

Quelques résultats sur l'équation de Cahn-Hilliard stochastique et déterministe

Ludovic Goudenège

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Ludovic Goudenège

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Quelques résultats sur l'équation de Cahn-Hilliard stochastique et déterministe

Thèse soutenue le 27 novembre 2009

devant le jury composé de :

Nicolas FOURNIER
Professeur à l'Université Paris - Est / *rapporteur*

Tadahisa FUNAKI
Professeur à l'Université de Tokyo/ *rapporteur*

Franck BOYER
Professeur à l'Université Paul Cézanne / *examinateur*

Ying HU
Professeur à l'Université de Rennes 1 / *examinateur*

Daniel MARTIN
Maître de conférences à l'Université de Rennes 1 / *examinateur*

Lorenzo ZAMBOTTI
Professeur à l'Université Pierre et Marie Curie / *examinateur*

Arnaud DEBUSSCHE
Professeur à l'ENS Cachan - Bretagne / *directeur de thèse*

Aux hérons des lacs de Ker Lann,

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Dès qu'on est deux, on va toujours quelque part.

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[1] Je parle des deux humains, et pas des deux grenouilles en plastique qui égaillent la vie des deux poissons Michel et Arnaud.

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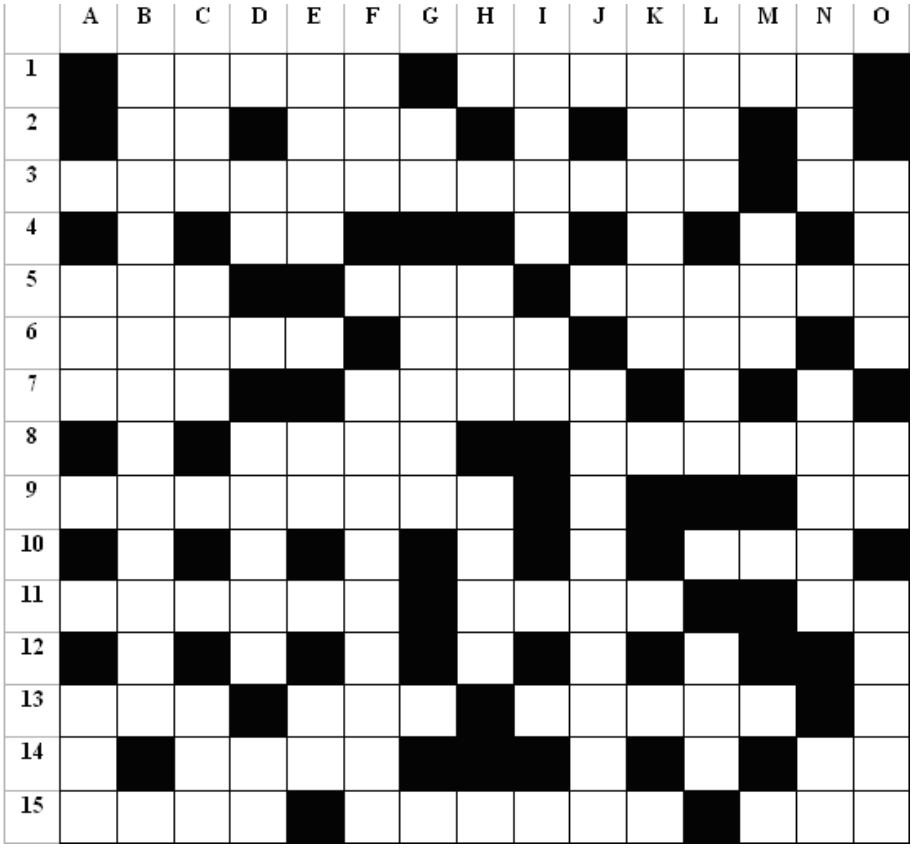
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HORIZONTAL :

1. Terme de politesse. Ne se lève pas avant 14h. 2. Assemblée générale. Oiseau. Aluminium. 3. Dans le domaine de l'aléatoire. Élement de boucle. 4. Marque le lieu. 5. Parasite intestinal. Principe spirituel. Motard et cuisinier. 6. Composé organique. Prière. Signal de détresse. 7. Rongeur. Ce pourquoi je suis là aujourd'hui. 8. Piquant, irritant. Indien sombre. 9. Physicien. Négation. 10. Magicien. 11. Petit homme à l'envers. Déchet de l'organisme. Ytterbium. 13. N'est pas dans la lune. Ride. Danceur de salsa. 14. Expulsion. Pronom personnel. 15. Ce qui est primordial sur terre. Ils sont deux. Solution.

VERTICAL :

A. Mesure de la puissance électrique réactive. Lentille. B. C'est toute ma vie ou presque. C. Le moi. Rigola. Langage d'écriture mathématique. D. Pronom démonstratif. Gentille roumaine. Xénon. E. Physicien. Marque la proximité. Constante mathématique. F. Se rendra. Bossa. G. Carte. Chope. H. Première femme. 1+1. I. Ce que nous serons toujours. Existes. J. Soutien. K. Ce que j'ai essayé de ne pas faire dans ma thèse. L. Exclamation destinée à encourager. Absence de lumière. Troublé. M. Pronom. N. Mèche rebelle. Mélinator. Pronom personnel. O. Ce que l'on fait tous les soirs. Feuillette. Globule d'air.

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Introduction

Mise en place de l'équation

Au milieu des années 50, les deux physiciens Cahn et Hilliard s'interrogent sur les résultats obtenus durant les cent dernières années à propos des énergies d'interface de systèmes non uniformes. En effet, durant cette décennie, les physiciens font des dizaines d'expériences pour répertorier les énergies de tension de surface de toutes sortes de systèmes. Vient alors la question de l'existence d'une formule générale qui pourrait expliquer tous ces résultats expérimentaux. Van der Waals, Guggenheim, Ramsay et Shields tentent de trouver un modèle pertinent permettant d'expliquer les résultats expérimentaux et ce qu'il se passe réellement au niveau d'une interface entre deux liquides. Malheureusement, ils ne trouvent pas de modèle qui soit à la fois général et physiquement satisfaisant. Ce sont ces objections qui amènent Cahn et Hilliard à recommencer les calculs.

En 1957, Cahn et Hilliard publient un premier article (voir [15]) dans lequel ils détaillent les calculs de l'énergie totale d'un système non uniforme. Ils montrent que cette énergie est la somme de deux contributions. La première concerne l'énergie effective du système qu'on peut appeler l'énergie libre réduite, mais qu'on appellera ici simplement l'énergie libre. La deuxième concerne une énergie "gradient" qui est un terme proportionnel au carré du gradient local :

$$\mathcal{E}_{totale} = \mathcal{E}_{libre\ reduite} + \mathcal{E}_{gradient}. \tag{0.0.1}$$

Grâce à cette approche, ils arrivent à retrouver les résultats expérimentaux sur les énergies d'interface, et arrivent même à faire des calculs numériques en considérant des solutions régulières.

On prend en compte un mélange isotherme et isotropique de deux espèces qu'on note A et B. On va noter $u \in [-1, 1]$ le ratio entre les deux composants. u vaut 1 (respectivement -1) dans le cas où le mélange est composé uniquement de l'espèce A (respectivement B). On verra plus tard que ces deux valeurs extrêmes ne sont en réalité jamais atteintes et représentent des zones de singularités. L'énergie totale peut alors s'écrire à l'aide de ce ratio comme une intégrale volumique sur le domaine Ω d'une densité de l'énergie $\bar{f}$:

$$\mathcal{E}_{totale} := \int_{\Omega} \bar{f}(u, \nabla u, \nabla^2 u, \dots)\ \mathrm{d}V. \tag{0.0.2}$$

∇u, $\nabla^2 u$, ... correspondent aux différentielles successives du ratio. Un développement de Taylor tronqué à l'ordre 2 autour d'un système de composition uniforme amène donc à la forme générale suivante :

$$\bar{f}(u) \sim f(u) + L \cdot \nabla u + K_1 \otimes \nabla^2 u + \nabla u \cdot K_2 \cdot \nabla u, \tag{0.0.3}$$

où L, K_1 et K_2 sont des tenseurs physiques d'interactions qui dépendent de $\bar{f}$. Si on note $(x_i)_{i=1,2,3}$

les coordonnées spatiales, on obtient les formules suivantes pour $i = 1, 2, 3$:

$$L_i = \left[\frac{\partial \bar{f}}{\partial\left(\frac{\partial c}{\partial x_i}\right)}\right]_{uniforme} \tag{0.0.4}$$

$$(K_1)_{i,j} = \left[\frac{\partial \bar{f}}{\partial\left(\frac{\partial^2 c}{\partial x_i \partial x_j}\right)}\right]_{uniforme} \tag{0.0.5}$$

$$(K_2)_{i,j} = \left[\frac{\partial^2 \bar{f}}{\partial\left(\frac{\partial c}{\partial x_i}\right)\partial\left(\frac{\partial c}{\partial x_j}\right)}\right]_{uniforme} \tag{0.0.6}$$

Par des arguments de symétrie ou d'invariance par rotation, on montre que $L = \vec{0}$ et que K_1 et K_2 sont des opérateurs homothétiques. On obtient alors :

$$\bar{f}(u) \sim f(u) + \kappa_1 \nabla^2 u + \kappa_2 (\nabla u)^2, \tag{0.0.7}$$

où κ_1 et κ_2 sont maintenant des données scalaires qui peuvent dépendre de la composition uniforme. On intègre sur Ω pour obtenir :

$$\mathcal{E}_{totale} := \int_\Omega \left(f(u) + \kappa_1 \nabla^2 u + \kappa_2 (\nabla u)^2\right) \mathrm{d}V. \tag{0.0.8}$$

On applique le théorème de la divergence pour obtenir :

$$\int_\Omega \kappa_1 \nabla^2 u \ \mathrm{d}V = -\int_\Omega \frac{d\kappa_1}{du} (\nabla u)^2 \ \mathrm{d}V + \int_{\partial\Omega} \kappa_1 \nabla u \cdot \nu \ \mathrm{d}S. \tag{0.0.9}$$

Des conditions de Neumann permettent alors de s'affranchir du dernier terme, et donc de faire disparaître la contribution de $\nabla^2 u$. On obtient donc finalement :

$$\mathcal{E}_{totale} := \int_\Omega \left(f(u) + \kappa (\nabla u)^2\right) \mathrm{d}V, \tag{0.0.10}$$

où

$$\kappa = -\frac{d\kappa_1}{du} + \kappa_2 = -\left[\frac{\partial^2 \bar{f}}{\partial c \, \partial \nabla^2 c}\right]_{uniforme} + \left[\frac{\partial^2 \bar{f}}{\left(\partial |\nabla c|\right)^2}\right]_{uniforme}. \tag{0.0.11}$$

Le paramètre κ est d'une grande importance. On l'appelle le *coefficient gradient*, et il est souvent noté $\varepsilon^2/2$. L'expression de l'énergie obtenue ci-dessus est appelée l'énergie fonctionnelle de Ginzburg-Landau. Le mélange des deux espèces A et B va donc évoluer en tentant de minimiser cette fonctionnelle d'énergie sous la contrainte de masse constante m :

$$\frac{1}{|\Omega|} \int_\Omega u(x, y, z) \ \mathrm{dxdydz} = m. \tag{0.0.12}$$

A l'aide de la fonctionnelle de Ginzburg-Landau, Cahn et Hilliard ont alors défini un potentiel chimique :

$$w := f'(u) - 2\kappa \Delta u, \tag{0.0.13}$$

où Δ est l'opérateur de Laplace. Ils ont ensuite utilisé cette définition dans une équation de transport classique (voir [13], [15] et [16]). Si on note J le flux de matière et $\mathcal{M}(u)$ la mobilité du fluide, les lois classiques de Fick fournissent les deux équations de transport suivantes :

$$\partial_t u = -\nabla \cdot J \text{ et } J = -\mathcal{M}(u) \nabla w. \tag{0.0.14}$$

Tout ceci nous amène enfin à la véritable forme de l'équation de Cahn et Hilliard :

$$\begin{cases} \partial_t u = \nabla \cdot \left[\mathcal{M}(u) \nabla w\right], & \text{sur } \Omega \subset \mathbb{R}^d, \\ w = f'(u) - 2\kappa \Delta u, & \text{sur } \Omega \subset \mathbb{R}^d, \\ \nabla u \cdot \nu = 0 = \nabla w \cdot \nu, & \text{sur } \partial\Omega, \end{cases} \tag{0.0.15}$$

qui se rencontre souvent avec d'autres notations :

$$\begin{cases} \partial_t u = \nabla \cdot [\mathcal{M}(u)\nabla w], & \text{sur } \Omega \subset \mathbb{R}^d, \\ w = \psi(u) - \varepsilon^2 \Delta u, & \text{sur } \Omega \subset \mathbb{R}^d, \\ \nabla u \cdot \nu = 0 = \nabla w \cdot \nu, & \text{sur } \partial\Omega, \end{cases} \tag{0.0.16}$$

où t représente le temps, ε $(= \sqrt{2\kappa})$ est un indicateur de l'épaisseur d'interface, ψ $(= f')$ est un terme non linéaire et ν est un vecteur unitaire normal à la surface $\partial\Omega$. La fonction de mobilité $\mathcal{M}$ ne joue pas un rôle très important. Les modélisations les plus courantes correspondent à $\mathcal{M}(u) = max(0, 1 - u^2)$.

On note $\dot{H}^{-1}$ l'espace des fonctions de masse nulle de H^{-1}. Alors la fonctionnelle d'énergie décrite en (0.0.10) définit un flot gradient dans $\dot{H}^{-1}$ pour l'équation de Cahn-Hilliard. De plus, elle correspond à une fonctionnelle de Lyapunov pour les solutions. En effet, on a le calcul suivant :

$$\nabla_{\dot{H}^{-1}} \mathcal{E}_{totale}(u) = \Delta \left(\varepsilon^2 \Delta u - f'(u) \right), \tag{0.0.17}$$

qui implique la relation suivante :

$$\frac{d}{dt} \mathcal{E}_{totale}(u(t)) = - \left\| \frac{d}{dt} u(t) \right\|_{-1}^2 \leq 0. \tag{0.0.18}$$

Les fonctions d'énergie libre

On a expliqué précédemment que la fonctionnelle de Ginzburg-Landau est composée de deux termes : le terme d'énergie libre (réduite) et le terme gradient. Ces deux termes vont avoir deux rôles distincts dans une évolution typique de Cahn-Hilliard. Le premier va contrôler les premiers instants de l'évolution, tandis que le deuxième est responsable du comportement en temps long de la solution. Le terme d'énergie libre est donc celui sur lequel doit se porter notre attention en premier lieu.

Dans le cas d'un mélange binaire uniforme, cette énergie libre est explicitement donnée par

$$f : u \mapsto T_c \frac{1 - u^2}{2} + T \left[\frac{1+u}{2} \ln\left(\frac{1+u}{2} \right) + \frac{1-u}{2} \ln\left(\frac{1-u}{2} \right) \right], \tag{0.0.19}$$

où T est la température du mélange et $T_c > T$ est la température critique à partir de laquelle le mélange serait purement homogène. Le terme non linéaire ψ est donc donné par :

$$\psi := f' : u \mapsto -T_c u + \frac{T}{2} \ln\left(\frac{1+u}{1-u} \right), \tag{0.0.20}$$

qui est singulier aux points $u = \pm 1$. Ces singularités représentent la première difficulté rencontrée durant une étude numérique. Sur la figure 1, on a représenté la fonction f pour $T = 300$ et $T_c = 320$ qui n'est pas singulière aux points $u = \pm 1$, seule sa dérivée ψ explose en ces points extrémaux. On remarque la forme caractéristique en double puits de cette fonction. Cette forme en double puits est une qualité essentielle qui doit être présente dans toutes les fonctions d'énergie libre. Elle comporte deux minima, et une zone concave située entre ces deux minima.

On a parlé d'une température critique T_c. Cette température correspond physiquement à la température à laquelle le mélange serait purement homogène. Mathématiquement, cela signifie que la fonction d'énergie libre n'a qu'un seul minimum. Les deux minima présents sur la figure 1 sont confondus en un seul point. Sur la figure 1, la courbe en traits pleins correspond à la fonction f lorsque $T = 300$ et $T_c = 320$, et la courbe en traits pointillés correspond à la fonction f lorsque $T = T_c = 320$.

Les singularités en $u = \pm 1$ posent des difficultés lors d'une étude mathématique de l'équation. C'est pourquoi la fonction d'énergie libre a souvent été remplacée par une fonction plus régulière, et

dont l'allure générale remplit les critères physiques désirés. La fonction la plus souvent rencontrée dans la littérature est la fonction suivante :

$$f : u \mapsto \frac{1}{4}\left(1-u^2\right)^2, \tag{0.0.21}$$

dont la dérivée est alors :

$$\psi : u \mapsto u^3 - u. \tag{0.0.22}$$

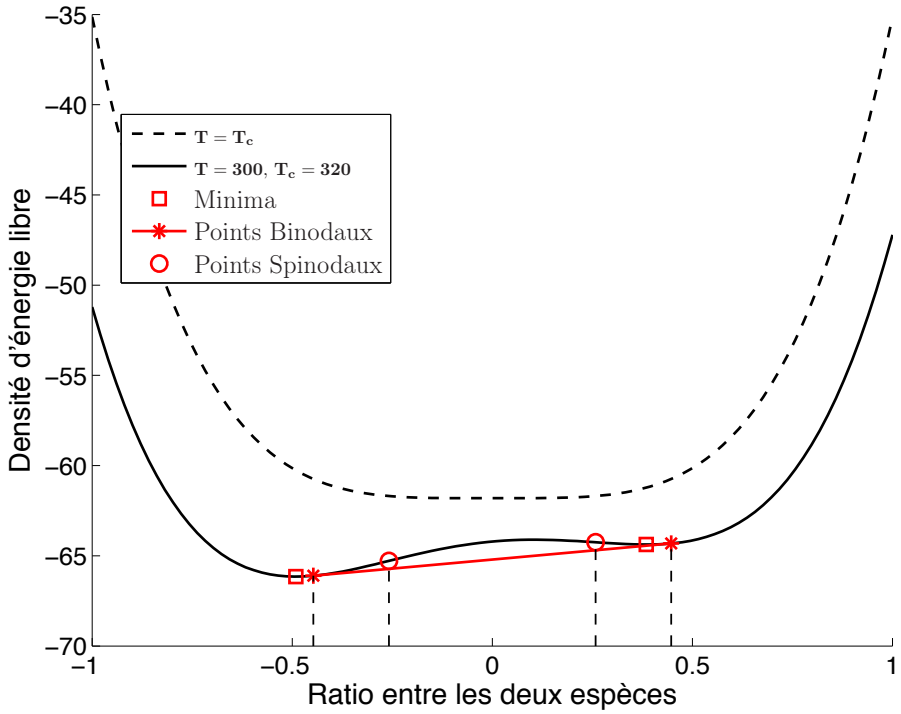


Figure 1: Densité d'énergie libre pour deux températures différentes.

Cette énergie en forme de double puits est souvent référencée dans les articles comme le classique "double well potential". De nombreux résultats ont d'abord été obtenus en considérant cette énergie particulière, et elle reste la plus étudiée à ce jour. Malheureusement, elle ne permet pas de prendre en compte les singularités aux points ± 1. En réalité, de nombreuses formes d'énergies peuvent être étudiées, en particulier du point de vue des simulations numériques. C'est exactement ce que nous ferons dans le chapitre 3 où nous considérerons des énergies polynomiales. Mais on aurait aussi pu considérer d'autres formes d'énergies (asymétrie, singularités aux points ± 1, ...).

Décomposition spinodale et minimisation d'interface

Les évolutions des mélanges binaires étudiés par Cahn, Hilliard et les physiciens de leur époque sont caractérisées par différentes phases. La forme en double puits de la fonction énergie assure que ces phases vont effectivement avoir lieu. En effet, quelque soit la fonction en forme de double puits qu'on choisisse, on retrouve les mêmes phases durant les simulations numériques.

La température joue également un rôle important sur l'évolution du mélange binaire. La fonction f est caractérisée par deux températures T et T_c. Quand la température T est plus grande que la température critiques T_c, la forme en double puits disparaît pour ne laisser qu'un seul puits. La fonction f est convexe et ne possède qu'un seul minimum. Ce minimum correspond physiquement à une état purement homogène où les deux espèces A et B sont confondues en une seule. Mais si on refroidit ce mélange homogène à une température T inférieure à la température critique, la forme en double puits de la fonction va empêcher le mélange de rester dans cette phase purement

homogène. La solution se sépare rapidement en deux phases distinctes. Chacune de ces phases est elle-même un mélange homogène des espèces A et B mais sous des proportions différentes. Ce phénomène est appelé la *décomposition spinodale*. Cette décomposition a lieu lorsque le ratio homogène de départ se situe dans la zone de concavité de la fonction f, car l'état homogène devient instable.

Cette zone de concavité est un intervalle situé entre les deux puits, plus précisément entre deux points particuliers σ_- et σ_+. On les appelle les *points spinodaux*.

Après quelques instants nécessaires à une stabilisation du mélange, les deux phases sont caractérisées par un ratio de concentration proche de certaines valeurs particulières qu'on appelle les *points binodaux* β_- and β_+. Ils sont définis mathématiquement par la relation suivante :

$$f'(\beta_-) = f'(\beta_+) = \frac{f(\beta_+) - f(\beta_-)}{\beta_+ - \beta_-}, \quad \text{avec } \beta_- < \beta_+. \tag{0.0.23}$$

Lorsque la fonction d'énergie libre est symétrique, les points binodaux correspondent aux minima de chaque puits. Mais dans le cas général, ils sont situés sur une droite doublement tangente à la courbe (voir [70]).

Finalement, on définit les *points singuliers* situés en $u = \pm 1$. Par définition, ce sont les points où la fonction ψ devient singulière. Mais dans le cas d'une fonction énergie régulière, on ne peut pas les définir ainsi. On considérera que les points singuliers correspondent aux points extrémaux où la fonction énergie f s'annule.

Une des expériences numériques les plus populaires sur les évolutions de Cahn-Hilliard concerne justement la décomposition spinodale. On peut voir ce phénomène sur la figure 2.

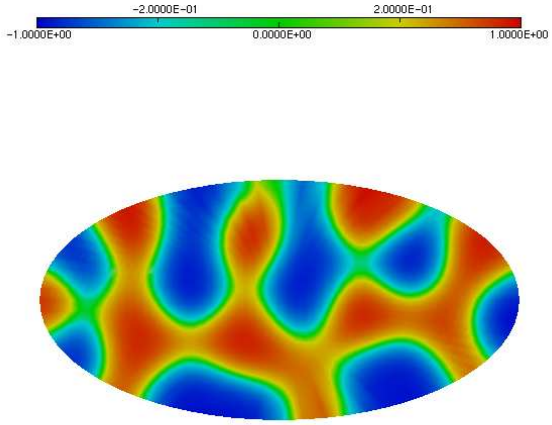


Figure 2: Décomposition spinodale.

Une fois que le mélange s'est séparé en deux phases distinctes, il va continuer d'évoluer mais de manière beaucoup plus lente. Les deux phases sont entièrement caractérisées par l'interface qui les sépare. Cette interface possède une *énergie d'interface*. Elle va donc évoluer pour tenter de minimiser cette énergie, et donc se déplacer pour réduire sa courbure et sa longueur (voir [14] et [64]).

Finalement, la solution atteint un équilibre caractérisé par une interface dont la forme dépend essentiellement du ratio initial entre les concentrations de chaque espèce (voir [52]). Néanmoins, cet équilibre est toujours caractérisé par une interface dont la longueur est minimale. Ce phénomène est classiquement expliqué à l'aide d'une simulation numérique. Cette "classical benchmark cross" permet en particulier de valider qualitativement la justesse du code numérique. On démarre la simulation numérique avec une donnée initiale qui a la forme d'une croix où l'interface est très fine.

En premier lieu, l'interface va diffuser légèrement pour atteindre son épaisseur d'équilibre. Puis, en essayant de minimiser son énergie d'interface, la croix va se transformer pour atteindre une forme circulaire.

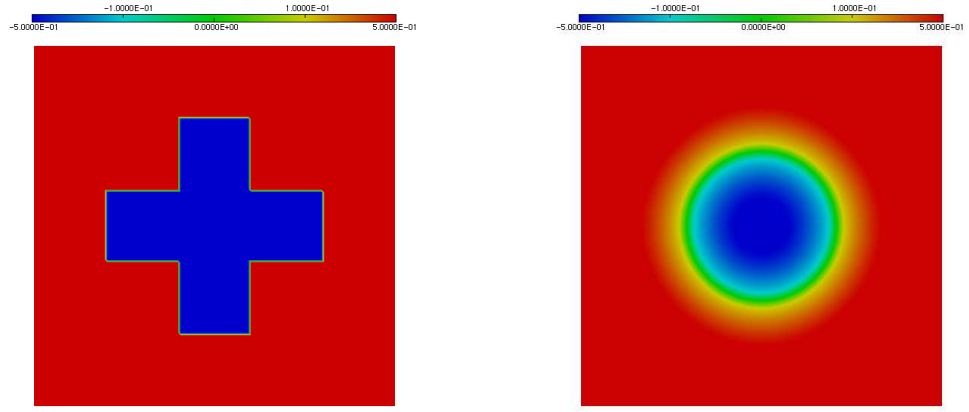


Figure 3: Évolution d'une condition initiale en forme de croix vers une bulle.

En conclusion, on mettra l'accent sur le fait que dans l'énergie totale, c'est la fonction d'énergie libre réduite f qui dirige la décomposition spinodale, et que son choix peut être crucial pour obtenir des résultats physiquement cohérents. Alors que c'est la partie gradient qui dirige l'évolution en temps long, et qui assure cette minimisation d'énergie d'interface. Pour des descriptions plus précises, on pourra consulter [11]. Dans le cas déterministe, et pour différents types de fonctions d'énergie libre on pourra consulter [11] ou [27]. De plus, ce modèle a également été utilisé avec succès pour décrire des phénomènes de séparations plus complexes. Voir en particulier l'étude de Novik-Cohen [58], et les références citées, ou d'autres résultats récents sur la décomposition spinodale et les phénomènes de nucléation dans [5], [9], [40], [53], [54], [66], [67] et [72].

Simulations numériques déterministes

Dans la première partie du chapitre 3 on va s'interroger sur la nécessité de choisir convenablement la fonction énergie f. Comme on l'a précisé précédemment, cette fonction énergie est très souvent la fonction décrite en (0.0.21). Toutefois, cette fonction ne possède aucune singularité. On peut donc se demander si cela a un sens physique de considérer une telle fonction. Le but essentiel est de réaliser une simulation avec la fonction singulière logarithmique décrite en (0.0.19). Mais l'étude théorique et les calculs numériques sont alors plus compliqués. Une idée qui se retrouve dans les démonstrations d'existence théorique est de considérer une suite de fonctions régulières qui converge vers la fonction singulière logarithmique. On va donc considérer une suite de fonctions polynomiales qui converge vers la fonction f. Comme cette fonction f est la limite d'une série entière, les fonctions régulières polynomiales considérées seront tout simplement les séries tronquées. Plus précisément, on note f_{2n} la fonction :

$$f_{2n} := u \mapsto T_c \left(\frac{1-u^2}{2} \right) + T \left[-\ln(2) + \sum_{p=1}^{n} \frac{u^{2p}}{2p(2p-1)} \right] + K_{2n}, \tag{0.0.24}$$

définie à une constante K_{2n} près et pour tout $n \geq 2$.

L'énergie d'un système est toujours définie à une constante près. Toutefois, pour déterminer cette constante, on peut physiquement considérer que cette fonction d'énergie doit être nulle à l'infini. Les raisons mathématiques qui justifient ce choix sont assez claires : dans le cas où on considère un domaine non borné ($\Omega = \mathbb{R}$ ou $\mathbb{R}^2$ par exemple), l'énergie globale est définie comme une intégrale sur tout le domaine. Pour que cette intégrale ne soit pas divergente, il est nécessaire que la limite en $\pm\infty$ soit nulle. La solution est égale à une des deux valeurs binodales à l'infini et

la condition nécessaire de non-divergence est alors donnée par les deux relations suivantes :

$$f\left(\beta^{+}\right) = 0 \text{ et } f\left(\beta^{-}\right) = 0. \tag{0.0.25}$$

Pour la fonction d'énergie logarithmique f, les points binodaux β_+ et β_- ne sont pas explicites mais ils sont tels que

$$T_c\beta_{\pm} = \frac{T}{2}\ln\left(\frac{1+\beta_{\pm}}{1-\beta_{\pm}}\right) \text{ avec } \beta_+ = -\beta_-. \tag{0.0.26}$$

On peut alors se servir de cette relation pour modifier la fonction d'énergie f par un facteur de translation. On note cette nouvelle énergie f définie par

$$f : u \mapsto -T_c\frac{u^2+\beta_{\pm}^2}{2} + \frac{T}{2}\ln\left(\frac{1-u^2}{1-\beta_{\pm}^2}\right) + \frac{T}{2}u\ln\left(\frac{1+u}{1-u}\right). \tag{0.0.27}$$

De la même manière, on peut définir les fonctions d'énergie f_{2n} en choisissant judicieusement les constantes K_{2n}. Une fois ces constantes définies, on peut comparer les énergies.

Une des premières idées consiste à simuler les solutions de Cahn-Hilliard pour les fonctions d'énergie polynomiales f_{2n} pour vérifier que ces solutions sont proches de la solution pour la fonction d'énergie logarithmique. Si on note u_{2n} la solution de Cahn-Hilliard pour la fonction d'énergie f_{2n} sous une même condition initiale, on peut tenter de comparer les énergies de ces solutions, ainsi que les erreurs L^2 avec une solution u pour la fonction d'énergie logarithmique f. En fait, les solutions u_{2n} sont assez proches de la solution u. De plus, on remarque qu'il existe une convergence de u_{2n} vers u. Ceci nous indique que les polynômes de bas degré ne sont pas forcément les mieux adaptés, même s'ils sont qualitativement corrects.

Toutefois, la fonction f_4 est d'un grand intérêt car on connaît explicitement les solutions pour $\Omega = \mathbb{R}$. On peut donc calculer les erreurs sur l'énergie, les pentes, les erreurs L^2 entre les solutions numériques et les solutions explicites. On a également pu mettre en évidence le ralentissement des évolutions pour le potentiel logarithmique (0.0.19) par rapport au potentiel (0.0.21). Lorsqu'on compare les simulations obtenues pour les fonctions polynomiales f_{2n} avec une simulation obtenue pour le potentiel logarithmique limite, on remarque que plus le degré est élevé, plus l'évolution est lente. L'évolution la plus lente étant de fait celle obtenue avec le potentiel logarithmique.

Dans un second temps, on a comparé les simulations obtenues pour des éléments finis dont le degré va de 1 à 10. Pour cela, on a utilisé la bibliothèque d'éléments finis MÉLINA [55] qui offrait l'avantage certain de permettre de modifier le degré d'interpolation des éléments finis sans modifier une seule ligne de code. On avait ainsi accès à des éléments $P1$, $Q1$, $Q3$, $Q7$, $P10$, $Q20$, $P32$ etc. On a toutefois décidé de ne pas dépasser le degré 10, et on va donc essentiellement comparer les simulations obtenues pour des éléments finis Q_i pour $i \in \{1,2,3,4,5,6,7,8,9,10\}$.

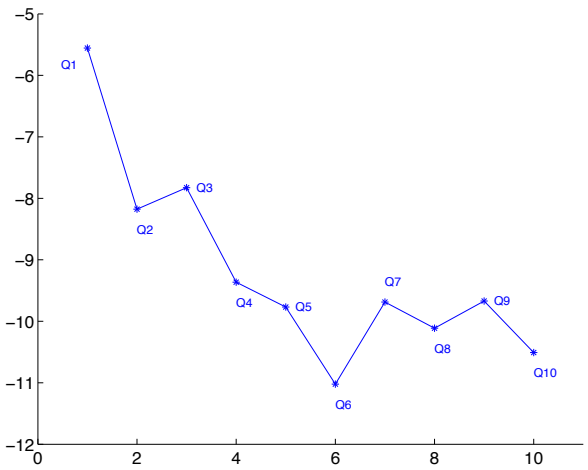


Figure 4: Logarithme décimal d'une erreur d'approximation à complexité fixée.

Il faut toutefois prendre garde lorsqu'on compare des simulations entre bas et hauts degrés. En effet, en dimension 1, sur un maillage de 100 points pour des éléments linéaires par morceaux Q_1, on doit calculer 100 inconnues. Mais sur le même maillage, pour des éléments quadratiques par morceaux Q_2, on doit calculer 200 inconnues. On ne compare donc pas les simulations à maillage fixé, mais à complexité fixée. Une simulation Q_1 sur un maillage de 100 points doit donc être comparée avec une simulation Q_2 sur un maillage de 50 points. En dimension 2, la complexité est encore différente, car elle dépend du carré de l'ordre des éléments.

Lors des simulations numériques, on a mis en évidence la décomposition spinodale, et les phénomènes de réduction d'énergie d'interface. Pour le potentiel en double puits, on a pu comparer les résultats obtenus numériquement avec les résultats exacts. On s'est intéressé plus particulièrement à l'énergie des systèmes, à la bonne approximation de l'interface en dimension 1 et aux erreurs L^2. Il ressort de nos simulations que pour une même complexité, les éléments de hauts-degrés permettent d'obtenir des résultats meilleurs que les éléments de bas degrés. Sur la figure 4, on a représenté les erreurs d'approximation de nos solutions numériques par rapport à une solution exacte pour des éléments $(Q_i)_{i\in[1,10]}$ sous une même complexité qui correspond à un maillage de 630 points en Q_1 et donc de 63 points en Q_{10}. On peut voir que les éléments de hauts degrés fournissent de meilleures approximations que les éléments de bas degrés. Une conclusion de notre étude est que les éléments Q_3 fournissent un bon compromis : ils sont assez précis qualitativement et quantitativement sans alourdir exagérement le temps de calcul.

Les comparaisons effectuées sur les bas et hauts degrés nous ont permis d'obtenir des simulations de qualité avec une très bonne précision. Il a ensuite été possible d'étudier des comportements déterministes assez fins sur les bifurcations de l'équation de Cahn-Hilliard. On a en particulier retrouvé les résultats décrits dans [50] et [52]. En particulier, sur un domaine rectangulaire $\Omega = [0,2]\times[0,1]$, on est dans un cas où les hypothèses du théorème 4.1 de [50] sont vérifiées. D'après ce théorème, si le paramètre d'interface $\varepsilon < \frac{2}{\pi}$, alors il existe exactement deux attracteurs $\pm u_\varepsilon$ qui peuvent s'écrire

$$\pm u_\varepsilon(x,y) = \pm\frac{2\varepsilon}{\sqrt{3}}\sqrt{\frac{1}{\varepsilon^2}-\frac{\pi^2}{4}}\cos\left(\frac{\pi x}{2}\right)+\sqrt{\varepsilon}\;o\left(\left|\frac{1}{\varepsilon^2}-\frac{\pi^2}{4}\right|^{1/2}\right),\quad x\in[0,2], y\in[0,1].$$

On définit alors l'attracteur approché $\pm v_\varepsilon$ par

$$\pm v_\varepsilon(x) := \pm\frac{2\varepsilon}{\sqrt{3}}\sqrt{\frac{1}{\varepsilon^2}-\frac{\pi^2}{4}}\cos\left(\frac{\pi x}{2}\right),\quad x\in[0,2], y\in[0,1].$$

Pour de multiples valeurs du paramètre ε proches de la valeur $\frac{2}{\pi}$, on a obtenu les états stationnaires numériques correspondants qu'on notera u'_ε. On peut alors étudier la convergence vers 0 de la quantité suivante :

$$\frac{\|u'_\varepsilon - v_\varepsilon\|_2}{\|v_\varepsilon\|_2} = \frac{\sqrt{3}\|u'_\varepsilon - v_\varepsilon\|_2}{2\varepsilon\sqrt{\left(\frac{1}{\varepsilon^2}-\frac{\pi^2}{4}\right)}}. \tag{0.0.28}$$

D'après le théorème, cette quantité doit tendre vers 0. Sur la Figure , on a tracé le logarithme décimal de la quantité (0.0.28) en fonction du logarithme décimal de $\frac{1}{\varepsilon^2}-\frac{\pi^2}{4}$.

On voit bien qu'il y a convergence vers 0, et on peut même calculer l'ordre de convergence pour obtenir un résultat plus fin que celui du théorème décrit dans [50].

On étudie aussi le cas d'un domaine non régulier, non couvert par la théorie de Ma et Wang. Dans ce cas, le développement ci-dessus est mis en défaut.

Équation aux dérivées partielles stochastique

Les modèles d'équations aux dérivées partielles déterministes correspondent à des modèles idéalisés. Ils font complètement abstraction des fluctuations thermiques qui sont présentes dans n'importe quel matériel. Afin d'y remédier, Cook a introduit un terme correctif dans l'équation de Cahn et Hilliard censé rendre compte des fluctuations thermiques aléatoires (Voir [19] ou [49]). Dans la littérature physique, le modèle résultant est dénommé *Modèle B* dans la classification de Hohenberg et Halperin (Voir [44]).

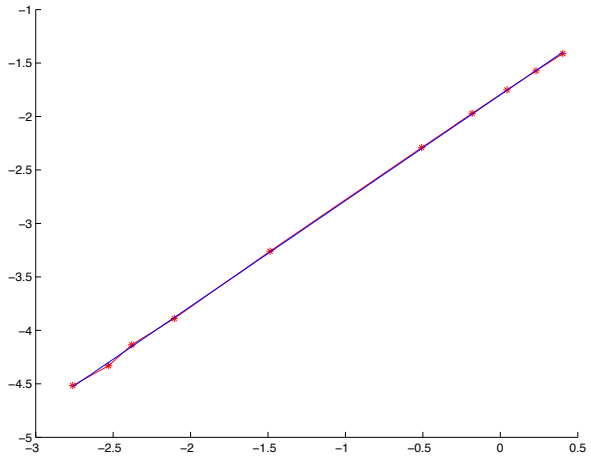


Figure 5: Vitesse de convergence de la quantité (0.0.28).

Si on fixe la mobilité $\mathcal{M}$ à la valeur $1/2$, on obtient l'équation de Cahn-Hilliard-Cook :

$$\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta\left(\varepsilon^2\Delta u - \psi(u)\right) + B\dot{W}, \tag{0.0.29}$$

sur le domaine spatial Ω, où t est le temps, B est un opérateur de corrélation spatiale et $\dot{W}$ est un bruit blanc en espace et en temps. Ce type d'équation a été étudié dans [8], [10], ou [21] où ils considèrent que Ψ, une primitive de ψ, est un polynôme de degré impair.

Etant donnée la structure gradient dans $\dot{H}^{-1}$ de l'équation, il est naturel de considérer un bruit blanc dans cet espace. Mathématiquement, cela revient à choisir pour B un opérateur de dérivation spatiale. On obtient alors un bruit très irrégulier qui ne peut avoir de solutions qu'en dimension d'espace égale à un. On verra plus tard que le cas des bruits réguliers en espace est beaucoup plus facile à traiter et qu'il n'y a pas alors de restriction sur la dimension.

Notre but est d'étudier l'équation de Cahn-Hilliard-Cook avec ce bruit très irrégulier pour le potentiel logarithmique avec les deux singularités en ± 1 en dimension 1. Pour cela, il convient étudier d'abord des équations plus simples. La première étape consiste à considérer une équation sans potentiel, mais avec une force de répulsion qui empêche la solution d'atteindre une valeur interdite. Dans un article récent, Debussche et Zambotti ont considéré une équation avec une mesure de réflexion qui empêche la solution de devenir négative.

Les équations aux dérivées partielles stochastiques avec un terme de réflexion permettent également de modéliser les problèmes d'évolution d'interface aléatoires près d'un mur. De nombreux travaux portent sur des équations d'ordre deux. Sur une telle équation de la chaleur stochastique avec réflexion, de nombreux résultats très fins ont été obtenus (Voir [26], [37], [57], [61], [73], [74] et [75]). Elle peut être décrite par le système suivant :

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2}\Delta v + \zeta + \dot{W}, \text{ dans } (0,1), \\ v(t,1) = v(t,0) = 0, \qquad\qquad \text{pour tout } \ t \geq 0, \\ v \geq 0, \quad d\zeta \geq 0 \text{ et } \displaystyle\int v d\zeta = 0, \end{cases} \tag{0.0.30}$$

où v est une fonction continue et ζ est une mesure positive sur le temps et l'espace. Il a été démontré que les fluctuations d'un modèle d'interface de la forme $\nabla\phi$ à proximité d'un mur convergent en loi vers une solution stationnaire de (0.0.30). De plus, la zone de contact $\{(t,\theta)/v(t,\theta) = 0\}$ et la mesure de réflexion ζ, ont été étudiées très en détail. Un outil crucial dans toutes ces études est la

propriété de monotonie suivante : si on définit v^ε comme l'unique solution de

$$\frac{\partial v^\varepsilon}{\partial t} = \frac{1}{2}\frac{\partial^2 v^\varepsilon}{\partial \theta^2} + \frac{1}{\varepsilon}(v^\varepsilon)^- + \dot{W}, \tag{0.0.31}$$

avec des conditions de Dirichlet au bord en 0 et 1, et pour une condition initiale $v^\varepsilon(0,\cdot) = u_0(\cdot)$, alors

$$0 < \varepsilon \leq \varepsilon' \Longrightarrow v^\varepsilon \geq v^{\varepsilon'}. \tag{0.0.32}$$

Pour d'autres résultats sur les interfaces aléatoires, on pourra consulter [38].

L'équation (0.0.30) n'est pas totalement satisfaisante du point de vue de la modélisation, en effet la masse totale n'est pas préservée au cours du temps. Si on introduit la contrainte de conservation de l'aire entre l'interface et le mur, on est amené à étudier l'équation de Cahn-Hilliard (voir [37] et [75]).

La difficulté qui apparaît alors est que la propriété de monotonie (0.0.32) n'est pas vérifiée pour notre problème. En effet, à cause de la contrainte de conservation de masse

$$\int_0^1 u^\varepsilon(t,\theta)d\theta = \int_0^1 u_0(\theta)d\theta = \int_0^1 u^{\varepsilon'}(t,\theta)d\theta, \quad \text{pour tout } t \geq 0. \tag{0.0.33}$$

Et si $u^\varepsilon \geq u^{\varepsilon'}$ alors nécessairement $u^\varepsilon = u^{\varepsilon'}$, ce qui entre en contradiction avec $\varepsilon \neq \varepsilon'$.

Dans [29], Debussche et Zambotti utilisent une méthode de pénalisation et introduisent une équation approchée pour laquelle cette propriété de monotonie n'est pas vérifiée.

Ils utilisent alors une autre technique pour prouver l'existence et l'unicité d'une solution. Ils utilisent le caractère *strong Feller* des semi-groupes de transition, prouvent le caractère tendu des solutions stationnaires à l'aide d'une décomposition de Lyong-Zheng, et étudient les mesures invariantes pour prouver l'existence de solutions dans $H^{-1}(0,1)$. L'autre point crucial est d'étudier les semi-groupes de transition des équations approchées pour prouver qu'ils convergent en un certain sens vers un semi-groupe de transition markovien d'une solution continue. Précisément, ils considèrent l'équation suivante :

$$\frac{\partial X}{\partial t} = -\frac{1}{2}\Delta\Big(\Delta X + \eta\Big) + B\dot{W}, \text{ dans } (0,1) \tag{0.0.34}$$

avec des conditions aux limites de Neumann et une condition initiale, où η est une mesure positive soumise à la condition de contact

$$\int X \mathrm{d}\eta = 0. \tag{0.0.35}$$

Les travaux de Debussche et Zambotti constituent donc une première approche dans la résolution du problème de Cahn-Hilliard-Cook.

La seconde étape est de considérer un potentiel avec seulement une seule singularité logarithmique. C'est ce que nous faisons dans le chapitre 1. Plus généralement, nous considérons une singularité (relocalisée en zéro pour simplifier les calculs) qui peut être de type logarithmique ou égale à une fonction puissance négative pour une réflexion en 0. On considère donc l'équation suivante :

$$\begin{cases} \dfrac{\partial X}{\partial t} = -\dfrac{1}{2}\Delta\Big(\Delta X + f(X) + \eta\Big) + B\dot{W}, & \text{dans } (0,1) \\ \nabla X \cdot \nu = 0 = \nabla(\Delta X)\cdot\nu, & \text{sur } \{0,1\}, \\ X(0,x) = x, \quad x \in L^2, \end{cases} \tag{0.0.36}$$

avec les mêmes notations qu'en (0.0.29) et (0.0.34), et où la fonction f est définie par

$$f(x) := f_{\ln}(x) := \begin{cases} -\ln x, & \text{pour tout } x > 0 \\ +\infty, & \text{pour tout } x \leq 0, \end{cases}$$

ou pour $\alpha > 0$ par

$$f(x) := f_\alpha(x) := \begin{cases} x^{-\alpha}, & \text{pour tout } x > 0 \\ +\infty, & \text{pour tout } x \leq 0. \end{cases}$$

Notre idée est d'utiliser des équations approchées définies à l'aide de fonctions approchées qui convergent vers la fonction f mais qui sont plus régulières. Avec les mêmes techniques que celles développées dans [29], on va montrer que les solutions de ces équations approchées convergent en un certain sens vers une solution de l'équation (0.0.36).

Finalement, dans le chapitre 2, on considérera le modèle complet avec le potentiel logarithmique et ses deux singularités en ± 1.

Une seule réflexion

Si on souhaite utiliser les techniques développées par Debussche et Zambotti, on doit considérer une suite de fonctions approchées qui soient globalement lipschitziennes. On note $\{f^n\}_{n\in\mathbb{N}}$ la suite de fonctions lipschitziennes qui converge vers f sur l'intervalle $]0,+\infty[$ définie pour tout $n \in \mathbb{N}$ par :

$$f^n : x \mapsto f(x^+ + 1/n), \qquad \text{pour tout } x \in \mathbb{R}. \tag{0.0.37}$$

Pour chaque fonction approchée f^n, on définit une équation approchée :

$$\begin{cases} dX^n + \frac{1}{2}(A^2X^n + Af^n(X^n))dt = BdW, \\ \\ X^n(0,x) = x, \end{cases} \tag{0.0.38}$$

où A est le Laplacien muni des conditions aux limites de Neumann i.e. :

$$D(A) = \text{ Domaine de } A = \{h \in W^{2,2}(0,1) : h'(0) = h'(1) = 0\}$$

où $W^{n,p}$ et $||.||_{W^{n,p}}$ représentent l'espace classique de Sobolev $W^{n,p}(0,1)$ et sa norme associée. On note $\bar{h}$ la moyenne d'une fonction $h \in L^2(0,1)$:

$$\bar{h} = \int_0^1 h(\theta)d\theta.$$

On définit pour tout $c \in \mathbb{R}$:

$$L^2_c = \{h \in L^2(0,1) : \bar{h} = c\},$$

ainsi que $L^2 = L^2(0,1)$. Pour tout $\gamma \in \mathbb{R}$, on définit $(-A)^{\gamma/2}$ par une méthode classique d'interpolation, et le domaine de $(-A)^{\gamma/2}$ est

$$V_\gamma := D((-A)^{\gamma/2}),$$

muni de la semi-norme classique :

$$|h|_\gamma = \left(\sum_{i=1}^{+\infty}(-\lambda_i)^\gamma h_i^2\right)^{1/2},$$

et de la norme :

$$\|h\|_\gamma = \left(|h|^2_\gamma + \bar{h}^2\right)^{1/2}$$

associée au produit scalaire défini pour tout $h, k \in V_\gamma$ par $(h,k)_\gamma$.
On notera simplement H l'espace V_{-1}. La moyenne joue un rôle important, et on travaillera souvent avec des fonctions de moyenne fixée $c \in \mathbb{R}$. Pour cela, on définit $H_c = \{h \in H, \bar{h} = c\}$ pour tout $c \in \mathbb{R}$. Enfin, on pose

$$D(B) = W_0^{1,2}(0,1), B = \frac{\partial}{\partial\theta}, D(B^*) = W^{1,2}(0,1) \text{ et } B^* = -\frac{\partial}{\partial\theta}.$$

On remarque que $BB^* = -A$. De plus, on note Π le projecteur orthogonal de V_{-1} sur H_0. Enfin, pour simplifier les notations, on note $O_{s,t} := [s,t] \times [0,1]$ pour $s,t \in [0,T]$ avec $s < t$ et $T > 0$, et $O_t = O_{0,t}$ pour $0 \le t \le T$. Étant donnée une mesure ζ sur $O_{s,t}$ et une fonction continue v sur $O_{s,t}$, on définit

$$\langle v, \zeta \rangle_{O_{s,t}} := \int_{O_{s,t}} v d\zeta.$$

Evidemment, la suite des constantes de Lipschitz des fonctions f^n tend vers $+\infty$ lorsque $n \to +\infty$. On peut toutefois obtenir un contrôle de la suite des solutions et montrer que cette suite de solutions converge en un certain sens vers une fonction u. Malheureusement, comme dans les travaux de Debussche et Zambotti, la limite des solutions approchées n'est pas solution de l'équation limite, et un terme supplémentaire de réflexion doit être ajouté. On obtient donc que u est solution de l'équation suivante :

$$\begin{cases} dX + \frac{1}{2}(A^2X + Af(X) + \eta)dt = BdW, \\ \\ X(0,x) = x, \end{cases} \tag{0.0.39}$$

Le terme de réflexion est une mesure qui n'agit que lorsque u touche le point singulier 0. Cela se traduit par la *condition de contact* :

$$\int X d\eta = 0. \tag{0.0.40}$$

Pour donner un sens à l'équation avec un terme mesure, on doit passer par une formulation faible. On dit que X est solution faible si pour tout $0 < \delta \leq t \leq T$, pour tout $h \in D(A)$

$$\begin{aligned} \langle X(t,x), h\rangle &= \langle X(\delta,x), h\rangle - \frac{1}{2}\int_\delta^t \langle X(s,x), A^2h\rangle \mathrm{ds} - \frac{1}{2}\int_\delta^t \langle Ah(\theta), f(X(s,x))\rangle \mathrm{ds} \\ &\quad -\frac{1}{2}\big\langle Ah, \eta\big\rangle_{O_{\delta,t}} - \int_\delta^t \langle Bh, \mathrm{d}W\rangle. \end{aligned}$$

Comme on a déjà besoin d'une formulation faible pour donner un sens à l'équation stochastique, cela ne pose pas de problème. Les crochets présents dans la formule doivent toutefois être compris tantôt comme étant des produits scalaires, tantôt comme des crochets de dualité. Toute la difficulté est en fait déjà contenue dans cette formulation faible, car il faut pouvoir donner un sens à tous les termes. Le premier terme non trivial est le terme :

$$\int_\delta^t \langle Ah(\theta), f(X(s,x))\rangle \mathrm{ds} \tag{0.0.41}$$

Les fonctions test $h \in D(A^2)$ sont telles que $Ah \in L^\infty([0,1])$. Donc si on montre que $f(X(\cdot,x)) \in L^1(O_T)$, les crochets pourront être considérés comme des crochets de dualité entre $L^1(O_T)$ et $(L^1(O_T))' = L^\infty(O_T)$. On devra donc montrer que $f(X(\cdot,x)) \in L^1(O_T)$. Le deuxième terme non trivial est le terme :

$$\big\langle Ah, \eta\big\rangle_{O_{\delta,t}} \tag{0.0.42}$$

La mesure η est une mesure sur $]0,T] \times [0,1]$. Pour que le terme soit bien défini, il faut que pour tout $h \in D(A^2)$, $|\langle Ah, \eta\rangle_{O_{s,t}}| < +\infty$ pour tout $0 < s \leq t \leq T$. Ceci impose en particulier que la masse totale de la mesure η est finie (i.e. $|\eta(O_{\delta,T})| < +\infty$ pour tout $\delta \in]0,T]$). Sous ces conditions, l'unicité n'est pas difficile à démontrer. En effet, on a le résultat suivant :

Proposition 1 *Soit $x \in \mathcal{C}([0,1], \mathbb{R}^+)$ avec $\overline{x} > 0$. Soit $(X^i, \eta^i, W), i = 1,2$ deux solutions faibles de l'équation (0.0.39) avec $X_0^1 = x = X_0^2$. Alors $(X^1, \eta^1) = (X^2, \eta^2)$.*

En considérant $Y(t) = X^1(t,x) - X^2(t,x)$ et $\zeta = \eta^1 - \eta^2$, alors formellement, en prenant le produit scalaire avec Y dans V_{-1}, on obtient pour tout $0 \leq t \leq T$:

$$\begin{aligned} (Y(t), Y(t)) &= (Y(0), Y(0)) + \frac{1}{2}\big\langle Y(s), \zeta\big\rangle_{O_t} \\ &\quad + \int_0^t \langle Y(s,.), f(X^1(s,x)) - f(X^2(s,x))\rangle ds. \end{aligned} \tag{0.0.43}$$

Puisque f est décroissante, le dernier terme est négatif. Enfin, grâce à la condition de contact, on obtient :

$$\|Y(t)\|_{-1}^2 - \|Y(0)\|_{-1}^2 \leq 0. \tag{0.0.44}$$

Ce qui prouve l'unicité des solutions. Il faut toutefois vérifier que les mesures η^1 et η^2 sont également égales. On utilise pour cela un lemme fondamental :

Lemme 1 *Soit ζ une mesure finie signée sur $O_{\delta,T}$ et $V \in \mathcal{C}(O_{\delta,T})$. On suppose qu'il existe une fonction positive continue $c_T : [0,T] \to \mathbb{R}^+$ telle que :*

i) pour tout $r \in [\delta, T]$, pour tout $h \in \mathcal{C}([0,1])$, tel que $\bar{h} = 0$, $\langle h, \zeta \rangle_{O_{r,T}} = 0$,

ii) pour tout $r \in [\delta, T]$, $\overline{V(r,\cdot)} = c_T(r)$ avec $\langle V, \zeta \rangle_{O_{r,T}} = 0$,

alors ζ est la mesure identiquement nulle.

Les conditions du lemme sont justement vérifiées par la mesure ζ qui est donc la mesure identiquement nulle, et l'unicité est donc vérifiée.

Pour chaque $c \in \mathbb{R}$, (0.0.38) définit un semi-groupe de transition $(P_t^{n,c})_{t\geq 0}$:

$$P_t^{n,c}\phi(x) = \mathbb{E}[\phi(X^n(t,x)],\ t \geq 0, x \in H_c,\ \phi \in \mathcal{B}_b(H_c),\ n \in \mathbb{N}^*,$$

où $\mathcal{B}_b(H_c)$ représente les fonctions boréliennes bornées dans H_c. La deuxième étape consiste à étudier les mesures invariantes des équations (0.0.38). La convergence de ces mesures invariantes sera cruciale pour démontrer l'existence d'une limite de la suite des solutions des équations approchées. On note

$$U^n(x) := \int_0^1 F^n(x(\theta))d\theta, \quad x \in L^2(0,1), \tag{0.0.45}$$

où les fonctions F^n sont des primitives des fonctions $-f^n$. Les mesures invariantes sur L^2_c des équations approchées sont alors données explicitement par :

$$\nu_c^n(\mathrm{dx}) = \frac{1}{Z_c^n} \exp\left(-U^n(x)\right) \mu_c(\mathrm{dx}), \tag{0.0.46}$$

où Z_c^n est une constante de normalisation. Si on note ∇ le gradient dans $L^2(0,1)$, alors $\nabla U^n(x) = -f^n(x)$. Et comme f^n est décroissante, U^n est un potentiel convexe.

Soit

$$K = \{x \in L^2(0,1), x \geq 0\},$$

on peut montrer que $\mu_c(K) > 0$ pour $c > 0$. On peut alors définir U le potentiel associé à la fonction f

$$U(x) := \begin{cases} \int_0^1 F(x(\theta))\mathrm{d}\theta & \text{si } \int_0^1 \left|F(x(\theta))\right|\mathrm{d}\theta < +\infty \text{ et } x \in K, \\ +\infty & \text{sinon,} \end{cases}$$

ce qui nous donne la convergence suivante :

Proposition 2 *Pour $c > 0$,*

$$\nu_c^n \rightharpoonup \nu_c := \frac{1}{Z_c} \exp^{-U(x)} \mathbb{1}_{x\in K}\mu_c(\mathrm{dx}),\ \textit{quand } n \to +\infty,$$

où Z_c est une constante de normalisation.

La mesure invariante limite ne charge que les fonctions positives qui sont dans L^2_c. Cette remarque est de la plus haute importance, car elle nous indique qu'il faut raisonner dans les espaces de masse constante H_c.

L'étape suivante concerne alors directement la convergence des solutions approchées vers la solution limite. On prouve l'existence de solutions stationnaires de l'équation (0.0.36) et que ces solutions sont limites des solutions stationnaires des équations approchées (0.0.38) en un certain sens (voir [7] ou [33] par exemple).

On fixe $c > 0$ et on considère l'unique solution stationnaire (en loi) de l'équation (0.0.38) notée $\hat{X}_c^n$. On va démontrer le théorème suivant :

Théorème 1 *Soit $c > 0$ et $T > 0$. $\hat{X}_c^n$ converge en probabilité lorsque n tend vers l'infini vers un processus $\hat{X}_c$ qui appartient à $\mathcal{C}(O_T)$. En outre, $f(\hat{X}_c) \in L^1(O_T)$ presque sûrement et il existe une mesure η sur O_T telle que $(\hat{X}_c, \eta, W)$ est une solution stationnaire forte de l'équation (0.0.39). De plus, si on note*

$$\mathrm{d}\eta^n = f^n(\hat{X}_c^n(t,\theta))\mathrm{dtd}\theta - f(\hat{X}_c(t,\theta))\mathrm{dtd}\theta,$$

alors $(\hat{X}_c^n, \eta^n, W)$ converge en loi vers $(\hat{X}_c, \eta, W)$.

La démonstration de ce théorème est découpée en 3 étapes. Dans la première étape, on fait l'hypothèse qu'une suite extraite de $(\hat{X}_c^n)_{n\in\mathbb{N}}$ converge en loi, et on montre que sa limite $\hat{X}_c$ vérifie $f(\hat{X}_c) \in L^1(O_T)$ presque sûrement. Dans la deuxième étape, toujours sous la même hypothèse de convergence, on prouve qu'à une extraction près les mesures $(\eta^n)_{n\in\mathbb{N}}$ convergent vers une mesure positive η, et que $(\hat{X}_c, \eta)$ est une solution faible dans un sens probabiliste. Il reste alors à prouver que la suite des lois de $(\hat{X}_c^n)_{n\in\mathbb{N}}$ est tendue, et d'utiliser l'unicité pour conclure dans une troisième étape.

Dans la première étape, on peut se ramener à l'aide d'un théorème de Skorohod au cas où $\hat{X}_c^n$ converge presque sûrement uniformément vers $\hat{X}_c$. On étudie alors la suite des mesures suivantes :

$$\mathrm{d}\xi^n := f^n(\hat{X}_c^n(t,\theta))\mathbb{1}_{\hat{X}_c^n<1}\mathrm{d}t\mathrm{d}\theta.$$

On a tronqué à la valeur 1, car la fonction logarithmique $f = f_{\ln}$ change de signe à cet endroit. Si on avait considéré une fonction puissance $f = f_\alpha$, on auru pu s'en passer et considérer simplement

$$\mathrm{d}\xi^n := f^n(\hat{X}_c^n(t,\theta))\mathrm{d}t\mathrm{d}\theta.$$

C'est ici que la positivité entre en jeu. En réalité pour la fonction logarithmique $f_{\ln}$, la partie négative qu'on peut noter

$$\mathrm{d}\rho^n := f_{\ln}^n(\hat{X}_c^n(t,\theta))\mathbb{1}_{\hat{X}_c^n\geq 1}\mathrm{d}t\mathrm{d}\theta.$$

converge simplement, et n'est pas difficile à traiter. De ce fait, si on considère des fonctions non linéaires possédant un signe constant près de leur singularité, nos résultats peuvent se généraliser.

