Étude de l'approximation hydrostatique de Stokes & d'une équation dégénérée

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Contents

Présentation générale

Cette thèse est consacrée à l'étude de l'approximation hydrostatique de Stokes ainsi qu'à l'étude d'une équation dégénérée posée dans le demi-espace $\mathbb{R}^3_+$, puis dans un ouvert Ω quelconque. Elle se divise en trois partie.

Dans la première partie de la thèse, nous donnons de nouveaux résultats sur l'approximation hydrostatique de Stokes. Plus précisément, nous justifions l'existence et l'unicité d'une solution faible lorsque l'on considère des conditions aux limites et des données non homogènes. Nous prouvons ce résultat en réduisant l'approximation hydrostatique de Stokes à un système de type Stokes, en établissant un lien très proche entre ces deux modèles. Dès lors, nous sommes amenés à étudier ce dernier modèle, et à démontrer l'existence et l'unicité d'une solution faible, en considérant, ici encore, de nouvelles situations.
Comme nous le justifierons plus tard, la difficulté liée à l'étude de ces deux modèles concerne la géométrie du domaine Ω que l'on considère, voir (1). Pour travailler dans un domaine le plus générale possible, c'est-à-dire avec une profondeur qui peut tendre vers 0, nous sommes amenés à considérer un cadre fonctionnel particulier, constitué d'espaces avec poids. Enfin, le travail effectué dans cette première partie s'inspire de résultats déjà existants que nous nous permettrons de visiter à nouveau.
Par ailleurs, nous nous sommes intéressés à la question de la régularité des solutions faibles de l'approximation hydrostatique de Stokes, toujours dans un domaine où la profondeur tend vers 0. Le problème reste ouvert, même en considérant le cas académique de conditions de Dirichlet homogènes. C'est alors en voulant adapter des résultats déjà existants, que nous somme naturellement conduit à étudier une équation dégénérée, comme celle que l'on présente dans la suite de cette thèse.
Nous ne dévoilerons plus de détails sur cette première partie, puisqu'une introduction très détaillée de cette partie est donné dans la suite.

Les parties 2 et 3 sont consacrées à l'étude d'un problème elliptique avec un coefficient de diffusion qui peut dégénérer. Ce type d'équations intervient également dans des problèmes géophysiques, que ce soit des questions de modélisation de circulation globale, mais aussi dans des problèmes d'infiltration et de milieux poreux. La partie 2 est la plus significative de ce travail. Elle est consacrée au demi-espace $\mathbb{R}^3_+$ avec un coefficient de diffusion en $1/x_3$, voir (4.38). Comme nous l'expliquerons dans une introduction détaillée, cette équation sort du cadre d'étude standard où le poids, ici x_3, est borné à l'infini. Nous construisons donc un cadre fonctionnel adéquat et qui prend en considération des poids non bornés à l'infini et tendant vers 0 au bord de $\mathbb{R}^3_+$. Ces espaces, et d'autres intervenant dans la résolution de l'équation de Laplace dans $\mathbb{R}^3_+$, sont alors adaptés à la recherche d'existence et d'unicité de solutions faibles de ce problème. Nous établissons aussi deux résultats de régularité des solutions faibles.
Certaines des idées mises en oeuvre dans la partie 2 nous ont permis, dans la troisième partie et dernière partie de la thèse, d'étudier le problème plus général (4.39). La difficulté du cadre que l'on choisit ne permet d'avoir des résultats aussi optimaux que ceux que l'on obtient dans le cas du demi-espace. Cela

étant, on établit un cas d'existence et d'unicité de solution faible et un résultat de régularité associé.

Le travail effectué dans la partie 1 fait l'objet d'une publication de niveau A accepté dernièrement dans la revue scientifique Differential Equations and Applications. Celui de la deuxième partie fait l'objet d'un article en cours de soumission. C'est pourquoi nous avons choisi d'écrire la suite de ce manuscrit en anglais.

Part I

New conditions for the hydrostatic Stokes approximation

Introduction

In the first part of the thesis, dedicated to the study of the hydrostatic Stokes approximation, let us consider a bounded domain $\Omega \subset \mathbb{R}^3$ defined by

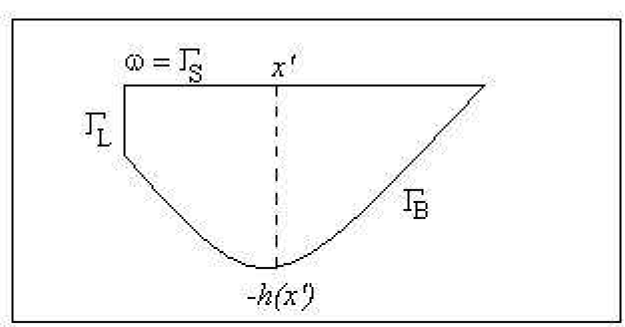
$$\Omega = \left\{ x = (x',\, x_3) \in \mathbb{R}^3 \,/\, x' \in \omega \text{ and } -h(x') < x_3 < 0 \right\}, \tag{1}$$

where $\omega \subset \mathbb{R}^2$ is a bounded Lipschitz-continuous domain and where h is a positive Lipschitz-continuous map over ω, chosen such that Ω has a Lipschitz-continuous boundary Γ.
The boundary Γ is split into three parts, each one with a non negative measure: the surface Γ_S, the bottom Γ_B , and sidewalls Γ_L, defined by:

$$\begin{aligned} &\Gamma_S = \omega \times \{0\}\,, \quad \Gamma_B = \{(x', -h(x'))\,/\, x' \in \omega\}\,, \\ &\Gamma_L = \left\{ x \in \mathbb{R}^3 \,/\, x' \in \partial\omega \text{ and } -h(x') \leqslant x_3 \leqslant 0 \right\}. \end{aligned} \tag{2}$$

Finally, we denote by $\boldsymbol{n}$ the unit outward vector normal to Γ.

Figure 1: The domain Ω



The hydrostatic Stokes approximation with homogeneous conditions, is a model used in oceanography. As we will see later, lots of study have been dedicated to this problem or to generalized versions combining non linear and time dependent cases, or with additional unknowns other than the ones we consider in this work. Given $\boldsymbol{f}' = (f_1,\, f_2) : \Omega \to \mathbb{R}^2$, it consists in finding a velocity

$\boldsymbol{u} = (\boldsymbol{u}', u_3) : \Omega \to \mathbb{R}^3$ and a pressure $p : \Omega \to \mathbb{R}$ solution to:

$$-\Delta \boldsymbol{u}' + \nabla' p = \boldsymbol{f}' \quad \text{in } \Omega, \tag{3}$$

$$\frac{\partial p}{\partial x_3} = 0 \quad \text{in } \Omega, \tag{4}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{5}$$

$$\boldsymbol{u}' = 0, \quad u_3 n_3 = 0 \quad \text{on } \Gamma. \tag{6}$$

In these equations, $\nabla' = (\partial_{x_1}, \partial_{x_2})$[1] is the gradient operator with respect to the variables x_1 and x_2. Then, we denote by $\Delta = \partial^2_{x_1^2} + \partial^2_{x_2^2} + \partial^2_{x_3^2}$ the Laplace operator. Equations (3) are the horizontal momentum equations. Equation (4) is called the hydrostatic pressure hypothesis.

Regarding to the usual Stokes problem, the disappearance of the term $-\Delta u_3$ in (4) confers to (3)-(6) a degenerate nature. Therefore, we expect from the vertical velocity u_3 to be less regular than the horizontal velocity $\boldsymbol{u}'$, which explains why u_3 and $\boldsymbol{u}'$ have different boundary conditions. This comment will be clearly justified in the following.

The study of system (3)-(6) is more or less difficult depending on whether the mapping h satisfies the additional property: *there is a constant* $\alpha > 0$, *such that*

$$\inf_{x' \in \omega} h(x') \geqslant \alpha. \tag{7}$$

If it is the case, we shall say that Ω has sidewalls, and then the piece of boundary Γ_L is not empty.

Let us briefly recall two approaches, that we can find in the literature, and that enable the study of (3)-(6). Firstly, Problem (3)-(6) is the limit, in a sense to be precised, of an anisotropic Stokes system set in a thin domain[2]. Indeed, let $\varepsilon \in]0, 1]$, and consider the thin domain Ω_ε defined as in (1) by

$$\Omega_\varepsilon = \left\{ (x', z) \in \mathbb{R}^3 \,/\, x' \in \omega \text{ and } -\varepsilon h(x') < z < 0 \right\}. \tag{8}$$

The open set Ω_ε is bounded, connected and has a Lipschitz-continuous boundary Γ_ε. Next, for a datum $\boldsymbol{F} = (\boldsymbol{F}', 0) : \Omega_\varepsilon \to \mathbb{R}^3$ let us introduce the following homogeneous Stokes system:

$$\left\{ \begin{aligned} -\Delta_\varepsilon \boldsymbol{v}_\varepsilon + \nabla \pi_\varepsilon = \boldsymbol{F}, \quad \nabla \cdot \boldsymbol{v} &= 0 \quad \text{in } \Omega_\varepsilon, \\ \boldsymbol{v} &= 0 \quad \text{on } \Gamma_\varepsilon, \end{aligned} \right. \tag{9}$$

where $\Delta_\varepsilon = \partial^2_{x_1^2} + \partial^2_{x_2^2} + \varepsilon^2 \partial^2_{x_3^2}$ stands for the anisotropic Laplacian operator. In the sequel, we want to derive from Problem (9) a limit problem when ε goes to 0. Therefore, let us proceed to the following scaling

$$x_3 = z/\varepsilon,$$

and let us set

$$\boldsymbol{u}'_\varepsilon(x) = \boldsymbol{v}'_\varepsilon(x', z), \quad u_3^\varepsilon(x) = \frac{v_3^\varepsilon(x', z)}{\varepsilon}, \quad p_\varepsilon(x) = \pi_\varepsilon(x', z), \quad \boldsymbol{f}'(x) = \boldsymbol{F}'(x', z).$$

The new unknowns $\boldsymbol{u}_\varepsilon$ and p_ε are defined in the domain Ω (see (1)), and are formally solution to the following system:

$$\left\{ \begin{aligned} -\Delta \boldsymbol{u}'_\varepsilon + \nabla' p_\varepsilon = \boldsymbol{f}', \quad -\varepsilon^2 \Delta u_3^\varepsilon + \frac{\partial p_\varepsilon}{\partial x_3} = 0, \quad \nabla \cdot \boldsymbol{u}_\varepsilon &= 0 \qquad \text{in } \Omega, \\ \boldsymbol{u}_\varepsilon &= 0 \qquad \text{on } \Gamma. \end{aligned} \right. \tag{10}$$

[1]Throughout the report, we denote by the symbol ∂_{x_i} the operator $\dfrac{\partial}{\partial x_i}$, in the text sentences.

[2]A thin domain has really different horizontal and vertical scales.

Then, by establishing sufficient energy inequalities, we get an asymptotic problem when ε goes to zero, that is problem (3)-(6).

The reader interested in the physical justification of such procedure can refer to the thesis of P. Azérad [3] where he studied the Navier-Stokes equations in thin domains. The reader will find a complete description of the above process in a more general situation (see Chapter 2 of [3]). In a few lines, let us mention that P. Azérad considers the anisotropic time dependent Navier-Stokes equations instead of (9), and renormalizes it, as we have done it to get (10) from (9). As he explains, the renormalization has also physical sense, as well as passing to the limits when ε goes to 0.

As mentioned above, P. Azérad has been closely interested in the asymptotic analysis of Navier-Stokes equations in a thin domain. In his thesis [3], he refers to the works of O. Besson, M. R. Laydi and O. Touzani [7, 9], who initiate the mathematical study of the approximation of Stokes and Navier-Stokes equations. In [7], the authors derive from the 2D version of (10) a limit problem characterized by two variational formulations, and therefore, prove the existence and uniqueness of a weak solution to the 2D hydrostatic Stokes equations. The previous work is improved in [9], where the anisotropic stationary Navier-Stokes equations is considered in a thin domain as well as the hydrostatic approximation of these equations. After establishing some estimates of the viscosity and convective terms, appearing in the momentum equations, they prove the existence of a solution of the hydrostatic approximation as a limit of the anisotropic Navier-Stokes equations.

The work of P. Azérad clearly enriches the mathematical and numerical analysis of the previous ones. In addition to give a complete and detailed physical background, the reader will see, in Chapter 3 of [3], the well-posed-ness of the hydrostatique Stokes problem by the mean of an analysis which is not based upon a limit process. The reader will also find a discretization of this problem relying on a stable hydrostatic finite element method. In another chapter the author considers the case of time-dependent linearized Navier-Stokes equations with hydrostatic approximation. The adding of the time dependence, even for the linearized version, brings real difficulties that P. Azérad handles by using a compacity method, based on energy *a priori* estimates. To finish, a complementary work is done with F. Guillén González [4, 5], using a limit process, on the hydrostatic approximation of the time-dependent incompressible Navier-Stokes equations. They prove a convergence and existence theorem for this model by means of anisotropic estimates and a new time-compactness criterion.

The second approach consists in reducing system (3)-(6) to the Stokes-type system (13). Indeed, the simplifications of (3)-(6) arise from the hydrostatic pressure hypothesis:

$$\frac{\partial p}{\partial x_3} = 0 \quad \text{in } \Omega, \tag{11}$$

ensuring that p_S, the pressure at $x_3 = 0$, is in fact the real unknown. Moreover, by integrating with respect to x_3 the incompressibility equation:

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{12}$$

and taking into account the boundary conditions over u_3, it appears that the vertical velocity u_3 is given by the horizontal velocity $\boldsymbol{u}'$. In this case, the equations of (3)-(6) are reduced to the following system:

$$\left\{ \begin{aligned} -\Delta \boldsymbol{u}' + \nabla' p_S = \boldsymbol{f}' & \qquad \text{in } \Omega, \\ \nabla' \cdot \int_{-h(x')}^{0} \boldsymbol{u}'(x',\, x_3)\, dx_3 = 0 & \qquad \text{in } \omega, \\ \boldsymbol{u}' = 0 & \qquad \text{on } \Gamma. \end{aligned} \right. \tag{13}$$

Then, we get back to u_3 and the global pressure p by setting:

$$x = (x',\, x_3) \in \Omega, \quad u_3(x) = \int_{x_3}^{0} \nabla' \cdot \boldsymbol{u}'(x',\, \xi)\, d\xi, \quad p(x) = p_S(x'). \tag{14}$$

Problem (13) is a simplified version of the more realistic model of the Primitive Equations, denoted by PEs in the sequel.

The study of the PEs is of real interest as we propose to detail it in the following lines. In a general way, studying system (13) or the PEs yields real difficulties when the mapping h vanishes on a non negligible part of $\partial\omega$. This explains why most of the work dedicated to their study uses assumption (7).

To our knowledge, the mathematical study of the PEs was initiated by J.L. Lions, R. Temam and S. Wang in [43], where the authors address the existence proof for global weak solutions and local strong solutions, and of time analyticity of these strong solutions.

The existence proof of weak solutions, given in [43], is reviewed by R. Lewandowski in [42] with another approach, using the principle of truncated transports. In addition to give a complete modelization of the PEs, R. Lewandowski studies other realistic turbulence models and introduces, again in [42], a coupled model of the PEs with the geostrophico barotropic system. We can also find in [42] the study of an improved hydrostatic equation, which is also the subject of the paper [20].

In the work of R. Temam and M. Ziane [48], we can find a uniqueness proof of the weak solution to the PEs, as well as a regularity result. More recently C. Cao and Edriss S. Titi address in [17], the global existence and uniqueness of strong solutions to the 3D viscous PEs, for which local-in-time strong solutions, with H^1 data of arbitrary size, were obtained earlier by F. Guillén González, N. Masmoudi and M. Á. Rodríguez-Bellido in [30]. See also [33, 34, 40, 41] for complementary results on the study of strong solutions.

Other works are dedicated to the study of the 2D case, see for example [12, 13, 14, 31, 32, 36]. To finish, we also refer to the PhD Thesis of M. E. Petcu [44] who has improved the above results.

We would not be complete without mentioning that other methods have been investigated to study the hydrostatic Navier-Stokes equations. In [27], F. O. Gallego uses a suitable monotone regularization from which follows a new proof of the existence of weak solutions to the original problem. The proof relies on a version of De Rham's lemma, that F. O. Gallego establishes in [26]. Then, E. Grenier gives in [29] another approach, consisting in deriving the homogeneous hydrostatic equations starting from 2D Euler equations, following for instance [10]. To finish, Chemin et al. [21] have considered the three-dimensional Navier-Stokes equations with vanishing vertical viscosity. Assuming that the initial velocity is square-integrable in the horizontal direction and H^s in the vertical direction, they prove existence of solutions for $s > 1/2$ and uniqueness of solutions for $s > 3/2$. In addition to this work, D. Iftimié [38] has closed the gap between the existence and uniqueness, proving uniqueness of solutions for $s > 1/2$.

Among the extensive work done on the hydrostatic approximation of the Stokes or Navier-Stokes equations, the problem is open to consider non homogeneous boundary conditions on the vertical velocity u_3 or/and a compressible condition. This is precisely to what is dedicated the first part of this thesis. Given $\boldsymbol{f}' = (f_1,\, f_2) : \Omega \to \mathbb{R}^2$, $\Phi : \Omega \to \mathbb{R}$, and $\boldsymbol{g} = (\boldsymbol{g}',\, g_3) : \Gamma \to \mathbb{R}^3$, Φ and $\boldsymbol{g}$ satisfying adequate compatibility condition, see (4.21) or (4.11), we are interested in the study of the following system:

$$(SH) \left\{ \begin{aligned} -\Delta \boldsymbol{u}' + \nabla' p = \boldsymbol{f}', \quad \frac{\partial p}{\partial x_3} = 0, \quad \nabla \cdot \boldsymbol{u} = \Phi \quad & \text{in } \Omega, \\ \boldsymbol{u}' = \boldsymbol{g}',\, B u_3 = g_3 \quad & \text{on } \Gamma, \end{aligned} \right.$$

where Bu_3 defines one of the following Dirichlet boundary conditions:

$$u_3 = g_3 \text{ on } \Gamma, \quad \text{or} \quad u_3 n_3 = g_3 \text{ on } \Gamma. \tag{15}$$

Our principal preoccupation is to establish the existence and uniqueness of a weak solution to the generalized System (SH). Our work enables us to consider a domain Ω with no sidewalls, and hence a mapping h vanishing on a non negligible part of $\partial\omega$. Therefore, we will consider an ingenious array of weighted spaces, where the weights depend on h. Our study is based on results already existing, such as the existence and uniqueness of the weak solution to (3)-(6) and (13) in the Hilbertian case, and we allow us to revisit it along this work.

As we shall see it later, it is possible to reduce (SH) to the Stokes type system $(SH)_M$, defined below. More precisely, we will prove that the unknowns $\boldsymbol{u}'$ and p are solutions to the following system:

$$(SH)_M \begin{cases} -\Delta \boldsymbol{u}' + \nabla' p_S = \boldsymbol{f}' & \text{in } \Omega, \\ \nabla' \cdot \displaystyle\int_{-h(x')}^{0} \boldsymbol{u}'(x', x_3)\, dx_3 = \phi & \text{in } \omega, \\ \boldsymbol{u}' = \boldsymbol{g}' & \text{on } \Gamma. \end{cases}$$

Moreover, we will also prove that we can recover u_3 and p thanks to this two unknowns only, by establishing a relation close to (14). This is why, a part of this work is dedicated to the existence and uniqueness of a weak solution to $(SH)_M$, always in a domain Ω with no sidewalls. Here again, the use of weighted functional spaces is to take into consideration.

The first part of the thesis is divided into four chapters. In order to make the reading of our work easier, we propose in the sequel to detail each chapter with a short sum up.

Chapter 1. In a first place, we give some notations and we set the basic functional framework of the Sobolev space $H^1(\Omega)$. We also rewind the definition of the space $H^{1/2}_{00}(\Gamma_0)$ when $\Gamma_0 \subset \Gamma$.
Throughout the work, we need to compute integrals over Γ. For given geometry of Ω see (1), any integral defined on Γ_B or Γ_S can be replaced by one defined on ω. We gather few elements of computation here.
Finally, we recall some properties of spaces related to the divergence operator and the well-known lemma of De Rham, see Lemma 1.2.

Chapter 2. This chapter is dedicated to the study of weak solutions to System $(SH)_M$. Our work enables to consider a domain Ω with no sidewalls along $\partial\omega$, hence to assume that h is identically equal to 0 on a non negligible part of $\partial\omega$. Indeed, we will introduce a framework based on weighted Lebesgue spaces (see (2.3) and (2.27)), with weights depending on the mapping h.
Firstly, we review in Proposition 2.12, the existence and uniqueness of a weak solution in the homogeneous boundary conditions case (13), by following the approach introduced in [27]. The proof we give remains essentially on the existence of De Rham-like lemma, see Lemma 2.9; it is therefore necessary to study basic properties of the operator M introduced in (2.4) and recall some notations on distributions independent on x_3 that we can find in [27].
Secondly, we consider the new situation to lift the conditions:

$$\nabla' \cdot M\boldsymbol{u}' = \phi \text{ in } \omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma,$$

where $\phi \in L^2_{1/\sqrt{h}}(\omega)$ and $\boldsymbol{g}' \in H^{1/2}_{1/\sqrt{h}}(\Gamma)^2$, see (2.3) and (2.27), and satisfy the good compatibility condition (2.34). To lift the above conditions, see Lemma 2.16, we will follow [28] by establishing beforehand an isomorphism related to the operator $\nabla' \cdot M$, see (2.13). Then, since the homogeneous case is already solved in Proposition 2.12, we deduce from the above lifting an existence and uniqueness proof of a weak solution to System $(SH)_M$, see Theorem 2.17.

Chapter 3. In Chapter 3, we study the weak solutions to (SH) in the case where u_3 satisfies the following boundary condition:

$$u_3 n_3 = 0 \text{ on } \Gamma. \tag{16}$$

To begin with, we define the boundary condition (16) by introducing the anisotropic space $H(\partial_{x_3}, \Omega)$ and by reviewing useful properties related to this space [47].
Then, we review the existence and uniqueness of a weak solution to (3)-(6), by following the two different approaches given in introduction. In the first one, which leads to Theorem 3.4, we give a complete and justified asymptotic analysis of the anisotropic Stokes system (10). In the second one, we give the mathematical justification to reduce Problem (3)-(6) to (13), see Proposition 3.16, which is already solved in Proposition 2.12.
To finish, we will reduce the more general conditions:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma, \quad u_3 n_3 = 0 \text{ on } \Gamma,$$

to the following ones:

$$\nabla' \cdot M\boldsymbol{u}' = \phi \text{ in } \omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma,$$

see Lemma 3.18, in order to solve (SH) always in the case of the boundary condition (16).

Chapter 4. It is devoted to the existence and uniqueness of a weak solution to the general model (SH) in the case where u_3 satisfies one of the non homogeneous conditions (15). Firstly, we investigate the natural case of the boundary condition:

$$u_3 n_3 = g_3 \text{ on } \Gamma. \tag{17}$$

By considering u_3 in the space $H_F(\partial_{x_3}, \Omega)$, see (4.14), we prove in Proposition 4.4 that $(u_3 n_3)|_\Gamma$ belongs to $L^2(\Gamma)$. Therefore, by establishing new properties on the operator M and F, we reduce, in a suitable situation, the conditions:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma, \quad u_3 n_3 = g_3 \text{ on } \Gamma,$$

to the following ones:

$$\nabla' \cdot M\boldsymbol{u}' = \phi \text{ in } \omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma. \tag{18}$$

As a consequence, we prove that solving (SH) reduces to solve $(SH)_M$, from which existence and uniqueness of a weak solution is already known in Theorem 2.17. This constitutes an existence and uniqueness proof of a weak solution to (SH) in the case of the boundary condition (17).
Then, we will consider the unexpected situation of the boundary condition:

$$u_3 = g_3 \text{ on } \Gamma. \tag{19}$$

In the same way that we have handled the case of the boundary condition (17), we would like to reduce the conditions:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma,$$

to (18), for a different datum ϕ, and then use Theorem 2.17. In Proposition 4.14 and under assumption (7) only, we will see that if $u_3 \in H(\partial_{x_3}, \Omega)$ then $u_3|_\Gamma$ belongs to the weighted space $L^2(\Gamma, |n_3|\, d\sigma)$ defined in (4.24). Then, we will make the above reduction in Lemma 4.18, by proving additional results such as Proposition 4.16 and Proposition 4.17, establishing one more time new properties for the operators M and F. We therefore give an existence and uniqueness proof of a weak solution to (SH) in the case of the boundary condition (19), and in the situation where Ω has sidewalls along $\partial\omega$.

Chapter 1

Basic functional framework

This chapter gathers notations, notions and results often used in this thesis. At first place, we recall basic properties related to the Sobolev space $H^1(\Omega)$ and its space of traces $H^{1/2}(\Gamma)$. We will also review the definition of the partial traces space $H^{1/2}_{00}(\Gamma_0)$ for any $\Gamma_0 \subset \Gamma$ with positive measure. Then, we will give some requirements to compute surface integrals on Γ, taking into account the geometry of Ω (2) and the properties of the unit outward normal vector $\boldsymbol{n}$. We will establish therefore the important relations (1.5) and (1.6). To finish, we will recall an essential result in the Stokes problem theory, see Lemma 1.2, coming with the isomorphisms given in (1.8) and a lift operator in Proposition 1.3.

1.1 Notations and spaces

We define $\mathcal{D}(\Omega)$ to be the linear space of infinitely differentiable functions, with compact support in Ω, and $\mathcal{D}'(\Omega)$ the space of distributions. Then we set:

$$\mathcal{D}(\overline{\Omega}) = \left\{ \varphi\,|_{\Omega} \ / \, \varphi \in \mathcal{D}(\mathbb{R}^3) \right\}.$$

Let us recall that $L^2(\Omega)$ denotes the space of the (almost everywhere classes of) measurable functions u such that $\int_\Omega u^2 \ dx < \infty$, which is a Hilbert space for the usual norm:

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega u^2 \ dx \right)^{1/2}$$

We also mention the space $L^2_0(\Omega) = \left\{ u \in L^2(\Omega) \, / \, \int_\Omega u \, dx = 0 \right\}$, which is a closed subspace of $L^2(\Omega)$, and isomorphic to the quotient space $L^2(\Omega)/\mathbb{R}$.

We denote by $H^1(\Omega)$ the Sobolev space $\left\{ u \in L^2(\Omega) \, / \, \nabla u \in L^2(\Omega)^3 \right\}$ which is a Hilbert space for the norm:

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)^3} \right)^{1/2}.$$

Since Ω has a Lipschitz-continuous boundary Γ, we can define a trace $u|_\Gamma$ for any function $u \in H^1(\Omega)$. To this purpose, we denote by $d\sigma$ the surface measure induced by the measure of Lebesgue dx, and we denote by $L^2(\Gamma)$ the space of $d\sigma$-measurable functions $\mu : \Gamma \to \mathbb{R}$ square integrable for the surface measure $d\sigma$, endowed with the norm:

$$\|\mu\|_{L^2(\Gamma)} = (\int_\Gamma \mu^2 \, d\sigma)^{1/2}.$$

Therefore, and since $\mathcal{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$, the mapping

$$u \mapsto u|_\Gamma,$$

defined on $\mathcal{D}(\overline{\Omega})$, can be extended in a unique way to a linear and continuous mapping, denoted in the same way, from $H^1(\Omega)$ into $L^2(\Gamma)$. Moreover, if u and v belong to $H^1(\Omega)$, it satisfy the following Green's formula:

$$\int_\Omega u\,\frac{\partial v}{\partial x_i}\,dx = -\int_\Omega v\,\frac{\partial u}{\partial x_i}\,dx + \int_\Gamma uvn_i\,d\sigma, \quad 1 \leqslant i \leqslant 3.$$

For convenience, we denote by $H^{1/2}(\Gamma)$ the space of traces of $H^1(\Omega)$ functions:

$$H^{1/2}(\Gamma) = \left\{\mu \in L^2(\Gamma) \,/\, \exists u \in H^1(\Omega) \text{ such that } u = \mu \text{ on } \Gamma\right\}.$$

Let us recall that $H^{1/2}(\Gamma)$ is a Hilbert space for the norm:

$$\|\mu\|_{H^{1/2}(\Gamma)} = \inf\left\{\|u\|_{H^1(\Omega)} \,/\, u \in H^1(\Omega),\, u = \mu \text{ on } \Gamma\right\},$$

and that it is continuously embedded in $L^2(\Gamma)$: there is a constant[1] $C > 0$ depending only on Ω such that:

$$\forall \mu \in H^{1/2}(\Gamma), \qquad \|\mu\|_{L^2(\Gamma)} \leq C\,\|\mu\|_{H^{1/2}(\Gamma)}.$$

Then, we consider the space

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H_0^1(\Omega)} = \left\{u \in H^1(\Omega) \,/\, u = 0 \text{ on } \Gamma\right\},$$

and we also introduce the space of distributions $H^{-1}(\Omega)$, as the dual space of $H_0^1(\Omega)$.

To finish, let us recall some definitions and properties of special spaces of traces. We refer to [22] for a detailed study of these spaces. For an open set $\Gamma_0 \subset \Gamma$ with a positive measure, we define the space:

$$H_{\Gamma_0}^1(\Omega) = \left\{u \in H^1(\Omega) \,/\, u = 0 \text{ on } \Gamma\backslash\Gamma_0\right\},$$

which is a Hilbert space endowed with the standard $H^1(\Omega)$ norm. Then, we denote by $H_{00}^{1/2}(\Gamma_0)$ the traces space of $H_{\Gamma_0}^1(\Omega)$ functions:

$$H_{00}^{1/2}(\Gamma_0) = \left\{g \in L^2(\Gamma_0) \,/\, \exists u \in H_{\Gamma_0}^1(\Omega) \text{ such that } u = g \text{ on } \Gamma_0\right\},$$

which is a Hilbert space for the norm:

$$\|g\|_{H_{00}^{1/2}(\Gamma_0)} = \inf\left\{\|u\|_{H^1(\Omega)} \,/\, u \in H_{\Gamma_0}^1(\Omega) \text{ and } u = g \text{ on } \Gamma_0\right\}.$$

Finally, we introduce $H^{-1/2}(\Gamma_0)$ as the dual space of $H_{00}^{1/2}(\Gamma_0)$. Let us also recall that $H_{00}^{1/2}(\Gamma_0)$ is dense in $L^2(\Gamma_0)$, so that $L^2(\Gamma_0)$ can be identified to a subspace of $H^{-1/2}(\Gamma_0)$.

[1] We will often use the character C to denote any positive constant depending only on Ω.

1.2 Surface computations

In this paragraph, we give some requirements to compute surface integrals such as:

$$\int_\Gamma \mu n_i \, d\sigma, \quad 1 \leqslant i \leqslant 3,$$

where the function μ belongs to the space $L^1(\Gamma)$ ($L^1(\Gamma)$ being defined similarly to $L^2(\Gamma)$). We introduce beforehand the following important notations. For any $d\sigma$-measurable function $\mu : \Gamma \to \mathbb{R}$, we define the functions μ_S and μ_B by setting:

$$x' \in \omega, \quad \mu_S(x') = \mu(x', 0), \quad \mu_B(x') = \mu(x', -h(x')). \tag{1.1}$$

Firstly, observe the following proposition, which enables to replace any integral defined on Γ_S or Γ_B by one defined on ω.

Proposition 1.1. *Let $p = 1$ or 2. The mapping $\mu \mapsto (\mu_S, \mu_B)$ is linear and continuous from $L^p(\Gamma)$ into $L^p(\omega)^2$. Moreover, one has:*

$$\int_{\Gamma_S} \mu \, d\sigma = \int_\omega \mu_S \, dx' \quad \textit{and} \quad \int_{\Gamma_B} \mu \, d\sigma = \int_\omega \mu_B (1 + |\nabla h|^2)^{1/2} dx'. \tag{1.2}$$

Proof. Let $p = 1$ or 2. Notice that the integrals in (1.2) are well defined since ω is bounded. Then, Proposition 1.1 follows from a change of variables, given the geometry of Ω, see (2). □

Secondly, and given the geometry of Ω, see (2), note that n_i satisfies for $i = 1, 2$:

$$n_i = 0 \text{ on } \Gamma_S, \quad n_i = 1 \text{ on } \Gamma_L, \quad (n_i)_B (1 + |\nabla h|^2)^{1/2} = -\frac{\partial h}{\partial x_i} \text{ in } \omega,$$

and also that the third component fulfills:

$$n_3 = 1 \text{ on } \Gamma_S, \quad n_3 = 0 \text{ on } \Gamma_L, \quad (n_3)_B (1 + |\nabla h|^2)^{1/2} = -1 \text{ in } \omega.$$

Therefore, if we combine these properties to Proposition 1.1, we obtain immediately, for $\mu \in L^1(\Gamma)$, the following relations:

$$\int_{\Gamma_B} \mu n_i \, d\sigma = -\int_\omega \mu_B \frac{\partial h}{\partial x_i} \, dx', \quad i = 1, 2, \tag{1.3}$$

$$\int_{\Gamma_S} \mu n_3 \, d\sigma = \int_\omega \mu_S \, dx', \quad \int_{\Gamma_B} \mu n_3 \, d\sigma = -\int_\omega \mu_B \, dx'. \tag{1.4}$$

and finally, we have proved that for $p = 1$ or 2:

$$\forall \mu \in L^p(\Gamma), \quad \int_\Gamma \mu n_i \, d\sigma = \int_{\Gamma_L} \mu \, d\sigma - \int_\omega \mu_B \frac{\partial h}{\partial x_i} \, dx', \quad i = 1, 2, \tag{1.5}$$

$$\forall \mu \in L^p(\Gamma), \quad \int_\Gamma \mu n_3 \, d\sigma = \int_\omega \mu_S \, dx' - \int_\omega \mu_B \, dx'. \tag{1.6}$$

1.3 Properties of spaces related to the divergence operator

For any vector field $\boldsymbol{v} = (v_1,\, v_2,\, v_3)$, we define the divergence operator by:

$$\operatorname{div} \boldsymbol{v} = \nabla \cdot \boldsymbol{v} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i}.$$

In this work, we will often use the lemma of De Rham, see Lemma 1.2, which is an important tool in the Stokes problem theory. This lemma is an adaptation of a powerful and difficult theorem, originally proved by G. De Rham in [23], and which says essentially:

If a distribution field $\boldsymbol{T} \in \mathcal{D}'(\Omega)^3$ satisfies:

$$\forall \boldsymbol{\varphi} \in \mathcal{D}(\Omega)^3 \text{ with } \nabla \cdot \boldsymbol{\varphi} = 0, \quad \langle \boldsymbol{T},\, \boldsymbol{\varphi} \rangle_{\mathcal{D}'(\Omega)^3,\, \mathcal{D}(\Omega)^3} = 0,$$

then $\boldsymbol{T} = \nabla p$ for some distribution $p \in \mathcal{D}'(\Omega)$.

The adaptation we give now is proved in [28], see Lemma 2.1 page 22 of [28], with a different approach inspired by L. Tatar in [46]. Beforehand, let us introduce the space:

$$\boldsymbol{V} = \left\{ \boldsymbol{v} \in H_0^1(\Omega)^3 \,/\, \nabla \cdot \boldsymbol{v} = 0 \text{ in } \Omega \right\}. \tag{1.7}$$

Lemma 1.2. *If $\boldsymbol{f} \in H^{-1}(\Omega)^3$ satisfies:*

$$\forall \boldsymbol{v} \in \boldsymbol{V}, \quad \langle \boldsymbol{f},\, \boldsymbol{v} \rangle_{H^{-1}(\Omega)^3, H_0^1(\Omega)^3} = 0,$$

then, there is $p \in L^2(\Omega)$ such that $\nabla p = \boldsymbol{f}$ in Ω. Since Ω is connected, p is unique up to an additive constant.

Observing that for any $\boldsymbol{v} \in H_0^1(\Omega)^3$ one has:

$$\int_\Omega \nabla \cdot \boldsymbol{v} \, dx = 0,$$

the range space of the divergence operator div is contained in $L_0^2(\Omega)$. This observation and Lemma 1.2 yield two isomorphisms:

$$\nabla : L^2(\Omega)/\mathbb{R} \longrightarrow \boldsymbol{V}^\circ \quad \text{and} \quad \operatorname{div} : \boldsymbol{V}^\perp \longrightarrow L_0^2(\Omega), \tag{1.8}$$

established in Corollary 2.4 page 24 in [28]. Here, $\boldsymbol{V}^\perp$ denotes the orthogonal of $\boldsymbol{V}$ for the scalar product:

$$\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, dx = \sum_{i=1}^{3} \int_\Omega \nabla u_i \cdot \nabla v_i \, dx,$$

and $\boldsymbol{V}^\circ$ denotes the polar space of $\boldsymbol{V}$:

$$\boldsymbol{V}^\circ = \Big\{ \boldsymbol{f} \in H^{-1}(\Omega)^3 \,/\, \forall \boldsymbol{v} \in \boldsymbol{V},\, \langle \boldsymbol{f},\, \boldsymbol{v} \rangle_{H^{-1}(\Omega)^3,\, H_0^1(\Omega)^3} = 0 \Big\}. \tag{1.9}$$

We finish this paragraph by giving a lift operator of boundary values, which is a consequence of the second isomorphism in (1.8). We repeat here the statement of Lemma 2.2 page 24 of [28].

Proposition 1.3. *Let $\Phi \in L^2(\Omega)$ and $\boldsymbol{g} \in H^{1/2}(\Gamma)^3$ satisfying the compatibility condition*

$$\int_\Omega \Phi \, dx = \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n} \, d\sigma. \tag{1.10}$$

Then, there is $\boldsymbol{u} \in H^1(\Omega)^3$ such that:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma, \tag{1.11}$$

and the following estimate holds:

$$\|\boldsymbol{u}\|_{H^1(\Omega)^3} \leqslant C \left(\|\Phi\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{H^{1/2}(\Gamma)^3} \right).$$

Chapter 2

Weak solutions to System $(SH)_M$

2.1 Introduction

Let $\boldsymbol{f}' : \Omega \to \mathbb{R}^2$ and consider the following problem to find $\boldsymbol{u}' : \Omega \to \mathbb{R}^2$ and $p_S : \omega \to \mathbb{R}$ such that:

$$\left\{ \begin{aligned} -\Delta \boldsymbol{u}' + \nabla' \widetilde{p_S} &= \boldsymbol{f}' && \text{in } \Omega, \\ \nabla' \cdot M\boldsymbol{u}' &= 0 && \text{in } \omega, \\ \boldsymbol{u}' &= 0 && \text{on } \Gamma, \end{aligned} \right. \tag{2.1}$$

and where the operator M is defined after in (2.4). Initial studies on Problem (2.1) prove that existence and uniqueness of a weak solution relies essentially on the existence of an inf-sup condition, see Remark 2.14. More recently, a new approach is given by F. O. Gallego in [26], where the authors establishes a version of De Rham's Lemma (see Lemma 1.2), adapted to this Problem or to a more general one, issued from the PEs, that we pose in (2.14).

In this chapter, we revisit the existence and uniqueness of a weak solution to (2.1), by following this new approach. Then, we go further by considering a new situation. More precisely, we establish an existence and uniqueness proof of a weak solution to the following problem:

$$\left\{ \begin{aligned} -\Delta \boldsymbol{u}' + \nabla' \widetilde{p_S} &= \boldsymbol{f}' && \text{in } \Omega, \\ \nabla' \cdot M\boldsymbol{u}' &= \phi && \text{in } \omega, \\ \boldsymbol{u}' &= \boldsymbol{g}' && \text{on } \Gamma, \end{aligned} \right. \tag{2.2}$$

where $\phi : \omega \to \mathbb{R}$ and $\boldsymbol{g}' : \Gamma \to \mathbb{R}^2$ are suitable given data. Recall that we have denoted this problem by $(SH)_M$ in the introduction of Part I.

An outline of Chapter 2 is as follows. In Section 2.2, we will give a complete and easier proof of a version of De Rham's Lemma, adapted from [26], and stated in Lemma 2.9. To this purpose, we will use an appropriate functional framework, adapted to a general domain Ω for which (7) does not hold. Firstly, and after having introduced some suitable weighted spaces (see (2.3)), we will present in (2.4) the operator M and will study in Proposition 2.2 some continuity properties of this operator. Then, we will need to recall some results already proved in [26] concerning distributions independent on the x_3-variable (see Definition 2.4), such as a representation result in Proposition 2.5; this result explains that we can associate to any distribution T independent on the x_3-variable , a distribution S on the surface ω. As we will see, in Proposition 2.7 or in Proposition 2.8, the regularity of the distribution S depends on the one of T. The result in Proposition 2.8 concerns a new situation, not investigated in [26], from which will follow the expected version of De Rham's Lemma stated in Lemma 2.9.

We will complete Section 2.2, by establishing new isomorphisms related to the operators ∇' and $\nabla' \cdot M$, see Lemma 2.9, in order to solve Problem (2.2), for which to our knowledge we initiate the study.

In Section 2.3, we will apply Lemma 2.9 to give an existence and uniqueness proof of weak solution to (2.1), in Proposition 2.12. Then, we will see how the second isomorphism of (2.13) and the particular choice of a weighted space for the datum $\boldsymbol{g}'$ (see (2.27)), will enable us to lift the following conditions:

$$\nabla' \cdot M\boldsymbol{u}' = \phi \text{ in } \omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Omega,$$

and therefore to solve the new problem (2.2).

2.2 Mathematical foundation

Let h be a positive Lipschitz-continuous mapping on ω. In the most part of the work, we do not need assumption (7), which enables us to consider a mapping h that vanishes on a non negligible part of $\partial\omega$. In this case, we offset the degeneracy of the domain Ω by working with weighted spaces. Let us start this section by introducing for $\alpha \in \{-1/2,\, 1/2\}$, the following weighted Lebesgue's space:

$$L^2_{h^\alpha}(\omega) = \left\{p : \omega \to \mathbb{R} \,/\, h^\alpha p \in L^2(\omega)\right\}, \tag{2.3}$$

which is a Hilbert space for the norm

$$\|p\|_{L^2_{h^\alpha}(\omega)} = \|h^\alpha p\|_{L^2(\omega)}.$$

Among the properties that possesses these spaces, note that the following continuous embeddings hold:

$$L^2_{1/\sqrt{h}}(\omega) \hookrightarrow L^2(\omega) \hookrightarrow L^2_{\sqrt{h}}(\omega),$$

and that the space $\mathcal{D}(\omega)$ is dense in $L^2_{h^\alpha}(\omega)$. As a consequence, the dual space of $L^2_{h^\alpha}(\omega)$ can be identified to $L^2_{h^{-\alpha}}(\omega)$.

Remark 2.1. Any measurable function p defined on ω can be defined in Ω by setting $p(x) = p(x')$, for $x \in \Omega$. In order to avoid confusions, we shall identify a function p defined on ω with one defined on Ω by setting

$$x \in \Omega, \quad \tilde{p}(x) = p(x').$$

Therefore, note that p belongs to $L^2_{\sqrt{h}}(\omega)$ if and only if $\widetilde{p}$ belongs to $L^2(\Omega)$ and also that

$$\|p\|_{L^2_{\sqrt{h}}(\omega)} = \|\tilde{p}\|_{L^2(\Omega)}.$$

Then, let u be a function defined in Ω. In order to study System (2.15) or (2.21), it is necessary to introduce and study the following operator:

$$x' \in \omega, \quad Mu(x') = \int_{-h(x')}^{0} u(x',\, x_3)\, dx_3. \tag{2.4}$$

Let us briefly recall that if $\varphi \in \mathcal{D}(\overline{\Omega})$, then $M\varphi \in C^1(\omega)$, since one has for $i = 1,\, 2$:

$$\frac{\partial}{\partial x_i}(M\varphi) = M(\frac{\partial \varphi}{\partial x_i}) + \frac{\partial h}{\partial x_i}(\varphi|_\Gamma)_B \text{ in } \omega. \tag{2.5}$$

Then, if $\varphi \in \mathcal{D}(\Omega)$, the following relation holds:

$$\frac{\partial}{\partial x_i}(M\varphi) = M(\frac{\partial \varphi}{\partial x_i}) \text{ in } \omega. \tag{2.6}$$

The following proposition gives the first properties of the operator M. In particular, we extend relation (2.5) and (2.6) to Lebesgue and Sobolev functions.

