Orthogonal polynomials with Hermitian matrix argument and associated semigroups

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Abstract

In this work we construct and study families of generalized orthogonal polynomials with hermitian matrix argument associated to a family of orthogonal polynomials on $\mathbb{R}$. Different normalizations for these polynomials are considered and we obtain some classical formulas for orthogonal polynomials from the corresponding formulas for the one–dimensional polynomials. We also construct semigroups of operators associated to the generalized orthogonal polynomials and we give an expression of the infinitesimal generator of this semigroup and, in the classical cases, we prove that this semigroup is also Markov.

For d–dimensional Jacobi expansions we study the notions of fractional integral (Riesz potentials), Bessel potentials and fractional derivatives. We present a novel decomposition of the L^2 space associated with the d–dimensional Jacobi measure and obtain an analogous of Meyer's multiplier theorem in this setting. Sobolev Jacobi spaces are also studied.

Résumé

Dans ce travail, nous construisons et étudions des familles de polynômes orthogonaux généralisés définis dans l'espace des matrices hermitiennes qui sont associées à une famille de polynômes orthogonaux sur $\mathbb{R}$. Nous considérons plusieurs normalisations pour ces polynômes, et obtenons des formules classiques à partir des formules correspondantes pour des polynômes définis sur $\mathbb{R}$. Nous construisons également des semi-groupes d'opérateurs associés aux polynômes orthogonaux généralisés, et donnons l'expression du générateur infinitésimal de ce semi-groupe ; nous prouvons que ce semi-groupe est markovien dans les cas classiques.

En ce qui concerne les expansions d-dimensionnelles de Jacobi nous étudions les notions d'intégrale fractionnelle (potentiel de Riesz), de potentiel de Bessel et de derivées fractionnelles. Nous donnons une nouvelle décomposition de l'espace L^2 associé à la mesure de Jacobi d-dimensionnelle, et obtenons un analogue du théorème du multiplicateur de Meyer dans ce cadre. Nous étudions aussi les espaces de Jacobi-Sobolev.

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Chapter 1

Introduction

Particular families of generalized orthogonal polynomials have been studied since the mid 1950's. The families of generalized Hermite and Laguerre polynomials were introduced by C. Hertz [Her55] in the context of special functions with matrix argument. Since then there has been an increasing interest in this branch of mathematics with multiple generalizations and applications. For example, the generalized Laguerre polynomials were defined in the context of symmetric cones by J. Faraut and A. Koranyi in [FK94] since they are very useful in the study of Harmonic Analysis on symmetric cones. These polynomials also have a large number of applications in multivariate statistics as can be seen in [Mui82].

Let us denote by H_n the space of Hermitian $n \times n$ matrices and by U_n the group of unitary matrices. We say that the function $f : H_n \to \mathbb{R}$ is central if it is U_n-invariant, that is, if $f(UXU^{-1}) = f(X)$ for all $U \in U_n$; thus, a central function is determined by its restriction to the subspace of diagonal matrices and this restriction is a symmetric function on $\mathbb{R}^n$. Moreover, a symmetric function on $\mathbb{R}^n$ uniquely determines a central function on H_n.

A generalized polynomial with Hermitian matrix argument is a central function on H_n whose restriction to the space of diagonal matrices is a symmetric polynomial. Plenty information about a generalized polynomial can be obtained from the study of associated symmetric polynomial.

Hermite and Laguerre symmetric polynomials associated to the corresponding families of generalized Hermite and Laguerre polynomials are special cases of generalized Hermite polynomials for Dunkl operators. These operators, defined by C. Dunkl ([Dun88, Dun89, Dun90]), have been extensively studied by O. Opdam, ([Opd93, Opd00]) and M. Rösler ([Ros98, Ros99, Ros02]), among others.

Dunkl operators are differential–reflection operators on an euclidian space associated with a finite reflection group. From an analytical point of view these operators extend notions of harmonic analysis and special functions in a symmetric space associated with a root system. Also, the theory of stochastic processes has gained considerable interest in the last years, see for example [RV98]. A comprehensive monograph on this subject is [GRY08], written in an accesible way by the researchers who have contributed to the development of this theory.

Dunkl operators are also of great interest in mathematical-physics due to its relation to certain quantum integrable models in physics; these are naturally related with certain Schrodinger operators for Caloguero-Moser-Sutherland models. The spectral properties of these operators can be determined using the Dunkl formalism, via the generalized Hermite polynomials. These polynomials are eigenfunctions of the Dunkl harmonic oscillator operator and have been extensively studied, e.g. [vDV00, Ros98, Ros99, BF97b, BF98].

From a combinatorial point of view, Hermite, Laguerre and Jacobi symmetric polynomials associated to the generalized Hermite, Laguerre and Jacobi polynomials with Hermitian matrix argument are special cases of symmetric orthogonal polynomials associated to the Jack symmetric polynomials studied by Lassalle in a series of notes [Las91a, Las91b, Las91c]. In these notes Lassalle defines these families of generalized (in Jack sense) polynomials as eigenfunctions of certain differential operators on $\mathbb{R}^n$ that generalize the Hermite, Laguerre and Jacobi one dimensional operators. In the Schur polynomials expansion case (parameter $\alpha = 1$ in this articles), the case we are interested in, he gives a Berezin–Karpelevich type formula that involves the one dimensional Hermite, Laguerre and Jacobi polynomials.

Let us recall that F.A. Berezin and F.I. Karpelevich [BK59] expressed the spherical functions on a complex Grassmann manifold as a quotient of a determinant containing Jacobi functions and a Vandermonde determinant. K.I. Gross and D.St.P. Richards [GR91, GR93] give similar formulas for some generalized hypergeometric functions.

The generalized Hermite and Laguerre polynomials in Jack sense introduced by Lassalle are also studied by T.H. Baker and P.J. Forrester [BF97a] in the context of Caloguero-Moser-Sutherland models in physics. For these polynomials they present analogous of classical results such as generating functions, differentiation and integration formulas and recursion relations.

In this work we construct families of generalized orthogonal polynomials with Hermitian matrix argument from a family of orthogonal polynomials on $\mathbb{R}$, obtaining, among others, the generalized Hermite, Laguerre and Jacobi polynomials. These generalized polynomials are given by a Berezin–Karpelevich type formula that involves a quotient of a determinant containing the corresponding orthogonal polynomials on $\mathbb{R}$ and a Vandermonde determinant. Alternatively, they can be obtained by the Gram-Schmidt orthogonalization process from the Schur polynomials.

Another object of interest in this work are the Markov semigroups associated to families of orthogonal polynomials. D. Bakry and O. Mazet [BM03] introduce the notion of a Markov generator sequence for a given family of orthogonal polynomials and characterize the Markov generator sequences for the classical orthogonal polynomials on $\mathbb{R}$. A Markov generator sequence for a family of orthogonal polynomials is equivalent to the existence of a Markov semigroup with spectral decomposition given by the family of orthogonal polynomials. Markov semigroups are intimately related to Markov processes, widely known stochastic processes with many applications in physics, economy and computer science, among many others. Among the Markov semigroups associated to orthogonal polynomials we have the Ornstein–Uhlenbeck, Laguerre and Jacobi semigroups. Besides the theoretical interest of obtaining an operator semigroup associated to the generalized polynomials, these semigroups are of interest because of their relation with the Dunkl theory. Perhaps the knowledge obtained from the study of these semigroups in this particular setting can help understanding further the behavior of the semigroups and processes in the Dunkl setting.

In this work we construct semigroups associated to the generalized orthogonal polynomials. For this, we consider a Markov generator sequence, and therefore a Markov semigroup, for the orthogonal polynomials on $\mathbb{R}$. We give an explicit expression of the infinitesimal generator of this semigroup and, under the hypothesis of diffusion, we prove that this semigroup on H_n is also Markov. We also have expressions for the kernels of this semigroup in terms of the one-dimensional kernels and give a probabilistic interpretation of the corresponding Markov process on H_n.

The behavior of the semigroup associated to a family of orthogonal polynomials is closely related to the harmonic analysis for the polynomial expansions. Harmonic analysis for expansions in terms of non-trigonometric orthogonal polynomials have various motivations. In first place, it arises as a theoretic extension of the known results for Fourier series. In 1965 B. Muckenhoupt and E.M. Stein [MS65] developed a harmonic analysis for non–trigonometric polynomial expansions in the case of ultraspherical polynomials. They introduced the notions of Poisson integral, conjugate function, Riesz potentials and H^p spaces for those expansions.

Expansions in terms of other classical families of polynomials have also been studied. In [FSU01] Riesz and Bessel potentials for the gaussian measure, that is, for Hermite expansions, were studied and, also in this case, in [LU04] fractional derivatives were studied and a characterization of the Gaussian Sobolev space was obtained. The Laguerre expansion case was treated in [GL$^+$05], were the authors obtained an analogous of P. A. Meyer's multiplier theorem for this expansions, introduced fractional derivatives and fractional integrals in this setting and studied Sobolev spaces and higher–order Riesz–Laguerre transforms.

Among the semigroups on $\mathbb{R}^n$ associated to the classical polynomials, the Jacobi semigroup is the most complicated and least studied of all. Before studying harmonic analysis properties for generalized orthogonal polynomials expansions on H_n, we study some notions of harmonic analysis for the d-dimensional Jacobi expansions, considering fractional differentiation and integral differentiation for these expansions.

In this work we obtain an analogous result of P.A. Meyer's multiplier theorem [Mey84] for Jacobi polynomial expansion, by considering an alternative decomposition of the L^2 space associated to the Jacobi measure. We then define study Riesz and Bessel potentials, associated to the d–dimensional Jacobi operator, proving that they can be extended continuously to L^p. Using fractional derivatives, we give a characterization of the Sobolev spaces associated to the Jacobi measure. With these results we complete the study of these notions for the classical families of orthogonal polynomials.

1.1 About the organization of this work

This thesis has four chapters. The present chapter corresponds to the introduction. We have chosen to include tree more sections in this introduction, sections 1.2, 1.3 and 1.4, that correspond to chapters 2, 3 and 4, where we shortly describe the results obtained in each one of the chapters. Each result referred in these sections corresponds to a result in the corresponding chapter. In the last section of this introduction we preset some open problems.

Chapter 2 includes the published paper [BU08] in which we define and study fractional integration and fractional differentiation for d–dimensional Jacobi expansions. We consider a novel decomposition of the L^2 space associated to the multidimensional Jacobi measure and obtain an analogue of Meyer's multipliers theorem for Jacobi expansions. We also obtain a characterization of the Sobolev spaces associated to the d–dimensional Jacobi measure.

Chapter 3 includes the published paper [BGU05]. In this, we construct and study families of generalized orthogonal polynomials on the space of Hermitian matrices from a family of orthogonal polynomials in $\mathbb{R}$. We give a Berezin–Karpelevich type formula for these polynomials. We also study different normalizations and expansions for these polynomials.

Chapter 4 includes the preprint [BGU09] accepted for publication, where we construct operators semigroups associated to the families of generalized orthogonal polynomials with hermitian matrix argument. We give an explicit expression of the infinitesimal generator of these semigroups and of the kernels that define it. Under the hypothesis of diffusions we prove that this semigroup is also Markov. We obtain some classical formulas for the generalized orthogonal polynomials from the corresponding formulas for the one–dimensional polynomials.

We have included references for each one of the chapters to facilitate the reading of this work. General references are also available at the end of the work. We also have included an appendix about the classical orthogonal polynomials, where properties and basic identities for these polynomials can be found.

1.2 About Chapter 2

This chapter includes the published paper [BU08] in which we define and study fractional integration and fractional differentiation for d–dimensional Jacobi expansions. For parameters $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$ in $\mathbb{R}^d$, satisfying $\alpha_i, \beta_i > -1$ and a multi-index $\kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}^d$, let $\vec{p}_\kappa^{\,\alpha,\beta}$ be the normalized Jacobi polynomial of order κ in $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$, defined on $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ by

$$\vec{p}_\kappa^{\,\alpha,\beta}(x) = \prod_{i=1}^{d} p_{\kappa_i}^{\alpha_i,\beta_i}(x_i),$$

where $p_n^{\alpha,\beta}$, for $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > -1$, is the normalized Jacobi polynomial of degree n in $\mathbb{R}$.

The family of Jacobi polynomials $\{\vec{p}_\kappa^{\,\alpha,\beta}\}$ is orthogonal in the space $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$, where

$$\mu^d_{\alpha,\beta}(dx) = \prod_{i=1}^{d} \left\{ \frac{1}{2^{\alpha_i+\beta_i+1}\, B(\alpha_i+1, \beta_i+1)} (1-x_i)^{\alpha_i}(1+x_i)^{\beta_i}\, dx_i \right\},$$

is the (normalized) d–dimensional Jacobi measure on $[-1,1]^d$.

The Jacobi polynomial $p_n^{\alpha,\beta}$ is an eigenfunction of the d-dimensional Jacobi operator,

$$\mathcal{L}^{\alpha,\beta} := \sum_{i=1}^{d} \left[(1-x_i^2)\frac{\partial^2}{\partial x_i^2} + \left(\beta_i - \alpha_i - (\alpha_i + \beta_i + 2)\, x_i\right) \frac{\partial}{\partial x_i} \right] \tag{1.1}$$

with eigenvalue $-\lambda_\kappa = -\sum_{i=1}^{d} \kappa_i(\kappa_i + \alpha_i + \beta_i + 1)$; that is,

$$\mathcal{L}^{\alpha,\beta} \vec{p}_\kappa^{\,\alpha,\beta} = -\lambda_\kappa \vec{p}_\kappa^{\,\alpha,\beta}.$$

The d-dimensional Jacobi semigroup $\{T_t^{\alpha,\beta}\}_{t\geq 0}$ is defined as the Markov semigroup associated to the Markov probability kernels

$$P^{\alpha,\beta}(t,x,dy) = \sum_{\kappa\in\mathbb{N}^d} e^{-\lambda_\kappa t} \vec{p}_\kappa^{\,\alpha,\beta}(x) \vec{p}_\kappa^{\,\alpha,\beta}(y) \mu^d_{\alpha,\beta}(dy) =: p_d^{\alpha,\beta}(t,x,y)\mu^d_{\alpha,\beta}(dy).$$

That is,

$$T_t^{\alpha,\beta} f(x) := \int_{[-1,1]^d} f(y)\, P^{\alpha,\beta}(t,x,dy) = \int_{[-1,1]^d} f(y)\, p_d^{\alpha,\beta}(t,x,y)\mu^d_{\alpha,\beta}(dy).$$

The d-dimensional Jacobi semigroup $\{T_t^{\alpha,\beta}\}_{t\geq 0}$ is a Markov diffusion semigroup, strongly continuous on $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$, with infinitesimal generator $-\mathcal{L}^{\alpha,\beta}$. Each of its operators is symmetric and is a contraction on $\mathcal{L}^p$ and has the Jacobi polynomials as eigenfunctions, with eigenvalue $e^{-\lambda_\kappa t}$, that is

$$T_t^{\alpha,\beta} \vec{p}_\kappa^{\,\alpha,\beta} = e^{-\lambda_\kappa t} \vec{p}_\kappa^{\,\alpha,\beta}.$$

For parameters $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in \mathbb{R}^d$ such that $\alpha_i, \beta_i \geq -\frac{1}{2}$, the Jacobi semigroup $\{T_t^{\alpha,\beta}\}_{t\geq 0}$ is not merely a contraction on $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$, but is hypercontractive, that is to say, for any $1 < p < \infty$ there exists an increasing function $q = q_{\alpha,\beta}\colon \mathbb{R}^+ \to [p,\infty)$, with $q(0) = p$, such that for every f and all $t \geq 0$,

$$\|T_t^{\alpha,\beta} f\|_{q(t)} \leq \|f\|_p \,.$$

The proof of this fact is an indirect one and it is due to the fact that the one–dimensional Jacobi operator satisfies a curvature–dimension inequality. This was proven by D. Bakry in [Bak96] noting that an operator on real interval I of the form

$$f''(x) - a(x)f'(x) \tag{1.2}$$

satisfies a curvature–dimension inequality with curvature constant ρ and dimension constant n if, and only if,

$$a'(x) \geq \rho + \frac{a^2(x)}{n-1}$$

and expressing the one–dimensional Jacobi operator in the form (1.2) by means of the change of variables $x = \sin(y)$, for $y \in [-\pi/2, \pi/2]$. As shown in [ABC+02], the curvature–dimension inequality yields a logarithmic Sobolev inequality for the one–dimensional Jacobi operator and, since this inequality is stable under tensorization [ABC+02], the d–dimensional Jacobi operator also satisfies a logarithmic Sobolev inequality. Using L. Gross's famous result [Gro75], which states the equivalence between the hypercontractivity property and the validity of a logarithmic Sobolev inequality, we have that the d–dimensional Jacobi semigroup is in fact hypercontractive.

For each $n \geq 0$, let $C_n^{\alpha,\beta}$ be the subspace of $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$ generated by linear combinations of $\{\vec{p}_\kappa^{\,\alpha,\beta} \colon |\kappa| = n\}$, where, as usual for a multi-index κ, $|\kappa| = \sum_{i=1}^d \kappa_i$. Then, we have the orthogonal decomposition

$$L^2([-1,1]^d, \mu^d_{\alpha,\beta}) = \bigoplus_{n=0}^{\infty} C_n^{\alpha,\beta}.$$

This is the Wiener–Jacobi decomposition of $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$, which is analogous to the Wiener decomposition of $L^2(\mathbb{R}^d, \gamma_d)$ in the Gaussian case. For $f \in L^2([-1,1]^d, \mu^d_{\alpha,\beta})$, the expansion of f in Jacobi polynomials is given by

$$f = \sum_{n=0}^{\infty} \sum_{|\kappa|=n} \hat{f}(\kappa)\vec{p}_\kappa^{\,\alpha,\beta},$$

with

$$\hat{f}(\kappa) = \int_{[-1,1]^d} f(y)\vec{p}_\kappa^{\,\alpha,\beta}(y)\mu^d_{\alpha,\beta}(dy),$$

the Jacobi–Fourier coefficient of f for the multi-index κ. This yields the spectral decomposition

$$\mathcal{L}^{\alpha,\beta} f = \sum_{n=0}^{\infty} \sum_{|\kappa|=n} (-\lambda_\kappa)\hat{f}(\kappa)\vec{p}_\kappa^{\,\alpha,\beta}.$$

Unlike in the Gaussian case, the Wiener–Jacobi decomposition of $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$ does not gives us an expression of the action of the Jacobi operator $\mathcal{L}^{\alpha,\beta}$ in terms of the ortogonal projections on the subspaces $C_n^{\alpha,\beta}$, as can be seen in the previous formula. This is due to the fact that the eigenvalue λ_κ of the Jacobi operator does not depends linearly on $|\kappa|$.

In the Gaussian and Laguerre cases and in the one dimensional Jacobi case the expressions of the action of the operators in terms of the orthogonal projections is crucial, so, in order to obtain an expression of this type, in Section 2.2 we consider a different decomposition of $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$ that we call modified Wiener–Jacobi decomposition.

With this decomposition in Section 2.3, we are able to obtain results in a similar way to the ones in the Hermite and Laguerre cases. Our first result is given in Proposition 2.1, where it is shown that the orthogonal projections $J_n^{\alpha,\beta}$ of the modified Wiener decomposition can be extended continuously to $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$. This is essentially a consequence of the hypercontractive property Jacobi semigroup.

In the Hermite expansion case a key result to extend Riesz and Bessel potentials continuously to L^p is P.A. Meyer's multiplier theorem. Before defining these potentials in the Jacobi setting, we establish an analogous result in this case. For any $\Phi\colon \mathbb{N} \to \mathbb{R}$, the multiplier operator T_Φ associated to Φ is defined for a polynomial f with expansion $f = \sum_{n=0}^{\infty} J_n^{\alpha,\beta} f$, by

$$T_\Phi f := \sum_{n=0}^{\infty} \Phi(n) J_n^{\alpha,\beta} f.$$

By Parseval's inequality it is immediate that T_Φ is bounded on $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$. After some technical lemmas, in Theorem 2.1 we give a condition on Φ under which the multiplier T_Φ can be extended to a continuous operator on $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$. This condition is similar to the one given by P.A. Meyer's multiplier theorem in the case of Hermite expansions.

Similarly to the classical case of the Laplacian, we define a fractional integral of order $\gamma > 0$, denoted $I_\gamma^{\alpha,\beta}$, which is called the *Riesz potential* of order γ, associated to the d-dimensional Jacobi operator by

$$I_\gamma^{\alpha,\beta} := (-\mathcal{L}^{\alpha,\beta})^{-\gamma/2}\Pi_0,$$

where $\Pi_0 = I - J_0^{\alpha,\beta}$, the orthogonal projection on the orthogonal complement of $G_0^{\alpha,\beta}$. By applying Meyer's multipliers theorem, in Theorem 2.2 we establish that the Riesz potential can be extended continuously to $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$.

The *Bessel potential* of order $\gamma > 0$ associated to the d-dimensional Jacobi operator, denoted $\mathcal{J}_\gamma^{\alpha,\beta}$, is defined by

$$\mathcal{J}_\gamma^{\alpha,\beta} := (I - \mathcal{L}^{\alpha,\beta})^{-\gamma/2},$$

and, again by the Meyer's multiplier theorem, we prove in Theorem 2.3 that $\mathcal{J}_\gamma^{\alpha,\beta}$ can be extended continuously to $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$.

The *fractional derivative* of order $\gamma > 0$, denoted $D_\gamma^{\alpha,\beta}$, associated to the d-dimensional Jacobi operator is defined by

$$D_\gamma^{\alpha,\beta} := (-\mathcal{L}^{\alpha,\beta})^{\gamma/2}$$

and for $1 < p < \infty$, the Jacobi Sobolev spaces (or *potential spaces*) of order$\gamma > 0$, namely $L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$, is defined as the image of $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$ under the Bessel potential $\mathcal{J}_\gamma^{\alpha,\beta}$, that is,

$$L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta}) = \mathcal{J}_\gamma^{\alpha,\beta} L^p([-1,1]^d, \mu^d_{\alpha,\beta}).$$

As in the classical case, the Jacobi Sobolev space $L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$ can also be defined as the completion of the set of polynomials with respect to the norm

$$\|f\|_{p,\gamma} := \|(I - \mathcal{L}^{\alpha,\beta})^{\gamma/2} f\|_p.$$

In Proposition 2.2 we present some inclusion properties among the Jacobi Sobolev spaces.

Let us consider the space

$$L_\gamma([-1,1]^d, \mu^d_{\alpha,\beta}) = \bigcup_{p>1} L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta}).$$

$L_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$ is the natural domain of $D^{\alpha,\beta}_\gamma$. We define it on this space as follows. Let $f \in L_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$; then there is $p > 1$ such that $f \in L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$ and a sequence of polynomials $\{f_n\}$ such that $\lim_{n\to\infty} f_n = f$ in $L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$. We define for $f \in L_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$,

$$D^{\alpha,\beta}_\gamma f = \lim_{n\to\infty} D^{\alpha,\beta}_\gamma f_n.$$

In Theorem 2.4 we shows that $D^{\alpha,\beta}_\gamma$ is well defined. In this theorem we also show that there exist positive constants $A_{p,\gamma}$ and $B_{p,\gamma}$ such that

$$B_{p,\gamma}\|f\|_{p,\gamma} \leq \|D^{\alpha,\beta}_\gamma f\|_p \leq A_{p,\gamma}\|f\|_{p,\gamma}.$$

which supplies a characterization of the Sobolev spaces.

Finally, in Proposition 2.3 we give an alternative integral representations for $I^{\alpha,\beta}_\gamma$, the Riesz potential, and for $D^{\alpha,\beta}_\gamma$, the fractional derivative, associated to the d-dimensional Jacobi operator.

1.3 About Chapter 3

This chapter includes the published paper [BGU05] in which we construct and study families of generalized orthogonal polynomials on the space of Hermitian matrices from a family of orthogonal polynomials in $\mathbb{R}$. For this we consider a measure μ on $\mathbb{R}$ such that the set of all polynomials is dense in $L^2(\mathbb{R}, \mu)$. By applying the Gram-Schmidt orthogonalization process to the monomials in $L^2(\mathbb{R}, \mu)$ we obtain a family $\{p_m\}$ of orthogonal polynomials on $L^2(\mathbb{R}, \mu)$.

We will consider a related measure on $\mathbb{R}^n$ defined by

$$\mu_n(d\underline{x}) = V^2(\underline{x})\mu^{\otimes n}(d\underline{x}) \tag{1.3}$$

where $\underline{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $V(\underline{x}) = \det(x_j^{n-i})_{i,j=1,\dots,n} = \prod_{1\le i<j\le n}(x_i - x_j)$ is the Vandermonde determinant. We will also require the set $\mathcal{P}_n$ of all of symmetric polynomials on $\mathbb{R}^n$ to be dense on $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$, the space of all symmetric functions in $L^2(\mathbb{R}^n, \mu_n)$.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition; that is, a non–increasing sequence of non–negative integers. For a partition λ, the Schur polynomials is defined on $\mathbb{R}^n$ as

$$S_\lambda(x_1, \dots, x_n) = \frac{\det(x_j^{\lambda_i+n-i})_{i,j=1,\dots n}}{V(x_1, \dots, x_n)}.$$

Each S_λ is a symmetric polynomial on $\mathbb{R}^n$ and it is well known that the family of Schur polynomials form an algebraic base of $\mathcal{P}_n$ (c.f. [Mac91]).

For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we define the following function on $\mathbb{R}^n$,

$$P_\lambda(x_1, \dots, x_n) = c_\lambda \frac{\det(p_{\lambda_i+n-i}(x_j))_{i,j=1,\dots,n}}{V(x_1, \dots, x_n)}, \tag{1.4}$$

where $c_\lambda$is a normalizing constant. In Theorem 3.1 we show that the functions P_λ are symmetric polynomials orthogonal in $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$ and that they have L^2 norm given by

$$\|P_\lambda\|^2_{L^2(\mathbb{R}^n,\mu_n)} = c_\lambda^2 n! \prod_{i=1}^{n} \|p_{\lambda_i+n-i}\|^2_{L^2(\mathbb{R},\mu)}.$$

Also we have that the family $\{P_\lambda\}$ is a Hilbert basis of $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$ that can be obtained by the Gram–Schmidt orthogonalization process, applied to the Schur polynomials $\{S_\lambda\}$, ordered in the graded lexicographical order.

Let us denote by H_n the space of $n \times n$ hermitian matrices an by U_n the group of unitary matrices. As said in the introduction, a function $f : H_n \to \mathbb{R}$ is said to be central if $f(UXU^{-1}) = f(X)$ for all $U \in U_n$. Since every hermitian matrix is diagonalizable by an unitary matrix, a central function only depends on its restriction to the space of diagonal matrices; thus, if f is a central function, then

$$\tilde{f}(x_1, \dots, x_n) = f(diag(x_1, \dots, x_n))$$

is a symmetric function en $\mathbb{R}^n$.

The Weyl integration formula ([Far06], p.13, [FK94], Th.VI.2.3) allows us to associate the finite measure μ on $\mathbb{R}^n$ to a central measure M on H_n in the following way

$$\int_{H_n} f(X)dM(X) = \int_{\mathbb{R}^n} \tilde{f}(x_1, \dots, x_n)V^2(x_1, \dots, x_n)d\mu(x_1)\dots d\mu(x_n) = \int_{\mathbb{R}^n} \tilde{f}(\underline{x})d\mu_n(\underline{x}), \tag{1.5}$$

for f a cental function on H_n and where μ_n is as in (1.3). If we denote by $L^2_{U_n}(H_n, M)$ the space of all central functions on $L^2(H_n, M)$, formula (1.5) shows that this space is isomorphic to $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$.

For a symmetric polynomial P on $\mathbb{R}^n$, let $\hat{P}(X)$ be the central function on H_n whose restriction to the diagonal matrices is equal to $P(\underline{x})$. The functions $\hat{P}$ are called (generalized) polynomials of Hermitian matrix argument.

By the definition of the mesure M on H_n, we have that the family of generalized polynomials $\{\hat{P}_\lambda\}$, associated with the family of symmetric polynomials defined in (1.4), form a Hilbert basis of $L^2_{U_n}(H_n, M)$ and for each partition $\lambda = (\lambda_1, \dots, \lambda_n)$ and $X \in H_n$

$$\hat{P}(X) = c_\lambda \frac{\det(p_{\lambda_i+n-i}(x_j))_{i,j=1,\dots,n}}{V(x_1, \dots, x_n)},$$

where $x_1, \dots, x_n$ are the eigenvalues of the matrix X. This result is stated in Corollary 3.1

As said before, the constant c_λ in the definition of P_λ is a normalizing constant and it depends on the normalization chosen for P_λ and p_m. In propositions 3.2 and 3.3 we give an expression of this value for two usually considered normalizations conditions for P_λ and p_m; namely, when the polynomials are monic and when the constant term of the polynomial is one.

Now that we have an orthogonal Hilbert basis of $L^2_{U_n}(H_n, M)$, it is useful to have a method to obtain the expansion of a central function in this basis. In Proposition 3.4 we give the explicit values of the coefficients in this expansion for functions of the form

$$\frac{\det(f_i(x_j))_{1\le i,j\le n}}{V(x_1, \dots, x_n)}$$

where $f_1, \dots, f_n$ are one variable functions. The value of this coefficients are given in terms of the coefficients of the expansion of each one of the functions f_i in terms of the basis $\{p_m\}$. A great numbers of central functions have this form and this proposition is very useful, as will be seen in Chapter 4, since it allows us to obtain a generating functions and a Christoffel-Darboux type formula for the generalized polynomials $\{\hat{P}_\lambda\}$.

Finally, in Section 4 we present as examples the families of generalized polynomials associated to the classical families of orthogonal polynomials; that is to say, the Hermite, Laguerre and Jacobi polynomials.

1.4 About Chapter 4

Chapter 4 includes the preprint [BGU09] accepted for publication, where we construct operators semigroups associated to the families of generalized orthogonal polynomials with hermitian matrix argument. As in the previous chapter, we begin with a measure μ on the real line such that the set of polynomials is dense in $L^2(\mathbb{R},\mu)$ and by applying the Gram–Schmidt ortogonalization process to the monomials, we obtain a family of orthogonal polynomials $\{p_m\}$ on $L^2(\mathbb{R},\mu)$. In this chapter we will fix the normalization $\|p_m\|_{L^2(\mathbb{R},\mu)} = 1$ for all $m \in \mathbb{N}$ and p_m has positive leading coefficient; thus, the family of generalized orthogonal polynomials on $L^2_{\text{sym}}(\mathbb{R}^n,\mu_n)$ associated to $\{p_m\}$ constructed in Theorem 3.1 are given by

$$P_\lambda(\underline{x}) = \frac{1}{\sqrt{n!}}\frac{\det(p_{\lambda_i+n-i}(x_j))}{V(\underline{x})},$$

for $\underline{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$, where

$$\mu_n(d\underline{x}) = V^2(x_1,\ldots,x_n)\mu(dx_1)\ldots\mu(dx_n),$$

the measure on $\mathbb{R}^n$ related to μ (c.f. (1.3) in this chapter).

In Section 4.2 we construct an operator semigroup associated to the generalized polynomials from a Markov generator sequence for the family $\{p_m\}$. This notion was introduced by D. Bakry and O. Mazet in [BM03] and a Markov generator sequence for $\{p_m\}$ is equivalent to the existence of a Markov semigroup with spectral decomposition given by the family of orthogonal polynomials. We say that $\{\gamma_m\}$ is a Markov generator sequence for the family $\{p_m\}$, if for every $t \geq 0$ there exists a Markov operator N_t (that is to say, N_t is conservative: $N_t 1 = 1$ and it preserves positivity: $N_t f \geq 0$ for all $f \geq 0$) on $L^2(\mathbb{R},\mu)$ such that $N_t(p_m) = e^{-\gamma_m t}p_m$. It result that each one of the operators N_t is a contraction with symmetric, and therefore invariant, measure μ.