Soit $y \in D(A)$ avec $\bar{y} = 0$, en prenant $h \in D(A^2)$ tel que $y = Ah$ comme fonction test dans la formulation faible (0.0.41), on obtient que pour tout $0 \leq t \leq T$, $\langle y, \xi^n\rangle_{O_t}$ possède une limite quand $n \to +\infty$. En effet, par l'uniforme convergence de $\hat{X}_c^n$ vers $\hat{X}_c^n$, on obtient

$$\begin{aligned}
\lim_{n\to+\infty}\langle y,\xi^n\rangle_{O_t} &= \lim_{n\to+\infty}\Bigg(\Big(\hat{X}_c^n(t,x),y\Big)_{-1} - (x,y)_{-1} \\
&\quad -\frac{1}{2}\int_0^t\langle\hat{X}_c^n(s,x),Ay\rangle\mathrm{d}s + \int_0^t(By,\mathrm{d}W)_{-1}\Bigg). \\
&= \Big(\hat{X}_c(t,x),y\Big)_{-1} - (x,y)_{-1} \\
&\quad -\frac{1}{2}\int_0^t\langle\hat{X}_c(s,x),Ay\rangle\mathrm{d}s + \int_0^t(By,\mathrm{d}W)_{-1}
\end{aligned}$$

Mais comme y est de moyenne nulle, on n'obtient pas la convergence de ξ^n. Il faudrait montrer que la masse totale $\xi^n(O_T)$ est bornée uniformément pour $n \in \mathbb{N}$. En effet, on aurait pour tout $h \in D(A)$

$$\lim_{n\to+\infty}\langle h,\xi^n\rangle = \lim_{n\to+\infty}\langle h-\bar{h},\xi^n\rangle + \lim_{n\to+\infty}\langle\bar{h},\xi^n\rangle, \tag{0.0.47}$$

où chaque terme serait borné. Donc, à extraction près d'une sous suite, on pourrait définir une mesure limite. La majoration uniforme de la masse totale se démontre à l'aide du lemme suivant :

Lemme 2 *Soit $T > 0$ et $\{\mu^n\}_{n\in\mathbb{N}}$ une suite de mesures finies positives sur O_T. On suppose qu'il existe $\{w^n\}_{n\in\mathbb{N}}$ une suite de fonctions de $\mathcal{C}(O_T)$ telles que w^n converge uniformément vers w quand n tend vers l'infini. On suppose alors qu'il existe une fonction $M_T : \mathcal{C}(O_T) \to \mathbb{R}^+$, une constante positive m_T et une fonction continue $c_T : [0,T] \to \mathbb{R}^+$ telles que*

$$\textit{pour tout } h \in D(A) \textit{ tel que } \bar{h} = 0, \quad \langle h,\mu^n\rangle_{O_T} \leq M_T(h), \quad \textit{pour tout } n \in \mathbb{N}, \tag{0.0.48}$$

$$\textit{pour tout } t \in [0,T], \int_0^1 w(t,\theta)d\theta = c_T(t) > 0 \tag{0.0.49}$$

et

$$\langle w^n,\mu^n\rangle_{O_T} \leq m_T. \tag{0.0.50}$$

Alors il existe une constante $\tilde{M}_T$ telle que

$$\text{pour tout } h \in \mathcal{C}(O_T), \quad \langle h, \mu^n \rangle_{O_T} \leq \tilde{M}_T \|h\|_\infty, \quad \text{pour tout } n \in \mathbb{N}. \tag{0.0.51}$$

et en particulier la suite $\mu^n(O_T)$ est bornée uniformément pour $n \in \mathbb{N}$.

On vient d'expliquer que la condition (0.0.48) est vérifiée par les mesures ξ^n. Il reste donc à trouver des fonctions w^n qui convergent uniformément vers une fonction w de moyenne positive. Le choix des fonction w^n dépend ici du type de singularité de la fonction f.

Dans le cas logarithmique, il suffit de remarquer que la fonction $(x)^+ f^n_{\ln}(x)\mathbb{1}_{x<1}$ est uniformément bornée pour $n \in \mathbb{N}$, alors il existe une constante positive m_T telle que

$$\langle (\hat{X}^n_c)^+, \xi^n \rangle_{O_T} \leq m_T. \tag{0.0.52}$$

Et puisque $\hat{X}_c$ est presque sûrement positif, $(\hat{X}^n_c)^+$ converge uniformément vers $\hat{X}_c$. Grâce au Lemme 2, on obtient bien $\limsup_{n\to+\infty} \xi^n(O_T) < +\infty$.

Grâce au lemme de Fatou, on peut écrire :

$$\begin{aligned} \int_{O_T} \left[f_{\ln}(\hat{X}_c(s,\theta))\mathbb{1}_{\hat{X}_c<1}\right] ds d\theta &= \int_{O_T} \liminf_{n\to+\infty} \left[f^n_{\ln}(\hat{X}^n_c(s,\theta))\mathbb{1}_{\hat{X}^n_c<1} ds d\theta\right] \\ &\leq \liminf_{n\to+\infty} \int_{O_T} \left[f^n_{\ln}(\hat{X}^n_c(s,\theta))\mathbb{1}_{\hat{X}^n_c<1}\right] ds d\theta \\ &< +\infty. \end{aligned} \tag{0.0.53}$$

On vient bien de montrer que presque sûrement $f_{\ln}(\hat{X}_c) \in L^1(O_T)$.

Dans le cas d'une fonction puissance f_α, remarquons cette fois-ci que $w^n := ((\hat{X}^n_c)^+ + 1/n)^\alpha$ converge uniformément vers $w := \hat{X}^\alpha_c$. De plus, puisque $w^n f^n_\alpha(\hat{X}^n_c(t,\theta)) = 1$, la relation (0.0.50) est vérifiée pour $m_T = T$. Comme dans le cas logarithmique, le lemme de Fatou nous assure que presque sûrement $f_\alpha(\hat{X}_c) \in L^1(O_T)$.

La deuxième étape consiste à vérifier que les solutions approchées convergent effectivement vers une solution de l'équation avec un terme de réflexion, et que la mesure de réflexion vérifie bien la condition de contact. On utilise alors fortement les caractéristiques des fonctions approchées f^n. D'après l'étape 1, les mesures

$$\mathrm{d}\xi^n := f^n(\hat{X}^n_c(t,\theta))\mathrm{d}t\mathrm{d}\theta$$

convergent vers une mesure ξ.
On note alors λ la mesure suivante:

$$\mathrm{d}\lambda := f(\hat{X}_c(t,\theta))\mathrm{d}t\mathrm{d}\theta, \tag{0.0.54}$$

et $\zeta^n := \xi^n - \lambda$. Ainsi ζ^n converge vers la mesure positive $\zeta := \xi - \lambda$.

En passant à la limite dans les équations approchées pour $n \to +\infty$ on obtient pour tout $h \in D(A^2)$ et pour tout $0 \leq t \leq T$:

$$\begin{aligned} \langle \hat{X}_c(t,.), h\rangle &= \langle x, h\rangle - \frac{1}{2}\int_{O_t} \hat{X}_c(s,\theta)A^2h(\theta)\mathrm{d}s\mathrm{d}\theta - \frac{1}{2}\int_{O_t} f(\hat{X}_c(s,\theta))Ah(\theta)\mathrm{d}s\mathrm{d}\theta \\ &\quad -\frac{1}{2}\langle Ah, \zeta\rangle_{O_t} - \int_0^t \langle Bh, \mathrm{d}W\rangle. \end{aligned}$$

C'est bien l'équation attendue, et il suffit donc de montrer que ζ vérifie la condition de contact. En fait, à l'aide d'un calcul technique on prouve que pour tout $\beta > 0$:

$$0 \leq \langle \hat{X}_c, \zeta\rangle_{O_T} \leq \beta. \tag{0.0.55}$$

La dernière étape est sans doute la plus complexe, car elle démontre le caractère tendu des lois des solutions approchées. Pour cela, on utilise fortement les résultats sur la décomposition de Lyons-Zheng décrites dans [35]. Il ne reste plus qu'à montrer la convergence des solutions approchées en probabilité. On utilise en particulier un lemme de Gyöngy et Krylov :

Lemme 3 *Soit $\{Z_n\}_{n\geq 1}$ une suite de variables aléatoires sur un espace Polonais E muni de sa tribu de Borel. Alors $\{Z_n\}_{n\geq 1}$ converge en probabilité vers une variable aléatoire de E si et seulement si pour toute paire de suites extraites $\{(Z_{n_k^1}, Z_{n_k^2})_{k\geq 1}$, on peut extraire une sous-suite qui converge faiblement vers une variable aléatoire supportée par la diagonale $\{(x,y)\in E\times E, x=y\}$.*

□

Le Corollaire 1 suivant est alors une conséquence directe du théorème 1.

Corollaire 1 *Soit $c>0$.*

i) Il existe un processus continu $(X(t,x), t\geq 0, x\in K\cap H_c)$ avec $X(0,x)=x$ et un ensemble K_0 dense dans $K\cap H_c$, tel que pour tout $x\in K_0$ il existe une unique solution forte de l'équation (0.0.39) donnée par $\left((X(t,x))_{t\geq 0}, \eta^x, W\right)$.

ii) La loi de $(X(t,x)_{t\geq 0}, \eta^x)$ est une loi conditionnelle régulière de la loi de $\left(\hat{X}_c, \eta\right)$ sachant $\hat{X}_c(0)=x\in K\cap H_c$.

On veut prouver que pour n'importe quelle condition initiale $x\in K\cap H_c$ avec $c>0$, il existe une solution forte de l'équation (0.0.39), nécessairement unique et que le processus X construit dans le Corollaire 1 est une réalisation d'une telle solution. On a prouvé ce résultat uniquement pour x dans un espace dense K_0, mais grâce à la convergence des semi-groupes de transition $P^{n,c}$, on va pouvoir conclure. Cette convergence est expliquée par la proposition suivante :

Proposition 3 *Soit $c>0$, alors pour tout $\phi\in\mathcal{C}_b(H)$ et $x\in K\cap H_c$:*

$$\lim_{n\to+\infty} P_t^{n,c}\phi(x) = \mathbb{E}[\phi(X(t,x))] =: P_t^c\phi(x). \tag{0.0.56}$$

De plus, le processus de Markov $(X(t,x), t\geq 0, x\in K\cap H_c)$ a la propriété de Feller forte et son semi-groupe de transition P^c est tel que :

$$|P_t^c\phi(x) - P_t^c\phi(y)| \leq \frac{\|\phi\|_\infty}{\sqrt{t}}\|x-y\|_H, \qquad \text{pour tout } x,y\in K\cap H_c, \text{pour tout } t>0. \tag{0.0.57}$$

A l'aide des solutions stationnaires, on peut alors définir par conditionnement des solutions pour des données initiales positives dans $K\cap H_c$ avec $c>0$. Le théorème final 2 décrit l'existence de solutions de l'équation (0.0.39).

Théorème 2 *Soit ξ une variable aléatoire sur K avec $\overline{\xi}>0$ presque sûrement et (ξ, W) indépendants. Alors il existe un processus continu noté $(X(t,\xi))_{t\geq 0}$ et une mesure η^ξ tels que :*

(a) $\left((X(t,\xi))_{t\geq 0}, \eta^\xi, W\right)$ est l'unique solution forte de (0.0.39) avec $X(0,\xi)=\xi$ presque sûrement.

(b) Le processus de Markov $(X(t,x), t\geq 0, x\in K\cap H_c)$ est continu et possède P^c pour semi-groupe de transition. De plus, P^c a la propriété de Feller forte sur H_c.

(c) Pour tout $c>0$, $x\in K\cap H_c$ et $0=t_0<t_1<\cdots<t_m$, $(X(t_i,x), i=1,\dots,m)$ est la limite au sens des distributions de $(X^n(t_i,x))_{i=1,\dots,m}$.

(d) Si ξ a la même loi que ν_c avec $c>0$, alors $(X(t,\xi))_{t\geq 0}$ est égal au sens des distributions à $\left(\hat{X}_c(t)\right)_{t\geq 0}$.

Mesures de réflexion et mesures de Revuz

On a prouvé l'existence d'une solution à l'équation (0.0.39) avec une mesure de réflexion. Dans [74], L. Zambotti utilise une formule d'intégration par parties pour prouver que dans certaines cas, la mesure de réflexion est identiquement nulle. De plus, il prouve que dans d'autres cas, la mesure ne peut pas être identiquement nulle. Il utilise pour cela la théorie des Fonctionnelles Additives Continues décrites dans [35]. On va adapter ses arguments pour prouver des résultats similaires, à savoir :

Théorème 3 *Pour tout $c > 0$, pour tout $x \in K \cap H_c$:*
i) Pour $\alpha \geq 3$, la mesure de réflexion η^x de la solution forte $((X(t,x))_{t\geq 0}, \eta^x, W)$ s'annule.
ii) Pour $\alpha < 3$, la mesure de réflexion η^x de la solution forte $((X(t,x))_{t\geq 0}, \eta^x, W)$ ne s'annule pas.

Ce théorème est intéressant. Il indique que pour des fonctions non linéaires de la forme x^α avec $\alpha \geq 3$, la non linéarité est assez forte pour empêcher la solution de toucher le point 0. On aurait donc pu considérer l'équation stochastique sans ajouter de terme de réflexion. Par contre, pour des non linéarités plus faibles (i.e. $\alpha < 3$), la mesure de réflexion ne s'annule pas. Puisque la solution touche effectivement le point 0, il ne pourrait y avoir de solution sans un terme de réflexion.

Pour prouver un tel résultat, on a besoin d'une formule d'intégration par parties sur la mesure invariante ν_c. On démarre avec la formule d'intégration par parties présente dans [29].

Pour tout Φ dans $\mathcal{C}_b^1(H,\mathbb{R})$ et $h \in D(A)$:

$$\begin{aligned}\mathbb{E}\left[\partial_h \Phi(Y) \mathbb{1}_{Y\in K}\right] &= -\mathbb{E}\left[\left(\langle Y, Ah\rangle - \overline{Y}\cdot\overline{h}\right)\Phi(Y)\mathbb{1}_{Y\in K}\right] \\ &\quad - \int_0^1 h(r)\frac{1}{\sqrt{2\pi^3 r(1-r)}}\mathbb{E}\left[\Phi(\mathcal{U}_r)e^{-(1/2)(\overline{\mathcal{U}}_r)^2}\right]\mathrm{d}r.\end{aligned} \tag{0.0.58}$$

où $\mathcal{U}_r$ est un processus particulier qu'on ne détaillera pas ici. Notons toutefois que c'est un processus positif.

On définit $\gamma^n : x \mapsto \frac{1}{Z_c^n}\exp(-U^n(x))$ pour tout $x \in H$, où Z_c^n est la constante de normalisation définie dans (0.0.46). Alors $\gamma^n \in \mathcal{C}_b^1(H)$ et pour tout $x, h \in K$:

$$\langle \nabla\gamma^n(x), h\rangle = \gamma^n(x)\langle \nabla \log\gamma^n(x), h\rangle = \gamma^n(x)\int_0^1 h(\theta) f^n(x(\theta))d\theta. \tag{0.0.59}$$

Soit ϕ dans $\mathcal{C}_b^1(H)$. On utilise (0.0.58), avec $\Phi = \phi\cdot\gamma^n$, et on dérive un produit. On obtient alors :

$$\begin{aligned}\int_H (\partial_{\Pi h}\phi)\cdot\gamma^n \, d\mu_c &= -\int_H \Big(\langle x, Ah\rangle + \langle \nabla\log\gamma^n(x), \Pi h\rangle\Big)\phi(x)\gamma^n(x)\mathbb{1}_{x\in K}\mu_c(dx) \\ &\quad - \int_0^1 \Pi h(r)\frac{p_{\overline{\mathcal{U}}_r}(c)}{\pi\sqrt{r(1-r)}}\mathbb{E}\left[\phi(\mathcal{U}_r)\gamma^n(\mathcal{U}_r)|\overline{\mathcal{U}}_r = c\right]\mathrm{d}r.\end{aligned} \tag{0.0.60}$$

en notant $p_{\overline{\mathcal{U}}_r} : \mathbb{R}^+ \to [0,1]$ la version continue de la densité de $\overline{\mathcal{U}}_r$.

Remarque 1 *On remarque que dans la formule (0.0.60), on utilise Πh au lieu de simplement h comme dans la formule (0.0.58). La condition de moyenne nulle sur h est essentielle, mais c'est également cette condition de moyenne nulle qui pose toutes les difficultés dans la suite.*

Chaque terme est convergent, mais le dernier terme est un terme résiduel qui va correspondre à la présence d'une mesure de réflexion. On fait tendre n vers l'infini dans la relation précédente, et on note Σ_r^c la mesure sur H_c telle que le théorème suivant soit vérifié :

Théorème 4 *Pour tout ϕ dans $\mathcal{C}_b^1(H)$ et $h \in D(A)$:*

$$\begin{aligned}\int_H \partial_{\Pi h}\phi(x)\mathbb{1}_{x\in K}\nu_c(dx) &= -\int_H \Big(\langle x, Ah\rangle + \langle f(x), \Pi h\rangle\Big)\phi(x)\nu_c(dx) \\ &\quad - \int_0^1 \Pi h(r)\int \phi\gamma d\Sigma_r^c \, dr.\end{aligned} \tag{0.0.61}$$

De plus, pour $\alpha \geq 3$, le dernier terme s'annule.

Le théorème 4 précise que le dernier terme s'annule si $\alpha \geq 3$. En effet, dans la formule d'intégration par parties approchée, le terme résiduel contient $\gamma^n(\mathcal{U}_r)$. Or lorsque γ^n correspond à la fonction f_α, $\mathcal{U}_r$ vérifie une loi des logarithmes itérés telle que presque sûrement pour tout $r \in (0,1)$:

$$\int_0^1 \frac{d\theta}{(\mathcal{U}_r(\theta))^{\alpha-1}} = +\infty.$$

Par convergence dominée, on obtient assez rapidement que le terme résiduel dans (0.0.61) converge vers 0.

On peut alors utiliser la formule d'intégration par parties (0.0.61) avec $\phi = 1$ pour prouver que la mesure de réflexion est nulle pour $\alpha \geq 3$. En effet, en prenant l'espérance de l'équation vérifiée par $\hat{X}_c$, on obtient pour tout $0 < \delta \leq s \leq t \leq T$, pour tout $h \in D(A^2)$:

$$\mathbb{E}\left[\int_s^t \langle \hat{X}_c(u), A^2 h\rangle du + \int_s^t \langle Ah(\theta), f(\hat{X}_c(u))\rangle du + \left\langle Ah, \eta\right\rangle_{O_{s,t}}\right] = 0. \tag{0.0.62}$$

Cette égalité est en parfaite correspondance avec la formule d'intégration par parties (0.0.61). Il faut toutefois montrer qu'on peut prendre l'espérance de l'équation. Le seul terme non trivial correspond à l'espérance du terme de réflexion. Rappelons que η est la limite de $\mathrm{d}\eta^n := f^n(\hat{X}_c^n(t,\theta))\mathrm{dtd}\theta - f(\hat{X}_c(t,\theta))\mathrm{dtd}\theta$. On prouve alors que pour tout $\delta \leq s \leq t \leq T$

$$\begin{aligned} \mathbb{E}\left[\eta(O_{s,t})\right] &\leq \liminf_{n\to+\infty} \mathbb{E}\left[\eta^n(O_{s,t})\right] \\ &\leq \liminf_{n\to+\infty} \mathbb{E}\left[\textstyle\int_{O_{s,t}} f^n(\hat{X}_c^n(u,\theta))\mathrm{d}u\ \mathrm{d}\theta\right]. \end{aligned} \tag{0.0.63}$$

En utilisant que ν_c^n est une mesure invariante pour l'équation approchée, il est facile de montrer que ce terme est borné. La masse totale de $O_{s,t}$ pour la mesure de réflexion η possède donc une espérance finie, et la formule (0.0.62) est valide.

On comprend alors que si le terme résiduel de l'intégration par parties (0.0.61) est nul, alors la mesure de réflexion est certainement nulle. Réciproquement, si le terme résiduel n'est pas nul, la mesure de réflexion joue un rôle et ne peut être identiquement nulle. En effet, en caractérisant la mesure de Revuz liée à la mesure de réflexion, on obtient la relation suivante :

$$\int_H \mathbb{E}\left[\int_0^{+\infty} \exp(-t)\int_0^1 \Pi h(\theta)\eta^x(\delta+dt, d\theta)\right]\nu_c(\mathrm{dx}) = \int_0^1\int_H \Pi h(r)\gamma d\Sigma_r^c\ \mathrm{d}r. \tag{0.0.64}$$

□

Deux réflexions

Puisque l'équation avec une seule réflexion vient d'être résolue, on se propose d'étudier l'équation suivante :

$$\begin{cases} \partial_t X = -\dfrac{1}{2}\Delta\Big(\Delta X + f(X) + \eta_- - \eta_+\Big) + \dot{\xi}, & \text{avec } \theta \in [0,1] = \Omega, \\ \nabla X \cdot \nu = 0 = \nabla(\Delta X)\cdot\nu, & \text{sur } \partial\Omega, \end{cases} \tag{0.0.65}$$

où les mesures sont soumises presque sûrement aux conditions de contact suivantes :

$$\int (1+X)\mathrm{d}\eta_- = \int (1-X)\mathrm{d}\eta_+ = 0. \tag{0.0.66}$$

Ici, on va considérer l'équation originale du modèle de Cahn-Hilliard-Cook. Pour $\lambda \in \mathbb{R}$, on définit la fonction non linéaire suivante :

$$f(x) := \begin{cases} +\infty, & \text{pour tout } x \leq -1, \\ \ln\left(\frac{1-x}{1+x}\right) + \lambda x, & \text{pour tout } x \in (-1,1), \\ -\infty, & \text{pour tout } x \geq 1, \end{cases} \tag{0.0.67}$$

et une primitive F de $-f$:

$$F(x) = (1+x)\ln(1+x) + (1-x)\ln(1-x) - \frac{\lambda}{2}x^2, \text{ pour tout } x \in (-1,1).$$

La fonction f possède donc une double singularité logarithmique en ± 1. La principale difficulté rencontrée était de comprendre comment gérer deux singularités. Dans [39], la positivité de la fonction f aux alentours de la singularité jouait un rôle essentiel. Ici la fonction n'a pas de signe, et les convergences obtenues par le lemme de Fatou ne fonctionnent plus.

Afin de justifier l'approximation polynomiale faite par de nombreux auteurs, plutôt que de considérer des fonctions approchées lipschitziennes, on a considéré des fonctions approchées polynomiales. On note $\{f^n\}_{n\in\mathbb{N}}$ la suite des fonctions polynomiales approchées qui converge vers la fonction f sur $(-1,1)$ définies pour $n \in \mathbb{N}$ par:

$$f^n(x) = -2\sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)} + \lambda x, \text{ pour tout } x \in \mathbb{R}.$$

On utilise également les primitives F^n de $-f^n$ définies par :

$$F^n(x) = 2\sum_{k=0}^{n} \frac{x^{2k+2}}{(2k+2)(2k+1)} - \frac{\lambda}{2}x^2, \text{ pour tout } x \in \mathbb{R},$$

Ainsi, pour $n \in \mathbb{N}$, on étudie les équations approchées polynomiales de (0.0.65) pour une condition initiale $x \in H$:

$$\begin{cases} \mathrm{d}X^n + \frac{1}{2}(A^2X^n + Af^n(X^n))\mathrm{dt} = B\mathrm{d}W, \\ \\ X^n(0,x) = x. \end{cases} \tag{0.0.68}$$

Cette équation a justement été étudiée dans [21] dans le cas $B = I_d$. Les résultats se généralisent immédiatement et on prouve aisément que pour tout $x \in H$ il existe une unique solution $X^n(\cdot,x)$ presque sûrement dans $\mathcal{C}([0,T];H) \cap L^{2n+2}((0,T)\times(0,1))$. C'est une solution au sens intégral ou faible, et de plus la moyenne de $X^n(t,x)$ ne dépend pas de t.

Pour chaque $c \in \mathbb{R}$, (0.0.65) définit un semi-groupe de transition $(P_t^{n,c})_{t\geq 0}$:

$$P_t^{n,c}\phi(x) = \mathbb{E}[\phi(X^n(t,x)],\ t \geq 0, x \in H_c,\ \phi \in \mathcal{B}_b(H_c),\ n \in \mathbb{N}^*.$$

L'existence d'une mesure invariante peut-être prouvée comme dans [21].

Comme dans le chapitre 1, on prouve que pour $c \in (-1,1)$ les mesures invariantes ν_c^n des équation approchées

$$\nu_c^n(dx) = \frac{1}{Z_c^n}\exp(-U^n(x))\mu_c(\mathrm{dx}),$$

où

$$U^n(x) := \int_0^1 F^n(x(\theta))\mathrm{d}\theta, \quad x \in L^2(0,1),$$

convergent vers la mesure

$$\nu_c(dx) = \frac{1}{Z_c}\exp(-U(x))\mathbb{1}_{x\in K}\mu_c(\mathrm{dx}),$$

où

$$U(x) := \int_0^1 F(x(\theta))\mathrm{d}\theta, \quad x \in L^2(0,1).$$

et

$$K = \{x \in L^2,\ 1 \geq x \geq -1\}.$$

Le théorème 2 adapté à l'équation (0.0.65) s'énonce ainsi :

Théorème 5 *Soit $c \in (-1,1)$. Soit $x \in K$ tel que $\bar{x} = c$, alors il existe un processus continu noté $(X(t,x))_{t\geq 0}$ et deux mesures positives η_+^x et η_-^x tels que*

(a) $\left((X(t,x))_{t\geq 0}, \eta_+^x, \eta_-^x, W\right)$ *est l'unique solution forte de* (0.0.66) *avec* $X(0,x) = x$ *presque sûrement.*

(b) Le processus de Markov $(X(t,x), t \geq 0, x \in K \cap H_c)$ *est continu et possède* P^c *pour semi-groupe de transition. De plus,* P^c *a la propriété de Feller forte sur* H_c*.*

(c) Pour tout $x \in K \cap H_c$ *et* $0 = t_0 < t_1 < \cdots < t_m$, $(X(t_i,x), i = 1, \ldots, n)$ *est la limite au sens des distributions de* $(X^n(t_i,x))_{i=1,\ldots,m}$*.*

Finalement ν_c *est une mesure invariante pour* P^c*.*

On fixe $-1 < c < 1$ et on considère l'unique solution stationnaire (en loi) de (0.0.68) qu'on note $\hat{X}_c^n$ et qui appartient à H_c. Le Théorème 5 repose sur la proposition suivante qui détaille la convergence des mesures de réflexion.

Proposition 4 *Soit* $-1 < c < 1$, $T > 0$. $\hat{X}_c^n$ *converge en probabilité quand* n *tend vers l'infini vers un processus* $\hat{X}_c$ *in* $\mathcal{C}(O_T)$*. En outre,* $f(\hat{X}_c) \in L^1(O_T)$ *presque sûrement et il existe deux mesures* (η_+, η_-) *sur* O_T *telles que* $(\hat{X}_c, \eta_+, \eta_-, W)$ *est une solution stationnaire forte de* (0.0.65)*. De plus, si on pose*

$$\mathrm{d}\eta_+^n = -f^n(\hat{X}_c^n(t,\theta))\mathbb{1}_{\hat{X}_c^n(t,\theta)>0}\mathrm{d}t\mathrm{d}\theta + f(\hat{X}_c(t,\theta))\mathbb{1}_{0<\hat{X}_c(t,\theta)\leq 1}\mathrm{d}t\mathrm{d}\theta,$$

et

$$\mathrm{d}\eta_-^n = f^n(\hat{X}_c^n(t,\theta))\mathbb{1}_{\hat{X}_c^n(t,\theta)\leq 0}\mathrm{d}t\mathrm{d}\theta - f(\hat{X}_c(t,\theta))\mathbb{1}_{-1\leq\hat{X}_c(t,\theta)\leq 0}\mathrm{d}t\mathrm{d}\theta,$$

alors $(\hat{X}_c^n, \eta_+^n, \eta_-^n, W)$ *converge en loi vers* $(\hat{X}_c, \eta_+, \eta_-, W)$*.*

En séparant les parties positives et négatives, on arrive à traiter le cas de deux réflexions de la même manière que pour le Théorème 1. Toutefois on arrive pas à montrer que $f(\hat{X}_c)$ appartient à $L^1(O_T)$ avec la méthode employée dans le Théorème 1. En effet, on veut contrôler les deux suites de mesures positives définies pour tout $n \in \mathbb{N}$ par

$$\mathrm{d}\xi_+^n := -f^n(\hat{X}_c^n(t,\theta))\mathbb{1}_{\hat{X}_c^n(t,\theta)>0}$$

et

$$\mathrm{d}\xi_-^n := f^n(\hat{X}_c^n(t,\theta))\mathbb{1}_{\hat{X}_c^n(t,\theta)\leq 0}.$$

Mais on a seulement accès à un contrôle sur la différence des deux mesures $\xi_-^n - \xi_+^n$. La différence pourrait converger, alors que chaque suite diverge.

On va utiliser un autre argument. Plaçons nous dans le cas d'un bruit très régulier, et pour simplifier, prenons $\lambda = 0$ et $c = 0$. Appliquons alors formellement la formule d'Itô pour obtenir

$$\mathbb{E}|X(t,x)|_{H^{-1}}^2 + 2\int_0^t \mathbb{E}|X(s,x)|_{H^{-1}}^2\mathrm{d}s + 2\int_0^t \mathbb{E}F(X(s,x))X(s,x)\mathrm{d}s = |x)|_{H^{-1}}^2 + \mathrm{Tr}Bt.$$

Comme on vérifie facilement que

$$|f(x)| \leq 2xf(x)(\mathbb{1}_{x\geq 1/2} + \mathbb{1}_{x\leq -1/2}) + C \leq 2xf(x) + C,$$

avec $C = \sum_{k=0}^{\infty} \frac{1}{2^{2k+2}(2k+1)}$, l'égalité ci-dessus montre que le terme non linéaire est bien intégrable. C'est cet argument qui fait que le cas des bruits réguliers est facile à traiter, ceci en toute dimension d'espace.

Dans notre cas, B n'est pas à trace finie. Cependant on peut obtenir une majoration a priori sur $f(\hat{X}_c)$ à l'aide de cet argument couplé à une approximation de Fejer de $\hat{X}_c$. La suite de la démonstration ne fait ensuite pas appel à de nouvelles techniques. Nous n'avons pas poussé l'étude jusqu'à l'obtention d'une formule d'intégration par parties qui permettrait de déterminer si les mesures de réflexion sont nulles ou non. Ce travail n'est pas une extension immédiate des résultats précédents en raison de la double contrainte. Nous étudierons cette question plus tard.

Lorsque le paramètre λ de (0.0.67) est petit, on peut montrer facilement que ν_c est l'unique mesure invariante et qu'elle est ergodique. En pratique, λ n'est pas petit et il est intéressant de savoir si ceci est encore vrai. On peut en fait prouver que c'est aussi le cas pour un λ quelconque. On notera que puisque $(P_t^c)_{t\geq 0}$ a la propriété de Feller forte, ces résultats découlent du théorème de Doob si on prouve que $(P_t^c)_{t\geq 0}$ est irréductible (voir par exemple [24]). Dans le cas d'un bruit additif pour les équations aux dérivées partielles stochastiques, on utilise généralement des arguments de contrôle et de continuité par rapport au bruit. Cette dernière propriété n'est pas complètement triviale dans notre situation mais on arrive à adapter les arguments pour obtenir :

Proposition 5 *Pour tout $c \in (-1,1)$, le semi-groupe $(P_t^c)_{t\geq 0}$ est irréductible.*

Corollaire 2 *Pour tout $c \in (-1,1)$, ν_c est l'unique mesure invariante du semi-groupe de transition $(P_t^c)_{t\geq 0}$. De plus, elle est ergodique.*

A l'aide d'arguments classiques, on pourrait montrer que, pour $\lambda = 0$, ν_c^n satisfait une inégalité log-Sobolev et de fait une inégalité de Poincaré. Les constantes présentes dans ces inégalités ne dépendant pas de n, on obtient les mêmes résultats pour ν_c. Pour $\lambda \neq 0$, on peut utiliser les arguments de [22] et prouver que c'est encore vrai.

Mais on peut également prouver un résultat encore plus fort : le mélange exponentiel. On utilise pour cela des argument de couplage développés par Odasso dans [62]. On montre en utilisant la formule de Bismut-Elworthy-Li que pour tout $\varphi \in \mathcal{B}_b(K \cap H_c)$, $T > 0$, $\varepsilon > 0$, on a

$$|P_T^c\varphi(x) - P_T^c\varphi(y)| \leq \frac{4e^{\lambda^2 T/4}}{\lambda\sqrt{T}}\varepsilon\|\varphi\|_\infty$$

si x, $y \in H_c$, $|x|_{-1} \leq \varepsilon$ et $|y|_{-1} \leq \varepsilon$.

On construit alors un couplage $(X_1(\cdot,x,y), X_2(\cdot,x,y))$ de $(X(\cdot,x), X(\cdot,y))$ défini uniquement aux instants $(kT)_{k\geq 1}$. A l'aide d'un temps d'arrêt τ possédant un moment exponentiel

$$\mathbb{E}(e^{\beta\tau}) = \mathcal{M} < \infty$$

pour β suffisamment petit, et de l'inégalité de Markov, on peut écrire que pour tout $k \geq 1$

$$|\mathbb{E}(\varphi(X(kT,x))) - \mathbb{E}(\varphi(X(kT,y)))| \leq 2\|\varphi\|_\infty \mathcal{M} e^{-\beta(k-1)T}.$$

On définit alors $k := \lfloor \frac{t}{T} \rfloor$ tel qu'on ait $P_t^c = P_{kT}^c P_{t-kT}^c$. On obtient alors facilement le théorème suivant :

Théorème 6 *Pour tout $c \in (-1,1)$, il existe $\beta > 0$ petit et une constante $C > 0$ tels que pour tout $\varphi \in \mathcal{B}_b(K \cap H_c)$, $t > 0$ et $x \in H_c$*

$$|\mathbb{E}[\varphi(X(t,x))] - \nu_c(\varphi)| \leq C\|\varphi\|_\infty e^{-\beta t}. \tag{0.0.69}$$

Simulations stochastiques

Pour réaliser des simulations stochastiques, la première difficulté consistait à trouver un bon moyen de simuler un bruit blanc. Ensuite, il fallait que ce bruit puisse être intégré au code déterministe déjà existant. Il s'est en particulier posé de nouvelles difficultés qui n'apparaissaient pas dans les simulations déterministes. Par exemple, la présence des points singuliers ne posaient a priori aucune difficulté lors des simulations, car on ne s'approchait pas des valeurs interdites, car les solutions numériques sont uniquement (ou presque) contenues dans l'intervalle délimité par les points binodaux $[\beta_-, \beta_+]$. Mais la présence du bruit de type gaussien risquait de faire sortir les solutions en dehors de l'intervalle admissible. Ce n'était évidemment pas étonnant puisque la théorie décrite dans les chapitres 1 et 2 démontre que les solutions touchent effectivement ces points singuliers. Sans terme numérique de réflexion, on devait donc s'attendre à une sortie de la solution. Des mesures numériques ont donc été prise pour prendre en compte ce terme de réflexion numérique.

Pour les simulations avec un terme de réflexion, on a pu mettre en évidence des phénomènes connus pour les équations d'ordre 2. Il s'agit de caractériser l'ensemble des points du domaine Ω où la solution va toucher un point singulier. En dimension 1 sur l'espace $[0,1]$, pour une seule mesure de réflexion en 0 et pour des conditions au bord homogènes de type Dirichlet, on sait que la solution touche 0 en un nombre fini de points, et plus exactement en moins de 4 points. De tels phénomènes se retrouvent également pour l'ordre 4, pour lequel le même résultat a été mis en évidence (voir figure 6).

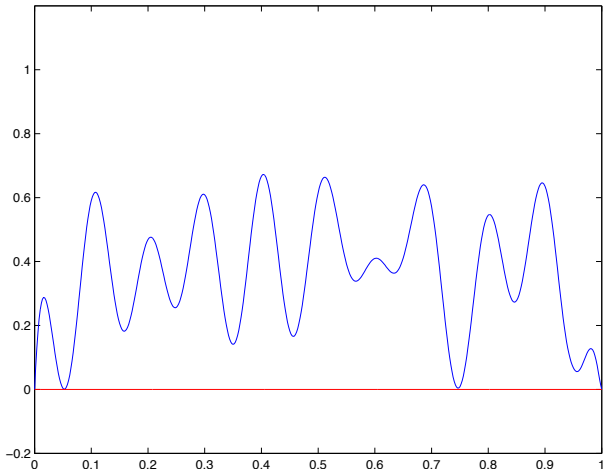


Figure 6: Processus contraint à être positif mais touchant la valeur singulière 0.

On a aussi étudié l'équation de Cahn-Hilliard-Cook :

$$\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta\left(\varepsilon^2\Delta u - \psi(u)\right) + \sigma B\dot{W}, \tag{0.0.70}$$

où $\sigma \in \mathbb{R}^+$ sera un paramètre décrivant la force du bruit. Lors de la décomposition spinodale, le bruit joue un rôle prédominant car il induit des micro-fluctuations dans le mélange. Ces micro-fluctuations accélèrent donc la séparation du mélange en deux phases. Une fois la décomposition spinodale réalisée, le bruit ne joue plus un rôle prédominant. De plus, si le facteur σ est petit, on ne voit pratiquement pas les oscillations du bruit. Pour un σ suffisamment grand, le bruit induit un comportement purement aléatoire qui permet des sauts de potentiels. De tels sauts de potentiels permettent à la solution de ne pas être piégée dans des minimas locaux, et d'atteindre des états de faibles énergies. Toutefois ces sauts de potentiels ont parfois l'effet inverse, et permettent à la solution de remonter dans des zones de plus fortes énergies. Il s'installe alors une sorte de va-et-vient entre les différents minima de la fonctionnelle d'énergie. On peut en particulier étudier les temps de présence de la solution dans chaque puits de potentiel.

Chapter 1

L'équation de Cahn-Hilliard stochastique avec une réflexion

Résumé

On considère une équation aux dérivées partielles stochastique possédant une non-linéarité de type logarithmique (ou une puissance négative), avec une réflexion en zéro sous la contrainte de conservation de masse. L'équation, dirigée par un bruit blanc en espace et en temps, contient un double Laplacien. L'absence de principe de maximum pour le double Laplacien pose des difficultés pour l'utilisation d'une méthode classique de pénalisation, pour laquelle une importante propriété de monotonie est utilisée. Etant inspiré par les travaux de Debussche et Zambotti, on emploie une méthode basée sur les équations en dimension infinie, utilisant l'approximation par des équations regulières et la convegence des semi-groupes de transition liés aux équations régularisées. On démontre l'existence et l'unicité de solutions pour des données initiales positives, et on donne plusieurs résultats sur les mesures invariantes et les mesures de réflexion.

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Stochastic Cahn-Hilliard equation with singular nonlinearity and reflection

C'est bien une habitude de l'homme
que de mettre de la pensée
là où la nature avait jeté du hasard.

Jérôme Touzalin

Introduction and main results

The Cahn-Hilliard-Cook equation is a model to describe phase separation in a binary alloy (see [13], [15] and [16]) in the presence of thermal fluctuations (see [19] and [49]). It takes the form:

$$\begin{cases} \partial_t u = -\frac{1}{2}\Delta\left(\Delta u - \psi(u)\right) + \dot{\xi}, & \text{on } \Omega \subset \mathbb{R}^n, \\ \nabla u \cdot \nu = 0 = \nabla(\Delta u) \cdot \nu, & \text{on } \partial\Omega, \end{cases} \tag{1.0.1}$$

where t denotes the time variable and Δ is the Laplace operator. Also $u \in [-1, 1]$ represents the ratio between the two species and the noise term $\dot{\xi}$ accounts for the thermal fluctuations. The nonlinear term ψ has the double-logarithmic form which was proposed by Cahn and Hilliard:

$$\psi : u \mapsto \ln\left(\frac{1+u}{1-u}\right) - \kappa u. \tag{1.0.2}$$

The deterministic equation has been extensively studied first in the case where ψ is replaced by a polynomial function (see [15], [49] and [60]) and then for non-smooth ψ (see [11] and [27]). Furthermore, this model has been used successfully for describing phase separation phenomena, see for example the survey [58], and the references therein, or others recent results on spinodal decomposition and nucleation in [5, 10, 40, 53, 54, 66, 67, 72]. The polynomial case is a simplification, and the concentration u is not constricted to remain between -1 and 1. Therefore the logarithmic nonlinearity might seem preferable.
Up to our knowledge, only the polynomial nonlinearity has been studied in the stochastic case (see [8, 9, 17, 18, 21, 31]). This article is a step toward the mathematical comprehension of the full model with double-logarithmic term and noise. We consider the one dimensional case and consider a nonlinear term with only one singularity. The model with two singularities at -1 and $+1$ involves further difficulties and will be the subject of a forthcoming article. Clearly, due to the noise, such an equation cannot have a solution, and a reflection measure should be added to the equation. Thus the right stochastic equation to study is:

$$\begin{cases} \partial_t X = -\frac{1}{2}\Delta\Big(\Delta X + f(X) + \eta\Big) + \partial_\theta \dot{W}, & \text{with } \theta \in [0,1] = \Omega, \\ \nabla X \cdot \nu = 0 = \nabla(\Delta X) \cdot \nu, & \text{on } \partial\Omega, \end{cases} \tag{1.0.3}$$

where W is a cylindrical Wiener process on $L^2(0,1)$, f is defined on $(0, +\infty)$ and described below, and where the measure is subject to the contact condition almost surely:

$$\int X d\eta = 0. \tag{1.0.4}$$

Stochastic partial differential equations with reflection can model the described problem or the evolution of random interfaces near a hard wall (see [37] and [75]). For other results on fluctuations of random interfaces, see [38]. For a detailed study of the contact set $\{(t,\theta) : X(t,\theta) = 0\}$ and of the reflection measure η, see [26], [73] and [74]. The equation (1.0.3) has been studied when no nonlinear term is taken into account in [29]. In this paper, the author has introduced various techniques needed to overcome the lack of comparison principle for fourth order equations. Indeed, the case of a second order equation was studied in [61] where an extensive use of monotonicity is used, as well as in all the articles treating with the second order case.
This article is in the spirit of [74] where a nonlinear term is taken into account for the second order equation. We study existence and uniqueness of solution for equation (1.0.3) with f of the form:

$$f(x) := f_{\ln}(x) := \begin{cases} -\ln x, & \text{for all } x > 0 \\ +\infty, & \text{for all } x \leq 0, \end{cases} \tag{1.0.5}$$

or for $\alpha > 0$:

$$f(x) := f_\alpha(x) := \begin{cases} x^{-\alpha}, & \text{for all } x > 0 \\ +\infty, & \text{for all } x \leq 0. \end{cases} \tag{1.0.6}$$

Moreover we characterize the case when the measure η vanishes. Our method mixes ideas from [29] and [74]. Additional difficulties are overcome, the main one being to understand how to deal with the nonlinear term. Again in [74], this term is not difficult to consider thanks to monotonicity arguments.
Our main results state that equations (1.0.3), (1.0.4) together with an initial condition have a unique solution (see sections 1.2.1 and 1.2.2). It is constructed thanks to the gradient structure of (1.0.3) and strong Feller property. Furthermore, we prove that the measure η vanishes only for f described in (1.0.6) with $\alpha \geq 3$ (see Theorem 1.3.1).

1.1 Preliminaries

1.1.1 Notation

We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(0,1)$. We denote by A the realization in $L^2(0,1)$ of the Laplace operator with the Neumann boundary condition, i.e.:

$$D(A) = \text{ Domain of } A = \{h \in W^{2,2}(0,1) : h'(0) = h'(1) = 0\}$$

where we use $W^{n,p}$ and $||.||_{W^{n,p}}$ to denote the Sobolev space $W^{n,p}(0,1)$ and its associated norm. Remark that A is self-adjoint on $L^2(0,1)$ and we have a complete orthonormal system of eigenvectors $(e_i)_{i\in\mathbb{N}}$ in $L^2(0,1)$. We denote by $\bar{h}$ the mean of $h \in L^2(0,1)$:

$$\bar{h} = \int_0^1 h(\theta) d\theta.$$

Then we define for all $c \in \mathbb{R}$:

$$L^2_c = \{h \in L^2(0,1) : \bar{h} = c\},$$

and $L^2 = L^2(0,1)$. We remark that $(-A)^{-1} : L^2_0 \to L^2_0$ is well defined. We denote by Q this operator. We can extend the definition of Q to $L^2(0,1)$ (we denote this operator $\bar{Q}$) by the formula:

$$\bar{Q}h = Q(h - \bar{h}) + \bar{h}, \text{ for all } h \in L^2(0,1)$$

For $\gamma \in \mathbb{R}$, we define $(-A)^\gamma$ by classical interpolation, and the domain of $(-A)^{\gamma/2}$ is

$$V_\gamma := D((-A)^{\gamma/2}).$$

It is endowed with the classical seminorm:

$$|h|_\gamma = \left(\sum_{i=1}^{+\infty} (-\lambda_i)^\gamma h_i^2 \right)^{1/2},$$

and with the norm

$$\|h\|_\gamma = \left(|h|_\gamma^2 + \bar{h}^2 \right)^{1/2},$$

associated to the scalar product defined for all $h, k \in V_\gamma$ by $(h,k)_\gamma$.
To lighten notations, we set $(\cdot,\cdot) := (\cdot,\cdot)_{-1}$ for the inner product of V_{-1}, and set $H := V_{-1}$. The average plays an important role and we often work with functions with a fixed average $c \in \mathbb{R}$. We define $H_c = \{h \in H, \bar{h} = c\}$ for all $c \in \mathbb{R}$. We set

$$D(B) = W_0^{1,2}(0,1), B = \frac{\partial}{\partial\theta}, D(B^*) = W^{1,2}(0,1) \text{ and } B^* = -\frac{\partial}{\partial\theta}.$$

We remark that $BB^* = -A$. Finally, we denote by Π the orthogonal projector of V_{-1} onto H_0. We have:

$$\begin{array}{rccc} \Pi : & V_{-1} & \to & H_0 \\ & h & \mapsto & h - \bar{h}. \end{array}$$

Notice that Π is also an orthogonal projector of L^2 onto L^2_0. Moreover:

$$-A\bar{Q}h = \Pi h, \text{ for all } h \in L^2(0,1). \tag{1.1.1}$$

We denote by $\mathcal{B}_b(H_c)$ the space of all Borel bounded functions and $\mathcal{C}_b(H_c)$ the space of continuous bounded functions. We set $O_{s,t} := [s,t] \times [0,1]$ for $s,t \in [0,T]$ with $s < t$ and $T > 0$, and $O_t = O_{0,t}$ for $0 \leq t \leq T$. Given a measure ζ on $O_{s,t}$ and a continuous function v on $O_{s,t}$, we set

$$\langle v, \zeta \rangle_{O_{s,t}} := \int_{O_{s,t}} v d\zeta.$$

In order to solve the equation (1.0.3), we use a Lipschitz approximation of this equation. We denote by $\{f^n\}_{n\in\mathbb{N}}$ the sequence of Lipschitz functions which converges to the function f on $(0,+\infty)$, defined for $n \in \mathbb{N}$ by:

$$f^n(x) := f(x^+ + 1/n), \text{ for all } x \in \mathbb{R}.$$

When $f = f_{\ln}$ is the logarithmic function (1.0.5), we use the following positive antiderivative of $-f^n = -f^n_{\ln}$

$$F^n(x) = F^n_{\ln}(x) := (x + 1/n)\ln(x^+ + 1/n) - x^+ + 1 - 1/n, \text{ for all } x \in \mathbb{R},$$

and the following positive antiderivative of $-f = -f_{\ln}$ defined only on $\mathbb{R}^+$ by:

$$F(x) = F_{\ln}(x) := x\ln(x) - x + 1, \text{ for all } x \in \mathbb{R}^+.$$

When $f = f_\alpha$ is the negative α-power function (1.0.6) with $\alpha \neq 1$, we use the following antiderivative of $-f^n = -f^n_\alpha$

$$F^n(x) = F^n_\alpha(x) := \frac{(x^+ + 1/n)^{1-\alpha}}{\alpha - 1} + n^\alpha x^-, \text{ for all } x \in \mathbb{R},$$

and the following antiderivative of $-f = -f_\alpha$ defined only on $\mathbb{R}^+$ by:

$$F(x) = F_\alpha(x) := \frac{x^{1-\alpha}}{\alpha - 1}, \text{ for all } x \in \mathbb{R}^+.$$

Finally when $\alpha = 1$, we use the following antiderivative of $-f^n = -f^n_\alpha$

$$F^n(x) = F^n_\alpha(x) := -\ln(x^+ + 1/n) + nx^-, \text{ for all } x \in \mathbb{R},$$

and the following antiderivative of $-f = -f_\alpha$ defined only on $\mathbb{R}^+$ by:

$$F(x) = F_\alpha(x) := -\ln x, \text{ for all } x \in \mathbb{R}^+.$$

We use the notation f, f^n, F, F^n when the result holds both for $f_{\ln}$ and f_α. Otherwise we use $f_{\ln}, f^n_{\ln}, F_{\ln}, F^n_{\ln}$ or $f_\alpha, f^n_\alpha, F_\alpha, F^n_\alpha$.
With these notations, we rewrite (1.0.3) in the abstract form:

$$\begin{cases} dX = -\frac{1}{2}(A^2X + Af(X) + \eta)dt + BdW, \\ \langle X, \eta \rangle_{O_T} = 0, \\ X(0,x) = x \text{ for } x \in V_{-1}, \end{cases} \tag{1.1.2}$$

where W is a cylindrical Wiener process on $L^2(0,1)$. Finally, in all the article, C denotes a constant which may depend on T and α and its value may change from one line to another.

1.1.2 The linear equation

The linear equation is given by

$$\begin{cases} dZ(t,x) = -\frac{1}{2}A^2Z(t,x)dt + BdW, \quad \text{for all } t \in [0,T], \\ Z(0,x) = x. \end{cases}$$

where $x \in V_{-1}$. Its solution is

$$Z(t,x) = e^{-tA^2/2}x + \int_0^t e^{-(t-s)A^2/2} B dW_s.$$

As easily seen this process is in $\mathcal{C}([0,+\infty[; L^2(0,1))$ (see [23]). In particular, the mean of Z is constant and the law of the process $Z(t,x)$ is the Gaussian measure:

$$Z(t,x) \sim \mathcal{N}\big(e^{-tA^2/2}x, Q_t\big),$$

where

$$Q_t = \int_0^t e^{-sA^2/2} BB^* e^{-sA^2/2} ds = (-A)^{-1}(I - e^{-tA^2}).$$

□

If we let $t \to +\infty$, the law of $Z(t,x)$ converges to the Gaussian measure on L^2_c:

$$\mu_c := \mathcal{N}(ce_0, Q), \text{ where } c = \bar{x}.$$

Notice that the kernel of Q is $\{te_0, t \in \mathbb{R}\}$ and μ_c is concentrated on L^2_c. It is important to remark that the measure μ_c is linked to the Brownian motion. Indeed, let $(\mathbf{B}_\theta)_{\theta\in[0,1]}$ be a Brownian motion, then the law of $Y_c(\theta) = \mathbf{B}(\theta) - \overline{\mathbf{B}} + c$ is μ_c (see [29]).

1.1.3 Lipschitz Approximation

For $n \in \mathbb{N}$, we study for the following Lipschitz approximation of (1.1.2) with an initial condition $x \in V_{-1}$:

$$\begin{cases} dX^n + \frac{1}{2}(A^2X^n + Af^n(X^n))dt = BdW, \\ X^n(0,x) = x. \end{cases} \tag{1.1.3}$$

We prove existence and uniqueness of solution in a suitable space for the equation (1.1.3). We then follow standard arguments to show existence and uniqueness of an invariant measure for the equation (1.1.3) with fixed $n \in \mathbb{N}$, and the strong Feller property of the semigroup. First we have to define the definition of a weak solution to (1.1.3).
We say X^n is a mild solution of (1.1.3) if it satisfies for all $t \geq 0$:

$$X^n(t,x) = Z(t,x) - \int_0^t Ae^{-(t-s)A^2/2} f^n(X^n(s,x))ds. \tag{1.1.4}$$

Thanks to a fixed point method, the following Lemma 1.1.1 is classical (see [21] for details).

Lemma 1.1.1 *Fix $n \in \mathbb{N}$, $0 < \varepsilon < 2/3$ and $p = 4(1-\varepsilon)$. For all $x \in L^2(0,1)$ there exists a unique adapted process $X^n \in \mathcal{C}([0,T]; V_{-1}) \cap L^p([0,T]; L^2(0,1))$ solution of equation (1.1.4). Moreover for all $t \geq 0$:*

$$\overline{X^n(t,x)} = \overline{x}. \tag{1.1.5}$$

Lemma 1.1.2 *For $n \in \mathbb{N}$ and $c \in \mathbb{R}$, for all $t > 0$:*

$$|X^n(t,x) - X^n(t,y)|_{-1} \leq \exp(-t\pi^4/2)|x-y|_{-1}, \quad \text{for all } x, y \in L^2_c. \tag{1.1.6}$$

Proof : The proof of Lemma 1.1.2 is standard and left to the reader.

□

It is classical that $X^n \in \mathcal{C}([0,T]; V_{-1}) \cap L^p([0,T]; L^2(0,1))$ satisfies (1.1.4) if and only if it is a weak solution of (1.1.3) in the sense

Definition 1.1.1 *For $n \in \mathbb{N}$, $0 < \varepsilon < 2/3$ and $p = 4(1-\varepsilon)$, let $x \in \mathcal{C}([0,1], \mathbb{R}^+)$ with $\overline{x} > 0$. We say that $(X^n(t,x))_{t\in[0,T]}$, defined on a stochastic basis linked to $(W(t))_{t\in[0,T]}$, is a solution to (1.1.3) on $[0,T]$ if :*

(a) almost surely $X^n(\cdot,x)\in\mathcal{C}([0,T];V_{-1})\cap L^p([0,T];L^2(0,1))$,

(b) for all $h\in D(A^2)$ *and for all* $0\leq t\leq T$ *:*

$$\begin{aligned}\langle X^n(t,x),h\rangle &= \langle x,h\rangle-\tfrac{1}{2}\int_0^t\langle X^n(s,x),A^2h\rangle ds\\ &\quad -\tfrac{1}{2}\int_0^t\langle Ah,f^n(X^n(s,x))\rangle ds-\int_0^t\langle Bh,dW\rangle.\end{aligned}$$

We now describe an important property of equation (1.1.3). It can be described as a gradient system in V_{-1} with a convex potential, and can be rewritten as:

$$\begin{cases} dX^n-\frac{1}{2}A(-AX^n+\nabla U^n(X^n))dt=BdW,\\ X^n(0,x)=x\in L^2(0,1),\end{cases}\tag{1.1.7}$$

where ∇ denotes the gradient in the Hilbert space $L^2(0,1)$, and :

$$U^n(x):=\int_0^1 F^n(x(\theta))d\theta,\quad x\in L^2(0,1).\tag{1.1.8}$$

Notice that $\nabla U^n(x)=-f^n(x)$ which is dissipative, then U^n is a convex potential. Finally, we define the probability measure on L^2_c:

$$\nu^n_c(dx)=\frac{1}{Z^n_c}\exp(-U^n(x))\mu_c(dx),\tag{1.1.9}$$

where Z^n_c is a normalization constant. By Lemma 1.1.2, we easily obtain that the equation (1.1.3) in H_c has a unique ergodic invariant measure and it is not difficult to prove that this measure is precisely ν^n_c. Since the potential U^n is convex, we can prove that the transition semigroup is strong Feller. Let $(P^{n,c}_t)_{n\in\mathbb{N}}$ be the sequence of transition semigroup for an initial condition in H_c such that

$$P^{n,c}_t\phi(x)=\mathbb{E}[\phi(X^{n,c}(t,x)],\quad \text{for all } t\geq 0, x\in H_c,\phi\in\mathcal{B}_b(H_c) \text{ and } n\in\mathbb{N}^*,$$

where $X^{n,c}(t,x)$ is the solution of the equation (1.1.7). Also it is classical and easy to justify by the Galerkin approximation (see [24]) that:

Proposition 1.1.1 *For arbitrary* $T>0$*, there exists a constant* $C_T>0$ *such that for all* $\phi\in\mathcal{B}_b(H_c)$*, for all* $n\in\mathbb{N}$ *and for all* $t\in[0,T]$*:*

$$|P^{n,c}_t\phi(x)-P^{n,c}_t\phi(y)|\leq\frac{\sqrt{C_T}}{\sqrt{t}}\|\phi\|_\infty\|x-y\|_{-1},\quad \textit{for all } x,y\in H_c.\tag{1.1.10}$$

1.2 Solutions of equation with a reflection measure

We want to know if the solutions of (1.1.3) converge to a solution of the equation (1.0.3). First we give the definition of a weak solution for (1.0.3) :

Definition 1.2.1 *Let* $x\in\mathcal{C}([0,1],\mathbb{R}^+)$ *and* $\overline{x}>0$*. We say that* $\left((X(t,x))_{t\in[0,T]},\eta,W\right)$*, defined on a filtered complete probability space* $\left(\Omega,\mathbb{P},\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]}\right)$*, is a weak solution to* (1.0.3) *on* $[0,T]$ *for the initial condition* x *if:*

(a) a.s. $X\in\mathcal{C}(]0,T]\times[0,1];\mathbb{R}^+)\cap\mathcal{C}([0,T];V_{-1})$ *and* $X(0,x)=x$*,*

(b) a.s. η *is a positive measure on* $(0,T]\times[0,1]$*, such that* $\eta(O_{\delta,T})<+\infty$ *for all* $\delta\in(0,T]$*,*

(c) W *is a cylindrical Wiener process on* $L^2(0,1)$*,*

(d) the process $(X(\cdot,x),W)$ *is* $(\mathcal{F}_t)$*-adapted,*

(e) a.s. $f(X(\cdot,x))\in L^1(O_T)$*,*

(f) for all $h \in D(A^2)$ *and for all* $0 < \delta \leq t \leq T$ *:*

$$\begin{aligned}\langle X(t,x),h\rangle &= \langle X(\delta,x),h\rangle - \tfrac{1}{2}\int_\delta^t \langle X(s,x),A^2h\rangle ds - \tfrac{1}{2}\int_\delta^t \langle Ah(\theta), f(X(s,x))\rangle ds \\ &\quad -\tfrac{1}{2}\big\langle Ah,\eta\big\rangle_{O_{\delta,t}} - \int_\delta^t \langle Bh, dW\rangle, \quad a.s.,\end{aligned}$$

(g) a.s. the contact property holds : $supp(\eta) \subset \{(t,\theta) \in O_T / X(t,x)(\theta) = 0\}$*, that is,*

$$\big\langle X,\eta\big\rangle_{O_T} = 0.$$

Finally, a weak solution (X,η,W) *is a strong solution if the process* $t \mapsto X(t,x)$ *is adapted to the filtration* $t \mapsto \sigma(W(s,.), s \in [0,t])$

Remark 1.2.1 *In (f), the only term where we use the function* f *is well defined. Indeed, by (e) we have* $f(X(\cdot,x)) \in L^1(O_T)$ *and by Sobolev embedding* $Ah \in D(A) \subset L^\infty(O_T)$*. Hence the notation* $\langle\cdot,\cdot\rangle$ *should be interpreted as a duality between* L^∞ *and* L^1*.*

1.2.1 Pathwise uniqueness

We want to prove that for any pair $(X^i,\eta^i,W), i = 1,2$, of weak solutions of (1.0.3) defined on the same probability space with the same driving noise W and with $X_0^1 = X_0^2$, we have $(X^1,\eta^1) = (X^2,\eta^2)$. This pathwise uniqueness will be used in the next subsection to construct stationary strong solutions of (1.0.3).

