_1 u_2(t)\|_{L^\infty(M)} \|u_3(t)\|_{L^2(M)} \|u_4(t)\|_{L^2(M)}\, dt \\ &\leq C \max(N_1, N_2)^\mu \prod_{j=1}^{4} \left\| u_j^{N_j} \right\|_{X^{0,1/4}(\mathbb{R}\times M)} \end{aligned} \tag{1.55}$$

where μ depends on the dimension and on ε.

Let us now begin the summation of the harmonics. As in [23], we decompose each function

$$u = \sum_K u_K, \quad u_K = \mathbf{1}_{K \leq \langle i\partial_t + \Delta \rangle < 2K}(u)$$

where K denotes the sequence of dyadic integers. Notice that

$$\|u\|_{X^{0,b}}^2 \approx \sum_K K^{2b} \|u_K\|_{L^2(\mathbb{R}\times M)}^2 \approx \sum_K \|u_K\|_{X^{0,b}}^2 .$$

Then, we decompose the integral in sum of the following elementary integrals

$$\begin{aligned} &I(N_1, \cdots, N_4, K_1, \cdots, K_4) \\ &= \int_{\mathbb{R}\times M} a_1^{N_1,K_1} a_2^{N_2,K_2} \left[((-\Delta^{\varepsilon/2}) u^{N_3,K_3}) \overline{v}^{N,K} - u^{N_3,K_3} (-\Delta^{\varepsilon/2}) \overline{v}^{N,K} \right] dt dx. \end{aligned}$$

Estimate (1.53) leads to (for every $b > 1/2$)

$$\begin{aligned} &|I(N_1, \cdots, N_4, K_1, \cdots, K_4)| \\ &\leq (N_1^\varepsilon + N_2^\varepsilon) m(N_1, \cdots, N_4)^{s_0} (K_1 K_2 K_3 K_4)^b \prod_{j=1}^{4} \left\| u_j^{N_j,K_j} \right\|_{L^2} . \end{aligned}$$

We will interpolate this estimate with different inequalities. We distinguish three cases : $N_4 \leq C(N_1 + N_2 + N_3)$ with $N_3 < \max(N_1, N_2)$ or $\max(N_1, N_2) \leq N_3$, and the last case $N_4 > C(N_1 + N_2 + N_3)$ with C large enough. Without loss of generality, we can assume $N_1 \geq N_2$.

First case : $N_3 < \max(N_1, N_2) = N_1$ and $N_4 \leq C(N_1 + N_2 + N_3)$
Estimate (1.54) gives

$$\begin{aligned} |I(N_1, \cdots, N_4, K_1, \cdots, K_4)| &\leq (N_3^\varepsilon + N_4^\varepsilon) m(N_1, \cdots, N_4)^{3/2} \\ &\quad (K_1 K_2 K_3 K_4)^{1/4} \prod_{j=1}^{4} \left\| u_j^{N_j, K_j} \right\|_{L^2}. \end{aligned}$$

Then, for every $\theta \in [0, 1]$

$$\begin{aligned} |I(N_1, \cdots, N_4, K_1, \cdots, K_4)| &\leq C(N_1^\varepsilon + N_2^\varepsilon)^{1-\theta} (N_3^\varepsilon + N_4^\varepsilon)^\theta m(N_1, \cdots, N_4)^{(1-\theta)s_0 + 3\theta/2} \\ &\quad (K_1 K_2 K_3 K)^{b(1-\theta)+\theta/4} \prod_{j=1}^{4} \left\| u_j^{N_j, K_j} \right\|_{L^2}. \end{aligned}$$

We denote $s(\theta) = (1-\theta)s_0 + 3\theta/2$ and $b(\theta) = b(1-\theta) + \theta/4$.

$$\begin{aligned} |I(N_1, \cdots, N, K_1, \cdots)| &\leq C(N_1^\varepsilon + N_2^\varepsilon)^{1-\theta} (N_3^\varepsilon + N_4^\varepsilon)^\theta m(N_1, \cdots, N_4)^{s(\theta)} \\ &\quad (K_1 K_2 K_3 K_4)^{b(\theta)-b'} \prod_{j=1}^{4} \left\| u_j^{N_j} \right\|_{X^{0,b'}} \end{aligned}$$

By choosing some appropriate θ and $b' < 1/2 < b$, we can make the serie in K convergent if $b(\theta) - b' < 0$. This yields :

$$\begin{aligned} &|I(N_1, \cdots, N_4)| \leq C(N_1^\varepsilon + N_2^\varepsilon)^{1-\theta} (N_3^\varepsilon + N_4^\varepsilon)^\theta m(N_1, \cdots, N_4)^{s(\theta)} \prod_{j=1}^{4} \left\| u_j^{N_j} \right\|_{X^{0,b'}} \\ &\leq C N_1^{(1-\theta)\varepsilon - s - \varepsilon} N_4^{s+\theta\varepsilon} N_2^{s(\theta)-s-\varepsilon} N_3^{s(\theta)+\theta\varepsilon - s} \prod_{j=1}^{2} \|u_j\|_{X^{s+\varepsilon,b'}} \|u_3\|_{X^{s,b'}} \|u_4\|_{X^{-s,b'}} \\ &\leq \left(\frac{N_4}{N_1}\right)^{s+\theta\varepsilon} N_2^{s(\theta)-s-\varepsilon} N_3^{s(\theta)+\theta\varepsilon-s} \prod_{j=1}^{2} \|u_j\|_{X^{s+\varepsilon,b'}} \|u_3\|_{X^{s,b'}} \|u_4\|_{X^{-s,b'}} \end{aligned}$$

The series is convergent thanks to $N_4 \leq C N_1$ and after choosing θ small enough such that $s(\theta) + \theta\varepsilon - s < 0$ with $b(\theta) - b' < 0$.

Second case : $N_1 = \max(N_1, N_2) \leq N_3$ and so $N_4 \leq C N_3$.
This time, N_3 is a large frequency and we can not have any loss $N_3^{\theta\varepsilon}$ from the interpolation. We proceed with the same interpolation procedure but between (1.53) and (1.55). After summation in K and a good choice of $b' < 1/2 < b$ and

$$\begin{aligned} &|I(N_1, \cdots, N_4)| \\ &\leq C N_1^{(1-\theta)(s_0+\varepsilon)+\theta\mu-s-\varepsilon} N_4^s N_2^{(1-\theta)s_0-s-\varepsilon} N_3^{-s} \prod_{j=1}^{2} \|u_j\|_{X^{s+\varepsilon,b'}} \|u_3\|_{X^{s,b'}} \|u_4\|_{X^{-s,b'}} \\ &\leq \left(\frac{N_4}{N_3}\right)^{s} N_1^{(1-\theta)(s_0+\varepsilon)+\theta\mu-s-\varepsilon} N_2^{(1-\theta)s_0-s-\varepsilon} \prod_{j=1}^{2} \|u_j\|_{X^{s+\varepsilon,b'}} \|u_3\|_{X^{s,b'}} \|u_4\|_{X^{-s,b'}}. \end{aligned}$$

We choose θ small enough such that $(1-\theta)(s_0+\varepsilon) + \theta\mu - s - \varepsilon \leq s_0 + \theta\mu - s < 0$ and $b(\theta) - b' < 0$. And we conclude by the same summation as in the first case.

Last case : $N_4 \geq C(N_1+N_2+N_3)$
This case is trivial in the particular case of $\mathbb{T}^3$, S^3 or $S^2\times S^1$ since this integral is zero for C large enough. In the general case, we apply the following lemma which is a variant of Lemma 2.6 in [9].

Lemma 3.A.4. *There exists $C>0$ such that, if for any $j=1,2,3$, $C\mu_{k_j}\leq \mu_{k_4}$, then for every $p>0$, there exists $C_p>0$ such that for every $w_j\in L^2(M)$, $j=1,2,3,4$*

$$\int_M \Pi_{k_1}w_1\Pi_{k_2}w_2\left[(-\Delta)^{\frac{\varepsilon}{2}}\Pi_{k_3}w_3\ \overline{\Pi_{k_4}w_4}-\Pi_{k_3}w_3\ (-\Delta)^{\frac{\varepsilon}{2}}\overline{\Pi_{k_4}w_4}\right]\leq C_p\mu_{k_4}^{-p}\prod_{j=1}^4\|w_j\|_{L^2}$$

where Π_k denotes the orthogonal projection on the eigenfunction e_k associated to the eigenvalue μ_k.

This ends the proof of the fist statement of Lemma 3.A.3. The second one is obtained by duality. □

3.A.3 S^3 and $S^2\times S^1$ fulfill Assumption 3.1.5

Lemma 3.A.5. *Assumption 3.1.5 holds true with any $s_0>1/2$ on S^3 and any $s_0>3/4$ on $S^2\times S^1$.*

Démonstration. We first treat the case of S^3 and follow the scheme of Proposition 3 of [23]. We write

$$f_j=\sum_{n_j}H^{(j)}_{n_j},$$

where $H^{(j)}_{n_j}$ are spherical harmonics of degree n_j, and where the sum on n_j bears on the domain

$$N_j\leq\sqrt{1+n_j(n_j+2)}<2N_j. \tag{1.56}$$

Then, the solution u_j are given by

$$u_j(t)=e^{it\Delta}f_j=\sum_{n_j}e^{-itn_j(n_j+2)}H^{(j)}_{n_j}$$

and we have to estimate

$$\begin{aligned}Q(f_1,\cdots,f_4,\tau)&=\int_{\mathbb{R}}\int_{S^3}\chi(t)e^{it\tau}u_1u_2\left[(-\Delta)^{\varepsilon/2}u_3\overline{u_4}-u_3(-\Delta)^{\varepsilon/2}\overline{u_4}\right]dxdt\\&=\sum_{n_1,\cdots,n_4}\widehat{\chi}(\sum_{j=1}^4\varepsilon_jn_j(n_j+2)-\tau)I(H^{(1)}_{n_1},\cdots,H^{(4)}_{n_4}),\end{aligned}$$

with $\varepsilon_j=-1$ or 1 depending on the position of conjugates and

$$I(H^{(1)}_{n_1},\cdots,H^{(4)}_{n_4})=(\sqrt{n_3(n_3+2)}^{\varepsilon}-\sqrt{n_4(n_4+2)}^{\varepsilon})\int_{S^3}H^{(1)}_{n_1}H^{(2)}_{n_2}H^{(3)}_{n_3}\overline{H}^{(4)}_{n_4}dx$$

We notice that $\int H_{n_1} H_{n_2} H_{n_3} \overline{H}_{n_4} \neq 0$ implies $n_4 \leq n_1 + n_2 + n_3$ and $n_3 \leq n_1 + n_2 + n_4$, that is $|n_4 - n_3| \leq n_1 + n_2$. Then, using Lemma 3.A.1 and fundamental theorem of calculus, we have

$$\begin{aligned}\left| \sqrt{n_3(n_3+2)}^{\varepsilon} - \sqrt{n_4(n_4+2)}^{\varepsilon} \right| &\leq \left| \sqrt{n_3(n_3+2)} - \sqrt{n_4(n_4+2)} \right|^{\varepsilon} \\ &\leq C\,|n_4 - n_3|^{\varepsilon} \leq C(N_1^{\varepsilon} + N_2^{\varepsilon}). \end{aligned} \tag{1.57}$$

Moreover, bilinear eigenfunctions estimates (see Theorem 2 of [13] or Theorem 2.5 of [12]) yield

$$\begin{aligned}\left| I(H_{n_1}^{(1)}, \cdots, H_{n_4}^{(4)}) \right| &\leq C(N_1^{\varepsilon} + N_2^{\varepsilon}) \left| \int_{S^3} H_{n_1}^{(1)} H_{n_2}^{(2)} H_{n_3}^{(3)} \overline{H}_{n_4}^{(4)} dx \right| \\ &\leq C(N_1^{\varepsilon} + N_2^{\varepsilon}) m(N_1, \cdots, N_4)^{1/2+} \prod_{j=1}^{4} \left\| H_{n_j}^{(j)} \right\|_{L^2}.\end{aligned}$$

Using the fast decay of $\widehat{\chi}$ at infinity, we infer

$$\begin{aligned}|Q(f_1, \cdots, f_4, \tau)| &\leq C(N_1^{\varepsilon} + N_2^{\varepsilon}) m(N_1, \cdots, N_4)^{\frac{1}{2}+} \sum_{l \in \mathbb{Z}} (1 + |l|^2)^{-1} \sum_{\Lambda([\tau]+l)} \prod_{j=1}^{4} \left\| H_{n_j}^{(j)} \right\|_{L^2} \\ &\leq C(N_1^{\varepsilon} + N_2^{\varepsilon}) m(N_1, \cdots, N_4)^{1/2+} \sup_{k \in \mathbb{Z}} \sum_{\Lambda(k)} \prod_{j=1}^{4} \left\| H_{n_j}^{(j)} \right\|_{L^2}\end{aligned}$$

where $\Lambda(k)$ denotes the set of $(n_1, \cdots, n_4)$ satisfying (1.56) for $j = 1, 2, 3, 4$ and

$$\sum_{j=1}^{4} \varepsilon_j n_j (n_j + 2) = k.$$

Now, we write

$$\{1, 2, 3, 4\} = \{\alpha, \beta, \gamma, \delta\}$$

with $m(N_1, \cdots, N_4) = N_\alpha N_\beta$ and we split the sum on $\Lambda(k)$ as

$$|Q(f_1, \cdots, f_4, \tau)| \leq C(N_1^{\varepsilon} + N_2^{\varepsilon}) m(N_1, \cdots, N_4)^{1/2+} \sup_{k \in \mathbb{Z}} \sum_{a \in \mathbb{Z}} S(a) S'(k - a)$$

where

$$S(a) = \sum_{\Gamma(a)} \left\| H_{n_\alpha}^{(\alpha)} \right\|_{L^2} \left\| H_{n_\gamma}^{(\gamma)} \right\|_{L^2}; \quad S'(a') = \sum_{\Gamma'(a')} \left\| H_{n_\beta}^{(\beta)} \right\|_{L^2} \left\| H_{n_\delta}^{(\delta)} \right\|_{L^2},$$

$$\Gamma(a) = \{(n_\alpha, n_\gamma) : (1.56) \text{ holds for } j = \alpha, \gamma, \sum_{j=\alpha,\gamma} \varepsilon_j n_j (n_j + 2) = a\},$$

$$\Gamma'(a') = \{(n_\beta, n_\delta) : (1.56) \text{ holds for } j = \beta, \delta, \sum_{j=\beta,\delta} \varepsilon_j n_j (n_j + 2) = a'\}.$$

Then, we use a number theoretic result involving the ring of Gauss integers (see Lemma 3.2 of [9]).

Lemma 3.A.6. *Let $\sigma \in \{\pm 1\}$. For every $\eta > 0$, there exists C_η such that, given $M \in \mathbb{Z}$ and a positive integer N,*

$$\#\{(k_1, k_2) \in \mathbb{N}^2 : N \leq k_1 \leq 2N, k_1^2 + \sigma k_2^2 = M\} \leq C_\eta N^\eta.$$

Noticing that $n_j(n_j + 2) = (n_j + 1)^2 - 1$, we get

$$\sup_a \#\Gamma(a) \leq C_\eta N_\alpha^\eta; \quad \sup_{a'} \#\Gamma'(a') \leq C_\eta N_\beta^\eta,$$

and consequently, by the Cauchy-Schwarz inequality and the orthogonality of the $H_{n_j}^{(j)}$

$$\sum_{a \in \mathbb{Z}} S(a) S'(k - a) \leq C_\eta (N_\alpha N_\beta)^{\eta/2} \times$$
$$\left(\sum_a \sum_{\Gamma(a)} \|H_{n_\alpha}^{(\alpha)}\|_{L^2}^2 \left\|H_{n_\gamma}^{(\gamma)}\right\|_{L^2}^2 \right)^{1/2} \left(\sum_a \sum_{\Gamma'(k-a)} \left\|H_{n_\beta}^{(\beta)}\right\|_{L^2}^2 \|H_{n_\delta}^{(\delta)}\|_{L^2}^2 \right)^{1/2}$$
$$\leq C_\eta (N_\alpha N_\beta)^{\eta/2} \prod_{j=1}^4 \|f_j\|_{L^2}.$$

This completes the proof for S^3.

For $S^2 \times S^1$, we adapt this argument with some slight modifications. First, the formulae should be changed to

$$u_j(t)(x, y) = e^{it\Delta} f_j = \sum_{n_j, p_j} e^{-itn_j(n_j+1) - ip_j^2 t} H_{n_j, p_j}^{(j)}(x) e^{ip_j y}$$

where $H_{n_j,p_j}^{(j)}$ are spherical harmonics on S^2 of degree n_j. Estimate (1.57) becomes

$$\begin{aligned}
&\left| \sqrt{n_3(n_3 + 1) + p_3^2}^{\varepsilon} - \sqrt{n_4(n_4 + 2) + p_4^2}^{\varepsilon} \right| \\
&\leq \left| \sqrt{n_3(n_3 + 1) + p_3^2} - \sqrt{n_4(n_4 + 1) + p_4^2} \right|^\varepsilon \\
&\leq \left| \left[\sqrt{n_3(n_3 + 1)} - \sqrt{n_4(n_4 + 1)} \right]^2 + (p_3 - p_4)^2 \right|^{\varepsilon/2} \\
&\leq \left| C(n_3 - n_4)^2 + (p_3 - p_4)^2 \right|^{\varepsilon/2} \\
&\leq C \left| (n_1 + n_2)^2 + (p_1 + p_2)^2 \right|^{\varepsilon/2} \leq C(N_1^\varepsilon + N_2^\varepsilon)
\end{aligned}$$

where we have used $|n_3 - n_4| \leq |n_1 + n_2|$ and $|p_3 - p_4| \leq |p_1| + |p_2|$ for the integral to be non zero. Bilinear eigenfunctions estimates for S^2 yield

$$\left| I(H_{n_1,p_1}^{(1)}, \cdots, H_{n_4,p_4}^{(4)}) \right| \leq C(N_1^\varepsilon + N_2^\varepsilon) m(N_1, \cdots, N_4)^{1/4} \prod_{j=1}^4 \left\| H_{n_j,p_j}^{(j)} \right\|_{L^2}.$$

We finish the proof similarly, replacing the formula for $\Gamma(a)$ by

$$\begin{aligned}
\Gamma(a) = \quad & \{(n_\alpha, p_\alpha, n_\gamma, p_\gamma) : N_j \leq \sqrt{1 + n_j(n_j + 2) + p_j^2} \leq 2N_j, j = \alpha, \gamma \\
& \text{and} \sum_{j=\alpha,\gamma} \varepsilon_j [n_j(n_j + 2) + p_j^2] = a\}.
\end{aligned}$$

In that case, the same number theoretic arguments yield $\sup_a \#\Gamma(a) \leq C_\eta N_\alpha^{1+\eta}$ and finally, after Cauchy-Schwarz inequality, we obtain

$$|Q(f_1, \cdots, f_4, \tau)| \leq C(N_1^\varepsilon + N_2^\varepsilon) m(N_1, \cdots, N_4)^{1/4+(1+\eta)/2} \prod_{j=1}^{4} \|f_j\|_{L^2}.$$

□

3.B Unique continuation

3.B.1 Carleman estimates

This section is only a variant in the Riemannian setting of some results of A. Mercado, A. Osses and L. Rosier [36]. We follow their proof very closely, sometimes line by line. For sake of simplicity, we will assume that u is supported in a fixed compact K of a Riemannian manifold Ω. Yet, the same reasonning as in [36] would allow to handle the case of Dirichlet boundary conditions for u. We have changed the notation of the manifold from M to Ω because the Carleman estimates will not be used on the whole compact manifold M but only on some open set Ω.
D denotes the Levi-Civita connection associated to the metric g. Then, it is torsion-free and the Hessian of the functions are symmetrics.
$\cdot\quad$, $|\quad|$, ∇ and Δ denote the scalar product, the norm, the gradient and the Laplacian with respect to the metric g. Moreover, the scalar product will be the real one : if $X = a + ib$ and $Y = c + id$, $X.Y = a \cdot c - b \cdot d + i(b \cdot c + a \cdot d)$ and $|X|^2 = X \cdot \overline{X}$.
v_g denotes the Riemannian volume form and all the integrals are defined with this (even if it will be often omitted).
First, we list a few formulae that will be used along the proof. For any functions $f, h \in C^\infty(\Omega)$ with h compactly supported and any vector fields X, Y and Z, we have

$$\begin{aligned}
D_Z(X \cdot Y) &= (D_Z X) \cdot Y + X \cdot (D_Z Y) \\
\nabla f \cdot Z &= D_Z f \\
(D_X \nabla f) \cdot Y &= Hess(f)(X, Y) \\
\int_\Omega (\Delta f) h \, dv_g &= -\int_\Omega \nabla f \cdot \nabla h \, dv_g \\
\nabla(fh) &= (\nabla f)h + f(\nabla h) \\
div(fX) &= f div(X) + X \cdot \nabla f.
\end{aligned}$$

For brevity, $\iint$ will denote the integral over $]-T, T[\times\Omega$ and $\iint_\omega$ the integral over $]-T, T[\times\omega$ where ω is an open subset of Ω.

Let $\Psi \in C^4(\Omega)$ real valued . We assume that Ψ satisfies the following properties

$$\nabla\Psi \neq 0 \text{ in } \Omega\setminus\,\omega \tag{2.58}$$
$$\Psi(x) \geq 2/3\,\|\Psi\|_{L^\infty}. \tag{2.59}$$

(2.59) is technical and is easily fulfilled by replacing Ψ by $\Psi + C$ with C large enough. We distinguish two cases : strong pseudoconvexity and weak pseudoconvexity.

The case of strong pseudoconvexity can be found in Isakov[25] but with local in time estimates, it reads

$$Hess(\Psi(x))(\xi,\xi) + |\nabla\Psi(x)\cdot\xi|^2 > 0 \quad \forall (x,\xi)\in T\Omega\setminus T\omega, \tag{2.60}$$

which implies since the support is compact that

$$Hess(\Psi(x))(\xi,\xi) + |\nabla\Psi(x)\cdot\xi|^2 > C\,|\xi|^2 \quad \forall (x,\xi)\in T\Omega\setminus T\omega,\ x\in K \tag{2.61}$$

Weak pseudoconvexity is defined by

$$Hess(\Psi(x))(\xi,\xi) + |\nabla\Psi(x)\cdot\xi|^2 \geq 0 \quad \forall (x,\xi)\in T\Omega\setminus T\omega. \tag{2.62}$$

Set $C_\Psi = 2\,\|\Psi\|_{L^\infty(\Omega)}$ and

$$\theta(t,x) := \frac{e^{\lambda\Psi(x)}}{(T-t)(T+t)}, \quad \varphi(t,x) := \frac{e^{\lambda C_\Psi} - e^{\lambda\Psi(x)}}{(T-t)(T+t)}, \quad \forall (t,x)\in]-T,T[\times\Omega$$

Denote by $L(q) = i\partial_t q + \Delta q$ the linear Schrödinger operator.

Proposition 3.B.1. *Let $T>0$. Let Ω be a Riemannian manifold and K a compact subset of Ω. Assume that there exists a function $\Psi\in C^4(\Omega)$ such that (2.58), (2.59) and (2.61) hold for some open set $\omega\subset\Omega$. Then, there exist constants λ_0, s_0 and C such that for all $\lambda\geq\lambda_0$, all $s\geq s_0$ and $q\in L^2(]-T,T[,H^1(\Omega))$, supported in K, with $L(q)\in L^2(]-T,T[\times\Omega)$ we have*

$$\begin{aligned}&\iint \left[s^3\lambda^4\theta^3|q|^2 + s\lambda\theta\,|\nabla q|^2\right]e^{-2s\varphi} \\ \leq\ & C\iint |L(q)|^2\,e^{-2s\varphi} + C\iint_\omega \left[s^3\lambda^4\theta^3|q|^2 + s\lambda\theta\,|\nabla q|^2\right]e^{-2s\varphi}.\end{aligned} \tag{2.63}$$

Proposition 3.B.2. *If in Proposition 3.B.1, we replace Assumption (2.61) by (2.62), we obtain the same result with*

$$\begin{aligned}&\iint \left[s^3\lambda^4\theta^3|q|^2 + s\lambda^2\theta\,|\nabla\Psi\cdot\nabla q|^2\right]e^{-2s\varphi} \\ \leq\ & C\iint |L(q)|^2\,e^{-2s\varphi} + C\iint_\omega \left[s^3\lambda^4\theta^3|q|^2 + s\lambda\theta\,|\nabla q|^2\right]e^{-2s\varphi}.\end{aligned} \tag{2.64}$$

Démonstration. Using regularisation in a standard way, we are reduced to consider $q\in C^\infty(]-T,T[\times\Omega)$. Denote $u = e^{-s\varphi}q$ and $w = e^{-s\varphi}L(q) = e^{-s\varphi}L(e^{s\varphi}u)$. We notice that u and all its time derivatives vanish at $t=-T$ and $t=T$. Thus, all the integrations by part in time do not create any boundary term. We compute

$$w = Pu = iu_t + is\varphi_t u + \Delta u + 2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u + s^2|\nabla\varphi|^2 u$$

We decompose $P = P_1 + P_2$ with

$$\begin{aligned}P_1 u &:= is\varphi_t u + 2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u\\ P_2 u &:= iu_t + \Delta u + s^2|\nabla\varphi|^2 u\end{aligned}$$

$$\|w\|^2_{L^2(-T,T[\times\Omega)} = \|P_1u + P_2u\|^2 = \|P_1u\|^2 + \|P_2u\|^2 + 2\Re(P_1u, P_2u).$$

As usual in Carleman estimates, we only use

$$2\Re(P_1u, P_2u) \leq \|w\|^2_{L^2(-T,T[\times\Omega)}\,.$$

We also decompose $2\Re(P_1u, P_2u) = I_1 + I_2 + I_3$ with

$$\begin{aligned}
I_1 &:= 2\Re\iint(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u)(-i\overline{u}_t + \Delta\overline{u} + s^2|\nabla\varphi|^2\overline{u}))\\
I_2 &:= 2\Re\iint is\varphi_t u(-i\overline{u}_t + \Delta\overline{u})\\
I_3 &:= 2\Re\iint is\varphi_t u(s^2|\nabla\varphi|^2\overline{u}) = 0.
\end{aligned}$$

We first deal with I_1.

$$\begin{aligned}
I_1 &= 2\Re\iint(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u)((\Delta\overline{u} + s^2|\nabla\varphi|^2\overline{u})\\
&\quad -2\Re\iint i(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u)\overline{u}_t\\
&= I_1^1 + I_1^2.
\end{aligned}$$

Set $J = \iint(\nabla\varphi\cdot\nabla u)\Delta\overline{u} = -\iint\nabla\overline{u}\cdot\nabla(\nabla\varphi\cdot\nabla u))$. We have

$$\begin{aligned}
\nabla\overline{u}\cdot\nabla(\nabla\varphi\cdot\nabla u)) &= D_{\nabla\overline{u}}(\nabla\varphi\cdot\nabla u) = (D_{\nabla\overline{u}}\nabla\varphi)\cdot\nabla u + \nabla\varphi\cdot(D_{\nabla\overline{u}}\nabla u)\\
&= Hess(\varphi)(\nabla u, \nabla\overline{u}) + Hess(u)(\nabla\overline{u}, \nabla\varphi).
\end{aligned}$$

Actually

$$\begin{aligned}
\nabla\varphi\cdot\nabla|\nabla u|^2 &= D_{\nabla\varphi}(\nabla u\cdot\nabla\overline{u}) = (D_{\nabla\varphi}\nabla u)\cdot\nabla\overline{u} + \nabla u\cdot(D_{\nabla\varphi}\nabla\overline{u})\\
&= 2\Re(D_{\nabla\varphi}\nabla u)\cdot\nabla\overline{u} = 2\Re Hess(u)(\nabla\varphi, \nabla\overline{u}).
\end{aligned}$$

Therefore,

$$2\Re J = -2\iint Hess(\varphi)(\nabla u, \nabla\overline{u}) + \iint\Delta\varphi\,|\nabla u|^2\,.$$

Expanding I_1^1, we obtain

$$\begin{aligned}
I_1^1 &= 2\Re\left\{2sJ + \iint s(\Delta\varphi)u\Delta\overline{u} + \iint 2s^3(\nabla\varphi\cdot\nabla u)|\nabla\varphi|^2\overline{u} + \iint s^3(\Delta\varphi)|u|^2|\nabla\varphi|^2\right\}\\
&= 4s\Re J - 2s\Re\iint((\nabla\Delta\varphi)u + \Delta\varphi\nabla u)\cdot\nabla\overline{u}\\
&\quad + \iint 2s^3|\nabla\varphi|^2\nabla\varphi\cdot\nabla|u|^2) + 2\iint s^3(\Delta\varphi)|u|^2|\nabla\varphi|^2.
\end{aligned}$$

Where we have used $\nabla|u|^2 = 2\Re(\overline{u}\nabla u)$. Then, we remark that

$$\begin{aligned}
-2s\Re\iint(\nabla\Delta\varphi)u\cdot\nabla\overline{u} &= -s\iint(\nabla\Delta\varphi)\cdot\nabla|u|^2\\
&= s\iint(\Delta^2\varphi)|u|^2,
\end{aligned}$$

$$2\iint s^3(\Delta\varphi)|u|^2|\nabla\varphi|^2 = -2s^3\iint \nabla\varphi\cdot(|\nabla\varphi|^2\nabla|u|^2 + |u|^2\nabla|\nabla\varphi|^2).$$

We simplify

$$\begin{aligned}
I_1^1 &= -4s\Re\iint Hess(\varphi)(\nabla u,\nabla\overline{u}) + 2s\iint \Delta\varphi\,|\nabla u|^2 \\
&\quad + s\iint(\Delta^2\varphi)|u|^2 - 2s\iint \Delta\varphi|\nabla u|^2 - 2s^3\iint |u|^2\nabla\varphi\cdot\nabla|\nabla\varphi|^2 \\
&= -4s\iint Hess(\varphi)(\nabla u,\nabla\overline{u}) + s\iint(\Delta^2\varphi)|u|^2 - 2s^3\iint(\nabla\varphi\cdot\nabla|\nabla\varphi|^2)|u|^2.
\end{aligned}$$

Expanding $2\Re a = a+\overline{a}$ for I_1^2 and performing integration by part in t for the first term, we get

$$\begin{aligned}
-I_1^2 &= \iint i(2s\nabla\varphi\cdot\nabla u + s(\Delta\varphi)u)\overline{u}_t - i\iint(2s\nabla\varphi\cdot\nabla\overline{u} + s(\Delta\varphi)\overline{u})u_t \\
&= \iint -i\left[2s\nabla\varphi_t\cdot\nabla u + 2s\nabla\varphi\cdot\nabla u_t + s(\Delta\varphi_t)u + s(\Delta\varphi)u_t\right]\overline{u} \\
&\quad - i\iint 2s(\nabla\varphi\cdot\nabla\overline{u})u_t - i\iint s(\Delta\varphi)\overline{u}u_t.
\end{aligned}$$

Integration by part in x yields

$$-i\iint 2s(\nabla\varphi\cdot\nabla\overline{u})u_t = 2is\iint(\Delta\varphi)\overline{u}u_t + 2is\iint(\nabla\varphi\cdot\nabla u_t)\overline{u}.$$

As a consequence

$$\begin{aligned}
-I_1^2 &= \iint -i2s(\nabla\varphi_t\cdot\nabla u)\overline{u} - is\iint(\Delta\varphi_t)|u|^2 \\
&= \iint -i2s(\nabla\varphi_t\cdot\nabla u)\overline{u} + is\iint \nabla\varphi_t\cdot\nabla|u|^2 \\
&= i\iint s\nabla\varphi_t\cdot(u\nabla\overline{u} - \overline{u}\nabla u) = 2s\Re i\iint \nabla\varphi_t\cdot(u\nabla\overline{u}).
\end{aligned}$$

Finally,

$$\begin{aligned}
I_1 &= -4s\Re\iint Hess(\varphi)(\nabla u,\nabla\overline{u}) + s\iint(\Delta^2\varphi)|u|^2 \\
&\quad - 2s^3\iint \nabla\varphi\cdot\nabla|\nabla\varphi|^2|u|^2 - 2s\Re i\iint \nabla\varphi_t\cdot(u\nabla\overline{u})
\end{aligned}$$

On the other hand, we have

$$\nabla\varphi\cdot\nabla|\nabla\varphi|^2 = D_{\nabla\varphi}(\nabla\varphi\cdot\nabla\varphi) = 2D_{\nabla\varphi}\nabla\varphi\cdot\nabla\varphi = 2Hess(\varphi)(\nabla\varphi,\nabla\varphi).$$

We now turn to the other term I_2 :

$$\begin{aligned}
I_2 &= 2\Re\iint is\varphi_t u(-i\overline{u}_t + \Delta\overline{u}) = s\iint \varphi_t\partial_t|u|^2 + 2s\Re i\iint \varphi_t u\Delta\overline{u} \\
&= -s\iint \varphi_{tt}|u|^2 - 2s\Re i\iint(\nabla\varphi_t u + \varphi_t\nabla u)\cdot\nabla\overline{u} \\
&= -s\iint \varphi_{tt}|u|^2 - 2s\Re\iint i(\nabla\varphi_t\cdot\nabla\overline{u})u.
\end{aligned}$$

Consequently, our final result is

$$\begin{aligned}
2\Re(M_1u, M_2u) &= \iint \left[-4s^3 Hess(\varphi)(\nabla\varphi, \nabla\varphi) - s\varphi_{tt} + s(\Delta^2\varphi)\right]|u|^2 && (2.65)\\
&\quad -4s\Re \iint Hess(\varphi)(\nabla u, \nabla \overline{u}) && (2.66)\\
&\quad -4s\Re \iint iu\nabla\varphi_t \cdot \nabla\overline{u}. && (2.67)
\end{aligned}$$

(2.65) and (2.66) are the main parts in $|u|^2$ and $|\nabla u|^2$ respectively. (2.67) is a remainder term that will be estimated from above.

In what follows, $\varepsilon > 0$ denote small constants (used in estimates from below) and C large ones (used for estimates from above). We observe the following indentities, that will be used along the proof,

$$\nabla\varphi = -\lambda\theta\nabla\Psi,$$

$$\begin{aligned}
Hess(\varphi)(X, Y) &= (D_X\nabla\varphi)\cdot Y\\
&= -\lambda D_X(\theta\nabla\Psi)\cdot Y = -\lambda\theta(D_X\nabla\Psi)\cdot Y - \lambda d\theta(X)\nabla\Psi\cdot Y\\
&= -\lambda\theta Hess(\Psi)(X,Y) - \lambda^2\theta(\nabla\Psi\cdot X)(\nabla\Psi\cdot Y)\\
&= -\theta\lambda\left[Hess(\Psi)(X,Y) + \lambda(\nabla\Psi\cdot X)(\nabla\Psi\cdot Y)\right].
\end{aligned}$$

Firstly, we estimate term (2.67),

$$\begin{aligned}
|(2.67)| &\leq Cs\iint |\nabla\varphi_t\cdot\nabla u||u| \leq Cs\iint \frac{t\lambda e^{\lambda\Psi}}{(T^2-t^2)^2}|\nabla\Psi\cdot\nabla u||u|\\
&\leq Cs\iint \frac{e^{\lambda\Psi}}{(T^2-t^2)}|\nabla\Psi\cdot\nabla u|^2 + Cs\iint\frac{(T\lambda)^2e^{\lambda\Psi}}{(T^2-t^2)^3}|u|^2\\
&\leq Cs\iint \theta\left|\nabla\Psi\cdot\nabla u\right|^2 + Cs\lambda^{-1}\iint\left|\nabla\varphi\right|^3|u|^2 + Cs\iint_\omega \lambda^2\theta^3|u|^2. \quad (2.68)
\end{aligned}$$

Then, we estimate term (2.65) using Assumptions (2.58) and (2.62) (or (2.61)). On $(\Omega\backslash\omega)\cap K$, we have

$$\begin{aligned}
-4s^3Hess(\varphi)(\nabla\varphi,\nabla\varphi) &= 4s^3\lambda\theta\left[Hess(\Psi)(\nabla\varphi,\nabla\varphi) + \lambda\left|\nabla\Psi\cdot\nabla\varphi\right|^2\right]\\
&\geq 4s^3\lambda\theta(\lambda-1)\left|\nabla\Psi\cdot\nabla\varphi\right|^2 \geq s^3\lambda^4\theta^3\left|\nabla\Psi\right|^4 \geq \varepsilon s^3\lambda\left|\nabla\varphi\right|^3.
\end{aligned}$$

Assumption (2.59) gives $\Psi(x) \leq C_\Psi \leq 3\Psi(x)$ and then, we have on $(\Omega\backslash\omega)\cap K$

$$|s\varphi_{tt}| \leq Cs\frac{e^{\lambda C_\Psi}}{((T^2-t^2))^3} \leq Cs\frac{e^{3\lambda\Psi(x)}}{((T^2-t^2))^3} \leq Cs\left|\nabla\varphi\right|^3.$$

Moreover, on $(\Omega\backslash\omega)\cap K$ we have

$$\left|s\Delta^2\varphi\right| \leq Cs\theta\lambda^4 \leq Cs\lambda\left|\nabla\varphi\right|^3.$$

Finally, for λ and s large enough

$$\iint_{\Omega\backslash\omega}\left[-4s^3Hess(\varphi)(\nabla\varphi,\nabla\varphi) - s\varphi_{tt} + s(\Delta^2\varphi)\right]|u|^2 \geq \iint_{\Omega\backslash\omega}\varepsilon s^3\lambda\left|\nabla\varphi\right|^3|u|^2.$$

For the domain ω, we have the estimate

$$\left|\iint_{\omega}\left[-4s^3Hess(\varphi)(\nabla\varphi,\nabla\varphi)-s\varphi_{tt}+s(\Delta^2\varphi)\right]|u|^2\right|\leq C\iint_{\omega}s^3\lambda^4\theta^3|u|^2.$$

The final estimate for (2.65) is

$$(2.65)\geq\iint_{\Omega\backslash\omega}\varepsilon s^3\lambda\left|\nabla\varphi\right|^3|u|^2-C\iint_{\omega}s^3\lambda^4\theta^3|u|^2. \tag{2.69}$$

Now, let us estimate (2.66). We begin with the integral on ω.

$$\begin{aligned}-4s\Re\iint_{\omega}Hess(\varphi)(\nabla u,\nabla\overline{u}) &= 4s\Re\iint_{\omega}\theta\lambda\left[Hess(\Psi)(\nabla u,\nabla\overline{u})+\lambda\left|\nabla\Psi\cdot\nabla u\right|^2\right]\\ &\geq -Cs\lambda\iint_{\omega}\theta\left|\nabla u\right|^2+4s\iint_{\omega}\theta\lambda^2\left|\nabla\Psi\cdot\nabla u\right|^2\\ &\geq -Cs\lambda\iint_{\omega}\theta\left|\nabla u\right|^2\end{aligned}$$

Now, for the integral on $\Omega\backslash\omega$, we distinguish the two cases described above :

Strong pseudoconvexity : end of the proof of Proposition 3.B.1
Using assumption (2.61), we can estimate the part of (2.66) on $\Omega\backslash\omega$ by

$$\begin{aligned}-4s\Re\iint_{\Omega\backslash\omega}Hess(\varphi)(\nabla u,\nabla\overline{u}) &= 4s\Re\iint_{\Omega\backslash\omega}\theta\lambda\left[Hess(\Psi)(\nabla u,\nabla\overline{u})+\lambda\left|\nabla\Psi\cdot\nabla u\right|^2\right]\\ &\geq \varepsilon s\lambda\iint_{\Omega\backslash\omega}\theta\left|\nabla u\right|^2.\end{aligned}$$

The final estimate for (2.66) is

$$(2.66)\geq\varepsilon s\lambda\iint_{\Omega\backslash\omega}\theta\left|\nabla u\right|^2-Cs\lambda\iint_{\omega}\theta\left|\nabla u\right|^2. \tag{2.70}$$

Putting together (2.68), (2.69) and (2.70), we get for s, λ large enough

$$\begin{aligned}(2.65)+(2.66)+(2.67) &\geq \varepsilon\iint_{\Omega\backslash\omega}s^3\lambda\left|\nabla\varphi\right|^3|u|^2-C\iint_{\omega}s^3\lambda^4\theta^3|u|^2-Cs\lambda\iint_{\omega}\theta\left|\nabla u\right|^2\\ &\quad+\varepsilon s\lambda\iint_{\Omega\backslash\omega}\theta\left|\nabla u\right|^2-Cs\iint\theta\left|\nabla\Psi\cdot\nabla u\right|^2\\ &\quad-Cs\lambda^{-1}\iint\left|\nabla\varphi\right|^3|u|^2-Cs\iint_{\omega}\lambda^2\theta^3|u|^2\\ &\geq \varepsilon\iint s^3\lambda^4\theta^3|u|^2+\varepsilon s\lambda\iint\theta\left|\nabla u\right|^2\\ &\quad-C\iint_{\omega}s^3\lambda^4\theta^3|u|^2-Cs\lambda\iint_{\omega}\theta\left|\nabla u\right|^2\end{aligned} \tag{2.71}$$

where we have used the decomposition $\iint_{\Omega\backslash\omega}=\iint-\iint_{\omega}$ for the second inequality.
Replacing u by $e^{-s\varphi}q$ and computing $\nabla q=e^{s\varphi}\left[\nabla u-s\lambda\theta u\nabla\Psi\right]$ we have, after absorption

$$\begin{aligned}&\iint\left[s^3\lambda^4\theta^3|q|^2+s\lambda\theta\left|\nabla q\right|^2\right]e^{-2s\varphi}\\ &\leq C\iint\left[s^3\lambda^4\theta^3|u|^2+s\lambda\theta|\nabla u|^2+s^3\lambda^3\theta^3|\nabla\psi|^2|u|^2\right]\\ &\leq C\iint\left[s^3\lambda^4\theta^3|u|^2+s\lambda\theta|\nabla u|^2\right]\end{aligned} \tag{2.72}$$

$$\begin{aligned}&\iint_\omega s^3\lambda^4\theta^3|u|^2+s\lambda\iint_\omega \theta\,|\nabla u|^2\\&\le C\iint_\omega\left[s^3\lambda^4\theta^3|q|^2+s\lambda\theta\,|\nabla q|^2+s^3\lambda^3\theta^3|\nabla\psi|^2|q|^2\right]e^{-2s\varphi}\\&\le C\iint_\omega\left[s^3\lambda^4\theta^3|q|^2+s\lambda\theta\,|\nabla q|^2\right]e^{-2s\varphi}.\end{aligned}\tag{2.73}$$

Combining (2.71), (2.72) and (2.73), we get the expected result :

$$\begin{aligned}&\iint\left[s^3\lambda^4\theta^3|q|^2+s\lambda\theta\,|\nabla q|^2\right]e^{-2s\varphi}\\ \le\ & C\iint|i\partial_t q+\Delta q|^2\,e^{-2s\varphi}+C\iint_\omega\left[s^3\lambda^4\theta^3|q|^2+s\lambda\theta\,|\nabla q|^2\right]e^{-2s\varphi}.\end{aligned}$$

Weak pseudoconvexity : end of the proof of Proposition 3.B.2
Assumption (2.62) yields that for λ large enough

$$-4s\Re\iint_{\Omega\backslash\omega}Hess(\varphi)(\nabla u,\nabla\overline{u})\ge\varepsilon s\iint_{\Omega\backslash\omega}\theta\lambda^2\,|\nabla\Psi\cdot\nabla u|^2$$

We finish the proof similarly to get

$$\begin{aligned}(2.65)+(2.66)+(2.67)\ \ge\ &\iint_{\Omega\backslash\omega}\varepsilon s^3\lambda\,|\nabla\varphi|^3\,|u|^2-C\iint_\omega s^3\lambda^4\theta^3|u|^2-Cs\lambda\iint_\omega\theta\,|\nabla u|^2\\&+\varepsilon s\iint_{\Omega\backslash\omega}\theta\lambda^2\,|\nabla\Psi\cdot\nabla u|^2-Cs\iint\theta\,|\nabla\Psi\cdot\nabla u|^2\\&-Cs\lambda^{-1}\iint|\nabla\varphi|^3\,|u|^2-Cs\iint_\omega\lambda^2\theta^3|u|^2\\ \ge\ &\varepsilon\iint s^3\lambda^4\theta^3|u|^2+\varepsilon s\lambda^2\iint\theta\,|\nabla\Psi\cdot\nabla u|^2\\&-C\iint_\omega s^3\lambda^4\theta^3|u|^2-Cs\lambda\iint_\omega\theta\,|\nabla u|^2\end{aligned}$$

and then

$$\begin{aligned}&\iint\left[s^3\lambda^4\theta^3|q|^2+s\lambda^2\theta\,|\nabla\Psi\cdot\nabla q|^2\right]e^{-2s\varphi}\\ \le\ & C\iint|i\partial_t q+\Delta q|^2\,e^{-2s\varphi}+C\iint_\omega\left[s^3\lambda^4\theta^3|q|^2+s\lambda\theta\,|\nabla q|^2\right]e^{-2s\varphi}.\end{aligned}$$

□

3.B.2 Carleman estimates with potential $L^\infty([-T,T],L^3)$

The following result proves that the strong pseudoconvexity allows to absorb some potential terms in $L^\infty([-T,T],L^3)$. This is in contrast with the weak pseudoconvexity which only absorbs terms in $L^\infty([-T,T]\times\Omega)$.

Proposition 3.B.3. *Assume $dim(\Omega)\le 3$. Let V_1, $V_2\in L^\infty([-T,T],L^3)$. Then, Proposition 3.B.1 holds with L replaced by $L(q)=i\partial_t q+\Delta q+V_1q+V_2\overline{q}$.*

Démonstration. We use the notation of Proposition 3.B.1. We write

$$\begin{aligned}\iint |i\partial_t q+\Delta q|^2 e^{-2s\varphi} &\leq 4\left\|e^{-s\varphi}L(q)\right\|^2_{L^2([0,T],L^2)}+4\left\|e^{-s\varphi}(V_1q)\right\|^2_{L^2([0,T],L^2)}\\ &\quad +4\left\|e^{-s\varphi}(V_2\overline{q})\right\|^2_{L^2([0,T],L^2)}.\end{aligned}$$

But, by Hölder inequality and Sobolev embedding, we have for $s>1$

$$\begin{aligned}\left\|e^{-s\varphi}V_1q\right\|^2_{L^2([0,T],L^2)} &\leq C\left\|V_1\right\|^2_{L^\infty(L^3)}\left\|e^{-s\varphi}q\right\|^2_{L^2(L^6)}\\ &\leq C\left(\left\|e^{-s\varphi}q\right\|^2_{L^2(L^2)}+\left\|\nabla(e^{-s\varphi}q)\right\|^2_{L^2(L^2)}\right)\\ &\leq C\left(\left\|e^{-s\varphi}q\right\|^2_{L^2(L^2)}+\left\|e^{-s\varphi}\nabla q\right\|^2_{L^2(L^2)}\right.\\ &\quad\left.+s^2\lambda^2\left\|\theta(\nabla\Psi)e^{-s\varphi}q\right\|^2_{L^2(L^2)}\right)\\ &\leq C\left(\iint\left[s^2\lambda^2\theta^3|q|^2+\theta\left|\nabla q\right|^2\right]e^{-2s\varphi}\right)\end{aligned}$$

where we have used $\theta\geq C$. We get the desired result using estimate (2.63) of Proposition 3.B.1 for s large enough. □

Remark 3.B.1. *The uniqueness results we will obtain from the former Proposition are not optimal with respect to the regularity of the potential. Indeed, some recent papers (see the work of H. Koch and D. Tataru [27] or D. Dos Santos Ferreira [19]) establish Carleman type estimates in L^p which are much better than what we get. They are more complicated and not required for our purpose. Yet, they would become necessary if we considered nonlinearities $|u|^\alpha u$ with $\alpha>2$.*

3.B.3 Application to uniqueness

Proposition 3.B.4. *Let Ω, T, ω, Ψ fulfilling the same assumptions as Proposition 3.B.1. Let $q\in L^\infty([-T,T],H^1(\Omega))$ compactly supported, solution of $i\partial_t q+\Delta q+V_1q+V_2\overline{q}=0$ with $V_i\in L^\infty([-T,T],L^3)$.*
Let D be an open subset of Ω such that $\widetilde{m}=\inf_{x\in D}\{\Psi(x)\}>\sup_{x\in\omega}\{\Psi(x)\}=m$.
Then, $q=0$ on $]-T,T[\times D$.

Remark 3.B.2. *By considering the maximum of Ψ, we see that the assumptions of Proposition 3.B.4 can not be fulfilled on a compact manifold. Therefore, we will only apply this result on an open set Ω of M, and the compact support of u becomes important.*

Since the previous Carleman estimates hold for every time interval (with constants depending on its length), we are reduced to the following lemma :

Lemma 3.B.1. *Under assumptions of Proposition 3.B.4, there exists one $\eta>0$ such that $q=0$ on $]-\eta,\eta[\times D$.*

Démonstration. Fix $\lambda \geq \lambda_0 > 1$ (the next constants could depend on λ but not on s). Let $T \geq \eta > 0$ to be chosen later. Denote $\lambda_1 = e^{\lambda C_\psi} - e^{\lambda \widetilde{m}}$ and $\lambda_1 + \varepsilon = e^{\lambda C_\psi} - e^{\lambda m}$ with $\lambda_1 > 0$ and $\varepsilon > 0$. By definition of $\widetilde{m}$ and m, we have for $s \geq 0$

$$\begin{aligned} e^{-2s\varphi} &\leq e^{-2s\frac{\lambda_1+\varepsilon}{T^2-t^2}} \quad \forall (t,x) \in]-T,T[\times\omega \\ e^{-2s\frac{\lambda_1}{T^2-\eta^2}} &\leq e^{-2s\varphi} \quad \forall (t,x) \in]-\eta,\eta[\times D \end{aligned}$$

Moreover, once λ_1 and ε are fixed, there exists some constant C such that $y^3 e^{-2(\lambda_1+\varepsilon)y} \leq C e^{-2(\lambda_1+\varepsilon/2)y}$ for $y \geq 0$. Therefore, for every $(t,x) \in]-T,T[\times\Omega$ with $x \in Supp\ q = K$, we have

$$(s\theta)^3 e^{-2s\frac{\lambda_1+\varepsilon}{T^2-t^2}} \leq C \left(\frac{s}{T^2-t^2}\right)^3 e^{-2s\frac{\lambda_1+\varepsilon}{T^2-t^2}} \leq C e^{-2s\frac{\lambda_1+\varepsilon/2}{T^2-t^2}} \leq C e^{-2s\frac{\lambda_1+\varepsilon/2}{T^2}}$$

Here, the constant C does not depend on s. Then, using Carleman estimate and $\theta \geq C > 0$, we get

$$\iint_{]-\eta,\eta[\times D} s^3 |q|^2 e^{-2s\frac{\lambda_1}{T^2-\eta^2}} \leq C \iint_{]-T,T[\times\omega} \left[|q|^2 + |\nabla q|^2\right] e^{-2s\frac{\lambda_1+\varepsilon/2}{T^2}}$$

Therefore,

$$s^3 e^{-2s\frac{\lambda_1}{T^2-\eta^2}} \iint_{]-\eta,\eta[\times D} |q|^2 \leq C e^{-2s\frac{\lambda_1+\varepsilon/2}{T^2}} \|q\|^2_{L^2(H^1)}$$

Then, to finish the proof, we just have to choose η such that $-2\frac{\lambda_1}{T^2-\eta^2} > -2\frac{\lambda_1+\varepsilon/2}{T^2}$, that is $\eta^2 < \frac{T^2\varepsilon/2}{\lambda_1+\varepsilon/2}$ and let s tend to $+\infty$. $\square$

3.B.4 Geometrical examples

We give some geometrical examples where Proposition 3.B.4 applies. Denote $q \in L^\infty([-T,T], H^1(\Omega))$ a solution of $i\partial_t q + \Delta q + V_1 q + V_2 \overline{q} = 0$ with $V_i \in L^\infty([-T,T], L^3)$. In these following cases, Assumptions 3.1.2 and 3.1.4 are fulfilled. For the convenience of the reader, we recall the problem :

Proposition 3.B.5. *Let $(M, \widetilde{\omega})$ be either*
- $(\mathbb{T}^3, \{x \in \mathbb{R}^3/(\theta_1\mathbb{Z} \times \theta_2\mathbb{Z} \times \theta_3\mathbb{Z}) \,|\exists i \in \{1,2,3\}, x_i \in]-\varepsilon,\varepsilon[+\theta_i\mathbb{Z}\})$
- $(S^3, \widetilde{\omega})$ *where $\widetilde{\omega}$ is a neighborhood of $S^3 \cap \{x_4 = 0\}$ in $S^3 \subset \mathbb{R}^4$.*
-$(S^2 \times S^1, (\omega_1 \times S^1) \cup (S^2 \times]0,\varepsilon[))$ *where ω_1 is a neighborhood of the equator of S^2.*
For every $T > 0$, the only solution in $C([0,T], H^1)$ to the system

$$\begin{cases} i\partial_t q + \Delta q + b_1(t,x) q + b_2(t,x)\overline{q} = 0 \text{ on } [0,T] \times M \\ q = 0 \text{ on } [0,T] \times \widetilde{\omega} \end{cases} \tag{2.74}$$

where $b_1(t,x)$ and $b_2(t,x) \in L^\infty([0,T], L^3)$ is the trivial one $q \equiv 0$.

3.B.4.1 $M = \mathbb{T}^3$

We assume $q = 0$ on $\widetilde{\omega} = \{x \in \mathbb{R}^3/(\theta_1\mathbb{Z} \times \theta_2\mathbb{Z} \times \theta_3\mathbb{Z}) \,|\exists i \in \{1,2,3\}, x_i \in]-\varepsilon, \varepsilon[+\theta_i\mathbb{Z}\}$.
We define $\widetilde{q}$ on $\mathbb{R}^3$ by $\widetilde{q}(x) = q(x)$ if $x \in [0,\theta_1] \times [0,\theta_2] \times [0,\theta_3]$ and $\widetilde{q}(x) = 0$ otherwise. $\widetilde{q}$ satisfies the same Schrödinger equation on $\mathbb{R}^3$ with compact support K. By translation, we can assume that 0 is the center of the rectangle.
We use the function $\Psi = \|(x,y,z)\|^2 + C$. C is chosen large enough so that (2.59) is fulfilled on K. Let $\delta > 0$ small. Outside of $\omega = B(0,\delta)$, Ψ is stricly convex (that is strongly pseudoconvex for the flat metric inherited from $\mathbb{R}^3$) and $\nabla\Psi \neq 0$. Then, assumptions (2.58) and (2.61) are fulfilled.
We can apply Theorem 3.B.4 with $\Omega = \mathbb{R}^3$, $\omega = B(0,\delta)$ and $D = B(0,2\delta)^c$. As δ is arbitrary, we get $\widetilde{q} = 0$ everywhere and so $q = 0$.

3.B.4.2 $M = S^3$

Lemma 3.B.2. *Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere. Then, the function $h : (x_1, \cdots, x_{n+1}) \mapsto x_{n+1}$ restricted to $S^n \cap \{x_{n+1} < 0\}$ has stricly positive Hessian for the metric induced by $\mathbb{R}^{n+1}$.*

Démonstration. h defined on $\mathbb{R}^{n+1}$ is linear. Then, using Exercice 2.65 b) of [20], we get $Hess(h) = -hg$ where g is the bilinear form of the Riemannian structure. Then, $Hess(h)$ is positive definite if and only if $h < 0$. □

We assume $q = 0$ on a neighborhood of $x_4 = 0$. Let $\delta > 0$ small. We choose $\Omega = \{x \in S^3 | x_4 < 0\}$, $D = \{x \in S^3 | x_4 \in]-1+2\delta, 0[\}$ and $\omega = S^3 \cap \{x_4 \in [-1, -1+\delta[\}$. We use the function $\Psi = x_4 + C$. C is chosen large enough so that (2.59) is fulfilled on the support of q. On $\Omega \backslash \omega$, Ψ is stricly convex thanks to Lemma 3.B.2 and $\nabla\Psi \neq 0$. Therefore, assumptions (2.58) and (2.61) are fulfilled. As the support of q is compact in Ω, Theorem 3.B.4 applies and we get $q = 0$ on D. Since δ is arbitrary, we get $q = 0$ on $S^3 \cap \{x_4 < 0\}$. The symmetry of the problem gives $q = 0$ on S^3.

3.B.4.3 $M = S^2 \times S^1$

Let $\omega_1 \subset S^2$ be a neighbourhood of the equator $\{x_3 = 0\}$ and $\varepsilon > 0$.
We assume $q = 0$ on $(\omega_1 \times S^1) \bigcup (S^2 \times]-\varepsilon, \varepsilon[)$.
The geometric situation is quite similar to the case of $\mathbb{T}^3$: this is a product of manifolds and the weight function Ψ will be the sum of two pseudoconvex weights in each coordinate.
The current point x of S^2 will be denoted by its coordinates in $\mathbb{R}^3$ and the current point y of $S^1 = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ by its coordinates in $\mathbb{R}$. Then, we can define $\widetilde{q}$ on the open set $\Omega = \{x \in S^2 | x_3 < 0\} \times \mathbb{R}$ by $\widetilde{q}(x,y) = q(x,y)$ if $y \in [0,1]$ and 0 otherwise. $\widetilde{q}$ is then compactly supported and is solution of the same Schrödinger equation.

We choose $\Psi(x,y) = x_3 + y^2 + C$ with C large enough. Ψ is definite positive everywhere and nonsingular everywhere outside of any $\omega = \{(x,y) \in S^2 \times \mathbb{R} | x_3 \in [-1, -1+\delta[\text{ and } y^2 < \delta\}$ for $\delta > 0$. Then, choosing

$$D = \left\{(x,y) \in S^2 \times \mathbb{R} \,\middle|\, x_3 \in]-1+3\delta, 0[\text{ or } y^2 > 3\delta \right\}$$

and applying Theorem 3.B.4 we get $\widetilde{q} = 0$ on D. Therefore, $q = 0$ on $S^2 \times S^1$.

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Chapitre 4

Contrôle et stabilisation de l'équation de Korteweg-de Vries en domaine périodique

Contents

Ce chapitre est la reprise d'un article écrit avec Lionel Rosier et Bing-Yu Zhang. Il est publié dans le journal Communications in Partial Differential Equations [23].

4.1 Introduction

The well-known Korteweg-de Vries (KdV) equation can be written as

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0, \tag{1.1}$$

where $u = u(x,t)$ denotes a real-valued function of two real variables x and t. The equation was first derived by Korteweg and de Vries [20] in 1895 (or by Boussinesq [4] in 1876 [1]) as a model for propagation of some surface water waves along a channel. The KdV equation has been intensively studied from various aspects of both mathematics and physics since the 1960s when solitons were discovered through solving the KdV equation, and the inverse scattering method, a so-called nonlinear Fourier transform, was invented to seek solitons [13, 27]. It turns out that the equation is not only a good model for some water waves but also a very useful approximation model in nonlinear studies

1. The interested readers are refereed to a nice article of de Jager [10] for the origin of the KdV equation.

whenever one wishes to include and balance a weak nonlinearity and weak dispersive effects [27]. In particular, the equation is now commonly accepted as a mathematical model for the unidirectional propagation of small-amplitude long waves in nonlinear dispersive systems.

In this paper, we consider the KdV equation posed on the periodic domain $\mathbb{T}$:

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R}. \tag{1.2}$$

The equation is known to possess an infinite set of conserved integral quantities, two of which are

$$I_1(t) = \int_{\mathbb{T}} u(x,t)dx$$

and

$$I_2(t) = \int_{\mathbb{T}} u^2(x,t)dx.$$

From the historical origins [20, 4, 27] of the KdV equation, involving the behavior of water waves in a shallow channel, it is natural to think of I_1 and I_2 as expressing conservation of volume (or mass) and energy, respectively. The Cauchy problem for the equation (1.2) has been intensively studied for many years (see [45, 41, 17, 18, 47, 3, 19, 7] and the references therein). The best known result so far [16] is that the Cauchy problem is well-posed in the space $H^s(\mathbb{T})$ for any $s \geq -1$:

Let $s \geq -1$ and $T > 0$ be given. For any $u_0 \in H^s(\mathbb{T})$, the equation (1.2) admits a unique solution $u \in C([0,T]; H^s(\mathbb{T}))$ satisfying

$$u(x,0) = u_0(x).$$

Moreover, the corresponding solution map ($u_0 \to u$) is continuous from the space $H^s(\mathbb{T})$ to the space $C([0,T]; H^s(\mathbb{T}))$. [2]

In this paper we will study the equation (1.2) from a control point of view with a forcing term $f = f(x,t)$ added to the equation as a control input :

$$\partial_t u + u\partial_x u + \partial_x^3 u = f, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{1.3}$$

where f is assumed to be supported in a given open set $\omega \subset \mathbb{T}$. The following exact control problem and stabilization problem are fundamental in control theory.

Exact control problem : *Given an initial state u_0 and a terminal state u_1 in a certain space, can one find an appropriate control input f so that the equation (1.3) admits a solution u which satisfies $u(.,0) = u_0$ and $u(.,T) = u_1$?*

Stabilization problem : *Can one find a feedback control law : $f = Ku$ so that the resulting closed-loop system*

$$\partial_t u + u\partial_x u + \partial_x^3 u = Ku, \quad x \in \mathbb{T},\ t \in \mathbb{R}^+$$

is asymptotically stable as $t \to +\infty$?

2. If $s > -\frac{1}{2}$, this solution map is, in fact, analytic.

These problems were first studied by Russell and Zhang for the KdV equation [37, 38, 39] (see also [46] for a unique continuation property of KdV). In their work, in order to keep the *mass* $I_1(t)$ conserved, the control input $f(x,t)$ is chosen to be of the form

$$f(x,t) = (Gh)(x,t) := g(x)\left(h(x,t) - \int_{\mathbb{T}} g(y)h(y,t)dy\right) \tag{1.4}$$

where h is considered as a new control input, and $g(x)$ is a given nonnegative smooth function such that $\{g > 0\} = \omega$ and

$$2\pi[g] = \int_{\mathbb{T}} g(x)dx = 1.$$

For the chosen g, it is easy to see that

$$\frac{d}{dt}\int_{\mathbb{T}} u(x,t)dx = \int_{\mathbb{T}} f(x,t)dx = 0 \qquad \text{for any } t \in \mathbb{R}$$

for any solution $u = u(x,t)$ of the system

$$\partial_t u + u\partial_x u + \partial_x^3 u = Gh; \tag{1.5}$$

thus the *mass* of the system is indeed conserved.

The following results are due to Russell and Zhang [39].

Theorem A : *Let $s \geq 0$ and $T > 0$ be given. There exists a $\delta > 0$ such that for any $u_0, u_1 \in H^s(\mathbb{T})$ with $[u_0] = [u_1]$ satisfying*

$$\|u_0\|_s \leq \delta, \qquad \|u_1\|_s \leq \delta,$$

one can find a control input $h \in L^2(0,T; H^s(\mathbb{T}))$ such that the system (1.5) admits a solution $u \in C([0,T]; H^s(\mathbb{T}))$ satisfying

$$u(x,0) = u_0(x), \qquad u(x,T) = u_1(x).$$

In order to stabilize system (1.5), Russell and Zhang employed a simple feedback control law

$$h(x,t) = -G^* u(x,t). \tag{1.6}$$

The resulting closed-loop system

$$\partial_t u + u\partial_x u + \partial_x^3 u = -GG^* u, \quad x \in \mathbb{T},\ t \in \mathbb{R}. \tag{1.7}$$

is *locally* exponentially stable.

Theorem B : *Let $s = 0$ or $s \geq 1$ be given. There exist positive constants M, δ and γ such that if $u_0 \in H^s(\mathbb{T})$ satisfies*

$$\|u_0 - [u_0]\|_s \leq \delta, \tag{1.8}$$

then the corresponding solution u of (1.7) satisfies

$$\|u(\cdot,t)-[u_0]\|_s \le Me^{-\gamma t}\|u_0-[u_0]\|_s$$

for any $t \ge 0$.

Thus one can always find an appropriate control input h to guide the system (1.5) from a given initial state u_0 to a terminal state u_1 so long as *their amplitudes are small* and $[u_0]=[u_1]$. A question arises naturally.

Question 1 : *Can one still guide the system by choosing an appropriate control input h (defined on a sufficiently long time interval) from a given initial state u_0 to a given terminal state u_1 when u_0 or u_1 have* large *amplitude ?*

According to Theorem B, solutions of system (1.7) issued from initial data close to their means converge at a uniform exponential rate to their means in the space $H^s(\mathbb{T})$ as $t \to \infty$.

One may ask naturally :

Question 2 : *Do any solution of the closed-loop system (1.7) converge exponentially to its mean as $t \to \infty$?*

A further question is :

Question 3 : *For any given number $\lambda > 0$, can we design a linear feedback control law such that the exponential decay rate of the resulting closed-loop system is λ ?*

One of the main results in this paper is a positive answer to Question 1 as given below.

Theorem 4.1.1. *Let $s \ge 0$, $R > 0$, and $\mu \in \mathbb{R}$ be given. There exists a time $T > 0$ such that if $u_0, u_1 \in H^s(\mathbb{T})$ with $[u_0]=[u_1]=\mu$ are such that*

$$\|u_0\|_s \le R, \qquad \|u_1\|_s \le R,$$

then one can find a control input $h \in L^2(0,T;H^s(\mathbb{T}))$ such that the system (1.5) admits a solution $u \in C([0,T];H^s(\mathbb{T}))$ satisfying

$$u(x,0)=u_0(x), \qquad u(x,T)=u_1(x).$$

So the system (1.5) is *globally* exactly controllable.

As for Question 2, we have the following affirmative answer.

Theorem 4.1.2. *Let $s \ge 0$ and $\mu \in \mathbb{R}$ be given. There exists a constant $\kappa > 0$ such that for any $u_0 \in H^s(\mathbb{T})$ with $[u_0]=\mu$, the corresponding solution u of the system (1.7) satisfies*

$$\|u(\cdot,t)-[u_0]\|_s \le \alpha_{s,\mu}(\|u_0-[u_0]\|_0)e^{-\kappa t}\|u_0-[u_0]\|_s \qquad \textit{for all } t \ge 0,$$

where $\alpha_{s,\mu}:\mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function depending on s and μ.

Note that Theorem 4.1.1 follows from Theorem 4.1.2 and a local control result around the state $u(x) = \mu$ (similar to Theorem A) thanks to the time reversibility of the KdV equation.

The decay rate κ in Theorem 4.1.2 has an upper bound

$$\kappa \leq \inf\{-\mathrm{Re}\,\lambda : \ \lambda \in \sigma_p(A_G)\}$$

where A_G is the operator defined by

$$A_G v = -v''' - \mu v' - GG^* v$$

with $\mathcal{D}(A_G) = H^3(\mathbb{T})$ as domain. In order to have the decay rate κ arbitrarily large, a different feedback control law is needed.

Theorem 4.1.3. *Let $\lambda > 0$, $s \geq 0$, and $\mu \in \mathbb{R}$ be given. There exists a number $\delta > 0$ and a linear bounded operator Q_λ from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$ such that if one chooses the feedback control law*

$$h = -Q_\lambda u$$

in system (1.5)-(1.4), then the solution u of the resulting closed-loop system

$$\partial_t u + u\partial_x u + \partial_x^3 u = -GQ_\lambda u, \quad u(x,0) = u_0(x), \ x \in \mathbb{T} \tag{1.9}$$

satisfies

$$\|u(\cdot,t) - [u_0]\|_s \leq Ce^{-\lambda t}\|u_0 - [u_0]\|_s \qquad \text{for all} \ \ t \geq 0,$$

whenever $\|u_0\|_s \leq \delta$ and $[u_0] = \mu$, $C > 0$ denoting a constant independent of u_0.