If we require that $\sum_m e^{-2\gamma_m t} < \infty$ for all $t \geq 0$, then each operator N_t is a Hilbert–Schmidt operator and can be represented as

$$N_t(f)(x) = \int f(y)\mathcal{N}_t(x,y)d\mu(y),$$

where

$$\mathcal{N}_t(x,y) = \sum_{m\in\mathbb{N}} e^{-\gamma_m t}p_m(x)p_m(y).$$

It is not difficult to see that the family of kernels $\{\mathcal{N}_t\}$ satisfies the Chapman–Kolmogorov equations; that is,

$$\int \mathcal{N}_t(x,y)\mathcal{N}_s(y,z)d\mu(y) = \mathcal{N}_{t+s}(x,z);$$

thus, $\{N_t\}$ is a Markov semigroup with invariant measure μ and spectral decomposition over the family of polynomials $\{p_m\}$. A thoughtful study of the Markov generating sequence for the classical families of orthogonal polynomials can be found in [BM03].

From an increasing Markov generator sequence $\{\gamma_m\}$ for $\{p_m\}$, such that $\sum_m e^{-2\gamma_m t} < \infty$ for all $t \geq 0$, we define a family of operators on $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$ related to the generalized polynomials, in the following way: for $f \in L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$, let

$$T_t(f)(\underline{x}) := \int_{\mathbb{R}^n} f(\underline{y}) \mathcal{T}_t(\underline{x}, \underline{y}) \mu_n(dy),$$

where

$$\mathcal{T}_t(\underline{x}, \underline{y}) := \sum_\lambda e^{-t(\sum_{j=1}^n \gamma_{\lambda_j+n-j} - \sum_{j=1}^n \gamma_{n-j})} P_\lambda(\underline{x}) P_\lambda(\underline{y}).$$

In Theorem 4.1 we prove that the family of operators $\{T_t\}$ is a conservative semigroup on $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$ with symmetric and invariant measure μ_n and that for each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ and each $t \geq 0$, the polinomial P_λ is an eigenfunction of T_t with eigenvalue $e^{-\varphi_\lambda t}$, where

$$\varphi_\lambda = \sum_{j=1}^n \gamma_{\lambda_j+n-j} - \sum_{j=1}^n \gamma_{n-j} \geq 0. \tag{1.6}$$

Once we have defined a semigroup associated to the polynomials $\{P_\lambda\}$ on $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$, we wish to do the same for the family of associated generalized polynomials on $L^2_{U_n}(H_n, M)$. For an operator T on $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$, let $\hat{T}$ be the operator on $L^2_{U_n}(H_n, M)$ such that for every $h \in L^2_{U_n}(H_n, M)$,

$$\hat{T}h\big|_{Diag} = T(h|_{Diag}) = T(\tilde{h}),$$

where $Diag$ stands for the space of all $n \times n$ diagonal matrices. This condition determines uniquely the operator $\hat{T}$.

Now, if $\hat{T}_t$ is the operator on $L^2_{U_n}(H_n, M)$ determined by the operator T_t on $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$, as a corollary to Theorem 4.1, we have that the family of operators $\{\hat{T}_t\}$ is a conservative semigroup on $L^2_{U_n}(H_n, M)$ and that for each $t \geq 0$, the generalized polynomial $\hat{P}_\lambda$ is an eigenfunction of $\hat{T}_t$ with eigenvalue $e^{-\varphi_\lambda t}$, with φ_λ given in (1.6). The measure M is the symmetric and invariant measure for this semigroup and we have the representation

$$\hat{T}h(X) = \int_{H_n} h(Y) \hat{\mathcal{T}}(X, Y) dM(Y),$$

where

$$\hat{\mathcal{T}}(X, Y) = \sum_\lambda e^{-t\varphi_\lambda} \hat{P}_\lambda(X) \hat{P}_\lambda(Y).$$

Now that we have defined a semigroup associated to the generalized polynomials we are interested in characterizing its infinitesimal generator. We deal with this in Section 4.3.

Let us denote by L the infinitesimal generator of the semigroup $\{T_t\}$ on $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$; that is,

$$Lf = \lim_{t \to 0} \frac{N_t f - f}{t},$$

on a dense subset $\mathcal{D}(L)$ of $L^2(\mathbb{R}, \mu)$. Then $L(p_m) = -\gamma_m p_m$, and thus L is a symmetric operator in $L^2(\mathbb{R}, \mu)$ with spectral decomposition over the family $\{p_m\}$. The invariance of the measure μ for $\{N_t\}$ can be expressed in terms of the operator L as $\int L f d\mu = 0$ for all f in $\mathcal{D}(L)$.

Let L_k be the operator on $L^2(\mathbb{R}^n, \mu_n)$ that acts as L on the $k-$th coordinate. For example, if $L = \frac{d}{dx}$, then $L_k = \frac{\partial}{\partial x_k}$. For a fixed symmetric polynomial q on $\mathbb{R}^n$, let D_q be the operator on $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$ defined by

$$D_q = \frac{1}{V} q(L_1, \ldots, L_n) V.$$

In Proposition 4.2 we prove that for each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ the polynomial P_λ is an eigenfunction of D_q with eigenvalue $q(-\gamma_{\lambda_1+n-1}, -\gamma_{\lambda_2+n-2}, \ldots, -\gamma_{\lambda_n})$, that is,

$$D_q P_\lambda = q(-\gamma_{\lambda_1+n-1}, -\gamma_{\lambda_2+n-2}, \ldots, -\gamma_{\lambda_n}) P_\lambda.$$

In particular, we consider the symmetric polynomial

$$q_0(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n + c$$

where $c = \sum_{j=1}^{n} \gamma_{n-j}$, a positive constant. Let us denote by

$$D := D_{q_0} = \frac{1}{V}(L_1 + \cdots + L_n)V + c.$$

In Theorem 4.2 we show that this operator is the infinitesimal generator of the semigroup $\{T_t\}$. This is a consequence of Proposition 4.2, that tells us that

$$DP_\lambda = -(\sum_{j=1}^{n} \gamma_{\lambda_j+n-j} - c)P_\lambda = -\varphi_\lambda P_\lambda, \quad \text{c.f (1.6)};$$

that is, the operator D and the infinitesimal generator of the semigroup $\{T_t\}$ have the same spectral decomposition. By spectral theory of semigroups and the density of the polynomials P_λ on $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$, we have that D is indeed the infinitesimal generator of the semigroup $\{T_t\}$. It is straightforward that the associated operator $\hat{D}$ on $L^2_{U_n}(H_n, M)$ is the infinitesimal generator of the semigroup $\{\hat{T}_t\}$.

We started with a Markov semigroup $\{N_t\}$ on $L^2(\mathbb{R}, \mu)$ associated to the orthogonal polynomials $\{p_m\}$ and constructed semigroups on $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$ and $L^2_{U_n}(H_n, M)$. We know that these semigroups are conservative, but, the question we want to answer now is whether or not they are also Markov semigroups, that is, if they also preserve positivity. In Section 4.4 we study the positive preserving property for these semigroups, under the hypothesis of diffusion. We use a characterization of Markov semigroups with invariant measure, given by O. Mazet in [Maz02], that involves the "carré du champ" operator of the infinitesimal generator of the semigroup.

For the infinitesimal generator L of the semigroup $\{N_t\}$, the carré du champ operator of L is the symmetric bilinear form defined by

$$\Gamma(f,g) = \frac{1}{2}(L(fg) - fLg - gLf),\ f,g \in \mathcal{A},$$

where $\mathcal{A}$ is an "standard algebra" in $\mathcal{D}(L)$ (see [ABC$^+$02] or [Bak06]). In our setting we can and will take $\mathcal{A}$ as the algebra of polynomials.

It is known that the carré du champ operator of the infinitesimal generator of a Markov semigroup is positive, in the sense that $\Gamma(f,f) \geq 0$ for all $f \in \mathcal{A}$. The converse implication is, in general, not true. But O. Mazet [Maz02] proved that if μ is the invariant measure of the semigroup $\{N_t\}$, then the positivity of the carré du champ operator does implies that the semigroup is Markov. Since this is our case, we shall use this result.

We consider the case when the infinitesimal generator L of the semigroup $\{N_t\}$ is a diffusion. In subsection 4.4.1 we treat the continuous case, that is, when μ is a non–atomic measure. In this case the operator L takes the form

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx},$$

where a and b are polynomials of degree at most 2 and 1, respectively. It is known that the Markov semigroup on $\mathbb{R}$ such that its infinitesimal generator is a diffusion and that have a family of orthogonal polynomials as eigenfunctions are the Ornstein-Uhlenbeck, Laguerre and Jacobi semigroups, associated to the Hermite, Laguerre and Jacobi polynomials, respectively (c.f. [Maz97]). This are the families of polynomials we are considering in this case.

In Proposition 4.3 we obtain an expression of the carré du champ of the operator D on $L^2_{\text{sym}}(\mathbb{R},\mu_n)$ in terms of the carré du champ of L, the infinitesimal generator of the semigroup on $L^2(\mathbb{R},\mu)$. By means of Mazet's result [Maz02], in Theorem 4.3 we show that in this case the semigroup $\{T_t\}$ on $L^2_{\text{sym}}(\mathbb{R},\mu_n)$ is indeed Markov. Also, in Proposition 4.4 we give an explicit expression of the operator D that coincides with the operators announced by Lassalle in [Las91a, Las91b, Las91c].

In subsection 4.4.2 we treat the discrete case, that is, when μ is a purely atomic measure. In this case that the infinitesimal generator of the semigroup $\{N_t\}$ on $L^2_{\text{sym}}(\mathbb{R},\mu_n)$ is a discrete diffusion means that

$$L = \sigma(x)\Delta\nabla + \tau(x)\Delta, \tag{1.7}$$

where σ and τ are polynomials of degree at most two one, respectively and Δ, ∇ are the difference operators

$$\begin{aligned} \Delta f(x) &= f(x+1) - f(x), \\ \nabla f(x) &= f(x) - f(x-1). \end{aligned}$$

In [MSU91] it is proven that the only families of discrete orthogonal polynomials that are eigenfunctions of operators of the form (1.7) are the Charlier, Meixner, Kravchuk and Hahn polynomials. Using Mazet's result [Maz02] it is not difficult to prove that the semigroups associated with these polynomials are Markov; thus, these are the families of polynomials we are considering in this case.

In Proposition 4.6 we give an expression of the carré du champ operator of D and in Theorem 4.4 we show that the semigroup $\{T_t\}$ on $L^2_{\text{sym}}(\mathbb{R}, \mu_n)$ in this case is also Markov. In Proposition 4.7 we give an explicit expression of the operador D.

In Section 4.5 we study the kernels $\mathcal{T}_t$ that define the operators of the semigroup $\{T_t\}$ on $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$. Using Proposition 3.4 (Chapter 3) we show in Theorem 4.5 that it is possible to express this kernels in terms of the kernels $\mathcal{N}_t$ that define the operators of the semigroup $\{N_t\}$ on $L^2(\mathbb{R}, \mu)$; explicitly we have

$$\mathcal{T}_t(\underline{x}, \underline{y}) = e^{t\sum_{j=1}^{n} \gamma_{n-j}} \frac{\det(\mathcal{N}_t(x_j, y_i))_{i,j}}{n! V(\underline{x}) V(\underline{y})}.$$

In subsections 4.5.1 and 4.5.2 we give a probabilistic view of the problems we have treated.

In Section 4.6 we obtain some classical formulas of orthogonal polynomials for the generalized orthogonal polynomials from the corresponding formulas for the polynomials on $\mathbb{R}$. Specifically, we obtain a Christoffel-Darboux type formula and a generating function for the generalized polynomials. We stress that this formulas are obtained as applications of Proposition 3.4 of Chapter 3.

Finally, in Section 4.7 we present as examples the Markov semigroups associated to the generalized polynomials corresponding to the classical families of orthogonal polynomials on $\mathbb{R}$, both of continuous and discrete variable.

1.5 Open problems

The results presented in this thesis open a wide range of research areas for the generalized polynomials such as the systematic study of orthogonal polynomials properties (distribution of zeros, asymptotic behavior).

A very interesting problem, from our point of view, is the study of expansions in terms of the generalized polynomials and the extensions of some classical notions of harmonic analysis to this context, such as maximal functions and singular integrals.

Another interesting question is whether it is possible to obtain a Meyer's multiplier theorem for expansions in terms of the generalized polynomials or not. Once a result of this type is known, it would be natural to study Riesz and Bessel potentials, fractional derivatives and Sobolev spaces in this context. A key result for the study of these notions in the classical cases on $\mathbb{R}$ is the hypercontractive property of the semigroup associated with the orthogonal polynomials. A very interesting question is if it is possible to obtain the hypercontractive property for the semigroup associated to the generalized polynomials from the hypercontractivity of the corresponding semigroups on $\mathbb{R}$.

Chapter 2

Fractional integration and fractional differentiation for d-dimensional Jacobi expansions

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Abstract: We define and study fractional integrals (i.e., Riesz potentials), Bessel potentials, and fractional derivatives, arising from multivariate expansions in Jacobi polynomials. One of our key results is an analogue of Meyer's multiplier theorem, for Jacobi polynomials rather than Hermite ones. To obtain it, we employ a novel orthogonal decomposition of the L^2 space associated to the multidimensional Jacobi measure. Using fractional derivatives, we obtain a characterization of the Sobolev spaces for the Jacobi measure. Studies have been conducted of the potentials and fractional derivatives arising from multivariate Hermite and Laguerre expansions. This paper completes the study of these notions for the classical families of orthogonal polynomials.

Key words: Fractional integration, fractional differentiation, multivariate Jacobi expansion, multiplier theorem, Riesz potential, Bessel potential, Sobolev space.

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2.1 Introduction

Let us consider $\mu^d_{\alpha,\beta}$, the (normalized) Jacobi measure on $[-1,1]^d$ with parameters $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $\beta = (\beta_1, \beta_2, \dots, \beta_d)$ in $\mathbb{R}^d$, satisfying $\alpha_i, \beta_i > -1$. It is defined by

$$\mu^d_{\alpha,\beta}(dx) = \prod_{i=1}^{d} \left\{ \frac{1}{2^{\alpha_i+\beta_i+1}\, B(\alpha_i+1, \beta_i+1)} \, (1-x_i)^{\alpha_i}(1+x_i)^{\beta_i}\, dx_i \right\}, \tag{2.1}$$

where B is the Euler beta function. Let us also consider $\mathcal{L}^{\alpha,\beta}$, the d-dimensional Jacobi operator,

$$\mathcal{L}^{\alpha,\beta} := \sum_{i=1}^{d} \left[(1-x_i^2)\frac{\partial^2}{\partial x_i^2} + \left(\beta_i - \alpha_i - (\alpha_i+\beta_i+2)\, x_i\right)\frac{\partial}{\partial x_i} \right]. \tag{2.2}$$

It is not difficult to see that $\mathcal{L}^{\alpha,\beta}$ is a formally symmetric operator on the space $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$.

For a multi-index $\kappa = (\kappa_1, \dots, \kappa_d) \in \mathbb{N}^d$, let $\vec{p}_\kappa^{\,\alpha,\beta}$ be the normalized Jacobi polynomial of order κ in $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$, defined on $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ by

$$\vec{p}_\kappa^{\,\alpha,\beta}(x) = \prod_{i=1}^{d} p_{\kappa_i}^{\alpha_i,\beta_i}(x_i),$$

where $p_n^{\alpha,\beta}$, for $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > -1$, is the normalized Jacobi polynomial of degree n in $\mathbb{R}$ that can be defined using the Rodrigues formula [Sz59]. That is,

$$p_n^{\alpha,\beta}(x) = c_n (1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\right\}, \qquad x \in (-1,1).$$

Here c_n is chosen so that $p_n^{\alpha,\beta}$ has unit norm in $L^2(\mathbb{R}, \mu_{\alpha,\beta})$. Since the Jacobi polynomials $\{p_n^{\alpha,\beta}\}$ on $\mathbb{R}$ are orthogonal with respect to the Jacobi measure on $[-1,1]$, the normalized Jacobi polynomials $\{\vec{p}_\kappa^{\,\alpha,\beta}\}$ are orthonormal with respect to $\mu^d_{\alpha,\beta}(dx)$. Moreover, $\{\vec{p}_\kappa^{\,\alpha,\beta}\}$ is an orthonormal basis of the Hilbert space $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$.

It is well known that the Jacobi polynomials are eigenfunctions of the Jacobi operator $\mathcal{L}^{\alpha,\beta}$ with eigenvalues $-\lambda_\kappa = -\sum_{i=1}^{d}\kappa_i(\kappa_i+\alpha_i+\beta_i+1)$; that is,

$$\mathcal{L}^{\alpha,\beta}\vec{p}_\kappa^{\,\alpha,\beta} = -\lambda_\kappa \vec{p}_\kappa^{\,\alpha,\beta}. \tag{2.3}$$

The d-dimensional Jacobi semigroup $\{T_t^{\alpha,\beta}\}_{t\geq 0}$ is defined as the Markov semigroup associated to the Markov probability kernels [Bal06, Bak06]

$$P^{\alpha,\beta}(t,x,dy) = \sum_{\kappa\in\mathbb{N}^d} e^{-\lambda_\kappa t}\vec{p}_\kappa^{\,\alpha,\beta}(x)\vec{p}_\kappa^{\,\alpha,\beta}(y)\mu^d_{\alpha,\beta}(dy) =: p_d^{\alpha,\beta}(t,x,y)\mu^d_{\alpha,\beta}(dy).$$

That is,

$$T_t^{\alpha,\beta} f(x) := \int_{[-1,1]^d} f(y)\, P^{\alpha,\beta}(t,x,dy) = \int_{[-1,1]^d} f(y)\, p_d^{\alpha,\beta}(t,x,y)\mu^d_{\alpha,\beta}(dy).$$

Unfortunately, no reasonable representation for the kernel is known, but such a representation will not be needed in what follows. The d-dimensional Jacobi semigroup $\{T_t^{\alpha,\beta}\}_{t\geq 0}$ is a Markov diffusion semigroup, strongly continuous on $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$, with infinitesimal generator $-\mathcal{L}^{\alpha,\beta}$. Each of its operators is symmetric and is a contraction on $\mathcal{L}^p$. By (2.3), for all $t \geq 0$,

$$T_t^{\alpha,\beta} \vec{p}_\kappa^{\,\alpha,\beta} = e^{-\lambda_\kappa t} \vec{p}_\kappa^{\,\alpha,\beta}. \tag{2.4}$$

It can be proved that for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{R}^d$ with $\alpha_i, \beta_i \geq -\frac{1}{2}$, $\{T_t^{\alpha,\beta}\}_{t\geq 0}$ is not merely a contraction on $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$, but is *hypercontractive*, that is to say, for any $1 < p < \infty$ there exists an increasing function $q = q_{\alpha,\beta} \colon \mathbb{R}^+ \to [p, \infty)$, with $q(0) = p$, such that for every f and all $t \geq 0$,

$$\|T_t^{\alpha,\beta} f\|_{q(t)} \leq \|f\|_p \, .$$

The proof of this fact is indirect and is not very well known. It is based on the fact that the one-dimensional Jacobi operator satisfies a Sobolev inequality, which is proved by checking that it satisfies a curvature-dimension inequality. (This result was obtained by D. Bakry [Bak96].) This yields a logarithmic Sobolev inequality for the one-dimensional Jacobi operator. As this inequality is stable under tensorization [ABC$^+$02], the d-dimensional Jacobi operator also satisfies a logarithmic Sobolev inequality; and using L. Gross's famous result [Gro75], which asserts the equivalence between the hypercontractivity property and the validity of a logarithmic Sobolev inequality, the result is obtained. All the implications between these functional inequalities and L. Gross's result can be found in [ABC$^+$02]. A detailed proof of the hypercontractivity property for the Jacobi semigroup can be found in [Bal06] (see also [Bak96]).

From now on, we shall consider only the Jacobi semigroups with parameters $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{R}^d$ satisfying $\alpha_i, \beta_i \geq -\frac{1}{2}$.

For $0 < \delta \leq 1$, the generalized d-dimensional Poisson–Jacobi semigroup of order δ, $\{P_t^{\alpha,\beta,\delta}\}$, is defined by

$$P_t^{\alpha,\beta,\delta} f(x) := \int_0^\infty T_s^{\alpha,\beta} f(x)\, \mu_t^\delta(ds), \tag{2.5}$$

where $\{\mu_t^\delta\}$ are the stable measures on $[0,\infty)$ of order δ. The generalized d-dimensional Poisson–Jacobi semigroup of order δ is a strongly continuous semigroup on $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$ with infinitesimal generator $(-\mathcal{L}^{\alpha,\beta})^\delta$. By (2.3), we have that

$$P_t^{\alpha,\beta,\delta} \vec{p}_\kappa^{\,\alpha,\beta} = e^{-\lambda_\kappa^\delta t} \vec{p}_\kappa^{\,\alpha,\beta}. \tag{2.6}$$

In particular, for $\delta = 1/2$ we obtain the d-dimensional Poisson–Jacobi semigroup $P_t^{\alpha,\beta,1/2}$, which will simply be written as $P_t^{\alpha,\beta}$. We can explicitly compute $\mu_t^{1/2}$, by

$$\mu_t^{1/2}(ds) = \frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2} ds,$$

and we have Bochner's subordination formula,

$$P_t^{\alpha,\beta} f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}}\, T_{t^2/4u}^{\alpha,\beta} f(x)\, du. \tag{2.7}$$

Measures and semigroups associated to other classical families of orthogonal polynomials, and to the corresponding expansions, have been studied previously. In [FSU01], Riesz and Bessel potentials were studied for the Gaussian measure associated to the family of Hermite polynomials; in [LU04], fractional derivatives for the Gaussian measure were studied, and a characterizations of the Gaussian Sobolev spaces was obtained. (See also [Urb98], for an extensive survey of Gaussian harmonic analysis.) In [GL$^+$05], semigroups associated to Laguerre polynomial expansions were studied, yielding an analogue for Laguerre expansions of P. A. Meyer's multiplier theorem. Fractional derivatives and fractional integrals were introduced in that setting, and various Sobolev spaces associated to Laguerre expansions, and also higher-order Riesz–Laguerre transforms, were studied.

In this paper, in which we focus on the Jacobi case, we complete the study of these notions related to the classical families of orthogonal polynomials. The $d = 1$ case, of univariate Jacobi polynomials, was treated in [BU07], and in the present article we extend our treatment to higher dimensions ($d > 1$). Due to the nonlinearity of the eigenvalues of the Jacobi operator, the way of obtaining the d-dimensional case from the one-dimensional one differs between the Hermite and Laguerre cases on the one hand, and the Jacobi case on the other.

The paper is organized as follows. In the next section we give an alternative decomposition of the space $L^2([-1,1]^d, \mu_{\alpha,\beta}^d)$, which we call a modified Wiener–Jacobi decomposition. In section 2.3, using that decomposition and the hypercontractivity property of the d-dimensional Jacobi semigroup, we obtain an analogue for d-dimensional Jacobi expansions of Meyer's multiplier theorem [Mey84], and define and study, as in the one-dimensional case [BU07], the fractional derivatives, the fractional integrals, and the Bessel potentials for the Jacobi operator, and the Sobolev spaces associated to the Jacobi measure.

To simplify notation, we shall not always make explicit the dependence on the dimension d. As usual, C will denote a constant; not necessarily the same in each occurrence. The symbol $\mathcal{P}$ will denote the set of polynomials with real coefficients.

2.2 A modified Wiener–Jacobi decomposition

Let us consider for each $n \geq 0$, the closed subspace $C_n^{\alpha,\beta}$ of $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$ generated by linear combinations of $\{\vec{p}_\kappa^{\,\alpha,\beta} \colon |\kappa| = n\}$, where, as usual for a multi-index κ, $|\kappa| = \sum_{i=1}^d \kappa_i$. Since $\{\vec{p}_\kappa^{\,\alpha,\beta}\}$ is an orthonormal basis of $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$, we have the orthogonal decomposition

$$L^2([-1,1]^d, \mu^d_{\alpha,\beta}) = \bigoplus_{n=0}^{\infty} C_n^{\alpha,\beta}. \tag{2.8}$$

This is the Wiener–Jacobi decomposition of $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$, which is analogous to the Wiener decomposition of $L^2(\mathbb{R}^d, \gamma_d)$ in the Gaussian case.

For $f \in L^2([-1,1]^d, \mu^d_{\alpha,\beta})$, the expansion of f in Jacobi polynomials is given by

$$f = \sum_{n=0}^{\infty} \sum_{|\kappa|=n} \hat{f}(\kappa) \vec{p}_\kappa^{\,\alpha,\beta},$$

with $\hat{f}(\kappa) = \int_{[-1,1]^d} f(y) \vec{p}_\kappa^{\,\alpha,\beta}(y) \mu^d_{\alpha,\beta}(dy)$, the Jacobi–Fourier coefficient of f for the multi-index κ. This yields the spectral decompositions

$$\mathcal{L}^{\alpha,\beta} f = \sum_{n=0}^{\infty} \sum_{|\kappa|=n} (-\lambda_\kappa) \hat{f}(\kappa) \vec{p}_\kappa^{\,\alpha,\beta}, \qquad T_t^{\alpha,\beta} f = \sum_{n=0}^{\infty} \sum_{|\kappa|=n} e^{-\lambda_\kappa t} \hat{f}(\kappa) \vec{p}_\kappa^{\,\alpha,\beta},$$

and

$$P_t^{\alpha,\beta,\delta} f = \sum_{n=0}^{\infty} \sum_{|\kappa|=n} e^{-\lambda_\kappa^\delta t} \hat{f}(\kappa) \vec{p}_\kappa^{\,\alpha,\beta}.$$

As the eigenvalues λ_κ of the d-dimensional Jacobi operator do not depend linearly on $|\kappa|$, we do not have an expression for the action of $\mathcal{L}^{\alpha,\beta}$, $T_t^{\alpha,\beta}$ or $P_t^{\alpha,\beta}$ on f, in terms of the orthogonal projections on the subspaces $C_n^{\alpha,\beta}$, as in the one-dimensional case (see [BU07]), or as in the case of d-dimensional expansions in Hermite or Laguerre polynomials (see [GL$^+$05, LU04]). For this reason, we are going to consider, in the same spirit as the Wiener–Jacobi decomposition, an alternative decomposition of $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$, so as to obtain expressions for $\mathcal{L}^{\alpha,\beta} f$, $T_t^{\alpha,\beta} f$, and $P_t^{\alpha,\beta} f$ in terms of orthogonal projections.

For fixed $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$ in $\mathbb{R}^d$, such that $\alpha_i, \beta_i > -\frac{1}{2}$, consider the set

$$R^{\alpha,\beta} = \left\{ r \in \mathbb{R}^+ \colon \exists\, (\kappa_1, \ldots, \kappa_n) \in \mathbb{N}^d, \text{ with } r = \sum_{i=1}^d \kappa_i(\kappa_i + \alpha_i + \beta_i + 1) \right\}.$$

$R^{\alpha,\beta}$ is a countable subset of $\mathbb{R}^+$, thus it can be written as $R^{\alpha,\beta} = \{r_n\}_{n=0}^{\infty}$ with $r_0 < r_1 < \ldots$. Let

$$A_n^{\alpha,\beta} = \left\{ \kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}^d \colon \sum_{i=1}^d \kappa_i(\kappa_i + \alpha_i + \beta_i + 1) = r_n \right\}.$$

Note that $A_0^{\alpha,\beta} = \{(0,\ldots,0)\}$, and that if $\kappa \in A_n^{\alpha,\beta}$, then it is the case that $\lambda_\kappa = \sum_{i=1}^d \kappa_i(\kappa_i + \alpha_i + \beta_i + 1) = r_n$.

Let $G_n^{\alpha,\beta}$ denote the closed subspace of $L^2([-1,1]^d, \mu_{\alpha,\beta}^d)$ generated by the linear combinations of $\{\vec{p}_\kappa^{\,\alpha,\beta} \colon \kappa \in A_n^{\alpha,\beta}\}$. By the orthogonality of the Jacobi polynomials with respect to $\mu_{\alpha,\beta}^d$, and the density of the polynomials, it is not difficult to see that $\{G_n^{\alpha,\beta}\}$ is an orthogonal decomposition of $L^2([-1,1]^d, \mu_{\alpha,\beta}^d)$, that is,

$$L^2([-1,1]^d, \mu_{\alpha,\beta}^d) = \bigoplus_{n=0}^{\infty} G_n^{\alpha,\beta}, \tag{2.9}$$

which we shall call a modified Wiener–Jacobi decomposition.

Let us denote by $J_n^{\alpha,\beta}$ the orthogonal projection of $L^2([-1,1]^d, \mu_{\alpha,\beta}^d)$ onto $G_n^{\alpha,\beta}$. Then, for $f \in L^2([-1,1]^d, \mu_{\alpha,\beta}^d)$ the Jacobi expansion of f can now be written as

$$f = \sum_{n=0}^{\infty} J_n^{\alpha,\beta} f, \qquad \text{where} \quad J_n^{\alpha,\beta} f = \sum_{\kappa \in A_n^{\alpha,\beta}} \hat{f}(\kappa) \vec{p}_\kappa^{\,\alpha,\beta}. \tag{2.10}$$

By (2.3), (2.4), (2.6), we have that for $f \in L^2([-1,1]^d, \mu_{\alpha,\beta}^d)$ with Jacobi expansion $f = \sum_{n=0}^\infty J_n^{\alpha,\beta} f$, the actions of $\mathcal{L}^{\alpha,\beta}, T_t^{\alpha,\beta}, P_t^{\alpha,\beta}$ on f are

$$\mathcal{L}^{\alpha,\beta} f = \sum_{n=0}^{\infty} (-r_n) J_n^{\alpha,\beta} f, \quad T_t^{\alpha,\beta} f = \sum_{n=0}^{\infty} e^{-r_n t} J_n^{\alpha,\beta} f \quad P_t^{\alpha,\beta,\delta} f = \sum_{n=0}^{\infty} e^{-r_n^\delta t} J_n^{\alpha,\beta} f.$$

Thus, using the modified Wiener–Jacobi decomposition (2.9) we are able to obtain expansions of $\mathcal{L}^{\alpha,\beta}$, $T_t^{\alpha,\beta}$ and $P_t^{\alpha,\beta,\delta}$ in terms of the orthogonal projections $J_n^{\alpha,\beta}$. As we have mentioned, this cannot be done with the usual Wiener–Jacobi decomposition (2.8).

With this decomposition of $L^2([-1,1]^d, \mu_{\alpha,\beta}^d)$, the proofs of our results are essentially similar to the ones in the one-dimensional Jacobi case [BU07], taking r_n instead of λ_n, and to the Hermite and Laguerre cases [LU04, Urb98, GL$^+$05], taking r_n instead of n. In order to make this article as self-contained as possible, in what follows we are going to give complete proofs of the most important results.