Proposition 1.2.1 *Let* $x \in \mathcal{C}([0,1],\mathbb{R}^+)$ *with* $\overline{x} > 0$*. Let* $(X^i,\eta^i,W), i = 1,2$ *be two weak solutions of (1.0.3) with* $X_0^1 = x = X_0^2$*. Then* $(X^1,\eta^1) = (X^2,\eta^2)$*.*

Proof : We use the following Lemma from [29]. For the sake of completeness, we recall the proof.

Lemma 1.2.1 *Let* ζ *be a finite signed measure on* $O_{\delta,T}$ *and* $V \in \mathcal{C}(O_{\delta,T})$*. Suppose that there exists a positive continuous function* $c_T : [0,T] \to \mathbb{R}^+$ *such that :*

i) for all $r \in [\delta,T]$*, for all* $h \in \mathcal{C}([0,1])$*, such that* $\bar{h} = 0$*,* $\langle h,\zeta\rangle_{O_{r,T}} = 0$*,*

ii) for all $r \in [\delta,T]$*,* $\overline{V(r,\cdot)} = c_T(r)$ *with* $\langle V,\zeta\rangle_{O_{r,T}} = 0$*,*

then ζ *is the null measure.*

Proof : Let $k \in \mathcal{C}([0,1])$. Since ζ is a finite measure, by $i)$ we obtain for all $\delta \leq s \leq t \leq T$:

$$\langle k,\zeta\rangle_{O_{s,t}} = \langle \bar{k},\zeta\rangle_{O_{s,t}} = \bar{k}\zeta(O_{s,t}), \qquad \text{for all } k \in \mathcal{C}([0,1]).$$

This implies ζ can be decomposed as $\zeta = \gamma \otimes d\theta$, where γ is a measure on $[0,T]$. By $ii)$, we obtain:

$$0 = \langle V,\zeta\rangle_{O_{s,t}} = \int_s^t \Big(\int_0^1 V(\cdot,\theta)d\theta\Big)d\gamma = \int_s^t c_T(u)\ \gamma(\mathrm{d}u).$$

We conclude that for all $\delta \leq s \leq t \leq T$, $\gamma([s,t]) = 0$, since $c_T > 0$. Thus ζ is the null measure.

□

We now prove the proposition. Let $Y(t) = X^1(t,x) - X^2(t,x)$ and $\zeta = \eta^1 - \eta^2$, Y is the solution of the following equation:

$$\begin{cases} dY = -\frac{1}{2}A\left(AY + (f(X^1) - f(X^2) + \zeta)\right)dt, \\ Y(0) = 0. \end{cases} \tag{1.2.1}$$

Using now the following approximation of Y :

$$Y^N(t,.) = \frac{1}{N}\sum_{n=0}^{N}\sum_{i=0}^{n}\langle Y(t),e_i\rangle e_i,$$

and taking the scalar product in V_{-1} between Y and Y^N, we have for all $0 < \delta \leq t \leq T$:

$$\begin{aligned}(Y(t),Y^N(t)) &\leq (Y(\delta),Y^N(\delta)) + \frac{1}{2}\langle Y^N(s),\zeta\rangle_{O_{\delta,t}} \\ &\quad + \frac{1}{2N}\sum_{n=0}^{N}\sum_{i=0}^{n}\int_\delta^t \langle Y(s,.),e_i\rangle\langle f(X^1(s,x)) - f(X^2(s,x)),e_i\rangle ds. \quad (1.2.2)\end{aligned}$$

For all $s \in [\delta,t]$,

$$\begin{aligned}&\frac{1}{N}\sum_{n=0}^{N}\sum_{i=0}^{n}\langle Y(s),e_i\rangle\langle f(X^1(s,x)) - f(X^2(s,x)),e_i\rangle - \langle Y(s), f(X^1(s,x)) - f(X^2(s,x))\rangle \\ =\ &\langle Y^N(s) - Y(s), f(X^1(s,x)) - f(X^2(s,x))\rangle \\ \leq\ &\|Y^N(s) - Y(s)\|_{L^\infty([0,1])}\|f(X^1(s,x)) - f(X^2(s,x))\|_{L^1([0,1])},\end{aligned}$$

where $\|\cdot\|_{L^\infty([0,1])}$ and $\|\cdot\|_{L^1([0,1])}$ are the classical norm on the space $[0,1]$. The latter term converges to zero since $Y^N(s)$ converges uniformly to $Y(s)$ on $[0,1]$. Taking the negative part, we have by the Fatou lemma:

$$\begin{aligned}&\liminf_{N\to+\infty}\int_\delta^t\left(\frac{1}{N}\sum_{n=0}^{N}\sum_{i=0}^{n}\langle Y(s),e_i\rangle\langle f(X^1(s,x)) - f(X^2(s,x)),e_i\rangle\right)^- ds \\ \geq\ &\int_\delta^t \liminf_{N\to+\infty}\left(\frac{1}{N}\sum_{n=0}^{N}\sum_{i=0}^{n}\langle Y(s),e_i\rangle\langle f(X^1(s,x)) - f(X^2(s,x)),e_i\rangle\right)^- ds \\ =\ &\int_\delta^t \left(\langle Y(s), f(X^1(s,x)) - f(X^2(s,x))\rangle\right)^- ds \\ =\ &0,\end{aligned}$$

since f is nonincreasing. Taking the limit in (1.2.2) as N grows to infinity, we obtain by the contact condition

$$\begin{aligned}\|Y(t)\|_{-1}^2 - \|Y(\delta)\|_{-1}^2 &\leq \frac{1}{2}\langle Y,\zeta\rangle_{O_{\delta,t}} \\ &= -\frac{1}{2}\langle X^1,\eta^2\rangle_{O_{\delta,t}} - \frac{1}{2}\langle X^2,\eta^1\rangle_{O_{\delta,t}} \\ &\leq 0.\end{aligned}$$

Letting $\delta \to 0$, we have $Y(t) = 0$ for all $t \geq 0$ and $X^1(t,x) = X^2(t,x)$ for all $t \geq 0$. Moreover, with the definition of a weak solution, we see that :

$$\text{for all } h \in D(A^2), \qquad \langle Ah,\zeta\rangle_{O_{\delta,t}} = 0.$$

By density, we obtain ζ and $V = X^1 = X^2$ satisfy the hypothesis of Lemma 1.2.1, and therefore ζ is the null measure, i.e. $\eta^1 = \eta^2$.

□

1.2.2 Convergence of invariant measures

Let :

$$K = \{x \in L^2(0,1), x \geq 0\},$$

then we know that μ_c is the law of $Y^c = \mathbf{B} - \overline{\mathbf{B}} + c$. We remark the following inclusion :

$$\{\mathbf{B}_\theta \in [-c/2, c/2], \text{ for all } \theta \in [0,1]\} \subset \{Y^c \in K\},$$

therefore $\mu_c(K) > 0$ with $c > 0$. Let us define U the potential associated to the function f.

$$U(x) := \begin{cases} \int_0^1 F(x(\theta))d\theta & \text{if } \int_0^1 \Big|F(x(\theta))\Big|d\theta < +\infty \text{ and } x \in K, \\ +\infty & \text{else.}\end{cases}$$

If $f = f_{\ln}$, we use $U_{\ln}$. If $f = f_\alpha$, we use U_α.

Remark 1.2.2 *Note that, for $\alpha < 1$, $F_\alpha(x(\theta)) = -\frac{1}{1-\alpha}x(\theta)^{1-\alpha}$. By the Hölder inequality:*

$$\int_0^1 |F_\alpha(x(\theta))|d\theta < +\infty, \text{ for all } x \in K.$$

We have the following result :

Proposition 1.2.2 *For $c > 0$,*

$$\nu_c^n \rightharpoonup \nu_c := \frac{1}{Z_c}\exp^{-U(x)}\mathbb{1}_{x\in K}\mu_c(dx), \text{ when } n \to +\infty,$$

where Z_c is a normalization constant.

Proof : Let $\psi \in \mathcal{C}_b^0(L^2, \mathbb{R})$. We want to prove that

$$\int_H \psi(x)\exp(-U^n(x))\mu_c(dx) \underset{n\to+\infty}{\longrightarrow} \int_H \psi(x)\exp(-U(x))\mathbb{1}_{x\in K}\mu_c(dx). \tag{1.2.3}$$

Case 1 $f = f_{\ln}$ is the logarithmic function.
We have

$$\exp(-U^n(x)) \underset{n\to+\infty}{\longrightarrow} \exp(-U(x))\mathbb{1}_{x\in K}, \quad \mu_c \text{ almost surely.} \tag{1.2.4}$$

Since μ_c is supported by $\mathcal{C}([0,1])$, we can restrict to $x \in \mathcal{C}([0,1])$. If x is not positive on $[0,1]$, there exists $\delta_x > 0$ small such that $\lambda(\{\theta \in [0,1] : x(\theta) \le -\delta_x\}) > 0$ and

$$\int_0^1 F_{\ln}^n(x(\theta))1_{\{x<0\}}d\theta > \int_0^1 F_{\ln}^n(x(\theta))1_{\{x\le-\delta_x\}}d\theta > 0, \quad \text{for all } n \ge 1.$$

Then, since $F_{\ln}^n$ is nonincreasing on $(-\infty, 0)$:

$$\begin{aligned}
0 \le \exp(-U_{\ln}^n(x)) &\le \exp\Big(-\int_0^1 F_{\ln}^n(x(\theta))1_{\{x\le-\delta_x\}}d\theta\Big)\\
&\le \exp\Big(-\int_0^1 F_{\ln}^n(-\delta_x)1_{\{x\le-\delta_x\}}d\theta\Big)\\
&\le \exp\Big(-F_{\ln}^n(-\delta_x)\lambda(\{x\le-\delta_x\})\Big)\\
&\le \exp\Big(\big((1/n-\delta_x)\ln n - 1 + 1/n\big)\lambda(\{x\le-\delta_x\})\Big).
\end{aligned}$$

And this latter term converges to zero as n grows to infinity.
Now for $x \in K$, $F_{\ln}^n(x(\theta))$ converges to $F_{\ln}(x(\theta))$ almost everywhere as n grows to infinity. Moreover $F_{\ln}^n(x(\theta)) \le \mathbb{1}_{x\le1} + F_{\ln}^1(x(\theta))\mathbb{1}_{x>1}$, and the right-hand side is clearly integrable. By the dominated convergence theorem, we deduce (1.2.4). Since $U_{\ln}^n \ge 0$, (1.2.3) follows by the dominated convergence theorem.
Case 2 $f = f_\alpha$ is the negative α-power function.
For a fixed $x \in L^2$, the potentials are increasing as n grows to infinity, we deduce:

$$\exp(-U_\alpha^n(x)) \le \exp(-U_\alpha^1(x)), \quad \text{for all } n \ge 1, \quad \text{for all } x \in L^2. \tag{1.2.5}$$

The right-hand side is integrable on H, thus it suffices to prove that

$$\exp(-U_\alpha^n(x)) \underset{n\to+\infty}{\longrightarrow} \exp(-U_\alpha(x))\mathbb{1}_{x\in K}, \quad \mu_c \text{ almost surely,} \tag{1.2.6}$$

where

$$\exp(-U_\alpha(x))\mathbb{1}_{x\in K} = \begin{cases} \exp(-U_\alpha(x)) & \text{if } \int_0^1 |F_\alpha(x(\theta))|\,d\theta < +\infty \text{ and } x \in K,\\ 0 & \text{else.}\end{cases} \tag{1.2.7}$$

By the same arguments as in case 1, (1.2.6) holds for all $x \notin K$.
For $x \in K$, such that $\int_0^1 |F_\alpha(x(\theta))|\,d\theta < +\infty$, $F_\alpha^n(x(\theta))$ converges almost everywhere to $F_\alpha(x(\theta))$

as n grows to infinity. Moreover $F^1_\alpha(x(\theta)) \leq F^n_\alpha(x(\theta)) \leq F_\alpha(x(\theta))$ for all $\theta \in [0,1]$, and by the dominated convergence theorem (1.2.6) holds.
If $\int_0^1 |F_\alpha(x(\theta))|\, d\theta = +\infty$, necessarily $\alpha \geq 1$. For $\alpha > 1$, $F^n_\alpha \geq 0$ and (1.2.6) follows from monotone convergence. If $\alpha = 1$, we write

$$\int_0^1 F^n_\alpha(x(\theta))d\theta = \int_0^1 F^n_\alpha(x(\theta))\mathbb{1}_{x(\theta)\leq 1/2}d\theta + \int_0^1 F^n_\alpha(x(\theta))\mathbb{1}_{x(\theta)>1/2}d\theta.$$

The first term converges to $\int_0^1 F_\alpha(x(\theta))\mathbb{1}_{x(\theta)\leq 1/2}$ by monotone convergence, and the second term converges to $\int_0^1 F_\alpha(x(\theta))\mathbb{1}_{x(\theta)<1/2}$ by uniform integrability. We have proved that (1.2.6) always holds, (1.2.3) follows.

□

1.2.3 Existence of stationary solutions

In this section, we prove the existence of stationary solutions of equation (1.0.3) and that they are limits of stationary solutions of (1.1.3), in some suitable sense (see [7] or [33] for instance). Fix $c > 0$ and consider the unique (in law) stationary solution of (1.1.3) denoted by $\hat{X}^n_c$ in H_c. We are going to prove that the laws of $\hat{X}^n_c$ weakly converge as n grows to infinity to a stationary strong solution of (1.0.3).

Theorem 1.2.1 *Let $c > 0$ and $T > 0$. $\hat{X}^n_c$ converges in probability as n grows to infinity to a process $\hat{X}_c$ in $\mathcal{C}(O_T)$, $f(\hat{X}_c) \in L^1(O_T)$ almost surely, and it exists a measure η on O_T such that $(\hat{X}_c, \eta, W)$ is a stationary strong solution of (1.0.3). Moreover, setting*

$$\mathrm{d}\eta^n = f^n(\hat{X}^n_c(t,\theta))\mathrm{d}t\mathrm{d}\theta - f(\hat{X}_c(t,\theta))\mathrm{d}t\mathrm{d}\theta,$$

then $(\hat{X}^n_c, \eta^n, W)$ converges in law to $(\hat{X}_c, \eta, W)$.

The proof of Theorem 1.2.1 requires arguments that differ significantly in the logarithmic case and in the negative α-power case. We thus have chosen to do two separated proofs. Some arguments however are similar and are not repeated.

Proof in the logarithmic case:

The proof is split in 3 steps. In step 1, assuming that a subsequence of $\hat{X}^n_c$ converges in law, its limit $\hat{X}_c$ is shown to satisfy $f_{\ln}(\hat{X}_c) \in L^1(O_T)$ almost surely. Then in step 2, under the same assumption as in step 1, we prove that up to a further extraction the measures η^n converges to a positive measure η and that $(\hat{X}_c, \eta)$ is a weak solution in the probabilistic sense. It then remains to prove tightness of $\hat{X}^n_c$ and to use pathwise uniqueness to conclude in step 3.

Step 1.

Let us assume that $(n_k)_{k\in\mathbb{N}}$ is a subsequence such that $(\hat{X}^{n_k}_c)_{n\in\mathbb{N}}$ converges in law in $\mathcal{C}(O_T)$ to a process $\hat{X}_c$.
By Skorohod's theorem, we can find a probability space and a sequence of processes $(V^k, \mathcal{W}^k)_{k\in\mathbb{N}}$ on that probability space such that $(V^k, \mathcal{W}^k) \to (V, \mathcal{W})$ in $\mathcal{C}(O_T)$ almost surely and $(V^k, \mathcal{W}^k)$ has the same distribution as $(\hat{X}^{n_k}_c, \mathcal{W})$ for all $k \in \mathbb{N}$. Notice that $V \geq 0$ almost surely since for all $t \leq T$ the law of $V(t,.)$ is ν_c which is concentrated on K. Let now ξ^k and ρ^k be the following measures on O_T:

$$d\xi^k := f^{n_k}_{\ln}(V^k(t,\theta))\mathbb{1}_{V^k<1}dtd\theta,$$

and

$$d\rho^k := f^{n_k}_{\ln}(V^k(t,\theta))\mathbb{1}_{V^k\geq 1}dtd\theta.$$

Let $y \in D(A)$ with $\bar{y} = 0$, taking $h \in D(A^2)$ such that $y = Ah$ as a test function in (b) of Definition 1.1.1, we deduce that, for all $0 \leq t \leq T$, $\langle y, \xi^k + \rho^k\rangle_{O_t}$ has a limit when $k \to +\infty$. Moreover by the uniform convergence in $\mathcal{C}(O_T)$ of V^k to V, we have

$$f^{n_k}_{\ln}(V^k(t,\theta))\mathbb{1}_{V^k\geq 1} \underset{k\to+\infty}{\longrightarrow} f_{\ln}(V(t,\theta))\mathbb{1}_{V\geq 1}, \quad \text{for all } (t,\theta) \in O_T, \tag{1.2.8}$$

and the convergence is uniform. We obtain for all $0 \leq t \leq T$ and for all $h \in D(A)$:

$$\langle h, \rho^k \rangle_{O_t} \underset{k\to+\infty}{\longrightarrow} \int_{O_t} h(\theta) f_{\ln}(V(s,\theta)) \mathbb{1}_{V\geq 1} ds d\theta. \tag{1.2.9}$$

Note that $f_{\ln}(x)\mathbb{1}_{x\geq 1}$ is a continuous function so that $f_{\ln}(V)\mathbb{1}_{V\geq 1} \in L^1(O_T)$. Moreover, for any $y \in D(A)$ with $\bar{y} = 0$, for all $0 \leq t \leq T$,

$$\langle y, \xi^k \rangle_{O_t} \text{ has a limit when } k \to +\infty. \tag{1.2.10}$$

Notice that almost surely:

$$f_{\ln}^{n_k}(V^k(t,\theta))\mathbb{1}_{V^k<1} \underset{k\to+\infty}{\longrightarrow} \begin{cases} +\infty & \text{if } V(t,\theta) \leq 0, \\ f_{\ln}(V(t,\theta)) & \text{if } V(t,\theta) \in (0,1]. \end{cases} \tag{1.2.11}$$

Thus the limit of this term is not trivial. Let us now prove that the total mass $\xi^n(O_T)$ is bounded. We use the following Lemma whose proof is postponed to the end of this section.

Lemma 1.2.2 *Let $T > 0$, and $\{\mu^k\}_{k\in\mathbb{N}}$ be a sequence of finite positive measures on O_T. Suppose there exists $\{w^k\}_{k\in\mathbb{N}}$ a sequence of functions in $\mathcal{C}(O_T)$ such that w^k converges uniformly to w, when k grows to infinity. Suppose also there exist a function $M_T : \mathcal{C}(O_T) \to \mathbb{R}^+$, a nonnegative constants m_T and a positive continuous function $c_T : [0,T] \to \mathbb{R}^+$ such that*

$$\text{for all } h \in D(A) \text{ such that } \bar{h} = 0, \quad \langle h, \mu^k \rangle_{O_T} \leq M_T(h), \quad \text{for all } k \in \mathbb{N}, \tag{1.2.12}$$

$$\text{for all } t \in [0,T], \int_0^1 w(t,\theta) d\theta = c_T(t) > 0 \tag{1.2.13}$$

and

$$\langle w^k, \mu^k \rangle_{O_T} \leq m_T. \tag{1.2.14}$$

Then there exists a constant $\tilde{M}_T$ such that

$$\text{for all } h \in \mathcal{C}(O_T), \quad \langle h, \mu^k \rangle_{O_T} \leq \tilde{M}_T \|h\|_\infty, \quad \text{for all } k \in \mathbb{N}. \tag{1.2.15}$$

and in particular $\mu^k(O_T)$ is bounded uniformly for $k \in \mathbb{N}$.

Let us denote by :

$$M_T(h) = \sup_{k\in\mathbb{N}} \left| \langle h, \xi^k \rangle_{O_T} \right| \tag{1.2.16}$$

for $h \in D(A)$ such that $\bar{h} = 0$. By (1.2.10), we know that M_T is well defined. Moreover we have

$$\langle (V^k)^+, \xi^k \rangle_{O_T} = \int_{O_T} (V^k(t,\theta))^+ f_{\ln}^{n_k}(V^k(t,\theta)) \mathbb{1}_{V^k<1} dt d\theta. \tag{1.2.17}$$

Since $(x)^+ f_{\ln}^{n_k}(x)\mathbb{1}_{x<1}$ is uniformly bounded in $k \in \mathbb{N}$, there exists a positive constant m_T such that

$$\langle (V^k)^+, \xi^k \rangle_{O_T} \leq m_T. \tag{1.2.18}$$

Since V is almost surely positive, $(V^k)^+$ converges uniformly to V. Moreover $\overline{V(t,.)} = c > 0$ for all $t \in [0,T]$. We use Lemma 1.2.2 and obtain $\limsup_{k\to+\infty} \xi^k(O_T) < +\infty$.
Thanks to the Fatou lemma, we can write :

$$\begin{aligned} \int_{O_T} \left[f_{\ln}(V(s,\theta)) \mathbb{1}_{V<1} \right] ds d\theta &= \int_{O_T} \liminf_{k\to+\infty} \left[f_{\ln}^{n_k}(V^k(s,\theta)) \mathbb{1}_{V^k<1} ds d\theta \right] \\ &\leq \liminf_{k\to+\infty} \int_{O_T} \left[f_{\ln}^{n_k}(V^k(s,\theta)) \mathbb{1}_{V^k<1} \right] ds d\theta \\ &< +\infty. \end{aligned} \tag{1.2.19}$$

It follows that almost surely $f_{\ln}(V) \in L^1(O_T)$.

□

Step 2.
We again assume that we have $(n_k)_{k\in\mathbb{N}}$ a subsequence such that $(\hat{X}_c^{n_k})_{k\in\mathbb{N}}$ converges in law to a process $\hat{X}_c$. Again, by Skorohod's theorem, we can find a probability space and a sequence of processes $(V^k,\mathcal{W}^k)_{k\in\mathbb{N}}$ such that almost surely $(V^k,\mathcal{W}^k)\to(V,\mathcal{W})$ in $\mathcal{C}(O_T)$ as k grows to infinity, and $(V^k,\mathcal{W}^k)$ has the same distribution as $(\hat{X}_c^{n_k},W)$ for all $k\in\mathbb{N}$.
By step 1, the total mass $\xi^k(O_T)$ is bounded and there exists $(n_{k_m})_{m\in\mathbb{N}}$ a sub-subsequence such that the measures

$$\xi^{k_m}:=f_{\ln}^{n_{k_m}}(V^{k_m}(t,\theta))\mathbb{1}_{V^{k_m}<1}dtd\theta$$

converge to a measure ξ.
We denote by λ the following measure:

$$d\lambda:=f_{\ln}(V(t,\theta))\mathbb{1}_{V<1}dtd\theta, \tag{1.2.20}$$

and $\zeta^m:=\xi^{k_m}-\lambda$. Thus ζ^m converges to the measure $\zeta:=\xi-\lambda$. Let u be a continuous nonnegative function on O_T, we have

$$\begin{aligned}\langle u,\zeta\rangle_{O_T} &= \lim_{m\to+\infty}\langle u,\zeta^m\rangle_{O_T}\\ &= \lim_{m\to+\infty}\langle u,\xi^{k_m}\rangle_{O_T}-\langle u,\lambda\rangle_{O_T}.\end{aligned}$$

And this is positive, thanks to (1.2.19). Therefore ζ is a positive measure. Taking the limit as m grows to infinity in the approximated equation, we obtain that for all $h\in D(A^2)$ and for all $0\le t\le T$:

$$\begin{aligned}\langle V(t,.),h\rangle &= \langle x,h\rangle-\frac{1}{2}\int_{O_t}V(s,\theta)A^2h(\theta)dsd\theta-\frac{1}{2}\int_{O_t}f_{\ln}(V(s,\theta))Ah(\theta)dsd\theta\\ &\quad-\frac{1}{2}\langle Ah,\zeta\rangle_{O_t}-\int_0^t\langle Bh,d\mathcal{W}\rangle.\end{aligned}$$

This is the expected equation. Let us now show that the contact condition holds for (V,ζ). We prove in fact that for all β nonnegative:

$$0\le\langle V,\zeta\rangle_{O_T}\le\beta. \tag{1.2.21}$$

The key is to study the behavior of $f_{\ln}^{n_{k_m}}(V^{k_m}(t,\theta))\mathbb{1}_{V^{k_m}<1}$ near points $(t,\theta)\in O_T$ such that $V(t,\theta)$ is small. Fix $\beta>0$, there exists $\varepsilon>0$ such that $-T\varepsilon\ln(\varepsilon)\le\beta$. Let us define the following measures for all $m\in\mathbb{N}$.

$$d\xi_\varepsilon^m:=f_{\ln}^{n_{k_m}}(V^{k_m}(t,\theta))\mathbb{1}_{V^{k_m}<\varepsilon}dtd\theta,\quad d\tau_\varepsilon^m:=f_{\ln}^{n_{k_m}}(V^{k_m}(t,\theta))\mathbb{1}_{\varepsilon\le V^{k_m}<1}dtd\theta,$$

$$d\lambda_\varepsilon:=f_{\ln}(V(t,\theta))\mathbb{1}_{V<\varepsilon}dtd\theta,\qquad d\tau_\varepsilon:=f_{\ln}(V(t,\theta))\mathbb{1}_{\varepsilon\le V<1}dtd\theta.$$

Clearly τ_ε^m converges to τ_ε, it follows

$$\begin{aligned}\limsup_{m\to+\infty}\langle V^{k_m},\zeta^m\rangle_{O_T} &= \limsup_{m\to+\infty}\left(\langle V^{k_m},\xi_\varepsilon^{k_m}\rangle_{O_T}-\langle V^{k_m},\lambda_\varepsilon\rangle_{O_T}+\langle V^{k_m},\tau_\varepsilon^m\rangle_{O_T}-\langle V^{k_m},\tau_\varepsilon\rangle_{O_T}\right)\\ &= \limsup_{m\to+\infty}\left(\int_{O_T}V^{k_m}f_{\ln}^{n_{k_m}}(V^{k_m})\mathbb{1}_{V^{k_m}<\varepsilon}dtd\theta-\int_{O_T}V^{k_m}f_{\ln}(V)\mathbb{1}_{V<\varepsilon}dtd\theta\right)\\ &\le \limsup_{m\to+\infty}\left(\int_{O_T}V^{k_m}f_{\ln}^{n_{k_m}}(V^{k_m})\mathbb{1}_{0\le V^{k_m}<\varepsilon}dtd\theta\right)\\ &\quad+\limsup_{m\to+\infty}\left(\int_{O_T}(V^{k_m})^-f_{\ln}(V)\mathbb{1}_{V<\varepsilon}dtd\theta\right)\end{aligned}$$

Since $(V^{k_m})^-$ converges uniformly to zero, we deduce:

$$\begin{aligned}\limsup_{m\to+\infty}\langle V^{k_m},\zeta^m\rangle_{O_T} &\leq T\limsup_{m\to+\infty}\sup_{x\in[0,\varepsilon]}\left|-x\ln\left(x+\frac{1}{n_{k_m}}\right)\right|\\ &\leq T\limsup_{m\to+\infty}\left(-\varepsilon\ln\left(\varepsilon+\frac{1}{n_{k_m}}\right)\right)\\ &\leq -T\varepsilon\ln(\varepsilon)\,.\end{aligned}$$

Thus the contact condition holds.

□

Step 3.
By the convergence of the family $(\nu_c^n)_{n\in\mathbb{N}}$, we know that the initial distribution of $\hat{X}_c^n$ converges to ν_c. As in Lemma 5.2 from [29], we obtain that for all $T>0$, the laws of $(\hat{X}_c^n)_{n\in\mathbb{N}}$ are tight in $\mathcal{C}(O_T)$.
We use a result form [42] that allows to get the convergence of the approximated solutions in probability in any space in which these approximated solutions are tight.

Lemma 1.2.3 *Let $\{Z_n\}_{n\geq1}$ be a sequence of random elements on a Polish space E endowed by its Borel σ-algebra. Then $\{Z_n\}_{n\geq1}$ converges in probability to an E-valued random element if and only if from every pair of subsequences $\{(Z_{n_k^1},Z_{n_k^2})_{k\geq1}$, one can extract a subsequence which converges weakly to a random element supported on the diagonal $\{(x,y)\in E\times E, x=y\}$.*

For any subsequence $(n_k)_{k\in\mathbb{N}}$, we have convergence of ξ^k to a finite measure ξ on O_T along some sub-subsequence $(k_m)_{m\in\mathbb{N}}$. Let $\xi_i, i=1..2$ be two such limits. By the second step, and the uniqueness of the reflexion measure, we know $\zeta_1:=\xi_1-\lambda$ and $\zeta_2:=\xi_2-\lambda$ are equals. So the limit of $(\xi^k)_{k\in\mathbb{N}}$ is unique, and ξ^k converges to its limit ξ.
Assume $(n_k^1)_{k\in\mathbb{N}}$ and $(n_k^1)_{k\in\mathbb{N}}$ are two arbitrary subsequences. In the notations of the second step and by the third step, the process $\left(\hat{X}_c^{n_k^1},\hat{X}_c^{n_k^2},W\right)$ is tight in a suitable space. By Skorohod's theorem, we can find a probability space and a sequence of processes $(V_1^k,V_2^k,\mathcal{W}^k)$ such that $(V_1^k,V_2^k,\mathcal{W}^k)\to(V_1,V_2,\mathcal{W})$ almost surely in $\mathcal{C}(O_T)$, and $(V_1^k,V_2^k,\mathcal{W}^k)$ as the same distribution as $\left(\hat{X}_c^{n_k^1},\hat{X}_c^{n_k^2},W\right)$ for all $k\in\mathbb{N}$. In Skorohod's space, the approximated measures respectively converge to two contact measures ζ_1 and ζ_2. By the second step, $(V_1,\zeta_1,\mathcal{W})$ and $(V_2,\zeta_2,\mathcal{W})$ are both weak solutions of (1.0.3). By uniqueness, necessarily $V_1=V_2$ and $\zeta_1=\zeta_2$. Therefore the subsequence $\left(\left(\hat{X}_c^{n_k^1},\eta^{n_k^1},W\right),\left(\hat{X}_c^{n_k^2},\eta^{n_k^2},W\right)\right)_{k\in\mathbb{N}}$ converges in probability to a process supported on the diagonal. We use Lemma 1.2.3 to prove that the sequence $(\hat{X}_c^n,\eta^n,W)$ converges in law to $(\hat{X}_c,\eta,W)$ stationary strong solution of (1.0.3).

□

Proof in the negative α-power case:
We again split the proof in three steps.
Step 1.
Let us assume that $(n_k)_{k\in\mathbb{N}}$ is a subsequence such that $(\hat{X}_c^{n_k})_{n\in\mathbb{N}}$ converges in law in $\mathcal{C}(O_T)$ to a process $\hat{X}_c$.
By Skorohod's theorem, we can find a probability space and a sequence of processes $(V^k,\mathcal{W}^k)_{k\in\mathbb{N}}$ on that probability space such that $(V^k,\mathcal{W}^k)\to(V,\mathcal{W})$ in $\mathcal{C}(O_T)$ almost surely and $(V^k,\mathcal{W}^k)$ has the same distribution as $(\hat{X}_c^{n_k},\mathcal{W})$ for all $k\in\mathbb{N}$. Notice that $V\geq0$ almost surely since for all $t\leq T$ the law of $V(t,.)$ is ν_c which is concentrated on K. Let now ξ^k be the following measure on O_T:

$$d\xi^k:=f_\alpha^{n_k}(V^k(t,\theta))dtd\theta.$$

Let $y\in D(A)$ with $\bar{y}=0$, taking $h\in D(A^2)$ such that $y=Ah$ as a test function in (b) of Definition 1.1.1, we deduce that, for all $0\leq t\leq T$,

$$\left\langle y,\xi^k\right\rangle_{O_t}\text{ has a limit when }k\to+\infty. \tag{1.2.22}$$

As in the logarithmic case, we now prove that the total mass $\xi^n(O_T)$ is bounded. Let us denote by :

$$M_T(h) = \sup_{k\in\mathbb{N}} \left| \langle h, \xi^k \rangle_{O_T} \right| \tag{1.2.23}$$

for $h \in D(A)$ such that $\bar{h} = 0$. By (1.2.22), we know that M_T is well defined. Therefore by Lemma 1.2.2 it suffices to find a function w, such that $\overline{w(t,.)}$ is continuous and nonnegative, and a sequence $(w^k)_{k\in\mathbb{N}}$ such that for a positive constant m_T, w^k converges uniformly to w such that

$$\langle w^k, \xi^k \rangle_{O_T} \leq m_T. \tag{1.2.24}$$

Denote by $w^k := ((V^k)^+ + 1/n_k)^\alpha$, it converges uniformly to $w := V^\alpha$. Since $w^k f_\alpha^{n_k}(V^k(t,\theta)) = 1$, (1.2.24) holds with $m_T = T$. As in the logarithmic case, by the Fatou Lemma, it follows that almost surely $f_\alpha(V) \in L^1(O_T)$.

□

Step 2. We again assume that we have $(n_k)_{k\in\mathbb{N}}$ a subsequence such that $(\hat{X}_c^{n_k})_{k\in\mathbb{N}}$ converges in law to a process $\hat{X}_c$. Again, by Skorohod's theorem, we can find a probability space and a sequence of processes $(V^k, \mathcal{W}^k)_{k\in\mathbb{N}}$ such that almost surely $(V^k, \mathcal{W}^k) \to (V, \mathcal{W})$ in $\mathcal{C}(O_T)$ as k grows to infinity, and $(V^k, \mathcal{W}^k)$ has the same distribution as $(\hat{X}_c^{n_k}, W)$ for all $k \in \mathbb{N}$.
By step 1, the total mass $\xi^k(O_T)$ is bounded and there exists $(n_{k_m})_{m\in\mathbb{N}}$ a sub-subsequence such that the measures

$$\xi^{k_m} := f_\alpha^{n_{k_m}}(V^{k_m}(t,\theta)) \mathbb{1}_{V^{k_m}<1} dtd\theta$$

converges to a measure ξ.
We denote by λ the following measure:

$$d\lambda := f_\alpha(V(t,\theta))dtd\theta, \tag{1.2.25}$$

and $\zeta^m := \xi^m - \lambda$. Thus ζ^m converges to the measure $\zeta := \xi - \lambda$. Thanks to the Fatou Lemma, ζ is a positive measure. Taking the limit as m grows to infinity in the approximated equation, we obtain that for all $h \in D(A^2)$ and for all $0 \leq t \leq T$:

$$\begin{aligned} \langle V(t,.), h\rangle &= \langle x, h\rangle - \frac{1}{2}\int_{O_t} V(s,\theta)A^2h(\theta)dsd\theta - \frac{1}{2}\int_{O_t} f_\alpha(V(s,\theta))Ah(\theta)dsd\theta \\ &\quad -\frac{1}{2}\langle Ah, \zeta\rangle_{O_t} - \frac{1}{2}\int_0^t \langle Bh, d\mathcal{W}\rangle. \end{aligned}$$

This is the expected equation. Let us now show that the contact condition holds for (V, ζ).
Case 1 : $0 \leq \alpha < 1$.
As in the second step of the logarithmic case, fix $\beta > 0$, so there exists $\varepsilon > 0$ such that $T\varepsilon^{1-\alpha} \leq \beta$. Let us define the following measures for all $m \in \mathbb{N}$:

$$d\xi_\varepsilon^m := f_\alpha^{n_{k_m}}(V^{k_m}(t,\theta))\mathbb{1}_{V^{k_m}<\varepsilon} dtd\theta, \quad d\tau_\varepsilon^m := f_\alpha^{n_{k_m}}(V^{k_m}(t,\theta))\mathbb{1}_{\varepsilon\leq V^{k_m}} dtd\theta,$$

$$d\lambda_\varepsilon := f_\alpha(V(t,\theta))\mathbb{1}_{V<\varepsilon} dtd\theta, \qquad d\tau_\varepsilon := f_\alpha(V(t,\theta))\mathbb{1}_{\varepsilon\leq V} dtd\theta.$$

Since τ_ε^m converges to τ_ε, we have

$$\begin{aligned} \limsup_{m\to+\infty} \langle V^{k_m}, \zeta^m\rangle_{O_T} &= \limsup_{m\to+\infty} \left(\langle V^{k_m}, \xi_\varepsilon^{k_m}\rangle_{O_T} - \langle V^{k_m}, \lambda_\varepsilon\rangle_{O_T} + \langle V^{k_m}, \tau_\varepsilon^m\rangle_{O_T} - \langle V^{k_m}, \tau_\varepsilon\rangle_{O_T} \right) \\ &= \limsup_{m\to+\infty} \left(\int_{O_T} V^{k_m} f_\alpha^{n_{k_m}}(V^{k_m}) \mathbb{1}_{V^{k_m}<\varepsilon} dtd\theta - \int_{O_T} V^{k_m} f_\alpha(V)\mathbb{1}_{V<\varepsilon} dtd\theta \right) \\ &\leq \limsup_{m\to+\infty} \left(\int_{O_T} V^{k_m} f_\alpha^{n_{k_m}}(V^{k_m}) \mathbb{1}_{0\leq V^{k_m}<\varepsilon} dtd\theta \right) \\ &\quad + \limsup_{m\to+\infty} \left(\int_{O_T} (V^{k_m})^- f_\alpha(V)\mathbb{1}_{V<\varepsilon} dtd\theta \right). \end{aligned}$$

It follows

$$\begin{aligned}\limsup_{m\to+\infty}\langle V^{k_m},\zeta^m\rangle_{O_T} &\leq T\limsup_{m\to+\infty}\sup_{x\in[0,\varepsilon]}\left|\frac{x}{\left(x+\frac{1}{n_{k_m}}\right)^{\alpha}}\right|\\ &\leq T\limsup_{m\to+\infty}\left|\frac{\varepsilon}{\left(\varepsilon+\frac{1}{n_{k_m}}\right)^{\alpha}}\right|\\ &\leq T\varepsilon^{1-\alpha}.\end{aligned}$$

Thus the contact condition holds.
Case 2 : $\alpha\geq 1$.
Let $\gamma>0$; we prove that for all nonnegative β, $0\leq\langle V^{\alpha+\gamma},\zeta\rangle_{O_T}\leq\beta$ and conclude that the contact condition holds by the Hölder inequality.
Fix $\beta>0$, so there exists $\varepsilon>0$ such that $T\varepsilon^{\gamma}\leq\beta$. Let us define the following measures for all $m\in\mathbb{N}$:

$$d\xi_\varepsilon^m := f_\alpha^{n_{k_m}}(V^{k_m}(t,\theta))\mathbb{1}_{V^{k_m}<\varepsilon}dtd\theta,\quad d\tau_\varepsilon^m := f_\alpha^{n_{k_m}}(V^{k_m}(t,\theta))\mathbb{1}_{\varepsilon\leq V^{k_m}}dtd\theta,$$

$$d\lambda_\varepsilon := f_\alpha(V(t,\theta))\mathbb{1}_{V<\varepsilon}dtd\theta,\qquad d\tau_\varepsilon := f_\alpha(V(t,\theta))\mathbb{1}_{\varepsilon\leq V}dtd\theta.$$

Since τ_ε^m converges to τ_ε, we have

$$\begin{aligned}\limsup_{m\to+\infty}\langle\left(V^{k_m}\right)^{\alpha+\gamma},\zeta^m\rangle_{O_T} &= \limsup_{m\to+\infty}\Big(\langle\left(V^{k_m}\right)^{\alpha+\gamma},\xi_\varepsilon^{k_m}\rangle_{O_T}-\langle\left(V^{k_m}\right)^{\alpha+\gamma},\lambda_\varepsilon\rangle_{O_T}\\ &\quad+\langle\left(V^{k_m}\right)^{\alpha+\gamma},\tau_\varepsilon^m\rangle_{O_T}-\langle\left(V^{k_m}\right)^{\alpha+\gamma},\tau_\varepsilon\rangle_{O_T}\Big)\\ &= \limsup_{m\to+\infty}\left(\int_{O_T}\left(V^{k_m}\right)^{\alpha+\gamma}f_\alpha^{n_{k_m}}(V^{k_m})\mathbb{1}_{V^{k_m}<\varepsilon}dtd\theta\right.\\ &\quad\left.-\int_{O_T}\left(V^{k_m}\right)^{\alpha+\gamma}f_\alpha(V)\mathbb{1}_{V<\varepsilon}dtd\theta\right)\\ &\leq \limsup_{m\to+\infty}\left(\int_{O_T}\left(V^{k_m}\right)^{\alpha+\gamma}f_\alpha^{n_{k_m}}(V^{k_m})\mathbb{1}_{0\leq V^{k_m}<\varepsilon}dtd\theta\right)\\ &\quad+\limsup_{m\to+\infty}\left(\int_{O_T}\left(\left(V^{k_m}\right)^-\right)^{\alpha+\gamma}f_\alpha(V)\mathbb{1}_{V<\varepsilon}dtd\theta\right).\end{aligned}$$

It follows

$$\begin{aligned}\limsup_{m\to+\infty}\langle\left(V^{k_m}\right)^{\alpha+\gamma},\zeta^m\rangle_{O_T} &\leq T\limsup_{m\to+\infty}\sup_{x\in[0,\varepsilon]}\left|\frac{x^{\alpha+\gamma}}{\left(x+\frac{1}{n_{k_m}}\right)^{\alpha}}\right|\\ &\leq T\limsup_{m\to+\infty}\left|\frac{\varepsilon^{\alpha+\gamma}}{\left(\varepsilon+\frac{1}{n_{k_m}}\right)^{\alpha}}\right|\\ &\leq T\varepsilon^{\gamma}.\end{aligned}$$

Thus the contact condition holds.

□

Step 3 is strictly identical to the logarithmic case and we do not repeat it. This ends the proof of Theorem 1.2.1. Now we give the proof of Lemma 1.2.2.

Proof of Lemma 1.2.2:
We prove this Lemma thanks to the previous Lemma 1.2.1. If $\mu^k(O_T)$ is bounded uniformly for $k \in \mathbb{N}$, then the constant

$$\tilde{M}_T = \sup_{k\in\mathbb{N}} \mu^k(O_T) \tag{1.2.26}$$

satisfies (1.2.15). Suppose $\mu^k(O_T)$ is unbounded, then there exists $k_0 \in \mathbb{N}$ such that $\mu^k(O_T) > 0$ for all $k \geq k_0$, we denote for all $k \geq k_0$

$$\nu^k := \mu^k/\mu^k(O_T).$$

$\{\nu^k\}_{k\geq k_0}$ is a sequence of probability measure on O_T, and we can extract a subsequence $\{\nu^{k_m}\}_{m\in\mathbb{N}}$ such that there exists a probability measure ν with $\nu^{k_m} \rightharpoonup \nu$ when m grows to infinity. Therefore, by the uniform convergence of w^k

$$\left\langle w^{k_m}, \nu^{k_m} \right\rangle_{O_T} \underset{m\to+\infty}{\longrightarrow} \left\langle w, \nu \right\rangle_{O_T}. \tag{1.2.27}$$

And by the uniform boundedness in (1.2.14), we have

$$\left\langle w^{k_m}, \nu^{k_m} \right\rangle_{O_T} \leq \frac{m_T}{\mu^{k_m}(O_T)} \underset{l\to+\infty}{\longrightarrow} 0, \tag{1.2.28}$$

therefore

$$\left\langle w, \nu \right\rangle_{O_T} = 0. \tag{1.2.29}$$

Moreover, for all $h \in D(A)$ such that $\bar{h} = 0$

$$\left\langle h, \nu^{k_m} \right\rangle_{O_T} \underset{m\to+\infty}{\longrightarrow} \left\langle h, \nu \right\rangle_{O_T}, \tag{1.2.30}$$

and by the uniform boundedness in (1.2.12), for all $h \in D(A)$ such that $\bar{h} = 0$, we have

$$\left\langle h, \nu^{k_m} \right\rangle_{O_T} \leq \frac{M_T(h)}{\mu^{k_m}(O_T)} \underset{m\to+\infty}{\longrightarrow} 0. \tag{1.2.31}$$

So that for all $h \in D(A)$ such that $\bar{h} = 0$, we have

$$\left\langle h, \nu \right\rangle_{O_T} = 0. \tag{1.2.32}$$

Since ν is a probability measure, we deduce that (1.2.32) holds in fact for any $h \in \mathcal{C}(O_T)$ such that $\bar{h} = 0$. The hypothesis of Lemma 1.2.1 are satisfied, and we can conclude that the measure ν is null. This is a contradiction since ν is a probability measure. Then the sequence $\mu^k(O_T)$ is bounded uniformly for $k \in \mathbb{N}$, and the constant $\tilde{M}_T$ in (1.2.26) fulfills (1.2.15).

□

1.2.4 Convergence of the semigroup

First we state the following result which is a corollary of Theorem 1.2.1.

Corollary 1.2.1 *Let $c > 0$.*

i) There exists a continuous process $(X(t,x), t \geq 0, x \in K \cap H_c)$ with $X(0,x) = x$ and a set K_0 dense in $K \cap H_c$, such that for all $x \in K_0$ there exists a unique strong solution of equation (1.0.3) given by $\left((X(t,x))_{t\geq 0}, \eta^x, W\right)$.

ii) The law of $(X(t,x)_{t\geq 0}, \eta^x)$ is a regular conditional distribution of the law of $\left(\hat{X}_c, \eta\right)$ given $\hat{X}_c(0) = x \in K \cap H_c$.

Proof : By Theorem 1.2.1, we have a stationary strong solution $\hat{X}_c$ in H_c, such that W and $\hat{X}_c(0)$ are independent. Conditioning $\left(\hat{X}_c, \eta\right)$ on the value of $\hat{X}_c(0) = x$, with $c = \overline{x}$, we obtain for ν_c-almost every x a strong solution that we denote $(X(t,x), \eta^x)$ for all $t \geq 0$ and for all $x \in K \cap H_c$. This process is the desired process. Indeed, since the support of ν_c is $K \cap H_c$, we have a strong solution for a dense set K_0 in $K \cap H_c$.
Notice that all processes $(X(t,x))_{t\geq 0}$ with $x \in K_0$ are driven by the same noise W and are continuous with values in H. Arguing as in the proof of Lemma 1.1.1 we see that for all $x, y \in K_0$, for all $t \geq 0$:

$$\|X(t,x) - X(t,y)\|_{-1} \leq \|x - y\|_{-1}.$$

Then by density, we obtain a continuous process $(X(t,x))_{t\geq 0}$ in H_c for all $x \in K \cap H_c$.

□

We want to prove that for any deterministic initial condition $x \in K \cap H_c$ where $c > 0$, there exists a strong solution of equation (1.0.3), necessarily unique and that the process X constructed in Corollary 1.2.1 is a realization of such solution. We have proved this result only for x in a dense set K_0, but thanks to the convergence of the transition semigroup $P^{n,c}$, we will be able to conclude. First we prove that the transition semigroup converges on $K \cap H_c$. This result is explained by the following proposition :

Proposition 1.2.3 *Let $c > 0$, for all $\phi \in \mathcal{C}_b(H)$ and $x \in K \cap H_c$:*

$$\lim_{n\to+\infty} P_t^{n,c}\phi(x) = \mathbb{E}[\phi(X(t,x))] =: P_t^c\phi(x). \tag{1.2.33}$$

Moreover the Markov process $(X(t,x), t \geq 0, x \in K \cap H_c)$ is strong Feller and its transition semigroup P^c is such that:

$$|P_t^c\phi(x) - P_t^c\phi(y)| \leq \frac{\|\phi\|_\infty}{\sqrt{t}}\|x - y\|_H, \quad \text{for all } x, y \in K \cap H_c, \text{for all } t > 0. \tag{1.2.34}$$

Proof : By proposition 1.2.1 X^n is strong Feller on H_c and for all $\phi : H_c \to \mathbb{R}$ bounded and Borel we have :

$$|P_t^{n,c}\phi(x) - P_t^{n,c}\phi(y)| \leq \frac{\|\phi\|_\infty}{\sqrt{t}}\|x - y\|_H, \quad \text{for all } x, y \in K \cap H_c, \text{ for all } t > 0. \tag{1.2.35}$$

Since $(\nu_c^n)_{n\geq 1}$ is tight in H_c, then there exists an increasing sequence of compact sets $(J^p)_{p\in\mathbb{N}}$ in H such that:

$$\lim_{p\to+\infty} \sup_{n\geq 1} \nu_c^n(H \setminus J^p) = 0. \tag{1.2.36}$$

Set $J := \cup_{p\in\mathbb{N}} J^p \cap K$. Since the support of ν_c is in $K \cap H_c$ and $\nu_c(J) = 1$, then J is dense in $K \cap H_c$. Fix $t > 0$, by (1.2.35), for any $\phi \in \mathcal{C}_b(H)$:

$$\sup_{n\in\mathbb{N}}(\|P_t^{n,c}\phi\|_\infty + [P_t^{n,c}\phi]_{Lip(H_c)}) < +\infty. \tag{1.2.37}$$

Let $(n_j)_{j\in\mathbb{N}}$ be any sequence in $\mathbb{N}$. With a diagonal procedure, by the Arzelà-Ascoli theorem, there exist $(n_{j_l})_{l\in\mathbb{N}}$ a subsequence and a function $\Theta_t : J \to \mathbb{R}$ such that:

$$\lim_{l\to+\infty} \sup_{x\in J^p} |P_t^{n_{j_l},c}\phi(x) - \Theta_t(x)| = 0, \quad \text{for all } p \in \mathbb{N}. \tag{1.2.38}$$

By density, Θ_t can be extended uniquely to a bounded Lipschitz function $\tilde{\Theta}_t$ on $K \cap H_c$ such that

$$\tilde{\Theta}_t(x) = \lim_{l\to+\infty} P_t^{n_{j_l},c}\phi(x), \quad \text{for all } x \in K \cap H_c. \tag{1.2.39}$$

Note that the subsequence depends on t. Therefore, we have to prove that the limit defines a semigroup and does not depend on the chosen subsequence.

By Theorem 1.2.1, we have for all $\phi, \psi \in \mathcal{C}_b(H)$:

$$\begin{aligned}
\mathbb{E}\left[\psi\left(\hat{X}_c(0)\right)\phi\left(\hat{X}_c(t)\right)\right] &= \lim_{l\to+\infty} \mathbb{E}\left[\psi\left(\hat{X}_c^{n_{j_l}}(0)\right)\phi\left(\hat{X}_c^{n_{j_l}}(t)\right)\right] \\
&= \lim_{l\to+\infty} \int_H \psi(y)\mathbb{E}\left[\phi\left(\hat{X}_c^{n_{j_l}}(t)\right)\Big|\hat{X}_c^{n_{j_l}}(0)=y\right]\nu_c^{n_{j_l}}(dy) \\
&= \lim_{l\to+\infty} \int_H \psi(y)P_t^{n_{j_l},c}\phi(y)\nu_c^{n_{j_l}}(dy) \\
&= \int_H \psi(y)\tilde{\Theta}_t(y)\nu_c(dy).
\end{aligned}$$

Thus, by Corollary 1.2.1, we have the following equality:

$$\mathbb{E}\left[\phi\left(X(t,x)\right)\right] = \tilde{\Theta}_t(x), \quad \text{for } \nu_c\text{-almost every } x. \tag{1.2.40}$$

Since $\mathbb{E}[\phi(X(t,.))]$ and $\tilde{\Theta}_t$ are continuous on $K \cap H_c$, and $\nu_c(K \cap H_c) = 1$, the equality (1.2.40) is true for all $x \in K \cap H_c$. Moreover the limit does not depend on the chosen subsequence, and we obtain (1.2.33). Since the semigroups are equi-Lipschitz, we deduce (1.2.34).

□

1.2.5 Existence of solutions

We have proved that there exists a continuous process X which is a strong solution of equation (1.0.3) for an x in a dense space. In this section, we prove existence for an initial condition in $K \cap H_c$ with $c > 0$.

Theorem 1.2.2 *Let ξ be a K-valued random value with $\overline{\xi} > 0$ almost surely and (ξ, W) independent, then there exists a continuous process denoted $(X(t,\xi))_{t\geq 0}$ and a measure η^ξ such that:*

(a) $\left((X(t,\xi))_{t\geq 0}, \eta^\xi, W\right)$ is the unique strong solution of (1.0.3) with $X(0,\xi) = \xi$ almost surely.

(b) The Markov process $(X(t,x), t \geq 0, x \in K \cap H_c)$ is continuous and has P^c for transition semigroup which is strong Feller on H_c.

(c) For all $c > 0$, $x \in K \cap H_c$ and $0 = t_0 < t_1 < \cdots < t_m$, $(X(t_i,x), i = 1,\ldots,n)$ is the limit in distribution of $(X^n(t_i,x))_{i=1,\ldots,m}$.

(d) If ξ has distribution ν_c with $c > 0$, then $(X(t,\xi))_{t\geq 0}$ is equal in distribution to $\left(\hat{X}_c(t)\right)_{t\geq 0}$.

Proof : By Corollary 1.2.1 we have a process $(X(t,x), t \geq 0, x \in K \cap H_c)$, such that for all x in a set K_0 dense in $K \cap H_c$ we have a strong solution $\left((X(t,x))_{t\geq 0}, \eta^x, W\right)$ of (1.0.3) with initial condition x. By proposition 1.2.3, we have that the Markov process X has transition semigroup P_c on H_c.
The strong Feller property of P^c implies that for all $x \in K \cap H_c$ and $s > 0$ the law of $X(s,x)$ is absolutely continuous with respect to the invariant measure ν_c. Indeed, if $\nu_c(\Gamma) = 0$, then $\nu_c(P_s^c(\mathbb{1}_\Gamma)) = \nu_c(\Gamma) = 0$. So $P_s^c(\mathbb{1}_\Gamma)(x) = 0$ for ν_c-almost every x and by continuity for all $x \in K \cap H_c$.
Therefore almost surely $X(s,x) \in K_0$ for all $s > 0$ and $x \in K \cap H_c$. Fix $s > 0$, denote for all $\theta \in [0,1]$:

$$\tilde{X} := t \mapsto X(t+s,x), \tilde{W}(\cdot,\theta) := t \mapsto W(t+s,\theta) - W(s,\theta)),$$

and the measure $\tilde{\eta}^x$ such that for all $T > 0$, and for all $h \in \mathcal{C}(O_T)$:

$$\left\langle h, \tilde{\eta}^x \right\rangle_{O_T} := \int_{O_s^{T+s}} h(t-s,\theta)\eta^x(dt,d\theta)$$

So we have a process $\tilde{X} \in \mathcal{C}([0,T];H) \cap \mathcal{C}(O_T)$ and a measure $\tilde{\eta}^x$ on O_T which is finite on $[\delta,T]\times[0,1]$ for all $\delta \geq 0$, such that $\left((\tilde{X}(t,x))_{t\geq 0}, \tilde{\eta}^x, \tilde{W}\right)$ is a strong solution of (1.0.3) with initial condition

$X(s,x)$. By continuity $X(s,x) \to x$ in H as $s \to 0$, so $((X(t,x))_{t\geq 0}, \eta^x, W)$ is a strong solution of (1.0.3) with initial condition x in the sense of the definition 1.2.1. Thanks to the previous results, (b), (c) and (d) are obvious.

□

1.3 Reflection and Revuz measures

We have proved the existence of solution to (1.0.3) with a reflection measure. In [74], L. Zambotti uses an integration by parts formula to prove that, in some cases, the reflection measure vanishes. Moreover, L. Zambotti proves that, in some other cases, the reflection measure does not vanish. He uses the theory of the Continuous Additive Functionals described in [35]. We adapt his arguments and prove similar results for our case. First we state our main result :

Theorem 1.3.1 *For all $c > 0$, for all $x \in K \cap H_c$:*
i) For $\alpha \geq 3$, the reflection measure η^x of the strong solution $((X(t,x))_{t\geq 0}, \eta^x, W)$ vanishes.
ii) For $\alpha < 3$, the reflection measure η^x of the strong solution $((X(t,x))_{t\geq 0}, \eta^x, W)$ does not vanish.

If the reflection measure η^x vanishes, then we have a classic solution. On the other hand, if the reflection measure does not vanish, we do not have a classic solution, and the additional term with the measure becomes necessary. We state that the reflection measure is a PCAF and its Revuz measure is explicit and appears in an integration by parts formula (see Section 1.3.4). We use the theory of Additive Functionals of a Markov process described in [35] to obtain the following result:

Proposition 1.3.1 *There exists $\mathcal{A}^{[\nu]}$ a linear combination of PCAFs in the strict sense of X_c and for all $\delta > 0$ there exists a process Y_c^δ such that:*

$$\frac{1}{2}\int_0^1 \Pi h(\theta)\eta^x([\delta, \delta+t], d\theta) = \mathcal{A}_t^{[\nu]}(Y_c^\delta(\cdot, x)), \qquad \text{for all } t \geq 0,\ x \in K,\ \text{almost surely,}$$

and the Revuz-measure of $\mathcal{A}^{[\nu]}$ is:

$$\frac{1}{2}\int_0^1 \Pi h(r)dr\ \gamma\ d\Sigma_r^c, \tag{1.3.1}$$

where Σ_r^c is a measure which appears in the integration by parts formula as a boundary term.

