Étude de Quelques Problèmes Quasilinéaires Elliptiques Singuliers

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Résumé

L'objet de cette thèse est l'étude de quelques problèmes elliptiques singuliers. Dans les problèmes que nous avons considérés, ce caractère singulier se caractérise par la présence d'une non-linéarité qui explose au bord du domaine où le problème est posé. Ceci pose un certain nombre de difficultés liées au manque de régularité des solutions et donc de compacité qui ne permettent pas d'utiliser directement des méthodes classiques de l'analyse non-linéaire. Nous avons, à travers les problèmes abordés dans les Chapitres 2, 3 et 4, montré comment ces difficultés peuvent être surmontées et apporté des résultats nouveaux en particulier sur le nombre de solutions et leur régularité. L'ingrédient principal est la détermination précise du comportement des solutions au bord du domaine. On peut alors adapter et généraliser un certain nombre de méthodes classiques comme les méthodes variationnelles, les méthodes de monotonie (en particulier la technique des sur et sous solutions), les méthodes liées à la théorie des équations différentielles, les méthodes liées à la théorie de la bifurcation. Il est important de remarquer que la présence de la singularité peut parfois (c'est le cas de certains résultats du Chapitre 4) donner lieu à des résultats nouveaux qui contrastent avec le cas "régulier". Nous tenons également à souligner que la plupart des résultats nouveaux ont été obtenus en combinant différentes méthodes d'analyse non-linéaire. Enfin, l'étude des problèmes singuliers a fait l'objet d'une abondante littérature traitant de questions très diverses. On pourra à cet effet consulter l'article revue de Hernandez-Mancebo [37] ou l'ouvrage de Ghergu-Rădulescu [31] pour une présentation générale de ces résultats. Nous nous sommes limités ici dans cette thèse au cas où la non-linéarité présente une croissance critique (Chapitre 2 et 4) et au cas où l'opérateur de diffusion est un opérateur quasilinéaire elliptique dégénéré (du type p-laplacien, Chapitre 3).

Précisément, dans le Chapitre 2, nous abordons la question de la multiplicité de solutions pour un problème elliptique critique singulier en dimension $N \geq 3$. Les résultats obtenus améliorent des résultats précédemment obtenus par Haitao [35] et Hirano-Saccon-Shioji [36].

Dans le Chapitre 3, nous discutons la validité de la propriété C^1 versus $W_0^{1,p}$ minimiseurs de l'énergie pour un problème quasilinéaire elliptique singulier. Ces résultats permettent alors par des méthodes variationnelles d'établir la multiplicité des solutions. Ce travail fait suite par ailleurs à Giacomoni-Schindler-Takáč [30] où est démontrée l'existence de solutions multiples pour un problème quasi-linéaire singulier du même type. En utilisant une méthode différente (introduite dans Brock-Iturriga-Ubilla [15]), nous simplifions la preuve et généralisons un des résultats énoncés dans ce travail.

Enfin, dans le Chapitre 4, nous présentons des résultats de bifurcation globale pour un problème semilinéaire elliptique singulier et critique en dimension 2 qui fait apparaître un paramètre de bifurcation. En utilisant des techniques liées à la théorie de la bifurcation qu'on adapte au cas singulier, on montre l'existence des branches globales (i.e. continua) de solutions pour ce problème. On détermine par ailleurs le comportement général de ces continua de solutions. Dans le cas à symétrie radiale, on établit précisément le comportement global de ces branches ainsi que le comportement des solutions près du point de bifurcation asymptotique. Les résultats obtenus montrent une grande variété de comportement qui est fonction du comportement asymptotique (à l'infini) de la non-linéarité. Pour certains, ces résultats sont liés au caractère singulier de la non-linéarité. Nous utilisons entre autres une méthode de tir et la transformation d'Emden-Fowler pour traiter en détail le cas radial.

Mots clefs : Problèmes quasilinéaires elliptiques avec coefficients singuliers, solutions positives multiples, multiplicité, unicité, régularité, méthodes variationnelles, méthode de bifurcation, méthode de tir, méthodes de monotonie.

Abstract

This thesis is concerned with the study of some singular elliptic problems. In these problems the singularity is due to a nonlinear term that blows up on the boundary. Therefore we encounter a certain number of difficulties linked to the lack of regularity for the approximate solutions and hence a lack of compactness. The lack of compactness makes classical methods of nonlinear analysis more delicate. In Chapters 2, 3, and 4, we show how these difficulties can be dealt with and establish new

results concerning the existence of multiple solutions to singular problems and the regularity of such solutions. The main tool is the study of the solutions near the boundary of the domain. This allows us to adapt a number of classical variational and monotonie methods as well as methods from the theory of differential equations and from the theory of bifurcation. Some of the results we establish contrast sharply with results to the analogous non singular problem. Most of the results we establish are obtained by combining several methods from nonlinear analysis. Singular problems have been studied in many recent works. We refer the reader to the survey article of Hernandez-Mancebo [37] as well as the work by Ghergu-Rădulescu [31] for a more general presentation of these results. In this thesis we have limited our presentation to singular problems involving critical growth (chapters 2 and 4) and to the case where the diffusion term is a degenerate quasilinear elliptic operator (the p-laplacian, Chapter 3).

Precisely, in Chapter 2, we investigate the question of multiplicity of solutions for a singular problem with critical growth in dimension $N \geq 3$. We obtain results that improve those of Haitao [35] and Hirano-Saccon-Shioji [36].

In Chapter 3, we investigate the validity of C^1 versus $W_0^{1,p}$ energy minimizers for a quasilinear elliptic singular problem. This allows us to use variational methods to extend results from Giacomoni-Schindler-Takáč [30]. We simplify the proof using methods introduced in Brock-Iturriga-Ubilla [15].

In Chapter 4, we present global bifurcation results for a semilinear elliptic singular problem with critical growth in dimension 2 which involves a bifurcation parameter. Using results from the bifurcation theory which we adapt to the singular case, we prove the existence of global branches (i.e. continua) of solutions for this problem. We determine the general behavior of these continua of solutions. In the radially symmetric case we give a precise description of these global branches. The results obtained show a great variety of behaviors as a function of the asymptotic behavior of the nonlinearity (at infinity). Some of these results

are linked to the singularity of the nonlinearity. We use the shooting method as well as the Emden-Fowler transformation for the radial case.

Key words: Quasilinear elliptic problems with singular coefficients, multiple positive solutions, multiplicity, uniqueness, regularity, variational methods, bifurcation, shooting methods, monotone methods.

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DÉDICACE

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Introduction générale

L'objet de cette thèse est l'étude des problèmes quasi-linéaires elliptiques singuliers. Par "singulier", nous entendons ici que la nonlinéarité qui apparait au second membre est singulière, plus précisément explose au bord. Ce caractère singulier ne permet pas d'appliquer directement les méthodes classiques d'analyse nonlinéaire. Ceci s'explique par le défaut de régularité des outils mis en oeuvre: fonctionnelle énergie, opérateur linéarisé. Cela génère un manque de compacité qui ne permet pas d'employer directement les méthodes classiques. En conséquence, dans le cadre des problèmes abordés dans cette thèse des estimations assez fines sur les solutions sont requises pour remédier au manque de compacité. Les estimations a priori des solutions au voisinage du bord du domaine sont tout particulièrement importantes. Pour établir ces estimations, on est amené à combiner des méthodes très diverses d'analyse nonlinéaire : méthodes variationnelles, méthodes topologiques en particulier de bifurcation, méthodes d'équations différentielles ordinaires.

Notons que les problèmes qui ont été abordés dans ce travail apparaissent dans de nombreux modèles issus de la physique (optique nonlinéaire), la dynamique des populations (modèles prédateurs-proies), l'industrie (industrie pétrolière). On pourra consulter avec profit l'article revue Hernandez-Mancebo [37] où est donnée une bibliographie importante qui présent-ent ces modèles.

Cette thèse est organisée de la façon suivante: nous commençons par rappeler quelques définitions et techniques utilisées dans ce travail. Ceci fait l'objet du premier chapitre. Ensuite, dans le second chapitre, nous abordons la question de la multiplicité de solutions pour un problème elliptique critique singulier en dimension supérieure ou égale à trois. Nous discutons également pour ce problème la régularité des solutions obtenues en fonction du paramètre qui mesure la singularité. Nous traitons la question de la multiplicité en utilisant différentes méthodes variationnelles. Un des avantages de ces méthodes par rapport à d'autres méthodes comme les méthodes de point fixe par exemple est d'obtenir l'existence de solutions dans des conditions relativement faibles de régularité de la fonctionnelle énergie et de l'espace des solutions. Ce travail améliore des résultats précédents obtenus par HAITAO [35] et HIRANO-SACCON-SHIOJI [36]. Ce travail a par ailleurs fait l'objet d'une publication dans Nonlinear Analysis, Theory, Methods and Applications.

Dans le chapitre suivant, nous considérons la version quasilinéaire du problème traité dans le Chapitre 2. L'opérateur quasi-linéaire elliptique du second ordre est ici un opérateur dégénéré de type p-laplacien. Dans ce cadre, l'existence de solutions multiples est liée aux propriétés géométriques de la fonctionnelle énergie. Une étape importante est notamment de démontrer l'existence d'un minimum local de l'énergie. Suite au travail pionnier de BREZIS-NIRENBERG [13], l'existence d'un minimum local peut être obtenue en combinant un résultat de régularité höldérienne et le principe de comparaison fort. La combinaison de ces deux ingrédients conduit à établir que tout C^1 minimiseur local de la fonctionnelle énergie est également un minimiseur local dans l'espace d'énergie. Nous discutons la validité de cette propriété pour le problème quasilinéaire elliptique singulier considéré dans ce chapitre. Ceci requiert de démontrer des résultats précis sur la régularité des solutions faibles pour ce problème singulier. Nous faisons appel pour cela à certains résultats établis dans GIACOMONI-SCHINDLER-TAKÁČ [30] et BROCK-ITURRAGA-UBILLA [15]. L'avantage de la méthode développée dans ce chapitre est qu'elle peut être appliquée à d'autres types d'opérateurs quasi-linéaires et ainsi conduire à des travaux ultérieurs.

Enfin, dans le dernier chapitre, nous présentons des résultats de bifurcation globale pour un problème semilinéaire elliptique singulier en dimension 2 faisant apparaitre un paramètre dit "paramètre de bifurcation". On utilise en premier lieu des méthodes topologiques pour montrer l'existence et le comportement des branches globales (i.e. continua) de solutions. Ces méthodes sont en effet adaptées pour démontrer la connexité d'ensembles de solutions. Ce qui peut être difficilement obtenu par des méthodes variationnelles. On est amené à généraliser dans le cas singulier les outils classiques de la théorie de la bifurcation comme l'utilisation du degré topologique de Leray-Schauder, la théorie locale de la bifurcation, en particulier la bifurcation à partir d'une valeur propre simple et isolée (résultat dans CRANDALL-RABINOWITZ [17]). Pour cela, nous utilisons quelques résultats démontrés dans HERNANDEZ-MANCEBO-VEGA [38] à propos de la différentiabilité de certains opérateurs singuliers et de l'existence de valeurs propres principales de l'opérateur linéarisé (lui-même singulier). Dans le cas à symétrie radiale, on est en mesure d'affiner les résultats obtenus dans le cas général. En particulier, on démontre dans un certain nombre de cas l'unicité du point de bifurcation asymptotique et on donne le comportement des solutions (analyse du blow-up) près de ce point. Les résultats obtenus dépendent du comportement asymptotique de la nonlinéarité à l'infini. Quand la nonlinéarité a un comportement critique, on obtient des résultats très différents suivant la nature de la perturbation. Ceci requiert des estimations à priori très fines obtenues via une méthode de tir et la transformation d'Emden-Fowler comme dans Atkinson-Peletier [8].

Nous allons dans la suite, faire une description générale des chapitres présenté plus haut.

Chapitre 1

L'objectif de ce chapitre est de rappeler l'essentiel des notations et des techniques utilisées dans les trois autres chapitres. Le chapitre est organisé comme suit: dans un premier paragraphe, nous rappelons quelques définitions et résultats sur les espaces de Sobolev. Le paragraphe suivant a pour objet de présenter différentes notions et techniques qui seront utilisées dans les chapitres suivants. Notons que certaines techniques et résultats concernant plus spéfiquement le p-laplacien se trouvent rappelées en annexes A et B à la fin de ce manuscrit.

Chapitre 2

Dans ce chapitre nous abordons la question de l'existence et de la multiplicité de solutions pour un problème singulier critique. On s'intéresse tout particulièrement au problème suivant:

$$(P_\lambda)\begin{cases} -\Delta u = \lambda(u^{-\delta} + u^q + \rho(u)) \quad \text{dans } \Omega \\ u|_{\partial\Omega} = 0,\, u > 0 \text{ dans } \Omega \end{cases}$$

où $\lambda > 0, 0 < \delta$ and $q = 2^* - 1 \stackrel{\text{def}}{=} \frac{N+2}{N-2}$, Ω est un domaine borné régulier de $\mathbb{R}^N$, $N \geq 3$.

Ce travail fait suite à des nombreuses publications concernant l'existence et la régularité des solutions pour un problème elliptique singulier. Un des premiers travaux est du à Crandall-Rabinowitz-Tartar [19] dans lequel les auteurs démontrent l'unicité de la solution u dans $C^2(\Omega) \cap C(\overline{\Omega})$ de

$$(P_1)\begin{cases} Lu = g(x,u) \text{ dans } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

où L est un opérateur du second ordre uniformément elliptique, Ω est un domaine borné régulier de $\mathbb{R}^N$ avec $N \geq 1$ et $g(x,.)$ une application positive et décroissante admettant une singularité en 0 (i.e. $\lim\limits_{t\to 0^+} g(x,t) = +\infty$ uniformément en $x \in \overline{\Omega}$). Dans le cas $g(x,t) = p(x)t^{-\delta}$, on peut discuter précisément la régularité des solutions en fonction du paramètre δ qui mesure la singularité, ce qui a fait l'objet des travaux successifs GOMES [32], LAZER-MC KENNA [40], GUI-HUA LIN [34]. En particulier, lorsque $p(x) = 1$, il découle

$$u \in C^1(\overline{\Omega}) \text{ si et seulement si } \delta < 1;$$

$$\text{si } \delta > 1 \text{ alors } u \in C^{\frac{2}{\delta+1}}(\overline{\Omega}) \text{ et}$$

$$u \in H_0^1(\Omega) \text{ si et seulement si } \delta < 3.$$

Le résultat d'existence de [19] a été ensuite étendu par COCLITE-PALMIERI [16] pour des problèmes du type suivant

$$(P_2)\begin{cases} -\Delta u = \frac{\lambda}{u^\delta} + u^q \text{ dans } \Omega \\ u|_{\partial\Omega} = 0,\, u > 0 \text{ dans } \Omega \end{cases}$$

où $\delta, \lambda > 0, 1 < q$ et Ω est un domaine borné régulier de $\mathbb{R}^N$. Plus précisément, les auteurs prouvent l'existence de $\tilde{\lambda} > 0$ tel que si $0 < \lambda < \tilde{\lambda}$, le problème (P_2) admet une solution et aucune solution si $\lambda > \tilde{\lambda}$. Le résultat de multiplicité est démontré pour le cas sous critique, i.e. pour $q < \frac{N+2}{N-2}$ (resp. ∞) si $N > 2$ (resp. $N = 1,2$) par YIJING-SHAOPING-YIMING [51] en utilisant une méthode variationnelle qui consiste à minimiser sur la variété naturelle de Nehari. Cette méthode a été précédemment utilisée avec succès dans TARANTELLO [47] pour des problèmes critiques non-homogènes. Précisément les auteurs montrent l'existence d'au moins deux solutions faibles dans $H_0^1(\Omega)$ du problème (P_2) pour $0 < \delta < 1$ et pour $\lambda > 0$ petit. Ce résultat est ensuite étendu au cas critique pour $N \geq 3$ simultanément par HAITAO [35] et par HIRANO-SACCON-SHIOJI [36] en utilisant deux méthodes différentes, la méthode de Perron et la méthode de Nehari évoquée plus haut. La restriction $\delta < 1$ demeure dans ces deux travaux. Notons que dans [36] les auteurs prouvent que les solutions faibles sont des solutions classiques (i.e appartiennent à $C^2(\Omega) \cap C(\overline{\Omega})$) en imposant des conditions de régularité sur le bord du domaine affaiblies. Dans ADIMURTHI-GIACOMONI [3] les auteurs prouvent que les résultats précédents restent vraies en dimension $N = 2$ lorsque le terme surlinéaire a une croissance sur-exponentielle critique.

Dans ce chapitre, nous discutons également la question de l'existence et de la multiplicité de solutions positives pour le problème (P_λ). Notons que la présence du terme sous linéaire $u^{-\delta}$ donne l'existence d'une solution minimale pour des valeurs petites de λ nonobstant les difficultés liées à la non différentiabilité de la fonctionnelle associée. La preuve de l'existence de la première solution ne fait donc intervenir que le comportement de ρ près de 0. La condition sur le comportement de ρ à l'infini garantit elle que la condition de Palais Smale est satisfaite et donc permet de démontrer la multiplicité de solutions comme dans le travail pionnier BREZIS-NIRENBERG [14]. Ce résultat de multiplicité est établi avec $0 < \delta < 3$ ce qui améliore en ce sens les résultats contenus dans [35], [36]. Notons que la restriction sur les valeurs de δ assure que la solution u est bien dans l'espace d'énergie.
Adoptant les hypothèses suivantes

(H1) ρ est $C^1(\mathbb{R}^+)$; $\rho(0) = 0$, $\rho'(0) = 0; \rho(t) + t^q \geq 0$ pour tout $t \geq 0$

(H2) $\exists\, \beta < 2^* - 2$ tel que $\rho^-(t)t^{-\beta} \to 0$, $\rho^+(t)t^{-2^*+1} \to 0$ quand $t \to +\infty$.

On démontre le résultat d'existence de solutions multiples suivant:

Théorème 1. *Supposons que* **(H1)** *et* **(H2)** *sont satisfaites et soit* $0 < \delta < 3$. *Alors il existe* $\Lambda \in (0,\infty)$ *tel que:*

(*i*) *Pour* $0 < \lambda < \Lambda$, *il existe au moins deux solutions faibles* u_λ *et* v_λ *dans* $H_0^1(\Omega)$ *de* (P_λ) *telles que* $u_\lambda < v_\lambda$ *dans* Ω;

(*ii*) *Pour* $\lambda = \Lambda$, *il existe au moins une solution faible dans* $H_0^1(\Omega)$ *de* (P_λ);

(*iii*) *Pour* $\lambda > \Lambda$, *il n'y a aucune solution faible de* (P_λ) *dans* $H_0^1(\Omega)$.

Esquisse de preuve du théorème 1 :

Etape 1: $\Lambda > 0$. On définit Λ comme suit:

$$\Lambda = \sup\{\lambda > 0/ \text{ il existe une solution faible de } (P_\lambda)\}$$

et on appelle solution faible toute fonction u dans $H_0^1(\Omega)$, vérifiant l'équation au sens des distributions, i.e. $\forall \phi \in C_c^\infty(\Omega)$,

$$\int_\Omega \nabla u \nabla \phi \,\mathrm{d}x = \lambda \left(\int_\Omega u^{-\delta} \phi \,\mathrm{d}x + \int_\Omega u^q \phi \,\mathrm{d}x \int_\Omega \rho(u) \phi \,\mathrm{d}x \right).$$

En utilisant une technique de sur et sous solution, on démontre l'existence de solutions faibles pour λ petit, ce qui implique $\Lambda > 0$.

Etape 2: Existence de solutions pour tout $\lambda \in]0,\Lambda]$
Puisque la fonctionnelle est non différentiable, on utilise la méthode de Perron qui est une version variationnelle de la méthode de sur et sous solutions. Elle consiste à minimiser la fonctionnelle associée E_λ, définie sur $H_0^1(\Omega)$ par la relation suivante

$$E_\lambda(u) = \begin{cases} \frac{1}{2}\int_\Omega \mid \nabla u \mid^2 \mathrm{d}x - \frac{\lambda}{1-\delta}\int_\Omega (u^+)^{1-\delta}\mathrm{d}x - \frac{\lambda}{q+1}\int_\Omega (u^+)^{q+1}\mathrm{d}x \\ -\lambda \int_\Omega \tilde{\rho}(u^+)\mathrm{d}x \ \text{ si } \ \delta \neq 1 \\ \frac{1}{2}\int_\Omega \mid \nabla u \mid^2 \mathrm{d}x - \lambda \int_\Omega \ln(u^+)\mathrm{d}x - \frac{\lambda}{q+1}\int_\Omega (u^+)^{q+1}\mathrm{d}x \\ -\lambda \int_\Omega \tilde{\rho}(u^+)\mathrm{d}x \ \text{ si } \ \delta = 1 \end{cases}$$

dans l'ensemble convexe M :

$$M \stackrel{\text{def}}{=} \{u \in H_0^1(\Omega) \backslash \underline{u} \leq u \leq \overline{u} \ \text{ presque partout dans } \ \Omega\}$$

où $\underline{u} = \underline{u}_\lambda$ est la solution unique (d'après [19]) de

$$(P_3) \begin{cases} -\Delta u = \frac{\lambda}{u^\delta} \ \text{ dans } \Omega \\ u|_{\partial\Omega} = 0, \, u > 0 \text{ dans } \Omega \end{cases}$$

et donc sous solution de (P_λ). On choisit $\overline{u}$ comme une solution de (P_λ) pour $\lambda_0 > \lambda$. La définition de Λ garantit l'existence d'un tel $\lambda_0 < \Lambda$. $\overline{u}_\lambda$ est alors une sur solution de (P_λ). On montre que le minimum de l'énergie E_λ sur M, u_λ, est une solution faible de (P_λ) pour $0 < \lambda < \Lambda$. Notons que le fait que $u_\lambda \geq \underline{u}$ dans M permet de surmonter le manque de régularité de E_λ. Par ailleurs l'estimation de l'énergie sur M indépendamment de λ assure l'existence d'une solution pour $\lambda = \Lambda$.

On montre ensuite que u_λ est un minimum local de l'énergie. La preuve utilise le fait que u_λ est un minimiseur de l'énergie sur M et le principe du maximum fort. Notons qu'on ne peut pas ici utiliser l'approche de AMBROSETTI-BREZIS-CERAMI [6] qui consiste d'abord à démontrer que u_λ est un minimiseur de l'énergie dans la topologie C^1 (via le lemme de Hopf) et ensuite à démontrer que tout C^1-minimiseur est un H_0^1-minimiseur (via un théorème de régularité). En effet, nous n'avons pas ici la régularité nécessaire sur $\underline{u}_\lambda, \overline{u}_\lambda$, et u_λ pour $\delta \geq 1$. Cette approche fonctionne cependant pour $\delta < 1$.
Etape 3: Existence d'une seconde solution
On utilise un argument variationnel de type lemme du col. u_λ étant un minimum local de l'énergie, on peut démontrer que E_λ a une géométrie du lemme du col. On ne peut cependant pas appliquer directement le théorème d'AMBROSETTI-RABINOWITZ (voir [7] dans le cas où u_λ est un minimum local strict) ou celui de GHOUSSOUB-PREISS [28] (dans le cas d'un minimum

local au sens large) du fait du manque de régularité de E_λ. Pour surmonter cette difficulté, on utilise l'approche dans BADIALE-TARANTELLO [11] où les auteurs prouvent des résultats de multiplicité pour des problèmes avec nonlinéarités critiques et discontinues. L'idée pour contourner le manque de régularité de la fonctionnelle associée est l'utilisation fine du principe variationnel d'Ekeland ([24]) qui permet de construire une suite $\{v_k\}$ se substituant aux suites de Palais Smale classiques et contenant suffisamment d'informations pour produire à la limite une solution faible v_λ du problème (P_λ). Notons que on est amené à distinguer entre deux géométries (voir cas (P1) et (P2)) au voisinage de u_λ. Il reste cependant à prouver que la solution faible v_λ est différente de u_λ. Pour cela, il faut une compacité forte de v_k dans $H_0^1(\Omega)$. Dans le cas (P1), la localisation de la suite $\{v_k\}$ garantit cette compacité. La proposition 2.3.1 montre l'existence d'une infinité de solutions aussi proches que l'on veut de u_λ. Dans le cas (P2), on utilise un argument de min-max:

$$\Gamma \stackrel{\text{def}}{=} \{\eta \in C([0,1],H_0^1(\Omega))/\eta(0) = u_\lambda,\eta(1) = e\}$$

et

$$\gamma_0 \stackrel{\text{def}}{=} \inf_{\eta\in\Gamma} \max_{t\in[0,1]} \overline{E}_\lambda(\eta(t)).$$

On démontre alors que si

$$\gamma_0 < E_\lambda(u_\lambda) + \frac{S^{\frac{N}{2}}}{N} \quad \text{(premier niveau critique, } S \text{ constante de Sobolev).} \quad (1)$$

alors $\{v_k\}$ est compacte dans $H_0^1(\Omega)$. (1) s'établit en utilisant les fonctions de Talenti comme fonctions tests et l'hypothèse **(H2)**. $\lim_{k\to\infty} v_k = v_\lambda$ implique alors que v_λ est solution faible avec $E_\lambda(v_\lambda) = \gamma_0 > E_\lambda(u_\lambda)$ et donc $u_\lambda \neq v_\lambda$.

On peut également montrer en reprenant la méthode développée dans [19] l'existence de solution au sens des distributions pour $\lambda > 0$ suffisamment petit et pour tout $\delta > 0$. Le contrôle de la solution au bord dit que pour $\delta \geq 3$, la solution ne peut être dans $H_0^1(\Omega)$. Pour $\delta < 3$, ces estimations au bord (dérivant de la construction d'une sursolution) montrent que toute solution faible est dans $H_0^1(\Omega) \cap C(\overline{\Omega})$. 3 est donc la valeur optimale de δ en dessous de laquelle l'existence de solutions faibles de (P_λ) est garantie pour $\lambda > 0$ petit. Pour $\delta < 1$ en utilisant la fonction de Green associée, on montre que toute solution faible est dans $C^{1,\alpha}(\overline{\Omega})$ pour $0 < \alpha \stackrel{\text{def}}{=} \alpha(\delta) < 1$.

Chapitre 3

Dans le cadre d'une classe d'équations quasilinéaires elliptiques singulières (de type p-laplacien) nous discutons le problème de multiplicité de solutions. Une approche pour démontrer la multiplicité de solutions est d'appliquer des méthodes variationnelles de type min-max en montrant l'existence d'une autre solution au voisinage d'une première solution. En particulier, cette multiplicité est assurée lorsque la fonctionnelle énergie possède une géométrie du lemme de col au voisinage d'une première solution qui est un minimum local de cette fonctionnelle et lorsque l'on a compacité des suites minimisantes associées. L'existence d'un minimum local de la fonctionnelle energie associée à des problèmes elliptiques nonlinéaires n'est pas en général une question aisée et a fait l'objet de plusieurs travaux. Un premier résultat qui concerne le cas de l'opérateur de Laplace est contenu dans BRÉZIS-NIRENBERG [13] qui démontre pour une classe de fonctionnelles "énergie" ayant une croissance critique qu'un C^1-minimiseur local de l'énergie est un minimum local dans l'espace d'énergie (ici $H_0^1(\Omega)$). Précisément, soit Ω un borné régulier de $\mathbb{R}^N$ et la fonctionnelle $E\ : H_0^1(\Omega) \to \mathbb{R}$ définie par

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x$$

où $F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s$ et f est une fonction de Carathéodory définie dans $\Omega \times \mathbb{R}$ qui vérifie la condition de croissance

$$|f(x,u)| \leq C(1 + |u|^p)$$

pour tout $1 \leq p \leq \frac{N+2}{N-2}$ si $N \geq 3$ et $1 \leq p < \infty$ si $N = 1$ où $N = 2$. On a alors

Théorème 2 (Brezis-Nirenberg)**.** *Supposons que $u_0 \in H_0^1(\Omega)$ est un minimum local de E pour la topologie C^1; c'est à dire qu'il existe $r > 0$ tel que*

$$E(u_0) \leq E(u_0 + v),\ \forall\, v \in C_0^1(\overline{\Omega})\ \ \textit{avec}\ \|v\|_{C^1} \leq r.$$

Alors u_0 est aussi un minimum local de E pour la topologie H_0^1, c'est à dire qu'il existe $\epsilon_0 > 0$ tel que

$$E(u_0) \leq E(u_0 + v),\ \forall\, v \in H_0^1(\overline{\Omega})\ \ \textit{avec}\ \|v\|_{H^1} \leq \epsilon_0.$$

L'intéret de ce résultat pionnier en la matière réside dans le fait qu'il est plus facile via le principe du maximum fort de démontrer l'existence de mimima locaux dans la topologie C^1. Mais il a un coût qui est de démontrer un résultat de régularité dans $C^{1,\alpha}(\overline{\Omega})$ des solutions faibles pour un problème perturbé de l'équation d'Euler-Lagrange. Dans AMBROSETTI-BREZIS-CERAMI [6], on trouve une application très intéressante de cette méthode aux problèmes avec nonlinéarité de type convexe-concave. Dans Guedda-Veron [33] puis dans Azorero Garcia-Manfredi-Peral [10], on trouve l'extension de cette méthode au cas d'une classe de problèmes quasilinéaires elliptiques avec un opérateur de type p-Laplacien. La difficulté supplémentaire réside dans ce cas dans l'extension du résultat de régularité höldérienne dans le cas d'opérateurs non-linéaires, dégénérés de type p-Laplacien. Elle s'appuie (sans être une simple généralisation) sur des travaux antérieurs comme Ladysenskaja-Ural'ceva [43], Di benedetto [22], Tolksdorf [49] (régularité intérieure) et Lieberman [45] (régularité près du bord). Les nonlinéarités considérées dans ces deux travaux ont une croissance critique. Le Théorème 2 a été étendu ensuite au cas quasilinéaire singulier dans Giacomoni-Schindler-Takáč [30] où les auteurs adaptent et suivent l'approche développée dans [10]. En particulier, le résultat de régularité $C^{1,\alpha}$ doit être montré pour les solutions d'une classe d'équation quasilinéaire faisant intervenir deux p-Laplaciens. Une autre approche a été récemment introduite par Brock-Iturraga-Ubilla [15] dans le cas sous critique. L'idée de cette nouvelle approche consiste à analyser un problème de minimisation sur un ensemble convexe plus grand et d'aboutir ainsi à une équation d'Euler-Lagrange plus simple avec un seul opérateur quasilinéaire de type p-laplacien. Les estimations a priori des minimiseurs locaux est cependant plus délicate dans ce cas (car la contrainte est plus faible). Dans De Figueiredo-Gossez-Ubilla [25], les auteurs prouvent la validité du résultat dans la cas critique. Ce résultat est ensuite étendu pour $p = N$ dans Giacomoni-Prashandh-Sreenadh [29] pour des fonctionnelles elliptiques avec croissance critique dans $\mathbb{R}^N$ et faisant intervenir un terme nonlinéaire défini sur le bord de l'ouvert Ω.

Dans ce chapitre nous étudions la validité du résultat pour des problèmes quasilinéaires singuliers un peu plus généraux que dans [30] en reprenant la démarche introduite dans [15]. Précisement nous étudions le problème suivant:

$$(\mathrm{P}_4)\left\{\begin{array}{ll} -\Delta_p u & = g(u) + f(x,u) \quad \text{dans } \Omega; \\ u|_{\partial\Omega} & = 0, \quad u > 0 \quad \text{dans}\, \Omega, \end{array}\right.$$

où $1 < p < \infty$, $p - 1 \leq p^* - 1$, $\lambda > 0$, $0 < \delta < 1$, Ω est un domaine borné régulier de $\mathbb{R}^N$, $N \geq 2$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ est une fonction C^1 satisfaisant:

(f1) $f(x,s) \geq 0$ pour $(x,s) \in \overline{\Omega} \times \mathbb{R}^+$ et $f(x,0) = 0$.

(f2) il existe $q > p - 1$ satisfaisant $q \leq p^* - 1 \overset{\text{def}}{=} \frac{Np}{N-p} - 1$ si $p < N$, $q < \infty$ sinon, tel que $f(x,s) \leq C(1 + s)^q$ pour tout $(x,s) \in \Omega \times \mathbb{R}$ et pour une certaine constante $C > 0$.

et $g : \mathbb{R}^+ \to \mathbb{R}^+$ continue sur $(0,+\infty)$ satisfaisant

(g1) g est décroissante sur $(0,+\infty)$,

(g2) $c_1 \leq \liminf_{t\to 0^+} g(t)t^\delta \leq \limsup_{t\to 0^+} g(t)t^\delta = c_2$ pour des constantes $c_1,c_2 > 0$.

Soit $F(x,u) \stackrel{\text{def}}{=} \int_0^u f(x,s)\text{ds}$ et $G(u) \stackrel{\text{def}}{=} \int_0^u g(s)\text{ds}$. On considère la fonctionnelle singulier $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ donnée par

$$I(u) \stackrel{\text{def}}{=} \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} - \int_\Omega F(x,u^+) - \int_\Omega G(u^+). \tag{2}$$

Adoptant la même technique que dans [15] nous montrons le résultat suivant

Théorème 3. *Soit* $u_0 \in C^1(\overline{\Omega})$ *satisfaisant*

$$u_0 \geq \eta\, d(x,\partial\Omega) \textit{ pour tout } \eta > 0 \tag{3}$$

un minimum local de I *dans la topologie* $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$; *i.e.,*

$$\exists\, \delta > 0 \textit{ tel que } u \in C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})\ ,\|u-u_0\|_{C^1(\overline{\Omega})} < \delta \Rightarrow I(u_0) \leq I(u).$$

Alors, u_0 *est un minimum local de* I *dans* $W_0^{1,p}(\Omega)$.