Note that this is still a *local* stabilization result. However, the feedback laws in Theorems 4.1.2 and 4.1.3 may be combined into a *time-varying* feedback law (as in [9]) ensuring a global stabilization with an arbitrary large decay rate.

Theorem 4.1.4. *Let $\lambda > 0$, $s \geq 0$, and $\mu \in \mathbb{R}$ be given. There exists a smooth map Q_λ from $H^s(\mathbb{T}) \times \mathbb{R}$ to $H^s(\mathbb{T})$ which is periodic with respect to the second variable (namely $t \in \mathbb{R}$), and such that for any $u_0 \in H^s(\mathbb{T})$ with $[u_0] = \mu$ the solution u of the closed-loop system*

$$\partial_t u + u\partial_x u + \partial_x^3 u = -GQ_\lambda(u,t), \qquad u(\cdot,0) = u_0$$

satisfies

$$||u(\cdot,t) - [u_0]||_s \leq \alpha_{s,\lambda,\mu}(||u_0 - [u_0]||_s)e^{-\lambda t}||u_0 - [u_0]||_s \qquad \text{for all } t \geq 0,$$

where $\alpha_{s,\lambda,\mu} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function depending on s, λ and μ.

The following remarks are in order.

Remark 4.1.1.

(i) In Theorem A, the control time T is independent of the initial state u_0 and the terminal state u_1 and can be, in fact, chosen arbitrarily small. By contrast, in Theorem 4.1.1, the control time T depends on the size of the initial state u_0 and the terminal state u_1 in the space $L^2(\mathbb{T})$. Whether the time T can be chosen independent of the size of u_0 and u_1 is an interesting open question.

(ii) While the decay rates κ in Theorem 4.1.2 and λ in Theorem 4.1.4 are independent of u_0, the constants $\alpha_{s,\mu}(\|u_0-[u_0]\|_0)$ or $\alpha_{s,\lambda,\mu}(\|u_0-[u_0]\|_s)$ are likely not uniformly bounded; i.e., it may happen that

$$\lim_{r\to\infty}\alpha_{s,\mu}(r)=\infty \ \text{or}\ \lim_{r\to\infty}\alpha_{s,\lambda,\mu}(r)=\infty.$$

To prove our global controllability and stabilization results described above, we will as usual consider first the associated linear open-loop system

$$u_t+u_{xxx}=Gh \tag{1.10}$$

and the associated linear closed-loop system

$$u_t+u_{xxx}=-GQ_\lambda. \tag{1.11}$$

Without much difficulty we can show by using a standard approach in control theory of linear systems that the system (1.10) is exactly controllable in the space $H^s(\mathbb{T})$ and that the closed-loop system (1.11) is exponentially stable in the space $H^s(\mathbb{T})$ with an arbitrarily large decay rate λ. However, how to extend the linear results to the corresponding nonlinear systems is a challenging task. Indeed, after having published their linear results [38], Russell and Zhang had to wait for several years to extend their results to the nonlinear systems [39] until Bourgain [3] discovered a subtle smoothing property of solutions of the KdV equation posed on a periodic domain $\mathbb{T}$, thanks to which he was able to show that the Cauchy problem (1.2) is well-posed in the space $H^s(\mathbb{T})$ for any $s\geq 0$. This newly discovered smoothing property of the KdV equation has played a crucial role in the proofs of Theorem A and Theorem B in [39]. By contrast, establishing the global exact controllability and stabilizability for the nonlinear system (1.7) is even more challenging. After all, the results presented in Theorem A and Theorem B are essentially linear in nature; they are more or less small perturbation of the linear results. The global results presented in Theorem 4.1.1, Theorem 4.1.2 and Theorem 4.1.4 are truly nonlinear and their proofs demand new tools. The needed help turns out to be certain propagation properties of compactness and regularity for the KdV equation which are inspired by those established by Laurent in [21] for the Schrödinger equation. Notice that this strategy has already been successfully applied by Dehman, Lebeau, and Zuazua [12] for the wave equation, and by Dehman, Gérard, and Lebeau [11] or Laurent [21, 22] for the Schrödinger equation.

Note that for any solution u of the systems in consideration, the mean value $[u]$ is invariant. Thus it is convenient to introduce the number $\mu:=[u]=[u_0]$, and to set

$$\tilde u=u-\mu.$$

Then $[\tilde u]=0$ and $\tilde u$ solves

$$\partial_t\tilde u+\partial_x^3\tilde u+(\mu+\tilde u)\partial_x\tilde u=Gh.$$

if u solves (1.5). Throughout the paper, μ will denote a given (real) constant, $H_0^s(\mathbb{T})=\{u\in H^s(\mathbb{T});\ [u]=0\}$, and $L_0^2(\mathbb{T})=\{u\in L^2(\mathbb{T});\ [u]=0\}$. We shall establish exponential stability results in $H_0^s(\mathbb{T})$ for the equation

$$\partial_t u+\partial_x^3u+\mu\partial_x u+u\partial_x u=-K_\lambda u$$

that will imply all the results stated above.

The paper is outlined as follows.

- In Section 2, the exact controllability and stabilizability are presented for the associated linear systems.

- In Section 3, some preliminary results in Bourgain spaces, including the propagation of compactness and the propagation of regularity for the KdV equation, are provided.

- In Section 4, the stabilization of the KdV equation by a time invariant feedback control law is studied.

- In Section 5, the stabilization of the KdV equation by a time-varying feedback control law is investigated.

Finally we end our introduction with a few comments on the boundary controllability of the KdV equation posed on a finite interval $(0, L)$:

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad x \in (0, L),\ t > 0, \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t). \end{cases} \tag{1.12}$$

The problem was first investigated by Rosier [31] and has been intensively studied in the past decade. (See [31, 48, 32, 30, 33, 8, 29, 34, 24, 14, 5, 28, 6, 25, 26] and the references therein.) In contrast to control problems of other equations (parabolic equation or hyperbolic equations for instance), the boundary control system (1.12) has some interesting properties.

(i) If

$$L \in \mathcal{N} := \left\{ 2\pi\sqrt{\frac{j^2 + l^2 + jl}{3}}; j, l \in \mathbb{N}^* \right\},$$

the linear system

$$\begin{cases} u_t + u_x + u_{xxx} = 0, \quad x \in (0, L),\ t > 0, \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t). \end{cases}$$

associated to (1.12) is **not** exactly controllable if $h_1 = h_2 \equiv 0$. However, the nonlinear system (1.12) is locally exactly controllable (still with $h_1 = h_2 \equiv 0$) [31, 8, 5, 6].

(ii) The system (1.12) is exactly controllable from the right (using h_3 as control input with $h_1 = h_2 \equiv 0$), but **only** null controllable from the left (using h_1 as a control input with $h_2 = h_3 \equiv 0$) [33, 14]. The system thus behaves like a parabolic system if control is acted only on the left end of the spatial domain and behaves like a hyperbolic system if control is allowed to act on the right end of the spatial domain.

4.2 Linear Systems

Consideration is first given to the associate linear open loop control system

$$\partial_t v + \partial_x^3 v + \mu \partial_x v = Gh, \quad v(x,0) = v_0(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{2.1}$$

where the operator G is as defined in Section 1 and h is the applied control function.

Let A denote the operator

$$Aw = -w''' - \mu w'$$

with its domain $\mathcal{D}(A) = H^3(\mathbb{T})$. The operator A generates a strongly continuous group $W(t)$ on the space $L^2(\mathbb{T})$; the eigenfunctions are simply the orthonormal Fourier basis functions in $L^2(\mathbb{T})$,

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \qquad k = 0, \pm 1, \pm 2, \cdots.$$

The corresponding eigenvalue of ϕ_k is

$$\lambda_k = ik^3 - i\mu k,\ k = 0, \pm 1, \pm 2, \cdots.$$

For any $l \in \mathbb{Z}$, let

$$m(l) = \#\{k \in \mathbb{Z};\ \lambda_k = \lambda_l\}.$$

In addition, $A^* = -A$, $G^* = G$ and $W^*(-t) = W(t)$ for any $t \in \mathbb{R}$. Using the gap condition

$$\lim_{|k|\to\infty} |\lambda_{k+1} - \lambda_k| = +\infty$$

and the fact that $m(l) \leq 3$ for any l and $m(l) = 1$ for $|l|$ large enough, we may deduce from Ingham lemma that the system (2.1) is exactly controllable in $H_0^s(\mathbb{T})$ in small time for any $s \geq 0$.

Theorem 4.2.1. *[39, Theorem 2.1 and Corollary 2.1] Let $s \geq 0$ and $T > 0$ be given. There exists a bounded linear operator*

$$\Phi : H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}) \mapsto L^2(0,T; H_0^s(\mathbb{T}))$$

such that for any $v_0,\ v_1 \in H_0^s(\mathbb{T})$,

$$W(T)v_0 + \int_0^T W(T-t)G(\Phi(v_0, v_1))(t)\, dt = v_1$$

and

$$\|\Phi(v_0, v_1)\|_{L^2(0,T;H^s(\mathbb{T}))} \leq C(\|v_0\|_s + \|v_1\|_s)$$

where $C > 0$ depends only on T and $\|g\|_s$.

The following estimate is a direct consequence of Theorem 4.2.1.

Corollary 4.2.1. *Let $T > 0$ be given. There exists $\delta > 0$ such that*

$$\int_0^T \|GW(t)\phi\|_0^2 dt \geq \delta \|\phi\|_0^2$$

for any $\phi \in L_0^2(\mathbb{T})$.

Note that the arguments presented in this paper give another proof of Corollary 4.2.1.

In addition, if one chooses the following simple feedback law

$$h(v) = -G^*v,$$

the resulting closed-loop system

$$\partial_t v + \partial_x^3 v + \mu \partial_x v = -GG^*v, \quad v(x,0) = v_0(x), \quad x \in \mathbb{T} \tag{2.2}$$

is exponentially stable.

Proposition 4.2.1. *Let $s \geq 0$ be given. There exists a number $\kappa > 0$ independent of s such that for any $v_0 \in H_0^s(\mathbb{T})$, the corresponding solution v of (2.2) satisfies*

$$\|v(.,t)\|_s \leq Ce^{-\kappa t}\|v_0\|_s$$

for any $t \geq 0$ where $C > 0$ is a constant depending only on s.

Démonstration. The case $s = 0$ was proved in [38, Theorem 2]. We only provide the proof for the case $s = 3$. The case of $0 < s < 3$ follows by interpolation. The other cases of s can be proved similarly.

Pick any $v_0 \in H_0^3(\mathbb{T})$ and let $w = \partial_t v$. Then w solves

$$\partial_t w + \partial_x^3 w + \mu \partial_x w = -GG^*w, \quad w(x,0) = w_0(x), \quad x \in \mathbb{T}$$

where $w_0(x) = -v_0^{'''}(x) - \mu v_0' - GG^*v_0(x)$ belongs to $L_0^2(\mathbb{T})$. Thus

$$\|w(\cdot,t)\|_0 = \|\partial_t v(\cdot,t)\|_0 \leq C_0 e^{-\kappa t}\|w_0\|_0$$

for any $t \geq 0$. Therefore it follows from

$$\partial_x^3 v + \mu \partial_x v + GG^*v = -w$$

that

$$\|v(\cdot,t)\|_3 \leq C_3 e^{-\kappa t}\|v_0\|_3$$

for any $t \geq 0$. The proof is complete. □

Next we show that it is possible to choose an appropriate linear feedback law such that the decay rate of the resulting closed-loop system is as large as one desires.

For given $\lambda > 0$, define

$$L_\lambda \phi = \int_0^1 e^{-2\lambda\tau} W(-\tau)GG^*W^*(-\tau)\phi \, d\tau$$

for any $\phi \in H^s(\mathbb{T})$. Clearly, L_λ is a bounded linear operator from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$. Moreover, L_λ is a self-adjoint positive operator on $L_0^2(\mathbb{T})$, and so is its inverse L_λ^{-1}. L_λ is therefore an isomorphism from $L_0^2(\mathbb{T})$ onto itself. The following result claims that the same is true on $H_0^s(\mathbb{T})$.

Lemma 4.2.1. *L_λ is an isomorphism from $H_0^s(\mathbb{T})$ onto $H_0^s(\mathbb{T})$ for all $s \geq 0$.*

Démonstration. Since the result is known for $s = 0$, and L_λ maps $H_0^s(\mathbb{T})$ into itself, we only have to prove that for any $v \in L_0^2(\mathbb{T})$, $L_\lambda v \in H_0^s(\mathbb{T})$ implies $v \in H_0^s(\mathbb{T})$, i.e. $D^s v \in L^2(\mathbb{T})$. Using the continuity of L_λ^{-1} on $L_0^2(\mathbb{T})$ and a commutator estimate similar to [21, Lemma A.1], we obtain

$$\begin{aligned}
||D^s v||_0 &\leq C||L_\lambda D^s v||_0 \\
&\leq C||\int_0^1 e^{-2\lambda\tau} W(-\tau)GG^* W^*(-\tau) D^s v\, d\tau||_0 \\
&\leq C||D^s \int_0^1 e^{-2\lambda\tau} W(-\tau)GG^* W^*(-\tau) v\, d\tau||_0 \\
&\quad + C||\int_0^1 e^{-2\lambda\tau} W(-\tau)[GG^*, D^s] W^*(-\tau) v\, d\tau||_0 \\
&\leq C||L_\lambda v||_s + C_s||v||_{s-1}.
\end{aligned}$$

The result follows at once for $s \in [0, 1]$. An induction yields the result for any $s \geq 0$. □

Choose the feedback control

$$h = -G^* L_\lambda^{-1} v.$$

The resulting closed-loop system reads :

$$\partial_t v + \partial_x^3 v + \mu \partial_x v = -K_\lambda v, \quad v(x,0) = v_0(x),\ x \in \mathbb{T}, \tag{2.3}$$

with

$$K_\lambda := GG^* L_\lambda^{-1}.$$

If $\lambda = 0$, we define $K_0 = GG^*$.

Proposition 4.2.2. *Let $s \geq 0$ and $\lambda > 0$ be given. For any $v_0 \in H_0^s(\mathbb{T})$, the system (2.3) admits a unique solution $v \in C(\mathbb{R}^+; H_0^s(\mathbb{T}))$. Moreover, there exists $M = M_s$ depending on s such that*

$$\|v(.,t)\|_s \leq M_s e^{-\lambda t} \|v_0\|_s$$

for any $t \geq 0$.

Démonstration. The case $s = 0$ follows from [42, Theorem 2.1]. The other cases of s are proved as for Proposition 4.2.1. □

4.3 Preliminaries

In this section we present some results which are essential to establish the exact controllability and stabilizability of the nonlinear systems.

4.3.1 The Bourgain space and its properties.

For given $b, s \in \mathbb{R}$, and a function $u : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, define the quantities

$$\begin{aligned} ||u||_{X_{b,s}} &:= \left(\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle k \rangle^{2s} \langle \tau - k^3 + \mu k \rangle^{2b} \left| \widehat{\widehat{u}}(k,\tau) \right|^2 d\tau \right)^{\frac{1}{2}}, \\ ||u||_{Y_{b,s}} &:= \left(\sum_{k=-\infty}^{\infty} \Big(\int_{\mathbb{R}} \langle k \rangle^{s} \langle \tau - k^3 + \mu k \rangle^{b} \left| \widehat{\widehat{u}}(k,\tau) \right| d\tau \Big)^2 \right)^{\frac{1}{2}} \end{aligned}$$

where $\widehat{\widehat{u}}(k,\tau)$ denotes the Fourier transform of u with respect to the space variable x and the time variable t (by contrast, $\widehat{u}(k,t)$ denotes the Fourier transform in space variable x) and $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. Moreover, denote by D^r the operator defined on $\mathcal{D}'(\mathbb{T}^1)$ by

$$\begin{aligned} \widehat{D^r u}(k) &= |k|^r \widehat{u}(k) \qquad \text{if} \quad k \neq 0, \\ &= \widehat{u}(0) \qquad \text{if} \quad k = 0. \end{aligned} \tag{3.1}$$

The Bourgain space $X_{b,s}$ (resp. $Y_{b,s}$) associated to the KdV equation on $\mathbb{T}$ is the completion of the space $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ under the norm $\|u\|_{X_{b,s}}$ (resp. $\|u\|_{Y_{b,s}}$). Note that for any $u \in X_{b,s}$,

$$\|u\|_{X_{b,s}} = \|W(-t)u\|_{H^b(\mathbb{R}, H^s(\mathbb{T}))}.$$

For given $b, s \in \mathbb{R}$, let

$$Z_{b,s} = X_{b,s} \cap Y_{b-\frac{1}{2},s}$$

be endowed with the norm

$$||u||_{Z_{b,s}} = ||u||_{X_{b,s}} + ||u||_{Y_{b-\frac{1}{2},s}}.$$

For a given interval I, let $X_{b,s}(I)$ (resp. $Z_{b,s}(I)$) be the restriction space of $X_{b,s}$ to the interval I with the norm

$$\begin{aligned} \|u\|_{X_{b,s}(I)} &= \inf \left\{ \|\widetilde{u}\|_{X^{s,b}} \;\middle|\; \widetilde{u} = u \text{ on } \mathbb{T}^1 \times I \right\} \\ (\text{resp.} \;\; \|u\|_{Z_{b,s}(I)} &= \inf \left\{ \|\widetilde{u}\|_{Z_{b,s}} \;\middle|\; \widetilde{u} = u \text{ on } \mathbb{T}^1 \times I \right\}). \end{aligned}$$

For simplicity, we denote $X_{b,s}(I)$ (resp. $Z_{b,s}(I)$) by $X_T^{s,b}$ (resp. $Z_{b,s}^T$) if $I = (0, T)$. The following properties of the spaces $X_{b,s}^T$ are $Z_{b,s}^T$ are easily verified.

(i) $X_{b,s}(I)$ is a Hilbert space.
(ii) $D^r u \in X_{b,s-r}(I)$ for any $u \in X_{b,s}(I)$.
(iii) If $b_1 \leq b_2$ and $s_1 \leq s_2$, then X_{b_2,s_2} is continuously imbedded in the space X_{b_1,s_1}.
(iv) For a given finite interval I, if $b_1 < b_2$ and $s_1 < s_2$, then the space $X_{b_2,s_2}(I)$ is compactly imbedded in the space $X_{b_1,s_1}(I)$.
(v) $Z_{\frac{1}{2},s}(I) \subset C(\bar{I}; H^s(\mathbb{T}))$ for any $s \in \mathbb{R}$.

Lemma 4.3.1. *Let $b, s \in \mathbb{R}$ and $T > 0$ be given. There exists a constant $C > 0$ such that*

(i) for any $\phi \in H^s(\mathbb{T})$,

$$\begin{aligned} \|W(t)\phi\|_{X_{b,s}^T} &\leq C\|\phi\|_s; \\ \|W(t)\phi\|_{Z_{b,s}^T} &\leq C\|\phi\|_s; \end{aligned}$$

(ii) for any $f \in X^T_{b-1,s}$,

$$\left\| \int_0^t W(t-\tau) f(\tau) d\tau \right\|_{X^T_{b,s}} \leq C \|f\|_{X^T_{b-1,s}}$$

provided that $b > \frac{1}{2}$;
(iii) for any $f \in Z^T_{-\frac{1}{2},s}$,

$$\| \int_0^t W(t-\tau) f(\tau)\, d\tau \|_{Z^T_{\frac{1}{2},s}} \leq C \|f\|_{Z^T_{-\frac{1}{2},s}}.$$

Démonstration. See e.g. [43] or [7]. □

Lemma 4.3.2. *(Strichartz estimates) The following estimates hold :*

$$\begin{aligned} || \sum_{k,l\in\mathbb{Z}} c_{k,l} e^{i(kx+lt)} ||_{L^4(\mathbb{T}^2)} &\leq C \left(\sum_{k,l\in\mathbb{Z}} (1+|l-k^3+\mu k|)^{\frac{2}{3}} |c_{k,l}|^2 \right)^{\frac{1}{2}}, & (3.2) \\ ||u||_{L^4(\mathbb{T}^2)} &\leq C||u||_{X_{\frac{1}{3},0}}, & (3.3) \\ ||u||_{L^4(\mathbb{T}\times(0,T))} &\leq C||u||_{X^T_{\frac{1}{3},0}}. & (3.4) \end{aligned}$$

Démonstration. (3.2) comes from [3, Proposition 7.15]. To prove (3.3), pick any $u \in X_{\frac{1}{3},0}$ decomposed as

$$u(x,t) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \widehat{\widehat{u}}(k,\tau) e^{i(kx+\tau t)} d\tau.$$

Writing $\tau = l + \sigma$ with $l \in \mathbb{Z}$, $\sigma \in [0,1)$, we have that

$$u(x,t) = \int_0^1 e^{i\sigma t} \sum_{k,l\in\mathbb{Z}} \widehat{\widehat{u}}(k, l+\sigma) e^{i(kx+lt)} d\sigma.$$

Using (3.2) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} ||u||_{L^4(\mathbb{T}^2)} &\leq \int_0^1 || \sum_{k,l\in\mathbb{Z}} \widehat{\widehat{u}}(k,l+\sigma) e^{i(kx+lt)} ||_{L^4(\mathbb{T}^2)}\, d\sigma \\ &\leq C \int_0^1 \left(\sum_{k,l\in\mathbb{Z}} (1+|l-k^3+\mu k|)^{\frac{2}{3}} |\widehat{\widehat{u}}(k,l+\sigma)|^2 \right)^{\frac{1}{2}} d\sigma \\ &\leq C \left(\sum_{k\in\mathbb{Z}} \int_0^1 \sum_{l\in\mathbb{Z}} (1+|l-k^3+\mu k|)^{\frac{2}{3}} |\widehat{\widehat{u}}(k,l+\sigma)|^2 d\sigma \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} (1+|\tau-k^3+\mu k|)^{\frac{2}{3}} |\widehat{\widehat{u}}(k,\tau)|^2 d\tau \right)^{\frac{1}{2}}. \end{aligned}$$

It remains to establish (3.4). Let $T > 0$ and $u \in X^T_{\frac{1}{3},0}$. Pick $p \in \mathbb{N}^*$ with $T \leq 2\pi p$, and an extension $\tilde{u} \in X_{\frac{1}{3},0}$ of u with $||\tilde{u}||_{X_{\frac{1}{3},0}} \leq 2||u||_{X^T_{\frac{1}{3},0}}$. Then

$$||u||^4_{L^4(\mathbb{T}\times(0,T))} \leq ||\tilde{u}||^4_{L^4(\mathbb{T}\times(0,2\pi p))} \leq p(C||\tilde{u}||_{X_{\frac{1}{3},0}})^4 \leq C'||u||^4_{X^T_{\frac{1}{3},0}}.$$

Note that C' depends only on T. □

Lemma 4.3.3 (Bilinear estimates)**.** *Let $s \geq 0$, $T \in (0,1)$, and $u, v \in X^T_{\frac{1}{2},s} \cap L^2(0,T;L^2_0(\mathbb{T}))$. Then there exist some constants $\theta > 0$ and $C > 0$ independent of T and u, v such that*

$$\|(uv)_x\|_{Z^T_{-\frac{1}{2},s}} \leq CT^\theta \|u\|_{X^T_{\frac{1}{2},s}} \|v\|_{X^T_{\frac{1}{2},s}} \tag{3.5}$$

The proof of Lemma 4.3.3 can be found in [3] with $\theta = 1/12$ (see also [7]).

To end this section, we prove a multiplication property of the Bourgain space $X^T_{b,s}$. If $\psi = \psi(t)$ is any C^∞ function, then $\psi u \in X^T_{b,s}$ for any $u \in X^T_{b,s}$. However, if $\phi = \phi(x) \in C^\infty(\mathbb{T})$, then ϕu may not belong to the space $X^T_{b,s}$ for $u \in X^T_{b,s}$. Some regularity in the index b is lost due to the fact that the multiplication by a (smooth) function of x does not keep the structure in time of the harmonics. This loss is, in fact, unavoidable. For instance, for $k \geq 1$, let $u_k = \psi(t)e^{ikx}e^{i(k^3-\mu k)t}$, where $\psi \in C_0^\infty(\mathbb{R})$ takes the value 1 on $[-1,1]$. The sequence $\{u_k\}$ is uniformly bounded in the space $X_{b,0}$ for every $b \geq 0$. However, multiplying u_k by $\phi(x) = e^{ix}$, we observe that $\|e^{ix}u_k\|_{X_{b,0}} \approx k^{2b}$.

The next lemma shows that this is the worst case.

Lemma 4.3.4. *Let $-1 \leq b \leq 1$, $s \in \mathbb{R}$ and $\varphi \in C^\infty(\mathbb{T}^1)$. Then, for any $u \in X^{s,b}$, $\varphi(x)u \in X_{b,s-2|b|}$. Similarly, the multiplication by φ maps $X^{s,b}_T$ into $X^T_{b,s-2|b|}$.*

Démonstration. We first consider the case of $b=0$ and $b=1$. The other cases of b will be derived later by interpolation and duality.

For $b = 0$, $X_{0,s} = L^2(\mathbb{R}, H^s(\mathbb{T}))$ and the result is obvious. For $b = 1$, note that $u \in X_{1,s}$ if and only if

$$u \in L^2(\mathbb{R}, H^s(\mathbb{T})) \text{ and } \partial_t u + \partial_x^3 u + \mu\partial_x u \in L^2(\mathbb{R}, H^s(\mathbb{T})),$$

and that

$$\|u\|^2_{X_{1,s}} = \|u\|^2_{L^2(\mathbb{R},H^s(\mathbb{T}))} + \left\|\partial_t u + \partial_x^3 u + \mu\partial_x u\right\|^2_{L^2(\mathbb{R},H^s(\mathbb{T}))}.$$

Thus,

$$\begin{aligned}
\|\varphi(x)u\|^2_{X_{1,s-2}} &= \|\varphi u\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))} + \left\|\partial_t(\varphi u) + \partial_x^3(\varphi u) + \mu\partial_x(\varphi u)\right\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))} \\
&\leq C\Big(\|u\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))} + \left\|\varphi\left(\partial_t u + \partial_x^3 u + \mu\partial_x u\right)\right\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))} \\
&\quad + \left\|\left[\varphi, \partial_x^3 + \mu\partial_x\right] u\right\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))}\Big) \\
&\leq C\left(\|u\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))} + \left\|\partial_t u + \partial_x^3 u + \mu\partial_x u\right\|^2_{L^2(\mathbb{R},H^{s-2}(\mathbb{T}))} + \|u\|^2_{L^2(\mathbb{R},H^s(\mathbb{T}))}\right) \\
&\leq C\,\|u\|^2_{X_{1,s}}.
\end{aligned}$$

Here, we have used the fact that

$$\left[\varphi, \partial_x^3 + \mu\partial_x\right] = -3(\partial_x\varphi)\partial_x^2 - 3(\partial_x^2\varphi)\partial_x - \partial_x^3\varphi - \mu\partial_x\varphi$$

is a differential operator of order 2. To conclude, we prove that the $X^{s,b}$ spaces are in interpolation. First, using Fourier transform, $X_{b,s}$ may be viewed as the weighted L^2 space $L^2(\mathbb{R}_\tau \times \mathbb{Z}_k, \langle k\rangle^{2s}\langle\tau - k^3 + \mu k\rangle^{2b}\lambda \otimes \delta)$, where λ is the Lebesgue measure on $\mathbb{R}$

and δ is the discrete measure on $\mathbb{Z}$. Then, we use the complex interpolation theorem of Stein-Weiss for weighted L^p spaces (see [1, p. 114]) : for $0<\theta<1$

$$(X_{0,s},X_{1,s'})_{[\theta]} \approx L^2\left(\mathbb{R}\times\mathbb{Z},\langle k\rangle^{2s(1-\theta)+2s'\theta}\left\langle \tau-k^3+\mu k\right\rangle^{2\theta}\mu\otimes\delta\right)\approx X_{\theta,s(1-\theta)+s'\theta}.$$

Since the multiplication by φ maps $X_{0,s}$ into $X_{0,s}$ and $X_{1,s}$ into $X_{1,s-2}$, we conclude that for $0\le b\le 1$, it maps $X^{s,b}=(X_{0,s},X_{1,s})_{[b]}$ into $(X_{0,s},X_{1,s-2})_{[b]}=X_{b,s-2b}$, which yields the $2b$ loss of regularity as announced.
Then, by duality, this also implies that for $0\le b\le 1$, the multiplication by $\varphi(x)$ maps $X_{-b,-s+2b}$ into $X_{-b,-s}$. As the number s may take arbitrary values in $\mathbb{R}$, we also have the result for $-1\le b\le 0$ with a loss of $-2b=2|b|$.
To get the same result for the restriction spaces $X_T^{s,b}$, we write the estimate for an extension $\tilde{u}$ of u, which yields

$$\|\varphi u\|_{X^T_{b,s-2|b|}}\le\|\varphi\tilde{u}\|_{X_{b,s-2|b|}}\le C\,\|\tilde{u}\|_{X_{b,s}}.$$

Taking the infimum on all the $\tilde{u}$, we get the claimed result. □

4.3.2 Propagation of compactness and regularity

In this subsection, we present some properties of propagation of compactness and regularity for the linear differential operator $L=\partial_t+\partial_x^3+\mu\partial_x$ associated with the KdV equation. Those propagation properties will play a key role when studying the global stabilizability of the KdV equation.

Proposition 4.3.1. *Let $T>0$ and $0\le b'\le b\le 1$ be given (with $b>0$) and suppose that $u_n\in X^T_{b,0}$ and $f_n\in X^T_{-b,-2+2b}$ satisfy*

$$\partial_t u_n+\partial_x^3u_n+\mu\partial_x u_n=f_n$$

for $n=1,2,...$ Assume that there exists a constant $C>0$ such that

$$\|u_n\|_{X^T_{b,0}}\le C\qquad \text{for all } n\ge 1, \tag{3.6}$$

and that

$$\|u_n\|_{X^T_{-b,-2+2b}}+\|f_n\|_{X^T_{-b,-2+2b}}+\|u_n\|_{X^T_{-b',-1+2b'}}\to 0 \text{ as } n\to\infty. \tag{3.7}$$

In addition, assume that for some nonempty open set $\Omega\subset\mathbb{T}$ it holds

$$u_n\to 0 \text{ strongly in } L^2(0,T;L^2(\Omega)).$$

Then

$$u_n\to 0 \text{ strongly in } L^2_{loc}((0,T);L^2(\mathbb{T}^1)).$$

Démonstration. Pick $\varphi\in C^\infty(\mathbb{T}^1)$ and $\psi\in C_0^\infty((0,T))$ real valued and set

$$B=\varphi(x)D^{-2} \text{ and } A=\psi(t)B.$$

Then
$$A^* = \psi(t)D^{-2}\varphi(x).$$
For $\varepsilon > 0$, let $A_\varepsilon = Ae^{\varepsilon\partial_x^2} = \psi(t)B_\varepsilon$ be a regularization of A. Then
$$\begin{aligned}\alpha_{n,\varepsilon} &:= ([A_\varepsilon, L]u_n, u_n)_{L^2(\mathbb{T}^1\times(0,T))} \\ &= ([A_\varepsilon, \partial_x^3 + \mu\partial_x]u_n, u_n) - (\psi'(t)B_\varepsilon u_n, u_n).\end{aligned}$$
On the other hand,
$$\alpha_{n,\varepsilon} = (f_n, A_\varepsilon^* u_n)_{L^2(\mathbb{T}^1\times(0,T))} + (A_\varepsilon u_n, f_n)_{L^2(\mathbb{T}^1\times(0,T))}$$
since $Lu_n = f_n$ and $L^* = -L$. By Lemma 4.3.4,
$$\begin{aligned}\left|(f_n, A_\varepsilon^* u_n)_{L^2(\mathbb{T}^1\times(0,T))}\right| &\leq \|f_n\|_{X^T_{-b,-2+2b}}\|A_\varepsilon^* u_n\|_{X^T_{b,2-2b}} \\ &\leq \|f_n\|_{X^T_{-b,-2+2b}}\|u_n\|_{X^T_{b,0}}.\end{aligned} \tag{3.8}$$
Consequently,
$$\lim_{n\to\infty}\sup_{0<\varepsilon\leq 1}\left|(f_n, A_\varepsilon^* u_n)_{L^2(\mathbb{T}^1\times(0,T))}\right| = 0.$$
Similarly, we have
$$\lim_{n\to\infty}\sup_{0<\varepsilon\leq 1}\left|(A_\varepsilon u_n, f_n)_{L^2(\mathbb{T}^1\times(0,T))}\right| = 0$$
and
$$\lim_{n\to\infty}\sup_{0<\varepsilon\leq 1}|(\psi'(t)B_\varepsilon u_n, u_n)| = 0.$$
Thus
$$\lim_{n\to\infty}\sup_{0<\varepsilon\leq 1}|\alpha_{n,\varepsilon}| = 0$$
and therefore
$$\lim_{n\to\infty}\sup_{0<\varepsilon\leq 1}\left|([A_\varepsilon, \partial_x^3 + \mu\partial_x]u_n, u_n)\right| = 0.$$
In particular,
$$\lim_{n\to\infty}([A, \partial_x^3 + \mu\partial_x]u_n, u_n)_{L^2(\mathbb{T}^1\times(0,T))} = 0.$$
Since D^{-2} commutes with ∂_x, we have
$$[A, \partial_x^3 + \mu\partial_x] = -3\psi(t)(\partial_x\varphi)\partial_x^2 D^{-2} - 3\psi(t)(\partial_x^2\varphi)\partial_x D^{-2} - \psi(t)(\partial_x^3\varphi + \mu\partial_x\varphi)D^{-2}. \tag{3.9}$$
Using the same argument as in (3.8), we get
$$(\psi(t)(\partial_x^3\varphi + \mu\partial_x\varphi)D^{-2}u_n, u_n)_{L^2(\mathbb{T}^1\times(0,T))} \to 0.$$
However, for the second term in (3.9), the loss of regularity is too large if we use the estimates with the same b. Using the index b' instead, we have
$$\begin{aligned}(\psi(t)(\partial_x^2\varphi)\partial_x D^{-2}u_n, u_n) &\leq \|\psi(t)(\partial_x^2\varphi)\partial_x D^{-2}u_n\|_{X^T_{b',1-2b'}}\|u_n\|_{X^T_{-b',-1+2b'}} \\ &\leq \|u_n\|_{X^T_{b',0}}\|u_n\|_{X^T_{-b',-1+2b'}}\end{aligned}$$

which tends to 0 as $n \to \infty$, by (3.6)-(3.7). Note that $-\partial_x^2 D^{-2}$ is the orthogonal projection on the subspace of functions with $\widehat{u}(0) = 0$. Using Rellich Theorem combined to the fact that $b > 0$, we easily see that $\widehat{u_n}(0,t)$ tends to 0 in $L^2(0,T)$ (strongly), and hence

$$(\psi(t)(\partial_x\varphi)\widehat{u_n}(0,t), u_n)_{L^2(\mathbb{T}^1\times(0,T))} \to 0.$$

We have thus proved that for any $\varphi \in C^\infty(\mathbb{T}^1)$ and any $\psi \in C_0^\infty((0,T))$

$$(\psi(t)(\partial_x\varphi)u_n, u_n)_{L^2(\mathbb{T}^1\times(0,T))} \to 0.$$

Note that a function $\phi \in C^\infty(\mathbb{T})$ can be written in the form $\partial_x\varphi$ for some function $\varphi \in C^\infty(\mathbb{T})$ if and only if $\int_{\mathbb{T}^1} \phi(x)\,dx = 0$. Thus, for any $\chi \in C_0^\infty(\Omega)$ and any $x_0 \in \mathbb{T}^1$, $\phi(x) = \chi(x) - \chi(x - x_0)$ can be written as $\phi = \partial_x\varphi$ for some $\varphi \in C^\infty(\mathbb{T})$.

Since u_n is strongly convergent to 0 in $L^2(0,T; L^2(\Omega))$,

$$\lim_{n\to\infty} (\psi(t)\chi u_n, u_n)_{L^2(\mathbb{T}^1\times(0,T))} = 0.$$

Therefore, for any $x_0 \in \mathbb{T}^1$,

$$\lim_{n\to\infty} (\psi(t)\chi(\cdot - x_0)u_n, u_n)_{L^2(\mathbb{T}^1\times(0,T))} = 0.$$

The proof is then completed by constructing a partition of unity of $\mathbb{T}^1$ involving functions of the form $\chi_i(\cdot - x_0^i)$ with $\chi_i \in C_0^\infty(\Omega)$ and $x_0^i \in \mathbb{T}^1$. □

Next we investigate the propagation of regularity for the operator $L = \partial_t + \partial_x^3 + \mu\partial_x$.

Proposition 4.3.2. *Let $T > 0$, $0 \le b < 1$, $r \in \mathbb{R}$ and $f \in X_{-b,r}^T$ be given. Let $u \in X_{b,r}^T$ be a solution of*

$$\partial_t u + \partial_x^3 u + \mu\partial_x u = f.$$

If there exists a nonempty open set Ω of $\mathbb{T}$ such that $u \in L^2_{loc}((0,T), H^{r+\rho}(\Omega))$ for some ρ with

$$0 < \rho \le \min\{1 - b, \frac{1}{2}\},$$

then $u \in L^2_{loc}((0,T), H^{r+\rho}(\mathbb{T}^1))$.

Démonstration. Set $s = r + \rho$ and for $n = 1, 2, \ldots$

$$u_n = e^{\frac{1}{n}\partial_x^2}u =: \Xi_n u,\ f_n = \Xi_n f = Lu_n.$$

There exists a constant $C > 0$ such that

$$\|u_n\|_{X_{b,r}^T} \le C, \quad \|f_n\|_{X_{-b,r}^T} \le C \qquad \forall\, n \ge 1.$$

Pick $\varphi \in C^\infty(\mathbb{T}^1)$ and $\psi \in C_0^\infty((0,T))$ as in the proof of Proposition 4.3.1, and set

$$B = D^{2s-2}\varphi(x) \text{ and } A = \psi(t)B.$$

We have

$$\begin{aligned} & (Lu_n, A^*u_n)_{L^2(\mathbb{T}^1\times(0,T))} + (Au_n, Lu_n)_{L^2(\mathbb{T}^1\times(0,T))} \\ = \;& ([A, \partial_x^3 + \mu\partial_x]u_n, u_n)_{L^2(\mathbb{T}^1\times(0,T))} - (\psi'(t)Bu_n, u_n) \end{aligned}$$

$$\begin{aligned}
|(Au_n, f_n)_{L^2(\mathbb{T}^1\times(0,T))}| &\le \|Au_n\|_{X^T_{b,-r}}\|f_n\|_{X^T_{-b,r}} \\
&\le \|u_n\|_{X^T_{b,r+2\rho-2+2b}}\|f_n\|_{X^T_{-b,r}} \\
&\le C\|u_n\|_{X^T_{b,r}}\|f_n\|_{X^T_{-b,r}} \\
&\le C
\end{aligned}$$

since $r+2\rho-2+2b \le r$. The same estimates for the other terms imply that

$$\left|([A,\partial_x^3+\mu\partial_x]u_n, u_n)_{L^2(\mathbb{T}^1\times(0,T))}\right| \le C.$$

Note that

$$[A,\partial_x^3+\mu\partial_x] = -3\psi(t)D^{2s-2}(\partial_x\varphi)\partial_x^2 - 3\psi(t)D^{2s-2}(\partial_x^2\varphi)\partial_x - \psi(t)D^{2s-2}(\partial_x^3\varphi+\mu\partial_x\varphi)$$

and $2s-2+1 = 2r+2\rho-1 \le 2r$. We have

$$\begin{aligned}
&|(\psi(t)D^{2s-2}(\partial_x^2\varphi)\partial_x u_n, u_n)_{L^2(\mathbb{T}^1\times(0,T))}| \\
&\quad\le C\|\psi(t)D^{2s-2}(\partial_x^2\varphi)\partial_x u_n\|_{L^2(0,T;H^{-r}(\mathbb{T}))}\|u_n\|_{L^2(0,T;H^r(\mathbb{T}))} \\
&\quad\le C\|u_n\|^2_{L^2(0,T;H^r(\mathbb{T}))} \\
&\quad\le C
\end{aligned}$$

and

$$\begin{aligned}
&|(\psi(t)D^{2s-2}(\partial_x^3\varphi+\mu\partial_x\varphi)u_n, u_n)_{L^2(\mathbb{T}^1\times(0,T))}| \\
&\quad\le C\|\psi(t)D^{2s-2}(\partial_x^3\varphi+\mu\partial_x\varphi)u_n\|_{L^2(0,T;H^{-r}(\mathbb{T}))}\|u_n\|_{L^2(0,T;H^r(\mathbb{T}))} \\
&\quad\le C\|u_n\|^2_{L^2(0,T;H^r(\mathbb{T}))} \\
&\quad\le C
\end{aligned}$$

for any $n \ge 1$. Thus

$$|(\psi(t)D^{2s-2}(\partial_x\varphi)\partial_x^2 u_n, u_n)| \le C. \tag{3.10}$$

For any $\chi \in C_0^\infty(\Omega)$,

$$\begin{aligned}
&(\psi(t)D^{2s-2}\chi^2\partial_x^2 u_n, u_n) \\
&\quad= (\psi(t)D^{s-2}\chi\partial_x^2 u_n, \chi D^s u_n) + (\psi(t)[D^{s-2},\chi]\chi\partial_x^2 u_n, D^s u_n) \\
&\quad= (\psi(t)D^{s-2}\chi\partial_x^2 u_n, D^s\chi u_n) + (\psi(t)D^{s-2}\chi\partial_x^2 u_n, [\chi, D^s]u_n) \\
&\qquad+(\psi(t)[D^{s-2},\chi]\chi\partial_x^2 u_n, D^s u_n) =: I_1+I_2+I_3.
\end{aligned}$$

We infer from the assumptions that $\chi u \in L^2_{loc}((0,T),H^s(\mathbb{T}))$ and that $\chi\partial_x^2 u \in L^2_{loc}((0,T),H^{s-2}(\mathbb{T}))$. Thus

$$\chi u_n = \Xi_n\chi u + [\chi,\Xi_n]u$$

is uniformly bounded in $L^2_{loc}((0,T),H^s(\mathbb{T}))$ by [21, Lemma A.3] and the fact that $s \le r+1$. Applying the same argument to $\chi\partial_x^2 u_n$, we obtain

$$|I_1| \le C.$$

It follows from [21, Lemma A.1] and the fact that $u \in L^2(0,T;H^r(\mathbb{T}))$ that

$$\begin{aligned}
|I_2| &\le C\left\|D^{r-2}\chi\partial_x^2 u_n\right\|_{L^2(0,T;L^2(\mathbb{T}))}\left\|D^\rho[\chi,D^s]u_n\right\|_{L^2(0,T;L^2(\mathbb{T}))} \\
&\le C\left\|u_n\right\|_{L^2(0,T;H^r(\mathbb{T}))}\left\|u_n\right\|_{L^2(0,T;H^{s-1+\rho}(\mathbb{T}))} \le C.
\end{aligned}$$

A similar bound may be obtained for $|I_3|$. Consequently,

$$\left|(\psi(t)D^{2s-2}\chi^2\partial_x^2 u_n, u_n)\right| \le C$$

for any $n \ge 1$. Then, using (3.10) with $\partial_x\varphi = \chi^2(x) - \chi^2(x-x_0)$ yields

$$\left|(\psi(t)D^{2s-2}\chi^2(\cdot - x_0)\partial_x^2 u_n, u_n)\right| \le C$$

for any $n \ge 1$. Using a partition of unity as in the proof of Proposition 4.3.1, we obtain

$$|(\psi(t)D^{2s-2}\partial_x^2 u, u)| \le C,$$

that is

$$\int_0^T \psi(t)\big(\sum_{k\neq 0} |k|^{2s}|\widehat{u}(k,t)|^2\big)\, dt \le C.$$

The proof is thus complete. □

Corollary 4.3.1. *Let $u \in X^T_{\frac{1}{2},0}$ be a solution of*

$$\partial_t u + \partial_x^3 u + \mu\partial_x u + u\partial_x u = 0 \text{ on } \mathbb{T}^1 \times (0,T). \tag{3.11}$$

Assume that $u \in C^\infty(\Omega \times (0,T))$, where Ω is a nonempty open set in $\mathbb{T}$. Then $u \in C^\infty(\mathbb{T}^1 \times (0,T))$.

Démonstration. Recall that the mean value $[u]$ is conserved. Changing μ into $\mu + [u]$ if needed, we may assume that $[u] = 0$. We have $u\partial_x u \in X^T_{-\frac{1}{2},0}$ by Lemma 4.3.3. It follows from Proposition 4.3.2 that $u \in L^2_{loc}((0,T), H^{\frac{1}{2}}(\mathbb{T}))$. Choose t_0 such that $u(t_0) \in H^{\frac{1}{2}}(\mathbb{T})$. We can then solve (3.11) in $X^T_{\frac{1}{2},\frac{1}{2}}$ with the initial data $u(t_0)$. By uniqueness of the solution in $X^T_{\frac{1}{2},0}$, we conclude that $u \in X^T_{\frac{1}{2},\frac{1}{2}}$. An iterated application of Proposition 4.3.2 yields that $u \in L^2(0,T; H^r(\mathbb{T}))$ for every $r \in \mathbb{R}$, and hence $u \in C^\infty(\mathbb{T}^1 \times (0,T))$. □

Corollary 4.3.2. *Let Ω be a nonempty open set in $\mathbb{T}^1$ and let $u \in X^T_{\frac{1}{2},0}$ be a solution of*

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu\partial_x u + u\partial_x u &= 0 \quad \text{on } \mathbb{T}^1 \times (0,T) \\ u &= c \quad \text{on } \Omega \times (0,T) \end{cases}$$

where $c \in \mathbb{R}$ denotes some constant. Then $u(x,t) = c$ on $\mathbb{T} \times (0,T)$

Démonstration. Using Corollary 4.3.1, we infer that $u \in C^\infty(\mathbb{T}^1 \times (0,T))$. It follows that $u \equiv c$ on $\mathbb{T} \times (0,T)$ by the unique continuation property for the KdV equation (see [40, 33]). □

4.4 Nonlinear systems

In this section, we are concerned with the stability properties of the closed loop system

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu \partial_x u + u\partial_x u = -K_\lambda u, & x \in \mathbb{T},\ 0 < t < T, \\ \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{4.1}$$

where $\lambda \geq 0$ is a given number and $u_0 \in L^2_0(\mathbb{T})$.

We first check that the system is globally well-posed in the space $H^s_0(\mathbb{T})$ for any $s \geq 0$.

Theorem 4.4.1. *Let $\lambda \geq 0$ and $s \geq 0$ be given. Then for any $T > 0$ and any $u_0 \in H^s_0(\mathbb{T})$, there exists a unique solution $u \in Z^T_{\frac{1}{2},s} \cap C([0,T]; L^2_0(\mathbb{T}))$ of (4.1). Furthermore, the following estimate holds*

$$\|u\|_{Z^T_{\frac{1}{2},s}} \leq \alpha_{T,s}(\|u_0\|_0)\|u_0\|_s \tag{4.2}$$

where $\alpha_{T,s} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function depending only on T and s.

Démonstration. We shall first establish the existence and uniqueness of a solution $u \in Z^T_{\frac{1}{2},s} \cap L^2(0,T; L^2_0(\mathbb{T}))$ of (4.1) for $T > 0$ small enough. Then we shall show that T can be taken as large as one wishes.

Let $u_0 \in H^s_0(\mathbb{T})$. Rewrite system (4.1) in its integral form

$$u(t) = W(t)u_0 - \int_0^t W(t-\tau)(u\partial_x u)(\tau)d\tau - \int_0^t W(t-\tau)[K_\lambda u](\tau)d\tau \tag{4.3}$$

where $W(t) = e^{-t(\partial_x^3 + \mu\partial_x)}$. For given u_0, define the map

$$\Gamma(v) = W(t)u_0 - \int_0^t W(t-\tau)(v\partial_x v)(\tau)d\tau - \int_0^t W(t-\tau)[K_\lambda v](\tau)d\tau.$$

The following estimate is needed.

Lemma 4.4.1. *For any $\varepsilon > 0$ there exists a positive constant $C(\varepsilon)$ such that*

$$|| \int_0^t W(t-\tau)[K_\lambda v](\tau)d\tau||_{Z^T_{\frac{1}{2},s}} \leq C(\varepsilon)T^{1-\varepsilon}||v||_{Z^T_{\frac{1}{2},s}}. \tag{4.4}$$

Proof of Lemma 4.4.1. Let $v \in Z^T_{\frac{1}{2},s}$. Pick an extension of v to $\mathbb{T} \times \mathbb{R}$, still denoted by v, and such that

$$||v||_{Z_{\frac{1}{2},s}} \leq 2||v||_{Z^T_{\frac{1}{2},s}}.$$

Pick any $\eta \in C^\infty(\mathbb{R})$ with $\eta(t) = 1$ for $|t| \leq 1$ and $\eta(t) = 0$ for $|t| \geq 2$. By Lemma 4.3.1, it is clearly sufficient to prove that

$$||\eta^2(t/T)K_\lambda v||_{Z_{-\frac{1}{2},s}} \leq CT^{1-\varepsilon}||v||_{Z_{\frac{1}{2},s}}. \tag{4.5}$$

Let us first estimate $||\eta^2(t/T)K_\lambda v||_{X_{-\frac{1}{2},s}}$. We have that

$$||\eta^2(t/T)K_\lambda v||_{X_{-\frac{1}{2},s}} \le ||\eta^2(t/T)K_\lambda v||_{X_{\frac{-1+\varepsilon}{2},s}} \le CT^{\frac{1-\varepsilon}{2}}||\eta(t/T)K_\lambda v||_{X_{0,s}} \le CT^{1-\varepsilon}||v||_{X_{\frac{1}{2},s}}$$

where we used [43, Lemma 2.11] twice and Lemma 4.2.1. This yields also

$$||\eta^2(t/T)K_\lambda v||_{Y_{-1,s}} \le ||\eta^2(t/T)K_\lambda v||_{X_{\frac{-1+\varepsilon}{2},s}} \le CT^{1-\varepsilon}||v||_{X_{\frac{1}{2},s}}.$$

and (4.5) follows. The proof of Lemma 4.4.1 is complete. □

It follows then from Lemmas 4.3.1, 4.3.3 and 4.4.1 that there exist some positive constants θ, C_1, C_2 and C_3 such that

$$||\Gamma(v)||_{Z^T_{\frac{1}{2},s}} \le C_1\|u_0\|_s + C_2T^\theta||v||^2_{Z^T_{\frac{1}{2},s}} + C_3T^{1-\varepsilon}||v||_{Z^T_{\frac{1}{2},s}} \tag{4.6}$$

$$||\Gamma(v_1)-\Gamma(v_2)||_{Z^T_{\frac{1}{2},s}} \le C_2T^\theta||v_1+v_2||_{Z^T_{\frac{1}{2},s}}||v_1-v_2||_{Z^T_{\frac{1}{2},s}} + C_3T^{1-\varepsilon}||v_1-v_2||_{Z^T_{\frac{1}{2},s}} \tag{4.7}$$

for any $v, v_1, v_2 \in Z^T_{\frac{1}{2},s} \cap L^2(0,T;L^2_0(\mathbb{T}))$. Pick $d = 2C_1\|u_0\|_s$ and $T>0$ such that

$$2C_2dT^\theta + C_3T^{1-\varepsilon} \le \frac{1}{2}. \tag{4.8}$$

Then

$$||\Gamma(v)||_{Z^T_{\frac{1}{2},s}} \le d$$

and

$$\|\Gamma(v_1)-\Gamma(v_2)\|_{Z^T_{\frac{1}{2},s}} \le \frac{1}{2}\|v_1-v_2\|_{Z^T_{\frac{1}{2},s}}$$

whenever $||v||_{Z^T_{\frac{1}{2},s}} \le d$, $||v_1||_{Z^T_{\frac{1}{2},s}} \le d$, and $||v_2||_{Z^T_{\frac{1}{2},s}} \le d$. Thus the map Γ is a contraction in the closed ball $B_d(0)$ of $Z^T_{\frac{1}{2},s} \cap L^2(0,T;L^2_0(\mathbb{T}))$ for the $||\cdot||_{Z^T_{\frac{1}{2},s}}$ norm. Its fixed point u is the desired solution of (4.1) in the space $Z^T_{\frac{1}{2},s} \cap L^2(0,T;L^2_0(\mathbb{T}))$. It follows from the property (v) of the Bourgain space $Z^T_{b,s}$ recalled in the previous section that $u \in C([0,T];H^s_0(\mathbb{T}))$ with

$$||u||_{L^\infty(0,T;H^s(\mathbb{T}))} \le C_4||u||_{Z^T_{\frac{1}{2},s}} \le 2C_1C_4||u_0||_s.$$

Let us now pass to the global existence of the solution. Assume first that $s=0$. The solution of (4.1) satisfies

$$||u(.,t)||^2_0 = ||u||^2_0 - \int_0^t (GL^{-1}_\lambda u, Gu)_0(\tau)d\tau \quad \forall t \ge 0$$

which yields with Gronwall lemma

$$||u(.,t)||^2_0 \le ||u_0||^2_0\, e^{Ct} \tag{4.9}$$

with $C = ||G||^2||L^{-1}_\lambda||$. A standard continuation argument shows that (4.1) is globally well-posed in $L^2_0(\mathbb{T})$. (Note that $||u(.,t)||_0 \le ||u_0||_0$ when $\lambda = 0$ and $t \ge 0$.) Next, we show that (4.1) is globally well-posed in the space $H_0^3(\mathbb{T})$. For a smooth solution u of (4.1), let $v = u_t$. Then

$$\begin{cases} \partial_t v + \partial_x^3 v + \mu\partial_x v + \partial_x(uv) = -K_\lambda v, & x \in \mathbb{T},\ 0 < t < T, \\ v(x,0) = v_0(x), \qquad x \in \mathbb{T}, \end{cases} \tag{4.10}$$

where

$$v_0 = -K_\lambda u_0 - u_0''' - \mu u_0' - u_0 u_0'.$$

For T fulfilling (4.8), we have

$$||u||_{Z^T_{\frac{1}{2},0}} \le d = 2C_1||u_0||_0.$$

The same computations as those leading to (4.6) yield

$$||v||_{Z^T_{\frac{1}{2},0}} \le C_1||v_0||_0 + (4C_1C_2T^\theta||u_0||_0 + C_3T^{1-\varepsilon})||v||_{Z^T_{\frac{1}{2},0}}$$

and hence

$$||v||_{Z^T_{\frac{1}{2},0}} \le 2C_1||v_0||_0$$

for $0 < T < T_1(||u_0||_0)$, where $T_1(\cdot)$ is a continuous nonincreasing function. Therefore,

$$||v||_{L^\infty(0,T;L^2(\mathbb{T}))} \le C_4||v||_{Z^T_{\frac{1}{2},0}} \le C_1'||v_0||_0$$

for $0 < T < T_1$ and $C_1' = 2C_1C_4$. From the equation

$$\partial_x^3 u = -K_\lambda u - v - \mu\partial_x u - u\partial_x u,$$

we infer that for $0 < t < T < T_1$

$$\begin{aligned} ||\partial_x^3 u||_0 &\le C_7||u||_0 + ||v||_0 + (C_8 + ||u||_0)||\partial_x u||_{L^\infty_x} \\ &\le C_7||u||_0 + ||v||_0 + C_9(1 + ||u||_0)||u||_0^{\frac{1}{2}}||\partial_x^3 u||_0^{\frac{1}{2}} \\ &\le \frac{1}{2}||\partial_x^3 u||_0 + ||v||_0 + C_{10}(||u||_0 + ||u||_0^3). \end{aligned}$$

Consequently,

$$||u||_{L^\infty(0,T;H^3(\mathbb{T}))} \le \alpha_{T,3}(||u_0||_0)||u_0||_3$$

for $T < T_1(||u_0||_0)$. Combined to (4.9), this shows that $u \in C(\mathbb{R}^+; H_0^3(\mathbb{T}))$ and that (4.2) holds true for $s = 3$. A similar result can be obtained for any $s \in 3\mathbb{N}^*$. For other values of s, the global well-posedness follows by nonlinear interpolation [44, 2]. The proof is complete. □

Next we prove a local exponential stability result when applying the feedback law $h = -K_\lambda u$.

Theorem 4.4.2. *Let $0 < \lambda' < \lambda$ and $s \ge 0$ be given. There exists $\delta > 0$ such that for any $u_0 \in H_0^s(\mathbb{T})$ with $||u_0||_s \le \delta$, the corresponding solution u of (4.1) satisfies*

$$||u(.,t)||_s \le Ce^{-\lambda' t}||u_0||_s \qquad \text{for all } t \ge 0$$

where $C > 0$ is a constant independent of u_0.

Démonstration. We proceed as in [35, 36]. System (4.1) can be rewritten in an equivalent integral form

$$u(t) = W_\lambda(t)u_0 - \int_0^t W_\lambda(t-\tau)(uu_x)(\tau)\,d\tau \tag{4.11}$$

where $W_\lambda(t) = e^{-t(\partial_x^3+\mu\partial_x+K_\lambda)}$. At this point we need to extend some estimates in Lemmas 4.3.1-4.3.3 for the C^0-group $W_\lambda(t)$.

Lemma 4.4.2. *Let $s \geq 0$, $\lambda \geq 0$ and $T > 0$ be given. Then there exists a constant $C > 0$ such that*
(i) for any $\phi \in H_0^s(\mathbb{T})$

$$||W_\lambda(t)\phi||_{Z^T_{\frac{1}{2},s}} \leq C||\phi||_s.$$

(ii) For any $u, v \in Z^T_{\frac{1}{2},s}$

$$||\int_0^t W_\lambda(t-\tau)(uv)_x(\tau)d\tau||_{Z^T_{\frac{1}{2},s}} \leq C||u||_{Z^T_{\frac{1}{2},s}}||v||_{Z^T_{\frac{1}{2},s}}.$$

Proof of Lemma 4.4.2 : An application of Duhamel formula gives

$$W_\lambda(t)\phi = W(t)\phi - \int_0^t W(t-\tau)[K_\lambda W_\lambda(\tau)\phi]d\tau. \tag{4.12}$$

Using Lemma 4.4.1, this yields

$$\begin{aligned} ||W_\lambda(t)\phi||_{Z^T_{\frac{1}{2},s}} &\leq ||W(t)\phi||_{Z^T_{\frac{1}{2},s}} + ||\int_0^t W(t-\tau)[K_\lambda W_\lambda(\tau)\phi]d\tau||_{Z^T_{\frac{1}{2},s}} \\ &\leq C||\phi||_s + CT^{1-\varepsilon}||W_\lambda(t)\phi||_{Z^T_{\frac{1}{2},s}} \end{aligned}$$

(i) follows at once if T is small enough, say $T < T_0$. For $T \geq T_0$, the result follows from an easy induction. To prove (ii), we use the identity

$$\int_0^t W_\lambda(t-\tau)f(\tau)\,d\tau = \int_0^t W(t-\tau)f(\tau)\,d\tau - \int_0^t W(t-\tau)K_\lambda\left(\int_0^\tau W_\lambda(\tau-\sigma)f(\sigma)\,d\sigma\right)d\tau$$

which gives with $f = (uv)_x$

$$\begin{aligned} & ||\int_0^t W_\lambda(t-\tau)(uv)_x(\tau)\,d\tau||_{Z^T_{\frac{1}{2},s}} \\ \leq\ & ||\int_0^t W(t-\tau)(uv)_x(\tau)d\tau||_{Z^T_{\frac{1}{2},s}} \\ & +||\int_0^t W(t-\tau)K_\lambda\left(\int_0^\tau W_\lambda(\tau-\sigma)(uv)_x(\sigma)\,d\sigma\right)d\tau||_{Z^T_{\frac{1}{2},s}} \\ \leq\ & C||u||_{Z^T_{\frac{1}{2},s}}||v||_{Z^T_{\frac{1}{2},s}} + CT^{1-\varepsilon}||\int_0^t W_\lambda(t-\tau)(uv)_x(\tau)\,d\tau||_{Z^T_{\frac{1}{2},s}} \end{aligned}$$

(ii) follows again if T is small enough, say $T < T_0$. For $T \geq T_0$, the result follows from (i) and an easy induction. □
For given $s \geq 0$, there exists by Proposition 4.2.2 some constant $C > 0$ such that

$$||W_\lambda(t)u_0||_s \leq Ce^{-\lambda t}||u_0||_s \qquad \forall t \geq 0.$$

Pick $T > 0$ such that

$$2Ce^{-\lambda T} \leq e^{-\lambda' T}.$$

We seek a solution u to the integral equation (4.11) as a fixed point of the map

$$\Gamma(v) = W_\lambda(t)u_0 - \int_0^t W_\lambda(t-\tau)(vv_x)(\tau)\, d\tau$$

in some closed ball $B_M(0)$ in the space $Z^T_{\frac{1}{2},s} \cap L^2(0,T;L^2_0(\mathbb{T}))$ for the $||v||_{Z^T_{\frac{1}{2},s}}$ norm. This will be done provided that $||u_0||_s \leq \delta$ where δ is a small number to be determined. Furthermore, to ensure the exponential stability with the claimed decay rate, the numbers δ and M will be chosen in such a way that

$$||u(T)||_s \leq e^{-\lambda' T}||u_0||_s.$$

By Lemma 4.4.2, there exist some positive constants C_1, C_2 (independent of δ and M) such that

$$||\Gamma(v)||_{Z^T_{\frac{1}{2},s}} \leq C_1||u_0||_s + C_2||v||^2_{Z^T_{\frac{1}{2},s}}$$

and

$$||\Gamma(v_1) - \Gamma(v_2)||_{Z^T_{\frac{1}{2},s}} \leq C_2||v_1 + v_2||_{Z^T_{\frac{1}{2},s}}||v_1 - v_2||_{Z^T_{\frac{1}{2},s}}.$$

On the other hand, since $Z^T_{\frac{1}{2},s} \subset C([0,T];H^s(\mathbb{T}))$, we have for some constant $C' > 0$ and all $v \in B_M(0)$

$$\begin{aligned} ||\Gamma(v)(T)||_s &\leq ||W_\lambda(T)u_0||_s + ||\int_0^T W_\lambda(T-t)(vv_x)(\tau)d\tau||_s \\ &\leq Ce^{-\lambda T}\delta + C'M^2. \end{aligned}$$

Pick $\delta = C_4M^2$, where C_4 and M are chosen so that

$$\frac{C'}{C_4} \leq Ce^{-\lambda T}, \quad (C_1C_4 + C_2)M^2 \leq M, \quad \text{and} \quad 2C_2M \leq \frac{1}{2}.$$

Then we have

$$\begin{aligned} ||\Gamma(v)||_{Z^T_{\frac{1}{2},s}} &\leq M \qquad \forall v \in B_M(0), \\ ||\Gamma(v_1) - \Gamma(v_2)||_{Z^T_{\frac{1}{2},s}} &\leq \frac{1}{2}||v_1 - v_2||_{Z^T_{\frac{1}{2},s}} \qquad \forall v_1, v_2 \in B_M(0). \end{aligned}$$

Therefore, Γ is a contraction in $B_M(0)$. Furthermore, its unique fixed point $u \in B_M(0)$ fulfills

$$||u(T)||_s = ||\Gamma(u)(T)||_s \leq e^{-\lambda' T}\delta.$$

Assume now that $0 < ||u_0||_s < \delta$. Changing δ into $\delta' := ||u_0||_s$ and M into $M' = (\delta'/\delta)^{\frac{1}{2}}M$, we infer that $||u(T)||_s \leq e^{-\lambda' T}||u_0||_s$, and an obvious induction yields $||u(nT)||_s \leq e^{-\lambda' nT}||u_0||_s$ for any $n \geq 0$. As $Z^T_{\frac{1}{2},s} \cap L^2(0,T;L^2_0(\mathbb{T})) \subset C([0,T];H^s_0(\mathbb{T}))$, we infer by the semigroup property that there exists some constant $C' > 0$ such that

$$||u(t)||_s \leq C'e^{-\lambda' t}||u_0||_s$$

provided that $||u_0||_s \leq \delta$. The proof is complete. □

The stability result presented in Theorem 4.4.2 was local. We extend it to a global stability result in the following theorem.

Theorem 4.4.3. *Assume $\lambda = 0$ in (4.1).*[3] *There exists a $\kappa > 0$ such that for any $R_0 > 0$, there exists a constant $C > 0$ such that for any $u_0 \in L_0^2(\mathbb{T})$ with*

$$\|u_0\|_0 \leq R_0,$$

the corresponding solution u of (4.1) (with $\lambda = 0$) satisfies

$$\|u(\cdot,t)\|_0 \leq Ce^{-\kappa t}\|u_0\|_0 \qquad \text{for all } \ t \geq 0. \tag{4.13}$$

Theorem 4.4.3 is a direct consequence of the following observability inequality.

Proposition 4.4.1. *Let $T > 0$ and $R_0 > 0$ be given. There exists a constant $\beta > 1$ such that for any $u_0 \in L_0^2(\mathbb{T})$ satisfying*

$$\|u_0\|_0 \leq R_0,$$

the corresponding solution u of (4.1) satisfies

$$\|u_0\|_0^2 \leq \beta \int_0^T ||Gu||_0^2(t)dt. \tag{4.14}$$

Indeed, if (4.14) holds, then it follows from the energy estimate

$$\|u(\cdot,t)\|_0^2 = \|u_0\|_0^2 - \int_0^t ||Gu||_0^2(\tau)d\tau \quad \forall t \geq 0 \tag{4.15}$$

that

$$\|u(\cdot,T)\|_0^2 \leq (1-\beta^{-1})\|u_0\|_0^2.$$

Thus

$$\|u(\cdot,mT)\|_0^2 \leq (1-\beta^{-1})^m\|u_0\|_0^2$$

which gives (4.13) by the semigroup property. We obtain a constant κ independent of R_0 by noticing that for $t > c(\|u_0\|_0)$, the L^2 norm of $u(.,t)$ is smaller than 1, so that we can take the κ corresponding to $R_0 = 1$. □

Now we present a proof of Proposition 4.4.1.

Proof of Proposition 2.6.1 : We prove the estimate (4.14) by contradiction. If (4.14) is not true, then for any $n \geq 1$, (4.1) admits a solution $u_n \in Z^T_{\frac{1}{2},0} \cap C([0,T]; L_0^2(\mathbb{T}))$ satisfying

$$\|u_n(0)\|_0 \leq R_0$$

and

$$\int_0^T ||Gu_n||_0^2 dt < \frac{1}{n}\|u_{0,n}\|_0^2 \tag{4.16}$$

where $u_{0,n} = u_n(0)$. Since $\alpha_n := \|u_{0,n}\|_0 \leq R_0$, one can choose a subsequence of $\{\alpha_n\}$, still denoted by $\{\alpha_n\}$, such that

$$\lim_{n\to\infty} \alpha_n = \alpha.$$

3. Recall that $K_0 = GG^*$.

There are two possible cases : (i) $\alpha > 0$ and (ii) $\alpha = 0$.