1.3.1 Integration by parts formula

For all $\phi \in \mathcal{C}_b^1(H_c)$ we denote by $\partial_h \phi$ the directional derivative of ϕ along $h \in H$:

$$\partial_h \phi : x \mapsto \lim_{t\to 0} \frac{1}{t}(\phi(x+th) - \phi(x)), \quad x \in H.$$

For all $\phi \in \mathcal{C}_b^1(H)$, we have:

$$\langle \nabla\phi(x), h\rangle = \partial_h \phi(x).$$

The starting point is Theorem 7.1 in [29] where the following formula has been proved for a process Y whose the law is μ. For all $r \in (0,1)$, there exists a process $\mathcal{U}_r : [0,1] \to \mathbb{R}$ linked to the standard Brownian meander (see [30] [65]) such that we have the following Theorem 1.3.2. This process is almost surely positive except at time $\theta = r$ where it attains its minimum 0. Moreover it admits a density, and for $\alpha \geq 3$, by the law of the iterated logarithm:

$$\int_0^1 \frac{d\theta}{(\mathcal{U}_r(\theta))^{\alpha-1}} = +\infty, \qquad \text{almost surely and for all } r \in (0,1).$$

Theorem 1.3.2 *For all Φ in $\mathcal{C}_b^1(H,\mathbb{R})$ and $h \in D(A)$:*

$$\begin{aligned}\mathbb{E}\left[\partial_h \Phi(Y)\mathbb{1}_{Y\in K}\right] &= -\mathbb{E}\left[\left(\langle Y, Ah\rangle - \overline{Y}\cdot\overline{h}\right)\Phi(Y)\mathbb{1}_{Y\in K}\right] \\ &\quad - \int_0^1 h(r)\frac{1}{\sqrt{2\pi^3 r(1-r)}}\mathbb{E}\left[\Phi(\mathcal{U}_r)e^{-(1/2)(\overline{\mathcal{U}}_r)^2}\right]dr.\end{aligned} \tag{1.3.2}$$

We denote by $p_{\overline{\mathcal{U}}_r} : \mathbb{R}^+ \to [0,1]$ the continuous version of the density of $\overline{\mathcal{U}}_r$. By conditioning on $\overline{Y} = c$, we obtain:

$$\begin{aligned}\mathbb{E}\left[\partial_{\Pi h}\Phi(Y_c)\mathbb{1}_{Y_c\in K}\right] &= -\mathbb{E}\left[\langle Y_c, Ah\rangle\Phi(Y_c)\mathbb{1}_{Y_c\in K}\right] \\ &\quad - \int_0^1 \Pi h(r)\frac{p_{\overline{\mathcal{U}}_r}(c)}{\pi\sqrt{r(1-r)}}\mathbb{E}\left[\Phi(\mathcal{U}_r)|\overline{\mathcal{U}}_r = c\right]dr,\end{aligned} \tag{1.3.3}$$

where Y_c has been defined in the section 1.1. Moreover, notice that we have the following classical and easy to prove integration by parts formula for the measures $(\nu_c^n)_{n\in\mathbb{N}}$. For all Φ in $\mathcal{C}_b^1(H)$ and $h \in D(A)$:

$$\int_H \partial_{\Pi h}\Phi\, d\nu_c^n = -\int_H \langle x, Ah\rangle\Phi(x)\nu_c^n(dx) - \int_0^1 \Pi h(r)\int_H \Phi(x) f^n(x(r))\nu_c^n(dx)dr. \tag{1.3.4}$$

We define $\gamma^n : x \mapsto \frac{1}{Z_c^n}\exp(-U^n(x))$ for all $x \in H$, where Z_c^n is the constant of normalization defined in (1.1.9). Then $\gamma^n \in \mathcal{C}_b^1(H)$ and for all $x, h \in K$:

$$\langle \nabla\gamma^n(x), h\rangle = \gamma^n(x)\langle \nabla \log\gamma^n(x), h\rangle = \gamma^n(x)\int_0^1 h(\theta) f^n(x(\theta))d\theta. \tag{1.3.5}$$

Let ϕ be in $\mathcal{C}_b^1(H)$. We use (1.3.3), with $\Phi = \phi\cdot\gamma^n$. So we obtain:

$$\begin{aligned}\int_H \partial_{\Pi h}(\phi\cdot\gamma^n)\, d\mu_c &= -\int_H \langle x, Ah\rangle\phi(x)\gamma^n(x)\mathbb{1}_{x\in K}\mu_c(dx) \\ &\quad - \int_0^1 \Pi h(r)\frac{p_{\overline{\mathcal{U}}_r}(c)}{\pi\sqrt{r(1-r)}}\mathbb{E}\left[\phi(\mathcal{U}_r)\gamma^n(\mathcal{U}_r)|\overline{\mathcal{U}}_r = c\right]dr.\end{aligned} \tag{1.3.6}$$

We compute the derivative of the product, and obtain:

$$\begin{aligned}\int_H (\partial_{\Pi h}\phi)\cdot\gamma^n\, d\mu_c &= -\int_H \Big(\langle x, Ah\rangle + \langle \nabla\log\gamma^n(x), \Pi h\rangle\Big)\phi(x)\gamma^n(x)\mathbb{1}_{x\in K}\mu_c(dx) \\ &\quad - \int_0^1 \Pi h(r)\frac{p_{\overline{\mathcal{U}}_r}(c)}{\pi\sqrt{r(1-r)}}\mathbb{E}\left[\phi(\mathcal{U}_r)\gamma^n(\mathcal{U}_r)|\overline{\mathcal{U}}_r = c\right]dr.\end{aligned} \tag{1.3.7}$$

In order to let n go to infinity, we have to study the convergence of all the terms. By section 1.2.2, the left-hand side converges to:

$$\int_H (\partial_{\Pi h}\phi)d\nu_c.$$

Denote now by I_r^n the following term:

$$I_r^n := \frac{p_{\overline{\mathcal{U}}_r}(c)}{\pi\sqrt{r(1-r)}}\mathbb{E}\left[\phi(\mathcal{U}_r)\gamma^n(\mathcal{U}_r)\big|\overline{\mathcal{U}}_r = c\right].$$

Since Z_c^n converges, there exists C such that for all $r \in (0,1)$:

$$|I_r^n| \le C\frac{p_{\overline{\mathcal{U}}_r(c)}}{\sqrt{r(1-r)}}\|\phi\|_\infty J_r^n,$$

where J_r^n is defined by:

$$J_r^n := \mathbb{E}\left[\exp\left(-\int_0^1 F^n(\mathcal{U}_r(\theta))d\theta\right)\right].$$

In the logarithmic case and in the negative α-power case, as in section 1.2.2 and by dominated convergence, we have for all $r \in (0,1)$:

$$\lim_{n\to+\infty} J_r^n = \mathbb{E}\left[\exp\left(-\int_0^1 F(\mathcal{U}_r(\theta))d\theta\right)\right]. \tag{1.3.8}$$

Therefore, in the logarithmic case and in the negative α-power case for $\alpha > 1$, since $|J_r^n| < 1$, by dominated convergence, the last term in (1.3.7) has a limit when n grows to infinity.
In the negative α-power case for $\alpha \leq 1$, since

$$|J_r^n| \leq \mathbb{E}\left[\exp\left(-\int_0^1 F_\alpha^1(\mathcal{U}_r(\theta))d\theta\right)\right],$$

by dominated convergence, the last term in (1.3.7) has a limit when n grows to infinity.
Moreover, if $\alpha \geq 3$, by the law of the iterated logarithm, almost surely and for all $r \in (0,1)$:

$$\int_0^1 \frac{d\theta}{(\mathcal{U}_r(\theta))^{\alpha-1}} = +\infty.$$

Thus, in this case

$$\lim_{n\to+\infty} J_r^n = 0, \tag{1.3.9}$$

and, by dominated convergence, the last term in (1.3.7) converges to 0.
Now we use the representation described in [29] in order to prove the convergence of the first term in the right-hand side of (1.3.7). Denote by S^n the following

$$\begin{aligned} S^n &:= -\int_H \left(\langle x, Ah\rangle + \langle \nabla \log \gamma^n(x), \Pi h\rangle\right)\phi(x)\gamma^n(x)\mathbb{1}_{x\in K}\mu_c(dx) \\ &= -\mathbb{E}\left[\left(\langle Y_c, Ah\rangle + \langle \nabla \log \gamma^n(Y_c), \Pi h\rangle\right)\phi(Y_c)\gamma^n(Y_c)\mathbb{1}_{Y_c\in K}\right] \end{aligned} \tag{1.3.10}$$

We use the following Theorem whose proof is in Appendix A in [29].

Theorem 1.3.3 *There exists $\rho : \mathcal{C}([0,1]) \to \mathbb{R}$, such that for all $\Psi : \mathcal{C}([0,1]) \to \mathbb{R}$ bounded and Borel*

$$\mathbb{E}[\Psi(Y)] = \int_{\mathbb{R}} \mathbb{E}\left[\Psi(y+\mathbf{B})\rho(y+\mathbf{B})\right]dy. \tag{1.3.11}$$

Thanks to this Theorem, we can write:

$$\begin{aligned} S^n &= -\int_{\mathbb{R}} \mathbb{E}\Big[\left(\langle y+\mathbf{B}, Ah\rangle + \langle \nabla \log \gamma^n(y+\mathbf{B}), \Pi h\rangle\right) \\ &\quad \times \phi(y+\mathbf{B})\gamma^n(y+\mathbf{B})\rho(y+\mathbf{B})\mathbb{1}_{y+\mathbf{B}\in K}\Big|\overline{\mathbf{B}} = c-y\Big]dy \end{aligned}$$

Notice that the process $\mathcal{U}_r(\cdot) - \mathcal{U}_r(0)$ is 0 at time 0 and attains its minimum $-\mathcal{U}_r(0)$ only at time r. Let τ be independent of all the processes which are necessary to construct $\mathcal{U}_r$ and such that τ has the arcsine law, then $\mathcal{U}_\tau(\cdot) - \mathcal{U}_\tau(0)$ has the same law as $\mathbf{B}$ (see [30]). We can write:

$$\begin{aligned} S^n &= -\int_0^1 \frac{1}{\pi\sqrt{r(1-r)}} \int_{\mathbb{R}} \mathbb{E}\Big[\left(\langle y+\mathcal{U}_r-\mathcal{U}_r(0), Ah\rangle + \left\langle f\left(\tfrac{1}{n}+y+\mathcal{U}_r-\mathcal{U}_r(0)\right), \Pi h\right\rangle\right) \\ &\quad \times \phi(y+\mathcal{U}_r-\mathcal{U}_r(0))\gamma\left(\tfrac{1}{n}+y+\mathcal{U}_r-\mathcal{U}_r(0)\right)\rho(y+\mathcal{U}_r-\mathcal{U}_r(0)) \\ &\quad \times \mathbb{1}_{y+\mathcal{U}_r-\mathcal{U}_r(0)\in K}\Big|\overline{\mathcal{U}_r-\mathcal{U}_r(0)} = c-y\Big]dy\ dr \\ &= -\int_0^1 \frac{1}{\pi\sqrt{r(1-r)}} \int_{\mathbb{R}} \mathbb{E}\Big[\left(\langle z-\tfrac{1}{n}+\mathcal{U}_r-\mathcal{U}_r(0), Ah\rangle + \langle f\left(z+\mathcal{U}_r-\mathcal{U}_r(0)\right), \Pi h\rangle\right) \\ &\quad \times \phi(z-\tfrac{1}{n}+\mathcal{U}_r-\mathcal{U}_r(0))\gamma\left(z+\mathcal{U}_r-\mathcal{U}_r(0)\right)\rho(z-\tfrac{1}{n}+\mathcal{U}_r-\mathcal{U}_r(0)) \\ &\quad \times \mathbb{1}_{z-\frac{1}{n}+\mathcal{U}_r-\mathcal{U}_r(0)\in K}\Big|\overline{\mathcal{U}_r-\mathcal{U}_r(0)} = c-z+\tfrac{1}{n}\Big]dz\ dr. \end{aligned}$$

Now we use Proposition 1.3.3 stated in the next section 1.3.3. Thus, we can use the Fatou lemma to prove that for all $h \in D(A)$:

$$\frac{1}{\pi\sqrt{r(1-r)}}\left[\langle f(z+\mathcal{U}_r-\mathcal{U}_r(0)),\Pi h\rangle\,\|\phi\|_\infty\gamma\left(z+\mathcal{U}_r-\mathcal{U}_r(0)\right)\mathbb{1}_{z+\mathcal{U}_r-\mathcal{U}_r(0)\in K}\right]$$

is integrable on $\Omega\times\mathbb{R}\times[0,1]$. Thus, we can use the dominated convergence theorem to see:

$$\begin{aligned}
\lim_{n\to+\infty} S^n &= -\int_0^1 \tfrac{1}{\pi\sqrt{r(1-r)}}\int_{\mathbb{R}}\mathbb{E}\Big[\Big(\langle z+\mathcal{U}_r-\mathcal{U}_r(0),Ah\rangle+\Big\langle f\left(z+\mathcal{U}_r-\mathcal{U}_r(0)\right),\Pi h\Big\rangle\Big)\\
&\quad\times\phi(z+\mathcal{U}_r-\mathcal{U}_r(0))\gamma\left(z+\mathcal{U}_r-\mathcal{U}_r(0)\right)\rho(z+\mathcal{U}_r-\mathcal{U}_r(0))\\
&\quad\times\mathbb{1}_{z+\mathcal{U}_r-\mathcal{U}_r(0)\in K}\Big|\overline{\mathcal{U}_r-\mathcal{U}_r(0)}=c-z\Big]dz\;dr\\
&= -\int_{\mathbb{R}}\mathbb{E}\Big[\Big(\langle z+\mathcal{U}_\tau-\mathcal{U}_\tau(0),Ah\rangle+\langle f\left(z+\mathcal{U}_\tau-\mathcal{U}_\tau(0)\right),\Pi h\rangle\Big)\\
&\quad\times\phi(z+\mathcal{U}_\tau-\mathcal{U}_\tau(0))\gamma\left(z+\mathcal{U}_\tau-\mathcal{U}_\tau(0)\right)\rho(z+\mathcal{U}_\tau-\mathcal{U}_\tau(0))\\
&\quad\times\mathbb{1}_{z+\mathcal{U}_\tau-\mathcal{U}_\tau(0)\in K}\Big|\overline{\mathcal{U}_\tau-\mathcal{U}_\tau(0)}=c-z\Big]dz\\
&= -\mathbb{E}\left[\Big(\langle Y,Ah\rangle+\langle f(Y),\Pi h\rangle\Big)\phi(Y)\gamma\left(Y\right)\mathbb{1}_{Y\in K}\Big|\overline{Y}=c\right]\\
&= -\mathbb{E}\left[\Big(\langle Y_c,Ah\rangle+\langle f(Y),\Pi h\rangle\Big)\phi(Y_c)\gamma\left(Y_c\right)\mathbb{1}_{Y_c\in K}\right]\\
&= -\int_H\Big(\langle x,Ah\rangle+\langle f(x),\Pi h\rangle\Big)\phi(x)\nu_c(dx)
\end{aligned}\tag{1.3.12}$$

For all $r\in(0,1)$, denote Σ_r^c such that:

$$\Sigma_r^c(d\omega):=\frac{1}{\mu_c(K)}\frac{p_{\overline{\mathcal{U}}_r}(c)}{\pi\sqrt{r(1-r)}}\mathbb{P}\left(\mathcal{U}_r\in d\omega|\overline{\mathcal{U}}_r=c\right),\tag{1.3.13}$$

thus we have the following Theorem:

Theorem 1.3.4 *For all ϕ in $\mathcal{C}_b^1(H)$ and $h\in D(A)$:*

$$\begin{aligned}
\int_H\partial_{\Pi h}\phi(x)\mathbb{1}_{x\in K}\nu_c(dx) &= -\int_H\Big(\langle x,Ah\rangle+\langle f(x),\Pi h\rangle\Big)\phi(x)\nu_c(dx)\\
&\quad-\int_0^1\Pi h(r)\int\phi\gamma d\Sigma_r^c\;dr.
\end{aligned}\tag{1.3.14}$$

Moreover, for $\alpha\geq 3$, the last term vanishes.

1.3.2 Dirichlet forms

We now describe the Dirichlet Forms and the resolvent associated to $\hat{X}_c^n$, in order to obtain the Dirichlet Forms and the resolvent associated to $\hat{X}_c$. The first result is the following description of the generator of Z. Let $\psi_h: x\mapsto\exp(i(x,h)_{-1})$ for $x\in H_c$ and $h\in D(A^2)$, then the generator of Z is such that

$$L\psi_h(x):=\frac{d}{dt}\mathbb{E}[\psi_h(Z(t,x))]\Big|_{t=0}=-\frac{1}{2}\psi_h(x)\Big(i(A^2h,x)_{-1}+\|\Pi h\|_{-1}^2\Big)$$

We define for all $\phi\in\mathcal{C}_b(H_c)$ the resolvent of $\hat{X}_c^n$ on H_c:

$$R_\lambda^{n,c}\phi(x):=\int_0^\infty e^{-\lambda t}\mathbb{E}\left[\phi(\hat{X}_c^n(t,x))\right]dt,\quad x\in H_c,\lambda>0.\tag{1.3.15}$$

We define $Exp_A(H_c) \subset \mathcal{C}_b(H_c)$ as the linear span of $\{cos((h,\cdot)); sin((h,\cdot)) : h \in D(A^2)\}$. Then we define the symmetric bilinear form:

$$\mathcal{E}^{n,c}(\phi,\psi) := \frac{1}{2}\int_H \langle -A\nabla\phi, \nabla\psi\rangle d\nu_c^n, \quad \text{for all } \phi,\psi \in Exp_A(H). \tag{1.3.16}$$

It is standard (see [25]) that $(\mathcal{E}^{n,c}, Exp_A(H_c))$ is closable in $L^2(\nu_c^n)$. We denote by $(\mathcal{E}^{n,c}, D(\mathcal{E}^{n,c}))$ the closure. $(R_\lambda^{n,c})_{\lambda>0}$ is the resolvent associated with $\mathcal{E}^{n,c}$, that is, for all $\lambda > 0$ and $\psi \in L^2(\nu_c^n)$, $R_\lambda^{n,c}\psi \in D(\mathcal{E}^{n,c})$ and:

$$\lambda\int_H R_\lambda^{n,c}\psi\phi\ d\nu_c^n + \mathcal{E}^{n,c}(R_\lambda^{n,c}\psi,\phi) = \int_H \psi\phi\ d\nu_c^n, \quad \text{for all } \phi \in D(\mathcal{E}^{n,c}). \tag{1.3.17}$$

Let $\psi_h : x \mapsto \exp(i(x,h))$ for $x \in H_c$ and $h \in D(A^2)$. By Itô formula

$$L^n\psi_h(x) := \frac{d}{dt}\mathbb{E}[\psi_h(\hat{X}_c^n(t,x))]\Big|_{t=0} = L\psi_h(x) + \frac{i}{2}\langle f^n(x), \Pi h\rangle\psi_h(x). \tag{1.3.18}$$

After an easy computation, we have $(L^n, Exp_A(H_c))$ is symmetric in $L^2(\nu_c^n)$ and:

$$\int_H L^n\phi\psi\ d\nu_c^n = -\frac{1}{2}\int_H \langle -A\nabla\phi,\nabla\psi\rangle d\nu_c^n, \quad \text{for all } \phi,\psi \in Exp_A(H_c). \tag{1.3.19}$$

Moreover we define for all $\phi \in \mathcal{C}_b(H_c)$ the resolvent of $\hat{X}_c$ on $K \cap H_c$:

$$R_\lambda^c\phi(x) := \int_0^\infty e^{-\lambda t}\mathbb{E}\left[\phi\left(\hat{X}_c(t,x)\right)\right]dt, \quad x \in K\cap H_c, \lambda > 0. \tag{1.3.20}$$

We also define the symmetric bilinear form:

$$\mathcal{E}^c(\phi,\psi) := \frac{1}{2}\int_H \langle -A\nabla\phi,\nabla\psi\rangle d\nu_c, \quad \text{for all } \phi,\psi \in \mathcal{C}_b^1(H). \tag{1.3.21}$$

Proceeding as in proposition 8.1 in [29], we can prove that for all $\phi,\psi \in \mathcal{C}_b^1(H_c)$, $\mathcal{E}^{n,c}(\phi,\psi) \to \mathcal{E}^c(\phi,\psi)$ and $R_\lambda^{n,c}\phi \to R_\lambda^c\phi$ uniformly as n grows to infinity. Let $\psi \in \mathcal{C}_b(H_c)$, we can write for all $h \in D(A^2)$:

$$\begin{aligned}\int_H \psi\phi_h d\nu_c &= \lim_{n\to+\infty}\int_H \psi\phi_h d\nu_c^n = \lim_{n\to+\infty}\int_H R_\lambda^{n,c}\psi(\lambda\phi_h - L^n\phi_h)d\nu_c^n \\ &= \int_H R_\lambda^c\psi(\lambda\phi_h - L\phi_h)d\nu_c - \tfrac{i}{2}\lim_{n\to+\infty}\int_H R_\lambda^{n,c}\psi(x)\phi_h(x)\langle f^n(x),\Pi h\rangle d\nu_c^n.\end{aligned} \tag{1.3.22}$$

Then, with Proposition 1.3.3 below:

$$\begin{aligned}\int_H \psi\phi_h d\nu_c &= \int_H R_\lambda^c\psi(\lambda\phi_h - L\phi_h)d\nu_c - \tfrac{i}{2}\int_H R_\lambda^c\psi(x)\phi_h(x)\langle f(x),\Pi h\rangle d\nu_c \\ &\quad -\tfrac{i}{2}\int_0^1 \Pi h(r)\int_H R_\lambda^c\psi\phi_h\gamma\ d\Sigma_r^c\ dr \\ &= \lambda\int_H R_\lambda^c\psi\phi_h d\nu_c - \tfrac{i}{2}\int_H R_\lambda^c\psi\phi_h\langle Ah,x\rangle d\nu_c + \tfrac{1}{2}\int_H R_\lambda^c\psi\phi_h\|\Pi h\|_{-1}^2 d\nu_c \\ &\quad -\tfrac{i}{2}\int_H R_\lambda^c\psi(x)\phi_h(x)\langle f(x),\Pi h\rangle d\nu_c - \tfrac{i}{2}\int_0^1 \Pi h(r)\int_H R_\lambda^c\psi\phi_h\gamma\ d\Sigma_r^c\ dr.\end{aligned} \tag{1.3.23}$$

Thanks to the integration by parts formula applied to $R_\lambda^c\psi\phi_h$, we have:

$$\int_H \psi\phi_h d\nu_c = \lambda\int_H R_\lambda^c\psi\phi_h d\nu_c + \mathcal{E}^c(R_\lambda^c\psi,\phi_h). \tag{1.3.24}$$

By linearity and by density, we obtain for all $\lambda > 0$ and $\psi \in \mathcal{C}_b(H)$:

$$\lambda\int_H R_\lambda^c\psi\phi\ d\nu_c + \mathcal{E}^c(R_\lambda^c\psi,\phi) = \int_H \psi\phi\ d\nu_c, \quad \text{for all } \phi \in \mathcal{D}, \tag{1.3.25}$$

where we denote $\mathcal{D} := \{R_\lambda^c\phi, \phi \in \mathcal{C}_b(H_c), \lambda > 0\}$. We use classical results from [51], and obtain the following proposition:

Proposition 1.3.2 *Let $c > 0$.*
i) $(\mathcal{E}^c, Exp_A(H_c))$ is closable in $L^2(\nu_c)$: we denote by $(\mathcal{E}^c, D(\mathcal{E}^c))$ the closure.
ii) $(\mathcal{E}^c, D(\mathcal{E}^c))$ is a symmetric Dirichlet form such that $Lip(H_c) \subset D(\mathcal{E}^c)$ and $\mathcal{E}^c(\phi,\phi) \leq |\phi|^2_{Lip(H_c)}$.
iii) $(R^c_\lambda)_{\lambda>0}$ is the resolvent associated with $\mathcal{E}^c$, that is, for all $\lambda > 0$ and $\psi \in L^2(\nu_c)$, $R^c_\lambda \psi \in D(\mathcal{E}^c)$ and:

$$\lambda \int_H R^c_\lambda \psi\phi \; d\nu_c + \mathcal{E}^c(R^c_\lambda \psi, \phi) = \int_H \psi\phi \; d\nu_c, \quad \text{for all } \phi \in D(\mathcal{E}^c). \tag{1.3.26}$$

iv) $(P^c_t)_{t\geq 0}$ is the semigroup associated with $(\mathcal{E}^c, D(\mathcal{E}^c))$.

1.3.3 Total mass of the reflection measure

We now state and prove Proposition 1.3.3 used above. This proposition is the key point of the proof of Theorem 1.3.1. Indeed, understanding the explosion of f_n as $n \to +\infty$ and bounding the total mass of the reflection measure are the principal difficulties.

Proposition 1.3.3 *For all $\phi \in \mathcal{C}_b(H_c)$, for all $h \in D(A)$:*

$$\int_H \langle f^n(x), h\rangle \phi(x) \gamma^n(x) \mu_c(dx) \tag{1.3.27}$$

has a limit when n grows to infinity.
Moreover for all $0 < \delta \leq s \leq t \leq T$, $\mathbb{E}[\eta(O_{s,t})] < +\infty$.

Proof : Denote $\sigma^n_{r,c}$ the measure such that for all $r \in [0,1]$, for all $c > 0$:

$$\sigma^n_{r,c}(dx) := f^n(x(r))\, \gamma^n(x) \mu_c(dx)$$

It suffices to prove that:

$$\limsup_{n\to+\infty} \left| \int_0^1 \int_H d\sigma^n_{r,c} \; dr \right| < +\infty. \tag{1.3.28}$$

In the logarithmic case, the measure is not positive. But it is the difference of two positive measures, corresponding to $\ln(x + 1/n) > 0$ or $\ln(x + 1/n) < 0$. However, the part where $\ln(x + 1/n) > 0$ is integrable, so we only consider the part where the singularity appears. By symmetry, it suffices to prove convergence of

$$\left| \int_0^{1/2} \int_H d\sigma^n_{r,c} \; dr \right| < +\infty. \tag{1.3.29}$$

The idea is to study an integration by parts formula for the law of Y_c on the path space

$$\tilde{K} := \{h \in \mathcal{C}([0,1]), h(\theta) \geq 0 \text{ for all } \theta \in [0,1/2]\}.$$

The crucial tool is that, on this space, the processes that we consider have no more fixed mean, and we can have an integration by parts formula without the constraint of zero mean. We set

$$\chi : \theta \mapsto \mathbb{1}_{[0,1/2]}(\theta),$$

and for $u \in \mathcal{C}([0,1/2])$

$$m(u) := \int_0^{1/2} (u(\theta) + u(1/2)) d\theta.$$

The starting point is the Lemma B.1 in [29] where the following formulae have been proved.

Lemma 1.3.1 *There exists $\tilde{\rho}_c : \mathcal{C}([0,1/2]) \to \mathbb{R}$ such that for all $\Psi : \mathcal{C}([0,1/2]) \to \mathbb{R}$ bounded and Borel:*

$$\mathbb{E}[\Psi(Y_c)] = \int_{\mathbb{R}} \mathbb{E}[\Psi(y + \mathbf{B})\tilde{\rho}_c(y + \mathbf{B})]\, dy. \tag{1.3.30}$$

Moreover, for all $r \in (0,1/2)$ and for all $c > 0$, there exists a measure $\mathcal{T}^c_r$ on Ω such that for all $\Phi \in \mathcal{C}^1_b(L^2(0,1/2))$:

$$\begin{aligned} \mathbb{E}\left[\partial_\chi \Phi(Y_c) \mathbb{1}_{Y_c \in \tilde{K}}\right] &= \mathbb{E}\left[24\,(m(Y_c) - c)\,\Phi(Y_c)\mathbb{1}_{Y_c\in\tilde{K}}\right] \\ &\quad - \int_0^{1/2} \int \Phi \; \mathrm{d}\mathcal{T}^c_r \; \mathrm{d}r. \end{aligned} \tag{1.3.31}$$

We have written $\Phi(Y_c)$ for $\Phi(Y_c|_{[0,1/2]})$ with a slight abuse of notation. We set now for $n \geq 1$, $r \in (0,1/2)$:

$$\tilde{U}^n(x) := \int_0^{1/2} F^n(x(\theta))d\theta, \quad x \in L^2(0,1).$$

We define $\tilde{\gamma}^n : x \mapsto \exp(-\tilde{U}^n(x))$ for all $x \in H$. Then $\tilde{\gamma}^n \in \mathcal{C}_b^1(L^2(0,1/2))$ and for all $x, h \in \tilde{K}$:

$$\langle \nabla \tilde{\gamma}^n(x), h\rangle = \tilde{\gamma}^n(x)\langle \nabla \log \tilde{\gamma}^n(x), h\rangle = \tilde{\gamma}^n(x)\int_0^{1/2} h(\theta) f^n(x(\theta))d\theta. \tag{1.3.32}$$

Moreover we define for $n \geq 1$, $r \in (0,1/2)$ and $\Psi \in \mathcal{C}_b^1(L^2(0,1/2))$:

$$\tilde{\Sigma}_r^{n,c}(\Psi) := \mathcal{T}_r^c(\Psi \cdot \gamma_n) := \int \Psi \gamma^n \, \mathrm{d}\mathcal{T}_r^c.$$

Let ϕ be in $\mathcal{C}_b^1(L^2(0,1/2))$. We use (1.3.31), with $\Phi = \phi \cdot \gamma^n$. So we obtain:

$$\int_H \partial_\chi(\phi \cdot \tilde{\gamma}^n)\mathbb{1}_{\tilde{K}} d\mu_c = \mathbb{E}\left[24\,(m(Y_c) - c)\,\phi(Y_c)\tilde{\gamma}^n(Y_c)\mathbb{1}_{Y_c \in \tilde{K}}\right] - \int_0^{1/2} \tilde{\Sigma}_r^{n,c}(\phi)dr. \tag{1.3.33}$$

We compute the derivative of the product, and take $\phi \equiv 1$, then we obtain:

$$\mathbb{E}\left[\langle \nabla \log \tilde{\gamma}^n(x), \chi\rangle \tilde{\gamma}^n(Y_c)\mathbb{1}_{Y_c \in \tilde{K}}\right] = \mathbb{E}\left[24\,(m(Y_c) - c)\,\tilde{\gamma}^n(Y_c)\mathbb{1}_{Y_c \in \tilde{K}}\right] - \int_0^{1/2} \tilde{\Sigma}_r^{n,c}(1)dr. \tag{1.3.34}$$

Define now for $n \geq 1$, $r \in (0,1/2)$:

$$\tilde{U}(x) := \int_0^{1/2} F(x(\theta))d\theta, \quad x \in L^2(0,1).$$

We also define $\tilde{\gamma} : x \mapsto \exp(-\tilde{U}(x))$ for all $x \in H$. Moreover we define for $n \geq 1$, $r \in (0,1/2)$ and $\Psi \in \mathcal{C}_b^1(L^2(0,1/2))$:

$$\tilde{\Sigma}_r^{c}(\Psi) := \mathcal{T}_r^c(\Psi \cdot \tilde{\gamma}) := \int \Psi \tilde{\gamma} \, \mathrm{d}\mathcal{T}_r^c.$$

Finally, we denote $\tilde{\sigma}_{r,c}^n$ the measure such that for all $r \in [0,1]$, for all $c > 0$:

$$\tilde{\sigma}_{r,c}^n(dx) := f^n\left(x(r)\right)\tilde{\gamma}^n(x)\mu_c(dx)$$

We easily prove the following result:

Lemma 1.3.2 *For all $c > 0$:*

$$\lim_{n \to +\infty} \int_0^{1/2} \int_H d\tilde{\sigma}_{r,c}^n = \mathbb{E}\left[24\,(m(Y_c) - c)\,\tilde{\gamma}(Y_c)\mathbb{1}_{Y_c \in \tilde{K}}\right] - \int_0^{1/2} \tilde{\Sigma}_r^c(1)dr.$$

Moreover, for $\alpha \geq 3$ the last term vanishes.

We set now for $n \geq 1$:

$$\tilde{U'}^n(x) := \int_{1/2}^1 F^n(x(\theta))d\theta = U^n(x) - \tilde{U}^n(x), \quad x \in L^2(0,1).$$

We also define $\tilde{\gamma'}^n : x \mapsto \exp(-\tilde{U'}^n(x))$ for all $x \in H$.
We notice now that we can compute explicitly the conditional distribution of Y_c given $(Y_c(\theta), \theta \in [0,1/2])$. Indeed, we have for all $u \in \mathcal{C}([0,1/2])$ and $\Psi \in \mathcal{C}_b(L^2(0,1))$

$$\mathbb{E}\left[\Psi(Y_c)|Y_c = u \text{ on } [0,1/2]\right] = \mathbb{E}[\Psi(\tilde{B}(c,u))],$$

where

$$\tilde{B}(c,u) := \begin{cases} u(\theta), & \theta \in [0,1/2], \\ u(1/2) + \mathbf{B}_{\theta - 1/2} - 12(1/2 - \theta)(\theta - 1/2)\left(\int_0^{1/2} \mathbf{B}(r)dr + m(u) - c\right), & \theta \in]1/2, 1]. \end{cases}$$

Then we have:

$$\begin{aligned}\int_0^{1/2}\int_H \Psi d\sigma^n_{r,c}\,dr &= \tfrac{1}{Z^n_c}\int_H\int_0^{1/2}\mathbb{E}\left[\Psi\times f^n\times\tilde\gamma^{,n}\left(\tilde B(c,u)\right)\right]\tilde\gamma^n(u)\mu_c(du)\,dr,\\ &= \tfrac{1}{Z^n_c}\int_H\int_0^{1/2}\mathbb{E}\left[\Psi\times\tilde\gamma^{,n}\left(\tilde B(c,u)\right)\right]f^n(u(r))\tilde\gamma^n(u)\mu_c(du)\,dr,\\ &= \tfrac{1}{Z^n_c}\int_H\int_0^{1/2}\mathbb{E}\left[\Psi\times\tilde\gamma^{,n}\left(\tilde B(c,u)\right)\right]\tilde\sigma^n_{r,c}(du).\end{aligned}$$

Arguing as in the proof of section 1.2.2, it is easy to conclude that the limit exists, which proves (1.3.29) and (1.3.27).
Recall η is the limit of $d\eta^n := f^n(\hat X^n_c(t,\theta))dtd\theta - f(\hat X_c(t,\theta))dtd\theta$. We just proved that for all $\delta\le s\le t\le T$

$$\begin{aligned}\mathbb{E}\left[\eta(O_{s,t})\right] &\le \liminf_{n\to+\infty}\mathbb{E}\left[\eta^n(O_{s,t})\right]\\ &\le \liminf_{n\to+\infty}\mathbb{E}\left[\int_{O_{s,t}} f^n(\hat X^n_c(u,\theta))du\;d\theta\right]\\ &\le \liminf_{n\to+\infty}\int_H\int_{O_{s,t}} f^n(x(\theta))du\;d\theta\gamma^n(x)\mu_c(dx)\\ &= (t-s)\liminf_{n\to+\infty}\int_H\int_0^1 d\sigma^n_{r,c}\,dr\\ &< +\infty.\end{aligned}\tag{1.3.35}$$

Thus the total mass of $O_{s,t}$ for the reflection measure η has a finite expectation.

□

1.3.4 Reflection and Revuz measures

We now give the proof of Theorem 1.3.1.
Proof : Let $c>0$, $x\in K\cap H_c$, and $\alpha\ge 3$. We take the expectation of equation (1.0.3) for the stationary solution. We obtain for all $0<\delta\le s\le t\le T$, for all $h\in D(A^2)$:

$$\mathbb{E}\left[\int_s^t\langle\hat X_c(u),A^2h\rangle du+\int_s^t\langle Ah(\theta),f(\hat X_c(u))\rangle du+\left\langle Ah,\eta\right\rangle_{O_{s,t}}\right]=0.\tag{1.3.36}$$

Thanks to Proposition 1.3.3, the expectation of each term of (1.3.36) is finite. So let $k\in D(A)$, taking $h\in D(A^2)$ such that $k-\bar k=Ah$ as a test function in (1.3.36), we obtain for all $0<\delta\le s\le t\le T$, for all $k\in D(A)$:

$$\begin{aligned}\mathbb{E}\left[\left\langle\Pi k,\eta\right\rangle_{O_{s,t}}\right] &= -\mathbb{E}\left[\int_s^t\langle\hat X_c(u),Ak\rangle du+\int_s^t\langle\Pi k(\theta),f(\hat X_c(u))\rangle du\right]\\ &= (s-t)\;\mathbb{E}\left[\langle\hat X_c(0),Ak\rangle+\langle\Pi k(\theta),f(\hat X_c(0))\rangle\right]\\ &= (s-t)\int_H\Big(\langle x,Ak\rangle+\langle f(x),\Pi k\rangle\Big)\nu_c(dx).\end{aligned}$$

We use (1.3.14) with $\phi=1$, and obtain that for all $k\in D(A)$, for all $0<\delta\le s\le t\le T$:

$$\mathbb{E}\left[\left\langle\Pi k,\eta\right\rangle_{O_{s,t}}\right]=0.\tag{1.3.37}$$

Now, as in Lemma 1.2.1, $\eta\otimes\mathbb{P}$ can be decomposed as $\eta\otimes\mathbb{P}=\Gamma\otimes d\theta$, where Γ is a measure on $[0,T]\times\Omega$, so we obtain that for all $0<\delta\le s\le t\le T$, for all $\mathcal{A}\subset\Omega$:

$$0=\mathbb{E}\left[\langle\hat X_c,\eta\rangle_{O_{s,t}}\mathbb{1}_{\mathcal{A}}\right]=\int_\Omega\left(\int_s^t\left(\int_0^1\hat X_c(u)(\theta)d\theta\right)\mathbb{1}_{\mathcal{A}}\right)d\Gamma(u,.)=c\times\Gamma([s,t]\times\mathcal{A}).$$

Since $c>0$, we conclude that for all $0<\delta\le s\le t\le T$, for all $\mathcal{A}\subset\Omega$, $\Gamma([s,t]\times\mathcal{A})=0$. Thus $\eta\otimes\mathbb{P}$ is the null measure. Since η is a positive measure, we obtain that η is the null measure almost surely. Since the law of $(X(t,x)_{t\ge0},\eta^x)$ is a regular conditional distribution of the law of $\left(\hat X_c,\eta\right)$ given $\hat X_c(0)=x\in K\cap H_c$, we have proved i) in Theorem 1.3.1.
We consider now the logarithmic case and the negative α-power case for $\alpha<3$.

Proposition 1.3.4 *The process $\{X_c(t,x), t \geq 0, x \in H_c \cap K\}$ is a continuous Hunt process on K with infinite life-time and strong Markov, properly associated with the Dirichlet Form $\mathcal{E}^c$. In particular, $\mathcal{E}^c$ is quasi-regular.*

The last assertion is a consequence of Theorem IV.5.1 in [51], which describes the necessity of quasi regularity of a Dirichlet Form associated with a Markov process. Now we use the theory of Additive Functionals of a Markov process (see [35]). Consider $\{Y_c(t,x), t \geq 0, x \in H_c \cap K\}$ a Hunt process with infinite life-time and strong Markov, properly associated with the Dirichlet Form $\mathcal{E}^c$. If $\mathcal{A}$ is a linear combination of PCAFs in the strict sense of Y_c , the Revuz-measure of $\mathcal{A}$ is a Borel signed measure m on K such that for all $\Phi, \Psi \in \mathcal{C}_b(H_c)$:

$$\int_{H_c} \Phi(x)\ \mathbb{E}\left[\int_0^{+\infty} \exp(-t)\Psi(Y_c(t,x))d\mathcal{A}_t\right] \nu_c(dx) = \int_{H_c} \mathbb{E}\left[\int_0^{+\infty} \exp(-t)\Phi(Y_c(t,x))dt\right] \Psi(x)m(dx).$$

Notice that there exists a correspondence between Revuz-measures and PCAF. We refer to Chapter 5 in [35] and Chapter VI in [51] for all basic definitions and details. In particular the definition of a martingale additive functional (MAF in abbreviation), the notion of the energy of an AF, and the quasi-sets.
X_c does not satisfy suitable properties to compute Revuz-measures of PCAFs in the strict sense of X_c. Thus we will use a family of process $(Y_c^\delta)_{\delta>0}$ such that:

$$Y_c^\delta(t,x) = X_c(t+\delta, x), \qquad \text{for all } x \in H_c \cap K, \text{ for all } t \geq 0, \text{ for all } \delta > 0.$$

Set $\delta > 0$. Let $k \in D(A^2)$, set $h \in D(A)$ such that $Ak = h$ and set $\mathcal{V} : H_c \cap K \mapsto \mathcal{V}(x) := \langle x, k\rangle$. Since the Dirichlet form $(\mathcal{E}_c, D(\mathcal{E}_c))$ is quasi-regular, we can apply the Fukushima decomposition (see Theorem VI.2.5 in [51]). We state that there exists a MAF of finite energy $M^{[\mathcal{V}]}$ and a CAF of zero energy $N^{[\mathcal{V}]}$ such that for $\mathcal{E}^c$-quasi every x:

$$\mathcal{V}(Y_c^\delta(0,x)) - \mathcal{V}(Y_c^\delta(t,x)) = M_t^{[\mathcal{V}]} + N_t^{[\mathcal{V}]}, \quad t \geq 0,\ \mathbb{P}_x^\delta - a.s, \tag{1.3.38}$$

with obvious notations for $\mathbb{P}_x^\delta$. $M^{[\mathcal{V}]}$ and $N^{[\mathcal{V}]}$ can be extended to CAF and MAF in the strict sense of X_c, which we still denote $M^{[\mathcal{V}]}$ and $N^{[\mathcal{V}]}$, such that $M^{[\mathcal{V}]}$ is a $\mathbb{P}_x$-martingale and (1.3.38) holds for all $x \in H_c \cap K$. We have the following expression:

$$\int_\delta^t \langle Bk, dW\rangle = M_t^{[\mathcal{V}]}(Y_c^\delta(\cdot, x)), \qquad \text{for all } t \geq 0, x \in K, \text{ almost surely.}$$

Moreover $N^{[\mathcal{V}]}$ is a linear combination of PCAFs in the strict sense of Y_c^δ such that for all $t \geq 0$, $x \in K$, almost surely:

$$\frac{1}{2}\int_\delta^t \left(\langle X_c(s,x), Ah\rangle + \langle f(X_c(s,x)), \Pi h\rangle\right) ds + \frac{1}{2}\int_0^1 \Pi h(\theta)\eta^x([\delta,t], d\theta) = N_t^{[\mathcal{V}]}(Y_c^\delta(\cdot,x))$$

and its Revuz measure is:

$$\frac{1}{2}\left(\langle z, Ah\rangle + \langle f(z), \Pi h\rangle\right)\nu_c(dz) + \frac{1}{2}\int_0^1 \Pi h(r)dr\ \gamma\ d\Sigma_r^c. \tag{1.3.39}$$

To prove the last assertion, it suffices to remark that for all $\Phi, \Psi \in \mathcal{C}_b(H_c)$:

$$\begin{aligned} & 2\int_{H_c} \Phi(x)\ \mathbb{E}\left[\int_0^{+\infty} \exp(-t)\Psi(Y_c^\delta(t,x))dN_t^{[\mathcal{V}]}\right]\nu_c(dx) \\ = & \int_{H_c} \Psi(x)\ \mathbb{E}\left[\int_0^{+\infty} \exp(-t)\Phi(Y_c^\delta(t))dt\right]\left(\langle x, Ah\rangle + \langle f(x), \Pi h\rangle + \int_0^1 \Pi h(r)dr\ \gamma\ d\Sigma_r^c\right)\nu_c(dx). \end{aligned}$$

Using the same arguments, we remark that there exists a CAF in the strict sense of Y_c^δ whose Revuz-measure is

$$\frac{1}{2}\left(\langle z, Ah\rangle + \langle f(z), \Pi h\rangle\right)\nu_c(dz). \tag{1.3.40}$$

Since X_c is a solution of the equation (1.0.3) in the sense of Definition 1.2.1, we obtain Proposition 1.3.1. Finally, we have the following equality:

$$\int_H \mathbb{E}\left[\int_0^{+\infty} \exp(-t) \int_0^1 \Pi h(\theta)\eta^x(\delta + dt, d\theta)\right] \nu_c(dx) = \int_0^1 \int_H \Pi h(r)\gamma d\Sigma_r^c \, dr, \tag{1.3.41}$$

and the reflection measure η^x cannot be identically equal to zero.

□

Chapter 2

L'équation de Cahn-Hilliard stochastique avec deux réflexions

Résumé

On considère une équation aux dérivées partielles stochastique possédant deux non-linéarités de type logarithmique, avec deux réflexions en 1 et -1 sous la contrainte de conservation de masse. L'équation, dirigée par un bruit blanc en espace et en temps, contient un double Laplacien. L'absence de principe de maximum pour le double Laplacien pose des difficultés pour l'utilisation d'une méthode classique de pénalisation, pour laquelle une importante propriété de monotonie est utilisée. Etant inspiré par les travaux de Debussche, Goudenège et Zambotti, on démontre l'existence et l'unicité de solutions pour des données initiales entre -1 et 1. Enfin, on démontre que l'unique mesure invariante est ergodique, et on énonce un résultat de mélange exponentiel.

Ce travail a été réalisé en collaboration étroite avec Arnaud Debussche.

Si le hasard fait si bien les choses,
ce n'est peut-être pas par hasard.

Introduction and main results

The Cahn-Hilliard-Cook equation is a model to describe phase separation in a binary alloy (see [13], [15] and [16]) in the presence of thermal fluctuations (see [19] and [49]). It takes the form:

$$\begin{cases} \partial_t u = -\frac{1}{2}\Delta\left(\Delta u - \psi(u)\right) + \dot{\xi}, & \text{on } \Omega \subset \mathbb{R}^n, \\ \nabla u \cdot \nu = 0 = \nabla(\Delta u - \psi(u)) \cdot \nu, & \text{on } \partial\Omega, \end{cases} \tag{2.0.1}$$

where t denotes the time variable and Δ is the Laplace operator. Also $u \in [-1,1]$ represents the ratio between the two species and the noise term $\dot{\xi}$ accounts for the thermal fluctuations. The nonlinear term ψ has the double-logarithmic form:

$$\psi : u \mapsto \frac{\theta}{2}\ln\left(\frac{1+u}{1-u}\right) - \theta_c u, \tag{2.0.2}$$

where θ and θ_c are temperatures with $\theta < \theta_c$.

The study of this equation presents several difficulties. First, the singularities at ± 1 have to be treated carefully. Also, since it is a fourth order equation, no comparison principle holds.

The deterministic equation where ψ is replaced by a polynomial function have first been studied (see [15], [49] and [60]). Then non smooth ψ have been considered (see [11] and [27]).

Phase separation have been analysed thanks to this model: see for example the survey [58], and the references therein, or others recent results on spinodal decomposition and nucleation in [5, 10, 40, 53, 54, 66, 67, 72].

In the case of a polynomial nonlinearity, some results have been obtained in the stochastic case (see [8, 9, 17, 18, 21, 31]).

Note that the solutions of the equation with polynomial nonlinearity do not remain in $[-1,1]$ in general, and their physical interpretation is not clear.

To our knowledge, the case of the logarithmic nonlinearity in the presence of noise have never been studied. The presence of noise has a strong effect and equation (2.0.1) cannot have a solution. Indeed, a solution should remain in $[-1,1]$ which is impossible with an additive noise. Two reflection measures have to be added to the model to remedy this problem. In this article, we propose to study:

$$\begin{cases} \partial_t u = -\frac{1}{2}\Delta\Big(\Delta u - \psi(u) + \eta_- - \eta_+\Big) + \dot{\xi}, & \text{with } \theta \in [0,1] = \Omega, \\ \nabla u \cdot \nu = 0 = \nabla(\Delta u) \cdot \nu, & \text{on } \partial\Omega, \end{cases} \tag{2.0.3}$$

where the measures are subject to the contact conditions almost surely:

$$\int (1+u)\mathrm{d}\eta_- = \int (1-u)\mathrm{d}\eta_+ = 0. \tag{2.0.4}$$

The stochastic heat equation with reflection, *i.e.* when the fourth order operator is replaced by the Laplace operator, is a model for the evolution of random interfaces near a hard wall. It has been extensively studied in the literature (see [26], [37], [38], [57], [61], [73], [74] and [75]). Essential tools in these articles are the comparison principle and the fact that the underlying Dirichlet form is symmetric so that the invariant measure is known explicitly.

In our case, we consider a noise which is obtained as the space derivative of the space-time white noise. In other words, the noise is the time derivative of a cylindrical Wiener process in $H^{-1}(0,1)$. This is physically reasonable since the Cahn-Hilliard equation can be interpreted as a gradient system in this space. With such noise, the system is still symmetric and the invariant measure is known explicitly. As in the second order case, we use this fact in an essential way.

However, as already mentioned, no comparison principle holds and new techniques have to be developed. The equation (2.0.3) has been studied with a single reflection and when no nonlinear term is taken into account in [29]. The reflection is introduced to enforce positivity of the solution. Various techniques have been introduced to overcome this lack of comparison principle. Moreover, as in the second order case, an integration by part formula for the invariant measure has been derived. Then, in [39], a singular nonlinearity of the form $u^{-\alpha}$ or $\ln u$ have been considered.

Existence and uniqueness of solutions have been obtained and using the integration by parts formula as in [74], it has been proved that the reflection measure vanishes if and only if $\alpha \geq 3$. In particular, for a logarithmic nonlinearity, the reflection is active.

Here, we consider the original Cahn-Hilliard-Cook model (2.0.1) with the double-logarithmic nonlinear term (2.0.2). The noise is as in the above mentioned articles and we still have an explicit invariant measure. Our method mixes ideas from [29], [39] and [74]. Additional difficulties are overcome, the main one being to understand how to deal with the nonlinear term. Indeed, in [39], the positivity of the nonlinear term was essential. We overcome this difficulty thanks to a delicate a priori estimate. Our main results state that equations (2.0.3), (2.0.4) together with an initial condition have a unique solution (see Proposition 2.2.3 and Theorem 2.1.1). As in [29], it is constructed thanks to the gradient structure of (2.0.3) and strong Feller property. Moreover, we prove that this solution is the limit of the solution of the Cahn-Hilliard-Cook equation with polynomial nonlinearity without reflections. This justifies the use of the polynomial models. We also prove that the invariant measure is unique and ergodic. Such property is very easy to obtain if θ_c is small (see [29]) or in the polynomial case (see [21]). Finally, a stronger result of exponential mixing is given in the last Theorem 2.3.1. It is based on coupling and arguments developped by Odasso in [62].

In future studies, we shall generalize the integration by part formula obtain in [29] to prove that the reflection measure does not vanish. The presence of two reflection measures introduces additional difficulties. In the second order case, this has been studied in [36]. For other results about the reflection measure and related integration by parts formulae: see [43] or [63].

2.1 Preliminaries

We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(0,1)$; A is the realization in $L^2(0,1)$ of the Laplace operator with Neumann boundary condition, i.e.:

$$D(A) = \text{ Domain of } A = \{h \in W^{2,2}(0,1) : h'(0) = h'(1) = 0\}$$

where we use $W^{n,p}$ and $||.||_{W^{n,p}}$ to denote the Sobolev space $W^{n,p}(0,1)$ and its associated norm. Remark that A is self-adjoint on $L^2(0,1)$ and we have a complete orthonormal system of eigenvectors $(e_i)_{i\in\mathbb{N}}$ in $L^2(0,1)$ for the eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$. We denote by $\bar{h}$ the mean of $h \in L^2(0,1)$:

$$\bar{h} = \int_0^1 h(\theta) \mathrm{d}\theta.$$

We remark that A is invertible on the space of functions with 0 average. In general, we define $(-A)^{-1}h = (-A)^{-1}(h - \bar{h}) + \bar{h}$.

For $\gamma \in \mathbb{R}$, we define $(-A)^\gamma$ by classical interpolation. We set $V_\gamma := D((-A)^{\gamma/2})$. It is endowed with the classical seminorm and norm :

$$|h|_\gamma = \left(\sum_{i=1}^{+\infty} (-\lambda_i)^\gamma h_i^2 \right)^{1/2}, \ \|h\|_\gamma = \left(|h|_\gamma^2 + \bar{h}^2 \right)^{1/2}, \text{ for } h = \sum_{i\in\mathbb{N}} h_i e_i.$$

$|\cdot|_\gamma$ is associated to the scalar product $(\cdot,\cdot)_\gamma$. To lighten notations, we set $(\cdot,\cdot) := (\cdot,\cdot)_{-1}$ and $H := V_{-1}$. The average can be defined in any V_γ by $\bar{h} = (h, e_0)$. It plays an important role and we often work with functions with a fixed average $c \in \mathbb{R}$. We define $H_c = \{h \in H : \bar{h} = c\}$ for all $c \in \mathbb{R}$.

We use the following regularization operators:

$$Q_N x = \frac{1}{N} \sum_{n=0}^{N} \sum_{i=0}^{n} (x, e_i) e_i.$$

It is defined on $L^2(0,1)$ and can extended to any V_γ. Clearly $Q_N x$ converges to x in V_γ if $x \in V_\gamma$. Moreover, it is well known that if $x \in C([0,1];\mathbb{R})$, then the converges holds in $C([0,1];\mathbb{R})$. Note also that Q_N is self-adjoint in V_γ and commutes with A.

The covariance operator of the noise is the operator B defined by

$$B = \frac{\partial}{\partial\theta}, D(B) = W_0^{1,2}(0,1).$$

Note that

$$B^* = -\frac{\partial}{\partial\theta}, D(B^*) = W^{1,2}(0,1), BB^* = -A.$$

We denote by $\mathcal{B}_b(H_c)$ the space of all Borel bounded functions on H_c. We set $O_{s,t} := [s,t] \times [0,1]$ for $s,t \in [0,T]$ with $s < t$ and $T > 0$, and $O_t = O_{0,t}$ for $0 \leq t \leq T$. Given a measure ζ on $O_{s,t}$ and a continuous function v on $O_{s,t}$, we write

$$\langle v, \zeta \rangle_{O_{s,t}} := \int_{O_{s,t}} v \, \mathrm{d}\zeta.$$

For $\lambda \in \mathbb{R}$, we define:

$$f(x) := \begin{cases} +\infty, & \text{for all } x \leq -1, \\ \ln\left(\frac{1-x}{1+x}\right) + \lambda x, & \text{for all } x \in (-1,1), \\ -\infty, & \text{for all } x \geq 1, \end{cases} \tag{2.1.1}$$

and the following antiderivative F of $-f$:

$$F(x) = (1+x)\ln(1+x) + (1-x)\ln(1-x) - \frac{\lambda}{2}x^2, \text{ for all } x \in (-1,1).$$

With these notations, we rewrite (2.0.3) in the abstract form:

$$\begin{cases} \mathrm{d}X = -\frac{1}{2}A\left(AX + f(X) + \eta_- - \eta_+\right)\mathrm{d}t + B\mathrm{d}W, \\ \langle (1+X), \eta_- \rangle_{O_T} = \langle (1-X), \eta_+ \rangle_{O_T} = 0, \\ X(0,x) = x \text{ for } x \in V_{-1}, \end{cases} \tag{2.1.2}$$

where W is a cylindrical Wiener process on $L^2(0,1)$.

Definition 2.1.1 *Let $x \in \mathcal{C}([0,1];[-1,1])$. We say that $\left((X(t,x))_{t\in[0,T]}, \eta_+, \eta_-, W\right)$, defined on a filtered complete probability space $\left(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}\right)$, is a weak solution to (2.0.3) on $[0,T]$ for the initial condition x if:*

(a) a.s. $X \in \mathcal{C}\left((0,T]\times[0,1];[-1,1]\right) \cap \mathcal{C}([0,T];H)$ and $X(0,x) = x$,

(b) a.s. $\eta_\pm$ are two positive measures on $(0,T]\times[0,1]$, such that $\eta_\pm(O_{\delta,T}) < +\infty$ for all $\delta \in (0,T]$,

(c) W is a cylindrical Wiener process on $L^2(0,1)$,

(d) the process $(X(\cdot,x), W)$ is $(\mathcal{F}_t)$-adapted,

(e) a.s. $f(X(\cdot,x)) \in L^1(O_T)$,

(f) for all $h \in D(A^2)$ and for all $0 < \delta \leq t \leq T$:

$$\begin{aligned} \langle X(t,x), h\rangle \;=\; & \langle X(\delta,x), h\rangle - \tfrac{1}{2}\int_\delta^t \langle X(s,x), A^2h\rangle \mathrm{d}s - \tfrac{1}{2}\int_\delta^t \langle Ah, f(X(s,x))\rangle \mathrm{d}s \\ & -\tfrac{1}{2}\langle Ah, \eta_+\rangle_{O_{\delta,t}} + \tfrac{1}{2}\langle Ah, \eta_-\rangle_{O_{\delta,t}} - \int_\delta^t \langle Bh, \mathrm{d}W\rangle, \quad a.s., \end{aligned}$$

(g) a.s. the contact properties hold :
$supp(\eta_-) \subset \{(t,\theta) \in O_T / X(t,x)(\theta) = -1\}$ and $supp(\eta_+) \subset \{(t,\theta) \in O_T / X(t,x)(\theta) = 1\}$, that is,

$$\langle (1+X), \eta_- \rangle_{O_T} = \langle (1-X), \eta_+ \rangle_{O_T} = 0.$$

Finally, a weak solution (X, η_+, η_-, W) is a strong solution if the process $t \mapsto X(t,x)$ is adapted to the filtration $t \mapsto \sigma(W(s,.), s \in [0,t])$

Remark 2.1.1 *In (f), the only term where we use the function f is well defined. Indeed, by (e) we have $f(X(\cdot,x)) \in L^1(O_T)$ and by Sobolev embedding $Ah \in D(A) \subset L^\infty(O_T)$. Hence the notation $\langle \cdot, \cdot \rangle$ should be interpreted as a duality between L^∞ and L^1.*

The solution of the linear equation with initial data $x \in H$ is given by

$$Z(t,x) = e^{-tA^2/2}x + \int_0^t e^{-(t-s)A^2/2} B \mathrm{d}W_s.$$

As easily seen this process is in $\mathcal{C}([0,+\infty[;H)$ (see [23]). In particular, the mean of Z is constant and the law of the process $Z(t,x)$ is the Gaussian measure:

$$Z(t,x) \sim \mathcal{N}\big(e^{-tA^2/2}x, Q_t\big),\ Q_t = \int_0^t e^{-sA^2/2} BB^* e^{-sA^2/2} \mathrm{ds} = (-A)^{-1}(I - e^{-tA^2}).$$

If we let $t \to +\infty$, the law of $Z(t,x)$ converges to the Gaussian measure on L^2:

$$\mu_c := \mathcal{N}(ce_0, (-A)^{-1}),\ \text{where } c = \bar{x}.$$

Notice that μ_c is concentrated on $H_c \cap C([0,T])$.

In order to solve equation (2.1.2), we use polynomial approximations of this equation. We denote by $\{f^n\}_{n\in\mathbb{N}}$ the sequence of polynomial functions which converges to the function f on $(-1,1)$, defined for $n \in \mathbb{N}$ by:

$$f^n(x) = -2\sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)} + \lambda x,\ \text{for all } x \in \mathbb{R}.$$

We use the following antiderivative F^n of $-f^n$ defined by:

$$F^n(x) = 2\sum_{k=0}^{n} \frac{x^{2k+2}}{(2k+2)(2k+1)} - \frac{\lambda}{2}x^2,\ \text{for all } x \in \mathbb{R}.$$

Then for $n \in \mathbb{N}$, we study for the following polynomial approximation of (2.1.2) with an initial condition $x \in H$:

$$\begin{cases} \mathrm{d}X^n + \frac{1}{2}(A^2X^n + Af^n(X^n))\mathrm{dt} = B\mathrm{d}W, \\ X^n(0,x) = x. \end{cases} \tag{2.1.3}$$

This equation has been studied in [21] in the case $B = I$. The results generalize immediately and it can be proved that for any $x \in H$, there exists a unique solution $X^n(\cdot,x)$ *a.s.* in $\mathcal{C}([0,T];H) \cap L^{2n+2}((0,T)\times(0,1))$. It is a solution in the mild or weak sense. Moreover the average of $X^n(t,x)$ does not depend on t.

For each $c \in \mathbb{R}$, (2.1.2) defines a transition semigroup $(P_t^{n,c})_{t\geq 0}$:

$$P_t^{n,c}\phi(x) = \mathbb{E}[\phi(X^n(t,x)],\ t \geq 0, x \in H_c,\ \phi \in \mathcal{B}_b(H_c),\ n \in \mathbb{N}^*.$$

Existence of an invariant measure can be proved as in [21].

