Domaines extrémaux pour la première valeur propre de l'opérateur de Laplace-Beltrami

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Résumé

Dans tout ce qui suit, nous considérons une variété riemannienne compacte de dimension au moins égale à 2. A tout domaine (suffisamment régulier) Ω, on peut associer la première valeur propre λ_Ω de l'opérateur de Laplace-Beltrami avec condition de Dirichlet au bord. Nous dirons qu'un domaine Ω est extrémal (sous entendu, pour la première valeur propre de l'opérateur de Laplace-Beltrami) si Ω est un point critique de la fonctionnelle $\Omega \to \lambda_\Omega$ sous une contrainte de volume $Vol(\Omega) = c_0$. Autrement dit, Ω est extrémal si, pour toute famille régulière $\{\Omega_t\}_{t\in(-t_0,t_0)}$ de domaines de volume constant, telle que $\Omega_0 = \Omega$, la dérivée de la fonction $t \to \lambda_{\Omega_t}$ en 0 est nulle. Rappelons que les domaines extrémaux sont caractérisés par le fait que la fonction propre, associée à la première valeur propre sur le domaine avec condition de Dirichlet au bord, a une donnée de Neumann constante au bord. Ce résultat a été démontré par A. El Soufi et S. Ilias en 2007. Les domaines extrémaux sont donc des domaines sur lesquels peut être résolu un problème elliptique surdéterminé.

L'objectif principal de cette thèse est la construction de domaines extrémaux pour la première valeur propre de l'opérateur de Laplace-Beltrami avec condition de Dirichlet au bord. Nous donnons des résultats d'existence de domaines extrémaux dans le cas de petits volumes ou bien dans le cas de volumes proches du volume de la variété. Nos résultats permettent ainsi de donner de nouveaux exemples non triviaux de domaines extrémaux.

Le premier résultat que nous avons obtenu affirme que si une variété admet un point critique non dégénéré de la courbure scalaire, alors pour tout volume petit il existe un domaine extrémal qui peut être construit en perturbant une boule géodésique centrée en ce point critique non dégénéré de la courbure scalaire. La méthode que nous utilisons pour construire ces domaines extrémaux revient à étudier l'opérateur (non linéaire) qui à un domaine associe la donnée de Neumann de la première fonction propre de l'opérateur de Laplace-Beltrami sur le domaine. Il s'agit d'un opérateur (hautement non linéaire), nonlocal, elliptique d'ordre 1.

Dans $\mathbb{R}^n \times \mathbb{R}/\mathbb{Z}$, le domaine cylindrique $B_r \times \mathbb{R}/\mathbb{Z}$, où B_r est la boule de rayon $r > 0$ dans $\mathbb{R}^n$, est un domaine extrémal. En étudiant le linéarisé de l'opérateur elliptique du premier ordre défini par le problème précédent et en utilisant un résultat de bifurcation, nous avons démontré l'existence de domaines extrémaux nontriviaux dans $\mathbb{R}^n \times \mathbb{R}/\mathbb{Z}$. Ces nouveaux domaines extrémaux sont proches de domaines cylindriques $B_r \times \mathbb{R}/\mathbb{Z}$. S'ils sont invariants par rotation autour de l'axe vertical, ces domaines ne sont plus invariants par translations verticales. Ce deuxième résultat donne un contre-exemple à une conjecture de Berestycki, Caffarelli et Nirenberg énoncée en 1997.

Pour de grands volumes la construction de domaines extrémaux est techniquement plus difficile et fait apparaître des phénomènes nouveaux. Dans ce cadre, nous avons dû distinguer deux cas selon que la première fonction propre ϕ_0 de l'opérateur de Laplace-Beltrami sur la variété est constante ou non. Les résultats que nous avons obtenus sont les suivants :

1. Si ϕ_0 a des points critiques non dégénérés (donc en particulier n'est pas constante), alors pour tout volume assez proche du volume de la variété, il existe un domaine extrémal obtenu en perturbant le complément d'une boule géodésique centrée en un des points critiques non dégénérés de ϕ_0.
2. Si ϕ_0 est constante et la variété admet des points critiques non dégénérés de la courbure scalaire, alors pour tout volume assez proche du volume de la variété il existe un domaine extrémal obtenu en perturbant le complément d'une boule géodésique centrée en un des points critiques non dégénérés de la courbure scalaire.

Mots clés. Domaines extrémaux, première valeur propre, Laplacien, opérateur de Laplace-Beltrami, boules géodésiques, courbure scalaire, profile de Faber-Krahn, conjecture de Berestycki-Caffarelli-Nirenberg, problèmes elliptiques surdétérminés.

Abstract

In what follows, we will consider a compact Riemannian manifold whose dimension is at least 2. Let Ω be a (smooth enough) domain and λ_Ω the first eigenvalue of the Laplace-Beltrami operator on Ω with 0 Dirichlet boundary condition. We say that Ω is extremal (for the first eigenvalue of the Laplace-Beltrami operator) if Ω is a critical point for the functional $\Omega \to \lambda_\Omega$ with respect to variations of the domain which preserve its volume. In other words, Ω is extremal if, for all smooth family of domains $\{\Omega_t\}_{t\in(-t_0,t_0)}$ whose volume is equal to a constant c_0, and $\Omega_0 = \Omega$, the derivative of the function $t \to \lambda_{\Omega_t}$ computed at $t = 0$ is equal to 0. We recall that an extremal domain is characterized by the fact that the eigenfunction associated to the first eigenvalue of the Laplace-Beltrami operator over the domain with 0 Dirichlet boundary condition, has constant Neumann data at the boundary. This result has been proved by A. El Soufi and S. Ilias in 2007. Extremal domains are then domains over which can be solved an elliptic overdeterminated problem.

The main aim of this thesis is the construction of extremal domains for the first eigenvalue of the Laplace-Beltrami operator with 0 Dirichlet boundary condition. We give some existence results of extremal domains in the cases of small volume or volume closed to the volume of the manifold. Our results allow also to construct some new nontrivial exemples

of extremal domains.

The first result we obtained states that if the manifold has a nondegenerate critical point of the scalar curvature, then, given a fixed volume small enough, there exists an extremal domain that can be constructed by perturbation of a geodesic ball centered in that nondegenerated critical point of the scalar curvature. The methode used is based on the study of the operator that to a given domain associes the Neumann data of the first eigenfunction of the Laplace-Beltrami operator over the domain. It is a highly nonlinear, non local, elliptic first order operator.

In $\mathbb{R}^n \times \mathbb{R}/\mathbb{Z}$, the circular-cylinder-type domain $B_r \times \mathbb{R}/\mathbb{Z}$, where B_r is the ball of radius $r > 0$ in $\mathbb{R}^n$, is an extremal domain. By studying the linearized of the elliptic first order operator defined in the previous problem, and using some bifurcation results, we prove the existence of nontrivial extremal domains in $\mathbb{R}^n \times \mathbb{R}/\mathbb{Z}$. Such extremal domains are closed to the circular-cylinder-type domains $B_r \times \mathbb{R}/\mathbb{Z}$. If they are invariant by rotation with respect to the vertical axe, they are not invariant by vertical translations. This second result gives a counterexemple to a conjecture of Berestycki, Caffarelli and Nirenberg stated in 1997.

For big volumes the construction of extremal domains is technically more difficult and shows some new phenomena. In this context, we had to distinguish two cases, according to the fact that the first eigenfunction ϕ_0 of the Laplace-Beltrami operator over the manifold is constant or not. The results obtained are the following :

1. If ϕ_0 has a nondegenerated critical point (in particular it is not constant), then, given a fixed volume closed to the volume of the manifold, there exists an extremal domain obtained by perturbation of the complement of a geodesic ball centered in a nondegenerated critical point of ϕ_0.
2. If ϕ_0 is constant and the manifold has some nondegenerate critical points of the scalar curvature, then, for a given fixed volume closed to the volume of the manifold, there exists an extremal domain obtained by perturbation of the complement of a geodesic ball centered in a nondegenerate critical point of the scalar curvature.

Key words. Extremal domains, first eigenvalue, Laplacian, Laplace-Beltrami operator, geodesic balls, scalar curvature, Faber-Krahn profile, Berestycki-Caffarelli-Nirenberg's conjecture, elliptic overdeterminated problems.

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Chapitre 1

Introduction

Considérons une variété riemannienne compacte (M, g) de dimension n au moins égale à 2. A tout domaine suffisamment régulier Ω, on peut associer la première valeur propre λ_Ω de l'opérateur de Laplace-Beltrami

$$\Delta_g = \mathrm{div}_g(\nabla)\,,$$

avec condition de Dirichlet au bord, c'est à dire la constante positive λ_Ω telle que le problème de Dirichlet

$$\left\{ \begin{array}{rcll} \Delta_g u + \lambda_\Omega\, u & = & 0 & \text{sur} \quad \Omega \\ u & = & 0 & \text{sur} \quad \partial\Omega \end{array} \right.$$

admette une solution strictement positive u (voir [6]). Nous dirons qu'un domaine Ω est extrémal (sous entendu, pour la première valeur propre de l'opérateur de Laplace-Beltrami) si Ω est point critique de la fonctionnelle $\Omega \to \lambda_\Omega$ sous la contrainte de volume $\mathrm{Vol}(\Omega) = c_0$. Autrement dit, Ω est extrémal si, pour toute famille régulière $\{\Omega_t\}_{t\in(-t_0,t_0)}$ de domaines de volume constant, telle que $\Omega_0 = \Omega$, la dérivée de la fonction $t \to \lambda_{\Omega_t}$ en 0 est nulle.

L'objectif principal de cette thèse est la construction de domaines extrémaux pour la première valeur propre de l'opérateur de Laplace-Beltrami avec condition de Dirichlet au bord. Il s'agit là d'un problème classique en optimisation de forme. Des méthodes directes du calcul des variations qui permettent de démontrer l'existence de minimiseurs de la première valeur propre du Laplacien euclidien, parmi les sous-ensembles de volume prescrit d'un domaine donné, ont été introduites par G. Buttazzo et G. Dal Maso en 1993, [9]. Ce résultat est obtenu dans une classe de domaines assez irréguliers, appelés *domaines quasi-ouverts*, définis de la manière suivante. Soit A un sous-ensemble ouvert et borné de $\mathbb{R}^n$. Pour tout sous-ensemble E de A, soit $\mathcal{U}_E$ l'ensemble de toutes les fonctions u de l'espace de Sobolev $H^1_0(A)$ telles que $u \geq 1$ presque partout dans un voisinage de E, et définissons la capacité de E par la formule

$$\mathrm{cap}\,(E) = \inf \left\{ \int_A |\nabla u|^2 \ : \ u \in \mathcal{U}_E \right\}.$$

On dit que un sous-ensemble Ω de A est quasi-ouvert si pour tout $\epsilon > 0$ il existe un sous-ensemble ouvert Ω_ϵ de A tel que $\mathrm{cap}\,(\Omega_\epsilon \triangleleft \Omega) < \epsilon$, où $\Omega_\epsilon \triangleleft \Omega$ désigne la différence symétrique entre les ensembles Ω_ϵ et Ω.

En 2000, M. Hayouni a amélioré le résultat de G. Buttazzo et G. Dal Maso (voir [22]), en démontrant l'existence d'un domaine ouvert qui minimise la première valeur propre du Laplacien euclidien, parmi les sous-ensembles de volume prescrit d'un domaine donné. Grâce à la technique de symétrisation de Schwarz (voir les travaux de Krahn [25] et [26] et de J. Serrin, [36]), on sait que si M est une boule de $\mathbb{R}^n$ ou si $M = \mathbb{R}^n$, alors toute boule $B \subseteq M$ minimise la première valeur propre du Laplacien euclidien, parmi les domaines ayant le même volume. En général, on ne dispose d'aucune information sur la régularité et la structure de ces domaines extrémaux obtenus par minimisation de $\Omega \to \lambda_\Omega$ sous une contrainte de volume.

Dans cette thèse, nous donnons des résultats d'existence de domaines extrémaux pour la première valeur propre de l'opérateur de Laplace-Beltrami dans certaines variétés riemanniennes, dans le cas de petits volumes ou bien dans le cas de volumes proches du volume de la variété. Nos résultats permettent aussi de donner de nouveaux exemples non triviaux de domaines extrémaux dans les tores plats.

Notre approche s'inspire fortement de travaux semblables qui ont été faits dans le cadre du problème isopérimétrique (où l'on regarde, entre tous les domaines d'un même volume, les domaines dont l'aire du bord est minimale) et de l'existence d'hypersurfaces à courbure moyenne constante.
Il est bien connu que les bords des domaines solutions du problème isopérimétrique sont, quand ils sont suffisamment réguliers, des hypersurface à courbure moyenne constante (voir les travaux de d'Almgren [1], Gruter [20], Gonzalez, Massari et Tamanini [18]).

Théorème 1.0.1. ***(d'Almgren [1], Gruter [20], Gonzalez, Massari et Tamanini [18]).*** *Soit M une variété riemannienne compacte de dimension n. Alors, pour tout t, $0 < t < Vol(M)$, il existe une région compacte $\Omega \subset M$ dont le bord $\Sigma = \partial\Omega$ minimise l'aire entre toutes les régions du même volume t. De plus, sauf pour un ensemble singulier fermé de dimension de Hausdorff au plus $n - 8$, le bord Σ de toute région minimisante est une hypersurface régulière plongée de courbure moyenne constante, et si $\partial M \cap \Sigma \neq \emptyset$ alors ∂M et Σ se rencontrent orthogonalement.*

Néanmoins, il n'existe pas de méthode générale qui permette la construction d'hypersurface à courbure moyenne constante dans une variété riemannienne. Un exemple de méthode qui permet de construire des hypersurfaces à courbure moyenne constante a été donné en 1991 par R. Ye, qui a démontré (voir [40]) le :

Théorème 1.0.2. ***(Ye [40]).*** *Le feuilletage d'un voisinage suffisamment petit d'un point critique non dégénéré de la fonction courbure scalaire, par des sphères géodésiques centrées en ce point, peut être perturbé en un feuilletage par des hypersurfaces à courbure moyenne constante.*

Une réciproque partielle à ce résultat de R. Ye a été donnée en 2002 par O. Druet (voir [10]).

Théorème 1.0.3. ***(Druet [10])*** *Soit* (M, g) *une variété riemannienne compacte de dimension au moins* 2*. Soit* $p \in M$*. On assume que*

$$\mathrm{Scal}(p) < n(n-1)\,K$$

où Scal *est la courbure scalaire de la variété et* $K \in \mathbb{R}$*. Il existe* $r_p > 0$ *tel que pour tout domaine* Ω *contenu dans la boule géodésique de centre* p *et rayon* r_p

$$\mathrm{Vol}_g(\partial\Omega) > \mathrm{Vol}_{g_K}(\partial B)$$

où B *est une boule de volume* $\mathrm{Vol}_g(\Omega)$ *dans l'espace modèle de courbure sectionnelle constante* K *(i.e. une variété riemannienne complète simplement connexe de courbure sectionnelle constante* K*).*

Comme corollaire de ce résultat, on obtient que, pour des volumes petits, les solutions du problème isopérimétrique ressemblent à des sphères géodésiques de rayon petit centrées en un point où la courbure scalaire est maximale, ce qui représente une réciproque partielle au résultat de R. Ye.

On note

$$I_\tau := \min_{\Omega \subset M \,:\, \mathrm{Vol}\,\Omega = \tau} \mathrm{Vol}\,\partial\Omega$$

le *profil isopérimétrique* de la variété M, où l'on prend Ω dans les domaines ouverts, relativement compacts et réguliers de la variété riemannienne M. S. Nardulli, en utilisant le résultat de R. Ye, a obtenu un développement asymptotique du profil isopérimétrique de M dans un voisinage de 0 :

$$I_\tau = \frac{c_n}{\omega_n^{\frac{n-1}{n}}}\,\tau^{\frac{n-1}{n}} \left(1 - \frac{S}{2n\,(n+2)} \left(\frac{\tau}{\omega_n}\right)^{\frac{2}{n}} + o\left(\tau^{\frac{2}{n}}\right)\right)$$

où S est un maximum de la courbure scalaire de M, ω_n est le volume euclidien de la boule unité de dimension n, et c_n est le volume euclidien de la sphère S^{n-1}.

La détermination du profil isopérimétrique est intimement liée à la détermination du *profil de Faber-Krähn*, où l'on cherche le minimum de la première valeur propre de l'opérateur de Laplace-Beltrami parmi les domaines de volume prescrit :

$$FK_\tau^g := \min_{\Omega \subset M \,:\, \mathrm{Vol}\,\Omega = \tau} \lambda_\Omega$$

L'inégalité de Faber-Krähn (voir [25] et [26]) stipule que, dans l'espace euclidien

$$FK_\tau^{\mathring{g}} = \lambda_{\mathring{B}_1} \left(\frac{n}{c_n} \tau \right)^{-\frac{2}{n}}$$

où $\mathring{B}_1$ est la boule unité euclidienne et $\mathring{g}$ la métrique euclidienne. Comme mentionné ci-dessus, on peut le démontrer en utilisant la technique de symétrisation de Schwarz. De manière plus générale, si (M_K, g_K) désigne l'espace-modèle de courbure sectionnelle constante K, on démontre de la même façon que

$$FK_\tau^{g_K} = \lambda_B$$

où B est une boule de volume τ dans M_K.

Il est bien connu qu'une estimation par valeur inférieure sur le profil isopérimétrique donne une estimation par valeur inférieure sur le profil de Faber-Krahn. Ce résultat représente essentiellement l'inégalité de Faber-Krähn ([13], [25], voir aussi [6]) :

Théorème 1.0.4. ***(Faber [13], Krähn [25]).*** *A chaque ensemble ouvert Ω, constitué par l'union disjointe de domaines réguliers de la variété riemannienne M, on associe la boule géodésique B_Ω de l'espace modèle M_K à courbure sectionnelle constante K qui a le même volume que Ω. Si pour tout Ω l'inegalité isopérimetrique*

$$\mathrm{Vol}_g(\partial\Omega) \geq \mathrm{Vol}_{g_K}(\partial B_\Omega)$$

est vérifiée, avec égalité si et seulement si Ω est isométrique à B_Ω, alors pour tout domaine connexe et régulier par morceaux Ω

$$\lambda_\Omega \geq \lambda_{B_\Omega}$$

avec égalité si et seulement si Ω est isométrique à B_Ω.

Etant donné le rôle crucial joué par les points critiques de la fonction courbure scalaire dans la construction d'hypersurfaces à courbure moyenne constante, il est raisonnable de penser que ces points jouent aussi un rôle essentiel dans la construction de domaines extrémaux pour la première valeur propre de l'opérateur de Laplace-Beltrami. Dans cette thèse, nous montrons que c'est effectivement le cas.

Comme mentionné ci-dessus, les méthodes directes du calcul des variations ne permettent pas de construire des domaines extrémaux d'une variété riemannienne ou alors ne donnent que très peu d'informations sur ces domaines. Pour cette raison, dans cette thèse, nous avons utilisé des méthodes perturbatives qui permettent de donner de nouveaux exemples de tels domaines. Comme nous l'expliquerons dans le deuxième chapitre, les domaines extrémaux sont caractérisés par le fait que la fonction propre associée à la première valeur propre du Laplacien sur le domaine avec condition de Dirichlet au bord, a une donnée de Neumann constante au bord. Ce résultat a été démontré dans le cas euclidien par P. R. Garabedian et M. Schiffer [16] en 1953, et, dans une variété riemannienne, par A. El Soufi et S. Ilias en 2007, [12]. Une démonstration différente de ce fait, basée sur un approche de D. Z. Zanger (voir [41]), est présentée dans le deuxième chapitre de cette thèse. Les domaines extrémaux sont donc des domaines sur lesquels peut être résolu le problème elliptique surdéterminé

$$\left\{ \begin{array}{rcll} \Delta_g u + \lambda_\Omega \, u & = & 0 & \text{sur} \quad \Omega \\ u & = & 0 & \text{sur} \quad \partial\Omega \\ \dfrac{\partial u}{\partial \nu} & = & \text{constante} & \text{sur} \quad \partial\Omega \, , \end{array} \right.$$

où ν est le vecteur normal extérieur au bord de Ω.

Le premier résultat que nous avons obtenu (paru en 2009 dans les Annales de l'Institut Fourier, [32]) affirme que si une variété admet un point critique non dégénéré de la courbure scalaire, alors pour tout volume suffisamment petit il existe un domaine extrémal, et ce domaine peut être construit en perturbant une boule géodésique centrée en ce point. Voici l'énoncé précis :

Théorème 1.0.5. *On suppose que* p_0 *est un point critique non dégénéré de* Scal*, la fonction courbure scalaire de* (M, g)*. Alors, pour tout* $\epsilon > 0$ *suffisamment petit,* $\epsilon \in (0, \epsilon_0)$*, il existe un domaine régulier* $\Omega_\epsilon \subset M$ *tel que :*

(i) Le volume de Ω_ϵ *est égal au volume euclidien de* $\mathring{B}_\epsilon$*, la boule euclidienne de rayon* ϵ*.*

(ii) Le domaine Ω_ϵ *est extrémal.*

De plus il existe une constante $c > 0$ et, pour tout $\epsilon \in (0, \epsilon_0)$, il existe un point $p_\epsilon \in M$ tel que le bord du domaine Ω_ϵ est le graphe normal sur $\partial B_\epsilon(p_\epsilon)$ pour une fonction w_ϵ qui vérifie

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}\partial B_\epsilon(p_\epsilon))} \leq c\,\epsilon^3. \qquad et \qquad \mathrm{dist}(p_\epsilon, p_0) \leq c\,\epsilon\,.$$

où dist *représente la distance entre deux points.*

Il s'agit là du pendant du résultat de R. Ye dans ce contexte. La méthode que nous utilisons pour construire ces domaines extrémaux revient à étudier l'opérateur qui, à un domaine, associe la donnée de Neumann de la première fonction propre de l'opérateur de Laplace-Beltrami sur le domaine. Un volume τ petit étant fxé, l'objectif est de trouver, pour tout ϵ suffisamment petit, un point p_ϵ et une fonction w_ϵ, définie sur le bord de la boule géodésique de centre p_ϵ et rayon ϵ, tels que le domaine $B^g_{\epsilon(1+w_\epsilon)}(p_\epsilon)$, domaine dont le bord est le graphe normal pour la fonction $\epsilon\, w_\epsilon$ sur le bord de la boule géodésique $B^g_\epsilon(p_\epsilon)$ de rayon ϵ et centre p_ϵ, soit extrémal et ait un volume égal à τ. Tout d'abord, on remarque que si l'on fixe τ égal au volume de la boule euclidienne de rayon ϵ, toute boule modifiée $B^g_{\epsilon(1+w)}(p)$ de volume τ peut s'écrire dans la forme $B^g_{\epsilon(1+w_0+\bar{w})}(p)$ pour une certaine constante w_0 et une certaine fonction $\bar{w}$, définie sur la sphère euclidienne S^{n-1} (après avoir identifié $\partial B^g_{\epsilon(1+w_0)}(p)$ avec S^{n-1}) et de moyenne euclidienne nulle. Des calculs directs montrent sans difficulté que $w_0 = \mathcal{O}(\epsilon^2)$. On utilise un système de coordonnées locales, qui permet de paramétrer la boule géodésique $B^g_\epsilon(p)$ avec les coordonnées y de la boule unité euclidienne $\mathring{B}_1$, et une métrique dilatée $\bar{g} = \epsilon^{-2}\, g$. Cela nous permet de traiter le problème sur des boules modifiées de $\mathbb{R}^n$,

$$\mathring{B}_{1+w_0+\bar{w}} = \{y \in \mathbb{R}^n \ : \ |y| < 1 + w_0 + \bar{w}(y/|y|)\}$$

où la métrique induite s'écrit en coordonnées $y = (y^1, ..., y^n) \in \mathring{B}_1$ sous la forme

$$\bar{g}_{ij}(y) = \delta_{ij} + \frac{1}{3}\,\epsilon^2 \sum_{k,l} R_{ikjl}\, y^k\, y^l + \frac{1}{6}\,\epsilon^3 \sum_{k,l,m} R_{ikjl,m}\, y^k\, y^l\, y^m + \mathcal{O}(\epsilon^4)$$

où les coefficients R_{ikjl} et $R_{ikjl,m}$ sont les coefficients du tenseur de Riemann et de sa dérivé covariante, et donc de donner un sens au problème pour $\epsilon = 0$. Après avoir identifié $\partial B^{\bar{g}}_{1+w_0+\bar{w}}$ avec S^{n-1}, nous pouvons alors définir l'opérateur

$$F : M \times (0, \infty) \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \longrightarrow \mathcal{C}^{2,\alpha}_m(S^{n-1})$$

qui à $(p, \epsilon, \bar{w})$ associe la donnée de Neumann de la première fonction propre $\bar{\phi}$ de l'opérateur de Laplace-Beltrami $-\Delta_{\bar{g}}$ sur la boule $B^{\bar{g}}_{1+w_0+\bar{w}}$ moins sa moyenne dans la métrique $\bar{g}$ (l'indice m dans $\mathcal{C}^{2,\alpha}_m(S^{n-1})$ est là pour indiquer que les fonctions ont moyenne nulle) :

$$F(p, \epsilon, \bar{w}) := \bar{g}(\nabla\phi, \bar{\nu})\,|_{\partial B^{\bar{g}}_{1+w_0+\bar{w}}} - \frac{1}{\mathrm{Vol}_{\bar{g}}(\partial B^{\bar{g}}_{1+w_0+\bar{w}})} \int_{\partial B^{\bar{g}}_{1+w_0+\bar{w}}} \bar{g}(\nabla\bar{\phi}, \bar{\nu})\, \mathrm{dvol}_{\bar{g}}$$

où $\bar{\nu}$ est le vecteur normal au bord dans la métrique $\bar{g}$. Grâce à la caractérisation des domaines extrémaux, il nous reste à résoudre l'équation $F(p,\epsilon,\bar{w}) = 0$. Il s'agit d'un opérateur non linéaire et non local. Son linéarisé par rapport à $\bar{w}$ au point $(0,0,0)$ est un opérateur auto-adjoint, elliptique d'ordre 1. Ceci représente une différence essentielle entre l'étude des domaines extrémaux et l'étude des hypersurfaces à courbure moyenne constante où l'opérateur qui apparaît est un opérateur différentiel elliptique d'ordre 2. Une étude détaillée de cet opérateur montre que son noyau est donné par l'ensemble des fonctions linéaires restreintes à la sphère unité. Ensuite, grâce au théorème des fonctions implicites, on conclut que pour tout $p \in M$ et tout ϵ suffisamment petit, il existe une fonction $\bar{w}(p,\epsilon)$ et un vecteur $a(p,\epsilon)$ de $\mathbb{R}^n$ tels que

$$F(p,\epsilon,\bar{w}) = \mathring{g}(a,\cdot)$$

où $\mathring{g}(\cdot,\cdot)$ représente le produit scalaire dans $\mathbb{R}^n$. Des estimations permettent de démontrer que $\bar{w} = \mathcal{O}(\epsilon^2)$. Enfin, il suffit maintenant de chercher, pour tout ϵ suffisamment petit, les points p tels que $a(p,\epsilon) = 0$. En utilisant le développement limité de la métrique $\bar{g}$ ci-dessus, nous montrons qu'il existe une constante C telle que

$$a(p,\epsilon) = C\,\epsilon^3\,\nabla^g \mathrm{Scal}(p) + \mathcal{O}(\epsilon^4)\,.$$

Il suffit donc d'appliquer le théorème des fonctions implicites pour conclure que si p_0 est un point critique non dégénéré de la courbure scalaire, alors pour tout ϵ suffisamment petit il existe un point $p_\epsilon \in M$, proche de p_0, tel que $a(p_\epsilon,\epsilon) = 0$. Ceci complète la preuve du résultat.

Il est important de remarquer que O. Druet a obtenu en 2008 une réciproque partielle de notre premier résultat (voir [11]) : en effet, il démontre que les solutions du profil de Faber-Krahn pour les petits volumes sont des perturbations de boules géodésiques centrées en un point où la courbure scalaire est maximale. Précisément :

Théorème 1.0.6. ***(Druet [11]).*** *Soit (M,g) une variété riemannienne compacte de dimension au moins 2. On note*

$$\max_{p\in M} \mathrm{Scal}(p) = n(n-1)\,K$$

où Scal *est la courbure scalaire de la variété. Pour tout $\eta > 0$, il existe $V_\eta > 0$ tel que*

$$FK_V^g \geq FK_V^{g_{K+\eta}}$$

pour tout $0 < V < V_\eta$, où $g_{K+\eta}$ est la métrique de courbure sectionnelle constante $K+\eta$. De plus

$$\lambda_{B_\epsilon^g(p)} = \lambda_{\mathring{B}_1}\,\epsilon^{-2} - \frac{1}{6}\,\mathrm{Scal}(p) + o(1)$$

quand ϵ tend vers 0 pour tout point p d'une variété riemannienne complète (M, g).

L'étude de l'opérateur linéarisé du problème précédent permet de donner des nouveaux exemples de domaines extrémaux dans $\mathbb{R}^n \times \mathbb{R}/T\,\mathbb{Z}$. Nous partons du fait que dans $\mathbb{R}^n \times \mathbb{R}/T\,\mathbb{Z}$ le domaine cylindrique $B_r \times \mathbb{R}/T\,\mathbb{Z}$, où B_r est la boule de rayon $r > 0$ dans $\mathbb{R}^n$, est un domaine extrémal. En étudiant le linéarisé de l'opérateur elliptique du premier ordre défini par le problème précédent et en utilisant un résultat de bifurcation, nous démontrons l'existence de domaines extrémaux non triviaux dans $\mathbb{R}^n \times \mathbb{R}/T\,\mathbb{Z}$ (résultat paru en 2009 dans la revue Calculus of Variations and Partial Differential Equations, [37]). Ces nouveaux domaines extrémaux sont proches de domaines cylindriques $B_r \times \mathbb{R}/T\,\mathbb{Z}$. Si ces domaines sont toujours invariants sous l'action de la rotation autour de l'axe vertical, ces domaines ne sont plus invariants sous l'action de toutes les translations verticales. Voici l'énoncé précis du résultat :

Théorème 1.0.7. *Il existe un nombre réel positif $T_* < \dfrac{2\pi}{\sqrt{n-1}}$, une suite de nombres réels positifs $T_j \longrightarrow T_*$ et une suite de fonctions non nulles $v_j \in \mathcal{C}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$ qui converge vers* 0 *dans $\mathcal{C}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$ tels que les domaines*

$$\Omega_j = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R}/T_j\,\mathbb{Z} \qquad , \qquad |x| < 1 + v_j\left(\frac{2\pi}{T_j}\,t\right) \right\}$$

sont extrémaux pour la première valeur propre du Laplacien dans la variété $\mathbb{R}^n \times \mathbb{R}/T_j\,\mathbb{Z}$ avec métrique plate.

On peut remarquer immédiatement que, grâce à la caractérisation des domaines extrémaux, ces domaines de $\mathbb{R}^n \times \mathbb{R}/T\,\mathbb{Z}$ peuvent être prolongés par périodicité à des domaines non compacts de $\mathbb{R}^n$ sur lesquels la première fonction propre du Laplacien euclidien avec donnée de Dirichlet 0 au bord a sa donnée de Neumann au bord constante ! Ce résultat donne alors un contre-exemple à une conjecture de Berestycki, Caffarelli et Nirenberg énoncée en 1997, [3]. Selon cette conjecture, si Ω est un domaine régulier tel que $\mathbb{R}^n \backslash \overline{\Omega}$ est connexe et pour lequel il existe une solution bornée et positive du problème surdéterminé

$$\begin{cases} \Delta u + f(u) = 0 & \text{sur } \Omega \\ u = 0 & \text{sur } \partial\Omega \\ \dfrac{\partial u}{\partial \nu} = \text{constante} & \text{sur } \partial\Omega\,, \end{cases} \tag{1.1}$$

pour une fonction Lipschitzienne f, alors Ω est nécessairement : un demi-espace, une boule, le complément d'une boule ou de $\mathbb{R}^j \times B$ où B est une boule. L'existence de ces contre-exemple à cette conjecture montre la nécessité de trouver les bonnes hypothèses portant sur la fonction f ou sur le domaine Ω pour que la conjecture soit vraie. A cet égard, il faut remarquer que A. Farina et E. Vadinoci ont publié des résultats en 2009 pour les petites dimensions (voir [14]) : en particulier, ils démontrent que dans le cas de dimension 2, et dans le cas de dimension 3 avec la condition $f(r) \geq 0$ pour tout $r \geq 0$, si u est une solution de (1.1) alors Ω n'est pas un épigraphe uniformement Lipschitzien coercif (i.e. la partie de $\mathbb{R}^n$ qui se trouve au dessus du graphe d'une certaine fonction Γ de $\mathbb{R}^{n-1}$ à valeurs dans $\mathbb{R}$ uniformement Lipschitzienne et telle que $\lim_{|x|\to+\infty} \Gamma(x) = 0$).

Pour décrire brièvement la preuve de notre théorème, on considère l'espace $\mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ des fonctions sont paires et de moyenne nulle. Pour toute fonction $v \in \mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ de norme suffisamment petite, soit $\phi = \phi_{v,T}$ l'unique solution positive du problème

$$\begin{cases} \Delta_{\mathring{g}}\, \phi + \lambda\, \phi &= 0 \quad \text{sur} \quad C^T_{1+v} \\ \phi &= 0 \quad \text{sur} \quad \partial C^T_{1+v} \end{cases}$$

où

$$C^T_{1+v} := \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R}/T\mathbb{Z} \qquad : \qquad 0 \leq |x| < 1 + v\left(\frac{2\pi t}{T}\right) \right\}.$$

et $\mathring{g}$ est la métrique plate. D'après la caractérisation des domaines extrémaux, il suffit de résoudre l'équation

$$N(v,T) := \mathring{g}(\nabla\phi, \nu)\,|_{\partial C^T_{1+v}} - \frac{1}{\text{Vol}_{\,\mathring{g}}(\partial C^T_{1+v})} \int_{\partial C^T_{1+v}} \mathring{g}(\nabla\phi,\nu)\,\text{dvol}_{\mathring{g}} = 0$$

On démontre que le linéarisé H_T de N au point $(0,T)$ est un opérateur auto-adjoint elliptique d'ordre 1 qui préserve chaque espace V_k engendré par la fonction $\cos(kt)$. De plus, en se servant de certains résultats de la théorie des fonctions de Bessel, on montre qu'il existe un certain $T_* < \frac{2\pi}{\sqrt{n-1}}$ pour lequel H_{T_*} a un noyau de dimension fini de la forme $V_{k_1} \oplus \cdots \oplus V_{k_l}$, la valeur propre de H_T associée à l'espace propre V_{k_1} change de signe en $T = T_*$, et les valeurs propres de H_T associées aux autres espaces propres $V_{k_2}, ..., V_{k_l}$ ne changent pas de signe en $T = T_*$. Le fait que le linéarisé a un noyau de dimension finie nous permet de faire une réduction de Lyapunov-Schmidt et de nous ramener à une équation en dimension finie. Le fait qu'il y ait une seule valeur propre de H_T qui change de signe en T_* permet d'appliquer le théorème de bifurcation de Krasnosel'skii, en démontrant que $(0, T_*)$ est un point de bifurcation de la solution de l'équation en dimension finie qui reste à résoudre, par rapport à la courbe de solutions

$$\{(0,T) \mid T \in (T_* - \epsilon, T_* + \epsilon)\}$$

et cela est équivalent à dire qu'il existe une suite (v_j, T_j), $v_j \not\equiv 0$, qui converge vers $(0, T_*)$ telle que la donnée de Neumann de la fonction propre ϕ_{v_j,T_j} est constante, et donc les domaines $C^{T_j}_{1+v_j}$ sont extrémaux pour la première valeur propre du Laplacien de la variété $\mathbb{R}^n \times \mathbb{R}/T_j\,\mathbb{Z}$ avec métrique plate.

Jusqu'à présent, nous avons seulement considéré le cas de contraintes de petits volumes. La construction de domaines extrémaux pour de grands volumes est techniquement plus difficile et fait apparaître des phénomènes nouveaux. A la lumière des résultats précédents, il est raisonnable de penser qu'il existe de domaines extrémaux qui ressemblent au complément d'une boule géodésique. Pour vérifier cela il faut calculer la donnée de Neumann de la première fonction propre de l'opérateur de Laplace-Beltrami hors d'un petit domaine d'une variété, ce qui fait apparaitre, d'une manière à préciser, la première fonction propre (positive) ϕ_0 sur toute la variété. Dans ce cadre, nous devons donc distinguer deux cas selon que la première fonction propre de l'opérateur de Laplace-Beltrami sur la variété est non constante (Cas 1) ou constante (Cas 2). Le premier cas correspond au cas où la variété ambiante est une variété à bord et où l'on considère des données de Dirichlet nulles sur le bord ; le deuxième cas correspond au cas où la variété ambiante n'a pas de bord ou bien au cas où l'on considère des données de Neumann nulles sur le bord de la variété. Dans ce cadre, les résultats que nous avons obtenus sont les suivants : si ϕ_0 a des points critiques non dégénérés (donc en particulier ϕ_0 n'est pas la fonction constante), alors pour tout volume assez proche du volume de la variété, il existe un domaine extrémal obtenu en perturbant le complément d'une boule géodésique centrée en un des points critiques non dégénérés de ϕ_0 ; si la première fonction propre de l'opérateur de Laplace-Beltrami est constante, alors pour tout volume assez proche du volume de la variété il existe un domaine extrémal pour la première valeur propre de l'opérateur de Laplace-Beltrami obtenu en perturbant le complément de boules géodésiques centrées en des points critiques non dégénérés de la fonction courbure scalaire. Voici l'énoncé précis du résultat démontré dans le chapitre 4.

Théorème 1.0.8. *Soit M une variété compacte. Dans le Cas 1, on suppose que p_0 est un point critique non dégénéré de la première fonction propre ϕ_0, et dans le Cas 2 on suppose que p_0 est un point critique non dégénéré de la fonction courbure scalaire* Scal *de M. Dans ce deuxième cas nous supposons également que la dimension n de la variété est au moins* 4. *Alors, pour tout ϵ suffisamment petit il existe un domaine régulier $\Omega_\epsilon \subset M$ tel que :*

(i) Le volume de Ω_ϵ est égale au volume euclidien de $\mathring{B}_\epsilon$.

(ii) Le domaine $M\backslash\Omega_\epsilon$ est extrémal.