(i) $\alpha > 0$

Note that the sequence $\{u_n\}$ is bounded in both spaces $L^\infty(0,T;L^2(\mathbb{T}))$ and $X^T_{\frac{1}{2},0}$. By Lemma 4.3.3, the sequence $\{\partial_x(u_n^2)\}$ is bounded in the space $X^T_{-\frac{1}{2},0}$. On the other hand, the space $X^T_{\frac{1}{2},0}$ is compactly imbedded in the space $X^T_{0,-1}$. Therefore, we can extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\begin{array}{rcll} u_n & \to & u & \text{weakly in } X^T_{\frac{1}{2},0}, \text{ and strongly in } X^T_{0,-1}, \\ -\dfrac{1}{2}\partial_x(u_n^2) & \to & f & \text{weakly in } X^T_{-\frac{1}{2},0}, \end{array}$$

where $u \in X^T_{\frac{1}{2},0}$ and $f \in X^T_{-\frac{1}{2},0}$. Furthermore, since $X^T_{\frac{1}{2},0}$ is continuously imbedded in $L^4(\mathbb{T}^1 \times (0,T))$ by (3.4), u_n^2 is bounded in $L^2(\mathbb{T}^1 \times (0,T))$. It follows that $\partial_x(u_n^2)$ is bounded in

$$L^2(0,T;H^{-1}(\mathbb{T})) = X^T_{0,-1}.$$

Conducting interpolation between $X^T_{-\frac{1}{2},0}$ and $X^T_{0,-1}$, we obtain that $\partial_x(u_n^2)$ is bounded in $X^T_{-\frac{1-\theta}{2},-\theta} = X^T_{-\frac{1}{2}+\frac{\theta}{2},-\theta}$ for $\theta \in [0,1]$. As $X^T_{-\frac{1}{2}+\frac{\theta}{2},-\theta}$ is compactly imbedded in $X^T_{-\frac{1}{2},-1}$ for $0 < \theta < 1$, we can extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $-\frac{1}{2}\partial_x(u_n^2)$ converges to f strongly in $X^T_{-\frac{1}{2},-1}$. It follows from (4.16) that

$$\int_0^T ||Gu_n||_0^2\, dt \longrightarrow \int_0^T ||Gu||_0^2\, dt = 0,$$

which implies that $u(x,t) = c(t)$ on $\omega \times (0,T)$ for some function $c(t)$. Thus, letting $n \to \infty$, we obtain from (4.1) that

$$\left\{ \begin{array}{rcll} \partial_t u + \partial_x^3 u + \mu\partial_x u & = & f & \text{on } \mathbb{T} \times (0,T), \\ u & = & c(t) & \text{on } \omega \times (0,T). \end{array} \right. \tag{4.17}$$

Let $w_n = u_n - u$ and $f_n = -\frac{1}{2}\partial_x(u_n^2) - f - K_0 u_n$. Note first that

$$\int_0^T ||Gw_n||_0^2 dt = \int_0^T ||Gu_n||_0^2 dt + \int_0^T ||Gu||_0^2 dt - 2\int_0^T (Gu_n, Gu)_0 dt \to 0 \tag{4.18}$$

Since $w_n \to 0$ weakly in $X^T_{\frac{1}{2},0}$, we infer from Rellich theorem that $\int_{\mathbb{T}} g(y) w_n(y,t)dy \to 0$ strongly in $L^2(0,T)$. Combined to (4.18), this yields

$$\int_0^T \int_{\mathbb{T}} g(x)^2 w_n(x,t)^2 dx dt \to 0.$$

Thus

$$\partial_t w_n + \partial_x^3 w_n + \mu\partial_x w_n = f_n$$

and

$$f_n \underset{X^T_{-\frac{1}{2},-1}}{\longrightarrow} 0, \quad w_n \underset{L^2(0,T;L^2(\tilde{\omega}))}{\longrightarrow} 0,$$

where $\tilde{\omega} := \{g > ||g||_{L^\infty(\mathbb{T})}/2\}$.

Applying Proposition 4.3.1 with $b = \frac{1}{2}$ and $b' = 0$ yields that

$$w_n \underset{L^2_{loc}((0,T);L^2(\mathbb{T}))}{\longrightarrow} 0.$$

Consequently, u_n^2 tends to u^2 in $L^1_{loc}((0,T);L^1(\mathbb{T}))$ and $\partial_x(u_n^2)$ tends to $\partial_x(u^2)$ in the distributional sense. Therefore $f = -\frac{1}{2}\partial_x(u^2)$ and $u \in X^T_{\frac{1}{2},0}$ satisfies

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu\partial_x u + \frac{1}{2}\partial_x(u^2) &= 0 \qquad \text{on } \mathbb{T}^1 \times (0,T), \\ u &= c(t) \qquad \text{on } \omega \times (0,T). \end{cases}$$

The first equation gives $c'(t) = 0$ which, combined to Corollary 4.3.2, yields that $u(x,t) \equiv c$ for some constant $c \in \mathbb{R}$. Since $[u] = 0$, $c = 0$, and u_n converges strongly to 0 in $L^2_{loc}((0,T), L^2(\mathbb{T}))$. We can pick some time $t_0 \in [0,T]$ such that $u_n(t_0)$ tends to 0 strongly in $L^2(\mathbb{T})$. Since

$$\|u_n(0)\|_0^2 = \|u_n(t_0)\|_0^2 + \int_0^{t_0} ||Gu_n||_0^2 dt,$$

it is inferred that $\alpha_n = \|u_n(0)\|_0 \to 0$ which is a contradiction to the assumption $\alpha > 0$.

(ii) $\alpha = 0$.

Note first that $\alpha_n > 0$ for all n. Set $v_n = u_n/\alpha_n$ for all $n \geq 1$. Then

$$\partial_t v_n + \partial_x^3 v_n + \mu\partial_x v_n + K_0 v_n + \frac{\alpha_n}{2}\partial_x(v_n^2) = 0$$

and

$$\int_0^T ||Gv_n||_0^2 dt < \frac{1}{n}. \tag{4.19}$$

Because of

$$\|v_n(0)\|_0 = 1, \tag{4.20}$$

the sequence $\{v_n\}$ is bounded in both spaces $L^\infty(0,T;L^2(\mathbb{T}))$ and $X^T_{\frac{1}{2},0}$. Indeed, $||v_n(t)||_0$ is a nonincreasing function of t, and the boundedness of $||v_n||_{X^T_{\frac{1}{2},0}}$ for small values of T follows from an estimate similar to (4.6) (since α_n is bounded). We can extract a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, such that $v_n \to v$ weakly in the space $X^T_{\frac{1}{2},0}$ and strongly in the spaces $X^T_{-\frac{1}{2},-1}$ and $X^T_{0,-1}$. Moreover, the sequence $\{\partial_x(v_n^2)\}$ is bounded in the space $X^T_{-\frac{1}{2},0}$, and therefore $\alpha_n\partial_x(v_n^2)$ tends to 0 in the space $X^T_{-\frac{1}{2},0}$. Finally, $\int_0^T ||Gv||_0^2 dt = 0$. Thus, v solves

$$\begin{cases} \partial_t v + \partial_x^3 v + \mu\partial_x v &= 0 \qquad \text{on } \mathbb{T}^1 \times (0,T) \\ v &= c(t) \qquad \text{on } \omega \times (0,T). \end{cases} \tag{4.21}$$

We infer that $v(x,t) = c(t) = c$ thanks to Holmgren Theorem (see e.g. [15]), and that $c = 0$ because of $[v] = 0$.

According to (4.19)

$$\int_0^T ||Gv_n||_0^2 dt \longrightarrow 0$$

and so $K_0 v_n$ converges strongly to 0 in $X^T_{-\frac{1}{2},-1}$. Then, an application of Proposition 4.3.1 as in (i) shows that v_n converges to 0 in $L^2_{loc}((0,T), L^2(\mathbb{T}))$. Thus we can pick a time $t_0 \in (0,T)$ such that $v_n(t_0)$ converges to 0 strongly in $L^2(\mathbb{T})$. Since

$$\|v_n(0)\|_0^2 = \|v_n(t_0)\|_0^2 + \int_0^{t_0} ||Gv_n||_0^2 dt,$$

we infer from (4.19) that $\|v_n(0)\|_0 \to 0$ which is a contradiction to (4.20). The proof is complete. □

Next we show that the solution u of (4.1) (with $\lambda = 0$) decays exponentially in any space $H^s(\mathbb{T})$.

Theorem 4.4.4. *Assume that $\lambda = 0$ in (4.1), and let $\kappa > 0$ be the infimum of the numbers κ given respectively in Proposition 4.2.1 and in Theorem 4.4.3. Let $s \geq 0$ and let $\kappa' \in (0,\kappa)$ be given. Then there exists a nondecreasing continuous function $\alpha_{s,\kappa'} : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $u_0 \in H_0^s(\mathbb{T})$, the corresponding solution u of (4.1) satisfies*

$$\|u(\cdot,t)\|_s \leq \alpha_{s,\kappa'}(\|u_0\|_0) e^{-\kappa' t} \|u_0\|_s$$

for all $t \geq 0$.

Démonstration. The result for $s = 0$ has already been established in Theorem 4.4.3 with $\kappa' = \kappa$. Let us consider now the case $s = 3$. Pick any number $R_0 > 0$ and any $u_0 \in H_0^3(\mathbb{T})$ with $||u_0||_0 \leq R_0$. Let u denote the solution of (4.1) emanating from u_0 at $t = 0$, and let $v = u_t$. Then v solves

$$\partial_t v + \partial_x^3 v + \mu \partial_x v + \partial_x(uv) = -K_0 v,\ v(x,0) = v_0(x),\ x \in \mathbb{T},\ t > 0, \tag{4.22}$$

where $v_0 = -K_0 u_0 - \mu u_0' - u_0 u_0' - u_0'''$. According to (4.2) and (4.13), for any $T > 0$ there exists a number $C > 0$ depending only on R_0 and T such that

$$\|u(\cdot,t)\|_{Z^{[t,t+T]}_{\frac{1}{2},0}} \leq C e^{-\kappa t} ||u_0||_0 \qquad \text{for all } t \geq 0.$$

Thus, for any $\epsilon > 0$, there exists a $t^* > 0$ such that if $t \geq t^*$, one has

$$\|u(\cdot,t)\|_{Z^{[t,t+T]}_{\frac{1}{2},0}} \leq \epsilon.$$

At this point we need an exponential stability result for the linearized system

$$\partial_t w + \partial_x^3 w + \mu \partial_x w + \partial_x(aw) = -K_0 w,\ w(x,0) = w_0(x),\ x \in \mathbb{T},\ t > 0 \tag{4.23}$$

where $a \in Z^T_{\frac{1}{2},s} \cap L^2(0,T; L_0^2(\mathbb{T}))$ is a given function.

Lemma 4.4.3. *Let $s \geq 0$ and $a \in Z^T_{\frac{1}{2},s} \cap L^2(0,T; L_0^2(\mathbb{T}))$ for all $T > 0$. Then for any $\kappa' \in (0,\kappa)$ there exist $T > 0$, $\beta > 0$ such that if*

$$\sup_{n \geq 1} \|a\|_{Z^{[nT,(n+1)T]}_{\frac{1}{2},s}} \leq \beta,$$

then

$$\|w(\cdot,t)\|_s \le Ce^{-\kappa' t}\|w_0\|_s \qquad \textit{for all} \;\; t\ge 0,$$

where $C>0$ is a constant independent of w_0.

Proof of Lemma 4.4.3 : First, a proof similar to that of Theorem 4.4.1 shows that for any $T>0$ and any $s\ge 0$, if $a\in Z^T_{\frac{1}{2},s}\cap L^2(0,T;L^2_0(\mathbb{T}))$, then (4.23) admits a unique solution $w\in Z^T_{\frac{1}{2},s}\cap L^2(0,T;L^2_0(\mathbb{T}))$ and

$$\|w\|_{Z^T_{\frac{1}{2},s}} \le \mu(\|a\|_{Z^T_{\frac{1}{2},s}})\|w_0\|_s \tag{4.24}$$

where $\mu:\mathbb{R}^+\to\mathbb{R}^+$ is a nondecreasing continuous function. Rewrite (4.23) in its integral form

$$w(t)=W_0(t)w_0-\int_0^t W_0(t-\tau)\partial_x(aw)(\tau)d\tau$$

where $W_0(t)=e^{-t(\partial_x^3+\mu\partial_x+K_0)}$. Thus, for any $T>0$, by Proposition 4.2.1, Lemma 4.4.2 and (4.24),

$$\begin{aligned}\|w(\cdot,T)\|_s &\le C_1e^{-\kappa T}\|w_0\|_s + C_2\|a\|_{Z^T_{\frac{1}{2},s}}\|w\|_{Z^T_{\frac{1}{2},s}}\\ &\le C_1e^{-\kappa T}\|w_0\|_s + C_2\|a\|_{Z^T_{\frac{1}{2},s}}\mu(\|a\|_{Z^T_{\frac{1}{2},s}})\|w_0\|_s\end{aligned}$$

where $C_1>0$ is independent of T while C_2 may depend on T. Let

$$y_n=w(\cdot,nT) \quad \text{for} \quad n=1,2,...$$

Then, using the semigroup property of the system (4.23),

$$\|y_{n+1}\|_s\le C_1e^{-\kappa T}\|y_n\|_s + C_2\|a\|_{Z^{[nT,(n+1)T]}_{\frac{1}{2},s}}\mu(\|a\|_{Z^{[nT,(n+1)T]}_{\frac{1}{2},s}})\|y_n\|_s$$

for $n\ge 1$. Choose $T>0$ large enough and $\beta>0$ small enough so that

$$C_1e^{-\kappa T}+C_2\beta\mu(\beta)=e^{-\kappa' T}$$

Then

$$\|y_{n+1}\|_s\le e^{-\kappa' T}\|y_n\|_s$$

for any $n\ge 1$ as long as

$$\sup_{n\ge 1}\|a\|_{Z^{[nT,(n+1)T]}_{\frac{1}{2},s}}\le\beta.$$

Thus

$$\|y_n\|_s\le e^{-n\kappa' T}\|y_0\|_s$$

for any $n\ge 1$, which implies that

$$\|w(\cdot,t)\|_s\le Ce^{-\kappa' t}\|w_0\|_s$$

for all $t\ge 0$. The proof is complete. □

Choose $\epsilon<\beta$, and then apply Lemma 4.4.3 to (4.22) to obtain

$$\|v(\cdot,t)\|_0\le Ce^{-\kappa'(t-t^*)}\|v(\cdot,t^*)\|_0$$

for any $t \geq t^*$, or

$$\|v(\cdot,t)\|_0 \leq C_1 e^{-\kappa' t}\|v_0\|_0$$

for any $t \geq 0$, where $C_1 > 0$ depends only on R_0. It then follows from the equation

$$\partial_x^3 u = -K_0 u - u\partial_x u - \mu\partial_x u - v$$

and Theorem 4.4.3 that

$$\|u(\cdot,t)\|_3 \leq Ce^{-\kappa' t}\|u_0\|_3$$

for any $t \geq 0$, where $C > 0$ depends only on R_0.

Thus the theorem has been proved for $s = 0$ and $s = 3$. Using the same argument for $u_1 - u_2$ and $a = u_1 + u_2$ for two different solutions u_1 and u_2, we obtain the Lipchitz stability estimate needed for interpolation :

$$\|(u_1 - u_2)(\cdot,t)\|_0 \leq Ce^{-\kappa' t}\|(u_1 - u_2)(\cdot,0)\|_0.$$

The case of $0 < s < 3$ follows by interpolation. The other cases can be proved similarly. □

4.5 Time-varying feedback law

In this section we prove that it is possible to design a smooth time-varying feedback law ensuring a semiglobal stabilization with an arbitrary large decay rate.

Let $\lambda > 0$ and $s \geq 0$ be given. According to Theorem 4.4.4, there exists a number $\kappa > 0$ and a nondecreasing function α_s such that any solution u of

$$\partial_t u + \partial_x^3 u + \mu\partial_x u + u\partial_x u = -GG^* u \tag{5.1}$$

emanating from $u_0 \in H_0^s(\mathbb{T})$ at $t = 0$ fulfills

$$||u(t)||_s \leq \alpha_s(||u_0||_0)e^{-\kappa t}||u_0||_s. \tag{5.2}$$

On the other hand, it follows from Theorem 4.4.2 that for any fixed $\lambda' \in (0,\lambda)$, any solution u of

$$\partial_t u + \partial_x^3 u + \mu\partial_x u + u\partial_x u = -K_\lambda u \tag{5.3}$$

emanating from $u_0 \in H_0^s(\mathbb{T})$ at $t = 0$ fulfills

$$||u(t)||_s \leq C_s e^{-\lambda' t}||u_0||_s \tag{5.4}$$

provided that $||u_0||_s \leq r_0$, for some constant C_s and some number $r_0 \in (0,1)$. Pick any function $\theta \in C^\infty(\mathbb{R};[0,1])$ fulfilling the following properties :

$$\theta(t) = 1 \qquad \text{for } \delta \leq t \leq 1-\delta \tag{5.5}$$
$$\theta(t) = 0 \qquad \text{for } 1 \leq t \leq 2 \tag{5.6}$$
$$\theta(t+2) = \theta(t) \qquad \text{for all } t \in \mathbb{R} \tag{5.7}$$

where $\delta \in (0, 1/10)$ is a number whose value will be specified later. Pick a function $\rho \in C^\infty(\mathbb{R}^+;[0,1])$ such that

$$\rho(r) = 1 \quad \text{for } r \leq r_0, \qquad \rho(r) = 0 \quad \text{for } r \geq 1. \tag{5.8}$$

Let $T > 0$ be given. We consider the following time-varying feedback law

$$\begin{aligned} K(u,t) &= \rho(||u||_s^2)\,[\theta(\frac{t}{T})K_\lambda u + \theta(\frac{t-T}{T})GG^*u] + (1-\rho(||u||_s^2))GG^*u \qquad (5.9)\\ &= GG^*\,\{\rho(||u||_s^2)\,[\theta(\frac{t}{T})L_\lambda^{-1}u + \theta(\frac{t-T}{T})u] + (1-\rho(||u||_s^2))u\}. \end{aligned}$$

K is chosen in order to have the following behavior. In a first time, when $||u||_s$ is large, we only have the damping GG^*, in order to get sure of the decay as long as the solution is large. Then, after a transition time, we have $\rho(||u||_s) = 1$ and we get into the oscillatory regime. During each period of length $2T$, we have three steps :
- a time where the damping K_λ is active, leading to a decay as $e^{-\lambda' t}$;
- a short transition time of order δ where there can be some loss ;
- a time where the damping GG^* is active, leading to a decay as $e^{-\kappa t}$.
The expected decay is a means of the two decays as stated in the following theorem.

Theorem 4.5.1. *Let $\lambda > 0$ and let $K = K(u,t)$ be as given in* (5.9)*. Pick any $\lambda' \in (0,\lambda)$ and any $\lambda'' \in (\lambda'/2, (\lambda'+\kappa)/2)$. Then there exists a time $T_0 > 0$ such that for $T > T_0$, $t_0 \in \mathbb{R}$ and $u_0 \in H_0^s(\mathbb{T})$, the unique solution of the closed-loop system*

$$\partial_t u + \partial_x^3 u + \mu\partial_x u + u\partial_x u = -K(u,t), \qquad u(t_0) = u_0 \qquad (5.10)$$

satisfies

$$||u(.,t)||_s \leq \gamma_s(||u_0||_s)e^{-\lambda''(t-t_0)}||u_0||_s \qquad \text{for all } t \geq t_0 \qquad (5.11)$$

where γ_s is a nondecreasing continuous function.

Démonstration. First, proceeding as for Theorem 4.4.1, we check that the system (5.10) is globally well-posed in $H_0^s(\mathbb{T})$. Next, rough estimates for $||u(.,t)||_s$ are established for the times t when both K_λ and GG^* are active.

Lemma 4.5.1. *Pick any pair $(t_0,u_0) \in \mathbb{R} \times H_0^s(\mathbb{T})$. Then the system* (5.10) *admits a unique solution $u : \mathbb{T} \times [t_0,+\infty) \to \mathbb{R}$ fulfilling*

$$u \in Z_{\frac12,s}^{[t_0,t_0+T]} \cap L^2(t_0,t_0+T;L_0^2(\mathbb{T})) \qquad \text{for all } T > 0.$$

The following a priori estimates hold true

$$\begin{aligned} &\text{If} \quad ||u_0||_s \leq 1, \quad ||u(.,t)||_s \leq \alpha_s(1) \quad \text{for all } t \geq t_0; &(5.12)\\ &\text{If} \quad ||u_0||_s > 1, \quad ||u(.,t)||_s \leq \alpha_s(||u_0||_0)||u_0||_s \quad \text{for all } t \geq t_0; &(5.13)\\ &\text{If} \quad ||u_0||_s \leq R, \quad ||u(.,t)||_s \leq K_s\, e^{d_s(t-t_0)}||u_0||_s \quad \text{for all } t \geq t_0, &(5.14) \end{aligned}$$

where K_s and d_s denote some positive constants depending only on s and R.

Proof of Lemma 4.5.1 : Let us begin with the local existence of a solution. Pick any pair $(t_0,u_0) \in \mathbb{R} \times H_0^s(\mathbb{T})$. It may be seen that

$$||K(v_1,t) - K(v_2,t)||_s \leq c||v_1 - v_2||_s \qquad \text{for all } \ v_1, v_2 \in H_0^s(\mathbb{T}),\ t \in \mathbb{R}$$

where c denotes a positive constant independent of v_1, v_2 and t. Defining the map

$$\Gamma(v)(t) = W(t-t_0)u_0 - \int_{t_0}^t W(t-\tau)(v\partial_x v)(\tau)\,d\tau - \int_{t_0}^t W(t-\tau)K(v(\tau),\tau)\,d\tau,$$

we infer as in the proof of Theorem 4.4.1 that (4.6) and (4.7) hold for all $v, v_1, v_2 \in Z^{[t_0,t_0+\tilde{T}]}_{\frac{1}{2},s} \cap L^2(t_0, t_0+\tilde{T}; L^2_0(\mathbb{T}))$. Moreover, the involved constants only depend on θ for its L^∞ norm and not on δ. Let $d = 2C_1||u_0||_s$ and $\tilde{T} > 0$ be such that

$$2C_2 d\tilde{T}^\theta + C_3\tilde{T}^{1-\varepsilon} \le \frac{1}{2}\cdot$$

Then the map Γ is a contraction in the closed ball $B_d(0)$ of $Z^{[t_0,t_0+\tilde{T}]}_{\frac{1}{2},s} \cap L^2(t_0, t_0+\tilde{T}; L^2_0(\mathbb{T}))$ for the $||v||_{Z^{[t_0,t_0+\tilde{T}]}_{\frac{1}{2},s}}$ norm. Its fixed point is the desired solution of (5.10). Note that for some constant $C_4 > 0$ we have that

$$||u||_{L^\infty(t_0,t_0+\tilde{T};H^s(\mathbb{T}))} \le C_4||u||_{Z^{[t_0,t_0+\tilde{T}]}_{\frac{1}{2},s}} \le 2C_1C_4||u_0||_s.$$

Noticing that $K(u,t) = GG^*u$ for $||u||_s > 1$ and using (5.2), we infer that the solution u of (5.10) is defined for all $t \ge t_0$. Moreover, (5.2) yields (5.12) and (5.13). Let

$$d' = 2C_1 \max(\alpha_s(1), \alpha_s(||u_0||_0)||u_0||_s).$$

Note that d' depends only on R and s. Replacing $\tilde{T}$ by T' satisfying

$$2C_2 d'T'^\theta + C_3 T'^{1-\varepsilon} \le \frac{1}{2}$$

in the application of the contraction mapping principle, we infer that the (unique) solution u of (5.10) fulfills

$$||u||_{Z^{[t_0+kT',t_0+(k+1)T']}_{\frac{1}{2},s}} \le 2C_1||u(.,t_0+kT')||_s.$$

This gives

$$||u||_{L^\infty(t_0+kT',t_0+(k+1)T',H^s(\mathbb{T}))} \le 2C_1C_4||u(.,t_0+kT')||_s$$

and

$$||u(.,t)||_s \le K_s e^{d_s(t-t_0)}||u_0||_s$$

for some constants $K_s > 0$, $d_s > 0$ depending only on s and R. □

Given λ'' as in the statement of the theorem, we pick $\delta > 0$ such that

$$\lambda'' < -2\delta d_s + (1-2\delta)\frac{\kappa+\lambda'}{2} \qquad \text{and} \qquad \delta d_s - (1-2\delta)\kappa < 0\cdot \tag{5.15}$$

Next, choose $r_1 \in (0, r_0)$ such that

$$\alpha_s(\alpha_s(1))C_s K_s^4 e^{4\delta T d_s} r_1 < r_0, \tag{5.16}$$

and $T_0 > 0$ such that

$$\alpha_s(1)\alpha_s(\alpha_s(1))K_s e^{[\delta d_s-(1-2\delta)\kappa]T} \le r_1, \tag{5.17}$$

$$\alpha_s(1)C_s K_s^4 e^{[4\delta d_s-(1-2\delta)(\kappa+\lambda')]T} \le e^{-2\lambda'' T} \tag{5.18}$$

for all $T \ge T_0$. Note that T_0 exists by (5.15). Pick any $u_0 \in H^s_0(\mathbb{T})$ and any time $t_0 \in \mathbb{R}$. The proof rests on a series of claims.

Claim 1. There exists a time $t_1 \in [t_0, t_0 + \kappa^{-1}\ln(\alpha_s(||u_0||_0)||u_0||_s)]$ such that

$$||u(t_1)||_s \le 1. \tag{5.19}$$

Without loss of generality we may assume that $||u(t_0)||_s \geq 1$. Then the dynamics of u is governed by (5.1) as long as $||u(t)||_s \geq 1$. By (5.2), we have (5.19) for some time t_1 with

$$\alpha_s(||u_0||_0)e^{-\kappa(t_1-t_0)}||u_0||_s \leq 1.$$

Therefore, Claim 1 holds. □

CLAIM 2. There exists a time $t_2 \in 2\mathbb{Z}T \cap [t_1, t_1+3T]$ such that

$$||u(t_2)||_s \leq r_1. \tag{5.20}$$

From the fact that $||u(t_1)||_0 \leq 1$ and (5.2) we have that

$$||u(t)||_s \leq \alpha_s(1) \qquad \text{for all } t \geq t_1.$$

Pick $R = \alpha_s(1)$ and let K_s and d_s be as given in Lemma 4.5.1 for that choice of R. Let $t_1' \geq t_1$ denote the first time of the form $t_1' = (2k+1)T + \delta$ with $k \in \mathbb{Z}$, and let $t_2 = (2k+2)T$. Then it follows from (5.2), (5.14) and (5.17) that

$$||u(t_2)||_s \leq K_s e^{\delta T d_s}\alpha_s(\alpha_s(1))e^{-\kappa(1-2\delta)T}||u(t_1')||_s \leq r_1.$$

CLAIM 3. $||u(t)||_s \leq r_0$ for all $t \geq t_2$ and $||u(t_2+2kT)||_s \leq e^{-2k\lambda'' T}||u(t_2)||_s$ for all $k \in \mathbb{N}$.

First, we notice that the dynamics of u is governed by (5.3) (resp. by (5.1)) when $t \in (t_2+\delta T, t_2+(1-\delta)T)$ (resp. when $t \in (t_2+(1+\delta)T, t_2+(2-\delta)T)$), as long as $||u(t)||_s \leq r_0$. Therefore, using (5.2), (5.4), (5.14), and (5.16) we obtain that

$$||u(t)||_s \leq (\alpha_s(\alpha_s(1))K_s^2 e^{2\delta T d_s})(C_s K_s^2 e^{2\delta T d_s})||u(t_2)||_s < r_0 \qquad \text{for all } t \in [t_2, t_2+2T].$$

On the other hand, by (5.18),

$$\begin{aligned}||u(t_2+2T)||_s &\leq (\alpha_s(1)e^{-\kappa(1-2\delta)T}K_s^2 e^{2\delta T d_s})(C_s e^{-\lambda'(1-2\delta)T}K_s^2 e^{2\delta T d_s})||u(t_2)||_s \\ &\leq e^{-2\lambda'' T}||u(t_2)||_s \\ &\leq r_1.\end{aligned}$$

The claim follows by an obvious induction. □

It follows from Claim 3 that for $t \geq t_2$

$$||u(t)||_s \leq c\, e^{-\lambda''(t-t_2)}||u(t_2)||_s$$

for some constant c independent of t and u_0. Since

$$t_2 - t_0 \leq 3T + \kappa^{-1}\ln(\alpha_s(||u_0||_0)||u_0||_s),$$

the theorem follows. □

Remark 4.5.1.

- *A natural idea to combine both feedback controls would be to consider a discontinuous feedback control which agrees with $K_0 u$ when $||u||_s$ is large, and with $K_\lambda u$ when $||u||_s$ is small. The main difficulty is then to define properly what we mean by a solution of the closed-loop system. In finite dimension, the Filippov solutions are widely used by the control community to deal with discontinuous systems. (See [9] for the definition of a Filippov solution.) The main advantage of the time-varying feedback law considered here is its regularity, which guarantees the existence and uniqueness of "classical" solutions for the closed-loop system.*

- *It would be interesting to see whether a smooth time-invariant feedback law ensuring a semi-global exponential stabilization with an arbitrary decay rate can be designed.*
- *A simpler, but less efficient, time-varying feedback law is*

$$K(u,t) := \theta(\frac{t}{T})\rho(||u||_s^2)\, K_\lambda u + \theta(\frac{t-T}{T})GG^* u.$$

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Chapitre 5

Stabilisation et contrôle de l'équation des ondes non linéaire critique sur une variété compacte de dimension 3

Contents

Cette partie est la reprise d'un article soumis.

Introduction

In this article, we study the internal stabilization and exact controllability for the defocusing critical nonlinear Klein-Gordon equation on some compact manifolds.

$$\left\{\begin{array}{rcl} \Box u = \partial_t^2 u - \Delta u & = & -u - |u|^4 u \quad \text{on} \quad [0,+\infty[\times M \\ (u(0),\partial_t u(0)) & = & (u_0,u_1) \in \mathcal{E}. \end{array}\right. \tag{0.1}$$

where Δ is the Laplace-Beltrami operator on M and $\mathcal{E}$ is the energy space $H^1(M) \times L^2(M)$. The solution displays a conserved energy

$$E(t) = \frac{1}{2}\left(\int_M |\partial_t u|^2 + \int_M |u|^2 + \int_M |\nabla u|^2\right) + \frac{1}{6}\int_M |u|^6. \tag{0.2}$$

This problem was already treated in the subcritical case by B. Dehman, G. Lebeau and E. Zuazua [11]. The problem is posed in a different geometry but their proof could easily be transposed in our setting. Yet, their result fails to apply to the critical problem for two main reasons, as explained in their paper :
(a) The boot-strap argument they employ to improve the regularity of solutions vanishing in the zone of control ω so that the existing results on unique continuation apply, does not work for this critical exponent.
(b) They can not use the linearizability results by P. Gérard [20] to deduce that the microlocal defect measure for the nonlinear problem propagates as in the linear case.

In this paper, we propose a strategy to avoid the second difficulty at the cost of an additional condition for the subset ω. It was already performed by B. Dehman and P. Gérard [8] in the case of $\mathbb{R}^3$ with a flat metric. In fact, in that case, this defect of linearisability is described by the profile decomposition of H. Bahouri and P. Gérard [2]. The purpose of this paper is to extend a part of this proof to the case of a manifold with a variable metric. This more complicated geometry leads to extra difficulties, in the profile decomposition and the stabilization argument. We also mention the recent result of L. Aloui, S. Ibrahim and K. Nakanishi [1] for $\mathbb{R}^d$. Their method of proof is very different and uses Morawetz-type estimates. They obtain uniform exponential decay for a damping around spatial infinity for any nonlinearity, provided the solution exists globally. This result is stronger than ours, but their method does not seem to apply to the more complicated geometries we deal with.

We will need some geometrical condition to prove controllability. The first one is the classical geometric control condition of Rauch and Taylor [33] and Bardos Lebeau Rauch [3], while the second one is more restrictive.

Assumption 5.0.1 (Geometric Control Condition)**.** *There exists $T_0 > 0$ such that every geodesic travelling at speed 1 meets ω in a time $t < T_0$.*

Definition 5.0.1. *We say that $(x_1, x_2, t) \in M^2 \times \mathbb{R}_+^*$ is a couple of focus at distance t if the set*

$$F_{x_1,x_2,t} := \left\{\xi \in S^*_{x_1}M \middle| exp_{x_1} t\xi = x_2\right\}$$

of directions of geodesics stemming from x_1 and reaching x_2 in a time t, has a positive surface measure.

We denote T_{focus} the infimum of the $t \in \mathbb{R}_+^*$ such that there exists a couple of focus at distance t.

If M is compact, we have necessarily $T_{focus} > 0$.

Assumption 5.0.2 (Geometric control before refocusing)**.** *The open set ω satisfies the Geometric Control Condition in a time $T_0 < T_{focus}$.*

For example, for $\mathbb{T}^3$, there is no refocusing and the geometric assumption is the classical Geometric Control Condition. Yet, for the sphere S^3, our assumption is stronger. For example, it is fulfilled if ω is a neighborhood of $\{x_4 = 0\}$. We can imagine some geometric situations where the Geometric Control Condition is fulfilled while our condition is not, for example if we take only a a neighborhood of $\{x_4 = 0, x_3 \geq 0\}$ (see Remark 5.0.1 and Figure 5.1 for S^2). We do not know if the exponential decay is true in this case.

The main result of this article is the following theorem.

Theorem 5.0.1. *Let $R_0 > 0$ and ω satisfying Assumption 5.0.2. Then, there exist $T > 0$ and $\delta > 0$ such that for any (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1)$ in $H^1 \times L^2$, with*

$$\begin{array}{ll} \|(u_0, u_1)\|_{H^1 \times L^2} \leq R_0; & \|(\tilde{u}_0, \tilde{u}_1)\|_{H^1 \times L^2} \leq R_0 \\ \|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq \delta; & \|(\tilde{u}_0, \tilde{u}_1)\|_{L^2 \times H^{-1}} \leq \delta \end{array}$$

there exists $g \in L^\infty([0,T], L^2)$ supported in $[0,T] \times \omega$ such that the unique strong solution of

$$\left\{ \begin{array}{rcll} \Box u + u + |u|^4 u &=& g & \text{on} \quad [0,T] \times M \\ (u(0), \partial_t u(0)) &=& (u_0, u_1). & \end{array} \right.$$

satisfies $(u(T), \partial_t u(T)) = (\tilde{u}_0, \tilde{u}_1)$.

Let us discuss the assumptions on the size. In some sense, our theorem is a high frequency controllability result and expresses in a rough physical way that we can control some "small noisy data". In the subcritical case, two similar kind of results were proved : in Dehman Lebeau Zuazua [11] similar results were proved for the nonlinear wave equation but without the smallness assumption in $L^2 \times H^{-1}$ while in Dehman Lebeau [10], they obtain similar high frequency controllability results for the subcritical equation but in a uniform time which is actually the time of linear controllability (see also the work of the author [30] for the Schrödinger equation). Actually, this smallness assumption is made necessary in our proof because we are not able to prove the following unique continuation result.

Missing theorem. *$u \equiv 0$ is the unique strong solution in the energy space of*

$$\left\{ \begin{array}{rcll} \Box u + u + |u|^4 u &=& 0 & \text{on} \quad [0,T] \times M \\ \partial_t u &=& 0 & \text{on} \quad [0,T] \times \omega. \end{array} \right.$$

In the subcritical case, this kind of theorem can be proved with Carleman estimates under some additional geometrical conditions and once the solution is known to be smooth. Yet, in the critical case, we are not able to prove this propagation of regularity. Note also that H. Koch and D. Tataru [27] managed to prove some unique continuation result in the critical case, but in the case $u = 0$ on ω instead of $\partial_t u = 0$. In the case of $\mathbb{R}^3$ with flat metric and ω the complementary of a ball, B. Dehman and P. Gérard [8] prove this theorem using the existence of the scattering operator proved by K. Nakanishi [32], which is not available on a manifold.

Moreover, as in the subcritical case, we do not know if the time of controllability does depend on the size of the data. This is actually still an open problem for several nonlinear evolution equations such as nonlinear wave or Schrödinger equation (even in the subcritical case). Note that for certain nonlinear parabolic equations, it has been proved that we can not have controllability in arbitrary short time, see [15] or [14].

The strategy for proving Theorem 5.0.1 consists in proving a stabilization result for a damped nonlinear Klein-Gordon equation and then, by a perturbative argument using the linear control, to bring the solution to zero once the energy of the solution is small enough. Namely, we prove

Theorem 5.0.2. *Let $R_0 > 0$, ω satisfying Assumption 5.0.2 and $a \in C^\infty(M)$ satisfying $a(x) > \eta > 0$ for all $x \in \omega$. Then, there exist $C, \gamma > 0$ and $\delta > 0$ such that for any (u_0, u_1) in $H^1 \times L^2$, with*

$$\|(u_0, u_1)\|_{H^1\times L^2} \leq R_0; \qquad \|(u_0, u_1)\|_{L^2\times H^{-1}} \leq \delta;$$

the unique strong solution of

$$\left\{\begin{array}{rcl} \Box u + u + |u|^4 u + a(x)^2 \partial_t u & = & 0 \quad \text{on} \quad [0,T]\times M \\ (u(0), \partial_t u(0)) & = & (u_0, u_1). \end{array}\right. \tag{0.3}$$

satisfies $E(u)(t) \leq Ce^{-\gamma t}E(u)(0)$.

This theorem is false for the classical nonlinear wave equation (see subsection 5.3.1.1) and it is why we have chosen the Klein-Gordon equation instead.

Let us now discuss the proof of Theorem 5.0.2, following B. Dehman and P. Gérard [8] for the case of $\mathbb{R}^3$. We have the energy decay

$$E(u)(t) = E(u)(0) - \int_0^t \int_M |a(x)\partial_t u|^2.$$

So, the exponential decay is equivalent to an observability estimate for the nonlinear damped equation. We prove it by contradiction. We are led to proving the strong convergence to zero of a normalized sequence u_n of solutions contradicting observability. In the subcritical case, the argument consisted in two steps

- to prove that the limit is zero by a unique continuation argument
- to prove that the convergence is actually strong by linearization and linear propagation of compactness thanks to microlocal defect measures of P. Gérard [19] and L. Tartar [36].

By linearization, we mean (according to the terminology of P. Gérard [20]) that we have $|||u_n - v_n||| \underset{n\to\infty}{\longrightarrow} 0$ where v_n is solution of the linear Klein-Gordon equation with same initial data :

$$\left\{\begin{array}{rcl} \Box v_n + v_n & = & 0 \quad \text{on} \quad [0,T]\times M \\ (v_n(0), \partial_t v_n(0)) & = & (u_n(0), \partial_t u_n(0)). \end{array}\right.$$

In our case, the smallness assumption in the lower regularity $L^2 \times H^{-1}$ makes that the limit is automatically zero, which allows to skip the first step. In the subcritical case, any sequence weakly convergent to zero is linearizable. Yet, for critical nonlinearity, there exists nonlinearizable sequences. Hopefully, in the case of $\mathbb{R}^3$, this defect can be precisely described. It is linked to the non compact action of the invariants of the equation : the dilations and translations. More precisely, the work of H. Bahouri and P. Gérard [2] states that any bounded sequence u_n of solutions to the nonlinear critical wave equation can be decomposed into an infinite sum of : the weak limit of u_n, a sequence of solutions to the free wave equation and an infinite sum of profiles which are translations-dilations of fixed nonlinear solutions. This decomposition was used by the authors of [8] to get the expected result in $\mathbb{R}^3$. Therefore, we are led to make an analog of this profile deomposition for compact manifolds. We begin by the definition of the profiles.

Definition 5.0.2. *Let $x_\infty \in M$ and $(f,g) \in \mathcal{E}_{x_\infty} = (\dot{H}^1 \times L^2)(T_{x_\infty}M)$. Given $[(f,g),\underline{h},\underline{x}] \in \mathcal{E}_{x_\infty} \times (\mathbb{R}_+^* \times M)^{\mathbb{N}}$ such that $lim_n(h_n,x_n) = (0,x_\infty)$ We call the associated concentrating data the class of equivalence, modulo sequences convergent to 0 in $\mathcal{E}$, of sequence in $\mathcal{E}$ that take the form*

$$h_n^{-\frac{1}{2}}\Psi_U(x)\left(f,\frac{1}{h_n}g\right)\left(\frac{x-x_n}{h_n}\right)+o(1)_{\mathcal{E}} \tag{0.4}$$

in some coordinate patch $U_M \approx U \subset \mathbb{R}^d$ containing x_∞ and for some $\Psi_U \in C_0^\infty(U)$ such that $\Psi_U(x) = 1$ in a neighborhood of x_∞. (Here, we have identified x_n, x_∞ with its image in U).

We will prove later (Lemma 5.1.3) that this definition does not depend on the coordinate charts and on Ψ_U : two sequences defined by (0.4) in different coordinate charts are in the same class. In what follows, we will often call concentrating data associated to $[(f,g),\underline{h},\underline{x}]$ an arbitrary sequence in this class.

Definition 5.0.3. *Let (t_n) a bounded sequence in $\mathbb{R}$ converging to t_∞ and (f_n,g_n) a concentrating data associated to $[(f,g),\underline{h},\underline{x}]$. A damped linear concentrating wave is a sequence v_n solution of*

$$\left\{\begin{array}{rcl} \Box v_n + v_n + a(x)\partial_t v_n & = & 0 \quad \text{on} \quad \mathbb{R}\times M \\ (v_n(t_n),\partial_t v_n(t_n)) & = & (f_n,g_n). \end{array}\right. \tag{0.5}$$

The associated damped nonlinear concentrating wave is the sequence u_n solution of

$$\left\{\begin{array}{rcl} \Box u_n + u_n + a(x)\partial_t u_n + |u_n|^4 u_n & = & 0 \quad \text{on} \quad \mathbb{R}\times M \\ (u_n(0),\partial_t u_n(0)) & = & (v_n(0),\partial_t v_n(0)). \end{array}\right. \tag{0.6}$$

If $a \equiv 0$, we will only write linear or nonlinear concentrating wave.

Energy estimates yields that two representants of the same concentrating data have the same associated concentrating wave modulo strong convergence in $L^\infty_{loc}(\mathbb{R},\mathcal{E})$. This is not obvious for the nonlinear evolution but will be a consequence of the study of nonlinear concentrating waves.

It can be easily seen that this kind of nonlinear solutions are not linearizable. Actually, it can be shown that this concentration phenomenon is the only obstacle to linearizability. We begin with the linear decomposition.

Theorem 5.0.3. *Let (v_n) be a sequence of solutions to the damped Klein-Gordon equation (0.5) with initial data at time $t = 0$ (φ_n, ψ_n) bounded in $\mathcal{E}$. Then, up to extraction, there exist a sequence of damped linear concentrating waves $(\underline{p}^{(j)})$, as defined in Definition 5.0.3, associated to concentrating data $[(\varphi^{(j)}, \psi^{(j)}), \underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)}]$, such that for any $l \in \mathbb{N}^*$,*

$$v_n(t,x) = v(t,x) + \sum_{j=1}^{l} p_n^{(j)}(t,x) + w_n^{(l)}(t,x), \tag{0.7}$$

$$\forall T > 0, \quad \overline{\lim_{n\to\infty}} \left\|w_n^{(l)}\right\|_{L^\infty([-T,T],L^6(M))\cap L^5([-T,T],L^{10})} \underset{l\to\infty}{\longrightarrow} 0 \tag{0.8}$$

$$\|(v_n, \partial_t v_n)\|_{\mathcal{E}}^2 = \sum_{j=1}^{l} \left\|(p_n^{(j)}, \partial_t p_n^{(j)})\right\|_{\mathcal{E}}^2 + \left\|(w_n^{(l)}), \partial_t w_n^{(l)})\right\|_{\mathcal{E}}^2 + o(1), \text{ as } n \to \infty, \tag{0.9}$$

where $o(1)$ is uniform for $t \in [-T, T]$.

The nonlinear flow map follows this decomposition up to an error term in the strong following norm

$$|||u|||_I = \|u\|_{L^\infty(I,H^1(M))} + \|\partial_t u\|_{L^\infty(I,L^2(M))} + \|u\|_{L^5(I,L^{10}(M))}.$$

Theorem 5.0.4. *Let $T < T_{focus}/2$. Let u_n be the sequence of solutions to damped non linear Klein-Gordon equation (0.6) with initial data, at time 0, (φ_n, ψ_n) bounded in $\mathcal{E}$. Denote $p_n^{(j)}$ (resp v the weak limit) the linear damped concentrating waves given by Theorem 5.0.3 and $q_n^{(j)}$ the associated nonlinear damped concentrating wave (resp u the associated solution of the nonlinear equation with $(u, \partial_t u)_{t=0} = (v, \partial_t v)_{t=0}$). Then, up to extraction, we have*

$$u_n(t,x) = u + \sum_{j=1}^{l} q_n^{(j)}(t,x) + w_n^{(l)}(t,x) + r_n^{(l)} \tag{0.10}$$

$$\overline{\lim_{n\to\infty}} \; |||r_n^{(l)}|||_{[-T,T]} \underset{l\to\infty}{\longrightarrow} 0 \tag{0.11}$$

where $w_n^{(l)}$ is given by Theorem 5.0.3.

The same theorem remains true if M is the sphere S^3 and $a \equiv 0$ (undamped equation) without any assumption on the time T.

The more precise result we get for the sphere S^3 will not be useful for the proof of our controllability result. Yet, we have chosen to give it because it is the only case where we are able to describe what happens when some refocusing occurs.

This profile decomposition has already been proved for the critical wave equation on $\mathbb{R}^3$ by H. Bahouri and P. Gerard [2] and on the exterior of a convex obstacle by I. Gallagher and P. Gérard [17]. The same decomposition has also been performed for the Schrödinger equation by S. Keraani [26] and quite recently for the wave maps by Krieger and Schlag [28]. Note that such decomposition has proved to be useful in different contexts : the understanding of the precise behavior near the threshold for well-posedness for focusing nonlinear wave see Kenig Merle [25] and Duyckaerts Merle[13], the study of the compactness of Strichartz estimates and maximizers for Strichartz estimates, (see Keraani [26]), global existence for wave maps [28]... May be our decomposition on manifolds could be useful in one of these contexts. Let us also mention that, this kind of decomposition appears for a long time in the context of Palais-Smale sequences for critical elliptic equation and optimal constant for Sobolev embedding, but with a finite number of profiles, see Brezis Coron [4], the book [12] and the references therein ...

Let us describe quickly the proof of the decomposition. The linear decomposition of Theorem 5.0.3 is made in two steps : first, we decompose our sequence in a sum of an infinite number of sequences oscillating at different rate $h_n^{(j)}$. Then, for each part oscillating at a fixed rate, we extract the possible concentration at certain points. We only have to prove that this process produces a rest w_n^l that gets smaller in the norm $L^\infty L^6$ at each stage. Once the linear decomposition is established, Theorem 5.0.4 says, roughly speaking, that the nonlinear flow map acts almost linearly on the linear decomposition. To establish the nonlinear decomposition we have to prove that each element of the decomposition do not interact with the others. For each element of the linear decomposition, we are able to describe the nonlinear solution arizing from this element as initial data. The linear rest w_n^l is small in $L^\infty([-T,T],L^6)$ for l large enough and so the associated nonlinear solution with same initial data is very close to the linear one. The behavior of nonlinear concentrating waves is described in [22] (see subsection 5.2.2.1 for a short review). Before the concentration, linear and nonlinear waves are very close. For times close to the time of concentration, the nonlinear rescaled solution behaves as if the metric was flat and is subject to the scattering of $\mathbb{R}^3$. After concentration, the solution is close to a linear concentrating wave but with a new profile obtained by the scattering operator on $\mathbb{R}^3$.

We finish this introduction by a discussion on the geometric conditions we imposed to get our main theorem. For the linear wave equation, the controllability is known to be equivalent to the so called Geometric Control Condition (Assumption 5.0.1). This was first proved by Rauch and Taylor [33] in the case of a compact manifold and by Bardos Lebeau Rauch [3] for boundary control (see Burq Gérard for the necessity [5]). For the nonlinear subcritical problem, the result of [10] only requires the classical Geometric Control Condition. Our assumption is stronger and we can naturally wonder if it is really necessary. It is actually strongly linked with the critical behavior and nonlinear concentrating waves. Removing this stronger assumption would require a better understanding of the scattering operator of the nonlinear equation on $\mathbb{R}^3$ (see Remark 5.0.1). However, we think that the same result could be obtained with the following weaker assumption.

Assumption 5.0.3. *ω satisfies the Geometric Control Condition. Moreover, for every couple of focus (x_1, x_2, t) at distance t, according to Definition 5.0.1, each geodesic starting from x_1 in direction ξ such that $exp_{x_1} t\xi = x_2$ meets ω in a time $0 \leq s < t$.*

Finally, we note that our theorem can easily be extended to the case of $\mathbb{R}^3$ with a metric flat at infinity. In this case, our stabilization term $a(x)$ should fulfill the both assumptions

- there exist $R > 0$ and $\rho > 0$ such that $a(x) > \rho$ for $|x| > R$
- $a(x) > \rho$ for $x \in \omega$ where ω satisfies Assumption 5.0.2.

The proof would be very similar. The only difference would come from the fact that the domain is not compact. So the profile decomposition would require the "compactness at infinity" (see property (1.6) of [2]). Moreover, the equirepartition of the energy could not be made only with measures but with an explicit computation (see (3.14) of [8])

Remark 5.0.1. *In order to know if our stronger Assumptions 5.0.2 or 5.0.3 are really necessary compared to the classical Geometric Control Condition, we need to prove that the following scenario can not happen. We take the example of S^3 with ω a neighborhood of $\{x_4 = 0, x_3 \geq 0\}$.*

Take some data concentrating on the north pole, with a Fourier transform (on the tangent plane) supported around a direction ξ_0. The nonlinear solution will propagate linearly as long as it does not concentrate : at time t it will be supported in an neighborhood of the point $x(t)$ where $x(t)$ follows the geodesic stemming from the north pole at time 0 in direction ξ_0. Then, if ξ_0 is well chosen, it can avoid ω during that time. Yet, at time π, the solution will concentrate again in the south pole. According to the description of S. Ibrahim [22], in a short time, the solution will be transformed following the nonlinear scattering operator on $\mathbb{R}^3$. So, at time $\pi + \varepsilon$ the solution is close to a linear concentrating wave but it concentrates with a new profile which is obtained with the nonlinear scattering operator on $\mathbb{R}^3$. This operator is strongly nonlinear and we do not know whether the new profile will be supported in Fourier near a new direction ξ_1. If it happens, the solution will then be supported near the point $y(t)$ where $y(t)$ follows the geodesic stemming from the south pole at time π in direction ξ_1. In this situation, it will be possible that the trajectory $y(t)$ still avoids ω. If this phenomenon happens several times, we would have a sequence that concentrates periodically on the north and south pole but always avoiding the region ω (which in that case satisfies Geometric Control Condition).

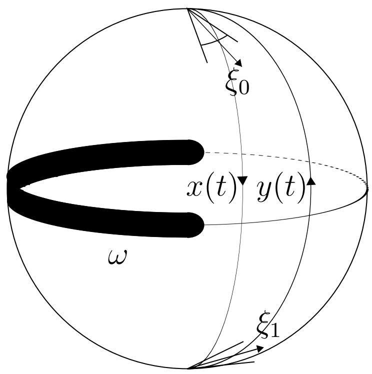

FIGURE 5.1 – Possible situation on the sphere

We are led to the following informal question. If S is the scattering operator on $\mathbb{R}^3$, is that possible that for some data $(f, g) \in \dot{H}^1 \times L^2$ supported in Fourier near a direction ξ_0, the Fourier transform of $S(f, g)$ is supported near another direction ξ_1. In other words, can the nonlinear wave operator change the direction of the light ?

The structure of the article is as follows. The first section contains some preliminaries that will be used all along the article : the existence theorem for damped nonlinear equation, the description of the main properties of concentrating waves and the useful properties of the scales necessary for the linear decomposition. The second section contains the proof of the profile decomposition of Theorem 5.0.3 and 5.0.4. It is naturally divided in two steps corresponding to the linear decomposition and the nonlinear one. We close this section by some useful consequences of the decomposition. The third section contains the proof of the main theorems : the control and stabilization.

Note that the main argument for the proof of stabilization is contained in the last section 5.3 : in Proposition 5.3.1 we apply the linearization argument to get rid of the profiles while Theorem 5.3.1 contains the proof of the weak observability estimates. We advise the hurried reader to have a first glance at these two proofs in order to understand the global argument.

5.0.1 Notation

For I an interval, denote

$$|||u|||_I = \|u\|_{L^\infty(I,H^1(M))} + \|\partial_t u\|_{L^\infty(I,L^2(M))} + \|u\|_{L^5(I,L^{10}(M))}\,.$$

Moreover, when we work in local coordinate, we will need the similar norm (except for $\dot{H}^1$ instead of H^1)

$$|||u|||_{I\times\mathbb{R}^3} = \|u\|_{L^\infty(I,\dot{H}^1(\mathbb{R}^3))} + \|\partial_t u\|_{L^\infty(I,L^2(\mathbb{R}^3))} + \|u\|_{L^5(I,L^{10}(\mathbb{R}^3))}\,.$$

Note that the if $I=\mathbb{R}$, $|||u|||_{I\times\mathbb{R}^3}$ is invariant by the translation and scaling $u\mapsto \frac{1}{\sqrt{h}}u\left(\frac{t-t_0}{h},\frac{x-x_0}{h}\right)$.

The energy spaces are denoted

$$\begin{aligned}\mathcal{E} &= H^1(M)\times L^2(M)\\ \mathcal{E}_{x_\infty} &= \dot{H}^1(T_{x_\infty}M)\times L^2(T_{x_\infty}M)\end{aligned}$$

with the respective norms

$$\begin{aligned}\|(f,g)\|^2_{\mathcal{E}} &= \|f\|^2_{L^2(M)} + \|\nabla f\|^2_{L^2(M)} + \|g\|^2_{L^2(M)}\\ \|(f,g)\|^2_{\mathcal{E}_\infty} &= \|\nabla f\|^2_{L^2(T_{x_\infty}M)} + \|g\|^2_{L^2(T_{x_\infty}M)}\end{aligned}$$

We will denote $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{E}_\infty}$ the associated scalar product.

When dealing with solutions of non linear wave equations on M (or on $T_{x_\infty}M$), "the unique strong solution" will mean the unique solution in the Strichartz space $L^5_{loc}(\mathbb{R},L^{10}(M))$ (or $L^5_{loc}(\mathbb{R},L^{10}(T_{x_\infty}M))$) such that $(u,\partial_t u)\in C(\mathbb{R},\mathcal{E})$ (or $C(\mathbb{R},\mathcal{E}_{x_\infty})$.

All along the article, for a point $x\in M$, we will sometime not distinguish x with its image in a coordinate patch and will write $\mathbb{R}^3$ instead of $T_{x_\infty}M$. M will always be smooth, compact and the number of coordinate charts we use is always assumed to be finite. We also assume that all the charts are relatively compact. In all the article, C will denote any constant, possibly depending on the manifold M and the damping function a. We will also write $\lesssim$ instead of $\leq C$ for a constant C.
$B^s_{2,\infty}(M)$ denotes the Besov space on M defined by

$$\|u\|_{B^s_{2,\infty}(M)} = \left\|\mathbf{1}_{[0,1[}(\sqrt{-\Delta_M})u\right\|_{L^2(M)} + \sup_{k\in\mathbb{N}}\left\|\mathbf{1}_{[2^k,2^{k+1}[}(\sqrt{-\Delta_M})u\right\|_{H^s(M)}\,.$$

We use the same definition for $B^s_{2,\infty}(\mathbb{R}^3)$ with Δ_M replaced by $\Delta_{\mathbb{R}^3}$ which can be expressed using the Fourier transform and Littlewood-Paley decomposition. Of course,

$B^s_{2,\infty}(M)$ is linked with $B^s_{2,\infty}(\mathbb{R}^3)$ by the expression in coordinate charts. This will be precised in Lemma 5.2.1.

From now on, $a = a(x)$ will always denote a smooth real valued function defined on M.

5.1 Preliminaries

5.1.1 Existence theorem

The existence of solutions to our equation is proved using two tools : Strichartz and Morawetz estimates. Strichartz estimates take the following form.

Proposition 5.1.1 (Strichartz and energy estimates). *Let $T > 0$ and (p, q) satisfying*

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{2}, \quad p > 2$$

Then, there exists $C > 0$ such that any solution u of

$$\begin{cases} \Box v + v + a(x)\partial_t v &= f \quad \text{on} \quad [-T, T] \times M \\ (v(0), \partial_t u(0)) &= (u_0, u_1). \end{cases}$$

satisfies the estimate

$$\|(v, \partial_t v)\|_{L^\infty([-T,T],\mathcal{E})} + \|v\|_{L^p([-T,T],L^q(M))} \leq C(\|(u_0, u_1)\|_{\mathcal{E}} + \|f\|_{L^1([-T,T],L^2)}).$$

Démonstration. The case with $a \equiv 0$ for the wave equation can be found in L.V. Kapitanski [24]. To treat the case of damped Klein-Gordon, we only have to absorb the additional terms and get the desired estimate for T small enough. We can then reiterate the operation to get the result for large times. □

Then, we are going to prove global existence for the equation

$$\begin{cases} \Box u + u + |u|^4 u &= a(x)\partial_t u + g \quad \text{on} \quad [-T, T] \times M \\ (u(0), \partial_t u(0)) &= (u_0, u_1) \in \mathcal{E} \end{cases} \tag{1.12}$$

with $g \in L^1([-T, T], L^2(M))$ and $a \in C^\infty(M)$.

The proof is now very classical, see for example [37] for a survey of the subject. The critical defocusing nonlinear wave equation on $\mathbb{R}^3$ was proved to be globally well posed by Shatah and Struwe [35, 34] using Morawetz estimates. Later, S. Ibrahim an M. Majdoub managed to apply this strategy in the case of variable coefficients in [23], but without damping and forcing term. In this subsection, we extend this strategy to the case with these additional terms. We also refer to the appendix of [2] where the computation of Morawetz estimates on $\mathbb{R}^3$ is made with a forcing term. We also mention the result of N. Burq, G. Lebeau and F. Planchon [7] in the case of 3-D domains.

We only have to check that the two additional terms do not create any trouble. Actually, the main difference is that the energy in the light cones is not decreasing, but it is locally "almost decreasing" (see formula (1.13)) and this will be enough to conclude with the same type of arguments.

As usual in critial problems, the local problem is well understood thanks to Strichartz estimates while we have to prove global existence. We only consider Shatah Struwe solutions, that is satisfying Strichartz estimates and we wave uniqueness for local solutions in this class. We assume that there is a maximal time of existence t_0 and we want to prove that it is infinite . The solution considered will be limit of smooth solutions of the nonlinear equation with smoothed initial data and nonlinearity. Therefore, the integrations by part are licit by a limiting argument.

We need some notations. To simplify the notations, the space-time point where we want to extend the solution will be $z_0 = (t_0, x_0) = (0, 0)$. φ is the geodesic distance on M to $x_0 = 0$ defined in a neighborhood U of 0. Denote for some small $\alpha < \beta < 0$

$$\begin{array}{rll} K_\alpha^\beta := & \{z = (t,x) \in [\alpha,\beta] \times U \,|\varphi \leq |t|\} & \text{backward truncated cone} \\ M_\alpha^\beta := & \{z = (t,x) \in [\alpha,\beta] \times U \,|\varphi = |t|\} & \text{mantle of the truncated cone} \\ D(t) := & \{x \in U \,|\varphi \leq |t|\} & \text{spacelike section of the cone at time } t \end{array}$$

In what follows, the gradient, norm, density are computed with respect to the Riemannian metric on M (for example, we have $|\nabla\varphi| = 1$). We also define

$$\begin{aligned} e(u)(t,x) &:= \frac{1}{2}\left(|\partial_t u|^2 + |\nabla u|^2\right) + \frac{1}{6}|u|^6 \quad \text{local energy} \\ E(u, D(t)) &:= \int_{D(t)} e(u)(t,x)dx \quad \text{energy at time } t \text{ in the section of the cone} \\ Flux(u, M_\alpha^\beta)) &:= \frac{1}{\sqrt{2}}\int_{M_\alpha^\beta} \frac{1}{2}|\partial_t u\nabla\varphi - \nabla u|^2 + \frac{1}{6}|u|^6 \; d\sigma \quad \text{flux exiting the truncated cone} \end{aligned}$$

Lemma 5.1.1. *Let u be a solution of equation (1.12). The function $E(u, D(t))$ satisfies for $\alpha < \beta < 0$*

$$\begin{aligned} E(u, D(\beta)) + Flux(u, M_\alpha^\beta) &= E(u, D(\alpha)) + \iint_{K_\alpha^\beta} a(x)|\partial_t u|^2 \\ &\quad -\Re\iint_{K_\alpha^\beta} u\partial_t\bar{u} + \Re\iint_{K_\alpha^\beta} g\partial_t\bar{u} \end{aligned}$$

and it has a left limit in $t = 0$ as a function of t.

Démonstration. The identity is obtained by multiplying the equation by $\partial_t\overline{u}$ to get $\partial_t e(u) - \Re div(\partial_t\overline{u}\nabla_x u) = a(x)|\partial_t u|^2 - u\partial_t\bar{u} + \Re g\partial_t\overline{u}$, then, we integrate over the truncated cone K_α^β and use Stokes formula. Denote $f(t) = E(u, D(t))$. Using the positivity of the flux and Hölder inequality, we estimate

$$\begin{aligned} \|f\|_{L^\infty([\alpha,\beta])} &\leq f(\alpha) + C(\beta-\alpha)\|f\|_{L^\infty([\alpha,\beta])} + C|\alpha|(\beta-\alpha)\|f\|_{L^\infty([\alpha,\beta])}^{2/3} \\ &\quad + \|g\|_{L^1([\alpha,\beta],L^2)}\|f\|_{L^\infty([\alpha,\beta])}^{1/2}. \end{aligned}$$

Using $C|\alpha|(\beta-\alpha)\|f\|_{L^\infty([\alpha,\beta])}^{2/3} \leq C(\beta-\alpha)(\|f\|_{L^\infty([\alpha,\beta])}^{1/2} + \|f\|_{L^\infty([\alpha,\beta])})$, we get for $\beta - \alpha$ small enough,

$$f(\beta)^{1/2} \leq \frac{1}{1 - 2C(\beta-\alpha)}\left[f(\alpha)^{1/2} + C(\beta-\alpha) + \|g\|_{L^1([\alpha,\beta],L^2)}\right] \tag{1.13}$$

This property will replace the decreasing of the energy that occurs without damping and forcing term in all the rest of the proof. It easily implies that f has a left limit. $\square$

Lemma 5.1.2. *For u and g a strong solution of*

$$\Box u + |u|^4 u = g \quad \text{on} \quad [-T, 0[\times M$$

we have the estimate

$$\begin{aligned}\int_{D(\alpha)} |u|^6 \leq\ & C\Bigg(\frac{\beta}{\alpha}\left(f(\beta)) + f(\beta)^{1/3}\right) + |f(\beta) - f(\alpha)| + \|g\|_{L^1L^2(K_\alpha^\beta)} \|\partial_t u\|_{L^\infty L^2(K_\alpha^\beta)} \\ & + \left(|f(\beta) - f(\alpha)| + \|g\|_{L^1L^2(K_\alpha^\beta)} \|\partial_t u\|_{L^\infty L^2(K_\alpha^\beta)}\right)^{1/3} \\ & + \|g\|_{L^1L^2(K_\alpha^\beta)} \left(\|\partial_t u\|_{L^\infty L^2(K_\alpha^\beta)} + \|\nabla u\|_{L^\infty L^2(K_\alpha^\beta)} + \|u\|_{L^\infty L^6(K_\alpha^\beta)}\right) \\ & + (\beta - \alpha) \sup_{t\in[\alpha,\beta]} \left[f(t) + f(t)^{1/3}\right]\Bigg)\end{aligned}$$

where we have used the notation $f(t) = E(u, D(t))$.

Démonstration. It is a consequence of Morawetz estimates. The only difference is the presence of the forcing term g and the metric. The case of flat metric is treated in [2]. The metric leads to the same estimates with an additional term $(\beta-\alpha) \sup_{t\in[\alpha,\beta]} f(t)+f(t)^{1/3}$ as treated in [23]. Another minor difference is that in the presence of a forcing term, the energy does not decrease and $f(\beta)+f(\beta)^{1/3}$ have to be replaced by the supremum on the interval. Note also that our estimate is made in the backward cone while the computation is made in the future cone in these references. We leave the easy modifications to the reader. $\square$

The previous estimates will be the main tools of the proof. It will be enough to prove some non concentration property in the light cone for $L^\infty L^6$, $L^5 L^{10}$ and finally in energy space. It is the object of the following three corollaries.

Corollary 5.1.1.

$$\int_{D(\alpha)} |u(\alpha, x)|^6 dx \underset{\alpha\to 0}{\longrightarrow} 0.$$

Démonstration. We are going to use the previous Lemma 5.1.2, replacing g by $g - u + a(x)\partial_t u$ and with $\beta = \varepsilon\alpha$, $0 < \varepsilon < 1$. Denote L the limit of $f(t)$ as t tends to 0 given by Lemma 5.1.1. So for α small enough, we have for a constant $C > 0$

$$\|\partial_t u\|_{L^\infty L^2(K_\alpha^\beta)} + \|\nabla u\|_{L^\infty L^2(K_\alpha^\beta)} + \|u\|_{L^\infty L^6(K_\alpha^\beta)} \leq 1 + C(L^{1/2} + L^{1/6}).$$

We also use

$$\begin{aligned}\|g - u + a(x)\partial_t u\|_{L^1L^2(K_\alpha^\beta)} &\leq \|g\|_{L^1L^2(K_\alpha^\beta)} + C(\beta - \alpha) \|u\|_{L^\infty L^2(K_\alpha^\beta)} \\ &\quad + C(\beta - \alpha) \|\partial_t u\|_{L^\infty L^2(K_\alpha^\beta)} \\ &\leq \|g\|_{L^1L^2(K_\alpha^\beta)} + C(\beta - \alpha)(1 + L^{1/6} + L^{1/2})\end{aligned}$$

which tends to 0 as β tends to 0. This yields

$$\overline{\lim_{\alpha\to 0}} \int_{D(\alpha)} |u(\alpha,x)|^6 dx \le C\varepsilon(L + L^{1/3}).$$

$\square$

Corollary 5.1.2.

$$u \in L^5L^{10}(K^0_{-T}).$$

Démonstration. Localized Strichartz estimates in cones (see Proposition 4.4 of [23]) give

$$\begin{aligned}
\|u\|_{L^4L^{12}(K^0_s)} &\le CE(u,D(s))^{1/2} + \|u\|^5_{L^5L^{10}(K^0_s)} + \|a(x)\partial_t u - u + g\|_{L^1L^2(K^0_s)} \\
&\le CE(u,D(s))^{1/2} + \|u\|_{L^\infty L^6(K^0_s)} (1 + \|u\|^4_{L^4L^{12}(K^0_s)}) \\
&\quad + \|\partial_t u\|_{L^\infty L^2(K^0_s)} + \|g\|_{L^1L^2(K^0_s)} .
\end{aligned}$$

A boot-strap argument and Corollary 5.1.1 give that for s sufficiently close to 0, $\|u\|_{L^4L^{12}(K^0_s)}$ is bounded. We get the announced result by interpolation between L^4L^{12} and $L^\infty L^6$. $\square$

Corollary 5.1.3.

$$E(u,D(s)) \underset{s\to 0}{\longrightarrow} 0.$$

Démonstration. Let $\varepsilon > 0$. Corollary 5.1.2 allows to fix $s < 0$ close to 0 so that $\|u\|_{L^5L^{10}(K^0_s)} \le \varepsilon$. Denote v_s the solution to the linear equation

$$\Box v_s + v_s + a(x)\partial_t v_s = 0, \quad (v_s, \partial_t v)_{t=s} = (u, \partial_t u)_{t=s}$$

then, the difference $w_s = u - v_s$ is solution of

$$\Box w_s + w_s + a(x)\partial_t w_s = -|u|^4 u, \quad (w_s, \partial_t w_s)_{t=s} = (0,0).$$

Then, for $s < t < 0$, linear energy estimates give

$$E_0(w_s, D(t))^{1/2} \le C \|u\|^5_{L^5L^{10}(K^0_s)} \le C\varepsilon^5$$

where we have set

$$E_0(w_s, D(t)) = \frac{1}{2}\int_{D(t,z_0)} \left[|\nabla w_s|^2 + |\partial_t w_s|^2\right] dx.$$

Triangular inequality yields

$$E_0(u, D(t))^{1/2} \le E_0(v_s, D(t))^{1/2} + C\varepsilon^5.$$

Since v_s is solution of the free damped linear equation, we have $E_0(v_s, D(t)) \underset{t\to 0}{\longrightarrow} 0$. This yields the result with E_0 instead of E. The final result is obtained thanks to Corollary 5.1.1. $\square$

We can now finish the proof of global existence.