Using Galerkin approximation and Bismut-Elworthy-Li formula, it can be seen that $(P_t^{n,c})_{t\geq 0}$ is Strong Feller. More precisely, for all $\phi \in \mathcal{B}_b(H_c)$, $n \in \mathbb{N}$ and $t > 0$:

$$|P_t^{n,c}\phi(x) - P_t^{n,c}\phi(y)| \leq \frac{2e^{\lambda^2 t/4}}{\lambda\sqrt{t}}\|\phi\|_\infty |x-y|_{-1}, \quad \text{for all } x, y \in H_c. \tag{2.1.4}$$

Irreducibility follows from a control argument. By Doob Theorem we deduce that there exists an unique and ergodic invariant measure ν_c^n.

It is classical that equation (2.1.3) is a gradient system in H_c and can be rewritten as:

$$\begin{cases} \mathrm{d}X^n + \frac{1}{2}A(AX^n - \nabla U^n(X^n))\mathrm{d}t = B\mathrm{d}W, \\ X^n(0,x) = x \in L^2(0,1), \end{cases} \tag{2.1.5}$$

where ∇ denotes the gradient in the Hilbert space $L^2(0,1)$, and:

$$U^n(x) := \int_0^1 F^n(x(\theta))\mathrm{d}\theta, \quad x \in L^2(0,1).$$

The measure ν_c^n is therefore given by:

$$\nu_c^n(\mathrm{d}x) = \frac{1}{Z_c^n}\exp(-U^n(x))\mu_c(\mathrm{d}x),$$

where Z_c^n is a normalization constant.

We prove in section 2.2 that, for $c \in (-1,1)$, the sequence $(\nu_c^n)_{n\in\mathbb{N}}$ converges to the measure

$$\nu_c(\mathrm{d}x) = \frac{1}{Z_c}\exp(-U(x))\mathbb{1}_{x\in K}\mu_c(\mathrm{d}x),$$

where

$$U(x) := \int_0^1 F(x(\theta))\mathrm{d}\theta, \quad x \in L^2(0,1).$$

and

$$K = \{x \in L^2 : \ 1 \geq x \geq -1\}.$$

In section 2.2, we prove the following result.

Theorem 2.1.1 *Let $c \in (-1,1)$. Let $x \in K$ such that $\bar{x} = c$, then there exist a continuous process denoted $(X(t,x))_{t\geq 0}$ and two nonnegative measures η_+^x and η_-^x such that $\left((X(t,x))_{t\geq 0}, \eta_+^x, \eta_-^x, W\right)$ is the unique strong solution of (2.0.3) with $X(0,x) = x$ a.s.*

The Markov process $(X(t,x), t \geq 0, x \in K \cap H_c)$ is continous and has P^c for transition semigroup which is strong Feller on H_c.

For all $x \in K \cap H_c$ and $0 = t_0 < t_1 < \cdots < t_m$, $(X(t_i,x), i = 1,\ldots,n)$ is the limit in distribution of $(X^n(t_i,x))_{i=1,\ldots,m}$.

Finally ν_c is an invariant measure for P^c.

In all the article, C denotes a constant which may depend on T and its value may change from one line to another.

2.2 Proof of Theorem 2.1.1

2.2.1 Pathwise uniqueness

We first prove that for any pair $(X^i, \eta_+^i, \eta_-^i, W), i = 1,2$, of weak solutions of (2.0.3) defined on the same probability space with the same driving noise W and with $X^1(0) = X^2(0)$, we have $\left(X^1, \eta_+^1, \eta_-^1\right) = \left(X^2, \eta_+^2, \eta_-^2\right)$. This pathwise uniqueness will be used in the next subsection to construct stationary strong solutions of (2.0.3).

Proposition 2.2.1 *Let $x \in \mathcal{C}([0,1];[-1,1])$. Let $\left(X^i, \eta_+^i, \eta_-^i, W\right), i = 1,2$ be two weak solutions of (2.0.3) with $X^1(0) = X^2(0) = x$. Then $\left(X^1, \eta_+^1, \eta_-^1\right) = \left(X^2, \eta_+^2, \eta_-^2\right)$.*

Proof : We use the following Lemma from [39].

Lemma 2.2.1 *Let ζ be a finite measure on $O_{\delta,T}$ and $V \in \mathcal{C}(O_{\delta,T})$. Suppose that there exists a positive continuous function $c_T : [0,T] \to \mathbb{R}^+$ such that :*

i) for all $r \in [\delta, T]$, *for all* $h \in \mathcal{C}([0,1])$, *such that* $\bar{h} = 0$, $\langle h, \zeta \rangle_{O_{r,T}} = 0$,

ii) for all $r \in [\delta, T]$, $\overline{V(r,\cdot)} = c_T(r)$ *with* $\langle V, \zeta \rangle_{O_{r,T}} = 0$,

then ζ *is the null measure.*

Let $Y(t) = X^1(t,x) - X^2(t,x)$, $\zeta_+ = \eta^1_+ - \eta^2_+$ and $\zeta_- = \eta^1_- - \eta^2_-$, Y is the solution of the following equation:

$$\begin{cases} \mathrm{d}Y = -\frac{1}{2}A\left(AY + \left(f(X^1) - f(X^2)\right) + \zeta_- - \zeta_+\right)\mathrm{dt}, \\ Y(0) = 0. \end{cases} \tag{2.2.1}$$

Taking the scalar product in H with $Y^N = Q_N Y$ and integrating in time, we obtain since Y has zero average:

$$|Y^N(t)|^2_{-1} - |Y^N(\delta)|^2_{-1} = -\int_\delta^t \left(|Y^N(s)|^2_1 - \langle f(X^1) - f(X^2), Y^N\rangle\right)\mathrm{ds} + \langle \zeta_- - \zeta_+, Y^N\rangle_{O_{\delta,t}}. \tag{2.2.2}$$

For all $s \in [\delta, t]$,

$$\begin{aligned} &\langle Y^N(s) - Y(s), f(X^1(s,x)) - f(X^2(s,x))\rangle \\ &\leq \|Y^N(s) - Y(s)\|_{L^\infty([0,1])}\|f(X^1(s,x)) - f(X^2(s,x))\|_{L^1([0,1])}, \end{aligned}$$

where $\|\cdot\|_{L^\infty([0,1])}$ and $\|\cdot\|_{L^1([0,1])}$ are the classical norms on the space $[0,1]$. The latter term converges to zero since $Y^N(s)$ converges uniformly to $Y(s)$ on $[0,1]$. Since $f(x) - \lambda x$ is nonincreasing,

$$\begin{aligned} \left(\langle Y(s), f(X^1(s,x)) - f(X^2(s,x))\rangle\right) &= \left(\langle Y(s), f(X^1(s,x)) - f(X^2(s,x)) - \lambda Y(s)\rangle\right) \\ &\quad + \left(\langle Y(s), \lambda Y(s)\rangle\right) \\ &\leq \lambda |Y(s)|^2_0. \end{aligned}$$

Taking the limit in (2.2.2) as N grows to infinity, we obtain:

$$|Y(t)|^2_{-1} - |Y(\delta)|^2_{-1} \leq \left\langle Y, \zeta_- - \zeta_+\right\rangle_{O_{\delta,t}} + \lambda \int_\delta^t |Y(s)|^2_0 \,\mathrm{ds}.$$

We now write

$$\begin{aligned} &\left\langle Y, \zeta_- - \zeta_+\right\rangle_{O_{\delta,t}} \\ &= \left\langle 1 + X^1, \eta^1_-\right\rangle_{O_{\delta,t}} - \left\langle 1 + X^2, \eta^1_-\right\rangle_{O_{\delta,t}} - \left\langle 1 + X^1, \eta^2_-\right\rangle_{O_{\delta,t}} + \left\langle 1 + X^2, \eta^2_-\right\rangle_{O_{\delta,t}} \\ &+ \left\langle 1 - X^1, \eta^1_+\right\rangle_{O_{\delta,t}} - \left\langle 1 - X^2, \eta^1_+\right\rangle_{O_{\delta,t}} - \left\langle 1 - X^1, \eta^2_+\right\rangle_{O_{\delta,t}} + \left\langle 1 - X^2, \eta^2_+\right\rangle_{O_{\delta,t}} \\ &\leq 0 \end{aligned}$$

by the contact condition and the positivity of the measures. It follows:

$$|Y(t)|^2_{-1} - |Y(\delta)|^2_{-1} \leq \lambda \int_\delta^t |Y(s)|^2_0 \,\mathrm{ds}.$$

By Gronwall Lemma, and letting $\delta \to 0$, we have $|Y(t)|_{-1} = 0$ for all $t \geq 0$. Since $\bar{Y}(t) = 0$, we deduce $X^1(t,x) = X^2(t,x)$ for all $t \geq 0$. Moreover, with the definition of a weak solution, we see that :

$$\text{for all } h \in D(A^2), \qquad \left\langle Ah, \zeta_+ - \zeta_-\right\rangle_{O_{\delta,t}} = 0.$$

By density, we obtain that $\zeta := \zeta_- - \zeta_+$ and $V := (1 - X^1)(1 + X^1) = (1 - X^2)(1 + X^2)$ satisfy the hypothesis of Lemma 2.2.1, and therefore $\zeta = \zeta_- - \zeta_+$ is the null measure. And since ζ_- and ζ_+ have disjoint supports, then ζ_- and ζ_+ are the null measure, i.e. $\eta^1_- = \eta^2_-$ and $\eta^1_+ = \eta^2_+$.

□

2.2.2 Convergence of invariants measures

We know (see [29]) that μ_c is the law of $Y^c = \mathbf{B} - \overline{\mathbf{B}} + c$, where B is brownian motion. Then for $0 \leq c < 1$, we remark the following inclusion :

$$\{\mathbf{B}_\theta \in \left[\frac{c-1}{2}, \frac{1-c}{2}\right], \text{ for all } \theta \in [0,1]\} \subset \{Y^c \in K\},$$

and we have a similar result for $-1 < c \leq 0$. Therefore $\mu_c(K) > 0$ with $-1 < c < 1$. Let us define U the potential associated to the function f :

$$U(x) = \begin{cases} \int_0^1 F(x(\theta))\mathrm{d}\theta & \text{if } x \in K, \\ +\infty & \text{else.} \end{cases}$$

We have the following result :

Proposition 2.2.2 *For $-1 < c < 1$,*

$$\nu_c^n \rightharpoonup \nu_c := \frac{1}{Z_c} \exp^{-U(x)} \mathbb{1}_{x \in K} \mu_c(\mathrm{dx}), \text{ when } n \to +\infty,$$

where Z_c is a normalization constant.

Proof : Let $\psi \in \mathcal{C}_b(L^2, \mathbb{R})$. We want to prove that

$$\int_H \psi(x) \exp(-U^n(x)) \mu_c(\mathrm{dx}) \underset{n \to +\infty}{\longrightarrow} \int_H \psi(x) \exp(-U(x)) \mathbb{1}_{x \in K} \mu_c(\mathrm{dx}). \tag{2.2.3}$$

We first prove,

$$\exp(-U^n(x)) \underset{n \to +\infty}{\longrightarrow} \exp(-U(x)) \mathbb{1}_{x \in K}, \ \mu_c \ a.s. \tag{2.2.4}$$

Since $\mu_c(C([0,1])) = 1$, we can restrict our attention to $x \in C([0,1])$. Then if $x \notin K$ there exists $\delta_x > 0$ such that $m(\{\theta \in [0,1] : x(\theta) \leq -1 - \delta_x\}) > 0$ or $m(\{\theta \in [0,1] : x(\theta) \geq 1 + \delta_x\}) > 0$, m being the Lebesgue measure. Suppose $m(\{\theta \in [0,1]/x(\theta) \leq -1 - \delta_x\}) > 0$, then we have since $\tilde{F}^n(x) = F^n(x) + \frac{\lambda}{2}x^2$ is positive and non increasing on $(-\infty, -1)$

$$\begin{aligned} 0 \leq \exp(-U^n(x)) &\leq \exp\left(-\int_0^1 \tilde{F}^n(x(\theta)) 1_{\{x \leq -1-\delta_x\}} - \frac{\lambda}{2} x(\theta)^2 \mathrm{d}\theta\right) \\ &\leq \exp\left(-\int_0^1 \tilde{F}^n(-1-\delta_x) 1_{\{x \leq -1-\delta_x\}} - \frac{\lambda}{2} x(\theta)^2 \mathrm{d}\theta\right) \\ &\leq \exp\left(-\tilde{F}^n(-1-\delta_x) m(\{x \leq -1-\delta_x\}) + \int_0^1 \frac{\lambda}{2} x(\theta)^2 \mathrm{d}\theta\right). \end{aligned}$$

And this latter term converges to zero as n grows to infinity.
Now for $x \in K$, $F^n(x(\theta))$ converges to $F(x(\theta))$ almost everywhere as n grows to infinity. Moreover $-\frac{\lambda}{2}x(\theta)^2 \leq F^n(x(\theta)) \leq \ln 2$, and by the dominated convergence Theorem, we deduce (2.2.4). Finally, (2.2.3) follows again by dominated convergence Theorem.

2.2.3 Existence of stationary solutions

In this section, we prove the existence of stationary solutions of equation (2.1.2) and that they are limits of stationary solutions of (2.1.3), in some suitable sense. Fix $-1 < c < 1$ and consider the unique (in law) stationary solution of (2.1.3) denote $\hat{X}_c^n$ in H_c. We are going to prove that the laws of $\hat{X}_c^n$ weakly converge as n grows to infinity to a stationary strong solution of (2.0.3).

Proposition 2.2.3 *Let $-1 < c < 1$ and $T > 0$. $\hat{X}_c^n$ converges in probability as n grows to infinity to a process $\hat{X}_c$ in $\mathcal{C}(O_T)$, $f(\hat{X}_c) \in L^1(O_T)$ almost surely, and it exists two measures (η_+, η_-) on O_T such that $(\hat{X}_c, \eta_+, \eta_-, W)$ is a stationary strong solution of (2.0.3). Moreover, setting*

$$\mathrm{d}\eta_+^n = -f^n(\hat{X}_c^n(t,\theta))\mathbb{1}_{\hat{X}_c^n(t,\theta)>0}\mathrm{dtd}\theta + f(\hat{X}_c(t,\theta))\mathbb{1}_{0<\hat{X}_c(t,\theta)\leq 1}\mathrm{dtd}\theta,$$

and

$$\mathrm{d}\eta_-^n = f^n(\hat{X}_c^n(t,\theta))\mathbb{1}_{\hat{X}_c^n(t,\theta)\leq 0}\mathrm{dtd}\theta - f(\hat{X}_c(t,\theta))\mathbb{1}_{-1\leq\hat{X}_c(t,\theta)\leq 0}\mathrm{dtd}\theta,$$

then $(\hat{X}_c^n, \eta_+^n, \eta_-^n, W)$ converges in law to $(\hat{X}_c, \eta_+, \eta_-, W)$.

Proof : Proceeding exactly as in [29] (see Lemma 5.2), we prove that the laws of $(\hat{X}_c^n, W^n)_{n\in\mathbb{N}}$ are tight in $C(O_T)\times C([0,T];V_\gamma)$, $\gamma < -1/2$. We have set $W^n = W$, $n \in \mathbb{N}$. We therefore can extract convergent subsequences. Let $(\hat{X}_c^{n_k}, W^{n_k})_{k\in\mathbb{N}}$ be such a subsequence. Using Skohorod theorem, one may find a probability space and a sequence of random variables $(\tilde{X}_c^k, \mathcal{W}^k)_{k\in\mathbb{N}}$ on this probability space with the same laws as $(\hat{X}_c^{n_k}, W^{n_k})_{k\in\mathbb{N}}$ which converge almost surely.

Below, we show in Step 1 that its limit $\tilde{X}_c$ satisfies $f(\tilde{X}_c) \in L^1(O_T)$ almost surely. Then in Step 2, we prove that the measures $\tilde{\eta}_\pm^k$, defined as above with $\hat{X}_c^{n_k}$ replaced by $\tilde{X}_c^k$, converges to two positive measures $\tilde{\eta}_\pm$ and that $(\tilde{X}_c, \tilde{\eta}_+, \tilde{\eta}_-)$ is a weak solution in the probabilistic sense. It then remains to use pathwise uniqueness to conclude in Step 3. In this proof, we only treat the case $\lambda = 0$. This assumption is not essential at all but lightens the computations. For $\lambda \neq 0$, an extra term has to be taken into account. It is very easy to deal with.

Step 1.
Applying Ito formula to $|Q_N\hat{X}_c^n(t)|_{-1}^2$, we obtain

$$\begin{aligned}
&|Q_N\hat{X}_c^n(T)|_{-1}^2 - |Q_N\hat{X}_c^n(0)|_{-1}^2 + \int_0^T |Q_N\hat{X}_c^n(t)|_1^2\mathrm{dt} - 2\int_{O_T} f_n(\hat{X}_c^n)\left(Q_N\hat{X}_c^n - c\right)\mathrm{dsd}\theta \\
&= 2\int_0^T (Q_N\hat{X}_n^c, B\mathrm{d}W(s)) + T\,\mathrm{Tr}(Q_NB)
\end{aligned}$$

Note that

$$\mathbb{E}\left(\left(\int_0^T (Q_N\hat{X}_c^n, B\mathrm{d}W(s))\right)^2\right) = \mathbb{E}\int_0^T |Q_N\hat{X}_c^n|_{-1}^2\mathrm{ds} = T\int_H |Q_Nx|_{-1}^2\nu_c^n(\mathrm{dx}) \leq C\,T$$

We set

$$\begin{aligned}
\varphi_n^N = |Q_N\hat{X}_c^n(T)|_{-1}^2 - |Q_N\hat{X}_c^n(0)|_{-1}^2 \quad &+\textstyle\int_0^T |Q_N\hat{X}_c^n(t)|_1^2\mathrm{dt} \\
&-2\textstyle\int_{O_T} f_n(\hat{X}_c^n)\left(Q_N\hat{X}_c^n - c\right)\mathrm{dsd}\theta - T\,\mathrm{Tr}(Q_NB)
\end{aligned}$$

and deduce

$$\mathbb{P}(|\varphi_n^N| \geq M) \leq \frac{C\,T}{M^2}.$$

Thus, for all $N \in \mathbb{N}$, the laws of $(\varphi_n^N)_{n\in\mathbb{N}}$ are tight. Therefore the laws of $(\hat{X}_c^n, W^n, (\varphi_n^N)_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ are tight and using Skohorod theorem on this sequence, we can assume that $\tilde{X}_c^k$, $\mathcal{W}^k$ and, for $N \in \mathbb{N}$, $\tilde{\varphi}_k^N$ converge almost surely. We have defined $\tilde{\varphi}_k^N$ as above with $\tilde{X}_c^k$ instead of $\hat{X}_c^n$. In particular, $\tilde{\varphi}_k^N$ is bounded almost surely:

$$\begin{aligned}
|Q_N\tilde{X}_c^k(T)|_{-1}^2 - |Q_N\tilde{X}_c^k(0)|_{-1}^2 \quad &+2\textstyle\int_0^T |Q_N\tilde{X}_c^k(t)|_1^2\mathrm{dt} \\
&-2\textstyle\int_{O_T} f_k(\tilde{X}_c^k)\left(Q_N\tilde{X}_c^k - c\right)\mathrm{dsd}\theta - T\,\mathrm{Tr}(Q_NB) \\
&\leq C(N,T,c)
\end{aligned}$$

where $C(N,T,c)$ is random. The first three terms are clearly also bounded almost surely. This uses the fact that Q_N is a bounded operator from H to V_1. Since Q_N has finite dimensional range, we obtain

$$-\int_{O_T} f_{n_k}(\tilde{X}_c^k)\left(Q_N\tilde{X}_c^k - c\right)\mathrm{d}s\mathrm{d}\theta \leq C(N,T,c) \tag{2.2.5}$$

for a different random constant $C(N,T,c)$.

Let us choose $\epsilon_0 = \min\left\{\frac{1-c}{4}, \frac{1+c}{4}\right\}$ and take $N \in \mathbb{N}$ such that

$$|Q_N\tilde{X}_c - \tilde{X}_c|_{C(O_T)} \leq \frac{1}{2}\epsilon_0$$

and K_0 such that for $k \geq K_0$

$$|\tilde{X}_c^k - \tilde{X}_c|_{C(O_T)} \leq \frac{1}{4}\epsilon_0.$$

Then, for all $k \geq K_0$,

$$|Q_N\tilde{X}_c^k - \tilde{X}_c^k|_{C(O_T)} \leq \epsilon_0.$$

Moreover, if $\tilde{X}_c^k \geq \frac{1+c}{2}$ then $f_{n_k}(\tilde{X}_c^k) \leq 0$ and

$$Q_N\tilde{X}_c^k - c \geq -\epsilon_0 + \frac{1+c}{2} - c \geq \frac{1-c}{4} \geq \epsilon_0.$$

Similarly, if $\tilde{X}_c^k \leq \frac{-1+c}{2}$ then $f_{n_k}(\tilde{X}_c^k) \geq 0$ and

$$Q_N\tilde{X}_c^k - c \leq -\epsilon_0.$$

Finally, noticing that f_n is uniformly bounded by a constant $K(c)$ on $[\frac{-1+c}{2}, \frac{1+c}{2}]$, we deduce

$$\begin{aligned}
\int_{O_T} |f_{n_k}(\tilde{X}_c^k)|\mathrm{d}s\mathrm{d}\theta \quad &\leq \quad -\frac{1}{\epsilon_0}\int_{\tilde{X}_c^k \geq \frac{1+c}{2}} f_{n_k}(\tilde{X}_c^k)\left(Q_N\tilde{X}_c^k - c\right)\mathrm{d}s\mathrm{d}\theta \\
&\quad -\frac{1}{\epsilon_0}\int_{\tilde{X}_c^k \leq \frac{-1+c}{2}} f_{n_k}(\tilde{X}_c^k)\left(Q_N\tilde{X}_c^k - c\right)\mathrm{d}s\mathrm{d}\theta + K(c) \\
&\leq \quad -\frac{1}{\epsilon_0}\int_{O_T} f_{n_k}(\tilde{X}_c^k)\left(Q_N\tilde{X}_c^k - c\right)\mathrm{d}s\mathrm{d}\theta \\
&\quad +\frac{1}{8}(\max\{1-c, 1+c\})^2K(c) + K(c).
\end{aligned}$$

Thanks to (2.2.5), we obtain

$$\int_{O_T} |f_{n_k}(\tilde{X}_c^k)|\mathrm{d}s\mathrm{d}\theta \leq C(N,T,c), \tag{2.2.6}$$

where the value of the random constant $C(N,T,c)$ has again changed. It easily deduced from this uniform bound that $|\tilde{X}_c| \leq 1$ almost everywhere with respect to t and ω and by Fatou Lemma that $f(\tilde{X}_c) \in L^1(O_T)$ almost surely.

□

Step 2.

Let now ξ^k be the following measure on O_T:

$$\mathrm{d}\xi^k := -f^{n_k}(\tilde{X}_c^k(t,\theta))\mathrm{d}t\mathrm{d}\theta.$$

and ξ_+^k and ξ_-^k the positive and negative parts:

$$\mathrm{d}\xi_+^k := -f^{n_k}(\tilde{X}_c^k(t,\theta))\mathbb{1}_{\tilde{X}_c^k>0}\mathrm{d}t\mathrm{d}\theta,\ \mathrm{d}\xi_-^k := f^{n_k}(\tilde{X}_c^k(t,\theta))\mathbb{1}_{\tilde{X}_c^k\leq 0}\mathrm{d}t\mathrm{d}\theta.$$

By step 1, $f(\tilde{X}_c) \in L^1(O_T)$ and we can define the following measure:

$$\mathrm{d}\lambda := -f((\tilde{X}_c(t,\theta))\mathbb{1}_{-1\le\tilde{X}_c\le 1}\mathrm{d}t\mathrm{d}\theta,$$

and the positive and negative parts:

$$\mathrm{d}\lambda_+ := -f((\tilde{X}_c(t,\theta))\mathbb{1}_{0<\tilde{X}_c\le 1}\mathrm{d}t\mathrm{d}\theta, \quad \mathrm{d}\lambda_- := f((\tilde{X}_c(t,\theta))\mathbb{1}_{-1\le\tilde{X}_c\le 0}\mathrm{d}t\mathrm{d}\theta.$$

By (2.2.6), $f^{n_k}(\tilde{X}_c^k) - f(\tilde{X}_c)$ is bounded in $L^1(O_T)$. We deduce that ξ^k has a subsequence ξ^{k_ℓ} which converges to a measure ζ. Note that this subsequence may depend on the random parameter ω. We set $\tilde{\eta} = \zeta - \lambda$.

Thanks to Fatou Lemma we have the following inequality for all $h \in \mathcal{C}(O_T)$ nonnegative:

$$\begin{aligned}\int_{O_T} h(s,\theta)\big[-f(\tilde{X}_c(s,\theta))\mathbb{1}_{0<\tilde{X}_c\le 1}\big]\mathrm{d}s\mathrm{d}\theta &= \int_{O_T} \liminf_{\ell\to+\infty}\big[-h(s,\theta)f^{n_{k_\ell}}(\tilde{X}_c^{k_\ell}(s,\theta))\mathbb{1}_{0<\tilde{X}_c^{k_\ell}\le 1}\big]\mathrm{d}s\mathrm{d}\theta \\ &\le \liminf_{\ell\to+\infty}\int_{O_T}\big[-h(s,\theta)f^{n_{k_\ell}}(\tilde{X}_c^{k_\ell}(s,\theta))\mathbb{1}_{0<\tilde{X}_c^{k_\ell}\le 1}\big]\mathrm{d}s\mathrm{d}\theta.\end{aligned}$$

Therefore $\tilde{\eta}_+^{k_\ell} = \xi_+^{k_\ell} - \lambda_+$ converges to a positive measure. Similarly, $\tilde{\eta}_-^{k_\ell} = \xi_-^{k_\ell} - \lambda_-$ converges to a positive measure. It follows:

$$\xi_+^{k_\ell} - \lambda_+ \rightharpoonup \tilde{\eta}_+ \quad \text{and} \quad \xi_-^{k_\ell} - \lambda_- \rightharpoonup \tilde{\eta}_-,$$

where $\tilde{\eta}_+$ and $\tilde{\eta}_-$ are the positive and negative parts of $\tilde{\eta}$.

Let us now show that the contact conditions holds for $\left(1-\tilde{X}_c, \tilde{\eta}_+\right)$ and $\left(1+\tilde{X}_c, \tilde{\eta}_-\right)$. Let us define the following measures for $\varepsilon > 0$ and $k \in \mathbb{N}$.

$$\mathrm{d}\xi_{+,\varepsilon}^k := -f^{n_k}(\tilde{X}_c^k(t,\theta))\mathbb{1}_{1-\varepsilon\le\tilde{X}_c^k}\mathrm{d}t\mathrm{d}\theta, \quad \mathrm{d}\tau_{+,\varepsilon}^k := -f^{n_k}(\tilde{X}_c^k(t,\theta))\mathbb{1}_{0<\tilde{X}_c^k<1-\varepsilon}\mathrm{d}t\mathrm{d}\theta,$$

$$\mathrm{d}\lambda_{+,\varepsilon} := -f(\tilde{X}_c(t,\theta))\mathbb{1}_{1-\varepsilon\le\tilde{X}_c}\mathrm{d}t\mathrm{d}\theta, \qquad \mathrm{d}\tau_{+,\varepsilon} := -f(\tilde{X}_c(t,\theta))\mathbb{1}_{0<\tilde{X}_c<1-\varepsilon}\mathrm{d}t\mathrm{d}\theta.$$

Clearly $\tau_{+,\varepsilon}^k$ converges to $\tau_{+,\varepsilon}$, it follows

$$\begin{aligned}\limsup_{\ell\to+\infty}\left\langle 1-\tilde{X}_c^{k_\ell}, \xi_+^{k_\ell}-\lambda_+\right\rangle_{O_T} &= \limsup_{\ell\to+\infty}\Big(\big\langle 1-\tilde{X}_c^{k_\ell}, \xi_{+,\varepsilon}^{k_\ell}\big\rangle_{O_T} - \big\langle 1-\tilde{X}_c^{k_\ell}, \lambda_{+,\varepsilon}\big\rangle_{O_T} \\ &\quad + \big\langle 1-\tilde{X}_c^{k_\ell}, \tau_{+,\varepsilon}^{k_\ell}\big\rangle_{O_T} - \big\langle 1-\tilde{X}_c^{k_\ell}, \tau_{+,\varepsilon}\big\rangle_{O_T}\Big) \\ &= \limsup_{\ell\to+\infty}\left(\int_{O_T}\left(\tilde{X}_c^{k_\ell}-1\right)f^{n_{k_\ell}}(\tilde{X}_c^{k_\ell})\mathbb{1}_{1-\varepsilon\le\tilde{X}_c^{k_\ell}}\mathrm{d}t\mathrm{d}\theta\right. \\ &\quad \left. + \int_{O_T}\left(1-\tilde{X}_c^{k_\ell}\right)f(\tilde{X}_c)\mathbb{1}_{1-\varepsilon\le\tilde{X}_c}\mathrm{d}t\mathrm{d}\theta\right) \\ &\le \limsup_{\ell\to+\infty}\left(\int_{O_T}\left(\tilde{X}_c^{k_\ell}-1\right)f^{n_{k_\ell}}(\tilde{X}_c^{k_\ell})\mathbb{1}_{1-\varepsilon\le\tilde{X}_c^{k_\ell}\le 1}\mathrm{d}t\mathrm{d}\theta\right) \\ &\quad + \limsup_{\ell\to+\infty}\left(\int_{O_T}\left(1-\tilde{X}_c^{k_\ell}\right)^- f(\tilde{X}_c)\mathbb{1}_{1-\varepsilon\le\tilde{X}_c}\mathrm{d}t\mathrm{d}\theta\right)\end{aligned}$$

Since $(1-\tilde{X}_c^{k_\ell})^-$ converges uniformly to zero, we deduce:

$$\begin{aligned}\limsup_{\ell\to+\infty}\big\langle 1-\tilde{X}_c^{k_\ell}, \xi_+^{k_\ell}-\lambda_+\big\rangle_{O_T} &\le T\sup_{x\in[1-\varepsilon,1]}|(x-1)f(x)| \\ &\le -T\varepsilon\ln\left(\frac{\varepsilon}{2-\varepsilon}\right).\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain the first contact condition since the left hand side clearly converges to $\langle 1-\tilde{X}_c, \tilde{\eta}_+\rangle$. The second is obtained similarly.

We now prove that $\xi^k - \lambda$ does not have more than one limit point so that in fact the whole sequence converge to $\tilde{\eta}$. Let $\tilde{\eta}_i$, $i = 1, 2$ be two limit points.

For all $h \in D(A^2)$ and for all $0 \leq t \leq T$:

$$\begin{aligned}\left\langle Ah, \xi^k - \lambda \right\rangle_{O_t} &= \langle \tilde{X}_c^k(t,.), h\rangle - \langle x, h\rangle + \int_{O_t} \tilde{X}_c^k(s,\theta) A^2 h(\theta) \mathrm{d}s\mathrm{d}\theta \\ &\quad + \int_0^t \langle Bh, \mathrm{d}\mathcal{W}^k\rangle + \int_{O_t} f(\tilde{X}_c(s,\theta)) Ah(\theta) \mathrm{d}s\mathrm{d}\theta.\end{aligned}$$

We deduce

$$\begin{aligned}\left\langle Ah, \tilde{\eta}_i \right\rangle_{O_t} &= -\langle \tilde{X}_c(t,.), h\rangle + \langle x, h\rangle - \int_{O_t} \tilde{X}_c(s,\theta) A^2 h(\theta) \mathrm{d}s\mathrm{d}\theta \\ &\quad - \int_0^t \langle Bh, \mathrm{d}\mathcal{W}\rangle - \int_{O_t} f(\tilde{X}_c(s,\theta)) Ah(\theta) \mathrm{d}s\mathrm{d}\theta.\end{aligned}$$

And by density

$$\left\langle h, \tilde{\eta}_1 \right\rangle_{O_t} = \left\langle h, \tilde{\eta}_2 \right\rangle_{O_t}$$

for any $h \in C([0,1])$ such that $\bar{h} = 0$. Since by the contact condition

$$\left\langle (1 - \tilde{X}_c)(1 + \tilde{X}_c), \tilde{\eta}_1 \right\rangle_{O_t} = \left\langle (1 - \tilde{X}_c)(1 + \tilde{X}_c), \tilde{\eta}_2 \right\rangle_{O_t}.$$

We deduce from Lemma 2.2.1 that $\tilde{\eta}_1 = \tilde{\eta}_2$.

□

Step 3.

We use a result form [42] that allows to get the convergence of the approximated solutions in probability in any space in which these approximated solutions are tight.

Lemma 2.2.2 *Let $\{Z_n\}_{n\geq 1}$ be a sequence of random elements on a Polish space E endowed by its borel σ-algebra. Then $\{Z_n\}_{n\geq 1}$ converges in probability to an E-valued random element if and any if from every pair of subsequences $\{(Z_{n_k^1}, Z_{n_k^2})_{k\geq 1}$, one can extract a subsequence which converges weakly to a random element supported on the diagonal $\{(x, y) \in E \times E, x = y\}$.*

Assume $(n_k^1)_{k\in\mathbb{N}}$ and $(n_k^1)_{k\in\mathbb{N}}$ are two arbitrary subsequences. Clearly, the process $\left(\hat{X}_c^{n_k^1}, \hat{X}_c^{n_k^2}, W^k\right)$ is tight in a suitable space. By Skorohod's theorem, we can find a probability space and a sequence of processes $\left(\tilde{X}_c^{1,k}, \tilde{X}_c^{2,k}, \mathcal{W}^k\right)$ such that $\left(\tilde{X}_c^{1,k}, \tilde{X}_c^{2,k}, \mathcal{W}^k\right) \to \left(\tilde{X}_c^1, \tilde{X}_c^2, \mathcal{W}\right)$ almost surely and $\left(\tilde{X}_c^{1,k}, \tilde{X}_c^{2,k}, \mathcal{W}^k\right)$ has the same distribution as $\left(\hat{X}_c^{n_k^1}, \hat{X}_c^{n_k^2}, W^k\right)$ for all $k \in \mathbb{N}$. In the Skorohod's space, the approximated measures respectively converge to two contact measures $\tilde{\eta}_1$ and $\tilde{\eta}_2$. By the second step, $(\tilde{X}_c^1, \tilde{\eta}_1, \mathcal{W})$ and $(\tilde{X}_c^2, \tilde{\eta}_2, \mathcal{W})$ are both weak solutions of (2.0.3). By uniqueness, necessarily $\tilde{X}_c^1 = \hat{X}_c^2$ and $\tilde{\eta}_1 = \tilde{\eta}_2$. Therefore the subsequence $\left(\hat{X}_c^{n_k^1}, \hat{X}_c^{n_k^2}\right)_{k\in\mathbb{N}}$ converges in distribution to a process supported on the diagonal. We use Lemma 2.2.2 to prove that the sequence $(\hat{X}_c^n)$ converges in probability to a process $\hat{X}_c$. Clearly $\hat{X}_c$ is stationary. Reproducing the argument of Step 1 and Step 2, we prove that it is a strong solution of (2.0.3) and the convergence of the contact measures.

□

2.2.4 Convergence of the semigroup

First we state the following result which is a corollary of Proposition 2.2.3.

Corollary 2.2.1 *Let $c > 0$.*

i) There exists a continuous process $(X(t,x), t \geq 0, x \in K \cap H_c)$ with $X(0,x) = x$ and a set K_0 dense in $K \cap H_c$, such that for all $x \in K_0$ there exists a unique strong solution of equation (2.0.3) given by $\left((X(t,x))_{t\geq 0}, \eta_+^x, \eta_-^x, W\right)$.

ii) The law of $\left(X(t,x)_{t\geq 0}, \eta_+^x, \eta_-^x\right)$ *is a regular conditional distribution of the law of* $\left(\hat{X}_c, \eta_+, \eta_-\right)$ *given* $\hat{X}_c(0) = x \in K \cap H_c$.

Proof : By Proposition 2.2.3, we have a stationary strong solution $\hat{X}_c$ in H_c, such that W and $\hat{X}_c(0)$ are independent. Conditioning $\left(\hat{X}_c, \eta_+, \eta_-\right)$ on the value of $\hat{X}_c(0) = x$, with $c = \overline{x}$, we obtain for ν_c-almost every x a strong solution that we denote $\left(X(t,x), \eta_+^x, \eta_-^x\right)$ for all $t \geq 0$ and for all $x \in K \cap H_c$. This process is the desired process. Indeed, since the support of ν_c is $K \cap H_c$, we have a strong solution for a dense set K_0 in $K \cap H_c$.
Notice that all processes $(X(t,x))_{t\geq 0}$ with $x \in K_0$ are driven by the same noise W and are continuous with values in H. Moreover, we have the following obvious identity:

$$|X^n(t,x) - X^n(t,y)|_{-1} \leq e^{\lambda t}|x-y|_{-1},\ x,y \in L^2_c,\ t \geq 0,$$

and by density we obtain a continuous process $(X(t,x))_{t\geq 0}$ in H_c for all $x \in K \cap H_c$.

□

Proposition 2.2.4 *Let* $c > 0$, *for all* $\phi \in \mathcal{C}_b(H)$ *and* $x \in K \cap H_c$:

$$\lim_{n\to+\infty} P_t^{n,c}\phi(x) = \mathbb{E}[\phi(X(t,x))] =: P_t^c\phi(x). \tag{2.2.7}$$

Moreover the Markov process $(X(t,x), t \geq 0, x \in K \cap H_c)$ *is strong Feller and its transition semigroup* P^c *is such that:*

$$|P_t^c\phi(x) - P_t^c\phi(y)| \leq \frac{2e^{\lambda^2 t/4}}{\lambda\sqrt{t}}|x-y|_{-1}, \quad \text{for all } x,y \in K \cap H_c, \text{for all } t > 0. \tag{2.2.8}$$

Proof : Since $(\nu_c^n)_{n\geq 1}$ is tight in H_c, then there exists an increasive sequence of compact sets $(J^p)_{p\in\mathbb{N}}$ in H such that:

$$\lim_{p\to+\infty} \sup_{n\geq 1} \nu_c^n(H \setminus J^p) = 0.$$

Set $J := \cup_{p\in\mathbb{N}} J^p \cap K$. Since the support of ν_c is in $K \cap H_c$ and $\nu_c(J) = 1$, then J is dense in $K \cap H_c$. Fix $t > 0$, by (2.1.4), for any $\phi \in \mathcal{C}_b(H)$:

$$\sup_{n\in\mathbb{N}}(\|P_t^{n,c}\phi\|_\infty + [P_t^{n,c}\phi]_{Lip(H_c)}) < +\infty.$$

Let $(n_j)_{j\in\mathbb{N}}$ be any sequence in $\mathbb{N}$. With a diagonal procedure, by Arzelà-Ascoli Theorem, there exists $(n_{j_l})_{l\in\mathbb{N}}$ a subsequence and a function $\Theta_t : J \to \mathbb{R}$ such that:

$$\lim_{l\to+\infty} \sup_{x\in J^p} |P_t^{n_{j_l},c}\phi(x) - \Theta_t(x)| = 0, \quad \text{for all } p \in \mathbb{N}.$$

By density, Θ_t can be extended uniquely to a bounded Lipschitz function $\tilde{\Theta}_t$ on $K \cap H_c$ such that

$$\tilde{\Theta}_t(x) = \lim_{l\to+\infty} P_t^{n_{j_l},c}\phi(x), \quad \text{for all } x \in K \cap H_c.$$

Note that the subsequence depends on t. Therefore, we have to prove that the limit defines a semigroup and does not depend on the chosen subsequence.
By Proposition 2.2.3, we have for all $\phi, \psi \in \mathcal{C}_b(H)$:

$$\begin{aligned}
\mathbb{E}\left[\psi\left(\hat{X}_c(0)\right)\phi\left(\hat{X}_c(t)\right)\right] &= \lim_{l\to+\infty} \mathbb{E}\left[\psi\left(\hat{X}_c^{n_{j_l}}(0)\right)\phi\left(\hat{X}_c^{n_{j_l}}(t)\right)\right] \\
&= \lim_{l\to+\infty} \int_H \psi(y)\mathbb{E}\left[\phi\left(\hat{X}_c^{n_{j_l}}(t)\right)\left|\hat{X}_c^{n_{j_l}}(0) = y\right.\right]\nu_c^{n_{j_l}}(\mathrm{dy}) \\
&= \lim_{l\to+\infty} \int_H \psi(y)P_t^{n_{j_l},c}\phi(y)\nu_c^{n_{j_l}}(\mathrm{dy}) \\
&= \int_H \psi(y)\tilde{\Theta}_t(y)\nu_c(\mathrm{dy}).
\end{aligned}$$

Thus, by Corollary 2.2.1, we have the following equality:

$$\mathbb{E}\left[\phi\left(X(t,x)\right)\right] = \tilde{\Theta}_t(x), \qquad \text{for } \nu_c\text{-almost every } x. \tag{2.2.9}$$

Since $\mathbb{E}[\phi(X(t,.))]$ and $\tilde{\Theta}_t$ are continuous on $K \cap H_c$, and $\nu_c(K \cap H_c) = 1$, the equality (2.2.9) is true for all $x \in K \cap H_c$. Moreover the limit does not depend on the chosen subsequence, and we obtain (2.2.7). Letting $n \to \infty$ in (2.1.4), we deduce (2.2.8).

□

2.2.5 End of the proof of Theorem 2.1.1

We have proved that there exists a continous process X which is a strong solution of equation (2.0.3) for an x in a dense space. In this section, we prove existence for an initial condition in $K \cap H_c$ with $c > 0$.

By Corollary 2.2.1 we have a process $(X(t,x), t \geq 0, x \in K \cap H_c)$, such that for all x in a set K_0 dense in $K \cap H_c$ we have a strong solution $\left((X(t,x))_{t\geq 0}, \eta_+^x, \eta_-^x, W\right)$ of (2.0.3) with initial condition x. By Proposition 2.2.3, the Markov process X has transition semigroup P^c on H_c.
The strong Feller property of P^c implies that for all $x \in K \cap H_c$ and $s > 0$ the law of $X(s,x)$ is absolutely continous with respect to the invariant measure ν_c. Indeed, if $\nu_c(\Gamma) = 0$, then $\nu_c(P_s^c(\mathbb{1}_\Gamma)) = \nu_c(\Gamma) = 0$. So $P_s^c(\mathbb{1}_\Gamma)(x) = 0$ for ν_c-almost every x and by continuity for all $x \in K \cap H_c$.
Therefore almost surely $X(s,x) \in K_0$ for all $s > 0$ and $x \in K \cap H_c$. Fix $s > 0$, denote for all $\theta \in [0,1]$:

$$\tilde{X} := t \mapsto X(t+s,x), \tilde{W}(\cdot,\theta) := t \mapsto W(t+s,\theta) - W(s,\theta)),$$

and the measures $\tilde{\eta}_\pm^x$ such that for all $T > 0$, and for all $h \in \mathcal{C}(O_T)$:

$$\left\langle h, \tilde{\eta_\pm}^x \right\rangle_{O_T} := \int_{O_s^{T+s}} h(t-s,\theta)\eta_\pm^x(\mathrm{d}t, \mathrm{d}\theta).$$

So we have a process $\tilde{X} \in \mathcal{C}([0,T];H) \cap \mathcal{C}(O_T)$ and two measures $\tilde{\eta}_+^x$ and $\tilde{\eta}_-^x$ on O_T which is finite on $[\delta, T] \times [0,1]$ for all $\delta \geq 0$, such that $\left((\tilde{X}(t,x))_{t\geq 0}, \tilde{\eta}_+^x, \tilde{\eta}_-^x, \tilde{W}\right)$ is a strong solution of (2.0.3) with initial condition $X(s,x)$. By continuity $X(s,x) \to x$ in H as $s \to 0$, so $\left((X(t,x))_{t\geq 0}, \eta_+^x, \eta_-^x, W\right)$ is a strong solution of (2.0.3) with initial condition x in the sense of the definition 2.1.1.

2.3 Ergodicity and mixing

When λ is small, it can be easily shown that ν_c is the unique invariant measure and is ergodic. We now prove that this is in fact true for any λ. Note that since $(P_t^c)_{t\geq 0}$ is Strong Feller, the results follows from Doob theorem if we prove that $(P_t^c)_{t\geq 0}$ is irreducible (see for instance [24]). For additive noise driven SPDEs, this is often proved by a control argument and continuity with respect to the noise. This latter property is not completely trivial in our situation but we are able to adapt the argument.

Proposition 2.3.1 *For any $c \in (-1,1)$, the semigroup $(P_t^c)_{t\geq 0}$ is irreducible.*

Proof :

Let x, $y \in C^\infty([0,1])$ be such that $|x|_{L^\infty(0,1)} \leq 1-\delta$ and $|y|_{L^\infty(0,1)} \leq 1-\delta$ for some $\delta > 0$ and $\bar{x} = \bar{y} = c$. We set

$$u(t) = \frac{t}{T}y + \left(1 - \frac{t}{T}\right)x$$

and define g_0 by

$$g_0(\theta,t) = \int_0^\theta \left(\frac{1}{T}(y-x) + \frac{1}{2}A(Au + f(u))\right)(\vartheta,t)\mathrm{d}\vartheta$$

Then g_0 is in $C^\infty([0,T]\times[0,1])$, $g_0(t)\in D(B)$, $t\in[0,T]$, and:

$$\frac{d}{dt}u=-\frac{1}{2}A(Au+f(u))+Bg_0.$$

Moreover

$$\frac{d}{dt}u=-\frac{1}{2}A(Au+f_\delta(u))+Bg_0 \tag{2.3.1}$$

where f_δ is any Lipschitz function equal to f on $[-1+\delta/2,1-\delta/2]$.

Let $X^\delta(\cdot,x)$ be the solution of (2.0.3) with f replaced by f_δ and set $Y^\delta(\cdot,x)=X^\delta(\cdot,x)-Z$, where $Z=Z(\cdot,0)$ is the solution of the linear equation with 0 as initial data. Then

$$\frac{d}{dt}Y^\delta=-\frac{1}{2}A\left(AY^\delta+f_\delta(Y^\delta+Z)\right),\ Y^\delta(0,x)=x.$$

Let also

$$z_0(t)=\int_0^t e^{-A^2(t-s)/2}Bg_0(s)\mathrm{ds}.$$

Since the gaussian process Z is almost surely continuous and has a non degenerate covariance, we clearly have

$$\mathbb{P}\left(|Z-z_0|_{C(O_T)}\le\varepsilon\right)>0$$

for any $\varepsilon>0$. Let us denote by Y^z the solution of

$$\frac{d}{dt}Y^z=-\frac{1}{2}A\left(AY^z+f_\delta(Y^z+z)\right),\ Y^z(0,x)=x. \tag{2.3.2}$$

We prove below that the mapping

$$\Phi_\delta\ :\ z\mapsto Y^z$$

is continuous from $C(O_T)$ into $C(O_T)$. Since $u=\Phi_\delta(z_0)+z_0$ and $X^\delta=\Phi_\delta(Z)+Z$, we deduce that there exists ε such that

$$\mathbb{P}\left(|X^\delta-u|_{C(O_T)}\le\delta/2\right)\ge\mathbb{P}\left(|Z-z_0|_{C(O_T)}\le\varepsilon\right)>0$$

Let us now observe that $|X^\delta-u|_{C(O_T)}\le\delta/2$ implies $|X^\delta|_{C(O_T)}\le 1-\delta/2$ so that $f_\delta(X^\delta)=f(X^\delta)$ and X^δ is fact solution of (2.0.3). By pathwise uniqueness, we deduce that $|X^\delta-u|_{C(O_T)}\le\delta/2$ implies $X^\delta=X$. It follows

$$\mathbb{P}\left(|X-u|_{C(O_T)}\le\delta/2\right)\ge\mathbb{P}\left(|X^\delta-u|_{C(O_T)}\le\delta/2\right)>0$$

In particular

$$\mathbb{P}\left(|X(T,x)-y|\le\delta/2\right)>0.$$

If we assume now that $x,y\in H_c$, we choose $\tilde{x},\tilde{y}\in C^\infty([0,1])$ such that

$$|x-\tilde{x}|\le\delta,\ |y-\tilde{y}|\le\delta,\ |\tilde{x}|_{L^\infty(0,1)}\le 1-\delta \text{ and } |\tilde{y}|_{L^\infty(0,1)}\le 1-\delta,$$

and $\bar{\tilde{x}}=\bar{\tilde{y}}=c$. We have

$$|X(T,x)-X(T,\tilde{x})|\le e^{\lambda T}|x-\tilde{x}|.$$

Therefore

$$\mathbb{P}\left(|X(T,x)-y|\le\delta/2+(1+e^{\lambda T})\delta\right)\ge\mathbb{P}\left(|X(T,\tilde{x})-\tilde{y}|\le\delta/2\right)>0.$$

This proves the results.

It remains to prove that Φ_δ is continuous. This follows form the mild form of equation (2.3.2):

$$Y^z(t)=e^{-tA^2/2}x+\int_0^t e^{-(t-s)A^2/2}Af_\delta(Y^z(s)+z(s))ds.$$

It is classical that, for $t>0$, $Ae^{-tA^2/2}$ maps $C([0,1])$ into itself and

$$\left|Ae^{-tA^2/2}\right|_{\mathcal{L}(C([0,1]))}\le Ct^{-1/2}.$$

This can be seen from the formula

$$Ae^{-tA^2/2}u = -\sum_{i\in\mathbb{N}} \lambda_i e^{-\lambda_i^2 t/2}\langle u, e_i\rangle e_i,$$

where $(e_i)_{i\in\mathbb{N}}$ and $(\lambda_i)_{i\in\mathbb{N}}$ are the eigenvectors and eigenvalues of $-A$. Since $|e_i|_{C([0,1]}$ are equi-bounded, we deduce

$$\begin{aligned}\left|Ae^{-tA^2/2}u\right|_{C([0,1]} &\leq C\left(\sum_{i\in\mathbb{N}} \lambda_i e^{-\lambda_i^2 t/2}\right)|u|_{L^1(0,1)}\\ &\leq C\, t^{-1/2}|u|_{C([0,1])}\end{aligned}$$

We deduce

$$\begin{aligned}&|Y^{z_1}(t) - Y^{z_2}(t)|_{C([0,1])}\\ &\leq C\, L_\delta \int_0^t (t-s)^{-1/2}\left(|Y^{z_1}(s) - Y^{z_2}(s)|_{C([0,1])} + |z_1(s) - z_2(s)|_{C([0,1])}\right) \mathrm{ds}\end{aligned}$$

where L_δ is the Lipschitz constant of f_δ. Gronwall Lemma implies the result for T sufficiently small. Iterating the argument we obtain the continuity of Φ_δ.

□

Corollary 2.3.1 *For every $c \in (-1,1)$, ν_c is the unique invariant measure of the transition semi-group $(P_t^c)_{t\geq 0}$. Moreover it is ergodic.*

Using classical arguments, it is easily seen that, for $\lambda = 0$, ν_c^n satisfies a log-Sobolev inequality and therefore a Poincaré inequality. The constant in these inequality do not depend on n so that we have the same result for ν_c. For $\lambda \neq 0$, we can argue as in [22] and prove that this is still true.

We now want to prove a stronger result : exponential mixing. We use coupling arguments developped by Odasso in [62].

Theorem 2.3.1 *For every $c \in (-1,1)$, there exist a small $\beta > 0$ and a constant $C > 0$ such that for all $\varphi \in \mathcal{B}_b(K \cap H_c)$, $t > 0$ and $x \in H_c$*

$$|\mathbb{E}[\varphi(X(t,x))] - \nu_c(\varphi)| \leq C\|\varphi\|_\infty e^{-\beta t}. \tag{2.3.3}$$

Proof : By (2.2.8), we know that for any $\varphi \in \mathcal{B}_b(K \cap H_c)$, $T > 0$, $\varepsilon > 0$,

$$|P_T^c\varphi(x) - P_T^c\varphi(y)| \leq \frac{4e^{\lambda^2 T/4}}{\lambda\sqrt{T}}\varepsilon\|\varphi\|_\infty$$

if x, $y \in H_c$, $|x|_{-1} \leq \varepsilon$ and $|y|_{-1} \leq \varepsilon$. By definition of the total variation norm, we deduce

$$\|\,(P_T^c)^*\,\delta_x - (P_T^c)^*\,\delta_y\|_{var} = \sup_{\|\varphi\|_\infty\leq 1} |P_T^c\varphi(x) - P_T^c\varphi(y)| \leq \frac{4e^{\lambda^2 T/4}}{\lambda\sqrt{T}}\varepsilon \tag{2.3.4}$$

for $T > 0$, x, $y \in H_c$, $|x|_{-1} \leq \varepsilon$ and $|y|_{-1} \leq \varepsilon$. We have denoted by δ_x the Dirac mass at $x \in H_c$ so that $(P_T^c)^*\,\delta_x$ is the law of $X(T,x)$.

Recall that a coupling of $((P_T^c)^*\,\delta_x, (P_T^c)^*\,\delta_y)$ is a couple of random variable (X_1, X_2) such that the law of X_1 is $(P_T^c)^*\,\delta_x$ and the law of X_2 is $(P_T^c)^*\,\delta_y$. By standard results on couplings (see for instance [48] section 4, or [56]), we know there exists a maximal coupling of $((P_T^c)^*\,\delta_x, (P_T^c)^*\,\delta_y)$. Let us denote by $(Y_1(x,y), Y_2(x,y))$ this maximal coupling, it satisfies

$$\mathbb{P}(Y_1(x,y) \neq Y_2(x,y)) = \|\,(P_T^c)^*\,\delta_x - (P_T^c)^*\,\delta_y\|_{var}. \tag{2.3.5}$$

Moreover $(Y_1(x,y), Y_2(x,y))$ depends measurably on (x,y).

By the Strong Feller property, we know that $x \mapsto \mathbb{P}(|X(T,x)|_{-1} \leq \varepsilon)$ is continuous on H_c. Therefore, thanks to Proposition 2.3.1, for any $x \in K \cap H_c$, there exists a $\eta_x > 0$ and a $\kappa_x > 0$ such that

$$\mathbb{P}(|X(T,y)|_{-1} \leq \varepsilon) > \kappa_x$$

for all $y \in K \cap H_c$ such that $|x-y|_{-1} \leq \eta_x$. By compactness of $K \cap H_c$ in H_c, we deduce that that there exits $\kappa_0 > 0$ such that

$$\mathbb{P}(|X(T,y)|_{-1} \leq \varepsilon) > \kappa_0 \tag{2.3.6}$$

for all $y \in K \cap H_c$.

Let $\tilde{W}$ a cylindrical Wiener process independent on W and denote by $\tilde{X}$ the associated solution of the stochastic Cahn-Hilliard equation which has the same law as X. For arbitrary x, $y \in K \cap H_c$, we define the coupling $(Z_1(x,y), Z_2(x,y)$ of $((P_T^c)^* \delta_x, (P_T^c)^* \delta_y)$ as follows

$$(Z_1(x,y), Z_2(x,y)) = \begin{cases} (X(T,x), X(T,y)) \text{ if } x = y, \\ (Y_1(x,y), Y_2(x,y)) \text{ if } |x|_{-1} \leq \varepsilon,\ |y|_{-1} \leq \varepsilon \text{ and } x \neq y, \\ (X(T,x), \tilde{X}(T,y)) \text{ otherwise.} \end{cases}$$

We now construct recursively $(X_1(kT,x,y), X_2(kT,x,y))$ a coupling of $((P_{kT}^c)^* \delta_x, (P_{kT}^c)^* \delta_y)$, the laws of $X(kT,x)$ and $X(kT,y)$. For $k=0$, we set $(X_1(kT,x,y), X_2(kT,x,y)) = (x,y)$. For $k \geq 0$, we define $(X_1((k+1)T,x,y), X_2((k+1)T,x,y))$ by

$$\begin{aligned} X_1((k+1)T,x,y) &= Z_1(X_1(kT,x,y), X_2(kT,x,y)), \\ X_2((k+1)T,x,y) &= Z_2(X_1(kT,x,y), X_2(kT,x,y)). \end{aligned}$$

Let us define

$$\tau = \inf\{kT :\ |X_1(kT,x,y)|_{-1} \leq \varepsilon,\ |X_2(kT,x,y)|_{-1} \leq \varepsilon\}$$

If $|x|_{-1} \leq \varepsilon$ and $|y|_{-1} \leq \varepsilon$, then $\tau = 0$ and $\mathbb{E}(e^{\alpha\tau}) = 1$.

If $\tau \neq 0$ *i.e.* if $|x|_{-1} \geq \varepsilon$ or $|y|_{-1} \geq \varepsilon$, then by construction of the coupling and (2.3.6)

$$\mathbb{P}(\tau > T) < 1 - \kappa_0^2.$$

More generally

$$\mathbb{P}(\tau > kT \big| \tau \geq kT) < 1 - \kappa_0^2.$$

We deduce

$$\mathbb{P}(\tau > kT) < (1-\kappa_0^2)^k$$

and

$$\mathbb{E}(e^{\alpha\tau}) = \sum_{k\in\mathbb{N}} e^{\alpha k T} \mathbb{P}(\tau = kT) \leq \sum_{k\in\mathbb{N}} e^{\alpha k T} (1-\kappa_0^2)^{k-1} = M < \infty$$

for α small enough. Similarly, if we define

$$\tau_n = \inf\{kT > \tau_{n-1} :\ |X_1(kT,x,y)|_{-1} \leq \varepsilon,\ |X_2(kT,x,y)|_{-1} \leq \varepsilon\},$$

for all $n \geq 2$ and with $\tau_1 := \tau$. We have

$$\mathbb{E}(e^{\alpha(\tau_n - \tau_{n-1})}) \leq M$$

so that

$$\mathbb{E}(e^{\alpha\tau_n}) \leq M^n.$$

Define

$$k_0 = \inf\{n \geq 1 :\ X_1(\tau_n + T, x, y) = X_2(\tau_n + T, x, y)\}.$$

By (2.3.4), (2.3.5), for all $n \geq 1$

$$\mathbb{P}(k_0 = n) \leq \mathbb{P}(k_0 > n-1) \leq \left(\frac{4e^{\lambda^2 T/4}}{\lambda\sqrt{T}} \varepsilon \right)^{n-1}.$$

We choose ε small enough such that

$$\left(\frac{4e^{\lambda^2 T/4}}{\lambda\sqrt{T}} \varepsilon \right) < 1.$$

Then we write

$$\begin{aligned} \mathbb{E}(e^{\beta\tau_{k_0}}) = \sum_{n\geq 1} \mathbb{E}\left(e^{\beta\tau_n} \mathbb{1}_{k_0=n}\right) &\leq \sum_{n\geq 1} (\mathbb{E}(e^{2\beta\tau_n}))^{1/2} (\mathbb{P}(k_0 = n))^{1/2} \\ &\leq \sum_{n\geq 1} M^{n\beta/\alpha} \left(\frac{4e^{\lambda^2 T/4}}{\lambda\sqrt{T}} \varepsilon \right)^{(n-1)/2} = \mathcal{M} < \infty \end{aligned}$$

for β small enough.