Proposition 2.2. *Let h be a positive Lipschitz-continuous mapping on ω. The operator M is linear and continuous from $L^2(\Omega)$ into $L^2_{1/\sqrt{h}}(\omega)$, from $H^1(\Omega)$ into $H^1(\omega) \cap L^2_{1/\sqrt{h}}(\omega)$, and one has for $i = 1,\, 2$:*

$$\forall u \in H^1(\Omega), \quad \frac{\partial}{\partial x_i}(Mu) = M(\frac{\partial u}{\partial x_i}) + \frac{\partial h}{\partial x_i}(u|_\Gamma)_B \ \text{ in } \omega; \tag{2.7}$$

$$\forall u \in H^1_0(\Omega), \quad \frac{\partial}{\partial x_i}(Mu) = M(\frac{\partial u}{\partial x_i}) \ \text{ in } \omega. \tag{2.8}$$

Proof. Let $u \in L^2(\Omega)$. The inequality of Cauchy-Schwarz gives for almost all x' in ω

$$\frac{|Mu(x')|^2}{h(x')} \leqslant \int_{-h(x')}^0 |u(x',\, x_3)|^2 \, dx_3.$$

As a consequence $Mu \in L^2_{1/\sqrt{h}}(\omega)$ and one has for any $u \in L^2(\omega)$

$$\|Mu\|_{L^2_{1/\sqrt{h}}(\omega)} \leqslant \|u\|_{L^2(\omega)}\,. \tag{2.9}$$

Next, let u in $H^1(\Omega)$ and $\psi \in \mathcal{D}(\omega)$. One has for $i = 1,\, 2$:

$$\begin{aligned}\int_\omega Mu\, \frac{\partial \psi}{\partial x_i}\, dx' = \int_\Omega u\, \widetilde{\frac{\partial \psi}{\partial x_i}}\, dx &= \int_\Omega u\, \frac{\partial \widetilde{\psi}}{\partial x_i}\, dx \\ &= -\int_\Omega \frac{\partial u}{\partial x_i}\, \widetilde{\psi}\, dx + \int_\Gamma u n_i \widetilde{\psi}\, d\sigma.\end{aligned}$$

Then, let us compute $\int_\Gamma u n_i \widetilde{\psi}\, d\sigma$. Since $\widetilde{\psi}$ does not depend on x_3, one has $\widetilde{\psi} = 0$ on Γ_L, from which follows by relation (1.5) that:

$$\int_\Gamma u n_i \widetilde{\psi}\, d\sigma = -\int_\omega (u|_\Gamma)_B \frac{\partial h}{\partial x_i}\, \psi\, dx'.$$

Consequently, one has proved that for any $\psi \in \mathcal{D}(\omega)$,

$$\int_\omega Mu\, \frac{\partial \psi}{\partial x_i}\, dx' = -\int_\omega \left[M(\frac{\partial u}{\partial x_i}) + (u|_\Gamma)_B \frac{\partial h}{\partial x_i} \right] \psi\, dx'.$$

Since $\partial_{x_i} u \in L^2(\Omega)$, one deduces that $M(\partial_{x_i} u) \in L^2(\omega)$, and since $u|_\Gamma \in H^{1/2}(\Gamma)$, Proposition 1.1 ensures that

$$(u|_\Gamma)_B \in L^2(\omega).$$

Thus (2.7) holds in $\mathcal{D}'(\omega)$ and in $L^2(\omega)$, since h is Lipschitz-continuous. The same arguments give the continuity of M from $H^1(\Omega)$ into $H^1(\omega)$.
When u belongs to $H^1_0(\Omega)$, the function $(u|_\Gamma)_B$ vanishes on ω. Therefore, we deduce (2.8) from relation (2.7). $\square$

Remark 2.3. The assumption $u \in H^1(\Omega)$ is not sufficient to get

$$(u|_\Gamma)_B \frac{\partial h}{\partial x_i} \in L^2_{1/\sqrt{h}}(\omega),$$

and to conclude that $\partial_{x_i}(Mu)$ belongs to $L^2_{1/\sqrt{h}}(\omega)$ *in (2.7).*

Next, let $\boldsymbol{v'} = (v_1, v_2)$ be a two-vector field defined on Ω. From the operator M, we define the divergence mean operator by setting:

$$\nabla' \cdot M\boldsymbol{v'} = \sum_{i=1,2} \frac{\partial}{\partial x_i}(Mv_i).$$

We begin this paragraph by establishing a De Rham-like lemma, which is an adaptation of Lemma 1.2. Here, we use the representation theorem of distributions independent on x_3, proved by F. O. Gallego in [26]. This result is at the heart of the resolution of problems related to the constraint $\nabla' \cdot M\boldsymbol{u'} = 0$, as in System (2.15).

We start off with giving the definition of distributions independent on x_3 and the representation theorem (see [26] Theorem 1 page 337).

Definition 2.4. Let $T \in \mathcal{D}'(\Omega)$. We say that the distribution T is independent on x_3 when

$$\frac{\partial T}{\partial x_3} = 0 \text{ in } \Omega.$$

Proposition 2.5. (Representation theorem)*Let h be a positive Lipschitz-continuous mapping on ω. Let $T \in \mathcal{D}'(\Omega)$. Then, the following assertions are equivalent.*
1. The distribution T does not depend on x_3.
2. There is a unique distribution $S \in \mathcal{D}'(\omega)$ such that

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \langle T, \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle S, M\varphi \rangle_{\mathcal{D}'(\omega),\mathcal{D}(\omega)} . \tag{2.10}$$

Remark 2.6. For any $\varphi \in \mathcal{D}(\Omega)$, one has $M\varphi \in \mathcal{D}(\omega)$. Indeed, for the first differentiation order, the compact support assumption kills the degenerated term

$$\frac{\partial h}{\partial x_i}(\varphi|_\Gamma)_B, .$$

As a consequence, $\partial_{x_i}(M\varphi) \in C^1(\omega)$ and has a compact support in ω. By a recursive argument on the order of differentiation, one deduces that $\varphi \in \mathcal{D}(\omega)$, with for any multi-index $\alpha \in \mathbb{N}^3$:

$$D^\alpha(M\varphi) = M(D^\alpha \varphi) \text{ in } \omega.$$

Besides, the regularity of the distribution S depends on the one of T (see [26] Lemma 1 page 339).

Proposition 2.7. *We keep the notations of* Proposition 2.5. *Let $T \in \mathcal{D}'(\Omega)$ and $S \in \mathcal{D}'(\omega)$ satisfying* (2.10). *If $T \in H^{-1}(\Omega)$, then*

$$S \in H^{-1}(\overset{\circ}{K}), \quad \textit{for any compact set } K \subset \omega.$$

If moreover, ess $\inf_\omega h > 0$, *then $S \in H^{-1}(\omega)$.*

Let us give, in the following proposition, the regularity of the distribution S when T belongs to $L^2(\Omega)$, a case that is not investigated in [26].

Proposition 2.8. *We keep the notations of* Proposition 2.5. *If $T \in L^2(\Omega)$ then $S \in L^2_{\sqrt{h}}(\omega)$, and one has*

$$T = \widetilde{S} \text{ in } \Omega.$$

Proof. Firstly, let us prove that S belongs to $L^2_{\mathrm{loc}}(\omega)$. Let us consider a compact set $K \subset \omega$, and set $\delta_K = \inf_{x' \in K} h(x')$.Then, let $a \in \mathcal{D}(\mathbb{R})$ such that

$$\operatorname{supp} a \subset (-\delta_K,\, 0), \quad \text{and} \quad \int_{-\delta_K}^{0} a(x_3)\, dx_3 = 1.$$

Therefore, one deduces from relation (2.10), that for any $\psi \in \mathcal{D}(\omega)$ with $\operatorname{supp} \psi \subset K$:

$$\begin{aligned} \left| \langle S,\, \psi \rangle_{\mathcal{D}'(\omega),\mathcal{D}(\omega)} \right| = \left| \langle S,\, M(a \otimes \psi) \rangle_{\mathcal{D}'(\omega),\mathcal{D}(\omega)} \right| &= \left| \int_\Omega T\, a \otimes \psi\, dx \right| \\ &\leqslant \|T\|_{L^2(\Omega)} \|a \otimes \psi\|_{L^2(\Omega)} \\ &\leqslant \|T\|_{L^2(\Omega)} \|a\|_{L^\infty(\mathbb{R})} \|\psi\|_{L^2(K)} . \end{aligned}$$

We deduce that $S \in L^2(K)$, hence the distribution S belongs to $L^2_{\mathrm{loc}}(\omega)$. Secondly, relation (2.10) also gives for any $\varphi \in \mathcal{D}(\Omega)$

$$\int_\Omega T\varphi\, dx = \int_\omega S\, M\varphi\, dx' = \int_\Omega \widetilde{S}\varphi\, dx,$$

from which we get $T = \widetilde{S}$ in $L^2(\Omega)$. To finish, we use Remark 2.1 and S belongs to $L^2_{\sqrt{h}}(\omega)$. □

From now on, let us introduce the space

$$\boldsymbol{V}_M = \left\{ \boldsymbol{v}' \in H^1_0(\Omega)^2 \,/\, \nabla' \cdot M\boldsymbol{v}' = 0 \text{ in } \omega \right\}.$$

Thanks to the previous results, we are ready to state the expected De Rham-like lemma. Note that a more general version is proved in [26] page 342 Lemma 2. However, we can prove our result in an easier way.

Lemma 2.9. *Let h be a positive Lipschitz-continuous mapping on ω. If $\boldsymbol{f}' \in H^{-1}(\Omega)^2$ satisfies*

$$\forall \boldsymbol{v}' \in \boldsymbol{V}_M, \quad \left\langle \boldsymbol{f}',\, \boldsymbol{v}' \right\rangle_{H^{-1}(\Omega)^2, H^1_0(\Omega)^2} = 0,$$

then, there is $q \in L^2_{\sqrt{h}}(\omega)$, unique up to an additive constant, such that $\nabla'\widetilde{q} = \boldsymbol{f}'$ in Ω.

Proof. Let us set $\boldsymbol{f} = (\boldsymbol{f}',\, 0)$. Thanks to (2.8) and (3.21), any $\boldsymbol{v} \in \boldsymbol{V}$ is such that $\boldsymbol{v}' \in \boldsymbol{V}_M$. As a consequence, the distribution $\boldsymbol{f}$ is exactly in the statement of Lemma 1.2. Therefore, there is a unique function p in $L^2(\Omega)/\mathbb{R}$ such that

$$\nabla' p = \boldsymbol{f}' \text{ and } \frac{\partial p}{\partial x_3} = 0 \text{ in } \Omega.$$

Then, since $\partial_{x_3} p = 0$ in Ω, we deduce from Proposition 2.5 and Proposition 2.8, that there is $q \in L^2_{\sqrt{h}}(\omega)$, unique up to an additive constant in this case, such that $p = \widetilde{q}$. Hence q satisfies $\nabla'\widetilde{q} = \boldsymbol{f}'$ in Ω. □

Lemma 2.9 yields the following isomorphism

$$\nabla' : L^2_{\sqrt{h}}(\omega)/\mathbb{R} \longrightarrow \boldsymbol{V}^\circ_M, \tag{2.11}$$

where $\boldsymbol{V}^\circ_M$ is defined as in relation (1.9). To finish the paragraph, we want to derive the dual isomorphism to (2.11). Firstly, note that

$$\nabla' \cdot M : (\boldsymbol{V}^\circ_M)' \longrightarrow (L^2_{\sqrt{h}}(\omega)/\mathbb{R})',$$

is the dual operator to (2.11). Next, one proves that the space $(\boldsymbol{V}^\circ_M)'$ can be identified to $\boldsymbol{V}^\perp_M \subset H^1_0(\Omega)^2$, by using similar arguments as in the proof of Corollary 2.4 in [28]. To finish, let us characterize the range space of the operator $\nabla' \cdot M$. One the one hand, observe that Green's formula and (2.8) yield:

$$\forall \boldsymbol{v}' \in H^1_0(\Omega)^2, \quad \int_\omega \nabla' \cdot M\boldsymbol{v}' \, dx' = 0.$$

Thus the range space of $\nabla' \cdot M$ is contained in a proper, closed subspace of $L^2_{1/\sqrt{h}}(\omega)$, that is

$$L^2_{1/\sqrt{h},\,0}(\omega) := L^2_{1/\sqrt{h}}(\omega) \cap L^2_0(\omega). \tag{2.12}$$

On the other hand, note that $(L^2_{\sqrt{h}}(\omega)/\mathbb{R})'$ can be identified with $L^2_{1/\sqrt{h}}(\omega) \perp \mathbb{R}$, the orthogonality taken in the following sense

$$\forall p \in L^2_{1/\sqrt{h}}(\omega), \quad \langle p,\, 1\rangle_{L^2_{1/\sqrt{h}}(\omega),\, L^2_{\sqrt{h}}(\omega)} = \int_\omega p \, dx' = 0,$$

and it follows that $L^2_{1/\sqrt{h}}(\omega) \perp \mathbb{R}$ can be identified to $L^2_{1/\sqrt{h},\,0}(\omega)$. Finally, one has proved the following lemma.

Lemma 2.10. *Let h be a positive Lipschitz-continuous mapping on ω. The following operators are isomorphisms:*

$$\nabla' : L^2_{\sqrt{h}}(\omega)/\mathbb{R} \longrightarrow \boldsymbol{V}^\circ_M \quad \text{and} \quad \nabla' \cdot M : \boldsymbol{V}^\perp_M \longrightarrow L^2_{1/\sqrt{h},\,0}(\omega). \tag{2.13}$$

Remark 2.11. Lemma 2.9 gives a characterization of the space $\boldsymbol{V}^\perp_M$. Indeed, one proves by adapting the proof of Corollary 2.3 page 23 in [28] that $\boldsymbol{u}' \in \boldsymbol{V}^\perp_M$ if and only if there is $q \in L^2_{\sqrt{h}}(\omega)$ such that

$$-\Delta \boldsymbol{u}' = \nabla \widetilde{q} \text{ in } \Omega.$$

By now, we are ready to study the weak solutions to System (2.21).

2.3 Existence and uniqueness of a weak solution

Thanks to the version of De Rham's Lemma established in [26], adapted to our problem in Lemma 2.9, F. O. Gallego gives an existence proof of weak solutions to PEs:

$$\begin{cases} -A\boldsymbol{u}' + \nabla' \widetilde{p_S} = \boldsymbol{f}' & \text{in } \Omega, \\ \nabla' \cdot M\boldsymbol{u}' = 0 & \text{in } \omega, \\ \boldsymbol{u}' = 0 \text{ on } \Gamma_B \cup \Gamma_L, \quad -\dfrac{\partial \boldsymbol{u}'}{\partial x_3} = \boldsymbol{\tau}' & \text{on } \Gamma_S, \end{cases} \tag{2.14}$$

where A is an adapted operator, and where the assumption (7) is not considered. Indeed, he proves that Problem (2.14) reduces to solve a variational formulation, for which only the existence of a solution can be obtained.

We would like to adapt this approach to our homogeneous problem:

$$\begin{cases} -\Delta \boldsymbol{u}' + \nabla' \widetilde{p_S} = \boldsymbol{f}' & \text{in } \Omega, \\ \nabla' \cdot M\boldsymbol{u}' = 0 & \text{in } \omega, \\ \boldsymbol{u}' = 0 & \text{on } \Gamma, \end{cases} \tag{2.15}$$

and therefore give a detailed existence and uniqueness proof of a weak solution to (2.15). This first result is important in a view to deal with another generalized problem, that we pose in (2.21), and for which to our knowledge, we initiate the study.

Proposition 2.12. *Let $\boldsymbol{f}'$ in $H^{-1}(\Omega)^2$. Then, there is a unique solution $(\boldsymbol{u}', p_S)$ in the space $H_0^1(\Omega)^2 \times (L^2_{\sqrt{h}}(\omega)/\mathbb{R})$ to Problem (2.15). Moreover, there is a constant $C > 0$ such that*

$$\|\boldsymbol{u}'\|_{H^1(\Omega)^2} + \|p_S\|_{L^2_{\sqrt{h}}(\omega)/\mathbb{R}} \leqslant C \left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2}. \tag{2.16}$$

Proof. Let $(\boldsymbol{u}', p_S) \in H_0^1(\Omega)^2 \times (L^2_{\sqrt{h}}(\omega)/\mathbb{R})$ be a solution to (2.15), and let $\boldsymbol{v}' \in \boldsymbol{V}_M$. Then, one has thanks to (2.8)

$$\begin{aligned} \langle \nabla' \widetilde{p_S}, \boldsymbol{v}' \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} = -\int_\Omega \widetilde{p_S}\, \nabla' \cdot \boldsymbol{v}' \, dx &= -\int_\omega p_S\, M(\nabla' \cdot \boldsymbol{v}') \, dx' \\ &= -\int_\omega p_S\, \nabla' \cdot M\boldsymbol{v}' \, dx' = 0, \end{aligned}$$

since $\nabla' \cdot M\boldsymbol{v}' = 0$ in ω. As a consequence, $\boldsymbol{u}'$ satisfies the following variational formulation:

$$\begin{cases} \text{Find } \boldsymbol{u}' \in \boldsymbol{V}_M \text{ such that:} \\ \forall \boldsymbol{v}' \in \boldsymbol{V}_M, \displaystyle\int_\Omega \nabla \boldsymbol{u}' : \nabla \boldsymbol{v}' \, dx = \left\langle \boldsymbol{f}', \boldsymbol{v}' \right\rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2}. \end{cases} \tag{2.17}$$

Conversely, any solution $\boldsymbol{u}' \in \boldsymbol{V}_M$ to (2.17) is such that

$$\forall \boldsymbol{v}' \in \boldsymbol{V}_M, \quad \left\langle \Delta \boldsymbol{u}' + \boldsymbol{f}', \boldsymbol{v}' \right\rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} = 0.$$

Hence $\Delta \boldsymbol{u}' + \boldsymbol{f}'$ is exactly in the statement of Lemma 2.9. Therefore, there is a unique p_S in $(L^2_{\sqrt{h}}(\omega)/\mathbb{R})$ such that

$$\nabla \widetilde{p_S} = \Delta \boldsymbol{u}' + \boldsymbol{f}' \text{ in } \Omega, \tag{2.18}$$

where $\boldsymbol{u}'$ is a solution to Problem (2.15). As a consequence, we have proved that a pair $(\boldsymbol{u}', p) \in H_0^1(\Omega)^2 \times (L^2_{\sqrt{h}}(\omega)/\mathbb{R})$ is a solution to (2.15) if and only if $\boldsymbol{u}'$ is a solution to (2.17). To prove the existence and uniqueness of a solution to (2.15), note that Lax-Milgram's lemma provides a unique $\boldsymbol{u}'$ in $\boldsymbol{V}_M$ satisfying (2.17). Then, by taking $\boldsymbol{v} = \boldsymbol{u}$ in (2.17), we deduce that

$$\|\nabla \boldsymbol{u}'\|_{L^2(\Omega)} \leqslant C \left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)}. \tag{2.19}$$

To finish, Lemma (2.10) and relation (2.19), give the following estimates

$$\begin{aligned} \|p_S\|_{L^2_{\sqrt{h}}(\omega)/\mathbb{R}} &\leqslant C \|\nabla \widetilde{p_S}\|_{L^2(\Omega)} \\ &\leqslant C(\left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2} + \|\Delta \boldsymbol{u}'\|_{H^{-1}(\Omega)^2}) \\ &\leqslant C \left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2}. \end{aligned}$$

$\square$

Remark 2.13. The reader is also referred to [27] for another application of the version of De Rham's Lemma established in [26]. Indeed, in [27], F. O. Gallego studies a modified version of the hydrostatic approximation: the differential equations for the horizontal velocity are perturbed with a certain monotone expression. This approach has lead to another existence proof of (2.14) and to analyze a one-equation hydrostatic turbulence model.

Remark 2.14. Another existence proof of weak solutions to (2.14) was given earlier by T. Chacón Rebollo and F. Guillén González in [19], where assumption (7) was not considered too, see Theorem 2 page 844. The authors perform a direct analysis of existence of solutions, based upon the existence of the following reduced inf-sup condition that we adapt to our notations:

$$\sup_{\boldsymbol{v}'\in H_0^1(\Omega)^2-\{0\}} \frac{\int_\omega p_S\, \nabla'\cdot M\boldsymbol{v}'\, dx'}{\|\boldsymbol{v}'\|_{H^1(\Omega)^2}} \geqslant C\,\|p_S\|_{L^2_{\sqrt{h}}(\omega)/\mathbb{R}},$$

and which is in fact equivalent to the version of De Rham's Lemma that we establish in Lemma 2.9.

F. Guillén González *and al.* has established in [35], an existence and uniqueness proof of a weak solution to the following modified version of (2.14):

$$\left\{\begin{aligned} -\Delta\boldsymbol{u}' + \nabla'\widetilde{p_S} &= \boldsymbol{f}' &&\text{in } \Omega,\\ \nabla'\cdot M\boldsymbol{u}' &= \phi &&\text{in } \omega,\\ \boldsymbol{u}' = 0 \text{ on } \Gamma_B\cup\Gamma_L, \quad -\frac{\partial\boldsymbol{u}'}{\partial x_3} &= 0 &&\text{on } \Gamma_S, \end{aligned}\right. \tag{2.20}$$

see Theorem 2.5 page 812. To prove such a result, the authors needed to impose assumption (7), and hence to consider a domain Ω with sidewalls along $\partial\omega$. In fact, their initial motivation was to establish the existence and uniqueness of very weak solutions to the dual problem to (2.20), so they also needed to study the strong solutions to (2.20), see Theorem 2.1 page 810.

Here, we want to generalize Theorem 2.5 of [35], to a case of inhomogeneous conditions exclusively, and with the supplementary difficulty of not having to impose the condition (7). In other words, we study now the following system:

$$\left\{\begin{aligned} -\Delta\boldsymbol{u}' + \nabla'\widetilde{p_S} &= \boldsymbol{f}' &&\text{in } \Omega,\\ \nabla'\cdot M\boldsymbol{u}' &= \phi &&\text{in } \omega,\\ \boldsymbol{u}' &= \boldsymbol{g}' &&\text{on } \Gamma. \end{aligned}\right. \tag{2.21}$$

To our knowledge, the conditions:

$$\nabla'\cdot M\boldsymbol{u}' = \phi \text{ in } \omega, \quad \text{and} \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma, \tag{2.22}$$

have never been treated simultaneously, and even when (7) holds. We propose to handle this new situation by building a lift operator related to the above conditions, and relying essentially on Lemma 2.10 established previously. As a consequence, we will prove that solving (2.21) reduces to solve (2.15) for a good datum $\boldsymbol{f}'$, and for which we can apply Theorem 2.17.

The following lines are, therefore, dedicated to the lift operator for the conditions (2.22). Firstly, since we want to find $\boldsymbol{u}' \in H^1(\Omega)^2$, we need to consider $\boldsymbol{g}' \in H^{1/2}(\Gamma)^2$ in (2.22). In this case, and since (2.7) yields:

$$\nabla'\cdot M\boldsymbol{u}' = M(\nabla'\cdot\boldsymbol{u}') + \nabla' h\cdot(\boldsymbol{g}')_B \text{ in } \omega,$$

we have to chose $\phi \in L^2(\omega)$ in (2.22), according to Proposition 2.2. Then, by integrating over ω the above relation, one has by (2.7), (1.3) and since $\boldsymbol{n}' = 0$ on Γ_S:

$$\begin{aligned}\int_\omega \nabla' \cdot M\boldsymbol{u}'\,dx' &= \int_\omega M(\nabla' \cdot \boldsymbol{u}')\,dx' + \int_\omega \nabla' h \cdot (\boldsymbol{g}'|_\Gamma)_B\,dx' \\ &= \int_\Omega \nabla' \cdot \boldsymbol{u}'\,dx - \int_{\Gamma_B} \boldsymbol{g}' \cdot \boldsymbol{n}'\,d\sigma \\ &= \int_\Gamma \boldsymbol{u}' \cdot \boldsymbol{n}'\,d\sigma - \int_{\Gamma_B} \boldsymbol{g}' \cdot \boldsymbol{n}'\,d\sigma, \\ &= \int_{\Gamma_L} \boldsymbol{g}' \cdot \boldsymbol{n}'\,d\sigma.\end{aligned}$$

As a consequence of these computations, we can conclude that:

$$\phi \in L^2(\omega), \quad \boldsymbol{g}' \in H^{1/2}(\Gamma)^2, \tag{2.23}$$

$$\int_{\Gamma_L} \boldsymbol{g}' \cdot \boldsymbol{n}'\,d\sigma = \int_\omega \phi\,dx, \tag{2.24}$$

are necessary conditions to the existence of a lifting $\boldsymbol{u}' \in H^1(\Omega)^2$ for the conditions (2.22).

Secondly, let us see if the conditions (2.23) and (2.24) are sufficient. Let $\boldsymbol{g}' \in H^{1/2}(\Gamma)^2$ and $\boldsymbol{v}' \in H^1(\Omega)^2$ such that $\boldsymbol{v}' = \boldsymbol{g}'$ on Γ. Then, thanks to the compatibility condition (2.24) and the above computations, one has:

$$\int_\omega \nabla' \cdot M\boldsymbol{v}'\,dx' = \int_{\Gamma_L} \boldsymbol{g}' \cdot \boldsymbol{n}'\,d\sigma = \int_\omega \phi\,dx, \tag{2.25}$$

and $\int_\omega (\nabla' \cdot M\boldsymbol{v}' - \phi)\,dx' = 0$. Then, we want to apply the second isomorphism of (2.13) to exhibit $\boldsymbol{w}' \in H_0^1(\Omega)^2$ such that $\nabla' \cdot M\boldsymbol{v}' - \phi = \nabla' \cdot M\boldsymbol{w}'$ on ω, and $\boldsymbol{u}' = \boldsymbol{w}' - \boldsymbol{v}'$ would be the expected lifting. Unfortunately, we can not use this isomorphism, since $\nabla' \cdot M\boldsymbol{v}' - \phi$ does not belong to $L^2_{1/\sqrt{h},0}(\omega)$, and even if we take $\phi \in L^2_{1/\sqrt{h}}(\omega)$. We refer here to Remark 2.3. As a conclusion, conditions (2.23) and (2.24) are not sufficient to establish the existence of the lifting $\boldsymbol{u}' \in H^1(\Omega)^2$.

It is therefore necessary to consider functions $\boldsymbol{v}' \in H^1(\Omega)^2$ having the following supplementary property:

$$\nabla' \cdot M\boldsymbol{v}' \in L^2_{1/\sqrt{h}}(\omega). \tag{2.26}$$

To this purpose, let us consider the following weighted subspace of $H^{1/2}(\Gamma)$, denoted and defined by:

$$H^{1/2}_{1/\sqrt{h}}(\Gamma) = \left\{ g \in H^{1/2}(\Gamma) \,/\, \frac{g}{\sqrt{h}} \in L^2(\Gamma) \right\}, \tag{2.27}$$

and endowed with the following hilbertian norm:

$$\|g\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)} = \left(\|g\|^2_{H^{1/2}(\Gamma)} + \left\| \frac{g}{\sqrt{h}} \right\|^2_{L^2(\Gamma)} \right)^{1/2}.$$

Then, observe the following proposition.

Proposition 2.15. *Let $\boldsymbol{u}' \in H^1(\Omega)^2$ such that $\boldsymbol{u}'|_\Gamma \in H^{1/2}_{1/\sqrt{h}}(\Gamma)^2$. Then, one has*

$$\nabla' \cdot M\boldsymbol{u}' \in L^2_{1/\sqrt{h}}(\omega).$$

Moreover, there is a constant $C > 0$ such that

$$\|\nabla' \cdot M\boldsymbol{u}'\|_{L^2_{1/\sqrt{h}}(\omega)} \leqslant C\left(\|\boldsymbol{u}'\|_{H^1(\Omega)^2} + \|\boldsymbol{u}'|_\Gamma\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)^2}\right). \tag{2.28}$$

Proof. Let $i = 1, 2$. The assumptions on $u_i|_\Gamma$ and Proposition 1.1 ensure that $(u_i|_\Gamma)_B \in L^2_{1/\sqrt{h}}(\omega)$, and since the mapping h is Lipschitz-continuous, one has

$$(u_i|_\Gamma)_B \frac{\partial h}{\partial x_i} \in L^2_{1/\sqrt{h}}(\omega). \tag{2.29}$$

Then, as $u \in H^1(\omega)$, it follows from relation (2.7) that

$$\frac{\partial M u}{\partial x_i} \in L^2_{1/\sqrt{h}}(\omega).$$

Next, since relation (1.3) gives

$$\int_\omega \frac{1}{h}\left[(u_i|_\Gamma)_B \frac{\partial h}{\partial x_i}\right]^2 dx' = -\int_{\Gamma_B} \frac{(u_i|_\Gamma)^2}{h} \frac{\partial h}{\partial x_i} n_i \, d\sigma \leqslant \left\|\frac{u_i|_\Gamma}{\sqrt{h}}\right\|^2_{L^2(\Gamma)} \leqslant \|u_i|_\Gamma\|^2_{H^{1/2}_{1/\sqrt{h}}(\Gamma)},$$

one obtains immediately by (2.9) that

$$\begin{aligned}\left\|\frac{\partial M u_i}{\partial x_i}\right\|_{L^2_{1/\sqrt{h}}(\omega)} &\leqslant \left\|M(\frac{\partial u_i}{\partial x_i})\right\|_{L^2_{1/\sqrt{h}}(\omega)} + \|u_i|_\Gamma\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)} \\ &\leqslant \|u_i\|_{H^1(\Omega)} + \|u_i|_\Gamma\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)}.\end{aligned}$$

By summation on $i = 1, 2$, we establish (2.28). □

We are now in position to prove the following lemma where we build the expected lifting of conditions (2.22) and which proof comes to correct the preliminary draft we have done before.

Lemma 2.16. *Let $\boldsymbol{g}' \in H^{1/2}_{1/\sqrt{h}}(\Gamma)^2$ and $\phi \in L^2_{1/\sqrt{h}}(\omega)$, satisfying the compatibility condition (2.34). Then, there is $\boldsymbol{u}' \in H^1(\Omega)^2$ such that:*

$$\nabla' \cdot M\boldsymbol{u}' = \phi \text{ in } \omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma. \tag{2.30}$$

Moreover, there is a constant $C > 0$ depending only on Ω such that:

$$\|\boldsymbol{u}'\|_{H^1(\Omega)^2} \leqslant C\left(\|\boldsymbol{g}'\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)^2} + \|\phi\|_{L^2_{1/\sqrt{h}}(\omega)}\right). \tag{2.31}$$

Proof. Let $\boldsymbol{g}' \in H^{1/2}_{1/\sqrt{h}}(\Gamma)^2$. Then, there is $\boldsymbol{v}'$ in $H^1(\Omega)^2$ such that $\boldsymbol{v}' = \boldsymbol{g}'$ on Γ, and

$$\frac{\boldsymbol{v}'}{\sqrt{h}} \in L^2(\Gamma)^2. \tag{2.32}$$

According to (2.25) one already has:

$$\int_\omega \nabla' \cdot M\boldsymbol{v}' \, dx' = \int_{\Gamma_L} \boldsymbol{g}' \cdot \boldsymbol{n}' \, d\sigma = \int_\omega \phi \, dx'.$$

Moreover, one deduces from Proposition (2.15) that $\nabla' \cdot M\boldsymbol{v}'$ belongs to $L^2_{1/\sqrt{h}}(\omega)$. As a consequence, the function $\nabla' \cdot M\boldsymbol{v}' - \phi$ is in $L^2_{1/\sqrt{h},0}(\omega)$ (see (2.12)), and the isomorphism given in (2.13) provides a unique $\boldsymbol{z}'$ in $\boldsymbol{V}_M^\perp$ satisfying

$$\nabla' \cdot M\boldsymbol{z}' = \nabla' \cdot M\boldsymbol{v}' - \phi \quad \text{in } \omega.$$

Moreover, there is a constant $C > 0$ depending only on Ω such that

$$\|\boldsymbol{z}'\|_{H^1(\Omega)^2} \leqslant C \,\|\nabla' \cdot M\boldsymbol{v}' - \phi\|_{L^2_{1/\sqrt{h}}(\omega)}.$$

Then, one has by (2.28) and since $\boldsymbol{v}' = \boldsymbol{g}'$ in $H^{1/2}_{1/\sqrt{h}}(\Gamma)$:

$$\begin{aligned}
\|\boldsymbol{z}'\|_{H^1(\Omega)^2} &\leqslant C \left(\|\boldsymbol{v}'\|_{H^1(\Omega)^2} + \|\boldsymbol{v}'|_\Gamma\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)^2} + \|\phi\|_{L^2_{1/\sqrt{h}}(\omega)} \right) \\
&\leqslant C \left(\|\boldsymbol{v}'\|_{H^1(\Omega)^2} + \|\boldsymbol{g}'\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)^2} + \|\phi\|_{L^2_{1/\sqrt{h}}(\omega)} \right).
\end{aligned}$$

By taking the infimum on the functions $\boldsymbol{v}' \in H^1(\Omega)^2$ such that $\boldsymbol{v}' = \boldsymbol{g}'$ on Γ, we deduce that

$$\|\boldsymbol{z}'\|_{H^1(\Omega)^2} \leqslant C \left(\|\boldsymbol{g}'\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)^2} + \|\phi\|_{L^2_{1/\sqrt{h}}(\omega)} \right). \tag{2.33}$$

Consequently, $\boldsymbol{u}' = \boldsymbol{v}' - \boldsymbol{z}'$ satisfies (2.30) and (2.31). □

Finally, we give an existence and uniqueness proof of a weak solution to Problem (2.21). Recall that we have precisely build the lifting of the conditions (2.22) in Lemma 2.16 to reach such result.

Theorem 2.17. *Let $\boldsymbol{f}' \in H^{-1}(\Omega)^2$, $\phi \in L^2_{1/\sqrt{h}}(\omega)$ and $\boldsymbol{g}' \in H^{1/2}_{1/\sqrt{h}}(\Gamma)^2$ such that*

$$\int_{\Gamma_L} \boldsymbol{g}' \cdot \boldsymbol{n}' \, d\sigma = \int_\omega \phi \, dx. \tag{2.34}$$

Then, there is a unique pair $(\boldsymbol{u}', p) \in H^1(\Omega)^2 \times (L^2_{\sqrt{h}}(\omega)/\mathbb{R})$ solution to (2.21), and a constant $C > 0$ depending only on Ω such that

$$\|\boldsymbol{u}'\|_{H^1(\Omega)^2} + \|p_S\|_{L^2_{\sqrt{h}}(\omega)/\mathbb{R}} \leqslant C \left(\left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2} + \|\phi\|_{L^2_{1/\sqrt{h}}(\omega)} + \|\boldsymbol{g}'\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)^2} \right).$$

Proof. Let $\boldsymbol{f}' \in H^{-1}(\Omega)^2$, $\phi \in L^2_{1/\sqrt{h}}(\omega)$ and $\boldsymbol{g}' \in H^{1/2}_{1/\sqrt{h}}(\Gamma)^2$ satisfying the compatibility condition (2.34). From Lemma 2.16, there is $\boldsymbol{v}' \in H^1(\Omega)^2$ satisfying the conditions (2.30). Moreover, from Proposition 2.12, there is a unique pair $(\boldsymbol{w}', p_S)$ in the space $H^1_0(\Omega)^2 \times (L^2_{\sqrt{h}}(\omega)/\mathbb{R})$ checking

$$-\Delta \boldsymbol{w}' + \nabla' \widetilde{p_S} = \boldsymbol{f}' - \Delta \boldsymbol{v}' \text{ in } \Omega, \quad \nabla' \cdot M\boldsymbol{w}' = 0 \text{ in } \omega,$$

and there is a constant $C > 0$ such that

$$\|\boldsymbol{w}'\|_{H^1(\Omega)^2} + \|p_S\|_{L^2_{\sqrt{h}}(\omega)/\mathbb{R}} \leqslant C \left\|\boldsymbol{f}' + \Delta \boldsymbol{v}'\right\|_{H^{-1}(\Omega)^2}.$$

More precisely, as $\boldsymbol{v}'$ satisfies the estimate (2.31), one deduces that

$$\begin{aligned} \|\boldsymbol{w}'\|_{H^1(\Omega)^2} + \|p_S\|_{L^2_{\sqrt{h}}(\omega)/\mathbb{R}} &\leqslant C \left\|\boldsymbol{f}' - \Delta \boldsymbol{v}'\right\|_{H^{-1}(\Omega)^2} \\ &\leqslant C \left(\left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2} + \|\nabla \boldsymbol{v}'\|_{L^2(\Omega)^2} \right) \\ &\leqslant C \left(\left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2} + \|\boldsymbol{g}'\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)^2} + \|\phi\|_{L^2_{1/\sqrt{h}}(\omega)} \right). \end{aligned} \tag{2.35}$$

Then, $\boldsymbol{u}' = \boldsymbol{w}' + \boldsymbol{v}'$ belongs to $H^1(\Omega)^2$ and is the unique solution, with p_S, to the non homogeneous Problem (2.21). We already have the estimates on p_S. The one on $\boldsymbol{u}'$ is immediate since (2.31) and (2.35) imply that

$$\begin{aligned} \|\boldsymbol{u}'\|_{H^1(\Omega)^2} = \|\boldsymbol{v}' + \boldsymbol{w}'\|_{H^1(\Omega)^2} &\leqslant \|\boldsymbol{v}'\|_{H^1(\Omega)^2} + \|\boldsymbol{w}'\|_{H^1(\Omega)^2} \\ &\leqslant C \left(\left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2} + \|\boldsymbol{g}'\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)^2} + \|\phi\|_{L^2_{1/\sqrt{h}}(\omega)} \right). \end{aligned}$$

□

Remark 2.18. When assumption (7) is considered, Theorem 2.17 can also be stated by replacing the space $L^2_{1/\sqrt{h}}(\omega)$ by $L^2(\omega)$ and $H^{1/2}_{1/\sqrt{h}}(\Gamma)$ by $H^{1/2}(\Gamma)$. In this case, the superficial pressure p_S is in $L^2(\omega)$.

Let us finish this chapter by summing up the work we have done. Firstly, any result we have proved are available in a general domain Ω, for which it is not necessary to impose (7). This is why we have defined a suitable functional framework, based on weighted spaces, see (2.3) and (2.27). Secondly, we have set the tools necessary to solve Problem (2.15) and its inhomogeneous version (2.21). We have solved Problem (2.15) by using a De Rham's like lemma, see Lemma 2.9, proper for the resolution of Stokes type system. Then, we have solved Problem (2.21) by considering the new situation to lift simultaneously the conditions (2.22), in order to bring back to the homogeneous case.

In the following chapter, we will investigate the uniqueness and existence of a weak solution to the hydrostatic Stokes approximation with homogeneous Dirichlet boundary conditions (3.1). We will also give the mathematical justification that enables to reduce (3.1) to (2.15), that we have already solved in Proposition 2.12; and this will constitute a second existence and uniqueness proof of a weak solution to (3.1). We will also see that we can solve a more general problem (3.23), by reducing its equations to (2.21) for a good datum ϕ, which we have solved previously in Theorem 2.17.

We foresee now that this reduction process is the key which has enabled us to go beyond the consideration of homogeneous conditions over u_3, as it the case in the literature, and to treat the new situation of non homogeneous boundary conditions over u_3, see Chapter 4.

Chapter 3

Weak solutions to (SH) with a homogeneous condition on u_3

3.1 Introduction

Let $\boldsymbol{f}' : \Omega \to \mathbb{R}^2$ and consider the homogeneous hydrostatic Stokes problem:

$$\begin{cases} -\Delta \boldsymbol{u}' + \nabla' p = \boldsymbol{f}', \quad \dfrac{\partial p}{\partial x_3} = 0, \quad \nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}' = 0, \quad u_3 n_3 = 0 \quad \text{on } \Gamma. \end{cases} \tag{3.1}$$

The existence and uniqueness of a weak solution to (3.1) is well known. A first approach consists in proving that (3.1) is the limit, in a precise sense, of an anisotropic Stokes system. This is what is proposed by O. Besson, R. Laydi and M. Touzani in [7] on the 2D case. A second approach is to solve (3.1) without using a limit process, and for example, P. Azérad proposed in [3] an existence and uniqueness proof based on the existence of an inf-sup condition, see after Remark 3.7.

In this chapter, we revisit the existence and uniqueness of a weak solution to (3.1) by following these two approaches. Instead of considering an inf-sup condition as in [3], we reduce the equations of (3.1) in order to bring back to (2.15), for which we know the existence and uniqueness of a weak solution, see Proposition 2.12. Then, we keep this second approach in order to solve the following more general problem:

$$\begin{cases} -\Delta \boldsymbol{u}' + \nabla' p = \boldsymbol{f}', \quad \dfrac{\partial p}{\partial x_3} = 0, \quad \nabla \cdot \boldsymbol{u} = \Phi \quad \text{in } \Omega, \\ \boldsymbol{u}' = \boldsymbol{g}', \quad u_3 n_3 = 0 \quad \text{on } \Gamma, \end{cases} \tag{3.2}$$

and for which we initiate the study.

An outline of this chapter is as follows. In Section 3.2, we will give a meaning to the boundary condition:

$$u_3 n_3 = 0 \text{ on } \Gamma, \tag{3.3}$$

by recalling some results on an anisotropic Sobolev space.

In Section 3.3, we will get interested in the study of weak solutions to Problem (3.1). Uniqueness of a weak solution is investigated at first in Lemma 3.2, and the proof we give uses the generalized Green's formula (3.6). Regarding to the existence of a weak solution, we will give a first proof based upon a limit process, see Theorem 3.4. More precisely, we give an asymptotic analysis of an anisotropic

Stokes system, which goes to Problem (3.1). Recall that we have sketched out this approach in the introduction of Part I.

In Section 3.4, we give another existence and uniqueness proof of a weak solution to Problem (3.1), by reducing the equations:

$$\nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega, \quad \boldsymbol{u}' = 0 \text{ on } \Gamma, \quad u_3 n_3 = 0 \text{ on } \Gamma,$$

to the following ones:

$$\nabla' \cdot M\boldsymbol{u}' = 0 \text{ in } \omega, \quad \boldsymbol{u}' = 0 \text{ on } \Gamma,$$

and this is precisely the purpose of Lemma 3.14. We therefore prove that solving Problem (3.1) reduces to solve (2.15), thing already done in Proposition 2.12. Our main preoccupation here is to give solid mathematical foundations which enable the reduction of the above equations. We therefore introduce the operator F defined in (3.18) for which we give some continuity properties in Proposition 3.9. Then, we give an essential result in Proposition 3.12 concerning the characterization of the trace $(Fu)n_3$ for functions u in $L^2(\Omega)$, and from which follows immediately Lemma 3.14, mentioned above.

As we will see later, this reduction process is the key to handle more general situations, such as dealing with Problem (3.23) where we consider the new conditions:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma, \quad u_3 n_3 = 0 \text{ on } \Gamma.$$

We will give an existence and uniqueness proof of a weak solution to this problem, by reducing the above conditions to the following ones:

$$\nabla' \cdot M\boldsymbol{u}' = \phi \text{ in } \omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma,$$

where ϕ is a function depending on Φ and $\boldsymbol{g}'$, see Lemma 3.18. We recognize here the last two equations of Problem (2.21) which is already solved in Theorem 2.17. To finish, as we will see in a later remark, it is also possible to solve Problem (3.23) with a proof based on an asymptotic analysis of an adapted Stokes system (see (3.31)).

For the moment, we will only consider the boundary condition (3.3) and we will save the case of non homogeneous condition on u_3 for the next chapter, since it requires an important mathematical investment.

3.2 The anisotropic space $H(\partial_{x_3}, \Omega)$

Let us begin with defining the boundary condition (3.3). For this purpose, we lean on the work done by R. Temam in [47]. Let us consider the space

$$H(\partial_{x_3}, \Omega) = \left\{ u \in L^2(\Omega) \,/\, \frac{\partial u}{\partial x_3} \in L^2(\Omega) \right\},$$

which is a Hilbert space for the norm

$$\|u\|_{H(\partial_{x_3},\Omega)} = (\|u\|^2_{L^2(\Omega)} + \|\partial_{x_3} u\|^2_{L^2(\Omega)})^{1/2}.$$

The space $\mathcal{D}(\overline{\Omega})$ is dense in $H(\partial_{x_3}, \Omega)$, and any function of $H(\partial_{x_3}, \Omega)$ has a trace, as it is stated in the following proposition.

Proposition 3.1. *The mapping*

$$\gamma_3 : u \mapsto u n_3|_\Gamma,$$

defined on $\mathcal{D}(\overline{\Omega})$ can be extended in a unique way to a linear and continuous mapping, still denoted γ_3, from $H(\partial_{x_3}, \Omega)$ into $H^{-1/2}(\Gamma)$.

Moreover, we derive the following Green's formula:

$$\left\{\begin{array}{l} \forall u \in H(\partial_{x_3},\, \Omega),\, \forall v \in H^1(\Omega), \\ \displaystyle\int_\Omega u\,\frac{\partial v}{\partial x_3}\,dx = -\int_\Omega v\,\frac{\partial u}{\partial x_3}\,dx + \langle u n_3,\, v\rangle_{H^{-1/2}(\Gamma),\, H^{1/2}(\Gamma)}\,. \end{array}\right. \tag{3.4}$$

As a consequence of Proposition 3.1, we understand now that the boundary condition (3.3) has to be taken in the sense of $H^{-1/2}(\Gamma)$.

Now, let us consider the space

$$H_0(\partial_{x_3},\, \Omega) = \overline{\mathcal{D}(\Omega)}^{H(\partial_{x_3},\,\Omega)}. \tag{3.5}$$

Then, it is clear that the following Green's formula holds:

$$\forall u \in H(\partial_{x_3},\, \Omega),\, \forall v \in H_0(\partial_{x_3},\, \Omega), \qquad \int_\Omega u\,\frac{\partial v}{\partial x_3}\,dx = -\int_\Omega v\,\frac{\partial u}{\partial x_3}\,dx, \tag{3.6}$$

as well as the following inequality of Poincaré: there is a constant $C > 0$, depending only on Ω, such that

$$\forall u \in H_0(\partial_{x_3},\, \Omega), \quad \|u\|_{L^2(\Omega)} \leqslant C\left\|\frac{\partial u}{\partial x_3}\right\|_{L^2(\Omega)}. \tag{3.7}$$

To finish, R. Temam has characterized functions of $H_0(\partial_{x_3},\, \Omega)$ as functions of null traces on Γ (see [47] pages 376 and 377 Theorem 2.2). More precisely, the following equality holds:

$$H_0(\partial_{x_3},\, \Omega) = \Big\{u \in H(\partial_{x_3},\, \Omega)\,/\, u n_3 = 0 \text{ in } H^{-1/2}(\Gamma)\Big\}.$$

3.3 Weak solution for the homogeneous case

We are now in position to prove an existence and uniqueness result of a weak solution to the hydrostatic Stokes Problem (3.1). Let us start by proving the uniqueness of a possible solution. For convenience, we denote by $\boldsymbol{X}_0$ the space

$$\boldsymbol{X}_0 = H_0^1(\Omega)^2 \times H_0(\partial_{x_3}, \Omega). \tag{3.8}$$

Lemma 3.2. *For any $\boldsymbol{f}' \in H^{-1}(\Omega)^2$, Problem* (3.1) *has at most one solution* $(\boldsymbol{u},\, p)$ *in the space* $\boldsymbol{X}_0 \times (L^2(\Omega)/\mathbb{R})$.