De plus il existe une constante $c > 0$ et pour tout $\epsilon \in (0, \epsilon_0)$ il existe un point $p_\epsilon \in M$ tel que le bord de $M\backslash\Omega_\epsilon$ est un graphe normal sur $\partial B_\epsilon(p_\epsilon)$ pour une certaine fonction w_ϵ, avec

$$\mathrm{dist}(p_\epsilon, p_0) \le c\,\epsilon\,.$$

et

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}\partial B_\epsilon(p_\epsilon))} \leq c\,\epsilon^2 \quad \text{dans le Cas 1 et } n \geq 3$$

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}\partial B_\epsilon(p_\epsilon))} \leq c\,\epsilon^2 \log \epsilon \quad \text{dans le Cas 1 et } n = 2$$

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}\partial B_\epsilon(p_\epsilon))} \leq c\,\epsilon^3 \quad \text{dans le Cas 2 et } n \geq 5$$

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}\partial B_\epsilon(p_\epsilon))} \leq c\,\epsilon^3 \log \epsilon \quad \text{dans le Cas 2 et } n = 4$$

La première différence que l'on remarque, par rapport au premier résultat, est que dans le cas des contraintes de grands volumes il se présentent au moins deux types de domaines extrémaux : ceux qui sont obtenus en supprimant de la variété un petit domaine centré en un maximum non dégénéré de la première fonction propre sur toute la variété (Cas 1), et ceux qui sont obtenus en supprimant de la variété un petit domaine centré en un point critique non dégénéré de la courbure scalaire (Cas 2). A première vue, on pourrait penser que ce résultat pour le Cas 2 puisse s'obtenir de manière assez facile à partir du premier résultat. En réalité, ce n'est pas le cas, et la construction de domaines extrémaux de grands volumes est techniquement plus difficile. Dans le premier résultat, revenant aux notations précédemment introduites, nous avons utilisé un système de coordonnées locales qui permet de paramétriser la boule géodésique $B^g_\epsilon(p)$ avec les coordonnées y de la boule unité euclidienne $\mathring{B}_1$. Cela, avec une métrique dilatée $\bar{g} = \epsilon^2\, g$, nous permettait de donner un sens au problème pour $\epsilon = 0$ et de travailler sur un domaine qui ne dépend pas de ϵ avec une métrique qui diffère de la métrique euclidienne par des termes d'ordre ϵ^2. Maintenant, comme le domaine en question est le complément de $B^g_\epsilon(p)$, c'est-à-dire $M \setminus B^g_\epsilon(p)$, on ne peut plus travailler sur un sous-ensemble de $\mathbb{R}^n$ avec une métrique dilatée proche de la métrique euclidienne. De plus, dans le premier résultat, pour construire la première fonction propre de l'opérateur de Laplace-Beltrami sur $B^g_{\epsilon(1+w_0+w)}(p)$ nous pouvions raisonner par perturbation de la première fonction propre du Laplacien euclidien sur la boule $\mathring{B}_\epsilon$, vu que la métrique g est localement euclidienne. En revanche, pour la première fonction propre de l'opérateur de Laplace-Beltrami dans le complément du domaine $B^g_{\epsilon(1+w_0+w)}(p)$, nous ne pouvons plus partir d'une fonction du problème limite. Nous sommes obligé alors de faire une étude auxiliaire, pour trouver une bonne approximation de la première fonction propre sur $M \setminus B^g_\epsilon(p)$, en utilisant les espace à poids $C^{k,\alpha}_\nu$. Dans cette étude, nous avons pu démontrer que si $n \geq 3$ pour tout ϵ suffisamment petit il existe $(\Lambda_\epsilon, \varphi_\epsilon, w_\epsilon)$ dans un voisinage de $(0,0,0)$ dans $\mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathcal{C}^{2,\alpha}_\nu(M \setminus \{p\})$, $\nu \in (2-n, \min\{4-n, 0\})$, de norme suffisamment petite tel que la première fonction propre sur $M \setminus B^g_\epsilon(p)$ peut se décompose sous la forme

$$\phi_\epsilon = \phi_0 - \epsilon^{n-2}\,(\phi_0(p) + \Lambda_\epsilon)\,\Gamma_p + w_\epsilon + \chi\, H_{\varphi_\epsilon}$$

où χ est une fonction troncature égale à 1 dans $B^g_{R_0}(p)$ et égale à 0 dans $M \setminus B^g_{2R_0}(p)$, Γ_p est la fonction de Green sur la variété M avec pôle au point p (voir [2]), et $H_\varphi(x)$ représente le prolongement harmonique borné de la fonction $\varphi \in C^{2,\alpha}(S^{n-1})$ sur $\mathbb{R}^n \setminus \mathring{B}_\epsilon$. Λ_ϵ et w_ϵ sont de norme suffisamment petite dans les espaces à poids $C^{2,\alpha}_\nu$. Quand $n = 2$, alors ϕ_ϵ se décompose comme

$$\phi_\epsilon = \phi_0 - (\log \epsilon)^{-1} (\phi_0(p) + \Lambda_\epsilon)\, \Gamma_p + w_\epsilon + \chi\, H_{\varphi_\epsilon}$$

et w_ϵ appartient à l'espace $\mathbb{R} \times C^{2,\alpha}_\nu(M \setminus B^g_\epsilon(p))$ pour $\nu \in (0,1)$. Comme on le voit facilement, si ϕ_0 n'est pas une fonction constante, alors le premier terme du développement limité en puissances de ϵ de la donnée de Neumann de ϕ_ϵ au bord $\partial B^g_\epsilon(p)$ fait intervenir le gradient de ϕ_0 au point p. Autrement, si ϕ_0 est constante alors nous sommes obligé de trouver un développement limité pour Γ_p et H_{φ_ϵ}. Quand la dimension de la variété est au moins 4 les premiers termes du développement limité de la fonction Γ_p, à partir duquel on peut reconstruire celui de H_{φ_ϵ}, ne dépendent pas de la géométrie globale de la variété et pour cette raison nous avons pu les calculer. Par contre en dimension 2 ou en dimension 3, les premier termes de ce développement limité dépendent de la géométrie globale de la variété et nous n'avons pas su les calculer explicitement.

Après cette étude auxiliaire, nous pouvons reprendre la stratégie du premier résultat, en en gardant les notations. Après avoir identifié $\partial B^g_{\epsilon(1+w_0+\bar{w})}(p) = \partial B^{\bar{g}}_{1+w_0+\bar{w}}(p)$ avec S^{n-1}, nous allons alors définir l'opérateur

$$F : M \times (0,\infty) \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \longrightarrow \mathcal{C}^{2,\alpha}(S^{n-1})$$

qui à $(p, \epsilon, \bar{w})$ associe la donnée de Neumann au bord de la première fonction propre de l'opérateur de Laplace-Beltrami sur le complément de la boule $B^{\bar{g}}_{1+w_0+\bar{w}}$ moins sa moyenne dans la métrique $\bar{g}$. La stratégie consiste alors à résoudre l'équation $F(p, \epsilon, \bar{w}) = 0$. Son linéarisé par rapport à $\bar{w}$ au point $(0,0,0)$ est un opérateur auto-adjoint, elliptique d'ordre 1, dont on connais bien le spectre. Il préserve les espaces propres du Laplacien sur la sphère unité et son noyau est donné par l'ensemble des fonctions linéaires restreintes à la sphère unité. Ensuite, grâce au théorème des fonctions implicites, on conclut que pour tout $p \in M$ et tout ϵ suffisamment petit, il existe une fonction $\bar{w}(p,\epsilon)$ et un vecteur $a(p,\epsilon)$ de $\mathbb{R}^n$ tels que

$$F(p, \epsilon, \bar{w}) = \mathring{g}(a, \cdot)$$

où $\mathring{g}(\cdot,\cdot)$ représente le produit scalaire dans $\mathbb{R}^n$. En utilisant le dévéloppement limité de la fonction ϕ_ϵ ci-dessus, nous montrons qu'il existe une constante C telle que

$$a(p,\epsilon) = -\epsilon\, C\, \nabla^g \phi_0(p) + \mathcal{O}(\epsilon^2) \qquad \text{si } \phi_0 \text{ est constante et } n \geq 3$$

$$a(p,\epsilon) = -\epsilon\, \log\epsilon\, C\, \nabla^g \phi_0(p) + \mathcal{O}(\epsilon) \qquad \text{si } \phi_0 \text{ est constante et } n = 2$$

$$a(p,\epsilon) = -\epsilon^3\, C\, \nabla^g \mathrm{Scal}(p) + \mathcal{O}(\epsilon^4) \qquad \text{si } \phi_0 \text{ n'est pas constante et } n \geq 5$$

$$a(p,\epsilon) = -\epsilon^3\, \log\epsilon\, C\, \nabla^g \mathrm{Scal}(p) + \mathcal{O}(\epsilon^3) \qquad \text{si } \phi_0 \text{ n'est pas constante et } n = 4$$

Il suffit donc d'appliquer le théorème des fonctions implicites à nouveau pour obtenir le résultat cherché.

Pour conclure et donner des perspectives de travail, l'étude des domaines extré-maux pour la première valeur propre de l'opérateur de Laplace-Beltrami peut aussi être posée dans le cadre des variétés à bord. Dans ce cadre on peut espérer démontrer qu'il existe des domaines extrémaux pour la première valeur propre de l'opérateur de Laplace-Beltrami, qui puissent se construit en perturbant une demi-boule géodésique centrée dans un certain point du bord de la variété pour les petits volumes, ou son complément pour les grands volumes. Avec un tel résultat on aurait un panorama global de résultats d'existence de domaines extrémaux pour les petits et les grands volumes, et par conséquence, on pourrait s'interroger sur leur unicité. Deux autres perspectives de travail sont les suivantes :

1. Généraliser le troisième résultat pour les dimensions 2 et 3 quand la première fonction propre de l'opérateur de Laplace-Beltrami sur toute la variété est constante. Cela se ramène à trouver le développement limité de la fonction de Green Γ_p autour du point p, qui devrait faire apparaitre des propriétés de la géométrie globale de la variété.
2. Etendre les résultats en s'affranchissant de l'hypothèse de non dégénéréscence des points critiques de la fonction courbure scalaire. Le meme problème qui se posait pour la construction de surfaces à courbure moyenne nulle après le résultat de R. Ye énoncé ci-dessus a été résolu par F. Pacard et X. Xu (voir [33]). En s'inspirant donc à ce résultat, on pourrait espérer montrer son pendant dans le cas des domaines extrémaux.

Chapitre 2

Extremal domains of small volume

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Extremal domains for the first eigenvalue of the Laplace-Beltrami operator

Résumé. Nous prouvons l'existence de domains éxtremaux de petit volume pour la première valeur propre de l'opérateur de Laplace-Beltrami dans certaines variétés riemanniennes. Ces domains ressemblent à des sphères géodésiques de rayon petit centrées en un point critique non dégénéré de la courbure scalaire.

Abstract. We prove the existence of extremal domains with small prescribed volume for the first eigenvalue of Laplace-Beltrami operator in some Riemannian manifold. These domains are close to geodesic spheres of small radius centered at a nondegenerate critical point of the scalar curvature.

2.1 Statement of the result

Assume that we are given (M, g) an n-dimensional Riemannian manifold. If Ω is a domain with smooth boundary in M, we denote by λ_Ω the first eigenvalue of $-\Delta_g$, the Laplace-Beltrami operator, in Ω with 0 Dirichlet boundary condition. A smooth domain $\Omega_0 \subset M$ is said to be extremal if $\Omega \longmapsto \lambda_\Omega$ is critical at Ω_0 with respect to variations of the domain Ω_0 which preserve its volume. In order to make this definition precise, we first introduce the definition of *deformation* of Ω_0.

Definition 2.1.1. We say that $\{\Omega_t\}_{t\in(-t_0,t_0)}$ is a deformation of Ω_0, if there exists a vector field Ξ such that $\Omega_t = \xi(t, \Omega_0)$ where $\xi(t, \cdot)$ is the flow associated to Ξ, namely

$$\frac{d\xi}{dt}(t,p) = \Xi(\xi(t,p)) \qquad \text{and} \qquad \xi(0,p) = p\,.$$

The deformation is said to be volume preserving if the volume of Ω_t does not depend on t.

If $\{\Omega_t\}_{t\in(-t_0,t_0)}$ is a domain deformation of Ω_0, we denote by λ_t the first eigenvalue of $-\Delta_g$ on Ω_t, with 0 Dirichlet boundary conditions. Observe that both $t \longmapsto \lambda_t$ and the associated eigenfunction $t \longmapsto u_t$ (normalized to be positive and have $L^2(\Omega_t)$ norm equal to 1) inherits the regularity of the deformation of Ω_0. These facts are standard and follow at once from the implicit function theorem together with the fact that the least eigenvalue of the Laplace-Beltrami operator with Dirichlet boundary condition is simple.

We can now give the definition of an extremal domain for the first eigenvalue of $-\Delta_g$ under Dirichlet boundary condition.

Definition 2.1.2. A domain Ω_0 is an *extremal domain* for the first eigenvalue of $-\Delta_g$ if for any volume preserving deformation $\{\Omega_t\}_t$ of Ω_0, we have

$$\frac{d\lambda_t}{dt}|_{t=0} = 0\,,$$

where λ_t is the first eigenvalue of $-\Delta_g$ on Ω_t, with 0 Dirichlet boundary condition.

For all $\epsilon > 0$ small enough, we denote by $B_\epsilon(p) \subset M$ the geodesic ball of center $p \in M$ and radius ϵ. We denote by $\mathring{B}_\epsilon \subset \mathbb{R}^n$ the Euclidean ball of radius ϵ centered at the origin.

Now we can state the main result of our paper :

Theorem 2.1.3. *Assume that p_0 is a nondegenerate critical point of* Scal*, the scalar curvature function of (M, g). Then, for all $\epsilon > 0$ small enough, say $\epsilon \in (0, \epsilon_0)$, there exists a smooth domain $\Omega_\epsilon \subset M$ such that :*

(i) The volume of Ω_ϵ is equal to the Euclidean volume of $\mathring{B}_\epsilon$.

(ii) The domain Ω_ϵ is extremal in the sense of definition 2.1.2.

Moreover, there exists $c > 0$ and, for all $\epsilon \in (0, \epsilon_0)$, there exists $p_\epsilon \in M$ such that the boundary of Ω_ϵ is a normal graph over $\partial B_\epsilon(p_\epsilon)$ for some function w_ϵ with

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}\partial B_\epsilon(p_\epsilon)} \leq c\,\epsilon^3. \qquad \textit{and} \qquad \operatorname{dist}(p_\epsilon, p_0) \leq c\,\epsilon\,.$$

To put this result in perspective let us digress slightly and recall a few facts about the existence of constant mean curvature hypersurfaces in Riemannian manifolds. It is well known that solutions of the isoperimetric problem

$$I_\tau := \min_{\Omega\subset M\,:\,\operatorname{Vol}\Omega=\tau} \operatorname{Vol}\partial\Omega$$

are (where they are smooth enough) constant mean curvature hypersurfaces. O. Druet [10] has proved that for small volumes (i.e. $\tau > 0$ small), the solutions of the isoperimetric problem are close (in a sense to be made precise) to geodesic spheres of small radius centered at a point where the scalar curvature function on (M, g) is maximal. Independently, R. Ye [40] has constructed constant mean curvature topological spheres which are close to geodesic spheres of small radius centered at a nondegenerate critical point of the scalar curvature function on (M, g). Building on these results and a result of F. Pacard and X. Xu [33], S. Narduli [28] has obtained an asymptotic expansion of I_τ as τ tends to 0.

It is well known ([13], [25], [26]) that the determination of the isoperimetric profile I_τ is related to the Faber-Krahn inequality where one looks for the least value of the first eigenvalue of the Laplace-Beltrami operator amongst domains with prescribed volume

$$FK_\tau := \min_{\Omega \subset M \,:\, \mathrm{Vol}\,\Omega = \tau} \lambda_\Omega$$

Observe that a solution to this minimizing problem (when it is smooth) is an extremal domain in the sense of Definition 2.1.2. Therefore, Theorem 2.1.3 can be understood as a first step in understanding the asymptotics of FK_τ as τ is close to 0.

Given the crucial rôle played by the critical points of the scalar curvature in the isoperimetric problem for small volumes, it is natural to expect that the critical points of the scalar curvature function will also be at the center of the study of FK_τ as τ is close to 0 and Theorem 2.1.3 is an illustration of such a link.

As a final remark, formal computations show that the estimate on p_ϵ can be improved into

$$\mathrm{dist}(p_\epsilon, p_0) \le c\, \epsilon^2 \, .$$

Since a rigorous proof of this estimate requires some extra technicalities which would have complicated the proof, we have chosen not to provide a proof of this fact.

2.2 Preliminary results

The following well known result gives a formula for the first variation of the first eigenvalue for the Dirichlet problem under variations of the domain. This formula has been obtained by P. R. Garabedian and M. Schiffer in [16] when the underlying manifold is the euclidean space and by A. El Soufi and S. Ilias [12] (see Corollary 2.1) when the underlying manifold is a Riemannian manifold. For the sake of completeness we give here a proof based on arguments contained in a paper by D. Z. Zanger in [41] where a corresponding formula is derived for the Neumann problem.

Let (M, g) be an n-dimensional Riemannian manifold. Assume that $\{\Omega_t\}_t$ is a perturbation of a domain Ω_0 using the vector field Ξ, as defined in Definition 2.1.2. The outward unit normal vector field to $\partial\Omega_t$ is denoted by ν_t. Let $u_t \in C^2(\Omega_t)$, be the corresponding smooth one-parameter family of Dirichlet first eigenfunctions of Laplace-Beltrami operator (normalized to be positive have $L^2(\Omega_t)$ norm equal to 1) with 0 Dirichlet boundary condition. The associated eigenvalue is denoted by λ_t.

We have the :

Proposition 2.2.1. *[12] The derivative of $t \longmapsto \lambda_t$ at $t = 0$ is given by*

$$\frac{d\lambda_t}{dt}|_{t=0} = -\int_{\partial\Omega_0} (g(\nabla u_0, \nu_0))^2 \; g(\Xi, \nu_0)\, \mathrm{dvol}_g,$$

where dvol_g is the volume element on $\partial\Omega_0$ for the metric induced by g and ν_0 is the normal vector field about $\partial\Omega_0$.

Proof : We denote by ξ the flow associated to Ξ. By definition, we have

$$u_t(\xi(t,p)) = 0 \tag{2.1}$$

for all $p \in \partial\Omega_0$. Differentiating (2.1) with respect to t and evaluating the result at $t = 0$ we obtain

$$\partial_t u_0 = -g(\nabla u_0, \Xi)\,,$$

on $\partial\Omega_0$. Now $u_0 \equiv 0$ on $\partial\Omega_0$, and hence only the normal component of Ξ plays a rôle in this formula. Therefore, we have

$$\partial_t u_0 = -\, g(\nabla u_0, \nu_0)\, g(\Xi, \nu_0)\,, \tag{2.2}$$

on $\partial\Omega_0$.

We differentiate with respect to t the identity

$$\Delta_g\, u_t + \lambda_t\, u_t = 0. \tag{2.3}$$

and again evaluate the result at $t = 0$. We obtain

$$\Delta_g \partial_t u_0 + \lambda_0\, \partial_t u_0 = -\partial_t \lambda_0\, u_0\,, \tag{2.4}$$

in Ω_0. Now we multiply (2.4) by u_0 and (2.3), evaluated the result at $t = 0$, by $\partial_t u_0$, subtract the results and integrate it over Ω_0 to get :

$$\begin{aligned} \partial_t\lambda_0 \int_{\Omega_0} u_0^2\, \mathrm{dvol}_g &= \int_{\Omega_0} (\partial_t u_0\, \Delta_g u_0 - u_0\, \Delta_g \partial_t u_0)\, \mathrm{dvol}_g \\ &= \int_{\partial\Omega_0} (\partial_t u_0\, g(\nabla u_0, \nu_0) - u_0\, g(\nabla \partial_t u_0, \nu_0))\, \mathrm{dvol}_g \\ &= -\int_{\partial\Omega_0} (g(\nabla u_0, \nu_0))^2 \; g(\Xi, \nu_0)\; \mathrm{dvol}_g\,, \end{aligned}$$

where we have used (2.2) and the fact that $u_0 = 0$ on $\partial\Omega_0$ to obtain the last equality. The result follows at once from the fact that u_0 is normalized to have $L^2(\Omega_0)$ norm equal to 1. □

This result allows us to state the problem of finding extremal domains into the solvability of an over-determined elliptic problem.

Proposition 2.2.2. *A smooth domain Ω_0 is extremal if and only if there exists a positive function u_0 and a constant λ_0 such that*

$$\left\{\begin{array}{rcll} \Delta_g u_0 + \lambda_0\, u_0 & = & 0 & \text{in} \quad \Omega_0 \\ u_0 & = & 0 & \text{on} \quad \partial\Omega_0 \\ g(\nabla u_0, \nu_0) & = & \text{constant} & \text{on} \quad \partial\Omega_0\,, \end{array}\right. \tag{2.5}$$

where ν_0 is the normal vector field about $\partial\Omega_0$.

Proof : Assume that u_0 is a positive solution of (2.5). Observe that for a volume preserving variation, we have

$$\int_{\partial\Omega_0} g(\Xi, \nu_0)\, \mathrm{dvol}_g = 0.$$

Now, if λ_0 is a solution of (2.5), it is the first eigenvalue of $-\Delta_g$ on Ω_0, under Dirichlet boundary condition. Moreover, we have

$$\int_{\partial\Omega_0} \left(g(\nabla u_0, \nu_0)\right)^2\, g(\Xi, \nu_0)\, \mathrm{dvol}_g = 0,$$

and the previous Proposition shows that the domain Ω_0 is extremal in the sense of Definition 2.1.2.

Conversely, assume that Ω_0 is extremal. Then given any function w such that

$$\int_{\partial\Omega_0} w\, \mathrm{dvol}_g = 0,$$

it is easy to check that there exists a vector field Ξ which generates a volume preserving deformation of Ω_0 and which satisfies

$$\Xi = w\, \nu_0$$

on $\partial\Omega_0$. The result of the previous Proposition implies that

$$\int_{\partial\Omega_0} \left(g(\nabla u_0, \nu_0)\right)^2\, w\, \mathrm{dvol}_g = 0.$$

The function w being arbitrary, we conclude that $g(\nabla u_0, \nu_0)$ is a constant function and hence u_0 is a solution of (2.5). This completes the proof of the result. □

Therefore, in order to find extremal domains, it is enough to find a domain Ω_0 (regular enough) for which the over-determined problem (2.5) has a nontrivial positive solution. We will not be able to solve this problem in full generality but we will be able to find solutions whose volumes are small.

2.3 Rephrasing the problem

To proceed, it will be useful to introduce the following notation. Given a point $p \in M$ we denote by $E_1, \ldots, E_n$ an orthonormal basis of the tangent plane to M at p. Geodesic normal coordinates $x := (x^1, \ldots, x^n) \in \mathbb{R}^n$ at p are defined by

$$X(x) := \mathrm{Exp}_p^g \left(\sum_{j=1}^n x^j \, E_j \right)$$

We recall the Taylor expansion of the coefficients g_{ij} of the metric $X^* g$ in these coordinates.

Proposition 2.3.1. *At the point of coordinate x, the following expansion holds :*

$$g_{ij} = \delta_{ij} + \frac{1}{3} \sum_{k,\ell} R_{ikj\ell} \, x^k \, x^\ell + \frac{1}{6} \sum_{k,\ell,m} R_{ikjl,m} \, x^k \, x^\ell \, x^m + \mathcal{O}(|x|^4), \tag{2.6}$$

Here R is the curvature tensor of g and

$$\begin{aligned} R_{ikj\ell} &= g\big(R(E_i, E_k) \, E_j, E_\ell\big) \\ R_{ikj\ell,m} &= g\big(\nabla_{E_m} R(E_i, E_k) \, E_j, E_\ell\big) \, , \end{aligned}$$

are evaluated at the point p.

The proof of this proposition can be found in [39], [27] or also in [35].

It will be convenient to identify $\mathbb{R}^n$ with T_pM and S^{n-1} with the unit sphere in T_pM. If $x := (x^1, \ldots, x^n) \in \mathbb{R}^n$, we set

$$\Theta(x) := \sum_{i=1}^n x^i \, E_i \in T_pM \, .$$

Given a continuous function $f : S^{n-1} \longmapsto (0, \infty)$ whose L^∞ norm is small (say less than the cut locus of p) we define

$$B_f^g(p) := \big\{ \mathrm{Exp}_p(\Theta(x)) \quad : \quad x \in \mathbb{R}^n \quad 0 \leq |x| < f(x/|x|) \big\} \, .$$

The superscript g is meant to remind the reader that this definition depends on the metric.

Our aim is to show that, for all $\epsilon > 0$ small enough, we can find a point $p \in M$ and a function $v : S^{n-1} \longrightarrow \mathbb{R}$ such that

$$\mathrm{Vol}\, B^g_{\epsilon(1+v)}(p) = \epsilon^n \ \mathrm{Vol}\, \mathring{B}_1$$

and the over-determined problem

$$\begin{cases} \Delta_g \phi + \lambda \phi = 0 & \text{in } B^g_{\epsilon(1+v)}(p) \\ \phi = 0 & \text{on } \partial B^g_{\epsilon(1+v)}(p) \\ g(\nabla \phi, \nu) = \text{constant} & \text{on } \partial B^g_{\epsilon(1+v)}(p) \end{cases} \tag{2.7}$$

has a nontrivial positive solution, where ν is the normal vector field about $\partial B^g_{\epsilon(1+v)}(p)$.

Observe that, considering the dilated metric $\bar g := \epsilon^{-2} g$, the above problem is equivalent to finding a point $p \in M$ and a function $v : S^{n-1} \longrightarrow \mathbb{R}$ such that

$$\mathrm{Vol}\, B^{\bar g}_{1+v}(p) = \ \mathrm{Vol}\, \mathring{B}_1$$

and for which the over-determined problem

$$\begin{cases} \Delta_{\bar g} \bar\phi + \bar\lambda \bar\phi = 0 & \text{in } B^{\bar g}_{1+v}(p) \\ \bar\phi = 0 & \text{on } \partial B^{\bar g}_{1+v}(p) \\ \bar g(\nabla \bar\phi, \bar\nu) = \text{constant} & \text{on } \partial B^{\bar g}_{1+v}(p) \end{cases} \tag{2.8}$$

has a nontrivial positive solution, where $\bar\nu$ is the normal vector field about $\partial B^{\bar g}_{1+v}(p)$. The relation between the solutions of the two problems is simply given by

$$\phi = \epsilon^{-n/2}\, \bar\phi$$

and

$$\lambda = \epsilon^{-2}\, \bar\lambda\,.$$

Let us denote by $\mathring{g}$ the Euclidean metric in $\mathbb{R}^n$ and λ_1 the first eigenvalue of $-\Delta_{\mathring{g}}$ in the unit ball $\mathring{B}_1$ with 0 Dirichlet boundary condition. We denote by ϕ_1 the associated eigenfunction

$$\begin{cases} \Delta_{\mathring{g}} \phi_1 + \lambda_1 \phi_1 = 0 & \text{in } \mathring{B}_1 \\ \phi_1 = 0 & \text{on } \partial \mathring{B}_1 \end{cases}. \tag{2.9}$$

which is normalized to be positive and have $L^2(\mathring{B}_1)$ norm equal to 1.

For notational convenience, given a continuous function $f : S^{n-1} \longmapsto (0, \infty)$, we set

$$\mathring{B}_f := \{x \in \mathbb{R}^n \quad : \quad 0 \le |x| < f(x/|x|)\}\ .$$

The following result follows from the implicit function theorem.

Proposition 2.3.2. *Given a point $p \in M$, there exists $\epsilon_0 > 0$ and for all $\epsilon \in (0, \epsilon_0)$ and all function $\bar{v} \in C^{2,\alpha}(S^{n-1})$ satisfying*

$$\|\bar{v}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leq \epsilon_0 \,,$$

and

$$\int_{S^{n-1}} \bar{v} \ \mathrm{dvol}_{\mathring{g}} = 0 \,,$$

there exists a unique positive function $\bar{\phi} = \bar{\phi}(\epsilon, p, \bar{v}) \in \mathcal{C}^{2,\alpha}(B^{\bar{g}}_{1+v}(p))$, a constant $\bar{\lambda} = \bar{\lambda}(\epsilon, p, \bar{v}) \in \mathbb{R}$ and a constant $v_0 = v_0(\epsilon, p, \bar{v}) \in \mathbb{R}$ such that

$$\mathrm{Vol}_{\bar{g}}(B_{1+v}) = \ \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1)$$

where $v := v_0 + \bar{v}$ and $\bar{\phi}$ is a solution to the problem

$$\left\{ \begin{array}{rcll} \Delta_{\bar{g}} \, \bar{\phi} + \bar{\lambda} \, \bar{\phi} & = & 0 & \text{in} \quad B^{\bar{g}}_{1+v} \\ \bar{\phi} & = & 0 & \text{on} \quad \partial B^{\bar{g}}_{1+v} \end{array} \right. \tag{2.10}$$

which is normalized by

$$\int_{B^{\bar{g}}_{1+v}(p)} \bar{\phi}^2 \, dvol_{\bar{g}} = 1. \tag{2.11}$$

In addition $\bar{\phi}$, $\bar{\lambda}$ and v_0 depend smoothly on the function $\bar{v}$ and the parameter ϵ and $\bar{\phi} = \phi_1$, $\bar{\lambda} = \lambda_1$ and $v_0 = 0$ when $\epsilon = 0$ and $\bar{v} \equiv 0$.

Proof : Instead of working on a domain depending on the function $v = v_0 + \bar{v}$, it will be more convenient to work on a fixed domain

$$\mathring{B}_1 := \{ y \in \mathbb{R}^n \quad : \quad |y| < 1 \},$$

endowed with a metric depending on both ϵ and the function v. This can be achieved by considering the parameterization of $B^{\bar{g}}_{1+v} (= B^{g}_{\epsilon(1+v)})$ given by

$$Y(y) := \mathrm{Exp}^{\bar{g}}_p \left(\left(1 + v_0 + \chi(y) \, \bar{v} \left(\frac{y}{|y|} \right) \right) \sum_i y^i \, E_i \right)$$

where χ is a cutoff function identically equal to 0 when $|y| \leq 1/2$ and identically equal to 1 when $|y| \geq 3/4$.

Hence the coordinates we consider from now on are $y \in \mathring{B}_1$ and in these coordinates the metric $\hat{g} := Y^* \bar{g}$ can be written as

$$\hat{g} = (1 + v_0)^2 \left(\mathring{g} + \sum_{i,j} C^{ij} \, dy_i \, dy_j \right) ,$$

where the coefficients $C^{ij} \in \mathcal{C}^{1,\alpha}(\mathring{B}_1)$ are functions of y depending on ϵ, $v = v_0 + \bar{v}$ and the first partial derivatives of v. Moreover, $C^{ij} \equiv 0$ when $\epsilon = 0$ and $\bar{v} = 0$.

Observe that

$$(\epsilon, v_0, \bar{v}) \longmapsto C^{ij}(\epsilon, v)\,,$$

are smooth maps.

Up to some multiplicative constant, the problem we want to solve can now be rewritten in the form

$$\left\{ \begin{array}{rcll} \Delta_{\hat{g}}\, \hat{\phi} + \hat{\lambda}\, \hat{\phi} & = & 0 & \text{in} \quad \mathring{B}_1 \\ \hat{\phi} & = & 0 & \text{on} \quad \partial \mathring{B}_1 \end{array} \right. \tag{2.12}$$

with

$$\int_{\mathring{B}_1} \hat{\phi}^2 \, \mathrm{dvol}_{\hat{g}} = 1 \tag{2.13}$$

and

$$\mathrm{Vol}_{\hat{g}}(\mathring{B}_1) = \ \mathrm{Vol}_{\mathring{g}}\,(\mathring{B}_1) \tag{2.14}$$

When $\epsilon = 0$ and $\bar{v} \equiv 0$, the metric $\hat{g} = (1 + v_0)^2\, \mathring{g}$ is nothing but the Euclidean metric and a solution of (2.9) is therefore given by $\hat{\phi} = \phi_1$, $\hat{\lambda} = \lambda_1$ and $v_0 = 0$. In the general case, the relation between the function $\bar{\phi}$ in the statement of the Proposition and the function $\hat{\phi}$ is simply given by

$$Y^* \bar{\phi} = \hat{\phi} \qquad \text{and} \qquad \bar{\lambda} = \hat{\lambda}$$

For all $\psi \in \mathcal{C}^{2,\alpha}(\mathring{B}_1)$ such that

$$\int_{\mathring{B}_1} \psi\, \phi_1 \, \mathrm{dvol}_{\mathring{g}} = 0$$

we define

$$N(\epsilon, \bar{v}, \psi, v_0) := \Big(\Delta_{\mathring{g}} \psi + \lambda_1\, \psi + (\Delta_{\hat{g}} - \Delta_{\mathring{g}} + \mu)\, (\phi_1 + \psi)\,, \ \ \mathrm{Vol}_{\hat{g}}(\mathring{B}_1) - \ \mathrm{Vol}_{\mathring{g}}\,(\mathring{B}_1) \Big)$$

where μ is given by

$$\mu = - \int_{\mathring{B}_1} \phi_1\, (\Delta_{\hat{g}} - \Delta_{\mathring{g}})\, (\phi_1 + \psi)\, \mathrm{dvol}_{\mathring{g}}$$

so that the first entry of M is $L^2(\mathring{B}_1)$-orthogonal to ϕ_1. Observe that N also depends on the choice of the point $p \in M$.

We have

$$N(0, 0, 0, 0) = (0, 0).$$

It should be clear that the mapping N is a smooth map from a neighborhood of $(0, 0, 0, 0)$ in $[0, \infty) \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathcal{C}^{2,\alpha}_{\perp,0}(\mathring{B}_1) \times \mathbb{R}$ into a neighborhood of $(0, 0)$ in $\mathcal{C}^{0,\alpha}_{\perp}(\mathring{B}_1) \times \mathbb{R}$. Here the subscript $\perp$ indicates that the functions in the corresponding space are $L^2(\mathring{B}_1)$-orthogonal to ϕ_1 (for the Euclidean metric) and the subscript 0 indicates that the functions vanish on the boundary of $\mathring{B}_1$. Finally, the subscript m indicates that the functions have mean 0 over the unit (Euclidean) sphere.

We claim that the partial differential of N with respect to ψ, computed at $(0,0,0,0)$, is given by

$$D_\psi N(0,0,0,0) = (\Delta_{\mathring{g}} + \lambda_1 \,,\, 0)$$

while the partial differential of N with respect to v_0, computed at $(0,0,0,0)$, is given by

$$\partial_{v_0} N(0,0,0,0) = \Big(0 \,,\, n \;\; \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1)\Big)$$

There is no difficulty in getting the expression of the first partial differential since $\hat{g} = \mathring{g}$ when $\epsilon = v_0 = 0$ and $\bar{v} = 0$ and hence

$$N(0,0,\psi,0) = (\Delta_{\mathring{g}}\psi + \lambda_1\,\psi + \mu\,(\phi_1 + \psi) \,,\, 0)$$

where $\mu = 0$. The derivation of the partial differential with respect to v_0 is not hard either but requires some care. Indeed, this time we have $\hat{g} = (1+v_0)^2\,\mathring{g}$ since $\bar{v} \equiv 0$ and $\epsilon = 0$ and hence

$$\begin{aligned} N(0,0,0,v_0) &= \Big(\big(((1+v_0)^{-2}-1)\,\Delta_{\mathring{g}} + \mu\big)\,\phi_1 \,,\, ((1+v_0)^n - 1)\;\; \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1)\Big) \\ &= \Big(\big(\mu - \lambda_1\,((1+v_0)^{-2}-1)\big)\,\phi_1 \,,\, ((1+v_0)^n - 1)\;\; \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1)\Big) \end{aligned}$$

where μ is given by

$$\mu = -((1+v_0)^{-2}-1)\int_{\mathring{B}_1} \phi_1\,\Delta_{\mathring{g}}\,\phi_1\,\mathrm{dvol}_{\mathring{g}} = \lambda_1\,((1+v_0)^{-2}-1).$$

So we get

$$\partial_{v_0} N(0,0,0,0) = \Big((\partial_{v_0}\mu|_{v_0=0} + 2\,\lambda_1)\,\phi_1 \,,\, n\;\; \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1)\Big)$$

and

$$\partial_{v_0}\mu|_{v_0=0} = -2\,\lambda_1$$

The claim then follows at once.

Hence the partial differential of N with respect to both ψ and v_0, computed at $(0,0,0,0)$ is precisely invertible from $\mathcal{C}^{2,\alpha}_{\perp,0}(\mathring{B}_1)\times\mathbb{R}$ into $\mathcal{C}^{0,\alpha}_{\perp}(\mathring{B}_1)\times\mathbb{R}$ and the implicit function theorem ensures, for all $(\epsilon,\bar{v})$ in a neighborhood of $(0,0)$ in $[0,\infty)\times\mathcal{C}^{2,\alpha}_m(S^{n-1})$, the existence of a (unique) $(\psi,v_0)\in\mathcal{C}^{2,\alpha}_{\perp,0}(\mathring{B}_1)\times\mathbb{R}$ such that $N(\epsilon,\bar{v},\psi,v_0)=0$. The function $\hat{\phi} := \phi_1 + \psi$ solves (2.12) and in order to have (2.13) fulfilled, it is enough to divide it by its L^2 norm.

The fact that the solution depends smoothly on the parameter ϵ, the function $\bar{v}$ and the point $p \in M$ is standard. This completes the proof of the result. $\square$

After canonical identification of $\partial B^{\bar{g}}_{1+v}(p)$ with S^{n-1}, we define, the operator F :

$$F(p,\epsilon,\bar{v}) = \bar{g}(\nabla\bar{\phi},\bar{\nu})\,|_{\partial B^{\bar{g}}_{1+v}} - \frac{1}{\mathrm{Vol}_{\bar{g}}(\partial B^{\bar{g}}_{1+v})} \int_{\partial B^{\bar{g}}_{1+v}} \bar{g}(\nabla\bar{\phi},\bar{\nu})\,\mathrm{dvol}_{\bar{g}}\,,$$

where $\bar{\nu}$ denotes the unit normal vector field to $\partial B^{\bar{g}}_{1+v}$ and $(\bar{\phi}, v_0)$ is the solution of (2.10) provided by the previous result. Recall that $v = v_0 + \bar{v}$. Schauder's estimates imply that F is well defined from a neighborhood of $M \times (0,0)$ in $M \times [0,\infty) \times \mathcal{C}^{2,\alpha}_m(S^{n-1})$ into $\mathcal{C}^{1,\alpha}_m(S^{n-1})$. The subscript m is meant to point out that the functions have mean 0. Our aim is to find $(p,\epsilon,\bar{v})$ such that $F(p,\epsilon,\bar{v}) = 0$. Observe that, with this condition, $\bar{\phi}$ will be the solution to the problem (2.8).

Following the proof of the previous result, we have the alternative expression for F.

$$F(p,\epsilon,\bar{v}) = \hat{g}(\nabla\hat{\phi},\hat{\nu})\,|_{\partial\mathring{B}_1} - \frac{1}{\mathrm{Vol}_{\hat{g}}(\partial\mathring{B}_1)} \int_{\partial\mathring{B}_1} \hat{g}(\nabla\hat{\phi},\hat{\nu})\,\mathrm{dvol}_{\hat{g}}\,,$$

where this time $\hat{\nu}$ is the the unit normal vector field to $\partial\mathring{B}_1$ using the metric $\hat{g}$.

We end this section by the proof of the :

Lemma 2.3.3. *There exists a constant $c > 0$ such that, for all $p \in M$ and all $\epsilon \geq 0$ small enough we have*

$$\|F(p,\epsilon,0)\|_{\mathcal{C}^{1,\alpha}} \leq c\,\epsilon^2$$

For all $a \in \mathbb{R}^n$, the following estimate holds

$$\left| \int_{S^{n-1}} \mathring{g}(a,\cdot)\,F(p,\epsilon,0)\;\mathrm{dvol}_{\mathring{g}} - C\,\epsilon^3\,g(\nabla\,\mathrm{Scal}(p),\Theta(a)) \right| \leq c\,\epsilon^4\,\|a\|\,,$$

where

$$C := \frac{1}{2n(n+2)}\,\frac{1}{\partial_r\phi_1(1)} \int_{\mathring{B}_1} r^2\,|\partial_r\phi_1|^2\;\mathrm{dvol}_{\mathring{g}}$$

Proof : We keep the notations of the proof of the previous result with $\bar{v} \equiv 0$. In order to prove these estimates, we follow the construction of $F(p,\epsilon,0)$ step by step. First of all, since $\bar{v} \equiv 0$, we have

$$N(\epsilon,0,0,0) = \Big((\Delta_{\hat{g}} - \Delta_{\mathring{g}} + \mu)\,\phi_1\,,\;\; \mathrm{Vol}_{\hat{g}}(\mathring{B}_1) - \;\mathrm{Vol}_{\mathring{g}}\,(\mathring{B}_1) \Big)\,,$$

and

$$\mu = -\int_{\mathring{B}_1} \phi_1\,(\Delta_{\hat{g}} - \Delta_{\mathring{g}})\,\phi_1\,\mathrm{dvol}_{\mathring{g}}\,.$$

If in addition $v_0 = 0$, we can estimate

$$\hat{g}_{ij} = \delta_{ij} + \mathcal{O}(\epsilon^2)\,,$$

hence

$$N(\epsilon, 0, 0, 0) = \mathcal{O}(\epsilon^2)\,.$$

The implicit function theorem immediately implies that the solution of

$$N(\epsilon, 0, \psi, v_0) = 0$$

satisfies

$$\|\psi(\epsilon, p, 0)\|_{\mathcal{C}^{2,\alpha}} + |v_0(\epsilon, p, 0)| \leq c\,\epsilon^2$$

To complete the proof, observe that $\hat{\nu} = (1+v_0)^{-1}\,\partial_r$ when $\bar{v} \equiv 0$. Therefore

$$\hat{g}(\nabla\hat{\phi}, \hat{\nu}) = \partial_r \phi_1 + \mathcal{O}(\epsilon^2)$$

(be careful that $\hat{g}$ is defined with $v_0 = v_0(\epsilon, p, 0)$ and $\bar{v} \equiv 0$). Since $\partial_r\phi_1$ is constant along $\partial\mathring{B}_1$, we conclude that

$$F(p, \epsilon, 0) = \mathcal{O}(\epsilon^2)$$

and this proves the first estimate.