Esquisse de preuve du théorème 3:

On commence par le cas sous critique (i.e. $r < p^* - 1$) ensuite on donne l'argument supplémentaire pour prouver le résultat dans le cas critique (i.e. $r = p^* - 1$). La preuve utilise un raisonnement par l'absurde.

Étape1: Cas sous critique

On définit le problème de minimisation

$$I_\epsilon = \inf_{v\in S_\epsilon} I(v)$$

où l'ensemble S_ϵ est donnée par

$$S_\epsilon \stackrel{\text{def}}{=} \{v \in W_0^{1,p}(\Omega) \backslash K(v) \leq \epsilon\}$$

et la contrainte K est donnée par

$$K(w) = \frac{1}{q+1}\int_\Omega |w(x) - u_0(x)|^{q+1}\text{d}x,\ \ (w \in W_0^{1,p}(\Omega)),$$

où $q \in (r,p^* - 1)$. En utilisant le fait que la contrainte est faiblement semi-continue inférieurement, on démontre que I_ϵ atteint son minimum en $v_\epsilon \in S_\epsilon$. On est alors amené à distinguer entre deux cas:

1) Cas $K(v_\epsilon) < \epsilon$
Puisque la fonctionnelle est non différentiable sur l'espace tout entier $W_0^{1,p}(\Omega)$, on démontre tout d'abord qu'il existe $\tilde{\eta} > 0$ tel que

$$v_\epsilon \geq \tilde{\eta}\, dist(x,\partial\Omega).$$

Ceci permet d'obtenir que la fonctionnelle est Gâteaux différentiable en v_ϵ et de dériver l'équation d'Euler-Lagrange satisfaite par v_ϵ. En utilisant cette équation, on montre ensuite que v_ϵ est borné dans $C^{1,\alpha}(\overline{\Omega})$, (étape délicate!) ce qui implique (via le théorème d'Ascoli!) que $v_\epsilon \to u_0$ dans $C^1(\overline{\Omega})$. On obtient alors une contradiction avec le fait que u_0 est un C^1-minimiseur.
2) Cas $K(v_\epsilon) = \epsilon$.
Comme dans le premier cas, on démontre que $v_\epsilon \geq \eta\varphi_1$ dans Ω pour un certain $\eta > 0$ afin de dériver l'équation d'Euler-Lagrange satisfaite par v_ϵ. En utilisant la théorie des multiplicateurs de Lagrange, ceci s'écrit sous la forme

$$I'(v_\epsilon) = \mu_\epsilon K'(v_\epsilon)$$

où $\mu_\epsilon \leq 0$ est le multiplicateur de Lagrange. On est amené une nouvelle fois à distinguer entre deux cas:
Cas (i): $\mu_\epsilon \in (-1,0)$
L'équation satisfaite par v_ϵ s'écrit

$$(P_\epsilon) - \Delta_p v_\epsilon = g(v_\epsilon) + f(x,v_\epsilon) + \mu_\epsilon(|v_\epsilon - u_0|^{q-1}(v_\epsilon - u_0)).$$

Par le principe de comparaison faible on montre alors qu'il existe $\eta > 0$, indépendante de ϵ tel que $\eta\varphi_1 \leq v_\epsilon$. Par ailleurs, puisque $|\mu_\epsilon|$ est borné, il existe $M,c > 0$ tel que

$$-\Delta_p(v_\epsilon - 1)^+ \leq M + c((v_\epsilon - 1)^+)^q.$$

En itérant des estimations L^r (méthode classique des itérations de Moser), on prouve que v_ϵ est uniformément bornée dans $L^\infty(\Omega)$ et par une estimation uniforme au bord du domaine Ω, on obtient que $v_\epsilon \leq k\varphi_1$ pour une constante $k > 0$ indépendament de ϵ. En combinant l'estimation $\eta\phi_1 \leq v_\epsilon \leq k\varphi_1$ et un résultat de régularité étendant le résultat de [45] dans le cas singulier (Théorème B.1 dans [30]) on démontre que v_ϵ est bornée dans $C^{1,\alpha}(\overline{\Omega})$ indépendament de ϵ. Donc v_ϵ converge vers u_0 dans $C^1(\overline{\Omega})$ et on conclue comme précédemment.
Cas (ii): $\mu_\epsilon \leq -1$
Comme dans le cas (i), puisque $\eta\varphi_1$ est une sous solution on a $v_\epsilon \geq \eta\varphi_1$ pour un certain $\eta > 0$. De plus, il existe $M > 0$, indépendant de ϵ, tel que pour

$$\gamma(s,x,t) \stackrel{\text{def}}{=} g(t) + f(x,t) + s|t - u_0(x)|^{q-1}(t - u_0(x)) \tag{4}$$

on a
$$\gamma(s,x,t) < 0, \quad \forall\,(s,x,t) \in (-\infty,-1] \times \Omega \times (M,+\infty).$$
Par le principe de comparaison faible on a $v_\epsilon \leq M$ et en utilisant à nouveau des itérations de Moser sur l'équation (P_ϵ), on obtient que $\mu_\epsilon|v_\epsilon - u|$ est uniformément bornée dans $L^\infty(\Omega)$. Il découle alors comme dans le premier cas que v_ϵ est uniformément bornée dans $C_0^{1,\alpha}(\overline{\Omega})$ indépendament de ϵ. Le théorème d'Ascoli-Arzela permet alors de conclure qu'il existe une suite $\epsilon_n \to 0^+$ tel que v_{ϵ_n} converge vers u dans $C^1(\overline{\Omega})$. Ce qui contredit que u_0 est un C^1-minimiseur.

Étape 2: Cas critique

Une première difficulté par rapport au cas sous critique réside dans le fait que la contrainte n'est pas faiblement semi-continue inférieurement. L'existence du minimiseur v_ϵ n'est donc pas garantie. Pour surmonter cette difficulté, on utilise un argument de cut-off (argument déja utilisé dans [13]). Précisement, on définit la fonctionnelle "tronquée "

$$I^j(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x - \int_\Omega G(v)\,\mathrm{d}x - \int_\Omega F_j(x,v)\,\mathrm{d}x \;\;\forall\, v \in W_0^{1,p}(\Omega). \quad (5)$$

pour $j = 1,2,....,$ où $f_j(x,s) := f(x,T_j(s)), F_j(x,s) = \int_0^s f_j(x,t)\,\mathrm{d}t$ et

$$T_j(s) = \begin{cases} -j & \text{si } \; s \leq -j, \\ s & \text{si } \; -j \leq s \leq j \\ +j & \text{si } \; s \geq j. \end{cases} \quad (6)$$

On définit le problème de minimisation

$$I_\epsilon^j = \inf_{v \in \mathcal{C}_\epsilon} I^j(v)$$

où l'ensemble $\mathcal{C}_\varepsilon$ est donnée par

$$\mathcal{C}_\varepsilon \overset{\text{def}}{=} \{v \in W_0^{1,p}(\Omega) \setminus \chi(v) \leq \epsilon\}$$

et la contrainte χ est donnée par

$$\chi(w) = \frac{1}{p^*}\int_\Omega |w(x) - u_0|^{p^*}\mathrm{d}x, \;\; (w \in W_0^{1,p}(\Omega)).$$

En utilisant la même technique que dans le cas sous critique on démontre que I_ϵ^j atteint son minimum en $u_\epsilon \in \mathcal{C}_\epsilon$. On peut par ailleurs définir $j_\epsilon \overset{\text{def}}{=} j(\epsilon)$ tel que $j_\epsilon \to +\infty$ quand $\epsilon \to 0^+$ et compte tenu du fait que $I^j(v) \to I(v)$

quand $j \to \infty$, $I^{j_\epsilon}(u_\epsilon) < I^{j_\epsilon}(u_0)$ pour $\epsilon > 0$ assez petit.
On montre ensuite qu'il existe $\eta > 0$ indépendament de ϵ tel que

$$u_\epsilon \geq \eta\varphi_1. \tag{7}$$

En utilisant (7), on prouve que I^{j_ϵ} est dérivable au sense de Gâteaux en u_ϵ et puisque u_ϵ est un minimum local de I^{j_ϵ}, on démontre que $(I^{j_\epsilon})'(u_\epsilon)$ est défini et par la théorie des multiplicateurs de Lagrange, il existe $\mu_\epsilon \in \mathbb{R}^-$ tel que

$$(I^{j_\epsilon})'(u_\epsilon) = \mu_\epsilon \chi'(v_\epsilon).$$

Comme dans le cas sous critique, on est amené à distinguer entre deux cas.
Cas (i) $\inf\limits_{\epsilon\in(0,1)} \mu_\epsilon > -\infty$
l'équation (P_ϵ) s'écrit

$$(P_\epsilon) - \Delta_p u = g(u) + f_{j_\epsilon}(x,u) + \mu_\epsilon(|u-u_0|^{p^*-2}(u-u_0)). \tag{8}$$

Comme dans le cas sous critique, on voit que pour une constante $M > 0$ assez grande et indépendante de ϵ on a

$$-\Delta_p(u_\epsilon - 1)^+ \leq M + c|(u_\epsilon - 1)^+|^{p^*-2}(u_\epsilon - 1)^+$$

On démontre alors que $\{u_\epsilon\}$ est uniformément bornée dans $L^\infty(\Omega)$ indépendamment de ϵ. La preuve utilise les itérations de Moser pour prouver que $\{u_\epsilon\}$ est bornée dans $L^{\beta p^*}(\Omega)$ pour un certain $\beta > 1$ indépendament de ϵ et le Théorème 7.1 dans Ladyzenskaja-Ural'ceva [43] p.263, pour déduire que $\{u_\epsilon\}$ est uniformément borné dans $L^\infty(\Omega)$. On en déduit alors comme précédemment que $\{u_\epsilon\}_{\epsilon>0}$ est uniformément borné dans $C^{1,\alpha}(\overline{\Omega})$ et donc relativement compacte dans $C^1(\overline{\Omega})$. On conclue en constatant que $I(u_0) = I^{j_\epsilon}(u_0) > I^{j_\epsilon}(u_\epsilon) = I(u_\epsilon)$ ce qui donne la contradiction désirée.
Cas (ii) $\inf\limits_{\epsilon\in(0,1)} \mu_\epsilon = -\infty$
Comme dans le cas sous critique, on démontre que $u_\epsilon \geq \eta\varphi_1$ pour un certain $\eta > 0$ et qu'il existe $M > 0$, indépendamment de ϵ, tel que on a

$$g(s) + f_{j_\epsilon}(x,s) + \mu_\epsilon|s - u_0(x)|^{p^*-2}(s - u_0(x)) < 0 \text{ si } s > M. \tag{9}$$

Par le principe de comparaison, on a que $u_\epsilon(x) \leq M$. Comme dans le cas sous critique, on démontre que $\{u_\epsilon\}_{\epsilon>0}$ est uniformément borné dans $C^{1,\alpha}(\overline{\Omega})$ et donc relativement compacte dans $C^1(\overline{\Omega})$ et on conclue comme précédemment.

Chapitre 4

Dans ce dernier chapitre, on s'intéresse au problème elliptique singulier suivant:

$$(\mathrm{P}_\lambda)\begin{cases} -\Delta u &= \lambda(\frac{1}{u^\delta} + h(u)e^{u^\alpha}) \quad \text{dans } \Omega \subset \mathbb{R}^2;\\ u|_{\partial\Omega} &= 0,\ u > 0 \quad \text{dans } \Omega, \end{cases}$$

où $\lambda > 0$ est un paramètre, $0 < \delta < 1$, $1 \leq \alpha \leq 2$, $\Omega \subset \mathbb{R}^2$ est un domaine borné régulier et h est une petite perturbation de e^{u^α} d'ordre inférieur à l'infini. Précisément, on adopte les hypothèses suivantes sur h:

(H1) $h(t) \in C^2(\mathbb{R}^+,\mathbb{R}^+)$ avec $h(0) = 0$.

(H2) $h(t)e^{-\epsilon t^\alpha} \underset{t\to\infty}{\to} 0;\quad h(t)e^{\epsilon t^\alpha} \underset{t\to\infty}{\to} +\infty,\quad \epsilon > 0.$

La nonlinéarité du second membre est donc singulière et de croissance surexponentielle. On discute pour (P_λ) la structure de l'ensemble des solutions faibles (telles qu'on les a définies dans le chapitre 2) et donc classiques (appartenant à $C^2(\Omega) \cap C(\overline{\Omega})$) d'après les résultats de régularité qui ont été démontrés au chapitre 2. On utilise pour cela la théorie de la bifurcation dans le contexte des problèmes singuliers. Ceci requiert de généraliser certains outils et techniques de base utilisés dans le cadre de cette théorie comme l'existence de valeurs propres principales, l'opérateur linéarisé, le degré topologique de Leray-Schauder. Pour cela, on a besoin de préciser la différentiabilité (pour pouvoir appliquer le théorème des fonctions implicites) et la compacité de certains opérateurs ainsi que la validité du principe du maximum fort (version lemme de Hopf). A cette fin, nous utilisons certains résultats contenus dans Hernandez-Mancebo-Vega [38]. En particulier, on fait appel aux Propositions 2.3 et 2.5 dans [38] qui donnent la compacité d'une classe d'opérateurs singuliers via un théorème de régularité. En utilisant ces différents outils, on démontre l'existence d'une branche connexe non bornée de solutions de (P_λ) émanant de (0,0) dans $\mathbb{R}^+ \times C^1(\overline{\Omega})$. Précisément, on prouve le résultat général suivant:

Théorème 4. *Supposons que* $1 \leq \alpha \leq 2$. *Alors, il existe une branche de solutions,* $\mathcal{C}$, *de* (P_λ) *avec* $\mathcal{C} \subset \mathcal{S} = \{(\lambda,u) \in [0,\Lambda] \times C^1(\overline{\Omega}) \cap C^2(\Omega) |\, u$ *solution de* $(\mathrm{P}_\lambda)\}$ *émanant de* (0,0) *telle que*

1) $\mathcal{C}$ *est non bornée, et il existe* $0 < \Lambda < +\infty$ *et* $0 < \beta = \beta(\delta) < 1$ *telle que* $\mathcal{S} \subset [0,\Lambda] \times C^{1,\beta}(\overline{\Omega}) \cap C_0(\overline{\Omega})$.
2) *Pour* $\lambda \in (0,\Lambda), \exists\, (\lambda,u_\lambda) \in \mathcal{C}$ *où* u_λ *est la solution minimale de* (P_λ) *pour* λ *fixé.*
3) *La courbe* $\mathbb{R}^+ \ni \lambda \to u_\lambda \in C^1(\overline{\Omega}) \cap C_0(\Omega)$ *a une regularité* C^1.
4) *(Bifurcation secondaire et multiplicité locale prés de* $\lambda = \Lambda$*)* $\lambda = \Lambda$ *est un point de bifurcation, ce qui signifie qu' il existe une unique courbe*

C^2 $(\lambda(s),u(s)) \in \mathcal{C}$ *avec*

$$\lambda(0) = \Lambda,\ \lambda'(0) = 0,\ \lambda''(0) < 0.$$

5) (point de bifurcation asymptotique) $\mathcal{C}$ admet un point de bifurcation asymptotique en un λ_2 satisfaisant $0 \leq \lambda_2 \leq \Lambda$.

Les étapes principales de la preuve de ce théorème sont les suivantes:

étape 1: Via le théorème des fonctions implicites et des arguments de continuation globale, on démontre l'existence d'une branche régulière (au moins C^1) de solutions minimales (λ,u_λ) de (P_λ), avec $\lambda \in [0,\Lambda)$ et $0 < \Lambda < +\infty$. Ces solutions sont par ailleurs stables pour $0 \leq \lambda < \Lambda$, ce qui signifie que la première valeur propre de l'opérateur linéarisé (qui existe via les résultats de [38]) est strictement positive. En $\lambda = \Lambda$, la solution minimale est semi-stable et il n'existe pas de solutions au dela de $\lambda = \Lambda$.

étape 2: $\lambda = \Lambda$ est un point de bifurcation secondaire. Ceci se démontre en utilisant la théorie de la bifurcation locale d'une valeur propre simple isolée, établie par M. G. Crandall et P. H. Rabinowitz dans [17]. Ceci permet d'avoir en particulier pour $\lambda < \Lambda$ et proche de Λ, multiplicité de solutions.

étape 3: On démontre ensuite en utilisant le degré de Leray-Schauder que la branche est non bornée et donc admet un point de bifurcation asymptotique. L'argument repose essentiellement sur la non-existence de solutions pour $\lambda > \Lambda$ et le fait que les solutions minimales (pour $\lambda < \Lambda$) du fait de leur stabilité sont localement isolées.

On discute ensuite en quel(s) point(s) la branche admet un point de bifurcation asymptotique, et éventuellement l'unicité du point de bifurcation asymptotique ainsi que le comportement des solutions au voisinage de ce point. Pour pouvoir décrire précisément le comportement des branches de solutions, on se place dans le cas à symétrie radiale: $\Omega = B_1(0)$ et on considère les solutions radiales symétriques (qui sont également radialement décroissantes d'après l'équation) du problème (P_λ). Notons que l'existence de solutions radiales symétriques peut être obtenu comme la limite de solutions d'un problème approché et du résultat dans Gidas-Ni-Nirenberg [27]. Dans cette étude, on distingue trois cas en fonction des valeurs de α. Le premier cas : $\alpha = 1$. Un résultat contenu dans Brezis-Merle [12] permet de conclure dans le cas $\alpha \leq 1$. Ce résultat donne l'analyse du blow-up pour les solutions u_n de:

$$(P_1)\left\{\begin{array}{lll} -\Delta u & = V_n(x)e^u & \text{dans } \Omega \subset \mathbb{R}^2; \\ u|_{\partial\Omega} & = 0,\ u > 0 & \text{dans } \ \Omega, \end{array}\right.$$

où Ω est un domaine borné régulier de $\mathbb{R}^2$ et $V_n(x)$ appartient à $L^p(\Omega)$ pour $1 \leq p \leq \infty$. Précisément, le résultat est le suivant:

Théorème 5 (Brezis-Merle). *Supposons que $\{u_n\}$ est une suite de solutions de*

$$-\Delta u_n = V_n(x)e^{u_n} \textit{ dans } \Omega$$

où Ω est un domaine borné et V_n,u_n satisfont

(i) $V_n \geq 0$,

(ii) $\|V_n\|_{L^p(\Omega)} \leq C_1$, $\|e^{u_n}\|_{L^{p'}(\Omega)} \leq C_2$ *avec p et p' conjuguées et C_1,$C_2 > 0$.*

Alors, il existe une sous suite $\{u_{n_k}\}$ satisfaisant l'alternative suivante :

(i) Soit $\{u_{n_k}\}$ est bornée dans $L^\infty_{loc}(\Omega)$,

(ii) ou $u_{n_k} \to -\infty$ quand $k \to \infty$ sur tout compact de Ω,

(iii) ou l'ensemble des points d'explosion $\mathcal{S}$ (relative à la suite $\{u_{n_k}\}$) est un ensemble fini d'éléments et $u_{n_k} \to -\infty$ dans $\Omega \backslash S$. De plus, $V_n e^{u_{n_k}}$ converge au sens de la mesure dans Ω vers $\sum \alpha_i \delta(a_i)$ avec $\alpha_i \geq \frac{4\pi}{p'}$, $\forall\, i$ et $\mathcal{S} = \cup\{a_i\}$.

$\delta(a_i)$ désigne la masse de Dirac centrée en a_i. En combinant une estimation des solutions dans L^∞_{loc} obtenue avec le théorème précédent et le fait que les solutions sont radialement décroissantes, on montre l'existence d'une branche connexe de solutions non bornée et que le seul point de bifurcation asymptotique pour cette branche est 0.

On étudie ensuite le cas $1 < \alpha < 2$. On montre dans le cas radial le résultat suivant:

Théorème 6. *Soit $\Omega = B_1$ et $1 < \alpha < 2$. Alors, il existe une branche non borné de solutions de (P_λ), $\mathcal{C}_r$ bifurquant de $(0,0)$ satisfaisant*

(i) Pour $(\lambda,u) \in \mathcal{C}_r$, u est à symétrie radiale.

(ii) $\lambda = 0$ est un point de bifurcation asymptotique et pour $(\lambda_n,v_{\lambda_n}) \in \mathcal{C}_r$ telle que $\lambda_n \to 0^+$ et $v_{\lambda_n}(0) \to +\infty$, on a $v_{\lambda_n}(x) \to 0$ quand $n \to +\infty$ pour $0 < |x|$.

Notons que (ii) généralise dans le cas singulier certains résultats donnés dans Ogawa-Suzuki [41]. Le cas $\alpha = 2$ est le cas critique. Il est relié à l'inégalité suivante démontrée successivement par Trudinger et Moser:

Théorème 7. *(Trudinger-Moser)*

1. *Soit $p < \infty$ et Ω un domaine borné. Alors $u \in H^1_0(\Omega)$ implique $e^{u^2} \in L^p(\Omega)$ et continue dans la topologie associée à la norme. De plus,*

2. $$4\pi = \max\left\{c\,;\ \sup_{\|w\|\leq 1}\int_\Omega e^{c|w|^2} < +\infty\right\}. \tag{10}$$

On voit d'après le résultat précédent que $H_0^1(\Omega) \ni u \to e^{4\pi u^2} \in L^1(\Omega)$ est continue pour la topologie forte. Mais de manière similaire à ce qui se produit en dimension supérieure pour l'exposant critique, $H_0^1(\Omega) \ni u \hookrightarrow e^{4\pi u^2} \in L^1(\Omega)$ n'est pas compact pour la topologie faible. Précisément, si on considère une suite $\{v_n\} \subset H_0^1(\Omega)$ telle que $\|v_n\|_{H_0^1(\Omega)} \leq 1$, on a par le théorème 7 $\sup_n \int_\Omega e^{4\pi v_n^2} < \infty$ mais pas a priori convergence de $e^{4\pi v_n^2}$ dans $L^1(\Omega)$. De ce fait, la question de l'existence de solutions pour

$$(P_4) \begin{cases} -\Delta u = f(x,u) \text{ dans } \Omega \subset \mathbb{R}^2 \\ u|_{\partial\Omega} = 0\,, u > 0 \end{cases}$$

où $f(x,u)$ a un comportement critique (voir **(H2)**) est loin d'être évidente. En particulier, la méthode adoptée dans [13] pour résoudre le problème (P_4) (en dimension supérieure) ne fonctionne pas à cause de la croissance critique de type exponentielle. Une autre similitude avec le cas de la dimension supérieure est que l'exposant 2 est une limite de compacité. En effet, $H_0^1(\Omega) \ni u \hookrightarrow e^{u^\alpha} \in L^1(\Omega)$ est compacte pour la topologie faible pour $\alpha < 2$. De même, la perte de compacité dans le cas $\alpha = 2$ fait apparaître des phénomènes de concentration comme l'ont mis en évidence Adimurthi-Struwe [5] pour des énergies petites (Théorème 1.1) et ensuite Druet (théorème 1 dans [23]) dans le cas général. Ce phénomène de concentration est démontré à partir d'une suite de solutions u_k de (P_4) mais contrairement au cas de la dimension $N \geq 3$ ne s'étend pas à une suite de Palais Smale quelconque comme il l'a été démontré dans Adimurthi-Prashanth [4]. Les suites de Palais Smale peuvent présenter différents types de perte de compacité.

Le cas $\alpha = 2$ est plus délicat à traiter et fait intervenir de manière cruciale le comportement asymptotique de h à l'infini. Précisément, on distingue selon les comportements suivants:
(H3) $h(t) = O(e^{-t^\beta})$ $1 < \beta < 2$.
(H4) $h(t) = O(e^{-t^\beta})$ $0 \leq \beta < 1 \cup \{h$ décroit exponentielement à l'infini $\}$.
(H5) $h(t)t \to +\infty$.
Si h est dans la classe **(H4)** ou **(H5)**, on a le résultat suivant:
Théorème 8. *Let $\Omega = B_1$. Supposons que les conditions* (**H1**)*,* (**H2**) *et soit* (**H4**) *ou* (**H5**) *sont satisfaites. Alors, considérant $\mathcal{C}_r \subset \mathcal{S}$ prouvé dans le Théorème 6, on obtient que $\lambda = 0$ est point de bifurcation asymptotique et quand $\lambda \to 0^+$*

$$|\nabla v_\lambda|^2 \rightharpoonup 4\pi\delta_0,$$

et il existe $\rho(\lambda) \to 0^+$ telle que $v_\lambda^2(\rho(\lambda)r) - v_\lambda^2(\rho(\lambda)) \to 2\log(\frac{2}{1+r^2})$ uniformément sur les ensembles compacts de $\mathbb{R}^2$.

D'après le résultat précédent, on obtient donc dans le cas radial et pour les classes de nonlinéarités **(H4)** et **(H5)** un résultat similaire au cas $1 < \alpha < 2$, c'est à dire l'existence de deux solutions pour tout $\lambda \in (0,\Lambda)$ et le fait que 0 est l'unique point de bifurcation asymptotique. Dans le cas où h est dans la classe **(H3)**, on a le résultat suivant qui décrit un phénomène radicalement différent:

Théorème 9. *Let $\Omega = B_1$. Supposons que les conditions* (**H1**), (**H2**) *et* (**H3**). *Alors considérant $\mathcal{C}_r \subset \mathcal{S}$ caractérisé dans le Théorème 6, on obtient que*

(i) Pour $\lambda > 0$ petit, $\mathcal{C}_r$ contient une unique courbe paramétrée $\{\lambda,u_\lambda\}$ bifurquant de $(0,0)$.

(ii) Il existe $\eta > 0$ telle que pour $\lambda \in (\Lambda - \eta,\Lambda)$ il existe v_λ solution de (P_λ) *telle que $u_\lambda < v_\lambda$ dans Ω et $(\lambda,u_\lambda), (\lambda,v_\lambda) \in \mathcal{C}_r$.*

(iii) $\mathcal{C}_r$ admet au moins un point de bifurcation asymptotique $0 < \lambda_0 \leq \Lambda$.

(iv) Il existe une suite $(\lambda_n,v_{\lambda_n}) \in \mathcal{C}_r$ telle que $n \to +\infty$, $\lambda_n \to \lambda_0$, $v_{\lambda_n}(0) \to +\infty$ et $v_{\lambda_n} \to v^*$ dans le domaine épointé $B_1\backslash\{0\}$ où v^* est une solution singulière de (P_{λ_0}).

D'après le théorème précédent, on obtient unicité de solutions pour $\lambda > 0$ petit et l'existence d'au moins une solution singulière ($\notin L^\infty(\Omega)$) qui correspond à la limite d'une suite de solutions près du point asymptotique de bifurcation $0 < \lambda_0 \leq \Lambda$. On voit également que près de $\lambda = \Lambda$, on a multiplicité de solutions. En effet puisque les solutions minimales sont stables, elles sont uniformément bornées et donc ne peuvent exploser quand $\lambda \to \Lambda$. Une seconde étape serait de décrire plus précisément le comportement de la branche près de λ_0 pour laquelle on conjecture la présence d'une infinité de bifurcations secondaires correspondant à des indices de Morse de plus en plus élevés. Ce phénomène a été mis en évidence par Dancer [21] pour des nonlinéarités à croissance exponentielle et en dimension quelconque supérieure à 3. Notons que certains résultats contenus dans les deux précédents théorèmes coincident avec ceux obtenus par méthodes variationnelles dans Adimurthi-Giacomoni [3].

Pour déterminer les résultats dans le cas à symétrie radiale et $\alpha > 1$, on utilise des méthodes d'équations différentielles, en particulier la méthode de tir. Pour cela, nous effectuons une série de transformations sur (P_λ) dont la transformation d'Emden-Fowler utilisé très judicieusement dans Atkinson-Peletier [8]. Précisément,

$$(P_\lambda) \qquad \left\{ \begin{array}{c} \left.\begin{array}{rcl} -(rw')' & = & \lambda r f(w) \\ w & > & 0 \end{array}\right\} \text{ dans } (0{,}1), \\ w'(0) = w(1) = 0. \end{array} \right.$$

En utilisant le changement d'échelle $R = \lambda^{\frac{1}{2}}$, (P_λ) est équivalent à

$$(P_R) \qquad \left\{ \begin{array}{c} \left.\begin{array}{rcl} -(rw')' & = & r f(w) \\ w & > & 0 \end{array}\right\} \text{ dans } (0{,}R), \\ w'(0) = w(R) = 0. \end{array} \right.$$

On discute la multiplicité des solutions de (P_R) pour des R petits et pour cela on utilise une méthode de tir

$$(P_\gamma) \qquad \left\{ \begin{array}{rcl} -(ru')' & = & r f(u) \\ u(0) & = & \gamma,\, u'(0) \;=\; 0. \end{array} \right.$$

On désigne par $R_0(\gamma)$ le premier zéro de la solution u prescrite en zéro par $u(0) = \gamma$. Compte tenu de la continuité de $\gamma \to R_0(\gamma)$, il est suffisant d'établir les comportements asymptotiques de $R_0(\gamma)$ dans les deux cas limites $\gamma \to 0$ et $\gamma \to +\infty$ pour estimer l'ensemble des valeurs prises par $R_0(\gamma)$ quand γ décrit $\mathbb{R}^+$. Pour cela, on utilise la transformation de Emden-Fowler $|x| = r = 2e^{-\frac{t}{2}}$, $t \in (2\log(2R^{-1}),\infty)$ et $y(t) = u(r)$ qui nous amène à étudier le problème suivant:

$$\begin{array}{c} \left.\begin{array}{rcl} -y'' & = & e^{-t} f(y) \\ y & > & 0 \end{array}\right\} \text{ in } (T,\infty), \\ y(T) = y'(\infty) = 0. \end{array}$$

et les comportements asymptotiques de $T_0(\gamma) = 2\log(2R_0(\gamma)^{-1})$. Pour établir les asymptotiques de $T_0(\gamma)$, on est amené à déterminer les asymptotiques à des points intermédaires particuliers. Ces développements permettent aussi de déterminer l'analyse du blow up des solutions près du point de bifurcation asymptotique. Les résultats précis ainsi que les conditions vérifiées par les nonlinéarités des classes **(H3)**, **(H4)** et **(H5)** sont donnés dans le chapitre 4. Nous présentons uniquement ici les résultats clés qui permettent d'aboutir aux théorèmes précédents:

Théorème 10. *Supposons* **(H1)-(H2)**. *Alors* $T_0(\gamma) \to +\infty$ *quand* $\gamma \to 0$.

Théorème 11. *Soit* $1 < \alpha < 2$. *Supposons* h *satisfait* **(H1)** *et* **(H2)**. *Alors* $T_0(\gamma)$ *satisfait*

$$T_0(\gamma) > m(\gamma) + log\{\frac{1}{2}g'(\gamma)\} - 1, \tag{11}$$

et $T_0(\gamma) \to +\infty$ *quand* $\gamma \to +\infty$.

Concernant la classe **(H3)**, on a

Théorème 12. *Soit f vérifiant les hypothèses* **(H1)-(H2)** *et appartenant à la classe* **(H3)**. *Alors,* $\limsup_{\gamma \to \infty} T_0(\gamma) < \infty$.

Concernant les classes **(H4)** et **(H5)**, on a

Théorème 13. *Soit f vérifiant les hypothèses* **(H1)-(H2)** *et appartenant aux classes* **(H4)** *et* **(H5)**. *Alors,* $\lim_{\gamma \to \infty} T_0(\gamma) = \infty$.

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Chapitre 1

Notation and preliminary results

Contents

In this chapter, we start by recalling some notations which will be frequently used throughout the rest of this work. Then, we review some basic methods for proving the existence of relative minimizers and saddle points. However, besides the classical lower semi-continuity method we also include the Mountain Pass Therorem due to AMBROSETTI-RABINOWITZ [2]. Moreover, we recall Ekeland's Variational principles (see [13]) and the Palais Smale conditions (see [11]). Finally, we state the Bifurcation result from an isolated and simple eigenvalue contained in CRANDALL-RABINOWITZ [6].