Proof. Let us consider a possible solution $(\boldsymbol{u},\, p)$ related to the data $\boldsymbol{f}' = 0$. By multiplying the first equation of (3.1) by $\boldsymbol{u}'$, and by using Green's formula, one has

$$\int_\Omega \nabla \boldsymbol{u}' : \nabla \boldsymbol{u}'\,dx = \int_\Omega p\,\nabla' \cdot \boldsymbol{u}'\,dx = -\int_\Omega p\,\frac{\partial u_3}{\partial x_3}\,dx,$$

since $\nabla \cdot \boldsymbol{u} = 0$ in Ω. As $u_3 \in H_0(\partial_{x_3}, \Omega)$ and $\partial_{x_3} p = 0$ in Ω, relation (3.6) gives:

$$\int_\Omega \nabla \boldsymbol{u}' : \nabla \boldsymbol{u}'\,dx = \int_\Omega u_3\,\frac{\partial p}{\partial x_3}\,dx = 0,$$

and, therefore, $\nabla \boldsymbol{u}' = 0$ in Ω. Since $\boldsymbol{u}' = 0$ on Γ and since Ω is connected, one has $\boldsymbol{u}' = 0$ in Ω. As $\nabla \cdot \boldsymbol{u} = 0$ in Ω, we deduce that $\partial_{x_3} u_3 = 0$ in Ω, and from the inequality (3.7) we get $u_3 = 0$ in Ω. Then, as $\nabla' p = \Delta \boldsymbol{u}' = 0$ in Ω, one obtains that $\nabla p = 0$ in Ω, hence $p = 0$ in $L^2(\Omega)/\mathbb{R}$, since we recall that Ω is connected.

As a conclusion, problem $(\mathcal{SH})$ has at most one solution when $\boldsymbol{f}' = 0$ which is is $\boldsymbol{u} = 0$ and $p = 0$. Consequently, Problem (3.1) has at most one solution in $\boldsymbol{X}_0 \times (L^2(\Omega)/\mathbb{R})$. $\square$

Then, let us prove the existence of a solution to (3.1). Here, we have decided to give an asymptotic analysis of the following anisotropic Stokes system:

$$\begin{cases} -\Delta \boldsymbol{u}'_\varepsilon + \nabla' p_\varepsilon = \boldsymbol{f}', \quad -\varepsilon^2 \Delta u_3^\varepsilon + \dfrac{\partial p_\varepsilon}{\partial x_3} = 0, \quad \nabla \cdot \boldsymbol{u}_\varepsilon = 0 \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \boldsymbol{u}_\varepsilon = 0 \quad \text{on } \Gamma, \end{cases} \tag{3.9}$$

where $\varepsilon \in]0,\, 1]$, previously mentioned in the introduction of Part I.

For any $(\boldsymbol{u}_\varepsilon,\, p_\varepsilon)$ in the space $H_0^1(\Omega)^3 \times (L^2(\Omega)/\mathbb{R})$ solution to (3.9), $\boldsymbol{u}_\varepsilon$ satisfies the following variational problem:

$$\text{Find } \boldsymbol{u}_\varepsilon \in \boldsymbol{V} \quad \text{such that:}$$
$$\begin{cases} \forall \boldsymbol{v} \in \boldsymbol{V} \\ \displaystyle\int_\Omega \nabla \boldsymbol{u}'_\varepsilon : \nabla \boldsymbol{v}'\, dx + \varepsilon^2 \int_\Omega \nabla u_3^\varepsilon \cdot \nabla v_3\, dx = \big\langle \boldsymbol{f}',\, \boldsymbol{v}' \big\rangle_{H^{-1}(\Omega)^2,\, H_0^1(\Omega)^2}. \end{cases} \tag{3.10}$$

(see (1.7) for the definition of $\boldsymbol{V}$.) Conversely, let us consider any solution $\boldsymbol{u}_\varepsilon \in \boldsymbol{V}$ to (3.10). One has for any $\boldsymbol{\varphi} \in \boldsymbol{V}$:

$$\big\langle \Delta \boldsymbol{u}'_\varepsilon + \boldsymbol{f}',\, \boldsymbol{\varphi}' \big\rangle + \big\langle \varepsilon^2 \Delta u_3^\varepsilon,\, \varphi_3 \big\rangle = 0,$$

where the duality is taken in the sense of $H^{-1}(\Omega)^2$, $H_0^1(\Omega)^2$. Therefore, the distribution $\boldsymbol{T} = (\Delta \boldsymbol{u}'_\varepsilon + \boldsymbol{f}',\, \varepsilon^2 \Delta u_3^\varepsilon)$ is exactly in the statement of Lemma 1.2, hence there is a unique function p_ε in $L^2(\Omega)/\mathbb{R}$ such that

$$\nabla p_\varepsilon = \boldsymbol{T} \text{ in } \Omega.$$

As a consequence, any pair $(\boldsymbol{u}_\varepsilon,\, p_\varepsilon)$ in the space $H_0^1(\Omega)^3 \times (L^2(\Omega)/\mathbb{R})$ is a solution to (3.9) if and only if $\boldsymbol{u}_\varepsilon$ satisfies (3.10). Therefore, we easily deduce, thanks to Lax-Milgram's lemma, that there is a unique $\boldsymbol{u}_\varepsilon$ in $\boldsymbol{V}$ satisfying (3.10). Hence, we have proved the first step of the asymptotic analysis consisting in proving that Problem (3.9) has a unique solution $(\boldsymbol{u}_\varepsilon,\, p_\varepsilon)$ in the space $H_0^1(\Omega)^3 \times (L^2(\Omega)/\mathbb{R})$.

Secondly, let us give some *a priori* estimates on the solution $(\boldsymbol{u}_\varepsilon,\, p_\varepsilon)$. More precisely, let us prove that there is a constant $C > 0$, independent on ε, such that:

$$\|\boldsymbol{u}'_\varepsilon\|_{H^1(\Omega)^2} + \varepsilon \|u_3^\varepsilon\|_{H^1(\Omega)} + \|u_3^\varepsilon\|_{H(\partial_{x_3},\Omega)} + \|p_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leqslant C \left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2}. \tag{3.11}$$

By taking $\boldsymbol{v} = \boldsymbol{u}_\varepsilon$ in (3.10), we deduce that

$$\|\nabla \boldsymbol{u}'_\varepsilon\|^2_{L^2(\Omega)} + \varepsilon^2 \|\nabla u_3^\varepsilon\|^2_{L^2(\Omega)} \leqslant C \left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)} \|\nabla \boldsymbol{u}'_\varepsilon\|_{L^2(\Omega)},$$

which leads to

$$\|\nabla \boldsymbol{u}'_\varepsilon\|_{L^2(\Omega)} + \varepsilon \|\nabla u_3^\varepsilon\|_{L^2(\Omega)} \leqslant C \left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2}. \tag{3.12}$$

Since $\nabla \cdot \boldsymbol{u}_{\boldsymbol{\varepsilon}} = 0$ in Ω, we deduce thanks to (3.12) that

$$\left\|\frac{\partial u_3^\varepsilon}{\partial x_3}\right\|_{L^2(\Omega)} = \|\nabla' \cdot \boldsymbol{u}'_\varepsilon\|_{L^2(\Omega)} \leqslant C \left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2}.$$

As u_3^ε belongs to $H_0(\partial_{x_3}, \Omega)$, Poincaré's inequality (3.7) and the above inequality imply that

$$\|u_3^\varepsilon\|_{L^2(\Omega)} \leqslant C \left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2}.$$

Then, we are interested in the estimate of $\|p_\varepsilon\|_{L^2(\Omega)/\mathbb{R}}$. The gradient operator is an isomorphism from $L^2(\Omega)/\mathbb{R}$ onto $\boldsymbol{V}^\circ$ (see (1.8)). As a consequence, since $p_\varepsilon \in L^2(\Omega)/\mathbb{R}$, there is a constant $C > 0$ such that

$$\begin{aligned}\|p_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} &\leqslant C\,\|\nabla p_\varepsilon\|_{H^{-1}(\Omega)}\\ &\leqslant C(\left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2} + \|\Delta \boldsymbol{u}'_\varepsilon\|_{H^{-1}(\Omega)^2} + \varepsilon^2\,\|\Delta u_3^\varepsilon\|_{H^{-1}(\Omega)})\\ &\leqslant C\,(\left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2} + \|\nabla \boldsymbol{u}'_\varepsilon\|_{L^2(\Omega)} + \varepsilon\,\|\nabla u_3^\varepsilon\|_{L^2(\Omega)}),\end{aligned}$$

by reading equations of (3.9), and since $\varepsilon \leqslant 1$. We deduce thanks to (3.12) that $(p_\varepsilon)_\varepsilon$ is bounded in $L^2(\Omega)/\mathbb{R}$, and we get (3.11).

Remark 3.3. By writing the third equation of (3.9) in the following way

$$\frac{\partial p_\varepsilon}{\partial x_3} = \varepsilon^2 \Delta u_3^\varepsilon,$$

we deduce thanks to (3.12) that there is a constant $C > 0$ independent on ε such that

$$\left\|\frac{\partial p_\varepsilon}{\partial x_3}\right\|_{H^{-1}(\Omega)} \leqslant \varepsilon\,\|\varepsilon \nabla u_3^\varepsilon\|_{L^2(\Omega)} \leqslant \varepsilon C. \tag{3.13}$$

Finally, let us finish the asymptotic analysis by passing to the limits as ε goes to 0 in (3.9). Thanks to the estimates (3.11) and (3.13), we proved that $(\boldsymbol{u}_\varepsilon,\, p_\varepsilon)$ is bounded in the space $\boldsymbol{X}_0 \times (L^2(\Omega)/\mathbb{R})$ (see (3.8) for the definition of $\boldsymbol{X}_0$). Since it is a reflexive space, there is a pair $(\boldsymbol{u},\, p)$ in the space $\boldsymbol{X}_0 \times (L^2(\Omega)/\mathbb{R})$ and a subsequence of $(\boldsymbol{u}_\varepsilon,\, p_\varepsilon)$, denoted in the same way, that converges weakly towards the couple $(\boldsymbol{u},\, p)$ in the space $\boldsymbol{X}_0 \times (L^2(\Omega)/\mathbb{R})$. In particular, one has

$$\begin{aligned}&\boldsymbol{u}'_\varepsilon \to \boldsymbol{u}' \quad \text{in } L^2(\Omega)^2 \quad \text{and} \quad \nabla \boldsymbol{u}'_\varepsilon \rightharpoonup \nabla \boldsymbol{u}' \quad \text{in } L^2(\Omega)^6,\\ &u_3^\varepsilon \rightharpoonup u_3 \quad \text{in } L^2(\Omega) \quad \text{and} \quad \frac{\partial u_3^\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial u_3}{\partial x_3} \quad \text{in } L^2(\Omega),\\ &p_\varepsilon \rightharpoonup p \quad \text{in } L^2(\Omega).\end{aligned}$$

Immediately, we see that

$$-\Delta \boldsymbol{u}'_\varepsilon + \nabla' p_\varepsilon \rightharpoonup -\Delta \boldsymbol{u}' + \nabla' p = \boldsymbol{f}' \text{ in } H^{-1}(\Omega)^2.$$

Then, we deduce from relation (3.13) that

$$\frac{\partial p_\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial p}{\partial x_3} = 0 \text{ in } H^{-1}(\Omega).$$

To finish, one has

$$\nabla \cdot \boldsymbol{u}_\varepsilon \rightharpoonup \nabla \cdot \boldsymbol{u} = 0 \text{ in } L^2(\Omega),$$

Then, the weak limit $(\boldsymbol{u},\, p)$ is a solution to (3.1) and satisfies (3.14) by taking the infimum limit in (3.11). We achieve the proof by using Lemma 3.2, ensuring that $(\boldsymbol{u},\, p)$ is the unique solution to (3.1). Consequently all the sequence $(\boldsymbol{u}_\varepsilon,\, p_\varepsilon)$ converges to $(\boldsymbol{u},\, p)$ in $\boldsymbol{X}_0 \times (L^2(\Omega)/\mathbb{R})$.

As a conclusion, we have proved the following result.

Theorem 3.4. *Let $\boldsymbol{f}'$ in $H^{-1}(\Omega)^2$. For any $\varepsilon > 0$, let $(\boldsymbol{u}_\varepsilon,\, p_\varepsilon) \in H_0^1(\Omega)^3 \times (L^2(\Omega)/\mathbb{R})$ the solution to Problem (3.9). Then, the sequence $(\boldsymbol{u}_\varepsilon,\, p_\varepsilon)_\varepsilon$ converges to $(\boldsymbol{u},\, p)$ in the space $\boldsymbol{X}_0 \times (L^2(\Omega)/\mathbb{R})$, unique solution to Problem (3.1). Moreover, there is a constant $C > 0$ such that*

$$\|\boldsymbol{u}'\|_{H^1(\Omega)^2} + \|u_3\|_{H(\partial_{x_3},\,\Omega)} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leqslant C\,\left\|\boldsymbol{f}'\right\|_{H^{-1}(\Omega)^2}. \tag{3.14}$$

Remark 3.5. By passing to the limits in the variational formulation (3.10), one obtains a variational formulation for Problem (3.1):

$$\left\{\begin{array}{l}\text{Find } \boldsymbol{u} \in \boldsymbol{V}_0 \text{ such that:}\\ \forall \boldsymbol{v} \in \boldsymbol{V}_0, \quad \displaystyle\int_\Omega \nabla \boldsymbol{u}' : \nabla \boldsymbol{v}'\, dx = \left\langle \boldsymbol{f}',\, \boldsymbol{v}' \right\rangle_{H^{-1}(\Omega)^2,\, H^1_0(\Omega)^2},\end{array}\right. \tag{3.15}$$

where the space $\boldsymbol{V}_0$ is defined by

$$\boldsymbol{V}_0 = \{\boldsymbol{v} \in \boldsymbol{X}_0 \,/\, \nabla \cdot \boldsymbol{v} = 0 \text{ in } \Omega\}.$$

Therefore, thanks to the generalized Poincaré's inequality (3.7), one has

$$\left\| \frac{\partial u_3}{\partial x_3} \right\|_{L^2(\Omega)} = \|\nabla' \cdot \boldsymbol{u}'\|_{L^2(\Omega)} \leqslant \sqrt{2}\, \|\nabla \boldsymbol{u}'\|_{L^2(\Omega)^3},$$

which proves that the bilinear form

$$\boldsymbol{u},\, \boldsymbol{v} \in \boldsymbol{V}_0, \quad (\boldsymbol{u},\, \boldsymbol{v}) \mapsto \int_\Omega \nabla \boldsymbol{u}' : \nabla \boldsymbol{v}'\, dx,$$

is coercive on $\boldsymbol{V}_0$ for the norm $(\|\boldsymbol{u}'\|^2_{H^1(\Omega)^2} + \|u_3\|^2_{H(\partial_{x_3},\, \Omega)})^{1/2}$. Thus, there is a unique function $\boldsymbol{u} \in \boldsymbol{V}_0$ such that $\boldsymbol{u}'$ satisfies relation (3.15). Hence, one has given another proof of Theorem 3.4.

Remark 3.6. Notice that the bilinear form in the variational formulation (3.15) does not act on the unknown u_3. Here, u_3 is implicitly contained in the space of test functions. This remark suggests to chose another space of test functions than $\boldsymbol{V}_0$, only related to $\boldsymbol{u}'$, hence to give an equivalent variational formulation to (3.15). Let us introduce the space

$$\mathcal{M} = \left\{ u \in L^2_0(\Omega) \,/\, \exists v \in H_0(\partial_{x_3},\, \Omega) \text{ such that } u = \frac{\partial v}{\partial x_3} \right\}.$$

The space $\mathcal{M}$ is a closed subspace of $L^2(\Omega)$, and thanks to the generalized Poincaré's inequality (3.7), there is a unique $v \in H_0(\partial_{x_3},\, \Omega)$ satisfying $u = \partial_{x_3} v$ in $\mathcal{M}$. Then, we consider the subspace of $H^1_0(\Omega)^2$ defined by

$$\boldsymbol{V}_{\mathcal{M}} = \left\{ \boldsymbol{u}' \in H^1_0(\Omega)^2 \,/\, \nabla' \cdot \boldsymbol{u}' \in \mathcal{M} \right\},$$

which is a closed subspace of $H^1_0(\Omega)^2$. Then, we prove that (3.15) is equivalent to

$$\left\{\begin{array}{l}\text{Find } \boldsymbol{u}' \in \boldsymbol{V}_{\mathcal{M}} \text{ such that:}\\ \forall \boldsymbol{v}' \in \boldsymbol{V}_{\mathcal{M}}, \quad \displaystyle\int_\Omega \nabla \boldsymbol{u}' : \nabla \boldsymbol{v}'\, dx = \left\langle \boldsymbol{f}',\, \boldsymbol{v}' \right\rangle_{H^{-1}(\Omega)^2,\, H^1_0(\Omega)^2}.\end{array}\right. \tag{3.16}$$

Indeed (3.16) is a formulation of Problem (3.1) as does prove the following reasoning. Let $(\boldsymbol{u},\, p)$ in the space $\boldsymbol{X}_0 \times (L^2(\Omega)/\mathbb{R})$. By using the arguments of the proof of the uniqueness of the pair $(\boldsymbol{u},\, p)$ (see Lemma 3.2), one proves that $\boldsymbol{u}'$ satisfies the variational formulation (3.16). Conversely, let $\boldsymbol{u}' \in \boldsymbol{V}_{\mathcal{M}}$ solution to (3.15). Firstly, by definition of the space $\boldsymbol{V}_{\mathcal{M}}$, there is $u_3 \in H_0(\partial_{x_3},\, \Omega)$ such that

$$\nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega, \quad u_3 n_3 = 0 \text{ on } \Gamma.$$

Secondly, one proves the existence of a p such that $-\Delta \boldsymbol{u}' + \nabla' p = \boldsymbol{f}'$ in Ω, by noticing that the distribution $\boldsymbol{T} = (\Delta \boldsymbol{u}' + \boldsymbol{f}',\, 0)$ is exactly in the statement of Lemma 1.2. Indeed, remark that for any $\boldsymbol{\varphi} \in \boldsymbol{V}$, one has $\boldsymbol{\varphi}' \in \boldsymbol{V}_{\mathcal{M}}$. Hence, one gets for any $\boldsymbol{\varphi} \in \boldsymbol{V}$:

$$\langle \boldsymbol{T},\, \boldsymbol{v} \rangle_{H^{-1}(\Omega)^3, H^1_0(\Omega)^3} = \left\langle \Delta \boldsymbol{u}' + \boldsymbol{f}',\, \boldsymbol{v}' \right\rangle_{H^{-1}(\Omega)^2,\, H^1_0(\Omega)^2} = \int_\Omega \nabla \boldsymbol{u}' : \nabla \boldsymbol{v}'\, dx = 0.$$

Remark 3.7. P. Azérad uses in [3] another different approach to establish Theorem 3.4. To begin with, he gives a variational formulation of (3.1) that takes into account the pressure. Note that in (3.15) or (3.16) the pressure disappears. Indeed, by considering the space $\boldsymbol{H}_p = H_0^1(\Omega)^2 \times H_0(\partial_{x_3}, \Omega) \times L_0^2(\Omega)$, the following bilinear form $a(\cdot, \cdot)$ defined for $\boldsymbol{U} = (\boldsymbol{u}, p)$ and $\boldsymbol{V} = (\boldsymbol{v}, q)$ in $\boldsymbol{H}_p$ by:

$$a(\boldsymbol{U}, \boldsymbol{V}) = \int_\Omega \nabla \boldsymbol{U}' : \nabla \boldsymbol{V}' \, dx - \int_\Omega p \, \nabla \cdot \boldsymbol{v} \, dx - \int_\Omega q \, \nabla \cdot \boldsymbol{u} \, dx,$$

he establishes that $\boldsymbol{U} \in \boldsymbol{H}_p$ is a solution to (3.1) if and only if:

$$\forall \boldsymbol{V} \in \boldsymbol{H}_p, \quad a(\boldsymbol{U}, \boldsymbol{V}) = \langle \boldsymbol{F}, \boldsymbol{V} \rangle_{\boldsymbol{H}_p', \boldsymbol{H}_p} . \tag{3.17}$$

Here, $\boldsymbol{F} = (\boldsymbol{F}', 0, 0)$. The weak formulation (3.17) traduces a mixed variational principle, since he proved in Theorem 3 page 27 of [3], that $\boldsymbol{U} \in \boldsymbol{H}_p$ satisfies (3.1) if and only if $\boldsymbol{U}$ is a sell point of of the following functional:

$$\mathcal{L}(\boldsymbol{U}) = \frac{1}{2} \|\nabla \boldsymbol{u}'\|^2_{L^2(\Omega)} - \int_\Omega p \, \nabla \cdot \boldsymbol{u} \, dx - \langle \boldsymbol{F}, \boldsymbol{U} \rangle_{\boldsymbol{H}_p', \boldsymbol{H}_p} .$$

The pressure p (resp u_3) can, therefore, be interpreted as the Lagrange multiplier related to the constraint $-\nabla \cdot \boldsymbol{u} = 0$ (resp $\partial_{x_3} p = 0$). Then, by establishing the following inf-sup condition:

$$\sup_{\boldsymbol{V} \in \boldsymbol{H}_p - \{0\}} \frac{a(\boldsymbol{U}, \boldsymbol{V})}{\|\boldsymbol{V}\|_{\boldsymbol{H}_p}} \geqslant C \, \|\boldsymbol{U}\|_{\boldsymbol{H}_p} ,$$

he solves (3.17) thanks to the generalized version of Lax-Milgram's Theorem in the non coercive case, which he reviews in Theorem 1 page 22.

Remark 3.8. Let us compare the above asymptotic analysis of (3.9) with the one we can find in [7]. In the note, the authors consider a mixed variational formulation, and they obtain the estimates on $\boldsymbol{u}_\varepsilon$ as we did. Then, to get the estimates on the pressure p_ε, they invoke the following inf-sup condition:

$$\sup_{\boldsymbol{v} \in H_0^1(\Omega)^3 - \{0\}} \frac{\int_\Omega p_\varepsilon \, \nabla \cdot \boldsymbol{v} \, dx}{\|\boldsymbol{v}\|_{H^1(\Omega)^3}} \geqslant C \, \|p_\varepsilon\|_{L^2(\Omega)} ,$$

where $C > 0$ is a constant depending only on Ω, which is in fact equivalent to the first isomorphism of (1.8) (see [28]), the one we actually use. Then, to go further into the work of [7], let us mention that the authors give a mixed variational formulation of the limit problem (3.1), where the vertical velocity disappears, in the same spirit than the one we give above, see (3.16).

3.4 Equivalence with (2.15) and applications

In the previous paragraph, we have seen a first approach of resolution of Problem (3.1), based pon a limit process. As we have mentioned in the introduction of Part I, it is possible to reduce Problem (3.1) to Problem (2.15), in an equivalent way, modulo relation (14). But, far from now, we have not brought any justifications. This is why, we propose at first place to give the mathematical explanation of such a result. We give, therefore, the second uniqueness and existence proof, announced in paragraph 3.1. In a second place, we will use the same process to reduce and solve a generalized version of Problem (3.1) (see (3.23)).

Let u be a function defined in Ω. Reducing the equations (3.1) to (2.15) requires to study the following operators:

$$
\begin{aligned}
x \in \Omega, \quad Fu(x) &= \int_{x_3}^{0} u(x',\, \xi)\, d\xi, \\
x \in \Omega, \quad Gu(x) &= \int_{-h(x')}^{x_3} u(x',\, \xi)\, d\xi.
\end{aligned} \tag{3.18}
$$

In the sequel, we will essentially focus on the operator F. The operator G is introduced only for technical reasons. We start off with giving some continuity properties of the operator F .

Proposition 3.9. *The operator F is linear and continuous from $L^2(\Omega)$ into $L^2(\Omega)$ and the operator G is the adjoint operator to F. Next,*

$$
\forall u \in L^2(\Omega), \quad Fu \in H(\partial_{x_3},\, \Omega) \text{ and } \frac{\partial}{\partial x_3}(Fu) = -u \text{ in } \Omega. \tag{3.19}
$$

As a consequence, there is a constant $C > 0$ depending at most on Ω such that

$$
\forall u \in L^2(\Omega), \quad \|u\|_{L^2(\Omega)} \leqslant \|Fu\|_{H(\partial_{x_3},\, \Omega)} \leqslant C\, \|u\|_{L^2(\Omega)}\,, \tag{3.20}
$$

and F is continuous from $L^2(\Omega)$ into $H(\partial_{x_3},\, \Omega)$.

Proof. Let $u \in L^2(\Omega)$. Thanks to the theorem of Fubini and since for almost all $x \in \Omega$

$$
\begin{aligned}
|Fu(x)|^2 &\leqslant |x_3| \int_{x_3}^{0} |u(x',\, x_3)|^2\, dx_3 \\
&\leqslant h(x') \int_{-h(x')}^{0} |u(x',\, x_3)|^2\, dx_3,
\end{aligned}
$$

we deduce that $Fu \in L^2(\Omega)$ and $\|Fu\|_{L^2(\Omega)} \leqslant \|h\|_\infty\, \|u\|_{L^2(\Omega)}$. Hence F is linear and continuous from $L^2(\Omega)$ into $L^2(\Omega)$.

Then, G is the adjoint to F for the scalar product of $L^2(\Omega)$ since, thanks to Fubini's theorem, one has for any u and v in $L^2(\Omega)$

$$
\begin{aligned}
\int_\Omega v\, Fu\, dx &= \int_\omega \left[\int_{-h(x')}^{0} \int_{x_3}^{0} u(x',\, \xi)\, v(x',\, x_3)\, d\xi\, dx_3 \right] dx' \\
&= \int_\omega \int_{-h(x')}^{0} u(x',\, \xi) \left[\int_{-h(x')}^{\xi} v(x',\, x_3)\, dx_3 \right] d\xi\, dx' \\
&= \int_\Omega u\, Gv\, dx.
\end{aligned}
$$

Next, one has for any $\varphi \in \mathcal{D}(\Omega)$

$$
\int_\Omega \frac{\partial \varphi}{\partial x_3}\, Fu\, dx = \int_\Omega u\, G(\frac{\partial \varphi}{\partial x_3})\, dx = \int_\Omega u\varphi\, dx.
$$

Hence (3.19) holds in $\mathcal{D}'(\Omega)$ and $\partial_{x_3}(Fu) \in L^2(\Omega)$. As a consequence, $Fu \in H(\partial_{x_3}, \Omega)$. As $u = -\partial_{x_3}(Fu)$, one has

$$\|u\|^2_{L^2(\Omega)} = \left\| \frac{\partial(Fu)}{\partial x_3} \right\|^2_{L^2(\Omega)} \leqslant \|Fu\|^2_{H(\partial_{x_3}, \Omega)},$$

and since F is continuous from $L^2(\Omega)$ into $L^2(\Omega)$, one has

$$\|Fu\|^2_{H(\partial_{x_3}, \Omega)} = \|u\|^2_{L^2(\Omega)} + \|Fu\|^2_{L^2(\Omega)} \leqslant C\, \|u\|^2_{L^2(\Omega)}.$$

Thus we establish (3.20). □

As a consequence of Proposition 3.9 and Proposition 3.1, one has $(Fu)n_3$ in $H^{-1/2}(\Gamma)$ for any $u \in L^2(\Omega)$. In the sequel, we want to characterize the trace $(Fu)n_3$, and before, we require the following technical lemma.

Proposition 3.10. *Let u in $H^1(\Omega)$, and set $\mathcal{G} = u|_\Gamma$. Then:*

$$\begin{aligned} &i. \quad M(\frac{\partial u}{\partial x_3}) = \mathcal{G}_S - \mathcal{G}_B \ \textit{in}\ \omega. \\ &ii. \quad F(\frac{\partial u}{\partial x_3}) = \widetilde{\mathcal{G}_S} - u \ \textit{in}\ \Omega. \\ &iii. \quad G(\frac{\partial u}{\partial x_3}) = u - \widetilde{\mathcal{G}_B} \ \textit{in}\ \Omega. \end{aligned}$$

Proof. Proposition 1.1 implies that $\mathcal{G}_S$ and $\mathcal{G}_B$ belongs to $L^2(\omega)$. Then, one gets from relation (1.6), that for any $\psi \in \mathcal{D}(\omega)$:

$$\begin{aligned} \int_\omega M(\frac{\partial u}{\partial x_3})\, \psi\, dx' = \int_\Omega \widetilde{\psi}\, \frac{\partial u}{\partial x_3}\, dx &= \int_\Gamma \widetilde{\psi}\, \mathcal{G}\, n_3\, d\sigma \\ &= \int_\omega \psi\, \mathcal{G}_S\, dx' - \int_\omega \psi\, \mathcal{G}_B\, dx'. \\ &= \int_\omega (\mathcal{G}_S - \mathcal{G}_B)\, \psi\, dx', \end{aligned}$$

and Claim i is proved. Let us deal with Claim ii. By Remark 2.1, the function $\widetilde{\mathcal{G}_S} - u$ belongs to $L^2(\Omega)$. Next, Proposition 3.9 and (1.6) again yield, for any $\varphi \in \mathcal{D}(\Omega)$:

$$\begin{aligned} \int_\Omega F(\frac{\partial u}{\partial x_3})\, \varphi\, dx &= \int_\Omega \frac{\partial u}{\partial x_3}\, G\varphi\, dx \\ &= -\int_\Omega u\varphi\, dx + \int_\omega \mathcal{G}_S (G\varphi)_S\, dx' - \int_\omega \mathcal{G}_B (G\varphi)_B\, dx. \end{aligned}$$

Since $(G\varphi)_S = M\varphi$ and $(G\varphi)_B = 0$ in ω, we deduce that

$$\int_\Omega F(\frac{\partial u}{\partial x_3})\, \varphi\, dx = -\int_\Omega u\varphi\, dx + \int_\Omega \widetilde{\mathcal{G}_S}\varphi\, dx,$$

from which one deduces Claim ii. To prove Claim iii, note that

$$Gu = \widetilde{Mu} - Fu \text{ in } \Omega.$$

□

Remark 3.11. We also prove, by adapting the previous reasoning with Green's formula (3.6) that

$$\forall u \in H_0(\partial_{x_3}, \Omega), \quad M(\frac{\partial u}{\partial x_3}) = 0 \text{ in } \omega, \quad F(\frac{\partial u}{\partial x_3}) = -u \text{ in } \Omega. \tag{3.21}$$

Let us give and prove, now, the trace characterization announced few lines above.

Proposition 3.12. *Let* $u \in L^2(\Omega)$*. Then, one has*

$$\begin{aligned} &1. \quad (Fu)\, n_3 = 0 \quad \text{in } H^{-1/2}(\Gamma \backslash \Gamma_B). \\ &2. \quad (Fu)\, n_3 = \widetilde{Mu}\, n_3 \quad \text{in } H^{-1/2}(\Gamma_B). \end{aligned}$$

Proof. Let $u \in L^2(\Omega)$. By Proposition 3.9, one has $Fu \in H(\partial_{x_3}, \Omega)$, and Proposition 3.1 ensures that

$$(Fu)\, n_3 \in H^{-1/2}(\Gamma).$$

Then, Proposition 3.10 gives for any $v \in H^1(\Omega)$:

$$\begin{aligned} \langle (Fu)\, n_3, v\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} &= \int_\Omega \frac{\partial v}{\partial x_3} Fu\, dx - \int_\Omega uv\, dx \\ &= \int_\Omega u\, G(\frac{\partial v}{\partial x_3})\, dx - \int_\Omega uv\, dx \\ &= \int_\Omega u\, (v - \widetilde{(v|_\Gamma)_B})\, dx - \int_\Omega uv\, dx. \\ &= -\int_\omega (v|_\Gamma)_B\, Mu\, dx'. \end{aligned}$$

Therefore we get

$$\forall v \in H^1(\Omega), \quad \langle (Fu)\, n_3, v\rangle_{H^{-1/2}(\Gamma),\, H^{1/2}(\Gamma)} = -\int_\omega (v|_\Gamma)_B\, Mu\, dx'.$$

As a consequence, one deduces that

$$\forall v \in H^1(\Omega), \text{ supp } v|_\Gamma \subset \Gamma_S, \quad \langle (Fu)\, n_3, v\rangle_{H^{-1/2}(\Gamma_S), H^{1/2}_{00}(\Gamma)} = 0,$$

and Claim 1 is proved. Let us focus on Claim 2. Since $\partial_{x_3} \widetilde{Mu} = 0$ in Ω, the function $\widetilde{Mu}$ belongs to $H(\partial_{x_3}, \Omega)$. Hence, one has

$$\widetilde{Mu}\, n_3 \in H^{-1/2}(\Gamma).$$

Moreover, from Green's formula (3.4), one gets for any $v \in H^1(\Omega)$

$$\begin{aligned} \left\langle \widetilde{Mu}\, n_3, v \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} &= \int_\Omega \widetilde{Mu}\, \frac{\partial v}{\partial x_3}\, dx \\ &= \int_\omega Mu\, M(\frac{\partial v}{\partial x_3})\, dx'. \end{aligned}$$

Then, one uses Proposition 3.10 to deduce

$$\begin{aligned} \left\langle \widetilde{Mu}\, n_3, v \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} &= -\int_\omega Mu\, (v|_\Gamma)_B\, dx' + \int_\omega Mu\, (v|_\Gamma)_S\, dx' \\ &= \langle (Fu)\, n_3, v\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} + \int_\omega Mu\, (v|_\Gamma)_S\, dx'. \end{aligned}$$

Therefore, one has for any $w \in H^1(\Omega)$ with supp $v|_\Gamma \subset \Gamma_B$

$$\langle (Fu)\, n_3, v \rangle_{H^{-1/2}(\Gamma_B), H^{1/2}_{00}(\Gamma_B)} = \left\langle \widetilde{Mu}\, n_3, v \right\rangle_{H^{-1/2}(\Gamma_B), H^{1/2}(\Gamma_{00})},$$

which proves Claim *ii*. □

We complete the study of the operator F with the following statement, which is an immediate consequence of Proposition 3.12. This last result establishes a link between the operators M (see (2.4)) and the operator F, which is the key of the equivalence between Problem (3.1) and Problem (2.15).

Corollary 3.13. *Let $u \in L^2(\Omega)$. Then, the following assertions are equivalent:*

1. $Mu = 0$ *in* $L^2_{1/\sqrt{h}}(\omega)$.
2. $(Fu)n_3 = 0$ *in* $H^{-1/2}(\Gamma)$.

From now on, we are in position to establish the equivalence between Problem (3.1) and Problem (2.15). Firstly, let us prove the following lemma.

Lemma 3.14. *Let $\boldsymbol{u} \in H^1_0(\Omega)^2 \times H(\partial_{x_3}, \Omega)$. Then the following assertions are equivalent*

1. $\nabla \cdot \boldsymbol{u} = 0$ *in* Ω, $\quad u_3 n_3 = 0$ *in* $H^{-1/2}(\Gamma)$.
2. $\nabla' \cdot (M\boldsymbol{u}') = 0$ *in* ω, $\quad u_3 = F(\nabla' \cdot \boldsymbol{u}')$ *in* Ω.

Proof. Assume claim 1. Then relation (3.21) give

$$M(\nabla' \cdot \boldsymbol{u}') = 0 \quad \text{and} \quad u_3 = F(\nabla' \cdot \boldsymbol{u}').$$

Moreover, thanks to (2.8) one has $M(\nabla' \cdot \boldsymbol{u}') = \nabla' \cdot M\boldsymbol{u}'$, which proves claim 2. Conversely, one has by relation (3.19)

$$\frac{\partial u_3}{\partial x_3} = -\nabla' \cdot \boldsymbol{u}' \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega.$$

Besides, and since $M(\nabla' \cdot \boldsymbol{u}') = 0$, Corollary 3.13 ensures that

$$F(\nabla' \cdot \boldsymbol{u}')\, n_3 = 0 \text{ in } H^{-1/2}(\Gamma),$$

hence $u_3 n_3 = 0$ in $H^{-1/2}(\Gamma)$. □

Remark 3.15. Firstly, note that Lemma 3.14 is also true when:

$$\boldsymbol{u}' \in H^1(\Omega)^2, \quad \boldsymbol{u}' = 0 \text{ on } \Gamma_B \cup \Gamma_L,$$

situation that can also be found in the paper [20], see Lemma 2 page 531. Instead of considering a density argument to prove the implication 2 ⇒ 1, as it is the case in this paper, we use the characterization of the trace $(Fu)n_3$ for functions $u \in L^2(\Omega)$, in accordance with Proposition 3.12, and therefore, we go further into the properties of the operator F. This approach appears to us more convenient, since by implication 1 ⇒ 2 we have necessarily: $u_3 = F(\nabla' \cdot \boldsymbol{u}')$. And therefore, the question to characterize the trace $u_3 n_3|_\Gamma$ naturally yields us to consider the more general situation of Proposition 3.12. Moreover, note that we do not need assumption (7), meanwhile it is the case in [20].

Secondly, according to Lemma 3.14 and the fact that p does not depend on x_3, see Proposition 2.8, solving Problem (3.1) reduces to solve Problem (2.15). Then, we get back to p and u_3 thanks to relation (3.22) given below. We state this important result in the following Proposition.

Proposition 3.16. *Let $\boldsymbol{f}' \in H^{-1}(\Omega)^2$. A pair $(\boldsymbol{u}, p) \in \boldsymbol{X}_0 \times (L^2(\Omega)/\mathbb{R})$ is a solution to Problem* (3.1) *if and only if $(\boldsymbol{u}', p_S) \in H_0^1(\Omega)^2 \times (L^2_{\sqrt{h}}(\omega)/\mathbb{R})$ is a solution to Problem* (2.15) *and*

$$x \in \Omega, \quad p(x) = \widetilde{p_S}(x'), \quad u_3(x) = F(\nabla' \cdot \boldsymbol{u}')(x). \tag{3.22}$$

Remark 3.17. As (2.15) is already solved in Proposition 2.12, we deduce from Proposition 3.16 a new proof of Theorem 3.4.

The reduction process that we have set, and that lead us to Proposition 3.16, enables in fact to solve the following more general Problem:

$$\begin{cases} -\Delta \boldsymbol{u}' + \nabla' p = \boldsymbol{f}', \quad \dfrac{\partial p}{\partial x_3} = 0, \quad \nabla \cdot \boldsymbol{u} = \Phi \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad \boldsymbol{u}' = \boldsymbol{g}', \quad u_3 n_3 = 0 \quad \text{on } \Gamma, \end{cases} \tag{3.23}$$

where $\Phi \in L^2(\Omega)$, $\boldsymbol{g}' \in H^{1/2}(\Gamma)^2$ and necessarily satisfying the compatibility condition:

$$\int_\Gamma \boldsymbol{g}' \cdot \boldsymbol{n}' \, d\sigma = \int_\Omega \Phi \, dx. \tag{3.24}$$

Indeed, observe the following lemma where we reduce the conditions:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma, \quad u_3 n_3 = 0 \text{ on } \Gamma, \tag{3.25}$$

to the following ones:

$$\nabla' \cdot M\boldsymbol{u}' = \phi \text{ in } \omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma. \tag{3.26}$$

Lemma 3.18. *Let $\Phi \in L^2(\Omega)$, let $\boldsymbol{g}' \in H^{1/2}_{1/\sqrt{h}}(\Gamma)^2$ satisfying the compatibility condition (3.24). Let us set:*

$$\phi = M\Phi + \boldsymbol{g}'_B \cdot \nabla' h \ \textit{in } \omega, \quad U = -F\Phi \ \textit{in } \Omega. \tag{3.27}$$

1. Then $\phi \in L^2_{1/\sqrt{h}}(\omega)$ and satisfies with $\boldsymbol{g}'$ the compatibility condition (2.34).
2. Any $\boldsymbol{u} \in H^1(\Omega)^2 \times H(\partial_{x_3}, \Omega)$ satisfies (3.25) *if and only if $\boldsymbol{u}'$ satisfies* (3.26) *and $u_3 = F(\nabla' \cdot \boldsymbol{u}') + U$ in Ω.*

Proof. Claim 1. By (7), one has $M\Phi \in L^2_{1/\sqrt{h}}(\omega)$ (see Proposition 2.2), and relation (2.29) ensures that $\boldsymbol{g}'_B \cdot \nabla' h \in L^2_{1/\sqrt{h}}(\omega)$. Thus $\phi \in L^2_{1/\sqrt{h}}(\omega)$. Next, one has by relation (1.3)

$$\begin{aligned} \int_\omega \phi \, dx' &= \int_\omega M\Phi \, dx + \int_\omega \boldsymbol{g}'_B \cdot \nabla' h \, dx' \\ &= \int_\Omega \Phi \, dx - \int_{\Gamma_B} \boldsymbol{g}' \cdot \boldsymbol{n}' \, d\sigma. \\ &= \int_\Gamma \boldsymbol{g}' \cdot \boldsymbol{n}' \, d\sigma - \int_{\Gamma_B} \boldsymbol{g}' \cdot \boldsymbol{n}' \, d\sigma = \int_{\Gamma_L} \boldsymbol{g}' \cdot \boldsymbol{n}' \, d\sigma, \end{aligned}$$

as (3.24) holds and since $\boldsymbol{n}' = 0$ on Γ_S.

Claim 2. As $\Phi \in L^2(\Omega)$, $\boldsymbol{g}' \in H^{1/2}_{1/\sqrt{h}}(\Gamma)^2$ and satisfy the compatibility condition (3.24), $\boldsymbol{g} = (\boldsymbol{g}', 0) \in H^{1/2}(\Gamma)^3$ (see (2.27)) and satisfies with Φ relation (1.10). Thus, relation (1.11) provides a $\boldsymbol{v} \in H^1(\Omega)^2 \times H^1_0(\Omega)$ such that

$$\nabla \cdot \boldsymbol{v} = \Phi \text{ in } \Omega, \quad \boldsymbol{v}' = \boldsymbol{g}' \text{ on } \Gamma.$$

Then, the function $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ is exactly in the statement of Lemma 3.14. As relation (3.25) is equivalent to:

$$\nabla \cdot \boldsymbol{w} = 0 \text{ in } \Omega, \quad w_3 n_3 = 0 \text{ in } H^{-1/2}(\Gamma),$$

one deduces that it is also equivalent to:

$$\nabla' \cdot (M\boldsymbol{w}') = 0 \text{ in } \omega, \quad w_3 = F(\nabla' \cdot \boldsymbol{w}') \text{ in } \Omega. \tag{3.28}$$

The first part of (3.28) becomes:

$$\begin{aligned} \nabla' \cdot M\boldsymbol{u}' = \nabla' \cdot Mv' &= M(\nabla' \cdot \boldsymbol{v}') + \boldsymbol{g}'_B \cdot \nabla' h \\ &= M(\Phi - \frac{\partial v_3}{\partial x_3}) + \boldsymbol{g}'_B \cdot \nabla' h \\ &= M(\Phi) + \boldsymbol{g}'_B \cdot \nabla' h = \phi, \end{aligned}$$

thanks to relations (2.7) and (3.21). Then, the second part of (3.28) becomes:

$$\begin{aligned} u_3 = F(\nabla' \cdot \boldsymbol{w}') + v_3 &= F(\nabla' \cdot \boldsymbol{u}') - F(\nabla' \cdot \boldsymbol{v}') + v_3 \\ &= F(\nabla' \cdot \boldsymbol{u}') - F(\Phi - \frac{\partial v_3}{\partial x_3}) + v_3 \\ &= F(\nabla' \cdot \boldsymbol{u}') - F\Phi, \end{aligned}$$

thanks to (3.21). Thus, we have proved the equivalence. □

Therefore, we are in position to give an existence and uniqueness proof of a weak solution to Problem (3.23). For convenience, let us denote by $\boldsymbol{Y}_0$ the space:

$$\boldsymbol{Y}_0 = H^1(\Omega)^2 \times H_0(\partial_{x_3}, \Omega). \tag{3.29}$$

Theorem 3.19. *Let $\boldsymbol{f}' \in H^{-1}(\Omega)^2$, let $\Phi \in L^2(\Omega)$ and $\boldsymbol{g}' \in H^{1/2}_{1/\sqrt{h}}(\Gamma)^2$ satisfying the compatibility condition* (3.24)*. Then, Problem* (3.23) *has a unique solution $(\boldsymbol{u}, p)$ in the space $\boldsymbol{Y}_0 \times (L^2(\Omega)/\mathbb{R})$, and there is a constant $C > 0$ such that:*

$$\|\boldsymbol{u}'\|_{H^1(\Omega)^2} + \|u_3\|_{H(\partial_{x_3},\Omega)} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leqslant C\left(\|\boldsymbol{f}'\|_{H^{-1}(\Omega)^2} + \|\Phi\|_{L^2(\Omega)} + \|\boldsymbol{g}'\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)^2}\right) \tag{3.30}$$

Proof. As above, Lemma 3.18 enables to reduce the equations of (3.23) to (2.21), for the datum ϕ defined by (3.27). Therefore, Problem (3.23) is solved by virtue of Theorem 2.17. □

Remark 3.20. For any $\varepsilon > 0$, let us consider the following system:

$$\begin{cases} -\Delta \boldsymbol{u}'_\varepsilon + \nabla' p_\varepsilon = \boldsymbol{f}', \quad -\varepsilon^2 \Delta u^\varepsilon_3 + \dfrac{\partial p_\varepsilon}{\partial x_3} = 0, \quad \nabla \cdot \boldsymbol{u}_\varepsilon = \Phi \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \boldsymbol{u}'_\varepsilon = \boldsymbol{g}', \quad u^\varepsilon_3 n_3 = 0 \quad \text{on } \Gamma. \end{cases} \tag{3.31}$$

Another proof of Theorem 3.19 consists in giving an asymptotic analysis of Problem (3.31), as we did in Section 4.3. Firstly, by relation (1.11), there is $\boldsymbol{v} \in H^1(\Omega)^2 \times H_0^1(\Omega)$ such that

$$\nabla \cdot \boldsymbol{v} = \Phi \text{ in } \Omega, \quad \boldsymbol{v}' = \boldsymbol{g}' \text{ on } \Gamma.$$

Then, by translation, the new unknown $\boldsymbol{w}_\varepsilon = \boldsymbol{u}_\varepsilon - \boldsymbol{v}$ satisfies (3.9) with $\boldsymbol{f}' - \Delta \boldsymbol{v}'$ in the right-hand side of the first equation, and with $-\varepsilon^2 \Delta v_3$ in the right-hand side of the second equation, instead of 0. Then, with similar techniques, we prove that there is a constant $C > 0$, independent on ε, such that

$$\|\boldsymbol{w}'_\varepsilon\|_{H^1(\Omega)^2} + \varepsilon \|w_3^\varepsilon\|_{H^1(\Omega)} + \|w_3^\varepsilon\|_{H(\partial_{x_3},\Omega)} + \|p_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leqslant C \left\|\boldsymbol{f}' - \Delta \boldsymbol{v}'\right\|_{H^{-1}(\Omega)^2}.$$

As a consequence, and since $\varepsilon^2 \Delta v_3$ goes to 0 as ε goes 0, a subsequence of $(\boldsymbol{w}_\varepsilon, p_\varepsilon)_\varepsilon$ converges weakly to $\boldsymbol{w} \in H_0^1(\Omega)^3$ and $p \in L^2(\Omega)/\mathbb{R}$, which are in fact the unique solution to

$$\begin{cases} -\Delta \boldsymbol{w}' + \nabla' p = \boldsymbol{f}' - \Delta \boldsymbol{v}', & \dfrac{\partial p_\varepsilon}{\partial x_3} = 0, \quad \nabla \cdot \boldsymbol{w} = 0 \quad \textit{in } \Omega, \\ & \boldsymbol{w}' = \boldsymbol{g}', \quad w_3 n_3 = 0 \quad \textit{on } \Gamma. \end{cases}$$

Then, $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{v}$ is solution with p to Problem (3.31). By passing to the limits in the above inequality, estimates (3.30) holds.