By Markov's inequality, we conclude that for all $k \geq 1$

$$\begin{aligned} &|\mathbb{E}(\varphi(X(kT,x))) - \mathbb{E}(\varphi(X(kT,y)))| \\ &= |\mathbb{E}(\varphi(X_1(kT,x,y))) - \mathbb{E}(\varphi(X_2(kT,x,y)))| \\ &\leq 2\|\varphi\|_\infty \mathbb{P}(X_1(kT,x,y) \neq X_2(kT,x,y)) \\ &\leq 2\|\varphi\|_\infty \mathbb{P}(kT > \tau_{k_0} + T) \\ &\leq 2\|\varphi\|_\infty \mathcal{M} e^{-\beta(k-1)T}. \end{aligned}$$

We define $k := \left\lfloor \frac{t}{T} \right\rfloor$ such that we have $P_t^c = P_{kT}^c P_{t-kT}^c$. Thus we can write

$$\begin{aligned} |\mathbb{E}[\varphi(X(t,x))] - \nu_c(\varphi)| &= \left| P_t^c \varphi(x) - \int_{H_c} \varphi(y)\nu_c(\mathrm{dy}) \right| \\ &= \left| \int_{H_c} P_t^c \varphi(x)\nu_c(\mathrm{dy}) - \int_{H_c} P_t^c \varphi(y)\nu_c(\mathrm{dy}) \right| \\ &\leq \left| \int_{H_c} \left(P_{kT}^c P_{t-kT}^c \varphi(x) - P_{kT}^c P_{t-kT}^c \varphi(y) \right) \nu_c(\mathrm{dy}) \right| \\ &\leq \int_{H_c} 2\|P_{t-kT}^c \varphi\|_\infty \mathcal{M} e^{-\beta(k-1)T} \nu_c(\mathrm{dy}) \\ &\leq 2\|\varphi\|_\infty \mathcal{M} e^{-\beta(k-1)T} \\ &\leq C\|\varphi\|_\infty e^{-\beta t}. \end{aligned}$$

□

Chapter 3

Simulations numériques déterministes et stochastiques

Résumé

Dans ce chapitre, on propose une méthode numérique basée sur des éléments finis continus pour l'équation de Cahn-Hilliard. L'augmentation du degré des éléments finis se révèle très efficace et favorable vis-à-vis des autres méthodes existantes (éléments $\mathcal{C}^1$, raffinement de maillage adaptatif, résolutions multi-grilles, etc.). En plus des benchmarks classiques, une étude numérique a été réalisée pour étudier l'influence de l'approximation polynomiale de l'énergie libre logarithmique, les bifurcations près de la première valeur propre du Laplacien, et pour illustrer quelques résultats sur l'équation stochastique.

Tout le monde peut faire des erreurs,
c'est pour ça qu'il y a des gommes au bout des crayons.

Lenny - The Simpsons

Introduction

We consider an isothermal binary alloy between two species A and B, and denote by $u \in [-1, 1]$ the ratio between the two components. By thermodynamic arguments, and under a mass conservation property, Cahn and Hilliard described a fourth-order model for the evolution of an isotropic system of nonuniform composition or density. They introduced a free energy density $\bar{f}$ to define a chemical potential, and use it in the classical transport equation (see [13], [15] and [16]). The total free energy $\mathcal{F}$ of the binary alloy is a volume integral on Ω of this free energy density (bulk free energy):

$$\mathcal{F} := \int_\Omega \bar{f}(u, \nabla u, \nabla^2 u, \dots) \, \mathrm{d}V. \tag{3.0.1}$$

They assumed $\bar{f}$ to be a function of u and its spatial derivatives. Then a truncated Taylor expansion of $\bar{f}$ has the following general form:

$$\bar{f}(u) \sim f(u) + L \cdot \nabla u + K_1 \otimes \nabla^2 u + \nabla u \cdot K_2 \cdot \nabla u, \tag{3.0.2}$$

where ∇ is the Nabla operator. By symmetry arguments, they showed that $L = \vec{0}$ and K_1 and K_2 are homothetic operators. Moreover they used Neumann boundary condition to cancel the term in $\nabla^2 u$ to obtain:

$$\mathcal{F} := \int_\Omega \left(f(u) + \kappa |\nabla u|^2 \right) \mathrm{d}V, \tag{3.0.3}$$

where κ is a parameter (often denoted $\varepsilon^2/2$) which is referred to as the *gradient coefficient*. Then, the chemical potential w is defined by:

$$w := f'(u) - 2\kappa \Delta u. \tag{3.0.4}$$

If we denote by J the flux and by $\mathcal{M}(u)$ the mobility, the classical Fick laws provide the following equations:

$$\partial_t u = -\nabla \cdot J \text{ and } J = -\mathcal{M}(u) \nabla w. \tag{3.0.5}$$

Finally, the Cahn-Hilliard equation takes the following general form:

$$\begin{cases} \partial_t u = \nabla \cdot \left[\mathcal{M}(u) \nabla w \right], & \text{on } \Omega \subset \mathbb{R}^n, \\ w = \psi(u) - \varepsilon^2 \Delta u, & \text{on } \Omega \subset \mathbb{R}^n, \\ \nabla u \cdot \nu = 0 = \nabla w \cdot \nu, & \text{on } \partial\Omega, \end{cases} \tag{3.0.6}$$

where t denotes the time variable, ε $(= \sqrt{2\kappa})$ is a measure of the interfacial thickness, ψ $(= f')$ is a nonlinear term, $\mathcal{M}$ is the mobility function, ν is a outward pointing unit normal on the boundary $\partial\Omega$, ∇ is the Nabla operator and Δ is the Laplace operator. It is well known that the Cahn-Hilliard equation is a gradient flow in H^{-1} with Lyapunov energy functional $\mathcal{F}$.

For a regular uniform alloy, the free energy f is explicitly given by :

$$f : u \mapsto N_m k_B T_c \frac{1 - u^2}{2} + N_m k_B T \left[\frac{1+u}{2} \ln\left(\frac{1+u}{2} \right) + \frac{1-u}{2} \ln\left(\frac{1-u}{2} \right) \right], \tag{3.0.7}$$

where k_B is the Boltzmann constant, N_m is a molecular density, T is the temperature and $T_c > T$ is the critical temperature. Thus the nonlinear term ψ is:

$$\psi := f' : u \mapsto -N_m k_B T_c u + \frac{N_m k_B T}{2} \ln\left(\frac{1+u}{1-u} \right), \tag{3.0.8}$$

which is singular at $u = \pm 1$. These singularities are the first difficulty in a numerical study, so this function ψ is often replaced by the derivative of the classic quartic double-well potential, where f takes the following form:

$$f : u \mapsto \frac{1}{4} \left(1 - u^2 \right)^2, \tag{3.0.9}$$

with derivative:

$$\psi : u \mapsto u^3 - u. \tag{3.0.10}$$

The Cahn-Hilliard equation has been extensively studied in this case where ψ is replaced by a polynomial function (see [15], [49] and [60]). Furthermore, this model has been used successfully for describing phase separation phenomena, see for example the survey [58], and the references therein, or other recent results on spinodal decomposition and nucleation in [5, 10, 40, 53, 54, 66, 67, 72]. Recently, Ma and Wang have studied the stationary solutions of the Cahn-Hiliard equation (see [50]). The case of non smooth ψ has been the object of much less research (see [11] and [27]).

Other frequent simplifications are often made. The mobility $\mathcal{M}$ is often taken to be constant and the physical parameters are set to 1 - as we have done above in (3.0.9). For a more physically relevant choice of mobility, we mention [71] where the following form is proposed $\mathcal{M}(u) = \max\{0, 1 - u^2\}$. Among the physical parameters, ε has a peculiar role since it may lead to different asymptotic behaviours and equilibria (see [52] and section 3.3). The study of evolution with $\varepsilon \to 0$ is of great importance: in particular a constant mobility leads to a Mullins-Sekerka evolution (nonlocal coupling) whereas a degenerate mobility leads to a purely local geometric motion (see [4]). Furthermore, when the interface thickness is on the order of a nanometer, an artificially large parameter ε is often used to regularize the numerical problem. When a fine resolution is out of reach, a change in the height of the barrier between wells in the free energy density, coupled with a change on ε, allows simulations with larger length scales (see[70] for details).

The evolution of the solution of (3.0.6) can essentially be split into two stages. The first one is the *spinodal decomposition* described in section 3.2 where the two species quickly separate from each other. In longer time, the evolution is slower, and the solution tends to reduce its interfacial energy. These two evolutions require different methods for a global efficient simulation. In the beginning, a very small time step and a precise grid resolution allow efficient computation. But this is not appropriate to get long-time behaviors. So an adaptative time stepping or/and an adaptative mesh can improve the efficiency of the algorithms. However, in the long-time evolution, the interfaces have to be precisely captured so that a global adaptative mesh cannot be used. In the literature, all these technical ideas have been studied: adaptive refinement of the time-stepping or of the mesh, $\mathcal{C}^1$ elements (see [70]), multigrid resolution (see [47]).

We propose here an alternative method using high degree $\mathcal{C}^0$ lagrangian finite elements under a constant mobility $\mathcal{M} \equiv 1$. The use of p-version (increasing polynomial degree, see [2]) instead of h-version (decrease mesh-step) has proved to be efficient for propagation [1, 45, 46], corner singularities [69], or oscillating problems [12]. The numerical results obtained here with the finite element library MÉLINA [55] show that this method is suitable in the Cahn-Hilliard framework as well.

Our paper is organized as follows: in section 3.1, we shortly describe the discretization (in both time and space) including the nonlinear solver and the high degree finite elements we used. Section 3.2 and section 3.3 are respectively devoted to the numerical results for the one-dimensional and the two dimensional problem. We investigate the performance of our method through different quantitative and qualitative aspects of the Cahn-Hilliard equation: comparison to explicit profile-solution in 1D (see section 3.2), spinodal decomposition (see section 3.2), discussion about polynomial approximations of the logarithmic potential (see section 3.2), impact of the temperature and the parameter ε (see section 3.2 and 3.3), long-time behavior and asymptotic stable states (see section 3.3). The numerical results are compared with existing ones in the literature, validating our approach.

We also consider the Cahn-Hilliard-Cook equation. This equation is obtained by adding an additive white noise on the right hand side of the equation. This noise term accounts for thermal effects. We investigate its effect on the spinodal decomposition and on the longtime behaviour of the solutions. We also take the opportunity to present simulations on a one dimensional equation where no nonlinear term is considered but a reflection term is added to force the solution to remain positive. This equation has been studied theoretically in [29] and models interfaces problems (see [77] Zambotti 41, funaki-olla, spohn).

3.1 Discretization

3.1.1 Space-Time schemes

We start with the description of the time discretization. Given a large integer N, a time step τ, and an initial data (w_0, u_0), we denote by $(w_n, u_n)_{n\leq N}$ the sequence of approximations at uniformly spaced times $t_n = n\tau$. The backward Euler scheme is given by:

$$\begin{cases} \frac{u_{n+1}-u_n}{\tau} = \Delta w_{n+1}, \\ \\ w_{n+1} = \psi(u_{n+1}) - \varepsilon^2 \Delta u_{n+1}. \end{cases} \tag{3.1.1}$$

A Crank-Nicolson scheme could easily be implemented but our experience is gives similar results as the one described hereafter. In section 3.5, we use variable time step to compute the reflection term, details are given in this section. The schemes are immediately generalized to our case. We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathrm{L}^2(\Omega)$. We use the standard Sobolev space $\mathrm{H}^1(\Omega)$. It is endowed with the seminorm

$$|h|_1 = \|\nabla h\|_{\mathrm{L}^2},$$

and with the norm

$$\|h\|_1 = \left(|h|_1^2 + \|h\|_{\mathrm{L}^2}^2\right)^{1/2}.$$

The weak form of the equation (3.1.1) reads:

$$\begin{cases} \langle u_{n+1} - u_n, \chi \rangle = -\tau \langle \mathcal{M}(u_{n+1}) \nabla w_{n+1}, \nabla \chi \rangle, \text{ for all } \chi \in X_1, \\ \\ \langle w_{n+1}, \xi \rangle = \langle \psi(u_{n+1}), \xi \rangle + \langle \varepsilon^2 \nabla u_{n+1}, \nabla \xi \rangle, \text{ for all } \xi \in X_2, \end{cases} \tag{3.1.2}$$

where X_1 and X_2 are the spaces of test functions ($\mathrm{H}^1(\Omega)$ for example). We discretize in space by continuous finite elements. Given a polygonal domain Ω, for a small parameter $h > 0$, we partition Ω into a set $\mathcal{T}^h$ of disjoint open elements K such that $h = \max_{K\in\mathcal{T}^h}(\mathrm{diam}(K))$ and $\bigcup_{K\in\mathcal{T}^h} \overline{K} = \overline{\Omega}$. Thus, we define the finite element space

$$V^h = \left\{ \chi \in \mathcal{C}(\bar{\Omega}) : \chi\big|_K \in \mathbb{P} \text{ for all } K \in \mathcal{T}^h \right\}, \tag{3.1.3}$$

where $\mathbb{P}$ is a space of polynomial functions, see section 3.1.3. We denote by $(\varphi_j)_{j\in J}$ the standard basis of nodal functions. Thus, for u and $v \in \mathcal{C}(\overline{\Omega})$, we define the *lumped scalar product* by:

$$\langle u, v \rangle^h := \sum_{i,j} \langle u, \varphi_i \rangle \langle v, \varphi_j \rangle \langle \varphi_i, \varphi_j \rangle.$$

The scheme (3.1.2) can be rewritten in the fully discrete form, just by replacing the continuous scalar product with the lumped scalar product.

We denote $\mathbf{u} = (u_j)_{j\in J}$ and $\mathbf{w} = (w_j)_{j\in J}$, the finite dimensional representation of u and w (we omit here the subscript n of the time scheme). Then we define the matrices $\mathbf{A}$ and $\mathbf{M}$, whose coefficients are given by the following relations:

$$\begin{aligned} [\mathbf{A}]_{ij} &:= \langle \nabla\varphi_i, \nabla\varphi_j \rangle, \quad \text{“stiffness” matrix, for all } i, j \in J \\ [\mathbf{M}]_{ij} &:= \langle \varphi_i, \varphi_j \rangle, \quad \text{“mass” matrix, for all } i, j \in J \end{aligned}$$

For each time-step, given a previous solution $(\mathbf{w}_n, \mathbf{u}_n)$, $(\mathbf{w}_{n+1}, \mathbf{u}_{n+1})$ is solution of the system

$$\begin{cases} \tau\mathbf{A}\mathbf{w}_{n+1} & +\mathbf{M}\mathbf{u}_{n+1} & = & \mathbf{M}\mathbf{u}_n, \\ \\ \mathbf{M}\mathbf{w}_{n+1} & -\varepsilon^2\mathbf{A}\mathbf{u}_{n+1} - \mathbf{M}\mathbf{\Psi}(\mathbf{u}_{n+1}) & = & 0, \end{cases} \tag{3.1.4}$$

where $\mathbf{\Psi}$ is a pointwise operator (*related to* ψ), and with $(\mathbf{w}_0, \mathbf{u}_0)$ the finite dimensional representation of the initial data. The system (3.1.4) is clearly block-symmetric. The proof of the convergence of this scheme can be found in [3].

3.1.2 Nonlinear solver

At each time step, we use a Newton procedure to solve the implicit nonlinear system (3.1.4) . For (3.1.4), we define the operator $\mathbf{L}$ by:

$$\mathbf{L} = \begin{pmatrix} \tau\mathbf{A} & \mathbf{M} \\ \mathbf{M} & -\varepsilon^2\mathbf{A} \end{pmatrix}.$$

Then denote by $\mathbf{S}$ the matrix of the left hand side of the backward Euler scheme,

$$\mathbf{S} = \begin{pmatrix} 0 & \mathbf{M} \\ 0 & 0 \end{pmatrix}.$$

Denote also by $\mathbf{G}$ the following operator:

$$G(\mathbf{w}, \mathbf{u}) := \begin{pmatrix} 0 \\ -\mathbf{M}\mathbf{\Psi}(\mathbf{u}) \end{pmatrix}.$$

Finally denote by $\mathbf{Y}_n$ the couple $(\mathbf{w}_n, \mathbf{u}_n)$ for each $n \leq N$. The backward Euler scheme at each time-step satisfies the following formula:

$$\mathbf{L}\mathbf{Y}_{n+1} + \mathbf{G}(\mathbf{Y}_{n+1}) - \mathbf{S}\mathbf{Y}_n = 0. \tag{3.1.5}$$

Denoting by $(\mathbf{Y}_n^k := (\mathbf{w}_n^k, \mathbf{u}_n^k))_{k\in\mathbb{N}}$ the iterations of Newton procedure for each $n \leq N$, they satisfy

$$\begin{cases} \mathbf{Y}_n^0 &= \mathbf{Y}_n, \\ \mathbf{Y}_n^{k+1} &= \mathbf{Y}_n^k - \left[\mathbf{L} + D_{\mathbf{G}}\left(\mathbf{Y}_n^k\right)\right]^{-1}\left[\left(\mathbf{L} + \mathbf{G} - \mathbf{S}\right)\left(\mathbf{Y}_n^k\right)\right], \text{ for all } k \in \mathbb{N}, \end{cases} \tag{3.1.6}$$

where $D_{\mathbf{G}}\left(\mathbf{Y}_n^k\right)$ is the differential of $\mathbf{G}$ at point $\mathbf{Y}_n^k$. Actually, we stop the procedure at $k = k_n$ when the residual is small, and define $\mathbf{Y}_{n+1} := \mathbf{Y}_n^{k_n}$. System (3.1.6) is an implicit linear system for each Newton-step, handled with a biconjugate gradient method.
When the nonlinear term is logarithmic, we should deal with the singularities at ± 1. However, in all our computations, the solution stays far from ± 1 so that no special care is needed. This is expected. Indeed, it is known that in the one dimensional case the solution satisfies an L^∞ bound which is strictly less than one (see [20]). The same result has not been proved in higher dimension but it is probably true. This is not the case when a noise is taken into account and we shall present our strategy to treat the singularities in section 3.5.
A simple remark shows the mass conservation through the total scheme. Indeed, if we multiply the first component of the second equation in the system (3.1.6) by the vector $\mathbb{I} := (1, 1, ..., 1)$ which belongs to V^h, we get for all $k \in \mathbb{N}$:

$$\mathbb{I}\,\mathbf{M}\mathbf{u}_n^k = \mathbb{I}\,\mathbf{M}\mathbf{u}_n.$$

3.1.3 Implementation with high degree finite elements

The finite element library MÉLINA [55] has the feature of providing lagrangian nodal elements with order up to 64 (the nodes may be chosen as the Gauss-Lobatto points to avoid Runge phenomenon for large degrees). It can thus be used as a p-version code – see [2] – or even to implement spectral methods – see [6]. In the following results, we use quadrangular elements for two-dimensional computations, with degree from 1 to 10. So we use the notation Q_i with $i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ to describe these elements. We justify this strategy by the fact that the expected solution is smooth but may present a thin interface ; since high degree polynomials are able to capture high frequencies they are well suited in such situations. Some comparisons are shown below between degree 1 with elements on a refined mesh, and degree 10 on a coarse mesh, justifying the efficiency of the method (in both terms of accuracy and computational cost).

3.2 Cahn-Hilliard evolution. Polynomial approximation of the logarithm

The temperature plays a crucial role in the evolution of the solution. The function ψ defined in (3.0.8) depends on two temperatures T and T_c. When the temperature T is greater than the critical temperature T_c, the second derivative of ψ is nonnegative, so the function ψ is convex and has only one minimum. We say that the function ψ has a single well profile. Thus the solution tends to this unique minimum and the alloy exists in a single homogeneous state.

But when the temperature T of the alloy is lowered under the critical temperature T_c, the function ψ changes from a single well to a double well (see figure 3.1), and the solution rapidly separates into two phases of nearly homogeneous concentration. This phenomenon is referred to as *spinodal decomposition*. If the initial concentration belongs to the region where the energy density is concave, i.e. between the two *spinodal points* σ_- and σ_+ (see figure 3.1), the homogeneous state becomes unstable.

The concentrations of the two regions composing the mixture after a short stabilization have value near the so called *binodal points* β_- and β_+ (see also figure 3.1), defined by

$$f'(\beta_-) = f'(\beta_+) = \frac{f(\beta_+) - f(\beta_-)}{\beta_+ - \beta_-}, \quad \text{with } \beta_- < \beta_+. \tag{3.2.1}$$

If the free energy is symmetric, the binodal points are the minima of each well, but in a more general case they are on a double tangent line (see [70]).

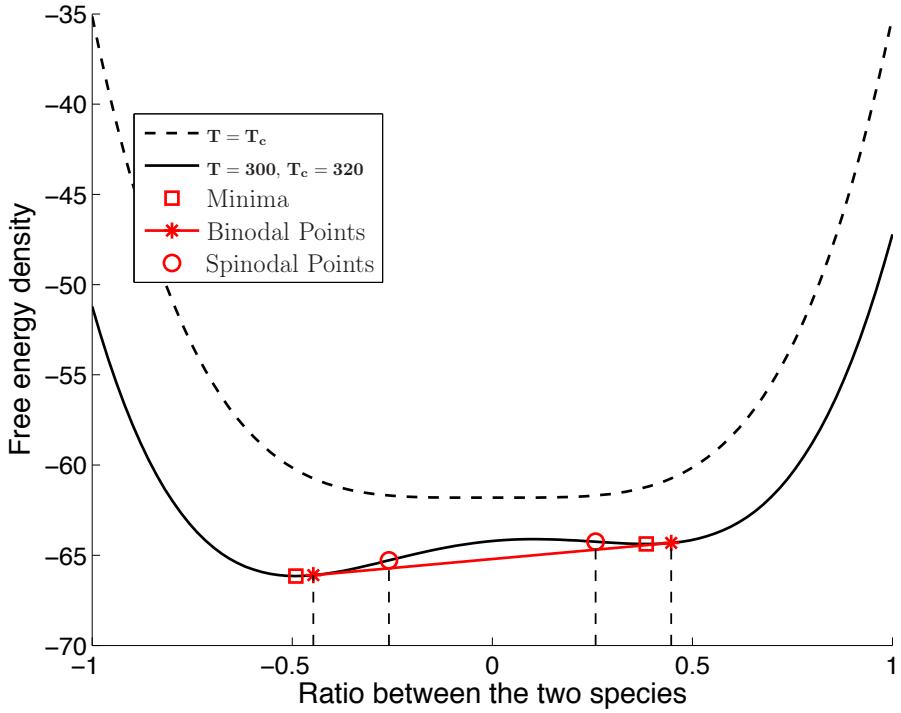
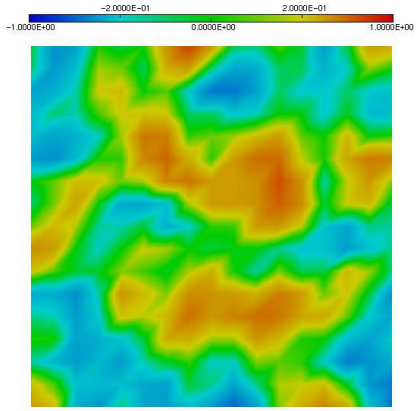


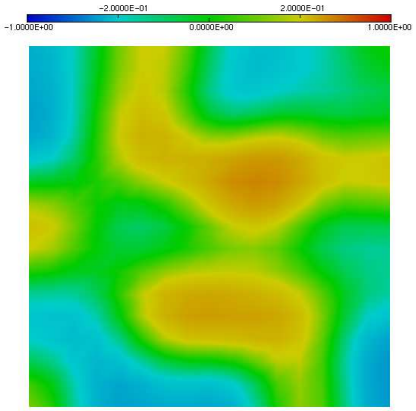
Figure 3.1: Free energy density for two different temperatures.

The spinodal decomposition is represented in the first two graphs of figure 3.2 or figure 3.3.

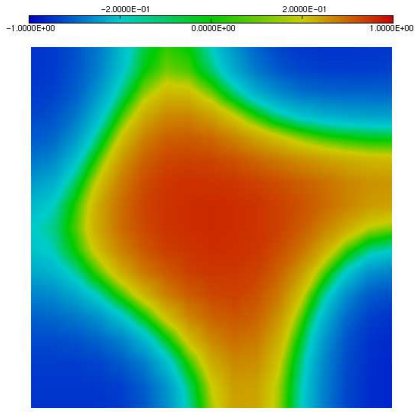
In longer time, the separated regions evolve to reduce their interfacial energies. These diffuse interfaces are shortened in an effect resembling surface tension on a sharp interface, as the material fronts move to reduce their own curvature (see [14] and [64]). Finally, the solution reaches an equilibrium whose location and form depend on total initial concentration (see [52]). Nevertheless this equilibrium is always a solution with an interface with minimal measure. On figures 3.2 and 3.3, this phenomenon is observed on the last four graphs.



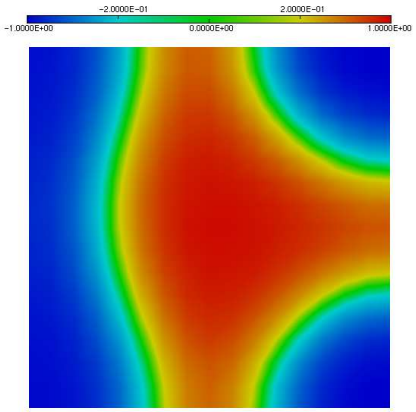
(a) t=0



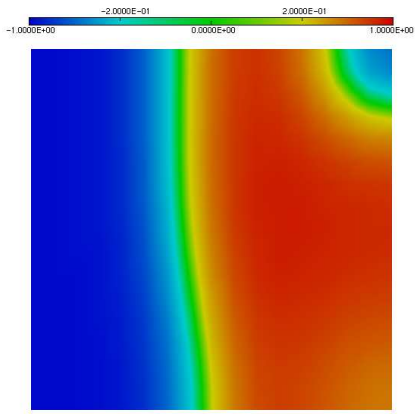
(b) t=0.01



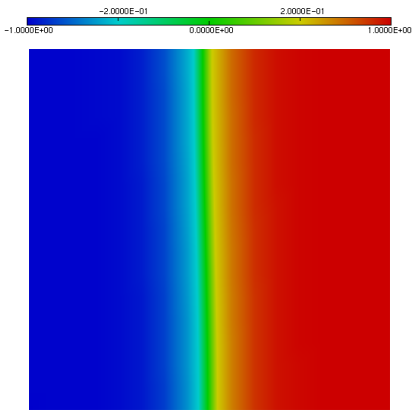
(c) t=0.05



(d) t=0.2

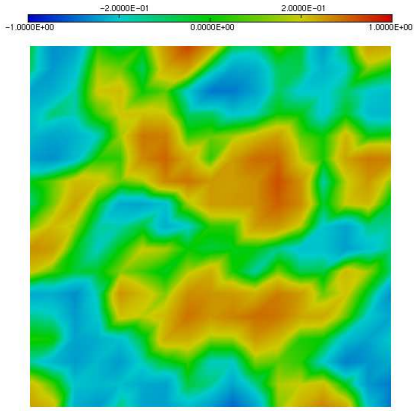


(e) t=0.6

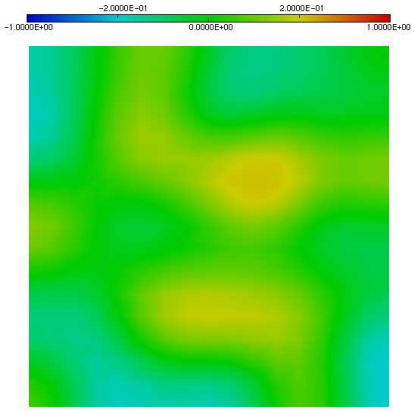


(f) t=1

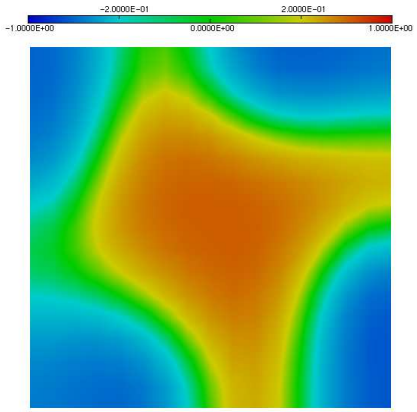
Figure 3.2: Spinodal decomposition under the classic quartic double-well potential.



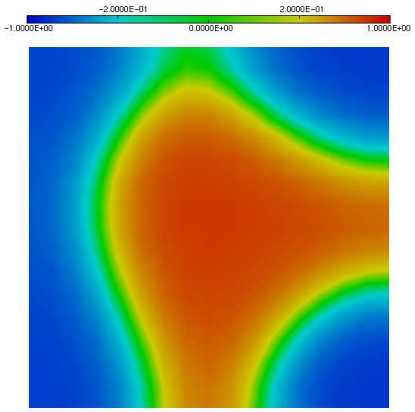
(a) t=0



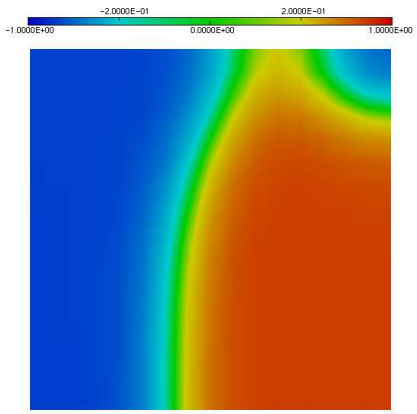
(b) t=0.01



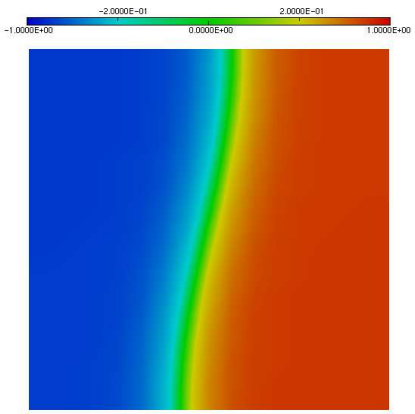
(c) t=0.05



(d) t=0.2



(e) t=0.6



(f) t=1

Figure 3.3: Spinodal decomposition under a logarithmic potential.

Figure 3.2 corresponds to an evolution under the classic quartic double-well potential (3.0.9) with non scaled coefficients, whereas figure 3.3 corresponds to an evolution under the logarithmic potential (3.0.7). They are both simulated on a mesh 12×12 under Q_1 polynomial elements. The ε parameter is such that $\varepsilon^2 = 0.07$. We see that the evolutions are quite similar and lead to the same stationnary state. On these two evolutions, we can compare the difference of the energies or the L^2 norm of the difference (see next paragraph). Note that the polynomial approximation of the logarithm does not change the qualitative behaviour. The same patterns appear and the longtime behaviour is very similar. The only notable difference is that with the logarithmic nonlinearity, the dynamic is slower. This is particularly clear on the graphs (b). The spinodal decomposition is almost finished only for the polynomial. Similarly, on graphs (f), we see that at time $t = 1$, the logarithmic evolution has not reached equilibrium yet. We have observed this in all our tests.

The second evolution is often illustrated by the classical benchmark cross. It can be considered as a qualitative validation of the numerical methods. This long time behavior is illustrated in figure 3.4. Starting from a cross-shaped initial condition, the interface first diffuses from the arbitrary width of the initial condition to the equilibrium interface width. Next, the solution tries to reduce its interfacial energy and tends to a circular form. In the total free energy (3.0.3), the term with the free energy function f is responsible to the spinodal decomposition, whereas the gradient term is responsible to the interfacial reduction. This phenomenon has been simulated on a mesh 256×256 under Q_3 polynomial elements. Figures 3.4 (a), (b) are obtained with the quartic nonlinearity. We see on figure 3.4 (c) and (d) that again the qualitative behavior is very similar with the logarithm.

It is difficult to measure precisely the qualitative difference between the two evolutions. The only physical quantity which can be measured in dimension two is the energy. A detailed study of this aspect is performed below. Also, in dimension one, we are able to measure the interface. We will see that the quartic nonlinearity tends to thicken the interface.

The replacement of the logarithmic free energy by the quartic one has been done by many author in order to avoid numerical and theoretical difficulties raised by the singular values ± 1. More generally, we can discuss the approximation of the logarithm by polynomial functions. We consider the $2n$-th order polynomial Taylor expansion f_{2n}:

$$f_{2n} := u \mapsto \left(T_c \left(\frac{1-u^2}{2} \right) + T \left[-\ln(2) + \sum_{p=1}^{n} \frac{u^{2p}}{2p(2p-1)} \right] \right) + K_{2n}. \tag{3.2.2}$$

It is defined up to an additive constant K_{2n}. The constant K_{2n} is apparently arbitrary. However, it is preferable to choose it in order that the energy of a solution u

$$\mathcal{F}_{2n}(u) := \int_\Omega \left(f_{2n}(u) + \kappa |\nabla u|^2 \right) \mathrm{d}V \tag{3.2.3}$$

is well defined on unbounded domains. Since it is expected that the solution converges to one of the binodal values, it is natural to choose K_{2n} so that f_{2n} vanishes at those points. We always consider this choice.

We have seen above that the quartic approximation does not seem to change drastically the qualitative behaviour, except that the evolution is faster. We now perform a quantitative study to measure more precisely the effect of the polynomial approximation.

The spinodal and binodal points are drawn on figure 3.5 for various n. When n increases, the spinodal and binodal points converge to the corresponding values for the logarithmic potential. However, the convergence is rather slow (see figures 3.5 and 3.6).

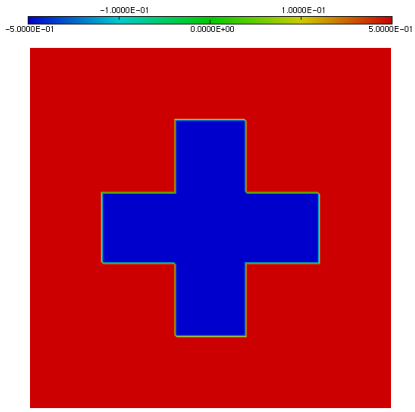
In the one dimensional case, it is possible to study the interface width.

Let us consider the domain $\Omega = \mathbb{R}$ and the quartic potential

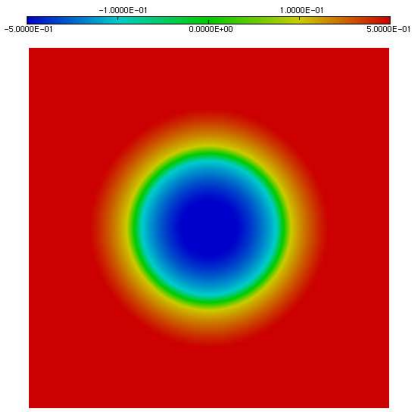
$$\psi_4 := u \mapsto -T_c u + T \left(u + \frac{u^3}{3} \right), \tag{3.2.4}$$

which is the derivative of

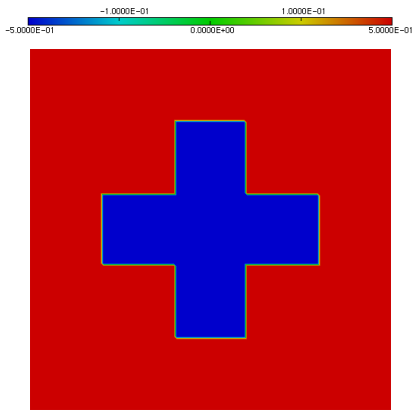
$$f_4 := u \mapsto T_c \left(\frac{1-u^2}{2} \right) + T \left[\frac{u^2}{2} + \frac{u^4}{12} \right] + K_4. \tag{3.2.5}$$



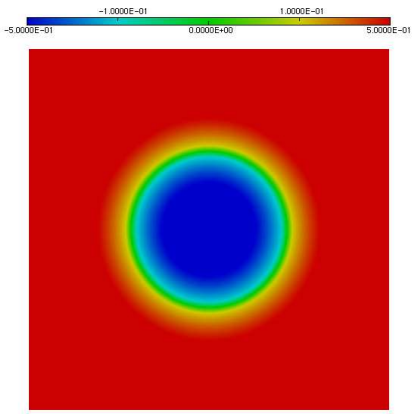
(a) Quartic potential - t=0



(b) t=1



(c) Logarithmic potential - t=0



(d) t=1

Figure 3.4: Evolution of a cross-shaped initial condition to a bubble.

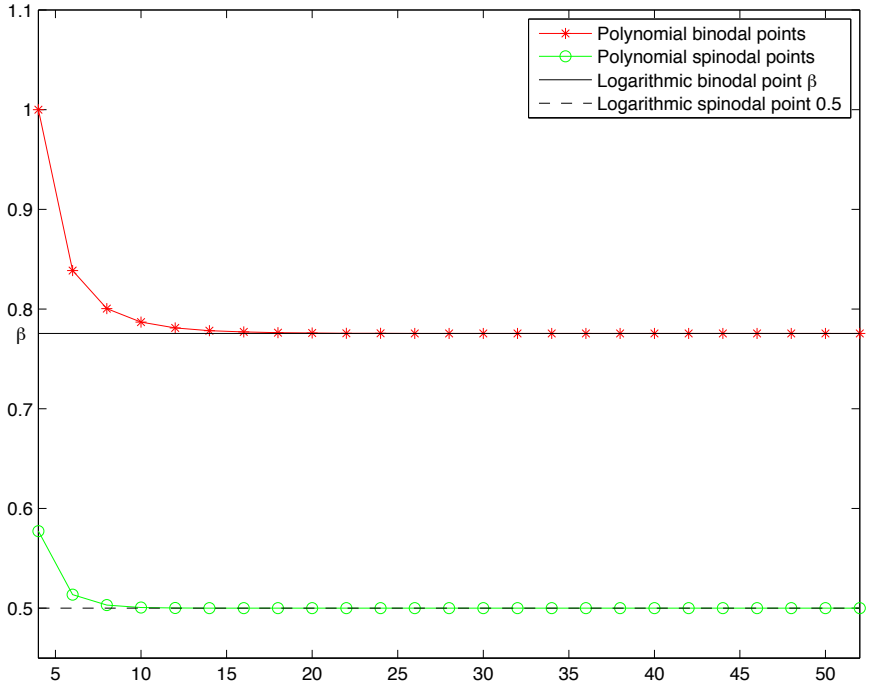


Figure 3.5: Polynomial and logarithmic spinodal and binodal points.

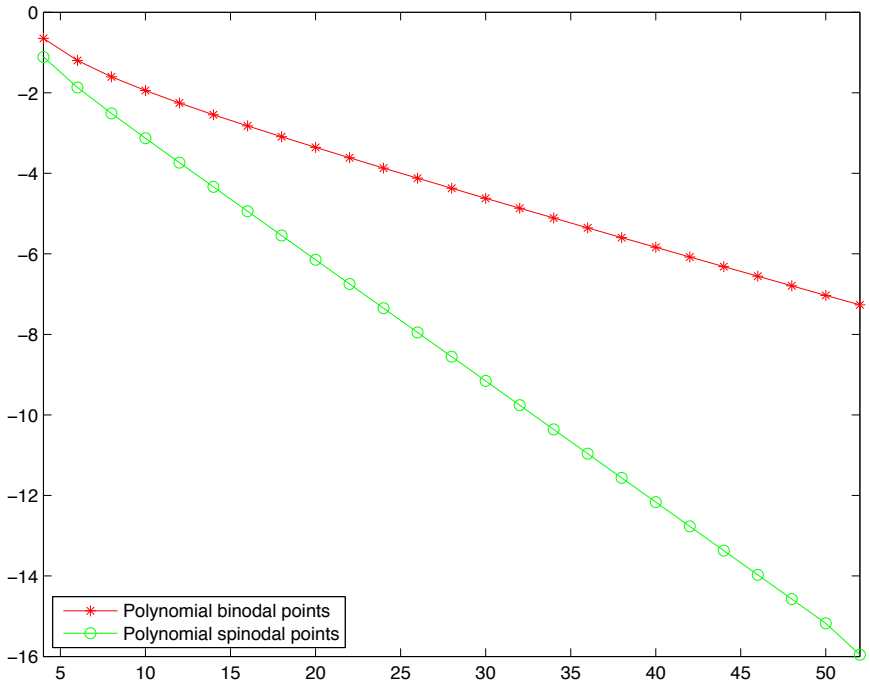


Figure 3.6: Speed rate of convergence of the polynomial points.

Then a stationary solution of the Cahn-Hilliard equation (3.0.6) can be explicitly computed (under a constant mobility $\mathcal{M}(u) \equiv 1$), see [32]:

$$u : x \mapsto u_+ \tanh(x\mu), \tag{3.2.6}$$

where

$$u_+ = \sqrt{3\left(\frac{T_c}{T} - 1\right)} \quad \text{and} \quad \mu = \frac{\sqrt{T_c - T}}{\varepsilon\sqrt{2}}. \tag{3.2.7}$$

Let us emphasize the fact that the solution is well constrained in $[-u_+, u_+]$. We can define a characteristic length ℓ (see figure 3.7), corresponding to the width of the region containing the main variations of a solution u :

$$\ell := \frac{|\lim_{x\to+\infty} u(x)| + |\lim_{x\to-\infty} u(x)|}{\text{Slope in interface point}},$$

where the interface point is the point x_0 where $u(x_0) = 0$. Thus we can compute explicitly this length and obtain:

$$\ell = \frac{2u_+}{u_4'(0)} = \frac{2\varepsilon\sqrt{2}}{\sqrt{T_c - T}}. \tag{3.2.8}$$

Cahn and Hilliard have defined a parameter $\lambda := \frac{2\varepsilon\sqrt{2}}{\sqrt{T_c}}$ in order to characterize the interface length. With this parameter λ we obtain the following expression for ℓ:

$$\ell = \frac{\lambda}{\sqrt{1 - \frac{T}{T_c}}}. \tag{3.2.9}$$

Cahn and Hilliard have shown that in the case of the logarithmic free density the interface length is of the same order. This suggests that the quartic double well approximation preserves important features of the solution.

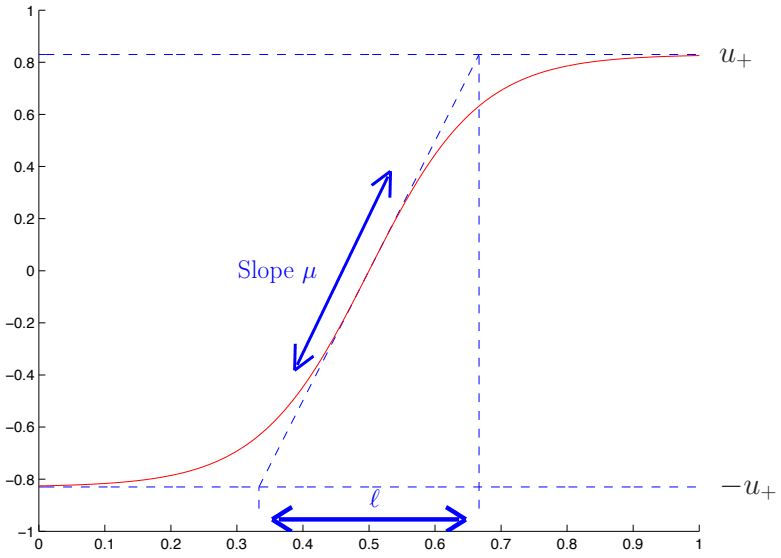


Figure 3.7: Interface length for the solution u_4.

On figure 3.8, we present the numerical solution for $\Omega = [0, 1]$ (blue stars), and the "tanh-profile" whose coefficients u_+ and μ have been fitted to the data. The fitting on u_+ corresponds to the value of the solution on the boundaries of the domain Ω. And the fitting on μ corresponds to a least square method between the numerical solution and a "tanh-profile" solution interpolated on the same meshes. The "tanh-profile" (defined over $\mathbb{R}$) may be considered as a good approximation of the solution on $\Omega = [0, 1]$ since the interface is very thin. The numerical solution is computed over a mesh grid 35 elements with elements Q_3.

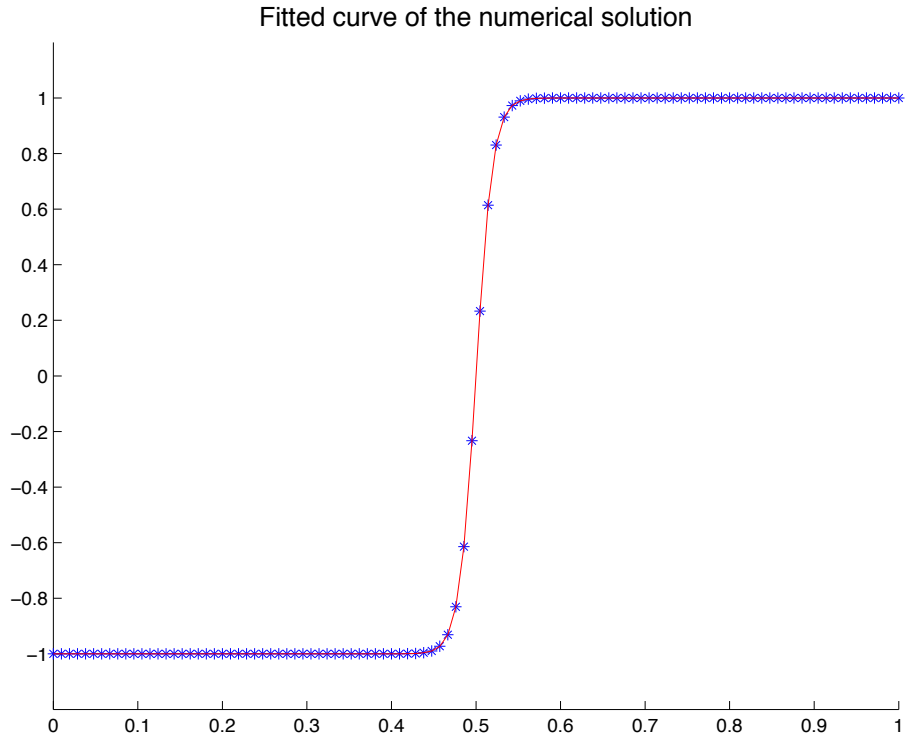


Figure 3.8: Fitted curve on the "tanh-profile".

However, we have measured numerically the interface width in the quartic and logarithmic cases. This width is plotted for various ε on figure 3.9. We see that as expected by the formula (3.2.8), it varies linearly with ε. But, for ε not too small, the interface width is thinner for the logarithmic equation. The quartic approximation introduces a non negligible extra diffusivity.

We can also compare the total free energies. Denote by u the solution of a simulation with the *logarithmic* function f and by $(u_{2n})_{n\geq 2}$ the family of solutions of the simulations with the polynomial functions $(f_{2n})_{n\geq 2}$. For the energy, we take as reference the logarithmic total free energy, and we study

$$|\mathcal{F}(u_{2n}) - \mathcal{F}(u)|. \tag{3.2.10}$$

On figure 3.10, the evolution of the logarithm of this quantity is plotted during a classical spinodal decomposition in dimension one.

We can see important peaks at the begining and smoother peaks between iterations 500 and 700. These peaks appear when the solution has a rapid evolution and when its topological form changes. For instance, these peaks correspond to the changes beetween the fourth and the fifth images of figure 3.2, and between the fifth and the sixth images. After the iteration 750, all the solutions are in an asymptotic stable state, and the energies do not change anymore.

For a quartic potential ($n = 2$ i.e. f_4 on figure 3.10), the energy error is significant and the polynomial approximation is not good in that respect.

We could also have shown the evolution of

$$|\mathcal{F}_{2n}(u_{2n}) - \mathcal{F}(u)|$$

In fact, it is very similar and does not bring new information.

On a mathematical point of view, it is interesting to study the error in L^2 norm:

$$\left(\int_\Omega |u_{2n} - u|^2 dx\right)^{1/2}.$$

We see on figure 3.11 that for $n = 2$, the error is important. It decreases with n but is still significant for $n = 3$. For $n \geq 6$, it is negligible.

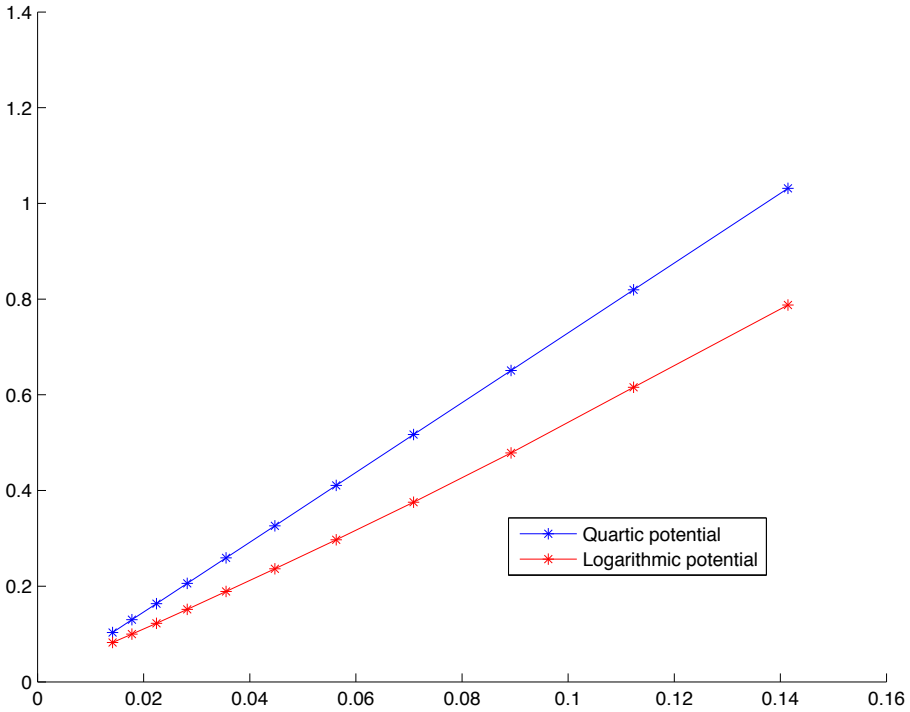


Figure 3.9: Length of the interface for the quartic and logarithmic potentials.

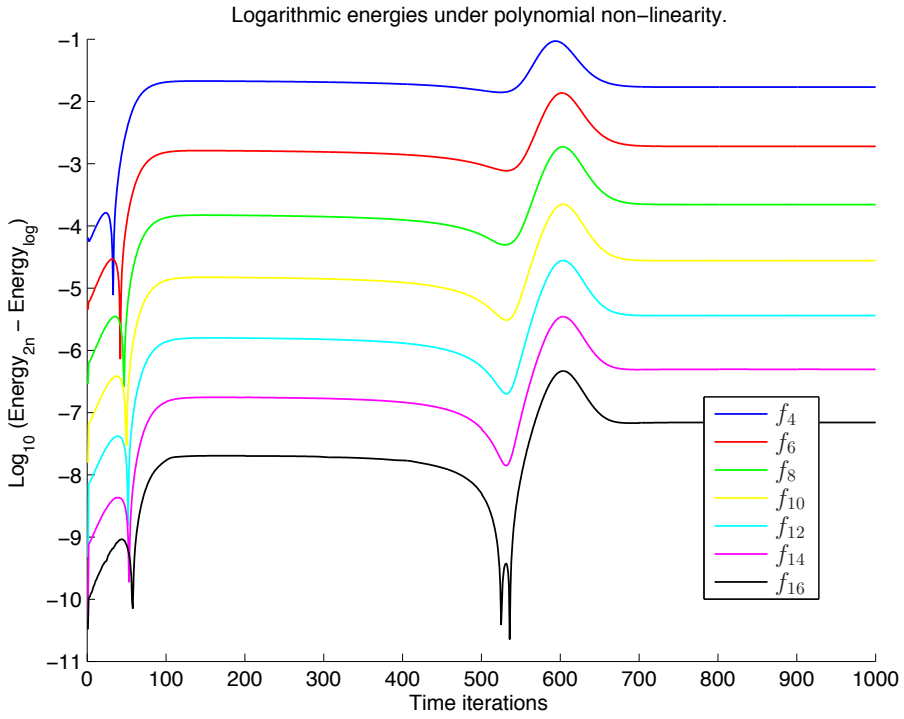


Figure 3.10: Polynomial energies versus logarithmic energy.

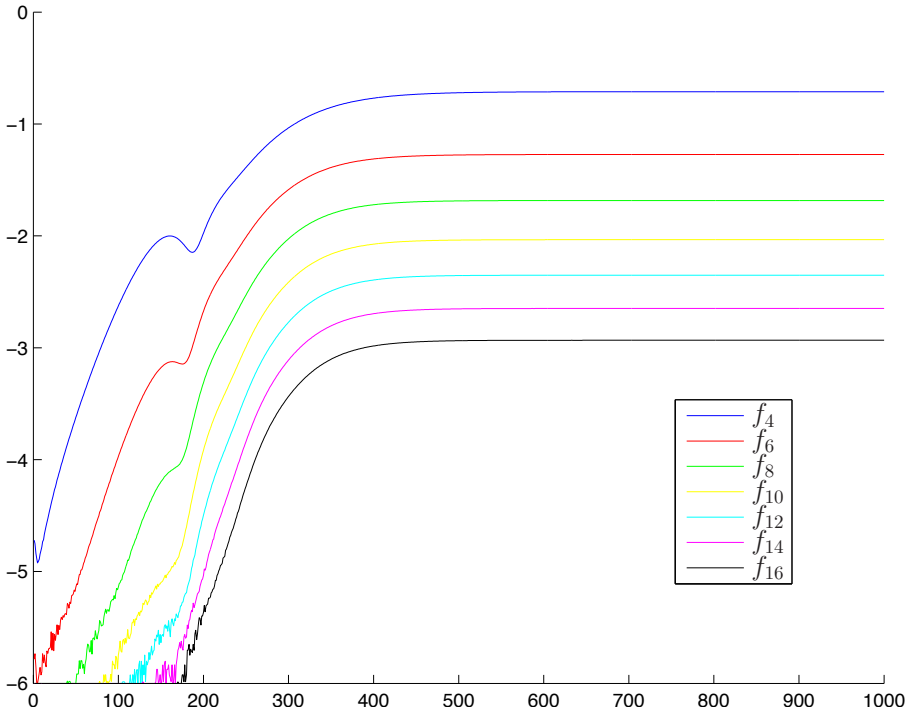


Figure 3.11: L^2 errors between the polynomial solutions and the logarithmic solution.

Figures 3.12 and 3.13 presents the same quantities for a two dimensional spinodal decomposition. We observe the same quantitative difference. Note that we clearly see on the energy evolution that when the degree n gets bigger the evolution slows down.

We conclude that the classical quartic approximation of the free energy may be considered as a good approximation for qualitative behaviour but it produces a significant error and accelerates the dynamic. If precision is sought, one should consider an approximation with a higher order polynomial.

3.3 Validation of the numerical method. Choice of the degree of the elements.

On the figure 3.14, we have drawn a numerical solution for different times. It is a Q_1 solution on a mesh with 100 elements under the quartic double-well potential. On the figure 3.14(f), the solution has reached its stable state and has binodal values ± 1 on the boundary.

In our first set of tests, we start with the same initial state near the "tanh profile" solution. The evolutions are driven by the quartic potential function. We wait for the stabilization of all the solutions and study the error on the energies and on the slopes of the interface.

Remark that we can explicitly compute the energy of the explicit solution. And since the energy of a numerical simulation is decreasing in time, this energy should converge to the energy of the explicit solution. The figure 3.15 shows the evolutions of the errors between the numerical energies and the explicit energy according to the degree of the polynomial space $\mathbb{P}$. Before the 800th iteration in time, the solutions are not stable. They try to minimize their energies. After the 800th iteration, all the solutions are in a stable state. We can see that the evolutions are qualitatively similar at the beginning, but the elements Q_1, Q_2 and Q_3 don't achieve the tolerance zone, whereas the other elements do. However, Q_2 and Q_3 give a very good result.

The slope of the interface is an essential physical quantity. So we have compared the errors on the slopes between the numerical solutions and the theoretical solution. Note that these slopes correspond to the values of the derivatives of the numerical solutions at the interface and our finite

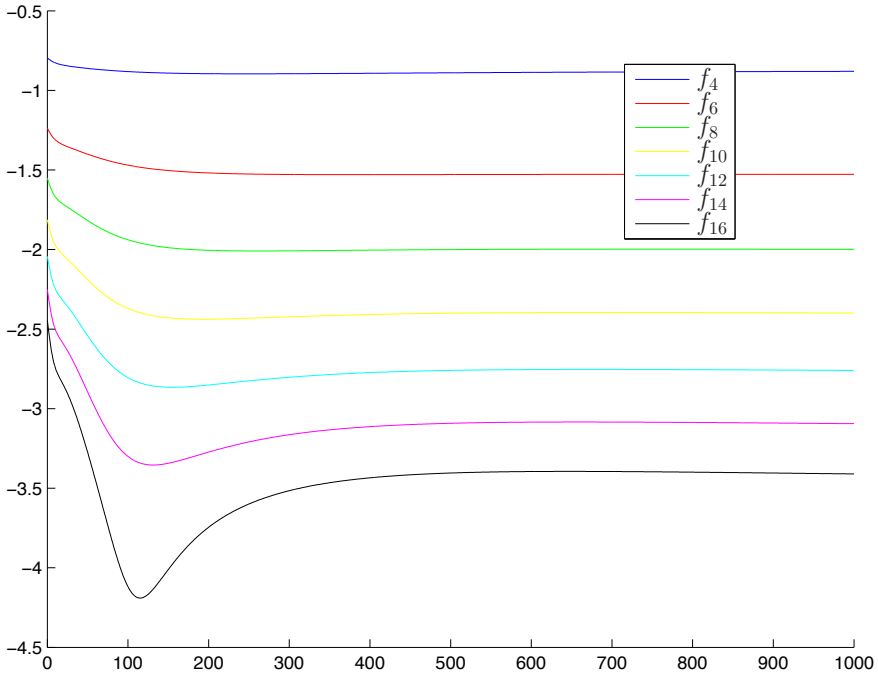


Figure 3.12: Polynomial energies versus logarithmic energy.