1.1 Notation

We denote by $\mathbb{R}^N$ the Euclidian space of dimension N, by $x = (x_1, ..., x_N)$ an element of $\mathbb{R}^N$ and by $\mid x \mid= (\sum_{i=1}^{N} x_i^2)^{\frac{1}{2}}$ the associated norm.
In all of this work, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth

boundary $\partial\Omega$. We denote by n the unit normal to $\partial\Omega$ outward to Ω and by σ the mesure on $\partial\Omega$.
When E is a measurable subset of $\mathbb{R}^N$, $|E|$ denotes the Lebesgue measure of E.
For $1 \leq q \leq \infty$, q' denotes the conjuguate exponent of q (that is to say $\frac{1}{q} + \frac{1}{q'} = 1$).
For $1 \leq p < \infty$, we denote $L^p(\Omega)$ the space defined by

$$L^p(\Omega) \stackrel{\text{def}}{=} \{u : \Omega \to \mathbb{R} \text{ mesurable; } \int_\Omega |u|^p \mathrm{d}x < \infty\}$$

with the norm

$$\|u\|_p = \left(\int_\Omega |u|^p \mathrm{d}x\right)^{\frac{1}{p}},$$

and for $p = \infty$, we denote

$$L^\infty(\Omega) \stackrel{\text{def}}{=} \{u : \Omega \to \mathbb{R} \text{ mesurable; } \operatorname*{ess\,sup}_\Omega |u| < \infty\}$$

with the norm

$$\|u\|_\infty = \operatorname*{ess\,sup}_\Omega |u| < \infty.$$

The Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) \stackrel{\text{def}}{=} \{u \in L^p(\Omega);\ \exists g_1, ..., g_N \in L^p(\Omega), \int_\Omega g_i \varphi = \int_\Omega u \varphi_{x_i}\ \forall \varphi \in D\}$$

with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{1,p,\Omega} = \|u\|_p + \|\nabla u\|_p, \text{ if } p < \infty,$$

and

$$\|u\|_{1,\infty} = \max(\|u\|_\infty, \|Du\|_\infty) \text{ if } p = \infty.$$

For $1 \leq p < \infty$

$$W_0^{1,p}(\Omega) \equiv \text{ the closure of } C_0^\infty(\Omega) \text{ in the space } W^{1,p}(\Omega).$$

We denote by $W^{-1,p'}(\Omega)$= $(W_0^{1,p}(\Omega))^*$ $p' = \frac{p}{p-1}$, $1 \leq p < \infty$ the topological dual space which can be characterized as follows:
if $F \in W^{-1,p'}(\Omega)$, then there exists $f_0, f_1, ..., f_N \in L^{p'}(\Omega)$ such that

$$\langle F, \phi \rangle = \int_\Omega f_0 \Phi + \sum_{i=1}^N \int_\Omega f_i \frac{\partial \Phi}{\partial x_i}\ \forall\ \Phi \in W_0^{1,p}(\Omega) \text{ and } \|F\| = \max_{1 \leq i \leq N} \|f_i\|_{L^{p'}(\Omega)}.$$

1.2 Preliminary results

In this section we recall some basic methods for proving the existence of relative minimizers and saddle points.

1.2.1 Lower semicontinuous functions

In this subsection, we give sufficient conditions for a functional to be bounded from below and attain its infimum. Precisely, we have the following classical results for minimization (see MAWHIN-WILLEM [10], BERGER [3] and STRUWE [13]).

Theorem 1.2.1. *Let f be a weakly lower semi-continuous functional on the reflexive Banach space X with a bounded minimizing sequence. Then f has a minimum on X.*

We prove the compactness of the minimizing sequence by the following results:

Theorem 1.2.2. *Let f be a weakly lower semi-continuous functional bounded from below on the reflexive Banach space X. If f is coercive, then $c = \inf f$ is attained at a point $x_0 \in X$.*

1.2.2 Ekeland's Variational principles

In general it is not clear that a bounded and lower semi-continuous functional E actually attains its infimum. The analytic function $f(x) = \arctan x$, for example, neither attains its infimum nor its supremum on the real line.
The idea of the variational principle of Ekeland is the following: Assume f lower semicontinuous real valued defined on the metric space (M, d) such that $f(x) \geq \beta$ for all $x \in M$. The principle deals with the construction of the minimizing "Palais Smale" sequence with some control, precisely, sequences verifying

$$\inf_{x \in M} \{f(x)\} + \epsilon > f(x_\epsilon),$$

and

$$f(y) \geq f(x_\epsilon) - \epsilon d(x_\epsilon, y).$$

We can read geometrically this condition saying that for all $\epsilon > 0$ the graph of f is above of this cone. Precisely we have the following result

Theorem 1.2.3. *(Ekeland Principle, strong form, 1979).*
Let M be a complete metric space and let $\psi : M \longrightarrow (-\infty, \infty)$ be a lower

semicontinuous functional which is bounded from below. Let $\varepsilon > 0$ and $u \in M$ be an minimum point of ψ. Then there exists $v \in M$ such that

1) $\psi(u) \geq \psi(v)$,

2) $d(u, v) \leq 1$,

3) If $v \neq w \in M$ then $\psi(w) \geq \psi(v) - \varepsilon d(v, w)$.

1.2.3 Palais Smale conditions

Minimizing sequences for differentiable functionals are convergent under certain compactness conditions. We shall use later the so called Palais-Smale [11] ((PS) for short) conditions that we formalize in the following definition.

Definition 1.2.1. *(Palais, 1970). Let X be a Banach space and $E : X \to \mathbb{R}$ a functional Gâteaux differentiable. E satisfies the Palais-Smale condition if every sequence (x_k) in X such that $E(x_k)$ is bounded and $E'(x_k) \to 0$ in X^* for $k \to \infty$ has a subsequence $x_{k_j} \to x \in X$.*

From (PS) condition, it follows that the set of critical points for a bounded functional is compact. A variant of (PS) condition, denoted as $(\text{PS})_c$, was introduced by Brézis-Coron-Nirenberg [4].

Definition 1.2.2. *(Brézis, Coron, Nirenberg, 1980). Let X be a Banach space and $E : X \to \mathbb{R}$ a functional Gâteaux differentiable. E satisfies the Palais-Smale condition to the level $c \in \mathbb{R}$ if and only if for all sequence $\{x_k\} \subset X$ verifying*

1) $E(x_k) \to c$ *as* $k \to \infty$,

2) $E'(x_k) \to 0$ *in* X^* *for* $k \to \infty$ *has a subsequence* $x_{k_j} \to x \in X$.

It is clear that (PS) condition implies the $(\text{PS})_c$ condition for every $c \in \mathbb{R}$. The $(\text{PS})_c$ condition implies the compactness of the set of critical points at a fixed level c. With this sequence we can formulate the Montain Pass Theorem.

1.2.4 The Mountain Pass Theorems

In this subsection, we present the Mountain Pass theorem of Ambrosetti-Rabinowitz [2] and some of its extensions. Their statements involve a compactness assumption, so the called Palais-Smale (PS) condition and variants. During the last two decades, minimax theorems have been extensivelly developed, generalizing the assumptions on differentiability of the functional,

the (PS) type conditions and geometric conditions on the functional. In a general form, the Mountain Pass theorem has been proved for continuous functionals by DEGIOVANNI-MARZOCCHI [8] . For discontinuous functionals it has been proved by RIBARSKA-TASACHEV-KRASTANOV [12].
Now, we start by formalizing the Mountain Pass Theorem due to Ambrosetti-Rabinowitz in the following Theorem.

Theorem 1.2.4. *(Ambrosetti, Rabinowitz, 1973). Let X be a real Banach space and $E \in C^1(X)$. Suppose that E satisfies (PS) condition, and*

1) $E(0) = 0$;

2) $\exists\, \rho > 0,\ \alpha > 0 :\ \|u\| = \rho \Longrightarrow E(u) \geq \alpha$;

3) $\exists\, u_1 \in V :\ \|u_1\| \geq \rho$ *and* $E(u_1) < \alpha$.

Then E has a critical value $c \geq \beta$ which can be characterized as

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} E(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1]; X)\ ; \gamma(0) = 0,\ \gamma(1) = u_1\}.$$

Now, we formulate a variant of Mountain Pass Theorem with $(\mathrm{PS})_{G,c}$ condition proved in GHOUSSOUB-PREISS [9] in order to get information about the location of critical points. They introduce $(\mathrm{PS})_{G,c}$ condition around a set G at the level c as follows.

Definition 1.2.3. *(Ghoussoub, Preiss, 1989). The differentiable functional $E : X \to \mathbb{R}$ satisfies the $(PS)_{G,c}$ condition around a set G at the level c if every sequence $\{v_n\}$ in X such that*

1) $\lim_{n\to\infty} dist(v_n, G) = 0$,

2) $\lim_{n\to\infty} E(v_n) = c$,

3) $\lim_{n\to\infty} \|E'(v_n)\| = 0$.

The following Mountain Pass Theorem has been proved in GHOUSSOUB-PREISS [9]

Theorem 1.2.5. *(Ghoussoub, Preiss, 1989). Let $E : X \to \mathbb{R}$ be a Gâteaux differentiable functional such that $E' : X \to X^*$ is continuous from X with norm topology to X^* with weak$*$ topology. Fix $e \neq 0$ and define*

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} E(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1];X) \; ; \gamma(0) = 0,\ \gamma(1) = u_1\}.$$

Let G be a closed subset such that $G \cap E_c$ separates 0 and e. Assume that E satisfies $(PS)_{G,c}$ condition. Then there exists $\overline{x} \in G$ such that $E(\overline{x}) = c$ and $E'(\overline{x}) = 0$.

1.2.5 Local Bifurcation for simple eigenvalue

In this subsection, we give the result of Crandall-Rabinowitz [6] (see also [7]) concerning the bifurcation from an isolted and simple eigenvalue. Before that we introduce some definitions

Definition 1.2.4. *Let X and Y be two Banach spaces on $\mathbb{R}$ and A, K bounded linear operators in $\mathcal{L}(X, Y)$. A is a Fredholm operator with index $p \in \mathbb{Z}$ if*

(1) $dim N(A) < \infty$,

(2) $R(A)$ is closed and $codim R(A) < \infty$,

(3) $p = dim N(A) - codim R(A)$,

where

$$N(A) = \{x \in X : Ax = 0\}$$

and

$$R(A) = \{Ax : x \in X\}$$

K is compact if for any subsequence bounded $(x_n; n \in \mathbb{N}) \subset X$, there exists a subsequence $(x_{n_k}; k \in \mathbb{N})$ such that $(Kx_{n_k}; k \in \mathbb{N})$ is a convergent sequence in Y.

Let us now give the definition of a bifurcation point:

Definition 1.2.5. *Let $F \in C^1(\mathbb{R} \times X, Y)$ where X and Y are as above. We assume also that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Then, $(\lambda_0, 0) \in \mathbb{R} \times X$ is a bifurcation point if there exists a sequence $\{(\mu_n, x_n); n \in \mathbb{N}\} \subset \mathbb{R} \times X$ such that*

(1) $F(\mu_n, x_n) = 0$ *and* $x_n \neq 0$ *for all* $n \in \mathbb{N}$,

(2) $\lim_{n\to\infty} \mu_n = \lambda_0$ *and* $\lim_{n\to\infty} x_n = 0$.

$\lambda_0 \in \mathbb{R}$ is a bifurcation value.

Theorem 1.2.6. *(Crandall-Rabinowitz) Let X and Y two Banach spaces, $k \in \mathbb{N}$ such that $k \geq 2$ and $F \in C^k(\mathbb{R} \times X, Y)$. We assume that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}, \partial_2 F(\lambda_0, 0)$ is a Fredholm operator of $0-$index and*

$$N(A) = \{s\xi_0 : s \in \mathbb{R}\} \text{ with } \xi_0 \in X \setminus \{0\}.$$

If the following condition (called the Transversality condition)

$$\partial^2_{1,2} F(\lambda_0, 0)\xi_0 \notin R(A)$$

is satisfied, then $(\lambda_0, 0) \in \mathbb{R} \times X$ is a bifurcation point. More precisely, there exists a bifurcating curve

$$\{\sigma \in \mathbb{R} : |\sigma| < \epsilon\} \ni s \to (\lambda(s), sx(s)) \in \mathbb{R} \times X$$

such that

(1) $\epsilon > 0, \lambda(0) = \lambda_0, x(0) = \xi_0,$

(2) $F(\lambda(s), sx(s)) = 0$ *if* $|s| < \epsilon,$

(3) *there exists an open set $U_0 \subset \mathbb{R} \times X$ such that $(\lambda_0, 0) \in U_0$ and*

$$\{(\lambda, x) \in U_0 : F(\lambda, x) = 0, x \neq 0\} = \{(\lambda(s), sx(s)) : 0 < |s| < \epsilon\},$$

(4) $s \to \lambda(s)$ *and* $s \to sx(s)$ *belong to* C^{k-1}, $s \to x(s)$ *to* C^{k-2},

(5) *If F is analytic, then $s \to \lambda(s)$ and $s \to x(s)$ are also analytic.*

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Chapitre 2

Multiplicity of positive solutions for a singular and critical problem

Contents

2.1 Introduction

In this chapter, we are interested in the following singular and critical problem:

$$(P_\lambda)\begin{cases} -\Delta u = \lambda(u^{-\delta} + u^q + \rho(u)) \quad \text{in } \Omega \\ u|_{\partial\Omega} = 0,\ u > 0 \text{ in } \Omega \end{cases}$$

where Ω is an open bounded domain with smooth boundary, $\lambda > 0, 0 < \delta$ and $1 < q \leq 2^* - 1$. As usual, $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N \leq 2$. Concerning ρ, we make the following assumptions:

(H1) ρ is $C^1(\mathbb{R}^+); \rho(0) = 0, \rho'(0) = 0; \rho(t) + t^q \geq 0$ for all $t \geq 0$.

(H2) (subcritical assumption)
If $q = 2^* - 1$, then $\exists\, \beta < 2^* - 2$ such that $\rho^-(t)t^{-\beta} \to 0$, $\rho^+(t)t^{-2^*+1} \to 0$ as $t \to +\infty$.

As usual, $r^+ = \max(r, 0)$ and $r^- = \max(-r, 0)$.
Such problems arise, for instance, in models of pseudo-plastic flows. Our main concern is the question of existence and multiplicity of "weak" solutions to the Dirichlet boundary problem (P_λ). By weak solution we mean a function $u \in H^1_{loc}(\Omega) \cap C_0(\overline{\Omega})$ satisfying $\operatorname{ess\,inf} u > 0$ over every compact $K \subset \Omega$ and

$$\int_\Omega \nabla u \nabla \phi \,\mathrm{d}x = \lambda \left(\int_\Omega u^{-\delta} \phi \,\mathrm{d}x + \int_\Omega u^q \phi \,\mathrm{d}x \int_\Omega \rho(u) \phi \,\mathrm{d}x \right)$$

for all $\phi \in C_c^\infty(\Omega)$. As usual, $C_c^\infty(\Omega)$ denotes the space of all C^∞ functions $\phi : \Omega \to \mathbb{R}$ with compact support.
Using a subsolutions and supersolutions technique, we first show the existence of at least one solution to (P_λ) for all small $\lambda > 0$. Precisely, we have the following result:

Proposition 2.1.1. *Let ρ satisfy assumptions* **(H1)-(H2)** *and $\delta > 0$. Then, for all $\lambda > 0$ small enough, there exists at least one weak solution, say u_λ, to (P_λ). Moreover, $u_\lambda \in C_0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies*

(i) *if $\delta < 1, u_\lambda \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha := \alpha(\delta) < 1$*

(ii) *if $\delta > 1$, there exists $c_0, c_1 > 0$ such that $c_0 \varphi_1^{\frac{2}{\delta+1}} \le u_\lambda \le c_1 \varphi_1^{\frac{2}{\delta+1}}$*

(iii) *$u_\lambda \in H^1_0(\Omega)$ if and only if $\delta < 3$.*

Here φ_1 is the normalized positive eigenvector associated to $\lambda_1(\Omega)$.
We point out that the above result does not depend on the growth of $g_\lambda(t) = \lambda(t^{2^*-1} + \rho(t))$ as $t \to \infty$. Moreover, the construction of appropriate subsolutions and supersolutions provides, for small λ, estimates about the behaviour of u_λ near the boundary and the regularity locally inside Ω.
Next, we discuss the question of multiplicity of weak solutions in $H^1_0(\Omega)$ which requires $\delta < 3$ from the conclusion (iii) of Proposition 2.1.1. To obtain multiple solutions, we use a modification of the Mountain Pass Lemma of Ambrosetti-Rabinowitz [3], cf. Ghoussoub-Preiss [13] and a cut-off argument. More precisely, we look for solutions to problem (P_λ) that are critical points of the energy functional E_λ defined by

$$E_\lambda(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{\lambda}{1-\delta}\int_\Omega (u^+)^{1-\delta}\mathrm{d}x - \frac{\lambda}{q+1}\int_\Omega (u^+)^{q+1}\mathrm{d}x \\ -\lambda \int_\Omega \tilde{\rho}(u^+)\mathrm{d}x \ \text{ if } \ \delta \neq 1 \\ \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \lambda \int_\Omega \ln(u^+)\mathrm{d}x - \frac{\lambda}{q+1}\int_\Omega (u^+)^{q+1}\mathrm{d}x \\ -\lambda \int_\Omega \tilde{\rho}(u^+)\mathrm{d}x \ \text{ if } \ \delta = 1, \end{cases}$$

$\tilde{\rho}(t) = \int_0^t \rho(s)ds$. Note that E_λ is not defined on the entire space $H_0^1(\Omega)$ when $\delta > 1$ because of the singular term $(u^+)^{1-\delta}$. Consequently, one cannot directly apply classical variational methods, such as the Mountain Pass Lemma of Ambrosetti-Rabinowitz [3].
First, we show that the number

$$0 < \Lambda = \inf\{\lambda > 0/(P_\lambda) \quad \text{has no weak solution } \} \tag{2.1}$$

satisfies $0 < \Lambda < \infty$. Then, we prove the existence of multiple (at least two distinct, positive) solutions of the problem (P_λ) for every $\lambda \in (0, \Lambda)$ a local minimizer and a saddle point for the functional E_λ. Indeed, the existence and multiplicity result is a consequence of competition between the quadratic term and two last terms in the energy functional E_λ. Notice that $1 - \delta < 2 < q + 1$.
Let $0 < \lambda < \Lambda$. The first term

$$-\frac{\lambda}{1-\delta} \int_\Omega (u^+)^{1-\delta} \mathrm{d}x$$

dominates provided $u > 0$ is "small", the quadratic term

$$\frac{1}{2} \int_\Omega \mid \nabla u \mid^2 \mathrm{d}x,$$

becomes dominatent for $u > 0$ "midsized", and the negative term

$$-\frac{\lambda}{q+1} \int_\Omega (u^+)^{q+1} \mathrm{d}x - \lambda \int_\Omega \tilde{\rho}(u^+) \mathrm{d}x$$

becomes dominant for $u > 0$ large, this intuitive picture clearly suggests two critical points of E_λ : a local minimizer between "small" and "midsized" and a saddle point between "midsized" and "large". As $\lambda \in (0, \Lambda)$ approaches Λ, the two critical points merge into a single one for $\lambda = \Lambda$ and disappear for $\lambda > \Lambda$.
Our main result is the following theorem:

Theorem 2.1.1. *Assume that* **(H1)** *and* **(H2)** *are satisfied and let* $0 < \delta < 3$. *Then there exists* $\Lambda \in (0, \infty)$ *with the following properties:*

> (i) *for every* $0 < \lambda < \Lambda$ *there exist at least two solutions of the problem* (P_λ) u_λ *and* v_λ *in* $H_0^1(\Omega)$ *such that* $u_\lambda < v_\lambda$ *in* Ω*;*
>
> (ii) *for* $\lambda = \Lambda$ *there exists at least one solution of* (P_λ) *in* $H_0^1(\Omega)$*;*
>
> (iii) *for every* $\lambda > \Lambda$*, there is no weak solution to* (P_λ)*.*

Note that, from Proposition 2.1.1, $0 < \delta < 3$ is the optimal condition to obtain solutions in $H_0^1(\Omega)$. Before giving the proof of the results stated above, let us briefly recall the literature concerning related singular problems. The following problem has been investigated in several previous works:

$$\begin{cases} -\Delta u = \frac{\lambda k(x)}{u^\delta} + \mu(x)u^q \text{ in } \Omega \\ u|_{\partial\Omega} = 0, \ u > 0 \text{ in } \Omega. \end{cases} \tag{2.2}$$

The weights $k, \mu : \Omega \to \mathbb{R}$ are assumed to be nonnegative and (essentially) bounded when $\mu = 0$ (the purely singular problem), Crandall-Rabinowitz-Tartar [9] show that, for any $\delta > 0$, problem (2.2) admits a unique solution $u_\lambda \in C^2(\Omega) \cap C(\overline{\Omega})$; if $\mu = 0$ furthermore, if $0 < \delta < 1$, then u_λ is in $C^2(\Omega) \cap C(\overline{\Omega})$ when $\mu > 0$ is small enough, Coclite-Palmieri [8] prove the existence of a solution to problem (2.2) for $0 < \lambda < \Lambda$ with Λ as in (2.1), $0 < \Lambda < \infty$. Assuming $0 < \delta < 1$, Yijing-Shaoping-Yiming [17] apply variational arguments based on Nehari's method to show the existence of at least two solutions for $q > 1$ subcritical: $q < \infty$ if $N = 1$ or 2, and $q < 2^* - 1$ if $N \geq 3$. The critical case $q = 2^* - 1$ and $N \geq 3$ was settled almost simultaneously in Haitao [10] and Hirano, Saccon, Shioji [12] by two different methods: Perron's method and Nehari's method, respectively. Still, these results concern the case $0 < \delta < 1$. In Adimurthi-Giacomoni [1], the existence of at least two solutions in dimension $N = 2$ is extended to critical nonlinearities of Trudinger-Moser type (see Moser [14]).

The outline of this chapter is as follows. In Section 2.2, we prove Proposition 2.1.1 and the existence of a local minimizer of E_λ in $H_0^1(\Omega)$ for $0 < \lambda < \Lambda$. In Section 2.3, using Ekeland's principle and minimax arguments, we prove the existence of a second solution and thus give the proof of Theorem 2.1.1.

2.2 Existence of one weak solution

We first recall the following well-known results:

Lemma 2.2.1. *Let $g : [0, \infty) \to (0, \infty)$ be such that $\frac{g(t)}{t}$ is decreasing. Let $v, w \in C^2_{loc}(\Omega) \cap C^1(\bar{\Omega})$ be weak solutions and supersolutions respectively of the problem*

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \ u > 0 & \text{in } \Omega \end{cases} \tag{2.3}$$

then $v \leq w$ a.e in Ω.

Proof. Let $\Omega^+ = \{v > w\}$. From the Hopf Lemma, there exists $c > 0$ such that $w, v \geq c\, dist(x, \partial\Omega)$ and from [7] appendix II

$$\int_{\Omega^+} (\frac{-\Delta v}{v} + \frac{\Delta w}{w})(v^2 - w^2)\mathrm{d}x = \int_{\Omega^+} \mid \nabla v - \frac{v}{w}\nabla w \mid^2 \mathrm{d}x$$

$$+ \int_{\Omega^+} \mid \nabla w - \frac{w}{v}\nabla v \mid^2 \mathrm{d}x \geq 0.$$

On the other hand, since $\frac{g(t)}{t}$ is decreasing, we get

$$\int_{\Omega^+} (\frac{-\Delta v}{v} + \frac{\Delta w}{w})(v^2 - w^2)\mathrm{d}x = \int_{\Omega^+} (\frac{g(v)}{v} - \frac{g(w)}{w})(v^2 - w^2)\mathrm{d}x \leq 0.$$

It follows that $g(v) = g(w)$ in Ω^+. Then, using (2.3), we see that $v = w$ in Ω^+ and $v \leq w$ in Ω. This completes the proof of the Lemma 2.2.1. □

Let $g : (0, +\infty) \to [0, +\infty[$ satisfying $\lim_{t \to 0^+} \frac{g(t)}{t^\delta} = c > 0$ for some $\delta > 0$, $t \to \frac{g(t)}{t}$ be nonincreasing and

$$(\overline{P}_\lambda) \begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \ u > 0 \text{ in } \Omega. \end{cases}$$

Then, we have the following result due to Crandall-Rabinowitz-Tartar.

Theorem 2.2.1. *[Crandall-Rabinowitz-Tartar] There exists one and only one solution, $\underline{u}_\lambda$, to $(\overline{P}_\lambda)$ in $C^{0,\alpha}(\overline{\Omega}) \cap C^2_{loc}(\Omega)$ with $0 < \alpha = \alpha(\delta) < 1$ and satisfying the following properties: there exists $0 < c_0 \leq c_1$ and $0 < c_0' \leq c_1'$ such that*

(i) If $0 < \delta < 1$, then $c_0 d \leq \underline{u}_\lambda \leq c_1 d$, in $\overline{\Omega}$ and $\mid \nabla \underline{u}_\lambda \mid \leq c_0'$

(ii) If $\delta > 1$, then $c_0 d^{\frac{2}{\delta+1}} \leq \underline{u}_\lambda \leq c_1 d^{\frac{2}{\delta+1}}$ in Ω and $c_0' \leq d^{\frac{\delta-1}{\delta+1}} \mid \nabla \underline{u}_\lambda(x) \mid \leq c_1'$ on $\partial\Omega$

From the above result, and the elliptic regularity theory, we can prove the following regularity result.

Theorem 2.2.2. *Let $\delta > 0$ and $\underline{u}_\lambda$ the unique solution to $(\overline{P}_\lambda)$; then*

(i) $\underline{u}_\lambda \in C^1(\overline{\Omega})$ if and only if $\delta < 1$,

(ii) $\underline{u}_\lambda \in H^1_0(\Omega)$ if and only if $\delta < 3$.

Proof. See Section 2.4 (Lemma 2.4.4 and Lemma 2.4.5). □

We are ready to prove Proposition 2.1.1.

Proof of Proposition 2.1.1. First, we show the existence of u_λ. We start with the case $0 < \delta < 1$. For this, we introduce the following cut-off function :

$$\overline{f}_\lambda(x,t) = \begin{cases} f_\lambda(t) & \text{if } t > \underline{u}_\lambda \\ f_\lambda(\underline{u}_\lambda) & \text{if } t \leq \underline{u}_\lambda \end{cases} \tag{2.4}$$

where $\underline{u}_\lambda$ is the solution given by Theorem 2.2.1 with $g(u) = \frac{1}{u^\delta}$ and $f_\lambda(u) = \lambda\left(\frac{1}{u^\delta} + u^q + \rho(u)\right)$.
Let $F_\lambda(x,s) = \int_0^s f_\lambda(x,t)\mathrm{d}s$. Define $\overline{E}_\lambda : H_0^1(\Omega) \to \mathbb{R}$ by

$$\overline{E}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega F_\lambda(x,u)\,\mathrm{d}x. \tag{2.5}$$

Then, from Lemma 2.4.3 in Section 2.4, we have that $\overline{E}_\lambda$ is C^1.

We consider the following minimization problem in the ball $\mathcal{B}_{r_0} = \{u \in H_0^1(\Omega)/ \parallel u \parallel_{H_0^1(\Omega)} \leq r_0\}$:

$$I_\lambda = \inf_{\mathcal{B}_{r_0}} \overline{E}_\lambda(u). \tag{2.6}$$

From Hölder's and Sobolev's inequalities and (**H1**), we have

$$\overline{E}_\lambda(u) = \frac{1}{2} \parallel u \parallel^2_{H_0^1(\Omega)} - \int_\Omega F_\lambda(x,u)\mathrm{d}x$$

$$\geq \frac{1}{2} \parallel u \parallel^2_{H_0^1(\Omega)} - \frac{\lambda}{1-\delta}\int_\Omega |u|^{1-\delta}\,\mathrm{d}x - \lambda C \parallel u \parallel^{q+1}_{L^{q+1}(\Omega)} - \lambda C \int_\Omega u^{2^*}\mathrm{d}x$$

$$\geq \frac{1}{2} \parallel u \parallel^2_{H_0^1(\Omega)} - \lambda C \parallel u \parallel^{1-\delta} - \lambda C \parallel u \parallel^{q+1}_{L^{q+1}(\Omega)} - \lambda C \parallel u \parallel^{2^*}_{L^{2^*}(\Omega)} .$$

Hence, for $r_0 > 0$ small enough ,$\exists\ \delta_0 > 0$ such that

$$\frac{1}{2} \parallel u \parallel^2_{H_0^1(\Omega)} - \frac{\lambda}{q+1}\int_\Omega |u|^{q+1}\,\mathrm{d}x - \lambda\int_\Omega \tilde{\rho}(u)\mathrm{d}x \geq 0 \ \forall\ u\ \in B_{r_0}, \tag{2.7}$$

$$\frac{1}{2} \parallel u \parallel^2_{H_0^1(\Omega)} - \frac{\lambda}{q+1}\int_\Omega |u|^{q+1}\,\mathrm{d}x - \lambda\int_\Omega \tilde{\rho}(u)\mathrm{d}x \geq \delta_0 \ \forall\ u\ \in \partial B_{r_0}. \tag{2.8}$$

Moreover, since $H_0^1(\Omega) \hookrightarrow L^{1-\delta}(\Omega)$, we have, for $\lambda > 0$, small enough,

$$\overline{E}_\lambda\backslash_{\partial B_{r_0}} \geq \frac{\delta_0}{2} > 0.$$

Set

$$c_0 = \inf_{u \in B_{r_0}} \overline{E}_\lambda(u).$$

For every $v \neq 0, \overline{E}_\lambda(tv) < 0$ provided $t > 0$ is sufficiently small. This implies that $c_0 < 0$.
Now, let $\{u_j\} \subset B_{r_0}$ be a minimizing sequence for c_0. Then, there exists u_λ such that up to a subsequence:

$$u_j \longrightarrow u_\lambda \quad \text{weakly in } H_0^1(\Omega), \tag{2.9}$$

and

$$u_j \longrightarrow u_\lambda \text{ strongly in } L^p(\Omega) \text{ for } 2 \leq p < \frac{2N}{N-2}. \tag{2.10}$$

Using the Hölder's inequality, we get that, as $j \to \infty$,

$$\begin{aligned}
\int_\Omega u_j^{1-\delta}\mathrm{d}x &\leq \int_\Omega u_\lambda^{1-\delta}\mathrm{d}x + \int_\Omega \mid u_j - u_\lambda \mid^{1-\delta} \mathrm{d}x \\
&\leq \int_\Omega u_\lambda^{1-\delta}\mathrm{d}x + C \parallel u_j - u_\lambda \parallel_{L^2(\Omega)}^{1-\delta} \\
&= \int_\Omega u_\lambda^{1-\delta}\mathrm{d}x + o(1).
\end{aligned}$$

Similarly

$$\begin{aligned}
\int_\Omega u_\lambda^{1-\delta}\mathrm{d}x &\leq \int_\Omega u_j^{1-\delta}\mathrm{d}x + \int_\Omega \mid u_j - u_\lambda \mid^{1-\delta} \mathrm{d}x \\
&\leq \int_\Omega u_j^{1-\delta}\mathrm{d}x + C \parallel u_j - u_\lambda \parallel_{L^2(\Omega)}^{1-\delta} \\
&= \int_\Omega u_j^{1-\delta}\mathrm{d}x + o(1).
\end{aligned}$$

Thus,

$$\int_\Omega u_j^{1-\delta}\mathrm{d}x = \int_\Omega u_\lambda^{1-\delta}\mathrm{d}x + o(1). \tag{2.11}$$

From (2.9) and (2.10), up to a subsequence, $u_j \to u_\lambda$ pointwise a.e. in Ω. Then, by the Brezis-Lieb Lemma ([4]),

$$\parallel u_j \parallel_{L^{q+1}(\Omega)}^{q+1} = \parallel u_\lambda \parallel_{L^{q+1}(\Omega)}^{q+1} + \parallel u_j - u_\lambda \parallel_{L^{q+1}(\Omega)}^{q+1} + o(1) \tag{2.12}$$

$$\parallel u_j \parallel_{H_0^1(\Omega)}^{2} = \parallel u_\lambda \parallel_{H_0^1(\Omega)}^{2} + \parallel u_j - u_\lambda \parallel_{H_0^1(\Omega)}^{2} + o(1). \tag{2.13}$$

Moreover, by $(\mathbf{H_2})$ and Vitali's Convergence Theorem we have, as $j \to \infty$,

$$\int_\Omega \tilde{\rho}(u_j)\,\mathrm{d}x \to \int_\Omega \tilde{\rho}(u_\lambda)\,\mathrm{d}x. \tag{2.14}$$

Indeed, from (**H2**), $\forall\, \epsilon > 0$, there exists $M > 0$ such that

$$\begin{aligned}\left|\int_\Omega \tilde{\rho}(u_j)\,\mathrm{d}x\right| &= \left|\int_{u_j\le M} \tilde{\rho}(u_j)\,\mathrm{d}x + \int_{u_j> M} \tilde{\rho}(u_j)\,\mathrm{d}x\right| \\ &\le \left|\int_{u_j\le M} \tilde{\rho}(u_j)\mathrm{d}x\right| + \epsilon\int_\Omega u_j^{2^*}\,\mathrm{d}x \\ &\le \left|\int_{u_j\le M} \tilde{\rho}(u_j)\mathrm{d}x\right| + \epsilon r_0^{2^*}.\end{aligned}$$

By dominated convergence, we get (2.14). In the same fashion, we can prove that, as $j \to \infty$,

$$\left|\int_\Omega (\tilde{\rho}(u_j) - \tilde{\rho}(u_\lambda))\mathrm{d}x\right| = o(1) \tag{2.15}$$

From (2.13)-(2.14), we get

$$\overline{E}_\lambda(u_j) = \overline{E}_\lambda(u_\lambda) + \frac{1}{2}\|u_j - u_\lambda\|^2_{H_0^1(\Omega)} - \frac{1}{q+1}\|u_j - u_\lambda\|^{q+1}_{L^{q+1}(\Omega)} + o(1).$$

Using (2.15), we have that as, $j \to \infty$,

$$\frac{1}{2}\|u_j - u_\lambda\|^2_{H_0^1(\Omega)} - \frac{1}{q+1}\|u_j - u_\lambda\|^{q+1}_{L^{q+1}(\Omega)} \ge o(1). \tag{2.16}$$