We finish by giving a summary of the main points of the work we have done in this chapter. The initial purpose was to give an existence and uniqueness proof of a weak solution to (3.1). The first proof we have given was based on a limit process, where (3.1) appears as the limit of the anisotropic Stokes system (3.9). Then, in order to give a second proof of this result, we have given the mathematical justification that enables to reduce (3.1) to (2.15), and this last problem was solved in the previous chapter, see Proposition 2.12. To finish, we have reduced the more general conditions (3.25) to (3.26). This enabled us, in accordance with Theorem 2.17, to solve the general problem (3.23).

Far from now, we have not been interested yet in the case of inhomogeneous conditions for the unknown u_3. We are lead up to wonder if such a reduction process is still applicable, and hence if it is reasonable, for example, to reduce the following conditions:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma, \quad u_3 n_3 = g_3 \text{ on } \Gamma,$$

and how to express in this case u_3 with respect to Φ, $\boldsymbol{g}'$ and g_3. We propose to answer this question in the next chapter.

Chapter 4

Weak solutions to (SH) with a non homogeneous condition on u_3

4.1 Introduction

In the previous chapter we have given an existence and uniqueness proof of a weak solution to the Stokes hydrostatic approximation, always by considering a homogeneous condition on the unknown u_3, see Theorem 3.19. Our proof was based on the reduction of the conditions (3.25) to (3.26), see Lemma 3.18. Let us study now what happens if we a consider a non homogeneous boundary condition over u_3. Given $\boldsymbol{f}' : \Omega \to \mathbb{R}^2$, $\Phi : \Omega \to \mathbb{R}$ and $\boldsymbol{g} : \Gamma \to \mathbb{R}^3$, we pose naturally the following problem:

$$\begin{cases} -\Delta \boldsymbol{u}' + \nabla' p = \boldsymbol{f}', \quad \dfrac{\partial p}{\partial x_3} = 0, \quad \nabla \cdot \boldsymbol{u} = \Phi \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad \boldsymbol{u}' = \boldsymbol{g}', \quad u_3 n_3 = g_3 \quad \text{on } \Gamma. \end{cases} \tag{4.1}$$

Intuitively, and in accordance with Proposition 3.1, we would like to find $\boldsymbol{u} \in H^1(\Omega)^2 \times H(\partial_{x_3}, \Omega)$ and $p \in L^2(\Omega)/\mathbb{R}$ solution to this problem for the data:

$$\boldsymbol{f}' \in H^{1/2}(\Gamma)^2, \quad \Phi \in L^2(\Omega), \quad \boldsymbol{g}' \in H^{1/2}(\Gamma)^2, \quad g_3 \in H^{-1/2}(\Gamma),$$

necessarily fulfilling the compatibility condition:

$$\int_\Omega \Phi \, dx = \int_\Gamma \boldsymbol{g}' \cdot \boldsymbol{n}' \, d\sigma + \langle g_3, 1 \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \, .$$

In a view to solve Problem (4.1) in the above situation, let us see if it is possible to reduce the more general conditions:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma, \quad u_3 n_3 = g_3 \text{ on } \Gamma, \tag{4.2}$$

to the following ones:

$$\nabla' \cdot M\boldsymbol{u}' = \phi, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma, \tag{4.3}$$

where ϕ is uniquely determined by Φ and $\boldsymbol{g}$ (as well as the unknown u_3). If we succeed, then we will solve Problem (4.1) by virtue of Theorem 2.17. Let us apply the operator M to the equality $\nabla \cdot \boldsymbol{u} = \Phi$ in order to identify the function ϕ in (4.3). We will then apply the operator F to the same equation to see how necessarily u_3 depends on $\boldsymbol{u}'$, Φ and $\boldsymbol{g}'$. We have by (2.7):

$$M(\nabla \cdot \boldsymbol{u}) = M(\nabla' \cdot \boldsymbol{u}') + M(\frac{\partial u_3}{\partial x_3}) = \nabla' \cdot M\boldsymbol{u}' - \nabla' h \cdot \boldsymbol{g}'_B + M(\frac{\partial u_3}{\partial x_3}),$$

from which we deduce, for the moment, that ϕ is necessarily equal to:

$$\phi = M\Phi + \nabla' h \cdot \boldsymbol{g}'_B - M(\frac{\partial u_3}{\partial x_3}) \text{ in } \omega.$$

Next, we would like to adapt Claim i of Proposition 3.10 and write $M(\frac{\partial u_3}{\partial x_3})$ with respect to g_3. Thanks to Green's formula (3.4), one has for any $\psi \in \mathcal{D}(\omega)$:

$$\int_\omega \psi \, M(\frac{\partial u_3}{\partial x_3}) \, dx' = \int_\Omega \widetilde{\psi} \, \frac{\partial u_3}{\partial x_3} \, dx = \left\langle g_3, \, \widetilde{\psi} \right\rangle_{H^{-1/2}(\Gamma), \, H^{1/2}(\Gamma)}, \tag{4.4}$$

and as we see, we are not able to go further on the computations without additional regularity on g_3. Note that the same problem appears when we apply F to the equality $\nabla \cdot \boldsymbol{u} = \Phi$. Indeed, one has

$$F(\nabla \cdot \boldsymbol{u}) = F(\nabla' \cdot \boldsymbol{u}') + F(\frac{\partial u_3}{\partial x_3}),$$

and if we apply Claim ii of Proposition 3.10, we obtain that for any $\varphi \in \mathcal{D}(\Omega)$:

$$\int_\Omega \varphi \, F(\frac{\partial u_3}{\partial x_3}) \, dx' = \int_\Omega \frac{\partial u_3}{\partial x_3} G\varphi \, dx = -\int_\Omega u_3 \, \frac{\partial G\varphi}{\partial x_3} \, dx + \langle g_3, \, G\varphi \rangle_{H^{-1/2}(\Gamma), \, H^{1/2}(\Gamma)} \, . \tag{4.5}$$

Here again, the $H^{-1/2}(\Gamma)$ regularity of g_3 is not sufficient to write $F(\frac{\partial u_3}{\partial x_3})$ with respect to g_3 and u_3.

As a conclusion of this discussion, the reduction process may not be the good approach to solve Problem (4.1) in the above situation, where we consider the optimal $H^{-1/2}(\Gamma)$ regularity for the datum g_3. Our purpose is not to study this optimal situation, but rather to deal with other ones for which the reduction process is successful. This is why an outline of Chapter 4 is as follows.

In Section 4.2, we prove that it is possible to reduce (4.2) to (4.3) with an additional regularity on g_3, compatible with the unknown u_3 we want to find. As a consequence, we give in Theorem 4.1 an existence and uniqueness proof of a weak solution to Problem (4.1).
More precisely, by considering the unknown u_3 in the space $H_F(\partial_{x_3}, \Omega)$ defined here after in (4.14), we prove in Proposition 4.4 that it is possible to define the trace $(u_3 n_3)|_\Gamma$ in $L^2(\Gamma)$.
Then, by extending Proposition 3.10 to any u in $H_F(\partial_{x_3}, \Omega)$, we are able to go further on the computations (4.4) and (4.5), and write $M(\frac{\partial u_3}{\partial x_3})$ and $F(\frac{\partial u_3}{\partial x_3})$ with respect to $g_3 = (u_3 n_3)|_\Gamma$, see Proposition 4.8.
Finally, and always by considering $u_3 \in H_F(\partial_{x_3}, \Omega)$ and $g_3 \in L^2(\Gamma)$, we are able to reduce (4.2) to (4.3) in Lemma 4.10. The combination of this lemma and Theorem 2.17 constitutes the existence and uniqueness proof announced few lines above.

In Section 4.3, we will consider another problem set in a domain with sidewalls along $\partial\omega$, where we consider a Dirichlet boundary condition on u_3. More precisely, we will give an existence and uniqueness proof of a weak solution to:

$$\begin{cases} -\Delta \boldsymbol{u}' + \nabla' p = \boldsymbol{f}', \quad \dfrac{\partial p}{\partial x_3} = 0, \quad \nabla \cdot \boldsymbol{u} = \Phi & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma, \end{cases} \tag{4.6}$$

for which the following boundary condition is *a priori* not defined:

$$u_3 = g_3 \text{ on } \Gamma, \tag{4.7}$$

Firstly, we will prove in Proposition 4.14 that it is possible to define, for any $u \in H(\partial_{x_3}, \Omega)$, the trace $u|_\Gamma$ in the weighted space $L^2(\Gamma, |n_3| \, d\sigma)$ defined in (4.24), and hence that (4.7) is meaningful. Then,

we will establish in Proposition 4.17 an essential result concerning the characterization of the trace $(Fu)|_\Gamma$ for functions $u \in L^2(\Omega)$.
As in the previous section, we will extend Proposition 3.10 to any u in $H(\partial_{x_3},\Omega)$, and we will go further on the computations (4.4) and (4.5), by writing $M(\frac{\partial u_3}{\partial x_3})$ and $F(\frac{\partial u_3}{\partial x_3})$ with respect to $g_3 = u_3|_\Gamma$, see Proposition 4.16. We will achieve Section 4.3 with Lemma 4.18, by proving that it is possible to reduce the conditions:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma, \tag{4.8}$$

to the following ones:

$$\nabla' \cdot M\boldsymbol{u}' = \phi, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma, \tag{4.9}$$

when we consider $u_3 \in H(\partial_{x_3},\Omega)$ and $g_3 \in L^2(\Gamma, |n_3|\, d\sigma)$. Finally, the combination of Lemma 4.18 and Theorem 2.17 will constitute the expected existence and uniqueness proof of a weak solution to (4.6).

4.2 A first case of inhomogeneous boundary conditions on u_3

In this section, we give an existence and uniqueness proof of a weak solution to the following problem:

$$\begin{cases} -\Delta \boldsymbol{u}' + \nabla' p = \boldsymbol{f}', \quad \dfrac{\partial p}{\partial x_3} = 0, \quad \nabla \cdot \boldsymbol{u} = \Phi \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\quad \boldsymbol{u}' = \boldsymbol{g}', \quad u_3 n_3 = g_3 \quad \text{on } \Gamma, \end{cases} \tag{4.10}$$

available in a general domain Ω where (7) is not necessary. In order to state this main result, we require the space $H_F(\partial x_3, \Omega)$ defined and studied in the next paragraph. Then, we set:

$$\boldsymbol{X}_F = H^1(\Omega)^2 \times H_F(\partial x_3, \Omega).$$

Theorem 4.1. *Let $\boldsymbol{f}' \in L^2(\Omega)^2$, $\Phi \in L^2(\Omega)$, $\boldsymbol{g}' \in H^{1/2}_{1/\sqrt{h}}(\Gamma)^2$ and $g_3 \in L^2(\Gamma)$ such that $g_3 = 0$ on $\Gamma_S \cup \Gamma_L$. Assume the following compatibility condition:*

$$\int_\Omega \Phi \, dx = \int_\Gamma \boldsymbol{g}' \cdot \boldsymbol{n}' \, d\sigma + \int_{\Gamma_B} g_3 \, d\sigma. \tag{4.11}$$

Then, there is a unique pair $(\boldsymbol{u}, p) \in \boldsymbol{X}_F \times (L^2(\Omega)/\mathbb{R})$ solution to Problem (4.10) and satisfying the estimate,

$$\begin{aligned} \|\boldsymbol{u}'\|_{H^1(\Omega)^2} + \|u_3\|_{H(\partial_{x_3},\Omega)} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leqslant C(\|\boldsymbol{f}'\|_{L^2(\Omega)^2} + \|\Phi\|_{L^2(\Omega)} \\ + \|\boldsymbol{g}'\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)^2} + \|g_3\|_{L^2(\Gamma)}), \end{aligned}$$

where $C > 0$ is a constant depending only on Ω.

The sequel is dedicated to the proof of Theorem 4.1. The one we give is essentially based on reducing the following conditions:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma, \quad u_3 n_3 = g_3 \text{ on } \Gamma, \tag{4.12}$$

in the situation where u_3 belongs to the new space $H_F(\partial_{x_3},\Omega)$ and where g_3 belongs to $L^2(\Gamma)$. As we prove it in the sequel, the space $H_F(\partial_{x_3},\Omega)$ introduced in (4.14), constitutes a sufficient functional framework to go further on the computations (4.4) and (4.5), see Proposition 4.8, and enables therefore to reduce (4.12) to the following conditions:

$$\nabla' \cdot M\boldsymbol{u}' = \phi, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma. \tag{4.13}$$

We refer here to Lemma 4.10. Next, since the unknown p of (4.10) does not depend on x_3, we deduce from Lemma 4.10 that solving (4.10) reduces to solve (2.21), for which we know from Theorem 2.17 the existence and uniqueness of a weak solution.

Let us introduce the following subspace of $H(\partial_{x_3}, \Omega)$ defined by:

$$H_F(\partial_{x_3}, \Omega) = \left\{ u \in H(\partial_{x_3}, \Omega) \,/\, \exists v \in L^2(\Omega) \text{ such that } u = Fv \text{ in } \Omega \right\}. \tag{4.14}$$

We start with two obvious results on the space $H_F(\partial_{x_3}, \Omega)$. In the first one, we state that $H_F(\partial_{x_3}, \Omega)$ is a Hilbert space for the standard $H(\partial_{x_3}, \Omega)$ norm. In the second one, we give a density property, useful to build a trace operator in Proposition 4.4. We state and prove these results in a row.

Proposition 4.2. *The space $H_F(\partial_{x_3}, \Omega)$ is a Hilbert space for the standard $H(\partial_{x_3}, \Omega)$ norm.*

Proof. Thanks to relation (3.20), F is a linear and continuous operator from $L^2(\Omega)$ into $H(\partial_{x_3}, \Omega)$. Moreover, the inequality

$$\|Fu\|_{H(\partial_{x_3}, \Omega)} \geqslant \|u\|_{L^2(\Omega)},$$

and classical completion arguments lead to the result. □

Lemma 4.3. *The space $F(\mathcal{D}(\Omega))$ is dense in $H_F(\partial_{x_3}, \Omega)$ for the standard $H(\partial_{x_3}, \Omega)$ norm.*

Proof. It is immediate since F is continuous from $L^2(\Omega)$ into $H(\partial_{x_3}, \Omega)$, see relation (3.20), and since $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$. □

Proposition 4.4. *The linear mapping*

$$\gamma_F : u \mapsto un_3|_\Gamma$$

defined on $F(\mathcal{D}(\Omega))$ can be extended in a unique way to a linear and continuous mapping, denoted in the same way, from $H_F(\partial x_3, \Omega)$ into $L^2(\Gamma)$.

Proof. Let $u \in F(\mathcal{D}(\Omega))$. Then, there is $v \in \mathcal{D}(\Omega)$ such that $u = Fv$ in Ω. Moreover, $Fv \in C^\infty(\Omega)$ and supp $(Fv)|_\Gamma \subset \Gamma_B$. As a consequence, one has by relation (1.4) and since $(Fv)_B = Mv$ in ω:

$$\begin{aligned}\int_\Gamma (un_3)^2 \, d\sigma = \int_{\Gamma_B} (Fv)^2 \, n_3^2 \, d\sigma = - \int_\omega (Mv)^2 \, (n_3)_B \, dx' \\ \leqslant \|Mv\|^2_{L^2(\omega)} \\ \leqslant C \, \|v\|^2_{L^2(\Omega)}.\end{aligned}$$

Next, note that v satisfies $v = -\partial_{x_3} u$ in Ω, thanks to relation (3.19). Therefore,

$$\int_\Gamma (un_3)^2 \, d\sigma \leqslant C \, \|u\|^2_{H(\partial_{x_3}, \Omega)}.$$

As a consequence, the mapping γ_F defined on $F[\mathcal{D}(\Omega)]$ is linear and continuous for the $H(\partial_{x_3}, \Omega)$ norm. By Lemma 4.3, we can extend γ_F in a unique way to a linear and continuous mapping, still denoted by γ_F, from $H_F(\partial x_3, \Omega)$ into $L^2(\Gamma)$. □

Remark 4.5. Let $u \in H_F(\partial_{x_3}, \Omega)$. By construction, and in accordance with Proposition 3.12, the trace $un_3|_\Gamma$ satisfies:

$$un_3 = 0 \text{ in } L^2(\Gamma_S \cup \Gamma_L). \tag{4.15}$$

We understand now why it is necessary to impose $g_3 = 0$ on $\Gamma_S \cup \Gamma_L$ in Theorem 4.1.

We give in the following proposition a representation formula of the trace $(un_3)|_\Gamma$, for $u \in H_F(\partial_{x_3}, \Omega)$, very useful in the proof of Lemma 4.10.

Proposition 4.6. *The following relation holds.*

$$\forall u \in H_F(\partial_{x_3}, \Omega), \forall v \in H^1_{\Gamma_B}(\Omega), \quad \int_{\Gamma_B} un_3\, v\, d\sigma = \int_\omega M(\frac{\partial u}{\partial x_3})\,(v|_\Gamma)_B\, dx'. \tag{4.16}$$

Proof. Let $u \in F(\mathcal{D}(\Omega))$ and $v \in H^1_{\Gamma_B}(\Omega)$. Then, there is $w \in L^2(\Omega)$ such that $u = Fw$. Moreover, one has by relation (1.4)

$$\begin{aligned}\int_{\Gamma_B} un_3\, v\, d\sigma = \int_{\Gamma_B} (Fw)\, v\, n_3\, d\sigma &= -\int_\omega Mw\, v_B\, dx', \\ &= \int_\omega M(\frac{\partial u}{\partial x_3})\, v_B\, dx'.\end{aligned}$$

As a consequence, one has proved that (4.16) holds for any $u \in F(\mathcal{D}(\Omega))$, and then, by density, for any $u \in H_F(\partial_{x_3}, \Omega)$, thanks to Lemma 4.3. □

Remark 4.7. From now on, we are in position to justify the compatibility condition (4.11). It comes from the following Stokes formula:

$$\forall \boldsymbol{u} \in \boldsymbol{X}_F, \quad \int_\Omega \nabla \cdot \boldsymbol{u}\, dx = \int_\Gamma \boldsymbol{g}' \cdot \boldsymbol{n}'\, d\sigma + \int_{\Gamma_B} g_3\, d\sigma,$$

which is satisfied thanks to Proposition 4.4 and Relation (4.16).

Next, recall that in the introduction, we have proved that is was not reasonable to consider $u_3 \in H(\partial_{x_3}, \Omega)$ and $g_3 \in H^{-1/2}(\Gamma)$ in (4.2), to reduce the conditions (4.2) to (4.3). Indeed, and according to (4.4) and (4.5), we were not able to express $M(\frac{\partial u_3}{\partial x_3})$ and $F(\frac{\partial u_3}{\partial x_3})$ with respect to $g_3 \in H^{-1/2}(\Gamma)$. In the following proposition, we establish that if $u \in H_F(\partial_{x_3}, \Omega)$ and $g_3 \in L^2(\Gamma)$, then it is possible to go further on the computations we have done in (4.4) and (4.5), and express $M(\frac{\partial u_3}{\partial x_3})$ with respect to g_3 and compute easily $F(\frac{\partial u_3}{\partial x_3})$.

Proposition 4.8. *Let u in $H_F(\partial_{x_3}, \Omega)$ and set $\mathcal{G} = un_3|_\Gamma$. Then, one has:*

$$\begin{aligned} &i) \quad M(\frac{\partial u}{\partial x_3}) = \mathcal{G}_B(1 + |\nabla h|^2)^{1/2} \text{ in } L^2_{1/\sqrt{h}}(\omega). \\ &ii) \quad F(\frac{\partial u}{\partial x_3}) = -u \text{ in } L^2(\Omega).\end{aligned}$$

Proof. Let $u \in H_F(\partial_{x_3}, \Omega)$ and $\psi \in \mathcal{D}(\omega)$. By Proposition 4.4, $\mathcal{G} \in L^2(\Gamma)$, hence $\mathcal{G}_B \in L^2(\omega)$ in accordance with Proposition 1.1. Then, since one has by relation (1.2):

$$\int_{\Gamma_B} \mathcal{G}\, \widetilde{\psi}\, d\sigma = \int_\omega \mathcal{G}_B\, \psi\, (1 + |\nabla h|^2)^{1/2}\; dx',$$

one deduces from (4.16) that for any $\psi \in \mathcal{D}(\omega)$:

$$\int_\omega M(\frac{\partial u}{\partial x_3})\,\psi\,dx' = \int_\omega \mathcal{G}_B(1+|\nabla h|^2)^{1/2}\,\psi\,dx'.$$

Consequently, Claim 1 holds since $\partial_{x_3} u \in L^2(\Omega)$ and thanks to Proposition 2.2. Next, let us establish Claim 2. Let $u \in L^2(\Omega)$ such that $u = Fv$. Therefore, one has by (3.19)

$$F(\frac{\partial u}{\partial x_3}) = -Fv = -u \text{ in } \Omega.$$

$\square$

Remark 4.9. Note that the main argument in Claim 1 of Proposition 4.8 is that when $u_3 \in H_F(\partial_{x_3}, \Omega)$ and $g_3 = (u_3 n_3)|_\Gamma \in L^2(\Gamma)$, the duality pairing in (4.4) is in fact an integral over Γ_B, and hence an integral over ω by relation (1.2). Besides, if we consider $u_3 \in H_F(\partial_{x_3}, \Omega)$ and $g_3 \in L^2(\Gamma)$ in (4.2), we see by completing the reasoning hold in introduction that we necessarily have in (4.3):

$$\phi = M\Phi + \nabla' h \cdot \boldsymbol{g}'_B - (g_3)_B(1+|\nabla h|^2)^{1/2} \text{ in } \omega,$$

and $u_3 = F(\nabla' \cdot u' - \Phi)$ in Ω.

We are now in position to reduce the conditions (4.12) to (4.13). Before observing the following, recall that the space $H^{1/2}_{1/\sqrt{h}}(\Gamma)$ is defined in (2.27).

Lemma 4.10. *Let $\Phi \in L^2(\Omega)$, $\boldsymbol{g}' \in H^{1/2}_{1/\sqrt{h}}(\Gamma)^2$ and $g_3 \in L^2(\Gamma)$ such that $g_3 = 0$ on $\Gamma_S \cup \Gamma_L$. Assume the compatibility condition* (4.11) *and set:*

$$\phi = M\Phi - (g_3)_B(1+|\nabla h|^2)^{1/2} + \boldsymbol{g}'_B \cdot \nabla h \quad \text{in } \omega. \tag{4.17}$$

1. Then $\phi \in L^2_{1/\sqrt{h}}(\omega)$ and satisfies with $\boldsymbol{g}'$ the compatibility condition:

$$\int_\omega \phi\,dx' = \int_{\Gamma_L} \boldsymbol{g}' \cdot \boldsymbol{n}'\,d\sigma. \tag{4.18}$$

2. Any function $\boldsymbol{u} \in \boldsymbol{X}_F$ satisfies (4.12) *if and only if $\boldsymbol{u}' \in H^1(\Omega)^2$ satisfies the reduced conditions* (4.13) *and $u_3 = F(\nabla' \cdot \boldsymbol{u}' - \Phi)$ in Ω.*

Proof. Claim 1. Thanks to Proposition 2.2, one has $M\Phi \in L^2_{1/\sqrt{h}}(\omega)$. Then, Corollary 4.8 gives

$$(g_3)_B(1+|\nabla h|^2)^{1/2} \in L^2_{1/\sqrt{h}}(\omega).$$

To finish, as $\boldsymbol{g}' \in H^{1/2}_{1/\sqrt{h}}(\Gamma)^2$, relation (2.28) ensures that $\boldsymbol{g}'_B \cdot \nabla h \in L^2(\omega)$. As a consequence, one deduces that ϕ belongs to $L^2_{1/\sqrt{h}}(\omega)$. Next, one has by relations (1.6) and (1.3)

$$\begin{aligned}\int_\omega \phi\,dx' &= \int_\omega M\Phi\,dx' - \int_\omega (g_3)_B(1+|\nabla h|^2)^{1/2}\,dx' + \int_\omega \boldsymbol{g}'_B \cdot \nabla h\,dx' \\ &= \int_\Omega \Phi\,dx - \int_\Gamma g_3\,d\sigma - \int_{\Gamma_B} \boldsymbol{g}' \cdot \boldsymbol{n}'\,d\sigma.\end{aligned}$$

As (4.11) holds, one obtains

$$\begin{aligned}\int_\omega \phi\, dx' &= \int_\Gamma \boldsymbol{g}' \cdot \boldsymbol{n}'\, d\sigma + \int_\Gamma g_3\, d\sigma - \int_\Gamma g_3\, d\sigma - \int_{\Gamma_B} \boldsymbol{g}' \cdot \boldsymbol{n}'\, d\sigma \\ &= \int_{\Gamma_S} \boldsymbol{g}' \cdot \boldsymbol{n}'\, d\sigma + \int_{\Gamma_L} \boldsymbol{g}' \cdot \boldsymbol{n}'\, d\sigma.\end{aligned}$$

To finish, since $\boldsymbol{n}'$ vanishes on Γ_S, one deduces that ϕ satisfies the compatibility condition (4.18).

Claim 2. Assume that $\nabla \cdot \boldsymbol{u} = \Phi$ and $u_3 = g_3$ in $L^2(\Gamma)$. Relation (2.7) gives

$$\begin{aligned}\nabla' \cdot M\boldsymbol{u}' &= M(\nabla' \cdot \boldsymbol{u}') - \boldsymbol{g}'_B \cdot \nabla h \\ &= M(\Phi - \frac{\partial u_3}{\partial x_3}) - \boldsymbol{g}'_B \cdot \nabla h \\ &= M\Phi - M(\frac{\partial u_3}{\partial x_3}) - \boldsymbol{g}'_B \cdot \nabla h.\end{aligned}$$

Moreover, one also has

$$F(\nabla' \cdot \boldsymbol{u}') = F\Phi - F(\frac{\partial u_3}{\partial x_3}).$$

By applying Proposition 4.8 in the previous equalities, one obtains

$$\nabla' \cdot M\boldsymbol{u}' = \phi, \quad u_3 = F(\nabla' \cdot \boldsymbol{u}' - \Phi).$$

Conversely, let $\boldsymbol{u}' \in H^1(\Omega)^2$ satisfying (4.13) and let $u_3 = F(\nabla' \cdot \boldsymbol{u}' - \Phi)$. Then, u_3 belongs to $H_F(\partial_{x_3}, \Omega)$ and satisfies, thanks to Proposition 3.9:

$$\frac{\partial u_3}{\partial x_3} = -\nabla' \cdot \boldsymbol{u}' + \Phi,$$

and $\nabla \cdot \boldsymbol{u} = \Phi$ in Ω. Finally, since $g_3 = 0$ on $\Gamma_S \cup \Gamma_L$, let us establish that $u_3 n_3 = g_3$ in $L^2(\Gamma_B)$. Let $\mu \in H^1_{\Gamma_B}(\Omega)$. Then, relation (4.16) give:

$$\int_{\Gamma_B} u_3 n_3\, \mu\, d\sigma = \int_\omega M(-\nabla' \cdot \boldsymbol{u}' + \Phi)\, (\mu|_\Gamma)_B\, d\sigma.$$

Next, by definition of ϕ, one has

$$\begin{aligned}M(\nabla' \cdot \boldsymbol{u}' - \Phi) &= \nabla' \cdot M\boldsymbol{u}' - \nabla h \cdot \boldsymbol{u}'_B - M\Phi \\ &= \phi - \nabla h \cdot \boldsymbol{g}'_B - M\Phi \\ &= -(g_3)_B(1 + |\nabla h|^2)^{1/2}.\end{aligned}$$

Hence, one deduces that

$$\begin{aligned}\int_{\Gamma_B} u_3 n_3\, \mu\, d\sigma &= \int_\omega (g_3)_B(1 + |\nabla h|^2)^{1/2}\, (\mu|_\Gamma)_B\, dx' \\ &= \int_{\Gamma_B} g_3\, \mu\, d\sigma.\end{aligned}$$

As a consequence, one has $u_3 n_3 = g_3$ in $H^{-1/2}(\Gamma_B)$, hence in $L^2(\Gamma_B)$. $\square$

Finally, let us give the proof of Theorem 4.1.

Proof. Thanks to Lemma 4.10, and since p does not depend on x_3 (see Proposition 2.8) we have proved that solving (4.10) reduces to solve (2.21) for the datum ϕ defined in (4.17), and for which we can apply Theorem 2.17. As a consequence, Theorem 4.1 is proved. $\square$

Let us finish this this section with additional comments.

Remark 4.11. Let us consider the subspace

$$L^2_{S,\,B}(\Gamma) := \left\{ g \in L^2(\Gamma) \,/\, g = 0 \text{ on } \Gamma_S \cup \Gamma_B \right\}.$$

As an immediate consequence of Theorem 4.1, we establish that the trace operator $\gamma_F : u \mapsto un_3|_\Gamma$, defined in Proposition 4.4, is onto from $H_F(\partial_{x_3},\,\Omega)$ into $L^2_{S,\,B}(\Gamma)$. Moreover, for any $g \in L^2_{S,\,B}(\Gamma)$ and any $u \in H_F(\partial_{x_3},\,\Omega)$ such that $un_3|_\Gamma = g$ in $L^2(\Gamma)$, one has:

$$\|u\|_{H(\partial_{x_3},\,\Omega)} \leqslant C\,\|g\|_{L^2(\Gamma)}\,.$$

Remark 4.12. Note that Theorem 4.1 give a lift operator for the conditions (4.2). More precisely, let $\Phi \in L^2(\Omega)$, $\boldsymbol{g}' \in H^{1/2}_{1/\sqrt{h}}(\Gamma)^2$ and $g_3 \in L^2(\Gamma)$ such that $g_3 = 0$ on $\Gamma_S \cup \Gamma_L$. Assume the compatibility condition (4.11). Then, there is $\boldsymbol{u}$ in $\boldsymbol{X}_F$ satisfying:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ in } \Gamma, \quad u_3 n_3 = g_3 \text{ in } \Gamma,$$

and a constant $C > 0$ depending at most on Ω such that:

$$\|\boldsymbol{u}'\|_{H^1(\Omega)^2} + \|u_3\|_{H(\partial_{x_3},\,\Omega)} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leqslant C\left(\|\Phi\|_{L^2(\Omega)} + \|\boldsymbol{g}'\|_{H^{1/2}_{1/\sqrt{h}}(\Gamma)^2} + \|g_3\|_{L^2(\Gamma)}\right).$$

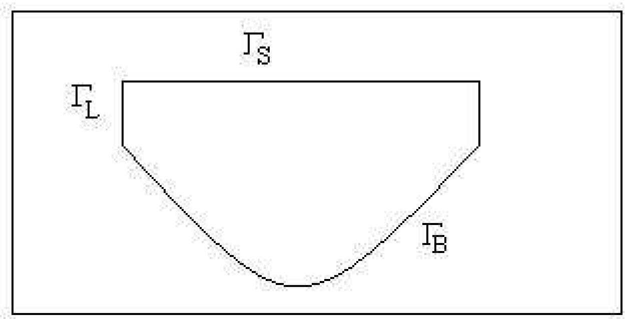
We see now, in the following section, an unexpected situation where we consider in (4.10) the Dirichlet boundary condition $u_3 = g_3$ instead of the condition $u_3 n_3 = g_3$ that we have considered in this section. We build an adapted functional framework that enables to reduce the conditions (4.8) to (4.9) and to solve a new problem. As it will be justified, such a situation is only available when (7) holds.

4.3 An unexpected inhomogeneous boundary conditions on u_3

Throughout this section we assume that Ω has sidewalls, see the representation above. More precisely, we make the assumption[1] (7) on the mapping h. Note, in this case, that the equalities $L^2_{h^\alpha}(\omega) = L^2(\omega)$ and $H^{1/2}_{1/\sqrt{h}}(\Gamma) = H^{1/2}(\Gamma)$ hold (we refer to (2.3) and (2.27) for the definition of these weighted spaces).

[1]Far from now, this assumption was not needed.

Figure 4.1: The domain Ω with sidewalls



In this section, we give an existence and uniqueness proof of a weak solution to the following problem:

$$\begin{cases} -\Delta \boldsymbol{u}' + \nabla' p = \boldsymbol{f}', \quad \dfrac{\partial p}{\partial x_3} = 0, \quad \nabla \cdot \boldsymbol{u} = \Phi \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma, \end{cases} \tag{4.19}$$

available only when (7) holds. As in the previous section, we state the main result at the beginning of the section, and we require, beforehand, the space $L^2(\Gamma, |n_3|\, d\sigma)$ defined in (4.24) and the following notations:

$$\boldsymbol{X} = H^1(\Omega)^2 \times H(\partial_{x_3}, \Omega), \quad \|\boldsymbol{u}\|_{\boldsymbol{X}} = \left(\|\boldsymbol{u}'\|^2_{H^1(\Omega)^2} + \|u_3\|^2_{H(\partial_{x_3}, \Omega)} \right)^{1/2}. \tag{4.20}$$

Theorem 4.13. *Assume* (7). *Let* $\boldsymbol{f}' \in H^{-1}(\Omega)^2$, $\Phi \in L^2(\Omega)$, $\boldsymbol{g}' \in H^{1/2}(\Gamma)^2$ *and* $g_3 \in L^2(\Gamma, |n_3|\, d\sigma)$ *satisfying the compatibility condition:*

$$\int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n}\, d\sigma = \int_\Omega \Phi\, dx. \tag{4.21}$$

Then, there is a unique pair $(\boldsymbol{u}, p) \in \boldsymbol{X} \times (L^2(\Omega)/\mathbb{R})$ *solution to Problem* (4.19) *and satisfying the estimate,*

$$\|\boldsymbol{u}\|_{\boldsymbol{X}} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leqslant C \left(\|\boldsymbol{f}'\|_{H^{-1}(\Omega)^2} + \|\Phi\|_{L^2(\Omega)} + \|\boldsymbol{g}'\|_{H^{1/2}(\Gamma)^2} + \|g_3\|_{L^2(\Gamma, |n_3|\, d\sigma)} \right),$$

where $C > 0$ *is a constant depending only on* Ω.

The sequel is dedicated to the proof of Theorem 4.13. We repeat the main lines of the proof of Theorem 4.1, adapted to the new framework we introduce after. Therefore, the proof we give is essentially based on reducing the following conditions:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma, \tag{4.22}$$

in the situation where u_3 belongs to $H(\partial_{x_3}, \Omega)$ and where g_3 belongs to $L^2(\Gamma, |n_3|\, d\sigma)$. As we prove it in the sequel, and under assumption (7) only, the space $H(\partial_{x_3}, \Omega)$ constitutes a sufficient functional framework to reduce (4.22) to the following conditions:

$$\nabla' \cdot M\boldsymbol{u}' = \phi, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma. \tag{4.23}$$

We refer here to Lemma 4.18. Next, since the unknown p of (4.19) does not depend on x_3, we deduce from Lemma 4.18 that solving (4.19) reduces to solve (2.21), for which we know from Theorem 2.17 the existence and uniqueness of a weak solution.

According to our schedule, we start by giving a meaning to the boundary condition:

$$u_3 = g_3 \text{ on } \Gamma,$$

To this purpose, let us introduce here after the weighted space $L^2(\Gamma, |n_3|\, d\sigma)$. Before, recall that $n_3 = 0$ on Γ_L, $n_3 = 1$ on Γ_S and $n_3 < 0$ on Γ_B. As a consequence, since n_3 has a constant sign on Γ_B, n_3 can be considered as a weight for the surface measure $d\sigma$. Therefore, it is meaningful to introduce the expected Lebesgue space:

$$L^2(\Gamma, |n_3|\, d\sigma) = \left\{ \mu : \Gamma \to \mathbb{R},\ |n_3|\, d\sigma - \text{measurable} \ / \ \mu\, |n_3|^{1/2} \in L^2(\Gamma) \right\}, \tag{4.24}$$

endowed with the hilbertian norm

$$\|\mu\|_{L^2(\Gamma, |n_3| d\sigma)} = \left\| \mu\, |n_3|^{1/2} \right\|_{L^2(\Gamma)}.$$

Then, let us observe in the following proposition that for $u_3 \in H(\partial_{x_3}, \Omega)$ it is possible to define a trace $u_3|_\Gamma$ in $L^2(\Gamma, |n_3|\, d\sigma)$.

Proposition 4.14. *The linear mapping*

$$\gamma : u \mapsto u|_\Gamma,$$

defined on $\mathcal{D}(\overline{\Omega})$, can be extended in a unique way to a linear and continuous mapping, denoted in the same way, from $H(\partial_{x_3}, \Omega)$ into $L^2(\Gamma, |n_3|\, d\sigma)$.

Proof. Let $\theta = \theta(x_3)$ in $\mathcal{D}(\mathbb{R})$ such that

$$\mathbb{I}_{[-\|h\|_{L^\infty(\omega)}, -2\alpha/3[} \leqslant \theta \leqslant \mathbb{I}_{[-\|h\|_{L^\infty(\omega)}, -\alpha/3[},$$

where α is defined in (7). Then, let us set

$$x \in \Omega, \quad \theta^B(x) = \theta(x_3), \quad \theta^S(x) = 1 - \theta(x_3).$$

Therefore $|n_3|\, \theta^S = n_3 \theta^S$ and $|n_3|\, \theta^B = -n_3 \theta^B$ over Γ. Next, let $u \in \mathcal{D}(\overline{\Omega})$. Then

$$\begin{aligned}
\int_\Gamma u^2\ |n_3|\, d\sigma &= \int_\Gamma u^2 (\theta^S + \theta^B)\ |n_3|\, d\sigma \\
&= \int_\Gamma u^2 \theta^S n_3\, d\sigma - \int_\Gamma u^2 \theta^B n_3\, d\sigma \\
&= \int_\Gamma u^2 (\theta^S - \theta^B) n_3\, d\sigma.
\end{aligned}$$

From Green's formula, we deduce

$$\begin{aligned}
\int_\Gamma u^2\ |n_3|\, d\sigma &= 2 \int_\Omega (\theta^S - \theta^B) u \frac{\partial u}{\partial x_3}\, dx + \int_\Omega u^2 \frac{\partial}{\partial x_3} (\theta^S - \theta^B)\, dx \\
&\leqslant C_\theta\ \|u\|^2_{H(\partial_{x_3}, \Omega)},
\end{aligned}$$

where $C_\theta > 0$ is a constant depending on θ. Consequently, the mapping γ is linear and continuous for the norm of $H(\partial_{x_3}, \Omega)$. The result holds by an extension argument. □

In order to derive from Proposition 4.14 the Green's formula given in (4.29), we need to prove that the integral:

$$\int_\Gamma uvn_3\,d\sigma,$$

is meaningful for any u, $v \in H(\partial_{x_3},\,\Omega)$. Moreover, it will be convenient to replace the above surface integral by one defined on ω, as done in relation (1.6). We state therefore the following proposition.

Proposition 4.15. *One has the following properties.*

1. Let $\mu \in L^2(\Gamma, |n_3|\,d\sigma)$. Then

$$i)\quad \mu n_3 \in L^2(\Gamma), \quad \textit{and} \quad \|\mu n_3\|_{L^2(\Gamma)} \leqslant \|\mu\|_{L^2(\Gamma,|n_3|d\sigma)}\,; \tag{4.25}$$

$$ii)\quad \mu_S \in L^2(\omega), \quad \textit{and} \quad \|\mu_S\|_{L^2(\omega)} \leqslant \|\mu\|_{L^2(\Gamma,|n_3|d\sigma)}\,; \tag{4.26}$$

$$iii)\quad \mu_B \in L^2(\omega), \quad \textit{and} \quad \|\mu_B\|_{L^2(\omega)} \leqslant \|\mu\|_{L^2(\Gamma,|n_3|d\sigma)}\,. \tag{4.27}$$

2. Let λ, $\mu \in L^2(\Gamma, |n_3|\,d\sigma)$. Then

$$\lambda\mu n_3 \in L^1(\Gamma), \quad \lambda\mu n_3 \in L^2(\Gamma). \tag{4.28}$$

Proof. Claim 1. Let $\mu \in L^2(\Gamma, |n_3|\,d\sigma)$. Since $|n_3| \leqslant 1$ almost every where on Γ, one has $n_3^2 \leqslant |n_3|$. Then

$$\int_\Gamma (\mu n_3)^2\,d\sigma \leqslant \int_\Gamma \mu^2\ |n_3|\,d\sigma = \|\mu\|_{L^2(\Gamma,|n_3|d\sigma)} < \infty,$$

and $i)$ is proved. Next, one deduces from relations (1.2) and (4.25) that

$$\begin{aligned}\int_\omega \mu_S^2\,dx' = \int_\omega \mu_S^2(n_{3,S})^2\,dx' = \int_{\Gamma_S} \mu^2 n_3^2\,d\sigma &\leqslant \|\mu n_3\|^2_{L^2(\Gamma)} \\ &\leqslant \|\mu\|^2_{L^2(\Gamma,|n_3|d\sigma)} < \infty.\end{aligned}$$

Moreover, one also has

$$\begin{aligned}\int_\omega \mu_B^2\,dx' = -\int_\omega \mu_B^2(n_3)_S(1+|\nabla h|^2)^{1/2} = -\int_{\Gamma_B} \mu^2 n_3\,d\sigma \\ \leqslant \|\mu\|^2_{L^2(\Gamma,|n_3|d\sigma)} < \infty.\end{aligned}$$

As a consequence, one has established $ii)$ and $iii)$. To finish, Claim 2 is a consequence of the Hölder's inequality, and of the fact that Γ is bounded. □

As expected, we derive from the density of $\mathcal{D}(\overline{\Omega})$ in $H(\partial_{x_3},\,\Omega)$ and from relation (4.28) the following Green's formula:

$$\forall u, v \in H(\partial_{x_3},\,\Omega), \quad \int_\Omega u\,\frac{\partial v}{\partial x_3}\,dx = -\int_\Omega v\,\frac{\partial u}{\partial x_3}\,dx + \int_\Gamma uvn_3\,d\sigma. \tag{4.29}$$

As a consequence, the following Stokes formula holds:

$$\forall \boldsymbol{u} \in H^1(\Omega)^2 \times H(\partial_{x_3},\,\Omega), \quad \int_\Omega \nabla\cdot\boldsymbol{u}\,dx = \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{n}\,d\sigma,$$

and we are now able to justify the compatibility condition (4.21). Moreover, we deduce from (4.26), (4.27) and Proposition 4.14, that for any u, $v \in H(\partial_{x_3},\,\Omega)$:

$$\int_\Gamma uvn_3\,d\sigma = \int_\omega (u|_\Gamma)_S(v|_\Gamma)_S\,dx' - \int_\omega (u|_\Gamma)_B(v|_\Gamma)_B\,dx'. \tag{4.30}$$

Next, let us give another consequence of relations (4.26) and (4.27), required in further computations. Given the properties of n_3 and since ω is bounded, we deduce from these relations that for any $\mu \in L^2(\Gamma, |n_3|\, d\sigma)$:

$$\int_\Gamma \mu\ |n_3|\, d\sigma = \int_{\Gamma_S} \mu\, d\sigma - \int_{\Gamma_B} \mu n_3\, d\sigma.$$

Then, (1.4) implies more generally that:

$$\forall \mu \in L^2(\Gamma, |n_3|\, d\sigma), \quad \int_\Gamma \mu\ |n_3|\, d\sigma \quad = \quad \int_\omega \mu_S\, dx' + \int_\omega \mu_B\, dx'. \tag{4.31}$$

In the following result, we observe a situation complementary to the one in Proposition 4.8, where it is possible to go further in the computations (4.4) and (4.5), and to write $M(\frac{\partial u_3}{\partial x_3})$ and $F(\frac{\partial u_3}{\partial x_3})$ with respect to the trace $u_3|_\Gamma$. The above result is, in fact, the generalization of Proposition 3.10 where we considered the case of $H^1(\Omega)$ functions.