We now turn to the proof of the second estimate. Instead of going through the construction of $\hat{\phi}$ step by step, we compute

$$\begin{aligned}
&\int_{S^{n-1}} \mathring{g}(\nabla\phi_1, a)\frac{\partial\hat{\phi}}{\partial r}\ \mathrm{dvol}_{\mathring{g}} \\
&\qquad = \int_{\mathring{B}_1} \mathring{g}(\nabla\phi_1, a)\,(\Delta_{\mathring{g}} + \lambda_1)\hat{\phi}\ \mathrm{dvol}_{\mathring{g}} - \int_{\mathring{B}_1} \hat{\phi}\,(\Delta_{\mathring{g}} + \lambda_1)\mathring{g}(\nabla\phi_1, a)\ \mathrm{dvol}_{\mathring{g}} \\
&\qquad = \int_{\mathring{B}_1} \mathring{g}(\nabla\phi_1, a)\,(\Delta_{\mathring{g}} + \lambda_1)\,\hat{\phi}\ \mathrm{dvol}_{\mathring{g}} \\
&\qquad = \int_{\mathring{B}_1} \mathring{g}(\nabla\phi_1, a)\,(\Delta_{\mathring{g}} - \Delta_{\hat{g}})\,\hat{\phi}\ \mathrm{dvol}_{\mathring{g}} + (\lambda_1 - \hat{\lambda})\int_{\mathring{B}_1} \mathring{g}(\nabla\phi_1, a)\,\hat{\phi}\ \mathrm{dvol}_{\mathring{g}} \\
&\qquad = \int_{\mathring{B}_1} \mathring{g}(\nabla\phi_1, a)\,(\Delta_{\mathring{g}} - \Delta_{\hat{g}})\,\phi_1\ \mathrm{dvol}_{\mathring{g}} + (\lambda_1 - \hat{\lambda})\int_{\mathring{B}_1} \mathring{g}(\nabla\phi_1, a)\,\phi_1\ \mathrm{dvol}_{\mathring{g}} \\
&\qquad\quad + \int_{\mathring{B}_1} \mathring{g}(\nabla\phi_1, a)\,(\Delta_{\mathring{g}} - \Delta_{\hat{g}})\,(\hat{\phi} - \phi_1)\ \mathrm{dvol}_{\mathring{g}} \\
&\qquad\quad + (\lambda_1 - \hat{\lambda})\int_{\mathring{B}_1} (\nabla\phi_1 \cdot a)\,(\hat{\phi} - \phi_1)\ \mathrm{dvol}_{\mathring{g}} \\
&\qquad = \int_{\mathring{B}_1} \mathring{g}(\nabla\phi_1, a)\,(\Delta_{\mathring{g}} - \Delta_{\hat{g}})\,\phi_1\ \mathrm{dvol}_{\mathring{g}} \\
&\qquad\quad + \int_{\mathring{B}_1} \mathring{g}(\nabla\phi_1, a)\,(\Delta_{\mathring{g}} - \Delta_{\hat{g}})\,(\hat{\phi} - \phi_1)\ \mathrm{dvol}_{\mathring{g}} \\
&\qquad\quad + (\lambda_1 - \hat{\lambda})\int_{\mathring{B}_1} \mathring{g}(\nabla\phi_1, a)\,(\hat{\phi} - \phi_1)\ \mathrm{dvol}_{\mathring{g}}
\end{aligned}$$

The last two terms can be estimated easily since $\hat{\lambda} - \lambda_1 = \mathcal{O}(\epsilon^2)$, $\hat{\phi} - \phi_1 = \mathcal{O}(\epsilon^2)$ and the coefficients of $\Delta_{\mathring{g}} - \Delta_{\hat{g}}$ are bounded by a constant times ϵ^2. Therefore, we conclude that there exists a constant c such that

$$\left| \int_{S^{n-1}} \mathring{g}(\nabla \phi_1, a) \frac{\partial \hat{\phi}}{\partial r} \; \mathrm{dvol}_{\mathring{g}} - \int_{\mathring{B}_1} \mathring{g}(\nabla \phi_1, a) \, (\Delta_{\mathring{g}} - \Delta_{\hat{g}}) \, \phi_1 \; \mathrm{dvol}_{\mathring{g}} \right| \leq c \, \epsilon^4 \, \|a\|$$

To proceed, we use the result of Proposition 2.3.1 to show that the coefficients of the metric $\hat{g}$ can be expanded as

$$\begin{aligned} \hat{g}_{ij}(y) &= (1+v_0)^2 \Bigg(\delta_{ij} + \frac{1}{3} \sum_{k,\ell} R_{ikj\ell} \, y^k \, y^\ell \, (1+v_0)^2 \, \epsilon^2 \\ &\quad + \frac{1}{6} \sum_{k,\ell,m} R_{ikjl,m} \, y^k \, y^\ell \, y^m \, (1+v_0)^3 \, \epsilon^3 + \mathcal{O}(\epsilon^4) \Bigg) \end{aligned}$$

Keeping in mind that $v_0 = \mathcal{O}(\epsilon^2)$, this simplifies slightly into

$$\hat{g}_{ij}(y) = (1+v_0)^2 \left(\delta_{ij} + \frac{1}{3} \sum_{k,\ell} R_{ikj\ell} y^k y^\ell \epsilon^2 + \frac{1}{6} \sum_{k,\ell,m} R_{ikjl,m} y^k y^\ell y^m \epsilon^3 + \mathcal{O}(\epsilon^4) \right)$$

This implies that

$$\begin{aligned} \hat{g}^{ij} &= (1+v_0)^{-2} \left(\delta_{ij} - \frac{1}{3} R_{ikj\ell} \, y^k \, y^\ell \, \epsilon^2 - \frac{1}{6} R_{ikj\ell,m} \, y^k \, y^\ell \, y^m \, \epsilon^3 \right) + \mathcal{O}(\epsilon^4) \\ \log |\hat{g}| &= 2n \, \log(1+v_0) + \frac{1}{3} R_{k\ell} \, y^k \, y^\ell \, \epsilon^2 + \frac{1}{6} R_{k\ell,m} \, y^k \, y^\ell \, y^m \, \epsilon^3 + \mathcal{O}(\epsilon^4) \end{aligned}$$

where

$$R_{k\ell} = \sum_{i=1}^{n} R_{iki\ell} \text{ and } R_{k\ell,m} = \sum_{i=1}^{n} R_{iki\ell,m}$$

Recall that

$$\Delta_{\hat{g}} := \sum_{i,j} \hat{g}^{ij} \, \partial_{y_i} \partial_{y_j} + \sum_{i,j} \partial_{y_i} \hat{g}^{ij} \, \partial_{y_j} + \frac{1}{2} \sum_{i,j} \hat{g}^{ij} \, \partial_{y_i} \log |\hat{g}| \, \partial_{y_j}$$

A straightforward calculation (still keeping in mind that $v_0 = \mathcal{O}(\epsilon^2)$) shows that

$$
\begin{aligned}
\big(\Delta_{\mathring{g}} - \Delta_{\hat{g}}\big)\phi_1 &= -\lambda_1 \left(1 - (1+v_0)^{-2}\right)\phi_1 + \\
&+ \frac{1}{3}\,\epsilon^2 \sum_{i,j,k,\ell} R_{ikj\ell}\left(\frac{y^i y^j y^k y^\ell}{r^2}\,\partial_r^2\phi_1 + \frac{y^k y^\ell}{r}\delta^i_j\,\partial_r\phi_1 - \frac{y^i y^j y^k y^\ell}{r^3}\,\partial_r\phi_1\right) + \\
&- \frac{2}{3}\,\epsilon^2 \sum_{i,j} R_{ij}\frac{y^i\,y^j}{r}\,\partial_r\phi_1 + \\
&+ \frac{1}{6}\,\epsilon^3 \sum_{i,j,k,\ell,m} R_{ikj\ell,m}\,\frac{y^i y^j y^k y^\ell y^m}{r^2}\left(\partial_r^2\phi_1 - \frac{\partial_r\phi_1}{r}\right) + \\
&+ \frac{1}{6}\,\epsilon^3 \sum_{k,j,\ell} R_{\cdot kj\ell,\cdot}\,\frac{y^j y^k y^\ell}{r}\,\partial_r\phi_1 - \frac{1}{4}\epsilon^3 \sum_{i,\ell,m} R_{i\ell,m}\frac{y^i y^\ell y^m}{r}\,\partial_r\phi_1 + \mathcal{O}(\epsilon^4),
\end{aligned}
$$

where $r := |y|$ and

$$
R_{\cdot kj\ell,\cdot} := \sum_{i=1}^{n} R_{ikj\ell,i}
$$

Observe that we have used the fact that $R(X,X) \equiv 0$ and the symmetries of the curvature tensor for which if either $i = k$ or $j = \ell$ then $R_{ikj\ell,m} = 0$.

Observe that, in the expansion of $(\Delta_{\mathring{g}} - \Delta_{\hat{g}})\,\phi_1$, terms which contain an even number of coordinates, such as $y^i y^j y^k y^\ell$ or $y^i y^j$ etc. do not contribute to the result since, once multiplied by $\mathring{g}(\nabla\phi_1, a)$, their average over S^{n-1} is 0. Therefore, we can write

$$
\begin{aligned}
\int_{\mathring{B}_1} \mathring{g}(\nabla\phi_1\, a)\,\big(\Delta_{\mathring{g}} - \Delta_{\hat{g}}\big)\,\phi_1 \quad \mathrm{dvol}_{\mathring{g}} &= \epsilon^3 \int_{\mathring{B}_1} \partial_r\phi_1\, a_\sigma \frac{y^\sigma}{r}\cdot \\
&\cdot\left(\frac{1}{6}\sum_{i,j,k,\ell,m} R_{ikj\ell,m}\,\frac{y^t y^k y^l y^i y^j}{r^2}\left(\partial_r^2\phi_1 - \frac{\partial_r\phi_1}{r}\right)\right. \\
&\left. + \frac{1}{2}\sum_{j,k,\ell}\big(\frac{1}{3}R_{\cdot kj\ell,\cdot} - \frac{1}{2}R_{kj,\ell}\big)\,\frac{y^k y^\ell y^j}{r}\,\partial_r\phi_1\right) \\
&+ \mathcal{O}(\epsilon^4),
\end{aligned}
$$

We make use of the identities in the Appendix to conclude that

$$
\begin{aligned}
&\int_{\mathring{B}_1} \mathring{g}(\nabla\phi_1, a)\,\big(\Delta_{\mathring{g}} - \Delta_{\hat{g}}\big)\,\phi_1 \quad \mathrm{dvol}_{\mathring{g}} = \\
&\qquad = \frac{1}{2n(n+2)}\,\epsilon^3\, g\big(\nabla\ \mathrm{Scal(p)}, \Theta(a)\big) \int_{\mathring{B}_1} r^2\,|\partial_r\phi_1|^2 \ \ \mathrm{dvol}_{\mathring{g}} + \mathcal{O}(\epsilon^4).
\end{aligned}
\tag{2.15}
$$

The second estimate follows at once from this computation together with the fact that, when $\bar{v} \equiv 0$, $\hat{\nu} = (1 + v_0)\, \partial_r$ as already mentioned and

$$\mathring{g}(\nabla \phi_1, a) = \partial_r \phi_1(1)\, \mathring{g}(a, \cdot)\,,$$

on $\partial \mathring{B}_1$ since this implies that

$$\int_{S^{n-1}} \mathring{g}(a, \cdot)\, \hat{g}(\nabla \hat{\phi}, \hat{\nu})|_{\partial \mathring{B}_1} \;\text{dvol}_{\mathring{g}} = \frac{1 + v_0}{\partial_r \phi_1(1)} \int_{S^{n-1}} \mathring{g}(\nabla \phi_1, a)\, \frac{\partial \hat{\phi}}{\partial r} \;\text{dvol}_{\mathring{g}}$$

This completes the proof of the result. □

Our next task will be to understand the structure of L_0, the operator obtained by linearizing F with respect to $\bar{v}$ at $\epsilon = 0$ and $\bar{v} = 0$. We will see that this operator is a first order elliptic operator which does not depend on the point p. Also, we will be interested in various properties of the expansion of $F(p, \epsilon, 0)$ in powers of ϵ.

2.4 The structure of L_0

We keep the notations of the previous section. We claim that, when $\epsilon = 0$, $\bar{g} = \mathring{g}$. Indeed, observe that, if we use coordinates

$$\bar{X}(y) := \text{Exp}_p^g \left(\epsilon \sum_i y^i\, E_i \right)$$

to parameterize a neighborhood of p in M, the coefficients $\bar{g}_{ij}$ of the metric $\bar{X}^* \bar{g} = \epsilon^{-2}\, \bar{X}^* g$ can be expanded as

$$\bar{g}_{ij}(y) = \delta_{ij} + \frac{1}{3} \sum_{k,\ell} R_{ikj\ell}\, y^k\, y^\ell\, \epsilon^2 + \frac{1}{6} \sum_{k,\ell,m} R_{ikjl,m}\, y^k\, y^\ell\, y^m\, \epsilon^3 + \mathcal{O}(\epsilon^4) \tag{2.16}$$

and, when $\epsilon = 0$, we conclude that $X^* \bar{g} = \mathring{g}$. Therefore, when $\epsilon = 0$ we have $\bar{g} = \mathring{g}$ and (2.10) becomes

$$\left\{ \begin{array}{rcll} \Delta_{\mathring{g}}\, \bar{\phi} + \bar{\lambda}\, \bar{\phi} & = & 0 & \text{in} \quad B^{\mathring{g}}_{1+v} \\[2mm] \bar{\phi} & = & 0 & \text{on} \quad \partial B^{\mathring{g}}_{1+v} \end{array} \right. \tag{2.17}$$

with the normalization

$$\int_{B^{\mathring{g}}_{1+v}} \bar{\phi}^2 \, \text{dvol}_{\mathring{g}} = 1 \tag{2.18}$$

and the volume constraint

$$\text{Vol}_{\mathring{g}}(B^{\mathring{g}}_{1+v}) = \text{Vol}_{\mathring{g}}(\mathring{B}_1)$$

Remember that we have set $v := v_0 + \bar{v}$.

We already have established the existence of a unique positive function $\bar{\phi} \in \mathcal{C}^{2,\alpha}(S^{n-1})$ (close to ϕ_1), a constant $\bar{\lambda} \in \mathbb{R}$ (close to λ_1) and a constant $v_0 \in \mathbb{R}$ (close to 0), solutions to the above problem so we are going to construct an expansion of $\bar{\phi}$, $\bar{\lambda}$ and v_0 in powers of $\bar{v}$ and its derivatives. This will lead to the structure of the linearized operator L_0.

Recall that λ_1 is the first eigenvalue of $-\Delta_{\mathring{g}}$ in the unit ball $\mathring{B}_1$ with 0 Dirichlet boundary condition and ϕ_1 is the associated eigenfunction which is normalized to be positive and have $L^2(\mathring{B}_1)$ norm equal to 1. Observe that in principle ϕ_1 is only defined in the unit ball, however, this function being radial, it is a solution of a second order ordinary differential equation and as such can be extended at least in a neighborhood of $\partial\mathring{B}_1$.

We start with the easy :

Lemma 2.4.1. *Assume that $\bar{v} \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$ is given. We define*

$$\phi_0(x) = \partial_r\phi_1(x)\, \bar{v}\, (x/|x|)$$

Then

$$\Delta_{\mathring{g}}\phi_0 + \lambda_1\, \phi_0 = \frac{1}{r^2}\, \partial_r\phi_1\, \left(\Delta_{S^{n-1}} + n - 1\right)\, \bar{v}. \tag{2.19}$$

Proof : This is a straightforward exercise. Using the fact that

$$\Delta_{\mathring{g}}\, \partial_r\phi_1 = -\lambda_1\, \partial_r\phi_1 + \frac{n-1}{r^2}\, \partial_r\phi_1\, ,$$

we find

$$\begin{aligned} \Delta_{\mathring{g}}\phi_0 &= v\, \Delta_{\mathring{g}}\partial_r\phi_1 + \partial_r\phi_1\, \Delta_{\mathring{g}}\bar{v} + 2\, \nabla\bar{v}\nabla\partial_r\phi_1 \\ &= -\lambda_1\, \phi_0 + \frac{1}{r^2}\, \partial_r\phi_1\, \left(\Delta_{S^{n-1}} + n - 1\right)\, \bar{v}\, , \end{aligned}$$

This completes the proof of the result. □

For all $\bar{v} \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$ let ψ be the (unique) solution of

$$\begin{cases} \Delta_{\mathring{g}}\psi + \lambda_1\, \psi = 0 & \text{in} \quad \mathring{B}_1 \\ \psi = -\partial_r\phi_1\, \bar{v} & \text{on} \quad \partial\mathring{B}_1 \end{cases} \tag{2.20}$$

which is $L^2(\mathring{B}_1)$-orthogonal to ϕ_1. We define

$$H(\bar{v}) := \left(\partial_r\psi + \partial_r^2\phi_1\, \bar{v}\right)\, |_{\partial\mathring{B}_1} \tag{2.21}$$

Recall that the eigenvalues of the operator $-\Delta_{S^{n-1}}$ are given by

$$\mu_j = j\, (n - 2 + j)$$

for $j \in \mathbb{N}$. The corresponding eigenspaces will be denoted by V_j.

We will need the following result :

Proposition 2.4.2. *The operator*

$$H : \mathcal{C}^{2,\alpha}_m(S^{n-1}) \longrightarrow \mathcal{C}^{1,\alpha}_m(S^{n-1}),$$

is a self adjoint, first order elliptic operator. (Recall that the subscript m is meant to point out that functions have mean 0 on S^{n-1}). The kernel of H is given by V_1, the eigenspace of $-\Delta_{S^{n-1}}$ associated to the eigenvalue $n-1$. Moreover there exists $c > 0$ such that

$$\|w\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leq c \, \|H(w)\|_{\mathcal{C}^{1,\alpha}(S^{n-1})} \, ,$$

provided w is $L^2(S^{n-1})$-orthogonal to $V_0 \oplus V_1$.

Proof : The fact that H is a first order elliptic operator is standard since it is the sum of the Dirichlet-to-Neumann operator for $\Delta_{\mathring{g}} + \lambda_1$ and a constant times the identity. In particular, elliptic estimates yield

$$\|H(w)\|_{\mathcal{C}^{1,\alpha}(S^{n-1})} \leq c \, \|w\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}$$

The fact that the operator H is (formally) self-adjoint is easy. Let ψ_1 (resp. ψ_2) the solution of (2.20) corresponding to the function w_1 (resp. w_2). We compute

$$\begin{aligned} \partial_r \phi_1(1) \int_{\partial \mathring{B}_1} (H(w_1)\, w_2 - w_1\, H(w_2))\, \mathrm{dvol}_{\mathring{g}} &= \partial_r \phi_1(1) \int_{\partial \mathring{B}_1} (\partial_r \psi_1\, w_2 - \partial_r \psi_2\, w_1)\, \mathrm{dvol}_{\mathring{g}} \\ &= \int_{\partial \mathring{B}_1} (\psi_1\, \partial_r \psi_2 - \psi_2\, \partial_r \psi_1)\, \mathrm{dvol}_{\mathring{g}} \\ &= \int_{\mathring{B}_1} (\psi_1\, \Delta_{\mathring{g}} \psi_2 - \psi_2 \Delta_{\mathring{g}}\, \psi_1)\, \mathrm{dvol}_{\mathring{g}} \\ &= 0 \end{aligned}$$

To prove the other statements, we define for all $w \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$, Ψ to be the continuous solution of

$$\left\{ \begin{array}{rcll} \Delta_{\mathring{g}} \Psi + \lambda_1\, \Psi &=& \dfrac{1}{r^2}\, \partial_r \phi_1 \; (\Delta_{S^{n-1}} + n - 1)\; w & \text{in} \quad \mathring{B}_1 \\ \Psi &=& 0 & \text{on} \quad \partial \mathring{B}_1 \, . \end{array} \right. . \tag{2.22}$$

Observe that $\partial_r \phi_1$ vanishes at first order at $r = 0$ and hence the right hand side is bounded by a constant times r^{-1} near the origin. Standard elliptic estimates then imply that the solution Ψ is at least continuous near the origin. A straightforward computation using the result of Lemma 2.4.1 and writing $\Psi = \psi + \partial_r \phi_1\, w$, shows that

$$H(w) := \partial_r \Psi|_{\partial \mathring{B}_1} \tag{2.23}$$

With this alternative definition, it should be clear that H preserves the eigenspaces V_j and in particular, H maps into the space of functions whose mean over S^{n-1} is 0. Moreover, it is clear that V_1 is included in the kernel of H since $(\Delta_{S^{n-1}} + n - 1)\, w = 0$ for any $w \in V_1$. We now prove that V_1 is the only kernel of this operator.

We consider

$$w = \sum_{j \geq 1} w_j$$

the eigenfunction decomposition of w. Namely $w_j \in V_j$. Then

$$H(w) = \sum_j \alpha_j\, w_j$$

where the constants α_j are given by

$$\alpha_j = \partial_r a_j(1)$$

where a_j is the continuous solution of

$$a_j'' + \frac{n-1}{r}\, a_j' + \lambda_1\, a_j - \frac{1}{r^2}\, \mu_j\, a_j = \frac{1}{r^2}\, (n - 1 - \mu_j)\, \partial_r \phi_1\,, \tag{2.24}$$

with $a_j(1) = 0$.

Observe that $\alpha_1 = 0$ and, in order to prove that the kernel of H is given by V_1, it is enough to show that $\alpha_j \neq 0$ for all $j \geq 2$.

We claim that

$$a_j \leq 0\,,$$

for all $j \geq 2$. This follows at once from the maximum principle since $n - 1 - \mu_j < 0$ for all $j \geq 2$ and $\partial_r \phi_1 \leq 0$.

Proof of the claim : By definition of λ_1, the operator $\Delta_{\mathring{g}} + \lambda_1$ is non-positive, in the sense that

$$-\int_{\mathring{B}_1} u\, (\Delta_{\mathring{g}} + \lambda_1)\, u \,\mathrm{dvol}_{\mathring{g}} = \int_{\mathring{B}_1} (|\nabla u|^2_{\mathring{g}} - \lambda_1\, u^2)\, \mathrm{dvol}_{\mathring{g}} \geq 0\,. \tag{2.25}$$

Specializing this inequality to radial functions, we get

$$\int_0^1 \left((\partial_r u)^2 - \lambda_1\, u^2\right)\, r^{n-1}\, \mathrm{dr} \geq 0$$

provided $u \in H^1_0(\mathring{B}_1)$ is radial.

Now, assume that $a_j \geq 0$ in $[r_1, r_2]$ with $a_j(r_i) = 0$, then multiplying (2.24) by $a_j\, r^{n-1}$ and integrating the result by parts between r_1 and r_2, we get

$$\int_{r_1}^{r_2} \left((\partial_r a_j)^2 - \lambda_1\, a_j^2 + \frac{1}{r^2}\, \mu_j\, a_j^2\right)\, r^{n-1}\, \mathrm{dr} \leq 0$$

and hence necessarily $a_j \equiv 0$ on $[r_1, r_2]$. This completes the proof of the claim.

The claim being proven, we use the fact that $a_j(1) = 0$ for all $j \geq 2$ to conclude that

$$0 \leq \partial_r a_j(1).$$

If $\partial_r a_j(1) = 0$ then necessarily $\partial_r^2 a_j(1) \leq 0$ but evaluation of (2.24) at $r = 1$ implies that

$$\begin{aligned} 0 = (n-1)\, a_j'(1) &= (n-1-\mu_j)\, \partial_r \phi_1(1) - a_j''(1) \\ &\geq (n-1-\mu_j)\, \partial_r \phi_1(1) \\ &> 0\,, \end{aligned}$$

which immediately leads to a contradiction. Hence, $\partial_r a_j(1) > 0$ for all $j \geq 2$ and this completes the proof of the fact that the kernel of the operator H is equal to V_1. □

The main result of this section is the following :

Proposition 2.4.3. *The operator L_0 is equal to H.*

Proof : By definition, the operator L_0 is the linear operator obtained by linearizing N with respect to $\bar{v}$ at $\epsilon = 0$ and $\bar{v} = 0$. In other words, we have

$$L_0(\bar{w}) = \lim_{s \to 0} \frac{F(p, 0, s\,\bar{w}) - F(p, 0, 0)}{s}.$$

Since $\epsilon = 0$, we have already seen that $\bar{g} = \mathring{g}$. Writing $\bar{v} = s\,\bar{w}$, we argue as in the proof of Proposition 2.3.2 and consider the parameterization of $\mathring{B}_{1+v}$ given by

$$Y(y) := \left(1 + v_0 + s\,\chi(y)\,\bar{w}\left(\frac{y}{|y|}\right)\right)\, y$$

where χ is a cutoff function identically equal to 0 when $|y| \leq 1/2$ and identically equal to 1 when $|y| \geq 3/4$. We set

$$\hat{g} := Y^* \mathring{g}$$

so that $\hat{\phi} = Y^* \bar{\phi}$, $\hat{\lambda} = \bar{\lambda}$ and v_0 are solutions (smoothly depending on the real parameter s) of

$$\left\{ \begin{array}{rcll} \Delta_{\hat{g}}\, \hat{\phi} + \hat{\lambda}\, \hat{\phi} &=& 0 & \text{in} \quad \mathring{B}_1 \\ \hat{\phi} &=& 0 & \text{on} \quad \partial \mathring{B}_1 \end{array} \right.$$

with

$$\int_{\mathring{B}_1} \hat{\phi}^2 \, \mathrm{dvol}_{\hat{g}} = 1$$

and

$$\mathrm{Vol}_{\hat{g}}(\mathring{B}_1) = \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1)$$

We remark that $\hat{\phi}_1 := Y^* \phi_1$ is a solution of

$$\Delta_{\hat{g}} \, \hat{\phi}_1 + \lambda_1 \, \hat{\phi}_1 = 0$$

since $\hat{g} = Y^* \mathring{g}$. Moreover

$$\hat{\phi}_1(y) = \phi_1((1 + v_0 + s\, \bar{w}(y))\, y)\,,$$

on $\partial \mathring{B}_1$. Writing $\hat{\phi} = \hat{\phi}_1 + \hat{\psi}$ and $\hat{\lambda} = \lambda_1 + \mu$, we find that

$$\left\{ \begin{array}{rcll} \Delta_{\hat{g}} \, \hat{\psi} + (\lambda_1 + \mu)\, \hat{\psi} + \mu \, \hat{\phi}_1 & = & 0 & \text{in} \quad \mathring{B}_1 \\ \hat{\psi} & = & -\hat{\phi}_1 & \text{on} \quad \partial \mathring{B}_1 \end{array} \right. \tag{2.26}$$

with

$$\int_{\mathring{B}_1} (2\, \hat{\phi}_1 \, \hat{\psi} + \hat{\psi}^2)\, \mathrm{dvol}_{\hat{g}} = \int_{\mathring{B}_1} \phi_1^2 \;\, \mathrm{dvol}_{\mathring{g}} - \int_{\mathring{B}_{1+v_0+s\bar{w}}} \phi_1^2 \;\, \mathrm{dvol}_{\mathring{g}} \tag{2.27}$$

and

$$\mathrm{Vol}_{\hat{g}}(\mathring{B}_1) = \; \mathrm{Vol}_{\mathring{g}}\,(\mathring{B}_1) \tag{2.28}$$

Obviously $\hat{\psi}$, μ and v_0 are smooth functions of s. When $s = 0$, we have $\bar{\phi} = \phi_1$, $\bar{\lambda} = \lambda_1$ and $v_0 = 0$. Therefore, $\hat{\psi}$, μ and v_0 all vanish and $\hat{\phi}_1 = \phi_1$, when $s = 0$. Moreover $\hat{g} = \mathring{g}$ when $s = 0$. We set

$$\dot{\psi} = \partial_s \hat{\psi}|_{s=0}, \qquad \dot{\mu} = \partial_s \mu|_{s=0}, \qquad \text{and} \qquad \dot{v}_0 = \partial_s v_0|_{s=0},$$

Differentiating (2.26) with respect to s and evaluating the result at $s = 0$, we obtain

$$\left\{ \begin{array}{rcll} \Delta_{\mathring{g}} \, \dot{\psi} + \lambda_1 \, \dot{\psi} + \dot{\mu}\, \phi_1 & = & 0 & \text{in} \quad \mathring{B}_1 \\ \dot{\psi} & = & -\partial_r \phi_1 \, (\dot{v}_0 + \bar{w}) & \text{on} \quad \partial \mathring{B}_1 \end{array} \right. \tag{2.29}$$

Observe that, as already mentioned, $\hat{\phi}_1(y) = \phi_1((1 + v_0 + s\, \bar{w}(y))\, y)$ on $\partial \mathring{B}_1$ and differentiation with respect to s at $s = 0$ yields $\partial_s \hat{\phi}_1|_{s=0} = \partial_r \phi_1 \, (\dot{v}_0 + \bar{w})$.

Differentiating (2.27) with respect to s and evaluating the result at $s = 0$, we obtain

$$\int_{\mathring{B}_1} \phi_1 \, \dot{\psi} \, \mathrm{dvol}_{\mathring{g}} = 0 \tag{2.30}$$

Indeed, the derivative of the right hand side of (2.27) with respect to s vanishes when $s = 0$ since ϕ_1 vanishes identically on $\partial \mathring{B}_1$.

Finally, differentiating (2.28) with respect to s and evaluating the result at $s = 0$, we obtain

$$\int_{S^{n-1}} (\dot{v}_0 + \bar{w}) \;\, \mathrm{dvol}_{\mathring{g}} = 0 \tag{2.31}$$

The last equality immediately implies (since, by definition, the average of $\bar{w}$ is 0) that $\dot{v}_0 = 0$. If we multiply the first equation of (2.29) by ϕ_1 and we integrate it, using the boundary condition and the fact that the average of $\bar{w}$ is 0 together with the fact that $\dot{v}_0 = 0$, we conclude that $\dot{\mu} = 0$. And hence $\dot{\psi}$ is precisely the solution of (2.20). To summarize, we have proven that

$$\hat{\phi} = \hat{\phi}_1 + s\,\psi + \mathcal{O}(s^2)$$

where ψ is the solution of (2.20) and we also know that

$$v_0 = \mathcal{O}(s^2)$$

In particular, in $\mathring{B}_1 \setminus \mathring{B}_{3/4}$, we have

$$\begin{array}{rcl} \hat{\phi}(y) & = & \phi_1\left(\left(1 + s\,\bar{w}(y/|y|)\right) y\right) + s\,\psi(y) + \mathcal{O}(s^2) \\ & = & \phi_1(y) + s\big(\bar{w}(y/|y|)\, r\,\partial_r\phi_1 + \psi\big) + \mathcal{O}(s^2) \end{array}$$

where we have set $r := |y|$.

To complete the proof of the result, it suffices to compute the normal derivative of the function $\hat{\phi}$ when the normal is computed with respect to the metric $\hat{g}$. We use polar coordinates $y = r\,z$ where $r > 0$ and $z \in S^{n-1}$. Then the metric $\hat{g}$ can be expanded in $\mathring{B}_1 \setminus \mathring{B}_{3/4}$ as

$$\hat{g} = (1 + v_0 + s\bar{w})^2\, dr^2 + 2\, s\, (1 + v_0 + s\bar{w})\, r\, d\bar{w}\, dr + r^2\, (1 + v_0 + s\bar{w})^2\, \mathring{h} + s^2\, r^2\, d\bar{w}^2$$

where $\mathring{h}$ is the metric on S^{n-1} induced by the Euclidean metric. It follows from this expression together with the fact that $v_0 = \mathcal{O}(s^2)$ that the unit normal vector field to $\partial\mathring{B}_1$ for the metric $\hat{g}$ is given by

$$\hat{\nu} = \left((1 + s\,\bar{w})^{-1} + \mathcal{O}(s^2)\right)\, \partial_r + \mathcal{O}(s)\, \partial_{z_j}$$

where ∂_{z_j} are vector fields induced by a parameterization of S^{n-1}. Using this, we conclude that

$$\hat{g}(\nabla\hat{\phi}_1, \hat{\nu}) = \partial_r\phi_1 + s\, \left(\bar{w}\, \partial_r^2\phi_1 + \partial_r\psi\right) + \mathcal{O}(s^2)$$

on $\partial\mathring{B}_1$. The result then follows at once from the fact that $\partial_r\phi_1$ is constant while the term $\bar{w}\, \partial_r^2\phi_1 + \partial_r\psi$ has mean 0 on the boundary $\partial\mathring{B}_1$. This completes the proof of the proposition. □

We denote by L_ϵ the linearization of F with respect to $\bar{v}$, computed at the point $(p, \epsilon, 0)$. Following the proof of the previous Proposition, it is easy to check the :

Lemma 2.4.4. *There exists a constant $c > 0$ such that, for all $\epsilon > 0$ small enough we have the estimate*

$$\|(L_\epsilon - L_0)\,\bar{v}\|_{\mathcal{C}^{1,\alpha}} \leq c\,\epsilon^2\,\|\bar{v}\|_{\mathcal{C}^{2,\alpha}}$$

Proof : Clearly both L_ϵ and L_0 are first order differential operators. To prove the estimate, we simply use the fact that, when $\epsilon \neq 0$, the difference between the coefficients of $\bar{g} = \epsilon^{-2}\,g$ and $\mathring{g}$ can be estimated by a constant times ϵ^2. This implies that the discrepancy between the linearized operator when $\epsilon = 0$ and when $\epsilon \neq 0$ is a first order differential operator whose coefficients can be estimated by a constant times ϵ^2. □

The main result of this section is the fact that the linearized operator L_0 is given by H. Observe that the kernel of L_0 is equal to V_1 which is the vector space spanned by the restriction of linear functions to the unit sphere. This is geometrically very natural since, when $\epsilon = 0$, a linear function $\bar{v} := \mathring{g}(a, \cdot) \in V_1$ correspond to infinitesimal translation of the unit ball in the direction $a \in \mathbb{R}^n$. Therefore we have

$$B^{\mathring{g}}_{1+s\,\bar{v}}(p) \sim \mathring{B}_1(p + s\,a),$$

This implies that the solution of (2.8) is given by ϕ_1 (modulo some $\mathcal{O}(s^2)$ term) and hence its normal data is constant (modulo some $\mathcal{O}(s^2)$ term). Therefore $F(p, 0, \bar{v}) = \mathcal{O}(s^2)$ which shows that $L_0\bar{v} = 0$.

2.5 The proof of Theorem 2.1.3

We shall now prove that, for $\epsilon > 0$ small enough, it is possible to solve the equation

$$F(p, \epsilon, \bar{v}) = 0$$

Unfortunately, we will not be able to solve this equation at once. Instead, we first prove the :

Proposition 2.5.1. *There exists $\epsilon_0 > 0$ such that, for all $\epsilon \in [0, \epsilon_0]$ and for all $p \in M$, there exists a unique function $\bar{v} = \bar{v}(p, \epsilon)$ and a vector $a = a(p, \epsilon) \in \mathbb{R}^n$ such that*

$$F(p, \epsilon, \bar{v}) + \mathring{g}(a, \cdot) = 0$$

The function $\bar{v}$ and the vector a depend smoothly on p and ϵ and we have

$$|a| + \|\bar{v}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leq c\,\epsilon^2$$

Proof : We fix $p \in M$ and define

$$\bar{F}(p, \epsilon, \bar{v}, a) := F(p, \epsilon, \bar{v}) + \mathring{g}(a, \cdot)$$

It is easy to check that $\bar{F}$ is a smooth map from a neighborhood of $(p, 0, 0, 0)$ in $M \times [0, \infty) \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathbb{R}^n$ into a neighborhood of 0 in $\mathcal{C}^{1,\alpha}(S^{n-1})$. Moreover,

$$\bar{F}(p, 0, 0, 0) = 0$$

and the differential of $\bar{F}$ with respect to $\bar{v}$, computed at $(p, 0, 0, 0)$ is given by H. Finally the image of the linear map $a \longmapsto \mathring{g}(a, \cdot)$ is just the vector space V_1. Thanks to the result of Proposition 2.4.2, the implicit function theorem applies to get the existence of $\bar{v}$ and a, smoothly depending on p and ϵ such that $F(p, \epsilon, \bar{v}) + \mathring{g}(a, \cdot) = 0$. The estimate for $\bar{v}$ and a follows at once from Lemma 2.3.3. □

In view of the result of the previous Proposition, it is enough to show that, provided that ϵ is small enough, it is possible to choose the point $p \in M$ such that $a(p, \epsilon) = 0$. We claim that, there exists a constant $\tilde{C} > 0$ (only depending on n) such that

$$\Theta(a(p, \epsilon)) = -\epsilon^3 \, \tilde{C} \, \nabla^g \, \mathrm{Scal}(p) + \mathcal{O}(\epsilon^4)$$

Indeed, for all $b \in \mathbb{R}^n$ we compute

$$\begin{aligned} \int_{S^{n-1}} \mathring{g}(a, \cdot) \, \mathring{g}(b, \cdot) \ \mathrm{dvol}_{\mathring{g}} &= -\int_{S^{n-1}} F(p, \epsilon, \bar{v}) \, \mathring{g}(b, \cdot) \ \mathrm{dvol}_{\mathring{g}} \\ &= -\int_{S^{n-1}} (F(p, \epsilon, 0) + L_0 \bar{v}) \, \mathring{g}(b, \cdot) \ \mathrm{dvol}_{\mathring{g}} \\ &\quad -\int_{S^{n-1}} (F(p, \epsilon, \bar{v}) - F(p, \epsilon, 0) - L_\epsilon \bar{v}) \, \mathring{g}(b, \cdot) \ \mathrm{dvol}_{\mathring{g}} \\ &\quad -\int_{S^{n-1}} (L_\epsilon - L_0) \bar{v} \, \mathring{g}(b, \cdot) \ \mathrm{dvol}_{\mathring{g}} \end{aligned}$$

Now, we use the fact that $\bar{v}$ is $L^2(S^{n-1})$-orthogonal to linear functions and hence so is $L_0 \, \bar{v}$. Therefore,

$$\int_{S^{n-1}} L_0 \, \bar{v} \, \mathring{g}(b, \cdot) \ \mathrm{dvol}_{\mathring{g}} = 0$$

Using the fact that $\bar{v} = \mathcal{O}(\epsilon^2)$, we get

$$F(p, \epsilon, \bar{v}) - F(p, \epsilon, 0) - L_\epsilon \bar{v} = \mathcal{O}(\epsilon^4)$$

Similarly, it follows from the result of Lemma 2.4.4 that

$$(L_\epsilon - L_0) \, \bar{v} = \mathcal{O}(\epsilon^4)$$

The claim then follows from the second estimate in Lemma 2.3.3 and the fact that

$$\int_{S^{n-1}} \mathring{g}(a,\cdot)\,\mathring{g}(b,\cdot)\ \mathrm{dvol}_{\mathring{g}} =$$

$$= g\big(\Theta(a),\Theta(b)\big)\int_{S^{n-1}} (x_1)^2\ \mathrm{dvol}_{\mathring{g}} = \frac{1}{n}\ \mathrm{Vol}_{\mathring{g}}(S^{n-1})\, g\big(\Theta(a),\Theta(b)\big).$$

Now if we assume that p_0 is a nondegenerate critical point of the scalar curvature function, we can apply once more the implicit function theorem to solve the equation

$$G(\epsilon,p) := \epsilon^{-3}\,\Theta(a(p,\epsilon)) = 0.$$

It should be clear that G depends smoothly on $\epsilon \in [0,\epsilon_0)$ and $p \in M$. Moreover, we have

$$G(0,p) = -\tilde{C}\,\nabla^g\ \mathrm{Scal}(p)$$

and hence $G(0,p_0) = 0$. By assumption the differential of G with respect to p, computed at p_0 is invertible. Therefore, for all ϵ small enough there exists p_ϵ close to p_0 such that

$$\Theta(a(p_\epsilon,\epsilon)) = 0$$

In addition we have

$$\mathrm{dist}(p_0,p_\epsilon) \le c\,\epsilon$$

This completes the proof the Theorem 2.1.3.