Then, from above, as $j \to \infty$ we get $\overline{E}_\lambda(u_j) \ge \underline{E}(u_\lambda) + o(1)$. Thus, from the definition of c_0, it follows that $\overline{E}_\lambda(u_\lambda) = c_0$. Hence $0 \le u_\lambda \ne 0$ is a local minimizer of $\overline{E}_\lambda$ in $H_0^1(\Omega)$. From, the strong maximum principle (see [5]), we can prove that u_λ is a weak solution to (P_λ).
Now, we study the problem (P_λ) with $\delta \ge 1$. In this case, for $\epsilon > 0$ we consider the following approximated problem:

$$(P_{\lambda,\epsilon})\begin{cases} -\Delta u = \lambda\left(\frac{1}{(u+\epsilon)^\delta} + (u+\epsilon)^q + \rho(u+\epsilon)\right) \text{ in } \Omega \\ u|_{\partial\Omega} = 0, u > 0 \text{ in } \Omega\end{cases}$$

where $\epsilon > 0$ small enough. To show the existence of a solution to $(P_{\lambda,\epsilon})$, we construct a subsolution $\underline{u}_{\lambda,\epsilon}$ and a supersolution $\overline{u}_{\lambda,\epsilon}$ such that $\underline{u}_{\lambda,\epsilon} \le \overline{u}_{\lambda,\epsilon}$.
We introduce the following problem

$$(\tilde{P}_{\lambda,\epsilon})\begin{cases} -\Delta u = \frac{\lambda}{(u+\epsilon)^\delta} \text{ in } \Omega \\ u|_{\partial\Omega} = 0, u > 0 \text{ in } \Omega.\end{cases}$$

From [9] (see also [7]), the problem $(\tilde{P}_{\lambda,\epsilon})$ has a unique positive solution $\underline{u}_{\lambda,\epsilon} \in C^{0,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$.
On the other hand, let v the solution to the following problem

$$\begin{cases} -\Delta v = 1 \text{ in } \Omega \\ v|_{\partial\Omega} = 0, v > 0 \text{ in } \Omega. \end{cases}$$

We will construct the supersolution of the problem $(P_{\lambda,\epsilon})$ in the form $\overline{u}_{\lambda,\epsilon} = \underline{u}_{\lambda,\epsilon} + Kv$ where λ is small enough and $K > 0$ depends on λ. Indeed,

$$-\Delta\overline{u}_{\lambda,\epsilon} = -\Delta(\underline{u}_{\lambda,\epsilon} + Kv) = \frac{\lambda}{(\underline{u}_{\lambda,\epsilon} + \epsilon)^\delta} + K$$

$$\geq \lambda\left[\frac{1}{(\underline{u}_{\lambda,\epsilon} + Kv + \epsilon)^\delta} + (\underline{u}_{\lambda,\epsilon} + Kv + \epsilon)^q + \rho(\underline{u}_{\lambda,\epsilon} + Kv + \epsilon)\right].$$

We Observe that

$$\left[\frac{1}{(\underline{u}_{\lambda,\epsilon} + Kv + \epsilon)^\delta} + (\underline{u}_{\lambda,\epsilon} + Kv + \epsilon)^q + \rho(\underline{u}_{\lambda,\epsilon} + Kv + \epsilon)\right]$$

$$\leq \frac{1}{\underline{u}^\delta_{\lambda,\epsilon}} + (\underline{u}_{\lambda,\epsilon} + Kv)^{2^*-1} + C$$

for $C > 0$ independent of $K, \epsilon > 0$ (from (**H2**) and since $\underline{u}_{\lambda,\epsilon} \leq \underline{u}_\lambda < \underline{u}_\Lambda$ for $\lambda \leq \Lambda$).
Then, $\overline{u}_{\lambda,\epsilon}$ is a supersolution provided that K satisfies the following inequality:

$$K \geq \lambda[K^{2^*-1} + C].$$

Now, for λ small enough and $\forall\ \epsilon \geq 0$, we have

$$g_\epsilon(t) = \lambda\left(\frac{1}{(t+\epsilon)^\delta} + (t+\epsilon)^q + \rho(t+\epsilon)\right),\ g(t) = \left(\frac{1}{t^\delta} + t^q + \rho(t)\right)$$

is nonincreasing for $t \in [0, |\ \overline{u}_\lambda\ |_{L^\infty}]$.
Then, for λ small enough we have

$$u_{\lambda,\epsilon} + \epsilon \leq u_{\lambda,\epsilon'} + \epsilon' \ \ \forall\ \ 0 < \epsilon \leq \epsilon'. \tag{2.17}$$

and

$$u_{\lambda,\epsilon} \geq u_{\lambda,\epsilon'}. \tag{2.18}$$

Indeed,

$$-\Delta(v_{\lambda,\epsilon'} - v_{\lambda,\epsilon}) = \lambda\left(g(v_{\lambda,\epsilon'}) - g(v_{\lambda,\epsilon})\right) \tag{2.19}$$

where $v_{\lambda,\epsilon'} = u_{\lambda,\epsilon'} + \epsilon'$ and $v_{\lambda,\epsilon} = u_{\lambda,\epsilon} + \epsilon$. Multiplying (2.19) by $(v_{\lambda,\epsilon'} - v_{\lambda,\epsilon})^+$ and integrating by parts over Ω, we get

$$\int_\Omega |\nabla(v_{\lambda,\epsilon'} - v_{\lambda,\epsilon})^+|^2 \,\mathrm{d}x = \lambda \int_\Omega (g(v_{\lambda,\epsilon'}) - g(v_{\lambda,\epsilon}))(v_{\lambda,\epsilon'} - v_{\lambda,\epsilon})^+ \mathrm{d}x \le 0.$$

Since $v_{\lambda,\epsilon} = \epsilon$ on $\partial\Omega$, $v_{\lambda,\epsilon'} = \epsilon'$ on $\partial\Omega$ and $Sup\{v_{\lambda,\epsilon'} - v_{\lambda,\epsilon}\}^+ \subset \Omega$, then we get (2.17). Now, let us prove (2.18)

$$\begin{aligned} -\Delta(u_{\lambda,\epsilon'} - u_{\lambda,\epsilon}) &= \lambda\left(g(u_{\lambda,\epsilon'} + \epsilon') - g(u_{\lambda,\epsilon} + \epsilon)\right) \\ &\le \lambda\left(g(u_{\lambda,\epsilon} + \epsilon') - g(u_{\lambda,\epsilon} + \epsilon)\right) \\ &\le g_\epsilon(u_{\lambda,\epsilon'}) - g_\epsilon(u_{\lambda,\epsilon}). \end{aligned} \tag{2.20}$$

Multiplying (2.20) by $(u_{\lambda,\epsilon'} - u_{\lambda,\epsilon})^+$ and integrating over Ω, we get

$$\int_\Omega |\nabla(u_{\lambda,\epsilon'} - u_{\lambda,\epsilon})^+|^2 \,\mathrm{d}x = \int_\Omega (g_\epsilon(u_{\lambda,\epsilon'}) - g_\epsilon(u_{\lambda,\epsilon}))(u_{\lambda,\epsilon'} - u_{\lambda,\epsilon})^+ \le 0.$$

From which we get (2.18). Hence, using (2.17) and (2.18), we obtain that there exists u_λ, a weak solution to (P_λ), such that

$$u_{\lambda,\epsilon} \to u_\lambda \text{ in } C(\overline{\Omega}) \text{ and } u_\lambda \in C^2_{loc}(\Omega) \cap C(\overline{\Omega}).$$

Then, using Lemma 2.4.5 in Section 2.4, (i) follows.
Moreover, from the behaviour of the solution $\underline{u}_{\lambda,\epsilon} \le u_{\lambda,\epsilon} \le \overline{u}_{\lambda,\epsilon}$, we get, as $\epsilon \to 0$, $\underline{u}_\lambda \le u_\lambda \le \underline{u}_\lambda + Kv$, and (ii) follows.
Finally, to prove (iii) we show that u_λ is a weak solution to (P_λ) in $H^1_0(\Omega)$. Indeed, for any $\phi \in H^1_0(\Omega), \phi \ge 0$, and using the estimates in (ii), we get

$$\begin{aligned} \sup_\epsilon \int_\Omega |\nabla u_{\lambda,\epsilon}|^2 \,\mathrm{d}x &= \sup_\epsilon \left(\lambda(\int_\Omega (u_{\lambda,\epsilon} + \epsilon)^{1-\delta} + u^{q+1}_{\lambda,\epsilon} + \rho(u_{\lambda,\epsilon})u_{\lambda,\epsilon})\mathrm{d}x\right). \\ &\le \lambda \int_\Omega \left(\underline{u}_\lambda^{1-\delta} + \overline{u}_\lambda^{q+1} + \rho(\overline{u}_\lambda)\overline{u}_\lambda\right) \mathrm{d}x < \infty. \end{aligned}$$

Therefore, as $\epsilon \to 0$ we get $u_{\lambda,\epsilon} \rightharpoonup u_\lambda$ weakly in $H^1_0(\Omega)$. Then, for any $\phi \in H^1_0(\Omega)$,

$$\int_\Omega \nabla u_\lambda \nabla \phi \mathrm{d}x = \lambda \int_\Omega \left(\frac{1}{u_\lambda^\delta}\phi + u_\lambda^q \phi + \rho(u_\lambda)\phi\right) \mathrm{d}x.$$

This completes the proof of the Proposition 2.1.1 for $\delta < 3$. □

Remark 2.2.1. *For $\delta > 1$, we can also show that $m(\epsilon + \phi_1)^{\frac{2}{\delta+1}}$ and $M(\epsilon + \phi_1)^{\frac{2}{\delta+1}}$ (m, M respectively small and large enough) are a subsolution and a supersolution of the problem $(P_{\lambda,\epsilon})$. Then $m\phi_1^{\frac{-2}{\delta+1}} \le u_\lambda \le M\phi_1^{\frac{-2}{\delta+1}}$.*

Lemma 2.2.2. $0 < \Lambda < \infty$ *for* $\delta < 3$.

Proof. Let $\lambda_1(\Omega)$ denote the principal eigenvalue of $-\Delta$ with the Dirichlet condition, and ϕ_1 the corresponding positive normalized eigenfunction.
Multiplying the equation in (P_λ) by ϕ_1 and integrating by parts over Ω, we get

$$\lambda_1(\Omega)\int_\Omega u\phi_1 \mathrm{d}x = \lambda \int_\Omega \left(\frac{1}{u^\delta}\phi_1 + u^q\phi_1 + \rho(u)\phi_1\right)\mathrm{d}x.$$

Using $(\mathbf{H_1})$, there exists $C_0 >$ such that

$$t^{-\delta} + t^q + \rho(t) \geq t^{-\delta} + t^q + C_0 + \epsilon t^q \geq \epsilon t.$$

This implies that $\lambda \leq \frac{\lambda_1(\Omega)}{\epsilon}$ and then $\Lambda < \frac{\lambda_1(\Omega)}{\epsilon} < \infty$. The proof of the Lemma 2.2.2 is completed. $\square$

Lemma 2.2.3. *Let* $0 < \delta < 3$. *Then, for* $\lambda \in (0, \Lambda)$ *there exists a solution* $\tilde{u}_\lambda$ *to* (P_λ) *which is a local minimizer of* $\overline{E}_\lambda$ *in* $H_0^1(\Omega)$. *Moreover,* $\overline{E}_\lambda(\tilde{u}_\lambda) = E_\lambda(\tilde{u}_\lambda) < 0$.

Proof. We show the existence of a local minimizer of the energy for $\delta < 3$. Since the functional E_λ is not differentiable, we use the cut-off argument. Define

$$h_\lambda(x,t) = \begin{cases} f_\lambda(\overline{u}_\lambda(x)) & \text{if } t \geq \overline{u}_\lambda(x) \\ f_\lambda(t) & \text{if } \underline{u}_\lambda(x) \leq t \leq \overline{u}_\lambda(x) \\ f_\lambda(\underline{u}_\lambda(x)) & \text{if } t \leq \underline{u}_\lambda(x) \end{cases}$$

and

$$\tilde{E}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega H_\lambda(x,u)\,\mathrm{d}x$$

where $H_\lambda(x,t) = \int_0^t h_\lambda(x,s)\,\mathrm{d}s$.
From dominated convergence and Sobolev Embedding, it is easy to see that $\tilde{E}_\lambda$ is bounded below and weakly lower semi-continuous in $H_0^1(\Omega)$.
Then, there exists $\tilde{u}_\lambda \in H_0^1(\Omega)$ such that

$$I_\lambda = \min_{u \in H_0^1(\Omega)} \tilde{E}_\lambda(u)$$

is achieved on $\tilde{u}_\lambda$.
Since $\tilde{E}_\lambda$ is C^1 on $H_0^1(\Omega)$ (see Lemma 2.4.3 in Section 2.4), $\tilde{u}_\lambda$ satisfies

$$\begin{cases} -\Delta u_\lambda = h_\lambda(x, u_\lambda) \text{ in } \Omega \\ u_\lambda|_{\partial\Omega} = 0, \quad u_\lambda > 0 \text{ in } \Omega. \end{cases}$$

Then, from the weak maximum principle we have $\underline{u}_\lambda \le \tilde{u}_\lambda \le \overline{u}_\lambda$ in Ω.
Then, $\tilde{u}_\lambda$ is a weak solution to

$$\begin{cases} -\Delta \tilde{u}_\lambda = f(\tilde{u}_\lambda) \ \text{ in } \Omega \\ \tilde{u}_\lambda|_{\partial\Omega} = 0, \ \ \tilde{u}_\lambda > 0 \text{ in } \Omega. \end{cases}$$

Let us prove that $\tilde{u}_\lambda$ is a local minimizer of $\overline{E}_\lambda$. As in [10], we argue by contradiction. There exists $(\phi_n)_{n=1}^\infty \subset H_0^1(\Omega)$ such that $\phi_n \to 0$ as $n \to \infty$ strongly in $H_0^1(\Omega)$ and

$$\overline{E}_\lambda(\tilde{u}_\lambda + \phi_n) < \overline{E}_\lambda(\tilde{u}_\lambda) = \tilde{E}_\lambda(\tilde{u}_\lambda).$$

Let $\tilde{u}_\lambda + \phi_n = u_n$, $v_n = \min(u_n, \overline{u}_\lambda), w_n = (u_n - \overline{u}_\lambda)^+$ and $S_n = \sup\{x \in \Omega / w_n \neq 0\}$. Then

$$u_n = v_n + w_n \ \text{ and } v_n \le \overline{u}_\lambda.$$

Therefore, using this notation we get

$$\overline{E}_\lambda(u_n) = \overline{E}_\lambda(v_n) + A_n = \tilde{E}(v_n) + A_n \ge \overline{E}_\lambda(\tilde{u}_\lambda) + A_n$$

with

$$A_n = \int_{S_n} \frac{1}{2}(|\nabla u_n|^2 - |\nabla \overline{u}|^2)\mathrm{d}x - \int_{S_n} [\overline{F}_\lambda(x, u_n) - \overline{F}_\lambda(x, \overline{u}_\lambda)]\mathrm{d}x$$

where $\overline{F}_\lambda(x,t) = \int_0^t \overline{f}_\lambda(x,s)\mathrm{d}s$ (see (2.4)). Now, we estimate A_n as follows. First we show that $meas(S_n) \to 0$ as $n \to \infty$.
For this, let $w = \overline{u}_\lambda(x) - \tilde{u}_\lambda(x)$. Then, we have

$$-\Delta w = f_\lambda(x, \overline{u}_\lambda) - f_\lambda(x, \tilde{u}_\lambda) = C(x)w$$

where $C \in C^1(\Omega)$. Since $w \ge 0$ in Ω and $w \neq 0$, the strong maximum principle (see Brezis-Nirenberg [6]) implies $w > 0$ in Ω. Therefore, for any $\eta > 0$, there exists a $c(\eta) > 0$ such that $\overline{u}_\lambda(x) - \tilde{u}_\lambda(x) = w \ge c(\eta)$ in $\Omega_\eta = \{x \in \Omega \backslash \ dist(x, \partial\Omega) \ge \eta\}$. In fact, for any $\epsilon > 0$ there exists a constant $\eta > 0$ such that meas$(\Omega \backslash \Omega_\eta) \le \frac{\epsilon}{2}$.
Moreover,

$$meas(S_n \cap \Omega_\eta) \le meas(\{x \in \Omega_\eta, |u_n - \tilde{u}_\lambda| \ge c(\eta)\})$$

$$\le \frac{|u_n - \tilde{u}_\lambda|}{c(\eta)}.$$

Since, $u_n \to \tilde{u}_\lambda$ in $H_0^1(\Omega)$, it follows that, for n sufficiently large,

$$meas(S_n) \le meas(\Omega \setminus \Omega_\eta) + meas(S_n \cap \Omega_\eta)$$

$$\leq \frac{\epsilon}{2} + \frac{| u_n - \tilde{u}_\lambda |}{c(\eta)}$$

i.e. $\text{meas}(S_n) \to 0$ as $n \to \infty$.
Then, it follows that

$$\| w_n \|^2_{H^1_0(\Omega)} = \int_{S_n} | \nabla(u_n - \overline{u}_\lambda) |^2 \, \mathrm{d}x$$

$$= 2 \int_{S_n} | \nabla u_n - \nabla \tilde{u}_\lambda |^2 \, \mathrm{d}x + \int_{S_n} | \nabla \overline{u}_\lambda - \nabla \tilde{u}_\lambda |^2 \, \mathrm{d}x \to 0 \text{ as } n \to \infty.$$

Then, we get

$$\begin{aligned}
A_n &= \frac{1}{2} \int_{S_n} \{| \nabla(\overline{u}_\lambda + w_n) |^2 - | \nabla \overline{u}_\lambda |^2\} \mathrm{d}x - \int_{S_n} [\overline{F}_\lambda(x, u_n) - \overline{F}_\lambda(x, \overline{u}_\lambda)] \mathrm{d}x \\
&= \frac{1}{2} \int_{S_n} | \nabla w_n |^2 \, \mathrm{d}x + \int_{S_n} \nabla w_n \nabla \overline{u}_\lambda \mathrm{d}x - \int_{S_n} [\overline{F}_\lambda(x, u_n) - \overline{F}_\lambda(x, \overline{u}_\lambda)] \mathrm{d}x \\
&\geq \int_{S_n} | \nabla w_n |^2 \, \mathrm{d}x - \int_{S_n} \rho(\overline{u}_\lambda + \theta w_n) w_n \mathrm{d}x + \int_{S_n} \rho(\overline{u}_\lambda) w_n \mathrm{d}x \\
&\quad - \int_{S_n} (\overline{u}_\lambda + \theta w_n)^q w_n \mathrm{d}x + \int_{S_n} \overline{u}_\lambda^q w_n \mathrm{d}x \\
&= \frac{1}{2} \| w_n \|^2_{H^1_0(\Omega)} - o(1) \| w_n \|^2_{H^1_0(\Omega)} \geq 0
\end{aligned}$$

by Sobolev Embedding and the Hölder inequality. Therefore, $\overline{E}_\lambda(u_n) \geq \overline{E}(\tilde{u}_\lambda)$ for n sufficiently large which provides the desired contradiction. This completes the proof of the Lemma 2.2.3. □

As a consequence of the Lemma 2.2.3, we have

Proposition 2.2.1. *For $\lambda = \Lambda$ there exists at least one positive solution to (P_λ).*

Proof. Let $\lambda_n \in (0, \Lambda)$ be an increasing sequence such that $\lambda_n \to \Lambda$ and u_{λ_n} be the solution of the problem (P_{λ_n}). Therefore, $\forall \varphi \in H^1_0(\Omega)$ we have

$$\int_\Omega \nabla u_{\lambda_n} \nabla \varphi \mathrm{d}x = \lambda \left(\int_\Omega u_{\lambda_n}^{-\delta} + \int_\Omega u_{\lambda_n}^q + \int_\Omega \rho(u_{\lambda_n}) \right) \varphi \mathrm{d}x \tag{2.21}$$

and for $\varphi = u_{\lambda_n}$

$$\int_\Omega | \nabla u_{\lambda_n} |^2 \, \mathrm{d}x = \lambda_n \left(\int_\Omega u_{\lambda_n}^{1-\delta} + \int_\Omega u_{\lambda_n}^{q+1} + \int_\Omega \rho(u_{\lambda_n}) u_{\lambda_n} \right) \mathrm{d}x. \tag{2.22}$$

Moreover, u_{λ_n} is a solution to the problem (P_{λ_n}), and from Lemma 2.2.3 the associated functional is bounded for all $0 < \delta < 3$. Therefore, for $0 < \delta < 1$,

$$E_{\lambda_n}(u_{\lambda_n}) = \frac{1}{2}\int_\Omega \mid \nabla u_{\lambda_n} \mid^2 \mathrm{d}x - \lambda_n \int_\Omega \frac{u_{\lambda_n}^{1-\delta}}{1-\delta}\mathrm{d}x - \lambda_n \int_\Omega \frac{u_{\lambda_n}^{q+1}}{q+1}\mathrm{d}x$$
$$-\lambda_n \int_\Omega \tilde{\rho}(u_{\lambda_n})\mathrm{d}x \le M. \tag{2.23}$$

Then, from (2.22) and (2.23), we get

$$\frac{\lambda_n}{2}\left(\int_\Omega u_{\lambda_n}^{1-\delta} + \int_\Omega u_{\lambda_n}^{q+1} + \int_\Omega \rho(u_{\lambda_n})u_{\lambda_n}\right)\mathrm{d}x - \lambda_n \int_\Omega \frac{u_{\lambda_n}^{1-\delta}}{1-\delta}\mathrm{d}x$$
$$-\lambda_n \int_\Omega \frac{u_{\lambda_n}^{q+1}}{q+1}\mathrm{d}x - \lambda_n \int_\Omega \tilde{\rho}(u_{\lambda_n})\mathrm{d}x \le M. \tag{2.24}$$

Moreover, by $(\mathbf{H_1})$ there exists $C_\epsilon > 0$ such that

$$\rho(t) \le C_\epsilon + \epsilon t^q \text{ for any } \epsilon > 0. \tag{2.25}$$

Hence, using (2.24) and (2.25), it follows that there exists $M_\epsilon > 0$ such that

$$\left(\frac{1}{2} - \frac{1}{q+1} - \frac{\epsilon}{q+1}\right)\int_\Omega u_{\lambda_n}^{q+1} \le M_\epsilon + \left(\frac{1}{1-\delta} - \frac{1}{2}\right)\int_\Omega u_{\lambda_n}^{1-\delta}\mathrm{d}x. \tag{2.26}$$

Inserting (2.26) in (2.22), we get $\sup\limits_{n\in\mathbb{N}} \|u_{\lambda_n}\|_{H_0^1(\Omega)} < \infty$ for $0 < \delta < 1$.
Now, let $\delta = 1$. First, we see that for any $\epsilon > 0$ there exists $K_\epsilon > 0$ such that

$$\ln(t) \le t^\epsilon + \frac{K_\epsilon}{|\Omega|} \;\; \forall\, t > 0. \tag{2.27}$$

Then, using (2.27), we get that from (2.23)

$$\frac{1}{2}\int_\Omega |\nabla u_{\lambda_n}|^2 \mathrm{d}x \le M + K_\epsilon + \lambda_n \left[\int_\Omega u_{\lambda_n}^{1+\epsilon}\mathrm{d}x + \frac{1}{q+1}\int_\Omega u_{\lambda_n}^{q+1}\mathrm{d}x + \int_\Omega \tilde{\rho}(u_{\lambda_n})\mathrm{d}x\right]. \tag{2.28}$$

From (2.22) and (2.28), there exists $\tilde{K}_\epsilon > 0$ such that

$$\left(\frac{1}{2} - \frac{1}{q+1} - \frac{\epsilon}{q+1}\right)\int_\Omega u_{\lambda_n}^{q+1}\mathrm{d}x \le \Lambda \int_\Omega u_{\lambda_n}^{1+\epsilon}\mathrm{d}x + \tilde{K}_\epsilon. \tag{2.29}$$

Inserting (2.29) in (2.22) and taking $\epsilon < 1$, we get $\sup\limits_{n\in\mathbb{N}} \|u_{\lambda_n}\|_{H_0^1(\Omega)} < \infty$ for $\delta = 1$.

Finally, let us consider $1 < \delta < 3$. Then, from Proposition 2.1.1 we see for $\delta > 1$ that

$$c_0\varphi_1(x)^{\frac{2}{1+\delta}} \leq \underline{u}_\lambda(x) \leq c_1\varphi_1(x)^{\frac{2}{1+\delta}} \tag{2.30}$$

where c_0, c_1 are two positive constants.
Hence, from (2.23) and (2.26), we have

$$(\frac{1}{2} - \frac{1}{q+1} - \frac{\epsilon}{q+1})\int_\Omega u_{\lambda_n}^{q+1}\mathrm{d}x \leq \frac{M}{\lambda_n} + (\frac{1}{1-\delta} - \frac{1}{2})\int_\Omega u_{\lambda_n}^{1-\delta}\mathrm{d}x. \tag{2.31}$$

Moreover, from (2.30), using $\lambda = \lambda_2 \leq \lambda_n$, we get

$$\int_\Omega u_{\lambda_n}^{1-\delta}\mathrm{d}x \leq \int_\Omega \underline{u}_{\lambda_2}^{1-\delta}\mathrm{d}x \leq c_0^{1-\delta}\int_\Omega \varphi^{\frac{2(1-\delta)}{1+\delta}}\mathrm{d}x < \infty \text{ if } \delta < 3. \tag{2.32}$$

Inserting (2.31) and (2.32) in (2.22), we get $\sup\limits_{n\in\mathbb{N}}\|u_{\lambda_n}\|_{H_0^1(\Omega)} < \infty$ for $1 < \delta < 3$.
Hence, $\sup\limits_{n\in\mathbb{N}}\|u_{\lambda_n}\|_{H_0^1(\Omega)} < \infty$ for all $0 < \delta < 3$.
It follows that $\{u_{\lambda_n}\}$ is bounded in $H_0^1(\Omega)$. Thus there exists $u_\Lambda \in H_0^1(\Omega)$ such that $u_{\lambda_n} \to u_\Lambda$ weakly in $H_0^1(\Omega)$. Using (2.26), the fact that

$$\limsup u_{\lambda_n}^{-\delta} \geq \underline{u}^{-\delta}$$

and letting $n \to +\infty$ in (2.21) and (2.22) we get that u_Λ is a weak solution of the problem (P_Λ) in $H_0^1(\Omega)$. The proof of the Proposition 2.2.1 is now completed. □

2.3 Multiplicity result

Since $\overline{E}_\lambda$ is C^1 on $H_0^1(\Omega)$, we construct a second solution v_λ as a Mountain Pass solution of $\overline{E}_\lambda$. Moreover, if v_λ is a critical point of $\overline{E}_\lambda$, then from the maximum principle (see [5]) v_λ is a critical point of E_λ.
For $0 < \lambda < \Lambda$ and considering $\omega(x) = \underline{u}_\lambda(x)$ in the Lemma 2.4.3, from Proposition 2.1.1 and the fact that $\lim\limits_{t\to+\infty} E_\lambda(t\varphi) = -\infty$ for $0 \leq \varphi \in H_0^1(\Omega)\backslash\{0\}$, $\overline{E}_\lambda$ has a Mountain Pass geometry close to u_λ. Hence, we may fix $e \in H_0^1(\Omega)\backslash\{0\}$ such that $\overline{E}_\lambda(e) < \overline{E}_\lambda(u_\lambda)$.
Let $R_0 = \| e - u_\lambda \|_{H_0^1(\Omega)}$, $l_0 > 0$ small enough such that u_λ is a minimizer of $\overline{E}_\lambda$ on $\{\| u - u_\lambda \|_{H_0^1(\Omega)} \leq l_0\}$.
Set

$$\Gamma \stackrel{\text{def}}{=} \{\eta \in C([0,1], H_0^1(\Omega))/\eta(0) = u_\lambda, \eta(1) = e\}$$

and define the Mountain Pass level

$$\gamma_0 \stackrel{\text{def}}{=} \inf_{\eta\in\Gamma}\max_{t\in[0,1]} \overline{E}_\lambda(\eta(t)).$$

Then, we have to deal with two situations:
(P_1) "zero altitude case"
$\inf\{\overline{E}_\lambda(u), u \in H_0^1(\Omega)$ and $\| u - u_\lambda \|_{H_0^1(\Omega)} = l\} \leq \overline{E}_\lambda(u_\lambda)$ for all $l < R_0$.
(P_2) there exists $l_1 < R_0$ such that

$$\inf\{\overline{E}_\lambda(u) : u \in H_0^1(\Omega) \text{ and } \| u - u_\lambda \| = l\} > \overline{E}_\lambda(u_\lambda).$$

Note that (P_1)(resp. (P_2)) implies that $\gamma_0 = \overline{E}_\lambda(u_\lambda)$ (resp. $\gamma_0 > \overline{E}_\lambda(u_\lambda)$). In the case where (P_1) occurs, we can localize the Palais Smale sequence, and we get at least a second weak solution to (P_λ).

More precisely we have the following result:

Proposition 2.3.1. *Let $0 < \delta < 3, \lambda \in (0, \Lambda)$ $q \in (1, 2^* - 1]$ and suppose that (P_1) holds. Then there exists a weak solution v_λ of (P_λ) such that $v_\lambda \neq u_\lambda$.*

Firstly, we define a generalized notion of Palais Smale sequence for $\overline{E}_\lambda$.

Definition 2.3.1. *Let $\mathfrak{F} \subset H_0^1(\Omega)$ be a closed set. We say that a sequence $\{v_n\}_{n=1}^\infty \subset H_0^1(\Omega)$ is a Palais Smale sequence for $\overline{E}_\lambda$ at the level c around $\mathfrak{F}$ (a ($PS_{\mathfrak{F},c}$) for short) if*

$$\lim_{n\to\infty} dist(v_n, \mathfrak{F}) = 0, \lim_{n\to\infty} \overline{E}_\lambda(v_n) = c, \lim_{n\to\infty} \| \overline{E}'_\lambda(v_n) \|_{H^{-1}(\Omega)} = 0.$$

Proof Proposition 2.3.1. From Theorem 1 in Ghoussoub-Preiss [13], for $l < l_0$ we get the existence of a $(PS_{\mathfrak{F},\gamma_0})$ bounded sequence, v_k up to a subsequence converging weakly in $H_0^1(\Omega)$ to v_λ, a weak solution to (P_λ). To prove that $u_\lambda \neq v_\lambda$, it is sufficient to prove that $v_k \to v_\lambda$ strongly in $H_0^1(\Omega)$ as $k \to \infty$. Since $v_k \rightharpoonup v_\lambda$ as $k \to \infty$, $v_k \to v_\lambda$ in L^p with $p < 2^* - 1$ and pointwise almost everywhere in Ω. Then, we have

$$\| v_k \|^2_{H_0^1(\Omega)} = \| v_k - v_\lambda \|^2_{H_0^1(\Omega)} + \| v_\lambda \|^2_{H_0^1(\Omega)} + o_k(1), \tag{2.33}$$

and from the Brezis-Lieb result [4],

$$\| v_k \|^{q+1}_{L^{q+1}(\Omega)} = \| v_k - v_\lambda \|^{q+1}_{L^{q+1}(\Omega)} + \| v_\lambda \|^{q+1}_{L^{q+1}(\Omega)} + o_k(1)$$

as $k \to \infty$.
From the Sobolev imbedding Theorem and by dominated convergence (if $\delta > 1$)

$$\int_{v_k \geq \underline{u}_\lambda} | v_k^{1-\delta} - v_\lambda^{1-\delta} | \, dx = o_k(1) \text{ as } k \to \infty.$$

Since v_λ is a weak solution to (P_λ), we have

$$\| v_\lambda \|^2_{H_0^1(\Omega)} - \lambda \| v_\lambda \|^{q+1}_{L^{q+1}(\Omega)} - \lambda \int_\Omega v_\lambda^{1-\delta} \mathrm{d}x - \lambda \int_\Omega \rho(v_\lambda) v_\lambda \mathrm{d}x = 0. \tag{2.34}$$

Therefore, as $k \to \infty$

$$\int_\Omega \nabla v_k \nabla (v_k - v_\lambda)\,\mathrm{d}x = \lambda \left[\int_{v_k \geq \underline{u}_\lambda} v_k^{-\delta}(v_k - v_\lambda)\,\mathrm{d}x + \int_\Omega v_k^q (v_k - v_\lambda)\,\mathrm{d}x \right.$$

$$\left. + \int (\rho(v_k)v_k - \rho(v_\lambda)v_\lambda)\,\mathrm{d}x \right] + o_k(1).$$

Since v_k is bounded in $H_0^1(\Omega)$ from Vitali's convergence Theorem, we obtain,

$$\left| \int (\rho(v_k)v_k - \rho(v_\lambda)v_\lambda)\,\mathrm{d}x \right| = o_k(1). \tag{2.35}$$

Using (2.33)-(2.35), it follows that

$$\int_\Omega \mid \nabla (v_k - v_\lambda) \mid^2 \ \mathrm{d}x = \int_\Omega \mid v_k - v_\lambda \mid^{q+1} \ \mathrm{d}x + o_k(1) \text{ as } k \to \infty. \tag{2.36}$$

Now, we consider two cases:
(1) $\overline{E}_\lambda(u_\lambda) \neq \overline{E}_\lambda(v_\lambda)$,
(2) $\overline{E}_\lambda(u_\lambda) = \overline{E}_\lambda(v_\lambda)$.
In case (1), we are done. If (2) holds, we set

$$\overline{E}(v_k - v_\lambda) = \overline{E}_\lambda(v_k) - \overline{E}_\lambda(v_\lambda) + o_k(1) \text{ as } k \to \infty. \tag{2.37}$$

Consequently, by (2.34) we obtain

$$\frac{1}{2} \parallel v_k - v_\lambda \parallel^2_{H_0^1(\Omega)} - \frac{1}{q+1} \parallel v_k - v_\lambda \parallel^{q+1}_{L^{q+1}(\Omega)} \leq o_k(1) \text{ as } k \to \infty.$$

Then, from (2.36) -(2.37), it follows that $\parallel v_k - v_\lambda \parallel_{H_0^1(\Omega)} \to 0$ as $k \to \infty$. Therefore
$\parallel v_\lambda - u_\lambda \parallel_{H_0^1(\Omega)} = l$ and $u_\lambda \neq v_\lambda$. The proof of the Proposition 2.3.1 is now completed. □

In case where (P_2) holds, we have the following result:

Proposition 2.3.2. *Let* $0 < \delta < 3, 1 < q < 2^* - 1$, *let* $\lambda \in (0, \Lambda)$, *and suppose that* (P_2) *holds. Then, there exists a weak solution* v_λ *such that* $u_\lambda \neq v_\lambda$.