Proposition 4.16. *Let u in $H(\partial_{x_3}, \Omega)$, and set $\mathcal{G} = u|_\Gamma$. Then:*

$$\begin{aligned} &i) \quad M(\frac{\partial u}{\partial x_3}) = \mathcal{G}_S - \mathcal{G}_B \ \textit{in } \omega. \\ &ii) \quad F(\frac{\partial u}{\partial x_3}) = \widetilde{\mathcal{G}_S} - u \ \textit{in } \Omega. \end{aligned}$$

Proof. Thanks to assumption (7) and Proposition 4.14, the function $u|_\Gamma$ belongs to $L^2(\Gamma, |n_3|\, d\sigma)$. Then, relations (4.26) and (4.27) and Remark 2.1 gives

$$\mathcal{G}_S - \mathcal{G}_B \in L^2(\omega) \qquad \text{and} \qquad \widetilde{\mathcal{G}_S} - u \in L^2(\Omega).$$

i. We deduce, from (4.29) and (4.30), that for any $\psi \in \mathcal{D}(\omega)$:

$$\begin{aligned} \int_\omega M(\frac{\partial u}{\partial x_3})\, \psi\, dx' = \int_\Omega \widetilde{\psi}\, \frac{\partial u}{\partial x_3}\, dx &= \int_\Gamma \widetilde{\psi}\, \mathcal{G}\, n_3\, d\sigma \\ &= \int_\omega \psi\, \mathcal{G}_S\, dx' - \int_\omega \psi\, \mathcal{G}_B\, dx'. \end{aligned}$$

ii. Proposition 3.9, relation (4.29) and (4.30), prove that for any $\varphi \in \mathcal{D}(\Omega)$:

$$\begin{aligned} \int_\Omega F(\frac{\partial u}{\partial x_3})\, \varphi\, dx &= \int_\Omega \frac{\partial u}{\partial x_3}\, G\varphi\, dx \\ &= -\int_\Omega u\varphi\, dx + \int_\omega \mathcal{G}_S (G\varphi)_S\, dx' - \int_\omega \mathcal{G}_B (G\varphi)_B\, dx. \end{aligned}$$

Since $(G\varphi)_S = M\varphi$ and $(G\varphi)_B = 0$ in ω, we deduce that

$$\int_\Omega F(\frac{\partial u}{\partial x_3})\, \varphi\, dx = -\int_\Omega u\varphi\, dx + \int_\Omega \widetilde{\mathcal{G}_S} \varphi\, dx.$$

□

Before establishing the reduction Lemma 4.18, we give a characterization of the improved trace $(Fu)|_\Gamma$ for $u \in L^2(\Omega)$ in a last proposition.

Proposition 4.17. *Let $u \in L^2(\Omega)$. Then, one has*

$$i) \quad (Fu)|_{\Gamma_S} = 0 \quad \text{in } L^2(\Gamma_S).$$
$$ii) \quad (Fu)|_{\Gamma_B} = \widetilde{Mu} \quad \text{in } L^2(\Gamma_B).$$

Proof. From Proposition 4.14, one deduces that $(Fu)|_\Gamma \in L^2(\Gamma, |n_3|\, d\sigma)$. Moreover, since $\widetilde{Mu} \in H(\partial_{x_3}, \Omega)$, one also has $(\widetilde{Mu})|_\Gamma \in L^2(\Gamma, |n_3|\, d\sigma)$. Hence relation (4.28) gives:

$$(Fu)|_\Gamma n_3 \in L^2(\Gamma), \quad (\widetilde{Mu})|_\Gamma n_3 \in L^2(\Gamma).$$

Thus, one deduces from Proposition 3.12, since $H^{1/2}_{00}(\Gamma_S)$ is dense in $L^2(\Gamma_S)$:

$$((Fu)|_\Gamma n_3)\,|_{\Gamma_S} = (Fu)|_{\Gamma_S} = 0 \quad \text{in } L^2(\Gamma_S),$$

and for the same reasons, one also has

$$((Fu)|_\Gamma n_3)\,|_{\Gamma_B} = \left((\widetilde{Mu})|_\Gamma n_3\right)|_{\Gamma_B} \quad \text{in } L^2(\Gamma_B).$$

To finish, as $n_3 \neq 0$ on Γ_B, one deduces

$$(Fu)|_{\Gamma_B} = \widetilde{Mu} \quad \text{in } L^2(\Gamma_B).$$

□

We are now in position to reduce the conditions (4.22) to (4.23). Observe the following lemma.

Lemma 4.18. *Let $\Phi \in L^2(\Omega)$, $\boldsymbol{g}' \in H^{1/2}(\Gamma)^2$ and $g_3 \in L^2(\Gamma, |n_3|\, d\sigma)$ satisfying (4.21). Let us set*

$$\phi = M\Phi + (g_3)_B - (g_3)_S + \boldsymbol{g}'_B \cdot \nabla' h \quad \text{in } \omega, \tag{4.32}$$

$$U = \widetilde{(g_3)_S} - F\Phi \quad \text{in } \Omega. \tag{4.33}$$

1. *Then $\phi \in L^2(\omega)$ and satisfies with $\boldsymbol{g}'$ the compatibility condition* (2.34).
2. *Any $\boldsymbol{u} \in \boldsymbol{X}$ satisfies* (4.22) *if and only if $\boldsymbol{u}'$ satisfies* (2.30) *and $u_3 = F(\nabla' \cdot \boldsymbol{u}') + U$.*

Proof. Claim 1. We use the arguments of the proof of Lemma 3.18 Claim 1, to prove that $M\Phi + \boldsymbol{g}'_B \cdot \nabla' h$ belongs to $L^2_{1/\sqrt{h}}(\omega)$, hence to $L^2(\omega)$ as (7) holds. Then, relation (4.26) and (4.27) imply that $(g_3)_B - (g_3)_S \in L^2(\omega)$, and $\phi \in L^2(\omega)$. Next, the same arguments combined with relation (4.30) give

$$\begin{aligned}\int_\omega \phi\, dx' &= \int_\Omega \Phi\, dx + \int_\omega (g_3)_B - (g_3)_S\, dx' + \int_{\Gamma_B} \boldsymbol{g}' \cdot \boldsymbol{n}'\, d\sigma \\ &= \int_\Gamma \boldsymbol{g}' \cdot \boldsymbol{n}'\, d\sigma + \int_\Gamma g_3 n_3\, d\sigma - \int_{\Gamma_B} \boldsymbol{g}' \cdot \boldsymbol{n}'\, d\sigma = \int_{\Gamma_L} \boldsymbol{g}' \cdot \boldsymbol{n}'\, d\sigma.\end{aligned}$$

Claim 2. Let $\boldsymbol{u} \in \boldsymbol{X}$ satisfying (4.22). Relation (2.7) gives

$$\begin{aligned}\nabla' \cdot M\boldsymbol{u}' = M(\nabla' \cdot \boldsymbol{u}') - \boldsymbol{g}'_B \cdot \nabla' h &= M(\Phi - \frac{\partial u_3}{\partial x_3}) - \boldsymbol{g}'_B \cdot \nabla' h \\ &= M\Phi - M(\frac{\partial u_3}{\partial x_3}) - \boldsymbol{g}'_B \cdot \nabla' h.\end{aligned}$$

Moreover, one also has

$$F(\nabla' \cdot \boldsymbol{u}') = F\Phi - F(\frac{\partial u_3}{\partial x_3}).$$

By applying Proposition 4.16 in the previous equalities, one obtains

$$\nabla' \cdot M\boldsymbol{u}' = \phi, \quad u_3 = F(\nabla' \cdot \boldsymbol{u}') + U.$$

Conversely, let $\boldsymbol{u}' \in H^1(\Omega)$ satisfying (2.30) and let $u_3 = F(\nabla' \cdot \boldsymbol{u}') + U$, with U defined in (4.33). Then, Proposition 3.9 ensures that $u_3 \in L^2(\Omega)$, and that

$$\frac{\partial u_3}{\partial x_3} = -\nabla' \cdot \boldsymbol{u}' + \frac{\partial U}{\partial x_3} = -\nabla' \cdot \boldsymbol{u}' + \Phi,$$

since one has noticed that $\widetilde{(g_3)_S}$ does not depend on x_3. As a consequence, u_3 belongs to $H(\partial_{x_3}, \Omega)$ and $\nabla \cdot \boldsymbol{u} = \Phi$ in Ω. Finally, let us establish that $u_3 = g_3$ in $L^2(\Gamma, |n_3|\, d\sigma)$. Firstly, let us compute $(u_3|_\Gamma)_S$ and $(u_3|_\Gamma)_B$. Note that $v = \nabla' \cdot \boldsymbol{u}' - \Phi$ belongs to $L^2(\Omega)$. Thus, one deduces from Proposition 4.17 the following equalities

$$\left(Fv|_\Gamma\right)_S = 0, \quad \left(Fv|_\Gamma\right)_B = M(\nabla' \cdot \boldsymbol{u}' - \Phi) \quad \text{on } \omega.$$

Then, by writing $u_3 = Fv + \widetilde{(g_3)_S}$ and since $\left[\widetilde{(g_3)_S}\right]_S = \left[\widetilde{(g_3)_S}\right]_B = (g_3)_S$, one obtains:

$$\left(u_3|_\Gamma\right)_S = (g_3)_S, \quad \left(u_3|_\Gamma\right)_B = M(\nabla' \cdot \boldsymbol{u}' - \Phi) + (g_3)_S. \tag{4.34}$$

As a consequence, one deduces from relation (4.34) and relation (4.31) that for any $\mu \in L^2(\Gamma, |n_3|\, d\sigma)$:

$$\begin{aligned} \int_\Gamma u_3\mu\, |n_3|\, d\sigma &= \int_\omega (u_3|_\Gamma)_S \mu_S\, dx' + \int_\omega (u_3|_\Gamma)_B \mu_B\, dx' \\ &= \int_\omega (g_3)_S \mu_S\, dx' + \int_\omega (g_3)_S \mu_B\, dx' + \int_\omega M(\nabla' \cdot \boldsymbol{u}' - \Phi)\mu_B\, dx'. \end{aligned}$$

Then, by using relations (2.7) and the definition of ϕ in (4.32), the following equalities hold:

$$\begin{aligned} M(\nabla' \cdot \boldsymbol{u}' - \Phi) &= \nabla' \cdot M\boldsymbol{u}' - \boldsymbol{u}'_B \cdot \nabla' h \\ &= \phi - \boldsymbol{u}'_B \cdot \nabla' h - M\Phi = \\ &= (g_3)_B - (g_3)_S. \end{aligned}$$

As a consequence, one deduces in the above equality that

$$\begin{aligned} \int_\Gamma u_3\mu\, |n_3|\, d\sigma &= \int_\omega (g_3)_S \mu_S\, dx' + \int_\omega (g_3)_B \mu_B\, dx' \\ &= \int_\Gamma g_3\mu\, |n_3|\, d\sigma. \end{aligned}$$

Therefore, $u_3 = g_3$ in $L^2(\Gamma, |n_3|\, d\sigma)$, which ends the proof. □

Finally, let us give the proof of Theorem 4.13.

Proof. Thanks to Lemma 4.18, and since p does not depend on x_3 (see Proposition 2.8) we have proved that solving (4.19) reduces to solve (2.21) for the datum ϕ defined in (4.32), and for which we can apply Theorem 2.17. As a consequence, Theorem 4.13 is proved. Let us finish by making the following comments. □

As in the previous section, let us finish with supplementary comments.

Remark 4.19. Let us consider the subspace

$$L^2_0(\Gamma, |n_3|\, d\sigma) := \left\{ g \in L^2(\Gamma, |n_3|\, d\sigma) \,/\, \int_\Gamma g n_3 \, d\sigma = 0 \right\}.$$

From Theorem 4.13, we have proved that the mapping $\gamma : u \mapsto u|_\Gamma$, given by Proposition 4.14, is onto from $H(\partial_{x_3}, \Omega)$ into $L^2_0(\Gamma, |n_3|\, d\sigma)$. Moreover, for any $g \in L^2_0(\Gamma, |n_3|\, d\sigma)$ and any $u \in H(\partial_{x_3}, \Omega)$ such that $u|_\Gamma = g$, one has

$$\|u\|_{H(\partial_{x_3}, \Omega)} \leqslant C \, \|g\|_{L^2(\Gamma, |n_3| d\sigma)} \, .$$

Remark 4.20. Also note that Theorem 4.13 gives a lift operator for the conditions (4.22). Let $\Phi \in L^2(\Omega)$, $\boldsymbol{g}' \in H^{1/2}(\Gamma)^2$ and $g_3 \in L^2(\Gamma,\, |n_3|\, d\sigma)$ satisfying the compatibility condition (4.21). Then, there is $\boldsymbol{u}$ in $\boldsymbol{X}$ satisfying the following conditions:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \text{ in } \Gamma,$$

Moreover, there is a constant $C > 0$ such that:

$$\|\boldsymbol{u}\|_{\boldsymbol{X}} \leqslant C \left(\|\Phi\|_{L^2(\Omega)} + \|\boldsymbol{g}'\|_{H^{1/2}(\Gamma)^2} + \|g_3\|_{L^2(\Gamma, |n_3| d\sigma)} \right). \tag{4.35}$$

Conclusion and prospect of Part I

As we have presented it, the aim of part I was to study the existence and uniqueness of weak solutions to the hydrostatic Stokes approximation in various new cases. Firstly, we have revisited the homogeneous case (3.1), see Theorem 3.4 and considered the new problem (3.2) solved in Theorem 3.19. In these two cases, we have always considered an homogeneous boundary condition on u_3. Then, we have studied two other situations where we considered a non homogeneous boundary condition on u_3, see (4.1) solved Theorem 4.1, and (4.6) solved in Theorem 4.13, and this last result is proved under assumption (7) only. In order to establish these results, we have reduced, in various situations, the following conditions:

$$\nabla \cdot \boldsymbol{u} = \Phi \text{ in } \Omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma, \quad Bu_3 = g_3 \text{ on } \Gamma,$$

where Bu_3 defines one of the boundary conditions (15), to the following ones:

$$\nabla' \cdot M\boldsymbol{u}' = \phi \text{ in } \omega, \quad \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma.$$

As a consequence, we have proved that any of the problems mentioned above reduce to Problem (2.2), solved in 2.17.

Far from now, we have not been interested in the study of the regularity of the weak solution to the hydrostatic Stokes approximation. In addition to the work already done, we have investigated the question in the case of the homogeneous Problem (3.1), and when (7) is not assumed.

By noticing that for any $\varepsilon > 0$ and for $\boldsymbol{f}' \in L^2(\Omega)^2$, the solution $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ given by Theorem 3.4 satisfies

$$\boldsymbol{u}_\varepsilon \in H^2(\Omega)^3, \quad p_\varepsilon \in H^1(\Omega), \tag{4.36}$$

we tried to improve the estimate (3.3) in order to get additional regularity on $(\boldsymbol{u}, p)$ solution to (3.1).

For technical reasons, we had to consider a domain Ω of class $C^{1,1}$, which excludes the definition (1). Then, we would adapt to the case of Ω as in (1). Thus, we tried to get local estimates on the pair $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$, along the lines of [6], by covering Ω with elementary open sets σ of thickness the parameter $0 < \lambda \leqslant 1$. This means that σ can be located in a horizontal band of thickness λ.

The following result concerns estimates inside Ω. Before, let us introduce some notations. We denote by θ a function of $\mathcal{D}(\mathbb{R}^3)$ with compact support in $\sigma \subset \Omega$. Then, we set $\boldsymbol{v}_\varepsilon = \theta \boldsymbol{u}_\varepsilon$ and $q_\varepsilon = \theta p_\varepsilon$. Therefore, introducing the quantities

$$L_\varepsilon(\sigma) := \|\boldsymbol{v}'_\varepsilon\|_{H^2(\sigma)^2} + \varepsilon \|v_3^\varepsilon\|_{H^2(\sigma)} + \|\nabla v_3^\varepsilon\|_{H(\partial x_3, \sigma)^3} + \|\nabla q_\varepsilon\|_{L^2(\sigma)^3},$$

$$R_{\varepsilon,\lambda}(\sigma) := \left\|\boldsymbol{f}'\right\|_{L^2(\Omega)^2} + \lambda \|\nabla p_\varepsilon\|_{L^2(\sigma)^3} + \lambda \|\nabla u_3^\varepsilon\|_{H(\partial x_3, \sigma)^3},$$

the following estimates hold.

Proposition 4.21. *Let $\varepsilon > 0$ and $\boldsymbol{f}' \in L^2(\Omega)^2$. Let $(\boldsymbol{u}_\varepsilon,\, p_\varepsilon)$ be the solution to* (3.9) *given by* Theorem 3.4 *and* (4.36)*. Let $0 < \lambda \leqslant 1$ and assume that θ has compact support in $\sigma \subset\subset \Omega$. Then, there is a constant $C_\theta > 0$ depending only on θ such that*

$$L_\varepsilon(\sigma) \leqslant C_\theta R_{\varepsilon,\,\lambda}(\sigma).$$

Then, we have obtained estimates near the boundary of Ω, by flattening the boundary of Ω. Recall that Ω is of class $C^{1,1}$. This means here that Ω is connected and locally situated on one side of its boundary Γ, a manifold of class $C^{1,1}$. More precisely, given a point $P \in \Gamma$, there are positive numbers λ and b, and an orthonormalized system of cartesian coordinates $x = (x',\, x_3)$ with origin at P, and a function $\psi(x')$ defined and $C^{1,1}$ on the sphere $\{x' \in \mathbb{R}^2 \,/\, |x'| \leqslant b\}$ such that: the point x for which $x_3 = \psi(x')$ belongs to Γ; the point x for which $\psi(x') < x_3 < \lambda + \psi(x')$ belong to Ω; the point x for which $-\lambda + \psi(x') < x_3 < \psi(x')$ belongs to $\mathbb{R}^3 \backslash \overline{\Omega}$. In other words, we define here the sets

$$\begin{aligned} &\sigma = \left\{x \in \mathbb{R}^3 \,/\, |x'| \leqslant b,\, -\lambda + \psi(x') < x_3 < \lambda + \psi(x')\right\}, \\ &\sigma^+ = \left\{x \in \sigma \,/\, x_3 > \psi(x')\right\}, \quad \dot{\sigma} = \left\{x \in \sigma \,/\, x_3 = \psi(x')\right\}. \end{aligned} \tag{4.37}$$

Moreover, the tangent plane to $\partial\Omega$ at P coincides with the x'-plane, so that $\nabla\psi(0) = 0$. To finish, we can assume with no loss of generality that $\lambda \leqslant 1$ and $b \leqslant 1$. Therefore we prove the following result.

Proposition 4.22. *Let $\varepsilon > 0$ and $\boldsymbol{f}' \in L^2(\Omega)^2$. Let $(\boldsymbol{u}_\varepsilon,\, p_\varepsilon)$ be the solution to* (3.9) *given by* Theorem 3.4 *and* (4.36)*. Then, there is $b > 0$ such that for any θ in $\mathcal{D}(\mathbb{R}^3)$ with compact support in $\sigma^+ \cup \dot{\sigma}$:*

$$L_\varepsilon(\sigma^+) \leqslant C_\theta R_{\varepsilon,\,\lambda}(\sigma^+),$$

where $C_\theta > 0$ is a constant depending only on θ.

Unfortunately, these optimal estimates prevent us to choose correctly the parameter λ, and get strong estimates on the pair $(\boldsymbol{u}_\varepsilon,\, p_\varepsilon)$. Therefore, it is not possible, by this way, to get some regularity on the pair $(\boldsymbol{u},\, p)$.

To our knowledge, the regularity of the solution to (3.1) in a domain Ω with a depth h vanishing on the shore, is still an open problem. The closest result concerns the regularity of weak solutions of (2.15) when the mapping h posses es the property (7). The following result is proved in the hand book of R. Temam and M. Ziane [48], see Theorem 4.4 page 121:

Theorem 4.23. *Assume that h satisfies* (7) *and let us consider the open set Ω_ε defined in* (8)*. Let $(\boldsymbol{u}'_\varepsilon,\, p_S)$ be the solution to* (2.15) *given by* Proposition 2.12*. Then*

$$\boldsymbol{u}'_\varepsilon \in H^2(\Omega_\varepsilon)^2, \quad p_S \in H^1(\Omega_\varepsilon),$$

and there is a constant $C > 0$ independent on ε such that

$$\|\boldsymbol{u}'_\varepsilon\|_{H^2(\Omega_\varepsilon)^2} + \varepsilon\, \|p_S\|_{H^1(\omega)} \leqslant C \left\|\boldsymbol{f}'\right\|_{L^2(\Omega)^2}.$$

Theorem 4.23, combined with the close link between (3.1) and (2.15), brings a regularity result in the case where the mapping h satisfies (7).

Proposition 4.24. *Assume that h satisfies* (7). *Let $\boldsymbol{f}' \in L^2(\Omega)^2$ and let $(\boldsymbol{u}, p)$ be the solution to* (3.1) *given by* Theorem 3.4. *Then*

$$\boldsymbol{u}' \in H^2(\Omega)^2, \quad \nabla u_3 \in H(\partial_{x_3}, \Omega)^3, \quad p \in H^1(\Omega),$$

and there is a constant $C > 0$ such that

$$\|\boldsymbol{u}'\|_{H^2(\Omega)^2} + \|\nabla u_3\|_{H(\partial_{x_3}, \Omega)^3} + \|p\|_{H^1(\Omega)} \leqslant C \left\|\boldsymbol{f}'\right\|_{L^2(\Omega)^2}.$$

Here, it would be interesting to improve Theorem 4.23 along the line of [48] by considering weighted functional spaces. This naturally leads us to the study of degenerate equations set on the surface ω, with a weight h blowing up. In the second part, we study one of these equations, set in the half-space $\mathbb{R}^3_+$.

Part II

A degenerate equation in the half space

Introduction of Part II

This work is devoted to the study of a degenerate elliptic equation set in the half-space $\mathbb{R}^3_+$. It is inspired by several models in oceanography when we consider a domain with a depth vanishing on the shore. More precisely, for a datum $g : \mathbb{R}^3_+ \to \mathbb{R}$, our problem consists in seeking $u : \mathbb{R}^3_+ \to \mathbb{R}$ formally solution to:

$$-\text{div}\left(\frac{1}{x_3}\nabla u\right) = g \text{ in } \mathbb{R}^3_+, \quad u = 0 \text{ on } \Gamma = \mathbb{R}^2 \times \{0\}\,. \tag{4.38}$$

We start from the study of a degenerate equation, made by D. Bresch, J. Lemoine and F. Guíllen-González in [11]. Let ω be a two-dimensional domain. Given $h : \omega \to \mathbb{R}$ and $g : \omega \to \mathbb{R}$, the authors are interested in finding ψ in $H(\omega)$ solution to

$$-\text{div}\left(\frac{1}{h}\nabla \Psi\right) = g \text{ in } \omega, \quad \Psi = 0 \text{ on } \partial\omega, \tag{4.39}$$

where

$$H(\omega) = \left\{\Psi \in L^2(\omega) \,/\, \frac{\nabla\Psi}{h^{1/2}} \in L^2(\omega)^2,\, \Psi = 0 \text{ on } \partial\omega\right\},$$

endowed with the norm $\|\Psi\|_{H(\omega)} = \left\|\frac{\nabla\Psi}{h^{1/2}}\right\|_{L^2(\omega)^2}$. Moreover, the weight h satisfies

$$h \in W^{1,\infty}(\omega),\, h > 0 \text{ in } \omega,$$

$$h(x) = \varphi(\delta(x')) \text{ in a neighborhood of } \partial\omega,$$

with $\delta(x') = \text{dist}(x', \omega)$. The function φ satisfies several assumptions, and we refer to [11] for the complete statement.

As do explain the authors in [11], such degenerate equation (4.39) naturally appears in different models issued from oceanography when hydrostatic pressure is assumed. It is the case of the Planetary geostrophic equation [16], the Vertical-geostrophic equations [15], and the hydrostatic Stokes or Navier-Stokes equations [5, 8, 18, 1, 48]. In particular, the authors are interested in obtaining information on the regularity of the fluid's velocity, and they prove the following result.

Theorem 4.25. *Let g be such that $\delta h^{1/2} g \in L^2(\omega)$. There exists a unique solution Ψ of (4.39) such that $\Psi \in H(\omega)$ and*

$$\|\Psi\|_{H(\omega)} \leqslant C \left\|\delta h^{1/2} g\right\|_{L^2(\omega)}.$$

Moreover, if $h^{1/2} g \in L^2(\omega)$, then

$$h^{1/2}\nabla(\frac{1}{h}\nabla\Psi) \in L^2(\omega)^4, \quad \left\|h^{1/2}\nabla(\frac{1}{h}\nabla\Psi)\right\|_{L^2(\omega)^4} \leqslant C \left\|h^{1/2} g\right\|_{L^2(\omega)}.$$

The assumptions carried on ω does not exclude to consider $\omega = \mathbb{R}^2_+$. In this case, it is natural to consider a mapping h having the same behavior than $x' \mapsto x_2$ in a neighborhood of $\partial\omega = \mathbb{R} \times \{0\}$. An additional difficulty would be to consider the mapping $h(x') = x_2$ that is no longer in $L^\infty(\mathbb{R}^2_+)$. This is what we propose to investigate in this work, by studying Problem (4.38). Note that it is for technical reasons, explained later, that we have decided to present the case of the dimension 3.

Our goal is to reach a similar result to Theorem 4.25, in our frame work. In particular, for an adapted weighted space $H(\mathbb{R}^3_+) := W^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$ see (5.9), we would like to prove the following theorem.

Theorem 4.26. *Let g be such that $x_3^{3/2} g \in L^2(\mathbb{R}^3_+)$. There exists a unique solution u of (4.39) such that $u \in H(\mathbb{R}^3_+)$ and*

$$\|u\|_{H(\mathbb{R}^3_+)} \leqslant C \left\| x_3^{3/2} g \right\|_{L^2(\mathbb{R}^3_+)}.$$

Moreover, if $\sqrt{x_3} g \in L^2(\mathbb{R}^3_+)$, then

$$\sqrt{x_3}\nabla(\frac{1}{x_3}\nabla u) \in L^2(\mathbb{R}^3_+)^9, \quad \left\| \sqrt{x_3}\nabla(\frac{1}{x_3}\nabla u) \right\|_{L^2(\mathbb{R}^3_+)^9} \leqslant c \left\| \sqrt{x_3} g \right\|_{L^2(\mathbb{R}^3_+)}.$$

For this, we propose to study (4.38) in a particular context of weighted spaces, where the weight depends on x_3. We will consider more general data g and give results of existence and uniqueness of weak and strong solutions to (4.38). We will also present some regularity results for data g as in [11], that includes Theorem 4.26.

The ideas contained in this work, can be adapted to the $\mathbb{R}^N_+$-case where $N \geqslant 3$. In the $\mathbb{R}^2_+$-case, it is necessary to introduce an additional logarithmic weight, which complicates a bit more the presentation of the results. This is why we limit our study to the case of the dimension 3.

In our sense, the ideas contained in this work would enable us to improve the result established in Theorem [11].

We propose the following organization. In Chapter 5, we briefly recall some properties of the weighted Sobolev spaces $W^{1,2}_\alpha(\mathbb{R}^3_+)$. Then, we introduce and study the family of spaces $W^{1,2}_{-\alpha,x_3}(\mathbb{R}^3_+)$ for non-negative α. In particular, we prove an inequality of Hardy in Proposition 5.8, useful for studying weak solutions to Problem (4.38) in the space $W^{1,2}_{-\alpha,x_3}(\mathbb{R}^3_+)$.

Chapter 6 is devoted to the study of weak solutions for Problem (4.38): in Theorem 6.4, existence and uniqueness in $W^{1,2}_{-\alpha,x_3}(\mathbb{R}^3_+)$ when $\alpha \in [0, 3/2[$, and in Proposition 6.9, existence and uniqueness in $W^{1,2}_{-\alpha}(\mathbb{R}^3_+)$ for $\alpha \in [0, 1[$.

To finish, we prove in Chapter 7, two regularity results for Problem (4.38). In the first one, stated in Theorem 7.2, we consider a datum g such that $x_3^{-\alpha+2} g \in L^2(\mathbb{R}^3_+)$, for $\alpha \in [0, 3/2[$. In the second one, see Theorem 7.6, we consider a datum g satisfying $x_3^{-\alpha+2} g \in L^2(\mathbb{R}^3_+) \cap W_0^{-1,2}(\mathbb{R}^3_+)$ for $\alpha \in [3/2, 5/2[$.

Chapter 5

Functional framework

5.1 Fundamental results on weighted Sobolev spaces

In this section, we recall the fundamental properties of the weighted Sobolev spaces $W^{1,\,2}_{\alpha}(\mathbb{R}^3_+)$. In order to control the behavior of our functions and distributions at infinity, we define the weight

$$x \in \mathbb{R}^3_+, \quad \rho(x) = \sqrt{1 + |x|^2}, \tag{5.1}$$

where $|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ is Euclidean norm of x. For $\alpha \in \mathbb{R}$, let us consider the following weighted Sobolev space

$$W^{1,\,2}_{\alpha}(\mathbb{R}^3_+) = \left\{ u \in \mathcal{D}'(\mathbb{R}^3_+) \,/\, \rho^{\alpha-1} u \in L^2(\mathbb{R}^3_+) \text{ and } \rho^{\alpha} \nabla u \in L^2(\mathbb{R}^3_+)^3 \right\}. \tag{5.2}$$

(see also [37] or [2]) It is a Hilbert space endowed with its natural norm:

$$\|u\|_{W^{1,\,2}_{\alpha}(\mathbb{R}^3_+)} = \left(\left\|\rho^{\alpha-1} u\right\|^2_{L^2(\mathbb{R}^3_+)} + \|\rho^{\alpha} \nabla u\|^2_{L^2(\mathbb{R}^3_+)^3} \right)^{1/2}.$$

The weight in (5.1) is chosen such that the space $W^{1,\,2}_{\alpha}(\mathbb{R}^3_+)$ satisfies two fundamental properties. On the one hand, $\mathcal{D}(\overline{\mathbb{R}^3_+})$ is dense in $W^{1,\,2}_{\alpha}(\mathbb{R}^3_+)$ (see [37], page 230 Theorem I.1). On the other hand, the following Poincaré-type inequality holds in $W^{1,\,2}_{\alpha}(\mathbb{R}^3_+)$.

Proposition 5.1. *There is a constant $C > 0$ such that*

$$\forall u \in W^{1,\,2}_{\alpha}(\mathbb{R}^3_+)/\mathcal{P}_q, \quad \|u\|_{W^{1,\,2}_{\alpha}(\mathbb{R}^3_+)} \leqslant C \,\|\rho^{\alpha} \nabla u\|_{L^2(\mathbb{R}^3_+)^3},$$

where $q = min(0, \left[-\alpha - \frac{1}{2}\right])$ is the highest degree of polynomials contained in $W^{1,\,2}_{\alpha}(\mathbb{R}^3_+)$, with the convention that $\mathcal{P}_q = \{0\}$ if $q < 0$.

Notation 5.2. Throughout this work, we denote by C any positive generic constant.

As $\frac{1}{\rho}$ is bounded, note that for any $\alpha \geqslant \beta$, we have the following embedding:

$$W^{1,\,2}_{\alpha}(\mathbb{R}^3_+) \hookrightarrow W^{1,\,2}_{\beta}(\mathbb{R}^3_+), \tag{5.3}$$

and, from [37] page 235 Corollary I.2, the mapping

$$u \in W^{1,\,2}_{\alpha}(\mathbb{R}^3_+) \mapsto \rho^{\alpha} u \in W^{1,\,2}_{0}(\mathbb{R}^3_+), \tag{5.4}$$

is an isomorphism.

In the following theorem, proved in [37] page 258 Theorem II.3, we give a trace operator for the space $W^{1,\,2}_{\alpha}(\mathbb{R}^3_+)$. The statement requires the space $W^{1/2,\,2}_{\alpha}(\Gamma)$ which is defined in [37] page 238 Definition II.1.

Theorem 5.3. *The mapping $\gamma : u \mapsto u(x',\,0)$ defined on $\mathcal{D}(\overline{\mathbb{R}^3_+})$, can be extended in a unique way to a linear and continuous mapping, still denoted by γ, from $W^{1,\,2}_{\alpha}(\mathbb{R}^3_+)$ into $W^{1/2,\,2}_{\alpha}(\Gamma)$.*

Then, let us define the space

$$\overset{\circ}{W}{}^{1,\,2}_{\alpha}(\mathbb{R}^3_+) := \overline{\mathcal{D}(\mathbb{R}^3_+)}^{\|\cdot\|_{W^{1,\,2}_{\alpha}(\mathbb{R}^3_+)}}, \tag{5.5}$$

which can be characterized in the following way:

$$\overset{\circ}{W}{}^{1,\,2}_{\alpha}(\mathbb{R}^3_+) = \left\{ u \in W^{1,\,2}_{\alpha}(\mathbb{R}^3_+) \,/\, u = 0 \text{ on } \Gamma \right\}.$$

From that, we can introduce the space of distributions $W^{-1,\,2}_{-\alpha}(\mathbb{R}^3_+)$ as the dual space of $\overset{\circ}{W}{}^{1,\,2}_{\alpha}(\mathbb{R}^3_+)$. Then, note that we have the isomorphism

$$u \in \overset{\circ}{W}{}^{1,\,2}_{\alpha}(\mathbb{R}^3_+) \mapsto \rho^{\alpha} u \in \overset{\circ}{W}{}^{1,\,2}_{0}(\mathbb{R}^3_+). \tag{5.6}$$

In the sequel, we shall need another weighted space that is

$$W^{2,\,2}_{0}(\mathbb{R}^3_+) = \left\{ u \in \mathcal{D}'(\mathbb{R}^3_+) \,/\, \frac{u}{\rho^2} \in L^2(\mathbb{R}^3_+),\ \frac{\nabla u}{\rho} \in L^2(\mathbb{R}^3_+)^3,\ D^2 u \in L^2(\mathbb{R}^3_+)^9 \right\}, \tag{5.7}$$

endowed with its natural norm:

$$\|u\|_{W^{2,\,2}_{0}(\mathbb{R}^3_+)} = \left(\left\| \frac{u}{\rho^2} \right\|^2_{L^2(\mathbb{R}^3_+)} + \left\| \frac{\nabla u}{\rho} \right\|^2_{L^2(\mathbb{R}^3_+)^3} + \left\| D^2 u \right\|^2_{L^2(\mathbb{R}^3_+)^9} \right)^{1/2}.$$

Here we denote by $D^2 u$ any second-order derivative of the distribution u. In [37], it is proved that the mapping $u \mapsto \frac{\partial u}{\partial x_3}(x',\,0)$ defines, in a sense to be precised, the first-order trace for distributions of $W^{2,\,2}_{0}(\mathbb{R}^3_+)$. Then, we also consider the space

$$\overset{\circ}{W}{}^{2,\,2}_{0}(\mathbb{R}^3_+) := \overline{\mathcal{D}(\mathbb{R}^3_+)}^{\|\cdot\|_{W^{2,\,2}_{0}(\mathbb{R}^3_+)}}, \tag{5.8}$$

that is also characterized by:

$$\overset{\circ}{W}{}^{2,\,2}_{0}(\mathbb{R}^3_+) = \left\{ u \in W^{1,\,2}_{0}(\mathbb{R}^3_+) \,/\, u = \frac{\partial u}{\partial x_3} = 0 \text{ on } \Gamma \right\}.$$

5.2 Weighted Sobolev spaces $W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$

5.2.1 Definition and first properties

Let α be a non negative real number. Let us define a new family of weighted Sobolev spaces

$$W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+) := \left\{ u \in \mathcal{D}'(\mathbb{R}^3_+) \,/\, \frac{u}{x_3^{\alpha+1}} \in L^2(\mathbb{R}^3_+) \text{ and } \frac{\nabla u}{x_3^{\alpha}} \in L^2(\mathbb{R}^3_+)^3 \right\}, \tag{5.9}$$

endowed with the norm

$$\|u\|_{W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)} = \left(\left\| \frac{u}{x_3^{\alpha+1}} \right\|^2_{L^2(\mathbb{R}^3_+)} + \left\| \frac{\nabla u}{x_3^{\alpha}} \right\|^2_{L^2(\mathbb{R}^3_+)^3} \right)^{1/2}.$$

Proposition 5.4. *Let $\alpha \geqslant 0$. The space $W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ is a Hilbert space for the norm $\|\cdot\|_{W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)}$.*

Proof. Let $(u_k)_k$ be a Cauchy sequence of $W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$. Then, there exists $v \in L^2(\mathbb{R}^3_+)$ and $\boldsymbol{w} \in L^2(\mathbb{R}^3_+)^3$ such that

$$\frac{u_k}{x_3^{\alpha+1}} \to v \text{ in } L^2(\mathbb{R}^3_+), \quad \frac{\nabla u_k}{x_3^{\alpha}} \to \boldsymbol{w} \text{ in } L^2(\mathbb{R}^3_+)^3. \tag{5.10}$$

Then, let us set $u = x_3^{\alpha+1} v$, and prove that $u \in W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ and that

$$u_k \to u \text{ in } W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+).$$

Firstly, it is clear that $\frac{u}{x_3^{\alpha+1}} = v \in L^2(\mathbb{R}^3_+)$. Moreover, $\frac{u_k}{x_3^{\alpha+1}} \to \frac{u}{x_3^{\alpha+1}}$ in $L^2(\mathbb{R}^3_+)$. Secondly, one has in the distributional sense

$$\frac{\nabla u}{x_3^{\alpha}} = \frac{\nabla(x_3^{\alpha+1} v)}{x_3^{\alpha}} = x_3 \nabla v + (\alpha+1) v \boldsymbol{e}_3. \tag{5.11}$$

Besides, one also has

$$\nabla \left(\frac{u_k}{x_3^{\alpha+1}} \right) = \frac{1}{x_3} \left(\frac{\nabla u_k}{x_3^{\alpha}} - (\alpha+1) \frac{u_k}{x_3^{\alpha+1}} \right),$$

which implies by relation (5.10) that

$$x_3 \nabla v = \boldsymbol{w} - (\alpha+1) v \boldsymbol{e}_3 \in L^2(\mathbb{R}^3_+).$$

Thus one deduces in (5.11) that $\frac{\nabla u}{x_3^{\alpha}} \in L^2(\mathbb{R}^3_+)$ and that $\frac{\nabla u_k}{x_3^{\alpha}} \to \frac{\nabla u}{x_3^{\alpha}}$ in $L^2(\mathbb{R}^3_+)$. Finally, one has proved that u_k converges to u $W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$, hence that $W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ is a complete space. To finish, we prove with no difficulty that

$$u,\, v \in W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+), \quad (u,\, v) \mapsto \int_{\mathbb{R}^3_+} \frac{u}{x_3^{\alpha+1}} \frac{v}{x_3^{\alpha+1}}\, dx + \int_{\mathbb{R}^3_+} \frac{\nabla u}{x_3^{\alpha}} \cdot \frac{\nabla v}{x_3^{\alpha}}\, dx,$$

defines a scalar product on $W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$. $\square$

Remark 5.5. To the difference with the spaces $W^{1,2}_{-\alpha}(\mathbb{R}^3_+)$, there is no embedding channel as in relation (5.3) for the family of spaces $W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$.

Remark 5.6. Let $\alpha \geqslant 0$. In the same way that we have the isomorphism (5.4), any distribution u belongs to $W^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ if and only if u satisfies $x_3^{-\alpha} u \in W^{1,\,2}_{0,\,x_3}(\mathbb{R}^3_+)$. Moreover, the following inequalities hold:

$$\frac{1}{(3+2\alpha^2)^{1/2}} \|u\|_{W^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)} \leqslant \|x_3^\alpha u\|_{W^{1,\,2}_{0,\,x_3}(\mathbb{R}^3_+)} \leqslant (3+2\alpha^2)^{1/2} \|u\|_{W^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)} . \tag{5.12}$$

5.2.2 A Hardy's inequality

Then, let us consider the space

$$\overset{\circ}{W}{}^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+) := \overline{\mathcal{D}(\mathbb{R}^3_+)}^{\|\cdot\|_{W^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)}}. \tag{5.13}$$

In the sequel, we establish a Hardy's inequality for functions of $\overset{\circ}{W}{}^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$, which is essential to the study of Problem (4.38). By definition of this space, we firstly prove the inequality for function of $\mathcal{D}(\mathbb{R}^3_+)$.

Lemma 5.7. *The following Hardy's inequality holds:*

$$\forall u \in \mathcal{D}(\mathbb{R}^3_+), \quad \left\| \frac{u}{x_3^{\alpha+1}} \right\|_{L^2(\mathbb{R}^3_+)} \leqslant C_\alpha \left\| \frac{\nabla u}{x_3^\alpha} \right\|_{L^2(\mathbb{R}^3_+)^3}, \tag{5.14}$$

where C_α is given by

$$C_\alpha = \frac{2}{2\alpha+1}. \tag{5.15}$$

Proof. For all $x' \in \mathbb{R}^2$, one has by integration by parts

$$\begin{aligned}
\int_0^{+\infty} \left| \frac{u}{x_3^{\alpha+1}} \right|^2 dx_3 &= \frac{1}{2\alpha+1} \int_0^{+\infty} \frac{1}{x_3^{2\alpha+1}} \frac{\partial u^2}{\partial x_3}\, dx_3 \\
&= \frac{2}{2\alpha+1} \int_0^{+\infty} \left(\frac{u}{x_3^{\alpha+1}} \right) \left(\frac{1}{x_3^\alpha} \frac{\partial u}{\partial x_3} \right) dx_3 \\
&\leqslant \frac{2}{2\alpha+1} \left(\int_0^{+\infty} \left| \frac{u}{x_3^{\alpha+1}} \right|^2 dx_3 \right)^{1/2} \left(\int_0^{+\infty} \left| \frac{1}{x_3^\alpha} \frac{\partial u}{\partial x_3} \right|^2 dx_3 \right)^{1/2},
\end{aligned}$$

and one deduces that

$$\int_0^{+\infty} \left| \frac{u}{x_3^{\alpha+1}} \right|^2 dx_3 \leqslant \left(\frac{2}{2\alpha+1} \right)^2 \int_0^{+\infty} \left| \frac{\partial u}{\partial x_3} \right|^2 dx_3.$$

Then, integrate with respect to x' to establish that

$$\forall u \in \mathcal{D}(\mathbb{R}^3_+), \quad \left\| \frac{u}{x_3^{\alpha+1}} \right\|_{L^2(\mathbb{R}^3_+)} \leqslant \frac{2}{2\alpha+1} \left\| \frac{\nabla u}{x_3^\alpha} \right\|_{L^2(\mathbb{R}^3_+)^3}.$$

□

Then, it follows immediately the following proposition.