2.6 Appendix

Lemma 2.6.1. *For all $\sigma = 1,\dots,n$, we have*

$$\sum_{i,j,k,\ell,m}\int_{S^{n-1}} R_{ikj\ell,m}\,x^i\,x^j\,x^k\,x^\ell\,x^m\,x^\sigma\ \mathrm{dvol}_{\mathring{g}} = 0.$$

Proof : To see that we consider all terms of the above sum, obtained fixing the 6-tuple (i,k,j,ℓ,m,σ). We observe that if in such a 6-tuple there is an element that appears an odd number of time then $\int_{S^{n-1}} x^i\,x^j\,x^k\,x^\ell\,x^m\,x^\sigma\ \mathrm{dvol}_{\mathring{g}} = 0$. Moreover, the symmetries of the curvature tensor imply that, if either $i=k$ or $j=\ell$, then $R_{ikj\ell,m} = 0$. Therefore, we have to compute

$$\sum_{i,k,\sigma}\int_{S^{n-1}} R^*\,(x^i)^2\,(x^k)^2\,(x^\sigma)^2\ \mathrm{dvol}_{\mathring{g}}$$

where

$$R^* := R_{ikik,\sigma} + R_{iki\sigma,k} + R_{ikki,\sigma} + R_{ik\sigma i,k} + R_{ikk\sigma,i} + R_{ik\sigma k,i} + R_{i\sigma ik,k} + R_{\sigma kik,i} + R_{i\sigma ki,k} + R_{\sigma kki,i}$$

Again, we apply the symmetries of Riemann curvature which imply that $R_{ikik,\sigma} + R_{ikki,\sigma} = 0$, $R_{iki\sigma,k} + R_{ik\sigma i,k} = 0$, $R_{ikk\sigma,i} + R_{ik\sigma k,i} = 0$, $R_{\sigma kik,i} + R_{\sigma kki,i} = 0$ and $R_{i\sigma ik,k} + R_{i\sigma ki,k} = 0$, and we conclude that the sum is equal to 0. □

Lemma 2.6.2. *For all $\sigma = 1, \dots, n$, we have*

$$\sum_{j,k,\ell} \int_{S^{n-1}} R_{\cdot kj\ell,\cdot}\, x^j\, x^k\, x^\ell\, x^\sigma \ \mathrm{dvol}_{\mathring{g}} = 0.$$

Proof : Arguing as in the previous proof, we find that $\int_{S^{n-1}} x^j\, x^k\, x^\ell\, x^\sigma \ \mathrm{dvol}_{\mathring{g}} = 0$ unless the indices j, k, ℓ, σ are pairwise equal. Hence, we can write

$$\sum_{j,k,\ell} \int_{S^{n-1}} R_{\cdot kj\ell,\cdot}\, x^j\, x^k\, x^\ell\, x^\sigma \ \mathrm{dvol}_{\mathring{g}} =$$
$$= \int_{S^{n-1}} R_{\cdot\sigma\sigma\sigma,\cdot}\, (x^\sigma)^4 \ \mathrm{dvol}_{\mathring{g}} + \sum_{j\neq\sigma} \int_{S^{n-1}} (R_{\cdot\sigma jj,\cdot} + R_{\cdot j\sigma j,\cdot} + R_{\cdot jj\sigma,\cdot})\ (x^\sigma)^2\, (x^j)^2 \ \mathrm{dvol}_{\mathring{g}}$$

Using the symmetries of the Riemann curvature tensor, we get $R_{\cdot\sigma\sigma\sigma,\cdot} = R_{\cdot\sigma jj,\cdot} = 0$ and $R_{\cdot j\sigma j,\cdot} + R_{\cdot jj\sigma,\cdot} = 0$. This completes the proof of the result. □

Lemma 2.6.3. *For all $\sigma = 1, \dots, n$, we have*

$$\sum_{i,\ell,m} \int_{S^{n-1}} R_{i\ell,m}\, x^i\, x^\ell\, x^m\, x^\sigma \ \mathrm{dvol}_{\mathring{g}} = \frac{2}{n(n+2)} \ \mathrm{Vol}_{\mathring{g}}(S^{n-1}) \ \mathrm{Scal}_{,\sigma}$$

Proof : Again, we find that $\int_{S^{n-1}} x^i\, x^\ell\, x^m\, x^\sigma \ \mathrm{dvol}_{\mathring{g}} = 0$ unless the indices i, ℓ, m, σ are pairwise equal. Hence we can write

$$\sum_{i,\ell,m,} \int_{S^{n-1}} R_{i\ell,m}\, x^i\, x^\ell\, x^m\, x^\sigma \ \mathrm{dvol}_{\mathring{g}} =$$
$$= R_{\sigma\sigma,\sigma} \int_{S^{n-1}} (x^\sigma)^4 \ \mathrm{dvol}_{\mathring{g}} + \sum_{j\neq\sigma} \int_{S^{n-1}} (R_{\sigma j,j} + R_{j\sigma,j} + R_{jj,\sigma})\ (x^\sigma)^2\, (x^j)^2 \ \mathrm{dvol}_{\mathring{g}}$$
$$= R_{\sigma\sigma,\sigma} \int_{S^{n-1}} (x^1)^4 \ \mathrm{dvol}_{\mathring{g}} + \sum_{j\neq\sigma} (R_{\sigma j,j} + R_{j\sigma,j} + R_{jj,\sigma}) \int_{S^{n-1}} (x^1)^2\, (x^2)^2 \ \mathrm{dvol}_{\mathring{g}}$$
$$= R_{\sigma\sigma,\sigma} \left(\int_{S^{n-1}} (x^1)^4 \ \mathrm{dvol}_{\mathring{g}} - 3 \int_{S^{n-1}} (x^1)^2\, (x^2)^2 \ \mathrm{dvol}_{\mathring{g}} \right)$$
$$+ \sum_{j} (R_{\sigma j,j} + R_{j\sigma,j} + R_{jj,\sigma}) \int_{S^{n-1}} (x^1)^2\, (x^2)^2 \ \mathrm{dvol}_{\mathring{g}}$$

Now we use the fact that

$$\int_{S^{n-1}} (x^1)^4 \;\, \mathrm{dvol}_{\mathring{g}} = 3 \int_{S^{n-1}} (x^1)^2 \, (x^2)^2 \;\, \mathrm{dvol}_{\mathring{g}} = \frac{3}{n(n+2)} \;\, \mathrm{Vol}_{\mathring{g}}(S^{n-1}) \, ,$$

to conclude that

$$\sum_{i,\ell,m,} \int_{S^{n-1}} R_{i\ell,m} \, x^i \, x^\ell \, x^m \, x^\sigma \;\, \mathrm{dvol}_{\mathring{g}} = \frac{1}{n(n+2)} \;\, \mathrm{Vol}_{\mathring{g}}(S^{n-1}) \, \sum_j \left(R_{\sigma j,j} + R_{j\sigma,j} + R_{jj,\sigma} \right)$$

Finally, the second Bianchi identity yields

$$\sum_j R_{\sigma j,j} = \sum_j R_{j\sigma,j} = \frac{1}{2} \;\, \mathrm{Scal}_{,\sigma}$$

and by definition $\sum_j R_{jj,\sigma} = \; \mathrm{Scal}_{,\sigma}$. Hence

$$\sum_{i,\ell,m,} \int_{S^{n-1}} R_{i\ell,m} \, x^i \, x^\ell \, x^m \, x^\sigma \;\, \mathrm{dvol}_{\mathring{g}} = \frac{2}{n(n+2)} \;\, \mathrm{Vol}_{\mathring{g}}(S^{n-1}) \;\, \mathrm{Scal}_{,\sigma}$$

This completes the proof of the result. □

Chapitre 3

New extremal domains in flat tori

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New extremal domains for the first eigenvalue of the Laplacian in flat tori

Résumé. On démontre l'existence de domaines extrémaux non trivaux et compacts pour la première valeur propre du Laplacien dans des variétés $\mathbb{R}^n \times \mathbb{R}/T\,\mathbb{Z}$ avec métrique plate, pour certains $T > 0$. Ces domaines sont proche du domaine de type cylindrique $B_1 \times \mathbb{R}/T\,\mathbb{Z}$, où B_1 est la boule unité dans $\mathbb{R}^n$, ils sont invariants par rotation autour de l'axe vertical et ils ne sont pas invariants par translations verticales.
Ces domaines peuvent etre prolongé par périodicité à des domaines non triviaux et non compacts de l'espace euclidien, sur lesquels la première fonction propre du Laplacien avec condition de Dirichlet nulle au bord a également donnée de Neumann constante.

Abstract. We prove the existence of nontrivial compact extremal domains for the first eigenvalue of the Laplacian in manifolds $\mathbb{R}^n \times \mathbb{R}/T\,\mathbb{Z}$ with flat metric, for some $T > 0$. These domains are close to the cylinder-type domain $B_1 \times \mathbb{R}/T\,\mathbb{Z}$, where B_1 is the unit ball in $\mathbb{R}^n$, they are invariant by rotation with respect to the vertical axe, and are not invariant by vertical translations.
Such domains can be extended by periodicity to nontrivial and noncompact domains in Euclidean spaces whose first eigenfunction of the Laplacian with 0 Dirichlet boundary condition has also constant Neumann data at the boundary.

3.1 Statement of the result

An open problem is to find the domains $\Omega \subseteq \mathbb{R}^{n+1}$, $n \geq 2$, for which the over-determined problem

$$\begin{cases} \Delta_{\mathring{g}} u + \lambda\, u &= 0 \qquad \text{in} \quad \Omega \\ u &= 0 \qquad \text{on} \quad \partial\Omega \\ \mathring{g}(\nabla u, \nu) &= \text{constant} \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{3.1}$$

has a positive solution $u \in C^{2,\alpha}(\Omega)$. Naturally $\mathring{g}$ is the Euclidean metric, λ is a positive constant (i.e. the first eigenvalue of the Laplacian), and ν is the normal unit outward vector about $\partial\Omega$. It is known (see [36]) that smooth bounded domains in Euclidean spaces for which the Laplace equation with right hand side constant, Dirichlet boundary data and constant Neumann data, are round balls. It is an open problem to study noncompact domains where this overdetermined problem is solvable.

We denote the coordinates of $\mathbb{R}^{n+1}$ as (x, t), $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We want to prove the following result.

Theorem 3.1.1. *There exist a real positive number* $T_* < \dfrac{2\pi}{\sqrt{n-1}}$*, a sequence of real positive numbers* $T_j \longrightarrow T_*$ *and a sequence of nonzero and nonconstant functions* $v_j \in \mathcal{C}^{2,\alpha}(\mathbb{R})$ *of period* T_j *converging to* 0 *in* $\mathcal{C}^{2,\alpha}(\mathbb{R})$ *such that the domains*

$$\Omega_j = \big\{(x,t) \in \mathbb{R}^{n+1} \qquad , \qquad |x| < 1 + v_j(t)\big\}$$

have a positive solution $u_j \in C^{2,\alpha}(\Omega_j)$ *to the problem (3.1). Moreover* $\displaystyle\int_0^{T_j} v_j\, dt = 0$.

We remark immediately that the domain Ω_j can be constructed starting from the cylinder-type domain

$$C_1 = \{(x,t) \in \mathbb{R}^{n+1} \qquad , \qquad |x| < 1\}$$

(where it is also known that there exists a positive solution $u \in C^{2,\alpha}(C_1)$ to the problem (3.1)) and modifying his boundary by a function only depending on the variable t, periodic and nonconstant (this implies that Ω_j is invariant by rotations with respect to the vertical axe and is not invariant by vertical translations). Then, if T_j is the period of the function v_j, such a domain arise to a compact domain homeomorphe to $B_1 \times \mathbb{R}/T_j\,\mathbb{Z}$ in the manifold $\mathbb{R}^n \times \mathbb{R}/T_j\,\mathbb{Z}$ with flat metric, where the problem (3.1), adapted to this new manifold, has a solution (naturally B_1 denotes the unit ball centered at 0). From the proposition 2.1 of [**?**], also proved in [16] and in [12], it is clear that such a domain Ω_j is extremal with respect to the first eigenvalue of the Laplacian in $\mathbb{R}^n \times \mathbb{R}/T_j\,\mathbb{Z}$ for the fixed volume $T_j\, vol(B_1)$, in the sens that for any volume preserving deformation $\{\Omega_s\}_{s\in(j-\epsilon, j+\epsilon)}$ of Ω_0, we have

$$\frac{d\lambda_s}{ds}|_{s=0} = 0\,,$$

where λ_s is the first eigenvalue of $-\Delta_g$ on Ω_s, with 0 Dirichlet boundary condition. Naturally we understood that $\{\Omega_s\}_{s\in(j-\epsilon,j+\epsilon)}$ is a deformation of Ω_0, if there exists a vector field Ξ such that $\Omega_s = \xi(s,\Omega_0)$ where $\xi(s,\cdot)$ is the flow associated to Ξ, and the deformation is said to be volume preserving if the volume of Ω_s does not depend on s.
On the other hand, given an extremal domain with respect to the first eigenvalue of the Laplacian in the manifold $\mathbb{R}^n \times \mathbb{R}/T\mathbb{Z}$, obtained by deformation of the boundary of the cylinder-type domain

$$C_1^T = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}/T\mathbb{Z} \qquad , \qquad |x| < 1\}$$

by a function $v(t) \in C^{2,\alpha}(\mathbb{R}/T\mathbb{Z})$ of mean 0, then such a domain arise to a noncompact domain homeomorphe to $B_1 \times \mathbb{R}$ defined by periodicity, where there exists a solution to the problem (3.1).

We can conclude that an alternative version of the theorem 3.1.1 is the following :

Theorem 3.1.2. *There exists a real positive number* $T_* < \dfrac{2\pi}{\sqrt{n-1}}$, *a sequence of real positive numbers* $T_j \longrightarrow T_*$ *and a sequence of nonzero functions* $v_j \in \mathcal{C}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$ *of mean* 0 *converging to* 0 *in* $\mathcal{C}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$ *such that the domain*

$$\Omega_j = \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R}/T_j\,\mathbb{Z} \qquad , \qquad |x| < 1 + v_j\left(\frac{2\pi}{T_j}\,t\right)\right\}$$

is extremal with respect to the first eigenvalue of the Laplacian in the manifold $\mathbb{R}^n \times \mathbb{R}/T_j\,\mathbb{Z}$ *with flat metric.*

This will be our main theorem, and we will prove it.

As a final remark, we observe that this result can give an answer, with a counterexample, to the conjecture of Berestycki, Caffarelli and Nirenberg proposed in [3], p. 1110. According to this conjecture, if Ω is a smooth domain such that $\mathbb{R}^{n+1}\backslash\overline{\Omega}$ is connected and there exists a bounded positive solution of

$$\begin{cases} \Delta_{\mathring{g}} u + f(u) = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \\ \mathring{g}(\nabla u, \nu) = \text{constant} & \text{on} \quad \partial\Omega\,, \end{cases} \tag{3.2}$$

for some Lipschitz function f, then Ω should be either a half-space, a ball, the complement of a ball, or a circular-cylinder-type domain $\mathbb{R}^j \times B$, with B a ball.

3.2 Rephrasing the problem

We want to show that for some $T > 0$ (we will see that we have to choose T close to a fixed real positive number T_*) we can modify the boundary of C_1^T in order to find an extremal domain, of volume $T\,vol(B_1)$, with respect to the first eigenvalue of the Laplacian in the manifold $\mathbb{R}^n \times \mathbb{R}/T\mathbb{Z}$ with flat metric. Given a continuous function $f : S^{n-1} \times \mathbb{R}/T\mathbb{Z} \longmapsto (0, \infty)$ we define

$$C_f^T := \{(x,t) \in \mathbb{R}^n \times \mathbb{R}/T\mathbb{Z} \qquad : \qquad 0 \leq |x| < f(x/|x|, t)\} \,.$$

Our aim, following the caracterization of extremal domains given in [**?**], is to show that there exists a $T > 0$ and a nonconstant function $v : S^{n-1} \times \mathbb{R}/T\mathbb{Z} \longrightarrow \mathbb{R}$ of mean 0 such that the over-determined problem

$$\left\{ \begin{array}{rcll} \Delta_{\mathring{g}}\, \phi + \lambda\, \phi & = & 0 & \text{in} \quad C_{1+v}^T \\ \phi & = & 0 & \text{on} \quad \partial C_{1+v}^T \\ \mathring{g}(\nabla \phi, \nu) & = & \text{constant} & \text{on} \quad \partial C_{1+v}^T \end{array} \right. \tag{3.3}$$

has a nontrivial positive solution, where ν is the normal vector field about ∂C_{1+v}^T, λ is a positive constant and $\mathring{g}$ represent the flat metric on $\mathbb{R}^n \times \mathbb{R}/T\mathbb{Z}$.

By symmetry, it is clear that the function v does not depend on the variable $x \in S^{n-1}$. Then $v = v(t)$. Moreover we can also require the function v to be even.

Let us denote by λ_1 the first eigenvalue of the euclidean Laplacian in the unit ball B_1 of $\mathbb{R}^n$ centered at the origin, with 0 Dirichlet boundary condition. We denote by $\tilde{\phi}_1$ the associated eigenfunction

$$\left\{ \begin{array}{rcll} \Delta \tilde{\phi}_1 + \lambda_1\, \tilde{\phi}_1 & = & 0 & \text{in} \quad B_1 \\ \tilde{\phi}_1 & = & 0 & \text{on} \quad \partial B_1 \end{array} \right. . \tag{3.4}$$

which is normalized to have $L^2(B_1)$-norm equal to $1/2\pi$. Then $\phi_1(x,t) = \tilde{\phi}_1(x)$ solve the problem

$$\left\{ \begin{array}{rcll} \Delta_{\mathring{g}} \phi_1 + \lambda_1\, \phi_1 & = & 0 & \text{in} \quad C_1^T \\ \phi_1 & = & 0 & \text{on} \quad \partial C_1^T \end{array} \right. \tag{3.5}$$

and

$$\int_{C_1^{2\pi}} \phi_1^2 \ \mathrm{dvol}_{\mathring{g}} = 1 \tag{3.6}$$

Because ϕ_1 do not depend on t, sometimes we will write simply $\phi_1(x)$.

Let $\mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ be the set of even functions on $\mathbb{R}/2\pi\mathbb{Z}$ of mean 0. For all $T>0$ and all $f \in \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ we set

$$C_f^T := \{(x,t) \in \mathbb{R}^n \times \mathbb{R}/T\mathbb{Z} \qquad : \qquad 0 \le |x| < f(2\pi t/T)\} \, .$$

For all function $v \in \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ whose norm is small enough, the domain C^T_{1+v} is well defined for all $T>0$. Standard results on Dirichlet eigenvalue problem (see [17]) apply to give the existence, for all $T>0$, of a unique positive function

$$\phi = \phi_{v,T} \in \mathcal{C}^{2,\alpha}\left(C^T_{1+v}\right)$$

and a constant $\lambda = \lambda_{v,T} \in \mathbb{R}$ such that ϕ is a solution to the problem

$$\begin{cases} \Delta_{\mathring{g}}\, \phi + \lambda\, \phi &= 0 \quad \text{in} \quad C^T_{1+v} \\ \phi &= 0 \quad \text{on} \quad \partial C^T_{1+v} \end{cases} \tag{3.7}$$

which is normalized by

$$\int_{C^{2\pi}_{1+v}} \left(\phi\left(x, \frac{T}{2\pi}\, t\right)\right)^2 \, \mathrm{dvol}_{\mathring{g}} = 1 \tag{3.8}$$

In addition ϕ and λ depend smoothly on the function v, and $\phi = \phi_1$, $\lambda = \lambda_1$ when $v \equiv 0$.

After canonical identification of ∂C^T_{1+v} with $S^{n-1} \times \mathbb{R}/T\mathbb{Z}$, we define, the operator N :

$$N(v,T) = \mathring{g}(\nabla\phi, \nu)\,|_{\partial C^T_{1+v}} - \frac{1}{\mathrm{Vol}\;_{\mathring{g}}(\partial C^T_{1+v})} \int_{\partial C^T_{1+v}} \mathring{g}(\nabla\phi,\nu)\,\mathrm{dvol}_{\mathring{g}}\,,$$

where ν denotes the unit normal vector field to ∂C^T_{1+v} and ϕ is the solution of (3.7). A priori $N(v,t)$ is a function defined over $S^{n-1} \times \mathbb{R}/T\mathbb{Z}$, but it is easy to see that it depends only on the variable $t \in \mathbb{R}/T\mathbb{Z}$ because v has such a property. For the same reason it is an even function, and moreover it is clear that its mean is 0. If now we operate a rescaling and we define

$$F(v,T)\,(t) \,=\, N(v,T)\,\left(\frac{T}{2\pi}\, t\right)$$

Schauder's estimates imply that F is well defined for v in a neighborhood of 0 in $\mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ and $T \in \mathbb{R}$, and takes its values in $\mathcal{C}^{1,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$. Our aim is to find (v,T) such that $F(v,T)=0$. Observe that, with this condition, ϕ will be the solution to the problem (3.3).

Our next task will be to understand the structure of L_0, the operator obtained by linearizing F with respect to v at a general point $(0,T)$.

3.3 The structure of the linearized operator

We already have recalled the existence of a unique positive function $\phi \in \mathcal{C}^{2,\alpha}(C^T_{1+v})$ (close to ϕ_1) and a constant $\lambda \in \mathbb{R}$ (close to λ_1), solutions to the problem (3.7) so we are going to construct an expansion of ϕ and λ in powers of v and its derivatives. This will lead to the structure of the linearized operator L_0.

Recall that λ_1 is the first eigenvalue of $-\Delta_{\mathring{g}}$ in C_1^T with 0 Dirichlet boundary condition and ϕ_1 is the associated eigenfunction which is normalized as in (3.6). Observe that in principle ϕ_1 is only defined in the cylinder, however, this function being radial in the first n variables and not depending on t, it is a solution of a second order ordinary differential equation and as such can be extended at least in a neighborhood of ∂C_1^T.

Let $v \in \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$. By Fourier expansion v can be written as

$$v = \sum_{k\geq 1} a_k \cos(kt) \tag{3.9}$$

We start with the easy :

Lemma 3.3.1. *Assume that $v \in \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ and write v as in (3.9). For $T > 0$ we define*

$$\phi_0(x,t) = \partial_r \phi_1(x)\, v(2\pi t/T)$$

where $r = |x|$. Then

$$\Delta_{\mathring{g}}\phi_0 + \lambda_1\, \phi_0 = \sum_{k\geq 1} a_k \, \frac{1}{r^2}\, \partial_r\phi_1 \, \cos\left(\frac{2\pi k t}{T}\right) \left[n-1-\left(\frac{2\pi k}{T}\right)^2 r^2\right] \tag{3.10}$$

Proof : This is a straightforward exercise. Using the fact that

$$\Delta_{\mathring{g}}\, \partial_r\phi_1 = -\lambda_1\, \partial_r\phi_1 + \frac{n-1}{r^2}\, \partial_r\phi_1\,,$$

we find

$$\begin{aligned}
\Delta_{\mathring{g}}\phi_0 &= v\, \Delta_{\mathring{g}}\partial_r\phi_1 + \partial_r\phi_1\, \Delta_{\mathring{g}} v + 2\,\nabla v \nabla \partial_r \phi_1 \\
&= v\left(-\lambda_1\, \partial_r\phi_1 + \frac{n-1}{r^2}\, \partial_r\phi_1\right) - \sum_{k\geq 1} a_k \left(\frac{2\pi k}{T}\right)^2 \partial_r\phi_1 \, \cos\left(\frac{2\pi k t}{T}\right) \\
&= -\lambda_1\, \phi_0 + \sum_{k\geq 1} a_k\, \frac{1}{r^2}\, \partial_r\phi_1\, \cos\left(\frac{2\pi k t}{T}\right)\left[n-1-\left(\frac{2\pi k}{T}\right)^2 r^2\right]
\end{aligned}$$

This completes the proof of the result. □

For all $v \in \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ and all $T > 0$ let ψ be the (unique) solution (periodic with respect to the variable t) of

$$\left\{ \begin{array}{rclcc} \Delta_{\mathring{g}}\psi + \lambda_1 \psi & = & 0 & \text{in} & C_1^T \\ \psi & = & -\partial_r \phi_1 \, v(2\pi t/T) & \text{on} & \partial C_1^T \end{array} \right. \tag{3.11}$$

which is $L^2(C_1^T)$-orthogonal to ϕ_1. We define

$$\tilde{H}_T(v) := \left(\partial_r \psi + \partial_r^2 \phi_1 \, v(2\pi t/T)\right)\big|_{\partial C_1^T} \tag{3.12}$$

By symmetry it is clear that $\tilde{H}_T(v)$ is a function only depending on t, then changing the variable we can define

$$H_T(v)(t) := \tilde{H}_T(v) \left(\frac{T}{2\pi} \, t \right) \tag{3.13}$$

Let V_k be the space spanned by the function $\cos(kt)$. We will need the following result :

Proposition 3.3.2. *The operator*

$$H_T : \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z}) \longrightarrow \mathcal{C}^{1,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z}),$$

is a self adjoint, first order elliptic operator that preserves the eigenspaces V_k for all k and all $T > 0$. There exists a positive real number $T_* < \dfrac{2\pi}{\sqrt{n-1}}$ such that the kernel of H_{T_*} is given by $V_{k_1} \oplus \cdots \oplus V_{k_l}$, with $1 = k_1 < k_2 < \cdots < k_l$. Moreover the eigenvalue associated to the eigenspace V_1, considered as a function on T, changes the sign at T_*, and the eigenvalues associated to the other eigenspaces $V_{k_2}, ..., V_{k_l}$, always considered as functions on T, do not change the sign at T_*. There exists a constant $c > 0$ such that

$$\|w\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})} \leq c \, \|H_{T_*}(w)\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}/2\pi\mathbb{Z})} \, ,$$

provided w is $L^2(\mathbb{R}/2\pi\mathbb{Z})$-orthogonal to $V_0 \oplus V_{k_1} \oplus \cdots \oplus V_{k_l}$, where V_0 is the space of constant functions.

Proof : The fact that H_T is a first order elliptic operator is standard since it is the sum of the Dirichlet-to-Neumann operator for $\Delta_{\mathring{g}} + \lambda_1$ and a constant times the identity. In particular, elliptic estimates yield

$$\|H_T(w)\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}/2\pi\mathbb{Z})} \leq c \, \|w\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})}$$

The fact that the operator H_T is (formally) self-adjoint is easy. Let ψ_1 (resp. ψ_2) the solution of (3.11) corresponding to the function w_1 (resp. w_2). Let $\tilde{\psi}_i(x,t) = \psi_i(x, Tt/2\pi)$. We compute

$$\begin{aligned}
\partial_r\phi_1(1) \int_0^{2\pi} &(H_T(w_1)\, w_2 - w_1\, H_T(w_2))\, dt = \\
&= \partial_r\phi_1(1) \int_0^{2\pi} (\partial_r\tilde{\psi}_1\, w_2 - \partial_r\tilde{\psi}_2\, w_1)\, dt \\
&= \int_0^{2\pi} (\tilde{\psi}_1\, \partial_r\tilde{\psi}_2 - \tilde{\psi}_2\, \partial_r\tilde{\psi}_1)\, dt \\
&= \frac{1}{\mathrm{Vol}_{\mathring{g}}(S^{n-1})} \int_{C_1^{2\pi}} (\tilde{\psi}_1\, \Delta_{\mathring{g}}\tilde{\psi}_2 - \tilde{\psi}_2\Delta_{\mathring{g}}\, \tilde{\psi}_1)\, \mathrm{dvol}_{\mathring{g}} \\
&= 0
\end{aligned}$$

To prove the other statements, we define for all $v \in \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ written as in (3.9), Ψ to be the continuous solution of

$$\left\{\begin{array}{rcll}
\Delta_{\mathring{g}}\Psi + \lambda_1\,\Psi &=& \displaystyle\sum_{k\geq 1} a_k\, \frac{1}{r^2}\, \partial_r\phi_1\, \cos\left(\frac{2\pi k t}{T}\right) \left[n - 1 - \left(\frac{2\pi k}{T}\right)^2 r^2\right] & \text{in} \quad C_1^T \\
\Psi &=& 0 & \text{on} \quad \partial C_1^T\,.
\end{array}\right. \tag{3.14}$$

Observe that $\partial_r\phi_1$ vanishes at first order at $r = 0$ and hence the right hand side is bounded by a constant times r^{-1} near the origin. Standard elliptic estimates then imply that the solution Ψ is at least continuous near the origin (the right side of (3.14) belongs to the space $L^p(C_1^T)$ for each $p < n$, then the solution Ψ belongs to the Sobolev space $W^{2,p}(C_1^T)$ for each $p < n$, and by the Sobolev embedding theorem for a compact domain Ω we have $W^{2,p}(\Omega) \subseteq C^{0,\alpha}(\Omega)$ for $p \geq \frac{n}{2-\alpha}$). A straightforward computation using the result of Lemma 3.3.1 and writing $\Psi(x,t) = \psi(x,t) + \partial_r\phi_1(x)\, v(2\pi t/T)$, shows that

$$\tilde{H}_T(v) := \partial_r\Psi|_{\partial C_1^T} \tag{3.15}$$

With this alternative definition, it should be clear that H_T preserves the eigenspaces V_k and in particular, H_T maps into the space of functions whose mean is 0.

Then

$$\tilde{H}_T(v) = \sum_{k\geq 1} \sigma_k(T)\, a_k\, \cos\left(\frac{2\pi k t}{T}\right) \tag{3.16}$$

where $\sigma_k(T)$ are the eigenvalues of H_T with respect to the eigenfunctions $\cos(kt)$ and are given by

$$\sigma_k(T) = \partial_r b_k(1) \tag{3.17}$$

where b_k is the continuous solution on $[0,1]$ of

$$\left(\partial_r^2 + \frac{n-1}{r}\,\partial_r + \lambda_1\right) b_k - \left(\frac{2\pi k}{T}\right)^2 b_k = \frac{1}{r^2}\left[n-1-\left(\frac{2\pi k}{T}\right)^2 r^2\right]\partial_r\phi_1\,, \qquad (3.18)$$

with $b_k(1) = 0$. From (3.12), (3.16) and (3.11) we deduce that

$$\psi = \sum_{k\geq 1} c_k(r)\, a_k\, \cos\left(\frac{2\pi k t}{T}\right)$$

where c_k is the continuous solution on $[0,1]$ of

$$\left(\partial_r^2 + \frac{n-1}{r}\,\partial_r + \lambda_1\right) c_k - \left(\frac{2\pi k}{T}\right)^2 c_k = 0 \qquad (3.19)$$

with $c_k(1) = -\partial_r\phi_1(1)$. Then an alternative characterization of the eigenvalue $\sigma_k(T)$ is

$$\sigma_k(T) = \partial_r c_k(1) + \partial_r^2\phi_1(1) \qquad (3.20)$$

We want to show that there exists $T_* > 0$ such that the kernel of H_{T_*} is finite-dimensional and contains V_1, the space of functions of the form $a_1\cos(t)$. For this aim we have to find a $T_* > 0$ such that $\sigma_1(T_*) = 0$ and $\sigma_k(T_*) \neq 0$ for almost $k > 1$.

To simplify the notation, we set $\sigma_1 = \sigma$. We need the following Lemma :

Lemma 3.3.3. *The function $\sigma(T)$ is analytic on $(0,+\infty)$ and has the following properties :*
- $\lim_{T\to 0^+}\sigma(T) = +\infty$
- $\sigma(T) < 0$ *for* $T \geq \dfrac{2\pi}{\sqrt{n-1}}$

In particular σ has at least a zero where it changes the sign, and the set of the zeros of σ is a discrete finite set.

Proof : The fact that σ is analytic comes from the following remark : if F is an invertible operator and I is the identity, then for $T > 0$ and any continuous function v the solution u of

$$\left(F - \frac{1}{T^2}\,\rho\, I\right) u = v$$

is analytic on T for each constant ρ; this comes from the equality

$$(I - sF)^{-1} = \sum_{n\geq 0} s^n\, F^n$$

for each $s \in \mathbb{R}$. Then to prove that c_1 is analytic on T it suffices to take

$$F = \left(\partial_r^2 + \frac{n-1}{r}\,\partial_r + \lambda_1\right) \qquad v = 0 \qquad \rho = (2\pi)^2$$

We conclude that $\partial_r c_1(1)$ is analytic with respect to T, and from (3.20) follows the analyticity of σ.

We study now the behaviour of σ when $T \to 0^+$. For T small enough, it is well defined the quantity

$$\xi = \sqrt{\frac{4\pi^2}{T^2} - \lambda_1}$$

Let $\tilde{c}(s) = c_1\left(\frac{s}{\xi}\right)$ for $s \in [0, \xi]$. We remark that

$$\lim_{T \to 0^+} \partial_r c_1(1) = \lim_{\xi \to +\infty} \partial_s \tilde{c}(\xi)$$

Then we are interested in the behaviour of $\partial_s \tilde{c}(\xi)$ at $+\infty$. From (3.19) (considered for $k = 1$) we obtain that $\tilde{c}$ is a continuous solution of the differential equation

$$\left(\partial_s^2 + \frac{n-1}{s}\,\partial_s - 1\right)\,\tilde{c} = 0 \tag{3.21}$$

in the interval $[0, \xi]$ with $\tilde{c}(\xi) = -\partial_r \phi_1(1)$. The previuos equation can be transformed into a well known Bessel's differential equation by the substitution

$$\tilde{c} = s^{\frac{2-n}{2}}\,\hat{c} \tag{3.22}$$

It follows that $\hat{c}$ satisfies

$$\left[\partial_s^2 + \frac{1}{s}\,\partial_s - \left(1 + \frac{\left(\frac{2-n}{2}\right)^2}{s^2}\right)\right]\,\hat{c} = 0 \tag{3.23}$$

that is the *modified Bessel's differential equation of order* $\frac{2-n}{2}$ (for an introduction to Bessel's equations see for example [4] or [19]), and its general solution is given by

$$\hat{c}(s) = A\, I_{\frac{2-n}{2}}(s) + B\, K_{\frac{2-n}{2}}(s) \tag{3.24}$$

for some constants $A, B \in \mathbb{R}$, where $I_m(s)$ and $K_m(s)$ (for $m \in \mathbb{R}$) are the well known *modified Bessel functions* given by

$$\begin{aligned} I_m(s) &= \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(m+k+1)}\left(\frac{s}{2}\right)^{m+2k} \\ K_m(s) &= \lim_{p \to m} \frac{\pi}{2}\left[\frac{I_{-p}(s) - I_p(s)}{\sin(p\,s)}\right] \end{aligned},$$

where Γ is the Gamma function

$$\Gamma(s) = \int_0^\infty e^{-t}\, t^{s-1}\, dt.$$

I_m and K_m are independent solutions of the *modified Bessel's differential equation of order* m

$$\partial_s^2 f + \frac{1}{s}\,\partial_s f - \left(1 + \frac{m^2}{s^2}\right) f = 0$$

It is well known that the behaviour of I_m at ∞ is

$$\lim_{s\to+\infty} \frac{I_m(s)}{\dfrac{1}{\sqrt{2\pi s}}\, e^s} = 1\ , \tag{3.25}$$

while that of K_m is

$$\lim_{s\to+\infty} \frac{K_m(s)}{\sqrt{\dfrac{\pi}{2s}}\, e^{-s}} = 1. \tag{3.26}$$

From (3.22) and (3.24) we have that

$$\tilde{c}(s) = s^{\frac{2-n}{2}} \left(A\, I_{\frac{2-n}{2}}(s) + B\, K_{\frac{2-n}{2}}(s)\right)$$

The behaviour of $\tilde{c}$ at ∞ can't be that of the function $s^{\frac{2-n}{2}}\, K_{\frac{2-n}{2}}(s)$ because of the continuity of $\tilde{c}$ at 0. In fact, if it was like that then we could integrate over $[0, +\infty]$ the function $\tilde{c}$ with its derivatives, and from (3.21), multiplying by $\tilde{c}\, s^{n-1}$ and integrating by parts, we would obtain

$$\int_0^{+\infty} ((\partial_s \tilde{c})^2 + \tilde{c}^2)\, s^{n-1}\, ds = 0$$

that would imply $\tilde{c} = 0$. Hence, by (3.25) and (3.26), the behaviour of $\tilde{c}$ when $s \to +\infty$ is that of the function $s^{\frac{2-n}{2}}\, I_{\frac{2-n}{2}}(s)$, i.e. there exixts a constant $\tilde{A} \neq 0$ such that

$$\lim_{s\to+\infty} \frac{\tilde{c}(s)}{\tilde{A}\, s^{\frac{1-n}{2}}\, e^s} = 1.$$

It follows that $\partial_s \tilde{c}(\xi) \to +\infty$ when $\xi \to +\infty$, and then, coming back to the definition of ξ, we conclude that

$$\lim_{T\to 0^+} \sigma(T) = +\infty.$$

Now we will show that for T big enough the function σ is negative. Let $A_T(r)$ be the solution of the following differential equation

$$\partial_r^2 A_T + \frac{n-1}{r}\,\partial_r A_T + \left[\lambda_1 - \left(\frac{2\pi}{T}\right)^2\right] A_T = \frac{1}{r^2}\,\partial_r \phi_1\,, \tag{3.27}$$

and $B_T(r)$ be the solution of this other differential equation

$$\partial_r^2 B_T + \frac{n-1}{r}\,\partial_r B_T + \left[\lambda_1 - \left(\frac{2\pi}{T}\right)^2\right] B_T = \partial_r \phi_1 \,, \tag{3.28}$$

with $A_T(1) = B_T(1) = 0$. It is easy to see that

$$b_1 = (n-1)\,A_T - \left(\frac{2\pi}{T}\right)^2 B_T$$

We have

$$\sigma(T) = (n-1)\,\partial_r A_T(1) - \left(\frac{2\pi}{T}\right)^2 \partial_r B_T(1)$$

We claim that $A_T \geq 0$, $B_T \geq 0$ and $A_T - B_T \geq 0$. Moreover $\partial_r A_T(1) < 0$, $\partial_r B_T(1) < 0$ and $\partial_r A_T(1) < \partial_r B_T(1)$. This follows from the maximum principle.
Proof of the claim : By definition of λ_1, the operator $\Delta_{\mathring{g}} + \lambda_1$ is non-positive, in the sense that

$$-\int_{B_1} u\,(\Delta_{\mathring{g}} + \lambda_1)\,u\,\mathrm{dvol}_{\mathring{g}} = \int_{B_1} (|\nabla u|^2_{\mathring{g}} - \lambda_1\,u^2)\,\mathrm{dvol}_{\mathring{g}} \geq 0\,. \tag{3.29}$$

Specializing this inequality to functions $u(x,t) = u(|x|)$, we get

$$\int_0^1 \left((\partial_r u)^2 - \lambda_1\,u^2\right)\,r^{n-1}\,\mathrm{dr} \geq 0 \tag{3.30}$$

where $r = |x|$, provided $u \in H_0^1(B_1)$. Assume now that $A_T \leq 0$ in $[r_1, r_2]$ with $A_T(r_i) = 0$, then multiplying (3.27) by $A_T\,r^{n-1}$ and integrating the result by parts between r_1 and r_2, we get

$$\int_{r_1}^{r_2} \left((\partial_r A_T)^2 - \lambda_1 A_T^2\right)\,r^{n-1}\,\mathrm{dr} + \left(\frac{2\pi}{T}\right)^2 \int_{r_1}^{r_2} A_T^2\,r^{n-1}\,\mathrm{dr} \leq 0$$

because $\partial_r\phi_1 \leq 0$. Hence, by (3.30), necessarily $A_T \equiv 0$ on $[r_1, r_2]$. This proves that $A_T \geq 0$ on $[r_1, r_2]$ and by the maximum priciple $A_T \geq 0$ on $[0,1]$. It follows from this fact that $\partial_r A_T(1) \leq 0$, because $A_T(1) = 0$. If it was $\partial_r A_T(1) = 0$ then necessarily we would have $\partial_r^2 A_T(1) \geq 0$ but evaluation of (3.27) at $r = 1$ implies that

$$0 = (n-1)\,\partial_r A_T(1) = \partial_r\phi_1(1) - \partial_r^2 A_T(1) \leq \partial_r\phi_1(1) < 0$$

which immediately leads to a contradiction. Hence, $\partial_r A_T(1) < 0$. The same reasoning applies starting from (3.28) to show that $B_T \geq 0$ and $\partial_r B_T(1) < 0$, and starting from the difference between (3.27) and (3.28) to show that $A_T - B_T \geq 0$ with $\partial_r(A_T - B_T) < 0$. This completes the proof of the claim.