Proof. From Brezis Nirenberg work [5] (see also Tarantello [16]), we know that the condition of Palais Smale is satisfied if

$$\gamma_0 < \overline{E}_\lambda(u_\lambda) + \frac{1}{N} S^{\frac{N}{2}} \tag{2.38}$$

where S is the best Sobolev constant.
We prove (2.38) using the following test functions

$$U_\epsilon(x) = \frac{C_N \epsilon^{\frac{N-2}{2}}}{(\epsilon^2 + \mid x-y\mid^2)^{\frac{N-2}{2}}}\phi(x),$$

where $\epsilon > 0$, C_N is a normalization constant, $y \in \Omega$, and $\phi \in C_c^\infty(\Omega)$ is a fixed function such that $\phi(x) = 1$ for x in a neighborhood of $y \in \Omega$.
We follow here the arguments of [16]. Let $R_0 \geq 1$ and consider any $t \in [0,1]$; then for a suitable $\beta \in (0, q+1)$ it follows that

$$\Bigg| \int_\Omega |u_\lambda + tR_0 U_\epsilon|^{q+1} \mathrm{d}x - \int_\Omega |u_\lambda|^{q+1}\mathrm{d}x - (tR_0)^{q+1}\int_\Omega |U_\epsilon|^{q+1}\mathrm{d}x$$

$$-(q+1)R_0 t\int_\Omega |u_\lambda|^q U_\epsilon \mathrm{d}x - (q+1)(tR_0)^q \int_\Omega U_\epsilon^q u_\lambda \mathrm{d}x\Bigg| = R_0^\beta o(\epsilon^{\frac{N-2}{2}}) \quad (2.39)$$

$$\| \nabla U_\epsilon \|_2^2 = \int_{\mathbb{R}^N} \mid \nabla U_1 \mid^2 \, \mathrm{d}x + O(\epsilon^{N-2})$$

and

$$\| U_\epsilon \|_{q+1}^{q+1} = \int_{\mathbb{R}^N} \frac{dx}{(1+\mid x\mid^2)^N} + O(\epsilon^N).$$

Using (2.39) we have

$$\overline{E}_\lambda(u_\lambda + tR_0U_\epsilon) = E_\lambda(u_\lambda + tR_0U_\epsilon)$$

$$= \frac{1}{2}\int_\Omega \mid \nabla(u_\lambda + tR_0U_\epsilon)\mid^2 \, \mathrm{d}x - \frac{\lambda}{1-\delta}\int_\Omega (u_\lambda + tR_0U_\epsilon)^{1-\delta}\,\mathrm{d}x$$

$$-\frac{\lambda}{q+1}\int_\Omega (u_\lambda + tR_0U_\epsilon)^{q+1}\mathrm{d}x - \lambda\int_\Omega \tilde{\rho}(u_\lambda + tR_0U_\epsilon)\mathrm{d}x$$

$$= \frac{1}{2}\int_\Omega \mid \nabla u_\lambda\mid^2 \, \mathrm{d}x + \frac{(tR_0)^2}{2}\int_\Omega \mid \nabla U_\epsilon\mid^2 \, \mathrm{d}x$$

$$+\lambda t R_0 \int_\Omega \nabla u_\lambda \nabla U_\epsilon \,\mathrm{d}x - \frac{\lambda}{1-\delta}\int_\Omega (u_\lambda + tR_0U_\epsilon)^{1-\delta}\,\mathrm{d}x$$

$$-\frac{\lambda}{q+1}\int_\Omega (u_\lambda + tR_0U_\epsilon)^{q+1}\,\mathrm{d}x - \lambda\int_\Omega \tilde{\rho}(u_\lambda + tR_0U_\epsilon)\,\mathrm{d}x.$$

$$= \frac{1}{2}\int_\Omega \mid \nabla u_\lambda\mid^2 \, \mathrm{d}x + \frac{(tR_0)^2}{2}\int_\Omega \mid \nabla U_\epsilon\mid^2 \, \mathrm{d}x$$

$$+\lambda t R_0 \int_\Omega (u_\lambda^{-\delta} + u_\lambda^q + \rho(u_\lambda))U_\epsilon \,\mathrm{d}x - \frac{\lambda}{1-\delta}\int_\Omega (u_\lambda + tR_0U_\epsilon)^{1-\delta}\,\mathrm{d}x - \frac{\lambda}{q+1}\int_\Omega u_\lambda^{q+1}\,\mathrm{d}x$$

$$-\frac{(tR_0)^{q+1}}{q+1}\int_{\mathbb{R}^N}\frac{dx}{(1+\mid x\mid^2)^N}-(tR_0)^q\int_\Omega U_\epsilon^q u_\lambda-\lambda\int_\Omega\tilde{\rho}(u_\lambda+tR_0U_\epsilon)\,\mathrm{d}x+R_0^\beta o(\epsilon^{\frac{N-2}{2}})$$

$$= E_\lambda(u_\lambda)+\frac{(tR_0)^2}{2}\int_\Omega\mid\nabla U_1\mid^2\,\mathrm{d}x-\frac{\lambda(tR_0)^{q+1}}{q+1}\int_{\mathbb{R}^N}\frac{\mathrm{d}x}{(1+\mid x\mid^2)^N}$$

$$-\lambda(tR_0)^q\int_\Omega U_\epsilon^q u_\lambda\,\mathrm{d}x-\lambda\left[\int_\Omega\tilde{\rho}(u_\lambda+tR_0U_\epsilon)-\int_\Omega\tilde{\rho}(u_\lambda)+tR_0\int_\Omega\rho(u_\lambda)U_\epsilon\right]\mathrm{d}x$$

$$+\lambda\left[tR_0\int_\Omega u_\lambda^{-\delta}U_\epsilon\mathrm{d}x+\frac{1}{1-\delta}\int_\Omega u_\lambda^{1-\delta}\,\mathrm{d}x-\frac{1}{1-\delta}\int_\Omega(u_\lambda+tR_0U_\epsilon)^{1-\delta}\mathrm{d}x\right]$$

$$+R_0^\beta o(\epsilon^{\frac{N-2}{2}}).$$

Now, we need to estimate the last terms as follows:

Claim . For $\mu>0$,

$$\left[tR_0\int_\Omega u_\lambda^{-\delta}U_\epsilon\,\mathrm{d}x+\frac{1}{1-\delta}\int_\Omega u_\lambda^{1-\delta}\,\mathrm{d}x-\frac{1}{1-\delta}\int_\Omega(u_\lambda+tR_0U_\epsilon)^{1-\delta}\,\mathrm{d}x\right]$$

$$\leq K\int_{B_\mu(y)}U_\epsilon\,\mathrm{d}x$$

and

$$-\left[\int_\Omega\tilde{\rho}(u_\lambda+tR_0U_\epsilon)-\int_\Omega\tilde{\rho}(u_\lambda)+tR_0\int_\Omega\rho(u_\lambda)U_\epsilon\right]\mathrm{d}x$$

$$\leq K'\int_{B_\mu(y)}U_\epsilon\,\mathrm{d}x.$$

Before giving the proof of the Claim, let us recall the following Lemma from [2].

Lemma 2.3.1.

$$\int_{B_\mu(y)}\mid\nabla U_\epsilon\mid^p\,\mathrm{d}x=S^{\frac{N}{2}}+O(\epsilon^{N-2}) \tag{2.40}$$

$$\int_{B_\mu(y)}U_\epsilon^{p^*}\mathrm{d}x=S^{\frac{N}{2}}+O(\epsilon^N) \tag{2.41}$$

$$\int_{B_\mu(y)}U_\epsilon^\beta\,\mathrm{d}x\leq\begin{cases}\epsilon^{N-\beta\frac{N-2}{2}}[O(1)+O(\epsilon^{\beta(N-2)-N})] & if\ \ \beta\neq\frac{N}{N-2},\\ O(\epsilon^{N-\beta\frac{N-2}{2}}\log\epsilon) & if\ \ \beta=\frac{N}{N-2}.\end{cases} \tag{2.42}$$

Proof of the Claim :

Let ξ be a constant such that $0<\xi<\frac{1}{4}$: then for $0<\delta<3$ and as $\epsilon\to 0$,

$$B_\epsilon=tR_0\int_\Omega u_\lambda^{-\delta}U_\epsilon\,\mathrm{d}x+\frac{1}{1-\delta}\int_\Omega u_\lambda^{1-\delta}\,\mathrm{d}x-\frac{1}{1-\delta}\int_\Omega(u_\lambda+tR_0U_\epsilon)^{1-\delta}\,\mathrm{d}x$$

$$\leq \int_{|x-y|\leq\epsilon^{\xi}} \left[tR_0 u_\lambda^{-\delta} U_\epsilon + \frac{1}{1-\delta} u_\lambda^{1-\delta} - \frac{\lambda}{1-\delta}(u_\lambda + tR_0 U_\epsilon)^{1-\delta} \right] \mathrm{d}x$$
$$+ \int_{|x-y|>\epsilon^{\xi}} \left[tR_0 u_\lambda^{-\delta} U_\epsilon + \frac{1}{1-\delta} u_\lambda^{1-\delta} - \frac{\lambda}{1-\delta}(u_\lambda + tR_0 U_\epsilon)^{1-\delta} \right] \mathrm{d}x$$
$$\leq CtR_0 \int_{|x-y|\leq\epsilon^{\xi}} U_\epsilon \,\mathrm{d}x + \lambda \int_{|x-y|>\epsilon^{\xi}} \delta (u_\lambda + tR_0\theta U_\epsilon)^{-(\delta+1)} (tR_0 U_\epsilon)^2 \,\mathrm{d}x$$

for $\theta \in (0,1)$. Therefore, using Lemma 2.3.1 we obtain

$$B_\epsilon \leq CtR_0 \epsilon^{\frac{N-2}{2}} \int_0^{\epsilon^{\xi}} r^{-(N-2)} r^{N-1} \mathrm{d}r + C(tR_0)^2 o(\epsilon^{\frac{N-2}{2}})$$
$$\leq CtR_0 \epsilon^{\frac{N-2}{2}} \int_0^{\epsilon^{\xi}} r \,\mathrm{d}r + C(tR_0)^2 o(\epsilon^{\frac{N-2}{2}})$$
$$\leq CtR_0 \; o(\epsilon^{\frac{N-2}{2}}) + C(tR_0)^2 o(\epsilon^{\frac{N-2}{2}}).$$

Now, we need to prove the second estimates

$$-\left[\int_\Omega \tilde{\rho}(u_\lambda + tR_0 U_\epsilon) \,\mathrm{d}x - \int_\Omega \tilde{\rho}(u_\lambda) \,\mathrm{d}x + tR_0 \int_\Omega \rho(u_\lambda) U_\epsilon \,\mathrm{d}x \right]$$
$$= -\left[\int_{|x-y|\leq\epsilon^{\xi}} \left(\tilde{\rho}(u_\lambda + tR_0 U_\epsilon) - \tilde{\rho}(u_\lambda) + tR_0 \rho(u_\lambda) U_\epsilon \right) \,\mathrm{d}x \right.$$
$$\left. + \int_{|x-y|>\epsilon^{\xi}} \left(\tilde{\rho}(u_\lambda + tR_0 U_\epsilon) - \tilde{\rho}(u_\lambda) + tR_0 \rho(u_\lambda) U_\epsilon \right) \mathrm{d}x \right]$$
$$= I + II.$$

We start by estimates I

$$I \leq |\; tR_0 \int_{|x-y|\leq\epsilon^{\xi}} \rho(u_\lambda) U_\epsilon \,\mathrm{d}x \;| + \int_{|x-y|\leq\epsilon^{\xi}} \rho(u_\lambda + \theta tR_0 U_\epsilon)(tR_0 U_\epsilon) \,\mathrm{d}x$$

for any $\theta \in (0,1)$. Hence, using (**H2**) we get

$$I \leq CtR_0 \; o(\epsilon^{\frac{N-2}{2}}) + \int_{|x-y|\leq\epsilon^{\xi}} \rho^-(u_\lambda + \theta tR_0 U_\epsilon)(tR_0 U_\epsilon) \,\mathrm{d}x$$
$$\leq CtR_0 \; o(\epsilon^{\frac{N-2}{2}}) + \int_{|x-y|\leq\epsilon^{\xi}} (k + (tR_0 U_\epsilon)^\beta) tR_0 U_\epsilon \,\mathrm{d}x$$
$$\leq CtR_0 \; o(\epsilon^{\frac{N-2}{2}}) + (tR_0)^{\beta+1} \int_{|x-y|\leq\epsilon^{\xi}} U_\epsilon^{\beta+1} \,\mathrm{d}x.$$

Then, using (2.42) in Lemma 2.3.1, we obtain

$$I \leq CtR_0\, o(\epsilon^{\frac{N-2}{2}}) + (tR_0)^{\beta+1}\, o(\epsilon^{\frac{N-2}{2}}).$$

Now, to estimate II we use the Taylor Lagrange expansion as follows:

$$II \leq -\int_{|x-y|>\epsilon^{\xi}} \rho'(u_\lambda + \tilde{\theta} tR_0 U_\epsilon)(tR_0U_\epsilon)^2 \mathrm{d}x, \quad \text{for any } \tilde{\theta} \in (0,1)$$

$$\leq k'(tR_0)^2 \int_{|x-y|>\epsilon^{\xi}} U_\epsilon^2 \mathrm{d}x \leq k'(tR_0)^2 \mid \Omega \mid \epsilon^{N-2-2\mu(N-2)}$$

$$\leq k'(tR_0)^2 \mid \Omega \mid o(\epsilon^{\frac{N-2}{2}})$$

for $\xi < \frac{1}{4}$. Who, prove the Claim. □

Thus

$$E_\lambda(u_\lambda + tR_0U_\epsilon) \leq E_\lambda(u_\lambda) + \frac{(tR_0)^2}{2} \int_{\mathbb{R}^N} \mid \nabla U_1 \mid^2 \, \mathrm{d}x$$

$$-\frac{\lambda(tR_0)^{q+1}}{q+1} \int_{\mathbb{R}^N} \frac{\mathrm{d}x}{(1+\mid x \mid^2)^N} - (tR_0)^q \int_\Omega u_\lambda U_\epsilon^q \, \mathrm{d}x + R_0^\beta o(\epsilon^{\frac{N-2}{2}}).$$

Then, we get

$$E_\lambda(u_\lambda) < \gamma_0 < E_\lambda(u_\lambda) + \frac{1}{N} S^{\frac{N}{2}} \tag{2.43}$$

where S is the best Sobolev constant.

Now, to prove that $u_\lambda \neq v_\lambda$, similar to Proposition 2.3.1, it is sufficient to prove that $v_k \to v_\lambda$ strongly in $H_0^1(\Omega)$ as $k \to \infty$. Then, since v_k converges weakly to a solution v_λ of problem (P_λ), from Brezis-Lieb [4] we have,

$$\| v_k - v_\lambda \|^2_{H_0^1(\Omega)} - \| v_k - v_\lambda \|^{q+1}_{L^{q+1}} \leq o_k(1) \text{ as } k \to \infty. \tag{2.44}$$

Moreover, by (2.43), one gets

$$\frac{1}{2} \| v_k - v_\lambda \|^2_{H_0^1(\Omega)} - \frac{1}{q+1} \| v_k - v_\lambda \|^{q+1}_{L^{q+1}}$$

$$\leq E_\lambda(v_k) - E_\lambda(u_\lambda) + o_k(1) \leq \gamma_0 - E_\lambda(u_\lambda) + o_k(1) \leq \frac{1}{N} S^{\frac{N}{2}}. \tag{2.45}$$

Therefore, from (2.44) and (2.45), we obtain

$$\| v_k - v_\lambda \|_{H_0^1(\Omega)} \underset{k\to\infty}{\longrightarrow} 0 \text{ and } E_\lambda(v_\lambda) = \gamma_0 \neq E_\lambda(u_\lambda).$$

Then, $u_\lambda \neq v_\lambda$. The proof of the Proposition 2.3.2 is now completed. □

2.4 Regularity

We start with the following inequality which enables us to estimate when $0 < \delta < 1$ the singularity in the Gâteaux derivative of the energy functional $E_\lambda : H_0^1(\Omega) \to \mathbb{R}$ where the proof can be found in P. TAKÁČ [15]

Lemma 2.4.1. *Let* $0 < \delta < 1$. *Then, there exists a constant* $c_\delta > 0$ *such that the inequality*

$$\int_0^1 \mid a + sb \mid^{-\delta} \mathrm{d}s \leq c_\delta (\max_{0\leq\delta\leq 1} \mid a + sb \mid)^{-\delta}$$

holds for all $a, b \in \mathbb{R}^N$ *with* $\mid a \mid + \mid b \mid > 0$.

Now, we study the Gâteaux-differentiability of the energy functional E_λ at a point $u \in H_0^1(\Omega)$.

Lemma 2.4.2. *Let* $0 < \delta < 1$. *Let* $u \in H_0^1(\Omega)$ *such that* $u \geq c\varphi_1$. *Then,* $\forall\, \varphi \in H_0^1(\Omega)$ *we have*

$$\lim_{t\to 0} \frac{1}{t} \left[E_\lambda(u + tv) - E_\lambda(u)\right] = \int_\Omega \nabla u \nabla v - \lambda \left[\int_\Omega u^{-\delta} v + \int_\Omega u^q v + \int_\Omega \rho(u) v\right].$$

Proof. We concentrate only in the singular term, the others being standard. Let

$$G(u) = \frac{1}{1-\delta} \int_\Omega (u^+)^{1-\delta} \mathrm{d}x \;\forall\, u \in H_0^1(\Omega).$$

Define

$$g(w) = \begin{cases} w^{-\delta} & if \;\; w > 0 \\ 0 \;\; if & w < 0 \end{cases}$$

then, we have

$$g(w) = \frac{1}{1-\delta} \frac{\mathrm{d}}{\mathrm{d}w} (w^+)^{1-\delta} = \begin{cases} w^{-\delta} & if \;\; w > 0 \\ 0 \;\; if & w < 0 \end{cases}$$

for $w \in \mathbb{R} \backslash \{0\}$.
Hence, it follows that

$$\frac{1}{t}\left(g(u + tv) - g(u)\right) = \frac{1}{t(1-\delta)} \left[\int_\Omega (u + tv)^{1-\delta} \mathrm{d}x - \int_\Omega (u^+)^{1-\delta} \mathrm{d}x\right]$$

$$= \int_\Omega \left(\int_0^1 g(u + stv) \mathrm{d}s\right) v \mathrm{d}x. \tag{2.46}$$

Then, note that for every $x \in \Omega$ we have $u(x) > 0$ and

$$\int_0^1 g(u + stv)\mathrm{d}s \to g(u) = u^{-\delta} \text{ as } t \to 0.$$

Moreover,

$$\left|\int_0^1 g(u + stv)\mathrm{d}s\right| \leq \int_0^1 \mid u + stv \mid^{-\delta} \mathrm{d}s.$$

Then, using the estimate in Lemma 2.4.1, we get

$$\begin{aligned} \left|\int_0^1 g(u + stv)\mathrm{d}s\right| &\leq K_\delta(\max_{0\leq s\leq 1} \mid u + stv \mid)^{-\delta} \\ &\leq K_\delta u^{-\delta} \leq K_\delta(\epsilon\varphi_1)^{-\delta} = K_{\delta,\epsilon}\varphi_1^{-\delta} \end{aligned}$$

where the constant $K_{\delta,\epsilon}$ is a positive constant independent of $x \in \Omega$. Moreover, by the Hardy inequality and $\forall\, v \in H_0^1(\Omega)$ we have $v\varphi_1^{-\delta} \in L^1(\Omega)$. Finally, using the Lebesgue dominated convergence and letting $t \to 0$ in (2.46), the Lemma 2.4.2 follows. The proof of the Lemma 2.4.2 is now completed. $\square$

We show now that using a cut-off nonlinearity, the associated enegy functional is C^1 on $H_0^1(\Omega)$.

Lemma 2.4.3. *Let $0 < \delta < 3$ and $\omega \in H_0^1(\Omega)$ such that $\omega \geq \epsilon\varphi^{\frac{2}{\delta+1}}$ with some $\epsilon > 0$. Setting for $x \in \Omega$*

$$f_\lambda(x, s) = \begin{cases} \lambda(s^{-\delta} + s^q + \rho(s)) & if \;\; s \geq \omega(x) \\ \lambda(\omega(x)^{-\delta} + \omega(x)^q + \rho(\omega(x))) & if \;\; s < \omega(x) \end{cases}$$

$F_\lambda(x, s) = \int_0^s f_\lambda(x, s)\,\mathrm{d}t$ and for $u \in W_0^{1,p}(\Omega)$

$$\overline{E}_\lambda(u) = \frac{1}{2}\int_\Omega \mid \nabla u \mid^2 \,\mathrm{d}x - \int_\Omega F_\lambda(x, u)\,\mathrm{d}x$$

we have that $\overline{E}_\lambda$ belongs to $C^1(H_0^1(\Omega), \mathbb{R})$.

Proof. We concentrate in the singular term, the others being standard. Let

$$h(x, s) = \begin{cases} s^{-\delta} & if \;\; s \geq \omega(x) \\ \omega(x)^{-\delta} & if \;\; s < \omega(x), \end{cases}$$

$H(x, s) = \int_0^s h(x, t)\mathrm{d}t$ and $S(u) = \int_\Omega H(x, u)\mathrm{d}x$.
First, we determine the Gâteaux derivative $S'(u)$. Let $v \in H_0^1(\Omega)$

$$S'(u)v = \lim_{t\to 0} \frac{S(u + tv) - S(u)}{t}$$

$$= \int_\Omega \max\{u(x), \omega(x)\}^{-\delta} v(x).$$

Indeed, let $U_1 = \{u(x) + tv(x) \geq \omega(x)\}, U_2 = \{u(x) + tv(x) < \omega(x)\}, V_1 = \{u(x) \geq \omega(x)\}$ and $V_2 = \{u(x) < \omega(x)\}$.
Using the above notation, we obtain

$$S(u + tv) = \int_{U_1} \left[\frac{(u + tv(x))^{1-\delta}}{1-\delta} - \frac{\omega(x)^{1-\delta}}{1-\delta} \right] \mathrm{d}x + \int_{U_2} \omega^{1-\delta}(x) \mathrm{d}x$$

and

$$S(u) = \int_{V_1} \left[\frac{u(x)^{1-\delta}}{1-\delta} - \frac{\omega(x)^{1-\delta}}{1-\delta} \right] \mathrm{d}x + \int_{V_2} \omega^{1-\delta}(x) \mathrm{d}x$$

Then, from above it follows that

$$\frac{S(u + tv) - S(u)}{t} = \int_{U_2 \cap V_1} \left(\frac{(u(x) + tv(x))^{1-\delta}}{1-\delta} - \frac{u(x)^{1-\delta}}{1-\delta} \right) \mathrm{d}x$$

$$+ \int_{U_1 \cap V_2} \left(\frac{(u(x) + tv(x))^{1-\delta} - \omega(x)^{1-\delta}}{1-\delta} \right) \mathrm{d}x$$

$$+ \int_{U_2 \cap V_1} \left(\frac{\omega(x)^{1-\delta}}{1-\delta} - \frac{u(x)^{1-\delta}}{1-\delta} \right) \mathrm{d}x.$$

Then, letting $t \to 0$, we get

$$\lim_{t \to 0} \frac{S(u + tv) - S(u)}{t} = \int_{V_1} u^{-\delta} v(x) \mathrm{d}x + \int_{V_2} \omega(x)^{-\delta} v(x) \mathrm{d}x + 0.$$

Now, let $u_k \in H_0^1(\Omega)$ such that $u_k \to u_\lambda$; it follows that

$$| \langle S'(u_k) - S'(u_\lambda), v \rangle |$$

$$= \left| \int_\Omega (\max\{u_k(x), \omega(x)\} v(x) - \max\{u_\lambda(x), \omega(x)\}^{-\delta} v(x)) \mathrm{d}x \right|$$

$$\leq 2 \int_\Omega \omega(x)^{-\delta} \mid v(x) \mid \mathrm{d}x.$$

Then, for $\delta < 1$

$$|\langle S'(u_k) - S'(u_\lambda), v \rangle| \leq 2\epsilon^{-\delta} \int_\Omega \varphi_1^{-\delta} \mid v \mid \mathrm{d}x$$

and for $\delta = 1$

$$|\langle S'(u_k) - S'(u_\lambda), v \rangle| \leq k_\epsilon \int_\Omega \varphi_1^{-(1+\epsilon)} \mathrm{d}x.$$

Finally, for $1 < \delta < 3$

$$|\langle S'(u_k) - S'(u_\lambda), v\rangle| \leq 2\epsilon^{-\delta} \int \varphi_1^{\frac{-2\delta}{\delta+1}} \varphi_1 \frac{|\ v(x)\ |}{\varphi_1} \mathrm{d}x.$$

Using the Hardy inequality, we get

$$\begin{aligned} |\ \langle S'(u_k) - S'(u_\lambda), v\rangle\ | &\leq 2k\epsilon^{-\delta}\Big(\int_\Omega \varphi_1^{2(\frac{1-\delta}{\delta+1})} \mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_\Omega (\frac{v}{d})^2 \mathrm{d}x\Big)^{\frac{1}{2}} \\ &\leq C \parallel v \parallel^2_{H_0^1(\Omega)} . \end{aligned}$$

Therefore, by the Lebesgue dominated convergence Theorem we conclude that the Gâteaux derivative is continuous which implies that $S \in C^1(H_0^1(\Omega), \mathbb{R})$. The proof of the Lemma 2.4.3 is now completed. $\square$

Lemma 2.4.4. *Each positive weak solution u_λ of (P_λ) belongs to $L^\infty(\Omega)$.*

Proof. Let $\phi : \mathbb{R} \to [0, 1]$ be a C^1 cut-off function such that

$$\phi(t) = \begin{cases} 0 & \text{if } \ t \leq 0, \\ 1 & \text{if } \ t \geq 1. \end{cases}$$

Then, for any $\epsilon > 0$, define $\phi_\epsilon(t) = \phi(\frac{(t-1)}{\epsilon})$ for $t \in \mathbb{R}$. Therefore, $\phi_\epsilon \circ u \in H_0^1(\Omega)$ with $\nabla(\phi_\epsilon \circ u) = (\phi'_\epsilon \circ u)\nabla u$.
Now, let $\psi = (\phi_\epsilon \circ u)w$ be a test function and use the weak form in the problem (P_λ); we obtain

$$\int_\Omega \nabla u.\nabla[(\phi_\epsilon \circ u)w]\mathrm{d}x = \int_\Omega \lambda(u^{-\delta} + u^q + \rho(u))(\phi_\epsilon \circ u)w\, \mathrm{d}x$$

where $w \in C_c^\infty(\Omega)$ satisfies $w \geq 0$. Hence, it follows that

$$\begin{aligned} &\int_\Omega |\ \nabla u\ |^2\ (\phi'_\epsilon \circ u)w\, \mathrm{d}x + \int_\Omega (\nabla u.\nabla w)(\phi_\epsilon \circ u)\, \mathrm{d}x \\ &\quad = \int_\Omega \lambda(u^{-\delta} + u^q + \rho(u))(\phi_\epsilon \circ u)w\, \mathrm{d}x \end{aligned}$$

with $\phi'_\epsilon \circ u \geq 0$. Therefore,

$$\int_\Omega (\nabla u.\nabla w)(\phi_\epsilon \circ u)\, \mathrm{d}x \leq \int_\Omega \lambda(u^{-\delta} + u^q + \rho(u))(\phi_\epsilon \circ u)w\, \mathrm{d}x.$$

Letting $\epsilon \to 0$, we obtain

$$\int_\Omega \nabla(u-1)^+.\nabla w\, \mathrm{d}x \leq \int_\Omega \lambda(1 + u^q + \rho(u))w\, \mathrm{d}x. \tag{2.47}$$

Finally the L^∞ bounded are obtained from equation (2.47), Lemma 10 and Theorem C in [12]. The proof of the Lemma 2.4.4 is now completed. $\square$

Lemma 2.4.5. *If* $\delta < 1, u_\lambda \in C^{1,\alpha}(\overline{\Omega})$ *for some* $0 < \alpha := \alpha(\delta) < 1$.

Proof. From Lemma 2.4.4 we see that $u_\lambda \in L^\infty(\Omega)$. Then, since $m\ d(x, \partial\Omega) \leq u_\lambda \leq M\ d(x, \partial\Omega)$ and using the Elliptic Regularity theory, we have that $u_\lambda \in C^2_{loc}(\Omega) \cap L^\infty(\Omega)$.
Moreover, we have

$$u_\lambda(x) = \lambda \int_\Omega G(x,y) \left(u_\lambda^{-\delta}(y) + u_\lambda^q(y) + \rho(u_\lambda(y))\right) \mathrm{d}y \tag{2.48}$$

where $G(x,y) = -\frac{1}{\alpha_N} \mid x - y \mid^{2-N} - \phi^x(y)$ is the Green function for the region Ω, and ϕ^x is the associated corrector (the regular part of the Green Function) and $\alpha_N = N(2-N)w_N$. Hence, since $-\frac{1}{\alpha_N} \mid x - y \mid^{2-N} \in L^1(\Omega)$ and by the dominated convergence Theorem we have that $u_\lambda \in C(\overline{\Omega})$.
Now, let $x \in \Omega$

$$\begin{aligned}\nabla u_\lambda(x) &= \lambda \int_\Omega \nabla_x(G(x,y))(u_\lambda^{-\delta}(y) + u_\lambda^q(y) + \rho(u_\lambda(y)))\,\mathrm{d}y \\ &= -\frac{\lambda}{N(2-N)w_N} \int_\Omega \frac{(x-y)}{\mid x-y \mid^N}(u_\lambda^{-\delta}(y) + u_\lambda^q(y) + \rho(u_\lambda(y)))\,\mathrm{d}y \\ &\quad -\lambda \int_\Omega \nabla_x(\phi^x(y))(u_\lambda^{-\delta}(y) + u_\lambda^q(y) + \rho(u_\lambda(y)))\,\mathrm{d}y \\ &\stackrel{\text{def}}{=} \phi^1(x) + \phi^2(x).\end{aligned} \tag{2.49}$$

Therefore, let $x_0 \in \partial\Omega$ and $\{x_k\} \subset \Omega$ such that $x_k \to x_0$ as $k \to \infty$. without loss of generality we assume $x_0 = 0$. Since ϕ^x is regular, $u_\lambda^{-\delta} \in L^1(\Omega)$ (which follows from $\delta < 1$) and $u_\lambda \in L^\infty(\Omega)$, we have

$$\phi^2(x_k) \to \phi^2(x_0)\ as\ k \to \infty \tag{2.50}$$

Furthermore, for $l > 0$ small, we have

$$\begin{aligned}&-\frac{\lambda}{\alpha_N} \int_\Omega \frac{(x_k-y)}{\mid x_k-y \mid^N}(u_\lambda^{-\delta}(y) + u_\lambda^q(y) + \rho(u_\lambda(y)))\,\mathrm{d}y \\ &= -\frac{\lambda}{\alpha_N} \int_{B_l(x_k)\cap\Omega} \frac{(x_k-y)}{\mid x_k-y \mid^N}(u_\lambda^{-\delta}(y) + u_\lambda^q(y) + \rho(u_\lambda(y)))\,\mathrm{d}y \\ &-\frac{\lambda}{\alpha_N} \int_{B_l^c(x_k)} \frac{(x_k-y)}{\mid x_k-y \mid^N}(u_\lambda^{-\delta}(y) + u_\lambda^q(y) + \rho(_\lambda(y)))\,\mathrm{d}y\end{aligned} \tag{2.51}$$

where $B_l^c(x_k)$ is the complement of $B_l^c(x_k)\cap\Omega$ in Ω.
Then, as $k\to\infty$, since $\mid x_k-y\mid\geq l$, the second term in (2.51) converges to

$$-\frac{\lambda}{\alpha_N}\int_{B_l^c(x_0)}\frac{(x_0-y)}{\mid x_0-y\mid^N}(u_\lambda^{-\delta}(y)+u_\lambda^q(y)+\rho(u_\lambda(y)))\,\mathrm{d}y.$$

Now, since $u_\lambda\geq Kd$, we have for some $K_0,K_1,K_2>0$

$$\begin{aligned}
&\mid\frac{\lambda}{\alpha_N}\int_{B_l(x_k)\cap\Omega}\frac{(x_k-y)}{\mid x_k-y\mid^N}(u_\lambda^{-\delta}(y)+u_\lambda^q(y)+\rho(u_\lambda(y)))\,\mathrm{d}y\mid\\
&\leq K_0\int_0^l\frac{\mathrm{d}y_n}{y_n^\delta}\int_{B^{N-1}(0,(l^2-y_n^2)^{\frac12})}\frac{\mathrm{d}y'}{(y_n^2+\mid x-y'\mid^2)^{\frac{N-1}{2}}}\\
&\leq K_1\int_0^l\frac{\mathrm{d}y_n}{y_n^\delta}\int_{B^{n-1}(0,\frac{(l^2-y_n^2)^{\frac12}}{|x_n^k-y_n|})}\frac{\mathrm{d}y'}{(1+\mid y'\mid^2)^{\frac{N-1}{2}}}\\
&\leq K_2\int_0^l\frac{\mathrm{d}y_n}{y_n^\delta}\int_{B^{N-1}(0,\frac{(l^2-y_n^2)^{\frac12}}{|y_n|})}\frac{\mathrm{d}y'}{(1+\mid y'\mid^2)^{\frac{N-1}{2}}}\\
&\leq K_2\int_0^l\frac{\mathrm{d}y_n}{y_n^\delta}\int_0^{\frac{k}{n}}\frac{r^{N-2}\mathrm{d}r}{(1+r^2)^{\frac{N-1}{2}}}\\
&\leq K_2\int_0^l\frac{\mathrm{d}y_n}{y_n^\delta}\mid\log y_n\mid<\infty.
\end{aligned}\tag{2.52}$$

since $\delta<1$. Therefore, from (2.50), (2.51) and (2.52) and the dominated convergence Theorem,

$$\nabla u_\lambda(x_n)\to\nabla u_\lambda(x_0)\text{ as }n\to\infty.\tag{2.53}$$

Then, $u_\lambda\in C^1(\overline{\Omega})$.
Finally, we use the Proposition 2.5 in [11] to conclude that $u_\lambda\in C^{1,\alpha}(\overline{\Omega})$, where $\alpha:=\alpha(\delta)$. The proof of the Lemma 2.4.5 is now completed. □

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Chapitre 3

$W_0^{1,p}$ Versus C^1 local minimizers for a singular and critical functional

Contents

3.1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded smooth domain, $1 < p < +\infty$, $0 < \delta < 1$. Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a C^1 function satisfying:

(f1) $f(x, s) \geq 0$ for $(x, s) \in \overline{\Omega} \times \mathbb{R}^+$ and $f(x, 0) = 0$.