Let $\varepsilon > 0$ to be chosen later. By Corollary 5.1.3, $E(u, D(s)) \leq \varepsilon$ for s close enough to 0. By dominated convergence, for any $s < 0$ close to 0, there exists $\eta > 0$ so that

$$\int_{\varphi(x) \leq t_0 - s + \eta} e(u)(s) = E(u, D(s, \eta)) \leq 2\varepsilon$$

where $E(u, D(s, \eta))$ is the spacelike energy at time s of the cone centered at $(t_0 = \eta, x_0 = 0)$ (see Figure 5.2). For s close enough to 0 and $s < s' < 0$, we apply estimate (1.13) in this cone. It gives

$$E(u, D(s', \eta))^{1/2} \leq C\left(E(u, D(s, \eta))^{1/2} + |s' - s| + \|g\|_{L^1([s,s'],L^2)}\right) \leq C\varepsilon^{1/2}.$$

In particular, $\|u\|_{L^\infty L^6(K)} \leq C\varepsilon^{1/2}$ on the truncated cone

$$K = \{(s', x) \,|\, \varphi(x) \leq \eta - s', s < s' < 0\}$$

Therefore, choosing ε small enough to apply the same proof as Corollary 5.1.2, we get

$$\|u\|_{L^5 L^{10}(K)} < +\infty.$$

Since $x_0 = 0$ is arbitrary, a compactness argument yields one $s < 0$ such that $\|u\|_{L^5([s,0[L^{10}(M))} < +\infty$. Therefore, by Duhamel formula, $(u(t), \partial_t u(t))$ has a limit in $\mathcal{E}$ as t tends to 0 and u can be extended for some small $t > 0$ using local existence theory.

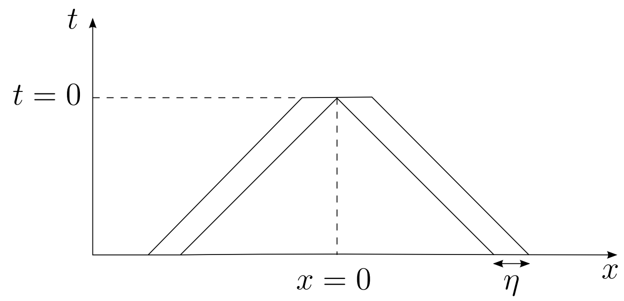

FIGURE 5.2 – The truncated cone K

Remark 5.1.1. *It is likely that global existence can also be proved using the Kenig-Merle argument [25] and the profile decomposition below (assuming only local existence) as is done for example in [28] for the wave maps.*

5.1.2 Concentration waves

In this section, we give details about concentrating waves that will be useful in the profile decomposition. The first lemma states that Definition 5.0.2 of concentrating data does not depend on the choice of coordinate patch and cut-off function Ψ_U.

Lemma 5.1.3. *Let $[(f, g), \underline{h}, \underline{x}] \in \mathcal{E} \times (\mathbb{R}_+^* \times M\times)^{\mathbb{N}}$ such that $lim_n(h_n, x_n) = (0, x_\infty)$ then, all the sequences defined by formula (0.4) in different coordinates charts and the cut-off function Ψ_U are equivalent, modulo convergence in $\mathcal{E}$.*

Démonstration. It is very close to the one of S. Ibrahim [22] where the concentrating data are given in geodesic coordinates. So, let $V_M \approx V$ be another coordinate patch and $\Phi : V \mapsto U$ the associated transition map. Without loss of generality, we can suppose that x_∞ is represented by 0 in U and V. We have to prove that the sequences

$$h_n^{-\frac{1}{2}}\Phi^*\Psi_U(x)(f, \frac{1}{h_n}g)\left(\frac{x-\Phi(x_n)}{h_n}\right) = h_n^{-\frac{1}{2}}\Psi_U(\Phi(x))(f, \frac{1}{h_n}g)\left(\frac{\Phi(x)-\Phi(x_n)}{h_n}\right)$$

and

$$h_n^{-1/2}\Psi_V(x)(f\circ D\Phi(0), \frac{1}{h_n}g\circ D\Phi(0))\left(\frac{x-x_n}{h_n}\right)$$

are equivalent in the energy space associated to M or $\mathbb{R}^3$ (the volume form and the gradient are not the same but the energies are equivalent). By approximation, we can assume $(f,g)\in(C_0^\infty(\mathbb{R}^3))^2$. We make the proof for the $\dot{H}^1$ part for f, the proof being simpler for g. We remark that the terms coming from derivatives hitting on $\Psi_U(x)$ tend to 0 in L^2. Therefore, we have to prove the convergence to 0 of

$$h_n^{-3}\left\|\Psi_U(\Phi(x))D\Phi(x)\nabla f\left(\frac{\Phi(x)-\Phi(x_n)}{h_n}\right) - \Psi_V(x)D\Phi(0)\nabla f\left(\frac{D\Phi(0)x - D\Phi(0)x_n}{h_n}\right)\right\|^2_{L^2(V)}.$$

First, we prove that the cut-off functions Ψ_U and Ψ_V can be replaced by a unique Ψ. Let δ so that $B(0,\delta)\subset V$. Let $\Psi\in C_0^\infty(B(0,\delta))$ such that $\Psi\equiv 1$ in a neighborhood of 0 and has a support included in the set of x such that $\Psi_V(x) = \Psi_U(\Phi(x)) = 1$, so that $\Psi\Psi_V = \Psi$ and $\Psi(\Psi_U\circ\Phi) = \Psi$. Then, on the support of $1-\Psi$, we have $\|\Phi(x)-\Phi(x_n)\| > \varepsilon$ for some $\varepsilon > 0$ and some n large enough. Therefore, we have

$$\begin{aligned}
& h_n^{-3}\left\|(1-\Psi(x))\Psi_U(\Phi(x))D\Phi(x)\nabla f\left(\frac{\Phi(x)-\Phi(x_n)}{h_n}\right)\right\|^2_{L^2(V)} \\
\leq\ & C h_n^{-3}\left\|\nabla f\left(\frac{\Phi(x)-\Phi(x_n)}{h_n}\right)\right\|^2_{L^2(\|\Phi(x)-\Phi(x_n)\|>\varepsilon)}
\end{aligned}$$

which is 0 for n large enough since f has compact support. Making the same proof for the other term, we are led to prove the convergence to 0 of

$$\begin{aligned}
& h_n^{-3}\left\|\Psi(x)D\Phi(x)\nabla f\left(\frac{\Phi(x)-\Phi(x_n)}{h_n}\right) - \Psi(x)D\Phi(0)\nabla f\left(\frac{D\Phi(0)x-D\Phi(0)x_n}{h_n}\right)\right\|^2_{L^2(B(0,\delta))} \\
\leq & \left\|D\Phi(h_nx+x_n)\nabla f\left(\frac{\Phi(h_nx+x_n)-\Phi(x_n)}{h_n}\right) - D\Phi(0)\nabla f\left(D\Phi(0)x\right)\right\|^2_{L^2(\{x:|x_n+h_nx|\leq\delta\})} \qquad (1.14)
\end{aligned}$$

By the fundamental theorem of calculus, there exists $z_n(x)\in[x_n, h_nx+x_n]$ such that $\left|\frac{\Phi(h_nx+x_n)-\Phi(x_n)}{h_n}\right| = |D\Phi(z_n)x| > C|x|$ for some uniform $C>0$. As ∇f is compactly supported, we deduce that for $|x|$ large enough, the integral is zero. So, we are led with the norm (1.14) with $L^2(B(0,C))$ instead of $L^2(\{x : |x_n+h_nx|\leq\delta\})$. We conclude by dominated convergence. □

Using the previous lemma in geodesic coordinates, we get that our definition of concentrating data is the same as Definition 1.2 of S. Ibrahim [22].

Remark that for a concentrating data, $x_n - x_\infty$ can not be defined invariantly on $T_{x_\infty}M$, we can only define the limit of $(x_n - x_\infty)/h_n$. The change of coordinates must act on x_n as an element of M and not $T_{x_\infty}M$ even if it converges to x_∞. Yet, the functions (f, g) of a concentrating data "live" on the tangent space. Moreover, the norm in energy of a concentrating data is the one of its data.

Lemma 5.1.4. *Let (u_n, v_n) a concentrating data associated to $[(\varphi, \psi), \underline{h}, \underline{x}]$, then, we have*

$$\|(u_n, v_n)\|_{\mathcal{E}} = \|(\varphi, \psi)\|_{\mathcal{E}_{x_\infty}} + o(1)$$

where ∇_{x_∞} and $L^2(T_{x_\infty}M)$ are computed with respect to the frozen metric.

The proof is a direct consequence of Lemma 5.1.5 and 5.1.6 below or by a direct computation in coordinates.

The next definition is the tool that will be used to "track" the concentrations.

Definition 5.1.1. *Let $x_\infty \in M$ and $(f, g) \in \mathcal{E}_{x_\infty}$. Given $[(f, g), \underline{h}, \underline{x}] \in \mathcal{E}_{x_\infty} \times (\mathbb{R}_+^* \times M)^{\mathbb{N}}$ such that $lim_n(h_n, x_n) = (0, x_\infty)$. Let (f_n, g_n) be a sequence bounded in $\mathcal{E}$, we set*

$$D_{h_n}(f_n, g_n) \rightharpoonup (f, g)$$

if in some coordinate patch $U_M \approx U \subset \mathbb{R}^d$ containing x_∞ and for some $\Psi_U \in C_0^\infty(U)$ such that $\Psi_U(x) = 1$ in a neighborhood of x_∞, we have

$$h_n^{\frac{1}{2}}(\Psi_U f_n, h_n \Psi_U g_n)(x_n + h_n x) \rightharpoonup (f, g) \quad \text{weakly in } \mathcal{E}_{x_\infty}$$

where we have identified $\Psi_U(f_n, g_n)$ with its representation on $T_{x_\infty}M$ in the local trivialisation.

If this holds for one (U, Ψ_U), it holds for any other coordinate chart with the induced transition map.

We denote $D^1_{h_n} f_n \rightharpoonup f$ if we only consider the first part concerning $\dot{H}^1$ and $D^2_{h_n} g_n \rightharpoonup g$ for the L^2 part convergence.

Of course, this definition depends on the core of concentration $\underline{h}$ and $\underline{x}$. In the rest of the paper, the rate $\underline{h}$ and $\underline{x}$ will always be implicit. When several rate of concentration $[\underline{h}^{(j)}, \underline{x}^{(j)}]$, $j \in \mathbb{N}$, are used in a proof, we use the notation $D_h^{(j)}$ to distinguish them.

The fact that this definition is independent of the choice of a coordinate chart can be seen with the following lemma which will also be useful afterward.

Lemma 5.1.5. *$D_{h_n}(f_n, g_n) \rightharpoonup (f, g)$ is equivalent to*

$$\int_M \nabla_M f_n \cdot \nabla_M u_n \underset{n\to\infty}{\longrightarrow} \int_{T_{x_\infty}M} \nabla_{x_\infty} f \cdot \nabla_{x_\infty} \varphi$$

$$\int_M g_n v_n \underset{n\to\infty}{\longrightarrow} \int_{T_{x_\infty}M} g\psi$$

where (u_n, v_n) is any concentrating data associated with $[(\varphi, \psi), \underline{h}, \underline{x}]$.

The ∇ is computed with respect to the metric on M when the integral is over M and with respect to the frozen metric in x_∞ when the integral is over $T_{x_\infty}M$.

Démonstration. We only compute the first term for the H^1 norm and assume $\varphi \in C_0^\infty(\mathbb{R}^3)$. $d\omega(y)$ denotes the Riemannian volume form at the point y, $\cdot_y$ the scalar product at the point y and $\nabla_{h_n x + x_n} = g(h_n x + x_n)^{-1}\nabla$.

We denote $V_h = \frac{V - x_n}{h}$ and $L_{n,V} = h_n^{\frac{1}{2}} \int_{V_h} \nabla_{x_\infty} \left[\Psi_V f_n\left(x_n + h_n x\right)\right] \cdot \nabla_{x_\infty}\varphi(x)\ d\omega(0)$.

$$\begin{aligned}
L_{n,V} &= h_n^{\frac{1}{2}} \int_{V_h} \nabla_{x_n + h_n x}\left[\Psi_V f_n\left(x_n + h_n x\right)\right] \cdot_{(x_n + h_n x)} \nabla_{x_n + h_n x}\varphi(x)\ d\omega(x_n + h_n x) + o(1) \\
&= h_n^{\frac{3}{2}} \int_{V_h} \Psi_V(x_n + h_n x)(\nabla_{x_n + h_n x} f_n)\left(x_n + h_n x\right) \cdot_{(x_n + h_n x)} \nabla_{x_n + h_n x}\varphi(x)\ d\omega(x_n + h_n x) + o(1) \\
&= h_n^{-\frac{3}{2}} \int_V \nabla_y f_n\left(y\right) \cdot_y \Psi_V(y)\nabla_y \varphi\left(\frac{y - x_n}{h_n}\right)\ d\omega(y) + o(1) \\
&= h_n^{-\frac{1}{2}} \int_V \nabla_y f_n\left(y\right) \cdot_y \nabla_y \left[\Psi_V(y)\varphi\left(\frac{y - x_n}{h_n}\right)\right]\ d\omega(y) + o(1) = \int_M \nabla_M f_n \cdot \nabla_M u_n\ + o(1).
\end{aligned}$$

Therefore, $L_{n,V}$ tends to $\int \nabla f(x) \cdot \nabla\varphi(x)\ d\omega(0)$ if and only if $\int_M \nabla_M f_n \cdot \nabla_M u_n$ has the same limit. □

An easy consequence of this lemma is the link with concentrating waves.

Lemma 5.1.6. *Let (f_n, g_n) be some concentrating data associated with $[(f,g), \underline{h}, \underline{x}]$, then, we have*

$$D_{h_n}(f_n, g_n) \rightharpoonup (f, g)$$

Démonstration. Lemma 5.1.3 permits to work in geodesic coordinates so that the metric g is the identity at the point x_∞. In this chart, we have $f_n\left(x_n + h_n x\right) = \Psi_U(x_n + h_n x)h_n^{-\frac{1}{2}} f$. So, the computation of Lemma 5.1.5 gives $\int \nabla_\infty f \cdot \nabla_\infty \varphi d\omega(0) = \int_M \nabla_M f_n \cdot \nabla_M u_n\ + o(1)$ which gives the result. □

We conclude this subsection by a definition of orthogonality that will discriminate concentrating data.

Definition 5.1.2. *We say that two sequences $[\underline{h}^{(1)}, \underline{x}^{(1)}, \underline{t}^{(1)}]$ and $[\underline{h}^{(2)}, \underline{x}^{(2)}, \underline{t}^{(2)}]$ are orthogonal if either*

- $\log \left|\frac{h_n^{(1)}}{h_n^{(2)}}\right| \underset{n\to\infty}{\longrightarrow} +\infty$
- $x_\infty^{(1)} \neq x_\infty^{(2)}$
- $h_n^{(1)} = h_n^{(1)} = h$ *and* $x_\infty^{(1)} = x_\infty^{(2)} = x_\infty$ *and in some coordinate chart around* x_∞*, we have*

$$\frac{\left|t_h^{(1)} - t_h^{(2)}\right|}{h} + \frac{\left|x_h^{(1)} - x_h^{(2)}\right|}{h} \underset{h\to 0}{\longrightarrow} +\infty$$

We note $[\underline{h}^{(1)}, \underline{x}^{(1)}, \underline{t}^{(1)}] \perp [\underline{h}^{(2)}, \underline{x}^{(2)}, \underline{t}^{(2)}]$ and $(\underline{x}^{(1)}, \underline{t}^{(1)}) \perp_h (\underline{x}^{(2)}, \underline{t}^{(2)})$ if $\underline{h}^{(1)} = \underline{h}^{(2)} = h$.

This definition does not depend on the coordinate chart. This can be seen because we have the estimate $\frac{1}{C}\left|x_h^{(1)}-x_h^{(2)}\right| \leq \left|\Phi(x_h^{(1)})-\Phi(x_h^{(2)})\right| \leq C\left|x_h^{(1)}-x_h^{(2)}\right|$ if Φ is the transition map.

5.1.3 Scales

In this subsection, we precise a few facts that will be useful in the first part of the proof of linear profile decomposition which consists of the extration of the scales of oscillation h_n^j.

On the Hilbert space $\mathcal{E} = H^1(M) \times L^2(M)$, we define the self-adjoint operator A_M by :

$$\begin{aligned} D(A_M) &= H_M^2 \times H_M^1 \\ A_M(u,v) &= ((-\Delta_M)^{1/2}v, (-\Delta_M)^{1/2}u) \end{aligned}$$

We define similarly $A_{\mathbb{R}^d}$ with the flat laplacian. We denote $A_{\mathbb{R}^d,N}$ the obvious operator on $(H^1(M)(\mathbb{R}^d) \times L^2(\mathbb{R}^d))^N$ obtained by applying $A_{\mathbb{R}^d}$ on each "coordinate".

The following definition is taken from Gallagher-Gérard[17].

Definition 5.1.3. *Let A be a selfadjoint (unbounded) operator on a Hilbert space H. Let (h_n) a sequence of positive numbers converging to 0. A bounded sequence (u_n) in H is said (h_n)-oscillatory with respect to A if*

$$\overline{\lim_{n\to\infty}} \left\| 1_{|A|\geq \frac{R}{h_n}} u_n \right\|_H \underset{R\to\infty}{\longrightarrow} 0. \tag{1.15}$$

(u_n) is said stricly (h_n)-oscillatory with respect to A if it satisfies (1.15) and

$$\overline{\lim_{n\to\infty}} \left\| 1_{|A|\leq \frac{\varepsilon}{h_n}} u_n \right\|_H \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

At the contrary, (u_n) is said (h_n)-singular with respect to A if we have

$$\left\| 1_{\frac{a}{h_n}|A|\leq \frac{b}{h_n}} u_n \right\|_H \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for all } 0<a<b.$$

Remark that $1_{|x|\leq 1}$ can easily be replaced by a well chosen function $\varphi \in C_0^\infty(\mathbb{R})$. Moreover, if a sequence (u_n) is stricly (h_n)-oscillatory while a second sequence (v_n) is (h_n)-singular, then we have the interesting property that $\langle u_n, v_n\rangle_H \underset{n\to\infty}{\longrightarrow} 0$.

Proposition 5.1.2. *Let $M = \cup_{i=1}^N U_i$ a finite covering of M with some associated local coordinate patch $\Phi_i : U_i \to V_i \subset \mathbb{R}^3$. Let $1 = \sum_i \Psi_i$ be an associated partition of the unity of M with $\Psi_i \in C_0^\infty(U_i)$. Let (u_n, v_n) a bounded sequence in the M energy space and h_n a sequence converging to 0. Then (u_n, v_n) is (stricly) (h_n)-oscillatory with respect to A_M, if and only if all the $\Phi_{i*}\Psi_i(u_n, v_n)$ are (strictly) (h_n)-oscillatory with respect to $A_{\mathbb{R}^d}$.*

Démonstration. First, we remark that a sequence is (strictly) (h_n)-oscillatory with respect to A if and only if it is (strictly) (h_n^2)-oscillatory with respect to A^2. So we can replace A_M and $A_{\mathbb{R}^3}$ by $-(\Delta_M, \Delta_M)$ and $-(\Delta_{\mathbb{R}^3}, \Delta_{\mathbb{R}^3})$. We apply a proposition taken from [17] that makes the link between oscillation with different operators.

Proposition 5.1.3 (Proposition 2.2.3 of [17]). *Let $\Lambda : H_1 \to H_2$ be a continuous linear map between Hilbert spaces H_1, H_2. Let A_1 be a selfadjoint operator on H_1, A_2 be a selfadjoint operator on H_2. Assume there exists $C > 0$ such that $\Lambda(D(A_1)) \subset D(A_2)$, $\Lambda^*(D(A_2)) \subset D(A_1)$ and for any $u \in D(A_1)$, $v \in D(A_2)$,*

$$\|A_2\Lambda u\| \leq C(\|A_1 u\| + \|u\|) \tag{1.16}$$
$$\|A_1\Lambda^* v\| \leq C(\|A_2 v\| + \|v\|). \tag{1.17}$$

If a bounded sequence (u_n) in H_1 is (strictly) (h_n)-oscillatory with respect to A_1, then (Λu_n) is (strictly) (h_n)-oscillatory with respect to A_2.

To prove the first implication, we apply the proposition with $\Lambda(u,v) = (\Phi_{1*}\Psi_1(u,v), \cdots, \Phi_{N*}\Psi_N(u,v))$. We only prove the necessary estimates, the inclusions of domains being a direct consequence of the inequalities and of the density of smooth functions. To simplify the notation, we denote $(u_i, v_i) = \Phi_{i*}\Psi_i(u,v)$. The proof of (1.16) mainly uses the equivalent definitions of the H^s norm on a manifold.

$$\begin{aligned}
\left\|A^2_{\mathbb{R}^3}(u_i,v_i)\right\|_{H^1_{\mathbb{R}^3}\times L^2_{\mathbb{R}^3}} &= \|\Delta_{\mathbb{R}^3} u_i\|_{H^1(M)_{\mathbb{R}^3}} + \|\Delta_{\mathbb{R}^3} v_i\|_{L^2_{\mathbb{R}^3}} \lesssim \|u_i\|_{H^3_{\mathbb{R}^3}} + \|v_i\|_{H^2_{\mathbb{R}^3}} \\
&\lesssim \|u\|_{H^3_M} + \|v\|_{H^2_M} \\
&\lesssim \|u\|_{H^1_M} + \|\Delta_M u\|_{H^1_M} + \|v\|_{L^2_M} + \|\Delta_M v\|_{L^2_M} \\
&\lesssim \left\|A^2_M(u,v)\right\|_{H^1_M\times L^2_M} + \|(u,v)\|_{H^1_M\times L^2_M}.
\end{aligned}$$

Let us prove (1.17) for the duality $H^1 \times L^2$ of the scalar product. Let $(f,g) = (f_i, g_i)_{i=1\cdots N} \in (C_0^\infty(\mathbb{R}^3)\times C_0^\infty(\mathbb{R}^3))^N$ and $(u,v) \in C^\infty(M)$.

$$\begin{aligned}
&((u,v), A^2_M\Lambda^*(f,g))_{H^1(M)\times L^2(M)} = (\Lambda A^2_M(u,v),(f,g))_{(H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3))^N} \\
= \;& \sum_i (\Phi_{i*}\Psi_i\Delta_M u, f_i)_{H^1_{\mathbb{R}^3}} + \sum_i (\Phi_{i*}\Psi_i\Delta_M v, g_i)_{L^2} \\
\lesssim \;& \sum_i \|\Phi_{i*}\Psi_i\Delta_M u\|_{H^{-1}_{\mathbb{R}^3}} \|f_i\|_{H^3_{\mathbb{R}^3}} + \sum_i \|\Phi_{i*}\Psi_i\Delta_M v\|_{H^{-2}_{\mathbb{R}^3}} \|g_i\|_{H^2_{\mathbb{R}^3}} \\
\lesssim \;& \|u\|_{H^1_M} \sum_i \|f_i\|_{H^3_{\mathbb{R}^3}} + \|\Delta_M v\|_{H^{-2}_M} \sum_i \|g_i\|_{H^2_{\mathbb{R}^3}} \\
\lesssim \;& \|(u,v)\|_{H^1_M\times L^2_M} \left(\sum_i \|(\Delta_{\mathbb{R}^3} f_i, \Delta_{\mathbb{R}^3} g_i)\|_{H^1_{\mathbb{R}^3}\times L^2_{\mathbb{R}^3}} + \|(f_i,g_i)\|_{H^1_{\mathbb{R}^3}\times L^2_{\mathbb{R}^3}}\right).
\end{aligned}$$

Therefore,
we get $\|A^2_M\Lambda^*(f,g)\|_{H^1_M\times L^2_M} \leq C\left(\left\|A^2_{\mathbb{R}^3,N}(f,g)\right\|_{(H^1_{\mathbb{R}^3}\times L^2_{\mathbb{R}^3})^N} + \|(f,g)\|_{(H^1_{\mathbb{R}^3}\times L^2_{\mathbb{R}^3})^N}\right)$ and Proposition 5.1.3 implies that (strict) (h_n)-oscillation of (u_n) with respect to A_M implies (strict) (h_n)-oscillation of Λu_n with respect to $A_{\mathbb{R}^3,N}$.

To prove the other implication, we use a quite similar operator. Denote φ_i some other cut-off functions in $C_0^\infty(V_i) \subset C_0^\infty(\mathbb{R}^3)$ such that $\varphi_i \equiv 1$ on $Supp(\Phi_{i*}\Psi_i)$. We define Γ the bounded operator from $(H^1_{\mathbb{R}^3}\times L^2_{\mathbb{R}^3})^N$ to $H^1_M \times L^2_M$ given by

$$\Gamma(f,g) = \sum_i \Phi_{i*}^{-1}\varphi_i(f_i,g_i)$$

Then, we have $\Gamma \circ \Lambda = Id$ and we only have to prove that (strict) (h_n)-oscillation of (f_n, g_n) with respect to $A_{\mathbb{R}^3,N}$ implies (strict) (h_n)-oscillation of $\Gamma(f_n, g_n)$ with respect to A_M. The needed estimates are quite similar and we omit them. $\square$

Remark 5.1.2. *Another way to prove Proposition 5.1.2 would have been to use the pseudodifferential operators $\varphi(h^2\Delta_M)$ as in [6].*

Now, we will prove that the (h_n)-oscillation is conserved by the equation, even with a damping term.

Proposition 5.1.4. *Let $T > 0$. Let (φ_n, ψ_n) a bounded sequence of $\mathcal{E}$ that is (stricly) (h_n)-oscillatory with respect to A_M. Let u_n be the solution of*

$$\left\{ \begin{array}{rcl} \Box u_n + u_n & = & a(x)\partial_t u_n \quad \text{on} \quad [0,T] \times M \\ (u_n(0), \partial_t u_n(0)) & = & (\varphi_n, \psi_n). \end{array} \right. \tag{1.18}$$

Then, $(u_n(t), \partial_t u_n(t))$ are (strictly) (h_n)-oscillatory with respect to A_M, uniformly on $[0,T]$.
At the contrary, if (φ_n, ψ_n) is (h_n)-singular with respect to A_M, $(u_n(t), \partial_t u_n(t))$ is (h_n)-singular with respect to A_M, uniformly on $[0,T]$.

Démonstration. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \chi(s) \leq 1$ and $\chi(s) = 1$ for $|s| \leq 1$. The (h_n)-oscillation (resp strict oscillation) is equivalent to $\overline{\lim\limits_{n\to\infty}} \|(1-\chi)(R^2 h_n^2 \Delta)(u_n, \partial_t u_n)\|_{\mathcal{E}} \underset{R\to\infty}{\longrightarrow} 0$
(resp $\overline{\lim\limits_{n\to\infty}} \left\|\chi(\frac{h_n^2\Delta}{R^2})(u_n, \partial_t u_n)\right\|_{\mathcal{E}} \underset{R\to\infty}{\longrightarrow} 0$).

$v_n = (1-\chi)(R^2 h_n^2 \Delta) u_n$ is solution of

$$\left\{ \begin{array}{rcl} \Box v_n + v_n & = & a(x)\partial_t v_n - [\chi(R^2 h_n^2 \Delta), a]\partial_t u_n \quad \text{on} \quad [0,T] \times M \\ (v_n(0), \partial_t v_n(0)) & = & (1-\chi)(R^2 h_n^2 \Delta)(\varphi_n, \psi_n). \end{array} \right. \tag{1.19}$$

and energy estimates give

$$\begin{array}{rcl} \|(v_n(t), \partial_t v_n(t))\|_{\mathcal{E}} & \leq & C_T \left\|(1-\chi)(R^2 h_n^2 \Delta)(\varphi_n, \psi_n)\right\|_{\mathcal{E}} + C_T \left\|[a, \chi(R^2 h_n^2 \Delta)]\partial_t u_n\right\|_{L^1([0,t],L^2)} \\ & \leq & C_T \left\|(1-\chi)(R^2 h_n^2 \Delta)(\varphi_n, \psi_n)\right\|_{\mathcal{E}} + C_T R h_n. \end{array}$$

where the last inequality comes from the fact that $\chi(-h^2\Delta)$ is a semiclassical pseudo-differential operator, as proved in Burq, Gérard and Tzvetkov [6], Proposition 2.1 using the Helffer-Sjöstrand formula.

Therefore, passing to the limitsup in n and using the oscillation assumption, we get the expected result uniformly in t for $0 \leq t \leq T$. The results for strict oscillation and singularity are proved similarly. $\square$

Proposition 5.1.5. *There exists $C_T > 0$ such that for every (φ_n, ψ_n) bounded sequence of $\mathcal{E}$ weakly convergent to 0, we have the estimate*

$$\overline{\lim_{n\to\infty}} \|(u_n, \partial_t u_n)\|_{L^\infty([0,T], B^1_{2,\infty}(M) \times B^0_{2,\infty}(M))} \leq C_T \overline{\lim_{n\to\infty}} \|(\varphi_n, \psi_n)\|_{B^1_{2,\infty}(M) \times B^0_{2,\infty}(M)}$$

where u_n is the solution of (1.18) .

Démonstration. Without loss of generality and since the equation is linear, we can assume that $\|(\varphi_n,\psi_n)\|_{\mathcal{E}}$ is bounded by 1. Let $\varepsilon > 0$. Let $\chi_0, \chi \in C_0^\infty(\mathbb{R})$ so that $1 = \chi_0 + \sum_{k=1}^\infty \chi(2^{-2k}x)$. We denote $u_n^k = \chi(2^{-2k}\Delta)u_n$. Using the same estimates as in the previous lemma, we get

$$\left\|(u_n^k(t), \partial_t u_n^k(t))\right\|_{\mathcal{E}} \leq C_T \left\|(u_n^k(0), \partial_t u_n^k(0))\right\|_{\mathcal{E}} + C_T 2^{-k}.$$

Take K large enough so that $C_T 2^{-k} \leq \varepsilon$ for $k \geq K$ so that we have .

$$\left\|(u_n^k(t), \partial_t u_n^k(t))\right\|_{\mathcal{E}} \leq C_T \left\|(\varphi_n, \psi_n)\right\|_{B^1_{2,\infty}(M)\times B^0_{2,\infty}(M)} + \varepsilon. \tag{1.20}$$

Then, for $k < K$, we use again some energy estimates for the equation verified by u_n^k, we get

$$\left\|(u_n^k(t), \partial_t u_n^k(t))\right\|_{\mathcal{E}} \leq C_T \left\|(u_n^k(0), \partial_t u_n^k(0))\right\|_{\mathcal{E}} + C_T \left\|[a, \chi(-2^{-2k}\Delta)]\partial_t u_n\right\|_{L^1([0,T],L^2)}.$$

Yet, for fixed k, $[a, \chi(-2^{-2k}\Delta)]$ is an operator from L^2 into H^1 (for instance) and we conclude by the Aubin-Lions Lemma that for fixed $k \leq K$

$$\overline{\lim_{n\to\infty}} \left\|(u_n^k(t), \partial_t u_n^k(t))\right\|_{\mathcal{E}} \leq C_T \overline{\lim_{n\to\infty}} \left\|(u_n^k(0), \partial_t u_n^k(0))\right\|_{\mathcal{E}}. \tag{1.21}$$

We get the expected result with an additional ε by combining (1.20) and (1.21). □

We end this subsection by two lemma that will be useful in the nonlinear decomposition. The first one is lemma 3.2 of [17].

Lemma 5.1.7. *Let h_n and $\tilde{h_n}$ be two orhogonal scales, and let (f_n) and $\tilde{f_n}$ be two sequences such that such ∇f_n (resp $\nabla \tilde{f_n}$)) is strictly (h_n) (resp $\tilde{h}_n$)-oscillatory with respect to $\Delta_{\mathbb{R}^3}$. Then, we have :*

$$\overline{\lim_{n\to\infty}} \left\|f_n \tilde{f}_n\right\|_{L^3(\mathbb{R}^3)} = 0$$

Then, we easily deduce the following result.

Lemma 5.1.8. *Let h_n and $\tilde{h_n}$ be two orhogonal scales and v_n, $\tilde{v}_n$ be two sequences that are strictly h_n (resp $\tilde{h_n}$) oscillatory with respect to Δ_M (considered on the Hilbert space H^1), uniformly on $[-T,T]$. Then, we have*

$$\|v_n \tilde{v}_n\|_{L^\infty([-T,T],L^3(M))} \underset{n\to\infty}{\longrightarrow} 0$$

Moreover, the same result remains true if $\tilde{v}_n$ is a constant sequence $v \in H^1$ and $\tilde{h}_n = 1$.

Démonstration. Using a partition of unity $1 = \sum_i \Psi_i^2$ adapted to coordinate charts, we have to compute

$$\|\Phi_{i*}\Psi_i v_n \Psi_i \tilde{v}_n\|_{L^\infty([-T,T],L^3(R^3))}.$$

Using Proposition 5.1.2, we infer that $\Phi_{i*}\Psi_i v_n$ is strictly (h_n)-oscillatory with respect to $\Delta_{\mathbb{R}^3}$ (defined on H^1) and the same result holds for $\nabla(\Phi_{i*}\Psi_i v_n)$ with respect to $\Delta_{\mathbb{R}^3}$ defined on L^2. We conclude by applying Lemma 5.1.7 to $\Phi_{i*}\Psi_i v_n$ and $\Phi_{i*}\Psi_i v_n$. □

5.1.4 Microlocal defect measure and energy

In this subsection, we state without proof the propagation of the measure for the damped wave equation. We refer to [19] for the definition and to [20] Section 4 or [16] in the specific context of the wave equation). It will be used several times in the article.

Lemma 5.1.9. *[Measure for the damped equation and equicontinuity of the energy] Let u_n, $\tilde{u}_n$ be two sequences of solution to*

$$\Box u_n + u_n = a(x)\partial_t u_n,$$

weakly convergent to 0 *in* $\mathcal{E}$*. Then, there exists a subsequence (still denoted u_n, $\tilde{u}_n$) such that for any $t \in [0,T]$ there exists a (nonnegative if $u_n = \tilde{u}_n$) Radon measure μ^t on S^*M such that for any classical pseudodifferential operator B of order* 0*, we have with a uniform convergence in t*

$$\big(B(-\Delta)^{1/2}u_n(t), (-\Delta)^{1/2}\tilde{u}_n(t)\big)_{L^2(M)} + (B\partial_t u_n(t), \partial_t\tilde{u}_n(t))_{L^2(M)} \underset{n\to\infty}{\longrightarrow} \int_{S^*M} \sigma_0(B)d\mu^t \quad (1.22)$$

Moreover, one can decompose

$$\mu^t = \frac{1}{2}(\mu^t_+ + \mu^t_-)$$

which satisfy the following transport equation

$$\partial_t\mu_\pm(t) = \pm H_{|\xi|_x}\mu_\pm(t) + a(x)\mu_\pm(t).$$

Furthermore, if $t_n \underset{n\to\infty}{\longrightarrow} t$, we have the same convergence with t replaced by t_n in (1.22).

The microlocal defect measure of a concentrating data $[(\varphi,\psi),\underline{h},\underline{x}]$ can be explicitely computed, as follows

$$\mu_\pm = (2\pi)^{-3}\delta_{x_\infty}(x) \otimes \int_0^{+\infty} \left|\hat{\psi}(r\xi) \pm i|r\xi|_\infty\hat{\varphi}(r\xi)\right|^2 r^2 dr.$$

This can be easily computed, for instance, with the next lemma.

Lemma 5.1.10. *Let $(\varphi_n,\psi_n) = [(\varphi,\psi),\underline{h},\underline{x}]$ be a concentration data and $A(x,D_x), B(x,D_x)$ two polyhomogeneous pseudodifferential operators of respective order* 0*. Then*

$$\|(A(x,D_x)\varphi_n, B(x,D_x)\psi_n) - [(A_0(x_\infty,D_x)\varphi, B_0(x_\infty,D_x)\psi),\underline{h},\underline{x}]\|_{H^1\times L^2} \underset{n\to\infty}{\longrightarrow} 0$$

where $A_0(x_\infty,D_x)$ is the Fourier multiplier of homogeneous symbol $a_0(x_\infty,\xi)$ defined on $T^*_{x_\infty}M$

Démonstration. We only give a sketch of proof for $B(x,D_x)\psi_n$. By approximation, we can assume that $\widehat{\psi} \in C_0^\infty(\mathbb{R}^3\backslash 0)$. In local coordinates centered at $x_\infty = 0$, we have for a $o(1)$ small in L^2

$$\begin{aligned} B(x,D_x)\psi_n &= h_n^{-\frac{3}{2}}B(x,D_x)\left[\Psi_U(x)(\chi\psi)\left(\frac{x-x_n}{h_n}\right)\right] + o(1) \\ &= h_n^{-\frac{3}{2}}\left[B_n(y,D_y)\psi\right]\left(\frac{x-x_n}{h_n}\right) + o(1) \end{aligned}$$

where $B_n(y, D_y)$ is the operator of symbol $b_n(y,\xi) = b_0(h_n y + x_n, \xi/h_n)$. Here b_0 is the principal symbol of B, homogeneous for large ξ. We write $b_0(h_n y + x_n, \xi/h_n) = b_0(x_n, \xi/h_n) + h_n y \int_0^1 (\partial_y b_0)(x_n + th_n, \xi/h_n)dt$. The first term converges to $b_0(0,\xi)$ by homogeneity while the second produces a term small in L^2. □

The previous lemma is made interesting when combined with the propagation of microlocal defect measure.

Lemma 5.1.11. *Let u_n a sequence of solutions of $\Box u_n + u_n = a(x)\partial_t u_n$ weakly convergent to 0 and $p_n = [(\varphi,\psi), \underline{h}, \underline{x}, \underline{t}]$ a linear damped concentrating wave. We assume $D_h(u_n, \partial_t u_n) \rightharpoonup 0$. Then, for any classical pseudodifferential operators $A(x, D_x)$ of order 0, we have uniformly for $t \in [-T,T]$*

$$\left(A(-\Delta)^{1/2}p_n(t), (-\Delta)^{1/2}u_n(t)\right)_{L^2(M)} + \left(A\partial_t p_n(t), \partial_t u_n(t)\right)_{L^2(M)} \underset{n\to\infty}{\longrightarrow} 0.$$

In particular, we have

$$\nabla p_n \cdot \nabla u_n + \partial_t p_n \partial_t u_n \rightharpoonup 0 \text{ in } \mathcal{D}'(]-T,T[\times M).$$

Démonstration. We first check the property for $t = t_n$. Using Lemma 5.1.10 several times, we are led to estimate

$$\left((-\Delta)^{1/2}\varphi_n, (-\Delta)^{1/2}u_n(t_n)\right)_{L^2(M)} + \left(\psi_n, \partial_t u_n(t_n)\right)_{L^2(M)}$$

where (φ_n, ψ_n) are the concentrating data associated with $[(A(x_\infty, D_x)\varphi, B(x_\infty, D_x)\psi), \underline{h}, \underline{x}]$. Then, the hypotheses $D_h(u_n, \partial_t u_n) \rightharpoonup 0$ and Lemma 5.1.5 yields the convergence to 0 for this particular case $t = t_n$. We conclude by equicontinuity and by the propagation of joint measures stated in Lemma 5.1.9. □

5.2 Profile Decomposition

5.2.1 Linear profile decomposition

The main purpose of this section is to establish Theorem 5.0.3. It is completed in two main steps : the first one is the extraction of the scales $h_n^{(j)}$ where we decompose v_n in an infinite sum of sequence $v_n^{(j)}$ which are respectively $h_n^{(j)}$-oscillatory and the second steps consists in decomposing each $v_n^{(j)}$ in an infinite sum of concentrating wave at the rate $h_n^{(j)}$. Actually, in order to perform the nonlinear decomposition, we will need that, in some sense, each profile of the decomposition do not interact with the other. It is stated in this orthogonality result.

Theorem 5.0.3'. *With the notation of Theorem 5.0.3, we have the additional following properties.*
If $2T < T_{focus}$, we have $(\underline{h}^{(k)}, \underline{x}^{(k)}, \underline{t}^{(k)}) \perp (\underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)})$ for any $j \neq k$, according to Definition 5.1.2.
If $M = S^3$ and $a \equiv 0$ (undamped solutions), but with T eventually large, we have $(\underline{h}^{(k)}, (-1)^m\underline{x}^{(k)}, \underline{t}^{(k)} + m\pi)$ orthogonal to $(\underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)})$ for any $m \in \mathbb{Z}$ and $j \neq k$.

5.2.1.1 Extraction of scales

Proposition 5.2.1. *Let $T>0$. Let (φ_n,ψ_n) a bounded sequence of $\mathcal{E}$ and v_n the solution of*

$$\left\{\begin{array}{rcll}\Box v_n+v_n&=&a(x)\partial_t v_n & \text{on} \quad [-T,T]\times M\\ (v_n(0),\partial_t v_n(0))&=&(\varphi_n,\psi_n).\end{array}\right. \tag{2.23}$$

Then, up to an extraction, v_n can be decomposed in the following way : for any $l\in\mathbb{N}^*$

$$v_n(t,x)=v(t,x)+\sum_{j=1}^{l} v_n^{(j)}(t,x)+\rho_n^{(l)}(t,x),$$

where $v_n^{(l)}$ is a strictly $(h_n^{(j)})$-oscillatory solution of the damped linear wave equation (2.23) on M. The scales $h_n^{(j)}$ satisfy $h_n^{(j)} \underset{n\to\infty}{\longrightarrow} 0$ and are orthogonal :

$$\left|\log\frac{h_n^{(k)}}{h_n^{(j)}}\right| \underset{n\to\infty}{\longrightarrow} +\infty \text{ if } j\neq k. \tag{2.24}$$

Moreover, we have

$$\overline{\lim_{n\to\infty}}\,\left\|\rho_n^{(l)}\right\|_{L^\infty(]-T,T[,L^6(M))} \underset{l\to\infty}{\longrightarrow} 0 \tag{2.25}$$

$$\begin{aligned}\|(v_n,\partial_t v_n)(t)\|_{\mathcal{E}}^2 &= \|(v,\partial_t v)(t)\|_{\mathcal{E}}^2+\sum_{j=1}^{l}\left\|(v_n^{(j)},\partial_t v_n^{(j)})(t)\right\|_{\mathcal{E}}^2\\ &+\left\|(\rho_n^{(l)},\partial_t\rho_n^{(l)})(t)\right\|_{\mathcal{E}}^2+o(1)(t),\end{aligned} \tag{2.26}$$

where $o(1)(t) \underset{n\to\infty}{\longrightarrow} 0$ uniformly for $t\in[-T,T]$.

Démonstration. We first make this decomposition for the initial data as done in [21] (see also [2]). Then, using the propagation of (h_n)-oscillation proved in Proposition 5.1.4, we extend it for all time.

More precisely, by applying the same procedure as in [21], with the operator A_M, we decompose

$$(\varphi_n,\psi_n)=(\varphi,\psi)+\sum_{j}^{l}(\varphi_n^{(j)},\psi_n^{(j)})+(\Phi_n^{(l)},\Psi_n^{(l)})$$

where $(\varphi_n^{(j)},\psi_n^{(j)})$ is $(h_n^{(j)})$-oscillatory for A_M, $h_n^{(j)} \underset{n\to\infty}{\longrightarrow} 0$, and

$$\overline{\lim_{n\to\infty}}\,\sup_{k\in\mathbb{N}}\left\|\mathbf{1}_{[2^k,2^{k+1}[}(A_M)(\Phi_n^{(l)},\Psi_n^{(l)})\right\|_{\mathcal{E}} \underset{l\to\infty}{\longrightarrow} 0. \tag{2.27}$$

Moreover, we have the orthogonality property :

$$\|(\varphi_n,\psi_n)\|_{\mathcal{E}}^2=\|(\varphi,\psi)\|_{\mathcal{E}}^2+\sum_{j}^{l}\left\|(\varphi_n^{(j)},\psi_n^{(j)})\right\|_{\mathcal{E}}^2+\left\|(\Phi_n^{(l)},\Psi_n^{(l)})\right\|_{\mathcal{E}}^2+o(1), \quad n\to\infty$$

and the $h_n^{(j)}$ are orthogonal each other as in (2.24). Moreover, $(\Phi_n^{(l)}, \Psi_n^{(l)})$ is $(h_n^{(j)})$- singular for $1 \leq j \leq l$.

This decomposition for the initial data can be extended to the solution by

$$v_n(t,x) = v(t,x) + \sum_j^l v_n^{(j)}(t,x) + \rho_n^{(l)}(t,x),$$

where each $v_n^{(j)}$ is solution of

$$\left\{ \begin{array}{rcll} \Box v_n^{(j)} + v_n^{(j)} &=& a(x)\partial_t v_n^{(j)} & \text{on} \quad \mathbb{R}_t \times M \\ (v_n^{(j)}(0), \partial_t v_n^{(j)}(0)) &=& (\varphi_n^{(j)}, \psi_n^{(j)}). & \end{array} \right.$$

Thanks to Proposition 5.1.4, each $(v_n^{(j)}(t), \partial_t v_n^{(j)}(t))$ is strictly $(h_n^{(j)})$-oscillatory and $(\rho_n^{(l)}(t), \partial_t \rho_n^{(l)}(t))$ is $(h_n^{(j)})$- singular for $1 \leq j \leq l$. So, we easily infer for instance that $\left\langle (\rho_n^{(l)}(t), \partial_t \rho_n^{(l)}(t)), (v_n^{(j)}(t), \partial_t v_n^{(j)}(t)) \right\rangle_{\mathcal{E}} \underset{n\to\infty}{\longrightarrow} 0$ uniformly on $[-T,T]$ where $\langle \ , \ \rangle_{\mathcal{E}}$ is the scalar product on $\mathcal{E}$. This is also true for the product between $v_n^{(j)}$ and $v_n^{(k)}$, $j \neq k$ thanks to the orthogonality (2.24). The same convergence holds for the product with v by weak convergence to 0 of the other terms. Then, we get

$$\|(v_n, \partial_t v_n)\|_{\mathcal{E}}^2 = \|(v, \partial_t v)\|_{\mathcal{E}}^2 + \sum_j^l \left\|(v_n^{(j)}, \partial_t v_n^{(j)})\right\|_{\mathcal{E}}^2 + \left\|(\rho_n^{(l)}, \partial_t \rho_n^{(l)})\right\|_{\mathcal{E}}^2 + o(1), \quad n \to \infty.$$

Let us now prove estimate (2.25) of the remaining term in $L^\infty(L^6)$. (2.27) gives the convergence to zero of $(\rho_n^{(l)}(0), \partial_t \rho_n^{(l)}(0))$ in $B^1_{2,\infty}(M) \times B^0_{2,\infty}(M)$. We extend this convergence for all time with Proposition 5.1.5 and get

$$\sup_{t\in[0,T]} \overline{\lim_{n\to\infty}} \left\| \left(\rho_n^{(l)}(t), \partial_t \rho_n^{(l)}(t)\right) \right\|_{B^1_{2,\infty} \times B^0_{2,\infty}} \underset{l\to\infty}{\longrightarrow} 0.$$

The following lemma will transfer this information in local charts.

Lemma 5.2.1. *There exists $C > 0$ such that*

$$\frac{1}{C} \|\Lambda f\|_{B^0_{2,\infty}(\mathbb{R}^3)^N} \leq \|f\|_{B^0_{2,\infty}(M)} \leq C \|\Lambda f\|_{B^0_{2,\infty}(\mathbb{R}^3)^N}$$
$$\frac{1}{C} \|\Lambda f\|_{B^1_{2,\infty}(\mathbb{R}^3)^N} \leq \|f\|_{B^1_{2,\infty}(M)} \leq C \|\Lambda f\|_{B^1_{2,\infty}(\mathbb{R}^3)^N}$$

where Λ is the operator described in Proposition 5.1.2 of cut-off and transition in N local charts.

We postpone the proof of this lemma and continue the proof of the proposition. Using this lemma, we get for every coordinate patch (U_i, Φ_i) and $\Psi_i \in C_0^\infty(U_i)$.

$$\overline{\lim_{n\to\infty}} \left\| \Phi_i^* \Psi_i \rho_n^{(l)} \right\|_{L^\infty([-T,T], B^1_{2,\infty}(\mathbb{R}^3))} \underset{l\to\infty}{\longrightarrow} 0$$

The refined Sobolev estimate, Lemma 3.5 of [2], yields for any $f \in H^1(M)(\mathbb{R}^3)$

$$\|f\|_{L^6(\mathbb{R}^3)} \leq \left\|(-\Delta_{\mathbb{R}^3})^{1/2} f\right\|_{L^2}^{1/3} \left\|(-\Delta_{\mathbb{R}^3})^{1/2} f\right\|_{\dot{B}^0_{2,\infty}}^{2/3} \leq \|f\|_{H^1(\mathbb{R}^3)}^{1/3} \|f\|_{B^1_{2,\infty}(\mathbb{R}^3)}^{2/3}$$

Therefore, we have
$$\overline{\lim_{n\to\infty}} \left\|\Phi_i^*\Psi_i\rho_n^{(l)}\right\|_{L^\infty([-T,T],L^6(\mathbb{R}^3))} \underset{l\to\infty}{\longrightarrow} 0$$
and finally
$$\overline{\lim_{n\to\infty}} \left\|\rho_n^{(l)}\right\|_{L^\infty([-T,T],L^6(M))} \underset{l\to\infty}{\longrightarrow} 0$$
This completes the proof of Proposition 5.2.1, up to the proof of Lemma 5.2.1. □

Proof of Lemma 5.2.1. We essentially use the following fact : see Lemma 3.1 of [2]. Let f_n be a sequence of $L^2(\mathbb{R}^3)$ weakly convergent to 0 and compact at infinity
$$\overline{\lim_{n\to+\infty}} \int_{|x|>R} |f(x)|^2 \ dx \underset{R\to+\infty}{\longrightarrow} 0.$$
Then, f_n tends to 0 in $\dot{B}^0_{2,\infty}(\mathbb{R}^3)$ if and only if f_n is h_n singular for every scale h_n.

Actually, the same result holds Δ_M, with the same demonstration. The compactness at infinity in $\mathbb{R}^3$ is only assumed to ensure
$$\overline{\lim_{n\to+\infty}} \left\|\mathbf{1}_{[-A,A](\Delta_{\mathbb{R}^3})} f_n\right\|_{L^2} = 0 \text{ for any } A > 0$$
which is obvious in the case of Δ_M because of weak convergence and discret spectrum.

Using Proposition 5.1.2, we obtain that f_n is (h_n)-singular with respect to Δ_M if and only if Λf_n is (h_n)-singular with respect to $\Delta_{\mathbb{R}^3}$. Combining both previous results, we obtain that the two norms we consider have the same converging sequences and are therefore equivalent. □

5.2.1.2 Description of linear concentrating waves (after S. Ibrahim)

In this subsection, we describe the assymptotic behavior of linear concentrating waves as described in [22] of S. Ibrahim. In [22], it is stated for the linear wave equation without damping. We give some sketch of proof when necessary to emphasize the tiny modifications.

The following lemma yields that for times close to concentration, the linear damped concentrating wave is close to the solution of the wave equation with flat metric and without damping. It is Lemma 2.2 of [22], except that there is an additional damping term which disappears after rescaling. We do not give the proof and refer to the more complicated nonlinear case (see estimate (2.53)).

Lemma 5.2.2. *Let $v_n = [(\varphi,\psi), \underline{h}, \underline{x}, \underline{t})]$ be a linear damped concentrating wave and v solution of*
$$\left\{\begin{array}{rcl} \Box_\infty v &=& 0 \quad \text{on} \quad \mathbb{R}\times T_{x_\infty}M \\ (v,\partial_t v)_{|t=0} &=& (\varphi,\psi) \end{array}\right. \tag{2.28}$$
Denote $\tilde{v}_n$ the rescaled function associated to v, that is $\tilde{v}_n = \Phi^*\Psi\frac{1}{\sqrt{h_n}} v\left(\frac{t-t_n}{h_n}, \frac{x-x_n}{h_n}\right)$ where (U,Φ) is a coordinate chart around x_∞ and $\Psi \in C_0^\infty(U)$ is constant equal to 1 around x_∞. Then, we have
$$\overline{\lim_{n\to\infty}} |||\tilde{v}_n - v_n|||_{[t_n-\Lambda h_n, t_n+\Lambda h_n]\times M} \underset{\Lambda\to\infty}{\longrightarrow} 0$$

Corollary 5.2.1. *With the notation of the Lemma, if* $\tilde{t_n} = t_n + (C + o(1))h_n$*, then* $(v_n, \partial_t v_n)_{|t=\tilde{t_n}}$ *is a concentrating data associated with* $[(v(C), \partial_t v), \underline{h}, \underline{x}]$.

Moreover, Lemma 2.3 of S. Ibrahim [22] yields the "non reconcentration" property for linear concentrating waves.

Lemma 5.2.3. *Let* $\underline{v} = [(\varphi, \psi, \underline{h}, \underline{x}, \underline{t})]$ *be a linear (possibly damped) concentrating wave. Consider the interval* $[-T, T]$ *containing* t_∞*, satisfying the following non-focusing property (see Definition 5.0.1)*

$$mes\,(F_{x,x_\infty,s}) = 0 \quad \forall x \in M \text{ and } s \neq 0 \text{ such that } t_\infty + s \in [-T, T]. \tag{2.29}$$

Then, if we set $I_n^{1,\Lambda} = [-T, t_n - \Lambda h_n]$ *and* $I_n^{3,\Lambda} =]t_n + \Lambda h_n, T]$*, we have*

$$\overline{\lim_n} \|v_n\|_{L^\infty(I_n^{1,\Lambda} \cup I_n^{3,\Lambda}, L^6(M))} \underset{\Lambda\to\infty}{\longrightarrow} 0$$
$$\overline{\lim_n} \|v_n\|_{L^5(I_n^{1,\Lambda} \cup I_n^{3,\Lambda}, L^{10}(M))} \underset{\Lambda\to\infty}{\longrightarrow} 0.$$

Sketch of the proof of Lemma 5.2.3 in the damped case. To simplify the notation, we can assume $t_n = 0$. In [22], the proof is made by contradiction, assuming the existence of a subsequence (still denoted v_n) such that $\|v_n(s_n)\|_{L^6(M)} \to C > 0$ and $\frac{|s_n|}{h_n} \to \infty$. If $s_n \to \tau \neq 0$, using Concentration-Compactness principle of [31], we are led to prove that the microlocal defect measure μ associated to $v_n(t_n)$ satisfies $\mu(\{y\} \times S^2) = 0$ for any $y \in M$. We use the same argument for the damped equation except that in that case, the measure μ^t associated to $v_n(t)$ is not solution of the exact transport equation but of a damped transport equation (see for Lemma 5.1.9). Yet, the non focusing assumption (2.29) still implies $\mu^t(\{y\} \times S^2) = 0$ for all $y \in M$ and $t \neq 0$, which allows to conclude similarly.

In the case $\tau = 0$, we use in local coordinates the rescaled function $\tilde{v}_n(s, y) = \sqrt{s_n} v_n(s_n s, s_n y + x_n)$. $\tilde{v}_n$ at time $s = 0$ is a concentrating data at scale h_n/s_n. We prove $\lim \|\tilde{v}_n(1, \cdot)\|_{L^6(\mathbb{R}^3)} = 0$. Again by concentration compactness, it is enough to prove that the microlocal defect measure μ^s of $\tilde{v}_n$ propagates along the curves of the hamiltonian flow with constant coefficient $H_{|\xi|}$. Since v_n is solution of $\Box v_n + v_n + a(x)\partial_t v_n = 0$, $\tilde{v}_n$ is solution of $\Box_n \tilde{v}_n + s_n^2 \tilde{v}_n + s_n a(s_n \cdot + x_n)\partial_t \tilde{v}_n = 0$ where $\Box_n$ is a suitably rescaled D'Alembert operator. Since the additional terms $s_n^2 \tilde{v}_n + s_n a(s_n \cdot + x_n)\partial_t \tilde{v}_n$ converges to 0 in $L^1 L^2$, we can finish the proof as in Lemma 2.3 of [22] by proving that μ^s propagates as if $\Box_n$ was replaced by $\Box_\infty$, that is along the hamiltonian $H_{|\xi|}$.

The estimate in norm $L^5 L^{10}$ is obtained by interpolation of $L^\infty L^6$ with another bounded Strichartz norm. $\square$

In the specific case of S^3, Lemma 4.2 of [22] allows to describe precisely the behavior of concentrating wave for large times, as follows.

Lemma 5.2.4. *Let* $\underline{p}$ *be a sequence of solutions of*

$$\begin{cases} \Box p_n &= 0 \quad \text{on} \quad [0, +\infty[\times M \\ (p_n(0), \partial_t p_n(0)) &= (\varphi_n, \psi_n) \end{cases}$$

where (φ_n, ψ_n) *is weakly convergent to* $(0, 0)$ *in* $\mathcal{E}$ *. Then, we have*

$$p_n(t + \pi, x) = -p_n(t, -x) + o(1)(t)$$

where the $o(1)(t)$ is small in the energy space. The same holds for solutions of $\Box u_n + u_n$.

In particular, if $\underline{p}$ is a concentrating wave associated with data $[(\varphi, \psi, \underline{h}, \underline{x}, \underline{t})]$, then, for any $j \in \mathbb{N}$, $p_n(t + j\pi, x)$ is a linear concentrating wave associated with $[(-1)^j(\varphi, \psi)((-1)^j.), \underline{h}, (-1)^j\underline{x}, \underline{t})]$

In the previous lemma, $-x$ refers to the embedding of S^3 into $\mathbb{R}^4$. Moreover, the notation $(\varphi, \psi)(-.)$ could be written more rigorously $(\varphi, \psi)(D_\infty I.)$ where $D_\infty I$ is the differential at the point x_∞ of the application $I : x \mapsto -x$ defined from S^3 into itself. Actually, we are identifying the tangent plane at the south pole with the one on the north pole by the application $x \mapsto -x$ on $\mathbb{R}^4$.

The fact that the result remains true for the equation $\Box u + u = 0$ comes from the fact that for initial data weakly convergent to zero, the solutions of $\Box u = 0$ and $\Box v + v = 0$ with same data are asymptotically close in the energy space. This can be proved by observing that for a weakly convergent sequence of solutions u_n the Aubin-Lions Lemma yields that u_n converges strongly to 0 in $L^\infty([-T, T], L^2)$. So $r_n = u_n - v_n$ is solution of $\Box r_n = u_n$ and converges strongly in $\mathcal{E}$.

5.2.1.3 Extraction of times and cores of concentration

In this section, h_n is a fixed sequence in $\mathbb{R}_+^*$ converging to 0. For simplicity, we will denote it by h and u_h for sequences of functions. The main purpose of this subsection is the proof of the following proposition, which is the profile decomposition for h-oscillatory sequences. It easily implies Theorem 5.0.3 when combined with Proposition 5.2.1.

Proposition 5.2.2. *Let (u_h) be a h-oscillatory sequence of solutions to the damped Klein-Gordon equation (2.23). Then, up to extraction, there exist damped linear concentrating waves p_h^k, as defined in Definition 5.0.3, associated to concentrating data $[(\varphi^{(k)}, \psi^{(k)}), \underline{h}, \underline{x}^{(k)}, \underline{t}^{(k)}]$, such that for any $l \in \mathbb{N}^*$, and up to a subsequence,*

$$v_h(t,x) = \textstyle\sum_{j=1}^l p_n^{(j)}(t,x) + w_n^{(l)}(t,x), \tag{2.30}$$

$$\forall T > 0, \quad \overline{\lim_{n\to\infty}} \left\| w_h^{(l)} \right\|_{L^\infty(]-T,T[,L^6(M))} \underset{l\to\infty}{\longrightarrow} 0 \tag{2.31}$$

$$\|(v_h, \partial_t v_h)\|_{\mathcal{E}}^2 = \textstyle\sum_{j=1}^l \left\| (p_h^{(j)}, \partial_t p_h^{(j)}) \right\|_{\mathcal{E}}^2 + \left\| (w_h^{(l)}, \partial_t w_h^{(l)}) \right\|_{\mathcal{E}}^2 + o(1), \text{ as } h \to \infty, \tag{2.32}$$
$$\text{uniformly for } t \in [-T, T].$$

Moreover, if $2T < T_{focus}$, for any $j \neq k$, we have $(\underline{x}^{(k)}, \underline{t}^{(k)}) \perp_h (\underline{x}^{(j)}, \underline{t}^{(j)})$ according to Definition 5.1.2.
If $M = S^3$ and $a \equiv 0$ (undamped solutions), but with T eventually large, $((-1)^m\underline{x}^{(k)}, \underline{t}^{(k)} + m\pi)$ is orthogonal to $(\underline{x}^{(j)}, \underline{t}^{(j)})$ for any $m \in \mathbb{Z}$ and $j \neq k$.

Remark 5.2.1. *The assumptions to get the orthogonality of the cores of concentration are related to our lack of understanding of the solutions concentrating in a point x_1 where (x_1, x_2, t) is a couple of focus at distance t. We know that the solution reconcentrates after a time t in the other focus x_2 but we do not know precisely how : can it split into several concentrating waves on x_2 with different "rate of concentration"? That is to say with some different x_n converging to x_2 but which are orthogonal.*

Before getting into the proof of the proposition, we state two lemmas that will be useful in the proof. Using the notation of Definition 5.1.1, denote

$$\delta^x(\underline{v}) = \sup_{\underline{x}} \left\{ \|\nabla\varphi\|^2_{L^2(T_{x_\infty}M)}, D^1_h v_h \rightharpoonup \varphi, \text{ up to a subsequence} \right\}$$

where the supremum is taken over all the sequences $\underline{x}$ in M.

If $v_h \in L^\infty([-T,T], H^1(M))$, we denote

$$\delta(\underline{v}) = \sup_{\underline{x},\underline{t}} \left\{ \|\nabla\varphi\|^2_{L^2(T_{x_\infty}M)}, D^1_h v_h(t_h) \rightharpoonup \varphi, \text{ up to a subsequence} \right\} = \sup_{\underline{t}} \delta^x(\underline{v}(t_h,\cdot))$$

where the supremum is taken over all the sequences $\underline{x} = (x_h)$ in M and $\underline{t} = (t_h)$ in $[-T,T]$.

Lemma 5.2.5. *Let $\Psi \in C^\infty(M)$. Then, there exists $C > 0$ such that for any $\underline{v}$, we have the estimate*

$$\delta^x(\Psi\underline{v}) \leq C\delta^x(\underline{v}).$$

The proof is left to the reader.

Lemma 5.2.6. *There exists $C > 0$ such that for any $\underline{v} = (v_h)$ a bounded strictly (h_n)-oscillatory sequence in $H^1(M)$*

$$\overline{\lim_{n\to+\infty}} \|v_h\|_{L^6} \leq C\delta^x(\underline{v})^{1/3} \overline{\lim_{n\to+\infty}} \|v_h\|^{1/6}_{H^1(M)}.$$

Démonstration. This lemma is already known in the case of $\mathbb{R}^3$ where the definition of $\delta^x_{\mathbb{R}^3}$ is the same except that D^1_h is only considered in the trivial coordinate chart. It is estimate (4.19) of [21] in the case of a 1-oscillatory sequence, which can be easily extended to (h_n)- oscillatory sequence by dilation.

Let $\Psi_i \in C^\infty_0(U_i)$ associated to a coordinate patch Φ_i. By Proposition 5.1.2, $\Phi_i^*\Psi_i v_h$ is still (h_n)-oscillatory and we can apply the estimate on $\mathbb{R}^3$. We get

$$\begin{aligned} &\overline{\lim_{n\to+\infty}} \|\Phi_i^*\Psi_i v_h\|_{L^6(\mathbb{R}^3)} \leq C\delta^x_{\mathbb{R}^3}(\Phi_{i*}\Psi_i\underline{v})^{1/3} \overline{\lim_{n\to+\infty}} \|\Phi_i^*\Psi_i v_h\|^{1/6}_{H^1(\mathbb{R}^3)} \\ \leq\ & C\delta^x_{\mathbb{R}^3}(\Phi_i^*\Psi_i\underline{v})^{1/3} \overline{\lim_{n\to+\infty}} \|v_h\|^{1/6}_{H^1(M)} \end{aligned}$$

Then, by definition of the convergence D_h, we easily get

$$\delta^x_{\mathbb{R}^3}(\Phi_i^*\Psi_i\underline{v}) \leq C\delta^x(\Psi_i\underline{v}).$$

We conclude using Lemma 5.2.5 and partition of unity. □

Lemma 5.2.7. *Let $T > 0$. There exists $C > 0$ such that for any sequence $\underline{v} = (v_h)$ (h_n)-oscillatory, solution of the damped linear Klein-Gordon equation on M with bounded energy, we have*

$$\overline{\lim_{n\to+\infty}} \|v_h\|_{L^\infty([-T,T],L^6(M))} \leq C\delta(\underline{v})^{1/3} \overline{\lim_{n\to+\infty}} \|(v_h(0), \partial_t v_h(0))\|^{1/6}_{\mathcal{E}}$$

Démonstration. Let t_h be an arbitrary sequence in $[-T, T]$. We apply Lemma 5.2.6 to the sequence $v_h(t_h)$ and get

$$\begin{aligned}\overline{\lim_{n\to+\infty}} \|v_h(t_h,\cdot)\|_{L^6} &\le C\delta^x(\underline{v}(t_h,\cdot))^{1/3} \overline{\lim_{n\to+\infty}} \|v_h(t_h)\|^{1/6}_{H^1(M)} \\ &\le C\delta(\underline{v})^{1/3} \overline{\lim_{n\to+\infty}} \|(v_h(0), \partial_t v_h(0))\|^{1/6}_{\mathcal{E}}\end{aligned}$$

by definition of δ and by energy estimates. □

Proof of Proposition 5.2.2. It is based on the same extraction argument as [2] and [17] : the concentration will be tracked using our tool D_h and we will extract concentrating waves so that $\delta(\underline{v})$ decreases. We conclude with Lemma 5.2.7 to estimate the $L^\infty(L^6)$ norm of the remainder term.

More precisely, if $\delta(\underline{v}) = 0$, Lemma 5.2.7 shows that there is nothing to be proved. Otherwise, pick $(x_h^{(1)}, t_h^{(1)})$ converging to $(x_\infty^{(1)}, t_\infty^{(1)})$ and $(\varphi^{(1)}, \psi^{(1)}) \in \mathcal{E}_{x_\infty}$, such that

$$\left\|\nabla\varphi^{(1)})\right\|^2_{L^2(T_{x_\infty}M)} + \left\|\psi^{(1)})\right\|^2_{L^2(T_{x_\infty}M)} \ge \left\|\nabla\varphi^{(1)})\right\|^2_{L^2(T_{x_\infty}M)} \ge \frac{1}{2}\delta(\underline{v})$$

and

$$D_h^{(1)}(v_h, \partial_t v_h)(t_h^{(1)}) \underset{h\to 0}{\rightharpoonup} (\varphi^{(1)}, \psi^{(1)}).$$

The existence of the weak limit $\psi^{(1)}$ (up to a subsequence) is ensured by the boundedness in $L^2(\mathbb{R}^3)$ of $\partial_t v_h$ (considered in a coordinate chart) by conservation of energy.

Then, we choose $p_h^{(1)}$ as the damped linear concentrating profile associated with $[(\varphi^{(1)}, \psi^{(1)}), \underline{h}, \underline{x}^{(1)}, \underline{t}^{(1)}]$ (actually, we pick one representant in the equivalence class modulo sequences converging to 0 in the energy space as in Definition 5.0.2). Remark here that the assumption $t_h^{(1)} \in [-T, T]$ ensures $t_\infty^{(1)} \in [-T, T]$, which will always be the case for all the concentrating waves we consider. Then, we give a lemma that will be the main step to the orthogonality of energies.