Let now

$$T \geq \frac{2\pi}{\sqrt{n-1}}$$

From the previous claim we have

$$-\left(\frac{2\pi}{T}\right)^2 \partial_r B_T \leq -(n-1)\partial_r B_T < -(n-1)\partial_r A_T$$

that means

$$\sigma(T) < 0$$

This completes the proof of the Lemma. □

Let $\{0_1, 0_2, ..., 0_p\}$ the finite set of the zeros of σ, and let T_* the smallest zero such that σ changes the sign at T_*, say that $T_* = 0_q$. It is clear then V_1 is in the Kernel of H_{T_*}. To prove that the kernel of H_{T_*} is finite-dimensional we have to show that if $T = T_*$ then $\partial_r b_k(1) \neq 0$ for almost $k > 1$. For this we set

$$\frac{k}{T} = \frac{1}{\tau}$$

for $T > 0$ and from (3.18) we obtain that

$$\sigma_k(T) = \sigma(\tau)$$

This implies that σ_k is analytic on T and the set of the zeros of σ_k is $\{k\,0_1, k\,0_2, ..., k\,0_p\}$. It is clear that if k is big enough, say $k > k_l$, then $T_* \notin \{k\,0_1, k\,0_2, ..., k\,0_p\}$, and this means that V_k is not a kernel of H_{T_*} for $k > k_l$. This implies that the kernel of H_{T_*} is of the form $V_{k_1} \oplus \cdots \oplus V_{k_l}$ with $1 = k_1 < \cdots < k_l$. Moreover if $V_{k_i} \in \mathrm{Ker}(H_{T_*})$ and $k_i \neq 1$ then the function $\sigma_{k_i}(T)$ does not change the sign at T_* because $\sigma_{k_i}(T_*) = \sigma(T_*/k_i)$ and $T_*/k_i < T_*$. This completes the proof of the proposition 3.3.2. □

The main result of this section is the following :

Proposition 3.3.4. *The operator L_0 is equal to H_T.*

Proof : By definition, the operator L_0 is the linear operator obtained by linearizing F with respect to v at $(0, T)$. In other words, we have

$$L_0(w) = \lim_{s \to 0} \frac{F(s\,w, T) - F(0, T)}{s}.$$

For $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$ we consider the parameterization of C^T_{1+v} given by

$$Y(y,t) := \left((1 + s\,\chi(y)\,w)\ y\,,\ \frac{T t}{2\pi} \right)$$

where χ is a cutoff function identically equal to 0 when $|y| \leq 1/2$ and identically equal to 1 when $|y| \geq 3/4$. We set

$$\hat{g} := Y^* \mathring{g}$$

so that $\hat{\phi} = Y^* \phi$ and $\hat{\lambda} = \lambda$ are solutions (smoothly depending on the real parameter s) of

$$\begin{cases} \Delta_{\hat{g}} \hat{\phi} + \hat{\lambda} \hat{\phi} = 0 & \text{in } C_1^{2\pi} \\ \hat{\phi} = 0 & \text{on } \partial C_1^{2\pi} \end{cases}$$

with

$$\int_{C_1^{2\pi}} \hat{\phi}^2 \, \mathrm{dvol}_{\hat{g}} = 1$$

We remark that $\hat{\phi}_1 := Y^* \phi_1$ is a solution of

$$\Delta_{\hat{g}} \hat{\phi}_1 + \lambda_1 \hat{\phi}_1 = 0$$

since $\hat{g} = Y^* \mathring{g}$. Moreover

$$\hat{\phi}_1(y,t) = \phi_1 \left((1 + s\, w)\ y\, ,\ \frac{Tt}{2\pi} \right), \tag{3.31}$$

on $\partial C_1^{2\pi}$. Writing $\hat{\phi} = \hat{\phi}_1 + \hat{\psi}$ and $\hat{\lambda} = \lambda_1 + \mu$, we find that

$$\begin{cases} \Delta_{\hat{g}} \hat{\psi} + (\lambda_1 + \mu)\, \hat{\psi} + \mu\, \hat{\phi}_1 = 0 & \text{in } C_1^{2\pi} \\ \hat{\psi} = -\hat{\phi}_1 & \text{on } \partial C_1^{2\pi} \end{cases} \tag{3.32}$$

with

$$\int_{C_1^{2\pi}} (2\, \hat{\phi}_1\, \hat{\psi} + \hat{\psi}^2)\, \mathrm{dvol}_{\hat{g}} = \int_{C_1^{2\pi}} \phi_1^2 \ \mathrm{dvol}_{\mathring{g}} - \int_{C_{1+sw}^{2\pi}} \phi_1^2 \ \mathrm{dvol}_{\mathring{g}} \tag{3.33}$$

Obviously $\hat{\psi}$ and μ are smooth functions of s. When $s = 0$, we have $\phi = \phi_1$ and $\lambda = \lambda_1$. Therefore, $\hat{\psi}$ and μ vanish and $\hat{\phi}_1 = \phi_1$, when $s = 0$. Moreover $\hat{g} = \mathring{g}$ when $s = 0$. We set

$$\dot{\psi} = \partial_s \hat{\psi}|_{s=0}, \qquad \text{and} \qquad \dot{\mu} = \partial_s \mu|_{s=0},$$

Differentiating (3.32) with respect to s and evaluating the result at $s = 0$, we obtain

$$\begin{cases} \Delta_{\mathring{g}} \dot{\psi} + \lambda_1 \dot{\psi} + \dot{\mu}\, \phi_1 = 0 & \text{in } C_1^{2\pi} \\ \dot{\psi} = -\partial_r \phi_1\, w & \text{on } \partial C_1^{2\pi} \end{cases} \tag{3.34}$$

because from (3.31), differentiation with respect to s at $s = 0$ yields $\partial_s \hat{\phi}_1|_{s=0} = \partial_r \phi_1\, w$.

Differentiating (3.33) with respect to s and evaluating the result at $s = 0$, we obtain

$$\int_{C_1^{2\pi}} \phi_1 \, \dot{\psi} \, \mathrm{dvol}_{\mathring{g}} = 0 \tag{3.35}$$

Indeed, the derivative of the right hand side of (3.33) with respect to s vanishes when $s = 0$ since ϕ_1 vanishes identically on $\partial C_1^{2\pi}$.

If we multiply the first equation of (3.34) by ϕ_1 and we integrate it over $C_1^{2\pi}$, using the boundary condition and the fact that the average of w is 0 we conclude that $\dot{\mu} = 0$. And hence $\dot{\psi}(2\pi t/T)$ is precisely the solution of (3.11). To summarize, we have proven that

$$\hat{\phi}(x,t) = \hat{\phi}_1(x,t) + s\,\psi(x, Tt/2\pi) + \mathcal{O}(s^2)$$

where ψ is the solution of (3.11). In particular, in $C_1^{2\pi} \setminus C_{3/4}^{2\pi}$, we have

$$\begin{aligned}\hat{\phi}(y,t) &= \phi_1\left((1 + s\,w)\; y\,,\; Tt/2\pi\right) + s\,\psi(y, Tt/2\pi) + \mathcal{O}(s^2)\\ &= \phi_1\left(y, Tt/2\pi\right) + s\left(w\, r\,\partial_r \phi_1 + \psi(y, Tt/2\pi)\right) + \mathcal{O}(s^2)\end{aligned}$$

where we have set $r := |y|$.

To complete the proof of the result, it suffices to compute the normal derivative of the function $\hat{\phi}$ when the normal is computed with respect to the metric $\hat{g}$. We use cylindrical coordinates $(y,t) = (r\,z, t)$ where $r > 0$ and $z \in S^{n-1}$. Then the metric $\hat{g}$ can be expanded in $C_1^{2\pi} \setminus C_{3/4}^{2\pi}$ as

$$\hat{g} \;=\; (1+sw)^2\, dr^2 + s\, r\, w'\,(1+sw)\, dr\, dt + \left((T/2\pi)^2 + s^2\, r^2\,(w')^2\right)\, dt^2 + r^2\,(1+sw)^2\, \mathring{h}$$

where $\mathring{h}$ is the metric on S^{n-1} induced by the Euclidean metric. It follows from this expression that the unit normal vector field to $\partial C_1^{2\pi}$ for the metric $\hat{g}$ is given by

$$\hat{\nu} = \left((1 + s\,w)^{-1} + \mathcal{O}(s^2)\right)\,\partial_r + \mathcal{O}(s)\,\partial_t$$

Using this, we conclude that

$$\hat{g}(\nabla \hat{\phi}_1, \hat{\nu}) = \partial_r \phi_1 + s\,\left(w\;\partial_r^2 \phi_1 + \partial_r \psi(y, Tt/2\pi)\right) + \mathcal{O}(s^2)$$

on $\partial C_1^{2\pi}$. The result then follows at once from the fact that $\partial_r \phi_1$ is constant while the term $w\,\partial_r^2 \phi_1 + \partial_r \psi(y, Tt/2\pi)$ has mean 0 on the boundary $\partial C_1^{2\pi}$. This completes the proof of the proposition. □

3.4 A Lyapunov-Schmidt argument

Our aim is to prove that for some $T \in (0, +\infty)$ there exists a nonzero (and obviously nonconstant) function v that solves the equation

$$F(v, T) = 0.$$

Unfortunately we will not be able to solve this equation at once. Instead we can split the image of the operator F into two spaces, one infinite-dimensional and one finite-dimensional, and solve the equation over the infinite-dimensional space.

It follows from the previous paragraph that the kernel of the operator H_{T_*} is finite-dimensional and given by $V_{k_1} \oplus \cdots \oplus V_{k_l}$. Based on this fact we will be interested in the method of Lyapunov-Schmidt, that is a procedure to reduce the dimension of the space in which we try to solve our equation $F(v, T) = 0$ near a singular point from infinite to finite dimension. The idea is to split the space into two subspaces and to project the equation into each one of them. One of the two equations obtained can be solved by the implicit function theorem.

Let Q be the projection operator onto the image of H_{T_*} and $Q \circ F$ the composition of operators F and Q. Let $v = v^{\parallel} + v^{\perp}$ with $v^{\perp} \in (\mathrm{Ker} H_{T_*})^{\perp}$ for a generical function in $\mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$. The next result follows from the implicit function theorem :

Proposition 3.4.1. *For all $v^{\parallel} \in (\mathrm{Ker} H_{T_*})$ which norm is small enough and all T sufficiently close to T_* there exists a unique function $v^{\perp} = v^{\perp}(v^{\parallel}, T)$ defined in a neighborhood of $(0, T_*)$ such that*

$$Q \circ F\left(v^{\parallel} + v^{\perp}, T\right) = 0.$$

Proof : We can define the operator

$$J(v^{\parallel}, v^{\perp}, T) = Q \circ F\left(v^{\parallel} + v^{\perp}, T\right)$$

The operator J maps from $\mathrm{Ker} H_{T_*} \times (\mathrm{Ker} H_{T_*})^{\perp} \times (0, +\infty)$ into the image of H_{T_*}. By the proposition 3.3.2 the implicit function theorem applies to get the existence of a unique function $v^{\perp}(v^{\parallel}, T) \in (\mathrm{Ker} H_{T_*})^{\perp}$ smoothly depending on $v^{\parallel}$ and T in a neighborhood of $(0, T_*)$ such that

$$J(v^{\parallel}, v^{\perp}(v^{\parallel}, T), T) = 0.$$

□

3.5 A bifurcation argument

We are now able to prove our main theorem 3.1.2. We will use a bifurcation argument. For the sake of completeness we recall the concept of bifurcation and bifurcation point (see

[24] and [38] for details). Let f be an operator on $\mathbb{B}_1 \times \Lambda$ into $\mathbb{B}_2$, where $\mathbb{B}_1$ and $\mathbb{B}_2$ are Banach spaces (or subspaces) and Λ is an interval of $\mathbb{R}$. Thus suppose that $\Gamma = (x(s), s)$ is a curve of solutions of the equation $f(x, s) = 0$. Let $(x_0, s_0) = (x(s_0), s_0)$ be an interior point on this curve with the property that every neighborhood of (x_0, s_0) in $\mathbb{B}_1 \times \Lambda$ contains solutions of the equation $f(x, s) = 0$ which are not in Γ. Then (x_0, s_0) is called a *bifurcation point* with respect to Γ and we say that in that point there is a *bifurcation* of the solution of $f(x, s) = 0$.

In relation to our problem we will show that $(0, T_*)$ is a bifurcation point with respect to the curve $\Gamma = (0, T)$ for the solution of the equation $F(v, T) = 0$. Here $\Lambda = (0, +\infty)$, $\mathbb{B}_1 = \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ and $\mathbb{B}_2 = \mathcal{C}^{1,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$. We remark that this is equivalent to the existence of a sequence of real positive numbers $T_j \longrightarrow T_*$ and a sequence of functions $v_j \in \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ converging to 0 in $\mathcal{C}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$ such that the points (v_j, T_j) are solutions of the equation $F(v, T) = 0$. And such a result is exactly our main theorem 3.1.2 (moreover the functions v_j are even).

We have proven that for each $(v^{\|}, T)$ in a neighborhood of $(0, T_*)$ there exists a function $v^{\perp} \in \mathrm{Ker}(H_{T_*})^{\perp}$ smoothly depending on $v^{\|}$ and T such that

$$Q \circ F\left(v^{\|} + v^{\perp}(v^{\|}, T), T\right) = 0.$$

Let I be the identity operator. We remark that if we find $(v^{\|}, T)$ such that

$$(I - Q) \circ F\left(v^{\|} + v^{\perp}(v^{\|}, T), T\right) = 0.$$

then it will be verified the equation

$$F\left(v^{\|} + v^{\perp}(v^{\|}, T), T\right) = 0.$$

Let us define

$$G(v^{\|}, T) = (I - Q) \circ F\left(v^{\|} + v^{\perp}(v^{\|}, T), T\right)$$

We will prove that $(0, T_*)$ is a bifurcation point with respect to the curve

$$\{(0, T) \qquad , \qquad T \text{ in a neighborhood of } T_*\}$$

for the finite dimensional operator G. It will follow the existence of a sequence of real positive numbers $T_j \longrightarrow T_*$ and a sequence of functions $v_j^{\|}$ converging to 0 such that the points $(v_j^{\|}, T_j)$ are solutions of the equation $G(v^{\|}, T) = 0$. Then the sequence (v_j, T_j) with v_j defined by

$$v_j = v_j^{\|} + v^{\perp}(v_j^{\|}, T)$$

will satisfy the statement of the main theorem 3.1.2.

We recall the concept of odd crossing number following the approach of [24]. Let we come back to the operator f, with finite-dimensional Banach spaces $\mathbb{B}_1$ and $\mathbb{B}_2$. A necessary condition for bifurcation is that 0 is an isolated eigenvalue of finite algebraic multiplicity, say l, of the operator obtained by linearizing f with respect to x at $(0, s_0)$, which can be denoted by $D_x f(0, s_0)$. It is crucial to know how the eigenvalue 0 perturbs for $D_x f(0, s_0)$ when s varies in a neighborhood of s_0. It is possible to show (see [23]) that the generalized eigenspace E_{s_0} of the eigenvalue 0 of $D_x f(0, s_0)$ having dimension l is perturbed to an invariant space E_s of $D_x f(0, s)$ of dimension l too, and all perturbed eigenvalues near 0 (the so-called 0-group) are eigenvalues of the finite-dimensional operator $D_x f(0, s)$ restricted to the l-dimensional invariant space E_s. Moreover the eigenvalues in that 0-group depend continuously on s. Motivated on these facts we can give the definition of *odd crossing number.*

Definition 3.5.1. Let $\Theta(s) = 1$ if there are no negative real eigenvalues in the 0-group of $D_x G(0, s)$, and

$$\Theta(s) = (-1)^{l_1 + \cdot + l_h}$$

if $\mu_1, ..., \mu_h$ are all the negative real eigenvalues of the 0-group having algebraic multiplicity $l_1, ..., l_h$, respectively. If $D_x f(0, s)$ is regular in a neighborhood of s_0 (naturally except in the point s_0) and $\Theta(s)$ changes the sign at s_0 then $D_x f(0, s)$ has an odd crossing number at s_0.

In presence of an odd crossing number there exists a standard result known as the Krasnosel'skii Bifurcation Theorem (see [24] for the proof) :

Theorem 3.5.2. *If $D_x f(0, s)$ has an odd crossing number at s_0, then $(0, s_0)$ is a bifurcation point for $f(x, s) = 0$ with respect to the curve* $\{(0, s) \mid s \text{ in a neighborhood of } s_0\}$.

Thank to this result, our main theorem 3.1.2 follows from the following proposition :

Proposition 3.5.3. *$D_{v^\parallel} G(0, T)$ has an odd crossing number at T_*.*

Proof : We observe that we can write

$$v^\parallel = \sum_{i=1}^{l} a_{k_i} \cos(k_i \, t)$$

where $1 = k_1 < \cdots < k_l$. It is clear, from the definition of G, that $D_{v^\parallel} G(0, T)$ preserves the eigenspaces, and

$$D_{v^\parallel} G(0, T) = H_T|_{V_{k_1} \oplus \cdots \oplus V_{k_l}}$$

Then the 0-group of eigenvalues is given by $\sigma_{k_1}(T), ..., \sigma_{k_l}(T)$, where $\sigma_{k_1}(T) = \sigma(T)$. For $T = T_*$ they are all equal to 0. Moreover, by the proposition 3.3.2 only $\sigma_{k_1}(T)$ changes sign at T_*, and its associated eigenspace has dimension 1. This means that $D_{v^\parallel} G(0, T)$ has a crossing number at T_* and completes the proof of the proposition. □

Chapitre 4

Extremal domains of big volume in compact manifolds

Prépublication

Résumé. On démontre l'existence de domaines extrémaux pour la première valeur propre de l'opérateur de Laplace-Beltrami, dans certaines variétés riemanniennes de dimension $n \geq 2$, de volume proche au volume de la variété. Si la première fonction propre (positive) ϕ_0 de l'opérateur de Laplace-Beltrami sur toute la variété est une fonction non constante, alors ces domaines sont proche au complément de boules géodésiques de rayon petit et centre proche au point où ϕ_0 atteint son maximum. Si ϕ_0 est une fonction constante et $n \geq 4$, ces domaines sont proche au complément de boules géodésiques de rayon petit et centre proche à un point critique non dégénéré de la courbure scalaire.

Abstract. We prove the existence of extremal domains for the first eigenvalue of the Laplace-Beltrami operator in some compact Riemannian manifolds of dimension $n \geq 2$, with volume close to the volume of the manifold. If the first (positive) eigenfunction ϕ_0 of the Laplace-Beltrami operator over the manifold is a nonconstant function, these domains are close to the complement of geodesic balls of small radius whose center is close to the point where ϕ_0 attains its maximum. If ϕ_0 is a constant function and $n \geq 4$, these domains are close to the complement of geodesic balls of small radius whose center is close to a nondegenerate critical point of the scalar curvature function.

4.1 Statement of the result

Article [32] is a first study on the possibility to construct extremal domains for the first eigenvalue of the Laplace-Beltrami operator in a Riemannian manifold. In that work are introduced all basic definitions and properties of extremal domains for the first eigenvalue of the Laplace-Beltrami operator, and the examples of domains obtained have small volume. In this article we will give an existence result for extremal domains of great volume in a compact riemannian manifold. We will be interested in domains obtained by taking the complement of small domains contained in the interior of the manifold : $M\backslash\Omega$, where Ω has a small volume and $\overline{\Omega} \subseteq \mathring{M}$. For the sake of completeness, we start recalling the basic facts of extremal domains (for other details see [32]).

Assume that we are given (M, g) a compact n-dimensional Riemannian manifold, $n \geq 2$, with or without boundary ∂M. In the case $\partial M \neq \emptyset$, then ∂M is supposing to be an $n-1$-dimensional Riemannian manifold. Let $\overline{\Omega}_0$ be a domain in the interior of M and let us consider the domain $M\backslash\Omega_0$.

Definition 4.1.1. We say that $\{M\backslash\Omega_t\}_{t\in(-t_0,t_0)}$, $\overline{\Omega}_t \subseteq \mathring{M}$, is a deformation of $M\backslash\Omega_0$ if there exists a vector field Ξ (such that $\Xi(\partial M) \subseteq \partial M$) for which $M\backslash\Omega_t = \xi(t, M\backslash\Omega_0)$ where $\xi(t, \cdot)$ is the flow associated to Ξ, namely

$$\frac{d\xi}{dt}(t,p) = \Xi(\xi(t,p)) \qquad \text{and} \qquad \xi(0,p) = p\,.$$

The deformation is said to be volume preserving if the volume of $M\backslash\Omega_t$ does not depend on t.

Let us denote by λ_t the first eigenvalue of $-\Delta_g$ on $M\backslash\Omega_t$ with 0 Dirichlet boundary condition on $\partial\Omega_t$. In the case where $\partial M \neq \emptyset$, then we ask also one of the following boundary condition :

1. 0 Dirichlet boundary condition on ∂M, or
2. 0 Neumann boundary condition on ∂M.

We will suppose the regularity of ∂M. Observe that both $t \mapsto \lambda_t$ and the associated eigenfunction $t \mapsto u_t$ (normalized to have $L^2(M\backslash\Omega_t)$ norm equal to 1) are continuously differentiable.

Definition 4.1.2. A domain $M\backslash\Omega_0$ is an *extremal domain* for the first eigenvalue of $-\Delta_g$ if for any volume preserving deformation $\{M\backslash\Omega_t\}_t$ of $M\backslash\Omega_0$, we have

$$\frac{d\lambda_t}{dt}|_{t=0} = 0\,.$$

According to the condition taken at the boundary (if the boundary is not empty) we will talk about extremal domains under the 0 Dirichlet boundary condition at ∂M or extremal domains under the 0 Neumann boundary condition at ∂M. Let ϕ_0 be the first eigenfunction of the Laplace-Beltrami operator over the manifold M

$$\Delta_g\,\phi_0 + \lambda_0\,\phi_0 \;=\; 0 \;\text{ in }\; M \tag{4.1}$$

with 0 Dirichlet or 0 Neumann boundary condition (if $\partial M \neq \emptyset$), normalized to have L^2-norm equal to 1. Here λ_0 is the first eigenvalue of $-\Delta_g$ on M. If the volume of Ω is very small, it is natural to expect that the first eigenfunction of the Laplace-Beltrami operator over $M \setminus \Omega$ will be close to ϕ_0. We remark that we have to distinguish two cases of behaviour of ϕ_0 (and then also of the first eigenfunction over $M \setminus \Omega$), according with the condition at the boundary :

- CASE 1. If $\partial M \neq \emptyset$ and ϕ_0 satisfy the 0 Dirichlet condition on ∂M then ϕ_0 is a positive non constant function, and then attains its maximum in at least a point of the manifold, say at p_0. Moreover $\lambda_0 > 0$.
- CASE 2. If $\partial M = \emptyset$, or if $\partial M \neq \emptyset$ and ϕ_0 satisfy the 0 Neumann condition on ∂M, then ϕ_0 is a constant function

$$\phi_0 = \frac{1}{\sqrt{\mathrm{Vol}_g(M)}}.$$

  and $\lambda_0 = 0$.

When we will consider the first eigenfunction of the Laplace-Beltrami operator over $M \setminus \Omega$, where $\Omega \subset \mathring{M}$, we will take at ∂M the same boundary condition of ϕ_0, distinguishing always the two cases.

For all $\epsilon > 0$ small enough, we denote by $B_\epsilon(p) \subset M$ the geodesic ball of center $p \in M$ and radius ϵ. We denote by $\mathring{B}_\epsilon \subset \mathbb{R}^n$ the Euclidean ball of radius ϵ centered at the origin.

Now we can state the main result of our paper :

Theorem 4.1.3. *In the CASE 1 assume that p_0 is a nondegenerate critical point of the first eigenfunction ϕ_0 of the Laplace-Beltrami operator over M, and in the CASE 2 assume that p_0 is a nondegenerate critical point of* Scal*, the scalar curvature function of (M, g). In the CASE 2 we will assume also $n \geq 4$. Then, for all $\epsilon > 0$ small enough, say $\epsilon \in (0, \epsilon_0)$, there exists a smooth domain $\Omega_\epsilon \subset M$ such that :*

(i) The volume of Ω_ϵ is equal to the Euclidean volume of $\mathring{B}_\epsilon$.

(ii) The domain $M \setminus \Omega_\epsilon$ is extremal in the sense of definition 4.1.2.

Moreover there exist a constant $c > 0$ and for all $\epsilon \in (0, \epsilon_0)$ there exists $p_\epsilon \in M$ such that the boundary of Ω_ϵ is a normal graph over $\partial B_\epsilon(p_\epsilon)$ for some function w_ϵ, with

$$\mathrm{dist}(p_\epsilon, p_0) \leq c\,\epsilon\,.$$

and

$$\begin{aligned} \|w_\epsilon\|_{\mathcal{C}^{2,\alpha}\partial B_\epsilon(p_\epsilon))} &\leq c\,\epsilon^2 && \text{in the CASE 1 and } n \geq 3 \\ \|w_\epsilon\|_{\mathcal{C}^{2,\alpha}\partial B_\epsilon(p_\epsilon))} &\leq c\,\epsilon^2 \log \epsilon && \text{in the CASE 1 and } n = 2 \\ \|w_\epsilon\|_{\mathcal{C}^{2,\alpha}\partial B_\epsilon(p_\epsilon))} &\leq c\,\epsilon^3 && \text{in the CASE 2 and } n \geq 5 \\ \|w_\epsilon\|_{\mathcal{C}^{2,\alpha}\partial B_\epsilon(p_\epsilon))} &\leq c\,\epsilon^3 \log \epsilon && \text{in the CASE 2 and } n = 4 \end{aligned}$$

We remark that the theorem do not give any information in the CASE 2 for the dimensions 2 and 3. In fact our methode to prove the main theorem is based, for the CASE 2, on the approximation of some Green function to the first eigenfunction of the Laplace-Beltrami operator outside a perturbed ball. When the dimension of M is at least 4, we are able to compute the first coefficients of the local expansion of that Green function and this allows us to obtain the estimations we need. But for the dimensions 2 and 3, global terms depending on the manifold do not allow us to obtain a local expansion of that Green function. This is the reason for which we didn't obtain information on extremal domains of big volume in the CASE 2 for the dimensions 2 and 3.

4.2 Characterization of the problem

In order to prove our theorem we need the following result that caracterizes extremal domains of the form $M\backslash\Omega$ in a Riemannian manifold M. The following result gives a formula for the first variation of the first eigenvalue for some mixte problems under variations of the domain.

We have the :

Proposition 4.2.1. *The derivative of* $t \longmapsto \lambda_t$ *at* $t = 0$ *is given by*

$$\frac{d\lambda_t}{dt}|_{t=0} = -\int_{\partial\Omega_0} \left(g(\nabla u_0, \nu_0)\right)^2 \; g(\Xi, \nu_0)\, \mathrm{dvol}_g,$$

where dvol_g *is the volume element on* $\partial\Omega_0$ *for the metric induced by* g *and* ν_0 *is the normal vector field about* $\partial\Omega_0$.

Proof : We denote by ξ the flow associated to Ξ. By definition, we have

$$u_t(\xi(t,p)) = 0 \tag{4.2}$$

for all $p \in \partial\Omega_0$. Moreover, if we take the 0 Dirichlet boundary condition on ∂M then equation (4.2) is valid also on ∂M. On the other hand, if we take the 0 Neumann condition on ∂M then we have

$$g(\nabla u_t(\xi(t,p)), \nu_t) = 0 \tag{4.3}$$

for all $p \in \partial M$, where ν_t is the unit normal vector about ∂M.

Differentiating (4.2) with respect to t and evaluating the result at $t = 0$ we obtain

$$\partial_t u_0 = -g(\nabla u_0, \Xi)\,,$$

on $\partial\Omega_0$. Now $u_0 \equiv 0$ on $\partial\Omega_0$, and hence only the normal component of Ξ plays a rôle in this formula. Therefore, we have

$$\partial_t u_0 = -\, g(\nabla u_0, \nu_0)\, g(\Xi, \nu_0)\,, \tag{4.4}$$

on $\partial\Omega_0$. The same reasoning is also valid on ∂M if we take the 0 Dirichlet boundary condition on ∂M. In this case, by the fact that $\Xi(\partial M) \subseteq T(\partial M)$ we have

$$\partial_t u_0 = 0 \tag{4.5}$$

on ∂M. On the other hand, if we take the 0 Neumann condition on ∂M then taking a system of coordinates $x = (x_1, ..., x_n)$ such that $\nu_t = -\partial_{x^0}$ on ∂M and differentiating (4.3) with respect to t and evaluating the result at $t = 0$ we obtain

$$0 = -\partial_{x^0}\partial_t u_0 - g(\nabla\partial_{x^0} u_0, \Xi) = -\partial_{x^0}\partial_t u_0 = g(\nabla\partial_t u_0, \nu_0) \tag{4.6}$$

on ∂M, where we used the fact that ν_t don't depend on t on ∂M together with the facts that $\partial_{x^0} u_0 = 0$ on ∂M and that $g(\Xi, \nu_0) = 0$ in ∂M because $\Xi(\partial M) \subseteq T(\partial M)$.

We differentiate now with respect to t the identity

$$\Delta_g\, u_t + \lambda_t\, u_t = 0. \tag{4.7}$$

and again evaluate the result at $t = 0$. We obtain

$$\Delta_g \partial_t u_0 + \lambda_0\, \partial_t u_0 = -\partial_t \lambda_0\, u_0\,, \tag{4.8}$$

in Ω_0. Now we multiply (4.8) by u_0 and (4.7), evaluated the result at $t = 0$, by $\partial_t u_0$, subtract the results and integrate it over Ω_0 to get :

$$\begin{aligned}
\partial_t\lambda_0 \int_{\Omega_0} u_0^2 \,\mathrm{dvol}_g &= \int_{M\setminus\Omega_0} (\partial_t u_0\, \Delta_g u_0 - u_0\, \Delta_g \partial_t u_0)\, \mathrm{dvol}_g \\
&= \int_{\partial M \cup \partial\Omega_0} (\partial_t u_0\, g(\nabla u_0, \nu_0) - u_0\, g(\nabla \partial_t u_0, \nu_0))\, \mathrm{dvol}_g \\
&= \int_{\partial\Omega_0} (\partial_t u_0\, g(\nabla u_0, \nu_0) - u_0\, g(\nabla \partial_t u_0, \nu_0))\, \mathrm{dvol}_g \\
&\quad + \int_{\partial M} (\partial_t u_0\, g(\nabla u_0, \nu_0) - u_0\, g(\nabla \partial_t u_0, \nu_0))\, \mathrm{dvol}_g \\
&= -\int_{\partial\Omega_0} (g(\nabla u_0, \nu_0))^2\; g(\Xi, \nu_0)\; \mathrm{dvol}_g\,,
\end{aligned}$$

where we have used (4.4), (4.5) or (4.6), the fact that $u_0 = 0$ on $\partial\Omega_0$, and the the fact that $u_0 = 0$ or $g(\nabla u_0, \nu_0) = 0$ on ∂M to obtain the last equality. The result follows at once from the fact that u_0 is normalized to have $L^2(\Omega_0)$ norm equal to 1. Observe that in the previous argument ∂M can be empty. □

This result allows us to characterize extremal domains for the first eigenvalue of the Laplace-Beltrami operator under some particular 0 mixte boundary conditions, and state the problem of finding extremal domains into the solvability of an over-determined elliptic problem. The proof of the following proposition is a consequence of the previous result ; because it is very similar to the proof of Proposition 2.2 in [32] we don't report it here.

Proposition 4.2.2. *Given a smooth domain Ω_0 contained in the interior of M, the domain $M \backslash \Omega_0$ is extremal if and only if there exists a constant λ_0 and a positive function u_0 (if $\partial M \neq \emptyset$ then we take 0 Dirichlet (CASE 1) or 0 Neumann (CASE 2) boundary condition on ∂M) such that*

$$\left\{ \begin{array}{rcll} \Delta_g u_0 + \lambda_0 \, u_0 & = & 0 & \text{in} \quad M\backslash\Omega_0 \\[2mm] u_0 & = & 0 & \text{on} \quad \partial\Omega_0 \\[2mm] g(\nabla u_0, \nu_0) & = & \text{constant} & \text{on} \quad \partial\Omega_0 \, , \end{array} \right. \tag{4.9}$$

where ν_0 is the normal vector field about $\partial\Omega_0$.

Therefore, in order to find extremal domains, it is enough to find a domain $M\backslash\Omega_0$ (regular enough) for which the over-determined problem (4.9) has a nontrivial positive solution. In this article we will solve this problem to find solutions whose volumes are close to the volume of our compact manifold.

4.3 Rephrasing the problem

Following the approach of [32], we introduce the following notation. Given a point $p \in M$ we denote by $E_1, \ldots, E_n$ an orthonormal basis of the tangent plane to M at p. Geodesic normal coordinates $x := (x^1, \ldots, x^n) \in \mathbb{R}^n$ at p are defined by

$$X(x) := \mathrm{Exp}_p^g \left(\sum_{j=1}^n x^j \, E_j \right)$$

We recall the Taylor expansion of the coefficients g_{ij} of the metric $X^* g$ in these coordinates.

Proposition 4.3.1. *At the point of coordinate x, the following expansion holds :*

$$g_{ij} = \delta_{ij} + \frac{1}{3} \sum_{k,\ell} R_{ikj\ell} \, x^k \, x^\ell + \frac{1}{6} \sum_{k,\ell,m} R_{ikjl,m} \, x^k \, x^\ell \, x^m + \mathcal{O}(|x|^4), \tag{4.10}$$

Here R is the curvature tensor of g and

$$\begin{array}{rcl} R_{ikj\ell} & = & g\big(R(E_i, E_k) \, E_j, E_\ell\big) \\[2mm] R_{ikj\ell;m} & = & g\big(\nabla_{E_m} R(E_i, E_k) \, E_j, E_\ell\big) \, , \end{array}$$

are evaluated at the point p.

The proof of this proposition can be found in [39] or also in [35].

It will be convenient to identify $\mathbb{R}^n$ with T_pM and S^{n-1} with the unit sphere in T_pM. If $x := (x^1, \dots, , x^n) \in \mathbb{R}^n$, we set

$$\Theta(x) := \sum_{i=1}^{n} x^i \, E_i \in T_pM \,.$$

Given a continuous function $f : S^{n-1} \longmapsto (0, \infty)$ whose L^∞ norm is small (say less than the cut locus of p) we define

$$B^g_f(p) := \left\{ \mathrm{Exp}_p(\Theta(x)) \qquad : \quad x \in \mathbb{R}^n \qquad 0 \leq |x| < f(x/|x|) \right\} \,.$$

The superscript g is meant to remind the reader that this definition depends on the metric.

Our aim is to show that, for all $\epsilon > 0$ small enough, we can find a point $p \in M$ and a function $v : S^{n-1} \longrightarrow \mathbb{R}$ such that

$$\mathrm{Vol}\, B^g_{\epsilon(1+v)}(p) = \epsilon^n \;\, \mathrm{Vol}\, \mathring{B}_1$$

and the over-determined problem

$$\left\{ \begin{array}{rclll} \Delta_g \, \phi + \lambda \, \phi & = & 0 & \text{in} & M \setminus B^g_{\epsilon(1+v)}(p) \\ \phi & = & 0 & \text{on} & \partial B^g_{\epsilon(1+v)}(p) \\ g(\nabla \phi, \nu) & = & \text{constant} & \text{on} & \partial B^g_{\epsilon(1+v)}(p) \end{array} \right. \tag{4.11}$$

with 0 Dirichlet (CASE 1) or 0 Neumann (CASE 2) boundary condition on ∂M if $\partial M \neq \emptyset$, has a nontrivial positive solution, where ν is the normal vector field about $\partial B^g_{\epsilon(1+v)}(p)$.

Observe that, considering the dilated metric $\bar{g} := \epsilon^{-2} \, g$, the above problem is equivalent to finding a point $p \in M$ and a function $v : S^{n-1} \longrightarrow \mathbb{R}$ such that

$$\mathrm{Vol}\, B^{\bar{g}}_{1+v}(p) = \;\mathrm{Vol}\, \mathring{B}_1$$

and for which the over-determined problem

$$\left\{ \begin{array}{rclll} \Delta_{\bar{g}} \, \bar{\phi} + \bar{\lambda} \, \bar{\phi} & = & 0 & \text{in} & B^{\bar{g}}_{1+v}(p) \\ \bar{\phi} & = & 0 & \text{on} & \partial B^{\bar{g}}_{1+v}(p) \\ \bar{g}(\nabla \bar{\phi}, \bar{\nu}) & = & \text{constant} & \text{on} & \partial B^{\bar{g}}_{1+v}(p) \end{array} \right. \tag{4.12}$$

with 0 Dirichlet (CASE 1) or 0 Neumann (CASE 2) boundary condition on ∂M if $\partial M \neq \emptyset$, has a nontrivial positive solution, where $\bar{\nu}$ is the normal vector field about $\partial B^{\bar{g}}_{1+v}(p)$. We can simply consider

$$\phi = \bar{\phi}$$

(naturally it will not have the norm equal to 1, but depending on ϵ) and

$$\lambda = \epsilon^{-2}\,\bar{\lambda}\,.$$

In what it follows we will consider sometimes the metric g and sometimes the metric $\bar{g}$, in order to simplify the computation we will meet.

4.4 The first eigenfunction of $-\Delta_g$ outside a small ball

We remark that the positive solution of the problem

$$\left\{\begin{array}{rcll} \Delta_g\,\phi_\epsilon + \lambda_\epsilon\,\phi_\epsilon & = & 0 & \text{in} \quad M\setminus B^g_\epsilon(p) \\ \phi_\epsilon & = & 0 & \text{on} \quad \partial B^g_\epsilon(p) \end{array}\right. \tag{4.13}$$

with 0 Dirichlet (CASE 1) or 0 Neumann (CASE 2) condition on ∂M if $\partial M \neq \emptyset$, normalized to have $L^2(M\setminus B^g_\epsilon(p))$-norm equal to 1, a priori is not known. In this section we will be interested in that solution.

Let $p\in M$, let c be a constant, and let Γ_p be a Green function over M with respect to the point p defined by

$$-\big(\Delta_g\; + \lambda_0\big)\Gamma_p = c_n\left(\delta_p - \phi_0(p)\,\phi_0\right) \qquad \text{in} \qquad M \tag{4.14}$$

with 0 Dirichlet boundary condition (for the CASE 1) or 0 Neumann boundary condition (for the CASE 2) at ∂M if $\partial M\neq\emptyset$,, and normalization

$$\int_M \Gamma\,\phi_0\;\;\mathrm{dvol}_g = 0,$$

where δ_p is the Dirac distribution for the manifold M with metric g at the point p, i.e.

$$\int_M \delta_p\,f\;\;\mathrm{dvol}_g = f(p) \text{ for all } f\in C^\infty_0(M) \qquad \text{and} \qquad \int_M \delta_p\;\;\mathrm{dvol}_g = 1$$

We remark that Γ_p exists because

$$\int_M \left[\delta_p - \phi_0(p)\,\phi_0\right]\,\phi_0\;\;\mathrm{dvol}_g = 0.$$

It is easy to check that for each dimension n of the manifold it is possible to chose the constant c_n in order to have the following expansions in local coordinates x of Γ_p in a neighborhood of the point p :

$$\begin{array}{lcrcll}
\text{for} \quad n=2 & : & \Gamma_p(x) & = & \log|x| + a + \mathring{g}(b,x) + \mathcal{O}(|x|^\alpha) & \forall\alpha<2 \\
\text{for} \quad n=3 & : & \Gamma_p(x) & = & |x|^{-1} + a' + \mathcal{O}(|x|^\alpha) & \forall\alpha<1 \\
\text{for} \quad n=4 & : & \Gamma_p(x) & = & |x|^{-2} + \mathcal{O}(|x|^\alpha) & \forall\alpha<0 \\
\text{for} \quad n\geq 5 & : & \Gamma_p(x) & = & |x|^{2-n} + \mathcal{O}(|x|^{4-n}) &
\end{array}$$

where $a, a' \in \mathbb{R}$ and $b \in \mathbb{R}^n$, and $\mathring{g}(\cdot,\cdot)$ is the scalar product in $\mathbb{R}$.

It is useful to consider the weighted space $\mathcal{C}^{k,\alpha}_\nu(M\setminus\{p\})$, defined as the space of functions in $\mathcal{C}^{k,\alpha}(M\setminus\{p\})$ such that, in the normal geodesic coordinates x around p,

$$\begin{aligned}\|u\|_{\mathcal{C}^{k,\alpha}_\nu(M\setminus\{p\})} &:= \sup_{\mathring{B}_{R_0}} |x|^{-\nu}\,|u| + \sup_{\mathring{B}_{R_0}} |x|^{1-\nu}\,|\nabla u| + \sup_{\mathring{B}_{R_0}} |x|^{2-\nu}\,|\nabla^2 u| + \cdots + \\ &+ \sup_{\mathring{B}_{R_0}} |x|^{k-\nu}\,|\nabla^k u| + \sup_{0<R\le R_0}\ \sup_{x,y\in \mathring{B}_R\setminus \mathring{B}_{R/2}} R^{k+\alpha-\nu}\left|\frac{\nabla^k u(x)-\nabla^k u(y)}{|x-y|^\alpha}\right| \le \infty.\end{aligned} \tag{4.15}$$

where R_0 is chosen in order to have the existence of the local coordinates $x \in \mathring{B}_{R_0}$.