(f2) There exists $q > p-1$ satisfying $q \leq p^* - 1 \stackrel{\text{def}}{=} \frac{Np}{N-p} - 1$ if $p < N$, $q < \infty$ otherwise, such that $f(x, s) \leq C(1 + s)^q$ for all $(x, s) \in \Omega \times \mathbb{R}^+$ and for some $C > 0$.

Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ continuous on $(0, +\infty)$ satisfying

(g1) g is nonincreasing on $(0, +\infty)$,

(g2) $c_1 \leq \liminf_{t \to 0^+} g(t)t^\delta \leq \limsup_{t \to 0^+} g(t)t^\delta = c_2$ for some $c_1, c_2 > 0$.

From **(g2)**, g is singular at the origin and $\lim_{t\to 0^+} g(t) = +\infty$.

Let $F(x,u) \stackrel{\text{def}}{=} \int_0^u f(x,s)\mathrm{d}s$ and $G(u) \stackrel{\text{def}}{=} \int_0^u g(s)\mathrm{d}s$. We consider the singular functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$I(u) \stackrel{\text{def}}{=} \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} - \int_\Omega F(x,u^+) - \int_\Omega G(u^+). \tag{3.1}$$

where as usual $t^+ \stackrel{\text{def}}{=} \max(t,0)$. Our aim in this chapter is to show the following

Theorem 3.1.1. *Suppose that the conditions* **(f1)-(f2)** *and* **(g1)-(g2)** *are satisfied. Let $u_0 \in C^1(\overline{\Omega})$ satisfying*

$$u_0 \geq \eta d(x,\partial\Omega) \text{ for some } \eta > 0 \tag{3.2}$$

be a local minimizer of I in $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ topology; that is,

$$\exists\, \epsilon > 0 \text{ such that } u \in C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})\,,\, \|u-u_0\|_{C^1(\overline{\Omega})} < \epsilon \Rightarrow I(u_0) \leq I(u).$$

Then, u_0 is a local minimum of I in $W_0^{1,p}(\Omega)$ also.

From Lemma 3.3.2 in Section 3.3, we remark that the conditions on u_0 in the above theorem imply that u_0 satisfies in the distributions sense the Euler-Lagrange equation associated to I, that is

$$(\mathrm{P}) \begin{cases} -\Delta_p u &= g(u) + f(x,u) \quad \text{in } \Omega; \\ u|_{\partial\Omega} &= 0, \quad u > 0 \quad \text{in } \Omega, \end{cases}$$

It means that $u_0 \in W_0^{1,p}(\Omega)$ is a weak solution to (P), i.e. satisfies $\operatorname{ess\,inf}_K u_0 > 0$ over every compact set $K \subset \Omega$ and

$$\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla\phi \,\mathrm{d}x = \int_\Omega g(u_0)\phi \,\mathrm{d}x + \int_\Omega f(x,u_0)\phi\,\mathrm{d}x \tag{3.3}$$

for all $\phi \in C_c^\infty(\Omega)$. As usual, $C_c^\infty(\Omega)$ denotes the space of all C^∞ functions $\phi : \Omega \to \mathbb{R}$ with compact support. We highlight that the condition (3.2) is natural. Indeed from Lemma 3.3.4 in the Section 3.3, any weak solution to (P) satisfies (3.2) for some $\eta > 0$ independent of u. In particular, $u_0 \geq \underline{u}$ where $\underline{u}$ is the unique weak solution to the "pure " singular problem (PS):

$$(\mathrm{PS}) \begin{cases} -\Delta_p u &= g(u) \quad \text{in } \Omega; \\ u|_{\partial\Omega} &= 0, \quad u > 0 \quad \text{in } \Omega. \end{cases}$$

From Lemma 3.3.4, $\underline{u}$ satisfies (3.2). Using the approach introduced in brezis-Nirenberg [5], used in Ambrosetti-Brezis-Cerami [1] and extended to the p-laplacian case in GUEDDA-VERON [17], AZORERO-MANFREDI-PERAL [7], Theorem 3.1.1 can be used to prove the existence of a second solution to (P) near u_0. Precisely, if f satisfies $\lim_{s\to+\infty} \frac{f(x,s)}{s^{p-1}} = +\infty$ uniformly in $x \in \overline{\Omega}$, Theorem 3.1.1 and assumptions **(f1)-(f2)**, **(g1)-(g2)** prove that I has the Moutain Pass geometry (see AMBROSETTI-RABINOWITZ [2]) around u_0 and then admits a second critical point (consequently a second weak solution to (P)) as it is shown in GIACOMONI-SCHINDLER-TAKAC [18] for the particular case $g(s) = s^{-\delta}$, $f(x,s) = s^q$ with $0 < \delta < 1$, $p-1 < q < p^*-1$. To apply Theorem 3.1.1 in this context, we need to prove the existence of a C^1-minimizer of I. This follows from the strong comparison principle we state below in the singular case (see Theorem 2.3 in [18]):

Theorem 3.1.2. *Suppose that the conditions* **(g1)-(g2)** *are satisfied. Let* $u, v \in C^{1,\beta}(\overline{\Omega})$, *for some* $0 < \beta < 1$, *satisfy* $0 \lneqq u$, $0 \lneqq v$ *and*

$$-\Delta_p u - g(u) = f, \tag{3.4}$$
$$-\Delta_p v - g(v) = h, \tag{3.5}$$

with $u = v = 0$ *on* $\partial\Omega$, *where* $f, h \in C(\Omega)$ *are such that* $0 \le f < h$ *pointwise everywhere in* Ω. *Then, the following strong comparison principle holds:*

$$0 < u < v \ \ in \ \ \Omega \quad \text{and} \quad \frac{\partial v}{\partial \nu} < \frac{\partial u}{\partial \nu} < 0 \ \ on \ \ \partial\Omega. \tag{3.6}$$

In respect to [18], we use different arguments to prove Theorem 3.1.1 which generalises Proposition 3.7 in [18].

Equation (P) appears in several models: non newtonial flows in porous media, chemical heterogeneous catalysts, nonlinear heat equations (see DÍAZ-MOREL-OSWALD [11], FULKS-MAYBEE [13], GAMBA-JUNGEL [14], GHERGU-RADULESCU [15], see also DÍAZ [10], LEACH-NEEDDHAM [24] and the overvie-ws about singular elliptic equations: HERNÁNDEZ-MANCEBO [22], GHERGU-RADULESCU [16]) and then have intrinsic mathematical interest.

For proving Theorem 3.1.1, we will need uniform L^∞-estimates for a family of solutions to (P_ϵ) as below.

Theorem 3.1.3. *Let* $\{u_\epsilon\}_{\epsilon\in(0,1)}$ *be a family of solutions to* (P_ϵ), *where* u_0 *satisfies* (3.2) *and solves* (P); *let* $\sup_{\epsilon\in(0,1)} (\|u_\epsilon\|_{W_0^{1,p}(\Omega)}) < \infty$. *Then, there exists* $C_1, C_2 > 0$ *(independent of* ϵ*) such that*

$$\sup_{\epsilon\in(0,1)} \|u_\epsilon\|_{L^\infty(\Omega)} < \infty \text{ and } C_1 d(x,\partial\Omega) \le u_\epsilon \le C_2 d(x,\partial\Omega).$$

The proof of the above theorem is a consequence of the results proved in Section 3.3.

Concerning the case where $p = N$, we know from the Trudinger-Moser inequality that the critical growth is given by $e^{bu^{\frac{N}{N-1}}}$ for any $b > 0$. Then setting $f(x,u) = h(x,u)e^{bu^{\frac{N}{N-1}}}$ and assuming **(g1)-(g2)**,

(h1) $h : \overline{\Omega} \times \mathbb{R}^+ \to [0,\infty)$ is a is C^1 and nonnegative with $h(x,0) = 0$,

(h2) $\liminf_{t\to\infty} h(x,t)e^{\epsilon|t|^{\frac{N}{N-1}}} = \infty$, $\liminf_{t\to\infty} h(x,t)e^{-\epsilon|t|^{\frac{N}{N-1}}} = 0$ uniformly in $x \in \overline{\Omega}$.

Theorem 3.1.1 holds. The proof is quite similar and uses in addition the Trudinger -Moser Inequality in a similar way as Giacomoni-Prashanth-Sreenadh [19]. We don't give the proof of this case in the present work.

Theorem 3.1.1 was proved first in [5] for the case of critical growth functionals $I : H^1_0(\Omega) \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$, and later for critical growth functionals $I : W^{1,p}_0(\Omega) \to \mathbb{R}$, $1 < p < N$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$ in [7] and for critical and singular functionals in [19] in the special case $g(t) = t^{-\delta}$ and $f(x,t) = t^q$, with $\delta < 1$ and $1 < q \leq p^* - 1$. A key feature of these latter works is the uniform $C^{1,\alpha}$ estimate they obtain for equations like (P_ϵ) but involving two p-Laplace operators. Using constraints based on L^p- norms rather than Sobolev norms as in [7], the equations for which uniform estimates are required can be simplified to a standard type involving only one p-Laplace operator. This approach was followed in [8] and also adopted in this work to slightly simplify and generalize the results in [18]. Since the quasilinear operator is not modified in (P_ϵ) in the proof of Theorem 3.1.1, Theorem 3.1.1 can be extended to more general quasilinear operators in divergence form and to nonisotropic operators (as the $p(x)$-laplacian operator appearing in heterogeneous porous media models). This would provide existence of multiple solutions for such quasilinear singular elliptic equations.

3.2 $W^{1,p}$ versus C^1 local minimizers

Proof of Theorem 3.1.1:
We first deal with the subcritical and then give the additional arguments to prove the result when $q = p^* - 1$. **Case 1:** $r < p^* - 1$. We use the arguments in [8].
Let $q \in (r, p^* - 1)$, define

$$K(w) = \frac{1}{q+1}\int_\Omega |w(x) - u_0(x)|^{q+1}\mathrm{d}x, \quad (w \in W^{1,p}_0(\Omega)),$$

and

$$S_\epsilon \stackrel{\text{def}}{=} \{v \in W_0^{1,p}(\Omega)\backslash K(v) \leq \epsilon\}.$$

We consider the following constraint minimization problem:

$$I_\epsilon = \inf_{v \in S_\epsilon} I(v).$$

Clearly, we have that $I_\epsilon > -\infty$. Then, from $q < p^* - 1$, the facts that I is weakly lower semicontinuous in $W_0^{1,p}(\Omega)$ and S_ϵ is closed and convex, it follows that I_ϵ is achieved on some $v_\epsilon \in S_\epsilon$, that is $I(v_\epsilon) = I_\epsilon$.
We now consider the following two cases:
1) Let $K(v_\epsilon) < \epsilon$. Then v_ϵ is also a local minimizer of I. We now show that I admits a gâteaux-derivative on v_ϵ to derive that v_ϵ satisfies the Euler-Lagrange equation associated to I. For this, according to Lemma 3.3.2 in Section 3.3, we need to prove that $\exists\ \tilde{\eta} > 0$ such that $v_\epsilon \geq \tilde{\eta}\, dist(x, \partial\Omega)$ or equivalently

$$\exists\ \eta > 0 \ \text{ such that } v_\epsilon \geq \eta\varphi_1; \tag{3.7}$$

[φ_1 is the normalized positive eigenfunction associated to

$$\lambda_1(\Omega) \stackrel{\text{def}}{=} \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega u^p}].$$

To prove (3.7), we argue by contradiction: $\forall\ \eta > 0$ let $\Omega_\eta = Supp\{(\eta\varphi_1 - v_\epsilon)^+\}$ and suppose that Ω_η has a non zero measure.
Let $v_\eta = (\eta\varphi_1 - v_\epsilon)^+$ and for $0 < t \leq 1$ set $\xi(t) = I(v_\epsilon + tv_\eta)$. Then, there exists $c(t) > 0$ such that $\inf \frac{v_\epsilon + tv_\eta}{\varphi_1} \geq c(t)$ for $t > 0$. Therefore ξ is differentiable for $0 < t \leq 1$ and $\xi'(t) = \langle I'(v_\epsilon + tv_\eta), v_\eta\rangle$. Thus,

$$\xi'(t) = \langle -\Delta_p(v_\epsilon + tv_\eta) - g(v_\epsilon + tv_\eta) - f(x, v_\epsilon + tv_\eta), v_\eta\rangle.$$

From **(f1)** and **(g2)**, we see that

$$\xi'(1) = \langle I'(\eta\varphi_1), v_\eta\rangle = \langle -\Delta_p(\eta\varphi_1) - g(\eta\varphi_1) - f(x, \eta\varphi_1), v_\eta\rangle < 0$$

for $\eta > 0$ small enough.
Moreover,

$$-\xi'(1) + \xi'(t) = \langle -\Delta_p(v_\epsilon + tv_\eta) + \Delta_p(v_\epsilon + v_\eta)$$
$$(g(v_\epsilon + tv_\eta) - g(v_\epsilon + v_\eta) - [f(x, v_\epsilon + tv_\eta) - f(x, v_\epsilon + v_\eta)]), v_\eta\rangle.$$

Since $g(s) + f(x, s)$ is non increasing for $0 < s$ small enough uniformly to $x \in \Omega$ (by **(f1)**, **(g1)-(g2)**) and from the monotonicity of $-\Delta_p$, we have that for $0 < \eta$ small enough $0 \leq \xi'(1) - \xi'(t)$. Moreover from Taylor's expansion, there exists $0 < \theta < 1$ such that

$$0 \leq I(v_\epsilon + v_\eta) - I(v_\epsilon) = \langle I'(v_\epsilon + \theta v_\eta), v_\eta \rangle = \xi'(\theta). \tag{3.8}$$

letting $t = \theta$ we have $\xi'(\theta) \leq \xi'(1) < 0$. We obtain a contradiction with (3.8) and then $v_\epsilon \geq \eta\varphi_1$ for some $\eta > 0$ (which depends a priori on ϵ). Since v_ϵ is a local minimizer of I, and I is Gâteaux differentiable in v_ϵ, we get $I'(v_\epsilon)$ is defined and $I'(v_\epsilon) = 0$. From the weak comparison principle, we have that $\eta\phi_1 \leq \underline{u} \leq v_\epsilon$ for some $\eta > 0$ (independent of ϵ). Since $v_\epsilon \in S_\epsilon$ and from the fact that v_ϵ satisfies (P), we get that $\{v_\epsilon\}_{\epsilon \geq 0}$ is uniformly bounded in $W_0^{1,p}(\Omega)$. Now, using Lemma 3.3.5 , Lemma 3.3.6 and Theorem 3.3.1 in the Section 3.3, we get

$$|v_\epsilon|_{C^{1,\alpha}(\overline{\Omega})} \leq C \tag{3.9}$$

and as $\epsilon \to 0^+$

$$v_\epsilon \to u_0 \text{ in } C^1(\overline{\Omega})$$

which contradicts the fact that u_0 is a local minimizer in $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$.
Now, we deal with the second case: 2) $K(v_\epsilon) = \epsilon$:
We again show that $v_\epsilon \geq \eta\varphi_1$ in Ω for some $\eta > 0$. Taking $v_\eta = (\eta\varphi - v_\epsilon)^+$, $\xi(t) = I(v_\epsilon + tv_\eta)$, we obtain as above that $\xi'(t) \leq \xi'(1) < 0$ for $0 < t < 1$ and $0 < \eta$ small enough.
Then $\xi(t) = I(v_\epsilon + tv_\eta)$ is decreasing. This implies that $I(v_\epsilon) > I(v_\epsilon + tv_\eta)$ for $t > 0$ and using (3.2)

$$K(v_\epsilon + tv_\eta) = \frac{1}{q+1}\int_\Omega |u_0 - (v_\epsilon + tv_\eta)|^{q+1}\mathrm{d}x \leq \int_\Omega |u_0 - v_\epsilon|^{q+1}\mathrm{d}x = \epsilon.$$

This yields a contradiction with the fact that v_ϵ is a global minimizer of I on S_ϵ.
In this case, from the Lagrange multiplier rule we have

$$I'(v_\epsilon) = \mu_\epsilon K'(v_\epsilon), \text{ for some } \mu_\epsilon \in \mathbb{R}.$$

We first show that $\mu_\epsilon \leq 0$:
We argue by contradiction. Suppose that $\mu_\epsilon > 0$, then there exists $\varphi \in W_0^{1,p}(\Omega)$ such that

$$\langle I'(v_\epsilon), \varphi \rangle < 0 \text{ and } \langle K'(v_\epsilon), \varphi \rangle < 0$$

and then for t small we have

$$\begin{cases} I(v_\epsilon + t\phi) < I(v_\epsilon), \\ K(v_\epsilon + t\phi) < K(v_\epsilon) = \epsilon. \end{cases} \tag{3.10}$$

This contradicts the fact that v_ϵ is a global minimizer of I in S_ϵ.
We deal now with two following cases:
case (i): $\mu_\epsilon \in (-1,0)$. In this case, using

$$(\mathrm{P}_\epsilon)\ -\Delta_p v_\epsilon = g(v_\epsilon) + f(x,v_\epsilon) + \mu_\epsilon(|v_\epsilon - u_0|^{q-1}(v_\epsilon - u_0))$$

it is easy from the weak comparison principle to show that $\eta\varphi_1 \leq v_\epsilon$ with some $\eta > 0$, independent of ϵ.
Indeed, for $0 < \eta$ small enough and for all $-1 \leq \mu_\epsilon \leq 0$, we have that $\eta\varphi_1$ is a strict subsolution to (P_ϵ) and

$$\int_\Omega (-\Delta_p(\eta\varphi) + \Delta_p v_\epsilon)(\eta\varphi - v_\epsilon)^+ \mathrm{d}x \leq \frac{1}{2}\int_\Omega (g(\eta\phi_1) - g(v_\epsilon))(\eta\varphi - v_\epsilon)^+ \mathrm{d}x \leq 0.$$

Now, since $\mu_\epsilon \leq 0$, there exists $M, c > 0$ independent of ϵ such that

$$-\Delta_p(v_\epsilon - 1)^+ \leq M + c((v_\epsilon - 1)^+)^q.$$

Using the Moser iterations technique as in Lemma 3.3.5 of Section 3.3, we get that $|v_\epsilon|_{L^\infty} \leq C$ for some C independent of ϵ.
Using Lemma 3.3.6 in Section 3.3, we deduce that $v_\epsilon \leq k\phi_1$ for some $k > 0$ independent of ϵ. From the uniform estimate $\eta\phi_1 \leq v_\epsilon \leq k\phi_1$, we can apply Theorem 3.3.1 in Section 3.3 and get $|v_\epsilon|_{C^{1,\alpha}(\overline{\Omega})} \leq C$ for some constant $C > 0$ independent of ϵ. Then we conclude as above.
Let us consider **the case (ii):** $\mu_\epsilon \leq -1$. As above, we have that $v_\epsilon \geq \eta\varphi_1$ for $\eta > 0$ small enough and independent of ϵ. Indeed, for $\eta > 0$ small enough, $\eta\phi_1$ is a subsolution to (P_ϵ) (using $\eta\phi_1 \leq u_0$ and $\mu_\epsilon \leq 0$). Then, from the weak comparison principle, we get that $\eta\phi_1 \leq v_\epsilon$. Furthermore, there exists a number $M > 0$, independent of ϵ, such that for

$$\gamma(s,x,t) \stackrel{\text{def}}{=} g(t) + f(x,t) + s|t - u_0(x)|^{q-1}(t - u_0(x)) \tag{3.11}$$

we have

$$\gamma(s,x,t) < 0, \quad \forall\,(s,x,t) \in (-\infty,-1] \times \Omega \times (M,+\infty).$$

Then, from the weak comparison principle we have that $v_\epsilon \leq M$. Now, using $(v_\epsilon - u)|v_\epsilon - u_0|^{\beta-1}$, with $\beta \geq 1$, as a test function we obtain,

$$\begin{aligned}
0 &\leq \beta \textstyle\int_\Omega \left(|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon - |\nabla u_0|^{p-2}\nabla u_0\right).\nabla(|v_\epsilon - u_0|^{\beta-1})\mathrm{d}x \\
&\leq \beta \textstyle\int_\Omega \left(|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon - |\nabla u_0|^{p-2}\nabla u_0\right).\nabla(|v_\epsilon - u_0|^{\beta-1})\mathrm{d}x \\
&\quad - \textstyle\int_\Omega \left(g(v_\epsilon) - g(u_0)\right)(v_\epsilon - u_0)|v_\epsilon - u_0|^{\beta-1}\mathrm{d}x \\
&\leq \textstyle\int_\Omega (f(x,v_\epsilon) - f(x,u_0))(v_\epsilon - u_0)|v_\epsilon - u_0|^{\beta-1} + \mu_\epsilon \int_\Omega |v_\epsilon - u_0|^{\beta+q}\mathrm{d}x.
\end{aligned}$$

Using the bounds about v_ϵ, u_0 and the Hölder inequality we get

$$-\mu_\epsilon \|v_\epsilon - u_0\|^q_{L^{\beta+q}(\Omega)} \leq C|\Omega|^{\frac{q}{q+\beta}},$$

where C does not depend on β and ϵ. Passing to the limit $\beta \to +\infty$ this leads to

$$-\mu_\epsilon \|v_\epsilon - u_0\|^q_{L^\infty(\Omega)} \leq C.$$

So the right-hand side of (3.11) is uniformly bounded in $L^\infty(\Omega)$ from which as in the first case, we obtain that v_ϵ, $(0 < \epsilon \leq 1)$ is bounded in $C^{1,\alpha}(\overline{\Omega})$ independently of ϵ. Finally, using Ascoli-Arzela Theorem we find a sequence $\epsilon_n \to 0^+$ such that

$$v_{\epsilon_n} \to u_0 \text{ in } C^1(\overline{\Omega}).$$

It follows that for $\epsilon > 0$ sufficiently small,

$$I(v_{\epsilon_n}) < I(u_0),$$

which contradicts the fact that u_0 is a local minimizer of I for the $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ topology. The proof of the Lemma 3.1.1 in the subcritical case is now complete.

Now, we deal with the **critical case**, i.e. $r = p^* - 1$. For this, as in [5], we first make a truncation argument to get the weak lower semicontinuity property of the energy functional. Precisely, assume by contradiction that u_0 is not a local minimizer of I in the $W^{1,p}_0(\Omega)$ topology. Let

$$\chi(w) = \frac{1}{p^*}\int_\Omega |w(x) - u_0|^{p^*}\,\mathrm{d}x, \quad (w \in W^{1,p}_0(\Omega)),$$

and

$$\mathcal{C}_\varepsilon \stackrel{\text{def}}{=} \{v \in W^{1,p}_0(\Omega)\setminus\ \chi(v) \leq \epsilon\}.$$

We now consider the truncated functional

$$I_j(v) = \frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x - \int_\Omega G(v)\,\mathrm{d}x - \int_\Omega F_j(x,v)\,\mathrm{d}x \ \ \forall\, v \in W^{1,p}_0(\Omega). \quad (3.12)$$

for $j = 1, 2, \ldots\ldots$, where $f_j(x,s) := f(x, T_j(s)), F_j(x,s) = \int_0^s f_j(x,t)\,\mathrm{d}t$ and

$$T_j(s) = \begin{cases} -j & \text{if } \ s \leq -j, \\ s & \text{if } \ -j \leq s \leq j \\ +j & \text{if } \ s \geq j \end{cases} \quad (3.13)$$

By the Lebesgue Theorem, we have that for any $v \in W^{1,p}_0(\Omega)$

$$I_j(v) \to I(v) \text{ as } j \to \infty.$$

It follows that for each $\epsilon > 0$, there is some j_ϵ (with $j_\epsilon \to +\infty$ as $\epsilon \to 0^+$) such that $I_{j_\varepsilon}(v_\epsilon) < I(u_0)$. On the other hand, since S_ϵ is closed and convex and from the fact that I_{j_ϵ} is weakly lower semicontinuous we deduce that I_{j_ϵ} achieves its infimum at some $u_\epsilon \in \mathcal{C}_\epsilon$. Therefore, for $0 < \epsilon$ small enough,

$$I_{j_\epsilon}(u_\epsilon) \leq I_{j_\epsilon}(v_\epsilon) < I_{j_\epsilon}(u_0) = I(u_0). \tag{3.14}$$

As above, we have that $\exists\ \eta > 0$ independent of ϵ such that

$$u_\epsilon \geq \eta\varphi_1. \tag{3.15}$$

From (3.15), I_{j_ϵ} admits a Gâteaux-derivative in u_ϵ and since u_ϵ a local minimizer of I_{j_ϵ}, we get that $I'_{j_\epsilon}(u_\epsilon)$ is defined and from the Lagrange multiplier rule, there exists $\mu_\epsilon \in \mathbb{R}^-$ such that $I'_{j_\epsilon}(u_\epsilon) = \mu_\epsilon \chi'(v_\epsilon)$.
By construction $u_\epsilon \to u_0$ in $L^{p^*}(\Omega)$ as $\epsilon \to 0$, and it follows from above that u_ϵ remains bounded in $W_0^{1,p}(\Omega)$.
Claim: $\{u_\epsilon\}$ are uniformly bounded in $L^\infty(\Omega)$ as $\epsilon \to 0$.
Assuming this claim, we can argue as above to derive a uniform bound of $\{u_\epsilon\}$ in $C^{1,\alpha}(\overline{\Omega})$. Then, it follows that for $\epsilon > 0$ sufficiently small,

$$I(u_\epsilon) = I_{j_\epsilon}(u_\epsilon) < I(u_0), \tag{3.16}$$

which contradicts the fact that u_0 is a local minimizer of I for the $C_0^1(\overline{\Omega})$ topology, and the proof of the Theorem 3.1.1 is complete.
Finally, let us prove the Claim. For that, we again distinguish between the following two cases : **case (i)** $\inf\limits_{\epsilon\in(0,1)} \mu_\epsilon > -\infty$ **case (ii)** $\inf\limits_{\epsilon\in(0,1)} \mu_\epsilon = -\infty$.
In Case (i) from the Euler equation (P_ϵ):

$$(P_\epsilon) - \Delta_p u = g(u) + f_{j_\epsilon}(x,u) + \mu_\epsilon(|u-u_0|^{p^*-2}(u-u_0)) \tag{3.17}$$

satisfied by u_ϵ we get that (see the first part of the proof of Lemma 3.3.6 in the Section 3.3)

$$-\Delta_p(u_\epsilon - 1)^+ \leq M + c|(u_\epsilon - 1)^+|^{p^*-2}(u_\epsilon - 1)^+$$

for some $M > 0$ independent of ϵ. Now using the Moser iterations (observe that the singular term involving g is monotone and then the proof works similarly) as in García Azorero-Peral [6] p.950-953 (see also Lemma 3.7 step 1 in De Figueiredo-Gossez-ubilla [9]), we get that $\{u_\epsilon\}$ are bounded in $L^{\beta p^*}(\Omega)$ for some $\beta > 1$ independently of ϵ. Then, using Theorem 7.1 in Ladyzenskaja-Ural'ceva [23] p.263, we obtain that $\{u_\epsilon\}$ is uniformly bounded in $L^\infty(\Omega)$. This proves the Claim in the case(i).
Let us consider now the Case (ii):

Again as in the subcritical case, we have that $u_\epsilon \geq \eta\varphi_1$ for some $\eta > 0$ independent of ϵ. Moreover, there exists a number $M > 0$, independent of ϵ, such that for

$$g(s) + f_{j_\epsilon}(x, s) + \mu_\epsilon |s - u_0(x)|^{p^*-2}(s - u_0(x)) < 0 \text{ if } s > M, \qquad (3.18)$$

Taking now $(u_\epsilon - M)^+$ as a testing function in (3.17), one concludes by the weak comparison principle that $u_\epsilon(x) \leq M$ in Ω.
Then, the proof of the claim follows from the same arguments as in the subcritical case. This concludes the proof of the Claim in case (ii). □

The results in Section 3.3 are some adaptations of results proved in [18] and used in the present work for proving $L^\infty(\Omega)$ and $C^{1,\alpha}(\overline{\Omega})$ estimates. In particular Theorem 3.1.3 is a consequence of Lemma 3.3.5, Lemma 3.3.6 and Theorem 3.3.1.

3.3 Regularity

We start with an important technical tool which enables us to estimate the singularity in the Gâteaux derivative of the energy functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined in (3.1).

Lemma 3.3.1. *Let $0 < \delta < 1$. Then there exists a constant $C_\delta > 0$ such that the inequality*

$$\int_0^1 |\mathbf{a} + s\mathbf{b}|^{-\delta} \, \mathrm{d}s \leq C_\delta \left(\max_{0 \leq s \leq 1} |\mathbf{a} + s\mathbf{b}| \right)^{-\delta} \qquad (3.19)$$

holds true for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ with $|\mathbf{a}| + |\mathbf{b}| > 0$.

An elementary proof of this Lemma can be found in Takáč [27, Lemma A.1, p. 233].

We continue by showing the Gâteaux-differentiability of the energy functional I at a point $u \in W_0^{1,p}(\Omega)$ satisfying $u \geq \varepsilon\varphi_1$ in Ω with a constant $\varepsilon > 0$.

Lemma 3.3.2. *Let the assumptions* **(f1)-(f2)** *and* **(g1)-(g2)** *be satisfied. Assume that $u, v \in W_0^{1,p}(\Omega)$ and u satisfies $u \geq \varepsilon\varphi_1$ in Ω with a constant $\varepsilon > 0$. Then we have*

$$\begin{aligned} &\lim_{t \to 0} \frac{1}{t} \left(I(u + tv) - I(u) \right) = \\ &\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}x - \int_\Omega g(u) v \, \mathrm{d}x - \int_\Omega f(x, u) v \, \mathrm{d}x \, . \end{aligned} \qquad (3.20)$$

Proof. We show the result only for the singular term $\int_\Omega g(u)v \,\mathrm{d}x$; the other two terms are treated in a standard way. So let

$$H(u) = \int_\Omega G(u(x)^+) \,\mathrm{d}x \quad \text{ for } u \in W_0^{1,p}(\Omega).$$

For $\xi \in \mathbb{R} \setminus \{0\}$ we define

$$z(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi} G(\xi^+) = \begin{cases} g(\xi) & \text{if } \xi > 0; \\ 0 & \text{if } \xi < 0. \end{cases}$$

Consequently,

$$\frac{1}{t}\left(H(u+tv) - H(u)\right) = \int_\Omega \left(\int_0^1 z(u+stv) \,\mathrm{d}s \right) v \,\mathrm{d}x \,. \tag{3.21}$$

Notice that for almost every $x \in \Omega$ we have $u(x) > 0$ and

$$\int_0^1 z(u(x) + stv(x)) \,\mathrm{d}s \;\longrightarrow\; z(u(x)) = g(u(x)) \quad \text{ as } t \to 0 \,.$$

Moreover, the integral on the left-hand side (with nonnegative integrand) is dominated by

$$\begin{aligned} &\int_0^1 z(u(x) + stv(x)) \,\mathrm{d}s \le C \int_0^1 |u(x) + stv(x)|^{-\delta} \,\mathrm{d}s \\ &\le C_\delta \left(\max_{0 \le s \le 1} |u(x) + stv(x)| \right)^{-\delta} \\ &\le C_\delta \, u(x)^{-\delta} \le C_\delta \, (\varepsilon\varphi_1(x))^{-\delta} = C_{\delta,\varepsilon} \, \varphi_1(x)^{-\delta} \end{aligned}$$

with constants $C, C_{\delta,\varepsilon} > 0$ independent of $x \in \Omega$. Here, we have used the estimate (3.19) from Lemma 3.3.1 above. Finally, we have $v\varphi_1^{-\delta} \in L^1(\Omega)$, by $v \in W_0^{1,p}(\Omega)$ and Hardy's inequality. Hence, we are allowed to invoke the Lebesgue dominated convergence theorem in (3.21) from which the Lemma follows by letting $t \to 0$. □

Corollary 3.3.1. *Let the assumptions* **(f1)-(f2)** *and* **(g1)-(g2)** *be satisfied. Then the energy functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ is Gâteaux-differentiable at every point $u \in W_0^{1,p}(\Omega)$ that satisfies $u \ge \varepsilon\varphi_1$ in Ω with a constant $\varepsilon > 0$. Its Gâteaux derivative $I'(u)$ at u is given by*

$$\begin{aligned} &\langle I'(u), v \rangle = \\ &\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \,\mathrm{d}x - \int_\Omega g(u)v \,\mathrm{d}x - \int_\Omega f(x,u)v \,\mathrm{d}x \end{aligned} \tag{3.22}$$

for $v \in W_0^{1,p}(\Omega)$.