Lemma 5.2.8. *Let $w_h^{(1)} = v_h - p_h^{(1)}$. Then,*

$$\|(v_h, \partial_t v_h)(t)\|^2_{\mathcal{E}} = \left\|(p_h^{(1)}, \partial_t p_h^{(1)})(t)\right\|^2_{\mathcal{E}} + \left\|(w_h^{(1)}, \partial_t w_h^{(1)})(t)\right\|^2_{\mathcal{E}} + o(1)$$

where the $o(1)$ is uniform for t in bounded intervals.

Démonstration. We first compute the energy at time $t_h^{(1)}$. We denote B the bilinear form associated with the energy :

$$B(a, b) = \int_M a\,\overline{b} + \nabla a\cdot\nabla\overline{b} + \partial_t a\,\partial_t\overline{b}$$

We have to prove

$$B\left((p_h^{(1)}(t_h^{(1)}), w_h^{(1)}(t_h^{(1)})\right) = B\left(p_h^{(1)}(t_h^{(1)}), v_h(t_h^{(1)}) - p_h^{(1)}(t_h^{(1)})\right) = o(1)$$

By weak convergence to 0 in H^1 of v_h, $p_h^{(1)}$ and $w_h^{(1)}$, we can omit the term $\int_M a\,\overline{b}$ of B. By construction and Lemma 5.1.6, we have $D_h^{(1)}(v_h, \partial_t v_h)(t_h^{(1)}) \underset{h\to 0}{\rightharpoonup} (\varphi^{(1)}, \psi^{(1)})$

and $D_h^{(1)}(p_h^{(1)}, \partial_t p_h^{(1)}) \underset{h\to 0}{\rightharpoonup} (\varphi^{(1)}, \psi^{(1)})$. Therefore, $D_h^{(1)}(w_h^{(1)}, \partial_t w_h^{(1)})(t_h^{(1)}) \underset{h\to 0}{\rightharpoonup} (0,0)$. Lemma 5.1.11 gives the expected result. Remark that if $a \equiv 0$, this is just a consequence of the conservation of scalar product for solution of linear wave equation. $\square$

We get the expansion of u_h announced in Proposition 5.2.2 by induction iterating the same process.

Let us assume that

$$
\begin{aligned}
v_h(t,x) &= \textstyle\sum_{j=1}^{l} p_n^{(j)}(t,x) + w_n^{(l)}(t,x),\\
\|(v_h, \partial_t v_h)\|_{\mathcal{E}}^2 &= \textstyle\sum_{j=1}^{l} \left\|(p_h^{(j)}, \partial_t p_h^{(j)})\right\|_{\mathcal{E}}^2 + \left\|(w_h^{(l)}, \partial_t w_h^{(l)})\right\|_{\mathcal{E}}^2 + o(1),
\end{aligned} \tag{2.33}
$$

where $o(1)$ is uniform in t as $h \to 0$ and where $p_h^{(j)}$ is a linear damped concentrating wave, associated with data $[(\varphi^{(k)}, \psi^{(k)}), \underline{h}, \underline{x}^{(k)}, \underline{t}^{(k)}]$ mutually orthogonal.

We argue as before : we can assume $\delta(\underline{w}^{(l)}) > 0$ and we can pick $(\varphi^{(l+1)}, \psi^{(l+1)}), \underline{x}^{(l+1)}, \underline{t}^{(l+1)}$ such that :

$$
\begin{gathered}
\left\|\nabla\varphi^{(l)}\right\|_{L^2(T_{x_\infty^{(l+1)}}M)}^2 + \left\|\psi^{(l)}\right\|_{L^2(T_{x_\infty^{(l+1)}}M)}^2 \geq \frac{1}{2}\delta(\underline{w}^{(l)})\\
D_h^{(l+1)}(w_h^{(l)}, \partial_t w_h^{(l)})(t_h^{(l+1))}) \underset{h\to 0}{\rightharpoonup} (\varphi^{(l+1)}, \psi^{(l+1)}).
\end{gathered} \tag{2.34}
$$

and we define $p_h^{(l+1)}$ as a linear damped concentrating wave, associated with data $[(\varphi^{(l+1)}, \psi^{(l+1)}), \underline{h}, \underline{x}^{(l+1)}, \underline{t}^{(l+1)}]$. Again, Lemma 5.2.8 applied to $w_h^{(l)}$ and $p_h^{(l+1)}$ implies estimates (2.32) with $w_h^{(l+1)} = w_h^{(l)} - p_h^{(l+1)}$.

Let us now deal with estimate (2.31). Lemma 5.1.4 combined with energy estimates gives for some $C > 0$ only depending on T and a

$$
\left\|\nabla\varphi^{(j)}\right\|_{L^2(T_{x_\infty^{(j)}}M)}^2 + \left\|\psi^{(j)}\right\|_{L^2(T_{x_\infty^{(j)}}M)}^2 \leq C\left\|(p_h^{(j)}, \partial_t p_h^{(j)})_{t=0}\right\|_{\mathcal{E}}^2 + o(1).
$$

From this and estimate (2.32), we infer

$$
\sum_{j=1}^{l}\left(\left\|\nabla\varphi^{(j)}\right\|_{L^2(T_{x_\infty^{(j)}}M)}^2 + \left\|\psi^{(j)}\right\|_{L^2(T_{x_\infty^{(j)}}M)}^2\right) \leq C\varlimsup_{h\to 0}\|(u_h, \partial_t u_h)\|_{\mathcal{E}}^2 \leq C.
$$

So, the series of general term $\left(\left\|\nabla\varphi^{(j)}\right\|_{L^2(T_{x_\infty^{(j)}}M)}^2 + \left\|\psi^{(j)}\right\|_{L^2(T_{x_\infty^{(j)}}M)}^2\right)$ converges. Using estimate (2.34), we get

$$
\lim_{l\to\infty}\delta(\underline{w}^{(l)}) = 0.
$$

Lemma 5.2.7 yields

$$
\varlimsup_{h\to 0}\left\|w_h^{(l)}\right\|_{L^\infty([-T,T],L^6(M)} \underset{l\to\infty}{\longrightarrow} 0.
$$

This completes the proof of the first part of Proposition 5.2.2. Let us now deal with the orthogonality result. We will need the following two lemmas.

Lemma 5.2.9. *Let* $(\underline{x}^{(1)}, \underline{t}^{(1)}) \not\perp_h (\underline{x}^{(2)}, \underline{t}^{(2)})$. *Let* v_h *be an* h*-oscillatory sequence solution of the damped linear wave equation such that*

$$D_h^{(1)}(v_h, \partial_t v_h)(t_h^{(1)}) \underset{h\to 0}{\rightharpoonup} (\varphi^{(1)}, \psi^{(1)}). \tag{2.35}$$

Then, there exists $(\varphi^{(2)}, \psi^{(2)})$ *such that, up to a subsequence*

$$D_h^{(2)}(v_h, \partial_t v_h)(t_h^{(2)}) \underset{h\to 0}{\rightharpoonup} (\varphi^{(2)}, \psi^{(2)}). \tag{2.36}$$

Moreover, we have

$$\left\|(\varphi^{(1)}, \psi^{(1)})\right\|_{\mathcal{E}_{x_\infty}} = \left\|(\varphi^{(2)}, \psi^{(2)})\right\|_{\mathcal{E}_{x_\infty}}. \tag{2.37}$$

Démonstration. First, we assume $\underline{x}^{(1)} = \underline{x}^{(2)}$. By translation in time, we can assume $\underline{t}^{(1)} = 0$. The non orthogonality assumption yields, up to extraction, $t_h^{(2)}/h = C + o(1)$ with C constant.

Let $(\varphi, \psi) \in \mathcal{E}_\infty$ arbitrary and p_h the linear damped concentrating wave associated with
$[(\varphi, \psi), \underline{h}, \underline{x}^{(1)}, 0]$. We use the equivalent definition stated in Lemma 5.1.5 : (2.35) is equivalent to

$$\begin{aligned}
\int_M \nabla v_h(0) \cdot \nabla p_h(0) &\underset{n\to\infty}{\longrightarrow} \int_{T_{x_\infty} M} \nabla \varphi^{(1)} \cdot \nabla \varphi \\
\int_M \partial_t v_h(0) \partial_t p_h(0) &\underset{n\to\infty}{\longrightarrow} \int_{T_{x_\infty} M} \psi^{(1)} \psi.
\end{aligned}$$

As both v_h and p_h are solutions of the damped wave equation on M and $t_h^{(2)} \underset{h\to 0}{\longrightarrow} 0$, we have by equicontinuity (see Lemma 5.1.9).

$$\int_M \nabla v_h(t_h^{(2)}) \cdot \nabla p_h(t_h^{(2)}) + \int_M \partial_t v_h(t_h^{(2)}) \partial_t p_h(t_h^{(2)}) \underset{n\to\infty}{\longrightarrow} \int_{T_{x_\infty} M} \nabla \varphi^{(1)} \cdot \nabla \varphi + \int_{T_{x_\infty} M} \psi^{(1)} \psi.$$

Let v, w satisfying on $T_{x_\infty} M$

$$\begin{aligned}
\Box_\infty v = 0, \quad (v, \partial_t v)_{|t=0} &= (\varphi^{(1)}, \psi^{(1)}) \\
\Box_\infty w = 0, \quad (w, \partial_t w)_{|t=0} &= (\varphi, \psi).
\end{aligned}$$

Conservation of the scalar product yields

$$\int_{T_{x_\infty} M} \nabla \varphi^{(1)} \cdot \nabla \varphi + \int_{T_{x_\infty} M} \psi^{(1)} \psi = \int_{T_{x_\infty} M} \nabla v(C) \cdot \nabla w(C) + \int_{T_{x_\infty} M} \partial_t v(C) \partial_t w(C).$$

But according to Corollary 5.2.1, $(p_h, \partial_t p_h)_{|t=t_h^{(2)}}$ is a concentrating data according to $[(w(C), \partial_t w(C)), \underline{h}, \underline{x}^{(1)}]$. Since the wave equation is reversible and (φ, ψ) is arbitrary, we have proved that for any concentrating data (f_h, g_h) associated with $[(\tilde{\varphi}, \tilde{\psi}, \underline{h}, \underline{x}^{(1)}]$, we have

$$\int_M \nabla v_h(t_h^{(2)}) \cdot \nabla f_h + \int_M \partial_t v_h(t_h^{(2)}) g_h \underset{n\to\infty}{\longrightarrow} \int_{T_{x_\infty} M} \nabla v(C) \cdot \nabla \tilde{\varphi} + \int_{T_{x_\infty} M} \partial_t v(C) \tilde{\psi}.$$

This gives the result for $x_h^{(1)} = x_h^{(2)}$ by taking $(\varphi^{(2)}, \psi^{(2)}) = (v(C), \partial_t v(C))$ which satisfies (2.37) by conservation of the energy.

In the general case $\underline{x}^{(1)} \not\perp_h \underline{x}^{(2)}$, we have in a local coordinate chart and up to a subsequence $x_h^{(2)} = x_h^{(1)} + (\vec{D} + o(1))h$ where $\vec{D} \in T_{x_\infty}M$ is a constant vector. We remark that if a bounded sequence (f_h, g_h) satisfies $D_h^{(1)}(f_h, g_h) \underset{h\to 0}{\rightharpoonup} (\varphi, \psi)$, it also fulfills $D_h^{(2)}(f_h, g_h) \underset{h\to 0}{\rightharpoonup} (\varphi(. + \vec{D}), \psi(. + \vec{D}))$. □

We will also need the following lemma which is the analog of Lemma 3.7 of [17]. We keep the notation of the algorithm of extraction for further use.

Lemma 5.2.10. *Let* $\{j, j'\} \in \{1, \cdots, K\}^2$ *be such that*

$$(\underline{x}^{(j)}, \underline{t}^{(j)}) \not\perp_h (\underline{x}^{(K+1)}, \underline{t}^{(K+1)}) \text{ and } (\underline{x}^{(j)}, \underline{t}^{(j)}) \perp_h (\underline{x}^{(j')}, \underline{t}^{(j')}).$$

Then, $D_h^{(K+1)}(w_h^{(K+1)}, \partial_t w_h^{(K+1)})(t_h^{(K+1)}) \rightharpoonup 0$ *implies* $D_h^{(j)}(w_h^{(K+1)}, \partial_t w_h^{(K+1)})(t_h^{(j)}) \rightharpoonup 0$. *Moreover, if we assume* $\left|t_\infty^{(j)} - t_\infty^{(j')}\right| < T_{focus}$ *(see Definition 5.0.1), then* $D_h^{(j)}(p_h^{(j')}, \partial_t p_h^{(j')})(t_h^{(j)}) \rightharpoonup (0,0)$ *for any concentrating wave* $p_h^{(j')}$ *associated with* $[(\varphi^{(j')}, \psi^{(j')}), \underline{h}, \underline{x}^{(j')}, \underline{t}^{(j')}]$.

Démonstration. The first result is a particular case of Lemma 5.2.9. The proof of the second part is very similar to Lemma 3.7 of [17]. To simplify the notation, we can assume by translation in time that $t_h^{(j')} = 0$. We have to distinguish two cases : time and space orthogonality.

In the case of time orthogonality, that is $\left|\frac{t_h^{(j)}}{h}\right| \underset{h\to 0}{\longrightarrow} +\infty$, we first prove $D_h^{1,(j)}(p_h^{(j')})(t_h^{(j)}) \rightharpoonup 0$ (recall that the exponent 1 in $D_h^{1,(j)}$ means that we only consider the H^1 part of the weak limit). Thanks to the nonfocusing assumption, Lemma 5.2.3 yields

$$\left\|p_h^{(j')}(t_h^{(j)}, .)\right\|_{L^6(M)} \underset{h\to 0}{\longrightarrow} 0$$

We choose (U, Φ_U) some local chart around $x_\infty^{(j)}$ and $\Psi_U \in C_0^\infty(U)$ equals to 1 around $x_\infty^{(j)}$. Then, $\left\|\Psi_U p_h^{(j')}(t_h^{(j)}, .)\right\|_{L^6(M)} \underset{h\to 0}{\longrightarrow} 0$ and $h^{\frac{1}{2}}\left\|\Psi_U p_h^{(j')}(t_h^{(j)}, x_h + hx)\right\|_{L^6(\mathbb{R}^3)} \underset{h\to 0}{\longrightarrow} 0$ (here, we have identified $\Psi_U p_h^{(j')}$ with its local representation in $\mathbb{R}^3$). In particular $h^{\frac{1}{2}}\Psi_U p_h^{(j')}(t_h^{(j)}, x_h + hx) \rightharpoonup 0$ and $D_h^{1,(j)}(p_h^{(j')})(t_h^{(j)}) \rightharpoonup 0$. Now, we want to prove more precisely $D_h^{(j)}(p_h^{(j')}, \partial_t p_h^{(j')})(t_h^{(j)}) \rightharpoonup 0$. Suppose $D_h^{(j)}(p_h^{(j')}, \partial_t p_h^{(j')})(t_h^{(j)}) \rightharpoonup (0, \psi)$. Take $s \in \mathbb{R}$ arbitrary. $\tilde{t}_h^{(j)} = t_h^{(j)} + sh$ fulfills the same assumption $\left|\frac{\tilde{t}_h^{(j)}}{h}\right| \underset{h\to 0}{\longrightarrow} +\infty$ and the nonfocusing property $|\tilde{t}_\infty^{(j)}| < T_{focus}$. So, we conclude similarly that $D_h^{1,(j)}(p_h^{(j')})(\tilde{t}_h^{(j)}) \rightharpoonup 0$. But the proof of Lemma 5.2.9 gives that $D_h^{(j)}(p_h^{(j')}, \partial_t p_h^{(j')})(\tilde{t}_h^{(j)}) \rightharpoonup (v, \partial_t v)(s)$ where v is solution of

$$\Box_\infty v = 0, \quad (v, \partial_t v)(0) = (0, \psi)$$

So, we have $v(s) = 0$ for any $s \in \mathbb{R}$, which gives $\psi = 0$ and $D_h^{(j)}(p_h^{(j')}, \partial_t p_h^{(j')})(t_h^{(j)}) \rightharpoonup (0, 0)$.

In the case of $t_h^{(j)} \not\perp_h t_h^{(j')}$ and space orthogonality, Lemma 5.2.9 allows us to assume that $t_h^{(j)} = t_h^{(j')} = 0$. In local coordinates, we have

$$(p_h^{(j')}, \partial_t p_h^{(j')})(0) = h^{-\frac{1}{2}} \Psi_U(x) \left(\varphi^{(j')}, \frac{1}{h} \psi^{(j')} \right) \left(\frac{x - x_h^{(j')}}{h} \right)$$

If $x_\infty^{(j')} \neq x_\infty^{(j)}$, the conclusion is obvious. If it is not the case, take $g \in C_0^\infty(\mathbb{R}^3)$. For the first part, we have to estimate

$$\int_{\mathbb{R}^3} \Psi_U^2(x_n^{(j)} + hy) \varphi^{(j')} \left(y + \frac{x_h^{(j)} - x_h^{(j')}}{h} \right) g(y) dy$$

which goes to 0 as h tends to 0 because g is compactly supported. The same result holds for the second part for $\partial_t p_h^{(j')}$. □

Let us come back to the proof of the orthogonality of cores in Proposition 5.2.2. Define :

$$j_K = \max \left\{ j \in \{1, \cdots K\} | \, (t_h^{(j)}, x_h^{(j)}) \not\perp_h (t_h^{(K+1)}, x_h^{(K+1)}) \right\}$$

assuming that such an index exists.

We list a few consequences of our algorithm

$$D_h^{(l+1)}(w_h^{(l)}, \partial_t w_h^{(l)})(t_h^{(l+1)}) \underset{h \to 0}{\rightharpoonup} (\varphi^{(l+1)}, \psi^{(l+1)}) \text{ with } \varphi^{(l+1)} \neq 0 \text{ if } l \leq K \tag{2.38}$$

$$w_h^{(l)} = p_h^{(l+1)} + w_h^{(l+1)} \tag{2.39}$$

$$w_h^{(j_K)} = \sum_{j=j_K+1}^{K+1} p_h^{(j)} + w_h^{(K+1)} \tag{2.40}$$

The definition of $p_h^{(l)}$ and Lemma 5.1.6 implies $D_h^{(l)}(p_h^{(l)}, \partial_t p_h^{(l)})(t_h^{(l)}) \rightharpoonup (\varphi^{(l)}, \psi^{(l)})$. Then, we get from (2.38) and (2.39) that $D_h^{(l+1)}(w_h^{(l+1)}, \partial_t w_h^{(l+1)})(t_h^{(l+1)}) \rightharpoonup (0,0)$. We apply this to $l+1 = j_K$ and it gives $D_h^{(K+1)}(w_h^{(j_K)}, \partial_t w_h^{(j_K)})(t_h^{(K+1)}) \rightharpoonup (0,0)$ thanks to the first part of Lemma 5.2.10 and the definition of j_K.

The definition of j_K and the second part of Lemma 5.2.10 gives $D_h^{1,(K+1)}(p_h^{(l)}, \partial_t p_h^{(l)})(t_h^{(K+1)}) \rightharpoonup (0,0)$ for $j_K + 1 \leq l \leq K$.

To conclude, we "apply" $D_h^{1,(K+1)}$ to equality (2.40) and get $D_h^{1,(K+1)} w_h^{(j_K)}(t_h^{(K+1)}) \rightharpoonup \varphi^{(K+1)}$ while we have just proved $D_h^{(K+1)}(w_h^{(j_K)}, \partial_t w_h^{(j_K)})(t_h^{(K+1)}) \rightharpoonup (0,0)$ which is a contradiction and complete the proof of the proposition for $2T < T_{focus}$.

In the case of S^3 and large times, the orthogonality result is a consequence of the orthogonality in short times and the almost periodicity. Denote

$$j_K = \max \left\{ j \in \{1, \cdots K\} | \, \exists m \in \mathbb{Z} \text{ s.t. } (t_h^{(j)} + m\pi, (-1)^m x_h^{(j)}) \not\perp_h (t_h^{(K+1)}, x_h^{(K+1)}) \right\}.$$

Then, for any $j_K + 1 \leq j \leq K$, we can find $m^{(j)} \in \mathbb{Z}$ such that

$$\left| t_\infty^{(j)} + m^{(j)}\pi - t_\infty^{(j_K)} \right| \leq \pi/2 < T_{focus}$$

$$(t_h^{(j)} + m^{(j)}\pi, (-1)^{m^{(j)}} x_h^{(j)}) \perp_h (t_h^{(K+1)}, x_h^{(K+1)}).$$

and we denote $m^{(j_K)} \in \mathbb{Z}$ such that $(t_h^{(j_K)} + m^{(j_K)}\pi, (-1)^{m^{(j_K)}} x_h^{(j_K)}) \not\perp_h (t_h^{(K+1)}, x_h^{(K+1)})$. We remark that $p_h^{(j)}\left(t_h^{(j)} + m^{(j)}\pi, .\right)$ is still a non zero concentrating data associated with
$[(-1)^{m^{(j)}}(\varphi,\psi)((-1)^{m^{(j)}}.), \underline{h}, (-1)^j \underline{x}]$ thanks to Lemma 5.2.4 (note that it is at this stage that we use $M = S^3$ and $a \equiv 0$: it is the only case where we are able to describe this phenomenon of reconcentration). So, we are in the same situation as before, and we get a contradiction.

This completes the proof of Proposition 5.2.2. □

Proof of Theorem 5.0.3. We only have to combine the both decompositions we made. Denote v_n^j (and the rest $\rho_n^{(l)}$ the $h_n^{(j)}$ oscillatory component obtained by decomposition (2.24) and $p_n^{(j,\alpha)}$ the concentrating waves obtained from decomposition (2.30) (and the rest $w_n^{(j,A_j)}$). We enumerate them by the bijection σ from $\mathbb{N}^2$ into $\mathbb{N}$ defined by

$$\sigma(j,\alpha) < \sigma(k,\beta) \text{ if } j+\alpha < k+\beta \text{ or } j+\alpha = k+\beta \text{ and } j<k.$$

For l and A_j fixed, $1 \leq j \leq l$, the rest can be written

$$w_n^{(l,A_1,\cdots,A_l)} = \rho_n^{(l)} + \sum_{j=1}^{l} w_n^{(j,A_j)}.$$

Let $\varepsilon > 0$. To get the result, it suffices to prove that for l_0 large enough, $\left\|w_n^{(l,A_1,\cdots,A_l)}\right\|_{L^\infty(L^6)} \leq \varepsilon$ for all $(l, A_1, \cdots, A_l)$ satisfying $l \geq l_0$ and $\sigma(j, A_j) \geq \sigma(l_0, 1)$.

(0.9) can easily be deduced from the same orthogonality result in the both other decomposition. In particular, it gives that the series of general term $\sum_{(j,\alpha)} \overline{\lim_{n\to\infty}} \left\|(p_n^{(j,\alpha)}, \partial_t p_n^{(j,\alpha)})_{t=0}\right\|_{\mathcal{E}}^2$ is convergent. In particular, we can find l_0 large enough such that we have

$$\sum_{\sigma(j,\alpha)>\sigma(l_0,1)} \overline{\lim_{n\to\infty}} \left\|(p_n^{(j,\alpha)}, \partial_t p_n^{(j,\alpha)})_{t=0}\right\|_{\mathcal{E}}^2 \leq \varepsilon. \tag{2.41}$$

Moreover, for l_0 large enough, we have for $l \geq l_0$

$$\overline{\lim_{n\to\infty}} \left\|\rho_n^{(l)}\right\|_{L^\infty(L^6)} \leq \varepsilon.$$

Then, for any $l \geq l_0$, one can find one B_l such that for any $1 \leq j \leq l$, $\tilde{A}_j \geq B_l$ implies

$$\overline{\lim_{n\to\infty}} \left\|w_n^{(j,\tilde{A}_j)}\right\|_{L^\infty(L^6)} \leq \varepsilon/l.$$

The rest can be decomposed by

$$w_n^{(l,A_1,\cdots,A_l)} = \rho_n^{(l)} + \sum_{j=1}^{l} w_n^{(j,\max(A_j,B_l))} + S_n^{(j,A_1,\cdots,A_l))}.$$

where

$$S_n^{(j,A_1,\cdots,A_l))} = \sum_{1\leq j\leq l, A_j<B_l} \left(w_n^{(j,A_j)} - w_n^{(j,B_l)}\right) = \sum_{j=1}^{l} \sum_{A_j<\alpha\leq B_l} p_n^{j,\alpha}.$$

Since $S_n^{(j,A_1,\cdots,A_l))}$ is solution of the damped wave equation, energy estimates and Sobolev embedding give

$$\begin{aligned}\overline{\lim_{n\to\infty}}\left\|S_n^{(j,A_1,\cdots,A_l))}\right\|^2_{L^\infty(L^6)} &\le C\,\overline{\lim_{n\to\infty}}\left\|(S_n^{(j,A_1,\cdots,A_l))},\partial_t S_n^{(j,A_1,\cdots,A_l))})_{t=0}\right\|^2_{\mathcal{E}}\\ &\le C\sum_{j=1}^{l}\sum_{A_j<\alpha\le B_l}\left\|(p_n^{(j,\alpha)},\partial_t p_n^{j,\alpha})_{t=0}\right\|^2_{\mathcal{E}}.\end{aligned}$$

where we have used almost orthogonality in the last estimate. But the sum is restricted to some (j,α) satisfying $\sigma(j,\alpha)>\sigma(j,\alpha_j)>\sigma(l_0,1)$ and is indeed smaller than $C\varepsilon$ thanks to (2.41).

Combining our estimates, we get that $\overline{\lim_{n\to\infty}}\left\|w_n^{(l,A_1,\cdots,A_l)}\right\|_{L^\infty(L^6)}$ is smaller than $(2+C)\varepsilon$ for all $(l,A_1,\cdots,A_l)$ satisfying $l\ge l_0$ and $\sigma(j,A_j)\ge\sigma(l_0,1)$. We get the same estimates with the $L^5(L^{10})$ norm by interpolation between $L^\infty(L^6)$ and $L^4(L^{12})$. The second norm being bounded by Strichartz estimates and the fact that $w_n^{(l,A_1,\cdots,A_l)}$ is uniformly bounded in the energy space. $\square$

We also state a few consequences of the algorithm of Theorem 5.0.3 that will be used below. The following both lemmas use the notation and the assumptions of Theorem 5.0.3 .

Lemma 5.2.11. *Let $2T<T_{focus}$. For any $l\in\mathbb{N}$ and $1\le j\le l$, we have, with the notation and assumptions of Theorem 5.0.3*

$$D_n^{(j)}(w_n^{(l)},\partial_t w_n^{(l)})(t_n^{(j)})\rightharpoonup(0,0).$$

Démonstration. Assume $D_n^{(j)}(w_n^{(l)},\partial_t w_n^{(l)})(t_n^{(j)})\rightharpoonup(\varphi,\psi)$. We directly use the decomposition of Theorem 5.0.3 to write for $L>l$

$$w_n^{(l)}=\sum_{i=l+1}^{L}p_n^{(i)}+w_n^{(L)}.$$

In case of scale orthogonality of $h_n^{(j)}$ and $h_n^{(i)}$, for $l+1\le i\le L$, we have directly $D_n^{(j)}(p_n^{(i)},\partial_t p_n^{(i)})(t_n^{(j)})\rightharpoonup(0,0)$. Otherwise, if $h_n^{(j)}=h_n^{(i)}$ and $(\underline{x}^{(j)},\underline{t}^{(j)})\perp_h(\underline{x}^{(i)},\underline{t}^{(i)})$, Lemma 5.2.10 gives the same result. Therefore, $D_n^{(j)}(w_n^{(L)},\partial_t w_n^{(L)})(t_n^{(j)})\rightharpoonup(\varphi,\psi)$. Since $\overline{\lim_{n\to\infty}}\left\|w_n^{(L)}\right\|_{L^\infty([-T,T],L^6)}\underset{L\to\infty}{\longrightarrow}0$, we have $\varphi=0$. We finish the proof as in Lemma 5.2.10. We use the same argument for times $t_n^{(j)}+sh_n^{(j)}$ and get $\psi\equiv0$ by the proof of Lemma 5.2.9. Remark that Lemma 5.2.9 requires that $w_n^{(l)}$ is strictly $h_n^{(j)}$-oscillatory, but this can be easily avoided by decomposing $w_n^{(l)}=f_n+g_n$ with f_n $(h_n^{(j)})$-oscillatory and g_n $(h_n^{(j)})$-singular. $\square$

Lemma 5.2.12. *With the notation and assumptions of Theorem 5.0.3, we have, for any $j\in\mathbb{N}$*

$$\overline{\lim_{n\to\infty}}\left\|p_n^{(j)}\right\|_{L^5([-T,T],L^{10})}\le C\,\overline{\lim_{n\to\infty}}\left\|v_n\right\|_{L^5([-T,T],L^{10})}$$

where C only depends on the manifold M.

Démonstration. We first assume $2T < T_{focus}$. Actually, in the case of $\mathbb{R}^3$, the result is proved using the fact that the $p_n^{(j)}$ are some concentration of some weak limit of a dilation of v_n. The proof for a manifold follows the same path with a little more care due to the fact that dilation only have a local meaning.

For any $\varepsilon > 0$, we prove

$$\overline{\lim_{n\to\infty}} \left\|p_n^{(j)}\right\|_{L^5([-T,T],L^{10})} \leq C \overline{\lim_{n\to\infty}} \|v_n\|_{L^5([-T,T],L^{10})} + C\varepsilon.$$

We use the decomposition of Theorem 5.0.3 and choose $l \geq j$ large enough such that

$$\overline{\lim_{n\to\infty}} \left\|w_n^l\right\|_{L^5([-T,T],L^{10})} \leq \varepsilon.$$

Let Ψ_U be a cut off function related to local charts (U, Φ_U) such that $\Psi_U(x) = 1$ around x_∞^j and $\Psi_U(x) = 0$ around any $x_\infty^i \neq x_\infty^j$.

For each $1 \leq i \leq l$, we decompose $[-T,T] = I_{n,i}^{1,\Lambda} \cup I_{n,i}^{2,\Lambda} \cup I_{n,i}^{3,\Lambda}$ according to Lemma 5.2.3.

For any i such that $x_\infty^i = x_\infty^j$, for Λ large enough, we have

$$\overline{\lim_{n\to\infty}} \left\|p_n^{(i)}\right\|_{L^5(I_{n,i}^{1,\Lambda}\cup I_{n,i}^{3,\Lambda},L^{10})} \leq \varepsilon/l. \tag{2.42}$$

Moreover, Lemma 5.2.2 yields for Λ large enough

$$\overline{\lim_{n\to\infty}} \left\|p_n^{(i)} - v_n^{(i)}\right\|_{L^5(I_{n,i}^{2,\Lambda},L^{10})} \leq \varepsilon/l \tag{2.43}$$

where $v_n^{(i)}(t,x) = \frac{1}{\sqrt{h_n^{(i)}}} \Phi_U^* \Psi_U(x) v^{(i)}\left(\frac{t-t_n^{(i)}}{h_n^{(i)}}, \frac{x-x_n^{(i)}}{h_n^{(i)}}\right)$ on a coordinate patch and $v^{(i)}$ solution of

$$\left\{\begin{array}{rcl} \square_{x_\infty^j} v^{(i)} & = & 0 \quad \text{on} \quad \mathbb{R}\times T_{x_\infty^j}M \\ (v^{(i)}(0), \partial_t v^{(i)}(0)) & = & (\varphi^{(i)}, \psi^{(i)}). \end{array}\right. \tag{2.44}$$

Thanks to (2.42) and (2.43), the conclusion of the lemma will be obtained if we prove

$$\left\|v^{(j)}\right\|_{L^5(\mathbb{R},L^{10}(T_{x_\infty^j}M))} \leq \overline{\lim_{n\to\infty}} \|v_n\|_{L^5([-T,T],L^{10})} + C\varepsilon.$$

We argue by duality. Take $f \in C_0^\infty(\mathbb{R}\times T_{x_\infty^j}M)$ with $\|f\|_{L^{5/4}(\mathbb{R},L^{10/9})} = 1$.

From now on, we work in local coordinates around $x_\infty^{(j)}$ and we will not distinguish a function defined on $U \subset M$ with its representant in $\mathbb{R}^3 \approx T_{x_\infty^j}M$. Denote W^j the operator defined on functions on $\mathbb{R}_t \times \mathbb{R}^3$ by

$$W^j g(s,y) := \sqrt{h_n^j} g(t_n^j + h_n^j s, t_n^j + h_n^j s).$$

The definition of $v_n^{(j)}$ in local coordinates yields

$$\int_{\mathbb{R}\times\mathbb{R}^3} (W^j 1_{[-T,T]} v_n^{(j)}) f \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}\times\mathbb{R}^3} v^{(j)} f.$$

On the other hand

$$\int_{\mathbb{R}\times\mathbb{R}^3}(W^j\Psi_U 1_{[-T,T]}p_n^j)f = \int_{\mathbb{R}\times\mathbb{R}^3} W^j\left[\Psi_U 1_{[-T,T]}(v_n - \sum_{x_\infty^{(i)}\neq x_\infty^{(j)}} p_n^i - \sum_{x_\infty^{(i)}=x_\infty^{(j)},i\neq j} p_n^i - w_n^l)\right]f.$$

For any $1\leq i\leq l$, with $x_\infty^{(i)}\neq x_\infty^{(j)}$, using again Lemma 5.2.3 and 5.2.2 and the fact that we can choose Ψ_U with $\Psi_U(x_\infty^{(i)})=0$, we easily get

$$\overline{\lim_{n\to\infty}}\left\|\Psi_U p_n^{(i)}\right\|_{L^5([-T,T],L^{10})}=0.$$

So for n large enough

$$\left|\int_{\mathbb{R}\times\mathbb{R}^3}(W^j\Psi_U p_n^{(j)})f\right|\leq C\left(\|v_n\|_{L^5([-T,T],L^{10})}+2\varepsilon\right)+\left|\int_{\mathbb{R}\times\mathbb{R}^3}W^j\left[\Psi_U 1_{[-T,T]}\sum_{x_\infty^{(i)}=x_\infty^{(j)},i\neq j}p_n^{(i)}\right]f\right|$$

But for $i\neq j$, $x_\infty^{(i)}=x_\infty^{(j)}$, using (2.42) and then (2.43), we have for Λ and n large enough

$$\begin{aligned}\left|\int_{\mathbb{R}\times\mathbb{R}^3}W^j\left[\Psi_U 1_{[-T,T]}p_n^{(i)}\right]f\right| &\leq \left|\int_{\mathbb{R}\times\mathbb{R}^3}W^j\left[\Psi_U 1_{I_{n,i}^{2,\Lambda}}p_n^{(i)}\right]f\right|+\varepsilon/l\\ &\leq \left|\int_{\mathbb{R}\times\mathbb{R}^3}W^j\left[\Psi_U 1_{I_{n,i}^{2,\Lambda}}v_n^{(i)}\right]f\right|+2\varepsilon/l.\end{aligned}$$

These terms are actually

$$\left|\int_{\mathbb{R}\times\mathbb{R}^3}W^j\left[\Psi_U 1_{I_{n,i}^{2,\Lambda}}v_n^{(i)}\right]f\right|=$$

$$\sqrt{\frac{h_n^j}{h_n^i}}\left|\int_{\mathbb{R}\times\mathbb{R}^3}\left[\Psi_U^2(h_n^j x+x_n^j)1_{\left[\frac{t_n^i-t_n^j-\Lambda h_n^i}{h_n^j},\frac{t_n^i-t_n^j+\Lambda h_n^i}{h_n^j}\right]}v^{(i)}\left(\frac{th_n^j+t_n^j-t_n^i}{h_n^i},\frac{xh_n^j+x_n^j-x_n^i}{h_n^i}\right)\right]f\right|.$$

Since this expression is uniformly continuous in $v^i\in L^5(\mathbb{R},L^{10}(\mathbb{R}^3))$, we may assume v^i in $C_0^\infty(\mathbb{R}\times\mathbb{R}^3)$. Then, if $\frac{h_n^j}{h_n^i}\underset{n\to\infty}{\longrightarrow}0$, we have

$$\left|\int_{\mathbb{R}\times\mathbb{R}^3}W^j\left[\Psi_U 1_{I_{n,i}^{2,\Lambda}}v_n^{(i)}\right]f\right|=\mathcal{O}(\sqrt{\frac{h_n^j}{h_n^i}}).$$

If $\frac{h_n^j}{h_n^i}\underset{n\to\infty}{\longrightarrow}\infty$, the change of variable $s=\frac{th_n^j+t_n^j-t_n^i}{h_n^j}$, $y=\frac{xh_n^j+x_n^j-x_n^i}{h_n^i}$ gives

$$\left|\int_{\mathbb{R}\times\mathbb{R}^3}W^j\left[\Psi_U 1_{I_{n,i}^{2,\Lambda}}v_n^i\right]f\right|=\mathcal{O}(\left(\frac{h_n^j}{h_n^i}\right)^{-7/2}).$$

If $h_n^j=h_n^i$, the space or time orthogonality yields that the integral is zero for n large enough.

In conclusion, for any $f\in C_0^\infty(\mathbb{R}\times\mathbb{R}^3)$ with $\|f\|_{L^{5/4}(\mathbb{R},L^{10/9})}=1$, we have proved :

$$\left|\int_{\mathbb{R}\times\mathbb{R}^3}v^j f\right|\leq C\,\overline{\lim_{n\to\infty}}\,\|v_n\|_{L^5([-T,T],L^{10})}+C\varepsilon$$

This gives the expected result by duality.

The case of S^3 is proved by considering subintervals of length smaller than T_{focus} where the former result can be applied. □

5.2.2 Nonlinear profile decomposition

5.2.2.1 Behavior of nonlinear concentrating waves (after S. Ibrahim)

In this subsection, we recall the description of nonlinear concentrating waves. As explained in the introduction, the behavior for times close to concentration is rulled by the scattering operator on $\mathbb{R}^3$ with a flat metric. So, we first state the existence of the wave operator on $\mathbb{R}^3$, following the notation of [2]. We state it for any constant metric on the tangent plane $T_{x_\infty}M \approx \mathbb{R}^3$.

Proposition 5.2.3 (Scattering operators on $\mathbb{R}^3$)**.** *Let $x_\infty \in M$ and $\Box_\infty$ the d'Alembertian operator (constant) on $T_{x_\infty}M \approx \mathbb{R}^3$ induced by the metric on M. To every solution of*

$$\left\{\begin{array}{rcl} \Box_\infty v & = & 0 \quad \text{on} \quad \mathbb{R} \times T_{x_\infty}M \\ (v(0), \partial_t v(0)) & = & (\varphi, \psi) \in \mathcal{E}_{x_\infty}. \end{array}\right.$$

there exists a unique strong solution $u_\pm$ of

$$\left\{\begin{array}{l} \Box_\infty u_\pm = -|u_\pm|^4 u_\pm \quad \text{on} \quad \mathbb{R} \times T_{x_\infty}M \\ \lim\limits_{t\to\pm\infty} \|(v - u_\pm, \partial_t(v - u_\pm))(t)\|_{\mathcal{E}_{x_\infty}} = 0. \end{array}\right.$$

The wave operators

$$\Omega_\pm : (v, \partial_t v)_{t=0} \mapsto (u_\pm, \partial_t u_\pm)_{t=0}$$

are bijective from $\mathcal{E}_{x_\infty}$ onto itself.

The scattering operator S is defined as $S = (\Omega_+)^{-1} \circ \Omega_-$.

The analysis of nonlinear concentrating waves computed by S. Ibrahim in [22] shows that there are three different periods to be considered : before, during and after the time of concentration. Roughly speaking, for times close to the concentrating time, the solution is closed to nonlinear concentrating waves on $\mathbb{R}^3$ with flat metric and without damping, as described in Bahouri-Gérard [2] : in the fast time $h_n t$, it follows the scattering on $\mathbb{R}^3$. Before and after the time of concentration, the nonlinear concentrating wave is "close" to some linear damped concentrating waves as defined in Table 5.2.1 below. This is precised in the following theorem whose proof can be found in S. Ibrahim [22]. Yet, in [22], the result is stated for an equation without damping and we give a sketch of the proof in the damped case in Section 5.2.2.2.

Theorem 5.2.1. *Let $\underline{v} = [(\varphi, \psi), \underline{h}, \underline{x}, \underline{t})]$ be a linear damped concentrating wave. We denote by $\underline{u}$ its associated nonlinear damped concentrating wave (same data at $t = 0$). There exist three linear damped concentrating waves denoted by $[(\varphi_i, \psi_i), \underline{h}, \underline{x}, \underline{t})]$, $i = 1, 2$ or 3 such that : for all interval $[-T, T]$ containing t_∞, satisfying the following nonfocusing property (see Definition 5.0.1)*

$$mes\,(F_{x,x_\infty,s}) = 0 \quad \forall x \in M \text{ and } s \neq 0 \text{ such that } t_\infty + s \in [-T, T] \tag{2.45}$$

we have

$$\overline{\lim_n}\, |||u_n - [(\varphi_1, \psi_1), \underline{h}, \underline{x}, \underline{t})]|||_{I_n^{1,\Lambda}} \underset{\Lambda\to+\infty}{\longrightarrow} 0 \tag{2.46}$$

$$\overline{\lim_n}\, |||u_n - [(\varphi_3, \psi_3), \underline{h}, \underline{x}, \underline{t})]|||_{I_n^{3,\Lambda}} \underset{\Lambda\to+\infty}{\longrightarrow} 0 \tag{2.47}$$

where, $I_n^{1,\Lambda} = [-T, t_n - \Lambda h_n]$ *and* $I_n^{3,\Lambda} =]t_n + \Lambda h_n, T]$.

Moreover, for times close to concentration $I_n^{2,\Lambda} = [t_n - \Lambda h_n, t_n + \Lambda h_n]$, *we have*

$$\forall \Lambda > 0, \quad \lim_n |||u_n - w_n|||_{I_n^{2,\Lambda}} = 0 \tag{2.48}$$

where $w_n(t,x) = \Psi_U(x) \frac{1}{\sqrt{h_n}} w\left(\frac{t-t_n}{h_n}, \frac{x-x_n}{h_n}\right)$ *on a coordinate patch and* w *solution of*

$$\left\{ \begin{array}{rcll} \Box_\infty w & = & -|w|^4 w & \text{on} \quad \mathbb{R} \times T_{x_\infty} M \\ (w(0), \partial_t w(0)) & = & (\varphi_2, \psi_2). \end{array} \right. \tag{2.49}$$

where $\Box_\infty$ *corresponds to the frozen metric on* $T_{x_\infty} M$.

The different functions (φ_i, ψ_i) *are defined according to Table 5.2.1, following the notation of Proposition 5.2.3.*

$\lim \frac{t_h}{h}$	(φ_1, ψ_1)	(φ_2, ψ_2)	(φ_3, ψ_3).
$-\infty$	$\Omega_-^{-1} \circ \Omega_+(\varphi, \psi)$	$\Omega_+(\varphi, \psi)$	(φ, ψ)
0	$\Omega_-^{-1}(\varphi, \psi)$	(φ, ψ)	$\Omega_+^{-1}(\varphi, \psi)$
∞	(φ, ψ)	$\Omega_-(\varphi, \psi)$	$\Omega_+^{-1} \circ \Omega_-(\varphi, \psi)$

TABLE 5.1 – Transformation of the profile through a focus

Remark 5.2.2. *Note that the transition from the first column to the third one represents the modification of profile due to the concentration and the concentrating functions are modified according to the scattering operator* S. *To go from the first column to the second one, we apply the operator* Ω_- *while we apply* Ω_+^{-1} *to get from the second to the third one.*

Remark 5.2.3. *The behavior for times close to concentration is not written this way in the article [22] of S. Ibrahim, but is a byproduct of its proof. We refer to the next section which contains a sketch of the proof.*

Corollary 5.2.2. *A nonlinear damped concentrating wave* q_h *is strictly (h)-oscillatory with respect to* A_M *and bounded in all Strichartz norms, uniformly on any bounded interval.*

Proof of Corollary 5.2.2. The boundedness of all Strichartz norms is a counsequence of Duhamel formula and Strichartz estimates once the result is known in the case of $L^5 L^{10}$. On the intervals $I_n^{1,\Lambda}$ and $I_n^{3,\Lambda}$ when q_h is closed to a linear concentrating wave, the result follows from Proposition 5.1.4 and linear Strichartz estimates. On $I_n^{2,\Lambda}$, q_h behaves like a concentration of a nonlinear solution on $T_{x_\infty} M$. The strict (h)-oscillation is obvious and the Strichartz estimates follow from global estimates on $\mathbb{R}^3$. $\square$

In the case of S^3, thanks to a better knowledge of the behavior of nonlinear concentrating waves we can avoid assumption (2.45). This is Theorem 1.8 from [22]. It will allow us to perform the profile decomposition for large times.

Theorem 5.2.2. *Let $\underline{v} = [(\varphi,\psi),\underline{h},\underline{x},\underline{t})]$ be a linear (not damped, that is $a(x)\equiv 0$) concentrating wave on S^3. We denote by $\underline{u}$ its nonlinear associated concentrating wave (same data at $t=0$). We assume that $t_\infty \in]0,\pi[$. Then, for all $j\in\mathbb{Z}$, we have*

$$\overline{\lim_n} \left|\left|\left| u_n - [\tilde{S}^{(j)}S(\varphi,\psi),\underline{h},(-1)^j\underline{x},\underline{t})]\right|\right|\right|_{]t_n+j\pi+\Lambda h_n, t_n+(j+1)\pi-\Lambda h_n]} \underset{\Lambda\to+\infty}{\longrightarrow} 0$$

where, $\tilde{S}=S\circ A$, $\tilde{S}^{(j)}=\tilde{S}\circ\tilde{S}\circ\cdots\circ\tilde{S}$, j times and $A(\varphi,\psi)(x) = -(\varphi,\psi)(-x)$.

Moreover, the cases $t_\infty \in]-\pi,0[$ and $t_\infty=0$ can be deduced similarly to Theorem 5.2.1 with some changes on the concentration data in the same spirit as Table 5.2.1.

5.2.2.2 Modification of the proof of S. Ibrahim for Theorem 5.2.1 in the case of damped equation

In this subsection, we give some sketch of proof for the behavior of nonlinear damped concentrating waves announced in subsection 5.2.2.1. These results are proved in [22] in the undamped case $a(x)\equiv 0$ and so we only briefly emphasize the main necessary modifications of proof. To simplify, we only treat the case $\frac{t_n}{h_n} \underset{n\to+\infty}{\longrightarrow} \infty$.

Sketch of the proof of estimate (2.46) of Theorem 5.2.1 :Behavior before concentration. The proof is exactly the same as Corollary 3.2 of [22]. $w_n = u_n - v_n$ is solution of

$$\begin{array}{rcll} \Box w_n + w_n + a(x)\partial_t w_n &=& -|w_n+v_n|^4(w_n+v_n) & \text{on} \quad I_n^{1,\Lambda}\times M \\ (w_n,\partial_t w_n)_{|t=0} &=& (0,0). & \end{array}$$

Using Strichartz and energy estimates, we are able to use a bootstrap argument if $\overline{\lim_{n\to\infty}} \|v_n\|_{L^5(I_n^{1,\Lambda},L^{10})}$ is small enough. This can be achieved thanks to Lemma 5.2.3 and gives the result. $\square$

Sketch of the proof of estimate (2.48) of Theorem 5.2.1 : Behavior close to concentration. By definition of v_n and finite propagation speed, the main energy part of v_n is concentrated near x_∞ for times close to t_∞. By estimate (2.46), it is also the case for u_n. Therefore, for times $t\in[t_n-\Lambda h_n, t_n+\Lambda h_n]$, we can neglect the energy outside of a fixed open set and work in local coordinates. Moreover, in that case, we can use the norm $|||\cdot|||_{I\times\mathbb{R}^3}$ instead of $|||\cdot|||_I$ and use the fact that is is invariant by translation and scaling up to a modification of the interval of time.

Denote $\tilde{u}_n$ (resp $\tilde{v}_n$) the rescaled function associated to u_n (resp v_n), so that $u_n(t,x) = \frac{1}{\sqrt{h_n}}\tilde{u}_n\left(\frac{t-t_n}{h_n},\frac{x-x_n}{h_n}\right)$. We need to prove $\overline{\lim_{n\to\infty}} |||\tilde{u}_n - w|||_{[-\Lambda,\Lambda]\times\mathbb{R}^3} \underset{\Lambda\to\infty}{\longrightarrow} 0$ where w is solution of

$$\left\{\begin{array}{rcl} \Box_\infty w &=& -|w|^4 w \quad \text{on} \quad \mathbb{R}\times\mathbb{R}^3 \\ (w,\partial_t w)_{|t=0} &=& (\varphi_2,\psi_2) = \Omega_-(\varphi,\psi). \end{array}\right.$$

By definition of Ω_-, w satisfies $\|(w-v,\partial_t(w-v))(t)\|_{\dot{H}^1\times L^2} \underset{t\to-\infty}{\longrightarrow} 0$ where v is solution of

$$\left\{\begin{array}{rcl} \Box_\infty v &=& 0 \quad \text{on} \quad \mathbb{R}\times\mathbb{R}^3 \\ (v,\partial_t v)_{|t=0} &=& (\varphi,\psi). \end{array}\right. \tag{2.50}$$

Moreover, it is known that $\Omega_-(\varphi,\psi) = \lim_{s\to-\infty} U(-s)U_0(s)(\varphi,\psi)$ where U and U_0 are the nonlinear and linear flow map. More precisely, by Lemma 3.4 of [22], we have $|||w_\Lambda - w|||_{[-\Lambda,\Lambda]\times\mathbb{R}^3} \underset{\Lambda\to\infty}{\longrightarrow} 0$ where w_Λ is the smooth solution of

$$\left\{\begin{array}{rcl} \Box_\infty w_\Lambda + |w_\Lambda|^4 w_\Lambda & = & 0 \quad \text{on} \quad [-\Lambda,\Lambda]\times\mathbb{R}^3 \\ (w_\Lambda, \partial_t w_\Lambda)_{|t=-\Lambda} & = & \chi_\Lambda(v,\partial_t v)_{|t=-\Lambda}. \end{array}\right.$$

where χ_Λ is an appropriate family of smoothing operator. So, we are left to prove $\overline{\lim_{n\to\infty}}\, |||\tilde{u}_n - w_\Lambda|||_{[-\Lambda,\Lambda]\times\mathbb{R}^3} \underset{\Lambda\to\infty}{\longrightarrow} 0.$

We introduce the auxiliary family of functions $\tilde{u}_n^\Lambda$ solution of

$$\left\{\begin{array}{rcl} \Box_n \tilde{u}_n^\Lambda + h_n^2\tilde{u}_n^\Lambda + |\tilde{u}_n^\Lambda|^4\tilde{u}_n^\Lambda & = & -h_n a(h_n x + x_n)\partial_t \tilde{u}_n^\Lambda \quad \text{on} \quad [-\Lambda,\Lambda]\times\mathbb{R}^3 \\ (\tilde{u}_n^\Lambda, \partial_t\tilde{u}_n^\Lambda)_{|t=-\Lambda} & = & (\tilde{v}_n, \partial_t\tilde{v}_n)_{|t=-\Lambda}. \end{array}\right.$$

where we have denoted $\Box_n$ the dilation of the operator $\Box$. So it can be written $\Box_n = \partial_t^2 - \sum_{i,j} g^{ij}(h_n x + x_n)\partial_{ij} + h_n V(h_n x + x_n)\cdot\nabla$ where V is a smooth vector field (note that it is only defined in an open set of size $\mathcal{O}(h_n^{-1})$ but it is also the case for $\tilde{u}_n$, $\tilde{u}_n^\Lambda$ and $\tilde{v}_n$, we omit the details). The proof is complete if we prove

$$\overline{\lim_{n\to\infty}}\, \left|\left|\left|\tilde{u}_n^\Lambda - w^\Lambda\right|\right|\right|_{[-\Lambda,\Lambda]\times\mathbb{R}^3} \underset{\Lambda\to\infty}{\longrightarrow} 0 \tag{2.51}$$

and

$$\overline{\lim_{n\to\infty}}\, \left|\left|\left|\tilde{u}_n^\Lambda - \tilde{u}_n\right|\right|\right|_{[-\Lambda,\Lambda]\times\mathbb{R}^3} \underset{\Lambda\to\infty}{\longrightarrow} 0. \tag{2.52}$$

We begin with (2.51). $r_{n,\Lambda} = \tilde{u}_n^\Lambda - w^\Lambda$ is solution of

$$\left\{\begin{array}{r} \Box_n r_{n,\Lambda} + h_n^2 r_{n,\Lambda} + h_n a(h_n x + x_n)\partial_t r_{n,\Lambda} = |w_\Lambda|^4 w_\Lambda - |r_{n,\Lambda} + w_\Lambda|^4(r_{n,\Lambda} + w_\Lambda) \\ -h_n^2 w_\Lambda - h_n a(h_n x + x_n)\partial_t w_\Lambda + (\Box_\infty - \Box_n)w_\Lambda, \\ (r_{n,\Lambda}, \partial_t r_{n,\Lambda})_{|t=-\Lambda} = (\tilde{v}_n - \chi_\Lambda v, \partial_t(\tilde{v}_n - \chi_\Lambda v))_{|t=-\Lambda}. \end{array}\right.$$

A quick scaling analysis easily yields that the operator $\Box_n + h_n^2 + h_n a(h_n x + x_n)\partial_t$ satisfies the same Strichartz and energy estimates as $\Box + 1 + a(x)\partial_t$ for some times of order Λ. Moreover, following the same argument as Lemma 2.1 of [22], we get that for fixed Λ

$$\overline{\lim_{n\to\infty}}\, \left\|-h_n^2 w_\Lambda - h_n a(h_n x + x_n)\partial_t w_\Lambda + (\Box_\infty - \Box_n)w_\Lambda\right\|_{L^1([-\Lambda,\Lambda],L^2)} = 0.$$

Thanks to Lemma 5.2.2, we know that $\overline{\lim_{n\to\infty}}\, \|(\tilde{v}_n - \chi_\Lambda v, \partial_t(\tilde{v}_n - \chi_\Lambda v))(-\Lambda)\|_{\dot{H}^1\times L^2}$ can be made arbitrary small for large Λ. Strichartz and energy estimates give for any $\eta > -\Lambda$

$$\begin{array}{rcl} |||r_{n,\Lambda}|||_{[-\Lambda,\eta]\times\mathbb{R}^3} & \leq & \|(\tilde{v}_n - \chi_\Lambda v, \partial_t(\tilde{v}_n - \chi_\Lambda v))(-\Lambda)\|_{\dot{H}^1\times L^2} \\ & & + \left\|-h_n^2 w_\Lambda - h_n a(h_n x + x_n)\partial_t w_\Lambda + (\Box_\infty - \Box_n)w_\Lambda\right\|_{L^1([-\Lambda,\eta],L^2)} \\ & & + \|r_{n,\Lambda}\|^5_{L^5([-\Lambda,\eta],L^{10})} + \|r_{n,\Lambda}\|_{L^5([-\Lambda,\eta],L^{10})}\|w_\Lambda\|^4_{L^5([-\Lambda,\eta],L^{10})}. \end{array}$$

If $\|w_\Lambda\|_{L^5([-\Lambda,\eta],L^{10})}$ is small enough, a bootstrap gives (2.51) on $[-\Lambda,\eta]$. We can iterate the process by dividing $[-\Lambda,\Lambda]$ in a finite number of intervals where the bootstrap can be performed.

For (2.52), we observe that $\tilde{u}_n^\Lambda$ and $\tilde{u}_n$ are solutions of the same equation but with different initial data which satisfy thanks to estimate (2.46)

$$\overline{\lim_{n\to\infty}}\, \left\|(\tilde{u}_n^\Lambda - \tilde{u}_n, \partial_t(\tilde{u}_n^\Lambda - \tilde{u}_n))(-\Lambda)\right\|_{\mathcal{E}} = \overline{\lim_{n\to\infty}}\, \|(\tilde{v}_n - \tilde{u}_n, \partial_t(\tilde{v}_n - \tilde{u}_n))(-\Lambda)\|_{\mathcal{E}} \underset{\Lambda\to\infty}{\longrightarrow} 0.$$

Then, Strichartz and energy estimates allow us to use a boot strap argument on subintervals I such that $\left\|\tilde{u}_n^\Lambda\right\|_{L^5(I,L^{10})}$ is small. (2.51) allows to complete the proof. $\Box$

5.2.2.3 Proof of the decomposition

This subsection is devoted to the proof of Theorem 5.0.4.

Let us define the function β in the following way :

$$\forall \omega \in \mathbb{C}, \quad \beta(\omega) \overset{def}{:=} |\omega|^4 \omega.$$

Proposition 5.2.4. *Let $0 < 2T < T_{focus}$ (see Definition 5.0.1). Let $p_n^{(j)}$, $1 \le j \le l$, linear damped concentrating waves, associated with data $[(\varphi^{(j)}, \psi^{(j)}), \underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)}]$ (we can have $h^{(j)} = 1$ for one of it), which are orthogonal according to Definition 5.1.2 and such that $t_\infty^{(j)} \in [-T, T]$. Denote $q_n^{(j)}$ the associated nonlinear damped concentrating waves (same data at $t = 0$).*

Then, we have

$$\overline{\lim_{n\to\infty}} \left\| \beta\left(\sum_{j=1}^{l} q_n^{(j)}\right) - \sum_{j=1}^{l} \beta(q_n^{(j)}) \right\|_{L^1([-T,T],L^2)} = 0. \tag{2.53}$$

Démonstration. We follow closely Lemma 4.2 of [17].

$$\left\| \beta\left(\sum_{j=1}^{l} q_n^{(j)}\right) - \sum_{j=1}^{l} \beta(q_n^{(j)}) \right\|_{L^1([-T,T],L^2)} \le \sum_{1\le j_1,\cdots,j_5\le l} \left\| \prod_{k=1}^{5} q_n^{(j_k)} \right\|_{L^1([-T,T],L^2)}$$

where at least two $q_n^{(j_k)}$ are different. In the case of othogonality of scales, we use Hölder inequality

$$\left\| \prod_{k=1}^{5} q_n^{(j_k)} \right\|_{L^1([-T,T],L^2)} \le C \left\| q_n^1 q_n^2 \right\|_{L^\infty([-T,T],L^3)} \prod_{k=3}^{5} \left\| q_n^{(j_k)} \right\|_{L^3([-T,T],L^{18})}.$$

Then, Corollary 5.2.2 and Lemma 5.1.8 yield the result (note that L^3L^{18} is a pair of Strichartz norm). So now, we can assume $h_n^1 = h_n^2 = h_n$. By Hölder and Corollary 5.2.2, we get

$$\begin{aligned} \left\| \prod_{k=1}^{5} q_n^{(j_k)} \right\|_{L^1([-T,T],L^2)} &\le C \left\| q_n^1 q_n^2 \right\|_{L^{5/2}([-T,T],L^5)} \prod_{k=3}^{5} \left\| q_n^{(j_k)} \right\|_{L^5([-T,T],L^{10})} \\ &\le C \left\| q_n^1 q_n^2 \right\|_{L^{5/2}([-T,T],L^5)}. \end{aligned}$$

We apply Theorem 5.2.1 to q_n^1. We obtain three couples (φ^i, ψ^i), $i = 1, 2, 3$ and split the interval $[-T, T] = \cup_{j=1}^{3} I_n^{j,\Lambda}$. We first deal with the interval $I_n^{1,\Lambda}$. Denote $\underline{v_1} = [(\varphi_1, \psi_1), \underline{h}, \underline{x}, \underline{t})]$ so that

$$\left\| q_n^1 q_n^2 \right\|_{L^{5/2}(I_n^{1,\Lambda},L^5)} \le \left\| q_n^1 \right\|_{L^5(I_n^{1,\Lambda},L^{10})} \le C \left\| q_n^1 - v_{1,n} \right\|_{L^5(I_n^{1,\Lambda},L^{10})} + \left\| v_{1,n} \right\|_{L^5(I_n^{1,\Lambda},L^{10})}$$

So, combining Theorem 5.2.1 and Lemma 5.2.3 yields

$$\overline{\lim_{n}} \left\| q_n^1 q_n^2 \right\|_{L^{5/2}(I_n^{1,\Lambda},L^5)} \underset{\Lambda\to\infty}{\longrightarrow} 0.$$

The same result holds for $I_n^{3,\Lambda}$ and we are led with the interval $I_n^{2,\Lambda}$. In the case of time orthogonality, say $\frac{|t_n^2-t_n^1|}{h_n} \underset{n\to\infty}{\longrightarrow} +\infty$, the two intervals $[t_n^1 - \Lambda h_n, t_n^1 + \Lambda h_n]$ and $[t_n^2 - \Lambda h_n, t_n^2 + \Lambda h_n]$ have empty intersection for fixed Λ and n large enough, which yields the result by the same estimates applied to q_n^2, once Λ is chosen large enough.

We can now assume, up to a translation in time, that $t_n^1 = t_n^2$. On $I_n^{2,\Lambda}$, Theorem 5.2.1 allows us to replace q_n^1 by $w_n^1(t,x) = \Psi_U^1(x) w^1\left(\frac{t-t_n^1}{h_n}, \frac{x-x_n^1}{h_n}\right)$ on a coordinate patch where w^1 is solution of a nonlinear wave equation on the tangent plane $T_{x_\infty^1}M$ and similarly for q_n^2. In the first case of space orthogonality, that is $x_\infty^1 \neq x_\infty^2$, the result is obvious on the interval $I_n^{2,\Lambda}$ by taking Ψ_U^1 and Ψ_U^2 with empty intersection. In the case $x_\infty^1 = x_\infty^2$, we are left with the estimate of

$$\int_{I_n^2} \left(\int_{\mathbb{R}^3} \left| w_n^1(t,x) w_n^2(t,x) \right|^5 \right)^{1/2} ds \leq \int_{[-\Lambda,\Lambda]} \left(\int_{\mathbb{R}^3} \left| w^1(t,x) w^2(t, x + \frac{x_n^1 - x_n^2}{h_n}) \right|^5 \right)^{1/2} ds$$

This yields the result in the last case of space orthogonality by approximating w^1 and w^2 by compactly supported functions. □

In the case of the sphere, we are able to state the same result without any restriction on the time.

Corollary 5.2.3. *Let $M = S^3$ and $T > 0$ (eventually large). We make the same assumptions as Proposition 5.2.4, except for the time T, with the additional hypothesis : $[\underline{h}^{(i)}, (-1)^m \underline{x}^{(i)}, \underline{t}^{(i)} + m\pi]$ is orthogonal to $[\underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)}]$ for any $m \in \mathbb{Z}$ and $i \neq j$. Moreover, we assume $a(x) \equiv 0$ (undamped equation).*
Then, the same conclusion as Proposition 5.2.4 is true.

Démonstration. We build a covering of the interval $[-T, T]$ with a finite number of intervals of length stricly less than $T_{focus} = \pi$ so that on each of this interval $I = [\alpha, \beta]$ and for any $1 \leq i \leq l$, there exists at most one $m^{(i)} \in \mathbb{Z}$ such that $t_\infty^{(i)} + m^{(i)}\pi \in I$. Moreover, one can also impose $\alpha \neq t_\infty^{(i)} + m^{(i)}\pi$.

Therefore, $\alpha \in]t_n^{(i)} + (m^i - 1)\pi + \Lambda h_n^{(i)}, t_n^{(i)} + m^{(i)}\pi - \Lambda h_n^{(i)}]$ for large fixed Λ and n large enough. Theorem 5.2.2 yields $\left\| (q_n^{(i)} - v_n^{(i)}, \partial_t(q_n^{(i)} - v_n^{(i)}))_{t=\alpha} \right\|_{\mathcal{E}} \underset{n\to\infty}{\longrightarrow} 0$ for a linear concentrating wave
$v_n^{(i)} = [\tilde{S}^{m^{(i)}} S(\varphi^{(i)}, \psi^{(i)}), \underline{h}^{(i)}, \underline{x}^{(i)}, \underline{t}^{(i)})]$. In each interval, we are in the same situation as in Proposition 5.2.4 which yields the desired result. □

Now, we are ready for the proof of the nonlinear profile decomposition. We give it in a quite sketchy way since it is very similar to the one of [2] or [17]. First, we obtain it in the particular case where the linear solution is small in Strichartz norm.

Lemma 5.2.13. *There exists $\delta_1 > 0$ such that if*

$$\overline{\lim_{n\to\infty}} \|v_n\|_{L^5([-T,T],L^{10})} \leq \delta_1$$

then the conclusion of Theorem 5.0.4 is true.

Démonstration. The proof is essentially the same as Lemma 4.3 of [2]. We have to estimate the rest $r_n^{(l)}$ solution of

$$\left\{\begin{array}{rcl}\Box r_n^{(l)}+r_n^{(l)}+a(x)\partial_t r_n^{(l)} & = & \beta(u)+\sum_{j=1}^{l}\beta(q_n^{(j)})-\beta\left(u+\sum_{j=1}^{l}q_n^{(j)}+w_n^{(l)}+r_n^{(l)}\right)\\ (r_n^{(l)},\partial_t r_n^{(l)})_{t=0} & = & (0,0).\end{array}\right.$$

We conclude as in [2] using Proposition 5.2.4 and Lemma 5.2.12 which is not immediate on a manifold. In the case of S^3 and $a\equiv 0$ for large T, we use Corollary 5.2.3 instead of Proposition 5.2.4 . $\square$

Once the result is obtained when Strichartz norms are small, we divide $[-T,T]$ in a finite number of intervals where the Strichartz norms are small enough. This is done in the following lemma.

Lemma 5.2.14. *Let $2T < T_{focus}$. Let $\delta > 0$ and $\tilde{q}_n$ be a sequence in $L^5([-T,T],L^{10}(M))$, such that*

$$\overline{\lim_{n\to\infty}}\,\|\tilde{q}_n\|_{L^5([-T,T],L^{10})}\leq\delta.$$

Fix also $l\in\mathbb{N}$ and l sequences of nonlinear concentrating wave $q_n^{(j)}$, $j=1,...,l$.

Then, for any $\delta'>\delta$, there exists $L\in\mathbb{N}$ such that for any $n\in\mathbb{N}$, we have the decomposition of $[-T,T]$ in closed intervals $I_n^{(i)}$

$$[-T,T]=\bigcup_{i=1}^{L}I_n^{(j)},$$

such that the sequence

$$\Gamma_n=\sum_{j=1}^{l}q_n^{(j)}+\tilde{q}_n$$

satisfies on each interval $I_n^{(i)}$

$$\overline{\lim_{n\to\infty}}\,\|\Gamma_n\|_{L^5(I_n^{(j)},L^{10})}\leq\delta'.$$

Démonstration. We first treat the case $l=1$. We divide $[-T,T]=I_n^{1,\Lambda}\cup I_n^{2,\Lambda}\cup I_n^{3,\Lambda}$ according to Theorem 5.2.1 (one of these intervals being possibly empty). Then, a combination of estimate (2.46) of Theorem 5.2.1 (comparison with linear concentrating wave) and Lemma 5.2.3 (non reconcentration) gives for Λ large enough

$$\overline{\lim_{n\to\infty}}\,\left\|q_n^{(1)}\right\|_{L^5(I_n^{1,\Lambda},L^{10})}\leq\delta'-\delta.$$

The same result holds for $I_n^{3,\Lambda}$ and we are left with the interval $I_n^{2,\Lambda}$. Once Λ is fixed, we can divide $[-\Lambda,\Lambda]$ in a finite number of intervals $I^{(i),\Lambda}$ such that $\|w\|_{L^5(I^{(i),\Lambda},L^{10}}\leq\delta-\delta'$ where w is the function defined by equation (2.49) of Theorem 5.2.1. Then, we replace each $I^{(i),\Lambda}$ by $I_n^{(i),\Lambda}$ obtained by translation dilation. We conclude by the approximation (2.48) of $q_n^{(1)}$ by translation dilation of w on the interval $I_n^{2,\Lambda}$. $\square$

Note that the previous lemma also applies for large times on S^3 with $a \equiv 0$ by doing a first decomposition of $[-T, T]$ in a finite number of intervals of length strictly less than π.