Let us consider $\varphi \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$, where m is meant to point out that functions have (euclidean) mean 0 over S^{n-1}, and let H_φ be a bounded harmonic extension of φ to $\mathbb{R}^n \setminus \mathring{B}_1$:

$$\begin{cases} \Delta_{\mathring{g}} H_\varphi = 0 & \text{in } \mathbb{R}^n\setminus \mathring{B}_1 \\ H_\varphi = \varphi & \text{on } \partial \mathring{B}_1 \end{cases} \tag{4.16}$$

where $\mathring{g}$ is the euclidean metric and we identified $\partial\mathring{B}_1$ with S^{n-1}. We have the :

Lemma 4.4.1. *The following inequality holds :*

$$\|H_\varphi(x)\|_{\mathcal{C}^{2,\alpha}_{1-n}(\mathbb{R}^n\setminus\mathring{B}_1)} \le c\,\|\varphi\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}$$

for some positive constant c. In particular

$$\lim_{|x|\to+\infty} H_\varphi(x) = 0.$$

Proof. Let us consider

$$\varphi = \sum_{j=1}^{\infty} \varphi_j$$

the eigenfunction decomposition of φ, i.e.

$$\Delta_{S^{n-1}}\varphi_j = -j(n-2+j)\,\varphi_j \tag{4.17}$$

It is easy to check that

$$H_\varphi(x) = \sum_{j=1}^{\infty} |x|^{2-n-j}\,\varphi_j(x/|x|)$$

is the solution of (4.16). Let us fix $|x|$. We have

$$|H_\varphi(x)| \le \sum_{j=1}^{\infty} |x|^{2-n-j}\,|\varphi_j(x/|x|)| = |x|^{1-n}\,|\varphi_1(x/|x|)| + \sum_{j=2}^{\infty} |x|^{2-n-j}\,|\varphi_j(x/|x|)| \tag{4.18}$$

Let us try to estimate $\|\varphi_j\|_{L^\infty(S^{n-1})}$. From (4.17) we have

$$\|\varphi_j\|_{W^{2k,2}(S^{n-1})} \leq c\, j^k\, (n-2+j)^k\, \|\varphi_j\|_{L^2(S^{n-1})}$$

and by the Sobolev embedding theorem we have that $W^{2k,2}(S^{n-1}) \subseteq L^\infty(S^{n-1})$ when $4k > n-1$. We conclude that there exists a positive number $P(n)$ depending only on the dimension n such that

$$\|\varphi_j\|_{L^\infty(S^{n-1})} \leq c\, j^{P(n)}\, \|\varphi_j\|_{L^2(S^{n-1})}$$

Moreover

$$\|\varphi_j\|^2_{L^2(S^{n-1})} \leq \|\varphi\|^2_{L^2(S^{n-1})} \leq \ \mathrm{Vol}_{\mathring{g}}(S^{n-1})\, \|\varphi\|^2_{L^\infty(S^{n-1})}$$

and we can conclude that there exists a constant c such that

$$\|\varphi_j\|_{L^\infty(S^{n-1})} \leq c\, j^{P(n)}\, \|\varphi\|_{L^\infty(S^{n-1})}$$

From (4.18) we get

$$|H_\varphi(x)| \leq c\, |x|^{1-n}\, \|\varphi\|_{L^\infty(S^{n-1})} \left(1 + \sum_{j=2}^{\infty} |x|^{1-j}\, j^{P(n)}\right)$$

It is easy to check that for $|x| \geq 2$

$$\sum_{j=2}^{\infty} |x|^{1-j}\, j^{P(n)} \leq \infty$$

and this allows us to conclude that for $|x| \geq 2$ there exists a constant c such that

$$|H_\varphi(x)| \leq c\, |x|^{1-n}\, \|\varphi\|_{L^\infty(S^{n-1})} \tag{4.19}$$

By the maximum principle this inequality is valid also for $1 \leq |x| \leq 2$. Standard elliptic estimates apply to give also

$$|\nabla H_\varphi(x)| \leq c\, |x|^{-n}\, \|\varphi\|_{L^\infty(S^{n-1})} \tag{4.20}$$

Finally, (4.19) and (4.20) give the following estimate

$$\|H_\varphi(x)\|_{\mathcal{C}^{2,\alpha}_{1-n}(\mathbb{R}^n \setminus \mathring{B}_1)} \leq c\, \|\varphi\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}$$

for some constant c. From (4.19) it is clear that

$$\lim_{|x| \to +\infty} H_\varphi(x) = 0.$$

This completes the proof of the Lemma. $\square$

Let us define a continuous extension of H_φ to $\mathbb{R}^n$ in this way :

$$\tilde{H}_\varphi(x) = \begin{cases} 0 & \text{for } \; |x| \leq \frac{1}{2} \\ (2|x|-1)\, H_\varphi\left(\frac{x}{|x|}\right) & \text{for } \; \frac{1}{2} \leq |x| \leq 1 \\ H_\varphi(x) & \text{for } \; \mathbb{R}^n \setminus \mathring{B}_1 \end{cases} \tag{4.21}$$

and let us denote

$$H_{\varphi,\epsilon} = H_\varphi\left(\frac{x}{\epsilon}\right)$$

and

$$\tilde{H}_{\varphi,\epsilon} = \tilde{H}_\varphi\left(\frac{x}{\epsilon}\right).$$

Let χ be a cutoff function identically equal to 1 for $|x| \leq R_0 << 1$ (R_0 is chosen in such a way that $B^g_{R_0}(p)$ belongs to the x-coordinate neighborhood of p) and identically equal to 0 in $M \setminus B^g_{2R_0}(p)$.

The main result of this section is the following :

Proposition 4.4.2. *Let us suppose $n \geq 3$ and $\nu \in (2-n, \min\{4-n, 0\})$. For all ϵ small enough there exist $(\Lambda_\epsilon, \varphi_\epsilon, w_\epsilon)$ in a neighborhood of $(0,0,0)$ in $\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})\times\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\}))$ such that the function*

$$\phi_\epsilon = \phi_0 - \epsilon^{n-2}\,(\phi_0(p) + \Lambda_\epsilon)\,\Gamma_p + w_\epsilon + \chi\,\tilde{H}_{\varphi_\epsilon,\epsilon} \tag{4.22}$$

(considered in $M \setminus B^g_\epsilon(p)$), is a positive solution of (4.13) where

$$\lambda = \lambda_0 + \epsilon^{n-2}\,\mu \tag{4.23}$$

with

$$\mu = c_n\,\phi_0(p)^2 + \mathcal{O}(\epsilon). \tag{4.24}$$

Moreover the following estimations hold :

- *If ϕ_0 is not a constant function (CASE 1) then there exists a positive constant c such that*

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\,\epsilon \qquad \text{and} \qquad \|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})} \leq c\,(\epsilon^{2n-4} + \epsilon^n + \epsilon^{3-\nu})$$

- *If ϕ_0 is a constant function (CASE 2) then there exists a positive constant c such that*

$$\begin{aligned} |\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} &\leq c\,\epsilon && \text{if } n = 3 \\ |\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} &\leq c\,\epsilon^\beta \;\; \forall\beta < 2 && \text{if } n = 4 \\ |\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} &\leq c\,\epsilon^2 && \text{if } n \geq 5 \end{aligned}$$

and

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})} \leq c\,\epsilon^2 \qquad \text{if } n = 3$$

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})} \leq c\,\left(\epsilon^4 + \epsilon^{\beta-\nu}\right) \quad \forall \beta < 4 \qquad \text{if } n = 4$$

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})} \leq c\,\left(\epsilon^{2n-4} + \epsilon^{1+n} + \epsilon^{4-\nu}\right) \qquad \text{if } n \geq 5$$

Proof. First we prove that

$$\lambda - \lambda_0 = \mathcal{O}(\epsilon^{n-2}). \tag{4.25}$$

By definition

$$\lambda = \min_{u \in H^1_0(M\setminus B^g_\epsilon(p))} \frac{\displaystyle\int_{M\setminus B^g_\epsilon(p)} |\nabla^g u|^2 \,\mathrm{dvol}_g}{\displaystyle\int_{M\setminus B^g_\epsilon(p)} u^2 \,\mathrm{dvol}_g} \tag{4.26}$$

Let us consider a sequence of functions $u_j \in H^1_0(M \setminus B^g_\epsilon(p))$ converging to the function

$$u_*(x) = \begin{cases} \left(\dfrac{|x|}{\epsilon} - 1\right)\phi_0\left(\dfrac{2\,\epsilon\, x}{|x|}\right) & \text{in} \quad B^g_{2\,\epsilon}(p) \setminus B^g_\epsilon(p) \\ \phi_0 & \text{in} \quad M \setminus B^g_{2\,\epsilon}(p) \end{cases}$$

It is easy to check that

$$\int_{M\setminus B^g_\epsilon(p)} u_*^2 \,\mathrm{dvol}_g = \int_M \phi_0^2 \,\mathrm{dvol}_g + \mathcal{O}(\epsilon^n)$$

while

$$\int_{M\setminus B^g_\epsilon(p)} |\nabla u_*|^2 \,\mathrm{dvol}_g = \int_M |\nabla \phi_0|^2 \,\mathrm{dvol}_g + \mathcal{O}(\epsilon^{n-2})$$

then, from the last two relations and (4.26), it follows (4.25). This allows us to search λ in the form

$$\lambda = \lambda_0 + \epsilon^{n-2}\,\mu \tag{4.27}$$

where $\mu = o(1)$.

Let us chose ϕ_ϵ in the form

$$\phi_\epsilon = \phi_0 - \epsilon^{n-2}\,(\phi_0(p) + \Lambda)\,\Gamma_p + w + \chi\,\tilde{H}_{\varphi,\epsilon}$$

for some $(\Lambda, \varphi, w) \in \mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathcal{C}^{2,\alpha}(M \setminus \{p\})$. Then ϕ_ϵ satisfy the first equation of (4.13) over $M \setminus B^g_\epsilon(p)$, with λ as in (4.27), if and only if :

$$\begin{aligned} &\left(\Delta_g + \lambda_0 + \epsilon^{n-2}\,\mu\right) w + \epsilon^{n-2}\left[\mu - c_n\,\phi_0(p)\,(\phi_0(p) + \Lambda)\right]\phi_0 + H_{\varphi,\epsilon}\,\Delta_g\chi + \\ &\quad + \chi\,\Delta_g H_{\varphi,\epsilon} + 2\,\nabla^g H_{\varphi,\epsilon}\,\nabla^g\chi - \epsilon^{2n-4}\,\mu\,(\phi_0(p) + \Lambda)\,\Gamma_p + (\lambda_0 + \epsilon^{n-2}\,\mu)\,\chi\,H_{\varphi,\epsilon} = 0 \end{aligned} \tag{4.28}$$

over $M \setminus B^g_\epsilon(p)$. This equation can be considered over $M \setminus \{p\}$ if we take $\tilde{H}_{\varphi,\epsilon}$ instead of $H_{\varphi,\epsilon}$, and a continuous extension $\widetilde{\Delta_g H}_{\varphi,\epsilon}$ of the function $\Delta_g H_{\varphi,\epsilon}$ (similarly to the continuous extension $\tilde{H}_{\varphi,\epsilon}$ of the function $H_{\varphi,\epsilon}$). Remark that the term $\nabla^g H_{\varphi,\epsilon} \nabla^g \chi$ is 0 in a neighborhood of $\partial B^g_\epsilon(p)$ then can be extended to 0 in $B^g_\epsilon(p)$.

We need the following :

Lemma 4.4.3. *Let $n \geq 3$. The operator*

$$\left(\Delta_g + \lambda_0 + \epsilon^{n-2}\,\mu\right) : \mathcal{C}^{2,\alpha}_{\nu,\perp,0}(M \setminus \{p\}) \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-2,\perp}(M \setminus \{p\}),$$

where the subscript $\perp$ is meant to point out that functions are L^2-orthogonal to ϕ_0 and the subscript 0 is meant to point out that functions satisfy the 0 Dirichlet (in the CASE 1) or 0 Neumann (in the CASE 2) boundary condition on ∂M if $\partial M \neq \emptyset$, is an isomorphism for $\nu \in (2-n, 0)$ and ϵ small enough.

Proof. Let us suppose that $\nu \in (2-n, 0)$ and $n \geq 3$. In [30] is proved that for all $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\mathring{B}_1 \setminus \{0\})$ there exists a unique solution $u \in \mathcal{C}^{2,\alpha}_{\nu}(\mathring{B}_1 \setminus \{0\})$ of

$$\left\{ \begin{array}{rcll} \Delta_{\mathring{g}}\, u & = & f & \text{in} \quad \mathring{B}_1 \setminus \{0\} \\ u & = & 0 & \text{on} \quad \partial \mathring{B}_1 \end{array} \right. . \tag{4.29}$$

Now let us consider the operator $\Delta_g + \lambda_0$. Let R_0 small enough, take the normal geodesic coordinates in $B^g_{R_0}(p)$, and let $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(M \setminus \{p\})$. Considering the dilated metric $R_0^{-2}\, g$, the parameterization of $B^g_{R_0}(p)$ given by

$$Y(y) := \mathrm{Exp}^g_p \left(R_0 \sum_i y^i\, E_i \right)$$

and the ball $\mathring{B}_1$ endowed with the metric $\breve{g} = Y^*(R_0^{-2}\, g)$, the problem

$$\left\{ \begin{array}{rcll} (\Delta_g + \lambda_0)\, u & = & f & \text{in} \quad B^g_{R_0} \setminus \{p\} \\ u & = & 0 & \text{on} \quad \partial B^g_{R_0} \end{array} \right.$$

is equivalent, in terms of existence and unicity of solutions, to the problem

$$\left\{ \begin{array}{rcll} (\Delta_{\breve{g}} + R_0^2\, \lambda_0)\, u & = & Y^* f & \text{in} \quad \mathring{B}_1 \setminus \{0\} \\ u & = & 0 & \text{on} \quad \partial \mathring{B}_1 \end{array} \right. .$$

Considering that the difference between the coefficients of the metric $\breve{g}$ and the metric $\mathring{g}$ can be estimate by a constant times R_0^2, the operator $\Delta_{\breve{g}} + R_0^2\, \lambda_0$ is a small perturbation of the operator $\Delta_{\mathring{g}}$ when R_0 is small. By the previous claim, we conclude that there exists a positive R_0 (small enough) such that, when $\nu \in (2-n, 0)$ and $n \geq 3$, for all $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(M \setminus \{p\})$ there exists a unique solution $u \in \mathcal{C}^{2,\alpha}_{\nu}(B^g_{R_0} \setminus \{p\})$ of

$$\begin{cases} (\Delta_g + \lambda_0)\, u &= f \quad \text{in} \quad B^g_{R_0} \setminus \{p\} \\ u &= 0 \quad \text{on} \quad \partial B^g_{R_0} \end{cases}$$

Let now consider the solution of

$$(\Delta_g + \lambda_0)\, v = f - (\Delta_g + \lambda_0)\, (\tilde{\chi}\, u) \tag{4.30}$$

with 0 Dirichlet boundary condition at ∂M, where $\tilde{\chi}$ is a cut-off function equal to 1 for $|x| \leq R_0/2$ and equal to 0 for $|x| \geq R_0$. We remark that this equation is well defined in M, because the singularity of f at p is balanced by $(\Delta_g + \lambda_0)\, (\tilde{\chi}\, u)$. Moreover the right hand side term is orthogonal to ϕ_0 if f has such a property. Hence, there exists a solution $v \in \mathcal{C}^{2,\alpha}_{\perp,0}(M)$ to (4.30), and we have that

$$(\Delta_g + \lambda_0)\, (\tilde{\chi}\, u + v) = f$$

in $M \setminus \{p\}$, with 0 Dirichlet condition at ∂M. Obviously $w = \tilde{\chi}\, u + v \in \mathcal{C}^{2,\alpha}_{\nu,\perp}(M \setminus \{p\})$. We proved then that for $\nu \in (2-n, 0)$ and $n \geq 3$ and for all $f \in \mathcal{C}^{0,\alpha}_{\nu-2,\perp}(M \setminus \{p\})$ there exists a unique solution $w \in \mathcal{C}^{2,\alpha}_{\nu,\perp}(M \setminus \{p\})$ of

$$(\Delta_g + \lambda_0)\, w = f$$

in $M \setminus \{p\}$ with 0 Dirichlet at ∂M. This result is still true for the operator $\Delta_g + \lambda_0 + \epsilon^{n-2}\, \nu$ when ϵ is small enough, because such an operator is a small perturbation of the operator $\Delta_g + \lambda_0$. The raisoning done do not change if we consider the 0 Neumann boundary condition on ∂M instead of the 0 Dirichlet boundary condition. This completes the proof of the Lemma. □

To semplify the notation let us define

$$\begin{aligned} A &:= \epsilon^{n-2} \Big[\mu - c_n\, \phi_0(p)\, (\phi_0(p) + \Lambda)\Big]\, \phi_0 \\ B &:= \tilde{H}_{\varphi,\epsilon}\, \Delta_g \chi + \chi\, \widetilde{\Delta_g H}_{\varphi,\epsilon} + 2\, \nabla^g H_{\varphi,\epsilon}\, \nabla^g \chi \\ C &:= -\epsilon^{2n-4}\, \mu\, (\phi_0(p) + \Lambda)\, \Gamma_p \\ D &:= (\lambda_0 + \epsilon^{n-2}\, \mu)\, \chi\, \tilde{H}_{\varphi,\epsilon} \end{aligned}$$

We remark that $\Gamma_p \in \mathcal{C}^{0,\alpha}_{\nu-2}(M \setminus \{p\})$ if $\nu < 4-n$. Equation (4.28), extended to $M \setminus \{p\}$, becomes

$$(\Delta_g + \lambda_0 + \epsilon^{n-2}\, \mu)\, w = -(A + B + C + D)$$

By the last result, if we chose μ in order to verify

$$\int_M (A+B+C+D)\,\phi_0 = 0 \tag{4.31}$$

there exists a solution $w(\epsilon,\Lambda,\varphi) \in \mathcal{C}^{2,\alpha}_{\nu,\perp,0}(M\setminus\{p\})$ to equation (4.28) for

$$\nu \in (2-n, \min\{0, 4-n\})\,,$$

for all $\Lambda \in \mathbb{R}$, for all $\varphi \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$, and for all ϵ small enough, and then

$$\phi_\epsilon = \phi_0 + \epsilon^{n-2}\,(\phi_0(p)+\Lambda)\,\Gamma_p + w(\epsilon,\Lambda,\varphi) + \chi\,H_{\varphi,\epsilon}$$

with $w(\epsilon,\Lambda,\varphi)$ restricted to $M\backslash B^g_\epsilon(p)$, satisfies the first equation of (4.13). From (4.31) we get

$$\mu = \frac{\epsilon^{n-2}\,c_n\,\phi_0(p)\,(\phi_0(p)+\Lambda) - \displaystyle\int_M B\,\phi_0 - \lambda_0 \int_M \chi\,\tilde{H}_{\varphi,\epsilon}\,\phi_0}{\epsilon^{n-2}\left(1 + \displaystyle\int_M \chi\,\tilde{H}_{\varphi,\epsilon}\,\phi_0\right)}$$

It is easy to check that

$$\int_M B\,\phi_0 \le c\,\epsilon^{n-1}\,\|\varphi\|_{L^\infty(S^{n-1})}$$

and

$$\int_M \chi\,\tilde{H}_{\varphi,\epsilon}\,\phi_0 \le c\,\epsilon^{n-1}\,\|\varphi\|_{L^\infty(S^{n-1})}$$

from which it follows the expansion of μ :

$$\mu = c_n\,\phi_0(p)\,(\phi_0(p)+\Lambda) + \mathcal{O}(\epsilon)\,\|\varphi\|_{L^\infty(S^{n-1})}. \tag{4.32}$$

We want now to give some estimations on the function w. By the previous results and Lemma 4.4.1 we have the following estimations :

- $\|A\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus\{p\})} \le c\,\epsilon^{n-1}\,\|\varphi\|_{L^\infty(S^{n-1})}$
- $\|B\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus\{p\})} \le c\,(\epsilon^{n-1}+\epsilon^{2-\nu})\,\|\varphi\|_{L^\infty(S^{n-1})}$
- $\|C\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus\{p\})} \le c\,\epsilon^{2n-4}$
- $\|D\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus\{p\})} \le c\,\left(\epsilon^{2-\nu}+\epsilon^{n-1}\right)\,\|\varphi\|_{L^\infty(S^{n-1})}$

In particular we get

$$\|A+B+C+D\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus\{p\})} \le c\,\left(\epsilon^{2n-4} + \epsilon^{n-1}\,\|\varphi\|_{L^\infty(S^{n-1})} + \epsilon^{2-\nu}\,\|\varphi\|_{L^\infty(S^{n-1})}\right)$$

This give us an estimation for the function w that we found before :

$$\|w\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})} \le c\,\left(\epsilon^{2n-4} + \epsilon^{n-1}\,\|\varphi\|_{L^\infty(S^{n-1})} + \epsilon^{2-\nu}\,\|\varphi\|_{L^\infty(S^{n-1})}\right).$$

We have proved the following :

First intermediate result. *Let $\nu \in (2-n, 4-n)$. For all $\Lambda \in \mathbb{R}$, for all $\varphi \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$, for all ϵ small enough, there exists a function $w(\epsilon, \Lambda, \varphi) \in \mathcal{C}^{2,\alpha}_{\nu,\perp,0}(M\setminus\{p\})$ such that (4.22) is a positive solution of the first equation of (4.13). Moreover there exists a positive constant c such that*

$$\|w\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})} \le c\,\left(\epsilon^{2n-4} + \epsilon^{n-1}\,\|\varphi\|_{L^\infty(S^{n-1})} + \epsilon^{2-\nu}\,\|\varphi\|_{L^\infty(S^{n-1})}\right). \tag{4.33}$$

Now we have to make attention to the second equation of (4.13). Let us define

$$N(\epsilon,\Lambda,\varphi) := \Big[\phi_0(\epsilon\, y) - \epsilon^{n-2}\,(\phi_0(p)+\Lambda)\,\Gamma(\epsilon\, y) + \big(w(\epsilon,\Lambda,\varphi)\big)(\epsilon\, y) + \varphi(y)\Big]_{y\in S^{n-1}}.$$

We remark that N represents the boundary value of the solution of the first equation of (4.13) that we found above, is well defined in a heighborhood of $(0,0,0)$ in $(0,+\infty)\times\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})$, and takes its values in $\mathcal{C}^{2,\alpha}(S^{n-1})$. It is easy to compute the differential of N with respect to Λ and φ at $(0,0,0)$:

$$\big(\partial_\Lambda N(0,0,0)\big)(\tilde\Lambda) = -\tilde\Lambda$$

$$\big(\partial_\varphi N(0,0,0)\big)(\tilde\varphi) = \tilde\varphi.$$

From the estimation of the function w it follows that

$$\begin{aligned}\|w\|_{L^\infty(\partial B^g_\epsilon(p))} &\le \epsilon^\nu\,\|w\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})}\\ &\le c\,\left(\epsilon^{2n-4+\nu} + \epsilon^{n-1+\nu}\,\|\varphi\|_{L^\infty(S^{n-1})} + \epsilon^2\,\|\varphi\|_{L^\infty(S^{n-1})}\right).\end{aligned}$$

Then we can estimate $N(\epsilon,0,0)$:

$$\|N(\epsilon,0,0)\|_{L^\infty(S^{n-1})} \le \big\|\phi_0(\epsilon\,x) - \epsilon^{n-2}\,\phi_0(p)\,\Gamma(\epsilon\,x)\big\|_{L^\infty(S^{n-1})} + \big\|\big(w(\epsilon,0,0)\big)(\epsilon\,x)\big\|_{L^\infty(S^{n-1})}$$

Here we have again to distinguish the different cases, according on the behaviour of the function ϕ_0. If ϕ_0 is not a constant function (CASE 1) we have (using the expansion (4.15) of Γ_p)

$$\|N(\epsilon,0,0)\|_{L^\infty(S^{n-1})} \le c\,\epsilon.$$

The same estimate is obtained if ϕ_0 is a constant function (CASE 2) and $n=3$. In the CASE 2 and $n=4$ we get

$$\|N(\epsilon,0,0)\|_{L^\infty(S^{n-1})} \le c\,\epsilon^\beta$$

$\forall \beta < 2$ and when $n \geq 5$:

$$\|N(\epsilon, 0, 0)\|_{L^\infty(S^{n-1})} \leq c\, \epsilon^2$$

The implicit function theorem applies to give the :

Second intermediate result. *Let $\nu \in (2-n, \min\{4-n, 0\})$, let w be the function found in the first intermediate result, and let ϵ be small enough. Then there exist $(\Lambda_\epsilon, \varphi_\epsilon)$ in a neighborhood of $(0,0)$ in $\mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1})$ such that $N(\epsilon, \Lambda_\epsilon, \varphi_\epsilon) = 0$ (i.e. (4.22) is a positive solution of (4.13)). Moreover the following estimations hold :*

- *If ϕ_0 is not a constant function (CASE 1) then*

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\, \epsilon$$

- *If ϕ_0 is a constant function (CASE 2) then*

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\, \epsilon \qquad \text{if } n = 3$$

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\, \epsilon^\beta \ \ \forall \beta < 2 \quad \text{if } n = 4$$

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\, \epsilon^2 \qquad \text{if } n \geq 5$$

Putting together the first and the second intermediate result, we get the following existence result : for all ϵ small enough there exist $(\Lambda_\epsilon, \varphi_\epsilon, w_\epsilon)$ in a neighborhood of $(0,0,0)$ in $\mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathcal{C}^{2,\alpha}(M \setminus \{p\})$ such that the function

$$\phi_\epsilon = \phi_0 - \epsilon^{n-2}\,(\phi_0(p) + \Lambda_\epsilon)\,\Gamma_p + w_\epsilon + \chi\, H_{\varphi_\epsilon, \epsilon}$$

considered in $M \setminus B^g_\epsilon(p)$, is a positive solution of (4.13) where

$$\lambda = \lambda_0 + \epsilon^{n-2}\,\mu \tag{4.34}$$

and from (4.43) we obtain

$$\mu = c_n\, \phi_0(p)^2 + \mathcal{O}(\epsilon). \tag{4.35}$$

Moreover :

- If ϕ_0 is not a constant function (CASE 1) then there exists a positive constant c such that

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\, \epsilon$$

  and from (4.33)

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\}))} \leq c\, \left(\epsilon^{2n-4} + \epsilon^n + \epsilon^{3-\nu}\right)$$

- If ϕ_0 is a constant function (CASE 2) then there exists a positive constant c such that

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\, \epsilon \qquad \text{if } n = 3$$

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\, \epsilon^\beta \ \ \forall \beta < 2 \quad \text{if } n = 4$$

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\, \epsilon^2 \qquad \text{if } n \geq 5$$

and from (4.33)

$$\begin{aligned} \|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})} &\leq c\,\epsilon^2 && \text{if } n=3 \\ \|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})} &\leq c\,\left(\epsilon^4+\epsilon^{\beta-\nu}\right) \quad \forall \beta<4 && \text{if } n=4 \\ \|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})} &\leq c\,\left(\epsilon^{2n-4}+\epsilon^{1+n}+\epsilon^{4-\nu}\right) && \text{if } n\geq 5 \end{aligned}$$

This completes the proof of the result. □

For the case $n=2$ we can generalise the previuos proposition, obtaining the :

Proposition 4.4.4. *Let we suppose $n=2$ and $\nu \in (0,1)$. For all ϵ small enough there exist $(\Lambda_\epsilon, \varphi_\epsilon, w_\epsilon)$ in a neighborhood of $(0,0,0)$ in $\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})\times\left(\tilde{\chi}\,\mathbb{R}\oplus\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})\right)$, where $\tilde{\chi}$ is some cut-off funtion equal to 1 in a neighborhood of the origin, such that the function*

$$\phi_\epsilon = \phi_0 - (\phi_0(p)+\Lambda_\epsilon)\,\Gamma_p + w_\epsilon + \chi\,\tilde{H}_{\varphi_\epsilon,\epsilon} \tag{4.36}$$

considered in $M\setminus B^g_\epsilon(p)$, is a positive solution of (4.13) where

$$\lambda = \lambda_0 + (\log\epsilon)^{-1}\,\mu \tag{4.37}$$

with

$$\mu = c_n\,\phi_0(p)^2 + \mathcal{O}(\epsilon). \tag{4.38}$$

Moreover the following estimations hold : there exists a positive constant c such that

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\,\epsilon \qquad \text{and} \qquad \|w_\epsilon\|_{\tilde{\chi}\,\mathbb{R}\oplus\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})} \leq c\,\epsilon^2\,\log\epsilon$$

Proof. We will follow the proof of the previous proposition, adapting it to the case of dimension 2. First we prove that

$$\lambda - \lambda_0 \leq c\,\log(\epsilon)^{-1}. \tag{4.39}$$

Let we consider a sequence of functions $u_j \in H^1_0(M\setminus B^g_\epsilon(p))$ converging to the function

$$u_*(x) = \begin{cases} K\,\log\dfrac{|x|}{\epsilon} & \text{in } \; B^g_{R_0}(p)\setminus B^g_\epsilon(p) \\ \phi_0 & \text{in } \; M\setminus B^g_{R_0}(p) \end{cases}$$

where K is a constant chosen in order to make the function u_* continuous. It is easy to check that for ϵ small enough

$$\int_{M\setminus B^g_\epsilon(p)} u_*^2\,\mathrm{dvol}_g = \leq c\,\log(\epsilon)^2$$

while

$$\int_{M\setminus B^g_\epsilon(p)} |\nabla u_*|^2 \, \mathrm{dvol}_g \leq c \, \log(\epsilon)$$

then, from the last two relations and (4.26), it follows (4.39). This allows us to search λ in the form

$$\lambda = \lambda_0 + \log(\epsilon)^{-1} \, \mu \tag{4.40}$$

where $\mu = o(1)$.

Let we chose ϕ_ϵ in the form

$$\phi_\epsilon = \phi_0 - (\log \epsilon)^{-1} \, (\phi_0(p) + \Lambda) \, \Gamma_p + w + \chi \, \tilde{H}_{\varphi,\epsilon}$$

for some $(\Lambda, \varphi, w) \in \mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathcal{C}^{2,\alpha}(M \setminus \{p\})$. Then ϕ_ϵ satisfy the first equation of (4.13), with λ as in (4.40), if and only if the quantity

$$\begin{aligned}\Big(\Delta_g + \lambda_0 + (\log \epsilon)^{-1} \, \mu\Big) \, w + (\log \epsilon)^{-1} \Big[\mu - c_n \, \phi_0(p) \, (\phi_0(p) + \Lambda)\Big] \, \phi_0 + \tilde{H}_{\varphi,\epsilon} \, \Delta_g \chi + \\ + \chi \, \Delta_g \tilde{H}_{\varphi,\epsilon} + 2 \, \nabla^g \tilde{H}_{\varphi,\epsilon} \, \nabla^g \chi - (\log \epsilon)^{-2} \, \mu \, (\phi_0(p) + \Lambda) \, \Gamma_p + (\lambda_0 + (\log \epsilon)^{-1} \, \mu) \, \chi \, \tilde{H}_{\varphi,\epsilon}\end{aligned} \tag{4.41}$$

is identically equal to 0 over $M \setminus B^g_\epsilon(p)$. This equation can be consider, after opportune extension of functions like in equation (4.28), over $M \setminus \{p\}$.

The operator

$$\big(\Delta_g + \lambda_0\big) : \tilde{\chi} \, \mathbb{R} \oplus \mathcal{C}^{2,\alpha}_{\nu,\perp,0}(M \setminus \{p\}) \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-2,\perp}(M \setminus \{p\}),$$

where the subscript $\perp$ is meant to point out that functions are L^2-orthogonal to ϕ_0 and the subscript 0 is meant to point out that functions satisfy the 0 Dirichlet (in the CASE 1) or 0 Neumann (in the CASE 2) boundary condition on ∂M if $\partial M \neq \emptyset$, and $\tilde{\chi}$ is some cut-off funtion equal to 1 in a neighborhood of the origin, is an isomorphism for $\nu \in (0, 1)$. The same result holds for the operator

$$\big(\Delta_g + \lambda_0 + (\log \epsilon)^{-1} \, \mu\big) : \tilde{\chi} \, \mathbb{R} \oplus \mathcal{C}^{2,\alpha}_{\nu,\perp,0}(M \setminus \{p\}) \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-2,\perp}(M \setminus \{p\}),$$

when ϵ is small enough. The proof of this fact can be obtained by comparison of the proof of the Lemma 4.4.3 and the analysis of the case of dimension 2 in [30].

To semplify the notation let we define

$$\begin{aligned} A &:= (\log \epsilon)^{-1} \Big[\mu - c_n \, \phi_0(p) \, (\phi_0(p) + \Lambda)\Big] \, \phi_0 \\ B &:= \tilde{H}_{\varphi,\epsilon} \, \Delta_g \chi + \chi \, \Delta_g \tilde{H}_{\varphi,\epsilon} + 2 \, \nabla^g \tilde{H}_{\varphi,\epsilon} \, \nabla^g \chi \\ C &:= -(\log \epsilon)^{-2} \, \mu \, (\phi_0(p) + \Lambda) \, \Gamma_p \\ D &:= (\lambda_0 + (\log \epsilon)^{-1} \, \mu) \, \chi \, \tilde{H}_{\varphi,\epsilon} \end{aligned}$$

We remark that $\Gamma \in \mathcal{C}^{0,\alpha}_{\nu-2}(M \setminus \{p\})$ when $\nu \in (0,1)$. Equation (4.41) becomes

$$(\Delta_g + \lambda_0 + (\log \epsilon)^{-1} \mu)\, w = -(A + B + C + D)$$

By the last result, if we chose μ in order to verify

$$\int_M (A + B + C + D)\, \phi_0 = 0 \tag{4.42}$$

there exists a solution $w(\epsilon, \Lambda, \varphi) = w^{(1)} + w^{(2)} \in \tilde{\chi}\, \mathbb{R} \oplus \mathcal{C}^{2,\alpha}_{\nu,\perp,0}(M \setminus \{p\})$ to equation (4.41) for $\nu \in (0,1)$, for all $\Lambda \in \mathbb{R}$, for all $\varphi \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$, and for all ϵ small enough, and then

$$\phi_\epsilon = \phi_0 - (\phi_0(p) + \Lambda)\, \Gamma_p + w(\epsilon, \Lambda, \varphi) + \chi\, H_{\varphi,\epsilon}$$

satisfy the first equation of (4.13). From (4.42) we get

$$\mu = \frac{(\log \epsilon)^{-1}\, c_n\, \phi_0(p)\, (\phi_0(p) + \Lambda) - \displaystyle\int_M B\, \phi_0 - \lambda_0 \int_M \chi\, \tilde{H}_{\varphi,\epsilon}\, \phi_0}{(\log \epsilon)^{-1} \left(1 + \displaystyle\int_M \chi\, \tilde{H}_{\varphi,\epsilon}\, \phi_0\right)}$$

It is easy to check that

$$\int_M B\, \phi_0 \leq c\, \epsilon\, \|\varphi\|_{L^\infty(S^{n-1})}$$

and

$$\int_M \chi\, \tilde{H}_{\varphi,\epsilon}\, \phi_0 \leq c\, \epsilon\, \|\varphi\|_{L^\infty(S^{n-1})}$$

from which it follows the expansion of μ :

$$\mu = c_n\, \phi_0(p)\, (\phi_0(p) + \Lambda) + \mathcal{O}(\epsilon\, \log \epsilon)\, \|\varphi\|_{L^\infty(S^{n-1}}. \tag{4.43}$$

We want now to give some estimations on the function w. By the previous facts and Lemma 4.4.1 we have the following estimations :

- $\|A\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus\{p\})} \leq c\, \epsilon\, \log \epsilon\, \|\varphi\|_{L^\infty(S^{n-1})}$
- $\|B\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus\{p\})} \leq c\, \epsilon\, \|\varphi\|_{L^\infty(S^{n-1})}$
- $\|C\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus\{p\})} \leq c\, (\log \epsilon)^{-2}$
- $\|D\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus\{p\})} \leq c\, (\epsilon + \epsilon^{2-\nu})\, \|\varphi\|_{L^\infty(S^{n-1})}$

In particular we get

$$\|A+B+C+D\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus\{p\})} \leq c\,\left((\log\epsilon)^{-2} + \epsilon\,\log\epsilon\,\|\varphi\|_{L^\infty(S^{n-1})}\right)$$

where we used the fact that for ϵ small enough and $\nu \in (0,1)$ we have $\epsilon^{2-\nu} < \epsilon < \epsilon\,\log\epsilon$. This give us an estimation on the function w that we found before :

$$\|w^{(1)}\| + \|w^{(2)}\|_{\mathcal{C}^{2,\alpha}_{\nu}(M\setminus\{p\})} \leq c\,\left((\log\epsilon)^{-2} + \epsilon\,\log\epsilon\,\|\varphi\|_{L^\infty(S^{n-1})}\right).$$

We have proved the following :

First intermediate result. *Let $\nu \in (0,1)$. For all $\Lambda \in \mathbb{R}$, for all $\varphi \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$, for all ϵ small enough, there exists a function $w(\epsilon,\Lambda,\varphi) = w^{(1)} + w^{(2)} \in \tilde{\chi}\,\mathbb{R} \oplus \mathcal{C}^{2,\alpha}_{\nu,\perp,0}(M \setminus \{p\})$ such that (4.36) is a positive solution of the first equation of (4.13). Moreover there exists a positive constant c such that*

$$\|w^{(1)}\| + \|w^{(2)}\|_{\mathcal{C}^{2,\alpha}_{\nu}(M\setminus\{p\})} \leq c\,\left((\log\epsilon)^{-2} + \epsilon\,\log\epsilon\,\|\varphi\|_{L^\infty(S^{n-1})}\right)$$

Now we have to make attention to the second equation of (4.13). Let we define

$$N(\epsilon,\Lambda,\varphi) := \Big[\phi_0(\epsilon\,y) - (\log\epsilon)^{-1}\,(\phi_0(p)+\Lambda)\,\Gamma(\epsilon\,y) + \big(w(\epsilon,\Lambda,\varphi)\big)(\epsilon\,y) + \varphi(y)\Big]_{y\in S^{n-1}}.$$

We remark that N represents the boundary value of the solution of the first of (4.13) we found above, is well defined in a heighborhood of $(0,0,0)$ in $(0,+\infty)\times\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})$, and takes its values in $\mathcal{C}^{2,\alpha}(S^{n-1})$. The differential of N with respect to Λ and φ at $(0,0,0)$ is :

$$\big(\partial_\Lambda N(0,0,0)\big)(\tilde{\Lambda}) = -\tilde{\Lambda}$$

$$\big(\partial_\varphi N(0,0,0)\big)(\tilde{\varphi}) = \tilde{\varphi}.$$

From the estimation of the function w it follows that

$$\|w\|_{L^\infty(\partial B^g_\epsilon(p))} \;\leq\; c\,\left((\log\epsilon)^{-2} + \epsilon\,\log\epsilon\,\|\varphi\|_{L^\infty(S^{n-1})}\right)$$

Then we can estimate $N(\epsilon,0,0)$:

$$\|N(\epsilon,0,0)\|_{L^\infty(S^{n-1})} \leq \big\|\phi_0(\epsilon\,y) - (\log\epsilon)^{-1}\,\phi_0(p)\,\Gamma(\epsilon\,y)\big\|_{L^\infty(S^{n-1})} + \big\|\big(w(\epsilon,0,0)\big)(\epsilon\,y)\big\|_{L^\infty(S^{n-1})}$$

and we have

$$\|N(\epsilon,0,0)\|_{L^\infty(S^{n-1})} \leq c\,\epsilon$$

The implicit function theorem applies to give the :

Second intermediate result. *Let $\nu \in (0,1)$ and let w be the function found in the first intermediate result, and let ϵ be small enough. Then there exist $(\Lambda_\epsilon, \varphi_\epsilon)$ in a neighborhood of $(0,0)$ is $\mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1})$ such that $N(\epsilon, \Lambda_\epsilon, \varphi_\epsilon) = 0$ (i.e. (4.36) is a positive solution of (4.13)). Moreover the following estimation holds :*

$$|\Lambda_\epsilon| + \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\,\epsilon$$

The two intermediate results complete the proof of the Proposition. □

Observe that, beeing the problem of finding eigenfunctions linear, we can consider as ϕ_ϵ in dimension 2 the following function

$$\phi_\epsilon = \log\epsilon\,\left[\phi_0 - (\log\epsilon)^{-1}\,(\phi_0(p) + \Lambda_\epsilon)\,\Gamma_p + w_\epsilon + \chi\, H_{\varphi_\epsilon,\epsilon}\right]$$

This will semplify our reasoning because that function, considered in the coordinates $y = \epsilon x$, converges, in a sense to be made precise, to the function $-\phi_0(p)\,\log|y|$ when ϵ tends to 0.