As a consequence of the hypercontractive property of the d-dimensional Jacobi operator, we have that the orthogonal projections $J_n^{\alpha,\beta}$ can be extended continuously to $L^p([-1,1]^d, \mu_{\alpha,\beta}^d)$; or more formally, we have

Proposition 2.1 *If $1 < p < \infty$ then for every $n \in \mathbb{N}$, $J_n^{\alpha,\beta}$, restricted to $\mathcal{P}$, can be extended to a continuous operator on $L^p([-1,1]^d, \mu_{\alpha,\beta}^d)$, which will also be denoted by $J_n^{\alpha,\beta}$. That is, there exists $C_{n,p} \in \mathbb{R}^+$ such that* *for* $f \in L^p([-1,1]^d, \mu^d_{\alpha,\beta})$,

$$\|J_n^{\alpha,\beta} f\|_p \leq C_{n,p}\|f\|_p.$$

Proof First, we note that $\mathcal{P}$ is dense in $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$ for any $1 \leq p < \infty$. (See [Sz59, Thm. 1.5.1].) Now let us consider the case $p > 2$. Since the Jacobi semigroup is hypercontractive, for the initial condition $q(0) = 2$, let t_0 be a positive number such that $q(t_0) = p$. Taking $f \in \mathcal{P}$, by the hypercontractive property, Parseval's identity, and Hölder's inequality, we obtain

$$\|T_{t_0}^{\alpha,\beta} J_n^{\alpha,\beta} f\|_p \leq \|J_n^{\alpha,\beta} f\|_2 \leq \|f\|_2 \leq \|f\|_p.$$

Now, since $T_{t_0}^{\alpha,\beta} J_n^{\alpha,\beta} f = e^{-t_0 r_n} J_n^{\alpha,\beta} f$ we get

$$\|J_n^{\alpha,\beta} f\|_p \leq C_{n,p}\|f\|_p,$$

with $C_{n,p} = e^{t_0 r_n}$. The general result now follows by density. Finally, for $1 < p < 2$ the result follows by duality. □

2.3 The results

For any $\Phi\colon \mathbb{N} \to \mathbb{R}$, the multiplier operator T_Φ associated to Φ is defined by

$$T_\Phi f := \sum_{n=0}^{\infty} \Phi(n) J_n^{\alpha,\beta} f, \qquad \text{for} \quad f = \sum_{n=0}^{\infty} J_n^{\alpha,\beta} f \in \mathcal{P}. \tag{2.11}$$

If Φ is a bounded function, then by Parseval's identity it is immediate that the operator T_Φ is bounded on $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$. In the case of Hermite expansions, the multiplier theorem of P. A. Meyer [Mey84] gives conditions on Φ under which the multiplier T_Φ can be extended to a continuous operator on L^p for $p \neq 2$. To establish an analogous result in this case, we need some previous results. First we note that for $n \in \mathbb{N}$, $r_n \geq n$. Then, as a consequence of the L^p continuity of the projections $J_n^{\alpha,\beta}$ and of the hypercontractivity of the d-dimensional Jacobi semigroup, we have

Lemma 2.1 *Let* $1 < p < \infty$*. Then, for each* $m \in \mathbb{N}$ *there exists a constant* C_m *such that for* $f \in L^p([-1,1]^d, \mu^d_{\alpha,\beta})$,

$$\|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - J_1^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p \leq C_m e^{-tm}\|f\|_p.$$

Proof Let $p > 2$ and for the initial condition $q(0) = 2$, let t_0 be a positive number such that $q(t_0) = p$.

If $t \le t_0$, since $T_t^{\alpha,\beta}$ is a contraction, using Proposition 2.1 we get

$$\|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p \le \|(I - J_0^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p$$
$$\le \|f\|_p + \sum_{n=0}^{m-1} \|J_n^{\alpha,\beta} f\|_p \le (1 + \sum_{n=0}^{m-1} e^{t_0 r_n})\|f\|_p.$$

But since $e^{t_0 r_n} \le e^{t_0 r_m}$ for all $0 \le n \le m-1$ and $r_m \ge m$ for all $m \ge 1$, we get

$$\|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p \le (1 + m e^{t_0 r_m})\|f\|_p \le C_m e^{-tm}\|f\|_p,$$

with $C_m = (1 + m e^{t_0 r_m}) e^{t_0 m}$.

Now, suppose $t > t_0$. For $f = \sum_{n=0}^{\infty} J_n^{\alpha,\beta} f$, by the hypercontractive property,

$$\|T_{t_0}^{\alpha,\beta} T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p^2 \le \|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_2^2$$
$$= \|\sum_{n=m}^{\infty} e^{-tr_n} J_n^{\alpha,\beta} f\|_2^2 = \sum_{n=m}^{\infty} e^{-2tr_n}\|J_n^{\alpha,\beta} f\|_2^2 \le \sum_{n=m}^{\infty} e^{-2tn}\|J_n^{\alpha,\beta} f\|_2^2,$$

as $r_n \ge n$ for all $n \ge 1$. Then, as $m \le n$,

$$\sum_{n=m}^{\infty} e^{-2tn}\|J_n^{\alpha,\beta} f\|_2^2 \le e^{-2tm} \sum_{n=0}^{\infty} \|J_n^{\alpha,\beta} f\|_2^2 = e^{-2tm}\|f\|_2^2 \le e^{-2tm}\|f\|_p^2.$$

Thus

$$\|T_{t_0}^{\alpha,\beta} T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - J_1^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p \le e^{-tm}\|f\|_p,$$

and therefore,

$$\|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p = \|T_{t_0}^{\alpha,\beta} T_{t-t_0}^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p$$
$$\le e^{-(t-t_0)m}\|f\|_p = C_m e^{-tm}\|f\|_p,$$

with $C_m = e^{t_0 m}$. For $1 < p < 2$ the result follows by duality. □

Using (2.5) and Minkowski's integral inequality, it is not difficult to see an analogous result for the generalized Poisson–Jacobi semigroup; that is, for $1 < p < \infty$ and each $m \in \mathbb{N}$, there exists C_m such that

$$\|P_t^{\alpha,\beta,\delta}(I - J_0^{\alpha,\beta} - J_1^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p \le C_m e^{-tm^{\delta}}\|f\|_p. \tag{2.12}$$

From the generalized Poisson–Jacobi semigroup we define a new family of operators $\{P_{k,\delta,m}^{\alpha,\beta}\}_{k\in\mathbb{N}}$ by the formula

$$P_{k,\delta,m}^{\alpha,\beta} f = \frac{1}{(k-1)!}\int_0^{\infty} t^{k-1} P_t^{\alpha,\beta,\delta}(I - J_0^{\alpha,\beta} - J_1^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\, dt.$$

By the preceding lemma and by Minkowski's integral inequality, we have the L^p-continuity of $P^{\alpha,\beta}_{k,\delta,m}$ for every $m \in \mathbb{N}$; that is to say, for $1 < p < \infty$ there is a constant C_m such that

$$\|P^{\alpha,\beta}_{k,\delta,m} f\|_p \leq \frac{C_m}{m^{\delta k}} \|f\|_p. \tag{2.13}$$

In particular, if we take $n \geq m$ and $\kappa \in A^{\alpha,\beta}_n$, then

$$P^{\alpha,\beta,\delta}_t (I - J^{\alpha,\beta}_0 - J^{\alpha,\beta}_1 - \cdots - J^{\alpha,\beta}_{m-1}) \vec{p}_\kappa^{\,\alpha,\beta} = e^{-r_n^\delta t} \vec{p}_\kappa^{\,\alpha,\beta}.$$

Thus, for all $k \in \mathbb{N}$,

$$P^{\alpha,\beta}_{k,\delta,m} \vec{p}_\kappa^{\,\alpha,\beta} = \frac{1}{r_n^{\delta k}} \vec{p}_\kappa^{\,\alpha,\beta}.$$

Therefore, for $f \in L^2([-1,1]^d, \mu^d_{\alpha,\beta})$ and $k \in A^{\alpha,\beta}_n$,

$$P^{\alpha,\beta}_{k,\delta,m} J^{\alpha,\beta}_n f = \begin{cases} \frac{1}{r_n^{\delta k}} J^{\alpha,\beta}_n f, & n \geq m; \\ 0, & n < m. \end{cases} \tag{2.14}$$

We are now ready to establish the multiplier theorem for d-dimensional Jacobi expansions. Our proof closely follows Watanabe's proof in the Hermite case [Wat84].

Theorem 2.1 *If for some $n_0 \in \mathbb{N}$ and $0 < \delta \leq 1$, $\Phi(k) = h\left(\frac{1}{r_k^\delta}\right)$, $k \geq n_0$, with h an analytic function in a neighborhood of zero, then T_Φ, the multiplier operator associated to Φ by* (2.11)*, admits a continuous extension to $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$.*

Proof Let $T_\Phi f = T^1_\Phi f + T^2_\Phi f = \sum_{k=0}^{n_0-1} \Phi(k) J^{\alpha,\beta}_k f + \sum_{k=n_0}^{\infty} \Phi(k) J^{\alpha,\beta}_k f$. By Lemma 2.1 we have that

$$\|T^1_\Phi f\|_p \leq \sum_{k=0}^{n_0-1} |\Phi(k)| \, \|J^{\alpha,\beta}_k f\|_p \leq \left(\sum_{k=0}^{n_0-1} |\Phi(k)| \, C_k \right) \|f\|_p,$$

that is, T^1_Φ is L^p-continuous. It remains to be shown that T^2_Φ is also L^p-continuous. By hypothesis, $h(x) = \sum_{n=0}^{\infty} a_n x^n$, for x in a neighborhood of zero; so,

$$T^2_\Phi f = \sum_{k=n_0}^{\infty} \Phi(k) J^{\alpha,\beta}_k f = \sum_{k=n_0}^{\infty} h\left(\frac{1}{r_k^\delta}\right) J^{\alpha,\beta}_k f = \sum_{k=n_0}^{\infty} \sum_{n=0}^{\infty} a_n \frac{1}{r_k^{\delta n}} J^{\alpha,\beta}_k f.$$

But by (2.14), for $k \geq n_0$, $\frac{1}{r_k^{\delta n}} J^{\alpha,\beta}_k f = P^{\alpha,\beta}_{n,\delta,n_0} J^{\alpha,\beta}_k f$, so we have

$$T^2_\Phi f = \sum_{k=n_0}^{\infty} \sum_{n=0}^{\infty} a_n P^{\alpha,\beta}_{n,\delta,n_0} J^{\alpha,\beta}_k f = \sum_{n=0}^{\infty} a_n P^{\alpha,\beta}_{n,\delta,n_0} \sum_{k=0}^{\infty} J^{\alpha,\beta}_k f = \sum_{n=0}^{\infty} a_n P^{\alpha,\beta}_{n,\delta,n_0} f.$$

Since by (2.13), $P^{\alpha,\beta}_{n,\delta,n_0}$ is L^p-continuous, we obtain

$$\|T^2_\Phi f\|_p \leq \sum_{n=0}^{\infty} |a_n|\, \|P^{\alpha,\beta}_{n,\delta,n_0} f\|_p \leq C_{n_0} \left(\sum_{n=0}^{\infty} |a_n| \frac{1}{n_0^{\delta n}} \right) \|f\|_p = C_{n_0} h \left(\frac{1}{n_0^{\delta}} \right) \|f\|_p.$$

Therefore, T_Φ is continuous on $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$. □

Similarly to the classical case of the Laplacian [Zyg59], and to the one-dimensional Jacobi case [BU07], a fractional integral of order $\gamma > 0$, denoted $I^{\alpha,\beta}_\gamma$, which is called the *Riesz potential* of order γ, can be formally defined in terms of the d-dimensional Jacobi operator by

$$I^{\alpha,\beta}_\gamma := (-\mathcal{L}^{\alpha,\beta})^{-\gamma/2} \Pi_0, \tag{2.15}$$

where $\Pi_0 = I - J^{\alpha,\beta}_0$. Here Π_0 is applied first, since zero is an eigenvalue of $\mathcal{L}^{\alpha,\beta}$ and therefore $(-\mathcal{L}^{\alpha,\beta})^{-\gamma/2}$ is not defined over all $L^2([-1,1]^d, \mu^d_{\alpha,\beta})$. For the Jacobi polynomial of order κ with $\kappa \in A^{\alpha,\beta}_n$, $n > 0$, we have

$$I^{\alpha,\beta}_\gamma \vec{p}_\kappa^{\,\alpha,\beta} = \frac{1}{\lambda_\kappa^{\gamma/2}} \vec{p}_\kappa^{\,\alpha,\beta} = \frac{1}{r_n^{\gamma/2}} \vec{p}_\kappa^{\,\alpha,\beta}. \tag{2.16}$$

Thus, for $f \in L^2([-1,1]^d, \mu^d_{\alpha,\beta})$ polynomial with Jacobi expansion $\sum_{n=0}^{\infty} J^{\alpha,\beta}_n f$,

$$I^{\alpha,\beta}_\gamma f = \sum_{n=1}^{\infty} \frac{1}{r_n^{\gamma/2}} J^{\alpha,\beta}_n f.$$

It is easy to get the following integral representation for the fractional integral of order $\gamma > 0$:

$$I^{\alpha,\beta}_\gamma f = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} P^{\alpha,\beta}_t f \, dt, \tag{2.17}$$

for f polynomial. As in the one-dimensional case, Meyer's multiplier theorem allows us to extend $I^{\alpha,\beta}_\gamma$ to a bounded operator on $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$.

Theorem 2.2 *The fractional integral of order γ admits a continuous extension to the space $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$, which will also be denoted by $I^{\alpha,\beta}_\gamma$.*

Proof If $\gamma/2 < 1$, then $I^{\alpha,\beta}_\gamma$ is a multiplier with associated function $\Phi(k) = \frac{1}{r_k^{\gamma/2}} = h\left(\frac{1}{r_k^{\gamma/2}}\right)$, where $h(z) = z$, which is analytic in a neighborhood of zero. Then the result follows immediately by Meyer's theorem.

Now, if $\gamma/2 \geq 1$, let $n \in \mathbb{N}$ such that $n > \gamma/2$ and $\delta = \frac{\gamma}{2n} < 1$. Then $\delta n = \frac{\gamma}{2}$. Let $h(z) = z^n$, which is analytic in a neighborhood of zero. Then we have $h\left(\frac{1}{r_k^{\delta}}\right) = \frac{1}{r_k^{\delta n}} = \frac{1}{r_k^{\gamma/2}} = \Phi(k)$. Again the result follows by applying Meyer's theorem. □

The *Bessel potential* of order $\gamma > 0$ associated to the d-dimensional Jacobi operator, denoted $\mathcal{J}_\gamma^{\alpha,\beta}$, is formally defined by

$$\mathcal{J}_\gamma^{\alpha,\beta} := (I - \mathcal{L}^{\alpha,\beta})^{-\gamma/2}. \tag{2.18}$$

This means that for a Jacobi polynomial of order κ with $\kappa \in A_n^{\alpha,\beta}$,

$$\mathcal{J}_\gamma^{\alpha,\beta} \vec{p}_\kappa^{\,\alpha,\beta} = \frac{1}{(1+\lambda_\kappa)^{\gamma/2}} \vec{p}_\kappa^{\,\alpha,\beta} = \frac{1}{(1+r_n)^{\gamma/2}} \vec{p}_\kappa^{\,\alpha,\beta},$$

and therefore if $f \in L^2([-1,1]^d, \mu_{\alpha,\beta}^d)$ is a polynomial with expansion $\sum_{n=0}^\infty J_n^{\alpha,\beta} f$,

$$\mathcal{J}_\gamma^{\alpha,\beta} f = \sum_{n=0}^\infty \frac{1}{(1+r_n)^{\gamma/2}} J_n^{\alpha,\beta} f. \tag{2.19}$$

As above, Meyer's theorem allows us to extend Bessel potentials to continuous operators on the spaces $L^p([-1,1]^d, \mu_{\alpha,\beta}^d)$.

Theorem 2.3 *The operator $\mathcal{J}_\gamma^{\alpha,\beta}$ admits a continuous extension to the space $L^p([-1,1]^d, \mu_{\alpha,\beta}^d)$, which will also be denoted by $\mathcal{J}_\gamma^{\alpha,\beta}$.*

Proof The Bessel potential of order γ is a multiplier operator associated to the function $\Phi(k) = \left(\frac{1}{1+r_k}\right)^{\gamma/2}$. Write $\left(\frac{1}{1+r_k}\right)^{\gamma/2} = \left(\frac{r_k}{1+r_k}\right)^{\gamma/2} \frac{1}{r_k^{\gamma/2}}$. Now, the result follows by twice applying Meyer's theorem, the first time taking the function $h(z) = \left(\frac{1}{1+z}\right)^{\gamma/2}$, and the second, taking the function $h(z) = z^n$ for $n \in \mathbb{N}$ such that $n\delta = \frac{\gamma}{2}$. Both functions are analytic in a neighborhood of zero. □

Again using the analogy to the classical Laplacian case [Zyg59], we formally define the *fractional derivative* of order $\gamma > 0$, denoted $D_\gamma^{\alpha,\beta}$, in terms of the d-dimensional Jacobi operator by

$$D_\gamma^{\alpha,\beta} := (-\mathcal{L}^{\alpha,\beta})^{\gamma/2}. \tag{2.20}$$

This means that for the Jacobi polynomial of order κ with $\kappa \in A_n^{\alpha,\beta}$, we have

$$D_\gamma^{\alpha,\beta} \vec{p}_\kappa^{\,\alpha,\beta} = \lambda_\kappa^{\gamma/2} \vec{p}_\kappa^{\,\alpha,\beta} = r_n^{\gamma/2} \vec{p}_\kappa^{\,\alpha,\beta}. \tag{2.21}$$

For the fractional derivative of order $0 < \gamma < 1$ we have the integral representation

$$D_\gamma^{\alpha,\beta} f = \frac{1}{c_\gamma} \int_0^\infty t^{-\gamma-1} (P_t^{\alpha,\beta} f - f)\, dt, \tag{2.22}$$

for f polynomial, where $c_\gamma = \int_0^\infty s^{-\gamma-1}(e^{-s} - 1)\, ds = \Gamma(-\gamma)$, for $0 < \gamma < 1$. This can be seen by using the change of variable $s = \lambda_\kappa^{1/2} t$, in $D_\gamma^{\alpha,\beta} \vec{p}_\kappa^{\,\alpha,\beta}$ for $\kappa \in A_n^{\alpha,\beta}$. If f is a polynomial, by (2.16) and (2.21) we then have

$$I_\gamma^{\alpha,\beta}(D_\gamma^{\alpha,\beta} f) = D_\gamma^{\alpha,\beta}(I_\gamma^{\alpha,\beta} f) = \Pi_0 f. \tag{2.23}$$

Let us now consider the Jacobi Sobolev spaces (or *potential spaces*). For $1 < p < \infty$, the Jacobi Sobolev space of order $\gamma > 0$, namely $L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$, is defined as the image of $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$ under the Bessel potential $\mathcal{J}^{\alpha,\beta}_\gamma$, that is,

$$L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta}) = \mathcal{J}^{\alpha,\beta}_\gamma L^p([-1,1]^d, \mu^d_{\alpha,\beta}).$$

The Jacobi Sobolev space $L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$ can also be defined as the completion of the set of polynomials $\mathcal{P}$ with respect to the norm

$$\|f\|_{p,\gamma} := \|(I - \mathcal{L}^{\alpha,\beta})^{\gamma/2} f\|_p.$$

That is to say, $f \in L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$ if and only if there is a sequence of polynomials $\{f_n\}$ such that $\lim_{n\to\infty} \|f_n - f\|_{p,\gamma} = 0$.

The next proposition gives us some inclusion properties among the Jacobi Sobolev spaces.

Proposition 2.2 *The Jacobi Sobolev spaces $L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$ have the following properties.*

(i) If $p < q$, then $L^q_\gamma([-1,1]^d, \mu^d_{\alpha,\beta}) \subseteq L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$, for each $\gamma > 0$.

(ii) If $0 < \gamma < \delta$, then $L^p_\delta([-1,1]^d, \mu^d_{\alpha,\beta}) \subseteq L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$, for each $1 < p < \infty$.

Proof (i) For γ fixed, this follows immediately by Hölder's inequality.

(ii) Let f be a polynomial and let $\phi = (I - \mathcal{L}^{\alpha,\beta})^{\delta/2} f = \sum_{n=0}^{\infty} (1 + r_n)^{\delta/2} J^{\alpha,\beta}_n f$, which is also a polynomial. Then $\phi \in L^p_\delta([-1,1]^d, \mu^d_{\alpha,\beta})$, $\|\phi\|_p = \|f\|_{p,\delta}$ and $\mathcal{J}^{\alpha,\beta}_{\gamma-\delta}\phi = (I - \mathcal{L}^{\alpha,\beta})^{(\gamma-\delta)/2}\phi = (I - \mathcal{L}^{\alpha,\beta})^{\gamma/2} f$, by the L^p-continuity of Bessel potentials,

$$\|f\|_{p,\gamma} = \|(I - \mathcal{L}^{\alpha,\beta})^{\gamma/2} f\|_p = \|\mathcal{J}_{\gamma-\delta}\phi\|_p \leq C_p \|f\|_{p,\delta}.$$

Now let $f \in L^p_\delta([-1,1]^d, \mu^d_{\alpha,\beta})$ and $g \in L^p([-1,1]^d, \mu^d_{\alpha,\beta})$ such that $f = \mathcal{J}^{\alpha,\beta}_\delta g$. There is a sequence of polynomials $\{g_n\}$ in $L^p([-1,1]^d, \mu^d_{\alpha,\beta})$ such that it is the case that $\lim_{n\to\infty} \|g_n - g\|_p = 0$. Set $f_n = \mathcal{J}^{\alpha,\beta}_\delta g_n$, then $\lim_{n\to\infty} \|f_n - f\|_{p,\delta} = 0$, and

$$\begin{aligned}\|f_n - f\|_{p,\gamma} &= \|(I - \mathcal{L}^{\alpha,\beta})^{\gamma/2}(f_n - f)\|_p = \|(I - \mathcal{L}^{\alpha,\beta})^{\gamma/2}(I - \mathcal{L}^{\alpha,\beta})^{-\delta/2}(g_n - g)\|_p \\ &= \|(I - \mathcal{L}^{\alpha,\beta})^{(\gamma-\delta)/2}(g_n - g)\|_p = \|\mathcal{J}_{\gamma-\delta}(g_n - g)\|_p,\end{aligned}$$

by the L^p-continuity of Bessel potentials, $\lim_{n\to\infty} \|f_n - f\|_{p,\gamma} = 0$.

Therefore, $\|f\|_{p,\gamma} \leq \|f_n - f\|_{p,\gamma} + \|f_n\|_{p,\gamma} \leq \|f_n - f\|_{p,\gamma} + \|f_n\|_{p,\delta}$. By taking the $n \to \infty$ limit, we obtain the result. □

Let us consider the space

$$L_\gamma([-1,1]^d, \mu^d_{\alpha,\beta}) = \bigcup_{p>1} L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta}).$$

$L_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$ is the natural domain of $D^{\alpha,\beta}_\gamma$. We define it on this space as follows. Let $f \in L_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$; then there is $p > 1$ such that $f \in L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$ and a sequence of polynomials $\{f_n\}$ such that $\lim_{n\to\infty} f_n = f$ in $L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$. We define for $f \in L_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$,

$$D^{\alpha,\beta}_\gamma f = \lim_{n\to\infty} D^{\alpha,\beta}_\gamma f_n.$$

The next theorem shows that $D^{\alpha,\beta}_\gamma$ is well defined; and its inequality (2.24) supplies a characterization of the Sobolev spaces.

Theorem 2.4 *Let $\gamma > 0$ and $1 < p, q < \infty$.*

(i) If $\{f_n\}$ is a sequence of polynomials such that $\lim_{n\to\infty} f_n = f$ in the space $L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$, then

$$\lim_{n\to\infty} D^{\alpha,\beta}_\gamma f_n \in L^p([-1,1]^d, \mu^d_{\alpha,\beta}),$$

and the limit does not depend on the choice of the sequence $\{f_n\}$.

If $f \in L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta}) \bigcap L^q_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$, then the limit does not depend on the choice of p or q. Thus $D^{\alpha,\beta}_\gamma$ is well defined on $L_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$.

(ii) $f \in L^p_\gamma([-1,1]^d, \mu^d_{\alpha,\beta})$ if and only if $D^{\alpha,\beta}_\gamma f \in L^p([-1,1]^d, \mu^d_{\alpha,\beta})$. Moreover, there exist positive constants $A_{p,\gamma}$ and $B_{p,\gamma}$ such that

$$B_{p,\gamma}\|f\|_{p,\gamma} \leq \|D^{\alpha,\beta}_\gamma f\|_p \leq A_{p,\gamma}\|f\|_{p,\gamma}. \tag{2.24}$$

Proof (ii) First, let us note that for $f = \sum_{n=0}^\infty J^{\alpha,\beta}_n f$ polynomial,

$$D^{\alpha,\beta}_\gamma \mathcal{J}^{\alpha,\beta}_\gamma f = \sum_{n=0}^\infty \left(\frac{r_n}{1+r_n}\right)^{\gamma/2} J^{\alpha,\beta}_n f,$$

that is, $D^{\alpha,\beta}_\gamma \mathcal{J}^{\alpha,\beta}_\gamma$ is a multiplier with associated function $\Phi(k) = \left(\frac{r_k}{1+r_k}\right)^{\gamma/2} = h(\frac{1}{r_k})$, where $h(z) = \left(\frac{1}{z+1}\right)^{\gamma/2}$; and therefore by Meyer's theorem, it is L^p-continuous.

Let f be a polynomial and let ϕ be a polynomial such that $f = \mathcal{J}_\gamma^{\alpha,\beta}\phi$. We have that $\|f\|_{p,\gamma} = \|\phi\|_p$, and by the continuity of the operator $D_\gamma^{\alpha,\beta}\mathcal{J}_\gamma^{\alpha,\beta}$

$$\|D_\gamma^{\alpha,\beta} f\|_p = \|D_\gamma^{\alpha,\beta}\mathcal{J}_\gamma^{\alpha,\beta}\phi\|_p \le A_{p,\gamma}\|\phi\|_p = A_{p,\gamma}\|f\|_{p,\gamma}.$$

To prove the converse, let us suppose that f is a polynomial. Then $D_\gamma^{\alpha,\beta} f$ is also a polynomial, and therefore $D_\gamma^{\alpha,\beta} f \in L^p([-1,1]^d, \mu_{\alpha,\beta}^d)$. Consider

$$\phi = (I - \mathcal{L}^{\alpha,\beta})^{\gamma/2} f = \sum_{k=0}^{\infty}(1+r_k)^{\gamma/2} J_k^{\alpha,\beta} f = \sum_{k=0}^{\infty}\left(\frac{1+r_k}{r_k}\right)^{\gamma/2} J_k^{\alpha,\beta}(D_\gamma^{\alpha,\beta} f).$$

The mapping $g = \sum_{k=0}^{\infty} J_k^{\alpha,\beta} g \mapsto \sum_{k=0}^{\infty}\left(\frac{1+r_k}{r_k}\right)^{\gamma/2} J_k^{\alpha,\beta} g$ is a multiplier operator with associated function $\Phi(k) = \left(\frac{1+r_k}{r_k}\right)^{\gamma/2} = h(\frac{1}{r_k})$ where $h(z) = (z+1)^{\gamma/2}$, so it is L^p-continuous by Meyer's theorem. Taking $g = D_\gamma^{\alpha,\beta} f$ we have

$$\|f\|_{p,\gamma} = \|\phi\|_p \le B_{p,\gamma}\|D_\gamma^{\alpha,\beta} f\|_p.$$

Thus we get (2.24) for polynomials.

For the general case, if $f \in L_\gamma^p([-1,1]^d, \mu_{\alpha,\beta}^d)$, there exists $g \in L^p([-1,1]^d, \mu_{\alpha,\beta}^d)$ such that $f = \mathcal{J}_\gamma^{\alpha,\beta} g$ and a sequence $\{g_n\}$ of polynomials such that it is the case that $\lim_{n\to\infty}\|g_n - g\|_p = 0$. Let $f_n = \mathcal{J}_\gamma^{\alpha,\beta} g_n$, so that $\lim_{n\to\infty}\|f_n - f\|_{p,\gamma} = 0$. Then, by the continuity of the operator $D_\gamma^{\alpha,\beta}\mathcal{J}_\gamma^{\alpha,\beta}$ and the fact $\lim_{n\to\infty}\|g_n - g\|_p = 0$,

$$\lim_{n\to\infty}\|D_\gamma^{\alpha,\beta}(f_n - f)\|_p = \lim_{n\to\infty}\|D_\gamma^{\alpha,\beta}\mathcal{J}_\gamma^{\alpha,\beta}(g_n - g)\|_p = 0.$$

Then, as $B_{p,\gamma}\|f_n\|_{p,\gamma} \le \|D_\gamma^{\alpha,\beta} f_n\|_p \le A_{p,\gamma}\|f_n\|_{p,\gamma}$, the result follows by taking the limit $n \to \infty$ in this inequality.

(i) Let $\{f_n\}$ be a sequence of polynomials such that $\lim_{n\to\infty} f_n = f$, in the space $L_\gamma^p([-1,1]^d, \mu_{\alpha,\beta}^d)$. Then for $n, m \in \mathbb{N}$, by (2.24)

$$\|D_\gamma^{\alpha,\beta} f_n - D_\gamma^{\alpha,\beta} f_m\|_p = \|D_\gamma^{\alpha,\beta}(f_n - f_m)\|_p \le B_{p,\gamma}\|f_n - f_m\|_{p,\gamma},$$

so $\{D_\gamma^{\alpha,\beta} f_n\}$ is a Cauchy sequence in $L^p([-1,1]^d, \mu_{\alpha,\beta}^d)$, and therefore it is the case that $\lim_{n\to\infty} D_\gamma^{\alpha,\beta} f_n \in L^p([-1,1]^d, \mu_{\alpha,\beta}^d)$.

Now, suppose that $\{q_n\}$ is another sequence of polynomials such that $\lim q_n = f$ in $L_\gamma^p([-1,1]^d, \mu_{\alpha,\beta}^d)$. Then $\lim_{n\to\infty} f_n - q_n = 0$. By (2.24),

$$B_{p,\gamma}\|f_n - q_n\|_{p,\gamma} \le \|D_\gamma^{\alpha,\beta} f_n - D_\gamma^{\alpha,\beta} q_n\|_p \le A_{p,\gamma}\|f_n - q_n\|_{p,\gamma},$$

and by taking the limit $n \to \infty$ we get that $\lim_{n\to\infty} D_\gamma^{\alpha,\beta} f_n = \lim_{n\to\infty} D_\gamma^{\alpha,\beta} q_n$ in $L^p([-1,1]^d, \mu_{\alpha,\beta}^d)$ and therefore the limit does not depend on the choice of the approximating sequence.

To finish the proof, let us suppose $f \in L_\gamma^p([-1,1]^d, \mu_{\alpha,\beta}^d) \bigcap L_\gamma^q([-1,1]^d, \mu_{\alpha,\beta}^d)$, and without loss of generality, let us assume that $p \leq q$. Then by Proposition 2.2(i), $L_\gamma^q([-1,1]^d, \mu_{\alpha,\beta}^d) \subseteq L_\gamma^p([-1,1]^d, \mu_{\alpha,\beta}^d)$, and therefore $f \in L_\gamma^q([-1,1]^d, \mu_{\alpha,\beta}^d)$. Now, if $\{f_n\}$ is a sequence of polynomials such that $\lim_{n\to\infty} f_n = f$ in $L_\gamma^q([-1,1]^d, \mu_{\alpha,\beta}^d)$ (hence in $L_\gamma^p([-1,1]^d, \mu_{\alpha,\beta}^d)$), we have

$$\lim_{n\to\infty} D_\gamma^{\alpha,\beta} f_n \in L^q([-1,1]^d, \mu_{\alpha,\beta}^d) = L^p([-1,1]^d, \mu_{\alpha,\beta}^d) \bigcap L^q([-1,1]^d, \mu_{\alpha,\beta}^d).$$

Therefore the limit does not depend on the choice of p or q. □

Finally, we shall give alternative representations of $D_\gamma^{\alpha,\beta}$ and $I_\gamma^{\alpha,\beta}$, but first we present a technical lemma, which illuminates the asymptotic behavior of the d-dimensional Poisson–Jacobi semigroup $\{P_t^{\alpha,\beta}\}$.

Lemma 2.2 *If $f \in C^2([-1,1]^d)$ then*

$$\left| \frac{\partial}{\partial t} P_t^{\alpha,\beta} f(x) \right| \leq C_{f,\alpha,\beta,d} e^{-d_{\alpha,\beta}^{1/2} t}, \tag{2.25}$$

with $d_{\alpha,\beta} = \min\{\alpha_j + \beta_j + 2 \colon j = 1, \ldots, d\}$.

As a consequence, the Poisson–Jacobi semigroup $\{P_t^{\alpha,\beta}\}_{t\geq 0}$ has exponential decay on $(C_0^{\alpha,\beta})^\perp = \bigoplus_{n=1}^\infty C_n^{\alpha,\beta}$. That is, if $f \in C^2([-1,1]^d)$, has the property that $\int f(y) \mu_{\alpha,\beta}^d(dy) = 0$, then

$$|P_t^{\alpha,\beta} f(x)| \leq C_{f,\alpha,\beta,d} e^{-d_{\alpha,\beta}^{1/2} t}. \tag{2.26}$$

Proof The proof of (2.25) is analogous to the one given in the Hermite case in [LU04]. Also, it is contained in the proof of Proposition 4.5 in [NS08].