Proposition 5.8. *The following Hardy's inequality holds:*

$$\forall u \in \mathring{W}^{1,2}_{-\alpha,\, x_3}(\mathbb{R}^3_+), \quad \left\| \frac{u}{x_3^{\alpha+1}} \right\|_{L^2(\mathbb{R}^3_+)} \leqslant C_\alpha \left\| \frac{\nabla u}{x_3^{\alpha}} \right\|_{L^2(\mathbb{R}^3_+)^3} . \tag{5.16}$$

To finish, let us characterize the space $\mathring{W}^{1,2}_{-\alpha,\, x_3}(\mathbb{R}^3_+)$ as a space of null traces functions. From the embedding

$$W^{1,2}_{-\alpha,\, x_3}(\mathbb{R}^3_+) \hookrightarrow W^{1,2}_{-\alpha}(\mathbb{R}^3_+), \tag{5.17}$$

holding since $x_3 \leqslant \rho$, any distribution $u \in W^{1,2}_{-\alpha,\, x_3}(\mathbb{R}^3_+)$ has a trace $u|_\Gamma$ in $W^{1/2,\,2}_{-\alpha}(\Gamma)$. Thus let us consider the space

$$W = \left\{ u \in W^{1,2}_{-\alpha,\, x_3}(\mathbb{R}^3_+) \,/\, u = 0 \text{ on } \Gamma \right\}.$$

Proposition 5.9. *The following identity holds algebraically*

$$\mathring{W}^{1,2}_{-\alpha,\, x_3}(\mathbb{R}^3_+) = W.$$

Proof. As W is a closed subspace of $W^{1,2}_{-\alpha,\, x_3}(\mathbb{R}^3_+)$, it is clear that

$$\mathring{W}^{1,2}_{-\alpha,\, x_3}(\mathbb{R}^3_+) \subset W.$$

Now let us prove the other inclusion. For $u \in W$, we prove in two steps that there is a sequence $(u_k) \subset \mathcal{D}(\mathbb{R}^3_+)$ converging to u in $W^{1,2}_{-\alpha,\, x_3}(\mathbb{R}^3_+)$. Let us introduce the space

$$\mathcal{H}_c(\mathbb{R}^3_+) = \left\{ u \in W \,/\, \operatorname{supp} u \text{ compact} \subset \overline{\mathbb{R}^3_+} \right\}.$$

In the first part of the proof we prove that $\mathcal{H}_c(\mathbb{R}^3_+)$ is dense in W. In the second part, we establish that $\mathcal{D}(\mathbb{R}^3_+)$ is dense in $\mathcal{H}_c(\mathbb{R}^3_+)$, which lead to the inclusion

$$W \subset \mathring{W}^{1,2}_{-\alpha,\, x_3}(\mathbb{R}^3_+).$$

Step one. Let us prove that $\mathcal{H}_c(\mathbb{R}^3_+)$ is dense in W. Let $\varphi \in \mathcal{D}(\mathbb{R}^3_+)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi(x) = 1$ if $|x| \leqslant 1$, $\varphi(x) = 0$ if $|x| \geqslant 2$. Then, let us set for any integer $k > 0$

$$x \in \mathbb{R}^3_+, \quad \varphi_k(x) = \varphi(\frac{x}{k}), \quad \text{and} \quad \psi_k = \varphi_k|_{\mathbb{R}^3_+}.$$

For $u \in W$, let us establish that $u_k = u\psi_k \in \mathcal{H}_c(\mathbb{R}^3_+)$ and

$$\lim_{k \to +\infty} \|u_k - u\|_{W^{1,2}_{-\alpha,\, x_3}(\mathbb{R}^3_+)} = 0. \tag{5.18}$$

Since $\frac{u}{x_3^{\alpha+1}}$ belongs to $L^2(\mathbb{R}^3_+)$, one deduces that

$$\frac{u_k}{x_3^{\alpha+1}} = \psi_k \frac{u}{x_3^{\alpha+1}} \in L^2(\mathbb{R}^3_+).$$

Then, let us prove that $\frac{\nabla u_k}{x_3^\alpha} \in L^2(\mathbb{R}^3_+)^3$. One has

$$\frac{\nabla u_k}{x_3^\alpha} = \psi_k \frac{\nabla u}{x_3^\alpha} + \frac{u}{k x_3^\alpha} (\nabla \psi)_k.$$

As $\frac{\nabla u}{x_3^\alpha} \in L^2(\mathbb{R}^3_+)^3$, one obtains $\psi_k \frac{\nabla u}{x_3^\alpha} \in L^2(\mathbb{R}^3_+)^3$. To finish, note that supp $(\nabla\psi)_k \subset \{k \leqslant |x| \leqslant 2k\}$, and since one has for any $x \in \mathbb{R}^3_+$ such that $k \leqslant |x| \leqslant 2k$

$$\left|\frac{u(x)}{kx_3^\alpha}\nabla\psi(\frac{x}{k})\right| \leqslant C\left|\frac{u(x)}{x_3^{\alpha+1}}\right|,$$

one deduces that $\frac{u}{kx_3^\alpha}(\nabla\psi)_k$ belongs to $L^2(\mathbb{R}^3_+)$. Thus $u_k \in W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$. Finally, one has $u_k = 0$ on Γ and supp $u_k \subset \overline{\mathbb{R}^3_+}$, which implies that $u_k \in \mathcal{H}_c(\mathbb{R}^3_+)$. Then, let us focus on (5.18). On the one hand,

$$\begin{aligned}\int_{\mathbb{R}^3_+}\frac{|u_k-u|^2}{x_3^{2(\alpha+1)}}\,dx = \int_{\mathbb{R}^3_+}|\psi_k-1|^2\,\frac{u^2}{x_3^{2(\alpha+1)}}\,dx &= \int_{\{|x|\geqslant k\}}|\psi_k-1|^2\,\frac{u^2}{x_3^{2(\alpha+1)}}\,dx\\ &\leqslant 2\int_{\{|x|\geqslant k\}}\left|\frac{u}{x_3^{\alpha+1}}\right|^2\,dx, \end{aligned}\tag{5.19}$$

and, in the same way, one has

$$\int_{\mathbb{R}^3_+}\frac{|\psi_k\nabla u-\nabla u|^2}{x_3^{2\alpha}}\,dx = \int_{\mathbb{R}^3_+}|\psi_k-1|^2\,\frac{|\nabla u|^2}{x_3^{2\alpha}}\,dx \leqslant 2\int_{\{|x|\geqslant k\}}\left|\frac{\nabla u}{x_3^{\alpha}}\right|^2\,dx. \tag{5.20}$$

On the other hand

$$\int_{\mathbb{R}^3_+}\frac{|u\nabla\psi_k|^2}{x_3^{2\alpha}}\,dx = \int_{\{k\leqslant|x|\leqslant 2k\}}|(\nabla\psi)_k|^2\,\frac{u^2}{k^2x_3^{2\alpha}}\,dx \leqslant C\int_{\{k\leqslant|x|\leqslant 2k\}}\left|\frac{u}{x_3^{\alpha+1}}\right|^2\,dx. \tag{5.21}$$

By collecting relations (5.19), (5.20) and (5.21), one deduces (5.18), thanks to the theorem of Lebesgue.

Step two. We prove that $\mathcal{D}(\mathbb{R}^3_+)$ is dense in $\mathcal{H}_c(\mathbb{R}^3_+)$. Let $u \in \mathcal{H}_c(\mathbb{R}^3_+)$. Then, let us consider the extension of u by 0 to $\mathbb{R}^3$ outside $\mathbb{R}^3_+$, that is the function $\widetilde{u}$. Then, as $u = 0$ on Γ, $\widetilde{u}$ possesses the following properties

$$\frac{\widetilde{u}}{|x_3|^{\alpha+1}} \in L^2(\mathbb{R}^3), \quad \frac{\nabla\widetilde{u}}{|x_3|^{\alpha}} \in L^2(\mathbb{R}^3)^3.$$

Then, for $\theta > 0$, we introduce the function u_θ defined by

$$x \in \mathbb{R}^3, \quad u_\theta(x) = \widetilde{u}(x-\theta\boldsymbol{e}_3). \tag{5.22}$$

Let us notice that, since $u = 0$ on Γ, then $(\nabla u)_\theta = \nabla u_\theta$. Therefore, thanks to Lemma 5.11, proved here after, applied to $v = u$ with $\alpha + 1$ and ∇u with α, one has $u_\theta \in W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ and

$$u_\theta \to u \text{ in } W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+), \text{ when } \theta \to 0.$$

Thus, if we consider a mollifier sequence $(\rho_\varepsilon)_\varepsilon$, the function $u_\theta \star \rho_\varepsilon|_{\mathbb{R}^3_+} \in \mathcal{D}(\mathbb{R}^3_+)$ for ε small enough and converges to u in $W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$, as ε and θ go to 0. Consequently, $\mathcal{D}(\mathbb{R}^3_+)$ is dense in $\mathcal{H}_c(\mathbb{R}^3_+)$. $\square$

Remark 5.10. It follows from Proposition 5.9 and from (5.17) that the following embedding holds

$$\overset{\circ}{W}{}^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+) \hookrightarrow \overset{\circ}{W}{}^{1,2}_{-\alpha}(\mathbb{R}^3_+). \tag{5.23}$$

Lemma 5.11. *Let $\alpha \geqslant 0$, let $v \in \mathcal{D}'(\mathbb{R}^3_+)$ and such that $\frac{v}{x_3^\alpha} \in L^2(\mathbb{R}^3_+)$. For any $\theta > 0$, let us consider the function v_θ defined in relation (5.22). Then, one has $\frac{v_\theta}{x_3^\alpha} \in L^2(\mathbb{R}^3_+)$, and the sequence $(\frac{v_\theta}{x_3^\alpha})_\theta$ converges to $\frac{v}{x_3^\alpha}$ in $L^2(\mathbb{R}^3_+)$ as θ goes to 0.*

Proof. Let $\alpha \geqslant 0$. Firstly, we check that $\frac{v_\theta}{x_3^\alpha} \in L^2(\mathbb{R}^3_+)$, since

$$\int_{\mathbb{R}^3_+} \left|\frac{v_\theta}{x_3^\alpha}\right|^2 dx = \int_{\mathbb{R}^3_+} \left|\frac{v}{(x_3+\theta)^\alpha}\right|^2 dx \leqslant \int_{\mathbb{R}^3_+} \left|\frac{v}{x_3^\alpha}\right|^2 dx < \infty.$$

Next, let us establish that $(\frac{v_\theta}{x_3^\alpha})_\theta$ converges to $\frac{v}{x_3^\alpha}$ in $L^2(\mathbb{R}^3_+)$. Introducing the integrals

$$I_{1\theta} = \int_{\mathbb{R}^3_+} \left|v_\theta\left(\frac{1}{x_3^\alpha} - \frac{1}{(x_3-\theta)^\alpha}\right)\right|^2 dx, \quad I_{2\theta} = \int_{\mathbb{R}^3_+} \left|\frac{v_\theta}{(x_3-\theta)^\alpha} - \frac{v}{x_3}\right|^2 dx,$$

one obtains

$$\int_{\mathbb{R}^3_+} \left|\frac{v_\theta - v}{x_3^\alpha}\right|^2 dx \leqslant 2(I_{1\theta} + I_{2\theta}).$$

Firstly, let us establish that $I_{1\theta} \to 0$. By change of variables , one has

$$I_{1\theta} = \int_{\mathbb{R}^3_+} \left|\frac{v}{x_3^\alpha}\left(\frac{x_3^\alpha - (x_3+\theta)^\alpha}{(x_3+\theta)^\alpha}\right)\right|^2 dx,$$

and $I_{1\theta} \to 0$ by the theorem of Beppo-Levi. Then, $I_{2\theta} \to 0$ by continuity of the translation in $L^2(\mathbb{R}^3_+)$. As a consequence, the sequence $(\frac{v_\theta}{x_3^\alpha})_\theta$ converges to $\frac{v}{x_3^\alpha}$ in $L^2(\mathbb{R}^3_+)$. □

5.2.3 The dual space $W^{-1,2}_{\alpha,x_3}(\mathbb{R}^3_+)$

Let us introduce the space of distributions $W^{-1,2}_{\alpha,x_3}(\mathbb{R}^3_+)$ as the dual space of $\overset{\circ}{W}{}^{1,2}_{-\alpha,x_3}(\mathbb{R}^3_+)$. The following proposition characterizes the distributions of $W^{-1,2}_{\alpha,x_3}(\mathbb{R}^3_+)$.

Proposition 5.12. *Any distribution $g \in \mathcal{D}'(\mathbb{R}^3_+)$ belongs to $W^{-1,2}_{\alpha,x_3}(\mathbb{R}^3_+)$, if and only if there is $f_0 \in L^2(\mathbb{R}^3_+)$ and $\boldsymbol{f} \in L^2(\mathbb{R}^3_+)^3$ such that*

$$g = \frac{f_0}{x_3^{\alpha+1}} + \nabla\cdot\left(\frac{\boldsymbol{f}}{x_3^\alpha}\right) \text{ in } \mathbb{R}^3_+.$$

Proof. Let $f_0 \in L^2(\mathbb{R}^3_+)$ and $\boldsymbol{f} \in L^2(\mathbb{R}^3_+)^3$. Then, one has for any $\varphi \in \mathcal{D}(\mathbb{R}^3_+)$

$$\begin{aligned}
\left|\left\langle \frac{f_0}{x_3^{\alpha+1}} + \nabla\cdot\left(\frac{\boldsymbol{f}}{x_3^\alpha}\right), \varphi\right\rangle_{\mathcal{D}'(\mathbb{R}^3_+),\mathcal{D}(\mathbb{R}^3_+)}\right| &= \left|\int_{\mathbb{R}^3_+} f_0\left(\frac{\varphi}{x_3^{\alpha+1}}\right) dx - \int_{\mathbb{R}^3_+} \boldsymbol{f}\cdot\left(\frac{\nabla\varphi}{x_3^\alpha}\right) dx.\right| \\
&\leqslant \|f_0\|_{L^2(\mathbb{R}^3_+)} \left\|\frac{\varphi}{x_3^{\alpha+1}}\right\|_{L^2(\mathbb{R}^3_+)} + \|\boldsymbol{f}\|_{L^2(\mathbb{R}^3_+)} \left\|\frac{\nabla\varphi}{x_3^\alpha}\right\|_{L^2(\mathbb{R}^3_+)^3} \\
&\leqslant C\,\|\varphi\|_{W^{1,2}_{-\alpha,x_3}(\mathbb{R}^3_+)}.
\end{aligned}$$

This proves that the mapping

$$\varphi \mapsto \left\langle \frac{f_0}{x_3^{\alpha+1}} + \nabla \cdot \left(\frac{\boldsymbol{f}}{x_3^\alpha} \right), \varphi \right\rangle_{\mathcal{D}'(\mathbb{R}^3_+), \mathcal{D}(\mathbb{R}^3_+)},$$

is linear and continuous on $\mathcal{D}(\mathbb{R}^3_+)$ for the norm $\|\cdot\|_{W^{1,2}_{-\alpha, x_3}(\mathbb{R}^3_+)}$, and one deduces by density that

$$\frac{f_0}{x_3^{\alpha+1}} + \nabla \cdot \left(\frac{\boldsymbol{f}}{x_3^\alpha} \right) \in W^{-1,2}_{\alpha, x_3}(\mathbb{R}^3_+).$$

Conversely, let $g \in W^{-1,2}_{\alpha, x_3}(\mathbb{R}^3_+)$, and let us consider the mapping

$$u \mapsto (\frac{u}{x_3^{\alpha+1}}, \frac{\nabla u}{x_3^\alpha}),$$

from $\overset{\circ}{W}{}^{1,2}_{-\alpha, x_3}(\mathbb{R}^3_+)$ into $L^2(\mathbb{R}^3_+)^4$. This mapping is isometric; thus one can consider $\overset{\circ}{W}{}^{1,2}_{-\alpha, x_3}(\mathbb{R}^3_+)$ as a closed sub-space of $L^2(\mathbb{R}^3_+)^4$. Thanks to the theorem of Hahn-Banach, if $g \in W^{-1,2}_{\alpha, x_3}(\mathbb{R}^3_+)$, one can extend g to a linear and continuous form on $L^2(\mathbb{R}^3_+)^4$. As a consequence, one deduces from Riesz representation, that there is $(f_0, \boldsymbol{f}) \in L^2(\mathbb{R}^3_+)^4$ such that:

$$\forall u \in \overset{\circ}{W}{}^{1,2}_{-\alpha, x_3}(\mathbb{R}^3_+), \quad g(u) = \int_{\mathbb{R}^3_+} f_0 \left(\frac{u}{x_3^{\alpha+1}} \right) dx - \int_{\mathbb{R}^3_+} \boldsymbol{f} \cdot \left(\frac{\nabla u}{x_3^\alpha} \right) dx,$$

and one deduces that:

$$g = \frac{f_0}{x_3^{\alpha+1}} + \nabla \cdot \left(\frac{\boldsymbol{f}}{x_3^\alpha} \right) \text{ in } \mathbb{R}^3_+.$$

□

Remark 5.13. Let $\alpha \geqslant 0$. From Proposition 5.12 and Remark 5.6, any distribution g belongs to $W^{-1,2}_{\alpha, x_3}(\mathbb{R}^3_+)$ if and only if g satisfies $x_3^\alpha g \in W^{-1,2}_0(\mathbb{R}^3_+)$. Moreover, the following inequalities hold:

$$\frac{1}{(3+2\alpha^2)^{1/2}} \|g\|_{W^{-1,2}_{\alpha, x_3}(\mathbb{R}^3_+)} \leqslant \|x_3^\alpha g\|_{W^{-1,2}_0(\mathbb{R}^3_+)} \leqslant (3+2\alpha^2)^{1/2} \|g\|_{W^{-1,2}_{\alpha, x_3}(\mathbb{R}^3_+)}. \tag{5.24}$$

Remark 5.14. From Proposition 5.8, the mapping $u \mapsto \left\| \frac{\nabla u}{x_3^\alpha} \right\|_{L^2(\mathbb{R}^3_+)^3}$, defines a norm on $\overset{\circ}{W}{}^{1,2}_{-\alpha, x_3}(\mathbb{R}^3_+)$ equivalent to $\|\cdot\|_{W^{1,2}_{-\alpha, x_3}(\mathbb{R}^3_+)}$ and in particular

$$\forall u \in \overset{\circ}{W}{}^{1,2}_{-\alpha, x_3}(\mathbb{R}^3_+), \quad \|u\|_{W^{1,2}_{-\alpha, x_3}(\mathbb{R}^3_+)} \leqslant (1 + C_\alpha^2)^{1/2} \left\| \frac{\nabla u}{x_3^\alpha} \right\|_{L^2(\mathbb{R}^3_+)^3}. \tag{5.25}$$

Therefore, a distribution g belongs to $W^{-1,2}_{\alpha, x_3}(\mathbb{R}^3_+)$ if and only if there is $\boldsymbol{f} \in L^2(\mathbb{R}^3_+)^3$ such that

$$g = \operatorname{div} \left(\frac{\boldsymbol{f}}{x_3^\alpha} \right) \text{ in } \mathbb{R}^3_+.$$

5.2.4 The case $\alpha = 0$

Proposition 5.15. *The space* $\overset{\circ}{W}{}^{1,\,2}_{0,\,x_3}(\mathbb{R}^3_+)$ *is equal to* $\overset{\circ}{W}{}^{1,\,2}_{0}(\mathbb{R}^3_+)$ *and their norm are equivalent.*

Proof. We already have the embedding (5.23) for $\alpha = 0$. Let us prove that $\overset{\circ}{W}{}^{1,\,2}_{0}(\mathbb{R}^3_+)$ is included in $\overset{\circ}{W}{}^{1,\,2}_{0,\,x_3}(\mathbb{R}^3_+)$. Let $u \in \overset{\circ}{W}{}^{1,\,2}_{0}(\mathbb{R}^3_+)$ and $(u_k)_k$ a sequence of $\mathcal{D}(\mathbb{R}^3_+)$ converging to u in $\overset{\circ}{W}{}^{1,\,2}_{0}(\mathbb{R}^3_+)$. Thanks to Hardy's inequality (5.14), the sequence $(\frac{u_k}{x_3})_k$ is a Cauchy sequence in $L^2(\mathbb{R}^3_+)$. Moreover, it is clear that $\frac{u_k}{x_3} \to \frac{u}{x_3}$ in $\mathcal{D}'(\mathbb{R}^3_+)$, thus that $\frac{u}{x_3} \in L^2(\mathbb{R}^3_+)$. As we also have $\nabla u \in L^2(\mathbb{R}^3_+)^3$, we have proved that $\overset{\circ}{W}{}^{1,\,2}_{0}(\mathbb{R}^3_+)$ is included in $\overset{\circ}{W}{}^{1,\,2}_{0,\,x_3}(\mathbb{R}^3_+)$. Thus, they are the same spaces. This means in particular that the identity

$$\overset{\circ}{W}{}^{1,\,2}_{0,\,x_3}(\mathbb{R}^3_+) \xrightarrow{i} \overset{\circ}{W}{}^{1,\,2}_{0}(\mathbb{R}^3_+),$$

is a continuous bijective linear mapping. As $\overset{\circ}{W}{}^{1,\,2}_{0,\,x_3}(\mathbb{R}^3_+)$ and $\overset{\circ}{W}{}^{1,\,2}_{0}(\mathbb{R}^3_+)$ are both Banach spaces, it follows from Banach theorem that i is an isomorphism, which ends the proof. $\square$

Remark 5.16. As a consequence, one deduces from Proposition 5.9 that $\mathcal{D}(\mathbb{R}^3_+)$ is dense in $\overset{\circ}{W}{}^{1,\,2}_{0}(\mathbb{R}^3_+)$ for the norm $\|\cdot\|_{W^{1,\,2}_{0,\,x_3}(\mathbb{R}^3_+)}$, and that any $u \in \overset{\circ}{W}{}^{1,\,2}_{0}(\mathbb{R}^3_+)$ satisfies the Hardy's inequality

$$\forall u \in \overset{\circ}{W}{}^{1,\,2}_{0}(\mathbb{R}^3_+), \quad \left\| \frac{u}{x_3} \right\|_{L^2(\mathbb{R}^3_+)} \leqslant \|\nabla u\|_{L^2(\mathbb{R}^3_+)^3}\,. \tag{5.26}$$

Remark 5.17. Recall that any distribution F belongs to $W^{-1,\,2}_{0}(\mathbb{R}^3_+)$ if and only if there is $f_0 \in L^2(\mathbb{R}^3_+)$ and $\boldsymbol{f} \in L^2(\mathbb{R}^3_+)$ such that

$$F = \frac{f_0}{\rho} + \nabla \cdot \boldsymbol{f} \text{ in } \mathbb{R}^3_+.$$

Thanks to Proposition 5.15, we have also proved that F belongs to $W^{-1,\,2}_{0}(\mathbb{R}^3_+)$ if and only if there is $f_0 \in L^2(\mathbb{R}^3_+)$ and $\boldsymbol{f} \in L^2(\mathbb{R}^3_+)$ such that

$$F = \frac{f_0}{x_3} + \nabla \cdot \boldsymbol{f} \text{ in } \mathbb{R}^3_+.$$

Remark 5.18. Let us define, the space

$$W^{2,\,2}_{0,\,x_3}(\mathbb{R}^3_+) = \left\{ u \in \mathcal{D}'(\mathbb{R}^3_+) \,/\, \frac{u}{x_3^2} \in L^2(\mathbb{R}^3_+),\ \frac{\nabla u}{x_3} \in L^2(\mathbb{R}^3_+)^3 \text{ and } D^2 u \in L^2(\mathbb{R}^3_+)^9 \right\}.$$

It is a Hilbert space for the norm

$$\|u\|_{W^{2,\,2}_{0,\,x_3}(\mathbb{R}^3_+)} = \left(\left\| \frac{u}{x_3^2} \right\|^2_{L^2(\mathbb{R}^3_+)} + \left\| \frac{\nabla u}{x_3} \right\|^2_{L^2(\mathbb{R}^3_+)^3} + \left\| D^2 u \right\|^2_{L^2(\mathbb{R}^3_+)^9} \right)^{1/2}.$$

Then, let us consider

$$\overset{\circ}{W}{}^{2,2}_{0,\,x_3}(\mathbb{R}^3_+) := \overline{\mathcal{D}(\mathbb{R}^3_+)}^{\|\cdot\|_{W^{2,\,2}_{0,\,x_3}(\mathbb{R}^3_+)}}.$$

Firstly, one can prove that

$$\overset{\circ}{W}{}^{2,2}_{0,\,x_3}(\mathbb{R}^3_+) = \left\{ u \in W^{2,\,2}_{0,\,x_3}(\mathbb{R}^3_+) \,/\, u = \frac{\partial u}{\partial x_3} = 0 \text{ on } \Gamma \right\},$$

by adapting the proof of Proposition 5.9. Secondly, the argument of the proof of Proposition 5.15 allows to establish that $\overset{\circ}{W}{}^{2,2}_{0,\,x_3}(\mathbb{R}^3_+)$ is equal to $\overset{\circ}{W}{}^{2,2}_{0}(\mathbb{R}^3_+)$ (see (5.8)). Moreover, their norm are equivalent.

5.3 The spaces $W^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon)$

For any $\varepsilon > 0$, let us consider the following subset of $\mathbb{R}^3_+$

$$B_\varepsilon = \left\{ x \in \mathbb{R}^3 \,/\, x_3 > \varepsilon \right\},$$

and extend the definition of $W^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ and $\overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ to the spaces $W^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon)$ and $\overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon)$. Obviously, every properties of $W^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ are true for $W^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon)$. In particular, the following Hardy's inequality holds

$$\forall u \in \overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon), \qquad \left\| \frac{u}{x_3^{\alpha+1}} \right\|_{L^2(B_\varepsilon)} \leqslant C_\alpha \left\| \frac{\nabla u}{x_3^{\alpha}} \right\|_{L^2(B_\varepsilon)^3}, \tag{5.27}$$

where C_α, given by (5.15), does not depend on ε. Moreover, we shall also require the following relation:

$$\forall u \in \overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon), \quad \|u\|_{W^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon)} \leqslant (1 + C_\alpha^2)^{1/2} \left\| \frac{\nabla u}{x_3^{\alpha}} \right\|_{L^2(B_\varepsilon)^3}. \tag{5.28}$$

To finish, $\overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon)$ can be characterized as a space of null traces functions:

$$\overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon) = \left\{ u \in W^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon) \,/\, u = 0 \text{ on } \partial B_\varepsilon \right\}.$$

Remark 5.19. Let $\alpha \geqslant 0$. Like in Remark 5.6, any u belongs to $W^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon)$ if and only if u satisfies $x_3^{-\alpha} u \in W^{1,\,2}_{0,\,x_3}(B_\varepsilon)$, with (5.12) adapted to B_ε.

Remark 5.20. As we have seen it in Remark 5.5, there is no hierarchy for the spaces $W^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$. It is in this sense that we introduce the family of spaces $W^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon)$, for the parameter α, since for any $\alpha \geqslant \beta \geqslant 0$ the following embeddings hold

$$W^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon) \hookrightarrow W^{1,\,2}_{-\beta,\,x_3}(B_\varepsilon). \tag{5.29}$$

$$\overset{\circ}{W}{}^{1,\,2}_{-\alpha}(B_\varepsilon) \hookrightarrow \overset{\circ}{W}{}^{1,\,2}_{-\beta,\,x_3}(B_\varepsilon). \tag{5.30}$$

Chapter 6

Weak solutions

6.1 Weak solutions in $W^{1,2}_{-\alpha,x_3}(\mathbb{R}^3_+)$

In this section, we study the weak solutions of Problem (4.38) in the space $W^{1,2}_{-\alpha,x_3}(\mathbb{R}^3_+)$, with $0 \leqslant \alpha < 3/2$. We will start off with the case $\alpha = 0$, which is equivalent to find weak solutions in $W^{1,2}_0(\mathbb{R}^3_+)$, according to Proposition 5.15. From the case $\alpha = 0$ follows the case $\alpha = 1$, and then we finish with the case $0 \leqslant \alpha < 3/2$. In fact, similarly to Lemma 6.6, an open problem is to find weak solutions in any space $W^{1,2}_{-\alpha,x_3}(\mathbb{R}^3_+)$ for $\alpha \geqslant 3/2$.

As mentioned above, let us start off with investigating the case $\alpha = 0$. Here, we seek weak solutions to Problem (4.38) in the space $W^{1,2}_0(\mathbb{R}^3_+)$ by writing Problem (4.38) in the following way:

$$\begin{cases} -\Delta u + \dfrac{1}{x_3}\dfrac{\partial u}{\partial x_3} = f & \text{in } \mathbb{R}^3_+, \\ \qquad\qquad\quad u = 0 & \text{on } \mathbb{R}^2, \end{cases} \tag{6.1}$$

where $f = x_3 g$.

Proposition 6.1. *Let $f \in W^{-1,2}_0(\mathbb{R}^3_+)$. Then, Problem* (6.1) *has a unique solution u in the space $\mathring{W}^{1,2}_{0,x_3}(\mathbb{R}^3_+)$ and satisfying the estimate*

$$\|u\|_{W^{1,2}_{0,x_3}(\mathbb{R}^3_+)} \leqslant \sqrt{5}\,\|f\|_{W^{-1,2}_0(\mathbb{R}^3_+)}. \tag{6.2}$$

Proof. For any $u \in \mathring{W}^{1,2}_0(\mathbb{R}^3_+)$, one has by Remark 5.17

$$\frac{1}{x_3}\frac{\partial u}{\partial x_3} \in W^{-1,2}_0(\mathbb{R}^3_+),$$

and for any $v \in \mathring{W}^{1,2}_0(\mathbb{R}^3_+)$,

$$\left\langle \frac{1}{x_3}\frac{\partial u}{\partial x_3}, v \right\rangle_{W^{-1,2}_0(\mathbb{R}^3_+),\, \mathring{W}^{1,2}_0(\mathbb{R}^3_+)} = \int_{\mathbb{R}^3_+} \frac{\partial u}{\partial x_3}\frac{v}{x_3}\,dx.$$

As a consequence, $u \in \mathring{W}^{1,2}_0(\mathbb{R}^3_+)$ is a solution to (6.1) if and only if u satisfies the formulation below

$$\begin{cases} \forall v \in \mathring{W}^{1,2}_0(\mathbb{R}^3_+), \\ \displaystyle\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla v\,dx + \int_{\mathbb{R}^3_+} \frac{\partial u}{\partial x_3}\frac{v}{x_3}\,dx = \langle f, v\rangle_{W^{-1,2}_0(\mathbb{R}^3_+),\, \mathring{W}^{1,2}_0(\mathbb{R}^3_+)}. \end{cases} \tag{6.3}$$

Next, let us focus on the bilinear form

$$u, v \in \overset{\circ}{W}{}_0^{1,2}(\mathbb{R}^3_+), \quad a(u, v) = \int_{\mathbb{R}^3_+} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^3_+} \frac{v}{x_3} \frac{\partial u}{\partial x_3} \, dx.$$

Thanks to (5.26), it is clearly continuous. Moreover, one has for any $u \in \mathcal{D}(\mathbb{R}^3_+)$

$$\int_{\mathbb{R}^3_+} \frac{u}{x_3} \frac{\partial u}{\partial x_3} \, dx = \frac{1}{2} \left\| \frac{u}{x_3} \right\|^2_{L^2(\mathbb{R}^3_+)} . \tag{6.4}$$

As $\mathcal{D}(\mathbb{R}^3_+)$ is dense in $\overset{\circ}{W}{}_0^{1,2}(\mathbb{R}^3_+)$ for the norm $\|\cdot\|_{W^{1,2}_{0,x_3}(\mathbb{R}^3_+)}$ (see Remark 5.16), relation (6.4) holds for any $u \in \overset{\circ}{W}{}_0^{1,2}(\mathbb{R}^3_+)$. Hence $a(\cdot, \cdot)$ is coercive on $\overset{\circ}{W}{}_0^{1,2}(\mathbb{R}^3_+)$. As a consequence, Lax-Milgram's theorem provides a unique $u \in \overset{\circ}{W}{}_0^{1,2}(\mathbb{R}^3_+)$ solution to (6.3), hence to Problem (6.1). To finish, note that u satisfies the following energy inequality

$$\|\nabla u\|^2_{L^2(\mathbb{R}^3_+)^3} + \frac{1}{2} \left\| \frac{u}{x_3} \right\|^2_{L^2(\mathbb{R}^3_+)} = \langle f, u \rangle_{W_0^{-1,2}(\mathbb{R}^3_+), \overset{\circ}{W}{}_0^{1,2}(\mathbb{R}^3_+)} , \tag{6.5}$$

which implies, by relation (5.25)

$$\begin{aligned}
\|\nabla u\|^2_{L^2(\mathbb{R}^3_+)^3} + \frac{1}{2} \left\| \frac{u}{x_3} \right\|^2_{L^2(\mathbb{R}^3_+)} &\leqslant \|f\|_{W_0^{-1,2}(\mathbb{R}^3_+)} \|u\|_{W^{1,2}_{0,x_3}(\mathbb{R}^3_+)} \\
&\leqslant \sqrt{5} \, \|f\|_{W_0^{-1,2}(\mathbb{R}^3_+)} \|\nabla u\|_{L^2(\mathbb{R}^3_+)^3} \\
&\leqslant \frac{5}{2} \|f\|^2_{W_0^{-1,2}(\mathbb{R}^3_+)} + \frac{1}{2} \|\nabla u\|^2_{L^2(\mathbb{R}^3_+)^3} ,
\end{aligned}$$

and one deduces estimate (6.2). □

As a consequence of Proposition 6.1, we study weak solutions in $W^{1,2}_{-1,x_3}(\mathbb{R}^3_+)$, by using a duality argument, as it is done in the following proposition.

Proposition 6.2. *Let $g \in W_0^{-1,2}(\mathbb{R}^3_+)$. Then, there is a unique $u \in \overset{\circ}{W}{}^{1,2}_{-1,x_3}(\mathbb{R}^3_+)$ solution to* (4.38) *and*

$$\|u\|_{W^{1,2}_{-1,x_3}(\mathbb{R}^3_+)} \leqslant 6\sqrt{10} \, \|g\|_{W_0^{-1,2}(\mathbb{R}^3_+)} .$$

Proof. From Proposition 6.1, the operator

$$\begin{array}{rccc}
P: & \overset{\circ}{W}{}_0^{1,2}(\mathbb{R}^3_+) & \to & W^{-1,2}_{1,x_3}(\mathbb{R}^3_+) \\
& u & \mapsto & -\operatorname{div}\left(\dfrac{1}{x_3} \nabla u\right)
\end{array}$$

is an isomorphism. Moreover, it is a self adjoint operator. Indeed, one has for any $u \in \overset{\circ}{W}{}_0^{1,2}(\mathbb{R}^3_+)$ and any $v \in \overset{\circ}{W}{}^{1,2}_{-1,x_3}(\mathbb{R}^3_+)$:

$$\begin{aligned}
\left\langle -\operatorname{div}\left(\frac{1}{x_3} \nabla u\right), v \right\rangle_{W^{-1,2}_{1,x_3}(\mathbb{R}^3_+), \overset{\circ}{W}{}^{1,2}_{-1,x_3}(\mathbb{R}^3_+)} &= \int_{\mathbb{R}^3_+} \nabla u \cdot \left(\frac{1}{x_3} \nabla v\right) dx \\
&= \left\langle u, -\operatorname{div}\left(\frac{1}{x_3} \nabla v\right) \right\rangle_{\overset{\circ}{W}{}_0^{1,2}(\mathbb{R}^3_+), W_0^{-1,2}(\mathbb{R}^3_+)} .
\end{aligned}$$

As a consequence P is an isomorphism from $\overset{\circ}{W}{}^{1,2}_{-1,x_3}(\mathbb{R}^3_+)$ into $W_0^{-1,2}(\mathbb{R}^3_+)$. Hence, for any $g \in W_0^{-1,2}(\mathbb{R}^3_+)$, there is a unique $u \in \overset{\circ}{W}{}^{1,2}_{-1,x_3}(\mathbb{R}^3_+)$ solution to Problem (4.38). Moreover, there is a constant $C > 0$ such that

$$\|u\|_{W^{1,2}_{-1,x_3}(\mathbb{R}^3_+)} \leqslant C \|g\|_{W_0^{-1,2}(\mathbb{R}^3_+)} . \tag{6.6}$$

To finish, we use Lemma 6.6, proved here after. By Remark 5.6, note that the function u/x_3 belongs to $\overset{\circ}{W}{}^{1,2}_0(\mathbb{R}^3_+)$. Then, the following Green's formulas are justified:

$$\begin{aligned}
\left\langle -\mathrm{div}\left(\frac{1}{x_3}\nabla u\right),\, \frac{u}{x_3}\right\rangle_{W_0^{-1,2}(\mathbb{R}^3_+),\,\overset{\circ}{W}{}^{1,2}_0(\mathbb{R}^3_+)} &= \int_{\mathbb{R}^3_+} \frac{\nabla u}{x_3}\cdot\nabla\left(\frac{u}{x_3}\right) dx \\
&= \left\|\frac{\nabla u}{x_3}\right\|^2_{L^2(\mathbb{R}^3_+)} - \frac{1}{2}\int_{\mathbb{R}^3_+} \frac{1}{x_3^3}\frac{\partial u^2}{\partial x_3}\, dx \\
&= \left\|\frac{\nabla u}{x_3}\right\|^2_{L^2(\mathbb{R}^3_+)} - \frac{3}{2}\left\|\frac{u}{x_3^2}\right\|^2_{L^2(\mathbb{R}^3_+)} .
\end{aligned}$$

Consequently, u satisfies the following energy equality:

$$\left\|\frac{\nabla u}{x_3}\right\|^2_{L^2(\mathbb{R}^3_+)} = \left\langle g,\, \frac{u}{x_3}\right\rangle_{W_0^{-1,2}(\mathbb{R}^3_+),\,\overset{\circ}{W}{}^{1,2}_0(\mathbb{R}^3_+)} + \frac{3}{2}\left\|\frac{u}{x_3^2}\right\|^2_{L^2(\mathbb{R}^3_+)} .$$

Thanks to Hardy's inequality (5.16) for $\alpha = 1$, one has $\left\|\frac{u}{x_3^2}\right\|^2_{L^2(\mathbb{R}^3_+)} \leqslant \frac{4}{9}\left\|\frac{\nabla u}{x_3}\right\|^2_{L^2(\mathbb{R}^3_+)}$, and it follows that

$$\left\|\frac{\nabla u}{x_3}\right\|^2_{L^2(\mathbb{R}^3_+)} \leqslant 3\,\|g\|_{W_0^{-1,2}(\mathbb{R}^3_+)} \left\|\frac{u}{x_3}\right\|_{W_0^{1,2}(\mathbb{R}^3_+)} .$$

Then, one has by relation (5.25)

$$\left\|\frac{u}{x_3}\right\|_{W_0^{1,2}(\mathbb{R}^3_+)} \leqslant \sqrt{5}\left\|\nabla\left(\frac{u}{x_3}\right)\right\|_{L^2(\mathbb{R}^3_+)^3} \leqslant \sqrt{5}(1+\frac{2}{3})\left\|\frac{\nabla u}{x_3}\right\|_{L^2(\mathbb{R}^3_+)^3} \leqslant 2\sqrt{5}\left\|\frac{\nabla u}{x_3}\right\|_{L^2(\mathbb{R}^3_+)^3} ,$$

and the above inequality becomes

$$\left\|\frac{\nabla u}{x_3}\right\|_{L^2(\mathbb{R}^3_+)} \leqslant 6\sqrt{5}\,\|g\|_{W_0^{-1,2}(\mathbb{R}^3_+)} .$$

To finish, we use relation (5.25) to prove (6.6). □

Then, let us study weak solutions in $W^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$, that is the case $\alpha = 1/2$.

Proposition 6.3. *Let $g \in W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+)$. Then, there is a unique $u \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$ solution to Problem* (4.38) *and satisfying*

$$\|u\|_{W^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)} \leqslant 3\,\|g\|_{W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+)} . \tag{6.7}$$

Proof. For any $u \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$, one has by Proposition 5.12

$$-\mathrm{div}\left(\frac{1}{x_3}\nabla u\right) \in W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+),$$

and for any $v \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$,

$$\left\langle -\mathrm{div}\left(\frac{1}{x_3}\nabla u\right),\, v\right\rangle = \int_{\mathbb{R}^3_+} \frac{\nabla u}{\sqrt{x_3}} \cdot \frac{\nabla v}{\sqrt{x_3}}\, dx,$$

where the duality is taken in the sense of $W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+)$, $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$. As a consequence, $u \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$ is a solution to (4.38) if and only if u satisfies the formulation below

$$\begin{cases} \forall v \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+), \\ \displaystyle\int_{\mathbb{R}^3_+} \frac{\nabla u}{\sqrt{x_3}} \cdot \frac{\nabla v}{\sqrt{x_3}}\, dx = \langle g,\, v\rangle_{W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+),\, \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)}. \end{cases} \tag{6.8}$$

Next, the bilinear form

$$u, v \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+), \quad a(u,\, v) = \int_{\mathbb{R}^3_+} \frac{\nabla u}{\sqrt{x_3}} \cdot \frac{\nabla v}{\sqrt{x_3}}\, dx,$$

is clearly continuous and coercive on $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$ thanks to Hardy's inequality (5.16). As a consequence, Lax-Milgram's theorem provides a unique $u \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$ satisfying (6.8), hence a solution to Problem (4.38). To finish, note that u satisfies the following energy inequality

$$\left\| \frac{\nabla u}{\sqrt{x_3}} \right\|^2_{L^2(\mathbb{R}^3_+)^3} = \langle g,\, u\rangle_{W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+),\, \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)}, \tag{6.9}$$

which implies thanks to Hardy's inequality (5.25)

$$\left\| \frac{\nabla u}{\sqrt{x_3}} \right\|_{L^2(\mathbb{R}^3_+)^3} \leqslant \sqrt{2}\, \|g\|_{W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+)},$$

and (6.2) by (5.24). □

More generally, we have the following theorem.

Theorem 6.4. *Let $0 \leqslant \alpha < 3/2$ and $g \in \mathcal{D}'(\mathbb{R}^3_+)$ such that*

$$x_3^{-\alpha+1} g \in W^{-1,2}_0(\mathbb{R}^3_+).$$

Then, there exists a unique $u \in \overset{\circ}{W}{}^{1,2}_{-\alpha,x_3}(\mathbb{R}^3_+)$ solution to Problem (4.38) *and there exists a constant $C > 0$, depending at most on α, such that*

$$\|u\|_{W^{1,2}_{-\alpha,x_3}(\mathbb{R}^3_+)} \leqslant C \left\| x_3^{-\alpha+1} g \right\|_{W^{-1,2}_0(\mathbb{R}^3_+)}. \tag{6.10}$$

More precisely, one can chose

$$C = \frac{2\left(1 + C_\alpha^2\right)^{1/2}}{1 - (2\alpha - 1)C_\alpha}\sqrt{5}.$$

Remark 6.5. Recall that $C_\alpha = \frac{2}{2\alpha+1}$ (see (5.15)) and note that $(2\alpha-1)C_\alpha < 1$ since $\alpha < 3/2$. Moreover, according to Remark 5.13, we can replace $x_3^{-\alpha+1} g \in W_0^{-1,2}(\mathbb{R}^3_+)$ by $g \in W^{-1,2}_{-\alpha+1,\,x_3}(\mathbb{R}^3_+)$ when $0 \leqslant \alpha < 1$, and in this case, the estimate (6.10) becomes

$$\|u\|_{W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)} \leqslant C\left(3+2\alpha^2\right)^{1/2} \|g\|_{W^{-1,2}_{-\alpha+1}(\mathbb{R}^3_+)}. \tag{6.11}$$

It is no more the case if $1 \leqslant \alpha < 3/2$, as the space $\mathring{W}^{1,2}_{\alpha-1,\,x_3}(\mathbb{R}^3_+)$ is not defined.

Before giving the proof of Theorem 6.4, let us give a technical lemma. This lemma uses the spaces $\mathring{W}^{1,2}_{-\alpha,\,x_3}(B_\varepsilon)$ which are defined and briefly studied in Section 5.3.

Lemma 6.6. *Let $g \in \mathcal{D}'(B_\varepsilon)$ such that*

$$x_3^{-\alpha+1} g \in W_0^{-1,2}(B_\varepsilon),$$

and let $u_\varepsilon \in \mathring{W}^{1,2}_{-\alpha,\,x_3}(B_\varepsilon)$ be a solution to

$$-\operatorname{div}\left(\frac{1}{x_3}\nabla u_\varepsilon\right) = g \ \text{in}\ B_\varepsilon.$$

Then, u_ε satisfies the following estimates

$$\|u\|_{W^{1,2}_{-\alpha,\,x_3}(B_\varepsilon)} \leqslant \frac{2\left(1+C_\alpha^2\right)^{1/2}}{1-(2\alpha-1)C_\alpha}\sqrt{5}\,\|x_3^{-\alpha+1} g\|_{W_0^{-1,2}(B_\epsilon)}. \tag{6.12}$$

Proof. By Remark 5.19, one has $\frac{u_\varepsilon}{x_3^\alpha} \in \mathring{W}^{1,2}_0(B_\varepsilon)$. Then, as $x_3^{-\alpha+1} g \in W_0^{-1,2}(B_\varepsilon)$, one has

$$\begin{aligned}\left\langle -x_3^{-\alpha+1}\operatorname{div}\left(\frac{1}{x_3}\nabla u_\varepsilon\right),\ \frac{u_\varepsilon}{x_3^\alpha}\right\rangle &= \int_{B_\varepsilon} \frac{\nabla u_\varepsilon}{x_3}\cdot\nabla\left(\frac{u_\varepsilon}{x_3^{2\alpha-1}}\right)dx\\ &= \left\|\frac{\nabla u_\varepsilon}{x_3^\alpha}\right\|^2_{L^2(B_\varepsilon)} - \frac{2\alpha-1}{2}\int_{\mathbb{R}^3_+}\frac{1}{x_3^{2\alpha+1}}\frac{\partial u_\varepsilon^2}{\partial x_3}\,dx\\ &= \left\|\frac{\nabla u_\varepsilon}{x_3^\alpha}\right\|^2_{L^2(B_\varepsilon)} - \frac{2\alpha-1}{C_\alpha}\left\|\frac{u_\varepsilon}{x_3^{\alpha+1}}\right\|^2_{L^2(B_\varepsilon)},\end{aligned}$$

where the duality is taken in the sense of $W_0^{-1,2}(B_\varepsilon)$, $\mathring{W}^{1,2}_0(B_\varepsilon)$, and where C_α is given by relation (5.15). Consequently, u_ε satisfies the following energy equality:

$$\left\|\frac{\nabla u_\varepsilon}{x_3^\alpha}\right\|^2_{L^2(B_\varepsilon)} = \left\langle x_3^{-\alpha+1} g,\ \frac{u_\varepsilon}{x_3^\alpha}\right\rangle_{W_0^{-1,2}(B_\varepsilon),\,\mathring{W}^{1,2}_0(B_\varepsilon)} + \frac{2\alpha-1}{C_\alpha}\left\|\frac{u_\varepsilon}{x_3^{\alpha+1}}\right\|^2_{L^2(B_\varepsilon)}. \tag{6.13}$$

From Hardy's inequality (5.27) one gets

$$[1-(2\alpha-1)C_\alpha]\left\|\frac{\nabla u_\varepsilon}{x_3^\alpha}\right\|^2_{L^2(B_\varepsilon)^3} \leqslant \|x_3^{-\alpha+1} g\|_{W_0^{-1,2}(B_\varepsilon)}\left\|\frac{u_\varepsilon}{x_3^\alpha}\right\|_{W_0^{1,2}(B_\varepsilon)}.$$

Then, one has thanks relations (5.28) and Hardy's inequality (5.27)

$$\left\|\frac{u_\varepsilon}{x_3^\alpha}\right\|_{W_0^{1,\,2}(B_\varepsilon)} \leqslant \sqrt{5}\left\|\nabla\left(\frac{u_\varepsilon}{x_3^\alpha}\right)\right\| \leqslant \sqrt{5}\left(\left\|\frac{\nabla u_\varepsilon}{x_3^\alpha}\right\|_{L^2(B_\varepsilon)^3} + \alpha\left\|\frac{u_\varepsilon}{x_3^{\alpha+1}}\right\|_{L^2(B_\varepsilon)}\right)$$
$$\leqslant (1+\alpha C_\alpha)\sqrt{5}\left\|\frac{\nabla u_\varepsilon}{x_3^\alpha}\right\|_{L^2(B_\varepsilon)^3},$$

and the above estimate becomes

$$\left\|\frac{\nabla u_\varepsilon}{x_3^\alpha}\right\|_{L^2(\mathbb{R}^3_+)} \leqslant \frac{(1+\alpha C_\alpha)}{1-(2\alpha-1)C_\alpha}\sqrt{5}\left\|x_3^{-\alpha+1} g\right\|_{W_0^{-1,\,2}(B_\varepsilon)}.$$

Then, notice that $1+\alpha C_\alpha \leqslant 2$, which implies (6.12) thanks to (5.28). □

Proof of Theorem 6.4. We divide it into two steps.