4.5 Perturbing the complement of a ball

The following result follows from the implicit function theorem.

Proposition 4.5.1. *Given a point $p \in M$, there exists $\epsilon_0 > 0$ and for all $\epsilon \in (0, \epsilon_0)$ and all function $\bar v \in C^{2,\alpha}(S^{n-1})$ satisfying*

$$\|\bar v\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leq \epsilon_0\,,$$

and

$$\int_{S^{n-1}} \bar v\ \operatorname{dvol}_{\mathring g} = 0\,,$$

there exists a unique positive function $\phi = \phi(\epsilon, p, \bar v) \in \mathcal{C}^{2,\alpha}(B^g_{1+v}(p))$, a constant $\lambda = \lambda(\epsilon, p, \bar v) \in \mathbb{R}$ and a constant $v_0 = v_0(\epsilon, p, \bar v) \in \mathbb{R}$ such that

$$\operatorname{Vol}_g(B^g_{\epsilon(1+v)}(p)) = \operatorname{Vol}_{\mathring g}(\mathring B_\epsilon)$$

where $v := v_0 + v$ and ϕ is a solution to the problem

$$\left\{\begin{array}{rcll} \Delta_g\,\phi + \lambda\,\phi & = & 0 & \text{in} \quad M \setminus B^g_{\epsilon(1+v)}(p) \\ \phi & = & 0 & \text{on} \quad \partial B^g_{\epsilon(1+v)}(p) \end{array}\right. \tag{4.44}$$

which is normalized by setting

$$\int_{M\setminus B^g_{\epsilon(1+v)}(p)} \phi^2\ \operatorname{dvol}_g = 1. \tag{4.45}$$

In addition ϕ, λ and v_0 depend smoothly on the function $\bar v$ and the parameter ϵ.

Proof : We begin by proving that given a point $p \in M$, there exists $\epsilon_0 > 0$ and for all $\epsilon \in (0, \epsilon_0)$ and all function $\bar{v} \in C^{2,\alpha}(S^{n-1})$ satisfying

$$\|\bar{v}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leq \epsilon_0 \,,$$

and

$$\int_{S^{n-1}} \bar{v} \;\, \mathrm{dvol}_{\mathring{g}} = 0 \,,$$

there exists a unique constant $v_0 = v_0(\epsilon, p, \bar{v}) \in \mathbb{R}$ such that

$$\mathrm{Vol}_g(B^g_{\epsilon(1+v)}(p)) = \;\mathrm{Vol}_{\mathring{g}}(\mathring{B}_\epsilon) = \epsilon^n \;\,\mathrm{Vol}_{\mathring{g}}(\mathring{B}_1) \tag{4.46}$$

where $v := v_0 + \bar{v}$. Let us define the dilated metric $\bar{g} = \epsilon^{-2}\, g$. Instead of working on a domain depending on the function $v = v_0 + \bar{v}$, it will be more convenient to work on a fixed domain

$$\mathring{B}_1 := \{y \in \mathbb{R}^n \quad : \quad |y| < 1\},$$

endowed with a metric depending on the function v. This can be achieved by considering the parameterization of $B^g_{\epsilon(1+v)}(p) = B^{\bar{g}}_{(1+v)}(p)$ given by

$$Y(y) := \mathrm{Exp}^{\bar{g}}_p \left(\left(1 + v_0 + \chi(y) \; \left(\bar{v} \left(\frac{y}{|y|} \right) \right) \right) \sum_i y^i \, E_i \right)$$

where χ is a cutoff function identically equal to 0 when $|y| \leq 1/2$ and identically equal to 1 when $|y| \geq 3/4$.

Hence (using the result of Proposition 4.3.1) the coordinates we consider from now on are $y \in \mathring{B}_1$ and in these coordinates the metric $\hat{g} := Y^*\bar{g}$ can be written as

$$\hat{g} = (1 + v_0)^2 \; \left(\mathring{g} + \sum_{i,j} C^{ij} \, dy_i \, dy_j \right) \,,$$

where the coefficients $C^{ij} \in \mathcal{C}^{1,\alpha}(\mathring{B}_\epsilon)$ are functions of y depending on ϵ $v = v_0 + \bar{v}$ and the first partial derivatives of v. Moreover, $C^{ij} \equiv 0$ when $\epsilon = 0$ and $\bar{v} = 0$.

Observe that

$$(\epsilon, v_0, \bar{v}) \longmapsto C^{ij}(\epsilon, v) \,,$$

are smooth maps.

Condition (4.46), when ϵ is small enough and not zero, is equivalent to

$$\mathrm{Vol}_{\hat{g}}(\mathring{B}_1) = \;\mathrm{Vol}_{\mathring{g}}\,(\mathring{B}_1) \tag{4.47}$$

that makes sense also for $\epsilon = 0$. When $\epsilon = 0$ and $\bar{v} \equiv 0$, the metric $\hat{g} = (1+v_0)^2 \, \mathring{g}$ is nothing but the Euclidean metric. We define

$$N(\epsilon, \bar{v}, v_0) := \operatorname{Vol}_{\hat{g}}(\mathring{B}_1) - \operatorname{Vol}_{\mathring{g}}(\mathring{B}_1)$$

Observe that N also depends on the choice of the point $p \in M$.

We have

$$N(0,0,0) = 0.$$

It should be clear that the mapping N is a smooth map from a neighborhood of $(0,0,0)$ in $[0,\infty) \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathbb{R}$ into a neighborhood of 0 in $\mathbb{R}$.

We claim that the partial differential of N with respect to v_0, computed at $(0,0,0,0)$, is given by

$$\partial_{v_0} N(0,0,0) = n \ \operatorname{Vol}_{\mathring{g}}(\mathring{B}_1)$$

Indeed, this time we have $\hat{g} = (1+v_0)^2 \, \mathring{g}$ since $\bar{v} \equiv 0$ and $\epsilon = 0$ and hence

$$N(0,0,v_0) = ((1+v_0)^n - 1) \ \operatorname{Vol}_{\mathring{g}}(\mathring{B}_1)$$

So we get

$$\partial_{v_0} N(0,0,0) = n \ \operatorname{Vol}_{\mathring{g}}(\mathring{B}_1)$$

The claim then follows at once.

Hence the partial differential of N with respect to both ψ and v_0, computed at $(0,0,0)$ is precisely invertible from $\mathbb{R}$ into $\mathbb{R}$ and the implicit function theorem ensures, for all $(\epsilon, \bar{v})$ in a neighborhood of $(0,0)$ in $[0,\infty) \times \mathcal{C}^{2,\alpha}_m(S^{n-1})$, the existence of a (unique) $v_0 \in R$ such that $N(\epsilon, \bar{v}, v_0) = 0$. The fact that v_0 depends smoothly on the parameter ϵ and the function $\bar{v}$ is standard.

Now that we have, for all $0 < \epsilon < \epsilon_0$ and all function $\bar{v}$ of mean 0, a function $v = v(\epsilon, p, \bar{v}) \in \mathcal{C}^{2,\alpha}(S^{n-1})$ such that

$$\operatorname{Vol}_{\bar{g}}(B^{\bar{g}}_{1+v}) = \operatorname{Vol}_{\mathring{g}}(\mathring{B}_1)$$

it is easy to find a solution $(\bar{\phi}, \bar{\lambda})$ to the problem (4.44) and to multiply it by a constant in order to verify the normalization condition. The fact that $\bar{\phi}$ and $\bar{\lambda}$ depend smoothly on the parameter ϵ and the function $\bar{v}$ is standard.

□

We will denote the function $\phi = \phi(\epsilon, p, \bar{v})$ found in the previous proof as $\phi_{\epsilon,\bar{v}}$, without noting the dependence on the point p. The same for the eigenvalue : $\lambda = \lambda_{\epsilon,\bar{v}}$, and $\bar{\lambda} = \bar{\lambda}_{\epsilon,\bar{v}} = \epsilon^2 \, \lambda_{\epsilon,\bar{v}}$. Let us denote

$$\hat{\phi} = \hat{\phi}_{\epsilon,\bar{v}} = Y^* \phi_{\epsilon,\bar{v}}$$

in a neighborhood of $\partial B^g_\epsilon(p)$. We keep the same notation over all the following paragraphs : for a general f considered in a neighborhood of $\partial B^g_\epsilon(p)$ we will denote

$$\hat{f} = Y^* f$$

We define the operator F :

$$F(p,\epsilon,\bar{v}) = \hat{g}(\nabla\hat{\phi},\hat{\nu})|_{\partial \mathring{B}_1} - \frac{1}{\mathrm{Vol}_{\hat{g}}(\partial \mathring{B}_1)} \int_{\partial \mathring{B}_1} \hat{g}(\nabla\hat{\phi},\hat{\nu}) \, \mathrm{dvol}_{\hat{g}} \, ,$$

where $\hat{\nu}$ denotes the unit normal vector field to $\partial \mathring{B}_1$ and (ϕ, v_0) is the solution of (4.44) provided by the previous result. Recall that $v = v_0 + \bar{v}$. Schauder's estimates imply that F is well defined from a neighborhood of $M\times(0,0)$ in $M\times[0,\infty)\times\mathcal{C}^{2,\alpha}_m(S^{n-1})$ into $\mathcal{C}^{1,\alpha}_m(S^{n-1})$. Our aim is to find $(p,\epsilon,\bar{v})$ such that $F(p,\epsilon,\bar{v}) = 0$. Observe that, with this condition, ϕ will be the solution to the problem (4.11).

4.6 Some estimates

We want now to give some estimates on $F(p,\epsilon,0)$. In other words we are considering the case when $\bar{v} = 0$. We keep the notations of the proof of the previous result. If in addition $v_0 = 0$, we can estimate

$$\hat{g}_{ij} = \delta_{ij} + \mathcal{O}(\epsilon^2) \, ,$$

hence

$$N(\epsilon,0,0) = \mathcal{O}(\epsilon^2) \, .$$

The implicit function theorem immediately implies that the solution of

$$N(\epsilon,0,v_0) = 0$$

satisfies

$$|v_0(\epsilon,p,0)| \le c\,\epsilon^2$$

Let us consider the normal coordinates x around p. Using the result of Proposition 4.3.1 is possible to show that

$$\begin{aligned} g^{ij} &= \delta_{ij} - \frac{1}{3} R_{ikj\ell}\, x^k\, x^\ell - \frac{1}{6} R_{ikj\ell,m}\, x^k\, x^\ell\, x^m + \mathcal{O}(|x|^4) \\ \log|g| &= \frac{1}{3} R_{k\ell}\, x^k\, x^\ell + \frac{1}{6} R_{k\ell,m}\, x^k\, x^\ell\, x^m + \mathcal{O}(|x|^4) \end{aligned}$$

where

$$R_{k\ell} = \sum_{i=1}^{n} R_{iki\ell} \text{ and } R_{k\ell,m} = \sum_{i=1}^{n} R_{iki\ell,m}$$

A straightforward calculation allows us to obtain the expansion of Γ_p. Recall that

$$\Delta_g := \sum_{i,j} g^{ij}\, \partial_{x_i}\partial_{x_j} + \sum_{i,j} \partial_{x_i} g^{ij}\, \partial_{x_j} + \frac{1}{2} \sum_{i,j} g^{ij}\, \partial_{x_i} \log |g|\, \partial_{x_j}$$

From the definition of the Green function Γ_p we can obtain its expansion near p. For $n \geq 5$ we have

$$\begin{aligned} \Gamma_p(x) &= |x|^{2-n} + \\ &+ \left(\frac{2-n}{18} R_{ikj\ell}\, x^i\, x^k\, x^j\, x^\ell\, |x|^{-n} - \frac{1}{12} R_{j\ell}\, x^j\, x^\ell\, |x|^{2-n} + \frac{\mathrm{Scal}(p) - 6\,\lambda_0}{12(4-n)}\, |x|^{4-n} \right) + \\ &+ \left(\frac{2-n}{48} R_{ikj\ell,t}\, x^i\, x^k\, x^j\, x^\ell\, x^t\, |x|^{-n} + \frac{1}{36} R_{\cdot kj\ell,\cdot}\, x^k\, x^j\, x^\ell\, |x|^{2-n} + \right. \\ &\qquad \left. - \frac{1}{24} R_{j\ell,t}\, x^j\, x^\ell\, x^t\, |x|^{2-n} + \frac{3\,\mathrm{Scal}_{,t}}{64(4-n)}\, x^t\, |x|^{4-n} \right) + a + \\ &+ \mathcal{O}(|x|^{6-n}). \end{aligned} \tag{4.48}$$

When $n = 4$ we have

$$\begin{aligned} \Gamma_p(x) &= |x|^{-2} + \\ &+ \left(-\frac{1}{9} R_{ikj\ell}\, x^i x^k x^j x^\ell\, |x|^{-4} - \frac{1}{12} R_{j\ell}\, x^j x^\ell\, |x|^{-2} + \frac{\mathrm{Scal}(p) - 6\,\lambda_0}{12} \log |x| \right) + \\ &+ \left(-\frac{1}{24} R_{ikj\ell,t}\, x^i x^k x^j x^\ell x^t\, |x|^{-4} + \frac{1}{36} R_{\cdot kj\ell,\cdot}\, x^k x^j x^\ell\, |x|^{-2} + \right. \\ &\qquad \left. - \frac{1}{24} R_{j\ell,t}\, x^j x^\ell x^t\, |x|^{-2} + \frac{3\,\mathrm{Scal}_{,t}}{64}\, x^t \log |x| \right) + \\ &+ a' + b \cdot x + \mathcal{O}(|x|^\alpha), \end{aligned} \tag{4.49}$$

for all $\alpha < 2$, where a, a' are constants and $b \in \mathbb{R}^n$. In the above expressions we used the notation

$$R_{\cdot kj\ell,\cdot} := \sum_{i=1}^{n} R_{ikj\ell,i}$$

Observe that to find such an expression we used the fact that $R(X, X) \equiv 0$, the symmetries of the curvature tensor for which if either $i = k$ or $j = \ell$ then $R_{ikj\ell,m} = 0$, and the second

Bianchi identity

$$\sum_j R_{tj,j} = \sum_j R_{jt,j} = \frac{1}{2}\ \mathrm{Scal}_{,t}.$$

Remark that for $n = 2$ or $n = 3$ is not possible to give a precise expansion of the Green function near p using only the local part of the equation that defines Γ_p.

The main result of this section is the :

Proposition 4.6.1. *In the CASE 1 (i.e. the case where ϕ_0 is not constant) there exists a constant $c > 0$ such that, for all $p \in M$ and all $\epsilon \geq 0$ small enough we have*

$$\|F(p,\epsilon,0)\|_{\mathcal{C}^{1,\alpha}} \leq c\,\epsilon \qquad \text{if} \quad n \geq 3$$

$$\|F(p,\epsilon,0)\|_{\mathcal{C}^{1,\alpha}} \leq c\,\epsilon\,\log\epsilon \quad \text{if} \quad n = 2$$

Moreover there exists a constant C_n depending only on n, such that for all $a \in \mathbb{R}^n$ the following estimates hold

$$\left| \int_{S^{n-1}} \mathring{g}(a,\cdot)\, F(p,\epsilon,0)\ \mathrm{dvol}_{\mathring{g}} - C_n\,\epsilon\, g(\nabla\phi_0(p),\Theta(a)) \right| \leq c\,\epsilon^2\,\|a\| \qquad \text{if} \quad n \geq 3$$

$$\left| \int_{S^{n-1}} \mathring{g}(a,\cdot)\, F(p,\epsilon,0)\ \mathrm{dvol}_{\mathring{g}} - C_n\,\epsilon\,\log\epsilon\, g(\nabla\phi_0(p),\Theta(a)) \right| \leq c\,\epsilon^2\,\log\epsilon\,\|a\| \quad \text{if} \quad n = 2$$

In the CASE 2 (i.e. the case where ϕ_0 is a constant function) and for $n \geq 4$ there exists a constant $c > 0$ such that, for all $p \in M$ and all $\epsilon \geq 0$ small enough we have

$$\|F(p,\epsilon,0)\|_{\mathcal{C}^{1,\alpha}} \leq c\,\epsilon^2 \qquad \text{if} \quad n \geq 5$$

$$\|F(p,\epsilon,0)\|_{\mathcal{C}^{1,\alpha}} \leq c\,\epsilon^2\,\log\epsilon \quad \text{if} \quad n = 4$$

Moreover there exists a constant C_n (depending only on n), such that for all $a \in \mathbb{R}^n$ the following estimates hold :

$$\left| \int_{S^{n-1}} \mathring{g}(a,\cdot)\, F(p,\epsilon,0)\ \mathrm{dvol}_{\mathring{g}} - C_n\,\epsilon^3\, g(\nabla\ \mathrm{Scal}(p),\Theta(a)) \right| \leq c\,\epsilon^4\,\|a\| \quad \text{if} \quad n \geq 5$$

$$\left| \int_{S^{n-1}} \mathring{g}(a,\cdot)\, F(p,\epsilon,0)\ \mathrm{dvol}_{\mathring{g}} - C_n\,\epsilon^3\log\epsilon\, g(\nabla\ \mathrm{Scal}(p),\Theta(a)) \right| \leq c\,\epsilon^3\,\|a\| \quad \text{if} \quad n = 4$$

Proof : Let ϵ be small enough, and $\bar{v} = 0$. We know that $v_0 = \mathcal{O}(\epsilon^2)$, then from proposition 4.4.2 it follows that for all ϵ small enough there exists $(\Lambda_\epsilon, \varphi_\epsilon, w_\epsilon)$ in a neighborhood of $(0,0,0)$ in $\mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathcal{C}^{2,\alpha}(M \setminus B^g_\epsilon(p))$ such that the first eigenfunction of $-\Delta_g$ over the complement of $B^g_{\epsilon(1+v_0)}(p)$ with 0 Dirichlet condition at $\partial B^g_{\epsilon(1+v_0)}(p)$ is given by

$$\phi_{\epsilon,0} = \phi_0 - \epsilon^{n-2}\,(1+v_0)^{n-2}\,(\phi_0(p) + \Lambda_\epsilon)\,\Gamma_p + w_\epsilon + \chi\,H_{\varphi_\epsilon,\epsilon}$$

if $n \geq 3$, and by

$$\phi_{\epsilon,0} = \log(\epsilon\,(1+v_0))\ \left[\phi_0 - (\log\epsilon\,(1+v_0))^{-1}\,(\phi_0(p)+\Lambda_\epsilon)\,\Gamma_p + w_\epsilon + \chi\,H_{\varphi_\epsilon,\epsilon}\right]$$

if $n = 2$, where estimations given in propositions 4.4.2 and 4.4.4 continue to hold because $v_0 = \mathcal{O}(\epsilon^2)$.

From the expression of $\phi_{\epsilon,0}$ it follows that in the CASE 1 we have

$$\begin{aligned}\int_{S^{n-1}} \mathring{g}(a,\cdot)\,\hat{g}(\nabla\hat{\phi}_{\epsilon,0},\hat{\nu})|_{\partial\mathring{B}_1}\ \mathrm{dvol}_{\mathring{g}} &= (1+\mathcal{O}(\epsilon)) \int_{S^{n-1}} \mathring{g}(a,\cdot)\,\frac{\partial\hat{\phi}_{\epsilon,0}}{\partial r}|_{\partial\mathring{B}_1}\ \mathrm{dvol}_{\mathring{g}}\\ &= (1+\mathcal{O}(\epsilon)) \int_{S^{n-1}} \mathring{g}(a,\cdot)\,\frac{\partial\hat{\phi}_0}{\partial r}|_{\partial\mathring{B}_1}\ \mathrm{dvol}_{\mathring{g}} + \mathcal{O}(\epsilon^2)\\ &= C_n\,\epsilon\,g(\nabla\phi_0(p),\Theta(a)) + \mathcal{O}(\epsilon^2)\end{aligned}$$

for $n \geq 3$, and

$$\int_{S^{n-1}} \mathring{g}(a,\cdot)\,\hat{g}(\nabla\hat{\phi}_{\epsilon,0},\hat{\nu})|_{\partial\mathring{B}_1}\ \mathrm{dvol}_{\mathring{g}} = C_2\,\epsilon\,\log\epsilon\,g(\nabla\phi_0(p),\Theta(a)) + \mathcal{O}(\epsilon^2\,\log\epsilon)$$

for $n = 2$. Then all the estimates for the CASE 1 follow at once from this computation together with the fact that, when $\bar{v} \equiv 0$, the unit normal vector $\hat{\nu}$ about the boundary is given by $(1+v_0)\,\partial_r\,(1+\mathcal{O}(\epsilon))$ because the metric $\hat{g}$ near p is the euclidean metric multiplied by $(1+v_0)^2$ and perturbed by some $\mathcal{O}(\epsilon^2)$ terms.

For the CASE 2 the situation is much more complex. We remark that if ϕ_0 is constant, then

$$\int_{S^{n-1}} \mathring{g}(a,\cdot)\,\hat{g}(\nabla\hat{\phi}_0,\hat{\nu})|_{\partial\mathring{B}_1}\ \mathrm{dvol}_{\mathring{g}} = 0.$$

Let us compute now

$$\epsilon^{n-2}\,(1+v_0)^{n-2}\,(\phi_0(p)+\Lambda_\epsilon) \int_{S^{n-1}} \mathring{g}(a,\cdot)\,\hat{g}(\nabla\hat{\Gamma}_p,\hat{\nu})|_{\partial\mathring{B}_1}\ \mathrm{dvol}_{\mathring{g}}$$

We remark that the previous term is equal to

$$(1+\mathcal{O}(\epsilon))\,\epsilon^{n-2}\,\phi_0(p) \int_{S^{n-1}} \mathring{g}(\cdot,a)\frac{\partial\hat{\Gamma}_p}{\partial r}\ \mathrm{dvol}_{\mathring{g}}$$

For this reason we will compute

$$\epsilon^{n-2}\,\phi_0(p) \int_{S^{n-1}} \mathring{g}(\cdot,a)\frac{\partial\hat{\Gamma}_p}{\partial r}\ \mathrm{dvol}_{\mathring{g}}$$

Recall that

$$\hat{\Gamma}_p(y) = \Gamma_p(\epsilon\,(1+v_0)\,y)$$

in a neighborhood of $\partial\mathring{B}_1$, then from (4.48) and (4.49) (keeping in mind that $v_0 = \mathcal{O}(\epsilon^2)$) we obtain easily the expression of $\hat{\Gamma}_p(y)$ in power of ϵ. Observe that, in the expansion of $\hat{\Gamma}_p$, terms which contain an even number of coordinates, such as $y^i y^j y^k y^\ell$ or $y^j y^\ell$ etc. do not contribute to the result since, once derived with respect to r they continue to contain an even number of coordinates, and multiplied then by $\mathring{g}(y,a)$, their average over S^{n-1} is 0. Then, considering only terms which contain an odd number of coordinates we have for $n \geq 5$:

$$\epsilon^{n-2} \int_{S^{n-1}} \mathring{g}(y,a)\,\frac{\partial \hat{\Gamma}_p}{\partial r}\,\mathrm{dvol}_{\mathring{g}} =$$

$$= \epsilon^3\, a_\sigma \Bigg[\int_{S^{n-1}} y^\sigma \cdot \frac{y^\tau}{|y|} \cdot \frac{\partial}{\partial y^\tau} \Bigg(\frac{2-n}{48}\, R_{ikj\ell,t}\, y^i y^k y^j y^\ell y^t\, |y|^{-n} + \frac{1}{36}\, R_{\cdot kj\ell,\cdot}\, y^k y^j y^\ell\, |y|^{2-n}$$

$$+ \frac{3\mathrm{Scal}_{,t}}{64(4-n)}\, y^t\, |y|^{4-n} - \frac{1}{24}\, R_{j\ell,t}\, y^j y^\ell y^t\, |y|^{2-n} \Bigg) \mathrm{dvol}_{\mathring{g}} \Bigg] + \mathcal{O}(\epsilon^4)$$

$$= \epsilon^3\,(5-n)\, a_\sigma \Bigg[\int_{S^{n-1}} y^\sigma \Bigg(\frac{2-n}{48}\, R_{ikj\ell,t}\, y^i y^k y^j y^\ell y^t + \frac{1}{36}\, R_{\cdot kj\ell,\cdot}\, y^k y^j y^\ell + \frac{3\mathrm{Scal}_{,t}}{64(4-n)}\, y^t$$

$$- \frac{1}{24}\, R_{j\ell,t}\, y^j y^\ell y^t \Bigg) \mathrm{dvol}_{\mathring{g}} \Bigg] + \mathcal{O}(\epsilon^4)$$

We make use of the identities in the Appendix to conclude that there exists a costant $C_n^{(1)}$ such that

$$\epsilon^{n-2}\,\phi_0(p) \int_{S^{n-1}} \mathring{g}(y,a)\,\frac{\partial \hat{\Gamma}_p}{\partial r}(y) = C_n^{(1)} \epsilon^3\, g\big(\nabla\ \mathrm{Scal(p)}, \Theta(a)\big) \,+\, \mathcal{O}(\epsilon^4). \tag{4.50}$$

where we have

$$C_n^{(1)} = \frac{5-n}{4n}\,\mathrm{Vol}_{\mathring{g}}(S^{n-1}) \left[-\frac{1}{3(n+2)} + \frac{3}{16(4-n)} \right] \phi_0(p)$$

Remark that when $n = 5$ such a constant is 0. For $n = 4$ we have

$$\epsilon^{n-2} \int_{S^{n-1}} \mathring{g}(y,a)\, \frac{\partial \hat{\Gamma}_p}{\partial r}\, \mathrm{dvol}_{\mathring{g}} =$$

$$= \epsilon^3 \log \epsilon\, a_\sigma \left[\int_{S^3} y^\sigma \cdot \frac{y^\tau}{|y|} \cdot \frac{\partial}{\partial y^\tau} \left(\frac{3\mathrm{Scal}_{,t}}{64}\, y^t \log |y| \right) \mathrm{dvol}_{\mathring{g}} \right] + \mathcal{O}(\epsilon^3)$$

$$= \frac{3}{256}\, \mathrm{Vol}_{\mathring{g}}(S^3)\, \epsilon^3 \log \epsilon\, g\big(\nabla\ \mathrm{Scal(p)}, \Theta(a)\big)\ + \mathcal{O}(\epsilon^3)$$

and then we set $C_4^{(1)} = \dfrac{3}{256}\, \mathrm{Vol}_{\mathring{g}}(S^3)$.

The last term we have to compute is

$$\int_{S^{n-1}} \mathring{g}(a,\cdot)\, \hat{g}(\nabla(\hat{w}_\epsilon + \hat{H}_{\varphi_\epsilon,\epsilon}), \hat{\nu})|_{\partial \mathring{B}_1}\ \mathrm{dvol}_{\mathring{g}}$$

As before we have

$$\int_{S^{n-1}} \mathring{g}(a,\cdot)\, \hat{g}(\nabla(\hat{w}_\epsilon + \hat{H}_{\varphi_\epsilon,\epsilon}), \hat{\nu})|_{\partial \mathring{B}_1}\ \mathrm{dvol}_{\mathring{g}} =$$

$$= (1 + \mathcal{O}(\epsilon)) \int_{S^{n-1}} \mathring{g}(a,\cdot)\, \frac{\partial(\hat{w}_\epsilon + \hat{H}_{\varphi_\epsilon,\epsilon})}{\partial r}|_{\partial \mathring{B}_1}\ \mathrm{dvol}_{\mathring{g}}$$

In the Proposition 4.4.2 we proved that in the CASE 2

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})} \leq c\, \left(\epsilon^4 + \epsilon^{\beta-\nu}\right) \quad \forall \beta < 4$$

for $n = 4$ and

$$\|w_\epsilon\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus\{p\})} \leq c\, \left(\epsilon^{2n-4} + \epsilon^{1+n} + \epsilon^{4-\nu}\right)$$

for $n \geq 5$. Hence

$$\|\nabla \hat{w}_\epsilon\|_{L^\infty(\partial \mathring{B}_1)} \leq c\,(\epsilon^{4+\nu} + \epsilon^\beta)\ \ \forall \beta < 4$$

for $n = 4$ and

$$\|\nabla \hat{w}_\epsilon\|_{L^\infty(\partial \mathring{B}_1)} \leq c\,(\epsilon^{2n-4+\nu} + \epsilon^{1+n+\nu} + \epsilon^4)$$

for $n \geq 5$ (keep in mind that we are estimating the gradient of the dilated function $\hat{w}_\epsilon$). Remember that $\nu \in (2-n, 4-n)$ because $n \geq 4$. It follows that we can choose ν in order to have

$$\int_{S^{n-1}} \mathring{g}(a,\cdot)\, \frac{\partial \hat{w}_\epsilon}{\partial r}|_{\partial \mathring{B}_1}\ \mathrm{dvol}_{\mathring{g}} = \mathcal{O}(\epsilon^\beta)$$

with $\beta = 4$ for $n \geq 5$ and $\beta = 3$ for $n = 4$. Let us consider now $\hat{H}_{\varphi_\epsilon,\epsilon}$. We do not know the expression of $\hat{H}_{\varphi_\epsilon,\epsilon}$ in a neighborhood of $\partial\mathring{B}_1$, but we can know its value on $\partial\mathring{B}_1$. From the equality $\hat{\phi}_\epsilon = 0$ on $\partial\mathring{B}_1$, using the the estimate on the function $\hat{w}_\epsilon$, we have that

$$\begin{aligned}\hat{H}_{\varphi,\epsilon} \;&=\; -\phi_0(p) + (1+v_0)^{n-2}\,(\phi_0(p)+\Lambda_\epsilon)\left[\epsilon^2\left(\frac{2-n}{18}\,R_{ikj\ell}\,y^iy^ky^jy^\ell \;-\; \frac{1}{12}R_{j\ell}\,y^jy^\ell + \right.\right.\\ &\quad\left. +\frac{\mathrm{Scal}(p)-6\lambda_0}{12\,(4-n)}\right) + \epsilon^3\left(\frac{2-n}{48}\,R_{ikj\ell,t}\,y^iy^ky^jy^\ell y^t \;+\; \frac{1}{36}\,R_{\cdot kj\ell,\cdot}\,y^ky^jy^\ell + \right.\\ &\quad\left.\left. -\frac{1}{24}\,R_{j\ell,t}\,y^jy^\ell y^t \;+\; \frac{3\,\mathrm{Scal}_{,t}}{64(4-n)}\,y^t\right)\right] + \mathcal{O}(\epsilon^4).\end{aligned}$$

on $\partial\mathring{B}_1$, for $n \geq 5$. For $n = 4$:

$$\begin{aligned}\hat{H}_{\varphi,\epsilon} \;&=\; -\phi_0(p) + (1+v_0)^{n-2}\,(\phi_0(p)+\Lambda_\epsilon)\left[\epsilon^2\,\log\epsilon\,\frac{\mathrm{Scal}(p)-6\lambda_0}{12} + \right.\\ &\quad\left.\epsilon^2\left(-\frac{1}{9}\,R_{ikj\ell}\,y^iy^ky^jy^\ell \;-\; \frac{1}{12}R_{j\ell}\,y^jy^\ell\right) \;+\; \epsilon^3\,\log\epsilon\,\frac{3\,\mathrm{Scal}_{,t}}{64}\,y^t\right)\right] + \mathcal{O}(\epsilon^3).\end{aligned}$$

Let us define an harmonic extension of $\mathring{g}(y,a)$ to $\mathbb{R}^n \setminus \mathring{B}_1$:

$$\begin{cases}\Delta_{\mathring{g}} G_a \;=\; 0 & \text{in} \quad \mathbb{R}^n \setminus \mathring{B}_1 \\ \quad G_a \;=\; \mathring{g}(y,a) & \text{on} \quad \partial\mathring{B}_1\end{cases} \tag{4.51}$$

It is easy to check that

$$G_a(y) = |y|^{-n}\,\mathring{g}(y,a)$$

We observe that the functions G_a and $\hat{H}_{\varphi_\epsilon,\epsilon}$, by Lemma 4.4.1 converge to 0 when $|y| \to +\infty$. Then

$$\begin{aligned}\int_{S^{n-1}} \mathring{g}(a,\cdot)\,\frac{\partial \hat{H}_{\varphi_\epsilon,\epsilon}}{\partial r}|_{\partial\mathring{B}_1}\;\mathrm{dvol}_{\mathring{g}} \;&=\; \int_{S^{n-1}} \hat{H}_{\varphi_\epsilon,\epsilon}\,\frac{\partial G_a}{\partial r}|_{\partial\mathring{B}_1}\;\mathrm{dvol}_{\mathring{g}} = \\ &= (1-n)\int_{S^{n-1}} \hat{H}_{\varphi_\epsilon,\epsilon}\,\mathring{g}(y,a)\;\mathrm{dvol}_{\mathring{g}}\end{aligned}$$

Using the expansion of $\hat{H}_{\varphi_\epsilon,\epsilon}$ that we found with the identities in the Appendix, we conclude that there exists a constant $C_n^{(2)} \neq 0$ such that

$$\int_{S^{n-1}} \mathring{g}(y,a)\,\hat{H}_{\varphi_\epsilon,\epsilon}\;\mathrm{dvol}_{\mathring{g}} = C_n^{(2)}\,\epsilon^3\,g\big(\nabla\,\mathrm{Scal(p)},\Theta(a)\big) \;+\; \mathcal{O}(\epsilon^4). \tag{4.52}$$

where

$$C_n^{(2)} = \frac{1}{4n}\,\mathrm{Vol}_{\mathring{g}}(S^{n-1})\,\left[-\frac{1}{3(n+2)} + \frac{3}{16(4-n)}\right]\,\phi_0(p)$$

for $n \geq 5$, and for $n = 4$

$$\int_{S^{n-1}} \mathring{g}(y,a)\,\hat{H}_{\varphi_\epsilon,\epsilon}\ \mathrm{dvol}_{\mathring{g}} = C_4^{(2)}\,\epsilon^3 \log\epsilon\, g\big(\nabla\ \mathrm{Scal(p)}, \Theta(a)\big)\ +\ \mathcal{O}(\epsilon^3). \tag{4.53}$$

with

$$C_4^{(2)} = \frac{3}{256}\,\mathrm{Vol}_{\mathring{g}}(S^3).$$

Summarizing we conclude that in the CASE 2

$$\|F(p,\epsilon,0)\|_{\mathcal{C}^{1,\alpha}} = \mathcal{O}(\epsilon^2)$$

and there exists a constant C_n depending only on n, such that for all $a \in \mathbb{R}^n$ the following estimates hold : for $n \geq 5$

$$\left|\int_{S^{n-1}} \mathring{g}(a,\cdot)\,F(p,\epsilon,0)\ \mathrm{dvol}_{\mathring{g}} - C_n\,\epsilon^3\, g(\nabla\ \mathrm{Scal}(p), \Theta(a))\right| \leq c\,\epsilon^4\,\|a\|\,.$$

where

$$C_n = \frac{6-2n}{n}\,\mathrm{Vol}_{\mathring{g}}(S^{n-1})\,\left[-\frac{1}{3(n+2)} + \frac{3}{16(4-n)}\right]\,\phi_0(p)$$

and for $n = 4$

$$\left|\int_{S^{n-1}} \mathring{g}(a,\cdot)\,F(p,\epsilon,0)\ \mathrm{dvol}_{\mathring{g}} - C_4\,\epsilon^3 \log\epsilon\, g(\nabla\ \mathrm{Scal}(p), \Theta(a))\right| \leq c\,\epsilon^3\,\|a\|\,.$$

where

$$C_4 = -\frac{3}{128}\,\mathrm{Vol}_{\mathring{g}}(S^3)\,\phi_0(p)$$

Remark that $C_n \neq 0$ for all $n \geq 4$. This completes the proof of the result. □

4.7 Linearizing the operator F

Our next task will be to understand the structure of L_0, the operator obtained by linearizing F with respect to $\bar{v}$ at $\epsilon = 0$ and $\bar{v} = 0$. We will see that this operator is a first order elliptic operator which does not depend on the point p.

Let us define in $\mathbb{R}^n \setminus \{0\}$

$$\phi_1(y) = \begin{cases} \phi_0(p)\,(1-|y|^{2-n}) & \text{if} \quad n \geq 3 \\ \phi_0(p)\ \log|y| & \text{if} \quad n = 2 \end{cases}$$

For all $\bar{v} \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$ let ψ be the (unique) bounded solution of

$$\left\{\begin{array}{rcll} \Delta_{\mathring{g}}\, \psi & = & 0 & \text{in} \quad \mathbb{R}^n \setminus \mathring{B}_1 \\ \psi & = & -\partial_r \phi_1\, \bar{v} & \text{on} \quad \partial \mathring{B}_1 \end{array}\right. \tag{4.54}$$

By the Lemma 4.4.1, $|\psi(y)| \longrightarrow 0$ when $|y| \to \infty$. We define

$$H(\bar{v}) := \left(\partial_r \psi + \partial_r^2 \phi_1\, \bar{v}\right)\, |_{\partial \mathring{B}_1} \tag{4.55}$$

We will need the following result :

Proposition 4.7.1. *The operator*

$$H : \mathcal{C}^{2,\alpha}_m(S^{n-1}) \longrightarrow \mathcal{C}^{1,\alpha}_m(S^{n-1}),$$

is a self adjoint, first order elliptic operator. The kernel of H is given by V_1, the eigenspace of $-\Delta_{S^{n-1}}$ associated to the eigenvalue $n-1$. Moreover there exists $c > 0$ such that

$$\|w\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \leq c\, \|H(w)\|_{\mathcal{C}^{1,\alpha}(S^{n-1})}\,,$$

provided w is $L^2(S^{n-1})$-orthogonal to $V_0 \oplus V_1$, where V_0 is the eigenspace associated to constant functions.