To prove (2.26), note that $\frac{\partial}{\partial t} P_t^{\alpha,\beta} f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{2u} \mathcal{L}^{\alpha,\beta} T_{t^2/4u}^{\alpha,\beta} f \, du$, thus performing the change of variable $u = d_{\alpha,\beta} s$ gives

$$\begin{aligned} \left| \frac{\partial}{\partial t} P_t^{\alpha,\beta} f(x) \right| &\leq C_{f,\alpha,\beta,d} \int_0^\infty e^{-u} \frac{t}{2\sqrt{\pi}} u^{-3/2} e^{-d_{\alpha,\beta} t^2/4u} \, du \\ &= C_{f,\alpha,\beta,d} \int_0^\infty e^{-d_{\alpha,\beta} s} \, \mu_t^{1/2}(ds) = C_{f,\alpha,\beta,d} e^{-d_{\alpha,\beta}^{1/2} t}. \end{aligned}$$

Since $\int_{[-1,1]^d} f(y) \mu_{\alpha,\beta}^d(dy) = 0$, we have $\lim_{t\to\infty} P_t^{\alpha,\beta} f(x) = 0$, and

$$\left| P_t^{\alpha,\beta} f(x) \right| \leq \int_t^\infty \left| \frac{\partial}{\partial s} P_s^{\alpha,\beta} f(x) \right| ds \leq C_{f,\alpha,\beta,d} \int_t^\infty e^{-d_{\alpha,\beta}^{1/2} s} \, ds = C_{f,\alpha,\beta,d} e^{-d_{\alpha,\beta}^{1/2} t}.$$

□

Now, since $\{P_t^{\alpha,\beta}\}_{t\geq 0}$ is a strongly continuous semigroup, we have

$$\lim_{t\to 0^+} P_t^{\alpha,\beta} f = f, \tag{2.27}$$

almost everywhere, by a well known result on the maximal function of the semigroup. Let us write

$$P_t^{\alpha,\beta} f(x) = \int_0^\infty T_s^{\alpha,\beta} f(x)\, \mu_t^{1/2}(ds) = \int_{[-1,1]^d} k_d^{\alpha,\beta}(t,x,y) f(y)\, \mu_{\alpha,\beta}^d(dy),$$

where $k_d^{\alpha,\beta}(t,x,y) = \int_0^\infty p_d^{\alpha,\beta}(s,x,y)\, \mu_t^{1/2}(ds)$. Define the operator $Q_t^{\alpha,\beta}$ by

$$Q_t^{\alpha,\beta} f(x) = -t\frac{d}{dt} P_t^{\alpha,\beta} f(x) = \int_{[-1,1]^d} q_d^{\alpha,\beta}(t,x,y) f(y)\, \mu_{\alpha,\beta}^d(dy), \tag{2.28}$$

with $q_d^{\alpha,\beta}(t,x,y) = -t\frac{d}{dt}k_d^{\alpha,\beta}(t,x,y)$. We can now give the promised alternative representations for $D_\gamma^{\alpha,\beta}$ and $I_\gamma^{\alpha,\beta}$.

Proposition 2.3 *Suppose that f is differentiable with continuous derivatives up to the second order, and that $\int_{[-1,1]^d} f(y)\mu_{\alpha,\beta}^d(dy) = 0$. Then*

$$-\gamma\, D_\gamma^{\alpha,\beta} f = \frac{1}{\Gamma(-\gamma)} \int_0^\infty t^{-\gamma-1} Q_t^{\alpha,\beta} f\, dt, \qquad 0<\gamma<1, \tag{2.29a}$$

$$\gamma\, I_\gamma^{\alpha,\beta} f = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} Q_t^{\alpha,\beta} f\, dt, \qquad 0<\gamma. \tag{2.29b}$$

Proof Let us start by proving (2.29a). Integrating by parts in (2.22) we have

$$D_\gamma^{\alpha,\beta} f = \frac{1}{\Gamma(-\gamma)} \lim_{\substack{a\to 0^+\\ b\to\infty}} \int_a^b t^{-\gamma-1}(P_t^{\alpha,\beta} f - f)\, dt = -\frac{1}{\gamma\,\Gamma(-\gamma)} \int_0^\infty t^{-\gamma-1} Q_t^{\alpha,\beta} f\, dt,$$

since, by (2.25),

$$\begin{aligned}\lim_{b\to\infty}\left|\frac{P_b^{\alpha,\beta} f(x) - f(x)}{b^\gamma}\right| &\leq \lim_{b\to\infty}\frac{1}{b^\gamma}\int_0^b \left|\frac{\partial}{\partial s} P_s^{\alpha,\beta} f(x)\right| ds \\ &\leq C_{f,\alpha,\beta,d} \lim_{b\to\infty} \frac{1-e^{-d_{\alpha,\beta}^{1/2} b}}{b^\gamma} = 0,\end{aligned}$$

and by (2.27), since $0<\gamma<1$, $\lim_{a\to 0^+}\frac{P_a^{\alpha,\beta} f - f}{a^\gamma} = 0$.

We can also prove (2.29b). Integrating by parts in (2.17) yields

$$I_\gamma^{\alpha,\beta} f = \frac{1}{\Gamma(\gamma)} \lim_{\substack{a\to 0^+\\ b\to\infty}} \int_a^b t^{\gamma-1} P_t^{\alpha,\beta} f dt = \frac{1}{\gamma\,\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} Q_t^{\alpha,\beta} f\, dt,$$

since by (2.26),

$$\lim_{b\to\infty}\left|b^{\gamma}P_t^{\alpha,\beta}f\right| \leq C_{f,\alpha,\beta,d}\lim_{b\to\infty}b^{\gamma}e^{-d_{\alpha,\beta}^{1/2}b}=0 \quad \text{and} \quad \lim_{a\to 0^+}a^{\gamma}P_a^{\alpha,\beta}f=0.$$

□

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Chapter 3

A formula for polynomials with Hermitian matrix argument

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Abstract: We construct and study orthogonal bases of generalized polynomials on the space of Hermitian matrices. They are obtained by the Gram–Schmidt orthogonalization process from the Schur polynomials. A Berezin–Karpelevich type formula is given for these multivariate polynomials. The normalization of the orthogonal polynomials of Hermitian matrix argument and expansions in such polynomials are investigated

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3.1 Introduction

We present first the main object of this paper, the generalized orthogonal polynomials with Hermitian matrix argument. They are orthogonal with respect to measures constructed in the following way from measures on $\mathbb{R}$.

Let H_n be the space of Hermitian matices and U_n the group of unitary matrices. We say that a function $f : H_n \to \mathbb{R}$ is central if $f(UXU^{-1}) = f(X)$ for all $U \in U_n$. Similarly, a Borel measure ν on H_n is called central if it is U_n-invariant: $\nu(UBU^{-1}) = \nu(B)$ for every Borel set $B \subset H_n$ and $U \in U_n$. If f is a central function, then f is determined by its restriction to the subspace of real diagonal matrices, which we denote D_n. Observe that $D_n \simeq \mathbb{R}^n$.

Let us define $\tilde{f}(x_1, \ldots, x_n) = f(diag(x_1, \ldots, x_n))$. Then $\tilde{f}$ is a symmetric function in $x_1, \ldots, x_n$ and the map $f \mapsto \tilde{f}$ is a bijection from the space of central functions on H_n to the space of symmetric functions on $\mathbb{R}^n$.

There is a natural and important way of generating central functions on H_n, starting from a function F on $\mathbb{R}$ by setting

$$f(diag(x_1, \ldots, x_n)) = F(x_1) \ldots F(x_n)$$

and then extending f to a central function on H_n. One denotes $f = \det F$.

Let m be the Lebesgue measure on H_n, treated as a real vector space and let f be a positive Borel central function on H_n. We normalize m in such a way that the Weyl integration formula ([Far06], p.13, [FK94], Th.VI.2.3) reads

$$\int_{H_n} f(X)dm(X) = \int_{\mathbb{R}^n} f(diag(x_1, \ldots, x_n))V^2(x_1, \ldots, x_n)dx_1 \ldots dx_n \tag{3.1}$$

where $V(x_1, \ldots, x_n)$ is the Vandermonde determinant. The formula (3.1) implies that if G is a positive Borel function on $\mathbb{R}$, then

$$\int_{H_n} f(X) \det G(X)dm(X) = \int_{\mathbb{R}^n} \tilde{f}(x_1, \ldots, x_n)V^2(x_1, \ldots, x_n) \prod_i G(x_i)dx_1 \ldots dx_n,$$

hence the measure $\det G(X)dm(X)$ on H_n corresponds to the permutation invariant measure on $\mathbb{R}^n$ given by $V^2(\underline{x}) \prod_i G(x_i)d\underline{x}$. Extending this remark by duality, to any Borel measure μ on $\mathbb{R}$ we associate a permutation invariant measure μ_n on $\mathbb{R}^n$ and a central measure M on H_n in the following way:

$$\mu_n(d\underline{x}) = V^2(\underline{x})\mu^{\otimes n}(d\underline{x}); \tag{3.2}$$

$$\int_{H_n} f(X)dM(X) = \int_{\mathbb{R}^n} \tilde{f}(x_1, \ldots, x_n)V^2(x_1, \ldots, x_n)d\mu(x_1) \ldots d\mu(x_n) \tag{3.3}$$

for any positive central function f on H_n.

For a symmetric polynomial P on $\mathbb{R}^n$, let $\hat{P}(X)$ be the central function on H_n whose restriction to the diagonal matrices is equal to $P(\underline{x})$. The functions $\hat{P}$ are called (generalized) polynomials of Hermitian matrix argument. In fact, $\hat{P}(X)$ is a symmetric polynomial in the eigenvalues of X.

In this article we construct and study orthogonal bases of generalized polynomials on H_n, with respect to measures M on H_n, defined in (4.4) for a given measure μ on $\mathbb{R}$. A Berezin–Karpelevich type formula is given for these multivariate polynomials in Theorem 3.1. The normalization of the orthogonal polynomials of Hermitian matrix argument and expansions in such polynomials are then investigated.

Let us recall that Berezin and Karpelevich ([BK59]) expressed the spherical functions on complex Grassmann manifolds $U(p,q)/U(p) \times U(q)$ as a quotient of a determinant containing Jacobi functions and a Vandermonde determinant. The Berezin–Karpelevich formula, studied by Takahashi ([Tak77]), was first proved by Hoogenboom ([Hoo82]). Similar formulas were given for hypergeometric functions of Hermitian matrix argument by Gross and Richards ([GR91], [GR93]).

Our formula in Theorem 3.1 expresses the generalized orthogonal polynomials on H_n as a quotient of a determinant containing corresponding orthogonal polynomials on $\mathbb{R}$ and a Vandermonde determinant.

Generalized Hermite and Laguerre polynomials of matrix argument were introduced and studied by Herz ([Her55]). In the Hermitian matrix case, they are orthogonal bases in the spaces $L^2_{U_n}(H_n, M)$, where the measure M is obtained as in (4.4) from the mesure $\mu(dx) = e^{-x^2}dx$ in the Hermite case and $\mu(dx) = x^\alpha e^{-x}\mathbf{1}_{(0,\infty)}(x)dx$ in the Laguerre case. The notation $L^2_{U_n}$ stands for central functions in L^2. The space $L^2_{U_n}(H_n, M)$ is isomorphic to the space $L^2_{\mathrm{sym}}(\mathbb{R}^n, \mu_n)$ of symmetric functions in $L^2(\mathbb{R}^n, \mu_n)$.

More generally, the Laguerre polynomials on symmetric cones were defined in [FK94]. The generalized Laguerre polynomials are very useful in harmonic analysis on symmetric cones ([FK94], [CF04]) and in multivariate statistics ([Mui82]).

Hermite and Laguerre polynomials of matrix argument are special cases of generalized Hermite polynomials for Dunkl operators (cf. [Ros98] and the references therein). They are also a special case of symmetric orthogonal polynomials associated to the Jack polynomials, studied by Lassalle in a series of notes [Las91b], [Las91a] and [Las91c]. In particular, Lassalle gave without proof the same formula as ours in Theorem 3.1, in the Jacobi, Laguerre and Hermite case, respectively. Our formula generalizes the formulas of Lassalle to the case of any orthogonal polynomial family on

H_n, and applies also to the Hermite polynomials generalized in the sense of Chihara, Kravtchouk polynomials, Charlier, Meixner and Pollaczek polynomials etc.

3.2 Preliminaries

In this section we introduce the needed notations and concepts; main references are [Mac91] and [DX01]. Let us fix $n \in \mathbb{N}$. We will use n-element partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, i.e. non–increasing sequences of non–negative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Let the length $l(\lambda)$ of a partition λ be the number of $\lambda_i \neq 0$ and its degree $|\lambda| = \sum_i \lambda_i$. Our partitions λ have the length smaller or equal to n.

We will consider the dominance order between partitions of the same degree. Given two partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ such that $|\lambda| = |\kappa|$, we have $\lambda \geq \kappa$ if

$$\lambda_1 + \ldots + \lambda_r \geq \kappa_1 + \ldots + \kappa_r$$

for all $1 \leq r \leq n$. The dominance order is not total. The graded lexicographic order $\succ_{gl}$ on partitions is total and will also appear in the sequel. We say that $\lambda \succ_{gl} \kappa$ if $|\lambda| > |\kappa|$ or if $|\lambda| = |\kappa|$ and $\lambda_i > \kappa_i$ for the first i such that $\lambda_i \neq \kappa_i$.

Let S_n be the symmetric group of permutations of n elements. If $x_1, x_2, \ldots, x_n$ are real variables, we will denote $m_\lambda(x_1, \ldots, x_n)$ the monomial symmetric function in n variables

$$m_\lambda(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} x_1^{\lambda_{\sigma(1)}} \ldots x_n^{\lambda_{\sigma(n)}}.$$

The family $\{m_\lambda\}_{l(\lambda) \leq n}$ is an algebraic basis of the vector space $\mathcal{P}_n$ of all symmetric polynomials in n variables.

Let $V(x_1, x_2, \ldots, x_n)$ be the Vandermonde determinant,

$$V(x_1, x_2, \ldots, x_n) = \det(x_j^{n-i})_{i,j=1,\ldots,n} = \prod_{1 \leq i < j \leq n} (x_i - x_j). \tag{3.4}$$

For each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, set

$$S_\lambda(x_1, \ldots, x_n) = \frac{\det(x_j^{\lambda_i + n - i})_{i,j=1,\ldots n}}{V(x_1, x_2, \ldots, x_n)}. \tag{3.5}$$

S_λ are called the Schur functions and are symmetric polynomials, homogeneous of degree $|\lambda|$. The family $\{S_\lambda\}_{l(\lambda) \leq n}$ is an algebraic basis of $\mathcal{P}_n$. Recall that (see [Mac91](7.2))

$$S_\lambda = m_\lambda + \sum_{\mu < \lambda, |\mu| = |\lambda|} k_{\lambda\mu} m_\mu, \quad k_{\lambda\mu} \in \mathbb{R} \tag{3.6}$$

so the Schur polynomials are monic, in the sense that they have a dominating term m_λ with coefficient 1. Note that λ is dominating both in the dominance order and in the graded lexicographic order.

We end the introduction with the following well known properties of polynomials of n variables. We have not found a proof in the literature so we include it for the sake of completness.

Proposition 3.1 *(a) If a polynomial $P(x_1,\dots,x_n)$ vanishes when $x_i = x_j$, then $P(x_1,\dots,x_n) = (x_i - x_j)R(x_1,\dots,x_n)$, where R is a polynomial.*

(b) If $P(x_1,\dots,x_n)$ is a polynomial vanishing when $x_i = x_j$, for all $i,j = 1,\dots,n$, $i \neq j$, then

$$P(x_1,\dots,x_n) = V(x_1,\dots,x_n)R(x_1,\dots,x_n),$$

where R is a polynomial.

Proof It is sufficient to consider $i = 1$ and $j = 2$. If $x_2,\dots,x_n$ are fixed, the polynomial P is a polynomial of one variable x_1:

$$\begin{aligned} P(x_1,x_2,\dots,x_n) &= a_k(x_2,\dots,x_n)x_1^k + a_{k-1}(x_2,\dots,x_n)x_1^{k-1} + \\ &\quad \cdots + a_1(x_2,\dots,x_n)x_1 + a_0(x_2,\dots,x_n) \end{aligned}$$

where $a_0,\dots,a_k$ are polynomials in $x_2,\dots,x_n$. By hypothesis,

$$\begin{aligned} 0 = P(x_2,x_2,\dots,x_n) &= a_k(x_2,\dots,x_n)x_2^k + a_{k-1}(x_2,\dots,x_n)x_2^{k-1} + \\ &\quad \cdots + a_1(x_2,\dots,x_n)x_2 + a_0(x_2,\dots,x_n), \end{aligned}$$

therefore

$$\begin{aligned} P(x_1,\dots,x_n) &= (x_1^k - x_2^k)a_k(x_2,\dots,x_n) + (x_1^{k-1} - x_2^{k-1})a_{k-1}(x_2,\dots,x_n) + \\ &\quad \cdots + (x_1 - x_2)a_1(x_2,\dots,x_n) = (x_1 - x_2)R(x_1,\dots,x_n), \end{aligned}$$

since $x_1^p - x_2^p = (x_1 - x_2)(x_1^{p-1} + x_1^{p-2}x_2 + \cdots + x_2^{p-1})$, for any $p \geq 1$. Part (b) follows immediately from part (a) of the Proposition. □

3.3 Symmetric orthogonal polynomials

Throughout all this section, we will suppose that μ is a positive Borel measure on $\mathbb{R}$, such that:

(i) The polynomials are a dense subset of $L^2(\mathbb{R},\mu)$

(ii) The symmetric polynomials $\mathcal{P}_n$ are dense subset of $L^2_{\mathrm{sym}}(\mathbb{R}^n, \mu_n)$, where the measure $\mu_n = V^2(x_1, \ldots, x_n)\mu(dx_1)\ldots\mu(dx_n)$ was introduced in (4.1).

This is always the case when μ has an exponential moment, that is there exists $\epsilon > 0$ such that $\int_{\mathbb{R}} e^{\epsilon|x|} d\mu(x) < \infty$. In fact, if μ has an exponential moment then so does μ_n and the density of polynomials in $L^2(\mathbb{R}, \mu)$ and $L^2(\mathbb{R}^n, \mu_n)$ is well known, see [BC81] and [DX01] for a short proof. These references contain much more information on the problem of density of the space of polynomials in L^p spaces. Note also that by (i), μ must be finite.

Now we are going to construct a family $\{P_\lambda\}_{l(\lambda)\le n}$ of symmetric orthogonal polynomials in n variables, starting from a family of orthogonal polynomials in one variable $\{p_m\}_{m\in\mathbb{N}}$, where p_m has degree m. This is the main result of this article.

Theorem 3.1 *Let μ be a finite positive Borel measure on $\mathbb{R}$ veryfing the conditions* (i) *and* (ii) *and $\{p_m\}_{m\in\mathbb{N}}$ an orthogonal family of polynomials in $L^2(\mathbb{R}, \mu)$, where p_m has degree m. For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and a normalizing constant $c_\lambda \neq 0$, let us define*

$$P_\lambda(x_1, \ldots, x_n) \quad = \quad c_\lambda \frac{\det(p_{\lambda_i+n-i}(x_j))_{i,j=1,\ldots,n}}{V(x_1, \ldots, x_n)}. \tag{3.7}$$

Then P_λ is a symmetric polynomials, and the family $\{P_\lambda\}$ is orthogonal in the Hilbert space $L^2(\mathbb{R}^n, \mu_n)$, where $\mu_n = V^2(x_1, \ldots, x_n)\mu(dx_1)\ldots\mu(dx_n)$. The L^2–norm of P_λ is equal to

$$\|P_\lambda\|^2_{L^2(\mathbb{R}^n,\mu_n)} = c_\lambda^2 n! \prod_{i=1}^{n} \|p_{\lambda_i+n-i}\|^2_{L^2(\mathbb{R},\mu)}. \tag{3.8}$$

The family $\{P_\lambda\}_{l(\lambda)\le n}$ is an orthogonal Hilbert basis of $L^2_{\mathrm{sym}}(\mathbb{R}^n, \mu_n)$, obtained by the Gram–Schmidt orthogonalization process, applied to the Schur polynomials family $\{S_\lambda\}_{l(\lambda)\le n}$, ordered in the graded lexicographic order.

The normalizing constant c_λ depends on the way of normalization of P_λ and p_m and will be specified in Propositions 3.2 and 3.3.

Proof First observe that

$$\det(p_{\lambda_i+n-i}(x_j))_{i,j=1,\ldots n} = \begin{vmatrix} p_{\lambda_1+n-1}(x_1) & p_{\lambda_1+n-1}(x_2) & \cdots & p_{\lambda_1+n-1}(x_n) \\ p_{\lambda_2+n-2}(x_1) & p_{\lambda_2+n-2}(x_2) & \cdots & p_{\lambda_2+n-2}(x_n) \\ \vdots & \vdots & & \vdots \\ p_{\lambda_n}(x_1) & p_{\lambda_n}(x_2) & \cdots & p_{\lambda_n}(x_n) \end{vmatrix}$$

is a polynomial in n variables, that vanishes for $x_i = x_j, i \neq j$. Hence, by Proposition 3.1, it is divisible by $V(x_1, \dots, x_n)$. Thus, for each λ, the function P_λ is a polynomial. Moreover P_λ is symmetric, since if $\sigma \in S_n$, denoting by $\underline{x} = (x_1, \dots, x_n)$ and $\sigma(\underline{x}) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$, we get

$$P_\lambda(\sigma(\underline{x})) = \frac{\det(p_{\lambda_i+n-i}(x_{\sigma(j)}))_{i,j=1,\dots,n}}{\det(x_{\sigma(j)}^{n-i})_{i,j=1,\dots,n}} = \frac{\epsilon(\sigma)\det(p_{\lambda_i+n-i}(x_j))_{i,j=1,\dots,n}}{\epsilon(\sigma)\det(x_j^{n-i})_{i,j=1,\dots,n}} = P_\lambda(\underline{x}),$$

where $\epsilon(\sigma)$ denotes the signature of the permutation σ.

Now let us consider two partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n)$. We have

$$\begin{aligned}
&\langle P_\lambda, P_\kappa\rangle_{L^2(\mathbb{R}^n,\mu_n)} = \int_{\mathbb{R}^n} P_\lambda(\underline{x})P_\kappa(\underline{x})d\mu_n(\underline{x}) \\
&= c_\lambda c_\kappa \int_{\mathbb{R}^n} \det(p_{\lambda_i+n-i}(x_j))_{i,j=1,\dots,n}\det(p_{\kappa_i+n-i}(x_j))_{i,j=1,\dots,n} d\mu(x_1)\dots d\mu(x_n) \\
&= c_\lambda c_\kappa \sum_{\sigma\in S_n}\sum_{\tau\in S_n}\epsilon(\sigma)\epsilon(\tau)\int_{\mathbb{R}^n}\prod_{i=1}^n p_{\lambda_{\sigma(i)}+n-\sigma(i)}(x_i)p_{\kappa_{\tau(i)}+n-\tau(i)}(x_i)d\mu(x_1)\dots d\mu(x_n) \\
&= c_\lambda c_\kappa \sum_{\sigma\in S_n}\sum_{\tau\in S_n}\epsilon(\sigma)\epsilon(\tau)\prod_{i=1}^n\int_{\mathbb{R}} p_{\lambda_{\sigma(i)}+n-\sigma(i)}(x)p_{\kappa_{\tau(i)}+n-\tau(i)}(x)d\mu(x)
\end{aligned}$$

and since $\{p_m\}_{m\in\mathbb{N}}$ is an orthogonal family in $L^2(\mathbb{R}, \mu)$, if a term in the last double sum is non–zero, we must have

$$\lambda_{\sigma(i)} - \sigma(i) = \kappa_{\tau(i)} - \tau(i), \quad i = 1, \dots, n.$$

Setting $\pi = \tau\sigma^{-1}$, this means that $\lambda_i - i = \kappa_{\pi(i)} - \pi(i)$ for all $i = 1, \dots, n$. It follows that if $j > i$ then $\pi(j) \geq \pi(i)$. In order to prove this, let us suppose that $j > i$ and $\pi(j) < \pi(i)$. We have $\kappa_{\pi(j)} \geq \kappa_{\pi(i)}$, that is $\lambda_j + \pi(j) - j \geq \lambda_i + \pi(i) - i$. This implies that $\lambda_j + \pi(j) > \lambda_i + \pi(i)$, which is contradictory with $\lambda_j \leq \lambda_i$ and $\pi(j) < \pi(i)$. Thus $\pi(i) \leq \pi(j)$ when $i < j$, what implies that necessarily π is the identity permutation. It follows that $\sigma = \tau$ and $\lambda = \kappa$. Therefore,

$$\langle P_\lambda, P_\kappa\rangle_{L^2(\mathbb{R}^n,\mu_n)} = d_\lambda\delta_{\lambda\kappa},$$

where

$$\begin{aligned}
d_\lambda &= c_\lambda^2\sum_{\sigma\in S_n}\prod_{i=1}^n\int_{\mathbb{R}} p^2_{\lambda_{\sigma(i)}+n-\sigma(i)}(x)d\mu(x) = c_\lambda^2\sum_{\sigma\in S_n}\prod_{i=1}^n\|p_{\lambda_{\sigma(i)}+n-\sigma(i)}\|^2_{L^2(\mathbb{R},\mu)} \\
&= c_\lambda^2 n!\prod_{i=1}^n\|p_{\lambda_i+n-i}\|^2_{L^2(\mathbb{R},\mu)}.
\end{aligned}$$

Thus $\{P_\lambda : l(\lambda) \leq n\}$ is an orthogonal family in $\mathcal{P}_n$ and the formula (4.3) follows. Now, $p_{\lambda_i+n-i}(x_j) = a_{\lambda_i+n-i}x_j^{\lambda_i+n-i} +$ terms of lower degree, $a_{\lambda_i+n-i} \neq 0$, so

$$P_\lambda(\underline{x}) = aS_\lambda(\underline{x}) + \sum_{|\kappa|<|\lambda|} b_{\lambda\kappa}m_\kappa(\underline{x}), \tag{3.9}$$

where $a = c_\lambda \prod_i a_{\lambda_i+n-i} \neq 0$. By formulas (3.6) and (3.9) it follows that

$$\mathrm{Vect}(\{P_\lambda\}_{l(\lambda)\leq n}) = \mathrm{Vect}(\{S_\lambda\}_{l(\lambda)\leq n}) = \mathrm{Vect}(\{m_\lambda\}_{l(\lambda)\leq n}) = \mathcal{P}_n$$

so, by hypothesis (ii), the family $\{P_\lambda\}_{l(\lambda)\leq n}$ is linearly dense in $L^2_{\mathrm{sym}}(\mathbb{R}^n, \mu_n)$.

In fact, the formula (3.9) implies a stronger fact

$$\mathrm{Vect}(\{P_\mu\}_{\mu\preceq_{gl}\lambda}) = \mathrm{Vect}(\{S_\mu\}_{\mu\preceq_{gl}\lambda}) \tag{3.10}$$

for any n-element partition λ. The formula (3.10) may be easily proved observing that $\mathrm{Vect}(\{P_\mu\}_{\mu\preceq_{gl}\lambda}) \subset \mathrm{Vect}(\{S_\mu\}_{\mu\preceq_{gl}\lambda})$ and that the dimensions of the two spaces are equal.

We order the Schur polynomials with respect to the graded lexicographic order. It follows that the family that one obtains by applying the Gram–Schmidt orthogonalization process to the family $\{S_\lambda\}$ is the family $\{P_\lambda\}$. □

Let us now extend Theorem 3.1 to polynomials of Hermitian matrix argument.

Corollary 3.1 *The generalized polynomials $\{\hat{P}_\lambda\}_{l(\lambda)\leq n}$ form an orthogonal Hilbert basis of the Hilbert space $L^2_{U_n}(H_n, M)$, with the measure M defined in (4.4).*

Now we will determine the value of the normalizing constant c_λ in the definition of P_λ, in relation with the normalization of the polynomials p_m and the required normalization of the polynomials P_λ. The formula (3.9) implies that in the Schur function decomposition, the polynomials P_λ have the leading term aS_λ, in the sense of the graded lexicographic order. Taking into account (3.6) we get

$$P_\lambda = am_\lambda + \sum_{\mu<\lambda, |\mu|=|\lambda|} ak_{\lambda\mu}m_\mu + \sum_{|\kappa|<|\lambda|} b_{\lambda\kappa}m_\kappa, \tag{3.11}$$

so, in monomial symmetric polynomial decomposition of P_λ, the leading term is am_λ, in the sense of both the graded lexicographic order and the graded dominance order. Consequently, it is natural to require that P_λ is monic, that is $a = 1$. From the formulas (3.9) and (3.11) we deduce the following

Proposition 3.2 *If the polynomials p_m and P_λ are monic, then $c_\lambda = 1$.*

Another frequently considered type of normalization of orthogonal polynomials consists in requiring the constant term of the polynomials to be equal to 1. It is not always possible (for example for even Hermite polynomials).

Proposition 3.3 *If we normalize the polynomials p_m and P_λ by requiring $p_m(0) = 1,\ m \in \mathbb{N}$, and $P_\lambda(\underline{0}) = 1,\ l(\lambda) \leq n$, then*

$$c_\lambda = \frac{1}{\det(c_{n-j}^{(\lambda_i+n-i)})_{1\leq i,j\leq n}},$$

where $c_k^{(m)}$ are the coefficients of the polynomials p_m in the monomial decomposition

$$p_m(x) = \sum_{k=0}^{m} c_k^{(m)} x^k.$$

Proof It is shown in [Far06], p.18 that if

$$f_i(x) = \sum_{k=0}^{\infty} c_k^{(i)} x^k,\ i = 1, \ldots, n, \quad |x| < r$$

then

$$\lim_{\underline{x}\to\underline{0}} \frac{\det(f_i(x_j))_{1\leq i,j\leq n}}{V(x_1, \ldots, x_n)} = \det(c_{n-j}^{(i)})_{1\leq i,j\leq n}.$$

In our case, taking

$$f_i(x) = p_{\lambda_i+n-i}(x) = \sum_{k=0}^{\lambda_i+n-i} c_k^{(\lambda_i+n-i)} x^k,$$

we have

$$P_\lambda(\underline{0}) = c_\lambda \lim_{\underline{x}\to\underline{0}} \frac{\det(f_i(x_j))_{1\leq i,j\leq n}}{V(x_1, \ldots, x_n)} = c_\lambda \det(c_{n-j}^{(\lambda_i+n-i)})_{1\leq i,j\leq n}. \tag{3.12}$$

□

Remark 3.1 *Observe that the property $p_m(0) \neq 0$ for all $m \in \mathbb{N}$ does not imply $P_\lambda(\underline{0}) \neq 0$. Indeed, by the formula (3.12), the value of $P_\lambda(\underline{0})$ depends not only on the coefficients $c_0 = p_m(0)$ appearing in the last column of the determinant in (3.12). A converse implication is however true: if λ is such that $p_{\lambda_i+n-i}(0) = 0,\ i = 1, \ldots, n$, then $P_\lambda(\underline{0}) = 0$.*

We end this section with a result on the expansion of some important central functions on H_n in the basis $\{P_\lambda\}$.