Step one. For $g \in \mathcal{D}(\mathbb{R}^3_+)$ we prove that there is a unique $u \in W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ solution to (4.38). As $g \in \mathcal{D}(\overline{B_\varepsilon})$ then $g \in W^{-1,\,2}_{1,\,x_3}(B_\varepsilon)$, and by adapting Proposition 6.1, there exists a unique $u_\varepsilon \in \overset{\circ}{W}{}^{1,\,2}_{0}(B_\varepsilon)$ such that

$$-\mathrm{div}\left(\frac{1}{x_3}\nabla u_\varepsilon\right) = g \text{ in } B_\varepsilon.$$

Therefore, by relation (5.30), the functions g and u_ε are exactly in the statement of Lemma 6.6. Thus, there is a constant $C > 0$ depending at most on α such that

$$\|u_\varepsilon\|_{W^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon)} \leqslant C\left\|x_3^{-\alpha+1} g\right\|_{W_0^{-1,\,2}(B_\varepsilon)} \leqslant C\left\|x_3^{-\alpha+1} g\right\|_{W_0^{-1,\,2}(\mathbb{R}^3_+)} \tag{6.14}$$

Then, let us consider $\widetilde{u_\varepsilon}$ the extension by 0 of u_ε outside B_ε to $\mathbb{R}^3_+$. As $u_\varepsilon = 0$ on ∂B_ε, one has

$$\nabla\widetilde{u_\varepsilon} = \widetilde{\nabla u_\varepsilon} \text{ in } \mathbb{R}^3_+.$$

Hence $\widetilde{u_\varepsilon} \in \overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ and

$$\|\widetilde{u_\varepsilon}\|_{W^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)} = \|u_\varepsilon\|_{W^{1,\,2}_{-\alpha,\,x_3}(B_\varepsilon)} \leqslant C\left\|x_3^{-\alpha+1} g\right\|_{W_0^{-1,\,2}(\mathbb{R}^3_+)}. \tag{6.15}$$

As a consequence, there is $u \in \overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ such that $\widetilde{u_\varepsilon}$ converges weakly to u in $\overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$. In the sequel, we want to prove that u is the solution to (4.38). Let $\varphi \in \mathcal{D}(\mathbb{R}^3_+)$. As φ has a compact support in $\mathbb{R}^3_+$, there is $\varepsilon_0 > 0$ such that supp $\varphi \subset B_{\varepsilon_0}$. Then, for any $\varepsilon < \varepsilon_0$, it is clear that

$$\int_{\mathbb{R}^3_+}\frac{1}{x_3}\nabla\widetilde{u_\varepsilon}\cdot\nabla\varphi\,dx = \int_{B_\varepsilon}\frac{1}{x_3}\nabla u_\varepsilon\cdot\nabla\varphi\,dx = \langle g|_{B_\varepsilon},\,\varphi\rangle_{\mathcal{D}'(B_\varepsilon),\,\mathcal{D}(B_\varepsilon)}$$
$$= \langle g,\,\varphi\rangle_{\mathcal{D}'(\mathbb{R}^3_+),\,\mathcal{D}(\mathbb{R}^3_+)}.$$

Since $(\frac{1}{x_3^\alpha}\nabla\widetilde{u_\varepsilon})_\varepsilon$ converges weakly to $\frac{1}{x_3^\alpha}\nabla u$ in $L^2(\mathbb{R}^3_+)^3$ as ε goes to 0, and since $\frac{1}{x_3^{\alpha-1}}\nabla\varphi \in L^2(B_\varepsilon)^3$, one deduces that

$$\int_{\mathbb{R}^3_+}\frac{1}{x_3}\nabla\widetilde{u_\varepsilon}\cdot\nabla\varphi\,dx \underset{\varepsilon\to 0}{\longrightarrow} \int_{\mathbb{R}^3_+}\frac{1}{x_3}\nabla u\cdot\nabla\varphi\,dx.$$

Finally, $u \in \overset{\circ}{W}{}^{1,2}_{-1,\,x_3}(\mathbb{R}^3_+)$ satisfies

$$-\mathrm{div}\left(\frac{1}{x_3}\nabla u\right) = g \text{ in } \mathbb{R}^3_+,$$

and by taking the infimum limit in (6.15), one gets

$$\|u\|_{W^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)} \leqslant C \left\|x_3^{-\alpha+1}g\right\|_{W_0^{-1,\,2}(\mathbb{R}^3_+)},$$

where C is exactly the constant appearing in (6.12). Therefore, u is the unique solution to Problem (4.38), since the above estimates yield $u = 0$ if we take $g = 0$.

Step two. Let $(g_k)_k \subset \mathcal{D}(\mathbb{R}^3_+)$ a sequence such that $(x_3^{-\alpha+1}g_k)_k$ converges to $x_3^{-\alpha+1}g$ in $W_0^{-1,\,2}(\mathbb{R}^3_+)$. Thanks to the step one of the proof, there is a unique $u_k \in \overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ such that

$$-\mathrm{div}\left(\frac{1}{x_3}\nabla u_k\right) = g_k \text{ in } \mathbb{R}^3_+,$$

and also satisfying

$$\|u_k\|_{W^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)} \leqslant C \left\|x_3^{-\alpha+1}g_k\right\|_{W_0^{-1,\,2}(\mathbb{R}^3_+)}.$$

Consequently $(u_k)_k$ is a Cauchy sequence of $\overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ and converges to a given $u \in \overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$. By taking the limit as k goes to 0 one deduces that

$$-\mathrm{div}\left(\frac{1}{x_3}\nabla u\right) = g \text{ in } \mathbb{R}^3_+,$$

with the precise estimates (6.10). □

Remark 6.7. If $\alpha \geqslant 3/2$ and $x_3^{-\alpha+1}g \in W_0^{-1,\,2}(\mathbb{R}^3_+)$, the existence of weak solutions to (4.38) is an open problem.

6.2 Weak solutions in $W^{1,\,2}_{-\alpha}(\mathbb{R}^3_+)$

Problem (6.1), that we recall below

$$\begin{cases} -\Delta u + \dfrac{1}{x_3}\dfrac{\partial u}{\partial x_3} = f & \text{in } \mathbb{R}^3_+, \\ \qquad\qquad\qquad u = 0 & \text{on } \mathbb{R}^2, \end{cases}$$

lends itself well to the study of weak solutions in the spaces $W^{1,\,2}_{-\alpha}(\mathbb{R}^3_+)$, for values of $\alpha \in]0,\,1[$ (note that the case $\alpha = 0$ is already treated in Proposition 6.1, according to Proposition 5.9). Indeed, one has the following estimates.

Lemma 6.8. *Let $0 < \alpha < 1$. Let $f \in W^{-1,\,2}_{-\alpha}(\mathbb{R}^3_+)$ and $u \in W^{1,\,2}_{-\alpha}(\mathbb{R}^3_+)$ a solution to Problem (6.1). Then u satisfies the estimate below*

$$\|u\|_{W^{1,\,2}_{-\alpha}(\mathbb{R}^3_+)} \leqslant \frac{1+\alpha}{\alpha(1-\alpha)}\,\|f\|_{W^{-1,\,2}_{-\alpha}(\mathbb{R}^3_+)},$$

Proof. Let $f \in W^{-1,2}_{-\alpha}$ and $u \in \overset{\circ}{W}{}^{1,2}_{-\alpha}(\mathbb{R}^3_+)$. Thanks to the isomorphism (5.6), one has $\frac{u}{\rho^{2\alpha}} \in \overset{\circ}{W}{}^{1,2}_{\alpha}(\mathbb{R}^3_+)$. Then, thanks to the following formulas

$$\Delta\left(\frac{1}{\rho^\alpha}\right) = -\frac{3\alpha}{\rho^{\alpha+2}} + \alpha(\alpha+2)\frac{|x|^2}{\rho^{\alpha+4}}, \quad \frac{\partial}{\partial x_3}\left(\frac{1}{x_3\rho^\alpha}\right) = -\frac{\rho^2+\alpha x_3^2}{x_3^2\rho^{\alpha+2}},$$

one deduces that

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla(\frac{u}{\rho^{2\alpha}})dx \\
&= \int_{\mathbb{R}^3_+} \left|\frac{\nabla u}{\rho^\alpha}\right|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3_+} \Delta(\frac{1}{\rho^{2\alpha}})\, u^2\, dx \\
&= \int_{\mathbb{R}^3_+} \left|\frac{\nabla u}{\rho^\alpha}\right|^2 dx + 3\alpha\int_{\mathbb{R}^3_+} \left|\frac{u}{\rho^{\alpha+1}}\right|^2 dx - 2\alpha(\alpha+1)\int_{\mathbb{R}^3_+} \frac{|x|^2}{\rho^{2\alpha+4}}u^2\, dx,
\end{aligned} \tag{6.16}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^3_+} \frac{u}{x_3\rho^{2\alpha}}\frac{\partial u}{\partial x_3}\, dx &= -\frac{1}{2}\int_{\mathbb{R}^3_+} \frac{\partial}{\partial x_3}\left(\frac{1}{x_3\rho^{2\alpha}}\right) u^2\, dx = \frac{1}{2}\int_{\mathbb{R}^3_+}\left(\frac{\rho^2+2\alpha x_3^2}{x_3^2\rho^{2\alpha+2}}\right)u^2\, dx \\
&= \frac{1}{2}\int_{\mathbb{R}^3_+} \frac{u^2}{x_3^2\rho^{2\alpha}}\, dx + \alpha\int_{\mathbb{R}^3_+}\left|\frac{u}{\rho^{\alpha+1}}\right|^2 dx.
\end{aligned} \tag{6.17}$$

Now, let us multiply by $\frac{u}{\rho^{2\alpha}} \in \overset{\circ}{W}{}^{1,2}_{\alpha}(\mathbb{R}^3_+)$ the equation of Problem (6.1) and integrate over $\mathbb{R}^3_+$. By collecting (6.16) and (6.17), one obtains the following energy equality

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} \left|\frac{\nabla u}{\rho^\alpha}\right|^2 dx + 4\alpha\int_{\mathbb{R}^3_+}\left|\frac{u}{\rho^{\alpha+1}}\right|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3_+}\left|\frac{u}{x_3\rho^\alpha}\right|^2 dx, \\
&= \left\langle f,\, \frac{u}{\rho^{2\alpha}}\right\rangle_{W^{-1,2}_{-\alpha}(\mathbb{R}^3_+),\, \overset{\circ}{W}{}^{1,2}_{\alpha}(\mathbb{R}^3_+)} + 2\alpha(\alpha+1)\int_{\mathbb{R}^3_+}\frac{|x|^2}{\rho^{2\alpha+4}}u^2\, dx.
\end{aligned} \tag{6.18}$$

Then, note that

$$\int_{\mathbb{R}^3_+} \frac{|x|^2}{\rho^{2\alpha+4}}u^2\, dx = \int_{\mathbb{R}^3_+}\frac{|x|^2}{\rho^2}\left|\frac{u}{\rho^{\alpha+1}}\right|^2 dx \leqslant \int_{\mathbb{R}^3_+}\left|\frac{u}{\rho^{\alpha+1}}\right|^2 dx,$$

and that α is such that $2\alpha(\alpha+1) \leqslant 4\alpha$. As a consequence, one deduces from (6.18) that

$$\int_{\mathbb{R}^3_+} \left|\frac{\nabla u}{\rho^\alpha}\right|^2 dx + 2\alpha(1-\alpha)\int_{\mathbb{R}^3_+}\left|\frac{u}{\rho^{\alpha+1}}\right|^2 dx \leqslant \|f\|_{W^{-1,2}_{-\alpha}(\mathbb{R}^3_+)}\left\|\frac{u}{\rho^{2\alpha}}\right\|_{W^{1,2}_{\alpha}(\mathbb{R}^3_+)}. \tag{6.19}$$

Then, as

$$\begin{aligned}
\left\|\frac{u}{\rho^{2\alpha}}\right\|_{W^{1,2}_{\alpha}(\mathbb{R}^3_+)} &\leqslant \left\|\frac{u}{\rho^{\alpha+1}}\right\|_{L^2(\mathbb{R}^3_+)} + \left\|\rho^\alpha\nabla\left(\frac{u}{\rho^{2\alpha}}\right)\right\|_{L^2(\mathbb{R}^3_+)} \\
&\leqslant \left\|\frac{u}{\rho^{\alpha+1}}\right\|_{L^2(\mathbb{R}^3_+)} + \left\|\frac{\nabla u}{\rho^\alpha}\right\|_{W^{1,2}_{\alpha}(\mathbb{R}^3_+)} + 2\alpha\left\|\frac{x}{\rho}\frac{u}{\rho^{\alpha+1}}\right\|_{L^2(\mathbb{R}^3_+)} \\
&\leqslant 2(1+\alpha)\,\|u\|_{W^{1,2}_{-\alpha}(\mathbb{R}^3_+)},
\end{aligned}$$

one deduces from (6.19) that

$$\|u\|_{W^{1,2}_{-\alpha}(\mathbb{R}^3_+)} \leqslant \frac{1+\alpha}{\alpha(1-\alpha)}\, \|f\|_{W^{-1,2}_{-\alpha}(\mathbb{R}^3_+)}\,,$$

since $2\alpha(1-\alpha) < 1$. □

The following proposition is a version of Theorem 6.4 (see the remark below) and concerns weak solutions having an isotropic behavior.

Proposition 6.9. *Let* $0 < \alpha < 1$ *and let* $f \in W^{-1,2}_{-\alpha}(\mathbb{R}^3_+)$. *Then, there is a unique* $u \in \overset{\circ}{W}{}^{1,2}_{-\alpha}(\mathbb{R}^3_+)$ *solution to Problem* (6.1) *and satisfying the estimate*

$$\|u\|_{W^{1,2}_{-\alpha}(\mathbb{R}^3_+)} \leqslant \frac{1+\alpha}{\alpha(1-\alpha)}\, \|f\|_{W^{-1,2}_{-\alpha}(\mathbb{R}^3_+)}\,.$$

Proof. Let $(f_k)_k \subset \mathcal{D}(\mathbb{R}^3_+)$ converging to f in $W^{-1,2}_{-\alpha}(\mathbb{R}^3_+)$. By Proposition 6.1, let $u_k \in \overset{\circ}{W}{}^{1,2}_{0}(\mathbb{R}^3_+) \hookrightarrow \overset{\circ}{W}{}^{1,2}_{-\alpha}(\mathbb{R}^3_+)$ (see (5.3)) such that

$$-\Delta u_k + \frac{1}{x_3}\frac{\partial u_k}{\partial x_3} = f_k \text{ in } \mathbb{R}^3_+.$$

One deduces from Lemma 6.8 that

$$\|u_k\|_{W^{1,2}_{-\alpha}(\mathbb{R}^3_+)} \leqslant \frac{1+\alpha}{\alpha(1-\alpha)}\, \|f_k\|_{W^{-1,2}_{-\alpha}(\mathbb{R}^3_+)}\,. \tag{6.20}$$

From this inequality, one deduces that the sequence $(u_k)_k$ is a Cauchy sequence of $\overset{\circ}{W}{}^{1,2}_{-\alpha}(\mathbb{R}^3_+)$. Hence, there is a unique $u \in \overset{\circ}{W}{}^{1,2}_{-\alpha}(\mathbb{R}^3_+)$ such that $(u_k)_k$ converges to u in $\overset{\circ}{W}{}^{1,2}_{-\alpha}(\mathbb{R}^3_+)$. Then, by taking the limit when k goes to $+\infty$ in the previous equation and in the estimates, one proves that u satisfies in the distributional sense

$$-\Delta u + \frac{1}{x_3}\frac{\partial u}{\partial x_3} = x_3 g \text{ in } \mathbb{R}^3_+,$$

and that u satisfies the expected estimates. By Lemma 6.8 u is the unique solution to Problem (6.1). □

Remark 6.10. Let us make the following comments.
i) If $\alpha \geqslant 1$ and $f \in W^{-1,2}_{-\alpha}(\mathbb{R}^3_+)$, the existence of weak solutions to (6.1) is an open problem.
ii) Let us note that the assumption $x_3^{-\alpha+1} g \in W_0^{-1,2}(\mathbb{R}^3_+)$ is equivalent to $x_3^{-\alpha} f \in W_0^{-1,2}(\mathbb{R}^3_+)$. And, if f satisfies this last assumption, we easily prove that $f \in W^{-1,2}_{-\alpha}(\mathbb{R}^3_+)$ (the converse assertion being not true). This means that when $0 < \alpha < 1$, the assumption given by Theorem 6.4 is stronger than the one in Proposition 6.9.

Chapter 7

Strong solutions for $x_3^{-\alpha+2} g$ in $L^2(\mathbb{R}^3_+)$

Let us consider the solution $u \in \overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ to Problem (4.38) given by Theorem 6.4, with $0 \leqslant \alpha < 3/2$. Then, let us multiply by $x_3^{-\alpha+2}$ the equation in the distributional sense. We readily see that u satisfies

$$x_3^{-\alpha+1}\Delta u + \frac{1}{x_3^{\alpha}}\frac{\partial u}{\partial x_3} = x_3^{-\alpha+2} g \text{ in } \mathbb{R}^3_+, \quad u = 0 \text{ on } \Gamma.$$

Since we already have $\frac{1}{x_3^{\alpha}}\frac{\partial u}{\partial x_3} \in L^2(\mathbb{R}^3_+)$, we deduce that

$$x_3^{-\alpha+2} g \in L^2(\mathbb{R}^3_+) \Rightarrow x_3^{-\alpha+1}\Delta u \in L^2(\mathbb{R}^3_+).$$

Thus, by taking such a distribution g, we expect that $x_3^{-\alpha+1} D^2 u \in L^2(\mathbb{R}^3_+)$, where $D^2 u$ denotes any second order derivative of u.

Remark 7.1. The choice of such a distribution g is relevant. Indeed, any distribution g on $\mathbb{R}^3_+$ such that

$$x_3^{-\alpha+2} g \in L^2(\mathbb{R}^3_+),$$

satisfies $x_3^{-\alpha+1} g \in W_0^{-1,\,2}(\mathbb{R}^3_+)$. Moreover, there is a generic constant $C > 0$ such that

$$\left\| x_3^{-\alpha+1} g \right\|_{W_0^{-1,\,2}(\mathbb{R}^3_+)} \leqslant C \left\| x_3^{-\alpha+2} g \right\|_{L^2(\mathbb{R}^3_+)}.$$

Theorem 7.2. *Let $0 \leqslant \alpha < 3/2$ and $g \in \mathcal{D}'(\mathbb{R}^3_+)$ such that*

$$x_3^{-\alpha+2} g \in L^2(\mathbb{R}^3_+).$$

Then, the solution $u \in W^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ given by Theorem 6.4 *satisfies*

$$x_3^{-\alpha+1} D^2 u \in L^2(\mathbb{R}^3_+),$$

and there is a constant $C > 0$ such that

$$\left\| x_3^{-\alpha+1} D^2 u \right\|_{L^2(\mathbb{R}^3_+)} \leqslant C \left\| x_3^{-\alpha+2} g \right\|_{L^2(\mathbb{R}^3_+)}.$$

Proof. Let us assume that $x_3^{-\alpha+2} g \in L^2(\mathbb{R}^3_+)$. According to Remark 7.1 the distribution g satisfies $x_3^{-\alpha+1} g \in W_0^{-1,\,2}(\mathbb{R}^3_+)$. Thus, thanks to Theorem 6.4, there is a unique $u \in \overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ solution to (4.38) and a constant $C > 0$ such that

$$\|u\|_{W^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)} \leqslant C \left\| x_3^{-\alpha+1} g \right\|_{W_0^{-1,\,2}(\mathbb{R}^3_+)} \leqslant C \left\| x_3^{-\alpha+2} g \right\|_{L^2(\mathbb{R}^3_+)}. \tag{7.1}$$

Now, we set $v = x_3^{-\alpha+1} u$. Then v satisfies in the distributional sense

$$\begin{aligned} -\Delta v &= -x_3^{-\alpha+1} \Delta u - \alpha(-\alpha+1) \frac{u}{x_3^{\alpha+1}} - 2(-\alpha+1) \frac{1}{x_3^{\alpha}} \frac{\partial u}{\partial x_3} \\ &= x_3^{-\alpha+2} g - \alpha(-\alpha+1) \frac{u}{x_3^{\alpha+1}} + (2\alpha-3) \frac{1}{x_3^{\alpha}} \frac{\partial u}{\partial x_3} \\ &= T. \end{aligned}$$

Since $u \in \overset{\circ}{W}{}^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ it is clear that $T \in L^2(\mathbb{R}^3_+)$. Therefore, one deduces from Corollary 3.4 of [2] page 274, that there is a unique $w \in W_0^{2,\,2}(\mathbb{R}^3_+)$ such that

$$-\Delta w = -\Delta v \text{ in } \mathbb{R}^3_+, \quad w = 0 \text{ on } \Gamma,$$

and a constant $C > 0$ satisfying according to the definition of T and thanks to (6.10):

$$\|w\|_{W_0^{2,\,2}(\mathbb{R}^3_+)} \leqslant C \, \|T\|_{L^2(\mathbb{R}^3_+)} \leqslant C \left\| x_3^{-\alpha+2} g \right\|_{L^2(\mathbb{R}^3_+)}. \tag{7.2}$$

Moreover, from Remark 5.6, one has $v \in \overset{\circ}{W}{}^{1,\,2}_{-1,\,x_3}(\mathbb{R}^3_+)$ and then $v \in \overset{\circ}{W}{}^{1,\,2}_{-1}(\mathbb{R}^3_+)$ by (5.23). Thus, $z = v - w$ satisfies

$$\Delta z = 0 \text{ in } \mathbb{R}^3_+, \quad z = 0 \text{ on } \Gamma,$$

which implies that $z = 0$ (see [2] page 272 Theorem 3.2). As a consequence, $v = w$ belongs to $W_0^{2,2}(\mathbb{R}^3_+)$, and in particular one has for $i,\, j = 1,\, 2,\, 3$

$$\frac{\partial^2}{\partial x_i \partial x_j} (x_3^{-\alpha+1} u) \in L^2(\mathbb{R}^3_+). \tag{7.3}$$

Moreover, from (7.2), one obtains:

$$\left\| \frac{\partial^2}{\partial x_i \partial x_j} (x_3^{-\alpha+1} u) \right\|_{L^2(\mathbb{R}^3_+)} \leqslant C \left\| x_3^{-\alpha+2} g \right\|_{L^2(\mathbb{R}^3_+)}. \tag{7.4}$$

As one has in the distributional sense

$$\begin{aligned} \frac{\partial^2}{\partial x_i \partial x_j} (x_3^{-\alpha+1} u) = x_3^{-\alpha+1} \frac{\partial^2 u}{\partial x_i \partial x_j} &+ (-\alpha+1)(\delta_{j3} \frac{1}{x_3^{\alpha}} \frac{\partial u}{\partial x_i} + \delta_{i3} \frac{1}{x_3^{\alpha}} \frac{\partial u}{\partial x_j}) \\ &- \delta_{j3} \delta_{i3} \alpha(-\alpha+1) \frac{u}{x_3^{\alpha+1}}, \end{aligned}$$

and since $u \in W^{1,\,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$, one deduces from (7.3) that

$$x_3^{-\alpha+1} \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\mathbb{R}^3_+),$$

and from (7.4) and (7.1) the expected estimates. $\square$

Theorem 7.2 does not hold for $\alpha = 3/2$. The following result gives a sufficient condition on g to provide the existence of strong solutions. Firstly, let us recall some basic results on the differential quotient technique.

Notation 7.3. For any $\eta \in \mathbb{R}^2 \times \{0\}$ with $\eta \neq 0$, and any measurable function u defined on $\mathbb{R}^3_+$, we set

$$x \in \mathbb{R}^3_+, \quad \tau_\eta u(x) = u(x+\eta), \quad D_\eta u(x) = \frac{\tau_\eta u(x) - u(x)}{|\eta|}.$$

For any measurable functions u and v defined on $\mathbb{R}^3_+$, one has the following relation:

$$D_\eta(uv) = \tau_\eta u \, D_\eta v + v \, D_\eta u. \tag{7.5}$$

and for any $u, v \in L^2(\mathbb{R}^3_+)$:

$$\int_{\mathbb{R}^3_+} u \, D_\eta v \, dx = \int_{\mathbb{R}^3_+} v \, D_{-\eta} u \, dx. \tag{7.6}$$

Beside, for any $u \in H^1(\mathbb{R}^3_+)$, the following estimate holds

$$\|D_\eta u\|_{L^2(\mathbb{R}^3_+)} \leqslant \|\nabla' u\|_{L^2(\mathbb{R}^3_+)^2} . \tag{7.7}$$

Finally, we also recall the following result whose proof is an exercise.

Proposition 7.4. *Let $u \in L^2(\mathbb{R}^3_+)$ such that*

$$\forall \eta \in \mathbb{R}^2 \times \{0\} \ \text{ with } \eta \neq 0, \quad \|D_\eta u\|_{L^2(\mathbb{R}^3_+)} \leqslant C,$$

where $C > 0$ is a constant not depending on η. Then

$$\nabla' u \in L^2(\mathbb{R}^3_+)^2 \quad \text{and} \quad \|\nabla' u\|_{L^2(\mathbb{R}^3_+)^2} \leqslant C\sqrt{2}.$$

Then, let us state the following result.

Theorem 7.5. *Let $g \in \mathcal{D}'(\mathbb{R}^3_+)$ such that*

$$\sqrt{x_3} g \in L^2(\mathbb{R}^3_+) \cap W_0^{-1,\,2}(\mathbb{R}^3_+).$$

Then, the solution u given by Proposition 6.3 *satisfies additionally:*

$$\frac{\nabla' u}{x_3^{3/2}} \in L^2(\mathbb{R}^3_+)^2, \quad \sqrt{x_3} \nabla(\frac{1}{x_3} \nabla u) \in L^2(\mathbb{R}^3_+)^9. \tag{7.8}$$

Moreover, there is a constant $C > 0$ such that

$$\left\| \frac{\nabla' u}{x_3^{3/2}} \right\|_{L^2(\mathbb{R}^3_+)^2} + \left\| \sqrt{x_3} \nabla(\frac{1}{x_3} \nabla u) \right\|_{L^2(\mathbb{R}^3_+)^9} \leqslant C \, \|\sqrt{x_3} g\|_{L^2(\mathbb{R}^3_+)} .$$

Proof. Recall that $\sqrt{x_3} g \in W_0^{-1,\,2}(\mathbb{R}^3_+)$ is equivalent to $g \in W^{-1,\,2}_{\frac{1}{2},\,x_3}(\mathbb{R}^3_+)$. Thus, by virtue of Proposition 6.3, there is a unique $u \in \overset{\circ}{W}{}^{1,\,2}_{-\frac{1}{2},\,x_3}(\mathbb{R}^3_+)$ solution to Problem (4.38). Moreover, u satisfies (6.8). Then, for any $\eta \in \mathbb{R}^2 \times \{0\}$ with $\eta \neq 0$, one proves that $D_\eta u \in \overset{\circ}{W}{}^{1,\,2}_{-\frac{1}{2},\,x_3}(\mathbb{R}^3_+)$. Indeed, one has

$$\frac{D_\eta u}{x_3^{3/2}} = D_\eta \left(\frac{u}{x_3^{3/2}} \right), \quad \frac{\nabla\left(D_\eta u\right)}{\sqrt{x_3}} = D_\eta \left(\frac{\nabla u}{\sqrt{x_3}} \right),$$

and $D_\eta u = 0$ on Γ. The same reasoning on $D_\eta u$ proves that $D_{-\eta} D_\eta u \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$, so that we can consider the test-function $v = D_{-\eta} D_\eta u$ in (6.8). As relation (7.6) gives:

$$\int_{\mathbb{R}^3_+} \frac{1}{x_3} \nabla u \cdot \nabla (D_{-\eta} D_\eta u)\, dx = \left\| \frac{D_\eta(\nabla u)}{\sqrt{x_3}} \right\|^2_{L^2(\mathbb{R}^3_+)^3},$$

the following equality holds

$$\left\| \frac{D_\eta(\nabla u)}{\sqrt{x_3}} \right\|^2_{L^2(\mathbb{R}^3_+)^3} = \langle g,\, D_{-\eta} D_\eta u \rangle_{W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+), \overset{\circ}{W}{}^{1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+)}. \tag{7.9}$$

Next, let us bound correctly the right-hand side of (7.9). Firstly, let us prove that $D_\eta g \in W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+)$ *i.e.* $\sqrt{x_3} D_\eta g \in W^{-1,2}_0(\mathbb{R}^3_+)$. Since $\sqrt{x_3} g \in L^2(\mathbb{R}^3_+)$ it is clear that $\sqrt{x_3} D_\eta g \in L^2(\mathbb{R}^3_+)$, and one has for any $\varphi \in \mathcal{D}(\mathbb{R}^3_+)$ and thanks to (7.7)

$$\begin{aligned}
\left| \int_{\mathbb{R}^3_+} \varphi \sqrt{x_3} D_\eta g \, dx \right| = \left| \int_{\mathbb{R}^3_+} \sqrt{x_3} g \, D_{-\eta} \varphi \, dx \right| &\leqslant \| \sqrt{x_3} g \|_{L^2(\mathbb{R}^3_+)} \, \| D_{-\eta} \varphi \|_{L^2(\mathbb{R}^3_+)} \\
&\leqslant \| \sqrt{x_3} g \|_{L^2(\mathbb{R}^3_+)} \, \| \nabla \varphi \|_{L^2(\mathbb{R}^3_+)^3} \\
&\leqslant \| \sqrt{x_3} g \|_{L^2(\mathbb{R}^3_+)} \, \| \varphi \|_{W^{1,2}_0(\mathbb{R}^3_+)},
\end{aligned}$$

and we get the result by definition of the space $\overset{\circ}{W}{}^{1,2}_0(\mathbb{R}^3_+)$. Hence, one has thanks to (5.24) for $\alpha = 1/2$

$$\| D_\eta g \|_{W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+)} \leqslant \sqrt{\frac{7}{2}} \, \| \sqrt{x_3} D_\eta g \|_{W^{-1,2}_0(\mathbb{R}^3_+)} \leqslant 2 \, \| \sqrt{x_3} g \|_{L^2(\mathbb{R}^3_+)}. \tag{7.10}$$

Since, in addition, g belongs to $W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+)$, let us prove that for any $v \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$

$$\langle D_\eta g,\, v \rangle_{W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+), \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)} = \langle g,\, D_{-\eta} v \rangle_{W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+), \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)}. \tag{7.11}$$

Let $v \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$ and $(v_k)_k \subset \mathcal{D}(\mathbb{R}^3_+)$ a sequence converging to v in $W^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$. Then, one obtains that

$$\begin{aligned}
\langle D_\eta g,\, v \rangle_{W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+), \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)} &= \lim_{k \to +\infty} \langle D_\eta g,\, v_k \rangle_{W^{-1,2}_{\frac{1}{2},x_3}(\mathbb{R}^3_+), \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)} \\
&= \lim_{k \to +\infty} \int_{\mathbb{R}^3_+} v_k \, D_\eta g \, dx \\
&= \lim_{k \to +\infty} \int_{\mathbb{R}^3_+} g \, D_{-\eta} v_k \, dx.
\end{aligned}$$

Next, as $\nabla v_k / \sqrt{x_3}$ converges to $\nabla v / \sqrt{x_3}$ in $L^2(\mathbb{R}^3_+)$, one deduces that $D_{-\eta}(\nabla v_k / \sqrt{x_3})$ converges to $D_{-\eta}(\nabla v / \sqrt{x_3})$ in $L^2(\mathbb{R}^3_+)$. As a consequence $D_{-\eta} v_k$ converges to $D_{-\eta} v$ in $W^{1,2}_{-\frac{1}{2},x_3}(\mathbb{R}^3_+)$ and relation (7.11) holds. Secondly, one deduces from relation (7.11), (7.10) and relation (5.16), the following estimates

$$\left|\langle g,\, D_{-\eta} D_\eta u\rangle_{W^{-1,\,2}_{\frac{1}{2},\,x_3}(\mathbb{R}^3_+),\mathring{W}^{1,\,2}_{-\frac{1}{2},\,x_3}(\mathbb{R}^3_+)}\right| = \left|\langle D_\eta g,\, D_\eta u\rangle_{W^{-1,\,2}_{\frac{1}{2},\,x_3}(\mathbb{R}^3_+),\mathring{W}^{1,\,2}_{-\frac{1}{2},\,x_3}(\mathbb{R}^3_+)}\right|$$
$$\leqslant \|D_\eta g\|_{W^{-1,\,2}_{\frac{1}{2},\,x_3}(\mathbb{R}^3_+)} \|D_\eta u\|_{W^{1,\,2}_{-\frac{1}{2},\,x_3}(\mathbb{R}^3_+)}$$
$$\leqslant 2\sqrt{2}\,\|\sqrt{x_3} g\|_{L^2(\mathbb{R}^3_+)} \left\|\frac{\nabla(D_\eta u)}{\sqrt{x_3}}\right\|_{L^2(\mathbb{R}^3_+)^3}.$$

To resume, by applying these estimates in (7.9), one proves that

$$\left\|\frac{\nabla(D_\eta u)}{\sqrt{x_3}}\right\|_{L^2(\mathbb{R}^3_+)^3} \leqslant 2\sqrt{2}\,\|\sqrt{x_3} g\|_{L^2(\mathbb{R}^3_+)}.$$

Hence, Hardy's inequality (5.16) yields

$$\forall \eta \in \mathbb{R}^2 \times \{0\} \text{ with } \eta \neq 0, \quad \left\|\frac{\nabla(D_\eta u)}{\sqrt{x_3}}\right\|_{L^2(\mathbb{R}^3_+)^3} + \left\|\frac{D_\eta u}{x_3^{3/2}}\right\|_{L^2(\mathbb{R}^3_+)} \leqslant 4\sqrt{2}\,\|\sqrt{x_3} g\|_{L^2(\mathbb{R}^3_+)}. \tag{7.12}$$

As a consequence of Proposition 7.4, one obtains for $i = 1, 2$

$$\frac{1}{\sqrt{x_3}}\nabla(\frac{\partial u}{\partial x_i}) \in L^2(\mathbb{R}^3_+)^3, \quad \frac{1}{x_3^{3/2}}\frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^3_+), \tag{7.13}$$

with the expected estimates. Moreover, by writing equation (4.38) in the following way:

$$\sqrt{x_3}\frac{\partial}{\partial x_3}\left(\frac{1}{x_3}\frac{\partial u}{\partial x_3}\right) = -\sum_{i=1,\,2}\frac{1}{\sqrt{x_3}}\frac{\partial^2 u}{\partial x_i^2} - \sqrt{x_3} g,$$

one deduces from (7.13) and what precedes that

$$\sqrt{x_3}\frac{\partial}{\partial x_3}\left(\frac{1}{x_3}\frac{\partial u}{\partial x_3}\right) \in L^2(\mathbb{R}^3_+),$$

with the matching estimates. □

More generally, we have the following result.

Theorem 7.6. *Let $3/2 \leqslant \alpha < 5/2$. Let $g \in \mathcal{D}'(\mathbb{R}^3_+)$ such that*

$$x_3^{-\alpha+2} g \in L^2(\mathbb{R}^3_+) \cap W_0^{-1,\,2}(\mathbb{R}^3_+).$$

Then, the solution $u \in \mathring{W}^{1,\,2}_{-\alpha+1,\,x_3}(\mathbb{R}^3_+)$ given by Theorem 6.4 *satisfies*

$$\frac{\nabla' u}{x_3^\alpha} \in L^2(\mathbb{R}^3_+)^2, \quad x_3^{-\alpha+2}\nabla(\frac{1}{x_3}\nabla u) \in L^2(\mathbb{R}^3_+)^9. \tag{7.14}$$

Moreover, there is a constant $C > 0$ such that

$$\left\|\frac{\nabla' u}{x_3^\alpha}\right\|_{L^2(\mathbb{R}^3_+)^2} + \left\|x_3^{-\alpha+2}\nabla(\frac{1}{x_3}\nabla u)\right\|_{L^2(\mathbb{R}^3_+)^9} \leqslant C\left\|x_3^{-\alpha+2} g\right\|_{L^2(\mathbb{R}^3_+)}. \tag{7.15}$$

Proof. We divide the proof into two steps.
Step one. We prove that for $g \in \mathcal{D}(\mathbb{R}^3_+)$ the solution u given by Theorem 6.4 satisfies (7.14) and (7.15). Note that $1/2 \leqslant \alpha - 1 < 3/2$. Then, from the proof of Theorem 6.4, Step one, there is a sequence $(u_\varepsilon)_\varepsilon \subset \overset{\circ}{W}{}^{1,2}_0(B_\varepsilon)$ of solutions to Problem (4.38), satisfying by relation (6.14) the following estimate

$$\|u_\varepsilon\|_{W^{1,2}_{-\alpha+1,\,x_3}(B_\varepsilon)} \leqslant C \left\|x_3^{-\alpha+2} g\right\|_{W_0^{-1,\,2}(B_\varepsilon)}, \tag{7.16}$$

and such that $(\widetilde{u_\varepsilon})_\varepsilon$ converges to u in $W^{1,2}_{-\alpha+1,\,x_3}(\mathbb{R}^3_+)$ as ε goes to 0. If, for $i = 1, 2$, we prove that $\widetilde{u_\varepsilon}$ satisfies

$$\left\|\frac{1}{x_3^\alpha}\frac{\partial \widetilde{u_\varepsilon}}{\partial x_i}\right\|_{L^2(\mathbb{R}^3_+)^2} + \left\|x_3^{-\alpha+1}\nabla(\frac{\partial \widetilde{u_\varepsilon}}{\partial x_i})\right\|_{L^2(\mathbb{R}^3_+)^6} \leqslant C \left\|x_3^{-\alpha+2} g\right\|_{L^2(\mathbb{R}^3_+)}, \tag{7.17}$$

then we prove (7.14) and (7.15). Indeed, if (7.17) holds, one has immediately

$$\frac{1}{x_3^\alpha}\frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^3_+), \quad x_3^{-\alpha+1}\nabla(\frac{\partial u}{\partial x_i}) \in L^2(\mathbb{R}^3_+)^3,$$

$$\left\|\frac{1}{x_3^\alpha}\frac{\partial u}{\partial x_i}\right\|_{L^2(\mathbb{R}^3_+)^2} + \left\|x_3^{-\alpha+1}\nabla(\frac{\partial u}{\partial x_i})\right\|_{L^2(\mathbb{R}^3_+)^6} \leqslant C \left\|x_3^{-\alpha+2} g\right\|_{L^2(\mathbb{R}^3_+)},$$

and it is clear that

$$x_3^{-\alpha+2}\frac{\partial}{\partial x_3}(\frac{1}{x_3}\frac{\partial u}{\partial x_3}) = -\sum_{i=1,\,2}\frac{1}{x_3^{\alpha-1}}\frac{\partial^2 u}{\partial x_i^2} - x_3^{-\alpha+2} g \in L^2(\mathbb{R}^3_+),$$

$$\left\|x_3^{-\alpha+2}\frac{\partial}{\partial x_3}(\frac{1}{x_3}\frac{\partial u}{\partial x_3})\right\|_{L^2(B_\varepsilon)} \leqslant C \left\|x_3^{-\alpha+2} g\right\|_{L^2(\mathbb{R}^3_+)}.$$

Therefore, let us establish estimates (7.17). As $\sqrt{x_3} g \in L^2(\mathbb{R}^3_+) \cap W_0^{-1,\,2}(\mathbb{R}^3_+)$, we automatically deduce, by adapting Theorem 7.5 to the open set B_ε , that u_ε satisfies

$$\frac{\nabla' u_\varepsilon}{x_3^{3/2}} \in L^2(B_\varepsilon)^2, \quad \sqrt{x_3}\nabla(\frac{1}{x_3}\nabla u_\varepsilon) \in L^2(B_\varepsilon)^6. \tag{7.18}$$

Then, let us set $v_\varepsilon = \frac{\partial u_\varepsilon}{\partial x_i}$, for $i = 1, 2$. Firstly, one has $v_\varepsilon \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},\,x_3}(B_\varepsilon) \hookrightarrow \overset{\circ}{W}{}^{1,2}_{-\alpha+1,\,x_3}(B_\varepsilon)$. Secondly, v_ε satisfies in the distributional sense

$$-\mathrm{div}\left(\frac{1}{x_3}\nabla v_\varepsilon\right) = \frac{\partial g}{\partial x_i} \text{ in } B_\varepsilon.$$

Moreover, $x_3^{-\alpha+2} g \in L^2(B_\varepsilon)$ implies $x_3^{-\alpha+2}\frac{\partial g}{\partial x_i} \in W_0^{-1,\,2}(B_\varepsilon)$, and there is a constant $C > 0$ independent on $\varepsilon > 0$ such that

$$\left\|x_3^{-\alpha+2}\frac{\partial g}{\partial x_i}\right\|_{W_0^{-1,\,2}(B_\varepsilon)} \leqslant C \left\|x_3^{-\alpha+2} g\right\|_{L^2(B_\varepsilon)}.$$

Note that the constant $C > 0$ comes from the Poincaré-type inequality given by Proposition (5.1). As a consequence, one deduces from Theorem 7.2 adapted to B_ε, that

$$\|v_\varepsilon\|_{W^{1,2}_{-\alpha+1,\,x_3}(B_\varepsilon)} \leqslant C \left\|x_3^{-\alpha+2}\frac{\partial g}{\partial x_i}\right\|_{W_0^{-1,\,2}(B_\varepsilon)} \leqslant C \left\|x_3^{-\alpha+2} g\right\|_{L^2(B_\varepsilon)} \leqslant C \left\|x_3^{-\alpha+2} g\right\|_{L^2(\mathbb{R}^3_+)},$$

thus one obtains for $i = 1, 2$

$$\left\| \frac{1}{x_3^\alpha} \frac{\partial u_\varepsilon}{\partial x_i} \right\|_{L^2(B_\varepsilon)} + \left\| \frac{1}{x_3^{\alpha-1}} \nabla(\frac{\partial u_\varepsilon}{\partial x_i}) \right\|_{L^2(B_\varepsilon)^3} \leqslant C \left\| x_3^{-\alpha+2} g \right\|_{L^2(\mathbb{R}^3_+)}.$$

Since one has in $\mathbb{R}^3_+$

$$\frac{1}{x_3^\alpha} \frac{\partial \widetilde{u_\varepsilon}}{\partial x_i} = \frac{1}{x_3^\alpha} \widetilde{\frac{\partial u_\varepsilon}{\partial x_i}}, \quad \frac{1}{x_3^{\alpha-1}} \nabla(\frac{\partial \widetilde{u_\varepsilon}}{\partial x_i}) = \frac{1}{x_3^{\alpha-1}} \nabla(\widetilde{\frac{\partial u_\varepsilon}{\partial x_i}}),$$

one deduces (7.17). This achieves the first step of the proof.
Step two. Let $(g_k)_k \subset \mathcal{D}(\mathbb{R}^3_+)$ a sequence such that $(x_3^{-\alpha+2} g_k)_k$ converges to $x_3^{-\alpha+2} g$ in $W_0^{-1,2}(\mathbb{R}^3_+) \cap L^2(\mathbb{R}^3_+)$. Thanks to the step one of the proof, there is a unique $u_k \in \overset{\circ}{W}{}^{1,2}_{-\alpha+1,\, x_3}(\mathbb{R}^3_+)$ such that

$$-\mathrm{div} \left(\frac{1}{x_3} \nabla u_k \right) = g_k \text{ in } \mathbb{R}^3_+,$$

and also satisfying (7.14) and (7.15). Consequently $(u_k)_k$ is a Cauchy sequence of $\overset{\circ}{W}{}^{1,2}_{-\alpha+1,\, x_3}(\mathbb{R}^3_+)$ and converges to $u \in \overset{\circ}{W}{}^{1,2}_{-\alpha+1,\, x_3}(\mathbb{R}^3_+)$ also satisfying (7.14) and (7.15). By taking the limit as k goes to 0 one deduces that

$$-\mathrm{div} \left(\frac{1}{x_3} \nabla u \right) = g \text{ in } \mathbb{R}^3_+,$$

with the precise estimates (7.15). $\square$

Conclusion and prospects of Part II

Let us briefly review the results obtained in this work. In order to study the degenerate equation (4.38), we have introduced, in Chapter 2, a suitable functional framework, constituted by the weighted Sobolev spaces $W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ and $\mathring{W}^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ (see (5.9) and (5.13)). Among the properties that possesses the space $\mathring{W}^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$, we have established a Hardy's inequality in Proposition 5.8, necessary to study the existence and uniqueness of a weak solution to (4.38). Besides, as the spaces $\mathring{W}^{1,2}_{0,\,x_3}(\mathbb{R}^3_+)$ and $\mathring{W}^{1,2}_{0}(\mathbb{R}^3_+)$ are the same (see Proposition 5.15), we have also proved a Hardy's inequality for functions of $\mathring{W}^{1,2}_{0}(\mathbb{R}^3_+)$ (see relation (5.26)).