Proof : The fact that H is a first order elliptic operator is standard since it is the sum of the Dirichlet-to-Neumann operator for $\Delta_{\mathring{g}}$ and a constant times the identity. In particular, elliptic estimates yield

$$\|H(w)\|_{\mathcal{C}^{1,\alpha}(S^{n-1})} \leq c\, \|w\|_{\mathcal{C}^{2,\alpha}(S^{n-1})}$$

The fact that the operator H is (formally) self-adjoint is easy. Let ψ_1 (resp. ψ_2) the solution of (4.54) corresponding to the function w_1 (resp. w_2). We compute

$$\begin{array}{rcl} \partial_r\phi_1(1) \displaystyle\int_{\partial \mathring{B}_1} (H(w_1)\, w_2 - w_1\, H(w_2))\, \mathrm{dvol}_{\mathring{g}} & = & \partial_r\phi_1(1) \displaystyle\int_{\partial \mathring{B}_1} (\partial_r\psi_1\, w_2 - \partial_r\psi_2\, w_1)\, \mathrm{dvol}_{\mathring{g}} \\ & = & \displaystyle\int_{\partial \mathring{B}_1} (\psi_1\, \partial_r\psi_2 - \psi_2\, \partial_r\psi_1)\, \mathrm{dvol}_{\mathring{g}} \\ & = & \displaystyle\lim_{R\to\infty} \int_{\mathring{B}_R \setminus \mathring{B}_1} (\psi_1\, \Delta_{\mathring{g}}\psi_2 - \psi_2 \Delta_{\mathring{g}}\, \psi_1)\, \mathrm{dvol}_{\mathring{g}} \\ & & - \displaystyle\lim_{R\to\infty} \int_{\partial \mathring{B}_R} (\psi_1\, \partial_r\psi_2 - \psi_2\, \partial_r\psi_1)\, \mathrm{dvol}_{\mathring{g}} \\ & = & 0 \end{array}$$

Let us consider

$$w = \sum_{j \geq 1} w_j$$

the eigenfunction decomposition of w. Namely $w_j \in V_j$. We can define ψ_j to be the bounded solution of

$$\left\{ \begin{array}{rclcl} \Delta_{\mathring{g}} \psi_j & = & 0 & \text{in} & \mathbb{R}^n \setminus \mathring{B}_1 \\ \psi_j & = & -\partial_r \phi_1 \, w_j & \text{on} & \partial \mathring{B}_1 \end{array} \right. \tag{4.56}$$

i.e.

$$\psi_j(y) = -|y|^{2-n-j} \, w_j(y/|y|) \, \partial_r \phi_1|_{\partial \mathring{B}_1}$$

Then

$$H(w) = \sum_j \partial_r \psi_j + \partial_r^2 \phi_1|_{\partial \mathring{B}_1} \, w = \sum_j \left[-(2-n-j) \, \partial_r \phi_1|_{\partial \mathring{B}_1} + \partial_r^2 \phi_1|_{\partial \mathring{B}_1} \right] \, w_j$$

With this alternative formula for H, it is clear that H preserves the eigenspaces V_j and in particular, H maps into the space of functions whose mean over S^{n-1} is 0. Moreover, it is easy to see that V_1 is the only kernel of the operator. In fact,

$$\partial_r \phi_1|_{\partial \mathring{B}_1} = \left\{ \begin{array}{lll} -(2-n) \, \phi_0(p) & \text{if} & n \geq 3 \\ \phi_0(p) & \text{if} & n = 2 \end{array} \right.$$

and

$$\partial_r^2 \phi_1|_{\partial \mathring{B}_1} = \left\{ \begin{array}{lll} -(2-n)(1-n) \, \phi_0(p) & \text{if} & n \geq 3 \\ -\phi_0(p) & \text{if} & n = 2 \end{array} \right.$$

and then $H(w_j) = 0$ if and only if $j = 1$. This completes the proof of the result. □

The main result of this section is the following :

Proposition 4.7.2. *The operator L_0 is equal to H.*

Proof : By definition, the operator L_0 is the linear operator obtained by linearizing F with respect to $\bar{v}$ at $\epsilon = 0$ and $\bar{v} = 0$. In other words, we have

$$L_0(\bar{w}) = \lim_{s \to 0} \frac{F(p, 0, s \, \bar{w}) - F(p, 0, 0)}{s}.$$

It is easy to see that $F(p, 0, 0) = 0$. In fact, we saw that the first eigenfunction $\phi_{\epsilon,0}$ over $M \setminus B^g_{\epsilon(1+v_0)}(p)$ is given by

$$\begin{array}{ll} \phi_{\epsilon,0} = \phi_0 - \epsilon^{n-2} \, (1+v_0)^{n-2} \, (\phi_0(p) + \Lambda_\epsilon) \, \Gamma_p + w_\epsilon + \chi \, H_{\varphi_\epsilon, \epsilon} & \text{if} \quad n \geq 3 \\ \phi_{\epsilon,0} = \log(\epsilon \, (1+v_0)) \, \left[\phi_0 - (\log(\epsilon \, (1+v_0)))^{-1} \, (\phi_0(p) + \Lambda_\epsilon) \, \Gamma_p + w_\epsilon + \chi \, H_{\varphi_\epsilon, \epsilon} \right] & \text{if} \quad n = 2 \end{array}$$

where $v_0 = v_0(p, \epsilon, 0) = \mathcal{O}(\epsilon^2)$, for some $(\Lambda_\epsilon, w_\epsilon, \varphi_\epsilon) \in \mathbb{R} \times C^{2,\alpha}(M \setminus B^{\bar{g}}_{1+v_0}(p)) \times C^{2,\alpha}_m(S^{n-1})$, where the estimations of Proposition 4.6.1 hold because $v_0 = \mathcal{O}(\epsilon^2)$. If we consider this expressions only in a neighborhood of $\partial B^g_{\epsilon(1+v_0)}(p)$ and the parameterization Y given in the proof of the Proposition 4.5.1 with coordinates y in a neighborhood of $\partial \mathring{B}_1$, it is easy to see that the function $\hat{\phi}_0 = Y^*\phi_0$ is equal to the constant function $\hat{\phi}_0 = \phi_0(p)$ when $\epsilon = 0$ and then, by the expansion of the function Γ_p and the estimations on $(\Lambda_\epsilon, w_\epsilon, \varphi_\epsilon)$, we have that when $\epsilon = 0$ the function $\hat{\phi}_{\epsilon,0}(y)$ is equal to $\phi_1(y)$. In a neighborhood of $\partial \mathring{B}_1$ the metric $\hat{g}$ converge, for $\epsilon = 0$, to the euclidean metric, and from this it follows that $F(p, 0, 0)$ is the normal derivative of ϕ_1 at $\partial \mathring{B}_1$ minus its euclidean mean, hence equal to 0.

Our next step is to compute $F(p, 0, s\bar{w})$, and for this we have to study $F(p, \epsilon, s\,\bar{w})$. Writing $\bar{v} = s\,\bar{w}$, we can consider a parameterization Y of $B^g_{2\epsilon}(p)$ given by the following expression :

$$Y(y) := \mathrm{Exp}^{\bar{g}}_p \left(\left(1 + \chi_1(y)\, v_0 + s\, \chi_2(y) \,\left(\bar{w} \left(\frac{y}{|y|} \right) \right) \right) \sum_i y^i\, E_i \right)$$

where $\bar{g}$ is the dilated metric $\epsilon^{-2}\, g$, y belongs to the euclidean ball $\mathring{B}_2$ of radius 2 centered at 0, χ_1 is a cutoff function identically equal to 1 when $0 < |y| \le 4/3$ and identically equal to 0 when $5/3 \le |y| \le 2$, χ_2 is a cutoff function identically equal to 1 when $3/4 \le |y| \le 4/3$ and identically equal to 0 when $0 < |y| \le 1/2$ and $5/3 \le |y| \le 2$, and $v_0 = v_0(p, \epsilon, s\,\bar{w})$. We set

$$\hat{g} := Y^*\bar{g}.$$

over $\mathring{B}_2$. It is an extension of the metric $\hat{g}$ that we defined before on $\mathring{B}_1$. We remark that $\hat{\phi}_{\epsilon,0} := Y^*\phi_{\epsilon,0}$ is a solution on $\mathring{B}_2$ of

$$\Delta_{\hat{g}}\, \hat{\phi}_{\epsilon,0} + \hat{\lambda}_{\epsilon,0}\, \hat{\phi}_{\epsilon,0} = 0$$

where $\hat{\lambda}_{\epsilon,0} = \bar{\lambda}_{\epsilon,0} = \epsilon^2\, \lambda_{\epsilon,0}$. If we set $\bar{\phi}_{\epsilon,0}(y) = \phi_{\epsilon,0}(\epsilon\, y)$ in a neighborhood of $\partial \mathring{B}_1$, where $x = \epsilon\, y$ are the normal coordinates near p that we defined in paragraph 3, we have

$$\hat{\phi}_{\epsilon,0}(y) = \bar{\phi}_{\epsilon,0}((1 + v_0 + s\, \bar{w}(y))\, y)\,, \tag{4.57}$$

on $\partial \mathring{B}_1$. Writing the first eigenfunction of $-\Delta_{\bar{g}}$ on $B^{\bar{g}}_{1+v}(p)$ as $\phi = \phi_{\epsilon,0} + \psi$ and $\bar{\lambda} = \bar{\lambda}_{\epsilon,0} + \tau$, we find that

$$\left\{ \begin{array}{rcll} (\Delta_{\bar{g}} + \bar{\lambda}_{\epsilon,0})\, \psi + \tau\, \psi + \tau\, \phi_{\epsilon,0} & = & 0 & \text{in} \quad M \setminus B^{\bar{g}}_{1+v}(p) \\ \psi & = & -\phi_{\epsilon,0} & \text{on} \quad \partial B^{\bar{g}}_{1+v}(p) \end{array} \right. \tag{4.58}$$

where we can normalize as

$$\int_{M \setminus B^{\bar{g}}_{1+v}(p)} (\phi_{\epsilon,0} + \psi)^2\ \mathrm{dvol}_{\bar{g}} = \int_{M \setminus B^{\bar{g}}_{1+v_0}(p)} \phi^2_{\epsilon,0}\ \mathrm{dvol}_{\bar{g}} \tag{4.59}$$

(the v_0 in the second integral is evaluated at $\bar{v} = 0$) and we have the condition on the volume of the domain

$$\mathrm{Vol}_{\hat{g}}(\mathring{B}_1) = \mathrm{Vol}_{\mathring{g}}(\mathring{B}_1) \tag{4.60}$$

Obviously ψ, τ and v_0 are smooth functions of s. When $s = 0$, we have $\phi = \phi_{\epsilon,0}$, $\bar{\lambda} = \bar{\lambda}_{\epsilon,0}$ and $v_0 = \mathcal{O}(\epsilon^2)$. Therefore, ψ and τ vanish when $s = 0$. We set

$$\dot{\psi} = \partial_s \psi|_{s=0}, \qquad \dot{\tau} = \partial_s \tau|_{s=0}, \qquad \text{and} \qquad \dot{v}_0 = \partial_s v_0|_{s=0},$$

Differentiating (4.58) with respect to s and evaluating the result at $s = 0$, we obtain

$$\begin{cases} (\Delta_{\bar{g}} + \bar{\lambda}_{\epsilon,0})\,\dot{\psi} + \dot{\tau}\,\phi_{\epsilon,0} = 0 & \text{in} \quad M \setminus B^{\bar{g}}_{1+v_0}(p) \\ \dot{\psi} = -\bar{g}(\nabla \phi_{\epsilon,0}, \bar{\nu})\,(\dot{v}_0 + \bar{w}) & \text{on} \quad \partial B^{\bar{g}}_{1+v_0}(p) \end{cases} \tag{4.61}$$

where v_0 is evaluated at $s = 0$. Observe that $\phi_{\epsilon,0}$ on $\partial B^{\bar{g}}_{1+v}(p)$ is equal to $\hat{\phi}_{\epsilon,0}$ on $\partial \mathring{B}_1$, then the second equation of (4.61) follows from (4.57).

Differentiating (4.59) with respect to s and evaluating the result at $s = 0$, we obtain that ψ is L^2-orthogonal to $\phi_{\epsilon,0}$ on $B^{\bar{g}}_{1+v_0}(p)$. Hence

$$\phi = \phi_{\epsilon,0} + s\,\dot{\psi} + \mathcal{O}(s^2)$$

where $\dot{\psi}$ is the solution of (4.61) L^2-orthogonal to $\phi_{\epsilon,0}$. Differentiating (4.60) with respect to s and evaluating the result at $s = 0$, we obtain

$$\int_{S^{n-1}} (\dot{v}_0 + \bar{w}) \;\mathrm{dvol}_{\hat{g}} = 0 \tag{4.62}$$

where the metric $\hat{g}$ is evaluated at $s = 0$. Since the discrepancy between the metric $\hat{g}$ and the euclidean metric $\mathring{g}$ at $\partial \mathring{B}_1$ can be estimated by a constant times ϵ^2 when $s = 0$, and the euclidean average of $\bar{w}$ is 0, we get that

$$\dot{v}_0 = \mathcal{O}(\epsilon^2)$$

and then from the Taylor expansion of v_0 with respect to s we get

$$v_0 = \mathcal{O}(\epsilon^2) + \mathcal{O}(s^2)$$

Now, in $\mathring{B}_{4/3} \setminus \mathring{B}_1$, we have

$$\begin{aligned} \hat{\phi}(y) &= \bar{\phi}_{\epsilon,0}\left(\left(1 + v_0(0) + s\,\bar{w}(y/|y|)\right) y\right) + s\,\dot{\psi}(y) + \mathcal{O}(s^2) \\ &= \bar{\phi}_{\epsilon,0}((1 + v_0(0))\,y) + s\,\left(\hat{g}\left(\nabla \bar{\phi}_{\epsilon,0}((1 + v_0(0))\,y), (\dot{v}_0 + \bar{w}(y/|y|))\,y\right) + \dot{\psi}\right) + \mathcal{O}(s^2) \end{aligned}$$

where we denoted $v_0(p,\epsilon,0) = v_0|_{s=0} = v_0(0)$. To complete the proof of the result, it suffices to compute the normal derivative of the function $\hat{\phi}$ when the normal is computed with respect to the metric $\hat{g}$. We use polar coordinates $y = r\,z$ where $r>0$ and $z\in S^{n-1}$. Then the metric $\hat{g}$ can be expanded in $\mathring{B}_{4/3}\setminus\mathring{B}_{3/4}$ as

$$\hat{g} = (1+v_0+s\bar{w})^2\,dr^2 + 2\,s\,(1+v_0+s\bar{w})\,r\,d\bar{w}\,dr + r^2\,(1+v_0+s\bar{w})^2\,\mathring{h} + s^2\,r^2\,d\bar{w}^2 + \mathcal{O}(\epsilon^2)$$

where $\mathring{h}$ is the metric on S^{n-1} induced by the Euclidean metric. It follows from this expression, together with the estimation of v_0, that the unit normal vector field to $\partial\mathring{B}_1$ for the metric $\hat{g}$ is given by

$$\hat{\nu} = \left((1+s\,\bar{w})^{-1} + \mathcal{O}(s^2)\right)\,\partial_r + \mathcal{O}(s)\,\partial_{z_j} + \mathcal{O}(\epsilon^2)$$

where ∂_{z_j} are vector fields induced by a parameterization of S^{n-1}. Using this, we conclude that

$$\hat{g}(\nabla\hat{\phi},\hat{\nu}) = \partial_r\bar{\phi}_{\epsilon,0}(y) + \mathcal{O}(s)\,\partial_{z_j}\,\bar{\phi}_{\epsilon,0}(y) + s\,\left(\bar{w}\,\partial_r^2\,\bar{\phi}_{\epsilon,0}(y) + \partial_r\dot{\psi}\right) + \mathcal{O}(s^2) + \mathcal{O}(\epsilon^2) \quad (4.63)$$

on $\partial\mathring{B}_1$. When $\epsilon=0$ we have that $\bar{\phi}_{\epsilon,0}(y) = \phi_1(y)$. It follows that $F(p,0,s\bar{w})$, up to terms of the order $\mathcal{O}(s^2)$, is given by the term

$$\partial_r\phi_1|_{\partial\mathring{B}_1} + s\,\bar{w}\,\partial_r^2\phi_1\big|_{\partial\mathring{B}_1} + s\,\lim_{\epsilon\to 0}\partial_r\dot{\psi}\Big|_{\partial\mathring{B}_1}$$

minus its euclidean mean, where the limit is understood in the pointwize sens. We need the following result.

Lemma 4.7.3. *Evaluate v_0 at $s=0$. Let $\nu\in(2-n,0)$ if $n\geq 3$ and $\nu\in(0,1)$ if $n=2$. Let H_φ be the function defined in the paragraph 4. For all ϵ small enough there exist a constant $\dot{\tau}$ and $(K_\epsilon,\varphi_\epsilon,\eta_\epsilon)$ in a neighborhood of $(0,0,0)$ in $\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})\times\mathcal{C}^{2,\alpha}_\nu(M\setminus B^{\bar{g}}_{1+v_0}(p))$ such that the function*

$$\dot{\psi} = K_\epsilon + \chi\,(\psi + H_{\varphi_\epsilon}) + \eta_\epsilon \quad (4.64)$$

defined in $M\setminus B^{\bar{g}}_{1+v_0}(p)$, is the solution of (4.61) L^2-orthogonal to $\phi_{\epsilon,0}$, where χ is a cut-off function equal to 1 in $B^{\bar{g}}_{R_0/\epsilon}(p)$ and equal to 0 out of $B^{\bar{g}}_{2R_0/\epsilon}(p)$ and ψ is defined by (4.54). Moreover the following estimations hold :

$$|K_\epsilon| \leq c\,\left(\epsilon^2+\epsilon^{n-1}\right) \qquad \text{and} \qquad \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\,\left(\epsilon^2+\epsilon^{n-1}\right)$$

$$\text{and} \qquad \|\eta_\epsilon\|_{\mathcal{C}^{2,\alpha}_\nu(M\setminus B^{\bar{g}}_{1+v_0}(p))} \leq c\,\left(\epsilon^2+\epsilon^{n-1}\right)$$

Proof. Let us choose $\dot{\psi}$ in the form

$$\dot{\psi} = K + \chi\,(\psi + H_\varphi) + \eta \quad (4.65)$$

for some $(K, \varphi, \eta) \in \mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathcal{C}^{2,\alpha}(M \setminus B^{\bar g}_{1+v_0}(p))$, where χ is a cut-off function equal to 1 in $B^{\bar g}_{R_0/\epsilon}(p)$ and equal to 0 out of $B^{\bar g}_{2R_0/\epsilon}(p)$. Then $\dot\psi$ satisfy the first equation of (4.61), if and only if :

$$\begin{aligned}\left(\Delta_{\bar g} + \bar\lambda_{\epsilon,0}\right) \eta \; &= \; -\psi\, \Delta_{\bar g}\chi - \chi\, \Delta_{\bar g}\psi - 2\, \nabla^{\bar g}\psi\, \nabla^{\bar g}\chi - H_\varphi\, \Delta_{\bar g}\chi - \chi\, \Delta_{\bar g} H_\varphi - 2\, \nabla^{\bar g} H_\varphi\, \nabla^{\bar g}\chi \\ & \quad -\bar\lambda_{\epsilon,0}\, \chi\, (\psi + H_\varphi) - \bar\lambda_{\epsilon,0}\, K - \dot\tau\, \phi_{\epsilon,0}\end{aligned} \tag{4.66}$$

For $n \geq 3$ and $\nu \in (2-n, 0)$, the operator

$$\left(\Delta_{\bar g} + \bar\lambda_{\epsilon,0}\right) : \mathcal{C}^{2,\alpha}_{\nu,\perp,0}(M \setminus B^{\bar g}_{1+v_0}(p)) \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-2,\perp}(M \setminus B^{\bar g}_{1+v_0}(p)),$$

where the subscript $\perp$ is meant to point out that functions are L^2-orthogonal to $\phi_{\epsilon,0}$, and the subscript 0 is meant to point out that functions satisfy the 0 Dirichlet (CASE 1) or 0 Neumann (CASE 2) condition on ∂M and the 0 Dirichlet condition on $\partial B^{\bar g}_{1+v_0}(p)$, is an isomorphism. For $n = 2$ and $\nu \in (0, 1)$ the same result holds for the operator

$$\left(\Delta_{\bar g} + \bar\lambda_{\epsilon,0}\right) : \tilde\chi\, \mathbb{R} \oplus \mathcal{C}^{2,\alpha}_{\nu,\perp,0}(M \setminus B^{\bar g}_{1+v_0}(p)) \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-2,\perp}(M \setminus B^{\bar g}_{1+v_0}(p)).$$

where $\tilde\chi$ is a cutoff function equal to 1 in a neighborhood of the origin. The proof of this facts has been given in paragraph 4.

To semplify the notation let us define

$$\begin{aligned} A \; &:= \; -\psi\, \Delta_{\bar g}\chi - 2\, \nabla^{\bar g}\psi\, \nabla^{\bar g}\chi - H_\varphi\, \Delta_{\bar g}\chi - 2\, \nabla^{\bar g} H_\varphi\, \nabla^{\bar g}\chi \\ B \; &:= \; -\chi\, \Delta_{\bar g}\psi - \bar\lambda_{\epsilon,0}\, \chi\, (\psi + H_\varphi) - \chi\, \Delta_{\bar g} H_\varphi - \bar\lambda_{\epsilon,0}\, K \\ C \; &:= \; -\dot\tau\, \phi_{\epsilon,0} \end{aligned}$$

Equation (4.66) becomes

$$(\Delta_{\bar g} + \bar\lambda_{\epsilon,0})\, \eta = A + B + C$$

By the last result, if we chose $\dot\tau$ in order to verify

$$\int_{M \setminus B^{\bar g}_{1+v_0}(p)} (A + B + C)\, \phi_{\epsilon,0} = 0 \tag{4.67}$$

there exists a solution $\eta = \eta(\epsilon, K, \varphi) \in \mathcal{C}^{2,\alpha}_{\nu,\perp,0}(M \setminus B^{\bar g}_{1+v_0}(p))$ (or $\tilde\chi\, \mathbb{R} \oplus \mathcal{C}^{2,\alpha}_{\nu,\perp,0}(M \setminus B^{\bar g}_{1+v_0}(p))$ if $n = 2$) to equation (4.66) for all ϵ small enough, for all constant K and all function φ, and then

$$\dot\psi = K + \chi\, (\psi + H_\varphi) + \eta$$

satisfy the first equation of (4.61).

We want now to give some estimations on the function η. By the previous results and Lemma 4.4.1 we have the following estimations :

– $\|A\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus B^{\bar{g}}_{1+v_0}(p))} \leq c\,\epsilon^{n-1}\,\left(1+\|\varphi\|_{L^\infty(S^{n-1})}\right)$

– $\|B\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus B^{\bar{g}}_{1+v_0}(p)} \leq c\,\epsilon^{2}\,\left(1+\|\varphi\|_{L^\infty(S^{n-1})}\right)$

In particular we get that

$$\dot{\tau} \leq c\,\left(\epsilon^2+\epsilon^{n-1}\right)\,\left(1+\|\varphi\|_{L^\infty(S^{n-1})}\right)$$

and then

$$\|A+B+C\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus B^{\bar{g}}_{1+v_0}(p)))} \leq c\,\left(\epsilon^2+\epsilon^{n-1}\right)\,\left(1+\|\varphi\|_{L^\infty(S^{n-1})}\right)$$

This give us an estimation on the function η that we found before :

$$\|\eta\|_{\mathcal{C}^{2,\alpha}_{\nu}(M\setminus B^{\bar{g}}_{1+v_0}(p))} \leq c\,\left(\epsilon^2+\epsilon^{n-1}\right)\,\left(1+\|\varphi\|_{L^\infty(S^{n-1})}\right).$$

Summarizing, we have proved the following : For all $\varphi \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$, for all constant K, for all ϵ small enough, there exists a function $\eta(\epsilon,K,\varphi) \in \mathcal{C}^{2,\alpha}_{\nu,\perp,0}(M\setminus B^{\bar{g}}_{1+v_0}(p))$ such that (4.65) is a positive solution of the first equation of (4.61). Moreover there exists a positive constant c such that

$$\|\eta\|_{\mathcal{C}^{2,\alpha}_{\nu}(M\setminus B^{\bar{g}}_{1+v_0}(p))} \leq c\,\left(\epsilon^2+\epsilon^{n-1}\right)\,\left(1+\|\varphi\|_{L^\infty(S^{n-1})}\right).$$

Now we have to make attention to the second equation of (4.61). Let us define

$$Z(\epsilon,K,\varphi) := \left[K+\chi(y)\,(\psi(y)+H_\varphi(y))+\eta(\epsilon,\varphi)(y)\right]_{y\in S^{n-1}}.$$

We remark that Z, that represents the boundary value of the solution of the first of (4.61) we found above, is well defined in a heighborhood of $(0,0,0)$ in $(0,+\infty)\times\mathbb{R}\times\mathcal{C}^{2,\alpha}_m(S^{n-1})$, and takes its values in $\mathcal{C}^{2,\alpha}(S^{n-1})$. It is easy to compute the differential of Z with respect to K and φ at $(0,0,0)$:

$$\big(\partial_\varphi Z(0,0,0)\big)(\tilde{K}) = \tilde{K}.$$

$$\big(\partial_\varphi Z(0,0,0)\big)(\tilde{\varphi}) = \tilde{\varphi}.$$

We can estimate $Z(\epsilon,0,0)$:

$$\|Z(\epsilon,0,0)+\partial_r\phi_1\,\bar{w}\|_{L^\infty(S^{n-1})} \leq c\,\left(\epsilon^2+\epsilon^{n-1}\right)$$

and then

$$\|Z(\epsilon,0,0)+\bar{g}(\nabla\phi_{\epsilon,0},\bar{\nu})\,(\dot{v}_0+\bar{w})\|_{L^\infty(S^{n-1})} \leq c\,\left(\epsilon^2+\epsilon^{n-1}\right)$$

The implicit function theorem applies to give the following : Let ϵ be small enough ; then there exist $(K_\epsilon, \varphi_\epsilon)$ in a neighborhood of $(0,0)$ in $\mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1})$ such that (4.65) is a positive solution of (4.61). Moreover the following estimations hold :

$$|K_\epsilon| \leq c\,\left(\epsilon^2 + \epsilon^{n-1}\right) \qquad \text{and} \qquad \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\,\left(\epsilon^2 + \epsilon^{n-1}\right)$$

Summarizing, we get the following existence result : for all ϵ small enough there exist a constant $\dot\tau$ and $(K_\epsilon, \varphi_\epsilon, \eta_\epsilon)$ in a neighborhood of $(0,0,0)$ in $\mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathcal{C}^{2,\alpha}_\nu(M \setminus B^{\bar g}_{1+v_0}(p))$ (or $\mathbb{R} \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \tilde\chi\,\mathbb{R} \oplus \mathcal{C}^{2,\alpha}_\nu(M \setminus \{p\})$ if $n = 2$) such that the function

$$\dot\psi = K_\epsilon + \chi\,(\psi + H_{\varphi_\epsilon}) + \eta_\epsilon$$

defined in $M \setminus B^{\bar g}_{1+v_0}(p)$, is solution of (4.61) Moreover :

$$|K_\epsilon| \leq c\,\left(\epsilon^2 + \epsilon^{n-1}\right) \qquad \text{and} \qquad \|\varphi_\epsilon\|_{L^\infty(S^{n-1})} \leq c\,\left(\epsilon^2 + \epsilon^{n-1}\right)$$

$$\text{and} \qquad \|\eta_\epsilon\|_{\mathcal{C}^{2,\alpha}_\nu(M \setminus B^{\bar g}_{1+v_0}(p))} \leq c\,\left(\epsilon^2 + \epsilon^{n-1}\right)$$

The last norm is over $\tilde\chi\,\mathbb{R} \oplus \mathcal{C}^{2,\alpha}_\nu(M \setminus \{p\})$ if $n = 2$. This completes the proof of the result. □

Using the previous lemma, we have that for ϵ small enough

$$\partial_r \dot\psi\Big|_{\partial\mathring{B}_1} = \partial_r \psi|_{\partial\mathring{B}_1} + \mathcal{O}(\epsilon^2)$$

for $n \geq 3$ and

$$\partial_r \dot\psi\Big|_{\partial\mathring{B}_1} = \partial_r \psi|_{\partial\mathring{B}_1} + \mathcal{O}(\epsilon)$$

for $n = 2$, because the estimation of η_ϵ is given on the weighted Holder spaces. The statement of the Proposition 4.7.2 then follows at once from the fact that $\partial_r \phi_1$ is constant while the term $\bar w\,\partial_r^2 \phi_1 + \partial_r \psi$ has mean 0 on the boundary $\partial\mathring{B}_1$. This completes the proof of the proposition. □

Now we denote by L_ϵ the linearization of F with respect to $\bar v$, computed at the point $(p, \epsilon, 0)$. It is easy to check the :

Lemma 4.7.4. *There exists a constant $c > 0$ such that, for all $\epsilon > 0$ small enough we have the estimate*

$$\|(L_\epsilon - L_0)\,\bar v\|_{\mathcal{C}^{1,\alpha}} \leq c\,\epsilon\,\|\bar v\|_{\mathcal{C}^{2,\alpha}} \qquad \text{in the CASE 1 and } n \geq 3$$

$$\|(L_\epsilon - L_0)\,\bar v\|_{\mathcal{C}^{1,\alpha}} \leq c\,\epsilon\,\log\epsilon\,\|\bar v\|_{\mathcal{C}^{2,\alpha}} \qquad \text{in the CASE 1 and } n = 2$$

$$\|(L_\epsilon - L_0)\,\bar v\|_{\mathcal{C}^{1,\alpha}} \leq c\,\epsilon^2\,\|\bar v\|_{\mathcal{C}^{2,\alpha}} \qquad \text{in the CASE 2 and } n \geq 5$$

$$\|(L_\epsilon - L_0)\,\bar v\|_{\mathcal{C}^{1,\alpha}} \leq c\,\epsilon^2\,\log\epsilon\,\|\bar v\|_{\mathcal{C}^{2,\alpha}} \qquad \text{in the CASE 2 and } n = 4$$

Proof : Clearly both L_ϵ and L_0 are first order differential operators. We already know the expression of L_0. We have

$$L_\epsilon(\bar{w}) = \lim_{s\to 0} \frac{F(p,\epsilon,s\,\bar{w}) - F(p,\epsilon,0)}{s}.$$

$F(p,\epsilon,s\,\bar{w})$ is given by (4.63) minus its mean, in the metric $\hat{g}$. $F(p,\epsilon,0)$, up to terms of order $\mathcal{O}(\epsilon^2)$, is given by $\partial_r\bar{\phi}_{\epsilon,0}(y)$ at $\partial\mathring{B}_1$ minus its the mean, in the metric $\hat{g}$ evaluated at $s=0$. The proof of the Lemma follows at once from Proposition 4.6.1 and Lemma 4.7.3. □

4.8 The proof of the main result

We shall now prove that, for $\epsilon > 0$ small enough, it is possible to solve the equation

$$F(p,\epsilon,\bar{v}) = 0$$

Unfortunately, we will not be able to solve this equation at once. Instead, we first prove the :

Proposition 4.8.1. *There exists $\epsilon_0 > 0$ such that, for all $\epsilon \in [0,\epsilon_0]$ and for all $p \in M$, there exists a unique function $\bar{v} = \bar{v}(p,\epsilon)$ and a vector $a = a(p,\epsilon) \in \mathbb{R}^n$ such that*

$$F(p,\epsilon,\bar{v}) + \mathring{g}(a,\cdot) = 0$$

The function $\bar{v}$ and the vector a depend smoothly on p and ϵ and we have

$$|a| + \|\bar{v}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \le c\,\epsilon \qquad \text{in the CASE 1 and } n \ge 3$$

$$|a| + \|\bar{v}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \le c\,\epsilon\,\log\epsilon \qquad \text{in the CASE 1 and } n = 2$$

$$|a| + \|\bar{v}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \le c\,\epsilon^2 \qquad \text{in the CASE 2 and } n \ge 5$$

$$|a| + \|\bar{v}\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \le c\,\epsilon^2\,\log\epsilon \qquad \text{in the CASE 2 and } n = 4$$

Proof : We fix $p \in M$ and define

$$\bar{F}(p,\epsilon,\bar{v},a) := F(p,\epsilon,\bar{v}) + \mathring{g}(a,\cdot)$$

It is easy to check that $\bar{F}$ is a smooth map from a neighborhood of $(p,0,0,0)$ in $M \times [0,\infty) \times \mathcal{C}^{2,\alpha}_m(S^{n-1}) \times \mathbb{R}^n$ into a neighborhood of 0 in $\mathcal{C}^{1,\alpha}(S^{n-1})$. Moreover,

$$\bar{F}(p,0,0,0) = 0$$

and the differential of $\bar{F}$ with respect to $\bar{v}$, computed at $(p,0,0,0)$ is given by H. Finally the image of the linear map $a \longmapsto \mathring{g}(a,\cdot)$ is just the vector space V_1. Thanks to the result of Proposition 4.7.1, the implicit function theorem applies to get the existence of $\bar{v}$ and a, smoothly depending on p and ϵ such that $\bar{F}(p, \epsilon, \bar{v}, a) = 0$. The estimates for $\bar{v}$ and a follow at once from Proposition 4.6.1. This completes the proof of the result. □

In view of the result of the previous Proposition, it is enough to show that, provided that ϵ is small enough, it is possible to choose the point $p \in M$ such that $a(p, \epsilon) = 0$. We claim that, there exists a constant $\bar{C} > 0$ (only depending on n) such that

$$\begin{array}{ll} \Theta(a(p,\epsilon)) = -\epsilon\, \tilde{C}\, \nabla^g \phi_0(p) + \mathcal{O}(\epsilon^2) & \text{in the CASE 1 and } n \geq 3 \\ \Theta(a(p,\epsilon)) = -\epsilon\, \log\epsilon\, \tilde{C}\, \nabla^g \phi_0(p) + \mathcal{O}(\epsilon) & \text{in the CASE 1 and } n = 2 \\ \Theta(a(p,\epsilon)) = -\epsilon^3\, \tilde{C}\, \nabla^g\, \mathrm{Scal}(p) + \mathcal{O}(\epsilon^4) & \text{in the CASE 2 and } n \geq 5 \\ \Theta(a(p,\epsilon)) = -\epsilon^3\, \log\epsilon\, \tilde{C}\, \nabla^g\, \mathrm{Scal}(p) + \mathcal{O}(\epsilon^3) & \text{in the CASE 2 and } n = 4 \end{array}$$

For all $b \in \mathbb{R}^n$ we compute

$$\begin{aligned} \int_{S^{n-1}} \mathring{g}(a,\cdot)\, \mathring{g}(b,\cdot)\ \mathrm{dvol}_{\mathring{g}} &= -\int_{S^{n-1}} F(p,\epsilon,\bar{v})\, \mathring{g}(b,\cdot)\ \mathrm{dvol}_{\mathring{g}} \\ &= -\int_{S^{n-1}} (F(p,\epsilon,0) + L_0\bar{v})\, \mathring{g}(b,\cdot)\ \mathrm{dvol}_{\mathring{g}} \\ &\quad -\int_{S^{n-1}} (F(p,\epsilon,\bar{v}) - F(p,\epsilon,0) - L_\epsilon\bar{v})\, \mathring{g}(b,\cdot)\ \mathrm{dvol}_{\mathring{g}} \\ &\quad -\int_{S^{n-1}} (L_\epsilon - L_0)\bar{v}\, \mathring{g}(b,\cdot)\ \mathrm{dvol}_{\mathring{g}} \end{aligned}$$

Now, we use the fact that $\bar{v}$ is $L^2(S^{n-1})$-orthogonal to linear functions and hence so is $L_0\, \bar{v}$. Therefore,

$$\int_{S^{n-1}} L_0\, \bar{v}\, \mathring{g}(b,\cdot)\ \mathrm{dvol}_{\mathring{g}} = 0$$

Using the fact that

$$\begin{array}{ll} \bar{v} = \mathcal{O}(\epsilon) & \text{in the CASE 1 and } n \geq 3 \\ \bar{v} = \mathcal{O}(\epsilon\, \log\epsilon) & \text{in the CASE 1 and } n = 2 \\ \bar{v} = \mathcal{O}(\epsilon^2) & \text{in the CASE 2 and } n \geq 5 \\ \bar{v} = \mathcal{O}(\epsilon^2\, \log\epsilon) & \text{in the CASE 2 and } n = 4 \end{array}$$

we get

$$F(p,\epsilon,\bar{v}) - F(p,\epsilon,0) - L_\epsilon \bar{v} = \mathcal{O}(\epsilon^2) \qquad \text{in the CASE 1 and } n \geq 3$$

$$F(p,\epsilon,\bar{v}) - F(p,\epsilon,0) - L_\epsilon \bar{v} = \mathcal{O}(\epsilon^2\,(\log\epsilon)^2) \qquad \text{in the CASE 1 and } n = 2$$

$$F(p,\epsilon,\bar{v}) - F(p,\epsilon,0) - L_\epsilon \bar{v} = \mathcal{O}(\epsilon^4) \qquad \text{in the CASE 2 and } n \geq 5$$

$$F(p,\epsilon,\bar{v}) - F(p,\epsilon,0) - L_\epsilon \bar{v} = \mathcal{O}(\epsilon^4\,(\log\epsilon)^2) \qquad \text{in the CASE 2 and } n = 4$$

Similarly, it follows from the result of Proposition 4.7.4 that

$$(L_\epsilon - L_0)\,\bar{v} = \mathcal{O}(\epsilon^2) \qquad \text{in the CASE 1 and } n \geq 3$$

$$(L_\epsilon - L_0)\,\bar{v} = \mathcal{O}(\epsilon^2\,(\log\epsilon)^2) \qquad \text{in the CASE 1 and } n = 2$$

$$(L_\epsilon - L_0)\,\bar{v} = \mathcal{O}(\epsilon^4) \qquad \text{in the CASE 2 and } n \geq 5$$

$$(L_\epsilon - L_0)\,\bar{v} = \mathcal{O}(\epsilon^4\,(\log\epsilon)^2) \qquad \text{in the CASE 2 and } n = 4$$

The claim then follows from the estimates in Proposition 4.6.1 and the fact that

$$\int_{S^{n-1}} \mathring{g}(a,\cdot)\,\mathring{g}(b,\cdot)\ \mathrm{dvol}_{\mathring{g}} = g\big(\Theta(a),\Theta(b)\big) \int_{S^{n-1}} (x_1)^2\ \mathrm{dvol}_{\mathring{g}} = \frac{1}{n}\ \mathrm{Vol}_{\mathring{g}}(S^{n-1})\, g\big(\Theta(a),\Theta(b)\big).$$

Now if we assume that p_0 is a nondegenerate critical point of the function ϕ_0 (CASE 1) or a nondegenerate critical point of the scalar curvature function (CASE 2), we can apply once more the implicit function theorem to solve the equations

$$G(\epsilon,p) := \epsilon^{-1}\,\Theta(a(p,\epsilon)) = 0 \qquad \text{in the CASE 1 and } n \geq 3$$

$$G(\epsilon,p) := (\epsilon\,\log\epsilon)^{-1}\,\Theta(a(p,\epsilon)) = 0 \qquad \text{in the CASE 1 and } n = 2$$

$$G(\epsilon,p) := \epsilon^{-3}\,\Theta(a(p,\epsilon)) = 0 \qquad \text{in the CASE 2 and } n \geq 5$$

$$G(\epsilon,p) := \epsilon^{-3}\,(\log\epsilon)^{-1}\,\Theta(a(p,\epsilon)) = 0 \qquad \text{in the CASE 2 and } n = 4$$

It should be clear that G depends smoothly on $\epsilon \in [0,\epsilon_0)$ and $p \in M$. Moreover we have

$$G(0,p) = -\tilde{C}\,\nabla^g \phi_0(p)$$

in the CASE 1 and

$$G(0,p) = -\tilde{C}\,\nabla^g\ \mathrm{Scal}(p)$$

in the CASE 2. Hence, under the hypothesis on p_0 of the main theorem, $G(0,p_0) = 0$ in both cases. By assumption the differential of G with respect to p, computed at p_0 is invertible. Therefore, for all ϵ small enough there exists p_ϵ close to p_0 such that

$$\Theta(a(p_\epsilon,\epsilon)) = 0$$

In addition we have

$$\mathrm{dist}(p_0, p_\epsilon) \le c\,\epsilon$$

This completes the proof the Theorem 4.1.3.

4.9 Appendix

We recall here some results demonstrated in [32].

Lemma 4.9.1. *For all $\sigma = 1, \ldots, n$, we have*

$$\sum_{i,j,k,\ell,m} \int_{S^{n-1}} R_{ikj\ell,m}\, x^i\, x^j\, x^k\, x^\ell\, x^m\, x^\sigma \;\; \mathrm{dvol}_{\mathring{g}} = 0.$$

Lemma 4.9.2. *For all $\sigma = 1, \ldots, n$, we have*

$$\sum_{j,k,\ell} \int_{S^{n-1}} R_{\cdot kj\ell,\cdot}\, x^j\, x^k\, x^\ell\, x^\sigma \;\; \mathrm{dvol}_{\mathring{g}} = 0.$$

Lemma 4.9.3. *For all $\sigma = 1, \ldots, n$, we have*

$$\sum_{i,\ell,m} \int_{S^{n-1}} R_{i\ell,m}\, x^i\, x^\ell\, x^m\, x^\sigma \;\; \mathrm{dvol}_{\mathring{g}} = \frac{2}{n(n+2)} \;\; \mathrm{Vol}_{\mathring{g}}(S^{n-1}) \;\; \mathrm{Scal}_{,\sigma}$$

Moreover we prove the

Lemma 4.9.4. *For all $\sigma = 1, \ldots, n$, we have*

$$\sum_{t} \int_{S^{n-1}} \;\; \mathrm{Scal}_{,t}\, x^t\, x^\sigma \;\; \mathrm{dvol}_{\mathring{g}} = \frac{1}{n} \;\; \mathrm{Vol}_{\mathring{g}}(S^{n-1}) \;\; \mathrm{Scal}_{,\sigma}$$

Proof : We find that $\int_{S^{n-1}} \mathrm{Scal}_{,t}\, x^t\, x^\sigma \;\; \mathrm{dvol}_{\mathring{g}} = 0$ unless the indices t and σ are equal. Then

$$\sum_{t} \int_{S^{n-1}} \;\; \mathrm{Scal}_{,t}\, x^t\, x^\sigma \;\; \mathrm{dvol}_{\mathring{g}} = \;\; \mathrm{Scal}_{,\sigma} \int_{S^{n-1}} (x^\sigma)^2 \;\; \mathrm{dvol}_{\mathring{g}} = \frac{1}{n} \;\; \mathrm{Vol}_{\mathring{g}}(S^{n-1}) \;\; \mathrm{Scal}_{,\sigma}$$

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