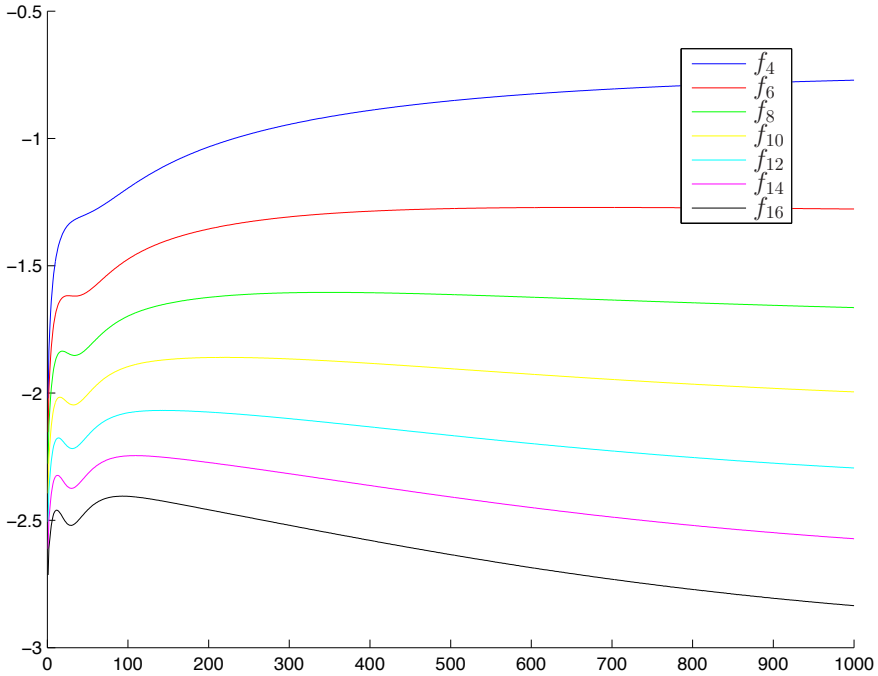
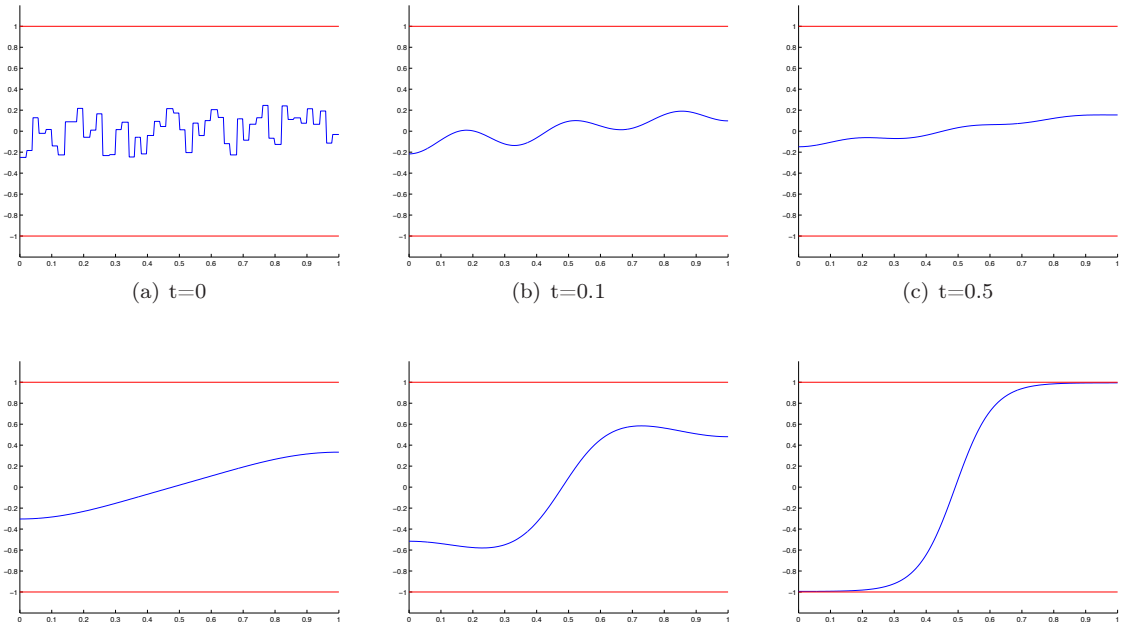


Figure 3.13: L^2 errors between the polynomial solutions and the logarithmic solution.



(a) t=0 (b) t=0.1 (c) t=0.5

(d) t=10 (e) t=20 (f) t=50

Figure 3.14: Q_1 solution on a mesh with 100 elements.

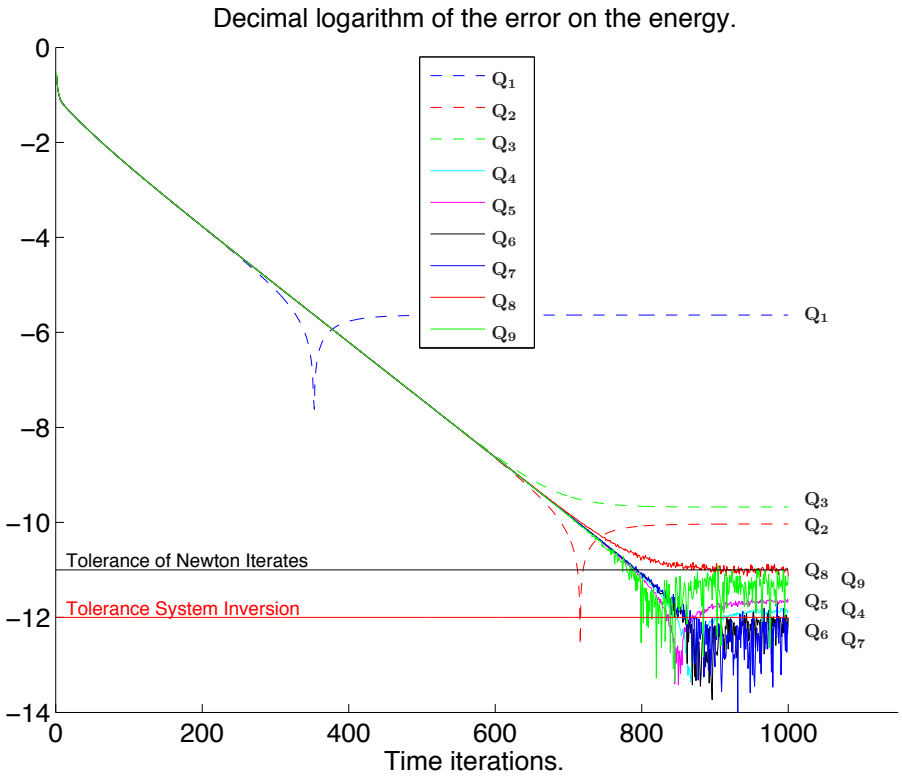
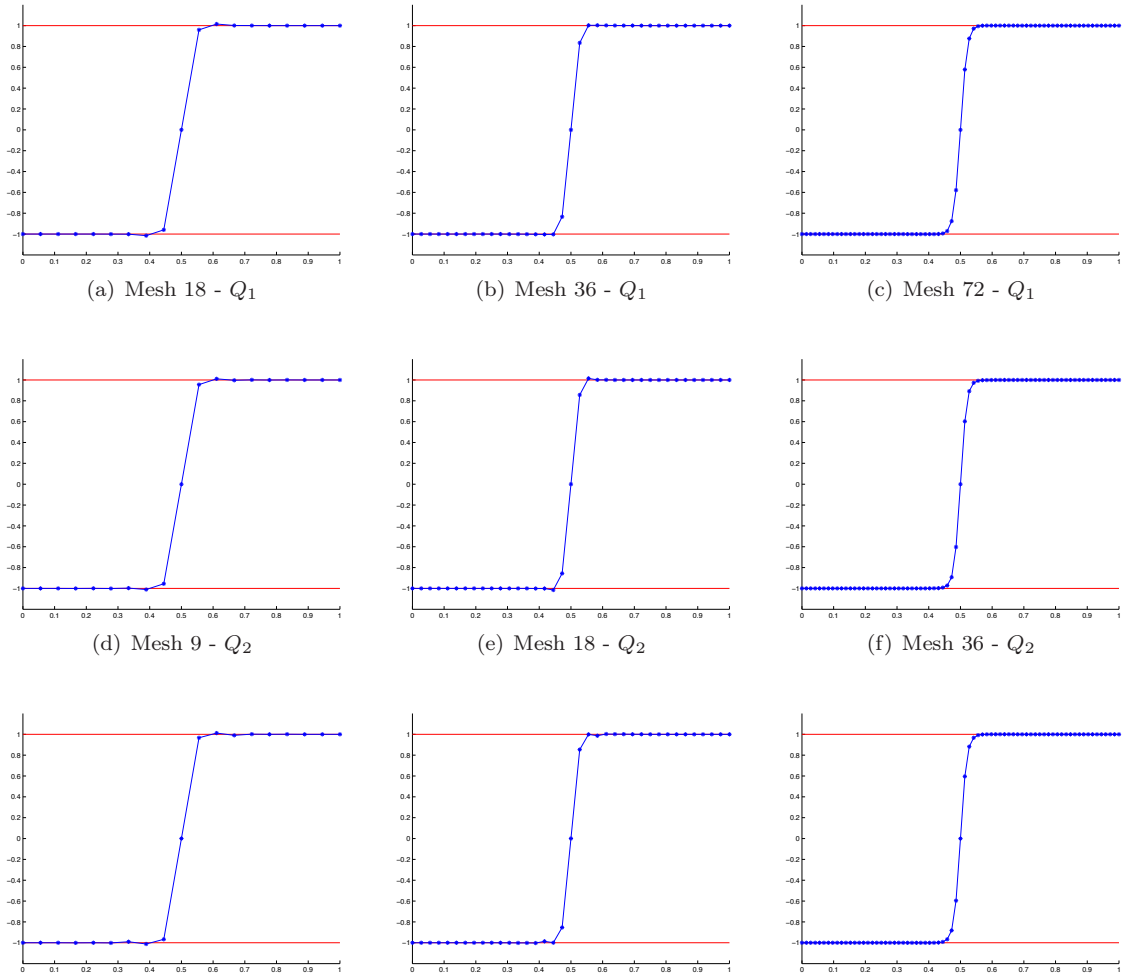


Figure 3.15: Energies during an evolution.

elements have not a $\mathcal{C}^1$ regularity.

Under the quartic double-well potential (3.2.5), we have an explicit slope μ for the stationnary solution. In figure 3.16, we present the numerical solution for $\Omega = [0, 1]$ (blue stars), and the "tanh-profile" whose coefficients u_+ and μ have been fitted to the data. The fitting on u_+ corresponds to the value of the solution on the boundaries of the domain Ω. And the fitting on μ corresponds to a least square method between the numerical solution and a "tanh-profile" solution interpolated on the same meshes. The "tanh-profile" (defined over $\mathbb{R}$) may be considered as a good approximation of the solution on $\Omega = [0, 1]$ since the interface is very thin.



(a) Mesh 18 - Q_1 (b) Mesh 36 - Q_1 (c) Mesh 72 - Q_1

(d) Mesh 9 - Q_2 (e) Mesh 18 - Q_2 (f) Mesh 36 - Q_2

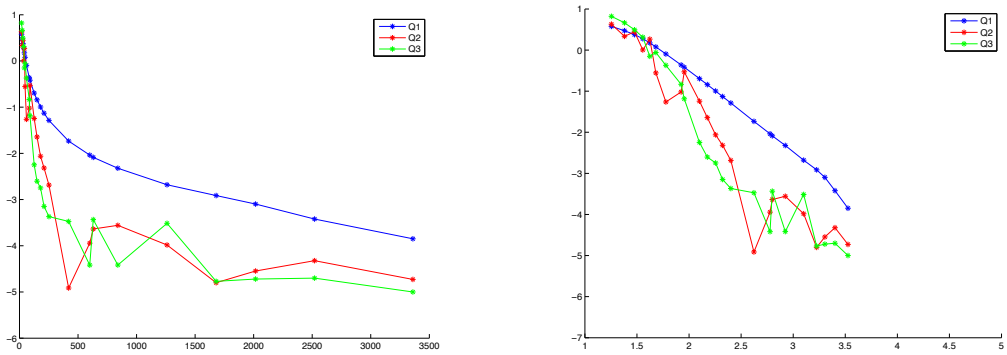
(g) Mesh 6 - Q_3 (h) Mesh 12 - Q_3 (i) Mesh 24 - Q_3

Figure 3.16: Fitted curves on the "tanh-profile".

If we want to compare the solutions between a Q_1 simulation and a Q_{10} simulation, we need to compare the two simulations under a same complexity which, up to the inversions of the linear systems, corresponds to a similar computational cost. In the one dimensional case, the complexity corresponds to the value $Degree \times Number\ of\ elements$. For a Q_{10} simulation, we need only a mesh with 10 times less elements than for a Q_1 simulation.

Figure 3.16 represents the numerical solution over mesh grids with three different complexities 18, 36 and 72, and under polynomial functions of degree 1, 2 and 3. For instance, for the elements Q_2, it corresponds to the mesh grids with 9, 18 and 36 elements. If we increase the number of elements or the degree of the polynomial space $\mathbb{P}$, then we obtain a better approximation of the slope of the "tanh-profile" solution. But for the same complexity, the curves are qualitatively similar. Figures 3.17(a) and 3.17(b) show the evolution of this approximation error according to the complexity for Q_1, Q_2 and Q_3 simulations. On figure 3.17(b), we have used a logarithmic scale

in order to compare the speed rate of the convergence. We obviously conclude that, for elements



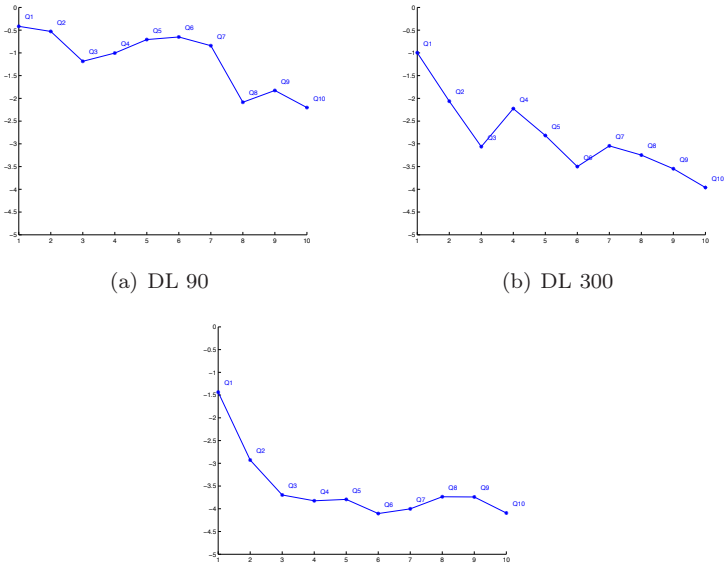
(a) Versus the number of elements of the mesh

(b) Versus the logarithm of the number of elements of the mesh

Figure 3.17: Comparison between the errors on the slope under the same complexities.

Q_1, Q_2 or Q_3, a fine mesh allows a better approximation. But the Q_2 and Q_3 elements seem to reach faster a saturation. They need only 500 elements on the mesh in order to reach a 10^{-5} precision, whereas the Q_1 elements need 5000 elements ! Figure 3.17(b) highlights this better speed on the approximation error of the slope. But Q_2 and Q_3 elements seems to have a similar speed before reaching the saturation zone.

If we fix the complexity, we can test which degree of the polynomial space $\mathbb{P}$ can provide the best speed. Figure 3.18 shows this approximation error according to the degree of the polynomial space $\mathbb{P}$ under the same complexity - denoted by DL.



(a) DL 90

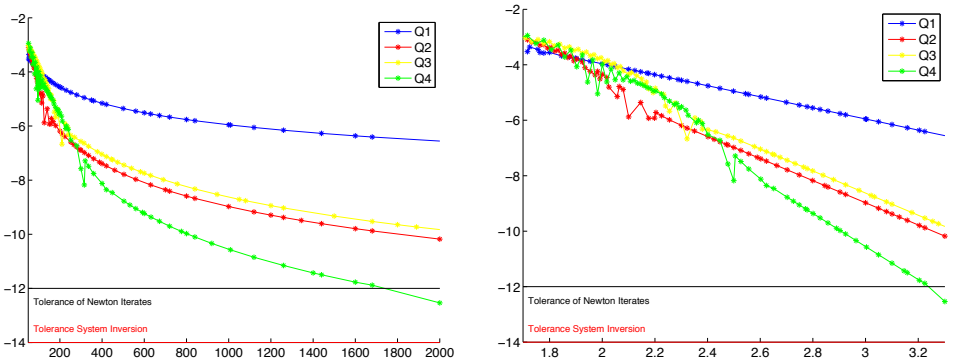
(b) DL 300

(c) DL 630

Figure 3.18: Error on the slope versus the degree of the polynomial space $\mathbb{P}$ under the same complexity.

Under the same complexity, we see on the figures 3.18 that high degree elements still provides better approximations than Q_1 elements. Although very high degree elements always provide better approximations than low degree elements, the slopes on the figure 3.18(c) of the curves for low degrees suggest that Q_3 elements are a good choice. Higher elements increases the computation time for matrix inversion and the gain is not valuable.

Figure 3.19(a) shows the error on the energies according to the complexity under Q_1, Q_2, Q_3 and Q_4 elements. As for the slopes, we see that the error is decreasing as the number of elements of the mesh is increasing. Whereas the error reaches a 10^{-4} precision for the slopes before saturation, the error on the energy reaches the tolerance zone for Q_4 elements on a mesh with 500 elements. Figure 3.19(b) shows the logarithm of the error according to the logarithm of the complexity. We see that the evolution is linear for the finest meshes with a good speed. We conclude in particular that we can compute an order of the speed of the convergence. For Q_1 elements, we find an order 2, for Q_2 elements, we find an order 4, for Q_3 elements, we find an order 4 and for Q_4 elements, we find an order 6. Note that the error on the energies should be of the order as the H^1 error.



(a) Errors according to the complexity

(b) Errors according to the logarithm of the complexity

Figure 3.19: Errors according to the logarithm of the complexity

Now, we fix the complexity and compare the approximation error on the energy according to the degree of the polynomial space $\mathbb{P}$. For the complexities 90, 300 and 630, we have drawn the decimal logarithm of the errors on figure 3.20.

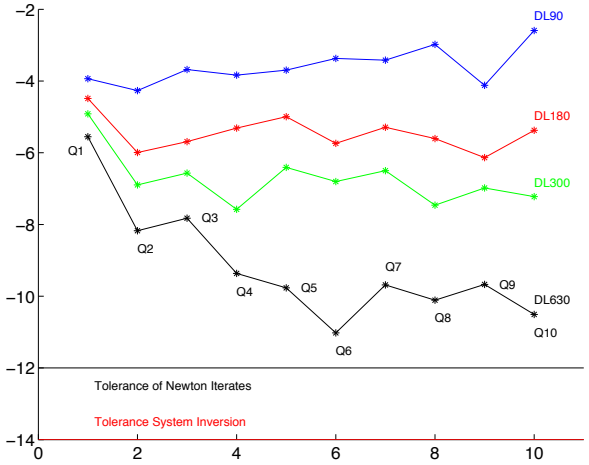
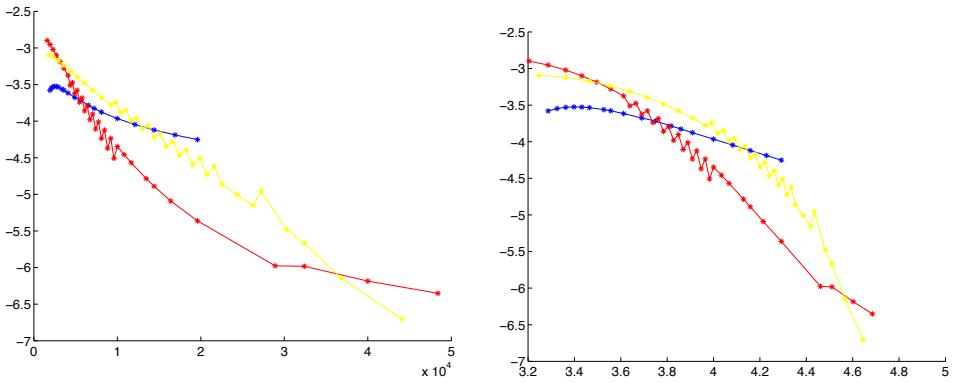


Figure 3.20: Errors on the energy according to the logarithm of the complexity

Again, under a same complexity, if we increase the degree of the polynomial space $\mathbb{P}$ the high degrees can provide better approximation, except on the coarse grids. We can conclude that for a fix mesh (fine enough), high degrees provide a better approximation. But for each complexity, it seems that we have a saturation because the Q_6, Q_7, Q_8, Q_9 and Q_{10} elements have almost the same errors. We conclude that we have to use elements with high degrees, but it is not necessary to choose the highest. We have to take into account the computational cost, and the precision of our inverse solver. Indeed, even if the complexity is the same, the matrix of the finite elements

have not the same profil. For instance, the length of the diagonal band of the "mass" matrix for Q_{10} elements is much larger than for Q_1 elements. Figures 3.19 and 3.20 indicate that Q_2 and Q_3 elements are a good compromise to ensure good results without increasing two much the computational cost.

In the two dimensional case, the results are drawn on figure 3.21. The behaviour is similar.



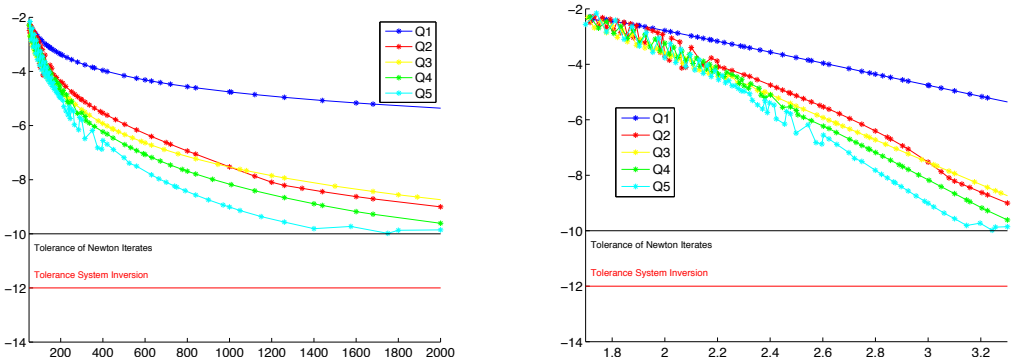
(a) Errors according to the complexity

(b) Errors according to the logarithm of the complexity

Figure 3.21: Errors on the energy according to the logarithm of the complexity

The energy and the interface are essential physical quantities. On a mathematical point of view, it is also important to study the L^2 error.

Figures 3.22(a) and 3.22(b) show the L^2 error according to the complexity for Q_1, Q_2, Q_3, Q_4 and Q_5 elements. On figure 3.22(b), we have used a logarithmic scale in order to compare the speed rate of the convergence. We have computed the order of the speed of the convergence. If we



(a) Versus the complexity.

(b) Versus the decimal logarithm of the complexity.

Figure 3.22: Comparison between the L^2 errors under the same complexity.

extrapolate the lines, we can find the necessary complexity in order to reach the saturation.

Degrees	Order	Complexity for saturation	Grid for saturation
1	1.9960	423829	423829
2	3.9682	3558	1779
3	4.0216	4110	1370
4	4.9119	2364	591
5	5.9040	1475	295

Again, Q_2 and Q_3 elements give very good results for a reasonable computational cost. We have decided to prefer Q_3 elements because it seems that they provide better results on the interface length as shown in figure 3.18.

3.4 Stationary states

The Cahn-Hilliard equation has a lot of asymptotic equilibria (see [41], [59] and [60]). In the one dimensional case, a state can be described by the number of interfaces and their positions. In figure 3.23, we show four states which are numerically stable. It is possible to observe more than one interface only for small ε. Only when the interface is very thin - *i.e.* for small ε, the interfaces do not interact. Note that the energy increases with the number of the interfaces.

In fact, this a bifurcation phenomenon. When ε crosses critical values, bifurcations happen and more stationary solutions appear.

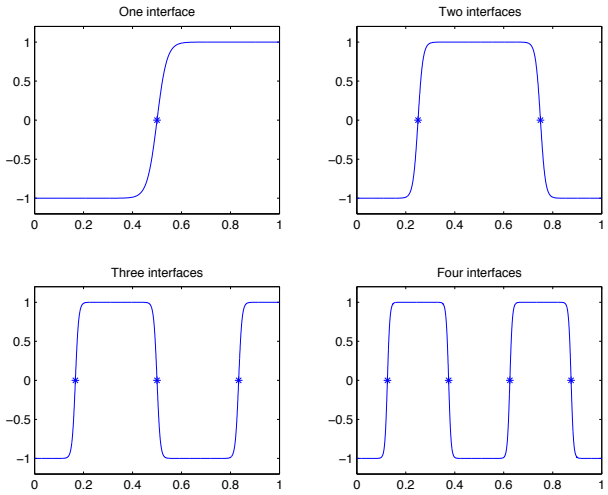


Figure 3.23: Four numerically stable states.

In [52], the authors consider the stationary states of (3.0.6) on the square. They numerically study the solutions of the following semi-linear elliptic equation.

$$\begin{cases} c = u - u^3 + \varepsilon^2 \Delta u, & \text{on } \Omega, \\ \nabla u \cdot \nu = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.4.1}$$

together with the mass constraint:

$$\frac{1}{|\Omega|} \int_\Omega u(x) \mathrm{d}x = m, \tag{3.4.2}$$

where $\Omega = [0,1]^2$ is the square, $c \in \mathbb{R}$ and $m \in \mathbb{R}$ are parameters. They study stationary solutions under the three-dimensional parameter space $\left(c, m, 1/\varepsilon^2\right)$. For this system and for all ε, a trivial solution is given by the constant solution $u \equiv m$ with $c = m - m^3$. The linearization around $u \equiv m$ of (3.4.1) under the mass constraint reads

$$\begin{cases} 0 = \left(1 - 3m^2\right) u + \varepsilon^2 \Delta u, & \text{on } \Omega, \\ \nabla u \cdot \nu = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.4.3}$$

Let v_r be an eigenfunction of the Laplace-Neumann operator in Ω defined in (3.4.6) with eigenvalue $r \in \mathbb{R}^+$, then v_r is also an eigenfunction of (3.4.3) when

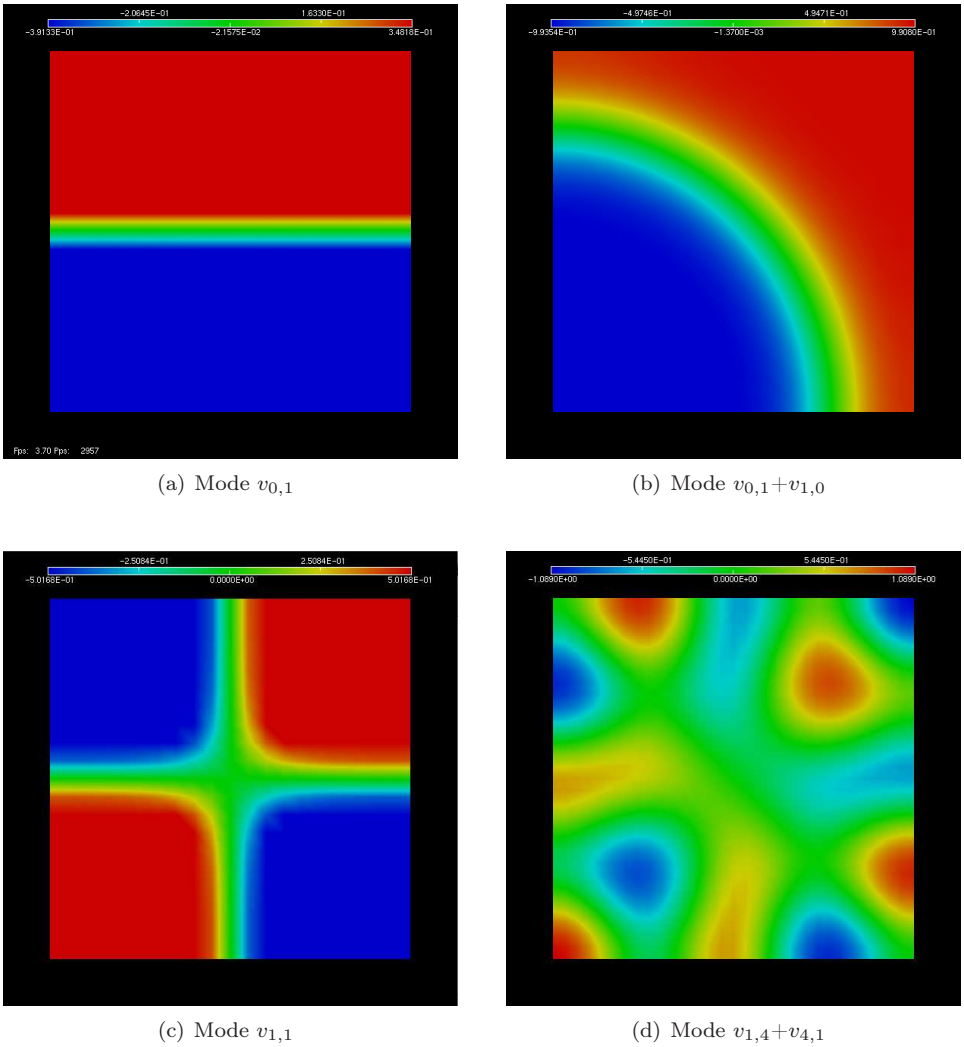
$$\frac{1}{\varepsilon^2} = \frac{r}{1 - 3m^2} \text{ for } |m| < \frac{1}{\sqrt{3}}. \tag{3.4.4}$$

But $\sigma_+ = 1/\sqrt{3}$ for the quartic double-well potential, so this equality shows that bifurcations may occur only for m in the spinodal region. For the square domain $\Omega = [0,1]^2$, the eigenfunctions are:

$$v_r(x,y) = v_{k,l}(x,y) := \cos(\pi k x)\cos(\pi l y) \quad \text{for } (x,y) \in [0,1]^2, \tag{3.4.5}$$

with $(k,l) \in \mathbb{N}^2$ such that $r = (k^2+l^2)\pi^2$. For the mode $v_{1,1}$ (i.e. $r = 2\pi^2$), we obtain non-trivial solutions bifurcating at $u \equiv \pm m^*$ with $m^* = \sqrt{(1-\varepsilon^2 r)/3}$. We fix $m = 0$, such that the bifurcations occur as $1 = \varepsilon^2 r$.

The previous asymptotic equilibria – described in [52] – are asymptotic solutions of the dynamical evolution. For instance, we have obtained the $v_{1,1}$ mode as a stationary solution of a dynamical evolution (See figure 3.24(c)). A random start may lead to different modes, and actually we only see the most stable of them in long time. Figures 3.24(a) and 3.24(b) show the stable states that we see most of the time. All the symetrical states are also stable. In [50], the authors have studied the global attractor on a square and they have proved that, after the first bifurcation, there exist 4 minimal attractors (see Theorem 4.2 in [50]) which are the symetrics of the figure 3.24(a). The other stable states shown here appear after subsequent bifurcations. Starting the simulation with well chosen initial data, we have been able to recover dynamically all the stable states described in [52]. If we choose the mode $v_{4,1} + v_{1,4}$ (which is the last mode studied by Maier-Paape and Miller), we see on figure 3.24(d) the asymptotic equilibria that we have obtained.



(a) Mode $v_{0,1}$ (b) Mode $v_{0,1}+v_{1,0}$

(c) Mode $v_{1,1}$ (d) Mode $v_{1,4}+v_{4,1}$

Figure 3.24: Asymptotic equilibria in the Maier-Paape-Miller nomenclature.

In general, the stable states are deeply dependent on the eigenvalues of the Laplace operator

on Ω. Let $(\rho_k)_{k\in\mathbb{N}}$ and $(v_{\rho_k})_{k\in\mathbb{N}}$ be the eigenvalues and eigenvectors of the following problem:

$$\begin{cases} -\Delta v_{\rho_k} = \rho_k v_{\rho_k}, & \text{on } \Omega \subset \mathbb{R}^n, \\ \nabla v_{\rho_k} \cdot \nu = 0, & \text{on } \partial\Omega, \\ \int_\Omega v_{\rho_k}(\theta)\mathrm{d}\theta = 0. \end{cases} \tag{3.4.6}$$

In [50], the authors have studied the bifurcations and the global attractors of the Cahn-Hilliard problem. In their nomenclature, they consider the following Cahn-Hilliard equation:

$$\begin{cases} \partial_t v = \Delta w, & \text{on } \Omega \subset \mathbb{R}^n, \\ w = -\lambda v + \gamma_2 v^2 + \gamma_3 v^3 - \Delta v, & \text{on } \Omega \subset \mathbb{R}^n, \\ \nabla v \cdot \nu = 0 = \nabla w \cdot \nu, & \text{on } \partial\Omega, \end{cases} \tag{3.4.7}$$

where λ, γ_2 and γ_3 are parameters. If u is a solution of the system (3.0.6) on Ω, then v is a solution on $\Omega/\sqrt{\varepsilon}$ of (3.4.7) if we define for all $t \in \mathbb{R}$ and $x \in \Omega$:

$$v : (t, x) \mapsto u(t, x\sqrt{\varepsilon}). \tag{3.4.8}$$

and the correpondance is given by the following equalities.

$$\lambda := \frac{1}{\varepsilon}, \quad \gamma_2 := 0 \text{ and } \gamma_3 := \frac{1}{\varepsilon}. \tag{3.4.9}$$

They prove that the first bifurcation occurs as their parameter λ is greater than a particular value. For our problem, this bifurcation occurs as $\frac{1}{\varepsilon^2} > \rho_1$.

Below, we study this first bifurcation and illustrate theoretical results of [50].

3.4.1 Asymptotic stable states on a rectangle

For a rectangular domain $\Omega = [0, 2] \times [0, 1]$, we are in the hypothesis of the Theorem 4.1 in [50]. According to this result, if $\frac{1}{\varepsilon^2} > \rho_1 := \frac{\pi^2}{4}$ then there exist exactly two attractors $\pm u_\varepsilon$ which can be expressed as

$$\pm\, u_\varepsilon(x, y) = \pm \frac{2\varepsilon}{\sqrt{3}} \sqrt{\frac{1}{\varepsilon^2} - \frac{\pi^2}{4}} \cos\left(\frac{\pi x}{2}\right) + \sqrt{\varepsilon}\; o\left(\left|\frac{1}{\varepsilon^2} - \frac{\pi^2}{4}\right|^{1/2}\right), \quad x \in [0, 2], y \in [0, 1]. \tag{3.4.10}$$

We define the approximated attractors $\pm v_\varepsilon$ by

$$\pm v_\varepsilon(x) := \pm C(\varepsilon) \sqrt{\frac{1}{\varepsilon^2} - \frac{\pi^2}{4}} \cos\left(\frac{\pi x}{2}\right), \quad x \in [0, 2], y \in [0, 1],$$

where $C(\varepsilon)$ is a constant depending on ε. It is chosen in order to minimize the L^2 norm of $u'_\varepsilon - v_\varepsilon$.

For multiple values of the parameter ε around the value $\frac{2}{\pi}$, we have obtained the corresponding numerical stationnary states u'_ε.

We have checked numerically the validity of formula (3.4.10). We study the following quantity

$$\frac{\|u'_\varepsilon - v_\varepsilon\|_2}{\|v_\varepsilon\|_2} = \frac{\|u'_\varepsilon - v_\varepsilon\|_2}{C(\varepsilon)\sqrt{\left(\frac{1}{\varepsilon^2} - \frac{\pi^2}{4}\right)}}.$$

This relative L^2 norm should converge to zero. On figure 3.25, we have plotted the decimal logarithm of this relative L^2 norm according to the decimal logarithm of $\frac{1}{\varepsilon^2} - \frac{\pi^2}{4}$. Figure 3.25 is in conformity with the theoretical results. We can see that the relative L^2 error converges to 0 as $\frac{1}{\varepsilon^2}$ converges to $\frac{\pi^2}{4}$. We even can improve formula (3.4.10) and find the exponent $\alpha_{rectangular}$ such that

$$\|u'_\varepsilon - v_\varepsilon\|_2 \sim \tilde{C} \left|\frac{1}{\varepsilon^2} - \frac{\pi^2}{4}\right|^{\alpha_{rectangular}}$$

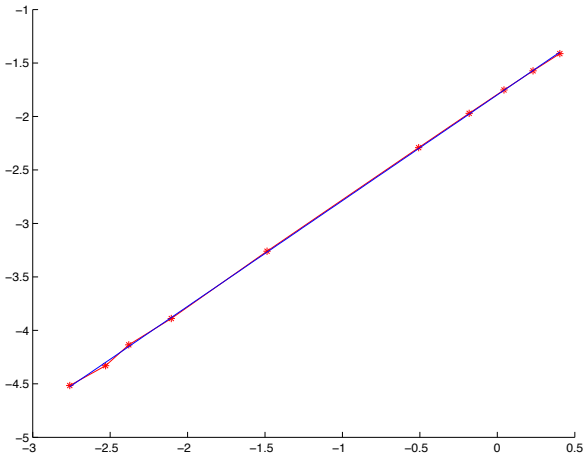


Figure 3.25: Convergence of the bifurcation solutions on a rectangular domain.

where $\tilde{C}$ is an unknown constant. We find that the exponent $\alpha_{rectangular} = 1/2 + 0.98908$, almost $3/2$. Moreover, since we know explicitly the attractor, we can verify that our minimal constant $C(\varepsilon)$ is near $\frac{2\varepsilon}{\sqrt{3}}$. On figure 3.26, we have drawn the logarithm of our minimal constant $C(\varepsilon)$ according to the logarithm of ε. We find

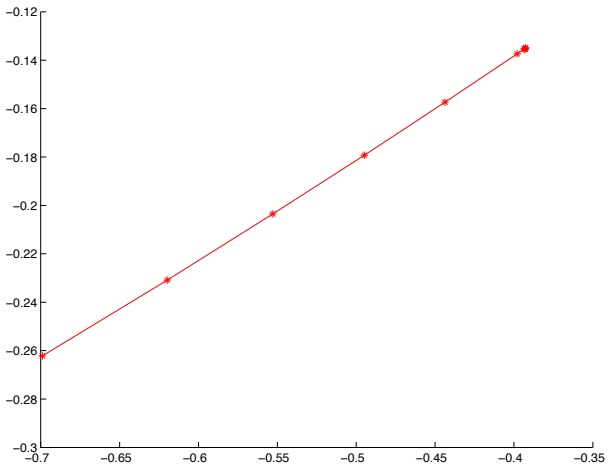


Figure 3.26: Order of convergence of the minimal constant $C(\varepsilon)$.

$$C(\varepsilon) \sim 1.0682 \ * \ \varepsilon^{0.83603} \tag{3.4.11}$$

this is in conformity with the fact that $C(\varepsilon)$ converges to $\lim_{\varepsilon \to \frac{2}{\pi}} \frac{2\varepsilon}{\sqrt{3}} = \frac{4}{\pi\sqrt{3}}$.

The segment may be seen as a degenerate rectangle. On the segment $[0, 1]$, the eigenvalues of (3.4.6) are $\rho_k := k^2\pi^2$ for all $k \in \mathbb{N}$. According to Theorem 4.2 in [50], there is a bifurcation at $\frac{1}{\varepsilon^2} > \pi^2$. Moreover, Remark 4.2 in [50] states that there exist two minimal attractors $\pm u_\varepsilon$ which can be expressed as

$$\pm\, u_\varepsilon(x) = \pm C(\varepsilon)\sqrt{\left(\frac{1}{\varepsilon^2} - \pi^2\right)}\cos(\pi x) + \sqrt{\varepsilon}\; o\left(\left|\frac{1}{\varepsilon^2} - \pi^2\right|^{1/2}\right), \quad x \in [0, 1], \tag{3.4.12}$$

where $C(\varepsilon)$ is a constant which can depend on ε. Again, we define the approximated attractors

$\pm v_\varepsilon$ by

$$\pm v_\varepsilon(x) := \pm C(\varepsilon)\sqrt{\left(\frac{1}{\varepsilon^2} - \pi^2\right)}\cos(\pi x), \quad x \in [0,1].$$

For multiple values of the parameter ε around the value $\frac{1}{\pi}$, we have obtained the corresponding numerical stationnary states u'_ε. We choose the constant $C(\varepsilon)$ in order to minimize the L^2 norm of $u'_\varepsilon - v_\varepsilon$ and study the convergence of u'_ε to v_ε using the quantity

$$\frac{\|u'_\varepsilon - v_\varepsilon\|_2}{\|v_\varepsilon\|_2} = \frac{2\|u'_\varepsilon - v_\varepsilon\|_2}{C(\varepsilon)\sqrt{\left(\frac{1}{\varepsilon^2} - \pi^2\right)}}.$$

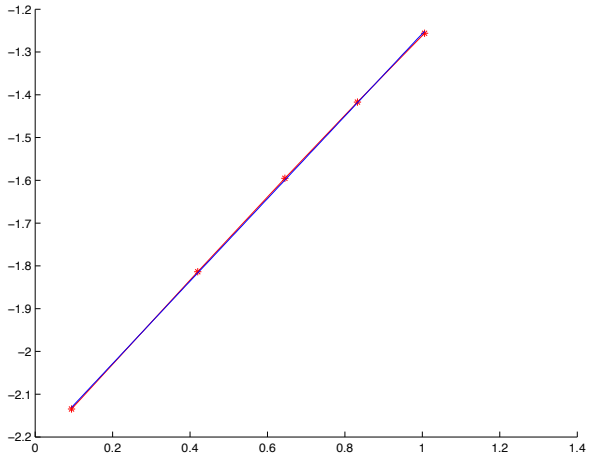


Figure 3.27: Convergence of the bifurcation solutions on a segment.

Figure 3.27 is in conformity with what we expect from the theoretical results. We can see that the relative L^2 error converges to 0 as $\frac{1}{\varepsilon^2}$ converges to π^2 and find the exponent $\alpha_{segment}$ such that

$$\|u'_\varepsilon - v_\varepsilon\|_2 \sim \tilde{C}\left|\frac{1}{\varepsilon^2} - \pi^2\right|^{\alpha_{segment}}$$

where $\tilde{C}$ is an unknown constant. We have found $\alpha_{segment} = 1/2 + 1.0254$, again almost $3/2$.

We have said that the constants $C(\varepsilon)$ have been numerically chosen in order to minimize the L^2 norm of $u'_\varepsilon - v_\varepsilon$. If we extend the results of [50], we expect that the constant $C(\varepsilon) \sim \frac{2}{\sqrt{3}}\varepsilon$. Thus, on the figure 3.28, we have drawn the logarithm of our minimal constants $C(\varepsilon)$ according to the logarithm of ε. If we study the slope, we find

$$C(\varepsilon) \sim 1.0844 \ * \ \varepsilon^{0.94594} \tag{3.4.13}$$

again in conformity with the theoretical formula.

3.4.2 Asymptotic stable states on smooth domains

For a smooth domain Ω, hypothesis of Theorem 3.1 in [50] hold. We have considered an ellipse. On the ellipse, the first eigenvalue $\rho_1 \simeq 0.8776$ is simple. On figure 3.29, we have drawn the corresponding first eigenvector. If $\frac{1}{\varepsilon^2} > \rho_1$, the problem (3.4.7) has two steady states $\pm u_\varepsilon$ which can be expressed as

$$\pm u_\varepsilon = \pm C(\varepsilon)\sqrt{\left(\frac{1}{\varepsilon^2} - \rho_1\right)}v_{\rho_1} + \sqrt{\varepsilon}\, o\left(\left|\frac{1}{\varepsilon^2} - \rho_1\right|^{1/2}\right),$$

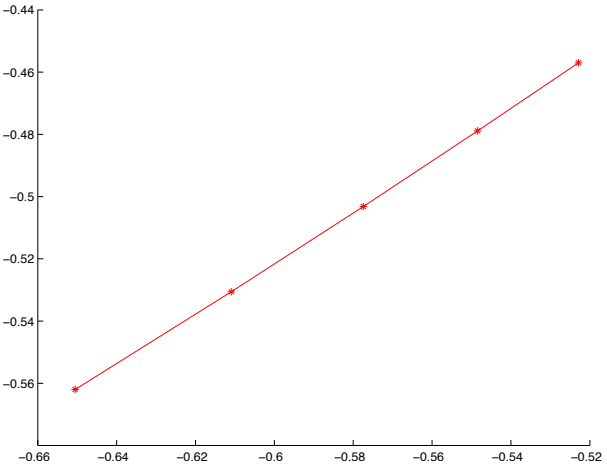


Figure 3.28: Convergence of the constant $C(\varepsilon)$.

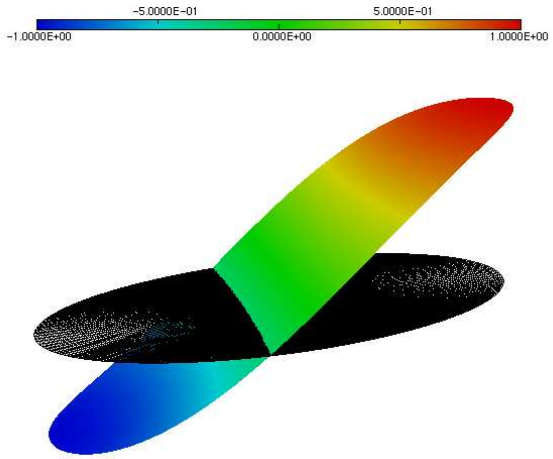


Figure 3.29: First eigenvector on the ellipse.

where $C(\varepsilon)$ is a constant which can depend on ε. We define the approximated attractors $\pm v_\varepsilon$ by

$$\pm v_\varepsilon := \pm C(\varepsilon)\sqrt{\left(\frac{1}{\varepsilon^2} - \rho_1\right)} v_{\rho_1},$$

where v_{ρ_1} is a fixed eigenvector. On figure 3.30, we have drawn a steady states. For multiple

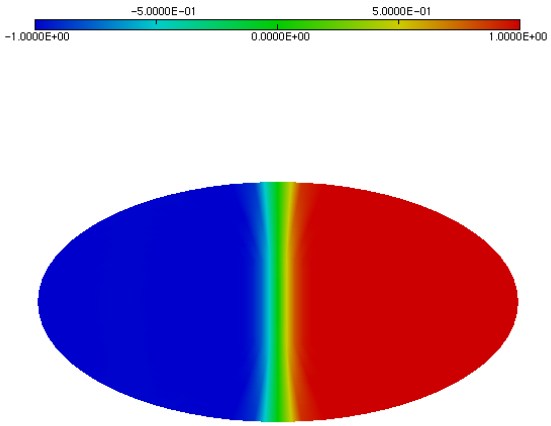


Figure 3.30: Steady state on the ellipse.

values of the parameter ε around the value $\frac{1}{\sqrt{\rho_1}}$, we have obtained the corresponding numerical stationnary states u'_ε. As in section 3.4.1, we choose the constant $C(\varepsilon)$ in order to minimize the L^2 norm of $u'_\varepsilon - v_\varepsilon$ and study the convergence of u'_ε to v_ε. We consider the quantity

$$\frac{\|u'_\varepsilon - v_\varepsilon\|_2}{\|v_\varepsilon\|_2} = \frac{\|u'_\varepsilon - v_\varepsilon\|_2}{C(\varepsilon)\sqrt{\left(\frac{1}{\varepsilon^2} - \rho_1\right)}\|v_{\rho_1}\|_2}.$$

According to Theorem 3.1 in [50], this relative L^2 norm should converge to zero. On figure 3.31, we have drawn the decimal logarithm of this relative L^2 norm according to the decimal logarithm of $\frac{1}{\varepsilon^2} - \rho_1$.

Figure 3.31 is in conformity with the theoretical results. We can see that the relative L^2 error converges to 0 as $\frac{1}{\varepsilon^2}$ converges to ρ_1. Then we compute the exponent $\alpha_{ellipse}$ such that

$$\|u'_\varepsilon - v_\varepsilon\|_2 \sim \tilde{C}\left|\frac{1}{\varepsilon^2} - \rho_1\right|^{\alpha_{ellipse}}$$

where $\tilde{C}$ is an unknown constant. We find that the exponent $\alpha_{ellipse} = 1/2 + 0.9580$.

3.4.3 Asymptotic stable states on a trapezoid

We try to see if the results of [50] extend to non smooth domains. We have tested a trapezoid where the first eigenvalue $\rho_1 \simeq 2.2417$ is simple. On figure 3.32, we have drawn the corresponding first eigenvector.

The two steady states $\pm u_\varepsilon$ should be expressed as

$$\pm u_\varepsilon(x) = \pm C(\varepsilon)\sqrt{\left(\frac{1}{\varepsilon^2} - \rho_1\right)} v_{\rho_1} + \sqrt{\varepsilon}\; o\left(\left|\frac{1}{\varepsilon^2} - \rho_1\right|^{1/2}\right), \quad x \in [0,1],$$

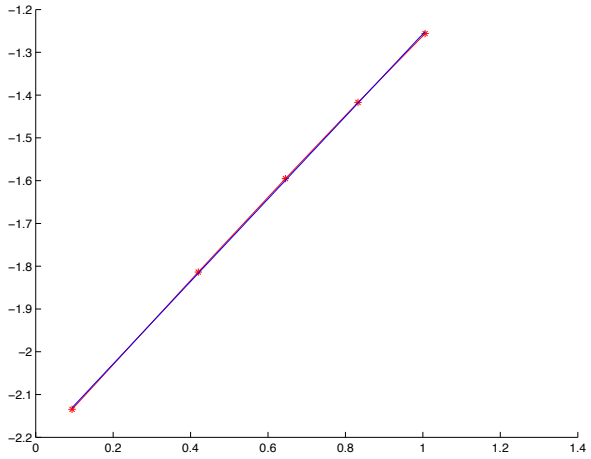


Figure 3.31: Convergence of the bifurcation solutions on the ellipse.

where $C(\varepsilon)$ is a constant which can depend on ε. We define the approximated attractors $\pm v_\varepsilon$ by

$$\pm v_\varepsilon(x) := \pm C(\varepsilon)\sqrt{\left(\frac{1}{\varepsilon^2} - \rho_1\right)}v_{\rho_1}, \quad x \in [0,1],$$

where v_{ρ_1} is a fixed eigenvector. On figure 3.33, we have drawn a steady state.

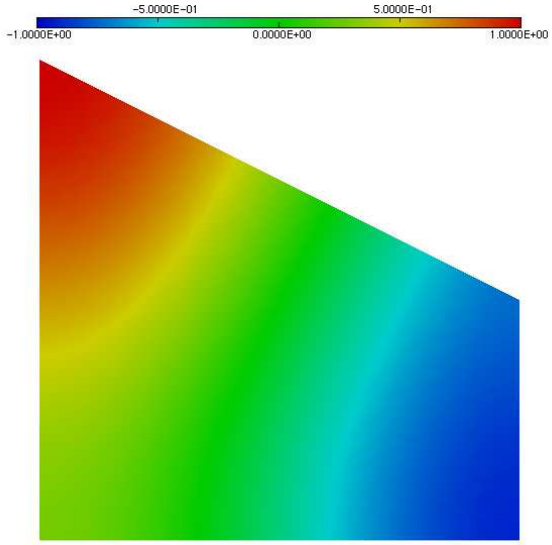


Figure 3.32: Eigenvector of the first eigenvalue on the trapezoid.

For multiple values of the parameter ε around the value $\frac{1}{\sqrt{\rho_1}}$, we have obtained the corresponding numerical stationnary states u'_ε. As in the section 3.4.1, if we choose the constant $C(\varepsilon)$ in order to minimize the L^2 norm of $u'_\varepsilon - v_\varepsilon$, we can study the convergence of u'_ε to v_ε. We have found that

$$\frac{\|u'_\varepsilon - v_\varepsilon\|_2}{\|v_\varepsilon\|_2} = \frac{\|u'_\varepsilon - v_\varepsilon\|_2}{C(\varepsilon)\sqrt{\left(\frac{1}{\varepsilon^2} - \rho_1\right)}\|v_{\rho_1}\|_2}$$

does not converge to 0. It seems that the bifurcation is different in this case. On figure 3.34, we have drawn the decimal logarithm of $\|u'_\varepsilon - v_\varepsilon\|_2$ according to the decimal logarithm of $\frac{1}{\varepsilon^2} - \rho_1$ We

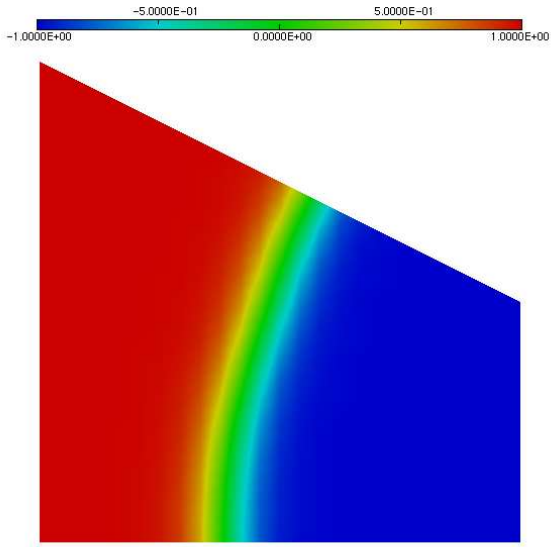


Figure 3.33: Steady state on the trapezoid.

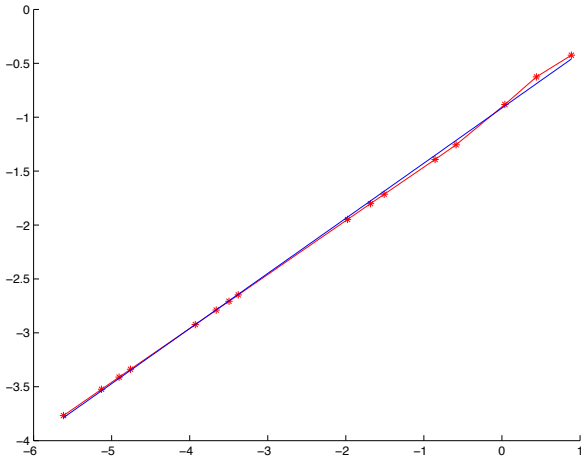


Figure 3.34: Convergence of the bifurcation solutions on the trapezoid domain.

find that

$$\|u'_\varepsilon - v_\varepsilon\|_2 \sim \tilde{C} \left| \frac{1}{\varepsilon^2} - \rho_1 \right|^{\alpha_{trapezoid}},$$

with $\alpha_{trapezoid} = 0.49937$. Thus this difference is of the same order as each term.

As in the case of the square, we can find numerically stable states corresponding to the next modes in the nomenclature of Maier-Paape and Miller in [52]. We have found 4 numerically stable states (see figure 3.35), whose corresponding energies have been drawn on the figure 3.36. We have drawn the energies according to the length of the interface. We can see the linear dependance between the interface length and the energy (the red line is the linear regression according to the least square method).

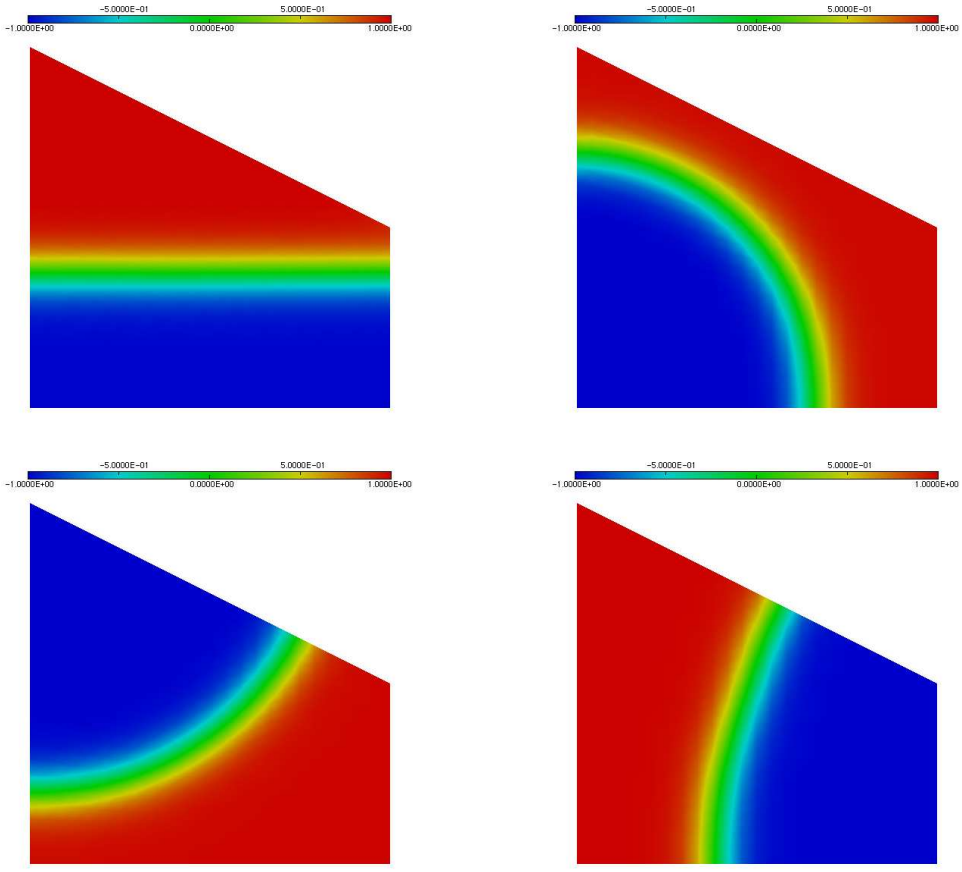


Figure 3.35: Numerically stable states on the trapezoid.

3.5 Stochastic numerical simulations

3.5.1 Simlulation of a white noise

We now consider the Cahn-Hilliard-Cook equation. A random forcing term of white noise type is added to account for thermal fluctuations:

$$\frac{\partial u}{\partial t} = \Delta\Big(-\varepsilon^2 \Delta u + \psi(u) + \eta \Big) + \sigma B \dot{W}, \tag{3.5.1}$$

where $\dot{W}$ is space-time white noise, $\sigma \in \mathbb{R}$ is a measure of the amplitude of the noise and B represents the correlation operator. The boundary conditions are Neumann again.

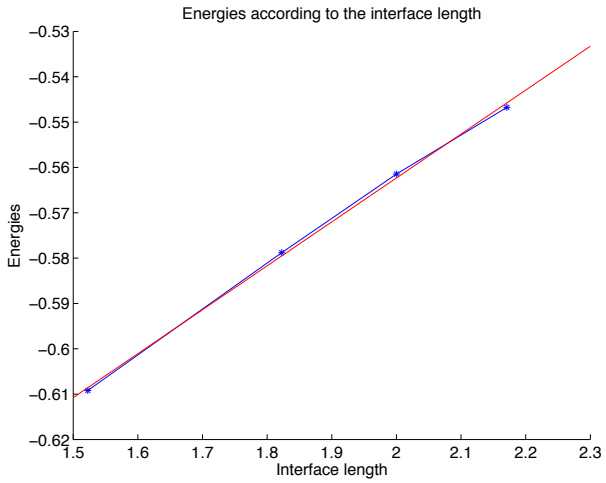


Figure 3.36: Energies of the numerically stable states on the trapezoid according to the length of the interface.

Let us recall that the space-time white noise is defined as the time derivative of a cylindrical Wiener process. In $L^2(\Omega)$, a cylindrical Wiener process W can be described by the following formula

$$W(t,x) = \sum_{i=0}^{+\infty} \beta_i(t) e_i(x), \quad x \in \Omega, t \geq 0, \tag{3.5.2}$$

where $(e_i)_{i\in\mathbb{N}}$ is any orthonormal basis of $L^2(\Omega)$ and $(\beta_i)_{i\in\mathbb{N}}$ is a sequence of independant brownian motions. It is classical that this series does not converge in $L^2(\Omega)$, and it is well known that the brownian motions are nowhere differentiable with respect to time. This reflects the irregularity in both space and time of the noise. The choice of the correlation operator B depends on the physical context. Since, the Cahn-Hilliard equation is a gradient flow in $H^{-1}(\Omega)$, a natural choice is to take B as the half power of the Laplace operator so that the gradient structure is preserved. Note that BW is defined by $BW = \sum_{i=0}^{+\infty} \beta_i(t) B e_i$.

When ψ is the logarithmic nonlinearity, a reflection term η has to be added in order that solutions may exist (see [28], [39]). It may be seen as a Lagrange multiplier associated to the constraint $u \in [-1,1]$. We have not used this nonlinearity below and all computation on the Cahn-Hilliard-Cook equation below have been done with the quartic nonlinearity.

We have performed some tests with the logarithmic nonlinearity and have not observed any notable difference.

A reflection is taken into account in section 3.5.3 for a different model. The algorithm described can easily be used in the case of the logarithmic nonlinearity.

The time discretization is again the backward Euler scheme. For each time step τ, the backward Euler scheme is written as

$$\begin{cases} \frac{u_{n+1}-u_n}{\tau} = \Delta w_{n+1} + \sigma B\left(\frac{W_{n+1}-W_n}{\tau}\right), \\ \\ w_{n+1} = \psi(u_{n+1}) - \varepsilon^2 \Delta u_{n+1}, \end{cases} \tag{3.5.3}$$

where $W_n = W(n\tau)$ for all $n \in \mathbb{N}$. This scheme depends on a initial value u_0 and a choice on σ.