We continue by proving the C^1-differentiability of the cut off energy functional $\overline{I}$ defined below:

Lemma 3.3.3. *Let the assumptions* **(f1)-(f2)** *and* **(g1)-(g2)** *be satisfied, and $w \in W_0^{1,p}(\Omega)$ such that $w \geq \epsilon\varphi_1$ with some $\epsilon > 0$. Setting for $x \in \Omega$*

$$h(x,s) = \begin{cases} g(s) + f(x,s) & \text{if } s \geq w(x), \\ g(w(x)) + f(x,w(x)) & \text{if } s < w(x), \end{cases}$$

$H(x,s) = \int_0^s h(x,t)\,\mathrm{d}t$ and for $u \in W_0^{1,p}(\Omega)$

$$\overline{I}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - \int_\Omega H(x,u)\,\mathrm{d}x,$$

we have that $\overline{I}$ belongs to $C^1(W_0^{1,p}(\Omega),\mathbb{R})$.

Proof. As in Lemma 3.3.2, we concentrate on the singular term, the others being standard. Let

$$h(x,s) = \begin{cases} g(s) & \text{if } s \geq w(x), \\ g(w(x)) & \text{if } s < w(x), \end{cases}$$

$H(x,s) = \int_0^s h(x,t)\,\mathrm{d}t$, and $S(u) = \int_\Omega H(x,u)\mathrm{d}x$. Proceeding as in Lemma 3.3.2, we obtain that for all $u \in W_0^{1,p}(\Omega)$, $S(u)$ has a Gâteaux derivative $S'(u)$ given by

$$\langle S'(u), v\rangle = \int_\Omega g((\max\{u(x),w(x)\})v(x)\,\mathrm{d}x.$$

Let $u_k \in W_0^{1,p}(\Omega)$, $u_k \to u_0$. Then

$$\begin{aligned} &|\langle S'(u_k) - S'(u_0), v\rangle| \\ &= \left|\int_\Omega \left(g(\max\{u_k(x),w(x)\})v(x) - g(\max\{u(x),w(x)\})v(x)\right)\,\mathrm{d}x\right| \\ &\leq 2C\int_\Omega w^{-\delta}|v|\,\mathrm{d}x \\ &\leq 2C\epsilon^{-\delta}\int_\Omega \varphi_1^{-\delta}|v|\,\mathrm{d}x \end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$. Again, as in Lemma 3.3.2, we use Hardy's inequality to deduce that $\varphi_1^{-\delta}v \in L^1(\Omega)$, so that by Lesbegue's dominated convergence theorem we conclude that the Gâteaux derivative of S is continuous which implies that $S \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$. $\square$

Next, we give some regularity results for weak solutions to problem (P). We start with the following Lemma which allows for test functions ϕ in (P) to be taken in $W_0^{1,p}(\Omega)$ rather than only in $C_c^\infty(\Omega)$ ($\subset W_0^{1,p}(\Omega)$).

Lemma 3.3.4. *Let the assumptions* **(f1)-(f2)** *and* **(g1)-(g2)** *be satisfied. Each positive weak solution u of problem* (P) *satisfies $u \geq \epsilon\phi_1$ a.e. in Ω, where $\epsilon > 0$ is a constant independent of u. Moreover, for every function $w \in W_0^{1,p}(\Omega)$ we have $g(u)w \in L^1(\Omega)$ and*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w \,\mathrm{d}x = \int_\Omega g(u)w \,\mathrm{d}x + \int_\Omega f(x,u)w \,\mathrm{d}x\,. \tag{3.23}$$

Proof. Let u be a positive weak solution of (P). Recall that u is required to satisfy $\operatorname{ess\,inf}_K u > 0$ over every compact set $K \subset \Omega$.

First, we establish the inequality

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w \,\mathrm{d}x \geq \int_\Omega g(u)w \,\mathrm{d}x + \int_\Omega f(x,u)w \,\mathrm{d}x \tag{3.24}$$

for every $w \in W_0^{1,p}(\Omega)$ satisfying $w \geq 0$ a.e. in Ω. Given $0 \leq w \in W_0^{1,p}(\Omega)$, there exists a sequence $\{w_k\}_{k=1}^\infty \subset C_c^\infty(\Omega)$ such that $w_k \geq 0$ in Ω and $w_k \to w$ strongly in $W_0^{1,p}(\Omega)$ as $k \to \infty$. Since $p < q+1 \leq p^*$, this entails $w_k \to w$ strongly also in $L^{q+1}(\Omega)$ as $k \to \infty$. Moreover, we can find a subsequence, denoted again by $\{w_k\}_{k=1}^\infty$, such that $w_k \to w$ almost everywhere in Ω as $k \to \infty$. In equation (3.23) we now replace w by w_k and apply Fatou's Lemma to the integral $\int_\Omega g(u)w_k \,\mathrm{d}x$ as $k \to \infty$, thus arriving at the desired inequality (3.24).

In particular, ineq. (3.24) implies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w \,\mathrm{d}x \geq \int_\Omega g(u)w \,\mathrm{d}x \tag{3.25}$$

whenever $0 \leq w \in W_0^{1,p}(\Omega)$. Now we are ready to compare u with the unique weak solution $\underline{u}$ of problem:

$$(\mathrm{PS}) \begin{cases} -\Delta_p u &= g(u) \quad \text{in } \Omega; \\ u|_{\partial\Omega} &= 0, \quad u > 0 \quad \text{in } \Omega, \end{cases}$$

which is a global minimizer of the convex functional

$$J(u) \stackrel{\text{def}}{=} \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)} - \int_\Omega G(u^+)$$

and then satisfies $\underline{u} \geq \epsilon_0 \phi_1$ for some $\epsilon_0 > 0$ (see the arguments in the beginning of the proof of Theorem 3.1.1). We apply the weak comparison principle to (the weak formulation of) problem (PS) (with $\underline{u}$ in place of u) and to inequality (3.25) (with u), thus obtaining $u \geq \underline{u}$ a.e. in Ω. This guarantees $u \geq \epsilon_0 \phi_1$ a.e. in Ω.

Next, there are constants $0 < \ell < L < \infty$ such that $\ell\, d(x, \partial\Omega) \leq \phi_1(x) \leq L\, d(x, \partial\Omega)$ for all $x \in \Omega$. It follows that $u \geq \epsilon_0 \ell\, d(\cdot, \partial\Omega)$ a.e. in Ω. Now, instead of using Fatou's Lemma in the limiting process above, we apply Hardy's inequality to the integral $\int_\Omega g(u) w_k \,\mathrm{d}x$ as $k \to \infty$, thus arriving at the desired equality (3.23) for every $w \in W_0^{1,p}(\Omega)$ satisfying $w \geq 0$ a.e. in Ω.

Finally, we make use of the polar decomposition $w = w^+ - w^-$ of an arbitrary function $w \in W_0^{1,p}(\Omega)$, where $w^+ = \max\{w,\, 0\}$ and $w^- = \max\{-w,\, 0\}$ satisfy $w^+, w^- \in W_0^{1,p}(\Omega)$ and $\nabla w = \nabla w^+ - \nabla w^-$. Since we have already verified (3.23) for w^+ and w^-, the desired equality (3.23) holds also for every $w \in W_0^{1,p}(\Omega)$. $\square$

Lemma 3.3.5. *Let the assumptions* **(f1)-(f2)** *and* **(g1)-(g2)** *be satisfied. Each positive weak solution u of* (P) *belongs to $L^\infty(\Omega)$ and*

$$\|u\|_{L^\infty(\Omega)} \leq C(\|u\|_{W_0^{1,p}(\Omega)}, N, p, q, \Omega).$$

Proof. First, we show that each positive weak solution u of (P) satisfies

$$\int_\Omega |\nabla(u-1)^+|^{p-2} \nabla(u-1)^+ \cdot \nabla w \,\mathrm{d}x \leq \int_\Omega (g(1) + f(x,u)) w \,\mathrm{d}x \qquad (3.26)$$

for every $w \in C_c^\infty(\Omega)$ with $w \geq 0$. Indeed, let $\psi : \mathbb{R} \to [0,1]$ be a C^1 cut-off function such that $\psi(s) = 0$ if $s \leq 0$, $\psi'(s) \geq 0$ if $0 \leq s \leq 1$, and $\psi(s) = 1$ if $s \geq 1$. Given any $\epsilon > 0$, define $\psi_\epsilon(t) \stackrel{\text{def}}{=} \psi((t-1)/\epsilon)$ for $t \in \mathbb{R}$. Hence, $\psi_\epsilon \circ u \in W_0^{1,p}(\Omega)$ with $\nabla(\psi_\epsilon \circ u) = (\psi_\epsilon' \circ u)\, \nabla u$. Using the weak form of problem (P), eq. (3.3), with the test function $\phi = (\psi_\epsilon \circ u) w$, where $w \in C_c^\infty(\Omega)$ satisfies $w \geq 0$, we obtain

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla[(\psi_\epsilon \circ u) w] \,\mathrm{d}x = \int_\Omega (g(u) + f(x,u))(\psi_\epsilon \circ u) w \,\mathrm{d}x\,.$$

Hence,

$$\begin{aligned} &\int_\Omega |\nabla u|^p \, (\psi_\epsilon' \circ u) w \,\mathrm{d}x + \int_\Omega |\nabla u|^{p-2} (\nabla u \cdot \nabla w)(\psi_\epsilon \circ u) \,\mathrm{d}x \\ &= \int_\Omega (g(u) + f(x,u))(\psi_\epsilon \circ u) w \,\mathrm{d}x \end{aligned}$$

with $\psi'_\epsilon \circ u \geq 0$, which yields

$$\int_\Omega |\nabla u|^{p-2}(\nabla u \cdot \nabla w)(\psi_\epsilon \circ u)\,\mathrm{d}x \leq \int_\Omega (g(u) + f(x,u))(\psi_\epsilon \circ u) w \,\mathrm{d}x\,.$$

Letting $\epsilon \to 0+$ we arrive at (3.26). Finally, the L^∞ bound and regularity of u are obtained directly from eq. (3.26) as follows: If $q < p^* - 1$, one applies Theorem A.1 from ANANE [4], and if $q = p^* - 1$, the bootstrapping arguments from the proof of Theorem A.1, pp. 950–953, in GARCÍA AZORERO and PERAL [6] to get a bound in $L^{\beta p^*}(\Omega)$ and Theorem 7.1 in LADYŽENSKAJA-URAAL'CEVA [23] yield the desired result. In both references [4, 6] the bootstrapping arguments use the technique due to SERRIN [26] (proof of Theorem 1). □

Finally, we are ready to bound any weak solution u of problem (P) by a positive scalar multiple of the eigenfunction ϕ_1 also from above. This result complements the corresponding bound from below, $u \geq \epsilon_0 \phi_1$ a.e. in Ω, stated in the first part of Lemma 3.3.4 above. Equivalently, these lower and upper bounds for u/ϕ_1 can be reformulated as follows, using the distance function d in place of ϕ_1:

Lemma 3.3.6. *Let the assumptions* **(f1)-(f2)** *and* **(g1)-(g2)** *be satisfied. Each positive weak solution u of problem* (P) *satisfies*

$$C_1\, d(x,\partial\Omega) \leq u \leq K_1\, d(x,\partial\Omega)$$

a.e. in Ω, where $0 < C_1 \leq K_1 < \infty$ are some constants independent of u, K_1 dependent of $\|u\|_{L^\infty(\Omega)}$.

Proof. Let $u \in W_0^{1,p}(\Omega)$ be a positive weak solution of problem (P). It follows from the first part of Lemma 3.3.4 and its proof that $u(x) \geq \underline{u}(x) \geq \epsilon_0 \phi_1(x) \geq \epsilon_0 \ell\, d(x,\partial\Omega)$ for a.e. $x \in \Omega$. Hence, we can take $C_1 = \epsilon_0 \ell > 0$ to get $u \geq C_1\, d(\cdot,\partial\Omega)$ a.e. in Ω.

Next, we take advantage of the inequality $u \geq C_1\, d(\cdot,\partial\Omega)$ to derive also $u \leq K_1\, d(\cdot,\partial\Omega)$. Recall that $u \in L^\infty(\Omega)$, by Lemma 3.3.5 above. First, we apply the estimate

$$f(x,u) \leq C\frac{u^\delta + u^{q+\delta}}{u^\delta} \leq C\frac{1 + \|u\|^{q+\delta}_{L^\infty(\Omega)}}{u^\delta} \quad \text{a.e. in } \Omega$$

to the right-hand side of the equation in problem (P) to conclude that

$$\begin{cases} -\Delta_p u \leq C\left(1 + \|u\|^{q+\delta}_{L^\infty(\Omega)}\right) u^{-\delta} & \text{in } \Omega; \\ u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega \end{cases} \tag{3.27}$$

for some constant $C > 0$ large enough. After the substitution

$$v = \left(C + C \|u\|^{q+\delta}_{L^\infty(\Omega)} \right)^{-1/(p-1+\delta)} u \,,$$

inequality (3.27) is equivalent with

$$\begin{cases} -\Delta_p v \le v^{-\delta} \quad \text{in } \Omega; \\ v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{in } \Omega. \end{cases} \tag{3.28}$$

Let $\bar{u}$ the unique weak solution to

$$\text{(PSD)} \begin{cases} -\Delta_p u &= u^{-\delta} \quad \text{in } \Omega; \\ u|_{\partial\Omega} &= 0, \quad u > 0 \quad \text{in } \Omega. \end{cases}$$

Now, in analogy with the proof of Lemma 3.3.4, we apply the weak comparison principle to problem (PSD) (with $\bar{u}$ in place of u) and to inequality (3.28) (with v), thus arriving at $v \le \bar{u}$ a.e. in Ω. Thus, it remains to verify $\bar{u} \le c' \, d(\cdot, \partial\Omega)$ a.e. in Ω, where $0 < c' < \infty$ is a constant. This will imply $u \le K_1 \, d(\cdot, \partial\Omega)$ a.e. in Ω with

$$K_1 = c' \left(C + C \|u\|^{q+\delta}_{L^\infty(\Omega)} \right)^{1/(p-1+\delta)} .$$

Thanks to $\ell \, d(x, \partial\Omega) \le \phi_1(x) \le L \, d(x, \partial\Omega)$ for all $x \in \Omega$, with some constants $0 < \ell < L < \infty$, the inequality $\bar{u} \le c' \, d(\cdot, \partial\Omega)$ in Ω is equivalent to $\bar{u} \le c'' \phi_1$ in Ω, where $0 < c'' < \infty$ is a constant. We now construct a supersolution w to problem (PSD) of the form $w = \beta \cdot \Theta_\alpha \circ \phi_1$ in Ω. Here, $\alpha, \beta > 0$ are suitable numbers and $\Theta_\alpha : [0, R_\alpha) \to \mathbb{R}_+$ is a C^1 function (where $0 < R_\alpha < \infty$ and $\mathbb{R}_+ = [0, \infty)$) that satisfies the initial value problem

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}r} \left(|\Theta'_\alpha(r)|^{p-2} \, \Theta'_\alpha(r) \right) = \Theta_\alpha(r)^{-\delta}, \quad 0 < r < R_\alpha; \\ \Theta_\alpha(0) = 0, \quad \Theta'_\alpha(0) = \alpha > 0. \end{cases} \tag{3.29}$$

The endpoint R_α is defined to be the supremum of all numbers $s \in (0, \infty)$ such that $\Theta'_\alpha(r) > 0$ holds for all $r \in [0, s)$. We will see that $0 < R_\alpha < \infty$ together with $\Theta'_\alpha(r) \searrow 0$ as $r \nearrow R_\alpha$.

Making use of the transformation

$$\begin{cases} \Theta_\alpha(r) = \alpha^{\frac{p}{1-\delta}} \cdot \Theta_1(\alpha^{-\frac{p}{p-1+\delta}} \, r), \quad 0 \le r \le R_\alpha; \\ R_\alpha = \alpha^{\frac{p}{p-1+\delta}} \, R_1, \end{cases} \tag{3.30}$$

we conclude that it suffices to treat the case $\alpha = 1$. Problem (3.29) with $\alpha = 1$ has the first integral

$$\begin{cases} -\dfrac{p-1}{p}\,|\Theta_1'(r)|^p - \dfrac{1}{1-\delta}\,\Theta_1(r)^{1-\delta} + C = 0, \quad 0 \le r < R_1; \\ \Theta_1(0) = 0, \quad \Theta_1'(0) = 1 > 0, \end{cases} \tag{3.31}$$

where the constant C is given by $C = (p-1)/p$. There exists precisely one C^1 function $\Theta_1 : [0, R_1) \to \mathbb{R}_+$ that satisfies (3.31) together with $\Theta_1'(r) > 0$ for all $r \in [0, R_1)$; it is determined from

$$\int_0^{\Theta_1(r)} \left(1 - \frac{p}{(p-1)(1-\delta)}\,\theta^{1-\delta}\right)^{-1/p} \mathrm{d}\theta = r, \quad 0 \le r < R_1, \tag{3.32}$$

where

$$\begin{aligned} R_1 &= \int_0^{[(p-1)(1-\delta)/p]^{1/(1-\delta)}} \left(1 - \frac{p}{(p-1)(1-\delta)}\,\theta^{1-\delta}\right)^{-1/p} \mathrm{d}\theta \\ &= \left(\frac{(p-1)(1-\delta)}{p}\right)^{1/(1-\delta)} \int_0^1 (1 - t^{1-\delta})^{-1/p}\,\mathrm{d}t < \infty \end{aligned} \tag{3.33}$$

is the maximal number such that $\Theta_1'(r) > 0$ for all $r \in [0, R_1)$.

Let us first fix $\alpha > 0$ large enough, such that $R_\alpha > M \overset{\text{def}}{=} \max_{\overline{\Omega}} \phi_1$. In the following calculations we make use of eqs. satisfied by ϕ_1 and (3.29) by Θ_α, respectively. The function $w(x) = \beta \cdot \Theta_\alpha(\phi_1(x))$ of $x \in \Omega$ satisfies

$$\begin{aligned} \nabla w(x) &= \beta \cdot \Theta_\alpha'(\phi_1(x))\,\nabla\phi_1(x), \\ |\nabla w(x)|^{p-2}\nabla w(x) &= \beta^{p-1}\,[\Theta_\alpha'(\phi_1(x))]^{p-1}\,|\nabla\phi_1(x)|^{p-2}\nabla\phi_1(x), \end{aligned}$$

whence

$$\begin{aligned} -\Delta_p w = &-\beta^{p-1}\left[\left((\Theta_\alpha')^{p-1}\right)' \circ \phi_1\right] |\nabla\phi_1|^p \\ &+ \beta^{p-1}\left[\left((\Theta_\alpha')^{p-1}\right) \circ \phi_1\right](-\Delta_p\phi_1) \\ &= \beta^{p-1}\,(\Theta_\alpha \circ \phi_1)^{-\delta}\,|\nabla\phi_1|^p \\ &+ \beta^{p-1}\,\lambda_1\left[\left((\Theta_\alpha')^{p-1}\right) \circ \phi_1\right] \cdot \phi_1^{p-1} \\ &= \beta^{p-1+\delta}\,|\nabla\phi_1|^p\,w^{-\delta} \\ &+ \beta^{p-1}\,\lambda_1\left[\left((\Theta_\alpha')^{p-1}\right) \circ \phi_1\right] \cdot \phi_1^{p-1}\,(\beta \cdot \Theta_\alpha \circ \phi_1)^\delta\,w^{-\delta} \end{aligned} \tag{3.34}$$

$$= \beta^{p-1+\delta}\left\{|\nabla\phi_1|^p + \lambda_1\left[\left((\Theta_\alpha')^{p-1}\right) \circ \phi_1\right] \cdot \phi_1^{p-1}\,(\Theta_\alpha \circ \phi_1)^\delta\right\} w^{-\delta}\,.$$

Recall $R_\alpha > M = \max_{\overline{\Omega}} \phi_1$. The function Θ_α being strictly increasing with strictly decreasing derivative Θ'_α on the interval $[0, R_\alpha]$, and $\Theta_\alpha(0) = 0$, $\Theta'_\alpha(0) = \alpha > \Theta'_\alpha(R_\alpha) = 0$, we can estimate

$$\begin{gathered}\left((\Theta'_\alpha)^{p-1}\right) \circ \phi_1 \geq \Theta'_\alpha(M)^{p-1} > 0\,, \\ \Theta_\alpha \circ \phi_1 \geq \Theta'_\alpha(M)\,\phi_1\,.\end{gathered}$$

We combine these inequalities to estimate the second summand in the curly brackets at the end of eq. (3.34) above, thus obtaining

$$-\Delta_p w \geq \beta^{p-1+\delta} \left\{ |\nabla \phi_1|^p + \lambda_1 \left(\Theta'_\alpha(M)\,\phi_1\right)^{p-1+\delta} \right\} w^{-\delta}\,. \tag{3.35}$$

Moreover, we have $w \in C^1(\overline{\Omega})$ together with $w = 0$ on $\partial\Omega$, $w > 0$ in Ω, and $\frac{\partial w}{\partial \nu} < 0$ on $\partial\Omega$. These claims follow from $\phi_1 \in C^1(\overline{\Omega})$ combined with the strong maximum and boundary point principles $\phi_1 > 0$ in Ω and $\frac{\partial \phi_1}{\partial \nu} < 0$ on $\partial\Omega$ (see Vázquez [30, Theorem 5, p. 200]). The same arguments render

$$\gamma \stackrel{\text{def}}{=} \min_{\overline{\Omega}} \left\{ |\nabla \phi_1|^p + \lambda_1 \left(\Theta'_\alpha(M)\,\phi_1\right)^{p-1+\delta} \right\} > 0\,.$$

We choose the number $\beta > 0$ large enough, such that $\beta^{p-1+\delta}\gamma \geq 1$. In particular, inequality (3.35) yields

$$-\Delta_p w \geq w^{-\delta} \quad \text{in } \Omega\,. \tag{3.36}$$

Finally, we apply the weak comparison principle to problem (PSD) (with $\bar{u}$ in place of u) and to inequality (3.36) (with w satisfying $w = 0$ on $\partial\Omega$), thus arriving at $w \geq \bar{u}$ a.e. in Ω. We have thus verified

$$v \leq \bar{u} \leq w = \beta \cdot \Theta_\alpha \circ \phi_1 \leq \alpha\beta\,\phi_1 \leq c'\,d(\cdot, \partial\Omega) \quad \text{a.e. in } \Omega\,,$$

where $c' \in (0, \infty)$ is a constant, as desired.

The proof of Lemma 3.3.6 is now complete. □

Now, we recall some results proved in [18]. Precisely, we consider the following quasilinear elliptic boundary value problem,

$$-\nabla \cdot (\mathbf{a}(x, \nabla u)) = f(x) \ \text{ in } \Omega; \qquad u = 0 \ \text{ on } \partial\Omega, \tag{3.37}$$

in a setting that is closely related to Lieberman's in [25, Theorem 1, p. 1203]. We assume that Ω is a (nonempty) bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a compact C^2 manifold. We denote by $x = (x_1, \ldots, x_N)$ a generic point in Ω and by u the unknown function of x, where $u \in W_0^{1,p}(\Omega)$ for $p \in (1,\infty)$. The quasilinear elliptic operator $(x,u) \mapsto \nabla \cdot (\mathbf{a}(x,\nabla u))$ is defined by

$$\nabla \cdot (\mathbf{a}(x,\nabla u)) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,\nabla u(x)) \quad \text{for } x \in \Omega \text{ and } u \in W_0^{1,p}(\Omega) \tag{3.38}$$

with values in $W^{-1,p'}(\Omega)$, the dual space of $W_0^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. The components a_i of the vector field $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathbf{a} = (a_1,\dots,a_N)$, are functions of x and $\eta = \nabla u \in \mathbb{R}^N$, such that $a_i \in C^0(\Omega \times \mathbb{R}^N)$ and $\partial a_i/\partial \eta_j \in C^0(\Omega \times (\mathbb{R}^N \setminus \{0\}))$. We assume that $\mathbf{a}$ satisfies the following *ellipticity* and *growth conditions*:

(**H1**) There exist some constants $\kappa \in [0,1]$, $\gamma, \Gamma \in (0,\infty)$, and $\alpha \in (0,1)$, such that

$$a_i(x,0) = 0; \quad i = 1,\dots,N, \tag{3.39}$$

$$\sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \eta_j}(x,\eta) \cdot \xi_i \xi_j \geq \gamma \cdot (\kappa + |\eta|)^{p-2} \cdot |\xi|^2, \tag{3.40}$$

$$\sum_{i,j=1}^{N} \left| \frac{\partial a_i}{\partial \eta_j}(x,\eta) \right| \leq \Gamma \cdot (\kappa + |\eta|)^{p-2}, \tag{3.41}$$

$$\sum_{i=1}^{N} |a_i(x,\eta) - a_i(y,\eta)| \leq \Gamma \cdot (1 + |\eta|)^p \cdot |x-y|^\alpha, \tag{3.42}$$

for all $x, y \in \Omega$, all $\eta \in \mathbb{R}^N \setminus \{0\}$, and all $\xi \in \mathbb{R}^N$.

We remark that conditions (3.39) through (3.42) are motivated by the elliptic boundary value problem

$$-\Delta_p u = f(x) \text{ in } \Omega; \qquad u = 0 \text{ on } \partial\Omega, \tag{3.43}$$

with the p-Laplacian defined by $\Delta_p u \stackrel{\text{def}}{=} \nabla \cdot (|\nabla u|^{p-2} \nabla u)$.

Finally, we impose the following *growth condition* on the function $f \in L^\infty_{\text{loc}}(\Omega)$:

(**H2**) There exist constants c and δ, $0 < c < \infty$ and $0 < \delta < 1$, such that

$$0 \leq f(x) \leq c\, d(x,\partial\Omega)^{-\delta} \quad \text{holds for almost all } x \in \Omega. \tag{3.44}$$

Then, we have the following analogue of a well-known regularity result for problem (3.37) due to Lieberman [25, Theorem 1, p. 1203] (regularity near the boundary). Interior regularity was established earlier independently by DiBenedetto [12, Theorem 2, p. 829] and Tolksdorf [29, Theorem 1, p. 127]. Theorem 3.3.1 is proved in [18].

Theorem 3.3.1. *Assume that* $\mathbf{a}(x,\eta)$ *satisfies the structural hypotheses* (3.39)-(3.42), *and* $f(x)$ *satisfies the growth hypothesis* (3.44). *Let* $u \in W_0^{1,p}(\Omega)$ *be the (unique) weak solution of problem* (3.37). *In addition, assume*

$$0 \le u(x) \le C\, d(x, \partial\Omega) \quad \textit{for almost all } x \in \Omega, \tag{3.45}$$

where C *is a constant,* $0 \le C < \infty$. *Then there exist constants* β *and* M, $0 < \beta < \alpha$ *and* $0 \le M < \infty$, *depending solely on* Ω, N, p, *on the constants* γ, Γ, α *in* (3.40) *through* (3.42), *on the constants* c, δ *in* (3.44), *and on the constant* C *in* (3.45), *but* not *on* $\kappa \in [0,1]$, *such that* u *satisfies* $u \in C^{1,\beta}(\overline{\Omega})$ *and*

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \le M. \tag{3.46}$$

3.4 Summary

Theorem 3.1.1 shows that for a class of quasilinear singular elliptic equations with boundary Dirichlet conditions, any $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ local minimizer for the associated energy functional is a $W_0^{1,p}(\Omega)$ minimizer. This result generalises Proposition 3.7 in [18] which only concerns the case $g(t) = t^{-\delta}$ and $f(x,t) = t^q$ with $0 < \delta < 1$ and $p - 1 < q < p^* - 1$. The proof used to prove Theorem 3.1.1 does not modify the quasilinear operator $-\Delta_p$ in (P) and then the $C^{1,\alpha}(\overline{\Omega})$-regularity is easier to get. Therefore, this approach could be considered for more general quasilinear operators in the form given in the Section 3.3 to get multiplicity results for corresponding singular equations. It will be interesting to get similar results for anisotropic operators as the $p(x)$-Laplacian operator.

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Chapitre 4

Bifurcation results for exponential type singular elliptic equation in $\mathbb{R}^2$

Contents

4.1 Introduction

In this chapter, we are dealing with the following problem

$$(\mathrm{P}_\lambda)\begin{cases} -\Delta u &= \lambda f(u) \stackrel{\text{def}}{=} \lambda(\frac{1}{u^\delta} + h(u)e^{u^\alpha}) \quad \text{ in } \Omega \subset \mathbb{R}^2; \\ u|_{\partial\Omega} &= 0, \quad u > 0 \quad \text{ in } \Omega, \end{cases}$$

where Ω is a bounded and smooth ($C^{3,\gamma}$, regular $0 < \gamma < 1$) domain of $\mathbb{R}^2$, $\lambda > 0$ a parameter, $0 < \delta < 1$ and $1 \le \alpha \le 2$.
We assume that $t \rightarrow h(t)e^{t^\alpha}$ is C^2, convex, increasing on $\mathbb{R}^+$ and that h is a perturbation of e^{u^α} at ∞. It means that h satisfies additionally the following assumptions:

(H1) $h(t) \in C^2(\mathbb{R}^+, \mathbb{R}^+)$ with $h(0) = 0$,

(H2) for any $\epsilon > 0$, $h(t)e^{-\epsilon t^\alpha} \underset{t\to\infty}{\rightarrow} 0$ and $h(t)e^{\epsilon t^\alpha} \underset{t\to\infty}{\rightarrow} +\infty$.

Concerning the behaviour of f near 0, we assume that f is singular; precisely, we have: there exists $0 < \delta < 1$; $f(t) = \frac{1}{t^\delta} + h(t)e^{t^\alpha}$ wehere h belongs to $C^2(\mathbb{R}^+)$ satisfying $h(0) = 0$.

Equation (P_λ) appears in several models: non newtonial flows in porous media, chemical heterogeneous catalysts, nonlinear heat equations (see [18], [20], [21], [22], see also [30] and the overviews about singular elliptic equations: [28], [23]) and then have intrinsic mathematical interest. In the frame of bifurcation theory, we show the existence of a connected branch of solutions to (P_λ). Precisely we prove the following main results

Theorem 4.1.1. *Suppose that* $1 \le \alpha \le 2$ *and* **(H1)**, **(H2)** *hold. Then, there exists a connected branch of solutions to* (P_λ), $\mathcal{C}$ *with* $\mathcal{C} \subset \mathcal{S} = \{(\lambda, u) \in [0, \Lambda] \times C^1(\overline{\Omega}) \cap C^2(\Omega) |\, u$ *solution to* $(\mathrm{P}_\lambda)\}$ *emanating from* $(0,0)$ *such that*

1) $\mathcal{C}$ *is unbounded, and there exists* $0 < \Lambda < +\infty$ *and* $0 < \beta = \beta(\delta) < 1$ *such that* $\mathcal{S} \subset [0, \Lambda] \times C^{1,\beta}(\overline{\Omega}) \cap C_0(\overline{\Omega})$.

2) for $\lambda \in (0, \Lambda], \exists\, (\lambda, u_\lambda) \in \mathcal{C}$ *where* u_λ *is the minimal solution.*

3) The curve $\mathbb{R}^+ \ni \lambda \to u_\lambda \in C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ *is* C^1.

4) (bending and local multiplicity near $\lambda = \Lambda$*)* $\lambda = \Lambda$ *is a bifurcation point, that is, there exists a unique* C^2*-curve* $(\lambda(s), u(s)) \in \mathcal{C}$ *with*

$$\lambda(0) = \Lambda,\ u(0) = u_\Lambda,\ \lambda'(0) = 0,\ \lambda''(0) < 0.$$

5) (asymptotic bifurcation point) $\mathcal{C}$ *admits an asymptotic bifurcation point at some* λ_2 *satisfying* $0 \le \lambda_2 \le \Lambda$.

Note that the local multiplicity near Λ can be also proved using the stability (i.e. $\lambda_1(L_\lambda) > 0$) of the minimal solutions u_λ for $\lambda \le \Lambda$ which provides uniform a priori estimates on u_λ.