End of the proof of Theorem 5.0.4 in the general case. We choose $l \in \mathbb{N}$ such that $\left\|w_n^{(l)}\right\| \leq \delta_1$ and use Lemma 5.2.14 in order to be able to apply Lemma 5.2.13 on each interval $I_n^{(j)}$. See [2] or, in the different context of Schrödinger equation, [26] . □

5.2.3 Applications

5.2.3.1 Strichartz estimates and Lipschitz bounds for the nonlinear evolution group

Proposition 5.2.5. *Let $T > 0$ be fixed. There exist a non-decreasing function, $A : [0, \infty[\to [0, \infty[$, such that any solution of*

$$\left\{ \begin{array}{rcl} \Box u + u + a(x)\partial_t u &=& -|u|^4 u \quad \text{on} \quad [-T, T] \times M \\ (u(0), \partial_t u(0)) &=& (u_0, u_1) \in \mathcal{E} \end{array} \right. \tag{2.54}$$

fulfills

$$\|u\|_{L^8([-T,T],L^8(M))} + \|u\|_{L^5([-T,T],L^{10}(M))} + \|u\|_{L^4([-T,T],L^{12}(M))} \leq A(\|(u_0, u_1)\|_{\mathcal{E}}).$$

Démonstration. The proof is exactly the same as Corollary 2 of [2]. Using Strichartz estimates, it is enough to get the result for $L^5 L^{10}$. We argue by contradiction and suppose that there exists a sequence u_n of strong solutions of equation (2.54) satisfying

$$\sup_n \|(u_{0,n}, u_{1,n})\|_{\mathcal{E}} < +\infty, \quad \|u_n\|_{L^5([-T,T],L^{10}(M))} \underset{n\to\infty}{\longrightarrow} +\infty.$$

We apply the profile decomposition of Theorem 5.0.4 to our sequence. We get a contradiction by the fact that the $L^5([-T, T], L^{10}(M))$ norm of a nonlinear concentrating wave is uniformly bounded thanks to Corollary 5.2.2. This argument works for times $2T < T_{focus}$ and can be reiterated since the nonlinear energy at times T can be bounded with respect to the one at time 0 thanks to almost conservation (we can also use energy estimates once we know u is uniformly bounded in $L^5 L^{10}$). □

Lemma 5.2.15. *Let $R_0 > 0$ and $T > 0$. Then, there exists $C > 0$ such any solution u satisfying*

$$\left\{ \begin{array}{rcl} \Box u + u + a(x)\partial_t u + |u|^4 u &=& 0 \quad \text{on} \quad [-T, T] \times M \\ (u(0), \partial_t u(0)) &=& (u_0, u_1) \in \mathcal{E} \\ \|(u_0, u_1)\|_{\mathcal{E}} &\leq& R_0. \end{array} \right. \tag{2.55}$$

fulfills

$$\|(u(t), \partial_t u(t))\|_{L^2 \times H^{-1}} \leq C \|(u(0), \partial_t u(0))\|_{L^2 \times H^{-1}} \quad \forall t \in [-T, T].$$

Démonstration. Proposition 5.2.5 yields a uniform bound for u in $L^4([-T,T],L^{12}(M))$ and so for $V=|u|^4$ in $L^1([0,T],L^3(M))$. We prove uniform estimates for some solutions of the linear equation

$$\left\{\begin{array}{r}\Box u+u+a(x)\partial_t u=Vu \quad \text{on} \quad [-T,T]\times M\\ (u(0),\partial_t u(0))=(u_0,u_1)\in L^2\times H^{-1}\end{array}\right. \tag{2.56}$$

where V satisfies $\|V\|_{L^1([-T,T],L^3(M))}\leq A(R_0)^4$. The product of functions in $L^\infty([-T,T],L^2)$ and $L^1([-T,T],L^3)$ is in $L^1([-T,T],L^{6/5})$ and so in $L^1([-T,T],H^{-1})$ by Sobolev embedding. Standard estimates yields

$$\begin{array}{rcl}\|(u,\partial_t u)\|_{L^\infty([0,t],L^2\times H^{-1})} & \leq & C\,\|(u(0),\partial_t u(0))\|_{L^2\times H^{-1}}\\ & & +C(t+\|V\|_{L^1([0,t],L^3)})\,\|(u,\partial_t u)\|_{L^\infty([0,t],L^2\times H^{-1})}\,.\end{array}$$

We can divide the interval $[-T,T]$ into a finite number of intervals $[a_i,a_{i+1}]_{i=1\cdots N}$ such that $C(t+\|V\|_{L^1([a_i,a_{i+1}],L^3(M))})<1/2$. N depends only on R_0 and T (not on V).

Then, on each of these intervals, we have

$$\|(u,\partial_t u)\|_{L^\infty([a_i,a_{i+1}],L^2\times H^{-1})}\leq 2C\,\|(u(a_i),\partial_t u(a_i))\|_{L^2\times H^{-1}}\,.$$

We obtain the expected result by iteration. The final constant C only depends on R_0 and T since it is also the case for N. $\square$

Corollary 5.2.4. *Let $R_0>0$ and $T>0$. For any $\varepsilon>0$, there exists $\delta>0$ such that any solution u satisfying (2.55) and $\|(u_0,u_1)\|_{L^2\times H^{-1}}\leq\delta$ satisfies*

$$\|(u(T),\partial_t u(T))\|_{L^2\times H^{-1}}\leq\varepsilon.$$

We will also need the following lemma which states the local uniform continuity of the flow map. Note that it can be proved to be locally Lipschitz with a slightly more complicated argument (see Corollary 2 of [17]). We will note need this for our purpose.

Lemma 5.2.16. *Let u_n, $\tilde{u}_n$ be two sequences of solutions of*

$$\left\{\begin{array}{rcl}\Box u_n+u_n+|u_n|^4u_n & = & g_n \quad \text{on} \quad [-T,T]\times M\\ (u_n,\partial_t u_n)_{t=0} & = & (u_{n,0},u_{n,1}) \text{ bounded in } \mathcal{E},\end{array}\right.$$

with $\|(u_{n,0}-\tilde{u}_{n,0},u_{n,1}-\tilde{u}_{n,1})\|_{\mathcal{E}}+\|g_n-\tilde{g}_n\|_{L^1([-T,T],L^2)}\underset{n\to\infty}{\longrightarrow}0$. Then, we have

$$|||u_n-\tilde{u}_n|||_{[-T,T]}\underset{n\to\infty}{\longrightarrow}0.$$

Démonstration. $r_n=u_n-\tilde{u}_n$ is solution of

$$\left\{\begin{array}{rcl}\Box r_n+r_n+|u_n|^4u_n-|\tilde{u}_n|^4\tilde{u}_n & = & g_n-\tilde{g}_n \quad \text{on} \quad [-T,T]\times M\\ (r_n,\partial_t r_n)_{t=0} & = & (u_{n,0}-\tilde{u}_{n,0},u_{n,1}-\tilde{u}_{n,1}).\end{array}\right.$$

Using energy and Strichartz estimates, we get

$$\begin{array}{rcl}|||r_n|||_{[-T,T]} & \leq & C\,\|(u_{n,0}-\tilde{u}_{n,0},u_{n,1}-\tilde{u}_{n,1})\|_{\mathcal{E}}+C\,\|g_n-\tilde{g}_n\|_{L^1([-T,T],L^2)}\\ & & +C\,\|r_n\|_{L^5([-T,T],L^{10})}\left(\|u_n\|^4_{L^5([-T,T],L^{10})}+\|\tilde{u}_n\|^4_{L^5([-T,T],L^{10})}\right).\end{array}$$

Using Proposition 5.2.5, we can divide the interval $[-T,T]$ in a finite number of intervals $I_{i,n}=[a_{i,n},a_{i+1,n}]$, $1\leq i\leq N$, such that $C\left(\|u_n\|^4_{L^5(I_{i,n},L^{10})}+\|\tilde{u}_n\|^4_{L^5(I_{i,n},L^{10})}\right)<1/2$ so that the third term can be absorbed. We iterate this estimate N times, which gives the result. $\square$

5.2.3.2 Profile decomposition of the limit energy

For u solution of the nonlinear wave equation, we denote its nonlinear energy density

$$e(u)(t,x) = \frac{1}{2}\left[|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 + |u(t,x)|^2\right] + \frac{1}{6}|u(t,x)|^6.$$

For a sequence u_n of solution with initial data bounded in $\mathcal{E}$, the corresponding non linear energy density is bounded in $L^\infty([-T,T],L^1)$ and so in the space of bounded measures on $[-T,T]\times M$. This allows to consider, up to a subsequence, its weak$*$ limit.

The following theorem is the equivalent of Theorem 7 in [8]. It proves that the energy limit follows the same profile decomposition as u_n. It will be the crucial argument that will allow to use microlocal defect measure on each profile and then to apply the linearization argument.

Theorem 5.2.3. *Assume $2T < T_{focus}$.*

Let u_n be a sequence of solutions of

$$\Box u_n + u_n + |u_n|^4 u_n = 0$$

with $(u_n, \partial_t u_n)(0)$ weakly convergent to 0 in $\mathcal{E}$.

The nonlinear energy density limit of u_n (up to subsequence) reads

$$e(t,x) = \sum_{j=1}^{+\infty} e^{(j)}(t,x) + e_f(t,x)$$

where $e^{(j)}$ is the limit energy limit density of $q_n^{(j)}$ (following the notation of Theorem 5.0.4) and

$$e_f = \lim_{l\to\infty}\lim_{n\to\infty} e(w_n^{(l)})$$

where both limit are considered up to a subsequence and in the weak $*$ sense.

In particular, e_f can be written

$$e_f(t,x) = \int_{\xi\in S_x^2} \mu(t,x,d\xi)$$

with

$$\mu(t,x,\xi) = \mu_-(G_t(x,\xi)) + \mu_+(G_{-t}(x,\xi))$$

where G_t is the flow map of the vector field $H_{|\xi|_x}$ on S^*M, that is the Hamiltonian of the Riemannian metric.

Moreover, e is also the limit of the linear energy density

$$e_{lin}(u_n)(t,x) = \frac{1}{2}\left[|\partial_t u_n(t,x)|^2 + |\nabla u_n(t,x)|^2\right].$$

Démonstration. Proposition 5.2.5 yields $\|u_n\|_{L^8([-T,T]\times M)} \leq C$. Then, compact embedding and Lemma 5.2.15 yields $\|u_n\|_{L^2([-T,T]\times M)} \underset{n\to\infty}{\longrightarrow} 0$ and so $\|u_n\|_{L^6([-T,T]\times M)} \underset{n\to\infty}{\longrightarrow} 0$ by interpolation. Therefore, e is the limit of $b(u_n, u_n)$, with

$$b(f,g) = \partial_t f(t,x)\overline{\partial_t g}(t,x) + \nabla f(t,x) \cdot \overline{\nabla g}(t,x)$$

Now, we have to compute the limit of $b(u_n, u_n)$ using decomposition (0.10) of Theorem 5.0.4. We set for any $l \in \mathbb{N}$

$$s_n^{(l)} = \sum_{j=1}^{l} q_n^{(j)}$$

and so

$$b(u_n, u_n) = b(s_n^{(l)}, s_n^{(l)}) + b(w_n^{(l)}, w_n^{(l)}) + 2b(s_n^{(l)}, w_n^{(l)}) + 2b(u_n, r_n^{(l)}) - b(r_n^{(l)}, r_n^{(l)}).$$

Because of (0.11), $\overline{\lim\limits_{n\to\infty}} \left\| 2b(u_n, r_n^{(l)}) - b(r_n^{(l)}, r_n^{(l)}) \right\|_{L^1([-T,T]\times M)}$ converges to zero as l tends to infinity. So, if we define $e_r^{(l)} = w^* \lim\limits_{n\to\infty} \left(2b(u_n, r_n^{(l)}) - b(r_n^{(l)}, r_n^{(l)})\right)$, we have

$$\|e_r^{(l)}\|_{TV} \underset{l\to\infty}{\longrightarrow} 0.$$

Let $\varphi \in C_0^\infty(]-T,T[\times M)$. For fixed l, it remains to estimate

$$\iint_{]-T,T[\times M} \varphi\, b(s_n^{(l)}, w_n^{(l)}) = \sum_{j=1}^{l} \iint_{]-T,T[\times M} \varphi\, b(q_n^{(j)}, w_n^{(l)}).$$

Since $b(q_n^{(j)}, w_n^{(l)})$ is bounded in $L^\infty(]-T,T[, L^1)$, we can assume, up to an error arbitrary small, that φ is supported in $\left\{t < t_\infty^{(j)}\right\}$ or $\left\{t > t_\infty^{(j)}\right\}$ (replace φ by $(1-\Psi)(t)\varphi$ with $\Psi(t_\infty^{(j)}) = 1$ and $\|\Psi\|_{L^1(]-T,T[)}$ small). On each interval, Theorem 5.2.1 allows to replace $q_n^{(j)}$ by a linear concentrating wave. Then, we combine Lemma 5.2.11 and Lemma 5.1.11 to get the weak convergence to zero of $b(s_n^{(l)}, w_n^{(l)})$ for fixed l.

Lemma 5.2.10 and the orthogonality of the cores of concentration give $D_h^{(j)}(p_h^{(j')}, \partial_t p_h^{(j')})(t_h^{(j)}) \rightharpoonup (0,0)$ for $j \neq j'$ and $p_h^{(j')}$ a concentrating wave at rate $[h^{(j')}, t^{(j')}, x^{(j')}]$. Then, the same argument as before yields

$$b(s_n^{(l)}, s_n^{(l)}) \underset{n\to\infty}{\rightharpoonup} \sum_{j=1}^{l} e^{(j)}.$$

So we have proved that for any $l \in \mathbb{N}$

$$b(u_n, u_n) \underset{n\to\infty}{\rightharpoonup} e = \sum_{j=1}^{l} e^{(j)} + e_w^{(l)} + e_r^{(l)}$$

where $e_w^{(l)}$ is the weak* limit of $b(w_n^{(l)}, w_n^{(l)})$ and $e_r^{(l)}$ satisfies $\left\|e_r^{(l)}\right\|_{TV} \underset{l\to\infty}{\longrightarrow} 0$. $e_w^{(l)}$ is the weak* limit of a sequence of solutions of the linear wave equation weakly convergent to 0 in energy space. Therefore, it has the announced form using the link with microlocal defect measure (see Lemma 5.1.9).

We get the final result by letting l tend to infinity. $\square$

Remark 5.2.4. *The fact that $|u_n|^6$ is weakly convergent to 0 is false if we consider the limit in $\mathcal{D}'(M)$ time by time. For example, for a nonlinear concentrating wave with $t_n = 0$, the weak limit in $\mathcal{D}'(]-T,T[\times M)$ of $|u_n|^6$ is of course still zero but the weak limit of $|u_n|^6(t)$ in $\mathcal{D}'(M)$ is zero if $t \neq 0$ and a multiple of a Dirac function if $t = 0$. So the limit in $\mathcal{D}'(M)$ of $e_{n|t=0}$ is not the same as the one of $b(u_n, u_n)_{|t=0}$. This comes from the fact that the limit of $b(u_n, u_n)(t)$ is not equicontinuous as a function of t while it is the case for the nonlinear energy. Yet, in the proof, we are only interested in its limit in the space-time distributional sense which will be continuous. Actually, the discontinuity at $t = 0$ of the limit of $b(u_n, u_n)(t)$ can be described explicitely from the scattering operator. At the contrary, the fact that the nonlinear energy density $e(t)$ is continuous in time can, in this case, be seen as a consequence of the conservation of the nonlinear energy of the scattering operator.*

5.3 Control and stabilization

5.3.1 Weak observability estimates, stabilization

5.3.1.1 Why Klein-Gordon and not the wave ?

In this subsection, we prove that the expected observability estimate

$$E(u)(0) \leq C \iint_{[0,T]\times M} |a\partial_t u|^2 \, dtdx.$$

does not hold for the nonlinear damped wave equation $\Box u + \partial_t u + u^5 = 0$ (in the simpler case $a \equiv 1$), even for small data. It explains why we have chosen the Klein-Gordon equation instead. The main point is that for small data, the nonlinear solution is close to the linear one which has the constants (in space-time) as undamped solutions (which is obviously false for $\Box u + u = 0$).

We take $a \equiv 1$ and initial data constant equal to $(\varepsilon, 0)$. The nonlinear wave equation takes the form of the following ODE

$$\begin{cases} \ddot{u} + \dot{u} + u^5 &= 0 \quad \text{on} \quad [0,T] \\ (u(0), \dot{u}(0)) &= (\varepsilon, 0). \end{cases}$$

Decreasing of energy yields for any $t \geq 0$

$$E(t) = \frac{1}{2}\dot{u}^2 + \frac{1}{6}u^6(t) \leq E(0) = \frac{1}{6}\varepsilon^6$$

and so

$$|u(t)| \leq \varepsilon \quad \forall t \geq 0.$$

Then, $c = \dot{u}$ is solution of

$$\begin{cases} \dot{c} + c + u^5 &= 0 \quad \text{on} \quad [0,T] \\ c(0) &= 0 \end{cases}$$

Therefore,

$$c(t) = -\int_0^t e^{-(t-s)} u^5(s)\, ds$$
$$\text{and} \qquad |\dot{u}(t)| = |c(t)| \leq \varepsilon^5.$$

For any $T > 0$, we have

$$\int_0^T |\dot{u}(s)|^2 \leq T\varepsilon^{10}.$$

Therefore, the observability estimate

$$T\varepsilon^{10} \geq \int_0^T |\dot{u}(s)|^2 \geq CE(0) = C\frac{1}{6}\varepsilon^6$$

can not hold if ε is taken small enough.

5.3.1.2 Weak observability estimate

As explained in the introduction, the proof of stabilization consists in the analysis of possible sequences contradicting an observability estimate. The first step is to prove that such sequence is linearizable in the sense that its behavior is close to solutions of the linear equation.

Proposition 5.3.1. *Let ω satisfying Assumption 5.0.2 and $a \in C^\infty(M)$ satisfying $a(x) > \eta > 0$ for all $x \in \omega$. Let $T > T_0$ and u_n be a sequence of solutions of*

$$\left\{ \begin{array}{rcl} \Box u_n + u_n + |u_n|^4 u_n + a(x)^2 \partial_t u_n & = & 0 \quad \text{on} \quad [0,T] \times M \\ (u_n, \partial_t u_n)_{t=0} & = & (u_{0,n}, u_{1,n}) \in \mathcal{E}. \end{array} \right. \tag{3.57}$$

satisfying

$$(u_{0,n}, u_{1,n}) \underset{n\to\infty}{\rightharpoonup} 0 \text{ weakly in } \mathcal{E}$$
$$\iint_{[0,T]\times M} |a(x)\partial_t u_n|^2 \ dtdx \underset{n\to\infty}{\longrightarrow} 0 \tag{3.58}$$

Then, u_n is linearizable on $[0,t]$ for any $t < T - T_0$, that is

$$|||u_n - v_n|||_{[0,t]} \underset{n\to\infty}{\longrightarrow} 0$$

where v_n is the solution of

$$\left\{ \begin{array}{rcl} \Box v_n & = & 0 \quad \text{on} \quad [0,T] \times M \\ (v_n, \partial_t v_n)_{t=0} & = & (u_{0,n}, u_{1,n}). \end{array} \right.$$

Démonstration. Denote $t_* = \sup\left\{ s \in [0,T] \,\middle|\, \lim\limits_{n\to\infty} |||u_n - v_n|||_{[0,s]} = 0 \right\}$ and we have to prove $t_* \geq T - T_0$. If it is not the case, we can find an interval $[t_* - \varepsilon, t_* - \varepsilon + L] \subset [0,T]$ with $T_0 < L < T_{focus}$ and $0 < 2\varepsilon < L - T_0$ (if $t_* = 0$, take the interval $[0,L] \subset [0,T]$). Then, Lemma 5.3.1 below gives that u_n is linearizable on $[t_* - \varepsilon, t_* + \varepsilon]$. We postpone the proof of Lemma 5.3.1 and finish the proof of the proposition. The definition of t_*

gives $\lim_{n\to\infty}|||u_n - v_n|||_{[0,t_*-\varepsilon]} = 0$ and we have proved that $\lim_{n\to\infty}|||u_n - \tilde{v}_n|||_{[t_*-\varepsilon,t_*+\varepsilon]} = 0$ where $\tilde{v}_n$ is solution of

$$\Box\tilde{v}_n = 0 \quad ; \quad (\tilde{v}_n, \partial_t\tilde{v}_n)_{t=t_*-\varepsilon} = (u_n, \partial_t u_n)_{t=t_*-\varepsilon}.$$

Since the norm $|||\cdot|||$ controls the energy norm, this easily yields $\lim_{n\to\infty}|||u_n - v_n|||_{[0,t_*+\varepsilon]} = 0$ which is a contradiction to the definition of t_*. $\Box$

Lemma 5.3.1. *With the assumptions of Proposition 5.3.1. Consider the profile decomposition according to Theorem 5.0.4 of u_n on a subinterval $[t_0, t_0 + L] \subset [0, T]$ with $T_0 < L < T_{focus}$.*

Then, for any $0 < \varepsilon < L - T_0$, this decomposition does not contain any non linear concentrating wave with $t_\infty^{(j)} \in [t_0, t_0 + \varepsilon]$ and u_n is linearizable on $[t_0, t_0 + \varepsilon]$.

Démonstration. To simplify the notation, we work on the interval $[0, L]$. Moreover, since $a(x)\partial_t u_n$ tends to 0 in L^1L^2, Lemma 5.2.16 allows to assume with the same assumptions that u_n is solution of the nonlinear equation without damping. Proposition 5.2.5 and Lemma 5.2.15 (with Rellich Theorem) give that u_n is bounded in $L^8([0,T] \times M)$ and convergent to 0 in $L^2([0,T]\times M)$. Therefore, u_n tends to 0 in $L^7([0,T]\times M)$ and so $|u_n|^4 u_n$ is convergent to 0 in $L^{7/5}([0,T] \times M) \hookrightarrow L^{4/3}([0,T] \times M) \hookrightarrow H^{-1}_{loc}(]0, l[\times M)$. Then, if we consider the (space-time) microlocal defect measure of u_n, the elliptic regularity and the equation verified by u_n gives that μ is supported in $\left\{\tau^2 = |\xi|_x^2\right\}$ as in the linearizable case. So, combining this with (3.58), we get

$$u_n \underset{n\to\infty}{\longrightarrow} 0 \text{ in } H^1_{loc}(]0, L[\times\omega).$$

Using the notation of Theorem 5.2.3, this gives $e = 0$ on $]0, L[\times\omega$. Since all the measures in the decomposition of e are positive, we get the same result for any nonlinear concentrating wave in the decomposition of u_n, that is

$$q_n^{(j)} \underset{n\to\infty}{\longrightarrow} 0 \text{ in } H^1_{loc}(]0, L[\times\omega)$$

and if $\mu^{(j)}$ is the microlocal defect measure of $q_n^{(j)}$, we have

$$\mu^{(j)} \equiv 0 \text{ in } S^*(]0, L[\times\omega). \tag{3.59}$$

Assume that $t_\infty^{(j)} \in [0, \varepsilon]$ for one $j \in \mathbb{N}$, so that the interval $[t_\infty^{(j)}, L]$ has length greater that T_0. Denote $p_n^{(j)}$ the linear concentrating wave approaching $q_n^{(j)}$ in the interval $I_n^{3,\Lambda}$ according to the notation of Theorem 5.2.1, so that for any $t_\infty^{(j)} < t < L$ (here, we use the fact that $L < T_{focus}$), we have

$$\left|\left|\left|q_n^{(j)} - p_n^{(j)}\right|\right|\right|_{[t,L]} \underset{n\to\infty}{\longrightarrow} 0.$$

In particular, $\mu^{(j)}$ is also attached to $p_n^{(j)}$ on the time interval $]t_\infty^{(j)}, L]$. Since $p_n^{(j)}$ is solution of the linear wave equation, its measure propagates along the hamiltonian flow. Assumption 5.0.2 and $\left|L - t_\infty^{(j)}\right| > T_0$ ensure that the geometric control condition is still verified on the interval $[t_\infty^{(j)}, L]$ which gives $\mu^{(j)} \equiv 0$ when combined with (3.59). That means $p_n^{(j)} \equiv 0$ and so $q_n^{(j)} \equiv 0$ as expected.

Then, for the profile decomposition of u_n on the interval $[0, L]$ (here the weak limit u is necessarily zero)

$$u_n = \sum_{j=1}^{l} q_n^{(j)} + w_n^{(l)} + r_n^{(l)},$$

we have proved that $t_n^{(j)} \in]\varepsilon, L]$. Then Theorem 5.2.1 and $L < T_{focus}$ provides a linear concentrating wave $p_n^{(j)}$ such that $\overline{\lim_{n\to\infty}} \left|\left|\left| q_n^{(j)} - p_n^{(j)} \right|\right|\right|_{[0,\varepsilon]} = 0$ while Lemma 5.2.3 give $\overline{\lim_{n\to\infty}} \left\| p_n^{(j)} \right\|_{L^5([0,\varepsilon],L^{10})} = 0$. Moreover, the conclusion of Theorem 5.2.1 give $\overline{\lim_{n\to\infty}} \left\| w_n^{(l)} + r_n^{(l)} \right\|_{L^5([0,\varepsilon],L^{10})} \underset{l\to\infty}{\longrightarrow} 0$. This finally yields $\overline{\lim_{n\to\infty}} \|u_n\|_{L^5([0,\varepsilon],L^{10})} = 0$ and therefore

$$\left\| |u_n|^4 u_n \right\|_{L^1([0,\varepsilon],L^2)} \underset{n\to\infty}{\longrightarrow} 0.$$

This gives exactly that u_n is linearizable on $[0, \varepsilon]$. $\square$

We are now ready for the proof of some weak observability estimates. We recall the notation $E(u)$ for the nonlinear energy defined in (0.2).

Theorem 5.3.1. *Let ω satisfying Assumption 5.0.2 with T_0 and $a \in C^\infty(M)$ satisfying $a(x) > \eta > 0$ for all $x \in \omega$. Let $T > 2T_0$ and $R_0 > 0$. Then, there exists $C > 0$ such that for any u solution of*

$$\left\{ \begin{array}{rcl} \Box u + u + |u|^4 u + a^2(x)\partial_t u & = & 0 \quad \text{on} \quad [0, T] \times M \\ (u, \partial_t u)_{t=0} & = & (u_0, u_1) \in \mathcal{E} \\ \|(u_0, u_1)\|_{\mathcal{E}} & \leq & R_0 \end{array} \right. \tag{3.60}$$

satisfies

$$E(u)(0) \leq C \left(\iint_{[0,T]\times M} |a(x)\partial_t u|^2 \ dtdx + \|(u_0, u_1)\|_{L^2\times H^{-1}} E(u)(0) \right).$$

Démonstration. We argue by contradiction : we suppose that there exists a sequence u_n of solutions of (3.60) such that

$$\left(\iint_{[0,T]\times M} |a(x)\partial_t u_n|^2 \ dtdx + \|(u_{0,n}, u_{1,n})\|_{L^2\times H^{-1}} E(u_n)(0) \right) \leq \frac{1}{n} E(u_n)(0).$$

Denote $\alpha_n = (E(u_n)(0))^{1/2}$. By Sobolev embedding for the L^6 norm, we have $\alpha_n \leq C(R_0)$. So, up to extraction, we can assume that $\alpha_n \longrightarrow \alpha \geq 0$.

We will distinguish two cases : $\alpha > 0$ and $\alpha = 0$.

- First case : $\alpha_n \longrightarrow \alpha > 0$

The second part of the estimate gives $\|(u_{0,n}, u_{1,n})\|_{L^2\times H^{-1}} \underset{n\to\infty}{\longrightarrow} 0$ and so $(u_{0,n}, u_{1,n}) \underset{n\to\infty}{\rightharpoonup} 0$ in $H^1 \times L^2$. Therefore, we are in position to apply Proposition 5.3.1 and get that u_n is linearizable on an interval $[0, L]$ with $L > T_0$. We get a contradiction to $\alpha > 0$ by applying the following classical linear proposition, which can be easily proved using microlocal defect measure as in Lemma 5.3.1.

Proposition 5.3.2. *Let ω satisfying Assumption 5.0.2 with T_0. Let $T > T_0$ and v_n be a sequence of solutions of*

$$\left\{\begin{array}{rcl}\Box v_n & = & 0 \quad \text{on} \quad [0,T]\times M\\ (v_n(0),\partial_t v_n(0)) & \underset{n\to\infty}{\rightharpoonup} & 0 \text{ in } \mathcal{E}\end{array}\right.$$

satisfying

$$\iint_{[0,T]\times M} |a(x)\partial_t v_n|^2 \ dtdx \underset{n\to\infty}{\longrightarrow} 0$$

Then, $(v_n(0),\partial_t v_n(0)) \underset{n\to\infty}{\longrightarrow} 0$ for the strong topology of $H^1\times L^2$. The same result holds with $\Box u_n$ replaced by $\Box u_n + u_n$.

- Second case : $\alpha_n \longrightarrow 0$

Let us make the change of unknown $w_n = u_n/\alpha_n$. w_n is solution of the system

$$\Box w_n + a^2(x)\partial_t w_n + w_n + \alpha_n^4|w_n|^4 w_n = 0 \tag{3.61}$$

and

$$\iint_{[0,T]\times M} |a(x)\partial_t w_n|^2 \ dtdx \leq \frac{1}{n}.$$

We have for a large constant $C>0$ depending on R_0 and for all $t\in[0,T]$,

$$\frac{1}{C}\left\|(u_n,\partial_t u_n)\right\|_{\mathcal{E}}^2 \leq E(u_n) \leq C\left\|(u_n,\partial_t u_n)\right\|_{\mathcal{E}}^2.$$

Therefore, we have

$$\begin{aligned}\|(w_n(t),\partial_t w_n(t))\|_{\mathcal{E}} &= \frac{\|(u_n(t),\partial_t u_n(t))\|_{\mathcal{E}}}{\sqrt{E(u_n(0))}} \leq C\frac{\sqrt{E(u_n(t))}}{\sqrt{E(u_n(0))}} \leq C\\ \|(w_n(0),\partial_t w_n(0))\|_{\mathcal{E}} &= \frac{\|(u_n(0),\partial_t u_n(0))\|_{\mathcal{E}}}{\sqrt{E(u_n(0))}} \geq \frac{1}{\sqrt{C}} > 0.\end{aligned} \tag{3.62}$$

Thus, we have $\|(w_n(0),\partial_t w_n(0))\|_{\mathcal{E}} \approx 1$ and $(w_n,\partial_t w_n)$ is bounded in $L^\infty([0,T],\mathcal{E})$.

Applying Strichartz estimates to equation (3.61), we get for $C = C(R_0) > 0$

$$\|w_n\|_{L^5([0,T],L^{10})} \leq C(1+\alpha_n^4\left\|w_n\right\|_{L^5([0,T],L^{10})}^5)$$

Then, using a bootstrap argument, we deduce that $\|w_n\|_{L^5([0,T],L^{10})}$ is bounded and therefore

$$\Box w_n + w_n \underset{n\to\infty}{\longrightarrow} 0 \text{ in } L^1([0,T],L^2).$$

Proposition 5.3.2 yields that w_n converges strongly to some w solution of

$$\Box w + w = 0; \quad \partial_t w \equiv 0 \text{ on } \omega.$$

We deduce as in J. Rauch and M. Taylor [33] or C. Bardos, G. Lebeau, J. Rauch [3] that the set of such solutions is finite dimensional and admits an eigenvector w for Δ. By unique continuation for second order elliptic operator, we get $\partial_t w \equiv 0$. Multiplying the equation by $\bar{w}$ and integrating, we obtain $w\equiv 0$ (note that, at this stage, the choice of the Klein-Gordon equation instead of the wave equation is crucial to avoid the constant solutions). We conclude that $(w_n(0),\partial_t w_n(0))$ tends to 0 strongly in $\mathcal{E}$ which gives a contradiction to (3.62). $\Box$

5.3.2 Controllability

5.3.2.1 Linear control

In this section, we recall some well known results about linear control theory and HUM method. Let $(\Phi_0, \Phi_1) \in L^2 \times H^{-1}$. We solve the system

$$\left\{ \begin{array}{rcl} \Box\Phi + \Phi & = & 0 \quad \text{on} \quad [0,T] \times M \\ (\Phi, \partial_t \Phi)_{|t=0} & = & (\Phi_0, \Phi_1). \end{array} \right. \tag{3.63}$$

and

$$\left\{ \begin{array}{rcl} \Box v + v & = & a^2\Phi \quad \text{on} \quad [0,T] \times M \\ (v, \partial_t v)_{|t=T} & = & (0,0). \end{array} \right. \tag{3.64}$$

The HUM operator S from $L^2 \times H^{-1}$ to $L^2 \times H^1$ is defined by

$$S(\Phi_0, \Phi_1) = (-\partial_t v(0), v(0)).$$

Lemma 5.3.2. *If ω satisfies the geometric control Assumption 5.0.1, then S is an isomorphism.*

Démonstration. Multiplying equation (3.64) by $\bar{\Phi}$, integrating over $[0,T] \times M$ and integrating by part, we get the formula

$$\int_0^T \int_M |a\Phi|^2 = -\int_M \partial_t v(0)\bar{\phi}(0) + \int_M v(0)\partial_t\bar{\phi}(0) = \langle S(\Phi_0, \Phi_1), (\Phi_0, \Phi_1)\rangle$$

where $\langle .,. \rangle$ denotes the duality between $L^2 \times H^1$ and $L^2 \times H^{-1}$. We get the conclusion thanks to the following observability estimate which can be proved by the same techniques used in the nonlinear problem

$$\|(\Phi_0, \Phi_1)\|^2_{L^2 \times H^{-1}} \leq \int_0^T \int_M |a\Phi|^2.$$

$\square$

5.3.2.2 Controllability for small data

Theorem 5.3.2. *Let ω satisfying Assumption 5.0.1 and $T > T_0$. Then, there exists $\delta > 0$ such that for any (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1)$ in $H^1 \times L^2$, with*

$$\|(u_0, u_1)\|_{\mathcal{E}} \leq \delta; \quad \|(\tilde{u}_0, \tilde{u}_1)\|_{\mathcal{E}} \leq \delta$$

there exists $g \in L^\infty([0, 2T], L^2)$ supported in $[0, 2T] \times \omega$ such that the unique strong solution of

$$\left\{ \begin{array}{rcl} \Box u + u + |u|^4 u & = & g \quad \text{on} \quad [0,2T] \times M \\ (u(0), \partial_t u(0)) & = & (u_0, u_1). \end{array} \right.$$

satisfies $(u(2T), \partial_t u(2T)) = (\tilde{u}_0, \tilde{u}_1)$

Démonstration. The proof is very similar to [11] except that the critical exponent do not allow to use compactness argument and we use the classical Picard fixed point instead of Schauder, as done in [9] or [29], [30] for NLS. By a compactness argument, we can select $a \in C_0^\infty(\omega)$ with $a(x) > \eta > 0$ for x in $\tilde{\omega}$ where $\tilde{\omega}$ satisfies Assumption 5.0.1. Since the equation is reversible, we can assume $(\tilde{u}_0, \tilde{u}_1) \equiv (0,0)$ and take the time T instead of $2T$. We seek g of the form $a^2(x)\Phi$ where Φ is solution of the free wave equation as in linear control theory with initial datum $(\Phi_0, \Phi_1) \in L^2 \times H^{-1}$. The purpose will be to choose the right $(\Phi_0, \Phi_1) \in L^2 \times H^{-1}$ to get the expected data. We consider the solutions of the two systems

$$\begin{cases} \Box\Phi + \Phi &= 0 \quad \text{on} \quad [0,T] \times M \\ (\Phi, \partial_t\Phi)_{|t=0} &= (\Phi_0, \Phi_1) \end{cases}$$

and

$$\begin{cases} \Box u + u + |u|^4 u &= a^2\Phi \quad \text{on} \quad [0,T] \times M \\ (u, \partial_t u)_{|t=T} &= (0,0). \end{cases} \tag{3.65}$$

Let us define the operator

$$\begin{array}{rccl} L: & L^2 \times H^{-1} & \to & H^1 \times L^2 \\ & (\Phi_0, \Phi_1) & \mapsto & L(\Phi_0, \Phi_1) = (u, \partial_t u)_{|t=0}. \end{array} \tag{3.66}$$

We split $u = v + \Psi$ with Ψ solution of

$$\begin{cases} \Box\Psi + \Psi &= a^2\Phi \quad \text{on} \quad [0,T] \times M \\ (\Psi, \partial_t\Psi)_{|t=T} &= (0,0). \end{cases} \tag{3.67}$$

This corresponds to the linear control, and $(-\partial_t\Psi, \Psi)_{|t=0} = S(\Phi_0, \Phi_1)$. As for function v, it is solution of

$$\begin{cases} \Box v + v &= -|u|^4 u \quad \text{on} \quad [0,T] \times M \\ (v, \partial_t v)_{|t=T} &= (0,0). \end{cases} \tag{3.68}$$

Φ belongs to $C([0,T], L^2)$. So, u, v and Ψ belong to $C([0,T], H^1) \cap C^1([0,T], L^2) \cap L^5([0,T], L^{10})$. We can write

$$L(\Phi_0, \Phi_1) = K(\Phi_0, \Phi_1) + S(\Phi_0, \Phi_1)$$

where $K(\Phi_0, \Phi_1) = (-\partial_t v, v)_{|t=0}$. $L(\Phi_0, \Phi_1) = (-u_1, u_0)$ is equivalent to $(\Phi_0, \Phi_1) = -S^{-1}K(\Phi_0, \Phi_1) + S^{-1}(-u_1, u_0)$. Defining the operator $B : L^2 \times H^{-1} \to L^2 \times H^{-1}$ by

$$B(\Phi_0, \Phi_1) = -S^{-1}K(\Phi_0, \Phi_1) + S^{-1}(-u_1, u_0),$$

the problem $L(\Phi_0, \Phi_1) = (-u_1, u_0)$ is equivalent to finding a fixed point of B. We will prove that if $\|(u_0, u_1)\|_{\mathcal{E}}$ is small enough, B is a contraction and reproduces a small ball B_R of $L^2 \times H^{-1}$.

Since S is an isomorphism, we have

$$\|B(\Phi_0, \Phi_1)\|_{L^2 \times H^{-1}} \leq C(\|K(\Phi_0, \Phi_1)\|_{L^2 \times H^1} + \|(u_0, u_1)\|_{\mathcal{E}})$$

So we are led to estimate $\|K(\Phi_0,\Phi_1)\|_{L^2\times H^1} = \left\|(v,\partial_t v)_{|t=0}\right\|_{\mathcal{E}}$. Energy estimates applied to equation (3.68) and Hölder inequality give

$$\left\|(v,\partial_t v)_{|t=0}\right\|_{\mathcal{E}} \leq C \left\||u|^4 u\right\|_{L^1([0,T],L^2)} \leq C \|u\|^5_{L^5([0,T],L^{10})}.$$

But Strichartz estimates applied to equation (3.65) give

$$\begin{aligned} \|u\|_{L^5([0,T],L^{10})} &\leq C\left(\left\|a^2\Phi\right\|_{L^1([0,T],L^2)} + \|u\|^5_{L^5([0,T],L^{10})}\right) \\ &\leq C\left(\|(\Phi_0,\Phi_1)\|_{L^2\times H^{-1}} + \|u\|^5_{L^5([0,T],L^{10})}\right). \end{aligned}$$

Using a bootstrap argument, we get that for $\|(\Phi_0,\Phi_1)\|_{L^2\times H^{-1}} \leq R$ small enough, we have

$$\|u\|_{L^5([0,T],L^{10})} \leq C\|(\Phi_0,\Phi_1)\|_{L^2\times H^{-1}}. \tag{3.69}$$

We finally obtain

$$\|B(\Phi_0,\Phi_1)\|_{L^2\times H^{-1}} \leq C\left(\|(\Phi_0,\Phi_1)\|^5_{L^2\times H^{-1}} + \|(u_0,u_1)\|_{\mathcal{E}}\right).$$

Choosing R small enough and $\|(u_0,u_1)\|_{H^1\times L^2} \leq R/2C$, we obtain $\|B(\Phi_0,\Phi_1)\|_{L^2\times H^{-1}} \leq R$ and B reproduces the ball B_R. Let us now prove that B is contracting. We examine the system

$$\left\{\begin{array}{rcll} \Box(u-\tilde{u}) + (u-\tilde{u}) + |u|^4u - |\tilde{u}|^4\tilde{u} &=& a^2(\Phi-\tilde{\Phi}) \quad \text{on} & [0,T]\times M \\ (u-\tilde{u},\partial_t(u-\tilde{u}))_{|t=T} &=& (0,0). & \end{array}\right. \tag{3.70}$$

$$\left\{\begin{array}{rcll} \Box(v-\tilde{v}) + (v-\tilde{v}) + |u|^4u - |\tilde{u}|^4\tilde{u} &=& 0 \quad \text{on} & [0,T]\times M \\ (v-\tilde{v},\partial_t(v-\tilde{v}))_{|t=T} &=& (0,0). & \end{array}\right. \tag{3.71}$$

We obtain similarly

$$\begin{aligned} \left\|B(\Phi_0,\Phi_1) - B(\tilde{\Phi}_0,\tilde{\Phi}_1)\right\|_{L^2\times H^{-1}} &\leq C\left\|(v-\tilde{v},\partial_t(v-\tilde{v}))_{|t=0}\right\|_{\mathcal{E}} \\ &\leq C\left\||u|^4u - |\tilde{u}|^4\tilde{u}\right\|_{L^1([0,T],L^2)} \\ &\leq C\|u-\tilde{u}\|_{L^5([0,T],L^{10})}\left(\|u\|^4_{L^5([0,T],L^{10})} + \|\tilde{u}\|^4_{L^5([0,T],L^{10})}\right) \\ &\leq CR^4\|u-\tilde{u}\|_{L^5([0,T],L^{10})} \end{aligned} \tag{3.72}$$

where we have used estimate (3.69) for the last inequality. Applying Strichartz estimates to equation (3.70), we get

$$\begin{aligned} \|u-\tilde{u}\|_{L^5([0,T],L^{10})} &\leq C(\left\||u|^4u - |\tilde{u}|^4\tilde{u}\right\|_{L^1([0,T],L^2)} + \left\|a^2(\Phi-\tilde{\Phi})\right\|_{L^1([0,T],L^2)}) \\ &\leq CR^4\|u-\tilde{u}\|_{L^5([0,T],L^{10})} + C\left\|(\Phi_0,\Phi_1) - (\tilde{\Phi}_0,\tilde{\Phi}_1)\right\|_{L^2\times H^{-1}} \end{aligned}$$

If R is taken small enough, it yields

$$\|u-\tilde{u}\|_{L^5([0,T],L^{10})} \leq C\left\|(\Phi_0,\Phi_1) - (\tilde{\Phi}_0,\tilde{\Phi}_1)\right\|_{L^2\times H^{-1}}. \tag{3.73}$$

Combining (3.72) and (3.73), we finally obtain for R small enough

$$\left\|B(\Phi_0,\Phi_1) - B(\tilde{\Phi}_0,\tilde{\Phi}_1)\right\|_{L^2\times H^{-1}} \leq CR^4\left\|(\Phi_0,\Phi_1) - (\tilde{\Phi}_0,\tilde{\Phi}_1)\right\|_{L^2\times H^{-1}}$$

and B is a contraction for R small enough, which completes the proof of Theorem 5.3.2. $\square$

5.3.2.3 Controllability of high frequency data

This subsection is devoted to the proof of the both main theorem of the article : Theorem 5.0.2 and 5.0.1.

Proof of Theorem 5.0.2. First, by decreasing of the energy and Sobolev embedding, there exists some constant $C(R_0)$ such that the assumption $\|(u_0,u_1)\|_{\mathcal{E}} \leq R_0$ implies

$$E(u)(t) \leq C(R_0) \text{ and } \|(u,\partial_t u)(t)\|_{\mathcal{E}} \leq C(R_0); \quad \forall t \geq 0. \tag{3.74}$$

Fix T such that Theorem 5.3.1 applies. Then, there exists $\varepsilon > 0$ such that for any (u_0,u_1) satisfying

$$\|(u_0,u_1)\|_{\mathcal{E}} \leq C(R_0); \quad \|(u_0,u_1)\|_{L^2\times H^{-1}} \leq \varepsilon, \tag{3.75}$$

we have the strong observability estimate

$$E(u)(0) \leq C \iint_{[0,T]\times M} |a(x)\partial_t u|^2 \; dtdx.$$

for any solution of the damped equation (0.3). That means that there exists $0 < C$ such that any solution of the damped equation satisfying (3.75) fulfills

$$E(u)(T) \leq (1-C)E(u)(0). \tag{3.76}$$

Pick $N \in \mathbb{N}$ large enough such that $(1-C)^N C(R_0) \leq \varepsilon^2/2$.

Corollary 5.2.4 and (3.74) allow us to choose δ small enough such that the assumption

$$\|(u_0,u_1)\|_{\mathcal{E}} \leq R_0; \qquad \|(u_0,u_1)\|_{L^2\times H^{-1}} \leq \delta$$

implies

$$\|(u(nT),\partial_t u(nT))\|_{L^2\times H^{-1}} \leq \varepsilon, \quad 0 \leq n \leq N. \tag{3.77}$$

So, with that choice, we have $E(u)(NT) \leq (1-C)^N E(u)(0)$. Then, by the energy decreasing, for any $t \geq NT$, we have

$$\|(u,\partial_t u)(t)\|^2_{L^2\times H^{-1}} \leq 2E(u)(t) \leq 2E(u)(NT) \leq \varepsilon^2.$$

Therefore, the decay estimate (3.76) is true on each interval $[nT,(n+1)T]$, $n \in \mathbb{N}$ and we have

$$E(u)(nT) \leq (1-C)^n E(u)(0)$$

which yields the result. □

Proof of Theorem 5.0.1. Since the equation is reversible, we can assume $(\tilde{u}_0,\tilde{u}_1) = (0,0)$. By a compactness argument, we can select $a \in C_0^\infty(\omega)$ with $a(x) > \eta > 0$ for x in $\tilde{\omega}$ where $\tilde{\omega}$ satisfies Assumption 5.0.2. We will first use the damping term $a(x)^2\partial_t u$ as a term of control. We apply Theorem 5.0.2 and Theorem 5.3.2 once the energy is small enough. □

5.A Inégalités de Morawetz pour les ondes sur une variété

On se propose dans cette section de donner une preuve des inégalités de Morawetz sur une variété, avec un terme source. Ces inégalités se trouvent dans [23] sur une variété mais sans terme source et dans [2] sur $\mathbb{R}^3$ avec terme source. Cette section ne prétend donc pas à la nouveauté. Elle présente aussi l'intérêt de présenter des calculs intrinsèques sans passer par les coordonnées locales comme cela est fait dans [23]. On veut une inégalité de type Morawetz pour une équation

$$\left\{ \begin{array}{rcl} \Box u + |u|^4 u & = & g \quad \text{on} \quad]0,T] \times M \\ (u(0), \partial_t u(0)) & = & (u_0, u_1) \in \mathcal{E}. \end{array} \right. \tag{1.78}$$

avec $g \in L^1([0,T], L^2(M))$. On centre tout en $t_0 = 0$ et $x_0 = 0$. On rappelle les notations de la sous-section 5.1.1 que l'on change légèrement pour raisonner dans le cône futur, ce qui simplifie un peu l'analyse. φ est la fonction distance géodesique à 0 définie dans un voisinage $U \subset M$ de 0. Soit $0 < a < b < \varepsilon$ assez petits pour définir

$$\begin{array}{rcll} K_a^b & := & \{z = (t,x) \in [a,b] \times U \,|\varphi \leq t\} & \text{cône future tronqué} \\ M_a^b & := & \{z = (t,x) \in [a,b] \times U \,|\varphi = t\} & \text{manteau du cône tronqué} \\ D(t) & := & \{x \in U \,|\varphi \leq |t|\} & \text{section en espace du cône au temps } t. \end{array}$$

Dans la suite, tous les gradients, normes, densitées sont calculées par rapport à la métrique Riemannienne sur M (par exemple, on a $\|\nabla\varphi\| = 1$ voir Lemme 5.A.2). La forme volume $d\sigma$ sur M_a^b est la métrique induite par la forme volume naturelle sur $\mathbb{R} \times M$.

$$\begin{array}{rcll} e(u)(t,x) & := & \frac{1}{2}\left(|\partial_t u|^2 + |\nabla u|^2\right) + \frac{1}{6}|u|^6 & \text{énergie locale} \\ E(u, D(t)) & := & \int_{D(t)} e(u)dx & \text{énergie au temps } t \text{ dans la section du cône} \\ Flux(u, M_a^b)) & := & \frac{1}{\sqrt{2}} \int_{M_a^b} \frac{1}{2}|\partial_t u \nabla\varphi + \nabla u|^2 + \frac{1}{6}|u|^6 \, d\sigma & \text{flux d'énergie sortant du cône} \end{array}$$

Lemma 5.A.1. *La fonction $E(u, D(t))$ satisfait pour $0 \leq a \leq b$*

$$E(u, D(b)) - E(u, D(a)) = Flux(u, M_a^b) + \Re \iint_{K_a^b} g\overline{\partial_t u}.$$

Démonstration. On obtient l'indentité en multipliant l'équation par le multiplicateur $\partial_t \overline{u}$ et en prenant la partie réelle pour avoir $\partial_t e(u) - \Re div(\partial_t \overline{u} \nabla_x u) = \Re g \partial_t \overline{u}$, ensuite, on intègre sur le cône tronqué K_a^b et on utilise la formule de Stokes (sur M_a^b, on a la normale sortante $\vec{N} = (N_t, N_x) = \frac{1}{\sqrt{2}}(-1, \nabla\varphi)$).

$$\Re \iint_{K_a^b} g\overline{\partial_t u} = E(u, D(b)) - E(u, D(a)) + \Re \frac{1}{\sqrt{2}} \int_{M_a^b} -e - \partial_t \overline{u} \nabla u \cdot \nabla\varphi.$$

Cela donne le résultat par l'identité $|\partial_t u \nabla\varphi + \nabla u|^2 = |\partial_t u|^2 + |\nabla u|^2 + 2\Re \partial_t \overline{u} \nabla u \cdot \nabla\varphi$. $\square$

La proposition principale de cette section est

Proposition 5.A.1. *Soit u une solution régulière de*

$$\Box u + |u|^4 u = g \quad \text{sur} \quad]0,T] \times M$$

alors, pour $0 < a < b \leq T$ petits, on a

$$\begin{aligned}
\int_{D(b)} |u|^6 \quad \leq \quad & \frac{a}{b}\left(f(a)) + f(a)^{1/3}\right) + |f(b) - f(a)| + \|g\|_{L^1L^2(K_a^b)} \|\partial_t u\|_{L^\infty L^2(K_a^b)} \\
& + \left(|f(b) - f(a)| + \|g\|_{L^1L^2(\tilde{K}_a^b)} \|\partial_t u\|_{L^\infty L^2(K_a^b)}\right)^{1/3} \\
& + \|g\|_{L^1L^2(K_a^b)} \left(\|\partial_t u\|_{L^\infty L^2(K_a^b)} + \|\nabla u\|_{L^\infty L^2(K_a^b)} + \|u\|_{L^\infty L^6(K_a^b)}\right) \\
& + C(b-a) \sup_{t\in[a,b]} \left[f(t) + f(t)^{1/3}\right]
\end{aligned}$$

où on a noté $f(t) = E(u, D(t))$.

Lemma 5.A.2. *Si φ est la distance géodesique à 0, on a*

$$|\nabla \varphi| = 1.$$

On définit la forme bilinéaire symétrique $B_x = \frac{1}{2}Hess(\varphi^2) - g$. Elle vérifie pour tout $\xi \in T_x M$.

$$|B_x(\xi, \xi)| \leq C\varphi |\xi|^2.$$

Démonstration. C'est le lemme de Gauss pour la première égalité : en coordonnées normales polaires, $g = dr^2 + h_{(r,\omega)}$ où $h_{(r,\omega)}$ est une métrique sur une sphère (on a bien sûr $\varphi = r$ mais on garde la notation r pour les coordonnées). Pour la deuxième, en coordonnées normales centrées en 0, on a $\varphi^2 = |x|^2$. De plus, les symboles de Christoffel vérifient $\Gamma_{i,j}^k(0) = 0$ et donc en coordonnées, $(Hess(f)(0))_{ij} = \partial_{ij}^2 f$ et donc $B_0 = 0$. Il reste donc à appliquer l'inégalité des accroissements finis. $\square$

Lemma 5.A.3. *Si $X = \varphi\nabla\varphi = \frac{1}{2}\nabla(\varphi^2)$, où φ est la distance géodesique à 0, on a*

$$\Re \nabla u \cdot \nabla(X \cdot \nabla \bar{u}) = div\left(X \frac{|\nabla u|^2}{2}\right) - div(X)\frac{|\nabla u|^2}{2} + |\nabla u|^2 + B(\nabla u, \nabla \bar{u}).$$

Démonstration. D est la connection de Levi-Civita. On fait les calculs

$$X \cdot \nabla \left(\frac{|\nabla u|^2}{2}\right) = D_X(\frac{|\nabla u|^2}{2}) = \Re D_X \nabla u \cdot \nabla \bar{u} = \Re Hess(u)(X, \nabla \bar{u})$$

$$\begin{aligned}
\Re \nabla \bar{u} \cdot (X \cdot \nabla u) &= \Re \frac{1}{2} \nabla \bar{u} \cdot \nabla \left[\nabla(\varphi^2) \cdot \nabla u\right] = \Re \frac{1}{2} D_{\nabla \bar{u}} \left[\nabla(\varphi^2) \cdot \nabla u\right] \\
&= \Re \frac{1}{2} D_{\nabla \bar{u}} \nabla(\varphi^2) \cdot \nabla u + \Re \frac{1}{2} \nabla(\varphi^2) \cdot D_{\nabla \bar{u}} \nabla u \\
&= \Re \frac{1}{2} Hess(\varphi^2)(\nabla u, \nabla \bar{u}) + \Re \frac{1}{2} Hess(u)(\nabla \bar{u}, \nabla(\varphi^2)) \\
&= |\nabla u|^2 + B(\nabla u, \nabla \bar{u}) + X \cdot \nabla \left(\frac{|\nabla u|^2}{2}\right).
\end{aligned}$$

Or,

$$div\left(X\frac{|\nabla u|^2}{2}\right) = div(X)\frac{|\nabla u|^2}{2} + X\cdot\nabla\left(\frac{|\nabla u|^2}{2}\right).$$

Ce qui donne le résultat. □

Preuve de la Proposition 5.A.1. On multiplie l'équation par le multiplicateur $L\bar{u} = t\partial_t\bar{u} + X\cdot\nabla\bar{u} + \bar{u}$.

$$\Re g t\partial_t\bar{u} = \Re(\partial_t^2 u - \Delta u + |u|^4 u)t\partial_t\bar{u} = t\partial_t e - t\Re div(\nabla u\partial_t\bar{u}) = \partial_t(te) - e - t\Re div(\nabla u\partial_t\bar{u})$$

$$\begin{aligned}
\Re(\partial_t^2 u)X\cdot\nabla\bar{u} &= \Re\partial_t(\partial_t u X\cdot\nabla\bar{u}) - \Re\partial_t u X\cdot\nabla\partial_t\bar{u}\\
&= \Re\partial_t(\partial_t u X\cdot\nabla\bar{u}) - \frac{1}{2}div(X|\partial_t u|^2) + \frac{1}{2}div(X)|\partial_t u|^2\\
\Re(\Delta u)X\cdot\nabla\bar{u} &= \Re div(\nabla u X\cdot\nabla\bar{u}) - \Re\nabla u\cdot\nabla(X\cdot\nabla\bar{u})\\
&= \Re div(\nabla u X\cdot\nabla\bar{u}) - div\left(X\frac{|\nabla u|^2}{2}\right) + div(X)\frac{|\nabla u|^2}{2}\\
&\quad -|\nabla u|^2 - \Re B(\nabla u,\nabla\bar{u})
\end{aligned}$$

où on a utilisé le Lemme 5.A.3 dans la dernière égalité.

$$\Re(|u|^4 u)X\cdot\nabla\bar{u} = div\left(X(\frac{1}{6}|u|^6)\right) - div(X)\left(\frac{1}{6}|u|^6\right)$$

$$\Re(\partial_t^2 u - \Delta u + |u|^4 u)\bar{u} = \Re\partial_t(\partial_t u\bar{u}) - |\partial_t u|^2 - \Re div(\bar{u}\nabla u) + |\nabla u|^2 + |u|^6$$

En sommant les égalité, on obtient

$$\begin{aligned}
\Re g L\bar{u} &= \Re\partial_t(te + \partial_t u X\cdot\nabla\bar{u} + \partial_t u\bar{u})\\
&\quad +\Re div\left[-t\nabla u\partial_t\bar{u} - \frac{1}{2}X|\partial_t u|^2 - \nabla u X\cdot\nabla\bar{u} + X\frac{|\nabla u|^2}{2} + X(\frac{1}{6}|u|^6) - \bar{u}\nabla u\right]\\
&\quad -e + \frac{1}{2}div(X)|\partial_t u|^2 - div(X)\frac{|\nabla u|^2}{2} + |\nabla u|^2 + B(\nabla u,\nabla\bar{u}) - div(X)\left(\frac{1}{6}|u|^6\right)\\
&\quad -|\partial_t u|^2 + |\nabla u|^2 + |u|^6\\
&= \partial_t\tilde{P} + div\tilde{Q} + \tilde{R}
\end{aligned}$$

avec

$$\begin{aligned}
\tilde{P} &= \Re(te + \partial_t u X\cdot\nabla\bar{u} + \partial_t u\bar{u})\\
&= \Re\frac{1}{2t}\left[|t\partial_t u|^2 + |t\nabla u|^2 + 2t\partial_t u X\cdot\nabla\bar{u} + 2t\partial_t u\bar{u}\right] + t\frac{|u|^6}{6}\\
&= \Re\frac{1}{2t}\left[|t\partial_t u|^2 + |X\cdot\nabla u|^2 + 2t\partial_t u X\cdot\nabla\bar{u} + 2t\partial_t u\bar{u} + 2uX\cdot\nabla\bar{u} + |u|^2\right] + t\frac{|u|^6}{6}\\
&\quad +\Re\frac{1}{2t}\left[|t\nabla u|^2 - |X\cdot\nabla u|^2 - |u|^2 - 2uX\cdot\nabla\bar{u}\right]\\
&= \Re\frac{1}{2t}|Lu|^2 + \Re\frac{t}{2}\left[|\nabla u|^2 - \left|\frac{X}{t}\cdot\nabla u\right|^2\right] + \Re\frac{1}{2t}\left[-|u|^2 - 2uX\cdot\nabla\bar{u}\right] + t\frac{|u|^6}{6}\\
&= P - \frac{|u|^2}{t} + \Re\frac{1}{2t}\left[-|u|^2 - 2uX\cdot\nabla\bar{u}\right] = P - \frac{3|u|^2}{2t} - \frac{1}{2t}X\cdot\nabla|u|^2\\
&= P + \frac{1}{2t}\left[-div(X|u|^2) + |u|^2(-3 + div(X))\right]
\end{aligned}$$

où on a noté

$$P(t,x) = \frac{1}{2t}|Lu|^2 + \frac{t}{2}\left[|\nabla u|^2 - \left|\frac{X}{t}\cdot\nabla u\right|^2\right] + t\frac{|u|^6}{6} + \frac{|u|^2}{t}.$$

Pour la dérivée en temps $\partial_t\tilde{P}$, on va garder $\partial_t P$. La dérivée en temps du deuxième terme de $\tilde{P}$ est un terme de divergence et va se rajouter à $\tilde{Q}$ pour former Q, le troisième se rajoute au reste $\tilde{R}$ pour former R. On obtient donc

$$\Re gL\bar{u} \quad = \quad \partial_t P + div(Q) + R$$

avec

$$\begin{aligned}
Q(t,x) &= \Re\left[-t\nabla u\partial_t\bar{u} - \frac{1}{2}X|\partial_t u|^2 - \nabla uX\cdot\nabla\bar{u} + X\frac{|\nabla u|^2}{2} + X(\frac{1}{6}|u|^6) - \bar{u}\nabla u - X\partial_t\frac{|u|^2}{2t}\right] \\
&= X\left[-\frac{1}{2}|\partial_t u|^2 + \frac{|\nabla u|^2}{2} + \frac{1}{6}|u|^6 - \partial_t\frac{|u|^2}{2t}\right] - \Re Lu\nabla\bar{u} \\
R(t,x) &= |\partial_t u|^2\left(-\frac{1}{2} + \frac{divX}{2} - 1\right) + |\nabla u|^2\left(-\frac{1}{2} - \frac{divX}{2} + 2\right) + |u|^6(-\frac{1}{6} - \frac{divX}{6} + 1) \\
&\quad + B(\nabla u, \nabla\bar{u}) + (-3 + div(X))\partial_t\frac{|u|^2}{2t}
\end{aligned}$$

Lemma 5.A.4. *En notant $X = \varphi\nabla\varphi$, on a*

$$div(X) = 3 + r\frac{J'(\omega,r)}{J(\omega,r)} \underset{not}{=} 3 + rj(\omega,r)$$

où $J(\omega,r)$ est la forme densité de volume en coordonnées normales ("polaires") : $x = \omega r$, $\omega \in S^2$, $r \in R^+$.

Démonstration. La Proposition 4.16 p 213 du livre de Gallot-Hulin-Lafontaine [18] affirme (après changement de signe)

$$\Delta r = \frac{n-1}{r} + \frac{J'(\omega,r)}{J(\omega,r)}$$

$$\begin{aligned}
div(X) &= \frac{1}{2}div(\nabla(\varphi^2)) = \frac{1}{2}\Delta(\varphi^2) = \frac{1}{2}(2\varphi\Delta\varphi + 2|\nabla\varphi|^2) \\
&= n - 1 + \varphi\frac{J'(\omega,r)}{J(\omega,r)} + 1 = n + \varphi\frac{J'(\omega,r)}{J(\omega,r)}.
\end{aligned}$$

□

On peut donc maintenant avoir une estimée de la "petitesse" de R lorsque on est proche de 0.

$$\begin{aligned}
R(t,x) &= |\partial_t u|^2\left(-\frac{1}{2}+\frac{divX}{2}-1\right)+|\nabla u|^2\left(-\frac{1}{2}-\frac{divX}{2}+2\right)\\
&\quad +|u|^6(-\frac{1}{6}-\frac{divX}{6}+1)+B(\nabla u,\nabla\bar{u})+(-3+div(X))\partial_t\frac{|u|^2}{2t}\\
&= r|\partial_t u|^2\frac{j(\omega,r)}{2}-r|\nabla u|^2\frac{j(\omega,r)}{2}+|u|^6(\frac{1}{3}-r\frac{j(\omega,r)}{6})\\
&\quad +B(\nabla u,\nabla\bar{u})+rj(\omega,r)\partial_t\frac{|u|^2}{2t}\\
&= R_1+R_2=\frac{|u|^6}{3}+R_2
\end{aligned}$$

R_1 est le même reste que sur $\mathbb{R}^3$ et a le bon signe. R_2 est un reste qui est nul pour une métrique plate et qui sera petit $\mathcal{O}(\varphi(x))$ pour des petits x.

Maintenant, on peut intégrer sur le cône tronqué K_a^b $(0<a<b)$, et appliquer la formule de Green

$$\Re\int_{K_a^b} gL\bar{u}=H(b)-H(a)+\frac{1}{\sqrt{2}}\Re\int_{M_a^b}(-P+\nabla\varphi\cdot Q)d\sigma+\Re\int_{K_a^b}R$$

en notant,

$$H(t)=\int_{D(t)}P(t,x)dx.$$

Mais si $(t,x)\in M_a^b$, on a $t=\varphi$ et $X=t\nabla\varphi$ et donc en utilisant $|\nabla\varphi|^2=1$.

$$\begin{aligned}
P(t,x) &= \frac{1}{2t}|Lu|^2+\frac{t}{2}\left[|\nabla u|^2-|\nabla\varphi\cdot\nabla u|^2\right]+t\frac{|u|^6}{6}+\frac{|u|^2}{t}\\
\nabla\varphi\cdot Q(t,x) &= t\left[-\frac{1}{2}|\partial_t u|^2+\frac{|\nabla u|^2}{2}+\frac{1}{6}|u|^6-\partial_t\frac{|u|^2}{2t}\right]-\Re Lu\nabla\bar{u}\cdot\nabla\varphi
\end{aligned}$$

Donc en sommant pour $(t,x)\in M_a^b$, on obtient

$$\begin{aligned}
-P+\nabla\varphi\cdot Q(t,x) &= -\frac{1}{2t}|Lu|^2+\frac{t}{2}\left[|\nabla\varphi\cdot\nabla u|^2-|\partial_t u|^2-\partial_t\frac{|u|^2}{t}\right]\\
&\quad -\frac{|u|^2}{t}-\Re Lu\nabla\bar{u}\cdot\nabla\varphi\\
&= -\frac{1}{2t}|Lu|^2+\frac{t}{2}\left[|\nabla\varphi\cdot\nabla u|^2-|\partial_t u|^2-2\Re\frac{\partial_t u\bar{u}}{t}+\frac{|u|^2}{t^2}\right]\\
&\quad -\frac{|u|^2}{t}-\Re Lu\nabla\bar{u}\cdot\nabla\varphi\\
&= -\frac{1}{2t}|Lu|^2-\frac{1}{2t}|Lu|^2=-\frac{1}{t}|Lu|^2.
\end{aligned}$$

Donc

$$H(b)-H(a)-\frac{1}{\sqrt{2}}\int_{M_a^b}\frac{1}{t}|Lu|^2\,d\sigma=\Re\int_{K_a^b}gL\bar{u}-\Re\int_{K_a^b}\frac{|u|^6}{3}-\Re\int_{K_a^b}R_2. \qquad (1.79)$$

Cette égalité est la même que sur $\mathbb{R}^3$ sauf que R_2 contient aussi des termes d'erreurs qui vont être petits si t est proche de 0.