In chapter 3, we have been interested in the study of weak solutions to (4.38) in the space $W^{1,2}_{-\alpha,\,x_3}(\mathbb{R}^3_+)$ and respectively in $W^{1,2}_{-\beta}(\mathbb{R}^3_+)$. We have proved the existence and uniqueness of a weak solution when $\alpha \in [0,\,3/2[$ and respectively when $\beta \in [0,\,1[$ (see Theorem 6.4 and Proposition 6.9). The problem is still open for $\alpha \geqslant 3/2$ and $\beta \geqslant 1$.

Finally, we have established in Chapter 4 two regularity results for the weak solution u given by Theorem 6.4. In the first one, see Theorem 7.2, we have considered a datum g such that $x_3^{-\alpha+2} g \in L^2(\mathbb{R}^3_+)$ for $\alpha \in [0,\,3/2[$. In the second one, see Theorem 7.6, we have used a datum g satisfying $x_3^{-\alpha+2} g \in L^2(\mathbb{R}^3_+) \cap W^{-1,\,2}_0(\mathbb{R}^3_+)$ for $\alpha \in [3/2,\,5/2[$.

Let us get back to the goal set in the Introduction, that was to establish Theorem 4.26. In fact, we have proved and improved Theorem 4.26. Firstly, the first part of Theorem 4.26 is a consequence of Proposition 6.3. Indeed, it suffices to notice that if g satisfies $x_3^{3/2} g \in L^2(\mathbb{R}^3_+)$, then $g \in W^{-1,\,2}_{\frac{1}{2},\,x_3}$ (see Proposition 5.12). This last condition is also equivalent to $\sqrt{x_3} g \in W^{-1,\,2}_0(\mathbb{R}^3_+)$, see Remark 5.13. Secondly, we have proved that if moreover g satisfies $\sqrt{x_3} g \in L^2(\mathbb{R}^3_+)$, then the regularity of the first-order horizontal derivatives of u is improved (see Theorem 7.5).

Remark 7.7. As a conclusion of this second part, we have proved a stronger result than Theorem 4.26, and we have extended this result to more general data g (see Theorem 7.6).

This study let us think that Theorem 4.25 is not optimal. We propose to investigate this in a forthcoming work, where we shall study Problem (4.39) along the lines of the work we have done here. Moreover, we shall also consider a weight h which does not belong to $L^\infty(\mathbb{R}^3_+)$.

Part III

A degenerate equation in a bounded domain

Introduction of Part III

The last part of the thesis is devoted to the resolution of a degenerate equation, set in a bounded and regular domain $\Omega \subset \mathbb{R}^3$. More precisely, for datum $g : \Omega \to \mathbb{R}$, and a weight $h : \Omega \to \mathbb{R}$ vanishing on the boundary $\Gamma = \partial\Omega$, our problem consists in finding $u : \Omega \to \mathbb{R}$ solution to:

$$-\text{div}\left(\frac{1}{h}\nabla u\right) = g \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma = \partial\Omega. \tag{7.19}$$

Recall that, in the introduction of Part II, we have already given the motivation of the study of Problem (7.19). As we have mentioned it, this problem is studied in [11] in the case of the dimension 2, for a general class of weights h.

In a view to improve the results in [11], we consider a weight h figuring among the ones proposed in this same paper. For the main assumptions on h, see (8.1)-(8.3). Among the improvements we propose, we give an existence and uniqueness result of a weak solution to (7.19), for a datum g in a negative weighted Sobolev space. We also prove a regularity result, by relaxing the assumptions on the weight h and the regularity of the domain Ω. It will be interesting to compare these results to the ones obtained in the $\mathbb{R}^3_+$-case of Part II.

Here, we present the case of dimension 3, in accordance with the $\mathbb{R}^3_+$-case. In fact, these results are also available in the case of the dimension $N \geqslant 2$. For this reason and the one stated above, it will be possible to compare our results to the ones proved in [11].

Part III is divided into two chapters. In Chapter 8, we set the functional framework, by introducing weighted spaces (see (8.4) and (8.5)), adapted to the study of (7.19). We establish for the space (8.5) a Hardy's inequality, and we characterize it as a space of null trace functions. In Chapter 9, we prove the existence and uniqueness result of a weak solution and the regularity result mentioned few lines above.

Chapter 8

Functional framework

Let us consider a bounded domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz-continuous boundary Γ. The weight we use in our work, is a mapping h defined on Ω and satisfying the following properties:

$$h > 0 \text{ in } \Omega, \tag{8.1}$$

$$h \in W^{1,\infty}(\Omega). \tag{8.2}$$

Moreover, we control the weight near the boundary by assuming that:

$$h = \delta \text{ in a neighborhood of } \Gamma, \tag{8.3}$$

where for $x \in \Omega$, $\delta(x)$ is the distance from x to Γ, defined as usual by:

$$\delta(x) = \inf_{y \in \Gamma} |x - y|\,.$$

Note that δ is Lipschitz-continuous near the boundary Γ, which is compatible with assumption (8.2).

Then, let us introduce the following weighted space:

$$W^{1,2}_{-\frac{1}{2},h}(\Omega) := \left\{ u \in \mathcal{D}'(\Omega) \,/\, \frac{u}{h^{3/2}} \in L^2(\Omega) \text{ and } \frac{\nabla u}{h^{1/2}} \in L^2(\Omega)^3 \right\}, \tag{8.4}$$

endowed with the hilbertian norm:

$$\|u\|_{W^{1,2}_{-\frac{1}{2},h}(\Omega)} = \left(\left\| \frac{u}{h^{3/2}} \right\|^2_{L^2(\Omega)} + \left\| \frac{\nabla u}{h^{1/2}} \right\|^2_{L^2(\Omega)^3} \right)^{1/2}.$$

Next, let us consider the space

$$\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{1,2}_{-\frac{1}{2},h}(\Omega)}}. \tag{8.5}$$

The first result we establish concerns functions of the space $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$. It is the matter of a Hardy's inequality which is essential to study Problem (7.19). A similar result is stated and proved in [11] see Lemma 2.3 page 6.

We prove our inequality with the local maps technique, which requires to introduce, beforehand, the following notations. Given two positive numbers a and b, a Lipschitz-continuous function ψ defined on the sphere $\left\{x' \in \mathbb{R}^2 \,/\, |x'| \leqslant b\right\}$ and satisfying $\nabla'\psi(0) = 0$, we define the following sets:

$$\begin{aligned} &\sigma = \left\{x \in \mathbb{R}^3 \,/\, |x'| \leqslant b,\, -a + \psi(x') < x_3 < a + \psi(x')\right\}, \\ &\sigma_+ = \{x \in \sigma \,/\, x_3 > \psi(x')\}\,, \quad \sigma_B = \{x \in \sigma \,/\, x_3 = \psi(x')\}\,. \end{aligned} \tag{8.6}$$

Then, according to the definition of a Lipschitz-continuous domain (see for example [22, 24]), any point $P \in \Gamma$ is the center of a neighborhood σ. Therefore, and since Γ is compact, we can cover Γ by a finite family of open sets $(\sigma_i)_{i=1,..,I}$, where any σ_i possesses its own system of cartesian coordinates $x_i = (x_i', x_i^3)$, with origin at P_i, and is defined by its own mapping ψ_i. Moreover, we shall assume that any σ_i has the same sizes a and b, less or equal to 1, with no loss of generality. We add to this covering of Γ an open set $\sigma_0 \subset\subset \Omega$, such that $(\cup\sigma_i)_{i=1,..,I} \cup \sigma_0$ covers $\overline{\Omega}$. To finish this short paragraph of notations, we shall enlighten the reading of the work by agreeing that σ refers to any open set of the family $(\sigma_i)_{i=1,..,I}$.

We are now in position to prove the following result.

Lemma 8.1. *The following Hardy's inequality holds:*

$$\forall u \in \mathcal{D}(\Omega), \quad \left\|\frac{u}{h^{3/2}}\right\|_{L^2(\Omega)} \leqslant C \left\|\frac{\nabla u}{h^{1/2}}\right\|_{L^2(\Omega)^3}, \tag{8.7}$$

where C is a constant depending only on Ω.

Proof. Let $u \in \mathcal{D}(\Omega)$. The interior estimate is obvious since for $\sigma_0 \subset\subset \Omega$, there are two positive constants h_0 and h_∞ such that $h_0 \leqslant h \leqslant h_\infty$ in σ_0. Therefore, let us focus on the estimate near the boundary Γ. Let us consider the open sets σ and σ_+ as in (8.6). Firstly, we prove the following inequality:

$$\int_{\sigma_+} \frac{u^2}{|x_3 - \psi(x')|^3}\, dx \leqslant \int_{\sigma_+} \frac{|\nabla u|^2}{|x_3 - \psi(x')|}\, dx. \tag{8.8}$$

For any $|x'| < b$ and any $z \leqslant \psi(x') + a$, one has thanks to Green's formula and Cauchy-Schwarz inequality:

$$\begin{aligned} \int_{\psi(x')}^{z} \frac{u^2}{|x_3 - \psi(x')|^3}\, dx_3 &= \frac{1}{2}\int_{\psi(x')}^{z} \frac{1}{|x_3 - \psi(x')|^2} \frac{\partial u^2}{\partial x_3}\, dx_3 \\ &= \int_{\psi(x')}^{z} \left(\frac{u}{|x_3 - \psi(x')|^{3/2}}\right)\left(\frac{1}{|x_3 - \psi(x')|^{1/2}} \frac{\partial u}{\partial x_3}\right) dx_3 \\ &\leqslant \left(\int_{\psi(x')}^{z} \frac{u^2}{|x_3 - \psi(x')|^3}\, dx_3\right)^{1/2} \left(\int_{\psi(x')}^{z} \frac{1}{|x_3 - \psi(x')|}\left|\frac{\partial u}{\partial x_3}\right|^2 dx_3\right)^{1/2}, \end{aligned}$$

from which we deduce that

$$\int_{\psi(x')}^{z} \frac{u^2}{|x_3 - \psi(x')|^3}\, dx_3 \leqslant \int_{\psi(x')}^{z} \frac{1}{|x_3 - \psi(x')|}\left|\frac{\partial u}{\partial x_3}\right|^2 dx_3.$$

Then, (8.8) follows by integrating with respect to x'. Secondly, as ψ is Lipschitz-continuous, it is proved in [39] (page 24 Remark 4.7) that for any $x \in \sigma_+$,

$$\frac{x_3 - \psi(x')}{1 + L_\psi} \leqslant \delta(x) \leqslant x_3 - \psi(x'), \tag{8.9}$$

implying that the quantities $h(x) = \delta(x)$ and $x_3 - \psi(x')$ are equivalent on σ_+. Therefore, one deduces that

$$\int_{\sigma_+} \frac{u^2}{h^3}\, dx \leqslant C \int_{\sigma_+} \frac{|\nabla u|^2}{h}\, dx,$$

and we get (8.7), by using the covering of $\overline{\Omega}$ introduced above. This completes the proof of Lemma 8.1. $\square$

Remark 8.2. From Lemma 8.1, there is a constant $C > 0$ depending only on Ω such that:

$$\forall u \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega), \quad \left\| \frac{u}{h^{3/2}} \right\|_{L^2(\Omega)} \leqslant C \left\| \frac{\nabla u}{h^{1/2}} \right\|_{L^2(\Omega)^3}. \tag{8.10}$$

Moreover, the mapping $u \mapsto \left\| \frac{\nabla u}{h^{1/2}} \right\|_{L^2(\Omega)^3}$ defines a norm on $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$ which is equivalent to the complete norm $\|\cdot\|_{W^{1,2}_{-\frac{1}{2},h}(\Omega)}$.

The following lines are dedicated to a characterization of functions of the space $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$. Firstly, it is convenient to note that if $u \in W^{1,2}_{-\frac{1}{2},h}(\Omega)$ has a compact support in Ω, then $u \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$. Indeed, for $u \in W^{1,2}_{-\frac{1}{2},h}(\Omega)$ with a compact support in Ω, let us consider $\widetilde{u}$ (resp. $\widetilde{h}$) the extension of u (resp. h) by 0 (resp. by 1) to $\mathbb{R}^3$ outside Ω. Then, $\widetilde{u}$ has the following properties:

$$\frac{\widetilde{u}}{\widetilde{h}^{3/2}} \in L^2(\mathbb{R}^3), \quad \frac{\nabla \widetilde{u}}{\widetilde{h}^{1/2}} \in L^2(\mathbb{R}^3)^3.$$

Therefore, if we consider a mollifier sequence $(\rho_\varepsilon)_\varepsilon$, one proves that for ε small enough $(\rho_\varepsilon \star \widetilde{u})|_\Omega$ converges to u in $W^{1,2}_{-\frac{1}{2},h}(\Omega)$. Secondly, let us characterize the space $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$ as a space of null traces functions. From the embedding:

$$W^{1,2}_{-\frac{1}{2},h}(\Omega) \hookrightarrow H^1(\Omega), \tag{8.11}$$

holding since $h \in L^\infty(\Omega)$, any distribution $u \in W^{1,2}_{-\frac{1}{2},h}(\Omega)$ has a trace $u|_\Gamma$ in $H^{1/2}(\Gamma)$. Therefore, it is meaningful to consider the following subspace of $H^1_0(\Omega)$:

$$W(\Omega) = \left\{ u \in W^{1,2}_{-\frac{1}{2},h}(\Omega) \,/\, u = 0 \text{ on } \Gamma \right\}.$$

We propose the following characterization of $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$.

Proposition 8.3. *The following identity holds algebraically:*

$$\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega) = W(\Omega). \tag{8.12}$$

Proof. As $W(\Omega)$ is a closed subspace of $W^{1,2}_{-\frac{1}{2},h}(\Omega)$, it is clear that

$$\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega) \subset W(\Omega).$$

By now, let us prove the inverse inclusion. Let $u \in W(\Omega)$ and, according to the notations we have adopted before, let $(\sigma_i)_{i=1,..,I} \cup \sigma_0$ be an open covering of $\overline{\Omega}$. Next, let us consider a partition of unity $(\theta_i)_{i=1,..,I} \cup \theta_0$ subordinate to the above covering. Then, we have:

$$u = \sum_{i=1}^{I} u\theta_i + u\theta_0 = \sum_{i=1}^{I} u_i + u_0 \text{ in } \Omega.$$

Therefore, the difficulty is reduced to prove that u_0 and any of the functions u_i belong to $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$. Let us start with u_0. It is clear that $u_0 \in W^{1,2}_{-\frac{1}{2},h}(\Omega)$ and also that u_0 has a compact support included in Ω. Therefore, as we have seen it few lines above, the function u_0 belongs to $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$. Then, let us deal with u_i, which is a beat more complicated. By sake of clarity, we drop the index i and denote by σ the open set σ_i. The idea we follow consists in flattening the piece of boundary σ_B (see (8.6)), as it is done for example in [6], with the change of independent variables $y = H(x)$ given by:

$$y' = x', \quad y_3 = x_3 - \psi(x').$$

Note that the inverse map $x = H^{-1}(y)$ is given by $x' = y'$ and $x_3 = y_3 + \psi(x')$. Also note that, since ψ is a Lipschitz-continuous function, H defines a Lipschitz-continuous diffeomorphism establishing a bijection between $\sigma_+ \cup \sigma_B$ and $Q_+ \cup Q_B$, where:

$$\begin{aligned} Q_+ &= H(\sigma_+) = \left\{ y \in \mathbb{R}^3 : \, |y'| < b \text{ and } 0 < y_3 < a \right\}, \\ Q_B &= H(\sigma_B) = \left\{ y \in \mathbb{R}^3 : \, |y'| < b \text{ and } y_3 = 0 \right\}. \end{aligned}$$

In the sequel, if φ is a function defined on σ_+, we denote by $\overline{\varphi}$ the function $\varphi(x)$ written with respect to the y variables, i.e. $\overline{\varphi}(y) = \varphi \circ H^{-1}(y)$.
Before getting back to our purpose, let us make the following general remark. Note that the function $\overline{h}$ satisfies assumptions (8.1)-(8.3), in the context of Q_+, since relation (8.9) holds and is equivalent to:

$$\frac{y_3}{1 + L_\psi} \leqslant \overline{h}(y) \leqslant y_3, \quad \text{for any } y \in Q_+. \tag{8.13}$$

Therefore, it is meaningful to consider the space $W^{1,2}_{-\frac{1}{2},\overline{h}}(Q_+)$. Then, we prove that for $\varphi \in W^{1,2}_{-\frac{1}{2},h}(\sigma_+)$ and $\psi \in W^{1,2}_{-\frac{1}{2},\overline{h}}(Q_+)$, one has $\overline{\varphi} \in W^{1,2}_{-\frac{1}{2},\overline{h}}(Q_+)$ and $\psi \circ H \in W^{1,2}_{-\frac{1}{2},h}(\sigma_+)$, with the related estimates:

$$\|\overline{\varphi}\|_{W^{1,2}_{-\frac{1}{2},\overline{h}}(Q_+)} \leqslant C\, \|\varphi\|_{W^{1,2}_{-\frac{1}{2},h}(\sigma_+)}, \quad \|\psi \circ H\|_{W^{1,2}_{-\frac{1}{2},h}(\sigma_+)} \leqslant C\, \|\psi\|_{W^{1,2}_{-\frac{1}{2},\overline{h}}(Q_+)}.$$

In other words, the diffeomorphism H switches the spaces $W^{1,2}_{-\frac{1}{2},h}(\sigma_+)$ and $W^{1,2}_{-\frac{1}{2},\overline{h}}(Q_+)$. This remark being made, let us consider $v = \overline{u_i}$. One has $v \in W^{1,2}_{-\frac{1}{2},\overline{h}}(Q_+)$ and also $v \in W^{1,2}_{-\frac{1}{2},y_3}(Q_+)$, always by (8.13), with the following norm equivalence on $W^{1,2}_{-\frac{1}{2},\overline{h}}(Q_+)$:

$$C_1\, \|v\|_{W^{1,2}_{-\frac{1}{2},\overline{h}}(Q_+)} \leqslant \|v\|_{W^{1,2}_{-\frac{1}{2},y_3}(Q_+)} \leqslant C_2\, \|v\|_{W^{1,2}_{-\frac{1}{2},\overline{h}}(Q_+)}. \tag{8.14}$$

Next, recall that in Proposition 5.9, we have proved the following equality:

$$\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},y_3}(\mathbb{R}^3_+) = \left\{ u \in W^{1,2}_{-\frac{1}{2},y_3}(\mathbb{R}^3_+) \,/\, u = 0 \text{ on } \mathbb{R}^2 \times \{0\} \right\}.$$

In fact, this equality still holds if we replace $\mathbb{R}^3_+$ by Q_+ (repeat step two of the proof of Proposition 5.9) and therefore, we have the following characterization:

$$\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},y_3}(Q_+) = \left\{ u \in W^{1,2}_{-\frac{1}{2},y_3}(Q_+) \,/\, u = 0 \text{ on } \partial Q \right\}. \tag{8.15}$$

Consequently, since $v = 0$ on ∂Q_+, we deduce from (8.15) that $v \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},y_3}(Q_+)$, and it follows from the first inequality in (8.14) that $v \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},\overline{h}}(Q_+)$. To finish, by transporting v on σ_+, one deduces, according to the above general remark, that $u_i = v \circ H$ comes from a converging sequence a functions of $W^{1,2}_{-\frac{1}{2},h}(\sigma_+)$ with a compact support in σ_+. Therefore, u_i belongs to $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\sigma_+)$, and $u_i \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$. This completes the proof of Proposition 8.3. □

Remark 8.4. If the open set Ω is not bounded, we bring back to the case of Ω bounded by truncation. Indeed, we can adapt with no difficulty the proof of Proposition 5.9 step one, to prove that for any $u \in W(\Omega)$, the sequence $u_k = u\varphi_k \in W(\Omega)$ has a compact support in Ω and that:

$$\lim_{k\to+\infty} \|u_k - u\|_{W^{1,2}_{-\frac{1}{2},h}(\Omega)} = 0,$$

where φ_k is a truncating function.

From the space $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$, let us introduce the space of distributions $W^{-1,2}_{\frac{1}{2},h}(\Omega)$ as the dual space of $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$. Let us characterize the distributions of $W^{-1,2}_{\frac{1}{2},h}(\Omega)$ with the following proposition.

Proposition 8.5. *Any distribution $g \in \mathcal{D}'(\Omega)$ belongs to $W^{-1,2}_{\frac{1}{2},h}(\Omega)$, if and only if there is $f_0 \in L^2(\Omega)$ and $\boldsymbol{f} \in L^2(\Omega)^3$ such that*

$$g = \frac{f_0}{h^{3/2}} + \nabla\cdot\left(\frac{\boldsymbol{f}}{h^{1/2}}\right) \text{ in } \Omega.$$

Moreover one can take $f_0 = 0$.

Proof. Let $f_0 \in L^2(\Omega)$ and $\boldsymbol{f} \in L^2(\Omega)^3$. Since $\frac{1}{h}$ belongs to $L^\infty_{\text{loc}}(\Omega)$, $\frac{f_0}{h^{3/2}}$ and $\frac{\boldsymbol{f}}{h^{1/2}}$ are distributions on Ω. Then, one has for any $\varphi \in \mathcal{D}(\mathbb{R}^3_+)$:

$$\begin{aligned}
\left| \left\langle \frac{f_0}{h^{3/2}} + \nabla\cdot\left(\frac{\boldsymbol{f}}{h^{1/2}}\right), \varphi \right\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \right| &= \left| \int_\Omega f_0 \left(\frac{\varphi}{h^{3/2}}\right) dx - \int_\Omega \boldsymbol{f}\cdot\left(\frac{\nabla\varphi}{h^{1/2}}\right) dx. \right| \\
&\leqslant \|f_0\|_{L^2(\Omega)} \left\| \frac{\varphi}{h^{3/2}} \right\|_{L^2(\Omega)} + \|\boldsymbol{f}\|_{L^2(\Omega)} \left\| \frac{\nabla\varphi}{h^{1/2}} \right\|_{L^2(\Omega)^3} \\
&\leqslant C \|\varphi\|_{W^{1,2}_{-\frac{1}{2},h}(\Omega)}.
\end{aligned}$$

As a consequence, the mapping

$$\varphi \mapsto \left\langle \frac{f_0}{h^{3/2}} + \nabla\cdot\left(\frac{\boldsymbol{f}}{h^{1/2}}\right), \varphi \right\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)},$$

is linear and continuous on $\mathcal{D}(\Omega)$ for the norm $\|\cdot\|_{W^{1,2}_{-\frac{1}{2},h}(\Omega)}$, and one deduces by density that:

$$\frac{f_0}{h^{3/2}} + \nabla\cdot\left(\frac{\boldsymbol{f}}{h^{1/2}}\right) \in W^{-1,2}_{\frac{1}{2},h}(\Omega).$$

Conversely, let $g \in W^{-1,2}_{\frac{1}{2},h}(\Omega)$, and let us consider the mapping

$$u \mapsto (\frac{u}{h^{3/2}}, \frac{\nabla u}{h^{1/2}}),$$

from $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$ into $L^2(\Omega)^4$. This mapping is isometric, and one can consider $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$ as a closed sub-space of $L^2(\Omega)^4$. Thanks to the theorem of Hahn-Banach, since $g \in W^{-1,2}_{\frac{1}{2},h}(\Omega)$, one can extend g to a linear and continuous form on $L^2(\Omega)^4$. Then, one deduces from Riesz representation that there is $(f_0, \boldsymbol{f}) \in L^2(\Omega)^4$ such that:

$$\forall u \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega), \quad g(u) = \int_\Omega f_0 \left(\frac{u}{h^{3/2}}\right) \, dx - \int_\Omega \boldsymbol{f} \cdot \left(\frac{\nabla u}{h^{1/2}}\right) \, dx,$$

which implies that:

$$g = \frac{f_0}{h^{3/2}} + \nabla \cdot \left(\frac{\boldsymbol{f}}{h^{1/2}}\right) \text{ in } \Omega.$$

To complete this proof, note that Remark 8.2 enables to chose $f_0 = 0$, by considering the isometric mapping:

$$u \mapsto \frac{\nabla u}{h^{1/2}},$$

instead of the one we consider above. □

Chapter 9

Theoretical study

9.1 Existence and uniqueness of a weak solution

We begin the theoretical study of Problem (7.19), by investigating a case of existence and uniqueness of a weak solution. A similar result is proved in [11] with a datum g satisfying $h^{3/2}g \in L^2(\Omega)$. Here, we consider a more general datum in a negative weighted space. As we will see later, we save the assumption $h^{3/2}g \in L^2(\Omega)$ to prove a regularity result, complementary to Proposition 9.1.

Proposition 9.1. *Let $g \in W^{-1,\,2}_{\frac{1}{2},\,h}(\Omega)$. Then, there is a unique $u \in \mathring{W}^{1,\,2}_{-\frac{1}{2},\,h}(\Omega)$ solution to Problem* (7.19) *and satisfying*

$$\|u\|_{W^{1,\,2}_{-\frac{1}{2},\,h}(\Omega)} \leqslant C\,\|g\|_{W^{-1,\,2}_{\frac{1}{2},\,h}(\Omega)}\,. \tag{9.1}$$

Proof. Firstly, the choice of g is compatible with a weak solution $u \in \mathring{W}^{1,\,2}_{-\frac{1}{2},\,h}(\Omega)$. Indeed, if we set $\boldsymbol{f} = \frac{\nabla u}{h^{1/2}}$, then $\boldsymbol{f} \in L^2(\Omega)^3$ and one deduces from Proposition 8.5 that

$$-\mathrm{div}\left(\frac{1}{h}\nabla u\right) = -\mathrm{div}\left(\frac{\boldsymbol{f}}{h^{1/2}}\right) \in W^{-1,\,2}_{\frac{1}{2},\,h}(\Omega).$$

Moreover, one has for any $v \in \mathring{W}^{1,\,2}_{-\frac{1}{2},\,h}(\Omega)$,

$$\left\langle -\mathrm{div}\left(\frac{1}{h}\nabla u\right),\, v\right\rangle_{W^{-1,\,2}_{\frac{1}{2},\,h}(\Omega),\,\mathring{W}^{1,\,2}_{-\frac{1}{2},\,h}(\Omega)} = \int_\Omega \frac{\nabla u}{h^{1/2}} \cdot \frac{\nabla v}{h^{1/2}}\,dx.$$

Therefore, $u \in \mathring{W}^{1,\,2}_{-\frac{1}{2},\,h}(\Omega)$ is a solution to (7.19) if and only if u satisfies the following variational formulation:

$$\forall v \in \mathring{W}^{1,\,2}_{-\frac{1}{2},\,h}(\Omega), \quad \int_\Omega \frac{\nabla u}{h^{1/2}} \cdot \frac{\nabla v}{h^{1/2}}\,dx = \langle g,\, v\rangle_{W^{-1,\,2}_{\frac{1}{2},\,h}(\Omega),\,\mathring{W}^{1,\,2}_{-\frac{1}{2},\,h}(\Omega)}\,. \tag{9.2}$$

Next, the bilinear form $a(\cdot,\,\cdot)$ defined by

$$u, v \in \mathring{W}^{1,\,2}_{-\frac{1}{2},\,h}(\Omega), \quad a(u,\,v) = \int_\Omega \frac{\nabla u}{h^{1/2}} \cdot \frac{\nabla v}{h^{1/2}}\,dx.$$

is clearly continuous and coercive on $\overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$ according to Remark 8.2. As a consequence, Lax-Milgram's theorem provides a unique $u \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$ satisfying (9.2), and hence a solution to Problem (7.19). To finish, if we denote by $C > 0$ the positive constant such that:

$$|a(u\,,u)| \geqslant C\,\|u\|^2_{W^{1,2}_{-\frac{1}{2},h}(\Omega)}\,,$$

one deduces the following estimate:

$$\|u\|_{W^{1,2}_{-\frac{1}{2},h}(\Omega)} \leqslant \frac{1}{C}\,\|g\|_{W^{-1,2}_{\frac{1}{2},h}(\Omega)}\,,$$

by taking $v = u$ in (9.2). This completes the proof of Proposition 9.1. □

9.2 A regularity result

Let us consider the solution $u \in \overset{\circ}{W}{}^{1,2}_{-\frac{1}{2},h}(\Omega)$ to Problem (7.19) given by Proposition 9.1. The lack of regularity of the mapping h, see assumption (8.2), prevent us to give a meaning to $\frac{1}{h}\Delta u$ in the classical distributional sense, and hence to write:

$$\operatorname{div}\left(\frac{1}{h}\nabla u\right) = \frac{1}{h}\Delta u - \frac{\nabla h}{h^{3/2}}\cdot\frac{\nabla u}{h^{1/2}} \text{ in } \Omega. \tag{9.3}$$

Thus, we need to give a weaker meaning to the above computations. Let us introduce the space $\mathcal{D}_1(\Omega)$ constituted with function $\varphi \in C^1(\Omega)$ with compact support in Ω. In the same way that we define the space of distributions $\mathcal{D}'(\Omega)$, we also define the space of first-order distributions $\mathcal{D}'_1(\Omega)$, as the set of the linear and continuous form on $\mathcal{D}_1(\Omega)$. The properties of $\mathcal{D}'(\Omega)$ are also true for $\mathcal{D}'_1(\Omega)$ (see [45]), and in particular, we can define the product θT for $\theta \in C^1(\Omega)$ and $T \in \mathcal{D}'_1(\Omega)$ by setting:

$$\forall \varphi \in \mathcal{D}_1(\Omega), \quad \langle \theta T,\, \varphi\rangle_{\mathcal{D}'_1(\Omega),\,\mathcal{D}_1(\Omega)} = \langle T,\, \theta\varphi\rangle_{\mathcal{D}'_1(\Omega),\,\mathcal{D}_1(\Omega)}\,.$$

Let us get back to the reasoning we were holding. Let us assume that Ω is of class $C^{1,1}$, and let us consider the extra assumption:

$$h \in W^{2,\infty}(\Omega). \tag{9.4}$$

Then, $h \in C^1(\Omega)$ and the equality (9.3) holds in $\mathcal{D}'_1(\Omega)$.

Remark 9.2. According to [24], the regularity of the distance function δ near Γ is given by the regularity of Γ. In this case, since Γ is $C^{1,1}$, δ is $C^{1,1}$ in a neighborhood of Γ which is compatible with assumption (9.4).

Next, if we multiply the equation of (7.19) by $h^{3/2}$, we readily see that u satisfies:

$$-h^{1/2}\Delta u + \nabla h\cdot\frac{\nabla u}{h^{1/2}} = h^{3/2} g \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma. \tag{9.5}$$

Since we already have $\frac{\nabla u}{h^{1/2}} \in L^2(\Omega)^3$, we deduce that

$$h^{3/2} g \in L^2(\Omega) \Rightarrow h^{1/2}\Delta u \in L^2(\Omega).$$

Thus, by taking such a distribution g, we expect that $h^{1/2}D^2 u \in L^2(\Omega)$, where $D^2 u$ denotes any second order derivative of u.

Remark 9.3. The choice of such a distribution g is relevant. Indeed, if g satisfies $h^{3/2}g \in L^2(\Omega)$, then $g \in W^{-1,\,2}_{\frac{1}{2},\,h}(\Omega)$, by Proposition 8.5. Moreover, there is a constant $C > 0$ depending only on Ω such that:

$$\|g\|_{W^{-1,\,2}_{\frac{1}{2},\,h}(\Omega)} \leqslant C \left\| h^{3/2} g \right\|_{L^2(\Omega)}.$$

The approach we have introduced here, enables us to prove a new regularity result concerning the weak solution u given by Proposition 9.1. The assumption we consider on g, that is $h^{3/2}g \in L^2(\Omega)$, is used in [11] to prove the existence and uniqueness of u. Whereas to get a regularity result, the authors need to consider $h^{1/2}g \in L^2(\Omega)$ and Ω of class C^3, see the remark below.

Theorem 9.4. *Assume that Ω is a of class $C^{1,1}$ and that h satisfies* (8.1)-(8.3) *and* (9.4). *Let $g \in \mathcal{D}'(\Omega)$ such that*

$$h^{3/2}g \in L^2(\Omega).$$

Then, the solution $u \in W^{1,\,2}_{-\frac{1}{2},\,h}(\Omega)$ given by Proposition 9.1 *satisfies additionally:*

$$h^{1/2}D^2u \in L^2(\Omega)^9, \quad \left\| h^{1/2}D^2u \right\|_{L^2(\Omega)^9} \leqslant C \left\| h^{3/2}g \right\|_{L^2(\Omega)},$$

where $C > 0$ is a constant depending only on Ω.

Proof. Let us assume that $h^{3/2}g \in L^2(\Omega)$. According to Remark 9.3, the distribution g belongs to $W^{-1,\,2}_{\frac{1}{2},\,h}(\Omega)$, and there is a constant $C > 0$ such that

$$\|g\|_{W^{-1,\,2}_{\frac{1}{2},\,h}(\Omega)} \leqslant C \left\| h^{3/2} g \right\|_{L^2(\Omega)}.$$

Thus, thanks to Proposition 9.1, there is a unique $u \in \overset{\circ}{W}{}^{1,\,2}_{-\frac{1}{2},\,h}(\Omega)$ solution to (7.19) and a constant $C > 0$ such that

$$\|u\|_{W^{1,\,2}_{-\frac{1}{2},\,h}(\Omega)} \leqslant C\, \|g\|_{W^{-1,\,2}_{\frac{1}{2},\,h}(\Omega)} \leqslant C \left\| h^{3/2} g \right\|_{L^2(\Omega)}. \tag{9.6}$$

Now, we set $v = h^{1/2}u$. Then $v \in H^1(\Omega)$ and we want to prove in the sequel that $\Delta v \in L^2(\Omega)$ in the sense of first-order distributions. Firstly, note that $\Delta v \in \mathcal{D}_1'(\Omega)$. Then, one has for any $\varphi \in \mathcal{D}_1(\Omega)$:

$$\begin{aligned}
\langle \Delta v,\, \varphi\rangle &= -\int_\Omega h^{1/2}\nabla u\cdot\nabla\varphi\,dx - \frac{1}{2}\int_\Omega \frac{u}{h^{1/2}}\nabla h\cdot\nabla\varphi\,dx \\
&= -\int_\Omega \nabla u\cdot\nabla(h^{1/2}\varphi)\,dx + \int_\Omega \nabla u\cdot\nabla(h^{1/2})\varphi\,dx - \frac{1}{2}\int_\Omega \frac{u}{h^{1/2}}\nabla h\cdot\nabla\varphi\,dx,
\end{aligned}$$

where the duality is taken in the sense of $\mathcal{D}_1'(\Omega)$, $\mathcal{D}_1(\Omega)$. Next, since we have $\Delta u \in \mathcal{D}_1'(\Omega)$, $h^{1/2}\varphi \in \mathcal{D}_1(\Omega)$ and $\frac{u}{h^{1/2}}\nabla h \in L^2_{\mathrm{loc}}(\Omega)$, one deduces that:

$$\langle \Delta v,\, \varphi\rangle = \left\langle \Delta u,\, h^{1/2}\varphi\right\rangle + \left\langle \nabla u\cdot\nabla(h^{1/2}),\, \varphi\right\rangle + \left\langle \ \mathrm{div}\,(\frac{u}{h^{1/2}}\nabla h),\, \varphi\right\rangle.$$

Finally, one obtains that:

$$\Delta v = h^{1/2}\Delta u + \Delta(h^{1/2})u + 2\nabla(h^{1/2})\cdot\nabla u =: T, \tag{9.7}$$

in the sense of first-order distributions on Ω. Next, observe from the following computations that $\Delta(h^{1/2})u$ and $\nabla(h^{1/2})\cdot\nabla u$ belong to $L^2(\Omega)$, since $u \in W^{1,\,2}_{-\frac{1}{2},\,h}(\Omega)$ and $h \in W^{2,\,\infty}(\Omega)$:

$$\Delta(h^{1/2})u = \frac{h\Delta h}{2}\frac{u}{h^{3/2}} - \frac{|\nabla h|^2}{4}\frac{u}{h^{3/2}}, \quad \nabla(h^{1/2})\cdot\nabla u = \frac{\nabla h}{2}\cdot\frac{\nabla u}{h^{1/2}}.$$

To finish, notice that $h^{1/2}\Delta u \in L^2(\Omega)$ thanks to relation (9.5), and $\Delta v \in L^2(\Omega)$. If we sum up, v satisfies the following properties:

$$v \in H^1(\Omega), \quad \Delta v \in L^2(\Omega), \quad v = 0 \text{ on } \Gamma,$$

implying finally that $v \in H^2(\Omega)$. In addition, there is a constant $C > 0$ depending only on Ω such that:

$$\begin{aligned} \|v\|_{H^2(\Omega)} \leqslant C\,\|T\|_{L^2(\Omega)} &\leqslant C\left(\left\|h^{3/2}g\right\|_{L^2(\Omega)} + \|u\|_{W^{1,2}_{-\frac{1}{2},h}(\Omega)}\right) \\ &\leqslant C\left\|h^{3/2}g\right\|_{L^2(\Omega)}, \end{aligned} \tag{9.8}$$

according to the definition of T in (9.7) and thanks to (9.6). In particular, one has for $i,\, j = 1,\, 2,\, 3$:

$$\frac{\partial^2}{\partial x_i \partial x_j}(h^{1/2}u) \in L^2(\Omega), \tag{9.9}$$

and from (9.8), it follows that:

$$\left\|\frac{\partial^2}{\partial x_i \partial x_j}(h^{1/2}u)\right\|_{L^2(\Omega)} \leqslant C\left\|h^{3/2}g\right\|_{L^2(\Omega)}. \tag{9.10}$$

Since one has in the first-order distributional sense:

$$\frac{\partial^2}{\partial x_i \partial x_j}(h^{1/2}u) = h^{1/2}\frac{\partial^2 u}{\partial x_i \partial x_j} + u\frac{\partial^2(h^{1/2})}{\partial x_i \partial x_j} + \frac{\partial h^{1/2}}{\partial x_i}\frac{\partial u}{\partial x_j} + \frac{\partial h^{1/2}}{\partial x_j}\frac{\partial u}{\partial x_i},$$

and since $u \in W^{1,2}_{-\frac{1}{2},h}(\Omega)$ and $h \in W^{2,\infty}(\Omega)$, one deduces from (9.9) that

$$h^{1/2}\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega),$$

and the expected estimate follows from (9.10) and (9.6). $\square$

Remark 9.5. From Theorem 9.4, we deduce that if $h^{3/2}g \in L^2(\Omega)$, u satisfies

$$h^{3/2}\nabla\left(\frac{1}{h}\nabla u\right) \in L^2(\Omega)^9, \quad \left\|h^{3/2}\nabla\left(\frac{1}{h}\nabla u\right)\right\|_{L^2(\Omega)^9} \leqslant C\left\|h^{3/2}g\right\|_{L^2(\Omega)}.$$

Remark 9.6. Let us assume now that Ω is of class C^3 and that $h^{1/2}g \in L^2(\Omega)$. By applying the difference quotient technique adapted to weighted space, we can prove as in [11] that the solution u given by Proposition 9.1 satisfies additionally:

$$h^{1/2}\nabla\left(\frac{1}{h}\nabla u\right) \in L^2(\Omega)^9, \quad \left\|h^{1/2}\nabla\left(\frac{1}{h}\nabla u\right)\right\|_{L^2(\Omega)^9} \leqslant C\left\|h^{3/2}g\right\|_{L^2(\Omega)},$$

which is a supplementary step of regularity for the weak solution u.

Prospects

Compared with the work done in the $\mathbb{R}^3_+$-case, some questions are still open. In the $\mathbb{R}^3_+$-case, we were able to write (4.38) in a different way, see (6.1), which enabled us to prove other existence and uniqueness cases of weak solutions to (4.38). We refer to Proposition 6.1 and Proposition 6.2. In a similar way, we are led to consider the following problem:

$$-\Delta u + \frac{\nabla h}{h} \cdot \nabla u = f \text{ in } \Omega, \tag{9.11}$$

where $f = hg$. Therefore, in accordance with Proposition 6.1, it would be interesting to find a weak solution to Problem (9.11) in the space $H^1_0(\Omega)$, and for a datum $f \in H^{-1}(\Omega)$.

Remark 9.7. Notice that the assumption $hg \in H^{-1}(\Omega)$ does not imply $g \in W^{-1,2}_{\frac{1}{2},h}(\Omega)$ otherwise the question is solved by Proposition 9.1, since $\mathring{W}^{1,2}_{-\frac{1}{2},h}(\Omega) \subset H^1_0(\Omega)$.

This question is still open, and we give in the sequel few elements of investigation. To begin with, there is no suitable formal estimates on u. Indeed, by multiplying (9.11) by u and by integrating over Ω, it appears the following term:

$$\int_\Omega \operatorname{div}\left(\frac{\nabla h}{h}\right) u^2 \, dx,$$

which is out of control. Besides, it does not seem reasonable to reinforce the assumptions on the weight h in order to ensure the positivity of such integral.

Therefore, we were interested in the work of J. Droniou, who studies in [25] a similar problem to (9.11), that is:

$$-\Delta u + \boldsymbol{B} \cdot \nabla u = f \text{ in } \Omega, \tag{9.12}$$

with $\boldsymbol{B} \in L^\infty(\Omega)^3$ and $f \in H^{-1}(\Omega)$. The author reaches to prove the existence and uniqueness of a weak solution $u \in H^1_0(\Omega)$ to (9.12) by studying the dual problem:

$$-\Delta v - \operatorname{div}\,(v\,\boldsymbol{B}) = f_* \text{ in } \Omega.$$

Still, we are not in the framework of Problem (9.12), since $\boldsymbol{B} = \frac{\nabla h}{h}$ is not in $L^\infty(\Omega)^3$ but only in $L^\infty_{\text{loc}}(\Omega)^3$, and his works do not guaranty yet the existence and uniqueness of a weak solution to our problem. Given that, by considering $h + \varepsilon$ instead of h in (9.11), for $\varepsilon > 0$, we obtain the existence and uniqueness of a weak solution u_ε to the following problem:

$$-\Delta u_\varepsilon + \frac{\nabla(h+\varepsilon)}{h+\varepsilon} \cdot \nabla u_\varepsilon = f \text{ in } \Omega.$$

The idea is then to pass to the limits in the above equation, which is still an open question.

Once this problem solved, it would be possible to foresee other cases of weak solutions in different weighted spaces, similar to the ones we have considered in Part II (see (5.9)). Moreover, we would adapt Theorem 9.4 to obtain new complementary regularity results.

In a second time, it would be conceivable to generalize this work to a class of weights of the kind of h. We have briefly thought about considering a mapping $h > 0$ that belongs to $W^{1,\infty}_{\text{loc}}(\Omega)$ and that satisfies:

$$h = \delta^{\alpha} \text{ in a neighborhood of } \Gamma,$$

for a given real power $\alpha > 0$. Then, for $\alpha \geqslant 1$ the mapping h is Lipschitz-continuous, and for $0 < \alpha < 1$ it is only continuous, which brings an additional difficulty. *A priori*, every results we have given in this part are true for this special weight, expected Theorem 9.4 where we need $\alpha \geqslant 1$.

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Résumé :

Dans ce travail, on étudie quelques problèmes d'équations aux dérivées partielles elliptiques que l'on rencontre dans la modélisation d'écoulements réels, comme par exemple la circulation océanique globale. La thèse est divisée en trois parties. La partie 1 est consacrée à l'étude du problème de Stokes dit « hydrostatique » en dimension trois posé dans un domaine borné non nécessairement cylindrique. L'originalité de ces travaux provient du fait que l'on considère des données non homogènes, tant dans l'équation de conservation de la masse que sur la condition aux limites portée sur la vitesse verticale. Pour traiter cette nouvelle situation, on se ramène par équivalence à résoudre un système d'équations primitives linéarisées non homogènes, que l'on résout avec une approche entièrement fonctionnelle et optimale grâce au cadre fonctionnel que l'on considère. Par conséquent, on montre deux cas d'existence et d'unicité d'une solution faible au problème de Stokes hydrostatique avec conditions non homogènes. Les partie 2 et 3 sont consacrées à l'étude d'un modèle elliptique avec un coefficient de diffusion qui peut dégénérer. Ce type d'équations intervient également dans des problèmes géophysiques, que ce soit dans des questions de modélisation de circulation globale, mais aussi dans des problèmes d'infiltration et de milieux poreux. On étudie le cas du demi-espace pour lequel on obtient une théorie optimale de régularité des solutions faibles. On traite enfin le cas général pour lequel on obtient un cas d'existence et d'unicité de solution faible et un résultat de régularité associé.

Mots clés : approximation hydrostatique de Stokes, conditions non homogènes, équations primitives linéarisées, inégalité de type Hardy, lemme de type De Rham, demi-espace, espaces à poids, théorie de régularité.

ON THE HYDROSTATIC STOKES APPROXIMATION AND A DEGENERATED EQUATION

Abstract :

In this work, we study some elliptic partial differential equations problems modelling fluid motion, such as global oceanographic circulation. The thesis is divided into three parts. Part 1 is dedicated to the so called « hydrostatic » Stokes problem in dimension three, set in a bounded domain non necessarily cylindrical. The originality of this work relies in the fact that we consider non homogeneous data, not only in the mass conservation equation but also in the boundary condition carried by the vertical velocity. To handle this new situation, we prove that the difficulty is reduced to solve a non homogeneous linearized primitive equations system, that we solve with an entirely functional and optimal approach given the framework we consider. Therefore, we give two cases of existence and uniqueness of a weak solution to the hydrostatic Stokes problem with non homogeneous conditions. Part 2 and 3 are dedicated to the study of an elliptic model with a diffusion coefficient having a possible degenerated behavior. We can also find these equations in geophysical problems, such as in global circulation modelling questions or seepage and porous media problems. We study the half-space case for which we obtain an optimal regularity theory of weak solutions. Finally, we deal with the general case for which we establish an existence and uniqueness proof of a weak solution, jointly with a regularity result.

Keywords : Stokes hydrostatic approximation, non homogeneous conditions, linearized primitive equations, Hardy type inequality, De Rham-like lemma, half-space, weighted spaces, regularity theory.