Proposition 3.4 *Given n functions $f_1, \ldots, f_n$ of one variable, with the expansions in the basis $\{p_m\}_{m\in\mathbb{N}}$*

$$f_i(x) = \sum_{k=0}^{\infty} c_k^{(i)} p_k(x),\ i = 1, \ldots, n,$$

convergent absolutely for $|x| < r$, we have

$$\det(f_i(x_j))_{1\le i,j\le n} = V(x_1,\ldots,x_n)\sum_{\lambda_1\ge\cdots\ge\lambda_n\ge 0} b_\lambda P_\lambda(\underline{x}),$$

where the series converges for $|x_j| < r$ *and*

$$b_\lambda = \frac{1}{c_\lambda}\det(c^{(i)}_{\lambda_j+n-j})_{1\le i,j\le n}.$$

Proof We follow the proof of a similar formula for monomials x^m instead of $p_m(x)$, given in [Far06], p.17. We have

$$\begin{aligned}
\det\left(f_i(x_j)\right)_{1\le i,j\le n} &= \sum_{\sigma\in S_n}\epsilon(\sigma) f_1(x_{\sigma(1)})\ldots f_n(x_{\sigma(n)}) \\
&= \sum_{\sigma\in S_n}\epsilon(\sigma)\left(\sum_{k_1=0}^{\infty} c^{(1)}_{k_1} p_{k_1}(x_{\sigma(1)})\right)\ldots\left(\sum_{k_n=0}^{\infty} c^{(n)}_{k_n} p_{k_n}(x_{\sigma(n)})\right) \\
&= \sum_{k_1,\ldots k_n=0}^{\infty} c^{(1)}_{k_1}\ldots c^{(n)}_{k_n}\sum_{\sigma\in S_n}\epsilon(\sigma) p_{k_1}(x_{\sigma(1)})\ldots p_{k_n}(x_{\sigma(n)}) \\
&= \sum_{k_1>\cdots>k_n\ge 0}\sum_{\tau\in S_n}\epsilon(\tau) c^{(1)}_{k_{\tau(1)}}\ldots c^{(n)}_{k_{\tau(n)}}\det((p_{k_i}(x_j))_{1\le i,j\le n}) \\
&= \sum_{k_1>\cdots>k_n\ge 0}\det(c^{(i)}_{k_j})\det((p_{k_i}(x_j))_{1\le i,j\le n})
\end{aligned}$$

Changing the sum indices $k_i = \lambda_i + n - i$, we get

$$\begin{aligned}
\det\left(f_i(x_j)\right)_{1\le i,j\le n} &= \sum_{\lambda_1\ge\cdots\ge\lambda_n\ge 0}\det((c^{(i)}_{\lambda_j+n-j})_{1\le i,j\le n})\det((p_{\lambda_i+n-i}(x_j))_{1\le i,j\le n}) \\
&= V(x_1,\ldots,x_n)\sum_{\lambda_1\ge\cdots\ge\lambda_n\ge 0} b_\lambda P_\lambda(x_1,\ldots,x_n).
\end{aligned}$$

□

We give now an application of the Proposition 3.4. Recall that the Legendre polynomials

$$P_m(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2-1)^m$$

form an orthogonal basis of $L^2(-1,1)$, i.e. μ is the Lebesgue measure restrained to $(-1,1)$. Denote by Q_m the Legendre functions of second kind. The Heine's formula ([Erd], 3.10(10), p. 168) says that for $x \ne y$, $x, y \in \mathbb{R}$

$$\frac{1}{y-x} = \sum_{k=0}^{\infty}(2k+1)P_k(x)Q_k(y). \tag{3.13}$$

We will generalize this formula to the Hermitian matrix setting. For a partition λ, let P_λ and Q_λ be the corresponding symmetric functions defined by

$$\begin{aligned}
P_\lambda(x_1,\ldots,x_n) &= \frac{\det(P_{\lambda_i+n-i}(x_j))_{i,j=1,\ldots,n}}{V(x_1,\ldots,x_n)}, \\
Q_\lambda(x_1,\ldots,x_n) &= \frac{\det(Q_{\lambda_i+n-i}(x_j))_{i,j=1,\ldots,n}}{V(x_1,\ldots,x_n)}.
\end{aligned}$$

According to the Corollary 3.1, the generalized Legendre polynomials $\hat{P}_\lambda$ form an orthogonal basis of the space $L^2_{U_n}(B(0,1), m(dx))$. The unit ball $B(0,1)$ is with respect to the norm $\|X\|$ of $X \in H_n$ considered as a linear functional on $\mathbb{R}^n$, i.e. $\|X\| = \max_i |x_i|$, where x_i are the eigenvalues of X.

Corollary 3.2 *Let $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$ be such that $x_i \neq y_j$, $i, j = 1, \dots, n$. Then*

$$\prod_{i,j=1}^{n} \frac{1}{y_i - x_j} = (-1)^{\frac{n(n-1)}{2}} \sum_{\lambda_1 \geq \cdots \geq \lambda_n \geq 0} d_\lambda P_\lambda(\underline{x}) Q_\lambda(\underline{y})$$

where

$$d_\lambda = \prod_{i=1}^{n} (2(\lambda_i + n - i) + 1).$$

Proof We apply the Proposition 3.4 to the functions

$$f_i(x) = \frac{1}{y_i - x}$$

and we use the expansion (3.13), thus we have $c_k^{(i)} = (2k+1) Q_k(y_i)$. We get that

$$\det\left(\frac{1}{y_i - x_j}\right)_{i,j} = V(x_1, \dots, x_n) V(y_1, \dots, y_n) \sum_{\lambda_1 \geq \cdots \geq \lambda_n \geq 0} d_\lambda P_\lambda(\underline{x}) Q_\lambda(\underline{y})$$

with $d_\lambda = \prod_{i=1}^{n} (2(\lambda_i + n - i) + 1)$. By the Cauchy's determinant formula (see e.g. [Mac91](7.6))

$$\det\left(\frac{1}{y_i - x_j}\right)_{i,j} = V(-\underline{x}) V(\underline{y}) \prod_{i,j=1}^{n} \frac{1}{y_i - x_j} = (-1)^{\frac{n(n-1)}{2}} V(\underline{x}) V(\underline{y}) \prod_{i,j=1}^{n} \frac{1}{y_i - x_j},$$

and we get the expansion formula of the Corollary. □

3.4 Examples

Example 3.1 Hermite polynomials.

Let us consider the family $\{h_m\}_{m \in \mathbb{N}}$ of monic Hermite orthogonal polynomials in $\mathbb{R}$. They are orthogonal polynomials with respect to the measure $\gamma(dy) = e^{-y^2} dy$. According to (3.7) and the Proposition 3.2, we define the monic Hermite polynomials on H_n by

$$\hat{H}_\lambda(X) = \frac{\det(h_{\lambda_i + n - i}(x_j))_{i,j=1,\dots,n}}{V(x_1, \dots, x_n)}$$

where $x_1, \dots, x_n$ are the eigenvalues of the matrix X. They form an orthogonal basis of the space $L^2_{U_n}(H_n, e^{-\mathrm{tr}(X^2)} dm(X))$, cf. the Corollary 3.1. Recall that the condition "H_λ monic" means that

$$H_\lambda = \sum_{\kappa \preceq_{gl} \lambda} c_{\kappa\lambda} S_\lambda,$$

with $c_{\lambda\lambda} = 1$, cf. (3.9), and also that the leading term coefficient in the monomial decomposition (3.11) of H_λ is equal to 1.

In [Las91b] Lassalle also considered generalized Hermite polynomials, but he used decompositions in the normalized Schur functions basis $S^*_\lambda = \frac{S_\lambda}{S_\lambda(1^n)}$, where $1^n = (1, \dots, 1)$. Let us call H^*_λ the Hermite polynomials in [Las91b]. It follows from [Las91b], (i) p.580 that the family $\{H^*_\lambda\}$ is obtained from the Schur polynomials $\{S_\lambda\}$ ordered in the graded lexicographic order, by the Gram–Schmidt orthogonalization process. The same is true for the family $\{H_\lambda\}$. Thus H^*_λ and H_λ differ only by a non–zero factor. If ones requires H^*_λ to be monic in the basis S^*_λ, we have $H^*_\lambda = c_\lambda H_\lambda$, with $c_\lambda = \frac{1}{S_{\lambda(1^n)}}$. In this way we prove the Théorème 6 of [Las91b], communicated by the author without proof.

Also, if $h^{(a)}_m$ are monic Hermite polynomials generalized in the sense of Chihara, i.e. they are orthogonal with respect to the measure $\gamma_a(dy) = |y|^{2a} e^{-y^2} dy$, $a \geq 0$, then the monic Hermite–Chihara polynomials of Hermitian matrix argument are defined by the formula

$$\hat{H}^{(a)}_\lambda(X) = \frac{\det(h^{(a)}_{\lambda_i+n-i}(x_j))_{i,j=1,\dots,n}}{V(x_1, \dots, x_n)},$$

where $x_1, \dots, x_n$ are the eigenvalues of the matrix X. By Corollary 3.1, this family forms an ortogonal basis of the space $L^2_{U_n}(H_n, |\det X|^{2a} e^{-\mathrm{tr}(X^2)} dm(X))$.

Example 3.2 Laguerre polynomials.

The Laguerre polynomials $L^{(\alpha)}_m$, $\alpha > -1$, are orthogonal polynomials on $(0, \infty)$ with respect to the measure $\mu_\alpha(dy) = y^\alpha e^{-y} \mathbf{1}_{(0,\infty)}(y) dy$. Let us normalize them by the condition $L^{(\alpha)}_m(0) = 1$. Then they have the following explicit representation, see [Sz59](5.1.6) and (5.1.7),

$$L^{(\alpha)}_m(y) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-y)^k}{(\alpha+1)_k}.$$

The Laguerre polynomials $\hat{L}^{(\alpha)}_\lambda(X)$ of Hermitian matrix argument are given, according to the formula (3.7), by

$$\hat{L}^{(\alpha)}_\lambda(X) = c_\lambda \frac{\det(L^{(\alpha)}_{\lambda_i+n-i}(x_j))_{i,j=1,\dots,n}}{V(x_1, \dots, x_n)} \tag{3.14}$$

where $x_1, \dots, x_n$ are the eigenvalues of the matrix X.

The measure $M_\alpha(dX) = (\det X)^\alpha e^{-\mathrm{tr} X} \prod_i \mathbf{1}_{(0,\infty)}(x_i) dm(X)$, corresponding to μ_α via the formula (4.4), is supported on the cone H^+_n of non–negative definite Hermitian matrices. The polynomials $\hat{L}^{(\alpha)}_\lambda$ form an orthogonal basis of $L^2_{U_n}(H^+_n, M_\alpha)$. We normalize them by setting $L_\lambda(\underline{0}) = 1$. The constant c_λ

may be determined using Proposition 3.3, for the coefficients $c_k^{(m)} = \binom{m}{k}\frac{(-1)^k}{(\alpha+1)_k}$. We compute

$$\begin{aligned}\det(c_{n-j}^{(\lambda_i+n-i)})_{1\le i,j\le n} &= \det\left(\binom{\lambda_i+n-i}{n-j}\frac{(-1)^{n-j}}{(\alpha+1)_{n-j}}\right)_{1\le i,j\le n}\\ &= \det\left(\frac{(\lambda_i+n-i)!}{(\lambda_i-i+j)!}\right)_{1\le i,j\le n}\prod_{j=0}^{n-1}\frac{(-1)^j}{(\alpha+1)_j j!}.\end{aligned}$$

Set $t_i = \lambda_i + n - i$. Then

$$R_j(t_i) := \frac{(\lambda_i+n-i)!}{(\lambda_i-i+j)!} = \frac{t_i!}{(t_i-n+j)!} = t_i(t_i-1)\dots(t_i-n+j+1)$$

when $j < n$. For $j = n$ we have $R_n = 1$. Thus

$$\det\left(\frac{(\lambda_i+n-i)!}{(\lambda_i-i+j)!}\right)_{1\le i,j\le n} = \det\left(R_j(t_i)\right)_{1\le i,j\le n}$$

where the polynomials R_j have the degree $n-j$ and are monic. Multilinearity properties of determinant imply that $\det\left(R_j(t_i)\right)_{1\le i,j\le n} = \det(t_i^{n-j})_{1\le i,j\le n}$, so it is equal to the Vandermonde determinant in t_i,

$$V(t_1,\dots,t_n) = \prod_{1\le i<j\le n}(t_i-t_j) = \prod_{1\le i<j\le n}(\lambda_i-\lambda_j-i+j).$$

Thus we find

$$c_\lambda = (-1)^{\frac{(n-1)n}{2}}\frac{\prod_{j=0}^{n-1}(\alpha+1)_j j!}{\prod_{1\le i<j\le n}(\lambda_i-\lambda_j-i+j)}$$

As $\prod_{j=0}^{n-1}(\alpha+1)_j j! = \prod_{1\le i<j\le n}(\alpha+j-i)i$, we can also write

$$c_\lambda = (-1)^{\frac{(n-1)n}{2}}\prod_{1\le i<j\le n}\frac{(\alpha+j-i)i}{\lambda_i-\lambda_j-i+j}.$$

The formula (3.14) for generalized symmetric Laguerre polynomials, with c_λ given by the last equality, was announced in [Las91c], Théorème 6, without proof.

Example 3.3 Jacobi polynomials.

For $a > -1$ and $b > -1$, the classical Jacobi polynomials $P_m^{(a,b)}$ are orthogonal polynomials on $[-1,1]$ with respect to the measure $\nu_{a,b}(dx) = (1-x)^a(1+x)^b\mathbf{1}_{[-1,1]}(x)dx$. We will consider a related family of Jacobi polynomials $J_m^{(a,b)}(y) := P_m^{(a,b)}(1-2y)$. The polynomials $J_m^{(a,b)}(y)$ are orthogonal on $[0,1]$, with respect to the measure $\mu_{a,b}(dx) = y^a(1-y)^b\mathbf{1}_{[0,1]}(y)dy$. If they are normalized by the condition $J_m^{(a,b)}(0) = 1$, then, using [Sz59](4.21.2), they have the monomial representation

$$J_m^{(a,b)}(y) = \sum_{k=0}^{m}\frac{(m+a+b+1)_k}{(a+1)_k}\binom{m}{k}(-y)^k.$$

The Jacobi polynomials $\hat{J}_\lambda^{(a,b)}(X)$ of Hermitian matrix argument are given, according to the formula (3.7), by

$$\hat{J}_\lambda^{(\alpha)}(X) = c_\lambda \frac{\det(J^{(a,b)}_{\lambda_i+n-i}(x_j))_{i,j=1,\dots,n}}{V(x_1,\dots,x_n)} \tag{3.15}$$

where $x_1,\dots,x_n$ are the eigenvalues of the matrix X.

The measure $M_{(a,b)}(dX) = (\det X)^a(\det(I-X))^b \prod_i \mathbf{1}_{[0,1]}(x_i)dm(X)$ corresponding to $\mu_{a,b}$ via the formula (4.4), is concentrated on the intersection $H_n^+ \cap \bar{B}(0,1)$ of the cone H_n^+ with the unit ball $\bar{B}(0,1)$ in H_n. The polynomials $\hat{J}_\lambda^{(a,b)}$ form an orthogonal basis of $L^2_{U_n}(H_n^+ \cap \bar{B}(0,1), M_{(a,b)})$. We normalize them by setting $J_\lambda^{(a,b)}(\underline{0}) = 1$. We apply the Proposition 3.3 in order to compute the constant c_λ in (3.15). The coefficients $c_k^{(m)}$ are equal $c_k^{(m)} = \frac{(m+a+b+1)_k}{(a+1)_k}\binom{m}{k}(-1)^k$. We obtain

$$\det(c^{(\lambda_i+n-i)}_{n-j})_{1\le i,j\le n} = \det\left(\frac{(\lambda_i+n-i+a+b+1)_{n-j}}{(a+1)_{n-j}}\binom{\lambda_i+n-i}{n-j}(-1)^{n-j}\right)_{1\le i,j\le n}$$
$$= \det\left(\frac{(\lambda_i+n-i+a+b+1)_{n-j}(\lambda_i+n-i)!}{(\lambda_i-i+j)!}\right)_{1\le i,j\le n}\prod_{j=1}^n\frac{(-1)^{n-j}}{(a+1)_{n-j}(n-j)!}.$$

Setting $t_i = \lambda_i + n - i$ and $A = a+b+1$, the previous determinant can be written as

$$D := \det\left(\frac{(t_i+A)_{n-j}t_i!}{(t_i-n+j)!}\right)_{1\le i,j\le n} = \det\left(\prod_{m=0}^{n-j-1}(t_i+A+m)(t_i-m)\right)_{1\le i,j\le n}.$$

Taking $t_i = t_j, 1 \le i,j \le n$, this determinant vanishes and therefore it is divisible by $\prod_{1\le i<j\le n}(t_i - t_j)$. If we take $t_i = -t_j - A, 1\le i,j\le n$, the determinant also vanishes, so it is divisible by $\prod_{1\le i<j\le n}(t_i + t_j + A)$. Thus

$$D = [\prod_{1\le i<j\le n}(t_i-t_j)(t_i+t_j+A)]R(t_1,\dots,t_n)$$

where R is a polynomial. When we fix $t_2,\dots,t_n$ and consider D as a polynomial of t_1, we see that it is monic and of degree $2n-2$. The same is true for $\prod_{1\le i<j\le n}(t_i-t_j)(t_i+t_j+A)$. Thus the polynomial R does not depend on t_1. Repeating this argument for all t_i we deduce that $R=1$. Finally we get

$$c_\lambda = (-1)^{\frac{(n-1)n}{2}}\prod_{1\le i<j\le n}\frac{(\alpha+j-i)i}{(\lambda_i-\lambda_j-i+j)(\lambda_i+\lambda_j+2n-i-j+a+b+1)}.$$

The formula (3.15) for generalized symmetric Jacobi polynomials, with c_λ as in the last equality, was given without proof in [Las91a], Théorème 10.

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Chapter 4

Semigroups associated to generalized Polynomials and some classical formulas

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Abstract: We study operator semigroups associated with a family of generalized orthogonal polynomials with Hermitian matrix entries. For this we consider a Markov generator sequence, and therefore a Markov semigroup, for the family of orthogonal polynomials on $\mathbb{R}$ related to the generalized polynomials. We give an expression of the infinitesimal generator of this semigroup and under the hypothesis of diffusion we prove that this semigroup is also Markov. We also give expressions for the kernel of this semigroup in terms of the one-dimensional kernels and obtain some classical formulas for the generalizad orthogonal polynomials from the correspondent formulas for orthogonal polynomials on $\mathbb{R}$.

Key words: Generalized orthogonal polynomials, Schur functions, Markov semigroups.

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4.1 Introduction

In a previous paper, [BGU05], we defined a family of generalized orthogonal polynomials with Hermitian matrix argument in the following way. Let μ be a finite measure on the real line such that the set of polynomials is dense in $L^2(\mathbb{R},\mu)$. This condition is satisfied if, for example, μ has an exponential moment, that is there exists $\epsilon > 0$ such that $\int_{\mathbb{R}} e^{\epsilon|x|}\mu(dx) < \infty$ (c.f. [BC81], [DX01]). We consider the measure μ_n on $\mathbb{R}^n$ defined by

$$\mu_n(d\underline{x}) = V^2(x_1,\dots,x_n)\mu(dx_1)\dots\mu(dx_n), \tag{4.1}$$

where

$$V(x_1,\dots,x_n) = \det(x_j^{n-i}) = \prod_{1\le i<j\le n}(x_i - x_j)$$

is the Vandermonde determinant. The measure μ_n is a permutation invariant measure on $\mathbb{R}^n$. We will also require the set of symmetric polynomials on $\mathbb{R}^n$ to be dense in $L^2_{\text{sym}}(\mathbb{R}^n,\mu_n)$, the space of all symmetric functions on $L^2(\mathbb{R}^n,\mu_n)$. If μ has an exponential moment, then this condition is guaranteed.

Let $\{p_m\}_{m\in\mathbb{N}}$ be a family of orthogonal polynomials in $L^2(\mathbb{R},\mu)$, with $deg(p_m) = m$. This can always be found by using the Gram-Schmidt orthogonalization process. Let us denote by Λ the set of all n-partitions, that is

$$\Lambda = \{\lambda = (\lambda_1,\dots,\lambda_n) : \lambda_i \in \mathbb{N}, \lambda_1 \ge \cdots \lambda_n \ge 0\}.$$

For $\lambda \in \Lambda$, in [BGU05] we defined a function on $\mathbb{R}^n$ by

$$P_\lambda(x_1,\dots,x_n) := c_\lambda \frac{\det(p_{\lambda_i+n-i}(x_j))}{V(x_1,\dots,x_n)}, \tag{4.2}$$

where c_λ is a normalizing constant that depends on the normalization chosen for the polynomials p_m and P_λ. Sometimes different normalizations of orthogonal polynomials are needed, see [BGU05], [Las91c]. That is why we maintain in this section the general notation c_λ for the normalization constant.

Theorem 3.1 in [BGU05] states that P_λ are symmetric polynomials, orthogonal in the Hilbert space $L^2_{\text{sym}}(\mathbb{R}^n,\mu_n)$ with norm

$$\|P_\lambda\|^2_{L^2_{\text{sym}}(\mathbb{R}^n,\mu_n)} = c_\lambda^2 n! \prod_{i=1}^n \|p_{\lambda_i+n-i}\|^2_{L^2(\mathbb{R},\mu)}, \tag{4.3}$$

and the family $\{P_\lambda\}_{\lambda\in\Lambda}$, is dense in $L^2_{\text{sym}}(\mathbb{R}^n,\mu_n)$, thus it forms an orthogonal Hilbert basis of this space. The polynomials $\{P_\lambda\}_{\lambda\in\Lambda}$ can be obtained by the Gram–Schmidt orthogonalization process, applied to the Schur polynomials family $\{S_\lambda\}_{\lambda\in\Lambda}$, ordered in the graded lexicographic order $\succ_{gl}$.

Let H_n be the space of Hermitian matrices and let $f : H_n \to \mathbb{R}$ be a central function on H_n, that is $f(UXU^{-1}) = f(X)$ for all unitary matrices U. Thus $f(X)$ depends only on the eigenvalues of the matrix X and therefore f is uniquely determinated by its values over the diagonal matrices. So, if f is central, the function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ defined by $\tilde{f}(x_1, \dots, x_n) = f(diag(x_1, \dots, x_n))$ is a symmetric function on $\mathbb{R}^n$. Moreover, this map is a bijection from the space of central functions on H_n onto the space of symmetric functions on $\mathbb{R}^n$.

If P is a symmetric polynomial in $\mathbb{R}^n$, let $\hat{P}$ be the central function on H_n such that its restriction to the diagonal matrices is equal to P. We call $\hat{P}$ a generalized polynomial with Hermitian matrix argument. Most properties of a generalized polynomial are derived from the corresponding properties of the associated symmetric polynomial.

To any Borel measure μ on $\mathbb{R}$, using Weyl's integral formula c.f. [Far06], we associate a measure M on H_n such that

$$\begin{aligned}\int_{H_n} f(X)dM(X) &= \int_{\mathbb{R}^n} f(diag(x_1, \dots, x_n))V^2(x_1, \dots, x_n)d\mu(x_1)\dots d\mu(x_n) \\ &= \int_{\mathbb{R}^n} \tilde{f}(\underline{x})d\mu_n(x) \qquad (4.4)\end{aligned}$$

for any positive central function f on H_n. For further reference on the construction of this measure, see [BGU05].

Let $L^2_{U_n}(H_n, M)$ be the space of all central functions on $L^2(H_n, M)$. It is clear that this space is isomorphic to the space $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$. The main result of [BGU05], Corollary 3.2, states that the family of generalized polynomials $\{\hat{P}_\lambda\}_{\lambda\in\Lambda}$ associated to the orthogonal polynomials over $\mathbb{R}^n$ given by (4.2) is an orthogonal Hilbert basis of this space. For more details of this construction see [BGU05].

A very useful result given in [BGU05] is Proposition 3.6, that allows us to find the coefficients in the expansion over the family $\{P_\lambda\}_{\lambda\in\Lambda}$ of an important class of central functions. Since it is going to be used repeatedly in what follows, we recall it here.

Proposition 4.1 (Proposition 3.6 **of [BGU05])** *Given n functions $f_1, \dots, f_n$ of one variable, with the expansions in the basis $\{p_m\}_{m\in\mathbb{N}}$*

$$f_i(x) = \sum_{k=0}^{\infty} c_k^{(i)} p_k(x), \ i = 1, \dots, n,$$

convergent absolutely for $|x| < r$, we have

$$\det(f_i(x_j))_{1\le i,j\le n} = V(x_1, \dots, x_n) \sum_{\lambda\in\Lambda} b_\lambda P_\lambda(\underline{x}),$$

where the series converges for $|x_j| < r$ and

$$b_\lambda = \frac{1}{c_\lambda} \det(c^{(i)}_{\lambda_j+n-j})_{1\leq i,j\leq n}.$$

As said before, the normalizing constant c_λ in the definition of P_λ given in (4.2) depends on the normalization chosen for the polynomials p_m and P_λ. In this article we are going to fix the following normalizations: $\|p_m\|_{L^2(\mathbb{R},\mu)} = 1$ for all $m \in \mathbb{N}$ and p_m has positive leading coefficient and $\|P_\lambda\|_{L^2_{\text{sym}}(\mathbb{R}^n,\mu_n)} = 1$ for all $\lambda \in \Lambda$. Then by (4.3), we have that $c_\lambda = \frac{1}{\sqrt{n!}}$ and

$$P_\lambda(\underline{x}) = \frac{1}{\sqrt{n!}} \frac{\det(p_{\lambda_i+n-i}(x_j))}{V(\underline{x})}.$$

The article is organized as follows. In Section 4.2 we define a semigroup associated to the family of orthogonal polynomials over H_n and we give an expression of the infinitesimal generator of this semigroup in Section 4.3. In Section 4.4 we see that for the classical families of orthogonal polynomials on $\mathbb{R}$, the associated semigroup is Markov. In order to provide this result we consider separately the continuous and discrete cases and we use a characterization result of O. Mazet [Maz02]. Then in Section 4.5 we present an expression of the kernels of the semigroup defined in Section 4.2 in terms of the kernels of the semigroups associated to the one–dimensional polynomials. Section 4.6 is devoted to generalize some classical formulas for orthogonal polynomials to this new family of polynomials. Finally, in Section 4.7 we present as examples the families of classical orthogonal polynomial of continuous and discrete variable, obtaining, in the continuous case, some expressions given without proof by M. Lassale in [Las91a], [Las91b] and [Las91c] and some new formulas, in the discrete case.

4.2 Semigroup associated to generalized polynomials

Let us begin this section with some preliminaries on semigroups associated to a family of polynomials and Markov semigroups. Let μ be a measure on $\mathbb{R}$ as before. We say that an operator S in $L^2(\mathbb{R},\mu)$ is Markov, or that satisfies the Markov condition, if $S(1) = 1$ and S maps positive functions into positive functions.

If $\{p_m\}_{m\in\mathbb{N}}$ is the family of ortogonal polynomials in $L^2(\mathbb{R},\mu)$, normalized so $\|p_m\|_{L^2(\mathbb{R},\mu)} = 1$, as in [BM03] we say that the sequence of real numbers $\{c_m\}_{m\in\mathbb{N}}$ is a Markov sequence for the family of polynomials $\{p_m\}_{m\in\mathbb{N}}$ if there exists a Markov operator S in $L^2(\mathbb{R},\mu)$ such that $S(p_m) = c_m p_m$ for all $m \in \mathbb{N}$. Then we have that the operator S has the family $\{p_m\}_{m\in\mathbb{N}}$ as spectral decomposition and it is symmetric in $L^2(\mathbb{R}, \mu)$, or equivalently, that μ is the invariant measure for the operator S. This, together with the fact that S is conservative, implies that μ is the invariant measure of S, that is

$$\int Sf d\mu = \int f d\mu, \ \forall f \in L^1(\mathbb{R}, \mu). \tag{4.5}$$

In consequence, S is a contraction operator in $L^p(\mathbb{R}, \mu)$ for all $1 \leq p \leq \infty$ and $c_\alpha \in [-1, 1]$ for all α.

If $\{c_m\}_{m\in\mathbb{N}}$ is square summable, that is $\sum_{m\in\mathbb{N}} c_m^2 < \infty$, then S is a Hilbert-Schmidt operator. Thus it can be represented as

$$S(f)(x) = \int f(y)\mathcal{S}(x, y)d\mu(y)$$

where

$$\mathcal{S}(x, y) = \sum_{m\in\mathbb{N}} c_m p_m(x)p_m(y)$$

is a positive kernel in $L^2(\mu \otimes \mu)$.

We say that the sequence of real numbers $\{\gamma_m\}_{m\in\mathbb{N}}$ is a Markov generator sequence for the family of polynomials $\{p_m\}_{m\in\mathbb{N}}$ if for every $t \geq 0$ the sequence $\{e^{-\gamma_m t}\}_{m\in\mathbb{N}}$ is a Markov sequence for $\{p_m\}_{m\in\mathbb{N}}$. Then, there exists a family of Markov operators $\{N_t\}_{t\geq 0}$ such that for each t the operator N_t is a contraction, $N_t(p_m) = e^{-\gamma_m t} p_m$ and $\gamma_m \geq 0$ for all $m \in \mathbb{N}$. If for each $t > 0$, we have that $\sum e^{-2\gamma_m t} < \infty$, as before, we have that each N_t is a Hilbert-Schmidt operator and can be represented as

$$N_t(f)(x) = \int f(y)\mathcal{N}_t(x, y)d\mu(y) \tag{4.6}$$

where

$$\mathcal{N}_t(x, y) = \sum_{m\in\mathbb{N}} e^{-\gamma_m t} p_m(x)p_m(y) \tag{4.7}$$

It is not difficult to see that

$$\int \mathcal{N}_t(x, y)\mathcal{N}_s(y, z)d\mu(y) = \mathcal{N}_{t+s}(x, z),$$

thus $\{N_t\}_{t\geq 0}$ is a Markov semigroup with invariant measure μ and spectral decomposition over the family of polynomials $\{p_m\}_{m\in\mathbb{N}}$.

If L is the infinitesimal generator of the semigroup $\{N_t\}_{t\geq 0}$, that is

$$Lf = \lim_{t\to 0} \frac{N_t f - f}{t}, \qquad f \in \mathcal{D}(L),$$

with $\mathcal{D}(L)$ a dense subset of $L^2(\mathbb{R}, \mu)$, then $L(p_m) = -\gamma_m p_m$, thus it is a symmetric operator in $L^2(\mathbb{R}, \mu)$ with spectral decomposition over the family $\{p_m\}_{m\in\mathbb{N}}$. The invariance of the measure μ for $\{N_t\}_{t\geq 0}$ can be expressed in terms of the operator L as $\int Lf d\mu = 0$ for all f in $\mathcal{D}(L)$.

A detailed study of the Markov generator sequences associated to the classical families of orthogonal polynomials can be found in [BM03].

We start with a Markov generator sequence (and therefore with a Markov semigroup) associated to the family of orthogonal polynomials $\{p_m\}_{m\in\mathbb{N}}$ and define a semigroup with spectral decomposition given by the generalized orthogonal polynomials. Later we will give conditions so that this semigroup is also Markov.

Theorem 4.1 *Let $\{\gamma_m\}_{m\in\mathbb{N}}$ be an increasing, square summable Markov generator sequence for the family $\{p_m\}_{m\in\mathbb{N}}$ of orthogonal polynomials on $L^2(\mathbb{R},\mu)$. Let us define*

$$\mathcal{T}_t(\underline{x},\underline{y}) := \sum_{\lambda\in\Lambda} e^{-t(\sum_{j=1}^n \gamma_{\lambda_j+n-j}-\sum_{j=1}^n \gamma_{n-j})} P_\lambda(\underline{x})P_\lambda(\underline{y}), \tag{4.8}$$

and for $f\in L^2_{\mathrm{sym}}(\mathbb{R}^n,\mu_n)$

$$T_t(f)(\underline{x}) := \int_{\mathbb{R}^n} f(\underline{y})\mathcal{T}_t(\underline{x},\underline{y})\mu_n(dy). \tag{4.9}$$

Then the family of operators $\{T_t\}_{t\geq 0}$ is a semigroup in $L^2_{\mathrm{sym}}(\mathbb{R}^n,\mu_n)$ with spectral decomposition over $\{P_\lambda\}_{\lambda\in\Lambda}$ with eigenvalues $e^{-t\varphi_\lambda}$ where

$$\varphi_\lambda = \sum_{j=1}^n \gamma_{\lambda_j+n-j} - \sum_{j=1}^n \gamma_{n-j} \geq 0 \tag{4.10}$$

and with symmetric measure μ_n.

We also have that $\{T_t\}_{t\geq 0}$ is a conservative semigroup, that is $T_t 1 = 1$ for all $t\geq 0$, where 1 is the constant function equal to 1 in $L^2_{\mathrm{sym}}(\mathbb{R}^n,\mu_n)$. Therefore μ_n is the invariant measure for $\{T_t\}_{t\geq 0}$.