The random term $B\left(W_{n+1} - W_n\right)$ can also be written as

$$B\left(W_{n+1} - W_n\right) = \sum_{i=0}^{+\infty} \left(\beta_i(n\tau+\tau) - \beta_i(n\tau)\right) B e_i. \tag{3.5.4}$$

And since β_i has independant increment whose laws is Gaussian, we can simulate the random term as

$$\sqrt{\tau} \sum_{i=0}^{+\infty} \xi_i B e_i, \tag{3.5.5}$$

where $(\xi_i)_{i\in\mathbb{N}}$ is a sequence of independant Gaussian variables with zero mean and a variance equal to one. The sum must be truncated, because we cannot numerically consider the whole hilbertian basis of $L^2(\Omega)$. The complete backward Euler scheme can be written as

$$\begin{cases} u_{n+1} = u_n + \tau\Delta w_{n+1} + \sigma\sqrt{\tau}\sum_{i=0}^{N}\xi_i B e_i, \\ \\ w_{n+1} = \psi(u_{n+1}) - \varepsilon^2\Delta u_{n+1}, \end{cases} \tag{3.5.6}$$

where N is a large integer which can depend on the mesh or the polynomial space $\mathbb{P}$.

In the one dimensional case, we choose the orthonormal basis of $L^2(0,1)$ as $e_i = (\cos(i\pi\cdot))_{i\in\mathbb{N}}$. This choice has been made for some technical reasons. Indeed, it has the advantage to satisfy the boundary conditions and is easier to compute numerically. Moreover, the operator B is diagonal in this basis. A classical choice is to consider the basis of the finite elements space, this is well suited for the implementation. But, in this case, the basis is not orthogonal and the resulting noise is not white in space. It has correlation length of the order of the mesh. If we denote $(\varphi_j)_{j\in J}$ the standard basis of nodal functions on the finite elements space, the fully discrete equation involves

$$\langle BW(t), \varphi_j\rangle = \sum_{0\leq i\leq N} \xi_i\mu_i\langle e_i, \varphi_j\rangle,$$

where μ_i are the eigenvalues of B. The matrix $(\langle e_i, \varphi_j\rangle)_{i=0,\dots,N;\, j\in J}$ is computed once for all at the beginning of the computation.

Note that, using fast Fourier transform, it is easy to verify that the noise is indeed white. For N large enough, the noise has a good response in all the frequencies. In practise, it is reasonable to choose N a little bigger than the inverse of the spatial mesh. Indeed, higher frequencies are not captured by the discretization.

3.5.2 Stochastic spinodal decompositions and long time evolution

In the beginning of the simulations, the spinodal decomposition drives the evolution of the alloy. During this spinodal decomposition, the alloy separates itself in multiple places, and we can see that some little bubbles appear. Then these bubbles grow and gather to form some larger bubbles.

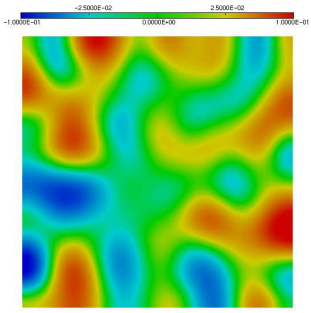
When a noise is added, the spinodal decomposition still takes place. The noise does not seem to change drastically the evolution. However, the noise induces an external fluctuation which accelerates the formation of the patterns.

On figure 3.37, we have drawn three evolutions for the same initial data and for different times. The first evolution is deterministic, the second is stochastic with a small parameter $\sigma = 0.1$, and the third is stochastic too but with a strong noise with a bigger paramater $\sigma = 0.35$.

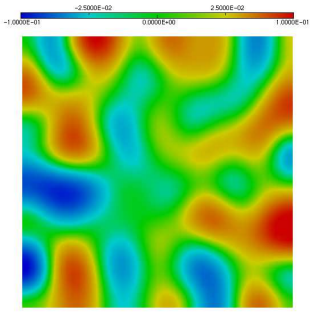
Clearly, the noise accelerates the spinodal decompositions, the growing of the bubbles, and the speed of the evolution in general. But, between a weak noise and a strong noise, the speed do not change a lot. A strong noise disturbs the normal evolution, and prevent the bubbles from gathering. For instance, on figure 3.37(f), we can see that the bubble in the top left corner, which is well formed on figure 3.37(e), is destroyed. We show the evolution for longer time on figure3.38. We see that at $t = 5$, the deterministic solution is stabilized. For a weak noise, it is almost stabilized but on a different state. But, for a stronger noise, stabilization is not reached. This indicates that the noise slows down the second part of the Cahn-Hilliard evolution when the interface is minimized.

In long time, the solutions of the Cahn-Hilliard equation evolves to minimize the energy. The simulation is stopped when the solution reachs a stable state. But, if we add a noise term, there do not exist stable states. If the noise is weak, the solutions evolves near a deterministic stable state with some oscillations. However, the noise makes it possible to reach some stable states with lower energies with a jump in the potential. Such a behavior is predicted by large deviation theory (see [34]). Figure 3.39 shows the evolution of the energy of a solution with $\sigma = 0.2$.

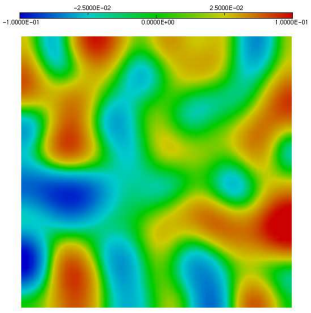
The solutions falls quickly in a stable potential well, then for some time the solution has some oscillations near this stable state. Finally, the noise permits a potential jump in order to join an other stable state with a lower energy. On the figures 3.40, we have drawn the solutions which correspond to the two corresponding potential wells. The second stable state is the one with



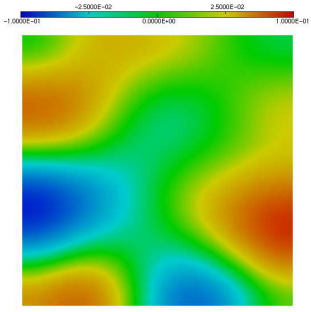
(a) Determinist - t=0.01



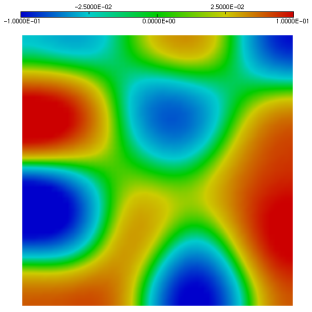
(b) Weak noise - t=0.01



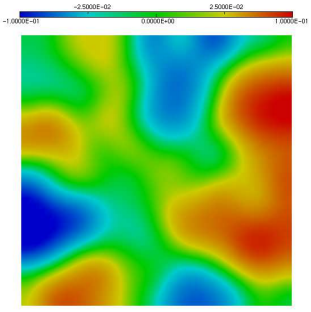
(c) Strong noise - t=0.01



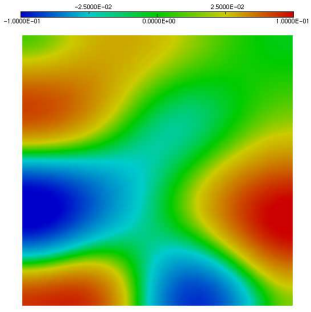
(d) Determinist - t=0.3



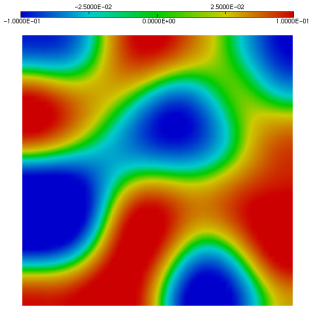
(e) Weak noise - t=0.3



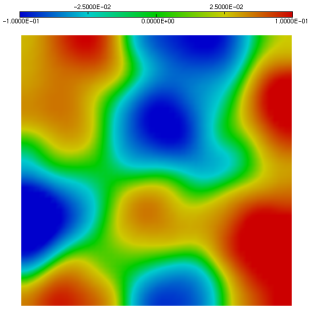
(f) Strong noise - t=0.3



(g) Determinist - t=0.6

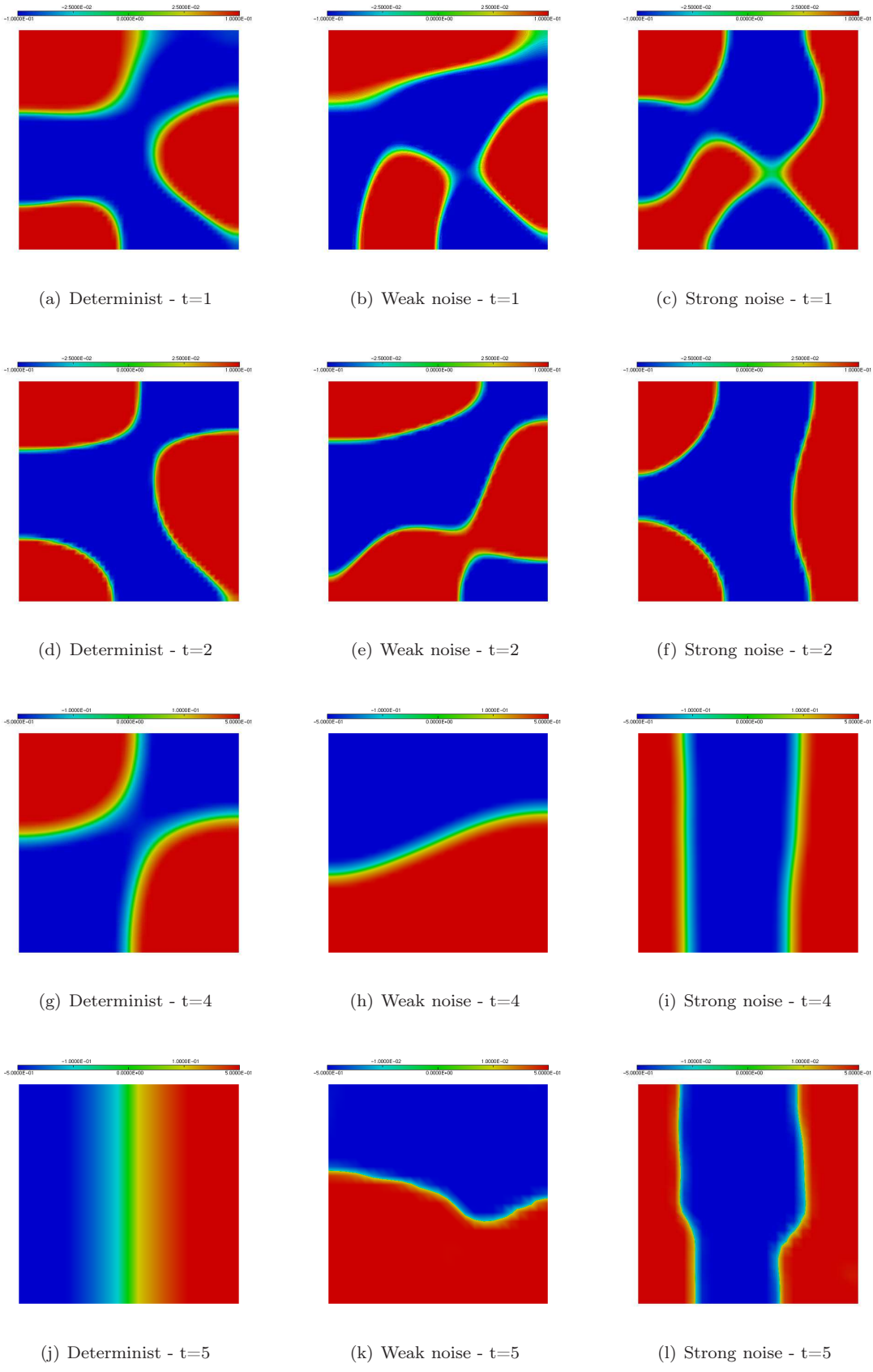


(h) Weak noise - t=0.6



(i) Strong noise - t=0.6

Figure 3.37: Spinodal decomposition under the classic quartic double-well potential.



(a) Determinist - t=1 (b) Weak noise - t=1 (c) Strong noise - t=1

(d) Determinist - t=2 (e) Weak noise - t=2 (f) Strong noise - t=2

(g) Determinist - t=4 (h) Weak noise - t=4 (i) Strong noise - t=4

(j) Determinist - t=5 (k) Weak noise - t=5 (l) Strong noise - t=5

Figure 3.38: Spinodal decomposition under the classic quartic double-well potential.

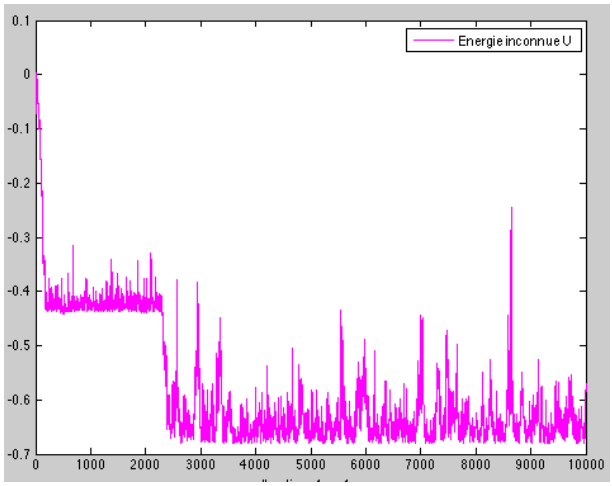
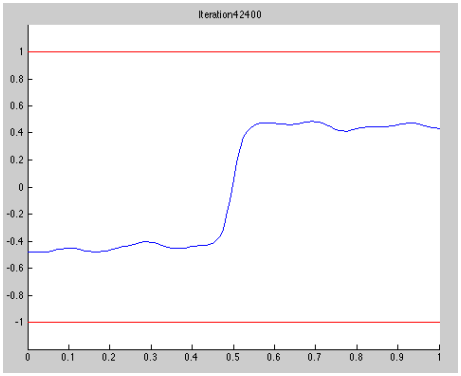
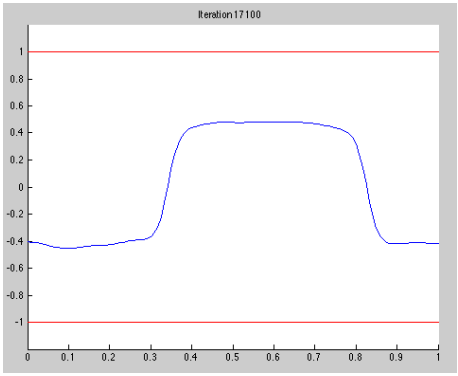


Figure 3.39: Jump of potential to a lower energy well.

minimal energy. It is very stable and the stochastic simulation does not leave its neighbourhood for a very long time.



(a) Well with energy -0.68



(b) Well with energy -0.42

Figure 3.40: States of the solution in the potential wells.

In order to see further transitions to other states, we need a stronger noise. Indeed, with small σ, the probability to leave the state with minimal energy is very small. On the figure 3.41, we have drawn the evolution of the energy of a solution with $\sigma = 1$ during a long time simulation. We can see that the solution has oscillations between at least three stable states.

The wells in the energy correspond to the solution represented on figure 3.40. The third well, which the energy is near -0.22, corresponds to the solution represented on the figure 3.42. As expected, the solution spends most of its time in the neighbourhood of the state minimal energy.

We have found the following repartition for the time spent near each stable state:

Number of interface	Energy	Time spent
3	-0.22	7.51%
2	-0.42	43.97%
1	-0.68	48.52%

In dimension two, we obtain similar results. We see on figure 3.43 the evolution in dimension two on a very long time. The parameters ε and σ are chosen not too small so that the dynamic is accelerated. We clearly see that the solution travels between the various stationary states. For instance, we observe transitions between figures 3.43(g) and 3.43(h), 3.43(j) and 3.43(k), or 3.43(k) and 3.43(l).

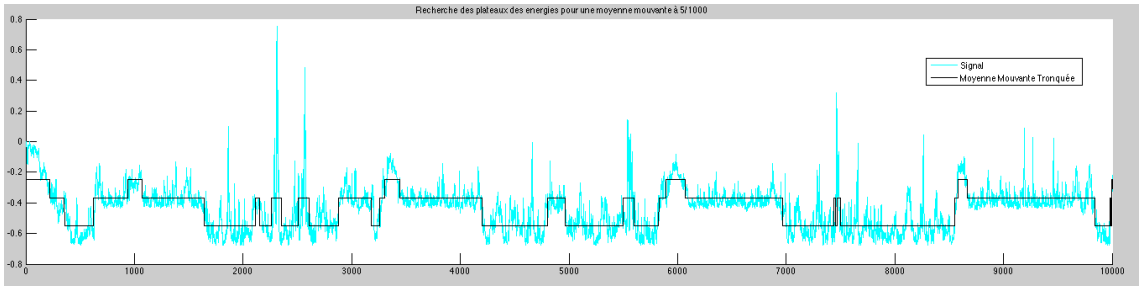


Figure 3.41: Oscillation between three stable states.

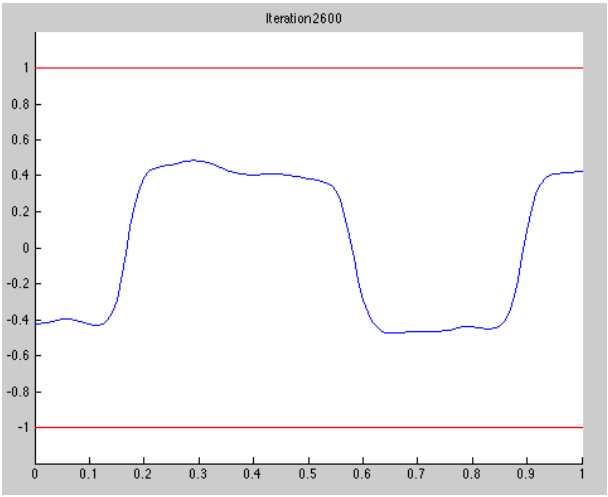
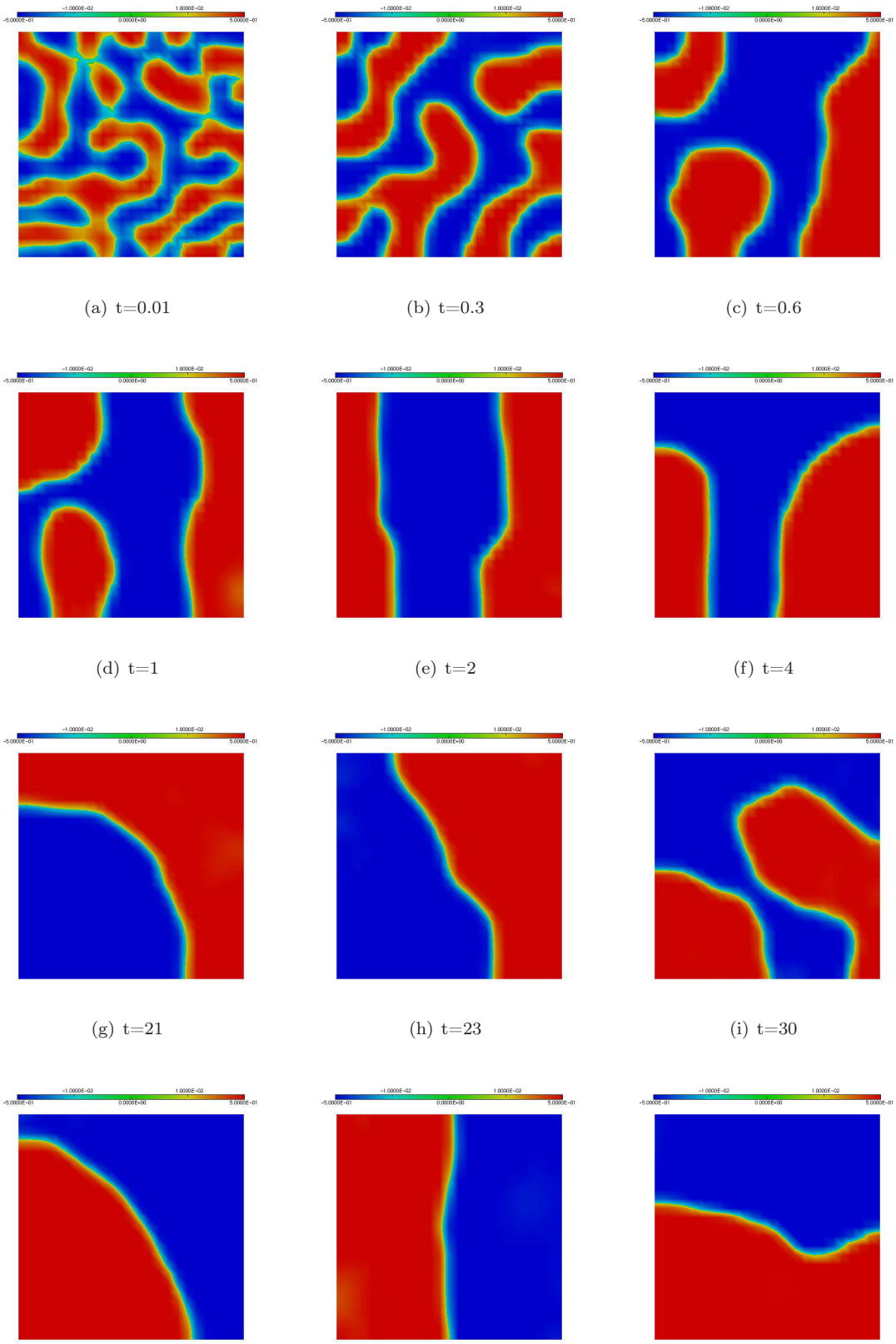


Figure 3.42: State of the solution in the third well with energy -0.22.



(a) t=0.01 (b) t=0.3 (c) t=0.6

(d) t=1 (e) t=2 (f) t=4

(g) t=21 (h) t=23 (i) t=30

(j) t=38 (k) t=47 (l) t=94

Figure 3.43: Spinodal decomposition for a strong noise.

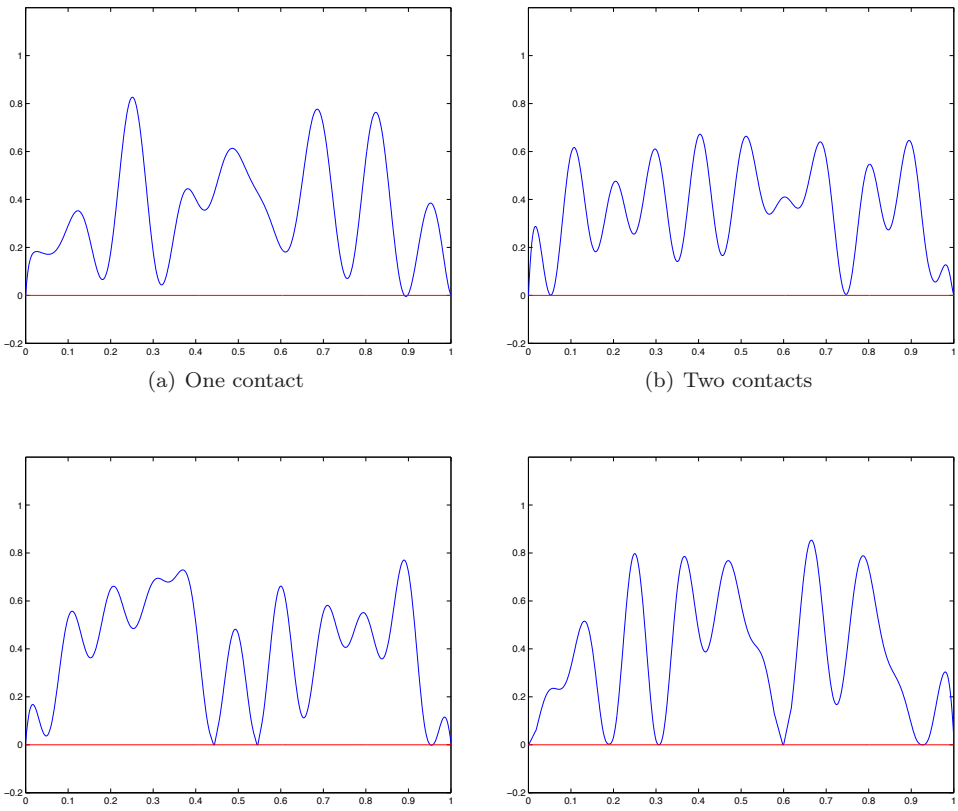
3.5.3 Model of interface on a wall

Interfaces on a hard wall can be modelled by stochastic partial differential equations with reflection. A first model is the following equation of order 2 on $\Omega = [0,1]$:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2}\Delta v + \zeta + \dot{W}, \\ v(t,.) = 0, \text{ sur } \partial\Omega, \qquad \text{pour tout } \ t \geq 0, \\ v \geq 0, \qquad \mathrm{d}\zeta \geq 0, \qquad \displaystyle\int v \mathrm{d}\zeta = 0, \end{cases} \tag{3.5.7}$$

where v is continuous and ζ is a positive measure on space and time. It has been shown that this equation is well posed and that the solution of (3.5.7) is a process which has contact with the forbidden value 0 in some points in the interior of the domain $]0,1[$. We denote $Z_t \subset]0,1[$ the set of the points where the processus has contact with the zero value at time t. A very fine result states that for almost all time t, the set Z_t is empty but almost surely there exist a time t such that Z_t has three points. Finally, Z_t has at most 4 points ([61], [26]).

We have performed a thousand of simulations of this process. We have indeed found trajectories with four contact points. On figure 3.44, we show the process with one, two, three or four contact points. Note that the larger n, the fewer instant with n contact have been found.



(a) One contact (b) Two contacts

(c) Three contacts (d) Four contacts

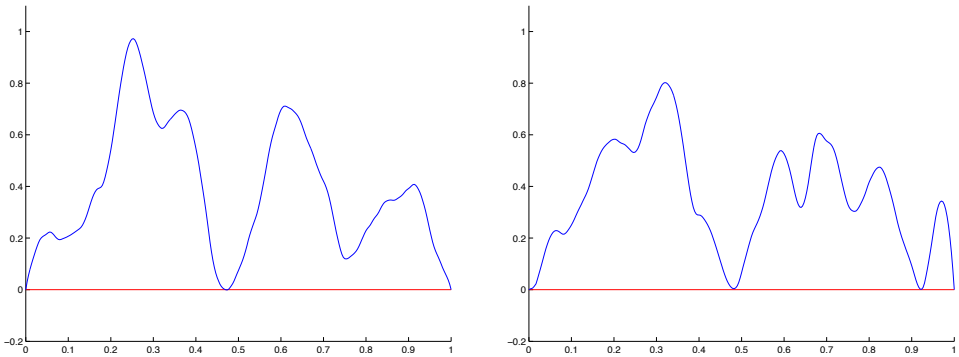
Figure 3.44: Process with contact with the zero value.

Equation (3.5.7) may seem not realistic. Indeed, the total mass is not conserved. The following

fourth order model of Cahn-Hilliard type is more realistic (see [37], [68], [77], [76]):

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{1}{2}\Delta\Big(\Delta u + \eta\Big) + B\dot{W}, \\ u(t,.) = 0,\ \nabla\Delta u \cdot \nu = 0 \text{ on } \partial\Omega, \quad \text{pour tout } t \geq 0, \\ u \geq 0, \qquad \mathrm{d}\eta \geq 0, \qquad\qquad \displaystyle\int u \mathrm{d}\eta = 0. \end{cases} \tag{3.5.8}$$

To our knowledge, the above result on contact points have not been generalized to this fourth order equation. We have simulated the above one dimensional Cahn-Hilliard equation with Dirichlet-Neumann conditions on the boundary. On figure 3.45, we can see the instants when the processus has contact with the zero value.



(a) One contact (b) Two contacts

(c) Three contacts (d) Four contacts

Figure 3.45: Process with contact with the zero value.

We have again observed trajectories with one, two, three or four contact points. This indicates that the result of [26] remains valid for the order four. Note however that we observed much fewer instants where the trajectories has a contact compared to the second order. We had to simulate on a very long time before we found an instant with four contacts. Note also that the trajectory seems to oscillates more with the second order. This is probably due to the mass conservation.

The numerical simulations required a particular care to compute the reflection term which is very irregular. It absolutely necessary to do a special treatment for the reflection measure which is fundamental to ensure positivity, both in the second and fourth order cases. We described three different algorithms we have used.

First algorithm: penalization.

The first idea is to use a penalization method as in the theoretical proof of existence ([29]). We

have simulated a process which is the solution of the following equation:

$$\begin{cases} \dfrac{\partial X}{\partial t} = -\dfrac{1}{2}\Delta\Big(\Delta X + f_\varepsilon(X)\Big) + B\dot{W}, \\ X(t,.) = 0, \text{ on } \partial\Omega, \qquad \text{for all } t \geq 0, \end{cases} \tag{3.5.9}$$

supplemented with Dirichlet-Neumann boundary conditions.

The reflection force is replaced by the function f_ε

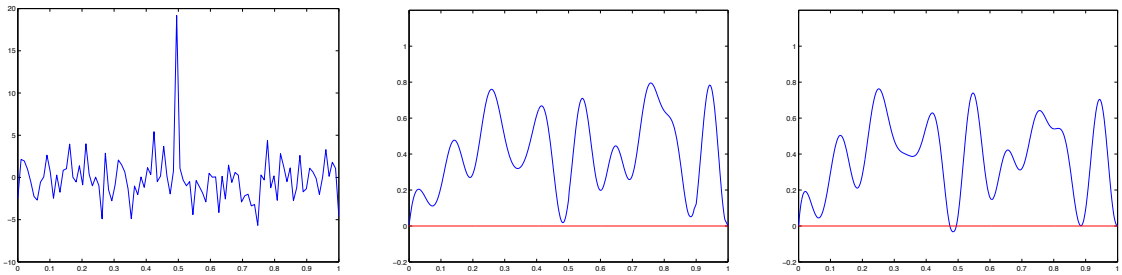
$$f_\varepsilon : x \mapsto -\frac{1}{\varepsilon}(x)^- \tag{3.5.10}$$

corresponding to a classical penalization method.

The parameter ε must be chosen with precaution. Indeed, a parameter ε too small leads to prohibitive computational time. A too large parameter does not simulate a good reflection force.

For any fixed value of the parameter ε, the process is almost surely negative at some instants. We denote $(t_i)_{0\leq i\leq N}$ the times of the simulation on the time intervall $[0,1]$. The time step is $\tau := 1/N$. The simulated process at the instant t_n is denoted X_{t_n}. If at the time t_{n-1}, the process $X_{t_{n-1}}$ is near the zero value in a point $x_0 \in]0,1[$, the penalization is not active. Then, the noise may be arbitrarily large and push the process out of the positive area, the penalization will become active during the iterations of the Newton algorithm described in (3.1.6). When the Newton iterates have converged, the process X_{t_n} can be in the negative area or completely push back in the positive area. We have observed that most of the time there exists a balance between the reflection force and the noise which enforces the process X_{t_n} to stay in the domain $[-\varepsilon, +\infty[$. The parameter ε may be interpreted as a tolerance on the negativity of the process.

However, it happens that the noise has either a very pronounced peak or several successive moderate peaks. Then the reflection force is not strong enough to prevent the process from becoming too negative. The process X_{t_n} does not remain in the domain $[-\varepsilon, +\infty[$. On figures 3.46, we have drawn a noise with a pronounced peak at the time t_n, the process $X_{t_{n-1}}$ and the process X_{t_n}. We can see that the process at the time t_n is too negative ($X_{t_n}(1/2) << -\varepsilon$).



(a) Bruit (b) $X_{t_{n-1}}$ (c) X_{t_n}

Figure 3.46: Noise with a prononced peak.

In conclusion, the simple penalization method is not adapted to the stochastic reflection problem.

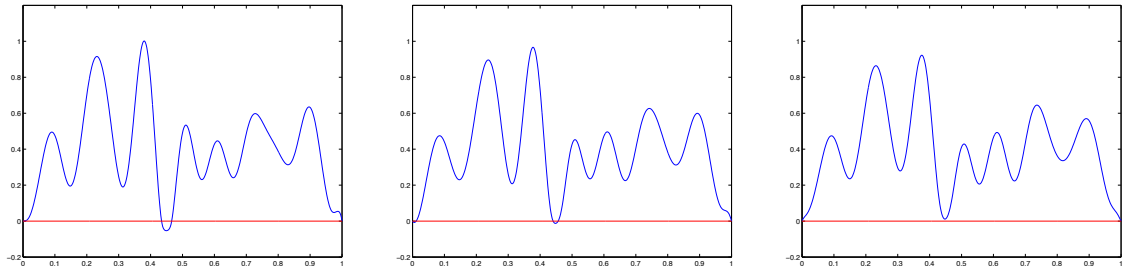
Second algorithm: adaptative penalization.

However, for a given process $X_{t_{n-1}}$ at the time t_{n-1} and a given noise at the time t_n, there exists a parameter ε small enough to prevent the process from becoming too negative at the time t_n. It is therefore natural to try a dynamical parameter ε. In order to chose the parameter ε at each time, we have to know in advance that the process will become too negative.

If at the time t_n the process X_{t_n} becomes negative, it is enough to restart the simulation from the time t_{n-1}, with the same noise, but with a smaller penalization parameter ε. We obtain a new process X'_{t_n} at the time t_n which is driven by a stronger reflection force in order to compensate the noise. If the process X'_{t_n} is still too negative, we restart again with an other smaller parameter

ε, else we continue and we give the default value to the parameter ε. This strategy depends on a negativity tolerance parameter fixed once for all at the beginning of the simulation.

On figure 3.47(a), we have drawn the process X obtained at the time t_n with a fixed parameter ε. The process is too negative, and a new strategy must be used. So we restart the simulation with a new process X'_{t_n} with ε divided by 2 (see X'_{t_n} on figure 3.47(b)). The negativity is still too important, the strategy is iterated in order to obtain a new process X''_{t_n} drawn on figure 3.47(c). For this latter process, the reflection force is strong enough to push back the process in the positive area.



(a) Fixed penalization parameter ε (b) $\varepsilon/2$ (c) $\varepsilon/4$

Figure 3.47: Adaptative penalization strategy.

On figure 3.48, we have drawn the evolution of the parameter ε - in logarithmic scale - according to the time. The default parameter is $\varepsilon = 10^{-2}$ and the tolerance is 10^{-3}. We see that most of the time or or two iterations are sufficient but it is sometimes necessary to go to four iterations. Theoretically, the number of iterations can be unbounded but in practise it stays moderate. This method is therefore satisfactory.

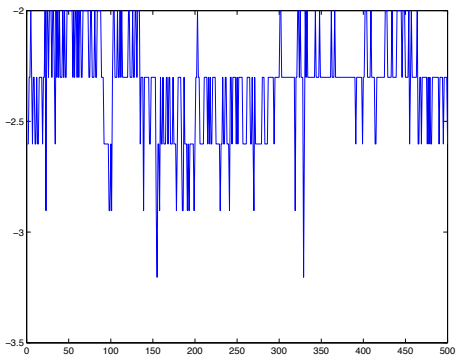


Figure 3.48: Evolution of the penalization parameter ε.

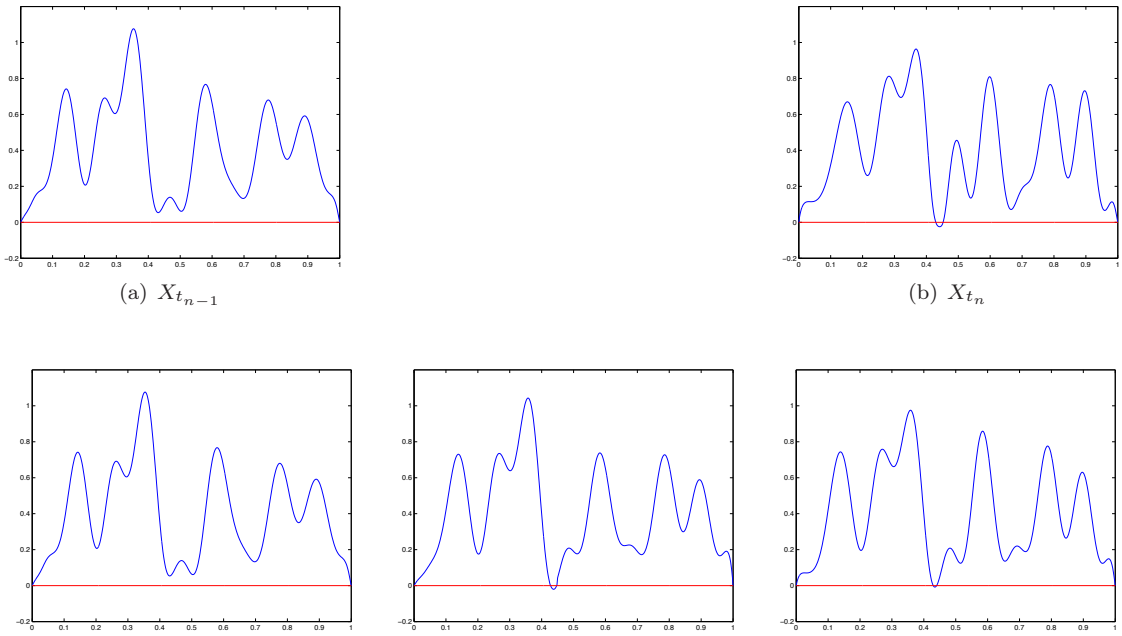
Third algorithm: adaptativity in time

In the third method, we fix ε but allow a dynamical time step. If at time t_{n-1}, the solution is non negative and becomes too negative after one time step. We restart from t_{n-1} but with half a time step. We thus obtain the value of the process at $t_{n-1/2}$, denoted by $X'_{t_{n-1/2}}$. We start again with half a time step and obtain X'_{t_n} which is supposed to be a better value of the process at time t_n. Figures 3.49(a) and 3.49(b), show X at time t_{n-1} and t_n. Figures 3.49(c), 3.49(d) and 3.49(e) show X' at t_{n-1}, $t_{n-1/2}$ and t_n.

The new process X'_{t_n} is much better than X_{t_n} and is non negative because the reflection force is active twice. This is this double action of the penalization which ensures the positivity of the process X'_{t_n}.

Again, if X'_{t_n} is negative, we start again the algorithm and divide the time step by four etc ...

Note that we have to be careful with simulation of the noise if we want to have a consistant simulation. Indeed, we start with $X_{t_{n-1}}(\omega)$, we simulate $X_{t_n}(\omega)$ with an increment of the noise



(a) $X_{t_{n-1}}$ (b) X_{t_n}

(c) $X'_{t_{n-1}} = X_{t_{n-1}}$ (d) $X'_{t_{n-1/2}}$ (e) X'_{t_n}

Figure 3.49: Strategy with dynamical time steps.

$\delta W_n(\omega)$. But, if we restart the simulation from $X_{t_{n-1}}(\omega)$, we need to simulate a new increment $\delta W'_{n-1/2}(\omega)$ in order to simulate $X'_{t_{n-1/2}}$. Then we simulate $X'_{t_n}(\omega)$ from $X_{t_{n-1/2}}(\omega)$ with another increment noise $\delta W'_n(\omega)$. But $\delta W'_{n-1/2}(\omega)$, $\delta W'_n(\omega)$ and $\delta W_n(\omega)$ are completely different. There is no reason that $X'(t_n)$ is close to $X(t_n)$ because the noise has been changed. Actually, the noise $\delta W'_n(\omega)$ corresponds to an other realization on the probabilistic space Ω and we must write it $\delta W'_n(\tilde{\omega})$ with $\tilde{\omega} \neq \omega$. In the same way, we must write $X'_{t_n}(\tilde{\omega})$ and not $X'_{t_n}(\omega)$. Since we do not compare the same realization, the distance between $X_{t_n}(\omega)$ and $X'_{t_n}(\tilde{\omega})$ can be arbitrary large.

It is easy to overcome this problem. The increments $\delta W'_{n-1/2}(\omega)$, $\delta W'_n(\omega)$ should be chosen in a consistent manner with respect to the previously simulated increment δW_n. So we fix the increment between the times t_{n-1} et t_n. Then we have to simulate $\delta W'_{n-1/2}$ and $\delta W'_n$ such that $\delta W_n = \delta W'_{n-1/2} + \delta W'_n$. The law of the half-increment $\delta W'_{n-1/2}$ and $\delta W'_n$ is a centered Gaussian law of variance $\tau/2$, and these increments are all independent. In a nut shell, we have the problem:

Problem :

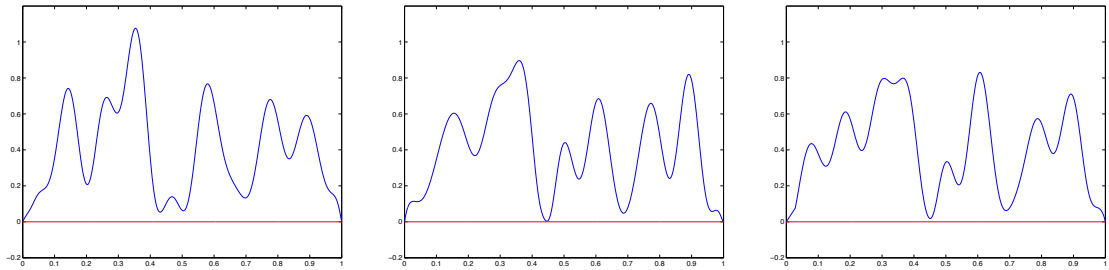
Let $U \sim \mathcal{N}(0, \tau)$, we want $R \sim \mathcal{N}\left(0, \frac{\tau}{2}\right)$ and $S \sim \mathcal{N}\left(0, \frac{\tau}{2}\right)$ such that $U = R + S$ with independent processes R and S.

Solution :

It is enough to simulate $V \sim \mathcal{N}(0, \tau)$ independent of U. Then we set $R = \frac{U+V}{2}$ and $S = \frac{U-V}{2}$.

On figure 3.49, we have drawn the results obtained with our third method with this correct simulation of the noise. If we do not take into account this subtility, we can obtain a completely different realization of the process (see figure 3.50). Note that without this good simulation of the noise, we in fact reject the paths of the noise which would have activated the reflection so that our algorithm would be a selection method of the trajectories driven by "gentle" noises.

Both the second and third algorithms give very good results.



(a) $X'_{t_{n-1}} = X_{t_{n-1}}$ (b) $X'_{t_{n-1/2}}(\tilde{\omega})$ (c) $X'_{t_n}(\tilde{\omega})$

Figure 3.50: Strategy with dynamical time steps with bad noise refinement.

Bibliography

[1] M. Ainsworth. Discrete dispersion relation for *hp*-version finite element approximation at high wave number. *SIAM J. Numer. Anal.*, 42(2):553–575 (electronic), 2004.

[2] I. Babuška, B. A. Szabo, and I. N. Katz. The *p*-version of the finite element method. *SIAM J. Numer. Anal.*, 18(3):515–545, 1981.

[3] J. W. Barrett and J. F. Blowey. Finite element approximation of the Cahn-Hilliard equation with concentration dependent mobility. *Math. Comp.*, 68(226):487–517, 1999.

[4] J. W. Barrett, J. F. Blowey, and H. Garcke. Finite element approximation of the Cahn-Hilliard equation with degenerate mobility. *SIAM J. Numer. Anal.*, 37(1):286–318 (electronic), 1999.

[5] P. W. Bates and P. C. Fife. The dynamics of nucleation for the Cahn-Hilliard equation. *SIAM J. Appl. Math.*, 53(4):990–1008, 1993.

[6] C. Bernardi and Y. Maday. Spectral methods. In *Handbook of numerical analysis, Vol. V*, Handb. Numer. Anal., V, pages 209–485. North-Holland, Amsterdam, 1997.

[7] D. Blömker and M. Hairer. Stationary solutions for a model of amorphous thin-film growth. *Stochastic Anal. Appl.*, 22(4):903–922, 2004.

[8] D. Blömker, S. Maier-Paape, and T. Wanner. Spinodal decomposition for the Cahn-Hilliard-Cook equation. *Communications in Mathematical Physics*, 223(3):553–582, 2001.

[9] D. Blömker, S. Maier-Paape, and T. Wanner. Phase separation in stochastic Cahn-Hilliard models. *Mathematical Methods and Models in Phase Transitions*, pages 1–41, 2005.

[10] D. Blömker, S. Maier-Paape, and T. Wanner. Second phase spinodal decomposition for the Cahn-Hilliard-Cook equation. *Trans. Amer. Math. Soc.*, 360(1):449–489 (electronic), 2008.

[11] J. F. Blowey and C. M. Elliott. The Cahn-Hilliard gradient theory for phase separation with nonsmooth free energy. I. Mathematical analysis. *European J. Appl. Math.*, 2(3):233–280, 1991.

[12] V. Bonnaillie-Noël, M. Dauge, D. Martin, and G. Vial. Computations of the first eigenpairs for the Schrödinger operator with magnetic field. *Comput. Methods Appl. Mech. Engrg.*, 196(37-40):3841–3858, 2007.

[13] J. W. Cahn. On spinodal decomposition. *Acta Metallurgica*, 9(9):795–801, September 1961.

[14] J. W. Cahn, C. M. Elliott, and A. Novick-Cohen. The Cahn-Hilliard equation with a concentration dependent mobility: motion by minus the Laplacian of the mean curvature. *European J. Appl. Math.*, 7(3):287–301, 1996.

[15] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system. i. interfacial free energy. *Journal of Chemical Physics*, 28(258), February 1958.

[16] J. W. Cahn and J. E. Hilliard. Spinodal decomposition: a reprise. *Acta Metallurgica*, 19(2):151–161, February 1971.

[17] C. Cardon-Weber. Cahn-Hilliard stochastic equation: existence of the solution and of its density. *Bernoulli*, 7(5):777–816, 2001.

[18] C. Cardon-Weber. Cahn-Hilliard stochastic equation: strict positivity of the density. *Stoch. Stoch. Rep.*, 72(3-4):191–227, 2002.

[19] H. Cook. Brownian motion in spinodal decomposition. *Acta Metallurgica*, 18:297–306, 1970.

[20] M. I. M. Copetti and C. M. Elliott. Numerical analysis of the Cahn-Hilliard equation with a logarithmic free energy. *Numer. Math.*, 63(1):39–65, 1992.

[21] G. Da Prato and A. Debussche. Stochastic Cahn-Hilliard equation. *Nonlinear Anal.*, 26(2):241–263, 1996.

[22] G. Da Prato, A. Debussche, and B. Goldys. Some properties of invariant measures of non symmetric dissipative stochastic systems. *Probab. Theory Related Fields*, 123(3):355–380, 2002.

[23] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.

[24] G. Da Prato and J. Zabczyk. *Ergodicity for infinite-dimensional systems*, volume 229 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1996.

[25] G. Da Prato and J. Zabczyk. *Second order partial differential equations in Hilbert spaces*, volume 293 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002.

[26] R. C. Dalang, C. Mueller, and L. Zambotti. Hitting properties of parabolic s.p.d.e.'s with reflection. *Ann. Probab.*, 34(4):1423–1450, 2006.

[27] A. Debussche and L. Dettori. On the Cahn-Hilliard equation with a logarithmic free energy. *Nonlinear Anal.*, 24(10):1491–1514, 1995.

[28] A. Debussche and L. Goudenège. Stochastic Cahn-Hilliard equation with double singular nonlinearities and two reflections. *To appear*, 2009.

[29] A. Debussche and L. Zambotti. Conservative stochastic Cahn-Hilliard equation with reflection. *Ann. Probab.*, 35(5):1706–1739, 2007.

[30] I. V. Denisov. Random walk and the Wiener process considered from a maximum point. *Teor. Veroyatnost. i Primenen.*, 28(4):785–788, 1983.

[31] N. Elezović and A. Mikelić. On the stochastic Cahn-Hilliard equation. *Nonlinear Anal.*, 16(12):1169–1200, 1991.

[32] C. M. Elliott and D. A. French. Numerical studies of the Cahn-Hilliard equation for phase separation. *IMA J. Appl. Math.*, 38(2):97–128, 1987.

[33] F. Flandoli and M. Romito. Statistically stationary solutions to the 3-D Navier-Stokes equation do not show singularities. *Electron. J. Probab.*, 6:no. 5, 15 pp. (electronic), 2001.

[34] M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, second edition, 1998. Translated from the 1979 Russian original by Joseph Szücs.

[35] M. Fukushima, Y. Ōshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[36] T. Funaki and K. Ishitani. Integration by parts formulae for Wiener measures on a path space between two curves. *Probab. Theory Related Fields*, 137(3-4):289–321, 2007.

[37] T. Funaki and S. Olla. Fluctuations for $\nabla\phi$ interface model on a wall. *Stochastic Process. Appl.*, 94(1):1–27, 2001.

[38] G. Giacomin, S. Olla, and H. Spohn. Equilibrium fluctuations for $\nabla\phi$ interface model. *Ann. Probab.*, 29(3):1138–1172, 2001.

[39] L. Goudenège. Stochastic Cahn-Hilliard equation with singular nonlinearity and reflection. *Stochastic Processes and their Applications*, 2009.

[40] C. P. Grant. Spinodal decomposition for the Cahn-Hilliard equation. *Comm. Partial Differential Equations*, 18(3-4):453–490, 1993.

[41] M. Grinfeld and A. Novick-Cohen. Counting stationary solutions of the Cahn-Hilliard equation by transversality arguments. *Proc. Roy. Soc. Edinburgh Sect. A*, 125(2):351–370, 1995.

[42] I. Gyöngy and N. Krylov. Existence of strong solutions for Itô's stochastic equations via approximations. *Probab. Theory Related Fields*, 105(2):143–158, 1996.

[43] Y. Hariya. Integration by parts formulae for Wiener measures restricted to subsets in $\mathbb{R}^d$. *J. Funct. Anal.*, 239(2):594–610, 2006.

[44] P. C. Hohenberg and B. I. Halperin. Theory of dynamic critical phenomena. *reviews of Modern Physics*, 49(3):435–479, 1977.

[45] F. Ihlenburg and I. Babuška. Finite element solution of the Helmholtz equation with high wave number. I. The h-version of the FEM. *Comput. Math. Appl.*, 30(9):9–37, 1995.

[46] F. Ihlenburg and I. Babuška. Finite element solution of the Helmholtz equation with high wave number. II. The h-p version of the FEM. *SIAM J. Numer. Anal.*, 34(1):315–358, 1997.

[47] D. Kay and R. Welford. A multigrid finite element solver for the Cahn-Hilliard equation. *J. Comput. Phys.*, 212(1):288–304, 2006.

[48] S. Kuksin and A. Shirikyan. A coupling approach to randomly forced nonlinear PDE's. I. *Comm. Math. Phys.*, 221(2):351–366, 2001.

[49] J. S. Langer. Theory of spinodal decomposition in alloys. *Annals of Physics*, 65:53–86, 1971.

[50] T. Ma and S. Wang. Cahn-Hilliard equations and phase transition dynamics for binary systems. *arXiv*, 0806.1286v1, 2008.

[51] Z. M. Ma and M. Röckner. *Introduction to the theory of (nonsymmetric) Dirichlet forms*. Universitext. Springer-Verlag, Berlin, 1992.

[52] S. Maier-Paape and U. Miller. Path-following the equilibria of the Cahn-Hilliard equation on the square. *Comput. Vis. Sci.*, 5(3):115–138, 2002.

[53] S. Maier-Paape and T. Wanner. Spinodal decomposition for the Cahn-Hilliard equation in higher dimensions. I. Probability and wavelength estimate. *Comm. Math. Phys.*, 195(2):435–464, 1998.

[54] S. Maier-Paape and T. Wanner. Spinodal decomposition for the Cahn-Hilliard equation in higher dimensions: nonlinear dynamics. *Arch. Ration. Mech. Anal.*, 151(3):187–219, 2000.

[55] D. Martin. The finite element library Mélina.
http://perso.univ-rennes1.fr/daniel.martin/melina, 2008.

[56] J. C. Mattingly. Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics. *Comm. Math. Phys.*, 230(3):421–462, 2002.

[57] T. Nishikawa. Hydrodynamic limit for the Ginzburg-Landau $\nabla\phi$ interface model with a conservation law. *J. Math. Sci. Univ. Tokyo*, 9(3):481–519, 2002.

[58] A. Novick-Cohen. The Cahn-Hilliard equation: mathematical and modeling perspectives. *Adv. Math. Sci. Appl.*, 8(2):965–985, 1998.

[59] A. Novick-Cohen and L. A. Peletier. Steady states of the one-dimensional Cahn-Hilliard equation. *Proc. Roy. Soc. Edinburgh Sect. A*, 123(6):1071–1098, 1993.

[60] A. Novick-Cohen and L. A. Segel. Nonlinear aspects of the Cahn-Hilliard equation. *Phys. D*, 10(3):277–298, 1984.

[61] D. Nualart and É. Pardoux. White noise driven quasilinear SPDEs with reflection. *Probab. Theory Related Fields*, 93(1):77–89, 1992.

[62] C. Odasso. Exponential mixing for the 3D stochastic Navier-Stokes equations. *Comm. Math. Phys.*, 270(1):109–139, 2007.

[63] Y. Otobe. Stochastic partial differential equations with two reflecting walls. *J. Math. Sci. Univ. Tokyo*, 13(2):129–144, 2006.

[64] R. L. Pego. Front migration in the nonlinear Cahn-Hilliard equation. *Proc. Roy. Soc. London Ser. A*, 422(1863):261–278, 1989.

[65] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1991.

[66] E. Sander and T. Wanner. Monte Carlo simulations for spinodal decomposition. *J. Statist. Phys.*, 95(5-6):925–948, 1999.

[67] E. Sander and T. Wanner. Unexpectedly linear behavior for the Cahn-Hilliard equation. *SIAM J. Appl. Math.*, 60(6):2182–2202 (electronic), 2000.

[68] H. Spohn. Interface motion in models with stochastic dynamics. *J. Statist. Phys.*, 71(5-6):1081–1132, 1993.

[69] E. P. Stephan and M. Suri. On the convergence of the p-version of the boundary element Galerkin method. *Math. Comp.*, 52(185):31–48, 1989.

[70] R. H. Stogner, G. F. Carey, and B. T. Murray. Approximation of Cahn-Hilliard diffuse interface models using parallel adaptive mesh refinement and coarsening with C^1 elements. *Internat. J. Numer. Methods Engrg.*, 76(5):636–661, 2008.

[71] J. E. Taylor and J. W. Cahn. Linking anisotropic sharp and diffuse surface motion laws via gradient flows. *J. Statist. Phys.*, 77(1-2):183–197, 1994.

[72] T. Wanner. Maximum norms of random sums and transient pattern formation. *Trans. Amer. Math. Soc.*, 356(6):2251–2279 (electronic), 2004.

[73] L. Zambotti. Integration by parts formulae on convex sets of paths and applications to SPDEs with reflection. *Probab. Theory Related Fields*, 123(4):579–600, 2002.

[74] L. Zambotti. Integration by parts on δ-Bessel bridges, $\delta > 3$ and related SPDEs. *Ann. Probab.*, 31(1):323–348, 2003.

[75] L. Zambotti. Fluctuations for a $\nabla\phi$ interface model with repulsion from a wall. *Probab. Theory Related Fields*, 129(3):315–339, 2004.

[76] L. Zambotti. A conservative evolution of the Brownian excursion. *Electron. J. Probab.*, 13:no. 37, 1096–1119, 2008.

[77] L. Zambotti. Fluctuations for a conservative interface model on a wall. *ALEA Lat. Am. J. Probab. Math. Stat.*, 4:167–184, 2008.

Résumé

Quelques résultats sur l'équation de Cahn-Hilliard stochastique et déterministe

Nous nous intéressons d'abord à l'équation aux dérivées partielles stochastique de Cahn-Hilliard en dimension 1 avec une seule singularité. C'est une équation d'ordre 4 dont la non linéarité est de type logarithmique ou en puissance négative $x^{-\alpha}$, à laquelle on ajoute la dérivée d'un bruit blanc en espace et en temps. On montre l'existence et l'unicité des solutions en utilisant les solutions d'équations approchées aux non linéarités Lipschitz. La présence d'une mesure de réflexion permet d'assurer l'existence de solutions. On étudie ces mesures à l'aide des mesures de Revuz associées et, grâce à une formule d'intégration par parties, on montre qu'elles sont identiquement nulles lorsque $\alpha \geq 3$.

Dans un deuxième temps, on considère la même équation mais avec deux singularités logarithmiques en ± 1. Il s'agit du modèle complet de l'équation de Cahn-Hilliard. Cette fois-ci on utilise des équations approchées aux non linéarités polynomiales pour montrer l'existence et l'unicité de solutions. Deux mesures de réflexion doivent ici être ajoutées pour assurer l'existence. De plus, on montrera que la mesure invariante est ergodique.

Enfin, on étudie l'équation déterministe : des simulations numériques basées sur une méthode d'éléments finis de hauts degrés permettent d'illustrer plusieurs résultats théoriques. La capture des interfaces et des états stationnaires requiert une attention particulière. On s'intéressera également aux bifurcations autour de la première valeur propre du Laplacien sur des domaines généraux.

Par ailleurs, quelques simulations stochastiques permettent de mettre en évidence les instants de contact avec les singularités, les évolutions stochastiques en temps long et les changements d'états stationnaires.

Mots clés : équations aux dérivées partielles stochastiques, Cahn-Hillliard, mesures invariantes, mesures de réflexion, mesures de Revuz, singularité, non linéarité logarithmique, formule d'intégration par parties, ergodicité, éléments finis, hauts degrés, bifurcations, interface, contact, états stationnaires, évolutions en temps long.

Abstract

Some results about the stochastic and deterministic Cahn-Hilliard equation.

In a first part, we are concerned with the stochastic Cahn-Hilliard partial differential equation in dimension 1 with one singularity. This is an equation of order 4 driven by the derivative of a space-time white noise. There is a logarithmic nonlinearity or a negative power $x^{-\alpha}$ nonlinearity. Thanks to Lipschitz approximated equations, we show existence and uniqueness of a solution. It is necessary to add a reflection measure to ensure existence. We study it with the associated Revuz measure. Thanks to the associated integration by parts formula, we can show that the reflection measure vanishes for $\alpha \geq 3$.

In a second part, we consider the same equation but with two logarithmic singularities in ± 1. This is the full Cahn-Hilliard's model. With polynomial approximated equations, we show the existence and uniqueness. We should add two reflection measures to ensure the existence. Moreover, we establish that the invariant measure is ergodic.

Finally, we are concerned with the deterministic equation. Some numerical simulations based on a high order finite elements method have been computed. We study the bifurcations around the first eigenvalue of the Laplace operator on general domains, the interfaces and the stationary states.

Furthermore, some stochastic simulations have been computed in order to show the contacts with the singular values. The long time evolutions and the jump between stationary states are also treated.

Keywords: Stochastic partial differential equation, Cahn-Hillliard, invariant measures, reflection measures, Revuz measures, singularity, logarithmic non-linearity, integration by parts formula, ergodicity, finite elements, high degrees, bifurcations, interface, contact, stationary states, long time evolutions.

N° d'ordre :

École normale supérieure de Cachan - Antenne de Bretagne

Campus de Ker Lann - Avenue Robert Schuman - 35170 BRUZ

Tél : +33(0)2 99 05 93 00 - Fax : +33(0)2 99 05 93 29