On va maintenant utiliser cette égalité pour obtenir des renseignements sur la non concentration de u en norme L^6. D'abord, on va estimer $H(b)$. Grâce au signe de $|u|^6$, on a

$$H(b) \leq H(a) + \frac{1}{\sqrt{2}} \int_{M_a^b} \frac{1}{t} |Lu|^2 \, d\sigma + \left| \int_{K_a^b} g L\bar{u} \right| + \left| \int_{K_a^b} R_2 \right|.$$

En utilisant le Lemme 5.A.2 de petitesse et que $j(\omega, r)$ est localement borné, on a

$$\begin{aligned}
\left| \int_{K_a^b} R_2 \right| &\leq Cb \int_{K_a^b} |\partial_t u|^2 + |\nabla u|^2 + |u|^6 + \frac{|u\partial_t u|}{t} + \frac{|u|^2}{t^2} \\
&\leq Cb \int_a^b \int_{D(t)} |\partial_t u|^2 + |\nabla u|^2 + |u|^6 + \frac{|u\partial_t u|}{t} + \frac{|u|^2}{t^2} \\
&\leq Cb(b-a) \sup_{t\in[a,b]} \left[E(u, D(t)) + E(u, D(t))^{2/3} + E(u, D(t))^{1/3} \right] \\
&\leq Cb(b-a) \sup_{t\in[a,b]} \left[E(u, D(t)) + E(u, D(t))^{1/3} \right]
\end{aligned}$$

où on a utilisé l'inégalité de Hölder $\|u\|_{L^2(D(t))} \leq Ct \|u\|_{L^6(D(t))}$. De même, on a

$$H(a) \leq a \left(E(u, D(a)) + E(u, D(a))^{1/3} \right).$$

Les autres termes sont estimés comme dans le cas de $\mathbb{R}^3$.

$$\begin{aligned}
\int_{M_a^b} \frac{1}{t} |Lu|^2 \, d\sigma &\leq \int_{M_a^b} b \, |\partial_t u + \nabla\varphi \cdot \nabla u|^2 + \frac{|u|^2}{t} d\sigma \\
&\leq C \int_{M_a^b} b \, |\nabla\varphi \cdot (\partial_t u \nabla\varphi + \nabla u)|^2 + \frac{|u|^2}{t} d\sigma \\
&\leq C \int_{M_a^b} b \, |\partial_t u \nabla\varphi + \nabla u|^2 \, d\sigma + b \left(\int_{M_a^b} |u|^6 d\sigma \right)^{1/3} \\
&\leq C \left(bFlux(u, M_a^b) + bFlux(u, M_a^b)^{1/3} \right).
\end{aligned}$$

où on a utilisé l'inégalité de Cauchy-Schwarz, $|\nabla\varphi| = 1$ et $\int_{M_a^b} \frac{|u|^2}{t} d\sigma \leq \left\| \frac{1}{t} \right\|_{L^{3/2}(M_a^b)} \|u\|^2_{L^6(M_a^b)} \leq Cb \|u\|^2_{L^6(M_a^b)}$.

Par l'inégalité de Hölder, on estime

$$\int_{K_a^b} |gL\bar{u}| \leq Cb \|g\|_{L^1 L^2(K_a^b)} \left(\|\partial_t u\|_{L^\infty L^2(K_a^b)} + \|\nabla u\|_{L^\infty L^2(K_a^b)} + \|u\|_{L^\infty L^6(K_a^b)} \right).$$

De plus, l'inégalité d'énergie donne

$$\begin{aligned}
Flux(u, M_a^b) &\leq |E(u, D(b)) - E(u, D(a))| + \int_{K_a^b} |g\partial_t \bar{u}| \\
&\leq |E(u, D(b)) - E(u, D(a))| + \|g\|_{L^1 L^2(K_a^b)} \|\partial_t u\|_{L^\infty L^2(K_a^b)}.
\end{aligned}$$

Finalement, en utilisant $\int_{D(t)} |u|^6 \leq \frac{H(t)}{t}$, on a pour la norme L^6 :

$$\begin{aligned}
\int_{D(b)} |u|^6 \quad \leq \quad & \frac{a}{b}\left(f(a)) + f(a)^{1/3}\right) + |f(b) - f(a)| + \|g\|_{L^1L^2(K_a^b)} \|\partial_t u\|_{L^\infty L^2(K_a^b)} \\
& + \left(|f(b) - f(a)| + \|g\|_{L^1L^2(K_a^b)} \|\partial_t u\|_{L^\infty L^2(K_a^b)}\right)^{1/3} \\
& + \|g\|_{L^1L^2(K_a^b)} \left(\|\partial_t u\|_{L^\infty L^2(K_a^b)} + \|\nabla u\|_{L^\infty L^2(K_a^b)} + \|u\|_{L^\infty L^6(K_a^b)}\right) \\
& + C(b-a) \sup_{t\in[a,b]} \left[f(t) + f(t)^{1/3}\right]
\end{aligned}$$

où on a noté $f(t) = E(u, D(t))$. $\square$

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Chapitre 6

Contrôlabilité locale pour des équations de Schrödinger et des ondes linéaires et non linéaires avec contrôle bilinéaire

Contents

Ce chapitre est la reprise d'un article écrit avec Karine Beauchard et accepté pour publication au Journal de Mathématiques Pures et Appliquées [19].

6.1 Introduction

6.1.1 Main result

Following [58], we consider a quantum particle, in a 1D infinite square potential well, subjected to an electric field. It is represented by the following Schrödinger equation

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = -\frac{\partial^2\psi}{\partial x^2}(t,x) - u(t)\mu(x)\psi(t,x), x \in (0,1), t \in (0,T), \\ \psi(t,0) = \psi(t,1) = 0, \end{cases} \tag{1.1}$$

where ψ is the wave function of the particle, u is the amplitude of the electric field and $\mu \in H^3((0,1),\mathbb{R})$ is the dipolar moment of the particle. The system (1.1) is a bilinear control system, in which

- the state is ψ, with $\|\psi(t)\|_{L^2(0,1)} = 1$, $\forall t \in (0,T)$,
- the control is the real valued function $u : [0,T] \to \mathbb{R}$.

Let us introduce some notations. The operator A is defined by

$$D(A) := H^2 \cap H^1_0((0,1),\mathbb{C}), \quad A\varphi := -\frac{d^2\varphi}{dx^2}. \tag{1.2}$$

Its eigenvalues and eigenvectors are

$$\lambda_k := (k\pi)^2, \varphi_k(x) := \sqrt{2}\sin(k\pi x), \forall k \in \mathbb{N}^*. \tag{1.3}$$

The family $(\varphi_k)_{k\in\mathbb{N}^*}$ is an orthonormal basis of $L^2((0,1),\mathbb{C})$ and

$$\psi_k(t,x) := \varphi_k(x)e^{-i\lambda_k t}, \forall k \in \mathbb{N}^*$$

is a solution of (1.1) with $u \equiv 0$ called eigenstate, or ground state, when $k = 1$. We define the spaces

$$H^s_{(0)}((0,1),\mathbb{C}) := D(A^{s/2}), \forall s > 0 \tag{1.4}$$

equipped with the norm

$$\|\varphi\|_{H^s_{(0)}} := \left(\sum_{k=1}^{\infty} |k^s\langle\varphi,\varphi_k\rangle|^2\right)^{1/2}.$$

We denote by $\langle .,. \rangle$ the $L^2((0,1),\mathbb{C})$ scalar product

$$\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}dx$$

and by $\mathcal{S}$ the unit $L^2((0,1),\mathbb{C})$-sphere. The first goal of this article is the proof of the following result.

Theorem 1. *Let $T > 0$ and $\mu \in H^3((0,1),\mathbb{R})$ be such that*

$$\exists c > 0 \text{ such that } \frac{c}{k^3} \leqslant |\langle\mu\varphi_1,\varphi_k\rangle|, \forall k \in \mathbb{N}^*. \tag{1.5}$$

There exists $\delta > 0$ and a C^1 map

$$\Gamma : \mathcal{V}_T \rightarrow L^2((0,T),\mathbb{R})$$

where

$$\mathcal{V}_T := \{\psi_f \in \mathcal{S} \cap H^3_{(0)}((0,1),\mathbb{C}); \|\psi_f - \psi_1(T)\|_{H^3} < \delta\},$$

such that, $\Gamma(\psi_1(T)) = 0$ *and for every* $\psi_f \in \mathcal{V}_T$*, the solution of (1.1) with initial condition*

$$\psi(0) = \varphi_1 \tag{1.6}$$

and control $u = \Gamma(\psi_f)$ *satisfies* $\psi(T) = \psi_f$*.*

Remark 1. *Thanks to the time reversibility of the system, Theorem 1 ensures the local controllability of the system (1.1) around the ground state : for every* $T > 0$*, there exists* $\delta > 0$ *such that, for every* $\psi_0, \psi_f \in \mathcal{S} \cap H^3_{(0)}((0,1),\mathbb{C})$ *with* $\|\psi_0 - \psi_1(0)\|_{H^3} + \|\psi_f - \psi_1(T)\|_{H^3} < \delta$*, there exists a control* $u \in L^2(0,T)$ *such that the solution of (1.1) with initial condition* $\psi(0) = \psi_0$ *satisfies* $\psi(T) = \psi_f$*.*

Remark 2. *The assumption (1.5) holds, for example, with* $\mu(x) = x^2$*, because*

$$\langle x^2\varphi_1, \varphi_k\rangle = \int_0^1 2x^2 \sin(k\pi x)\sin(\pi x)dx = \begin{cases} \frac{(-1)^{k+1}8k}{\pi^2(k^2-1)^2} \text{ if } k \geqslant 2, \\ \frac{-3+2\pi^2}{6\pi^2} \text{ if } k = 1. \end{cases} \tag{1.7}$$

But it does not hold when $\langle \mu\varphi_1, \varphi_k\rangle = 0$*, for some* $k \in \mathbb{N}^*$*, or when* μ *has a symmetry with respect to* $x = 1/2$*. However, the assumption (1.5) holds generically with respect to* $\mu \in H^3((0,1),\mathbb{R})$ *because*

$$\langle \mu\varphi_1, \varphi_k\rangle = \frac{4[(-1)^{k+1}\mu'(1) - \mu'(0)]}{k^3\pi^2} - \frac{\sqrt{2}}{(k\pi)^3}\int_0^1 (\mu\varphi_1)'''(x)\cos(k\pi x)dx, \forall k \in \mathbb{N}^*. \tag{1.8}$$

(see Appendix 6.A for a proof). Thus, Theorem 1 is very general.

6.1.2 A simpler proof

The local exact controllability of 1D Schrödinger equations, with bilinear control, has already been investigated in [14, 15, 17], (see also [16] for a similar result on a 1D beam equation). In these articles, three different models are studied. The local controllability of the nonlinear system is proved thanks to the linearization principle :

- first, we prove the controllability of a linearized system,
- then, we prove the local controllability of the nonlinear system, by applying an inverse mapping theorem.

This strategy is coupled with the return method and quasi-static deformations in [14, 17] and with power series expansions in [15, 17] (see [31, 33] by Coron for a presentation of these technics). In these articles, the most difficult part of the proof is the application of the inverse mapping theorem. Indeed, because of an a priori loss of regularity, we were led to apply the Nash-Moser implicit function theorem (see, for instance [6] by Alinhac, Gérard and [39] by Hörmander), instead of the classical inverse mapping theorem. The Nash-Moser theorem requires, in particular, the controllability of an infinite number of linearized systems, and tame estimates on the corresponding controls. These two points are difficult to prove and lead to long technical developments in [14, 15, 17].

In this article, we propose a simpler proof, that uses only the classical inverse mapping theorem (needing the controllability of only one linearized system), because we emphasize a hidden regularizing effect (see Proposition 2).

Therefore, the controllability result of Theorem 1 enters the classical framework of local controllability results for nonlinear systems, proved with fixed point arguments (see, for instance, [56] by Rosier, [29] by Cerpa and Crépeau, [59] by Russell and Zhang, [64] by Zhang, [65] by Zuazua ; this list is not exhaustive).

6.1.3 Additionnal results

The proof we developed for Theorem 1 is quite robust, thus we could apply it to other situations : other linear PDEs and also nonlinear PDEs, that are presented in the next subsections. This shows that the strategy proposed in this article works for a wide range of bilinear systems.

6.1.3.1 Generalization to higher regularities

The first situation is the analogue result of Theorem 1, but with higher regularities : we prove the local exact controllability of (1.1) in smoother spaces and with smoother controls. Namely, we prove the following result.

Theorem 2. *Let $T > 0$ and $\mu \in H^5((0,1),\mathbb{R})$ be such that (1.5) holds. There exists $\delta > 0$ and a C^1 map*

$$\begin{array}{rccl} \Gamma : & \mathcal{V}_T & \rightarrow & H^1_0((0,T),\mathbb{R}) \\ & \psi_f & \mapsto & \Gamma(\psi_f) \end{array}$$

where

$$\mathcal{V}_T := \{\psi_f \in \mathcal{S} \cap H^5_{(0)}((0,1),\mathbb{C}); \|\psi_f - \psi_1(T)\|_{H^5} < \delta\},$$

such that, $\Gamma(\psi_1(T)) = 0$ and for every $\psi_f \in \mathcal{V}_T$, the solution of (1.1), (1.6) with control $u = \Gamma(\psi_f)$ satisfies $\psi(T) = \psi_f$.

Of course, the strategy may be used to go further and prove the local exact controllability of (1.1) around the ground state

- in $H^7_{(0)}(0,1)$ with controls in $H^2_0((0,T),\mathbb{R})$,
- in $H^9_{(0)}(0,1)$ with controls in $H^3_0((0,T),\mathbb{R})$, etc.

6.1.3.2 On the 3D ball with radial data

The second situation is the analogue result of Theorem 1, but for the Schrödinger equation posed on the three dimensional unit ball B^3 for radial data. In polar coordinates, the Laplacian for radial data can be written

$$\Delta u(r) = \partial_r^2 u(r) + \frac{2}{r}\partial_r u(r).$$

In particular, we have $\Delta\left(\frac{g(r)}{r}\right) = \frac{\partial_r^2 u(r)}{r}$. The eigenfunctions of the Dirichlet operator $A = -\Delta$ with domain $D(A) := H^2_{radial} \cap H^1_0(B^3)$ are $\varphi_k = \frac{sin(k\pi r)}{r\sqrt{2\pi}}$ with eigenvalues $\lambda_k = (k\pi)^2$. Thus, we study the Schrödinger equation

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,r) = -\Delta\psi(t,r) - u(t)\mu(r)\psi(t,r), r \in (0,1), \\ \psi(t,1) = 0. \end{cases} \tag{1.9}$$

The theorem we obtain is very similar to Theorem 1.

Theorem 3. *Let $T>0$ and $\mu \in H^3(B^3,\mathbb{R})$ radial be such that*

$$\exists c>0 \text{ such that } \frac{c}{k^3} \leqslant |\langle \mu\varphi_1,\varphi_k\rangle|, \forall k \in \mathbb{N}^*. \tag{1.10}$$

There exists $\delta>0$ and a C^1 map

$$\Gamma: \ \mathcal{V}_T \ \rightarrow \ L^2((0,T),\mathbb{R})$$

where

$$\mathcal{V}_T := \{\psi_f \in \mathcal{S} \cap H^3_{(0),rad}(B^3,\mathbb{C}); \|\psi_f - \psi_1(T)\|_{H^3} < \delta\},$$

such that, $\Gamma(\psi_1(T))=0$ and for every $\psi_f \in \mathcal{V}_T$, the solution of (1.9) with initial condition

$$\psi(0)=\varphi_1 \tag{1.11}$$

and control $u=\Gamma(\psi_f)$ satisfies $\psi(T)=\psi_f$.

The analysis is very close to the 1D case since for this particular data, the Laplacian behaves as in dimension 1. We refer to Appendix A for the proof of the genericity of the assumption (1.10). Note that this simpler situation has also been used by Anton for proving global existence for the nonlinear Schrödinger equation [8].

6.1.3.3 Nonlinear Schrödinger equations

The third situation concerns nonlinear Schrödinger equations. More precisely we study the following nonlinear Schrödinger equation with Neumann boundary conditions

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = -\frac{\partial^2\psi}{\partial x^2}(t,x) + |\psi|^2\psi(t,x) - u(t)\mu(x)\psi(t,x), x\in(0,1), t\in(0,T), \\ \frac{\partial\psi}{\partial x}(t,0) = \frac{\partial\psi}{\partial x}(t,1) = 0. \end{cases} \tag{1.12}$$

It is a nonlinear control system where

- the state is ψ, with $\|\psi(t)\|_{L^2(0,1)}=1, \forall t\in[0,T]$,
- the control is the real valued function $u:[0,T]\to\mathbb{R}$.

We study its local controllability around the reference trajectory

$$(\psi_{ref}(t,x) := e^{-it}, u_{ref}(t)=0).$$

More precisely, we prove the following result.

Theorem 4. *Let $T>0$ and $\mu\in H^2(0,1)$ be such that*

$$\exists c>0 \text{ such that } \left|\int_0^1 \mu(x)\cos(k\pi x)dx\right| \geqslant \frac{c}{\max\{1,k\}^2}, \forall k\in\mathbb{N}. \tag{1.13}$$

There exists $\eta>0$ and a C^1-map

$$\Gamma: \mathcal{V}_T \to L^2((0,T),\mathbb{R})$$

where

$$\mathcal{V}_T := \{\psi_f \in \mathcal{S}\cap H^2(0,1); \psi_f'(0)=\psi_f'(1)=0 \text{ and } \|\psi_f - e^{-iT}\|_{H^2} < \eta\}$$

such that, for every $\psi_f\in\mathcal{V}_T$, the solution of (1.12) with initial condition

$$\psi(0,x)=1, \forall x\in(0,1) \tag{1.14}$$

and control $u:=\Gamma(\psi_f)$ is defined on $[0,T]$ and satisfies $\psi(T)=\psi_f$.

Remark 3. *The assumption (1.13) holds generically in $H^2(0,1)$. Indeed, integrations by part give*

$$\int_0^1 \mu(x)\cos(k\pi x)dx = \frac{1}{(k\pi)^2}\left((-1)^{k+1}\mu'(1)+\mu'(0)+\int_0^1 \mu''(x)\cos(k\pi x)dx\right), \forall k \in \mathbb{N}^*.$$

Other versions of this result, with higher regularities may be proved : the system is exactly controllable, locally around the reference trajectory
– in $H^4(0,1)$ with controls in $H_0^1(0,T)$,
– in $H^6(0,1)$ with controls in $H_0^2(0,T)$, etc.
Focusing nonlinearities may also be considered.

6.1.3.4 Nonlinear wave equations

The third situation concerns nonlinear wave equations. More precisely we study the following wave equation with Neumann boundary conditions

$$\begin{cases} w_{tt} = w_{xx} + f(w, w_t) + u(t)\mu(x)(w + w_t), x \in (0,1), t \in (0,T), \\ w_x(t,0) = w_x(t,1) = 0, \end{cases} \tag{1.15}$$

where f is an appropriate nonlinearity, that satisfies, in particular, $f(1,0) = 0$. It is a nonlinear control system where
– the state is (w, w_t),
– the control is the real valued function $u : [0,T] \to \mathbb{R}$.
We study its exact controllability, locally around the reference trajectory

$$(w_{ref}(t,x) = 1, u_{ref}(t) = 0).$$

More precisely, we prove the following result.

Theorem 5. *Let $T > 2$, $\mu \in H^2((0,1),\mathbb{R})$ be such that (1.13) holds and $f \in C^3(\mathbb{R}^2,\mathbb{R})$ be such that $f(1,0) = 0$ and $\nabla f(1,0) = 0$. There exists $\eta > 0$ and a C^1-map*

$$\Gamma : \mathcal{V}_T \to L^2((0,T),\mathbb{R})$$

where

$$\mathcal{V}_T := \{(w_f, \dot{w}_f) \in H^3 \times H^2((0,1),\mathbb{R});\ \ w_f'(0) = w_f'(1) = \dot{w}_f'(0) = \dot{w}_f'(1) = 0 \\ \text{and } \|w_f - 1\|_{H^3} + \|\dot{w}_f\|_{H^2} < \eta\}$$

such that $\Gamma(1,0) = 0$ and for every $(w_f, \dot{w}_f) \in \mathcal{V}_T$, the solution of (1.15) with initial condition

$$(w, w_t)(0,x) = (1,0), \forall x \in (0,1) \tag{1.16}$$

and control $u := \Gamma(w_f, \dot{w}_f)$ is defined on $[0,T]$ and satisfies $(w, w_t)(T) = (w_f, \dot{w}_f)$.

Other versions of this result, with higher regularities may be proved : the system is exactly controllable, locally around the reference trajectory
– in $H^4 \times H^3(0,1)$ with controls in $H_0^1(0,T)$,
– in $H^5 \times H^4(0,1)$ with controls in $H_0^2(0,T)$, etc.

6.1.4 A brief bibliography

6.1.4.1 A previous negative result

First, let us recall an important negative controllability result, for the equation (1.1), proved by Turinici [62]. It is a corollary of a more general result due to Ball, Marsden and Slemrod [10].

Proposition 1. *Let $\psi_0 \in \mathcal{S} \cap H^2_{(0)}((0,1),\mathbb{C})$ and $U[T;u,\psi_0]$ be the value at time T of the solution of (1.1) with initial condition $\psi(0)=\psi_0$. The set of attainable states from ψ_0,*

$$\{U[T;u,\psi_0]; T>0, u \in L^2((0,T),\mathbb{R})\}$$

has an empty interior in $\mathcal{S} \cap H^2_{(0)}((0,1),\mathbb{C})$. Thus (1.1) is not controllable in $\mathcal{S} \cap H^2_{(0)}((0,1),\mathbb{C})$ with controls in $L^2_{loc}([0,+\infty),\mathbb{R})$.

Proposition 1 is a rather weak negative controllability result, because it does not prevent from positive controllability results, in different spaces. This had already been emphasized for the particular cases studied in [14, 15, 17], in which the reachable set is proved to contain $H^7_{(0)}$ or $H^{5+}_{(0)}$. In this article, we prove that the reachable set (at least locally, with small controls in $L^2((0,T),\mathbb{R})$), coincides with $\mathcal{S} \cap H^3_{(0)}$, (which has, indeed, an empty interior in $\mathcal{S} \cap H^2_{(0)}$). Therefore, sometimes, Ball, Marsden and Slemrod's negative result is only due to an 'unfortunate' choice of functional spaces, that does not allow the controllability. It may not be due to a deep non controllability (such as, for example, when a subsystem evolves independently of the control).

6.1.4.2 Iterated Lie brackets

Now, let us quote some articles about the controllability of quantum systems.

First, the controllability of *finite dimensional* quantum systems (i.e. modelled by an ordinary differential equation) is well understood. Let us consider the quantum system

$$i\frac{dX}{dt} = H_0 X + u(t) H_1 X, \tag{1.17}$$

where $X \in \mathbb{C}^n$ is the state, H_0, H_1 are $n*n$ hermitian matrices, and $t \mapsto u(t) \in \mathbb{R}$ is the control. The controllability of (1.17) is linked to the rank of the Lie algebra spanned by H_0 and H_1 (see for instance [5] by Albertini and D'Alessandro, [7] by Altafini, [27] by Brockett, see also [3] by Agrachev and Sachkov, [33] by Coron for a more general discussion).

In *infinite dimension*, there are cases where the iterated Lie brackets provide the right intuition. For instance, it holds for the non controllability of the harmonic oscillator (see [50] by Mirrahimi and Rouchon). However, the Lie brackets are often less powerful in infinite dimension than in finite dimension. It is precisely the case of our system. Indeed, let us define the operators

$$\begin{array}{ll} D(f_0) := H^2 \cap H^1_0(0,1), & f_0(\psi) := -\psi'', \\ D(f_1) := L^2(0,1), & f_1(\psi) := x^2\psi, \end{array}$$

which correspond to $\mu(x) = x^2$. Let us compute the iterated Lie brackets at the point $\varphi_1(x) = \sqrt{2}\sin(\pi x)$. Since $\varphi_1 \in D(f_0)$, we can compute

$$\begin{aligned}[f_0, f_1](\varphi_1) &= -4x\varphi_1' - 2\varphi_1,\\ [f_1, [f_0, f_1]](\psi) &= 8x^2\varphi_1 = 8f_1(\varphi_1).\end{aligned}$$

Notice that $[f_0, f_1](\varphi_1)$ does not belong to $D(f_0)$ because $[f_0, f_1](\varphi_1)(1) = 4\sqrt{2}\pi \neq 0$. Thus, in order to give a sense to the Lie bracket $[f_0, [f_0, f_1]]$, one needs to extend the definition of f_0 to functions that do not vanish at $x = 0, 1$. A natural choice is

$$f_0(\psi) := -\psi'' + \psi(0)\delta_0' - \psi(1)\delta_1' \tag{1.18}$$

because, with this choice, we have

$$\langle f_0(\psi), \widetilde{\psi}\rangle = \langle \psi, f_0(\widetilde{\psi})\rangle, \forall \psi \in D(f_0), \forall \widetilde{\psi} \in H^2(0,1),$$

in the sense

$$-\int_0^1 \psi''(x)\widetilde{\psi}(x)dx = -\int_0^1 \psi(x)\widetilde{\psi}''(x)dx - \psi'(1)\widetilde{\psi}(1) + \psi'(0)\widetilde{\psi}(0).$$

With the definition (1.18), we get

$$[f_0, [f_0, f_1]](\psi) = -8f_0(\psi) + 4\psi'(1)\delta_1'$$

But then, again, $[f_0, [f_0, [f_0, f_1]]]$ is not well defined. Moreover, even if we could give a sense to any iterated Lie bracket, because of the presence of Dirac masses, it would not be clear which space the Lie algebra should generate in case of local controllability. Therefore, the way the Lie algebra rank condition could be used directly in infinite dimension is not clear (see also [33] for the same discussion on other examples). This is why we develop completely analytic methods in this article.

Finally, let us quote important articles about the controllability of PDEs, in which positive results are proved by applying geometric control methods to the (finite dimensional) Galerkin approximations of the equation. In [4] by Sarychev and Agrachev and [60] by Shirikyan, the authors prove exact controllability results for dissipative equations. In [30], by Boscain, Chambrion, Mason and Sigalotti, the authors prove the approximate controllability in L^2, for bilinear Schrödinger equations such as (1.1).

We also refer to the following works about the controllability of finite dimensional quantum systems [2, 21, 22, 23, 24, 25, 26], by Agrachev, Boscain, Chambrion, Charlot, Gauthier, Guérin, Jauslin and Mason, [41] by Khaneja, Glaser and Brockett, [54] by Ramakrishna, Salapaka, Dahleh, Rabitz, [61] by Sussmann and Jurdjevic, [63] by Turinici and Rabitz. Let us also mention [51] by Mirrahimi, Rouchon, Turinici and [18] for explicit feedback controls, inspired by Lyapunov technics.

6.1.4.3 Controllability results for Schrödinger and wave equations

The controllability of Schrödinger equations with distributed and boundary controls, that act linearly on the state, is studied since a long time.

For linear equations, the controllability is equivalent to an observability inequality that may be proved with different technics : multiplier methods (see [37] by Fabre, [48] by Machtyngier), microlocal analysis (see [47] by Lebeau, [28] by Burq), Carleman estimates (see [43, 44] by Lasiecka, Triggiani, Zhang), or number theory (see [55] by Ramdani, Takahashi, Tenenbaum and Tucsnak).

For nonlinear equations, we refer to [34] by Dehman, Gérard, Lebeau, [42] by Lange Teismann, [45, 46] by Laurent, [57] by Rosier, Zhang.

6.1.4.4 Other results about bilinear quantum systems

The study of the controllability of Schrödinger PDEs with bilinear controls started later.

The first result is negative and it is due to Turinici (see [62] and Proposition 1). It is a corollary of a more general result by Ball, Marsden and Slemrod [10]. Because of this noncontrollability result, such equations have been considered as non controllable for a long time. However, important progress have been made in the last years and this question is now better understood (see section 6.1.4.1). Let us also mention that this negative result has been adapted to nonlinear Schrödinger equations in [40] by Ilner, Lange and Teismann.

Concerning exact controllability issues, local results for 1D models have been proved in [14, 15] by Beauchard ; almost global results have been proved in [17], by Coron and Beauchard. In [32], Coron proved that a positive minimal time was required for the local controllability of the 1D model (1.1) with $\mu(x) = x - 1/2$.

Now, let us quote some approximate controllability results. In [20] Mirrahimi and Beauchard proved the global approximate controllability, in infinite time, for a 1D model and in [49] Mirrahimi proved a similar result for equations involving a continuous spectrum. Approximate controllability, in finite time, has been proved for particular models by Boscain and Adami in [1], by using adiabatic theory and intersection of the eigenvalues in the space of controls. Approximate controllability, in finite time, for more general models, have been studied by 3 teams, with different tools : by Boscain, Chambrion, Mason, Sigalotti [30], with geometric control methods ; by Nersesyan [53, 52] with feedback controls and variational methods ; and by Ervedoza and Puel [36] thanks to a simplified model.

Let us emphasize that the local exact controllability result of this article and the global approximate controllability of [53, 52] can be put together in order to get the global exact controllability of 1D models (see [52]).

Optimal control techniques have also been investigated for Schrödinger equations with a non linearity of Hartee type in [11, 12] by Baudouin, Kavian, Puel and in [35] by Cances, Le Bris, Pilot. An algorithm for the computation of such optimal controls is studied in [13] by Baudouin and Salomon.

6.1.5 Structure of this article

This article is organized as follows.

Section 6.2 aims at proving the controllability for the linear Schrödinger equations. The Subsections 6.2.1, 6.2.2, 6.2.3 and 6.2.4 are dedicated to the different steps of the proof of Theorem 1, where the equation is posed on a bounded interval. The Subsection 6.2.5 is dedicated to the proof of the same result with higher regularities, i.e. Theorem 2. The Subsection 6.2.6 is dedicated to the Schrödinger equation for radial data on the three dimensional ball, i.e. the proof of Theorem 3.

In Section 6.3, we prove Theorem 4 concerning the nonlinear Schrödinger equation (1.12).

In Section 6.4, we prove Theorem 5 concerning the nonlinear wave equation (1.15).

Finally, in Section 6.5, we state some conclusions, open problems and perspectives.

6.1.6 Notations

Let us introduce some conventions and notations that are valid in all this article. Unless otherwise specified, the functions considered are complex valued and, for example, we write $H^1_0(0,1)$ for $H^1_0((0,1),\mathbb{C})$. When the functions considered are real valued, we specify it and we write, for example, $L^2((0,T),\mathbb{R})$. We use the spaces

$$h^s(\mathbb{N}^*,\mathbb{C}) := \left\{ a = (a_k)_{k\in\mathbb{N}^*} \in \mathbb{C}^{\mathbb{N}^*}; \sum_{k=1}^{\infty} |k^s a_k|^2 < +\infty \right\}$$

equipped with the norm

$$\|a\|_{h^s} := \Big(\sum_{k=1}^{\infty} |k^s a_k|^2 \Big)^{1/2}.$$

The same letter C denotes a positive constant, that can change from one line to another one. If $(X, \|.\|)$ is a normed vector space and $R > 0$, $B_R[X]$ denotes the open ball $\{x \in X; \|x\| < R\}$ and $\overline{B}_R[X]$ denotes the closed ball $\{x \in X; \|x\| \leqslant R\}$.

6.2 Linear Schrödinger equations

The goal of this section is the proof of controllability results for linear Schrödinger equations, with bilinear controls.

The Subsections 6.2.1, 6.2.2, 6.2.3 and 6.2.4 are dedicated to the different steps of the proof of Theorem 1, where the equation is posed on a bounded interval. In Subsection 6.2.1, we prove existence, uniqueness, regularity results and bounds on the solution of the Cauchy problem (1.1), (1.6). In Subsection 6.2.2, we prove the C^1-regularity of the end-point map associated to our control problem. In Subsection 6.2.3, we prove the controllability of the linearized system around the ground state. Finally, in Subsection 6.2.4, we deduce Theorem 1 by applying the inverse mapping theorem.

The Subsection 6.2.5 is dedicated to the proof of the same result with higher regularities, i.e. Theorem 2.

The Subsection 6.2.6 is dedicated to the Schrödinger equation for radial data on the three dimensional ball, i.e. the proof of Theorem 3.

In all this section (except in Subsection 6.2.6), the operator A is defined by (1.2), the spaces $H^s_{(0)}(0,1)$ are defined by (1.4) and e^{-iAt} denotes the group of isometries of $H^s_{(0)}(0,1)$, $\forall s \geqslant 0$ generated by $-iA$,

$$e^{-iAt}\varphi = \sum_{k=1}^{\infty} \langle \varphi, \varphi_k \rangle e^{-i\lambda_k t} \varphi_k, \forall \varphi \in L^2(0,1). \tag{2.19}$$

We use few classical results concerning trigonometric moment problems that are recalled in Appendix B.

6.2.1 Well posedness of the Cauchy problem

This subsection is dedicated to the statement of existence, uniqueness, regularity results, and bounds for the weak solutions of the Cauchy problem

$$\begin{cases} i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} - u(t)\mu(x)\psi - f(t,x), \ x \in (0,1), t \in \mathbb{R}_+, \\ \psi(t,0) = \psi(t,1) = 0, \\ \psi(0,x) = \psi_0(x). \end{cases} \tag{2.20}$$

Proposition 2. *Let $\mu \in H^3((0,1),\mathbb{R})$, $T > 0$, $\psi_0 \in H^3_{(0)}(0,1)$, $f \in L^2((0,T), H^3 \cap H^1_0)$ and $u \in L^2((0,T),\mathbb{R})$. There exists a unique weak solution of (2.20), i.e. a function $\psi \in C^0([0,T], H^3_{(0)})$ such that the following equality holds in $H^3_{(0)}(0,1)$ for every $t \in [0,T]$,*

$$\psi(t) = e^{-iAt}\psi_0 + i\int_0^t e^{-iA(t-\tau)}[u(\tau)\mu\psi(\tau) + f(\tau)]d\tau. \tag{2.21}$$

Moreover, for every $R > 0$, there exists $C = C(T,\mu,R) > 0$ such that, if $\|u\|_{L^2(0,T)} < R$, then this weak solution satisfies

$$\|\psi\|_{C^0([0,T],H^3_{(0)})} \leqslant C\Big(\|\psi_0\|_{H^3_{(0)}} + \|f\|_{L^2((0,T),H^3 \cap H^1_0)}\Big). \tag{2.22}$$

If $f \equiv 0$ then

$$\|\psi(t)\|_{L^2(0,1)} = \|\psi_0\|_{L^2(0,1)}, \forall t \in [0,T]. \tag{2.23}$$

The main difficulty of the proof of this result is that $f(s)$ is not assumed to belong to $H^3_{(0)}(0,1)$ (i.e. $f''(s,.)$ may not vanish at $x=0$ and $x=1$), and μ is not assumed to satisfy $\mu'(0) = \mu'(1) = 0$ (and thus the operator $\varphi \mapsto \mu\varphi$ does not preserve $H^3_{(0)}(0,1)$ because for $\varphi \in H^3_{(0)}(0,1)$, we have $(\mu\varphi)'' = 2\mu'\varphi'$ at $x=0$ and $x=1$). The argument for proving Proposition 2 comes from the following Lemma.

Lemma 1. *Let $T > 0$ and $f \in L^2((0,T), H^3 \cap H^1_0)$. The function $G : t \mapsto \int_0^t e^{iAs} f(s)ds$ belongs to $C^0([0,T], H^3_{(0)})$, moreover*

$$\|G\|_{L^\infty((0,T),H^3_{(0)})} \leqslant c_1(T)\|f\|_{L^2((0,T),H^3 \cap H^1_0)} \tag{2.24}$$

where the constants $c_1(T)$ are uniformly bounded for T lying in bounded intervals.

Proof of Lemma 1 : By definition, we have

$$G(t)=\sum_{k=1}^{\infty}\left(\int_0^t \langle f(s),\varphi_k\rangle e^{i\lambda_k s}ds\right)\varphi_k.$$

For almost every $s\in(0,T)$, $f(s)\in H^3\cap H^1_0$, and we have

$$\begin{aligned}\langle f(s),\varphi_k\rangle &= \tfrac{1}{\lambda_k}\langle Af(s),\varphi_k\rangle\\ &= -\tfrac{\sqrt{2}}{\lambda_k}\int_0^1 f''(s,x)\sin(k\pi x)dx\\ &= \tfrac{\sqrt{2}}{(k\pi)^3}\Big((-1)^k f''(s,1)-f''(s,0)\Big)-\tfrac{\sqrt{2}}{(k\pi)^3}\int_0^1 f'''(s,x)\cos(k\pi x)dx.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|G(t)\|_{H^3_{(0)}} &= \Big\|\int_0^t\langle f(s),\varphi_k\rangle e^{i\lambda_k s}ds\Big\|_{h^3}\\ &\leqslant \tfrac{\sqrt{2}}{\pi^3}\Big(\Big\|\int_0^t f''(s,1)e^{i\lambda_k s}ds\Big\|_{l^2}+\Big\|\int_0^t f''(s,0)e^{i\lambda_k s}ds\Big\|_{l^2}\Big)\\ &\quad+\tfrac{1}{\pi^3}\Big\|\int_0^t\langle f'''(s),\sqrt{2}\cos(k\pi x)\rangle e^{i\lambda_k s}ds\Big\|_{l^2}.\end{aligned}$$

The family $(\sqrt{2}\cos(k\pi x))_{k\in\mathbb{N}^*}$ is orthonormal in $L^2(0,1)$, thus

$$\begin{aligned}\Big\|\int_0^t\langle f'''(s),\sqrt{2}\cos(k\pi x)\rangle e^{i\lambda_k s}ds\Big\|_{l^2} &= \left(\sum_{k=1}^{\infty}\Big|\int_0^t\langle f'''(s),\sqrt{2}\cos(k\pi x)\rangle e^{i\lambda_k s}ds\Big|^2\right)^{1/2}\\ &\leqslant \left(\sum_{k=1}^{\infty} t\int_0^t|\langle f'''(s),\sqrt{2}\cos(k\pi x)\rangle|^2 ds\right)^{1/2}\\ &\leqslant \sqrt{t}\left(\int_0^t\|f'''(s)\|^2_{L^2}ds\right)^{1/2}\\ &\leqslant \sqrt{t}\|f\|_{L^2((0,t),H^3)}.\end{aligned}$$

Thanks to Corollary 4 (in Appendix B), we get

$$\begin{aligned}\|G(t)\|_{H^3_{(0)}} &\leqslant \tfrac{\sqrt{2}C(t)}{\pi^3}\Big(\|f''(.,0)\|_{L^2(0,t)}+\|f''(.,1)\|_{L^2(0,t)}\Big)+\tfrac{\sqrt{t}}{\pi^3}\|f\|_{L^2((0,t),H^3)}\\ &\leqslant c_1(t)\|f\|_{L^2((0,t),H^3\cap H^1_0)}\end{aligned}$$

where $c_1(t)$ is uniformly bounded for t lying in bounded intervals. This bound shows that $G(t)$ belongs to $H^3_{(0)}(0,1)$ for every $t\in[0,T]$ and that the map $t\in[0,T]\mapsto G(t)\in H^3_{(0)}$ is continuous at $t=0$ (because $c_1(t)$ is uniformly bounded when $t\to 0$ and $\|f\|_{L^2((0,t),H^3\cap H^1_0)}\to 0$ when $t\to 0$, thanks to the dominated convergence theorem). The continuity of G at any $t\in(0,T)$ can be proved similarly. $\Box$

Proof of Proposition 2 : Let $\mu\in H^3((0,1),\mathbb{R})$, $T>0$, $\psi_0\in H^3_{(0)}(0,1)$, $f\in L^2((0,T),H^3\cap H^1_0)$ and $u\in L^2((0,T),\mathbb{R})$. We consider the map

$$\begin{array}{rccc}F: & C^0([0,T],H^3_{(0)}) & \to & C^0([0,T],H^3_{(0)})\\ & \psi & \mapsto & \xi\end{array}$$

where $\xi:=F(\psi)$ is defined by

$$\xi(t):=e^{-iAt}\psi_0+i\int_0^t e^{-iA(t-s)}\Big(u(s)\mu\psi(s)+f(s)\Big)ds,\forall t\in[0,T]. \tag{2.25}$$

We have assumed that $f \in L^2((0,T), H^3 \cap H^1_0)$ and $u \in L^2(0,T)$, thus, for every $\psi \in C^0([0,T], H^3_{(0)})$, the map $(u\mu\psi + f)$ belongs to $L^2((0,T), H^3 \cap H^1_0)$ and Lemma 1 ensures that F takes values in $C^0([0,T], H^3_{(0)})$. We have also used that in dimension 1, H^3 is an algebra.

Thanks to (2.24), we get, for every $t \in [0,T]$,

$$\begin{aligned}\|F(\psi_1)(t) - F(\psi_2)(t)\|_{H^3_{(0)}} &= \left\| \int_0^t e^{iAs} u(s)\mu(\psi_1 - \psi_2)(s)ds \right\|_{H^3_{(0)}} \\ &\leqslant c_1(t)\|u\mu(\psi_1 - \psi_2)\|_{L^2((0,t),H^3\cap H^1_0)} \\ &\leqslant c_1(t)\|u\|_{L^2(0,t)}\|\mu(\psi_1 - \psi_2)\|_{L^\infty((0,t),H^3\cap H^1_0)} \\ &\leqslant c_1(t)\|u\|_{L^2(0,t)}C(\mu)\|\psi_1 - \psi_2\|_{L^\infty((0,t),H^3_{(0)})}\end{aligned}$$

thus

$$\|F(\psi_1) - F(\psi_2)\|_{L^\infty((0,T),H^3_{(0)})} \leqslant c_2(T,\mu)\|u\|_{L^2(0,T)}\|\psi_1 - \psi_2\|_{L^\infty((0,T),H^3_{(0)})}. \qquad (2.26)$$

If $\|u\|_{L^2(0,T)}$ is small enough, then F is a contraction. Thanks to the Banach fixed point theorem, there exists $\psi \in C^0([0,T], H^3_{(0)})$ such that $F(\psi) = \psi$. The previous arguments show that, for this fixed point, we have

$$\|\psi\|_{L^\infty((0,T),H^3_{(0)})} \leqslant \|\psi_0\|_{H^3_{(0)}} + c_2(T,\mu)\|u\|_{L^2(0,T)}\|\psi\|_{L^\infty((0,T),H^3_{(0)})} + c_1(T)\|f\|_{L^2((0,T),H^3\cap H^1_0)}.$$

Thus, if $c_2(T,\mu)\|u\|_{L^2(0,T)} \leqslant 1/2$, then, we get (2.22).

We have proved Proposition 2 when $\|u\|_{L^2(0,T)}$ is small enough. If it is not the case, one may consider $0 = T_0 < T_1 < ... < T_N = T$ such that $\|u\|_{L^2(T_j,T_{j+1})}$ is small and apply the previous result on $[T_0,T_1], ..., [T_{N-1},T_N]$ in order to get the conclusion. Since our constant $c_1(t)$ is uniform on bounded sets, we easily get that N only depends on R, so that the constant in Proposition 2 does only depend on T, μ and R as claimed.

Now, let us prove that (2.23) holds when $f = 0$. Classical arguments allow to prove that, when $u \in C^0([0,T],\mathbb{R})$, then $\psi \in C^1([0,T], L^2)$ and the first equality of (1.1) holds in L^2 for every $t \in [0,T]$. Thus, when $u \in C^0([0,T],\mathbb{R})$, we can take the L^2-scalar product of this equation with ψ; and the imaginary part of the resulting equality gives

$$\frac{d}{dt}\|\psi(t)\|^2_{L^2} = 0.$$

Thus, we have (2.23) when $u \in C^0([0,T],\mathbb{R})$. A density argument allows to prove (2.23) when u only belongs to $L^2((0,T),\mathbb{R})$. $\square$

6.2.2 C^1-regularity of the end-point map

For $T > 0$ we introduce the tangent space of $\mathcal{S}$ at $\psi_1(T)$

$$V_T := \{\xi \in L^2(0,1); \Re\langle \xi, \psi_1(T)\rangle = 0\}$$

and the orthogonal projection

$$P_T : L^2(0,1) \to V_T.$$

Proposition 2 allows to consider the map

$$\begin{array}{rccc} \Theta_T: & L^2((0,T),\mathbb{R}) & \to & V_T \cap H^3_{(0)}(0,1) \\ & u & \mapsto & P_T[\psi(T)] \end{array} \tag{2.27}$$

where ψ is the solution of (1.1), (1.6). The goal of this section is the proof of the following result.

Proposition 3. *Let $T>0$ and $\mu \in H^3((0,1),\mathbb{R})$. The map Θ_T defined by (2.27) is C^1. Moreover, for every $u,v \in L^2((0,T),\mathbb{R})$, we have*

$$d\Theta_T(u).v = P_T[\Psi(T)] \tag{2.28}$$

where Ψ is the weak solution of the linearized system

$$\begin{cases} i\frac{\partial \Psi}{\partial t} = -\Psi'' - u(t)\mu(x)\Psi - v(t)\mu(x)\psi, x \in (0,1), t \in (0,T), \\ \Psi(t,0) = \Psi(t,1) = 0, \\ \Psi(0,x) = 0, \end{cases} \tag{2.29}$$

and ψ is the solution of (1.1),(1.6).

Proof of Proposition 3 : Let $T>0$, $\mu \in H^3((0,1),\mathbb{R})$ and $u \in L^2((0,T),\mathbb{R})$. First, let us emphasize that the linear map $v \mapsto \Psi(T)$ is continuous from $L^2((0,T),\mathbb{R})$ to $H^3_{(0)}(0,1)$ thanks to Proposition 2.

First step : We prove that Θ_T is differentiable and that (2.28) holds. Let ψ be the weak solution of (1.1),(1.6), Ψ solution of (2.29) and $\widetilde{\psi}$ solution of

$$\begin{cases} i\frac{\partial \widetilde{\psi}}{\partial t} = -\widetilde{\psi}'' - (u+v)(t)\mu(x)\widetilde{\psi}, x \in (0,1), t \in (0,T), \\ \widetilde{\psi}(t,0) = \widetilde{\psi}(t,1) = 0, \\ \widetilde{\psi}(0,x) = \varphi_1. \end{cases} \tag{2.30}$$

Then $\Delta := \widetilde{\psi} - \psi - \Psi$ is the weak solution of

$$\begin{cases} i\frac{\partial \Delta}{\partial t} = -\Delta'' - (u+v)(t)\mu(x)\Delta - v(t)\mu\Psi, x \in (0,1), t \in (0,T), \\ \Delta(t,0) = \Delta(t,1) = 0, \\ \Delta(0,x) = 0. \end{cases} \tag{2.31}$$

Let us prove that

$$\|\Delta\|_{C^0([0,T],H^3_{(0)})} = o(\|v\|_{L^2}) \text{ when } \|v\|_{L^2} \to 0, \tag{2.32}$$

which gives the conclusion. Let $R>0$ be such that $\|u\|_{L^2(0,T)} < R$ and $\|u+v\|_{L^2(0,T)} < R$. Thanks to Proposition 2, there exists $C_j = C_j(T,\mu,R) > 0$ for $j=0,1$ such that

$$\|\Delta\|_{C^0([0,T],H^3_{(0)})} \leqslant C_0\|v\mu\Psi\|_{L^2((0,T),H^3\cap H^1_0)} \leqslant C_1\|v\|_{L^2}\|\Psi\|_{C^0([0,T],H^3_{(0)})},$$

$$\begin{aligned} \|\Psi\|_{C^0([0,T],H^3_{(0)})} &\leqslant C_0\|v\mu\psi\|_{L^2((0,T),H^3\cap H^1_0)} \\ &\leqslant C_1\|v\|_{L^2}\|\psi\|_{C^0([0,T],H^3_{(0)})} \\ &\leqslant C_0C_1\|v\|_{L^2}\|\varphi_1\|_{H^3_{(0)}}, \end{aligned}$$

which proves (2.32).

Second step : We prove that $d\Theta_T$ is continuous. Actually, we prove that this map is locally Lipschitz. Let $u, \widetilde{u} \in L^2((0,T),\mathbb{R})$ and $v \in L^2((0,T),\mathbb{R})$. Let ψ be the solution of (1.1),(1.6), Ψ solution of (2.29) and $\widetilde{\psi}$, $\widetilde{\Psi}$ solution of

$$\left\{\begin{array}{l} i\frac{\partial\widetilde{\psi}}{\partial t} = -\widetilde{\psi}'' - \widetilde{u}(t)\mu(x)\widetilde{\psi},\\ \widetilde{\psi}(t,0)=\widetilde{\psi}(t,1)=0,\\ \widetilde{\psi}(0,x)=\varphi_1, \end{array}\right. \qquad \left\{\begin{array}{l} i\frac{\partial\widetilde{\Psi}}{\partial t} = -\widetilde{\Psi}'' - \widetilde{u}(t)\mu(x)\widetilde{\Psi} - v(t)\mu(x)\widetilde{\psi},\\ \widetilde{\Psi}(t,0)=\widetilde{\Psi}(t,1)=0,\\ \widetilde{\Psi}(0,x)=0, \end{array}\right.$$

We have

$$[d\Theta_T(u) - d\Theta_T(\widetilde{u})].v = P_T[\Psi(T) - \widetilde{\Psi}(T)] = P_T[\Xi(T)]$$

where Ξ is the weak solution of

$$\left\{\begin{array}{l} i\frac{\partial\Xi}{\partial t} = -\frac{\partial^2\Xi}{\partial x^2} - u(t)\mu\Xi - (u-\widetilde{u})\mu\widetilde{\Psi} - v\mu(\psi-\widetilde{\psi}),\\ \Xi(t,0)=\Xi(t,1)=0,\\ \Xi(0)=0. \end{array}\right.$$

Let $R>0$ be such that $\|u\|_{L^2(0,T)} < R$, $\|\widetilde{u}\|_{L^2(0,T)} < R$. Let us prove that

$$\|\Xi\|_{C^0([0,T],H^3_{(0)})} \leqslant \mathcal{C}\|v\|_{L^2}\|u-\widetilde{u}\|_{L^2}$$

where $\mathcal{C} = \mathcal{C}(T,\mu,R) > 0$, which gives the conclusion. Thanks to Proposition 2, we have

$$\begin{aligned} \|\Xi\|_{C^0([0,T],H^3_{(0)})} &\leqslant C_2\|(u-\widetilde{u})\mu\widetilde{\Psi} + v\mu(\psi-\widetilde{\psi})\|_{L^2((0,T),H^3\cap H^1_0)}\\ &\leqslant C_3\Big(\|u-\widetilde{u}\|_{L^2}\|\widetilde{\Psi}\|_{C^0([0,T],H^3_{(0)})} + \|v\|_{L^2}\|\psi-\widetilde{\psi}\|_{C^0([0,T],H^3_{(0)})}\Big)\\ &\leqslant C_4\Big(\|u-\widetilde{u}\|_{L^2}\|v\mu\widetilde{\psi}\|_{L^2((0,T),H^3\cap H^1_0)} + \|v\|_{L^2}\|(\widetilde{u}-u)\mu\widetilde{\psi}\|_{L^2((0,T),H^3\cap H^1_0)}\Big)\\ &\leqslant C_5\Big(\|u-\widetilde{u}\|_{L^2}\|v\|_{L^2}\|\widetilde{\psi}\|_{C^0([0,T],H^3_{(0)})} + \|v\|_{L^2}\|\widetilde{u}-u\|_{L^2}\|\widetilde{\psi}\|_{C^0([0,T],H^3_{(0)})}\Big)\\ &\leqslant C_6\|u-\widetilde{u}\|_{L^2}\|v\|_{L^2}, \end{aligned}$$

where $C_j = C_j(T,\mu,R) > 0$ for $j = 2,...,6$. $\square$

6.2.3 Controllability of the linearized system

The goal of this section is the proof of the following result.

Proposition 4. *Let $T>0$ and $\mu \in H^3((0,1),\mathbb{R})$ be such that (1.5) holds. The linear map $d\Theta_T(0) : L^2((0,T),\mathbb{R}) \rightarrow V_T \cap H^3_{(0)}(0,1)$ has a continuous right inverse $d\Theta_T(0)^{-1}$: $V_T \cap H^3_{(0)}(0,1) \rightarrow L^2((0,T),\mathbb{R})$.*

The proof of Proposition 4 relies on an Ingham inequality, due to Haraux (see [38] and Appendix B).

Proof of Proposition 4 : We have $d\Theta_T(0).v = \Psi(T)$ where

$$\left\{\begin{array}{l} i\frac{\partial\Psi}{\partial t} = -\Psi'' - v(t)\mu\psi_1,\\ \Psi(t,0)=\Psi(t,1)=0,\\ \Psi(0)=0, \end{array}\right. \tag{2.33}$$

thus

$$\Psi(T)=\sum_{k=1}^{\infty} i\langle\mu\varphi_1,\varphi_k\rangle\left(\int_0^T v(t)e^{i(\lambda_k-\lambda_1)t}dt\right)e^{-i\lambda_k T}\varphi_k.$$

Let $\Psi_f\in V_T\cap H^3_{(0)}(0,1)$. If Ψ is the solution of (2.33) for some $v\in L^2((0,T),\mathbb{R})$, then, the equality $\Psi(T)=\Psi_f$ is equivalent to the trigonometric moment problem

$$\int_0^T v(t)e^{i(\lambda_k-\lambda_1)t}dt = d_{k-1}(\Psi_f) := \frac{\langle\Psi_f,\varphi_k\rangle e^{i\lambda_k T}}{i\langle\mu\varphi_1,\varphi_k\rangle}, \forall k\in\mathbb{N}^*. \tag{2.34}$$

Now, we apply Corollary 1 (see Appendix B) with $\omega_k:=\lambda_{k+1}-\lambda_1, \forall k\in\mathbb{N}$, and we get the conclusion with

$$d\Theta_T(0)^{-1}(\Psi_f):=L[d(\Psi_f)],$$

where $d(\Psi_f):=(d_k(\Psi_f))_{k\in\mathbb{N}}$. Indeed, for $\Psi_f\in V_T\cap H^3_{(0)}(0,1)$, the sequence $d(\Psi_f)$ belongs to $l^2_r(\mathbb{N},\mathbb{C})$ thanks to the assumption (1.5). $\square$

6.2.4 Proof of Theorem 1

Let $T>0$ and $\mu\in H^3((0,1),\mathbb{R})$ be such that (1.5) holds. Let $R_1>0$ and $\delta_1>0$ be such that,

$$\forall u\in B_{R_1}[L^2((0,T),\mathbb{R})], \text{ the solution of } (1.1),(1.6) \text{ satisfies } \Re\langle\psi(T),\psi_1(T)\rangle>0,$$

(see Proposition 2) and

$$\forall\psi_f\in\mathcal{S}\cap H^3_{(0)}(0,1) \text{ with } \|\psi_f-\psi_1(T)\|_{H^3_{(0)}}<\delta_1, \text{ we have } \Re\langle\psi_f,\psi_1(T)\rangle>0.$$

The spaces $\overline{B}_{R_1}[L^2((0,T),\mathbb{R})]$ and $V_T\cap H^3_{(0)}(0,1)$ are Banach spaces. The map $\Theta_T:\overline{B}_{R_1}[L^2((0,T),\mathbb{R})]\to V_T\cap H^3_{(0)}(0,1)$ is C^1 (see Proposition 3), its differential at 0 has a continuous right inverse $d\Theta_T(0)^{-1}:V_T\cap H^3_{(0)}(0,1)\to L^2((0,T),\mathbb{R})$ (see Proposition 4). Thanks to the inverse mapping theorem, there exists $\delta\in(0,\delta_1)$ and a C^1 map

$$\Theta_T^{-1}:B_\delta[V_T\cap H^3_{(0)}(0,1)]\to\overline{B}_{R_1}[L^2((0,T),\mathbb{R})]$$

such that $\Theta_T(\Theta_T^{-1}(\widetilde{\psi_f}))=\widetilde{\psi_f}$ for every $\widetilde{\psi_f}\in B_\delta[V_T\cap H^3_{(0)}(0,1)]$.

For $\psi_f\in\mathcal{S}\cap H^3_{(0)}(0,1)$ with $\|\psi_f-\psi_1(T)\|_{H^3_{(0)}}<\delta$, we have $\|P_T\psi_f\|_{H^3_{(0)}}<\delta$, thus we can define

$$\Gamma(\psi_f)=:\Theta_T^{-1}[P_T\psi_f].$$

Thanks to the choice of R_1 and δ_1 we know that the solution of (1.1), (1.6) with $u=\Gamma(\psi_f)$ satisfies

$$\begin{aligned}\psi(T)&=P_T(\psi(T))+\sqrt{1-\|P_T\psi(T)\|^2_{L^2}}\psi_1(T)\\&=P_T(\psi_f)+\sqrt{1-\|P_T\psi_f\|^2_{L^2}}\psi_1(T)=\psi_f.\end{aligned}$$

6.2.5 Generalization to higher regularities

The goal of this section is the proof of Theorem 2. The first step of the proof consists in adapting Proposition 2.

Proposition 5. *Let $\mu \in H^5((0,1),\mathbb{R})$, $T>0$, $\psi_0 \in H^5_{(0)}(0,1)$, $f \in H^1_0((0,T), H^3 \cap H^1_0)$ and $u \in H^1_0((0,T),\mathbb{R})$. There exists a unique function $\psi \in C^1([0,T], H^3_{(0)})$ such that the equality (2.21) holds in $C^1([0,T],H^3_{(0)})$. Moreover, for every $R>0$ there exists $C=C(T,\mu,R)>0$ such that, if $\|u\|_{H^1_0(0,T)} < R$, then, this weak solution satisfies*

$$\|\psi\|_{C^1([0,T],H^3_{(0)})} \leqslant C\Big(\|\psi_0\|_{H^5_{(0)}} + \|f\|_{H^1((0,T),H^3\cap H^1_0)}\Big). \tag{2.35}$$

The proof of Proposition 5 is the same as the one of Proposition 2, except that we use the following Lemma, instead of Lemma 1.

Lemma 2. *Let $T>0$, $u_0 \in H^5 \cap H^3_{(0)}$ and $f \in H^1((0,T),H^3\cap H^1_0)$ be such that $-iAu_0 + f(0) \in H^3_{(0)}$. The function $G : t \mapsto e^{-iAt}u_0 + \int_0^t e^{-iA(t-s)}f(s)ds$ belongs to $C^1([0,T],H^3_{(0)})$, moreover*

$$\|G\|_{C^1([0,T],H^3_{(0)})} \leqslant c_1(T)\left(\|u_0\|_{H^3_{(0)}} + \|f\|_{H^1((0,T),H^3\cap H^1_0)} + \|-iAu_0+f(0)\|_{H^3_{(0)}}\right)$$

where the constants $c_1(T)$ are uniformly bounded for T lying in bounded intervals. We also have

$$\|-iAG(T)+f(T)\|_{H^3_{(0)}} \leqslant c_1(T)\left(\|u_0\|_{H^3_{(0)}} + \|f\|_{H^1((0,T),H^3\cap H^1_0)} + \|-iAu_0+f(0)\|_{H^3_{(0)}}\right).$$

Proof of Lemma 2 : We already know that $G \in C^0([0,T],H^3_{(0)})$. First let us write

$$G(t) = e^{-iAt}u_0 + \int_0^t e^{-iA\tau} f(t-\tau)d\tau.$$

Since $u_0 \in H^4_{(0)}$ and $f \in H^1((0,T),H^2_{(0)})$, we know that $G \in C^1([0,T],H^2_{(0)})$ and the following equality holds in $H^2_{(0)}$ for every $t\in[0,T]$,

$$\begin{aligned}\frac{\partial G}{\partial t}(t) &= -iAe^{-iAt}u_0 + e^{-iAt}f(0) + \int_0^t e^{-iA\tau}\frac{\partial f}{\partial t}(t-\tau)d\tau \\ &= e^{-iAt}\left[-iAu_0 + f(0)\right] + \int_0^t e^{-iA(t-s)}\frac{\partial f}{\partial t}(s)ds\end{aligned}$$

(the proof of this result involves classical technics). Thanks to this expression and Lemma 1, we get

$$\frac{\partial G}{\partial t} \in C^0([0,T],H^3_{(0)}).$$

Now, let us prove that $G \in C^1([0,T],H^3_{(0)})$, i.e. for every $t\in[0,T]$,

$$\Big\|\frac{G(t+h)-G(t)}{h} - \frac{\partial G}{\partial t}(t)\Big\|_{H^3_{(0)}} \to 0 \text{ when } h\to 0.$$

We have

$$\begin{aligned}\tfrac{G(t+h)-G(t)}{h}-\tfrac{\partial G}{\partial t}(t)= \;& e^{-iAt}\left[\tfrac{e^{-iAh}-Id}{h}u_0+iAu_0-f(0)\right]+\tfrac{1}{h}\int_t^{t+h}e^{-iA\tau}f(t+h-\tau)d\tau\\ &+\int_0^t e^{-iA\tau}\left[\tfrac{f(t+h-\tau)-f(t-\tau)}{h}-\tfrac{\partial f}{\partial t}(t-\tau)\right]d\tau.\end{aligned}\tag{2.36}$$

By applying Lemma 1, we see that the $H^3_{(0)}(0,1)$-norm of the term on the second line of the right hand side of (2.36) tends to zero when $h\to 0$ because $f\in H^1((0,T),H^3\cap H^1_0)$. Thanks to several changes of variables, the term on the first line of the right hand side of (2.36) may be decomposed in the following way

$$\begin{aligned}&e^{-iAt}\left[\tfrac{e^{-iAh}-Id}{h}\Big(u_0+iA^{-1}f(0)\Big)+iA\Big(u_0+iA^{-1}f(0)\Big)\right]\\ &+e^{-iAt}\tfrac{1}{h}\int_0^h e^{-iAs}\Big(f(h-s)-f(0)\Big)ds.\end{aligned}\tag{2.37}$$

The $H^3_{(0)}(0,1)$-norm of the first term of (2.37) tends to zero when $h\to 0$ because $u_0+iA^{-1}f(0)\in H^5_{(0)}(0,1)$. The $H^3_{(0)}(0,1)$-norm of the second term of (2.37) also tends to zero when $h\to 0$ because, thanks to Lemma 1 and Cauchy-Schwarz inequality, it is bounded by

$$\begin{aligned}&\left\|\int_0^h e^{iAs}\Big(\tfrac{f(s)-f(0)}{h}\Big)ds\right\|_{H^3_{(0)}}\leqslant c_1(h)\left\|\tfrac{f(.)-f(0)}{h}\right\|_{L^2((0,h),H^3\cap H^1_0)}\\ \leqslant\;& \tfrac{c_1(h)}{h}\left\|\int_0^{\cdot}\tfrac{\partial f}{\partial t}(\tau)d\tau\right\|_{L^2((0,h),H^3\cap H^1_0)}\leqslant \tfrac{c_1(h)\sqrt{h}}{h}\left\|\int_0^{\cdot}\tfrac{\partial f}{\partial t}(\tau)d\tau\right\|_{L^\infty((0,h),H^3\cap H^1_0)}\\ \leqslant\;& c_1(h)\left\|\tfrac{\partial f}{\partial t}\right\|_{L^2((0,h),H^3\cap H^1_0)}.\end{aligned}$$

The estimate (2.24) of Lemma 1 gives the first inequality of Lemma 2. Moreover, by integration by part in time, we get

$$\begin{aligned}-iAG(t)&=-iAe^{-iAt}u_0-\int_0^t iAe^{-iA\tau}f(t-\tau)d\tau\\ &=-iAe^{-iAt}u_0+e^{-iAt}f(0)-f(t)+\int_0^t e^{-iA\tau}\frac{\partial f}{\partial t}(t-\tau)d\tau.\end{aligned}$$

We get the second estimate thanks to the identity

$$-iAG(t)+f(t)=e^{-iAt}\left[-iAu_0+f(0)\right]+\int_0^t e^{-iA(t-\tau)}\frac{\partial f}{\partial t}(\tau)d\tau.\Box$$

The following statement is the appropriate adaptation of Propositon 3.

Proposition 6. *Let $T>0$ and $\mu\in H^5((0,1),\mathbb{R})$. The map Θ_T defined by (2.27) is C^1 from $H^1_0((0,T),\mathbb{R})$ to $V_T\cap H^5_{(0)}(0,1)$.*

Proof of Proposition 6 :

First step : we prove that Θ_T maps $H^1_0((0,T),\mathbb{R})$ into $V_T\cap H^5_{(0)}(0,1)$. Let $u\in H^1_0((0,T),\mathbb{R})$ and ψ be the weak solution of (1.1), (1.6). Then $\psi\in C^1([0,T],H^2_{(0)})\cap$

$C^0([0,T],H^4_{(0)})$ and the first equality of (1.1) holds in $H^2_{(0)}$ for every $t \in [0,T]$ (the proof of this result involves classical technics). In particular, we have

$$\begin{aligned}\|\psi(T)\|_{H^5_{(0)}} &= \|\psi''(T)\|_{H^3_{(0)}}\\ &= \left\|\frac{\partial\psi}{\partial t}(T)\right\|_{H^3_{(0)}} \text{ because } u(T)=0\end{aligned}$$

which is finite, thanks to Proposition 5.

Second step : We prove that $\Theta_T : H^1_0((0,T),\mathbb{R}) \to V_T \cap H^5_{(0)}$ is differentiable. Let $u, v \in H^1_0((0,T),\mathbb{R})$, ψ, Ψ, $\widetilde{\psi}$ be the weak solutions of (1.1),(1.6), (2.29), (2.30). Then, $\Delta := \widetilde{\psi} - \psi - \Psi$ is the weak solution of (2.31). Let us prove that

$$\|\Delta(T)\|_{H^5_{(0)}} = o(\|v\|_{H^1_0}) \text{ when } \|v\|_{H^1_0} \to 0,$$

which gives the conclusion. Let $R > 0$ be such that $\|u\|_{H^1_0} < R$ and $\|u+v\|_{H^1_0} < R$. Thanks to Proposition 5, there exists $C = C(T,\mu,R) > 0$, $C_1 = C_1(\mu) > 0$ such that

$$\begin{aligned}\|\Delta(T)\|_{H^5_{(0)}} &= \|\Delta''(T)\|_{H^3_{(0)}}\\ &= \left\|\frac{\partial\Delta}{\partial t}(T)\right\|_{H^3_{(0)}} \text{ because } u(T)=v(T)=0\\ &\leqslant C\|v\mu\Psi\|_{H^1_0((0,T),H^3\cap H^1_0)}\\ &\leqslant CC_1\|v\|_{H^1_0}\|\Psi\|_{C^1([0,T],H^3_{(0)})}\\ &\leqslant C^2C_1\|v\|_{H^1_0}\|v\mu\psi\|_{H^1_0((0,T),H^3\cap H^1_0)}\\ &\leqslant C^2C_1^2\|v\|^2_{H^1_0}\|\psi\|_{C^1([0,T],H^3_{(0)})}.\end{aligned}$$

The proof of the continuity of the map $d\Theta_T : H^1_0((0,T),\mathbb{R}) \to \mathcal{L}(H^1_0, V_T \cap H^5_{(0)})$ involves similar arguments. $\Box$

Remark 4. *With the same kind of arguments, we could get that $A\psi(t) - u(t)\mu\psi(t) \in C^0([0,T],H^3_{(0)})$. Therefore, $\psi(t)$ does not, in general, belong to $H^5_{(0)}(0,1)$ for $t \in (0,T)$.*

The following statement is the appropriate generalization of Proposition 4.

Proposition 7. *Let $T > 0$, $\mu \in H^5((0,1),\mathbb{R})$ be such that (1.5) holds and Θ_T be defined by (2.27). The linear map $d\Theta_T(0) : H^1_0((0,T),\mathbb{R}) \to V_T \cap H^5_{(0)}(0,1)$ has a continuous right inverse $d\Theta_T(0)^{-1} : V_T \cap H^5_{(0)}(0,1) \to H^1_0((0,T),\mathbb{R})$.*

Proof of Proposition 7 : Let $\Psi_f \in V_T \cap H^5_{(0)}(0,1)$. If Ψ is the solution of (2.33) for some $v \in H^1_0((0,T),\mathbb{R}$, then, the equality $\Psi(T) = \Psi_f$ is equivalent to the trigonometric moment problem (2.34), or equivalently

$$\begin{cases}\int_0^T \dot{v}(t)dt = 0,\\ \int_0^T (T-t)\dot{v}(t)dt = \frac{1}{i\langle\mu\varphi_1,\varphi_1\rangle}\langle\Psi_f,\varphi_1\rangle e^{i\lambda_1 T},\\ \int_0^T \dot{v}(t)e^{i(\lambda_k-\lambda_1)t}dt = \frac{\lambda_1-\lambda_k}{\langle\mu\varphi_1,\varphi_k\rangle}\langle\Psi_f,\varphi_k\rangle e^{i\lambda_k T}, \forall k \geqslant 2.\end{cases} \tag{2.38}$$

The conclusion comes from Corollary 2 (in Appendix B). $\Box$

Now, Theorem 2 may be proved exactly as Theorem 1.