Proof

First note that since for each $t\geq 0$ the sequence $\{e^{-\gamma_m t}\}_{m\in\mathbb{N}}$ is square summable, then also for each $t\geq 0$ we have that $\sum_{\lambda\in\Lambda} e^{-2t\varphi_\lambda} < \infty$. Thus, the kernel $\mathcal{T}_t(\underline{x},\underline{y})$ is in $L^2(\mu_n\otimes\mu_n)$ and therefore, T_t is a bounded operator.

By the orthogonality of the family $\{P_\lambda\}_{\lambda\in\Lambda}$ on $L^2_{\mathrm{sym}}(\mathbb{R}^n,\mu_n)$, we have that

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{T}_t(\underline{x},\underline{y})\mathcal{T}_s(\underline{y},\underline{z})d\mu_n(\underline{y}) &= \int_{\mathbb{R}^n}\left(\sum_{\lambda\in\Lambda} e^{-t\varphi_\lambda}P_\lambda(\underline{x})P_\lambda(\underline{y})\right)\left(\sum_{\kappa\in\Lambda} e^{-t\varphi_\kappa}P_\kappa(\underline{y})P_\kappa(\underline{z})\right)d\mu_n(\underline{y})\\ &= \sum_{\lambda\in\Lambda}\sum_{\kappa\in\Lambda} e^{-t\varphi_\lambda}e^{-t\varphi_\kappa}P_\lambda(\underline{x})P_\kappa(\underline{z})\int_{\mathbb{R}^n} P_\lambda(\underline{y})P_\kappa(\underline{y})d\mu_n(\underline{y})\\ &= \sum_{\lambda\in\Lambda} e^{-(t+s)\varphi_\lambda}P_\lambda(\underline{x})P_\lambda(\underline{z}) = \mathcal{T}_{t+s}(\underline{x},\underline{y}),\end{aligned}$$

that is, the family of kernels $\{\mathcal{T}_t\}_{t\geq 0}$ satisfy the Chapman–Kolmogorov equation. Thus $\{T_t\}_{t\geq 0}$ satisfies the semigroup property. Moreover, also from the orthogonality of $\{P_\lambda\}_{\lambda\in\Lambda}$ on $L^2_{\text{sym}}(\mathbb{R}^n,\mu_n)$ it is immediate that

$$T_t(P_\lambda)(\underline{x}) = e^{-t\varphi_\lambda}P_\lambda(\underline{x}) \tag{4.11}$$

that is, P_λ is an eigenfunction of T_t with eigenvalues $e^{-t\varphi_\lambda}$.

On the other hand, clearly $\mathcal{T}_t(\underline{x},\underline{y}) = \mathcal{T}_t(\underline{y},\underline{x})$ and therefore, T_t is a symmetric operator in $L^2_{\text{sym}}(\mathbb{R}^n,\mu_n)$. Or equivalently, μ_n is the symmetric measure for the semigroup $\{T_t\}_{t\geq 0}$.

It remains to be proven that T_t maps 1 into 1. Because of the normalization chosen for P_λ we have that $P_{\underline{0}} = \frac{1}{\sqrt{\mu_n(\mathbb{R}^n)}}$, with $\underline{0}$ the partition $\underline{0} = (0,\ldots,0)$; and also $\varphi_{\underline{0}} = 0$. So by (4.11)

$$T_t(1) = \sqrt{\mu_n(\mathbb{R}^n)}T_t(P_{\underline{0}}) = e^{-t\varphi_{\underline{0}}}\sqrt{\mu_n(\mathbb{R}^n)}P_{\underline{0}} = 1.$$

□

Now let us define a semigroup in $L^2_{U_n}(H_n,M)$ with spectral decomposition given by the generalized polynomials $\{\hat{P}_\lambda\}_{\lambda\in\Lambda}$. For an operator T on $L^2_{\text{sym}}(\mathbb{R}^n,\mu_n)$ let $\hat{T}$ be the operator on $L^2_{U_n}(H_n,M)$ such that for all $h\in L^2_{U_n}(H_n,M)$

$$\hat{T}h\big|_{Diag} = T(h|_{Diag}) = T(\tilde{h}),$$

where $Diag$ is the space of all $n\times n$ diagonal matrices.

Note that if $T = f_1(\alpha T_1+\beta T_2)f_2$ with f_1,f_2 symmetric functions, T_1,T_2 operators on $L^2_{\text{sym}}(\mathbb{R}^n,\mu_n)$ and α,β constants, then $\hat{T} = \hat{f}_1(\alpha\hat{T}_1+\beta\hat{T}_2)\hat{f}_2$ where $\hat{f}_i$ is the central function whose restriction to the diagonal matrices is equal to f_i.

Corollary 4.1 *The family of operators $\{\hat{T}_t\}_{t\geq 0}$ on $L^2_{U_n}(H_n,M)$ is a conservative semigroup with spectral decomposition given by the generalized polynomials $\{\hat{P}_\lambda\}_{\lambda\in\Lambda}$ with eigenvalues $e^{-t\varphi_\lambda}$ with φ_λ given in (4.10). The measure M is the symmetric and invariant measure for this semigroup. We have the representation*

$$\hat{T}h(X) = \int_{H_n} h(Y)\hat{\mathcal{T}}(X,Y)dM(Y) \tag{4.12}$$

where

$$\hat{\mathcal{T}}(X,Y) = \sum_{\lambda\in\Lambda} e^{-t\varphi_\lambda}\hat{P}_\lambda(X)\hat{P}_\lambda(Y). \tag{4.13}$$

4.3 Infinitesimal Generator

In this Section we are going to find an expression for the infinitesimal generator of the semigroup $\{T_t\}_{t\geq 0}$ defined in (4.9) in terms of the infinitesimal generator L of the semigroup $\{N_t\}_{t\geq 0}$. Let L_k be the operator on $L^2(\mathbb{R}^n, \mu_n)$ that acts as L over the k-th coordinate. For a symmetric polynomial q on $\mathbb{R}^n$ define the operator D_q on $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$ by

$$D_q = \frac{1}{V} q(L_1, \ldots, L_n) V. \tag{4.14}$$

Proposition 4.2 *For each partition $\lambda \in \Lambda$ the polynomial P_λ is an eigenfunction of the operator D_q with associated $q(-\gamma_{\lambda_1+n-1}, -\gamma_{\lambda_2+n-2}, \ldots, -\gamma_{\lambda_n})$, that is*

$$D_q P_\lambda = q(-\gamma_{\lambda_1+n-1}, -\gamma_{\lambda_2+n-2}, \ldots, -\gamma_{\lambda_n}) P_\lambda. \tag{4.15}$$

Proof Since any symmetric polynomial q in $\mathbb{R}^n$ can be expressed as a linear combination of monomial symmetric functions (see [Mac91]), it is enough to verify (4.15) for the operator D_{m_κ} with $m_\kappa = \sum_{\tau\in S_n} x_1^{\kappa_{\tau(1)}} \ldots x_n^{\kappa_{\tau(n)}}$, the monomial symmetric function associated to a partition $\kappa = (\kappa_1, \ldots, \kappa_n) \in \Lambda$, (here S_n is the symmetric group of $n-$permutations). In this case we have that

$$\begin{aligned}
\sqrt{n!}V(\underline{x})D_{m_\kappa}P_\lambda(\underline{x}) &= m_\kappa(L_1, \ldots, L_n)det(p_{\lambda_i+n-i}(x_j)) \\
&= \sum_{\tau\in S_n} L_1^{\kappa_{\tau(1)}} \ldots L_n^{\kappa_{\tau(n)}} \left(\sum_{\sigma\in S_n} \epsilon(\sigma) \prod_{j=1}^n p_{\lambda_{\sigma(j)}+n-\sigma(j)}(x_j) \right) \\
&= \sum_{\tau\in S_n} \sum_{\sigma\in S_n} \epsilon(\sigma) \prod_{j=1}^n (-\gamma_{\lambda_{\sigma(j)}+n-\sigma(j)})^{\kappa_{\tau(j)}} p_{\lambda_{\sigma(j)}+n-\sigma(j)}(x_j) \\
&= \sum_{\sigma\in S_n} \epsilon(\sigma) \prod_{j=1}^n p_{\lambda_{\sigma(j)}+n-\sigma(j)}(x_j) \left(\sum_{\tau\in S_n} \prod_{i=1}^n (-\gamma_{\lambda_{\sigma(i)}+n-\sigma(i)})^{\kappa_{\tau(i)}} \right) \\
&= \sum_{\sigma\in S_n} \epsilon(\sigma) \prod_{j=1}^n p_{\lambda_{\sigma(j)}+n-\sigma(j)}(x_j) \left(\sum_{\tau\in S_n} \prod_{i=1}^n (-\gamma_{\lambda_i+n-i})^{\kappa_{\tau\circ\sigma^{-1}(j)}} \right) \\
&= \left(\sum_{\nu\in S_n} \prod_{i=1}^n (-\gamma_{\lambda_i+n-i})^{\kappa_{\nu(i)}} \right) \sum_{\sigma\in S_n} \epsilon(\sigma) \prod_{j=1}^n p_{\lambda_{\sigma(j)}+n-\sigma(j)}(x_j) \\
&= m_\kappa(-\gamma_{\lambda_1+n-1}, \ldots, -\gamma_{\lambda_n})det(p_{\lambda_i+n-i}(x_j)),
\end{aligned}$$

that is

$$D_{m_\kappa}P_\lambda(\underline{x}) = m_\kappa(-\gamma_{\lambda_1+n-1}, \ldots, -\gamma_{\lambda_n})P_\lambda(\underline{x}),$$

as wanted. □

In what follows we will work with the symmetric polynomial $q_0(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n + c$ with $c = \sum_{j=1}^{n} \gamma_{n-j}$, a positive constant. The reason of this choice of c will be clear later. From now on we will denote by D the operator D_{q_0}. We have

Theorem 4.2 *The operator D on $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$ defined as*

$$D = \frac{1}{V}(L_1 + \cdots + L_n)V + c \tag{4.16}$$

with $c = \sum_{j=1}^{n} \gamma_{n-j}$ is the infinitesimal generator of the semigroup $\{T_t\}_{t \geq 0}$.

Proof By Proposition 4.2

$$DP_\lambda = -(\sum_{j=1}^{n} \gamma_{\lambda_j + n - j} - c)P_\lambda,$$

that is, in this case the associated eigenvalue is $-\varphi_\lambda$, c.f. (4.10). So, the operator D and the infinitesimal generator of the semigroup $\{T_t\}_{t \geq 0}$ have the same spectral decomposition. By spectral theory of semigroups (see [EN00], Lemma 1.9) and the density of $\{P_\lambda\}_{\lambda \in \Lambda}$ in $L^2_{\text{sym}}(\mathbb{R}^n, \mu_n)$ the result follows.
□

Remark 4.1 *Since $\{T_t\}_{t \geq 0}$ is a conservative semigroup, we have that $D1 = 0$, consequently $(L_1 + \cdots + L_n)V = -cV$ and*

$$c = -\sum_{k=1}^{n} \frac{L_k V}{V}. \tag{4.17}$$

On the cone $C = \{\underline{y} \in \mathbb{R}^n \cap supp(\mu) : y_1 > y_2 > \cdots > y_n\}$, the function V is positive, so $(L_1 + \cdots + L_n)V < 0$ on this cone. The Vandermonde determinant is a positive superharmonic (excessive) function for $L_1 + \cdots + L_n$ on C.

For the semigroup associated to the generalized polynomials we have

Corollary 4.2 *The operator $\hat{D}$ on $L^2_{U_n}(H_n, M)$ is the infinitesimal generator of the semigroup $\{\hat{T}_t\}_{t \geq 0}$.*

4.4 Positivity preserving

We already know that the semigroup $\{T_t\}_{t \geq 0}$ is conservative. Now we are going to see that, in certain cases, it also preserve positivity and therefore, it satisfy the Markov condition. For this we will use a characterization of Markov semigroups with invariant measure given in [Maz02] that involves the "carré du champ" operator associated to the infinitesimal generator of the semigroup.

For the infinitesimal generator L of the semigroup $\{N_t\}_{t\geq 0}$ on $L^2(\mathbb{R},\mu)$, the carré du champ operator of L is the symmetric bilinear form defined by

$$\Gamma(f,g) = \frac{1}{2}(L(fg) - fLg - gLf),\ f,g \in \mathcal{A},$$

for f,g in $\mathcal{A}$, a "standard algebra" in $Dom(L)$. In our setting we can, and will, take $\mathcal{A}$ as the algebra of polynomials. For more details see [ABC$^+$02] and [Bak06].

It is known that for a Markov semigroup, the associated carré du champ operator is positive for all f in $\mathcal{A}$ in the sense that $\Gamma(f,f) \geq 0$ for $f \in \mathcal{A}$. In [Maz02] it is proven that if μ is the invariant measure for the semigroup $\{N_t\}_{t\geq 0}$ then the converse implication is also true, that is, the positivity of the carré du champ operator implies that the semigroup is Markov. We will use this result several times in what follows.

4.4.1 Continuous case: diffusions

Suppose that the measure μ is non-atomic. We say that an operator T on $L^2(\mathbb{R},\mu)$ is a diffusion (see [Maz97] and [ABC$^+$02]) if for polynomials Φ and f

$$T(\Phi(f)) = \Phi''(f)\Gamma(f,f) + \Phi'(f)T(f),$$

where Γ is the carré du champ operator of T.

If the infinitesimal generator L of the semigroup $\{N_t\}_{t\geq 0}$ in $L^2(\mathbb{R},\mu)$ is a diffusion, as a consequence having the family of orthogonal polynomials $\{p_m\}_{m\in\mathbb{N}}$ as eigenfunctions, it can be proven (see [Maz97]) that L has the form

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \tag{4.18}$$

where a and b are polynomials of degree at most 2 and 1, respectively.

It is shown in [Maz97] that the only Markov semigroups on $L^2(\mathbb{R},\mu)$ with an infinitesimal generator that is a diffusion and that has a family of orthogonal polynomials as eigenfunctions are the Ornstein-Uhlenbeck, Laguerre and Jacobi semigroups, associated to the Hermite, Laguerre and Jacobi polynomials, respectively. For more details on these polynomials, see the examples, Section 4.7.

First let us find an expression for the carré du champ operator of the infinitesimal generator D of the semigroup $\{T_t\}_{t\geq 0}$, given in (4.16). Let $\mathcal{A} = alg\{p_0, p_1, \ldots\}$ the algebra in $\mathbb{R}$ generated by the polynomials $\{p_m\}_{m\in\mathbb{N}}$ and $\mathcal{A}^n = alg\{P_\lambda\}_{\lambda\in\Lambda}$ the algebra in $\mathbb{R}^n$ generated by the polynomials $\{P_\lambda\}_{\lambda\in\Lambda}$. The carré du champ Γ for the infinitesimal generator L of the semigroup $\{N_t\}_{t\geq 0}$ acts over $\mathcal{A}$. Denote by Γ^k the bilinear form on $\mathcal{A}^n$ that acts like Γ over the k-th coordinate. Note that in this case,

$$L_k = a(x_k)\frac{\partial^2}{\partial x_k^2} + b(x_k)\frac{\partial}{\partial x_k}. \tag{4.19}$$

Proposition 4.3 *If L is a diffusion, then the carré du champ operator associated to D, denoted by Γ^D, has the expression*

$$\Gamma^D(f,f) = \sum_{k=1}^{n}\left(\frac{1}{2}L_k f^2 - fL_k f\right) = \sum_{k=1}^{n}\Gamma^k(f,f). \tag{4.20}$$

Proof For $f \in \mathcal{A}^n$, we have that

$$\frac{1}{V}L_k(Vf) = L_k f + f\frac{L_k V}{V} + \frac{1}{V}2a(x_k)\frac{\partial}{\partial x_k}V\frac{\partial}{\partial x_k}f,$$

so, by (4.17)

$$Df = \sum_{k=1}^{n}\frac{L_k(Vf)}{V} + cf = \sum_{k=1}^{n}L_k f + \frac{2}{V}\sum_{k=1}^{n}a(x_k)\frac{\partial}{\partial x_k}V\frac{\partial}{\partial x_k}f. \tag{4.21}$$

Hence

$$2\Gamma^D(f,f) = Df^2 - 2fDf = \sum_{k=1}^{n}(L_k f^2 - 2fL_k f) = 2\sum_{k=1}^{n}\Gamma^k(f,f).$$

□

Then we have

Theorem 4.3 *If the infinitesimal generator L of the Markov semigroup $\{N_t\}_{t\geq 0}$ on $L^2(\mathbb{R},\mu)$ is a diffusion, then the semigroup $\{T_t\}_{t\geq 0}$ on $L^2_{\text{sym}}(\mathbb{R}^n,\mu_n)$ is also Markov.*

Proof

Since μ_n is the invariant measure for $\{T_t\}_{t\geq 0}$, by the result given in [Maz02], we only have to check that $\Gamma^D(f,f) \geq 0$ for $f \in \mathcal{A}^n$.

If we fix all the variables except x_k, by the definition of $P_\lambda(x_1,\ldots,x_n)$, it is clear that it is in $\mathcal{A}$. So, if $f \in \mathcal{A}^n$, it is also in $\mathcal{A}$ (considering f as a function of x_k only).

Now, since $\{N_t\}_{t\geq 0}$ is a Markov semigroup, we have that $\Gamma(f,f) \geq 0$ for $f \in \mathcal{A}$ and therefore $\Gamma^k(f,f) \geq 0$ for each k and $f \in \mathcal{A}^n$. The result now follows from the expression of Γ^D given in Proposition 4.3. □

For the semigroup $\{\hat{T}_t\}_{t\geq 0}$ associated to the generalized polynomials we have

Corollary 4.3 *If the infinitesimal generator L of the Markov semigroup $\{N_t\}_{t\geq 0}$ on $L^2(\mathbb{R},\mu)$ is a diffusion, then the semigroup $\{\hat{T}_t\}_{t\geq 0}$ on $L^2_{U_n}(H_n, M)$ is also Markov.*

If we substitute L_k in the expression of D given in (4.27) and to note that

$$\frac{\partial}{\partial x_k} V(\underline{x}) = V(\underline{x}) \left[\sum_{i=1 i\neq k}^{n} \frac{1}{x_k - x_i} \right],$$

we have

Proposition 4.4 *If L is a diffusion, then*

$$Df = \sum_{k=1}^{n} \left[a(x_k) \frac{\partial^2}{\partial x_k^2} f + 2 \sum_{i\neq k} \frac{a(x_k)}{x_k - x_i} \frac{\partial}{\partial x_k} f + b(x_k) \frac{\partial}{\partial x_k} f \right]. \tag{4.22}$$

Remark 4.2 *Using this expression for D we can also verify the Markov property in the diffusion case, using the positive maximum principle (Th.* 17.11*, p.* 321 *in [Kal97]), which is a Hille-Yosida-type condition that characterizes infinitesimal generators of Markov semigroups.*

4.4.2 Discrete case: discrete diffusions

Suppose now that the measure μ is purely atomic and that $supp(\mu) \subseteq \mathbb{N}$, then $L^2(\mathbb{R},\mu) = l^2(\mathbb{N},\mu)$. Note that when $supp(\mu)$ is finite, this space is a finite dimensional vector space and therefore the case in study simplifies.

Orthogonal polynomials with respect to discrete measures are the subject of the monograph [MSU91]. Following the notations in [MSU91], we consider the following difference operators

$$\Delta f(x) = f(x+1) - f(x),$$
$$\nabla f(x) = f(x) - f(x-1).$$

Definition 4.1 *We say that an operator T on $l^2(\mathbb{N},\mu)$ is a* discrete diffusion operator *if, and only if, for all polynomials Φ and f in $l^2(\mathbb{N},\nu)$, it satisfies*

$$T(\Phi(f)) = \Delta\nabla\Phi(f)\Gamma(f,f) + \Delta\Phi(f)Tf, \tag{4.23}$$

where Γ is the carré du champ operator of T.

In analogy to the continuous case we have

Proposition 4.5 *If the infinitesimal generator L of the semigroup $\{N_t\}_{t\geq 0}$ on $L^2(\mathbb{R}, \mu)$ is a discrete diffusion, then*

$$L = \sigma(x)\Delta\nabla + \tau(x)\Delta \tag{4.24}$$

where σ and τ are polynomials of degree at most two one, respectively.

Proof Considering $\Phi = p_m$ and $f = x$, we get that the relation

$$L = \Gamma(x, x)\Delta\nabla + Lx\Delta$$

holds for any polynomial.

The fact that $Lp_m = -\gamma_m p_m$ for $m = 1, 2$ implies that

$$\Gamma(x, x) = Ax^2 + Bx + C \quad \text{and} \quad Lx = ax + b.$$

Therefore, by the density of the polynomials the result follows. □

In [MSU91], Ch. 2 it is proven that the only families of discrete orthogonal polynomials that are eigenfunctions of operators of the form (4.24) are the Charlier, Meixner, Kravchuk and Hahn polynomials. For more on these polynomials, see Section 4.7 and [MSU91].

Now, if L is of the form (4.24), it is not difficult to see that

$$\Gamma(f, f) = \frac{1}{2}\left[\sigma(x)(\nabla f)^2 + (\sigma(x) + \tau(x))(\Delta f)^2\right]. \tag{4.25}$$

Since in all the abovementioned classical cases $\sigma \geq 0$ and $\sigma + \tau \geq 0$, ([MSU91], p. $42 - 44$) we have $\Gamma(f, f) \geq 0$. Then, again by the characterization given in [Maz02], the semigroups such that its infinitesimal generator is a discrete diffusion, associated to a family of orthogonal polynomials are indeed Markov and, by the result proven in [MSU91], the only discrete diffusion Markov semigroups are the ones associated to the Charlier, Meixner, Kravchuk and Hahn polynomials.

Suppose then that the infinitesimal generator L of the semigroup $\{N_t\}_{t\geq 0}$ on $L^2(\mathbb{R}, \mu)$ is a discrete diffusion. Using Mazet's result in [Maz02] we are going to see that in this case the semigroup $\{T_t\}_{t\geq 0}$ is also Markov. As seen in Theorem 4.2, the infinitesimal generator of this semigroup is $D = \frac{1}{V}(L_1 + \cdots + L_n)V + c$ where, in this case,

$$L_k = \sigma(x_k)\Delta_k\nabla_k + \tau(x_k)\Delta_k,$$

with

$$\Delta_k f(\underline{x}) = f(\underline{x} + e_k) - f(\underline{x}) \text{ and } \nabla_k f(\underline{x}) = f(\underline{x}) - f(\underline{x} - e_k),$$

and $\{e_k\}$ the canonical basis of $\mathbb{R}^n$. For these operators we have

$$\Delta_k f(\underline{x}) = \nabla_k f(\underline{x}+e_k),\ \ \nabla_k f(\underline{x}) = \Delta_k f(\underline{x}-e_k),$$
$$\Delta_k[f(\underline{x})g(\underline{x})] = f(\underline{x})\Delta_k g(\underline{x}) + g(\underline{x}+e_k)\Delta_k f(\underline{x}),$$
$$\nabla_k[f(\underline{x})g(\underline{x})] = f(\underline{x})\nabla_k g(\underline{x}) + g(\underline{x}-e_k)\nabla_k f(\underline{x}),$$

thus

$$\Delta_k\nabla_k[f(\underline{x})g(\underline{x})] = f(\underline{x})\Delta_k\nabla_k g(\underline{x}) + g(\underline{x}+e_k)\Delta_k\nabla_k f(\underline{x}) + \Delta_k f(\underline{x})(g(\underline{x}+e_k) - g(\underline{x}-e_k)).$$

Observe that Γ^k, the operator on $L^2_{\text{sym}}(\mathbb{R}^n,\mu_n)$ that acts like Γ in the k-th coordinate, is given by

$$\Gamma^k(f,f) = \frac{1}{2}\left[\sigma(x_k)(\nabla_k f)^2 + (\sigma(x_k)+\tau(x_k))(\Delta_k f)^2\right]. \tag{4.26}$$

Proposition 4.6 *If L is a discrete diffusion, then the carré du champ operator associated to D, denoted by Γ^D, has the expression*

$$2\Gamma^D(f,f)(x) = \sum_{k=1}^n \sigma(x_k)\frac{V(\underline{x}-e_k)}{V(\underline{x})}(\nabla_k f(\underline{x}))^2 + \sum_{k=1}^n(\sigma(x_k)+\tau(x_k))\frac{V(\underline{x}+e_k)}{V(\underline{x})}(\Delta_k f(\underline{x}))^2.$$

Proof By formula (4.26) we have

$$\Delta_k\nabla_k[V(\underline{x})f(\underline{x})] = f(\underline{x})\Delta_k\nabla_k V(\underline{x}) + V(\underline{x}+e_k)\Delta_k\nabla_k f(\underline{x}) + \nabla_k f(\underline{x})[V(\underline{x}+e_k) - V(\underline{x}-e_k)].$$

Together with (4.17) it implies that

$$Df(\underline{x}) = \sum_{k=1}^n \frac{V(\underline{x}+e_k)}{V(\underline{x})} L_k f(\underline{x}) + \sum_{k=1}^n \sigma(x_k)\nabla_k f(\underline{x})\frac{[V(\underline{x}+e_k) - V(\underline{x}-e_k)]}{V(\underline{x})}. \tag{4.27}$$

Since $\nabla_k f^2(\underline{x}) = [f(\underline{x}) + f(\underline{x}-e_k)]\nabla_k f(\underline{x})$, we have

$$\begin{aligned}2\Gamma^D(f,f)(\underline{x}) &= Df^2(\underline{x}) - 2f(\underline{x})Df(\underline{x}) = \\ &= \sum_{k=1}^n \frac{V(\underline{x}+e_k)}{V(\underline{x})}2\Gamma^k(f,f)(\underline{x}) - \sum_{k=1}^n \sigma(x_k)(\nabla_k f(\underline{x}))^2\frac{[V(\underline{x}+e_k) - V(\underline{x}-e_k)]}{V(\underline{x})}.\end{aligned}$$

Substituting the expression of Γ^k given in (4.26) we get the desired result. □

Theorem 4.4 *Suppose that the infinitesimal generator L of the Markov semigroup $\{N_t\}_{t\geq 0}$ on $l^2(\mathbb{N},\mu)$ is a discrete diffusion, then the semigroup $\{T_t\}_{t\geq 0}$ on $l^2_{\text{sym}}(\mathbb{N}^n,\mu_n)$ is also Markov.*

Proof

Again, by the result given in [Maz02], it is enough to verify that $\Gamma(f,f) \geq 0$ for all $f \in \mathcal{A}^n$. Let f be a symmetric polynomial in $l^2(\mathbb{N}^n,\mu_n)$, then f, f^2, Df and Df^2 are also symmetric polynomials and therefore $\Gamma^D(f,f) = Df^2 - 2fDf$ is also a symmetric polynomial. Thus, to verify that $\Gamma^D(f,f)(\underline{x}) \geq 0$ for $\underline{x} \in \mathbb{N}^n$, it is enough to prove that $\Gamma^D(f,f)(\underline{x}) \geq 0$ for $\underline{x} \in C := \{\underline{y} \in \mathbb{N}^n : y_1 > y_2 > \cdots > y_n\}$, since, if $\underline{x} \in \mathbb{N}^n$, then we can always find a permutation $\sigma \in S_n$ such that $\sigma(\underline{x}) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in C$ and therefore $\Gamma^D(f,f)(\underline{x}) = \Gamma^D(f,f)(\sigma(\underline{x})) \geq 0$.

Now, if $\underline{x} \in C$, then $V(\underline{x}) > 0$, $V(\underline{x} + e_k) \geq 0$ and $V(\underline{x} - e_k) \geq 0$, which implies, by the expression of $\Gamma(f,f)$ given in Proposition 4.6, that $\Gamma^D(f,f)(\underline{x}) \geq 0$, as wanted.

So, again in this case the semigroup $\{T_t\}_{t\geq 0}$ is Markov. □

Corollary 4.4 *If the infinitesimal generator L of the Markov semigroup $\{N_t\}_{t\geq 0}$ on $l^2(\mathbb{N}, \mu)$ is a discrete diffusion, then the semigroup $\{\hat{T}_t\}_{t\geq 0}$ on $L^2_{U_n}(H_n, M)$ is also Markov.*

We can also give an expression of the operator D in terms of the difference operators Δ_k and ∇_k. For this it is enough to substitute $L_k f$ in the expression of Df given in (4.27) and to note that

$$\Delta_k \nabla_k f(\underline{x}) + \nabla_k f(\underline{x}) = \nabla_k[\Delta_k f(\underline{x}) + f(\underline{x})] = \Delta_k f(\underline{x}).$$

Proposition 4.7 *If L is a discrete diffusion, then*

$$Df(\underline{x}) = \sum_{k=1}^{n} (\sigma(x_k) + \tau(x_k)) \frac{V(\underline{x} + e_k)}{V(\underline{x})} \Delta_k f(\underline{x}) - \sum_{k=1}^{n} \sigma(x_k) \frac{V(\underline{x} - e_k)}{V(\underline{x})} \nabla_k f(\underline{x}).$$

4.5 Explicit formulas for the kernels $\mathcal{T}_t(x, y)$

Using Proposition 4.1 it is possible to give and expression of the kernels $\mathcal{T}_t(x,y)$ defined by (4.8), that define the operator T_t, in terms of the kernels $\mathcal{N}_t(x,y)$ on $L^2(\mu \otimes \mu)$ defined in (4.7).

Theorem 4.5 *The kernel $\mathcal{T}_t$ can be expressed as*

$$\mathcal{T}_t(\underline{x}, \underline{y}) = e^{t \sum_{j=1}^{n} \gamma_{n-j}} \frac{\det(\mathcal{N}_t(x_j, y_i))_{i,j}}{n! V(\underline{x}) V(\underline{y})}. \tag{4.28}$$

Proof To verify (4.28), let us consider the functions

$$f_i(t,x) = \mathcal{N}_t(x, y_i) = \sum_{m=0}^{\infty} e^{-\gamma_m t} p_m(x) p_m(y_i),$$

then by Proposition 4.1

$$\det(\mathcal{N}_t(x_j, y_i))_{i,j} = \det(f_i(t, x_j))_{i,j} = V(\underline{x}) \sum_{\lambda \in \Lambda} b_\lambda(\underline{y}) P_\lambda(\underline{x}),$$

with

$$b_\lambda(\underline{y}) = \sqrt{n!}\,\det\left(e^{-t\gamma_{\lambda_j+n-j}} p_{\lambda_j+n-j}(y_i)\right)_{ij} = n! e^{-\sum_{j=1}^n \gamma_{\lambda_j+n-j}} P_\lambda(\underline{y}) V(\underline{y}).$$

Therefore

$$\begin{aligned}\det(\mathcal{N}_t(x_j,y_i))_{i,j} &= n! V(\underline{x}) V(\underline{y}) \sum_{\lambda\in\Lambda} e^{-\sum_{j=1}^n \gamma_{\lambda_j+n-j}} P_\lambda(\underline{y}) P_\lambda(\underline{x}) \\ &= n! e^{-t\sum_{j=1}^n \gamma_{n-j}} V(\underline{x}) V(\underline{y}) \mathcal{T}_t(\underline{x},\underline{y}),\end{aligned}$$

obtaining formula (4.28). □

For the kernel $\hat{\mathcal{T}}_t$ given by (4.13) associated to the semigroup $\{\hat{T}_t\}_{t\geq 0}$ in $L^2_{U_n}(H_n, M)$ we have an immediate result

Corollary 4.5 *The kernel $\hat{\mathcal{T}}_t$ can be expressed as*

$$\hat{\mathcal{T}}_t(X,Y) = e^{t\sum_{j=1}^n \gamma_{n-j}} \frac{\det(\mathcal{N}_t(x_j,y_i))_{i,j}}{n! V(x_1,\ldots,x_n) V(y_1,\ldots,y_n)} \tag{4.29}$$

where $x_1,\ldots,x_n,y_1,\ldots,y_n$ are the eigenvalues of the matrixes X and Y, respectively.

4.5.1 Probabilistic proof of the positivity preserving

Theorem 4.5 implies that the positivity preserving property of the semigroup $\{T_t\}_{t\geq 0}$ is equivalent to the positivity of the determinants $det(\mathcal{N}_t(x_i,y_j))_{i,j=1}^n$ of dimension $n\times n$.

The property of positivity of all the determinants $det(N_t(x_i,y_j))_{i,j=1}^k$, $k = 1,\ldots,n$ is called *total positivity* of the 1-dimensional kernel $N_t(x,y)$. Karlin and McGregor ([KM59], [KM60]) showed by probabilistic methods that if N_t is the kernel of a continuous diffusion semigroup then it is totally positive.

In the discrete case, the diffusion property (4.23) corresponds to the fact the the underlying Markov process X_t on a subset of $\mathbb{Z}$ only jumps by +1 or -1. In such cases the total positivity was also proved ([KM59]).

4.5.2 Probabilistic interpretation of the semigroup T_t

We can also obtain the semigroup $\{T_t\}_{t\geq 0}$ as the semigroup associated to a stochastic process. Consider a 1–dimensional Markov process $X = (X_t)_{t\geq 0}$ such that its associated semigroup equals $\{N_t\}_{t\geq 0}$ and take n independent copies $X(i)$, $i = 1,\ldots,n$ of this process. The n–dimensional process $\underline{X} = (X(1),\ldots,X(n))$ has the generator $L_1 + \ldots + L_n$. Kill the process $\underline{X}$ when leaving the Weyl Chamber

$C = \{\underline{y} \in \mathbb{R}^n : y_1 > y_2 > \cdots > y_n\}$. To the killed process $\underline{X}^C$ apply the Doob h–transform for $h = V$, the Vandermonde determinant. As observed in Remark 4.1, this function h is positive and excessive for the process $\underline{X}^C$.

The resulting process $Y = (\underline{X}^C)^h$ identifies with the process $\underline{X}$ conditioned to remain in the Weyl chamber C. On the other hand, the generator of Y equals $D = \frac{1}{V}(L_1 + \ldots + L_n)V + c$, the constant c ensuring that $D1 = 0$. Thus the associated semigroup of this process is $\{T_t\}_{t \geq 0}$.

The n–dimensional stochastic processes conditioned to stay in a Weyl chamber are intensely studied in recent years (see e.g. [JO06] and references therein).

4.6 Some Classical Formulas for generalized polynomials

4.6.1 Christoffel-Darboux Formula

It is well known that, as a consequence of their three term recursion formula, the family of orthogonal polynomials $\{p_m\}_{m\in\mathbb{N}}$ on $L^2(\mathbb{R}, \mu)$ satisfies the Christoffel-Darboux formula, see [Sz59], §3.2,

$$\sum_{k=0}^{m} \frac{p_k(x)p_k(y)}{\|p_k\|_2^2} = \frac{p_{m+1}(x)p_m(y) - p_m(x)p_{m+1}(y)}{A_m\|p_m\|_2^2(x-y)}, \tag{4.30}$$

for $x \neq y$ in $\mathbb{R}$ where $A_m = \frac{a_{m+1}}{a_m}$, and a_m is the leading coefficient of the polynomial p_m.

We are going to generalize the Christoffel-Darboux formula for the orthogonal polynomials $\{P_\lambda\}_{\lambda\in\Lambda}$, as an application of Proposition 4.1.

Proposition 4.8 *Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$ be such that $x_i \neq y_j$, $i, j = 1, \ldots, n$ and $m \in \mathbb{N}$ such that $m > n-1$. Then*

$$V(\underline{x})V(\underline{y}) \sum_{\{\lambda\in\Lambda:\lambda_1+n-1\leq m\}} \frac{P_\lambda(\underline{x})P_\lambda(\underline{y})}{\|P_\lambda\|^2_{L^2_{\text{sym}}(\mathbb{R}^n,\mu_n)}} = \frac{n!}{A_m^n\|p_m\|^{2n}_{L^2(\mathbb{R},\mu)}} \times$$
$$\times \det\left(\frac{p_{m+1}(x_j)p_m(y_i) - p_m(x_j)p_{m+1}(y_i)}{x_j - y_i}\right)_{i,j}, \tag{4.31}$$

Proof Let us consider the functions

$$f_i(x) = \frac{p_{m+1}(x)p_m(y_i) - p_m(x)p_{m+1}(y_i)}{A_m\|p_m\|^2_{L^2(\mathbb{R},\mu)}(x-y_i)} = \sum_{k=0}^{m_n} \frac{p_k(y_i)p_k(x)}{\|p_k\|_2 L^2(\mathbb{R},\mu)^2}.$$

Applying Proposition 4.1, for the coefficients

$$c_k^{(i)}(\underline{y}) = \begin{cases} \frac{p_k(y_i)}{\|p_k\|^2_{L^2(\mathbb{R},\mu)}} & \text{if } k \leq m \\ 0 & \text{if } k \geq m, \end{cases}$$

we obtain

$$\det\left(f_i(x_j)\right)_{i,j} = V(\underline{x}) \sum_{\lambda\in\Lambda} b_\lambda(\underline{y}) P_\lambda(\underline{x}),$$

where $b_\lambda(\underline{y}) = \sqrt{n!}\det\left(c^{(i)}_{\lambda_j+n-j}(\underline{y})\right)_{i,j}$. Let us study this last determinant more carefully. Call $a_{ij} = c^{(i)}_{\lambda_j+n-j}(\underline{y})$. We then have that $a_{ij} = \frac{p_k(y_i)}{\|p_k\|^2_{L^2(\mathbb{R},\mu)}}$ if, and only if, $\lambda_j + n - j \le m$.

Now, if $\lambda_1 + n - 1 \le m$ then, for every $k \ge 1$ we have $\lambda_k + n - k \le m$, as $\lambda_k \le \lambda_1$. Then $a_{ik} = \frac{p_k(y_i)}{\|p_k\|^2_{L^2(\mathbb{R},\mu)}}$ for all i,k. On the other hand, if $\lambda_1 + n - 1 > m$, then $a_{i1} = 0$ for all i.

This means that

$$b_\lambda(\underline{y}) = \sqrt{n!}\det\left(c^{(i)}_{\lambda_j+n-j}(\underline{y})\right)_{i,j} = \begin{cases} \sqrt{n!}\det\left(\frac{p_{\lambda_j+n-j}(y_i)}{\|p_{\lambda_j+n-j}\|^2_{L^2(\mathbb{R},\mu)}}\right)_{i,j} & \text{if } \lambda_1+n-1\le m \\ 0 & \text{if } \lambda_1+n-1 > m. \end{cases}$$

Therefore, if $\lambda_1 + n - 1 \le m$,

$$b_\lambda(\underline{y}) = \sqrt{n!}\det\left(\frac{p_{\lambda_j+n-j}(y_i)}{\|p_{\lambda_j+n-j}\|^2_{L^2(\mathbb{R},\mu)}}\right)_{i,j} = \frac{V(\underline{y})}{n!\prod_{j=1}^n \|p_{\lambda_j+n-j}\|^2_{L^2(\mathbb{R},\mu)}}\sqrt{n!}\frac{\det\left(p_{\lambda_j+n-j}(y_i)\right)_{i,j}}{V(\underline{y})} =$$

$$= \frac{V(\underline{y})}{n!\|P_\lambda\|^2_{L^2(\mathbb{R}^n,\mu_n)}} P_\lambda(\underline{y}),$$

from where

$$\det\left(f_i(x_j)\right)_{i,j} = V(\underline{x})V(\underline{y}) \sum_{\{\lambda\in\Lambda:\lambda_1+n-1\le m\}} \frac{P_\lambda(\underline{x})P_\lambda(\underline{y})}{n!\|P_\lambda\|^2_{L^2(\mathbb{R}^n,\mu_n)}}.$$

Also

$$\begin{aligned}\det\left(f_i(x_j)\right)_{i,j} &= \det\left(\frac{p_{m+1}(x_j)p_m(y_i) - p_m(x_j)p_{m+1}(y_i)}{A_m\|p_m\|^2_{L^2(\mathbb{R},\mu)}(x_j - y_i)}\right) \\ &= \frac{1}{A_m^n\|p_m\|^{2n}_{L^2(\mathbb{R},\mu)}}\det\left(\frac{p_{m+1}(x_j)p_m(y_i) - p_m(x_j)p_{m+1}(y_i)}{x_j - y_i}\right),\end{aligned}$$

thus

$$\begin{aligned}V(\underline{x})V(\underline{y}) \sum_{\{\lambda\in\Lambda:\lambda_1+n-1\ge m\}} \frac{P_\lambda(\underline{x})P_\lambda(\underline{y})}{\|P_\lambda\|^2_{L^2(\mathbb{R}^n,\mu_n)}} &= \frac{n!}{A_m^n\|p_m\|^{2n}_{L^2(\mathbb{R},\mu)}} \times \\ &\times \det\left(\frac{p_{m+1}(x_j)p_m(y_i) - p_m(x_j)p_{m+1}(y_i)}{x_j - y_i}\right)_{i,j}.\end{aligned}$$

□

For the generalized polynomials we have an analogous result

Corollary 4.6 *Let X, Y be Hermitian matrices with different eigenvalues $x_1, \dots x_n$, $y_1, \dots, y_n$ respectively and let $m \in \mathbb{N}$ be such that $m > n-1$. Then*

$$V(x_1, \dots x_n)V(y_1, \dots y_n) \sum_{\{\lambda\in\Lambda:\lambda_1+n-1\le m\}} \frac{\hat{P}_\lambda(X)\hat{P}_\lambda(Y)}{\|\hat{P}_\lambda\|^2_{L^2_{U_n}(H_n,M)}} = \frac{n!}{A^n_m \|p_m\|^{2n}_{L^2(\mathbb{R},\mu)}} \times$$
$$\times \det\left(\frac{p_{m+1}(x_j)p_m(y_i) - p_m(x_j)p_{m+1}(y_i)}{x_j - y_i}\right)_{i,j}. \quad (4.32)$$

4.6.2 Generating Function

Let us assume that the family $\{p_m\}_{m\in\mathbb{N}}$ has a generating function

$$f(x, w) = \sum_{m=0}^{\infty} d_m p_m(x) w^m, \quad x \in supp\{\mu\}, \quad |w| < r.$$

We are able to obtain a generating function for the polynomials $\{P_\lambda\}_{\lambda\in\Lambda}$, again as an application of Proposition 4.1,

Proposition 4.9 *Let $x_1, \dots, x_n, w_1, \dots, w_n \in \mathbb{R}$. If $f(\cdot,\cdot)$ is a generating function of $\{p_m\}_{m\in\mathbb{N}}$, then*

$$\frac{\det\left(f(x_j, w_i)\right)_{ij}}{V(\underline{x})V(\underline{w})} = \sum_{\lambda\in\Lambda} d_\lambda P_\lambda(\underline{x}) S_\lambda(\underline{w}), \quad (4.33)$$

where $d_\lambda = \prod_{j=1}^n d_{\lambda_j+n-j}$ and S_λ is the Schur polynomial associated to the partition λ. That is

$$\frac{\det\left(f(x_j, w_i)\right)_{ij}}{V(\underline{x})V(\underline{w})} \quad (4.34)$$

is a generating function of $\{P_\lambda\}_{\lambda\in\Lambda}$.

Proof Apply Proposition 4.1 to the functions

$$f_i(x) = f(x, w_i) = \sum_{m=0}^{\infty} d_m w_i^m p_m(x).$$

Then, for $c^{(i)}_m = d_m w_i^m$, we get

$$\det\left(f_i(x_j)\right)_{ij} = \det\left(f(x_j, w_j)\right)_{ij} = \sum_{\lambda\in\Lambda} b_\lambda(\underline{w}) P_\lambda(\underline{x})$$

with

$$b_\lambda(\underline{w}) = \det\left(c^{(i)}_{\lambda_j+n-j}\right)_{ij} = \det\left(d_{\lambda_j+n-j} w_i^{\lambda_j+n-j}\right)_{ij} = \prod_{j=1}^n d_{\lambda_j+n-j} V(\underline{w}) S_\lambda(\underline{w}).$$

Thus

$$\frac{\det\left(f(x_j,w_i)\right)_{ij}}{V(\underline{x})V(\underline{w})} = \sum_{\lambda\in\Lambda} d_\lambda P_\lambda(\underline{x}) S_\lambda(\underline{w}).$$

□

For the generalized polynomials we have

Corollary 4.7 *Let X and W be Hermitian matrixes with eigenvalues $x_1,\ldots,x_n$, $w_1,\ldots,w_n$ respectively, then*

$$\frac{\det\left(f(x_j,w_i)\right)_{ij}}{V(x_1,\ldots,x_n)V(w_1,\ldots,w_n)} = \sum_{\lambda\in\Lambda} d_\lambda \hat{P}_\lambda(X)\hat{S}_\lambda(W), \tag{4.35}$$

where $d_\lambda = \prod_{j=1}^n d_{\lambda_j+n-j}$ and $\hat{S}_\lambda$ is the central function in $L^2_{U_n}(H_n, M)$ such that its restriction to the diagonal matrices is the Schur polynomial S_λ.

4.7 Examples

Example 4.1 Hermite polynomials

Let us consider the family $\{H_m\}_{m\in\mathbb{N}}$ of normalized Hermite polynomials on $\mathbb{R}$. The normalized Hermite polynomial can be defined using the Rodrigues' formula,

$$e^{x^2} H_m(x) = \frac{(-1)^m}{(\sqrt{\pi}2^m m!)^{1/2}} \frac{d^m}{dx^m} e^{x^2}.$$

They are orthogonal polynomials with respect to the Gauss measure $\mu(dx) = e^{-x^2}dx$ and the Hermite polynomial H_m is an eigenfunction of the diffusion operator, called Ornstein-Ulenbeck operator, given by

$$L = \frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x}, \tag{4.36}$$

with eigenvalue $-2m$. Then, for the Markov generator sequence for this family, given by $\gamma_m = 2m$, the Markov semigroup $\{N_t\}_{t\geq 0}$ is the Ornstein-Uhlenbeck semigroup with generator L. By Mehler's formula (see [Urb06], (B.12)) the kernels $\mathcal{N}_t$ defining this semigroup can be expressed as

$$\mathcal{N}_t(x,y) = \frac{1}{\sqrt{2\pi(1-e^{-2t})}} \exp\left(\frac{e^{-2t}(x^2+y^2) - 2e^{-t}xy}{2(1-e^{-2t})}\right). \tag{4.37}$$

The normalized generalized Hermite polynomial for $\lambda \in \Lambda$ is defined by

$$\hat{H}_\lambda(X) = \frac{1}{\sqrt{n!}} \frac{\det(H_{\lambda_i+n-i}(x_j))}{V(x_1,\ldots,x_n)}, \tag{4.38}$$

where $x_1, \ldots, x_n$ are the eigenvalues of the Hermitian matrix X. The family of generalized Hermite polynomials $\{\hat{H}_\lambda\}_{\lambda\in\Lambda}$ form an orthogonal basis of the space $L^2_{U_n}(H_n, e^{-Tr(X^2)}dm(X))$, where m is the Lebesgue measure on H_n, treated as a real vector space.

By Corollary 4.1, the generalized polynomial $\hat{H}_\lambda$ is an eigenfunction of the Markov semigroup $\{\hat{T}_t\}_{t\geq 0}$, with eigenvalues $e^{-t\varphi_\lambda}$ where $\varphi_\lambda = 2\sum_{j=1}^n \lambda_j$. and by Corollary 4.5 and equation (4.37),

$$\hat{T}_t(X,Y) = \frac{1}{(2\pi(1-e^{2t}))^{n/2}} \frac{e^{-t(n-1)n/2}}{n!V(x_1,\ldots,x_n)V(y_1,\ldots,y_n)} \times \\ \times \det\left(\exp\frac{e^{-2t}(x_j^2+y_i^2) - 2e^{-t}x_iy_j}{2(1-e^{-2t})}\right). \tag{4.39}$$

By Proposition 4.4, the operator D, that is, the restriction to the space of diagonal matrices of the infinitesimal generator $\hat{D}$ of the semigroup $\{\hat{T}_t\}_{t\geq 0}$, is given by

$$D = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + \sum_{k,i=1 k\neq i}^n \frac{2}{x_k - x_i}\frac{\partial}{\partial x_k} - \sum_{k=1}^n 2x_k\frac{\partial}{\partial x_k}. \tag{4.40}$$

This operator coincides with the operator given in [Las91b] for the case of Schur function expansion (the parameter $\alpha = 1$ in that article). The constant c in the definition of the operator D (see (4.17)) is in this case $c = \sum_{j=1}^n \gamma_{n-j} = n(n-1)$.

Since the generating function of the one dimensional normalized Hermite polynomials is given by (see [Sz59], formula (5.5.7))

$$\sum_{m=0}^\infty d_m H_m(x) w^m = e^{2xw-w^2},$$

with $d_m = \frac{2^m}{\sqrt{m!\sqrt{\pi}}}$, by Corollary 4.7 for X and W Hermitian matrices with eigenvalues $x_1, \ldots, x_n$ and $w_1, \ldots, w_n$ respectively

$$\sum_{\lambda\in\Lambda} d_\lambda \hat{H}_\lambda(X)\hat{S}_\lambda(W) = \frac{\det(e^{2x_jw_i - w_i^2})}{V(x_1,\ldots,x_n)V(w_1,\ldots,w_n)},$$

where $d_\lambda = \pi^{-n/4}\prod_{j=1}^n \frac{2^{\lambda_j+n-j}}{\sqrt{(\lambda_j+n-j)!}}$ and $\hat{S}_\lambda$ is the central function such that its restriction to the space of diagonal matrices is the Schur polynomial S_λ. By Proposition II 3.2 of [Far06],

$$\sum_{\lambda\in\Lambda} d_\lambda \hat{H}_\lambda(X)\hat{S}_\lambda(W) = 2^{(n-1)n/2}e^{-Tr(W)}\sum_{\kappa\in\Lambda}\prod_{i=1}^n \frac{1}{(\kappa_i+n-i)!}\hat{S}_\kappa(2X)\hat{S}_\kappa(W). \tag{4.41}$$

This generating function coincides, up to a constant, with the one given in [BF97a], Proposition 3.1.

Example 4.2 Laguerre polynomials

For $\alpha > -1$, let $\{L_m^\alpha\}_{m\in\mathbb{N}}$ be the family of normalized Laguerre polynomials on $\mathbb{R}$. The polynomial L_m^α can be defined using the Rodrigues' formula,

$$e^{-x}x^\alpha L_m^\alpha(x) = \frac{1}{\sqrt{m!\Gamma(\alpha+m+1)}}\frac{d^m}{dx^m}(e^{-x}x^{\alpha-m}).$$

They are orthogonal with respect to the measure $\mu_\alpha(dx) = x^\alpha e^{-x}\mathbf{1}_{(0,\infty)}(x)dx$ and the Laguerre polynomial L_m^α is an eigenfunction of the Laguerre operator

$$L^\alpha = x\frac{\partial^2}{\partial x^2} + (\alpha+1-x)\frac{\partial}{\partial x}, \tag{4.42}$$

with eigenvalue $-m$. Then, for the Markov generator sequence for this family, given by $\gamma_m = m$, the Markov semigroup $\{N_t^\alpha\}_{t\geq 0}$ is the Laguerre semigroup with generator L^α. By the Hille-Hardy's formula (see [Urb06], (B.26)) the kernels $\mathcal{N}_t^\alpha$ defining this semigroup can be expressed as

$$\mathcal{N}_t^\alpha(x,y) = \frac{k_\alpha}{1-e^{-t}}e^{-\frac{(x+y)e^{-t}}{1-e^{-t}}}(-xye^{-t})^{\alpha/2}\mathcal{J}_\alpha\left(2\sqrt{\frac{-xye^{-t}}{1-e^{-t}}}\right), \tag{4.43}$$

where $\mathcal{J}_\alpha$ is the Bessel function of order α and k_α is a constant.

The normalized generalized Laguerre polynomial for $\lambda \in \Lambda$ is defined by

$$\hat{L}_\lambda^\alpha(X) = \frac{1}{\sqrt{n!}}\frac{\det(L^\alpha_{\lambda_i+n-i}(x_j))_{i,j=1,\ldots,n}}{V(x_1,\ldots,x_n)}, \tag{4.44}$$

where $x_1,\ldots,x_n$ are the eigenvalues of the Hermitian matrix X. The generalized Laguerre polynomials $\{\hat{L}_\lambda^\alpha\}_{\lambda\in\Lambda}$ are orthogonal basis of the space $L^2_{U_n}(H_n^+, M_\alpha)$, where the measure $M_\alpha(dX) = \det X^\alpha e^{-TrX}\prod_{i=1}^n \mathbf{1}_{(0,\infty)}(x_i)dm(X)$, with m the Lebesgue measure on H_n, supported in the cone H_n^+ of non-negative definite Hermitian matrices.

By Corollary 4.1, the generalized polynomial $\hat{L}_\lambda^\alpha$ is an eigenfunction of the Markov semigroup $\{\hat{T}_t^\alpha\}_{t\geq 0}$ on $L^2_{U_n}(H_n^+, M_\alpha)$, with eigenvalue $e^{-t\varphi_\lambda}$ where $\varphi_\lambda = \sum_{j=1}^n \lambda_j$. By Corollary 4.5 and equation (4.43),

$$\hat{\mathcal{T}}_t^\alpha(X,Y) = \frac{k_\alpha^n e^{\frac{t(n-1)n}{2}}}{(1-e^{-t})^n}\frac{\det\left(e^{-\frac{(x_j+y_i)e^{-t}}{1-e^{-t}}}(-x_jy_ie^{-t})^{\alpha/2}\mathcal{J}_\alpha\left(2\sqrt{\frac{-x_jy_ie^{-t}}{1-e^{-t}}}\right)\right)}{n!V(x_1,\ldots,x_n)V(y_1,\ldots,y_n)}. \tag{4.45}$$

By Proposition 4.4, the operator D^α, that is, the restriction to the space of diagonal matrices of the infinitesimal generator $\hat{D}^\alpha$ of the semigroup $\{\hat{T}_t^\alpha\}_{t\geq 0}$, is given by

$$D^\alpha = \sum_{k=1}^n x_k\frac{\partial^2}{\partial x_k^2} + \sum_{k,i=1 k\neq i}^n \frac{2x_k}{x_k-x_i}\frac{\partial}{\partial x_k} + \sum_{k=1}^n(\alpha+1-x_k)\frac{\partial}{\partial x_k}. \tag{4.46}$$

This operator coincides with the operator given in [Las91b] for the case of Schur function expansion (the parameter $\alpha = 1$ in that article). The constant c in the definition of the operator D (see (4.17)) is in this case $c = \sum_{j=1}^{n} \gamma_{n-j} = \frac{n(n-1)}{2}$.

The generating function for the one dimensional normalized Laguerre polynomials is given by (see [Sz59], formula (5.1.1))

$$\sum_{m=1}^{\infty} d_m L_m^{\alpha}(x) w^m = (1-w)^{-\alpha-1} e^{-\frac{xw}{1-w}},$$

with $d_m = \frac{(-1)^m}{\sqrt{m!}} \sqrt{\Gamma(m+\alpha+1)}$ so, by Corollary 4.7 for X and W Hermitian matrices with eigenvalues $x_1, \ldots, x_n$ and $w_1, \ldots, w_n$ respectively

$$\sum_{\lambda \in \Lambda} d_\lambda \hat{L}_\lambda^{\alpha}(X) \hat{S}_\lambda(W) = \frac{\det((1-w_i)^{-\alpha-1} e^{-\frac{x_j w_i}{1-w_i}})}{V(x_1, \ldots, x_n) V(w_1, \ldots, w_n)},$$

where $d_\lambda^2 = \prod_{j=1}^{n} \frac{(-1)^{\lambda_j+n-j}}{\sqrt{(\lambda_j+n-j)!}} \sqrt{\Gamma(\lambda_j+n-j+\alpha+1)}$. Again, by Proposition II 3.2 of [Far06]

$$\sum_{\lambda \in \Lambda} d_\lambda \hat{L}_\lambda^{\alpha}(X) \hat{S}_\lambda(W) = (-1)^{\frac{(n-1)n}{2}} \det(I-W)^{-\alpha-n} \times \\ \times \sum_{\kappa \in \Lambda} \prod_{i=1}^{n} \frac{1}{(\kappa_i+n-1)!} \hat{S}_\kappa(-X) \hat{S}_\kappa(W(I-W)^{-1}). \tag{4.47}$$

This generating function coincides up to a constant with the one given in [BF97a], Proposition 4.1, (4.4).

Example 4.3 Jacobi polynomials

For $\alpha, \beta > -1$, let $\{J_m^{\alpha,\beta}\}_{m \in \mathbb{N}}$ be the family of normalized Jacobi polynomials on $\mathbb{R}$. The normalized Jacobi polynomial can be defined using the Rodrigues' formula,

$$(1-x)^{\alpha}(1+x)^{\beta} J_m^{\alpha,\beta}(x) = k_m^{\alpha,\beta} \frac{d^m}{dx^m}((1-x)^{\alpha+m}(1+x)^{\beta+m}),$$

where $k_m^{\alpha,\beta}$ is a normalizing constant. They are orthogonal with respect to the measure $\mu_{\alpha,\beta}(dx) = (1-x)^{\alpha}(1+x)^{\beta} \mathbf{1}_{[-1,1]}(x) dx$ and the polynomial $J_m^{\alpha,\beta}$ is an eigenfunction of the Jacobi operator

$$L^{\alpha,\beta} = (1-x^2) \frac{\partial^2}{\partial x^2} + [\beta - \alpha - (\alpha+\beta+2)x] \frac{\partial}{\partial x}, \tag{4.48}$$

with eigenvalue $-m(m+\alpha+\beta+1)$. For the Markov generator sequence $\gamma_m^{\alpha,\beta} = m(m+\alpha+\beta+1)$ for this family, the semigroup $\{N_t^{\alpha,\beta}\}_{t \geq 0}$ is the Jacobi semigroup, with infinitesimal generator $L^{\alpha,\beta}$. Unfortunately, there is no reasonable explicit expression for the kernel $\mathcal{N}_t^{\alpha,\beta}$ of this semigroup.

The normalized generalized Jacobi polynomial for $\lambda \in \Lambda$ is defined by

$$\hat{J}_\lambda^{\alpha,\beta}(X) = \frac{1}{\sqrt{n!}} \frac{\det(J_{\lambda_i+n-i}^{\alpha,\beta}(x_j))_{i,j=1,\dots,n}}{V(x_1,\dots,x_n)}, \tag{4.49}$$

where $x_1,\dots,x_n$ are the eigenvalues of the Hermitian matrix X. The family $\{\hat{J}_\lambda^{\alpha,\beta}\}_{\lambda\in\Lambda}$ forms an orthogonal basis of the space $L^2_{U_n}(\bar{B}(0,1), M_{\alpha,\beta})$ where $\bar{B}(0,1)$ is the unit ball in H_n and $M_{\alpha,\beta}(dX) = \det(I-X)^\alpha \det(I+X)^\beta \prod_i \mathbf{1}_{[-1,1]}(x_i)dm(X)$, with m the Lebesgue measure on H_n.

By Corollary 4.1, the generalized polynomial $\hat{J}_\lambda^{\alpha,\beta}$ is an eigenfunction of the semigroup $\{\hat{T}_t^{\alpha,\beta}\}_{t\geq 0}$, with eigenvalue $e^{-t\varphi_\lambda}$ where $\varphi_\lambda = \sum_{j=1}^n [\lambda_j(\lambda_j+\alpha+\beta+1)+2\lambda_j(n-j)] + (\alpha+\beta+1)\frac{(n-1)n}{2}$. By Proposition 4.4, the operator $D^{\alpha,\beta}$, the restriction to the space of diagonal matrices of the infinitesimal generator $\hat{D}^{\alpha,\beta}$ of this semigroup, is given by

$$D^{\alpha,\beta} = \sum_{k=1}^n (1-x_k^2)\frac{\partial^2}{\partial x_k^2} + \sum_{k,i=1 k\neq i}^n \frac{2(1-x_k^2)}{x_k - x_i}\frac{\partial}{\partial x_k} + \\ + \sum_{k=1}^n [\alpha-\beta-(\alpha+\beta+1)x_k]\frac{\partial}{\partial x_k}. \tag{4.50}$$

This operator coincides with the operator given in [Las91b] for the case of Schur function expansion (the parameter $\alpha = 1$ in that article). The constant c in the definition of the operator D (see (4.17)) is in this case $c = \sum_{j=1}^n \gamma_{n-j} = \frac{2n(n-1)(n-2)}{3} + (a+b+2)\frac{n(n-1)}{2}$.

The generating function for the one dimensional Jacobi polynomials is (see [Sz59], formula (4.4.5))

$$\sum_{m=0}^\infty d_m J_m^{\alpha,\beta}(x) w^m = 2^{\alpha+\beta} R^{-1/2}\{1-w+R^{1/2}\}^{-\alpha}\{1+w+R^{1/2}\}^{-\beta},$$

where $R = 1-2xw+w^2$ and $d_m^2 = \frac{2^{\alpha+\beta+1}}{2m+\alpha+\beta+1}\frac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{m!\Gamma(n+\alpha+\beta+1)}$. By Corollary 4.7 for X, W Hermitian matrices with eigenvalues $x_1,\dots,x_n$ and $w_1,\dots,w_n$ respectively,

$$\sum_{\lambda\in\Lambda} d_\lambda \hat{J}_\lambda^{\alpha,\beta}(X)\hat{S}_\lambda(W) = \frac{\det\left(R_{ij}^{-1/2}\{1-w_i+R_{ij}^{1/2}\}^{-\alpha}\{1+w_i+R_{ij}^{1/2}\}^{-\beta}\right)}{2^{-n(\alpha+\beta)}V(x_1,\dots,x_n)V(w_1,\dots,w_n)}, \tag{4.51}$$

where $d_\lambda^2 = 2^{n(\alpha+\beta+1)}\prod_{i=1}^n \frac{\Gamma(\lambda_i+n-i+\alpha+1)\Gamma\lambda_i+n-i+\beta+1}{(2(\lambda_i+n-i)+\alpha+\beta+1)(\lambda_i+n-i)!\Gamma(\lambda_i+n-i+\alpha+\beta+1)}$ and $R_{i,j} = 1-2x_jw_j+w_i^2$.

Example 4.4 Charlier polynomials

For $a>0$, let $\{c_m^a\}_{m\in\mathbb{N}}$ the family of normalized Charlier polynomials. They are orthogonal with respect to the Poisson measure μ_a with atoms at $x \in \mathbb{N}$, given by

$$\mu_a(\{x\}) = \frac{e^{-a}a^x}{x!}.$$

The polynomial c_m^a is an eigenfunction of the discrete diffusion operator

$$L^a = x\Delta\nabla + (a-x)\Delta, \tag{4.52}$$

with eigenvalue $-m$, that is, L^a is the infinitesimal generator of the Markov semigroup $\{N_t^a\}_{t\geq 0}$ associated to the Charlier polynomials for the Markov generator sequence $\gamma_m = m$.