6.2.6 Case of the three dimensional ball with radial data

The goal of this section is the proof of Theorem 3. This proof is very similar to the case of the interval and we only give the necessary modifications. The equivalent of Lemma 1 is proved with a similar computation for $f \in L^2((0,T), H^3_{rad} \cap H^1_{(0)})$. More precisely, for almost every $s \in (0,T)$, we have

$$\begin{aligned} \langle f(s), \varphi_k \rangle &= \int_{B^3} f(s)\varphi_k = \frac{1}{\lambda_k^2}\int_{B^3} f(s)\Delta^2\varphi_k = \frac{1}{\lambda_k^2}\int_{B^3} \Delta f(s)\Delta\varphi_k \\ &= -\frac{1}{\lambda_k^2}\int_{B^3} \nabla\Delta f(s)\cdot\nabla\varphi_k + \frac{1}{\lambda_k^2}\int_{S^2} \Delta f(s)\frac{\partial\varphi_k}{\partial n}d\sigma. \end{aligned}$$

To bound the first term, we use $\nabla\Delta f \in L^2((0,T), L^2(B^3)^3)$ and the fact that the functions $(\nabla\varphi_k/\sqrt{\lambda_k})_{k\in\mathbb{N}^*}$ form an orthonormal family of $L^2(B^3)^3$ because

$$\int_{B^3} \nabla\varphi_i \cdot \nabla\varphi_j = -\int_{B^3} \varphi_i\Delta\varphi_j = \lambda_j\delta_{i,j}.$$

For the second term, since f and φ_k are radial, we have

$$\frac{1}{\lambda_k^2}\int_{S^2} \Delta f(s)\frac{\partial\varphi_k}{\partial n}d\sigma = \frac{2^{3/2}\sqrt{\pi}(-1)^k}{\lambda_k^{3/2}}\Delta f(s, r=1).$$

We conclude as in Lemma 1 for this term since the eigenvalues are the same and Corollary 4 still applies. The genericity of assumption (1.10) is detailed in the Appendix A, Proposition 17.

Remark 5. *It is very likely that the same analysis would work in any dimension $n \leqslant 5$, provided that H^3 remains an algebra. However, this would require the analysis of the zeros of the Bessel functions and we have chosen to present the simplest result.*

6.3 Nonlinear Schrödinger equations

In this section, we study the nonlinear Schrödinger equation with Neumann boundary conditions (1.12). The goal is the proof of Theorem 4

First, let us introduce the following notations, that will be valid in all the section 6.3. The operator A is defined by

$$D(A) = H^2_{(0)}(0,1) := \{\varphi \in H^2(0,1); \varphi'(0) = \varphi'(1) = 0\}, \quad A\varphi = -\varphi''. \tag{3.39}$$

Its eigenvectors $(\varphi_k)_{k\in\mathbb{N}}$ and eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ are

$$\begin{array}{ll} \varphi_0 := 1, & \lambda_0 := 0 \\ \varphi_k(x) := \sqrt{2}\cos(k\pi x) & \lambda_k := (k\pi)^2, \forall k \in \mathbb{N}^*. \end{array} \tag{3.40}$$

We introduce the spaces

$$H^s_{(0)}(0,1) := D(A^{s/2}), \forall s > 0 \tag{3.41}$$

and the notation

$$k_* := \max\{k, 1\}, \forall k \in \mathbb{N}. \tag{3.42}$$

6.3.1 Well posedness of the Cauchy problem

The goal of this subsection is the proof of the following result.

Proposition 8. *Let $\mu \in H^2((0,1),\mathbb{R})$ and $T > 0$. There exists $\delta > 0$ such that, for every $u \in B_\delta[L^2(0,T)]$, there exists a unique weak solution $\psi \in C^0([0,T], H^2_{(0)})$ of (1.12), (1.14). Moreover, we have*

$$\|\psi(t)\|_{L^2(0,1)} = \|\psi_0\|_{L^2(0,1)}, \forall t \in [0,T].$$

We search ψ in the form $\psi(t,x) = e^{-it}(1+\zeta(t,x))$, where ζ is a weak solution of

$$\left\{ \begin{array}{l} i\frac{\partial \zeta}{\partial t} = -\zeta'' + (|1+\zeta|^2 - 1)(1+\zeta) - u\mu(1+\zeta), \\ \zeta'(t,0) = \zeta'(t,1) = 0, \\ \zeta(0,x) = 0. \end{array} \right. \tag{3.43}$$

Proposition 8 will be the consequence of the existence and uniqueness of a weak solution ζ for (3.43) (the conservation of the L^2-norm may be proved as in the linear case). In order to precise the definition of such a weak solution, let us introduce the operator $\mathcal{A}$ defined by

$$D(\mathcal{A}) := H^2_{(0)}(0,1), \quad \mathcal{A}\zeta := -\zeta'' + 2\Re(\zeta).$$

Then for every $\zeta \in H^2_{(0)}(0,1)$ and every $t \in \mathbb{R}$, we have

$$e^{-i\mathcal{A}t}\zeta = \sum_{k=0}^{\infty} (a_k(t) + ib_k(t))\,\varphi_k$$

where

$$a_0(t) := \Re(\langle \zeta, \varphi_0\rangle); \quad b_0(t) := \Im(\langle \zeta, \varphi_0\rangle) - 2t\Re(\langle \zeta, \varphi_0\rangle),$$

$$a_k(t) := \Re(\langle \zeta, \varphi_k\rangle)\cos[\sqrt{\lambda_k(\lambda_k+2)}t] + \sqrt{\frac{\lambda_k}{\lambda_k+2}}\Im(\langle \zeta, \varphi_k\rangle)\sin[\sqrt{\lambda_k(\lambda_k+2)}t], \forall k \in \mathbb{N}^*,$$

$$b_k(t) := -\sqrt{\frac{\lambda_k+2}{\lambda_k}}\Re(\langle \zeta, \varphi_k\rangle)\sin[\sqrt{\lambda_k(\lambda_k+2)}t] + \Im(\langle \zeta, \varphi_k\rangle)\cos[\sqrt{\lambda_k(\lambda_k+2)}t], \forall k \in \mathbb{N}^*.$$

Remark that these formulae are only the result of the diagonalization of the matrix $\begin{pmatrix} 0 & \Delta \\ -\Delta+2 & 0 \end{pmatrix}$ obtained by the decomposition in real and imaginary part. Then Proposition 8 is equivalent to the following statement.

Proposition 9. *Let $\mu \in H^2((0,1),\mathbb{R})$ and $T > 0$. There exists $\delta > 0$ such that, for every $u \in B_\delta[L^2((0,T),\mathbb{R})]$, there exists a unique weak solution of (3.43), i.e. a function $\zeta \in C^0([0,T], H^2_{(0)})$ such that the following equality holds in $H^2_{(0)}$ for every $t \in [0,T]$*

$$\zeta(t) = \int_0^t e^{-i\mathcal{A}(t-s)}\Big([|1+\zeta(s)|^2 - 1][1+\zeta(s)] - 2\Re[\zeta(s)] - u(s)\mu[1+\zeta(s)]\Big)ds. \tag{3.44}$$

The proof of Proposition 9 relies on the following Lemma.

Lemma 3. *Let $T > 0$ and $f \in L^2((0,T), H^2)$. The function $G : t \mapsto \int_0^t e^{-i\mathcal{A}(t-s)} f(s)ds$ belongs to $C^0([0,T], H^2_{(0)})$, moreover*

$$\|G\|_{L^\infty((0,T),H^2_{(0)})} \leqslant c_0(T)\|f\|_{L^2((0,T),H^2)}$$

where the constants $c_0(T)$ are uniformly bounded for T lying in bounded intervals.

Proof of Lemma 3 : The proof of this Lemma is similar to the one of Lemma 1. By definition, we have

$$G(t) = \sum_{k=0}^{\infty}\sum_{a=1}^{4}\left(\int_0^t y_k^a(t,s)ds\right)\varphi_k$$

where

$$y_k^1(t,s) := \Re(\langle f(s), \varphi_k\rangle)\cos[\sqrt{\lambda_k(\lambda_k+2)}(t-s)], \forall k \in \mathbb{N},$$
$$y_k^2(t,s) := \sqrt{\frac{\lambda_k}{\lambda_k+2}}\Im(\langle f(s), \varphi_k\rangle)\sin[\sqrt{\lambda_k(\lambda_k+2)}(t-s)], \forall k \in \mathbb{N}^*,$$
$$y_k^3(t,s) := -i\sqrt{\frac{\lambda_k+2}{\lambda_k}}\Re(\langle f(s), \varphi_k\rangle)\sin[\sqrt{\lambda_k(\lambda_k+2)}(t-s)], \forall k \in \mathbb{N}^*,$$
$$y_k^4(t,s) := i\Im(\langle f(s), \varphi_k\rangle)\cos[\sqrt{\lambda_k(\lambda_k+2)}(t-s)], \forall k \in \mathbb{N},$$
$$y_0^2(t,s) := 0, y_0^3(t,s) := -2t\Re(\langle f(s), \varphi_k\rangle).$$

We have

$$\|G(t)\|_{H^2_{(0)}} \leqslant \sum_{a=1}^{4}\left(\sum_{k=1}^{\infty}\left|k_*^2\int_0^t y_k^a(t,s)ds\right|^2\right)^{1/2}.$$

Let us prove that there exists a constant $c = c(t) > 0$ (uniformly bounded on bounded intervals of t) such that

$$\left(\sum_{k=1}^{\infty}\left|k_*^2\int_0^t y_k^1(t,s)ds\right|^2\right)^{1/2} \leqslant c(t)\|f\|_{L^2((0,t),H^2)}. \tag{3.45}$$

(the other terms may be treated in the same way). Integrations by part give, for almost every $s \in (0,T)$,

$$\langle f(s), \varphi_k\rangle = \frac{\sqrt{2}}{(k\pi)^2}\left((-1)^k f'(s,1) - f'(s,0) - \int_0^1 f''(s,x)\cos(k\pi x)dx\right), \forall k \in \mathbb{N}^*.$$

Thus, we have, for every $k \in \mathbb{N}^*$,

$$\begin{aligned} k^2\int_0^t y_k^1(t,s)ds = \ & \tfrac{\sqrt{2}(-1)^k}{(\pi)^2}\int_0^t f'(s,1)\cos[\sqrt{\lambda_k(\lambda_k+2)}(t-s)]ds \\ & +\tfrac{\sqrt{2}}{(\pi)^2}\int_0^t f'(s,0)\cos[\sqrt{\lambda_k(\lambda_k+2)}(t-s)]ds \\ & -\tfrac{\sqrt{2}}{(\pi)^2}\int_0^t \langle f''(s), \varphi_k\rangle\cos[\sqrt{\lambda_k(\lambda_k+2)}(t-s)]ds. \end{aligned}$$

We get (3.45) thanks to Corollary 4, as in the proof of Lemma 1. □

Proof of Proposition 9 : We introduce the function $g : \mathbb{C} \to \mathbb{C}$ defined by

$$g(z) := [|1+z|^2 - 1][1+z]. \tag{3.46}$$

We have $dg(0).\zeta = 2\Re(\zeta)$. Let $c_0 = c_0(T)$ be as in Lemma 3. Let $c_1, c_2, c_3 > 0$ be such that

$$\|g(\zeta) - dg(0).\zeta\|_{H^2} \leqslant c_1\Big(\|\zeta\|^2_{H^2_{(0)}} + \|\zeta\|^3_{H^2_{(0)}}\Big), \forall \zeta \in H^2_{(0)}, \tag{3.47}$$

$$\|g(\tilde{\zeta}) - g(\zeta) - dg(0).(\tilde{\zeta}-\zeta)\|_{H^2} \leqslant c_2\|\zeta-\tilde{\zeta}\|_{H^2_{(0)}} \max\{\|\xi\|_{H^2_{(0)}}, \|\xi\|^2_{H^2_{(0)}}; \xi \in \{\zeta,\tilde{\zeta}\}\}, \forall \zeta,\tilde{\zeta} \in H^2_{(0)}, \tag{3.48}$$

$$\|\mu\zeta\|_{H^2} \leqslant c_3\|\zeta\|_{H^2_{(0)}}, \forall \zeta \in H^2_{(0)}. \tag{3.49}$$

Let $R > 0$ be small enough so that

$$c_0 c_1 \sqrt{T}(R^2 + R^3) < \frac{R}{2} \text{ and } c_0 c_2 \sqrt{T} \max\{R, R^2\} < \frac{1}{4}. \tag{3.50}$$

Let $\delta > 0$ be small enough so that

$$c_0 \delta c_3 (1 + R) < \frac{R}{2} \text{ and } c_0 \delta c_3 < \frac{1}{4}. \tag{3.51}$$

Let $u \in L^2((0,T),\mathbb{R})$ be such that $\|u\|_{L^2(0,T)} < \delta$. We consider the map

$$\begin{array}{rccc} F: & \overline{B}_R[C^0([0,T],H^2_{(0)})] & \to & \overline{B}_R[C^0([0,T],H^2_{(0)})] \\ & \zeta & \mapsto & \xi \end{array}$$

where $\xi := F(\zeta)$ is defined by

$$\xi(t) = -i\int_0^t e^{-i\mathcal{A}(t-s)}\Big([g(\zeta(s)) - dg(0).\zeta(s) - u(s)\mu[1+\zeta(s)]\Big)ds.$$

For $\zeta \in \overline{B}_R[C^0([0,T],H^2_{(0)})]$, the function $g(\zeta) - dg(0).\zeta - u\mu[1+\zeta]$ belongs to $L^2((0,T),H^2)$, thus ξ belongs to $C^0([0,T],H^2_{(0)})$ thanks to Lemma 3. Moreover, using (3.47), (3.49), (3.50), (3.51), we get

$$\begin{aligned} \|\xi\|_{L^\infty((0,T),H^2_{(0)})} &\leqslant c_0\Big\|g(\zeta) - dg(0).\zeta - u\mu[1+\zeta]\Big\|_{L^2((0,T),H^2)} \\ &\leqslant c_0\Big[\sqrt{T}\|g(\zeta) - dg(0).\zeta\|_{L^\infty((0,T),H^2)} + \|u\|_{L^2(0,T)}\|\mu[1+\zeta]\|_{L^\infty((0,T),H^2)}\Big] \\ &\leqslant c_0\Big[\sqrt{T}c_1(R^2+R^3) + \delta c_3(1+R)\Big] \\ &\leqslant R. \end{aligned}$$

Thus, F takes values in $\overline{B}_R[C^0([0,T],H^2_{(0)})]$.

For $\zeta,\tilde{\zeta} \in \overline{B}_R[C^0([0,T],H^2_{(0)})]$, using (3.48), (3.49), (3.50), (3.51), we get

$$\begin{aligned} & \|\xi - \tilde{\xi}\|_{L^\infty((0,T),H^2_{(0)})} \\ \leqslant\ & c_0\Big\|g(\zeta) - g(\tilde{\zeta}) - dg(0).(\zeta-\tilde{\zeta}) - u\mu(\zeta-\tilde{\zeta})\Big\|_{L^2((0,T),H^2)} \\ \leqslant\ & c_0\Big[\sqrt{T}c_2\|\zeta-\tilde{\zeta}\|_{L^\infty((0,T),H^2_{(0)})}\max\{R,R^2\} + \delta c_3\|\zeta-\tilde{\zeta}\|_{L^\infty((0,T),H^2_{(0)})}\Big] \\ \leqslant\ & \tfrac{1}{2}\|\zeta-\tilde{\zeta}\|_{L^\infty((0,T),H^2_{(0)})}. \end{aligned}$$

Thus F is a contraction. $\square$

6.3.2 C^1-regularity of the end-point map

Let $T>0$ and $\delta>0$ be as in Proposition 8. Let

$$V_T := \left\{\varphi \in L^2(0,1); \Re\left(e^{iT}\int_0^1 \varphi(x)dx\right)=0\right\},$$

and $P_T : L^2(0,1) \to V_T$ be the associated orthogonal projection. Then, the following map is well defined

$$\begin{array}{rccc} \Theta_T: & B_\delta[L^2((0,T),\mathbb{R})] & \to & H^2_{(0)}(0,1) \\ & u & \mapsto & P_T[\psi(T)], \end{array} \tag{3.52}$$

where ψ solves (1.12), (1.14). We want to prove that the map Θ_T is C^1 on a neighborhood of zero. We have seen that $\psi(t)=e^{-it}(1+\zeta(t))$, where ζ solves (3.43). Thus, it is sufficient to prove the following statement.

Proposition 10. *Let $\mu \in H^2((0,1),\mathbb{R})$, $T>0$, δ be as in Proposition 9, and*

$$\begin{array}{rccc} \tilde{\Theta}_T: & B_\delta[L^2((0,T),\mathbb{R})] & \to & H^2_{(0)}(0,1) \\ & u & \mapsto & \zeta(T), \end{array}$$

where ζ solves (3.43). There exists $\delta' \in (0,\delta)$ such that the map Θ_T is C^1 on $B_{\delta'}[L^2((0,T),\mathbb{R})]$. Moreover, for every $u \in B_{\delta'}[L^2((0,T),\mathbb{R})]$ and $v \in L^2((0,T),\mathbb{R})$ we have

$$d\tilde{\Theta}_T(u).v = \xi(T) \tag{3.53}$$

where ξ solves

$$\left\{\begin{array}{l} i\frac{\partial \xi}{\partial t} = -\xi'' + dg(\zeta).\xi - u\mu\xi - v\mu(1+\zeta), \\ \xi'(t,0)=\xi'(t,1)=0, \\ \xi(0,x)=0, \end{array}\right. \tag{3.54}$$

g is defined by (3.46) and ζ solves (3.43).

Proof of Proposition 10 : We use the same notations $c_0, c_1, c_2, c_3, R, \delta$ as in the proof of Proposition 9, in particular, the relations (3.47), (3.48), (3.49), (3.50), (3.51) are satisfied. We introduce constants $c_4, c_5 > 0$ such that

$$\|[dg(\zeta)-dg(0)].h\|_{H^2} \leqslant c_4 \|h\|_{H^2_{(0)}} \max\{\|\zeta\|_{H^2_{(0)}}, \|\zeta\|^2_{H^2_{(0)}}\}, \forall \zeta, h \in H^2_{(0)}, \tag{3.55}$$

$$\|g(\tilde{\zeta})-g(\zeta)-dg(0).(\tilde{\zeta}-\zeta)\|_{H^2} \leqslant c_5 \|\tilde{\zeta}-\zeta\|_{H^2_{(0)}} \max\{\|\xi\|_{H^2_{(0)}}, \|\xi\|^2_{H^2_{(0)}}; \xi \in \{\zeta,\tilde{\zeta}\}\}, \forall \zeta,\tilde{\zeta} \in H^2_{(0)}. \tag{3.56}$$

Moreover, we assume that

$$c_0\sqrt{T}\max\{c_4,c_5\}\max\{R,R^2\} < \frac{1}{4} \tag{3.57}$$

(this additional assumption may change δ into a smaller value δ').

Let $u, v \in B_\delta[L^2((0,T),\mathbb{R})]$ be such that $(u+v) \in B_\delta[L^2(0,T)]$. Let ζ, ξ and $\tilde{\zeta}$ be the solutions of (3.43), (3.54) and

$$\begin{cases} i\frac{\partial\tilde{\zeta}}{\partial t} = -\tilde{\zeta}'' + (|1+\tilde{\zeta}|^2 - 1)(1+\tilde{\zeta}) - (u+v)\mu(1+\tilde{\zeta}), \\ \tilde{\zeta}'(t,0) = \tilde{\zeta}'(t,1) = 0, \\ \tilde{\zeta}(0,x) = 0. \end{cases}$$

The existence of ξ may be proved in a similar way as the existence of ζ.

First step : Let us prove that

$$\|\tilde{\zeta} - \zeta\|_{L^\infty((0,T),H^2_{(0)})} \leqslant 2c_0c_3\|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})}\|v\|_{L^2}. \tag{3.58}$$

Thanks to Lemma 3, (3.56),(3.49), (3.57) and (3.51), we have

$$\begin{aligned} & \|\zeta - \tilde{\zeta}\|_{L^\infty((0,T),H^2_{(0)})} \\ \leqslant\ & c_0\left\| g(\tilde{\zeta}) - g(\zeta) - dg(0).(\tilde{\zeta} - \zeta) - (u+v)\mu(\tilde{\zeta} - \zeta) - v\mu(1+\zeta)\right\|_{L^2((0,T),H^2)} \\ \leqslant\ & c_0\Big[\sqrt{T}c_5\|\tilde{\zeta} - \zeta\|_{L^\infty((0,T),H^2_{(0)})}\max\{R,R^2\} + \delta c_3\|\tilde{\zeta} - \zeta\|_{L^\infty((0,T),H^2_{(0)})} \\ & +\|v\|_{L^2}c_3\|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})}\Big] \\ \leqslant\ & \tfrac{1}{2}\|\zeta - \tilde{\zeta}\|_{L^\infty((0,T),H^2_{(0)})} + c_0\|v\|_{L^2}c_3\|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})} \end{aligned}$$

which gives (3.58).

Second step : Let us prove that the linear map

$$\begin{array}{|ccc} L^2(0,T) & \to & H^2_{(0)}(0,1) \\ v & \mapsto & \xi(T) \end{array}$$

is continuous. Thanks to Lemma 3, (3.55), (3.49), (3.57) and (3.51), we have

$$\begin{aligned} \|\xi\|_{L^\infty((0,T),H^2_{(0)})} & \leqslant c_0\left\|[dg(\zeta) - dg(0)].\xi - u\mu\xi - v\mu(1+\zeta)\right\|_{L^2((0,T),H^2)} \\ & \leqslant c_0\Big[\sqrt{T}c_4\|\xi\|_{L^\infty((0,T),H^2_{(0)})}\max\{R,R^2\} + \delta c_3\|\xi\|_{L^\infty((0,T),H^2_{(0)})} \\ & \quad +\|v\|_{L^2}c_3\|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})}\Big] \\ & \leqslant \tfrac{1}{2}\|\xi\|_{L^\infty((0,T),H^2_{(0)})} + c_0\|v\|_{L^2}c_3\|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})}, \end{aligned}$$

which gives

$$\|\xi\|_{L^\infty((0,T),H^2_{(0)})} \leqslant 2c_0c_3\|v\|_{L^2}\|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})}. \tag{3.59}$$

Third step : Let us prove that $\tilde{\Theta}_T$ is differentiable and that (3.53) holds. Let $\Delta := \tilde{\zeta} - \zeta - \xi$. We want to prove that

$$\|\Delta(T)\|_{H^2_{(0)}} = o(\|v\|_{L^2}) \text{ when } \|v\|_{L^2} \to 0.$$

Let $\epsilon > 0$. There exists $\eta > 0$ such that, for every $f \in L^\infty((0,T),H^2_{(0)})$ with $\|f - \zeta\|_{L^\infty((0,T),H^2_{(0)})} < \eta$, we have

$$\|g(f) - g(\zeta) - dg(\zeta).(f-\zeta)\|_{L^\infty((0,T),H^2_{(0)})} < \epsilon\|f-\zeta\|_{L^\infty((0,T),H^2_{(0)})}.$$

Let us assume that v is small enough so that

$$2c_0c_3\|1+\zeta\|_{L^\infty((0,T),H^2_{(0)})}\|v\|_{L^2} < \eta.$$

Then, thanks to Lemma 3 and (3.58), (3.55) and (3.49), we have

$$\begin{aligned}
&\|\Delta\|_{L^\infty((0,T),H^2_{(0)})} \\
\leqslant\ & c_0\Big\|g(\tilde{\zeta})-g(\zeta)-dg(\zeta).(\tilde{\zeta}-\zeta)+[dg(\zeta)-dg(0)].\Delta-(u+v)\mu\Delta-v\mu\xi\Big\|_{L^2((0,T),H^2)} \\
\leqslant\ & c_0\Big[\sqrt{T}\epsilon\|\tilde{\zeta}-\zeta\|_{L^\infty((0,T),H^2_{(0)})}+\sqrt{T}c_4(R+R^2)\|\Delta\|_{L^\infty((0,T),H^2_{(0)})} \\
& +\delta c_3\|\Delta\|_{L^\infty((0,T),H^2_{(0)})}+\|v\|_{L^2}c_3\|\xi\|_{L^\infty((0,T),H^2_{(0)})}\Big].
\end{aligned}$$

Thanks to (3.57) and (3.49), we get

$$\|\Delta\|_{L^\infty((0,T),H^2_{(0)})} \leqslant 2c_0\Big[\sqrt{T}\epsilon\|\tilde{\zeta}-\zeta\|_{L^\infty((0,T),H^2_{(0)})}+\|v\|_{L^2}c_3\|\xi\|_{L^\infty((0,T),H^2_{(0)})}\Big],$$

which gives the conclusion, thanks to (3.58) and (3.59).

The continuity of the map $d\tilde{\Theta}_T$ may be proved with similar arguments. □

6.3.3 Controllability of the linearized system

The goal of this section is the proof of the following result.

Proposition 11. *Let $T>0$ and $\mu\in H^2((0,1),\mathbb{R})$ be such that (1.13) holds. Let $\delta>0$ be as in Proposition 8 and Θ_T be defined by (3.52). The linear map $d\Theta_T(0): L^2((0,T),\mathbb{R})\to V_T\cap H^2_{(0)}(0,1)$ has a continuous right inverse $d\Theta_T(0)^{-1}$: $V_T\cap H^2_{(0)}(0,1)\to L^2((0,T),\mathbb{R})$.*

Proof of Proposition 11 : It is equivalent to prove that the continuous linear map $d\tilde{\Theta}_T(0): L^2((0,T),\mathbb{R})\to \tilde{V}\cap H^2_{(0)}(0,1)$ has a continuous right inverse, where

$$\tilde{V} := \left\{\varphi\in L^2(0,1); \Re\int_0^1\varphi(x)dx=0\right\}.$$

We have $d\tilde{\Theta}_T(0).v=\xi(T)$ where ξ is the weak solution of

$$\begin{cases} i\frac{\partial\xi}{\partial t}=-\xi''+2\Re(\xi)-v(t)\mu(x), x\in(0,1), t\in(0,T),\\ \xi'(t,0)=\xi'(t,1)=0,\\ \xi(0,x)=0.\end{cases}$$

In particular, we have

$$\xi(T)=i\int_0^T e^{-i\mathcal{A}(T-s)}v(s)\mu ds = i\sum_{k=0}^{\infty}[a_k(T)+ib_k(T)]\varphi_k$$

where

$$a_0(T)=\langle\mu,\varphi_0\rangle\int_0^T v(s)ds,$$

$$b_0(T) = -2\langle \mu, \varphi_0 \rangle \int_0^T (T-t)v(s)ds,$$

$$a_k(T) = \langle \mu, \varphi_k \rangle \int_0^T v(s)\cos[\sqrt{\lambda_k(\lambda_k+2)}(T-s)]ds, \forall k \in \mathbb{N}^*,$$

$$b_k(T) = -\sqrt{\frac{\lambda_k+2}{\lambda_k}}\langle \mu, \varphi_k \rangle \int_0^T v(s)\sin[\sqrt{\lambda_k(\lambda_k+2)}(T-s)]ds, \forall k \in \mathbb{N}^*.$$

For $\xi_f \in \tilde{V} \cap H^2_{(0)}(0,1)$, the equality $\xi(T) = \xi_f$ is equivalent to the following trigonometric moment problem

$$\left\{ \begin{array}{l} \int_0^T v(s)ds = d_0(\xi_f) := \Im \frac{\langle \xi_f, \varphi_0 \rangle}{\langle \mu, \varphi_0 \rangle}, \\ \int_0^T v(s)e^{-i\sqrt{\lambda_k(\lambda_k+2)}s}ds = d_k(\xi_f) := \frac{e^{-i\sqrt{\lambda_k(\lambda_k+2)}T}}{\langle \mu, \varphi_k \rangle}\Big(\Im\langle \xi_f, \varphi_k \rangle + i\sqrt{\frac{\lambda_k}{\lambda_k+2}}\Re\langle \xi_f, \varphi_k \rangle\Big), \forall k \in \mathbb{N}^*, \\ \int_0^T sv(s)ds = \tilde{d}(\xi_f) := Td_0(\xi_f). \end{array} \right.$$

We conclude thanks to Corollary 2 (in Appendix B). □

The proof of Theorem 4 is completed using the same arguments as in Section 6.2.4 using the inverse mapping theorem and the conservation of the L^2 norm.

Remark 6. *With the same method, one may prove the local exact controllability of the focusing nonlinear Schrödinger equation*

$$\left\{ \begin{array}{l} i\frac{\partial \psi}{\partial t}(t,x) = -\frac{\partial^2 \psi}{\partial x^2}(t,x) - |\psi|^2\psi(t,x) - u(t)\mu(x)\psi(t,x), x \in (0,1), t \in (0,T), \\ \frac{\partial \psi}{\partial x}(t,0) = \frac{\partial \psi}{\partial x}(t,1) = 0, \end{array} \right.$$

around the reference trajectory $(\psi_{ref}(t,x) = e^{it}, u_{ref}(t) = 0)$. *The only difference in the proof is that we get the frequencies* $\sqrt{\lambda_k(\lambda_k-2)}$ *(instead of* $\sqrt{\lambda_k(\lambda_k+2)}$*) in the moment problem. When the space domain is the interval* $(0,1)$, *then all the quantities* $\lambda_k(\lambda_k-2)$, *for* $k \in \mathbb{N}^*$, *are positive (because* $\lambda_k = (k\pi)^2$*), thus there is no additional difficulty. When the space domain is different, for instance* $(0,a)$ *with* a *large, then* $\lambda_k = (k\pi/a)^2$, *thus a finite number of the quantities* $\lambda_k(\lambda_k-2)$ *are negative : we get a new moment problem with a finite number of moments with real valued exponentials, and an infinite number of trigonometric moments, that can be easily solved by adapting the tools used in this article.*

6.4 Nonlinear wave equations

In this section, we study the nonlinear wave equation with Neumann boundary conditions (1.15). The goal is the proof of Theorem 5. In all this section, we use the notations defined in (3.39), (3.40), (3.41), (3.42) and all the functions are real valued.

First, let us check that the Cauchy problem is well posed in $H^3_{(0)} \times H^2_{(0)}(0,1)$, when $u \in L^2(0,T)$. In order to write the system (1.15) in first order form, let us introduce

$$\begin{array}{ll} D(\mathcal{A}) := H^2_{(0)} \times H^1(0,1), & \mathcal{A} := \begin{pmatrix} 0 & Id \\ -A & 0 \end{pmatrix}, \\ D(\mathcal{B}) := L^2 \times L^2(0,1), & \mathcal{B} := \mu(x)\begin{pmatrix} 0 & 0 \\ Id & Id \end{pmatrix} \end{array} \qquad (4.60)$$

and $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(y_1, y_2) := (0, f(y_1, y_2))$. The operator $\mathcal{A}$ generates a C^0-group of bounded operators of $H^2_{(0)} \times H^1(0,1)$ defined by

$$e^{\mathcal{A}t} \begin{pmatrix} w_0 \\ \dot{w}_0 \end{pmatrix} = \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix},$$

where

$$w(t) = (\langle w_0, \varphi_0\rangle + \langle \dot{w}_0, \varphi_0\rangle t)\, \varphi_0 + \sum_{k=1}^{\infty} \left(\langle w_0, \varphi_k\rangle \cos(\sqrt{\lambda_k} t) + \frac{1}{\sqrt{\lambda_k}} \langle \dot{w}_0, \varphi_k\rangle \sin(\sqrt{\lambda_k} t) \right) \varphi_k,$$

$$\dot{w}(t) = \langle \dot{w}_0, \varphi_0\rangle \varphi_0 + \sum_{k=1}^{\infty} \left(-\sqrt{\lambda_k} \langle w_0, \varphi_k\rangle \sin(\sqrt{\lambda_k} t) + \langle \dot{w}_0, \varphi_k\rangle \cos(\sqrt{\lambda_k} t) \right) \varphi_k.$$

With the notation

$$\mathcal{W} := \begin{pmatrix} w \\ \frac{\partial w}{\partial t} \end{pmatrix}, \quad \mathcal{W}_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

the equation (1.15) may be written

$$\frac{\partial \mathcal{W}}{\partial t}(t,x) = \mathcal{A}\mathcal{W}(t,x) + F(\mathcal{W}) + u(t)\mathcal{B}\mathcal{W}(t,x), x \in (0,1). \tag{4.61}$$

Proposition 12. *Let $\mu \in H^2(0,1)$, $T > 0$, $f \in C^3(\mathbb{R}^2, \mathbb{R})$ be such that $f(1,0) = 0$ and $\nabla f(1,0) = 0$. There exists $\delta > 0$ such that, for every $u \in B_\delta[L^2(0,T)]$, there exists a unique weak solution of (4.61), (1.16), i.e. a function $\mathcal{W} \in C^0([0,T], H^3_{(0)} \times H^2_{(0)})$ such that the following equality holds in $H^3_{(0)} \times H^2_{(0)}(0,1)$, for every $t \in [0,T]$,*

$$\mathcal{W}(t) = e^{\mathcal{A}t}\mathcal{W}_0 + \int_0^t e^{\mathcal{A}(t-\tau)} \Big(F(\mathcal{W}(\tau)) + u(\tau)\mathcal{B}\mathcal{W}(\tau) + \mathcal{F}(\tau) \Big) d\tau. \tag{4.62}$$

The proof of this proposition relies on the following Lemma.

Proposition 13. *Let $T > 0$ and $g \in L^2((0,T), H^2)$. The function G defined by*

$$G(t) := \int_0^t e^{\mathcal{A}s} \begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds$$

belongs to $C^0([0,T], H^3_{(0)} \times H^2_{(0)})$. Moreover, there exists a constant $c_0(T) > 0$, uniformly bounded for T lying in bounded intervals, such that, for every $g \in L^2((0,T), H^2)$,

$$\|G\|_{L^\infty((0,T), H^3_{(0)} \times H^2_{(0)})} \leqslant c_0(T) \|g\|_{L^2((0,T),H^2)}. \tag{4.63}$$

Proof of Proposition 13 : We have, for every $t \in [0,T]$,

$$G(t) = \int_0^t \begin{pmatrix} \langle g(s), \varphi_0\rangle s \varphi_0 + \sum\limits_{k=1}^{\infty} \frac{\langle g(s), \varphi_k\rangle}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} s)\varphi_k \\ \langle g(s), \varphi_0\rangle \varphi_0 + \sum\limits_{k=1}^{\infty} \langle g(s), \varphi_k\rangle \cos(\sqrt{\lambda_k} s)\varphi_k \end{pmatrix} ds.$$

Thus, there exists $C > 0$ such that

$$\|G(t)\|_{H^3_{(0)} \times H^2_{(0)}} \leqslant C \left[\sum_{k=0}^{\infty} \left| k_*^2 \int_0^t \langle g(s), \varphi_k\rangle e^{ik\pi s} ds \right|^2 + \left| \int_0^t s \langle g(s), \varphi_0\rangle ds \right|^2 \right].$$

We get the conclusion as in the previous sections. □

Proof of Proposition 12 : Let us introduce the constants c_1, c_2, c_3 such that

$$\|f(w,w_t)\|_{H^2} \leqslant c_1\|(w-1,w_t)\|^2_{H^3_{(0)}\times H^2_{(0)}}, \forall (w,w_t)\in(1,0)+B_1[H^3_{(0)}\times H^2_{(0)}], \quad (4.64)$$

$$\begin{array}{rl} & \|f(w,w_t)-f(\tilde{w},\tilde{w}_t)\|_{H^2} \\ \leqslant & c_2\|(w-\tilde{w},w_t-\tilde{w}_t)\|_{H^3_{(0)}\times H^2_{(0)}} \max\{\|(w-1,w_t)\|_{H^3_{(0)}\times H^2_{(0)}}, \|(\tilde{w}-1,\tilde{w}_t)\|_{H^3_{(0)}\times H^2_{(0)}}\}, \\ & \forall (w,w_t),(\tilde{w},\tilde{w}_t)\in(1,0)+B_1[H^3_{(0)}\times H^2_{(0)}], \end{array} \quad (4.65)$$

and (3.49) holds. Let $R\in(0,1)$ be small enough so that

$$\sqrt{T}c_0c_1R^2 \leqslant \tfrac{R}{2}, \quad \sqrt{T}c_0c_2R \leqslant \tfrac{1}{4} \quad (4.66)$$

Let $\delta>0$ be small enough so that

$$\delta c_0 c_3 < \frac{1}{4}, \quad \delta c_0 c_3(1+R) < \frac{R}{2}. \quad (4.67)$$

Let $u\in B_\delta[L^2(0,T)]$. We consider the map

$$\begin{array}{rrcl} \mathcal{F}: & (1,0)+\overline{B}_R[C^0([0,T],H^3_{(0)}\times H^2_{(0)})] & \to & (1,0)+\overline{B}_R[C^0([0,T],H^3_{(0)}\times H^2_{(0)})] \\ & \zeta & \mapsto & \xi \end{array}$$

where

$$\xi(t)=e^{\mathcal{A}t}\mathcal{W}_0+\int_0^t e^{\mathcal{A}(t-\tau)}\Big(F(\zeta(\tau))+u(\tau)\mathcal{B}\zeta(\tau)\Big)d\tau, \forall t\in[0,T].$$

For $\zeta=(w,w_t)\in(1,0)+\overline{B}_R[C^0([0,T],H^3_{(0)}\times H^2_{(0)})]$, the second component of $F(\zeta)+u\mathcal{B}\zeta$ belongs to $L^2((0,T),H^2)$, thus ξ belongs to $C^0([0,T],H^3_{(0)}\times H^2_{(0)})$ thanks to Proposition 13. Moreover, thanks to (4.63), (4.64), (3.49), (4.66), and (4.67), we have, for every $t\in[0,T]$,

$$\begin{array}{rl} \|\xi(t)-(1,0)\|_{H^3_{(0)}\times H^2_{(0)}} & \leqslant c_0\|f(w,w_t)+u\mu[w+w_t]\|_{L^2((0,T),H^2)} \\ & \leqslant c_0\Big(\sqrt{T}\|f(w,w_t)\|_{L^\infty((0,T),H^2)}+\|u\|_{L^2(0,T)}\|\mu[w+w_t]\|_{L^\infty((0,T),H^2)}\Big) \\ & \leqslant c_0(\sqrt{T}c_1R^2+\delta c_3(R+1)) \\ & \leqslant R. \end{array}$$

Thus, $\mathcal{F}$ takes values in $(1,0)+\overline{B}_R[C^0([0,T],H^3_{(0)}\times H^2_{(0)})]$.

For $\zeta=(w,w_t),\tilde{\zeta}=(\tilde{w},\tilde{w}_t)\in(1,0)+\overline{B}_R[C^0([0,T],H^3_{(0)}\times H^2_{(0)})]$, thanks to (4.65), (3.49), (4.66) and (4.67), we have

$$\begin{array}{rl} & \|\mathcal{F}(\zeta)-\mathcal{F}(\tilde{\zeta})\|_{L^\infty((0,T),H^3_{(0)}\times H^2_{(0)})} \\ \leqslant & c_0\|f(w,w_t)-f(\tilde{w},\tilde{w}_t)+u\mu[w-\tilde{w}+w_t-\tilde{w}_t]\|_{L^2((0,T),H^2)} \\ \leqslant & c_0\Big[\sqrt{T}\|f(w,w_t)-f(\tilde{w},\tilde{w}_t)\|_{L^\infty((0,T),H^2)}+\delta c_3\|\zeta-\tilde{\zeta}\|_{L^\infty((0,T),H^3_{(0)}\times H^2_{(0)})}\Big] \\ \leqslant & c_0\Big[\sqrt{T}c_2R\|\zeta-\tilde{\zeta}\|_{L^\infty((0,T),H^3_{(0)}\times H^2_{(0)})}+\delta c_3\|\zeta-\tilde{\zeta}\|_{L^\infty((0,T),H^3_{(0)}\times H^2_{(0)})}\Big] \\ \leqslant & \frac{1}{2}\|\zeta-\tilde{\zeta}\|_{L^\infty((0,T),H^2_{(0)})} \end{array}$$

Thus $\mathcal{F}$ is a contraction. $\square$

Let $T>0$, $\mu \in H^2(0,1)$, $f \in C^3(\mathbb{R}^2,\mathbb{R})$ be such that $f(1,0)=0$, $\nabla f(1,0)=0$ and $\delta>0$ be as in Proposition 12. Then, the following map is well defined

$$\begin{array}{rccc} \Theta_T: & B_\delta[L^2(0,T)] & \rightarrow & H^3_{(0)} \times H^2_{(0)} \\ & u & \mapsto & (w,w_t)(T) \end{array} \tag{4.68}$$

where (w,w_t) is the weak solution of (1.15), (1.16). Working as in the previous section, one may prove the following statements.

Proposition 14. *Let $\mu \in H^2(0,1)$, $T>0$, $f \in C^3(\mathbb{R}^2,\mathbb{R})$ be such that $f(1,0)=0$, $\nabla f(1,0)=0$ and $\delta>0$ be as in Proposition 12. The map Θ_T defined by (4.68) is C^1. Moreover, for every $u \in B_\delta[L^2(0,T)]$ and $v \in L^2(0,T)$, we have $d\Theta_T(u).v=(W,W_t)(T)$, where (W,W_t) is the weak solution of*

$$\left\{\begin{array}{l} W_{tt}=W_{xx}+\frac{\partial f}{\partial y_1}(w,w_t).W+\frac{\partial f}{\partial y_2}(w,w_t).W_t+u(t)\mu[W+W_t]+v(t)\mu(x)[w+w_t], \\ W_x(t,0)=W_x(t,1)=0, \\ (W,W_t)(0,x)=0, \end{array}\right. \tag{4.69}$$

and (w,w_t) is the weak solution of (1.15), (1.16).

Proposition 15. *Let $T>2$, $\mu \in H^2(0,1)$ be such that (1.13) holds and $f \in C^3(\mathbb{R}^2,\mathbb{R})$ be such that $f(1,0)=0$, $\nabla f(1,0)=0$. The linear map $d\Theta_T(0): L^2(0,T) \rightarrow H^3_{(0)} \times H^2_{(0)}$ has a continuous right inverse $d\Theta_T(0)^{-1}: H^3_{(0)} \times H^2_{(0)} \rightarrow L^2(0,T)$.*

The proof is the same except that the gap between the eigenvalues does not tend to infinity and we use Corollary 3.

6.5 Conclusion, open problems, perspectives

In this article, we have proposed a method for the proof of the local exact controllability for linear and nonlinear bilinear systems. We have applied it to Schrödinger and wave equations, showing it works for a wide range of problems. It also works on other equations (for instance it may prove an optimal version of the controllability result proved in [16] for a 1D Beam equation).

In this article, we have presented various examples of application of the method. However, they all have in common that the linearized system fulfills a gap condition on the eigenvalues of the operator. This condition is not necessarily realized for the Schrödinger equation in higher space dimensions. Even in two dimension, we do not know any example of domain where it is true. So, one challenging question is the extension (or the impossibility to do it) of these results to other dimensions.

6.A Genericity of the assumption on μ

The goal of this section is the proof of the following result.

Proposition 16. *The set* $\{\mu \in H^3((0,1),\mathbb{R}); (1.5)$ *holds*$\}$ *is dense in* $H^3((0,1),\mathbb{R})$.

Proof : First, let us notice that

$$\mathcal{V} := \{\mu \in H^3((0,1),\mathbb{R}); \mu'(1) \pm \mu'(0) \neq 0\}$$

is a dense open subset of $H^3((0,1),\mathbb{R})$. Now, let us prove that the set

$$\mathcal{U} := \{\mu \in \mathcal{V}; \langle \mu\varphi_1, \varphi_k \rangle \neq 0, \forall k \in \mathbb{N}^*\}$$

is dense in $H^3((0,1),\mathbb{R})$. It is sufficient to prove that this set is dense in $\mathcal{V}$. For $n \in \mathbb{N}$, we introduce the set

$$\mathcal{U}_n := \{\mu \in \mathcal{V}; \langle \mu\varphi_1, \varphi_k \rangle \neq 0, \forall k \in \{1, ..., n\}\},$$

with the convention $\mathcal{U}_0 := \mathcal{V}$. Then the sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is decreasing and

$$\mathcal{U} = \bigcap_{n=0}^{\infty} \mathcal{U}_n.$$

Thanks to Baire Lemma, it is sufficient to check that, for every $n \in \mathbb{N}$, $\mathcal{U}_{n+1}$ is dense in $\mathcal{U}_n$ for the $H^3((0,1),\mathbb{R})$-topology. Let $n \in \mathbb{N}$ and let $\mu \in \mathcal{U}_n - \mathcal{U}_{n+1}$. Then $\mu \in \mathcal{V}$, $\langle \mu\varphi_1, \varphi_k \rangle \neq 0$ for $k = 1, ..., n$ and $\langle \mu\varphi_1, \varphi_{n+1} \rangle = 0$. Thanks to (1.7), $\mu + \epsilon x^2 \in \mathcal{U}_{n+1}$ for every $\epsilon \in \mathbb{R}$ such that

$$\epsilon \neq -\frac{\langle \mu\varphi_1, \varphi_j \rangle}{\langle x^2\varphi_1, \varphi_j \rangle}, \forall j \in \{1, ..., n\}.$$

Thus $\mathcal{U}_{n+1}$ is dense in $\mathcal{U}_n$.

Finally, thanks to (1.8), we have

$$\mathcal{U} \subset \{\mu \in H^3((0,1),\mathbb{R}); (1.5) \text{ holds}\},$$

which gives the conclusion. □

Proposition 17. *The set* $\{\mu \in H^3_{rad}(B^3,\mathbb{R}); (1.10)$ *holds*$\}$ *is dense in* $H^3(B^3,\mathbb{R})$.

Proof : We make the same proof. We use the formula

$$\langle \mu\varphi_1, \varphi_k \rangle = \frac{4\pi(-1)^{k+1}}{\lambda_k^{3/2}} \partial_r \mu(1) - \frac{1}{\lambda_k^2} \int_{B^3} \nabla\Delta(\mu\varphi_1) \cdot \nabla\varphi_k$$

instead of (1.8). Moreover, we can find one $\mu(r) = r^2$ that fulfills (1.10). □

6.B Moment problems

In this section, we recall classical results about moment problems (see, for instance [9]). The proofs are given for sake of completeness.

6.B.0.1 Families of vectors in Hilbert spaces

Let H be a separable Hilbert vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\Theta := (\xi_j)_{j\in\mathbb{Z}}$ be a family of vectors of H with $\xi_j \neq 0, \forall j \in \mathbb{Z}$.

Definition 1. *The family Θ is minimal in H if, for every $j \in \mathbb{Z}$, $\xi_j \notin \overline{Span\{\xi_i; i \in \mathbb{Z} - \{j\}\}}$.*

Proposition 18. *The family Θ is minimal in H if and only if there exists a biorthogonal family $\Theta' = (\xi'_j)_{j\in\mathbb{Z}}$, i.e. Θ' is a family of vectors of H such that*

$$\langle \xi_i, \xi'_j \rangle = \delta_{i,j}, \forall i, j \in \mathbb{Z}. \tag{2.70}$$

Proof of Proposition 18 : We assume Θ is minimal. For $j \in \mathbb{Z}$, let v_j be the orthogonal projection of ξ_j over the closed vector space $\overline{\text{Span}\{\xi_i, i \neq j\}}$ i.e.

$$v_j \in \overline{\text{Span}\{\xi_i, i \neq j\}} \text{ and } \langle \xi_j - v_j, \xi_i \rangle = 0, \forall i \neq j.$$

Let

$$\xi'_j := \frac{\xi_j - v_j}{\|\xi_j - v_j\|^2}, \forall j \in \mathbb{Z}.$$

Then, the families (ξ_j) and (ξ'_j) are biorthogonal.

Now, we assume that there exists a biorthogonal family $\Theta' = (\xi'_j)_{j\in\mathbb{Z}}$. Let us assume that there exists $j \in \mathbb{Z}$ such that $\xi_j \in \overline{\text{Span}\{\xi_i; i \in \mathbb{Z} - \{j\}\}}$. Then (2.70) implies $\langle \xi_j, \xi'_j \rangle = 1$ which is a contradiction. $\Box$

Remark 7. *If Θ is minimal, then there exists a unique biorthogonal family Θ' such that $\Theta' \subset \overline{Span\{\xi_i; i \in \mathbb{Z}\}}$. In the end of this appendix, the expression "the"biorthogonal family of Θ, refers to this unique biorthogonal family in $\overline{Span\{\xi_i; i \in \mathbb{Z}\}}$.*

Definition 2. *The family Θ is a Riesz basis of $\overline{Span\Theta}$ if Θ is the image of some orthonormal family by an isomorphism.*

Remark 8. *It is clear that, if Θ is a Riesz basis of $\overline{Span\Theta}$, then Θ is minimal in H.*

Proposition 19. *(1) If Θ is a Riesz basis of $\overline{Span\Theta}$, then its biorthogonal family Θ' is also a Riesz basis of $\overline{Span\Theta}$.*

(2) Θ is a Riesz basis of $\overline{Span\Theta}$ if and only if there exists $C_1, C_2 \in (0, +\infty)$ such that, for every scalar sequence $(c_j)_{j\in\mathbb{Z}}$ with finite support,

$$C_1 \left(\sum_{j=-\infty}^{\infty} |c_j|^2 \right)^{1/2} \leqslant \Big\| \sum_{j=-\infty}^{\infty} c_j \xi_j \Big\| \leqslant C_1 \left(\sum_{j=-\infty}^{\infty} |c_j|^2 \right)^{1/2}. \tag{2.71}$$

(3) If Θ is a Riesz basis of $\overline{Span\Theta}$ then there exists $C > 0$ such that, for every $f \in H$, we have

$$\left(\sum_{j\in\mathbb{Z}} |\langle f, \xi_j \rangle|^2 \right)^{1/2} \leqslant C\|f\|.$$

Proof of Proposition 19 :

(1) We assume Θ is a Riesz basis of $\overline{\text{Span}\Theta}$. Let $\mathcal{H}$ be an Hilbert space, $(\zeta_j)_{j\in\mathbb{Z}}$ be an orthonormal family of $\mathcal{H}$, $V:\mathcal{H}\to\overline{\text{Span}\Theta}$ an isomorphism such that $\xi_j=V(\zeta_j),\forall j\in\mathbb{Z}$. Then the adjoint operator $V^*:\overline{\text{Span}\Theta}\to\mathcal{H}$ is also an isomorphism and we have $\xi_j'=(V^*)^{-1}(\zeta_j),\forall j\in\mathbb{Z}$. Indeed, for every $j,k\in\mathbb{Z}$,

$$\delta_{j,k}=\langle\xi_j,\xi_k'\rangle_H=\langle V(\zeta_j),\xi_k'\rangle_H=\langle\zeta_j,V^*(\xi_k')\rangle_{\mathcal{H}}.$$

Thus Θ' is a Riesz basis of $\overline{\text{Span}\Theta}$.

(2) We assume Θ is a Riesz basis of $\overline{\text{Span}\Theta}$. Let $\mathcal{H}$ be an Hilbert space, $(\zeta_j)_{j\in\mathbb{Z}}$ be an orthonormal family of $\mathcal{H}$, $V:\mathcal{H}\to\overline{\text{Span}\Theta}$ an isomorphism such that $\xi_j=V(\zeta_j),\forall j\in\mathbb{Z}$ and $(c_j)_{j\in\mathbb{Z}}$ a scalar sequence with finite support. We have

$$\Big\|\sum_{j=-\infty}^{\infty}c_j\xi_j\Big\|=\Big\|V\Big[\sum_{j=-\infty}^{\infty}c_j\zeta_j\Big]\Big\|\leqslant\|V\|\Big\|\sum_{j=-\infty}^{\infty}c_j\zeta_j\Big\|=\|V\|\left(\sum_{j=-\infty}^{\infty}|c_j|^2\right)^{1/2}$$

and

$$\left(\sum_{j=-\infty}^{\infty}|c_j|^2\right)^{1/2}=\Big\|\sum_{j=-\infty}^{\infty}c_j\zeta_j\Big\|=\Big\|V^{-1}\Big[\sum_{j=-\infty}^{\infty}c_j\xi_j\Big]\Big\|\leqslant\|V^{-1}\|\Big\|\sum_{j=-\infty}^{\infty}c_j\xi_j\Big\|,$$

thus, we have (2.71) with $C_1=1/\|V^{-1}\|$ and $C_2=\|V\|$.

Now, we assume that (2.71) holds. Then the linear map $V:l^2(\mathbb{Z},\mathbb{K})\to\overline{\text{Span}\Theta}$ defined by $V[(c_j)_{j\in\mathbb{Z}}]=\sum_{j=-\infty}^{\infty}c_j\xi_j$ is well defined and injective. Let $h\in\overline{\text{Span}\Theta}$. There exists $(h_N)_{N\in\mathbb{N}}$ such that $h_N\to h$ in H when $N\to+\infty$ and for every $N\in\mathbb{N}$, there exists a sequence $c^{(N)}=(c_j^{(N)})_{j\in\mathbb{Z}}$ with finite support such that $h_N=\sum_{j=-\infty}^{\infty}c_j^{(N)}\xi_j$. Then $(h_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in H, thus, thanks to (2.71), $(c^{(N)})_{N\in\mathbb{N}}$ is a Cauchy sequence in $l^2(\mathbb{Z})$ and there exists $c=(c_j)_{j\in\mathbb{Z}}\in l^2(\mathbb{N})$ such that $c^N\to c$ in $l^2(\mathbb{Z})$. Then, (2.71) proves that $\sum_{j=-\infty}^{\infty}(c_j-c_j^{(N)})\xi_j\to0$ in H, i.e. $h=\sum_{j=-\infty}^{\infty}c_j\xi_j$. We have proved that V is an isomorphism, thus Θ is a Riesz basis of $\overline{\text{Span}\Theta}$.

(3) $\overline{\text{Span}\Theta}$ is a close vector subspace of H thus we have the orthogonal decomposition $H=\overline{\text{Span}\Theta}+\overline{\text{Span}\Theta}^{\perp}$ and the associated orthogonal projection $P:H\to\overline{\text{Span}\Theta}$. For $f\in H$, we have

$$\begin{aligned}\left(\sum_{j\in\mathbb{Z}}|\langle f,\xi_j\rangle|^2\right)^{1/2}&=\left(\sum_{j\in\mathbb{Z}}|\langle Pf,\xi_j\rangle|^2\right)^{1/2}\\&\leqslant\tfrac{1}{C_1}\Big\|\sum_{j\in\mathbb{Z}}\langle Pf,\xi_j\rangle\xi_j'\Big\|\\&=\tfrac{1}{C_1}\|Pf\|_H\leqslant\tfrac{1}{C_1}\|f\|.\Box\end{aligned}$$

Remark 9. *We have proved that, if Θ is a Riesz basis of $\overline{Span\Theta}$, then, for every $h\in\overline{Span\Theta}$ there exists $c=(c_j)_{j\in\mathbb{Z}}\in l^2(\mathbb{Z},\mathbb{K})$ such that $h=\sum_{j=-\infty}^{\infty}c_j\xi_j$. Moreover, if Θ' and Θ are biorthogonal families, then necessarily $c_j=\langle h,\xi_j'\rangle,\forall j\in\mathbb{Z}$. Thus, every $h\in\overline{Span\Theta}$ can be decomposed in the following way*

$$h=\sum_{j=-\infty}^{\infty}\langle h,\xi_j'\rangle\xi_j=\sum_{j=-\infty}^{\infty}\langle h,\xi_j\rangle\xi_j' \qquad (2.72)$$

where the series converge in H and the coefficients $(\langle h,\xi_j'\rangle)_{j\in\mathbb{Z}}$, $(\langle h,\xi_j\rangle)_{j\in\mathbb{Z}}$, belong to $l^2(\mathbb{Z},\mathbb{K})$.

6.B.0.2 Abstract moment problems

Now, we move to the investigation of abstract moment problems : given a scalar sequence $(d_j)_{j\in\mathbb{Z}}$ is it possible to find $f \in H$ such that

$$\langle f, \xi_j \rangle = d_j, \forall j \in \mathbb{Z}.$$

Let us introduce the operator

$$\begin{array}{rccc} J_\Theta: & H & \to & l^2(\mathbb{Z},\mathbb{K}) \\ & f & \mapsto & (\langle f, \xi_j\rangle)_{j\in\mathbb{Z}} \end{array}$$

with domain $D_\Theta := \{f \in H; J_\Theta(f) \in l^2(\mathbb{Z})\}$. It is clear that, if the family Θ is not complete in H, then the operator J_Θ has a non trivial null space $\overline{\text{Span}\Theta}^\perp$. This motivates the introduction of the operator $J^0_\Theta := J_\Theta\Big|_{\overline{\text{Span}\Theta}}$.

Proposition 20. *The operator $J^0_\Theta : \overline{Span\Theta} \to l^2(\mathbb{Z},\mathbb{K})$ is an isomorphism if and only if Θ is a Riesz basis of $\overline{Span\Theta}$.*

Proof of Proposition 20 : We assume $J^0_\Theta : \overline{\text{Span}\Theta} \to l^2(\mathbb{Z},\mathbb{K})$ is an isomorphism. Let $(\zeta_j)_{j\in\mathbb{Z}}$ be the canonical orthonormal basis of $l^2(\mathbb{Z})$. Then, the family

$$\Big((J^0_\Theta)^{-1}(\zeta_j)\Big)_{j\in\mathbb{Z}}$$

is a Riesz basis of $\overline{\text{Span}\Theta}$. Moreover, it is the biorthogonal family to Θ in $\overline{\text{Span}\Theta}$. Thanks to Proposition 19 **(1)**, Θ is also a Riesz basis of $\overline{\text{Span}\Theta}$.

We assume Θ is a Riesz basis of $\overline{\text{Span}\Theta}$. Thanks to the Remark 9, it is clear that $J^0_\Theta : \overline{\text{Span}\Theta} \to l^2(\mathbb{Z},\mathbb{K})$ is an isomorphism. $\Box$

6.B.0.3 Trigonometric moment problems

In this section, we recall important results on trigonometric moment problems. The following Ingham inequality is due to Haraux [38].

Theorem 6. *Let $N \in \mathbb{N}$, $(\omega_k)_{k\in\mathbb{Z}}$ be an increasing sequence of real numbers such that*

$$\omega_{k+1} - \omega_k \geqslant \gamma > 0, \forall k \in \mathbb{Z}, |k| \geqslant N,$$

$$\omega_{k+1} - \omega_k \geqslant \rho > 0, \forall k \in \mathbb{Z},$$

and $T > 2\pi/\gamma$. There exists $C_1 = C_1(\gamma,\rho,N,T), C_2 = C_2(\gamma,\rho,N,T) \in (0,+\infty)$ such that, for every sequence $(c_k)_{k\in\mathbb{Z}} \in \mathbb{C}^\mathbb{Z}$ with finite support, we have

$$C_1 \sum_{k\in\mathbb{Z}} |c_k|^2 \leqslant \int_0^T \Big| \sum_{k=-\infty}^{+\infty} c_k e^{-i\omega_k t}\Big|^2 dt \leqslant C_2 \sum_{k\in\mathbb{Z}} |c_k|^2.$$

Let us introduce the space

$$l^2_r(\mathbb{N},\mathbb{C}) := \{(d_k)_{k\in\mathbb{N}} \in l^2(\mathbb{N},\mathbb{C}); d_0 \in \mathbb{R}\}.$$

Thanks to Proposition 19 and Theorem 6, we have the following statement, which is used in the proof of Proposition 4.

Corollary 1. *Let $T > 0$ and $(\omega_k)_{k\in\mathbb{N}}$ be an increasing sequence of $[0,+\infty)$ such that $\omega_0 = 0$ and*

$$\omega_{k+1} - \omega_k \to +\infty \text{ when } k \to +\infty.$$

There exists a continuous linear map

$$\begin{array}{rccc} L: & l^2_r(\mathbb{N},\mathbb{C}) & \to & L^2((0,T),\mathbb{R}) \\ & d & \mapsto & L(d) \end{array}$$

such that, for every $d = (d_k)_{k\in\mathbb{N}} \in l^2_r(\mathbb{N},\mathbb{C})$, the function $v := L(d)$ solves

$$\int_0^T v(t)e^{i\omega_k t}dt = d_k, \forall k \in \mathbb{N}.$$

Proof of Corollary 1 : We define $\omega_{-k} := -\omega_k, \forall k \in \mathbb{N}^*$. Theorem 6 ensures that the family $(e^{i\omega_k t})_{k\in\mathbb{Z}}$ is a Riesz basis of $F := \text{Adh}_{L^2(0,T)}(\text{Span}\{e^{i\omega_k t}; k \in \mathbb{Z}\})$. Thanks to Proposition 20, the map

$$\begin{array}{rccc} J: & F & \to & l^2(\mathbb{Z},\mathbb{C}) \\ & v & \mapsto & \left(\int_0^T v(t)e^{i\omega_k t}dt\right)_{k\in\mathbb{Z}} \end{array}$$

is an isomorphism. For $d = (d_k)_{k\in\mathbb{N}} \in l^2_r(\mathbb{N},\mathbb{C})$, we define $\tilde{d} := (\tilde{d}_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z},\mathbb{C})$ by $\tilde{d}_k := d_k$ if $k \geqslant 0$ and $\overline{d_{-k}}$ if $k < 0$. Now, we define $L : l^2_r(\mathbb{N},\mathbb{C}) \to L^2((0,T),\mathbb{R})$ by $L(d) = J^{-1}(\tilde{d})$. The map L takes values in real valued functions because $\tilde{d}_{-k} = \overline{\tilde{d}_k}, \forall k \in \mathbb{N}$ for every $d \in l^2_r(\mathbb{N},\mathbb{C})$. □

Theorem 6 is also crucial in the proof of the following statement, used in the proof of Proposition 7.

Corollary 2. *Let $T > 0$ and $(\omega_k)_{k\in\mathbb{N}}$ be an increasing sequence of $[0,+\infty)$ such that $\omega_0 = 0$ and*

$$\omega_{k+1} - \omega_k \to +\infty \text{ when } k \to +\infty. \tag{2.73}$$

There exists a continuous linear map

$$\begin{array}{rccc} L: & \mathbb{R}\times l^2_r(\mathbb{N},\mathbb{C}) & \to & L^2((0,T),\mathbb{R}) \\ & (\tilde{d},d) & \mapsto & L(\tilde{d},d) \end{array}$$

such that, for every $\tilde{d} \in \mathbb{R}$, $d = (d_k)_{k\in\mathbb{N}} \in l^2_r(\mathbb{N},\mathbb{C})$, the function $v := L(\tilde{d},d)$ solves

$$\begin{array}{l} \int_0^T v(t)e^{i\omega_k t}dt = d_k, \forall k \in \mathbb{N}, \\ \int_0^T tv(t)dt = \tilde{d}. \end{array} \tag{2.74}$$

Proof of Corollary 2 : Let $\omega_k := -\omega_{-k}$, for every $k \in \mathbb{Z}$ with $k < 0$. From Proposition 6, $\Theta := (e^{i\omega_k t})_{k\in\mathbb{Z}}$ is a Riesz basis of $\text{Adh}_{L^2(0,T)}(\text{Span}\Theta)$.

First step : We prove that the family $\widetilde{\Theta} := \{t, e^{i\omega_k t}; k \in \mathbb{Z}\}$ is minimal in $L^2(0,T)$.

Working by contradiction, we assume that $\widetilde{\Theta}$ is not minimal in $L^2(0,T)$. Then, necessarily

$$t \in \text{Adh}_{L^2(0,T)}\text{Span}\Theta. \tag{2.75}$$

With successive integrations, we get

$$t^j \in \mathrm{Adh}_{C^0[0,T]}\Big(\mathrm{Span}\widetilde{\Theta}\Big), \forall j \in \mathbb{N} \text{ with } j \geqslant 2.$$

The Stone Weierstrass theorem ensures that $\{1, t^j; j \in \mathbb{N}, j \geqslant 2\}$ is dense in $C^0([0,T],\mathbb{C})$, thus, it is also dense in $L^2(0,T)$. From (2.75), we deduce that $\mathrm{Span}\Theta$ is dense in $L^2(0,T)$. This is a contradiction, because, thanks to Theorem 6, for every $\omega \in \mathbb{R} - \{\omega_k, k \in \mathbb{Z}\}$, the family $\{e^{i\omega t}, e^{i\omega_k t}; k \in \mathbb{Z}\}$ is minimal, i.e.

$$e^{i\omega t} \notin \mathrm{Adh}_{L^2(0,T)}\Big(\mathrm{Span}\Theta\Big).$$

Second step : We conclude.

For $k < 0$, we define $d_k := \overline{d_{-k}}$. Let $\{\tilde{\xi}, \xi_k; k \in \mathbb{Z}\}$ be the biorthogonal family to $\{t, e^{i\omega_k t}; k \in \mathbb{Z}\}$. From Theorem 6, there exists $C > 0$ and a unique solution $v \in \mathrm{Adh}_{L^2(0,T)}(\mathrm{Span}\Theta)$ of

$$\int_0^T v(t) e^{i\omega_k t} dt = d_k, \forall k \in \mathbb{Z}$$

and it satisfies

$$\|v\|_{L^2(0,T)} \leqslant C \left(\sum_{k\in\mathbb{Z}} |d_k|^2 \right)^{1/2}.$$

The uniqueness guarantees that v is real valued. Let us define

$$L(\tilde{d}, d) := u := v + \Big(\tilde{d} - \int_0^T t v(t) dt\Big)\tilde{\xi}.$$

Then, u is real valued (because v and $\tilde{\xi}$ are), u solves (2.74) and

$$\begin{aligned} \|u\|_{L^2} &\leqslant \|v\|_{L^2} + \Big(|\tilde{d}| + \Big|\int_0^T t v(t) dt\Big|\Big)\|\tilde{\xi}\|_{L^2} \\ &\leqslant \|v\|_{L^2}\Big(1 + \sqrt{\tfrac{T^3}{3}}\|\tilde{\xi}\|_{L^2}\Big) + |\tilde{d}|\|\tilde{\xi}\|_{L^2} \\ &\leqslant \Big(C\Big(1 + \sqrt{\tfrac{T^3}{3}}\|\tilde{\xi}\|_{L^2}\Big) + \|\tilde{\xi}\|_{L^2}\Big)\Big(|\tilde{d}|^2 + \textstyle\sum_{k\in\mathbb{Z}} |d_k|^2\Big)^{1/2}. \square \end{aligned}$$

For the wave equation, the gap between two successive frequencies does not tend to infinity, so we will need the following Corollary which is proved similarly.

Corollary 3. *Let $T > 2$. We make the same assumptions as in Corollary 2 except that we assume*

$$\omega_{k+1} - \omega_k \geqslant \pi$$

instead of (2.73). Then, we have the same conclusion as Corollary 2.

Corollary 4. *Let $(\omega_k)_{k\in\mathbb{N}}$ be an increasing sequence of $[0,+\infty)$ such that $\omega_0 = 0$ and*

$$\omega_{k+1} - \omega_k > \gamma > 0.$$

There exists a nondecreasing function

$$\begin{array}{rccc} C: & [0,+\infty) & \to & \mathbb{R}_+^* \\ & T & \mapsto & C(T) \end{array}$$

such that, for every $T > 0$ and for every $g \in L^2(0,T)$, we have

$$\left(\sum_{k=0}^{\infty} \left| \int_0^T g(t) e^{i\omega_k t} dt \right|^2 \right)^{1/2} \leqslant C(T) \|g\|_{L^2(0,T)}.$$

Proof of Corollary 4 : The existence of $C(T)$, for large $T \geqslant 2\pi/\gamma+1$, is a consequence of Theorem 6 and Proposition 19 **(3)**. Let us choose for $C(T)$ the smallest value possible for this constant. For $T \leq 2\pi/\gamma+1$, we choose $C(T) = C(2\pi/\gamma+1)$. Let $0 < T_1 < T_2 < +\infty$, $g \in L^2(0,T_1)$ and $\tilde{g} \in L^2(0,T_2)$ be defined by $\tilde{g} = g$ on $(0,T_1)$ and 0 on (T_1,T_2). By applying the inequality on $\tilde{g}$, we get $C(T_1) \leqslant C(T_2)$. $\Box$

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