The normalized generalized Charlier polynomial for $\lambda \in \Lambda$, is defined by

$$\hat{C}_\lambda^a(X) = \frac{1}{\sqrt{n!}}\frac{\det(c^a_{\lambda_j+n-j}(x_i))}{V(x_1,\ldots,x_n)}, \tag{4.53}$$

where $x_1,\ldots,x_n$ are the eigenvalues of the Hermitian matrix X. It is an eigenfunction of the Markov semigroup $\{\hat{T}_t^a\}$ defined in Corollary 4.1, with eigenvalues $e^{-t\varphi_\lambda}$ where $\varphi_\lambda = \sum_{j=1}^n \lambda_j$. By Proposition 4.7, the operator D^a, the restriction to the space of diagonal matrices of the infinitesimal generator $\hat{D}^a$ of this semigroup, is given by

$$D^a f(\underline{x}) = \sum_{k=1}^n a\frac{V(\underline{x}+e_k)}{V(\underline{x})}\Delta_k f(\underline{x}) - \sum_{k=1}^n x_k \frac{V(\underline{x}-e_k)}{V(\underline{x})}\nabla_k f(\underline{x}). \tag{4.54}$$

The one dimensional Charlier polynomials have generating function given by

$$\sum_{m=0}^{\infty}\frac{a^{-m/2}}{\sqrt{m!}}c_m^a(x)w^n = e^{-w}\left(1+\frac{w}{a}\right)^x,$$

then, according to Corollary 4.7 for X, W Hermitian matrices with eigenvalues $x_1,\ldots,x_n$ and $w_1,\ldots w_n$ respectively,

$$\sum_{\lambda\in\Lambda} d_\lambda \hat{C}_\lambda^a(X)\hat{S}_\lambda(W) = \frac{\det(e^{-w_i}(1+\frac{w_i}{a})^{x_j})}{V(x_1,\ldots,x_n)V(w_1,\ldots,w_n)},$$

with $d_\lambda = \prod_{i=1}^n \frac{a^{\frac{\lambda_i+n-i}{2}}}{\sqrt{(\lambda_i+n-i)!}}$. Since

$$\det(e^{-w_i}(1+\frac{w_i}{a})^{x_j}) = e^{-\sum_{i=1}^n w_i}\det((1+\frac{w_i}{a})^{x_j}),$$

the generating function for the generalized Charlier polynomials is

$$\sum_{\lambda\in\Lambda} d_\lambda \hat{C}^a(X)\hat{S}_\lambda(W) = e^{-TrW}\frac{\det((1+\frac{w_i}{a})^{x_j})}{V(x_1,\ldots,x_n)V(w_1,\ldots,w_n)}. \tag{4.55}$$

Example 4.5 Meixner polynomials

For $a>0$ and $0<b<1$, the family of normalized Meixner polynomials $\{m_m^{a,b}\}_{m\in\mathbb{N}}$ are orthogonal with respect to the discrete measure $\mu_{a,b}$ with atoms at $x\in\mathbb{N}$ given by

$$\mu_{a,b}(\{x\}) = \frac{b^x\Gamma(a+x)}{x!\Gamma(a)}.$$

The polynomial $m_m^{a,b}$ is an eigenfunction of the discrete diffusion operator

$$L^{a,b} = x\Delta\nabla + [ab - x(1-b)]\Delta, \tag{4.56}$$

with eigenvalue $-m(1-b)$, that is, $L^{a,b}$ is the infinitesimal generator of the Markov semigroup $\{N_t^{a,b}\}_{t\geq 0}$ associated to the Meixner polynomials for the Markov generator sequence $\gamma_m^{a,b} = m(1-b)$.

The normalized generalized Meixner for $\lambda \in \Lambda$, is defined by

$$\hat{\mathrm{M}}_\lambda^{a,b}(X) = \frac{1}{\sqrt{n!}} \frac{\det(m_{\lambda_j+n-j}^{a,b}(x_i))}{V(x_1,\ldots,x_n)}, \tag{4.57}$$

where $x_1,\ldots,x_n$ are the eigenvalues of the Hermitian matrix X. This generalized polynomial is an eigenfunction of the Markov semigroup $\{\hat{T}_t^{a,b}\}_{t\geq 0}$ defined in Corollary 4.1, with eigenvalue $e^{-t\varphi_\lambda^{a,b}}$ where $\varphi_\lambda^{a,b} = (1-b)\sum_{j=1}^n \lambda_j$. By Proposition 4.7, the operator $D^{a,b}$, the restriction to the space of diagonal matrices of the infinitesimal generator $\hat{D}^{a,b}$ of this semigroup, is given by

$$D^{a,b} f(\underline{x}) = \sum_{k=1}^n b(a+x_k) \frac{V(\underline{x}+e_k)}{V(\underline{x})} \Delta_k f(\underline{x}) - \sum_{k=1}^n x_k \frac{V(\underline{x}-e_k)}{V(\underline{x})} \nabla_k f(\underline{x}). \tag{4.58}$$

The generating function for the one dimensional Meixner polynomials is given by

$$\sum_{m=0}^\infty (-1)^m (l_m^{a,b})^{1/2} m_m^{a,b}(x) w^n = \left(1 - \frac{w}{b}\right)^x (1-w)^{-(a+x)},$$

where $l_m^{a,b} = \frac{(a)_m}{m! b^{m/2}}$. So, according to Corolary 4.7, for $X, W \in H_n$ with eigenvalues $x_1,\ldots,x_n$ and $w_1,\ldots,w_n$ respectively

$$\sum_{\lambda\in\Lambda} d_\lambda \hat{\mathrm{M}}_\lambda^{a,b}(X) \hat{S}_\lambda(W) = \frac{\det\left(\left(1-\frac{w_i}{b}\right)^{x_j} (1-w_i)^{-(a+x_j)}\right)}{V(x_1,\ldots,x_n)V(w_1,\ldots,w_n)}, \tag{4.59}$$

where $d_\lambda = \prod_{i=1}^n (-1)^{\lambda_1+n-i} \left(\frac{(a)_{\lambda_i+n-i}}{(\lambda_i+n-i)! b^{\frac{\lambda_i+n-i}{2}}} \right)^{1/2}$.

Example 4.6 Kravchuk polynomials

For $N \in \mathbb{N}$ and $0 \leq p \leq 1$, the family of normalized Kravchuk polynomials $\{k_m^p\}_{m=0}^N$ is orthogonal in the space $l^2(\mathbb{N} \cap [0,N], \mu_p)$, where μ_p is the binomial measure with atoms at $x = 0,\ldots,N$ given by

$$\mu_p(\{x\}) = \binom{N}{x} p^x (1-p)^{N-x}.$$

The polynomial k_m^p is an eigenfunction of the discrete diffusion operator

$$L^p = x\Delta\nabla + \frac{Np-x}{1-p}\Delta \frac{m}{1-p}, \tag{4.60}$$

with eigenvalue $-\frac{m}{1-p}$, that is, L^p is the infinitesimal generator of the Markov semigroup $\{N_t^p\}_{t\geq 0}$ associated to the Kravchuk polynomials for the Markov generator sequence $\gamma_m^p = \frac{m}{1-p}$.

The normalized generalized generalized Kravchuk polynomial, for $\lambda \in \Lambda$ such that $\lambda_1 + n - 1 \leq N$, is defined by

$$\hat{\mathrm{K}}_\lambda^p(X) = \frac{1}{\sqrt{n!}} \frac{\det(k_{\lambda_j+n-j}^p(x_i))}{V(x_1, \ldots, x_n)}, \tag{4.61}$$

where $x_1, \ldots, x_n$ are the eigenvalues of $X \in H_n$. Note that the condition $\lambda_1 + n - 1 \leq N$ for the partition λ implies that $\lambda_j + n - j \leq N$, for all $1 \leq j \leq n$ and therefore $k_{\lambda_j+n-j}^p$ is always defined. The family of generalized Kravchuk polynomials is finite. The generalized polynomial $\hat{\mathrm{K}}_\lambda^p$ is an eigenfunction of the Markov semigroup $\{\hat{T}_t^p\}_{t\geq 0}$ defined in Corollary 4.1, with eigenvalues $e^{-t\varphi_\lambda^p}$ where $\varphi_\lambda^p = \frac{1}{1-p}\sum_{j=1}^n \lambda_j$. By Proposition 4.7, the operator D^p, the restriction to the space of diagonal matrices of the infinitesimal generator $\hat{D}^p$ of this semigroup, is given by

$$D^p f(\underline{x}) = \sum_{k=1}^n \frac{p}{1-p}(1-x_k)\frac{V(\underline{x}+e_k)}{V(\underline{x})}\Delta_k f(\underline{x}) - \sum_{k=1}^n x_k \frac{V(\underline{x}-e_k)}{V(\underline{x})}\nabla_k f(\underline{x}). \tag{4.62}$$

The generating function for the one dimensional Kravchuk polynomials is

$$\sum_{m=0}^N \binom{N}{m}^{m/2} (p(1-p))^{m/2} k_m^p(x) w^n = (1+(1-p)w)^x (1-pw)^{N-x}.$$

So, according to Corollary 4.7, for $X, W \in H_n$ with eigenvalues $x_1, \ldots, x_n$ and $w_1, \ldots, w_n$ respectively

$$\sum_{\{\lambda:\lambda_1+n-1\leq N\}} d_\lambda \hat{\mathrm{K}}_\lambda^p(X)\hat{S}_\lambda(W) = \frac{\det((1+(1-p)w_i)^{x_j}(1-pw_i)^{N-x_j})}{V(x_1,\ldots,x_n)V(w_1,\ldots,w_n)}, \tag{4.63}$$

where $d_\lambda = \prod_{i=1}^n \binom{N}{\lambda_i+n-i}^{\frac{\lambda_i+n-1}{2}} (p(1-p))^{\frac{\lambda_i+n-1}{2}}$.

Example 4.7 Hahn polynomials

For $N \in \mathbb{N}$ and $\alpha, \beta > -1$, the family of normalized Hahn polynomials $\{h_m^{\alpha,\beta}\}_{m=0}^N$ is orthogonal in the space $l^2(\mathbb{N} \cap [0, N], \mu_{\alpha,\beta})$, where $\mu_{\alpha,\beta}$ is the discrete measure with atoms in $x = 0, \ldots, N$ given by

$$\mu_{\alpha,\beta}(\{x\}) = \frac{\Gamma(N+1+\alpha-x)\Gamma(\beta+1+x)}{x!(N-x)!}.$$

The polynomial $h_m^{\alpha,\beta}$ is an eigenfunction of the discrete diffusion operator

$$L^{\alpha,\beta} = x(N+1+\alpha-x)\Delta\nabla + [(\beta+1)N - (\alpha+\beta+2)x]\Delta, \tag{4.64}$$

with eigenvalue $-m(m+\alpha+\beta+2)$, that is, $L^{\alpha,\beta}$ is the infinitesimal generator of the Markov semigroup $\{N_t^{\alpha,\beta}\}_{t\geq 0}$ associated to the Hahn polynomials for the Markov generator sequence $\gamma_m^{\alpha,\beta} = m(m+\alpha+\beta+2)$.

The normalized generalized generalized Hahn polynomial, for $\lambda \in \Lambda$ such that $\lambda_1 + n - 1 \leq N$, is defined by

$$\hat{\mathrm{H}}_\lambda^{\alpha,\beta}(X) = \frac{1}{\sqrt{n!}} \frac{\det(h_{\lambda_j+n-j}^{\alpha,\beta}(x_i))}{V(x_1,\ldots,x_n)}, \tag{4.65}$$

where $x_1,\ldots,x_n$ are the eigenvalues of $X \in H_n$. The family of generalized Hahn polynomials is also finite. The generalized polynomial $\hat{\mathrm{H}}_\lambda^{\alpha,\beta}$ is an eigenfunction of the Markov semigroup $\{\hat{T}_t^{\alpha,\beta}\}_{t\geq 0}$ defined in Corollary 4.1, with eigenvalues $e^{-t\varphi_\lambda^{\alpha,\beta}}$ where $\varphi_\lambda^{\alpha,\beta} = \sum_{j=1}^n [\lambda_j(\lambda_j+\alpha+\beta+2) + 2\lambda_j(n-j)] + (\alpha+\beta+2)\frac{(n-1)n}{2}$. By Proposition 4.7, the operator $D^{\alpha,\beta}$, the restriction to the space of diagonal matrices of the infinitesimal generator $\hat{D}^{\alpha,\beta}$ of this semigroup, is given by

$$\begin{aligned} D^{\alpha,\beta} f(\underline{x}) = \sum_{k=1}^n \left[(\alpha+\beta+1)N - (\alpha+\beta+2)x_k + \alpha\right] \frac{V(\underline{x}+e_k)}{V(\underline{x})} \Delta_k f(\underline{x}) - \\ - \sum_{k=1}^n x_k(N+1+\alpha-x_k) \frac{V(\underline{x}-e_k)}{V(\underline{x})} \nabla_k f(\underline{x}). \end{aligned} \tag{4.66}$$

As an special case of this family, when $\alpha = \beta = 0$, we obtain the family of discrete Chebyshev polynomials $\{t_m\}_{m=1}^N$.

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Appendix A

Orthogonal polynomials

Let μ be a finite Borel measure on $\mathbb{R}$ with finite moments. By applying the Gram–Schmidt orthogonalization process on $L^2(\mathbb{R}, \mu)$ to the monomials, we obtain a unique, except for a constant, family of orthogonal polynomials $\{p_m\}_{m\in\mathbb{N}}$ in $L^2(\mathbb{R}, \mu)$. We call this family, the **orthogonal polynomials associated to the measure** μ, or with respect to the measure μ.

If μ is a non–atomic measure, we will say that the polynomials p_m are of a continuous variable and if μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, that is, if there exists a function $w : \mathbb{R} \to \mathbb{R}^+$ such that $\mu(A) = \int_A w(x)dx$ for any Borel set $A \subseteq \mathbb{R}$, then we will refer to the family $\{p_m\}_{m\in\mathbb{N}}$ as the **orthogonal polynomials associated to the weight function** w.

If μ is a purely atomic measure, then we will say that the polynomials p_m are of a discrete variable. If μ has finite support, let us say, in $\mathbb{N}$, then $L^2(\mathbb{R}, \mu) = l^2(\mathbb{N}, \mu)$. If the support of μ is finite, let us say $\{0, 1, \ldots, N-1\}$, then the functions $1, x, x^2, \ldots, x^{N-1}$ are still linearly independent and by applying the Gram–Schmidt orthogonalization process we obtain a finite system of orthogonal polynomials $p_0, p_1, \ldots, p_{N-1}$, and therefore the L^2 space is, in this case, a finite dimensional space.

We are interested in measures μ such that this orthogonal system is complete in $L^2(\mathbb{R}, \mu)$. There are many conditions over μ that guarantees the density of the polynomials; see for example Th. 1.5.2 in [Sz59] for the absolutely continuous case. A very general condition is that μ has an exponential moment, that is, there exists $\epsilon > 0$ such that $\int_{\mathbb{R}} e^{\epsilon|x|}d\mu(x) < \infty$. The density of the monomials $\{x^m\}_{m\in\mathbb{N}}$, and therefore of the orthogonal polynomials, in $L^2(\mathbb{R}, \mu)$ is well known (Th. 3.1.18, [DX01] or [BC81]).

When we apply the Gram–Schmidt orthogonalization process to the monomials, we obtain a unique system, except for a constant. The system is unique if we fix a condition, for example, if we ask normality. This is not the only normalization condition that can be asked; indeed, among the frequently considered type of normalization conditions for orthogonal polynomials we have to require each polynomial to be monic or to require the constant term of the polynomial to be equal to one.

By the construction of the family $\{p_m\}_{m\in\mathbb{N}}$ by the Gram-Schmidt process, we have that each polynomial of degree m can be written as a linear combination of $p_1,\ldots,p_m$ and therefore, for each $m \in \mathbb{N}$, the polynomial p_m is orthogonal in $L^2(\mathbb{R},\mu)$ to any polynomial of degree $k < m$; in particular

$$\int_{\mathbb{R}} p_m(x)x^k d\mu(x) = 0, \quad \text{para todo } m = 0,1,\ldots,m-1. \tag{A.1}$$

In fact, this condition determines the polynomial p_m except for a constant. If we think the coefficients of the polynomial p_m as variables, then equation (A.1) determines a linear system with m equations and $m+1$ variables in which the coefficients are the moments of the measure μ. When we fix a normalization constant, we are adding one more equation to this system and therefore, when we solve it, we obtain an unique solution in term of the moments of the measure μ.

The orthogonality property implies a series of important properties for the polynomials. In what follows we will name some of them. For details we refer to [Sz59], Ch. 3.

For any three consecutive orthogonal polynomials, we have

$$p_m(x) = (A_m x + B_m)p_{m-1} - C_m p_{m-2}, \quad n = 2,3,\ldots, \tag{A.2}$$

where A_m, B_m, C_m are constants, $A_m = \frac{a_m}{a_{m-1}}$, $C_m = \frac{A_m\|p_{m-1}\|^2_{L^2(\mathbb{R},\mu)}}{A_{m-1}\|p_{m-2}\|^2_{L^2(\mathbb{R},\mu)}}$ and a_m is the principal coefficient of the polynomial p_m. This formula is known as the **three term recurrence relation** and it is very useful both in theory and in practice.

As a consequence of relation we have the **Christoffel–Darboux formula**: for any $m \in \mathbb{N}$ and $x, y \in \mathbb{R}$ such that $x \neq y$,

$$\sum_{k=0}^{m} \frac{p_k(x)p_k(y)}{\|p_k\|^2_{L^2(\mathbb{R},\mu)}} = \frac{p_{m+1}(x)p_m(y) - p_m(x)p_{m+1}(y)}{A_m\|p_m\|^2_{L^2(\mathbb{R},\mu)}(x-y)}, \tag{A.3}$$

where A_m is as before. This identity is very useful in the study of the distribution of the zeros of orthogonal polynomials.

If the family of orthogonal polynomials has a **generating function**, that is, a function $f(x,w)$ such that the m–est coefficient of its development in terms of $1, w^1, w^2, \ldots$ is, except for a constant, the

polynomial $p_m(x)$
$$f(x,w) = \sum_{m=0}^{\infty} d_m p_m(x) w^m, \quad |w| < r. \tag{A.4}$$
This function determines, except for a constant, the family of polynomials $\{p_m\}_{m\in\mathbb{N}}$.

In what follows we will name this and other identities for the families of classical orthogonal polynomials.

A.1 Classical orthogonal polynomials of a continuous variable

The usually known as classical orthogonal polynomials are the Hermite, Laguerre and Jacobi polynomials. They have in common that they are solutions of differential equations of the form
$$a(x)y'' + b(x)y' + \lambda y = 0, \tag{A.5}$$
where a and b are polynomials of degree at most 2 and 1 respectively, and λ is a constant.

Diverse models in atomic, molecular and nuclear physics, electrodynamics and acoustics can be reduced to an equation of this type. The solutions of this type of equation, among them, the classical orthogonal polynomials, are extensively used in mathematical physics.

Some of the identities for this classical polynomials are shown bellow.

Hermite polynomials

The Hermite polynomials $\{\mathsf{H}_m\}_{m\in\mathbb{N}}$, are the orthogonal polynomials on $\mathbb{R}$ with respect to the Gaussian measure $\mu(dx) = e^{-x^2}dx$. If we ask for each polynomial H_m to be monic, then they have the explicit representation
$$\frac{\mathsf{H}_m(x)}{m!} = \sum_{k=0}^{[\frac{m}{2}]} \frac{(-1)^k}{2^{2k}k!} \frac{x^{m-2k}}{(m-2k)!}. \tag{A.6}$$
From this formula we have that for m odd, $\mathsf{H}_m(0) = 0$, and therefore it is not possible to normalize by requiring the constant term to be equal to one. The norm of this polynomials is given by
$$\|\mathsf{H}_m\|^2_{L^2(\mathbb{R},\mu)} = \frac{m!}{\sqrt{\pi}}. \tag{A.7}$$

The Hermite polynomial of degree m is the unique polynomial solutions of the differential equations of hypergeometric type
$$y'' - 2xy' + 2my = 0. \tag{A.8}$$

For each $m \in \mathbb{N}$ the Christoffel–Darboux formula for these polynomials is given by

$$\sum_{k=0}^{m} \frac{2^k}{k!} \mathsf{H}_k(x)\mathsf{H}_k(y) = \frac{2^m}{m!} \frac{\mathsf{H}_{m+1}(x)\mathsf{H}_m(y) - \mathsf{H}_m(x)\mathsf{H}_{m+1}(y)}{x-y} \tag{A.9}$$

and their generating function is

$$\sum_{m=0}^{\infty} \frac{2^m}{m!} \mathsf{H}_m(x) w^m = e^{2xw - w^2}. \tag{A.10}$$

The Rodrigues' formula for the Hermite polynomials is

$$e^{x^2} \mathsf{H}_m(x) = (-1)^m 2^m \frac{d^m}{dx^m} e^{x^2}. \tag{A.11}$$

Laguerre polynomimals

The Laguerre polynomials $\{\mathsf{L}_m^\alpha\}_{m\in\mathbb{N}}$, $\alpha > -1$, are the orthogonal polynomials on $[0,\infty)$ with respect to the measure $\mu_\alpha(dx) = x^\alpha e^{-x} \mathbf{1}_{[0,\infty)}(x) dx$. If we normalize by requiring $\mathsf{L}_m^\alpha(0) = 1$, then we have the explicit representation

$$\mathsf{L}_m^\alpha(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-x)^k}{(\alpha+1)_k}. \tag{A.12}$$

The norm if this polynomials is given by

$$\|\mathsf{L}_m^\alpha\|^2_{L^2(\mathbb{R},\mu_\alpha)} = l_m^\alpha = \frac{m!\Gamma^2(\alpha+1)}{\Gamma(\alpha+m+1)}. \tag{A.13}$$

The Laguerre polynomial of degree m is the unique polynomial solution of the differential equations of hypergeometric type

$$xy'' + (\alpha + 1 - x)y' + my = 0. \tag{A.14}$$

For each $m \in \mathbb{N}$ the Christoffel–Darboux formula for these polynomials is given by

$$\sum_{k=0}^{m} \binom{k+\alpha}{k} \mathsf{L}_k^\alpha(x)\mathsf{L}_k^\alpha(y) = (m+1)\binom{m+\alpha+1}{m+1} \frac{\mathsf{L}_{m+1}^\alpha(x)\mathsf{L}_m^\alpha(y) - \mathsf{L}_m^\alpha(x)\mathsf{L}_{m+1}^\alpha(y)}{x-y} \tag{A.15}$$

and their generating function is

$$\sum_{m=0}^{\infty} \binom{m+\alpha}{m}^2 \mathsf{L}_m^\alpha(x) w^m = (1-w)^{-\alpha-1} e^{-\frac{xw}{1-w}}. \tag{A.16}$$

The Rodrigues' formula for the Laguerre polynomials is

$$e^{-x} x^\alpha \binom{m+\alpha}{m} \mathsf{L}_m^\alpha(x) = \frac{1}{m!} \frac{d^m}{dx^m} (e^{-x} x^{m+\alpha}). \tag{A.17}$$

Jacobi polynomials

For $\alpha, \beta > -1$ the Jacobi polynomials $\{\mathsf{J}_m^{\alpha,\beta}\}_{m\in\mathbb{N}}$, are the orthogonal polynomials on $[-1,1]$ with respect to the measure $\mu_{\alpha,\beta}(dx) = (1-x)^\alpha(1+x)^\beta \mathbf{1}_{[-1,1]}(x)dx$. For each $m \in \mathbb{N}$, the polynomial $\mathsf{J}_m^{\alpha,\beta}$ is a solution of the differential equation

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + m(m+\alpha+\beta+1)y = 0. \tag{A.18}$$

Among some especial cases we have:

(i) Legendre polynomials, for $\alpha = \beta = 0$.

(ii) Chebyshev polynomials of the first kind, for $\alpha = \beta = -1/2$.

(iii) Chebyshev polynomials of the second kind, for $\alpha = \beta = 1/2$.

(iv) Gegenbauer or ultraspherical polynomials, for $\alpha = \beta = \lambda - 1/2$, with $\lambda > 0$.

We will also consider a related family of Jacobi polynomials

$$\mathsf{P}_m^{\alpha,\beta}(x) := \mathsf{J}_m^{\alpha,\beta}(1-2x). \tag{A.19}$$

The family of Jacobi polynomials $\{\mathsf{P}_m^{\alpha,\beta}\}_{m\in\mathbb{N}}$ are orthogonal on $[0,1]$ with respecto to the measure $\nu_{\alpha,\beta}(dx) = x^\alpha(1-x)^\beta \mathbf{1}_{[0,1]}(x)dx$. If we normalize by requiring $\mathsf{P}_m^{\alpha,\beta}(0) = 1$, they have the explicit representation

$$\mathsf{P}_m^{\alpha,\beta}(x) = \sum_{k=0}^{m} \frac{(m+\alpha+\beta+1)_k}{(\alpha+1)_k} \binom{m}{k} (-x)^k. \tag{A.20}$$

The norm of the polynomial $\mathsf{P}_m^{\alpha,\beta}$ is given by

$$\|\mathsf{P}_m^{\alpha,\beta}\|^2_{L^2(\mathbb{R},\mu_{\alpha,\beta})} = h_m^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}}{2m+\alpha+\beta+1} \frac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{\Gamma(m+1)\Gamma(m+\alpha+\beta+1)}. \tag{A.21}$$

For each $m \in \mathbb{N}$ the Christoffel–Darboux formula for these polynomials is given by

$$\sum_{k=0}^{m} \frac{1}{h_k^{\alpha,\beta}} \mathsf{P}_k^{\alpha,\beta}(x)\mathsf{P}_k^{\alpha,\beta}(y) = \frac{2^{-\alpha-\beta-1}}{2m+\alpha+\beta+2} \frac{\Gamma(m+2)\Gamma(m+\alpha+\beta+2)}{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)} \times \\ \times \frac{\mathsf{P}_{m+1}^{\alpha,\beta}(x)\mathsf{P}_m^{\alpha,\beta}(y) - \mathsf{P}_m^{\alpha,\beta}(x)\mathsf{P}_{m+1}^{\alpha,\beta}(y)}{y-x} \tag{A.22}$$

and their generating function is

$$\sum_{m=0}^{\infty} \mathsf{P}_m^{\alpha,\beta}(x)w^m = 2^{\alpha+\beta}R^{-1/2}\{1-w+R^{1/2}\}^{-\alpha}\{1+w+R^{1/2}\}^{-\beta}, \tag{A.23}$$

where $R = (1-w)^2 + 4xw$.

The Rodrigues' formula for these Jacobi polynomials is

$$x^{\alpha}(1-x)^{\beta}\mathsf{P}_m^{\alpha,\beta}(x) = \frac{1}{m!}\frac{d^m}{dx^m}(x^{\alpha+m}(1-x)^{\beta+m}). \tag{A.24}$$

A.2 Classical orthogonal polynomials of a discrete variable

The families of Charlier, Mexnier, Kravchuk and Hahn polynomials, called classical orthogonal polynomials of a discrete variable in [MSU91], are an important class of special function that arises naturally in various problems of mathematics, theoretical physics and engineering; this field is in extensive development. The study of classical orthogonal polynomials of a discrete variable was initiated by P.L. Chebyshev in the middle of the last century and continued with great interest.

The orthogonal polynomials of a discrete variable can be characterized in terms of the difference equation of hypergeometric type

$$\sigma(x)\Delta\nabla y + \tau(x)\Delta y + \lambda y = 0, \tag{A.25}$$

where σ and τ are polynomials of degree at most 2 and 1 respectively, and the difference operator Δ and ∇ are given by

$$\Delta f(x) = f(x+1) - f(x),$$
$$\nabla f(x) = f(x) - f(x-1).$$

The equation (A.25) may be obtained by approximating the differential equation (A.5).

Among one of the few books that address this topic extensively we find [MSU91]. In his monograph we can find a systematic, concise presentation of the theory of classical orthogonal polynomials of a discrete variable and of its applications.

Charlier polynomials

For $a > 0$, the family of normalized Charlier polynomials $\{c_m^a\}_{m\in\mathbb{N}}$ is orthogonal in the space $l^2(\mathbb{N}, \mu_a)$, where the measure μ_a is the Poisson measure with atoms at $x \in \mathbb{N}$, given by

$$\mu_a(\{x\}) = \frac{e^{-a}a^x}{x!}. \tag{A.26}$$

For $m \in \mathbb{N}$, the polynomial c_m^a is a solutions of the difference equation

$$x\Delta\nabla y + (a - x)\Delta y + my = 0. \tag{A.27}$$

The generating function for this polynomials is given by

$$\sum_{m=0}^{\infty} \frac{a^{-m/2}}{\sqrt{m!}} c_m^a(x) w^n = e^{-w} \left(1 + \frac{w}{a}\right)^x. \tag{A.28}$$

A Rodrigues's type formula for this polynomials is given by

$$c_m^a(x) = \frac{(-1)^m a^{-m/2}}{e^{-a}\sqrt{m!}} \frac{x!}{a^x} \nabla^m \left[\frac{e^{-a} a^{x+m}}{x!}\right]. \tag{A.29}$$

Mexnier polynomials

For $a > 0$ and $0 < b < 1$, the family of normalized Meixner polynomials $\{m_m^{a,b}\}_{m\in\mathbb{N}}$ is orthogonal in the space $l^2(\mathbb{N}, \mu_{a,b})$ where $\mu_{a,b}$ is the discrete measure with atoms at $x \in \mathbb{N}$ given by

$$\mu_{a,b}(\{x\}) = \frac{b^x \Gamma(a+x)}{x!\Gamma(a)}. \tag{A.30}$$

For each $m \in \mathbb{N}$, the polynomial $m_m^{a,b}$ is a solution of the difference equation

$$x\Delta\nabla y + [ab - x(1-b)]\Delta y + m(1-b)y = 0. \tag{A.31}$$

The generating function for this polynomials is given by

$$\sum_{m=0}^{\infty} (-1)^m (l_m^{a,b})^{1/2} m_m^{a,b}(x) w^n = \left(1 - \frac{w}{b}\right)^x (1-w)^{-(a+x)}, \tag{A.32}$$

where

$$l_m^{a,b} = \frac{(a)_m}{m! b^{m/2}}. \tag{A.33}$$

A Rodrigues's type formula for this polynomials is given by

$$m_m^{a,b}(x) = \frac{(-1)^m b^{-m/2} (1-b)^{a/2}}{\sqrt{m!}(a)_m^{1/2}} \frac{x!\Gamma(a)}{b^x \Gamma(a+x)} \nabla^m \left[b^{x+m} \frac{\Gamma(x+a+n)}{\Gamma(a)x!}\right]. \tag{A.34}$$

Kravchuk polynomials

For $N \in \mathbb{N}$ and $0 \leq p \leq 1$, the family of normalized Kravchuk polynomials $\{k_m^p\}_{m=0}^N$ is orthogonal in the space $l^2(\mathbb{N} \cap [0,N], \mu_p)$ where μ_p is the binomial measure with atoms at $x = 0, \dots, N$ given by

$$\mu_p(\{x\}) = \binom{N}{x} p^x (1-p)^{N-x}. \tag{A.35}$$

They are solution of the difference equation

$$x\Delta\nabla y + \frac{Np - x}{1-p}\Delta y + \frac{m}{1-p} y = 0. \tag{A.36}$$

The generating function for this polynomials is given by

$$\sum_{m=0}^{N} \binom{N}{m}^{m/2} (p(1-p))^{m/2} k_m^p(x) w^n = (1 + (1-p)w)^x (1-pw)^{N-x} \tag{A.37}$$

A Rodrigues's type formula for k_m^p is given by

$$k_m^p(x) = \frac{(-1)^m p^{m/2}(1-p)^{m/2}}{m!} \binom{N}{m}^{1/2} \binom{N}{x}^{-1} p^{-x}(1-p)^{-N+m+x} \times \\ \times \nabla^m \left[\frac{N!}{x!(N-m-x)!} p^{x+m} (1-p)^{N-m-x} \right]. \tag{A.38}$$

Hahn polynomials

For $N \in \mathbb{N}$ and $\alpha, \beta > -1$, the family of normalized Hahn polynomials $\{h_m^{\alpha,\beta}\}_{m=0}^N$ is orthogonal in the space $l^2(\mathbb{N} \cap [0,N], \mu_{\alpha,\beta})$, where $\mu_{\alpha,\beta}$ is the discrete measure with atoms at $x = 0, \dots, N$ given by

$$\mu_{\alpha,\beta}(\{x\}) = \frac{\Gamma(N+1+\alpha-x)\Gamma(\beta+1+x)}{x!(N-x)!}. \tag{A.39}$$

They are solution of the difference equation

$$x(N+1+\alpha-x)\Delta\nabla y + [(\beta+1)N - (\alpha+\beta+2)x]\Delta y + m(m+\alpha+\beta+1)y = 0. \tag{A.40}$$

A Rodrigues's type formula for $h_m^{\alpha,\beta}$ is given by

$$h_m^{\alpha,\beta}(x) = \frac{(-1)^m (d_m^{\alpha,\beta})^{1/2}}{m!} \frac{x!(N-x)!}{\Gamma(N+\alpha-x+1)\Gamma(\beta+x+1)} \times \\ \times \nabla^m \left[\frac{\Gamma(N+\alpha-x+1)\Gamma(m+\beta+x+1)}{x!(N-m-x)!} \right]. \tag{A.41}$$

As a special case of this family, when $\alpha = \beta = 0$, we have the discrete Chebyshev polynomials $\{t_m\}_{m=0}^N$.

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