In the radial case , i.e. $\Omega = B_1 \stackrel{\text{def}}{=} \{x \in \mathbb{R}^2 |\, |x| < 1\}$ and restricting to radial symetric solutions to (P_λ), we can analyze more precisely the asymptotic behaviour of the solutions near the asymptotic bifurcation point. First, when $\alpha = 1$, we can use some results from [9] and [10] to get that the unique asymptotic bifurcation point is $\lambda = 0$. Precisely, from [9] and [10], we have

Theorem 4.1.2 (Brezis-Merle, Li-Shaffrir)**.** *Let* $\{u_n\}$ *be a sequence of (weak) solutions to*

$$-\Delta u_n = V_n(x)e^{u_n} \quad in\ \Omega'$$

where Ω' *is a bounded domain in* $\mathbb{R}^2$ *and* V_n, u_n *satisfy*

(i) $V_n \ge 0$,

(ii) *there exists* $1 \leq p \leq \infty$ *such that* $\|V_n\|_{L^p(\Omega')} \leq C_1$, $\|e^{u_n}\|_{L^{p'}(\Omega')} \leq C_2$ *for some constants* $C_1, C_2 > 0$ *and with* p' *the conjugate of* p.

Then, there exists a subsequence (u_{n_k}) *satisfying the following alternative :*

(i) *either* u_{n_k} *is uniformly bounded in* $L^\infty_{loc}(\Omega')$,

(ii) *or* $u_{n_k} \to -\infty$ *as* $k \to \infty$ *uniformly in any compact subset of* Ω',

(iii) *or the blow-up set* S *(relative to* u_{n_k}*) is finite non empty and* $u_{n_k} \to -\infty$ *in* Ω/S. *In addition,* $V_{n_k} e^{u_{n_k}}$ *converge in the sense of distributionS in* Ω' *to* $\sum_i \alpha_i \delta(a_i)$ *with* $\alpha_i \geq \frac{4\pi}{p'}$, $\forall\, i$ *and* $S = \cup_i \{a_i\}$.

Using Theorem 4.1.2, we can show that

Theorem 4.1.3. *Let* $\Omega = B_1$ *and* $\alpha = 1$. *Assume that* **(H1)**,**(H2)** *hold. Then, there exists a connected and unbounded branch of solutions to* (P_λ), $\mathcal{C}_r$ *emanating from* $(0,0)$ *satisfying the following*

(i) *For* $(\lambda, u) \in \mathcal{C}_r$, u *is radially symmetric and radially decreasing.*

(ii) $\lambda = 0$ *is the unique asymptotic bifurcation point.*

(iii) *For* $\lambda \in (0, \Lambda)$, *there exists at least two distinct solutions* u_λ, v_λ *to* (P_λ) *such that* (λ, u_λ), (λ, v_λ) *belongs to* $\mathcal{C}_r$.

When $\alpha < 1$ (which is not under the scope of this work), similar results as Theorems 4.1.1 ([1]-[3], [5]) and 4.1.3 (in the radial symmetric case) hold with similar proofs. the local multiplicity of solutions near $\lambda = \Lambda$ holds also using the remark following the statement of Theorem 4.1.1.

When $1 < \alpha < 2$, we can extend in the singular case some results in [32]. Precisely, we show that

Theorem 4.1.4. *Let* $\Omega = B_1$ *and* $1 < \alpha < 2$. *Assume that* **(H1)**,**(H2)** *hold. Then, there exists a connected and unbounded branch of solutions to* (P_λ), $\mathcal{C}_r$ *emanating from* $(0,0)$ *satisfying the following*

(i) *For* $(\lambda, u) \in \mathcal{C}_r$, u *is radially symmetric and radially decreasing.*

(ii) $\lambda = 0$ *is the unique asymptotic bifurcation point and for* $(\lambda_n, v_{\lambda_n}) \in \mathcal{C}_r$ *such that* $\lambda_n \to 0^+$ *and* $v_{\lambda_n}(0) \to +\infty$, *we have* $v_{\lambda_n}(x) \to 0$ *as* $n \to +\infty$ *for* $0 < |x|$.

(iii) *For* $\lambda \in (0, \Lambda)$, *there exists at least two distinct solutions* u_λ, v_λ *to* (P_λ) *such that* (λ, u_λ), (λ, v_λ) *belongs to* $\mathcal{C}_r$.

We then analyze further the critical case $\alpha = 2$. In this case, we are in the non compact situation which is related to the Trudinger-Moser inequality:

Theorem 4.1.5. *(Trudinger-Moser)*

1. *Let $p < \infty$ then $u \in H_0^1(\Omega)$ implies $e^{u^2} \in L^p(\Omega)$ and is continuous in the norm topology.*

2.
$$4\pi = \max\left\{c\,;\ \sup_{\|w\|\leq 1}\int_\Omega e^{c|w|^2} < +\infty\right\}. \tag{4.1}$$

But similary to higher dimensions case, $H_0^1(\Omega) \ni u \hookrightarrow e^{4\pi u^2} \in L^1(\Omega)$ is not compact for the weak topology. According to this, the question of existence of solutions for

$$(P_4)\begin{cases} -\Delta u = f(x,u) \text{ in } \Omega \subset \mathbb{R}^2 \\ u|_{\partial\Omega} = 0\,,\ u > 0 \end{cases}$$

where $f(x,u)$ has a critical behaviour (see **(H2)**) is far from obvious. In particular, the method adopted to solve (P_4) in [8] (for higher dimensions) does not work because of the critical growth of exponential type.

The existence of a connected branch of solutions to (P_λ) can be proved in a similar way as in the subcritical case. However, the global behaviour of the branch depends on the asymptotics of h. Precisely, we distinguish between the following cases:

(H3) $h(t) = O(e^{-t^\beta})$ $1 < \beta < 2$.

(H4) $h(t) = O(e^{-t^\beta})$ $0 \leq \beta < 1 \cup \{h$ decays polynomially at $\infty\}$.

(H5) $h(t)t \to +\infty$.

For h belonging to class **(H4)** or **(H5)**, we have the following result proved in [26]. The set of hypotheses **(A6)-(A7)** (resp. **(A8)**) correspond to nonlinearities in **(H4)** (resp. **(H5)**).

Theorem 4.1.6. *Let conditions* (**H1**), (**H2**), *and* $g(s) \stackrel{\text{def}}{=} \log((f(s))$. *In addition suppose that either*

(A6) $\limsup_{s\to\infty}(g(s) - (\frac{1}{2})sg'(s) + \log((\frac{1}{2})g'(s))) < \infty$,

(A7) $\limsup_{s\to\infty}|g(s) - (\frac{1}{2})sg'(s)|s < \infty$, *or*

(A8) $\limsup_{s\to\infty} g(s) - (\frac{1}{2})sg'(s) + \log((\frac{1}{2})g'(s)) = \infty$

hold. Then there exists Λ *such that*

1) *For* $0 < \lambda < \Lambda$, *there exists at least two solutions* u_λ, v_λ *such that* $0 < u_\lambda < v_\lambda$ *in* Ω *and* $|\nabla v_\lambda|^2 \rightharpoonup 4\pi\delta_0,\ E(v_\lambda) \to 2\pi$ *as* $\lambda \to 0^+$.

2) *For* $\lambda = \Lambda$, *there exists at least one solution,* u_Λ *to* (P_Λ).

3) *For* $\lambda > \Lambda$, *there is no solution to* (P_λ).

In the radial case, using bifurcation theory we have a clear picture of the solutions set to (P_λ). Precisely, for the classes of nonlinearities **(H4)** and **(H5)** we can extend some results from [3]:

Theorem 4.1.7. *Let* $\Omega = B_1$. *Suppose that conditions* (**H1**), (**H2**) *and either* **(A6)**, **(A7)** *or* **(A8)** *hold. Then, considering* $\mathcal{C}_r \subset \mathcal{S}$ *proved in Theorem 4.1.4, we have that* $\lambda = 0$ *is an asymptotic bifurcation point and as* $\lambda \to 0^+$

$$|\nabla v_\lambda|^2 \rightharpoonup 4\pi\delta_0,$$

and there exists $\rho(\lambda) \to 0^+$ *as* $\lambda \to 0^+$ *such that*

$$v_\lambda^2(\rho(\lambda)r) - v_\lambda^2(\rho(\lambda)) \to 2\log(\frac{2}{1+r^2}) \quad \textit{uniformly on compact subset of } \mathbb{R}^2.$$

Theorem 4.1.7 implies that the "limiting equation" to our problem is

$$-\Delta u = 2e^u \quad \text{in } \mathbb{R}^2.$$

A similar blow up analysis has been proved in [32] (see Theorem B) for a smaller class in the regular case and for $\alpha \in (1, 2]$. The blow up behaviour for a family of solutions in the non radial case (and regular case) can be found in [5] and [19]. In case of the class of nonlinearities satisfying **(H3)**, in [2] the following result was proved :

Theorem 4.1.8. *Let* $\Omega = B_1$. *Suppose that the conditions* (**H1**), (**H2**) *and assume in addition that*

(A3) $g_1(s) \stackrel{\text{def}}{=} \log(h(s)) - \alpha \log s) < 0\ \forall$ *large* $s > 0$,

(A4) $\lim_{s\to\infty} \frac{sg_1'(s)}{g_1(s)} \in (0, 2), \lim_{s\to\infty} \left(g_1'(s)s^{-1}, \frac{g_1(s)}{g(s)}\right) = (0, 0)$,

(A5) $\lim_{s\to\infty} 2g(s)g''(s) - (g'(s))^2 > 0$.

Then, for $0 < \lambda$ *small,* (P_λ) *admits a unique radial symmetric solution.*

Theorems 4.1.6 and 4.1.8 have been extended to the N-laplacian case ($N \geq 2$) and for any $0 < \delta < \infty$ in [26]. Concerning the global behaviour of $\mathcal{C}_r$ proved in Theorem 4.1.4 we show in this case the following theorem:

Theorem 4.1.9. *Let* $\Omega = B_1$. *Suppose that the conditions* (**H1**), (**H2**) *and* **(A3)-(A5)**. *Then considering* $\mathcal{C}_r \subset \mathcal{S}$ *proved in Theorem 4.1.4, we have that*

(i) *For* $\lambda > 0$ *small,* $\mathcal{C}_r$ *contains a unique parametrized curve* $\{\lambda, u_\lambda\}$ *emanating from* $(0, 0)$.

(ii) *there exists* $\eta > 0$ *such that for* $\lambda \in (\Lambda - \eta, \Lambda)$ *there exists* v_λ *solution to* (P_λ) *such that* $u_\lambda < v_\lambda$ *in* Ω *and* (λ, u_λ), $(\lambda, v_\lambda) \in \mathcal{C}_r$.

(iii) $\mathcal{C}_r$ *admits at least one asymptotic bifurcation point* $0 < \lambda_0 \leq \Lambda$.

(iv) *There exists a sequence* $(\lambda_n, v_{\lambda_n}) \in \mathcal{C}_r$ *such that as* $n \to +\infty$, $\lambda_n \to \lambda_0$, $v_{\lambda_n}(0) \to +\infty$ *and* $v_{\lambda_n} \to v^*$ *on compact set of* $B_1 \setminus \{0\}$ *where* v^* *is a singular solution to* (P_{λ_0}).

For proving the above theorems, we can not use the classical Bifurcation Theory since the nonlinearity is singular at the origin. To overcome this difficulty, we use some results in [29] which provide a strong maximum principle and the regularity of some functionals in the singular case. In particular this allows us to define a principal eigenvalue to some linearized problem associated to problem (P_λ) and to adapt some classical tools in Bifurcation Theory (bifurcation from simple eigenvalue, Leray Schauder degree, see [11], [12] and [33]). Precisely, we will use the results in Section 2 in [29] related to the properties of the linearized operator (Propositions 2.3 and 2.5) and the strong maximum principle. These results can be used to prove the existence of the branch $\mathcal{C}$ of solutions to (P_λ) via the implicit function theorem and the local bifurcation result of Crandall Rabinowitz (see [11]).The continuation and the global behaviour of the branch will follow from the theory of global bifurcation theory adapted to the singular case. In a second part of this chapter, we analyze the global behaviour of the branch $\mathcal{C}_r$ near the asymptotic bifurcation point. We highlight that in the critical case ($\alpha = 2$) the different results we obtain are strongly related to the asymptotic behaviour at infinity of h. These results used some previous results proved in [16] and stated in the Section 4.4.

4.2 Bifurcation results

In this section, we show Theorem 4.1.1.
Proof of Theorem 4.1.1:
For this, we first show the existence and the regularity of solutions branch: From [14], we know that the following problem

$$(\bar{\mathrm{P}}_\lambda)\begin{cases} -\Delta u &= \lambda u^{-\delta} \ \text{ in } \Omega \\ u|_{\partial\Omega} &= 0, \ u > 0 \text{ in } \Omega \end{cases}$$

admits a unique solution, $\underline{u}_\lambda$, in $C^2(\Omega)\cap C_0(\overline{\Omega})$. If in addition $\delta < 1$, we have the following result:

Lemma 4.2.1. *i) There exists $k_\lambda, K_\lambda > 0$ such that*

$$k_\lambda d(x,\partial\Omega) \le \underline{u}_\lambda(x) \le K_\lambda d(x,\partial\Omega) \ \ \textit{for } x \in \Omega.$$

ii) $\bar{u}_\lambda \in C^{1,\beta}(\overline{\Omega})$ with $0 < \beta = \beta(\delta) < 1$.

Proof : See Theorem 2.2. in [14] for the proof of i) and Lemma A.5 in [25] for ii) (and for a more general situation). Then, we have the following result:

Proposition 4.2.1. *1) There exists Λ, $0 < \Lambda < \infty$ such that for $\lambda < \Lambda$ there is a minimal solution u_λ to (P_λ) satisfying*

$$\lambda \to u_\lambda \in C^1\left([0,\Lambda),\ C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})\right)$$

2) For $\lambda > \Lambda$ there is no solution to (P_λ).

Proof :
For any weak solution u to (P_λ), i.e. $u \in H^1_0(\Omega)$ satisfying $ess\inf_K u > 0$ for all $K \subset \Omega$ and for all $0 \le \phi \in C_c^\infty(\Omega)$:

$$\int_\Omega \nabla u \nabla \phi \, \mathrm{d}x = \int_\Omega \lambda(\frac{1}{u^\delta} + h(u)e^{u^\alpha})\phi \, \mathrm{d}x,$$

it is easy to see that [by multiplying the equation in (P_λ) by $(\underline{u}_\lambda - u)^+$] $\underline{u}_\lambda \le u$ and by Lemma A.5 in [25] that $u \in C^{1,\alpha}(\overline{\Omega})$.
Letting, $\underline{u} = \underline{u}_\lambda$ and $\overline{u} = \underline{u}_\lambda + Mv$ where

$$\begin{cases} -\Delta v &= 1 \ \text{ in } \Omega \\ v|_{\partial\Omega} &= 0, \ v > 0 \text{ in } \Omega \end{cases}$$

and $M > 0$ large enough. We see that $\underline{u}$ and $\overline{u}$ are sub and supersolutions respectively if λ is small enough. Indeed,

$$-\Delta\overline{u} = \frac{\lambda}{\underline{u}^{\delta}} + M \geq \frac{\lambda}{\overline{u}^{\delta}} + \lambda h(\bar{u})e^{\bar{u}^{\alpha}}$$

for $\lambda = \lambda(M)$ small enough.
We introduce the following iterative scheme with $K_0 = K_0(M) > 0$ large enough:

$$\begin{cases} -\Delta u_n - \lambda u_n^{-\delta} + K_0 u_n = K_0 u_{n-1} + \lambda h(u_{n-1})e^{u_{n-1}^{\alpha}} & \text{in } \Omega \\ u_0 = \underline{u}_\lambda. \end{cases} \tag{4.2}$$

Using that $u \to -\Delta u - \frac{\lambda}{u^{\delta}}$ is monotone operator in the cone of positive functions $C \stackrel{\text{def}}{=} \{u \subset C^1(\overline{\Omega}) \cap C_0(\overline{\Omega}) / u > 0 \ \frac{\partial u}{\partial n} < 0\})$, we have that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is non decreasing and bounded in $C^{1,\beta}(\overline{\Omega})$. Then, by Ascoli-Arzela Theorem, $u_n \to u_\lambda$ with $\underline{u}_\lambda \leq u_\lambda \leq \overline{u}$ in $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ and u_λ is the minimal solution to (P_λ). Setting

$$0 < \Lambda \stackrel{\text{def}}{=} \sup\{\lambda > 0/(P_\lambda) \quad \text{has a weak solution } \}, \tag{4.3}$$

we have proved above that $\Lambda > 0$ and from the superlinear behaviour of $h(u)e^{u^{\alpha}}$ we have clearly that $\Lambda < \infty$.
Using again the iterative scheme (4.2) with $\underline{u} = \underline{u}_\lambda$ and $\overline{u} = u_{\lambda'}$ for $\lambda \leq \lambda' < \Lambda$, we show the existence of the minimal solution to (P_λ) for any $\lambda < \Lambda$. From the strong Maximum Principle (see Lemma 2.7 in [29] or Theorem 2.3 in [27]), we deduce that

$$\underline{u}_\lambda < u_\lambda \quad \text{in } \Omega \ \text{ and } \frac{\partial \underline{u}_\lambda}{\partial n} > \frac{\partial u_\lambda}{\partial n} \ \text{ on } \partial\Omega. \tag{4.4}$$

Setting $k(t) \stackrel{\text{def}}{=} h(t)e^{t^{\alpha}}$, from Theorem 2.6 and Proposition 2.8 in [29], and the fact that $u_\lambda \geq k_\lambda d(\cdot, \partial\Omega)$ with $k_\lambda > 0$, we see that the linearized operator $L_\lambda = -\Delta + \frac{\lambda\delta}{u_\lambda^{\delta+1}} - \lambda k'(u_\lambda)$ admits a principal eigenvalue given by

$$\lambda_1(L_\lambda) \stackrel{\text{def}}{=} \sup_{v\in C}\{\inf_{x\in\Omega} \frac{L_\lambda v}{v}\}$$

and

$$\lambda_1(L_\lambda) \stackrel{\text{def}}{=} \inf_{\phi\in H_0^1(\Omega), \int_\Omega \phi^2=1} \int_\Omega |\nabla\phi|^2 \mathrm{d}x + \int_\Omega \frac{\lambda\delta}{u_\lambda^{\delta+1}}\phi^2 \mathrm{d}x - \int_\Omega \lambda k'(u_\lambda)\phi^2 \mathrm{d}x.$$

Since u_λ is the minimal solution, $\lambda_1(L_\lambda) \geq 0$. Indeed, arguing by contradiction and assume that $\lambda_1(L_\lambda) < 0$ then the function $w_\lambda = u_\lambda - c\varphi_\lambda$, where φ_λ is the normalized positive eigenfunction associated to $\lambda_1(L_\lambda)$ and for $c > 0$ small enough is a supersolution to (P_λ) with $w_\lambda \geq \underline{u}_\lambda$. It follows that there exists a solution, v_λ, to (P_λ) satisfying $v_\lambda < u_\lambda$ in Ω. This contradicts that u_λ is the minimal solution to (P_λ). We now prove that for $0 < \lambda < \Lambda$, $\lambda_1(L_\lambda) > 0$. Again we argue by contradiction, suppose that for some $0 \leq \lambda_0 < \Lambda$, $\lambda_1(L_{\lambda_0}) = 0$. From the Implicit Function Theorem, observe that $\lambda_0 > 0$. Then, for $0 < \lambda < \lambda_0$

$$u_\lambda - \lambda(-\Delta)^{-1}\{\frac{1}{u_\lambda^\delta} + k(u_\lambda)\} \stackrel{\text{def}}{=} [Id - \lambda F](u_\lambda).$$

Set

$$\begin{array}{rcccl} G & : & \mathbb{R}^+ \times \mathcal{C} & \to & C_0^1(\overline{\Omega}) \\ & & (\lambda, u) & \mapsto & u - \lambda F(u) \end{array}$$

from Theorem 3.1 in [29], we get that G is differentiable and $\frac{\partial}{\partial u}G(\lambda, u) = Id - \lambda F'(u)$ with $F'(u)$ is a compact operator by proposition 2.3 in [29].
Then from the Fredhom Alternative, $I - \lambda F'(u_\lambda)$ is invertible iff $N(I - \lambda F'(u_\lambda)) = \{0\}$ which follows from $\lambda(L_\lambda) > 0$. Then, from the Implicit Function Theorem, $\lambda \to u_\lambda$ is C^1 in $(0, \lambda_0)$.
We will now show that from $\lambda_1(L_{\lambda_0}) = 0$, we get that for $\lambda > \lambda_0$, $\lambda_1(L_\lambda) < 0$. Indeed,

$$\inf_{\phi \in H_0^1(\Omega), \int_\Omega \phi^2 = 1} \left\{ \int_\Omega |\nabla \phi|^2 \mathrm{d}x + \int_\Omega \frac{\lambda_0 \delta}{u_{\lambda_0}^{\delta+1}} \phi^2 \mathrm{d}x - \int_\Omega \lambda_0 k'(u_{\lambda_0}) \phi^2 \mathrm{d}x \right\}$$

$$= \int_\Omega |\nabla \phi_{\lambda_0}|^2 \mathrm{d}x + \int_\Omega \frac{\lambda_0 \delta}{u_{\lambda_0}^{\delta+1}} \phi_{\lambda_0}^2 \mathrm{d}x - \int_\Omega \lambda_0 k'(u_{\lambda_0}) \phi_{\lambda_0}^2 \mathrm{d}x = 0.$$

Since $\lambda \to \frac{\delta}{u_\lambda^{\delta+1}} - k'(u_\lambda)$ is decreasing and $\int_\Omega (\frac{\delta}{u_{\lambda_0}^{\delta+1}} - k(u_{\lambda_0}))\phi_{\lambda_0}^2 \mathrm{d}x < 0$, we get for $\lambda > \lambda_0$ that

$$\int_\Omega |\nabla \phi_{\lambda_0}|^2 \mathrm{d}x + \lambda \left[\int_\Omega \frac{\delta}{u_{\lambda_0}^{\delta+1}} \phi_{\lambda_0}^2 \mathrm{d}x - \int_\Omega k'(u_{\lambda_0}) \phi_{\lambda_0}^2 \mathrm{d}x \right] < 0.$$

Then it follows that $\lambda_1(L_\lambda) < 0$ for $\lambda > \lambda_0$ which contradicts $\lambda_1(L_\lambda) \geq 0 \; \forall \, \lambda \in (0, \Lambda)$.
Now, we prove that $u_\lambda \to u_\Lambda$ as $\lambda \to \Lambda$ where u_Λ is the minimal solution to (P_Λ).
Indeed, from

$$\int_\Omega |\nabla u_\lambda|^2 \mathrm{d}x - \lambda \int_\Omega f(u_\lambda) u_\lambda = 0 \;\; \text{and} \;\; \int_\Omega |\nabla u_\lambda|^2 \mathrm{d}x - \lambda \int_\Omega f'(u_\lambda) u_\lambda^2 \geq 0$$

we get that

$$\int_\Omega |\nabla u_\lambda|^2 \mathrm{d}x \le C \quad \text{from which we get} \quad u_\lambda \rightharpoonup u_\Lambda$$

where u_Λ is a weak solution (by Theorem 4.1.5) in $H_0^1(\Omega)$ and by the elliptic regularity theory $u_\Lambda \in L^\infty(\Omega)$.
By Vitali's convergence Theorem, from $\lambda \int_\Omega f'(u_\lambda)u_\lambda^2 \le C$ we get

$$f(u_\lambda)u_\lambda \to f(u_\Lambda)u_\Lambda \quad \text{in} \quad L^1(\Omega)$$

and $u_\lambda \to u_\Lambda$ in $H_0^1(\Omega)$, and then $u_\lambda \to u_\Lambda$ in $C^1(\overline{\Omega})$ (see Lemma A.5 in [25]).
Let $L_\Lambda = -\Delta - \Lambda f'(u_\Lambda)$. From above we have that $\lambda_1(L_\Lambda) \ge 0$ and from the Implicit Function Theorem and the non existence result to (P_λ) for $\lambda > \Lambda$, we get $\lambda_1(L_\Lambda) = 0$.
We now apply the local bifurcation result of Crandall-Rabinowitz [11] (see also [12]).
Let $G(\lambda, v) = v - (-\Delta)^{-1}\{\lambda f(v)\}$ acting from $\mathbb{R}^+ \times C \to C_0^1(\overline{\Omega})$. (We recall that $C \subset \{C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})\}$ is the cone of positive solutions i.e., $u > 0$ in Ω and $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$).
From Theorem 3.1 in [29], we know that G is C^1 in $\mathbb{R}^+ \times C$ and that

$$G'_v(\lambda, v) = I - (-\Delta)^{-1}(\lambda f'(v))$$

and

$$G'_\lambda(\lambda, v) = -(-\Delta)^{-1}(f(v)).$$

From Theorem 2.6 in [29], $N(G'_{u_\Lambda}(\Lambda, u_\Lambda))$ is one dimensional and spanned by $\varphi_\Lambda \subset C$. Moreover from the Fredholm Alternative (since G'_v is a compact perturbation of the identity (see Proposition 2.3 [29])) codim $R(G'_{u_\Lambda}) = 1$. In view of applying the implicit function theorem, we now prove that

$$G'_\lambda(\Lambda, u_\Lambda) \notin R(G'_v(\Lambda, u_\Lambda)).$$

For that, we argue by contradiction. Suppose that there exist $w \in C_0^1(\overline{\Omega})$

$$w - (-\Delta)^{-1}\{\lambda f'(v_\Lambda)w\} = -(-\Delta)^{-1}\{f(u_\Lambda)\}$$

from proposition 2.3 in [29], $w \in C^{1,\delta}(\overline{\Omega}) \cap C_0(\overline{\Omega}) \cap C^2(\Omega)$ and

$$-\Delta w = \Lambda f'(u_\Lambda)w - f(u_\Lambda).$$

multiplying by ϕ_Λ, and integrating by parts we get

$$-\int_\Omega f(u_\Lambda)\phi_\Lambda \mathrm{d}x = 0$$

which yields a contradiction since $f(u_\Lambda) > 0$ and $\phi_\Lambda > 0$ in Ω.
We can now apply Theorem 1.7 in [11]. Precisely, letting X any complement of span $\{\phi_\Lambda\}$ in $C_0^1(\overline{\Omega})$ then the solutions of $G(\lambda, v) = 0$ near (Λ, u_Λ) form a curve $(\lambda(s), u(s)) = (\Lambda + \tau(s), u_\Lambda + s\phi_\Lambda + x(s))$ where $s \to (\tau(s), x(s)) \in \mathbb{R} \times X$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $x(0) = x'(0) = 0$. Moreover τ is C^2 near 0.
Indeed, define $g : \mathbb{R} \times \mathbb{R} \times X \to C_0^1(\overline{\Omega})$ by $g(s, \tau, x) \stackrel{\text{def}}{=} G(s + \Lambda, u_\Lambda + s\phi_\Lambda + x)$, $g'_{(\tau,x)}(0, 0, 0)$ is an isomorphism from $\mathbb{R} \times X$ on to $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$. From the Fredholm alternative, it is enough to show that $g'_{(\tau,x)}(0, 0, 0)$ is injective. Indeed since $R(G'_v(\Lambda, u_\Lambda))$ is one codimensional and that $G'_\lambda(\Lambda, u_\Lambda) = (-\Delta)^{-1}(f(u_\Lambda)) \notin R(G'_v(\Lambda, u_\Lambda)$

$$\mathbb{R} \times X \ni (\tilde{\tau}, \tilde{x}) \to \tilde{\tau} G'_\lambda(\Lambda, u_\Lambda) + G'_v(\Lambda, u_\Lambda)\tilde{x}$$

is clearly injective. From the Implicit Function Theorem, we get that there exists $\epsilon > 0$, a neighborhood V of 0 in $\mathbb{R}$ and a unique $h : (-\epsilon, \epsilon) \to V \times X$ a C^2 function such that $h(s) \stackrel{\text{def}}{=} (\tau(s), x(s))$ for $s \in (-\epsilon, \epsilon)$, $\tau(0) = 0$, $x(0) = 0$ and $g(\tau(s), x(s)) = G(\tau(s) + \Lambda, u_\lambda + s\phi_\Lambda + x(s)) = 0$. Differentiating the above expression with respect to s at $s = 0$, we get

$$\tau'(s) G'_\lambda(\Lambda, u_\Lambda) + G'_v(\Lambda, u_\Lambda)(\phi_\Lambda + x'(0))$$

which implies that $\tau'(0) = 0$ and $x'(0) = 0$. Next, we show that $\tau''(0) < 0$. Differentiating again the expression

$$\begin{gathered}\tau'(s) G'_\lambda(\tau(s) + \Lambda, u_\Lambda + s\phi_\Lambda + x(s)) + \\ G'_v(\tau(s) + \Lambda, u_\Lambda + s\phi_\Lambda + x(s))(\phi_\Lambda + x'(s)) = 0\end{gathered}$$

we get for $s = 0$

$$\tau''(0) G'_\lambda(\Lambda, u_\Lambda) + G'_v(\Lambda, u_\Lambda)(x''(0)) + G''_{vv}(\Lambda, u_\Lambda)(\phi_\Lambda, \phi_\Lambda) = 0.$$

Using the dual product with $-\Delta\phi_\Lambda$, we get from the convexity of f that $\tau''(0) < 0$.
We now show that $\mathcal{C}$, the maximal component set in $\mathcal{S} = \{(\lambda, u) \in \mathbb{R}^+ \times C^1(\overline{\Omega}) \cap C_0(\overline{\Omega}) | u \text{ solves } \mathrm{P}_\lambda\}$ of $\cup_{0 \le \lambda \le \Lambda}(\lambda, u_\lambda)$ is unbounded in $\mathbb{R} \times C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$. From above, we have that $\mathcal{C}$ contains $\cup_{(-\epsilon,\epsilon)}(\Lambda + \tau(s), u_\Lambda + s\phi_\Lambda + x(s))$. We argue by contradiction: let us assume that $\mathcal{C}$ is bounded. We use the topological degree of Leray-Schauder to derive the contradiction. Precisely, as in [33] (see Lemma 1.2 and Theorem 3.2), we show that there exists Θ, a bounded open set in $\mathbb{R}^+ \times C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ with $\mathcal{C} \subset \Theta$. Moreover, Θ satisfies

$$\Theta \cap \{0\} \times C^1(\overline{\Omega}) \cap C_0(\overline{\Omega}) \subset \{0\} \times B_1 \quad \text{and} \quad \partial\Theta \cap \mathcal{S} = \emptyset$$

where B_1 denotes the unit ball of $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$. Furthermore, there exists $\lambda_0 > 0$ small enough such that for $0 \leq \lambda \leq \lambda_0$,

$$(\lambda, u) \in \mathcal{S} \Rightarrow \|u\|_{C^1(\overline{\Omega})} < 1.$$

Then, let

$$\Phi(\lambda, u) \stackrel{\text{def}}{=} u - \lambda(-\Delta)^{-1}(f(u)) \stackrel{\text{def}}{=} u - \lambda T(u).$$

Φ is defined from C to $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ and T is a compact operator. Then from above $d(\Phi(\lambda, \cdot), B_1)$ is well defined for $0 \leq \lambda \leq \lambda_0$ where d denotes the Leray Schauder Degree. Moreover, $d(\Phi(0), B_1) = d(I, B_1) = 1$. Let $\Theta_\lambda = \{u \in C^1(\overline{\Omega}) \cap C_0(\overline{\Omega}) \,|\, (\lambda, u) \in \Theta\}$. By the homotopy invariance, $d(\Phi(\lambda), \Theta_\lambda) = 1$ for $\lambda \in (0, \Lambda + \epsilon)$ with $\epsilon > 0$ small enough. Using that there is no solution to (P_λ) for $\lambda > \Lambda$, we get a contradiction. Therefore, $\mathcal{C}$ is unbounded. Now using that $(\lambda, u) \in \mathcal{S} \Rightarrow 0 \leq \lambda \leq \Lambda$, we get the existence of an asymptotic bifurcation point. This achieves the proof of Theorem 4.1.1. $\square$

4.3 The radial symmetric case

We now consider the radial symmetric case, i.e. $\Omega = B_1(0) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^2 | |x| < 1\}$ and solutions satisfying $u(x) = w(|x|)$. We analyze in this case more precisely the corresponding unbounded branch of solutions $\mathcal{C}$ obtained in Theorem 4.1.1. We denote by $\mathcal{C}_r$, the maximal component of $\mathcal{C}$ containing radial symmetric solutions. Using O.D.E. techniques and shooting arguments, we are able to give further informations about the behaviour of $\mathcal{C}_r$ near the asymptotic bifurcation points. Precisely, we prove Theorems 4.1.4, 4.1.7 and 4.1.9. For that, we perform some transformations that will put (P_λ) into the equivalent form of the classical Emden-Fowler equations. The existence of radial symmetric solutions can be obtained by the limit of approximate solutions which are radially symmetric and radially decreasing by Gidas-Ni-Nirenberg [24] result. In particular, the minimal solutions u_λ are radially symmetric. From the corresponding O.D.E., we note that any radially symmetric solution u to (P_λ) is also radially decreasing. Let us denote $\mathcal{C}_r$ the maximal component in $\mathcal{C}$ of radial symmetric solutions. We have that $\mathcal{C}_r$ contains $\cup_{\lambda \leq \Lambda}(\lambda, u_\lambda) \cup_{(-\epsilon,\epsilon)} (\lambda(s), u(s))$ and $\mathcal{C}_r$ is unbounded from above. We first prove that when $\alpha = 1$, $\lambda = 0$ is the unique asymptotic bifurcation point from which Theorem 4.1.3 follows. For this, we use the following Lemma:

Lemma 4.3.1. *Let* $\lambda_0 \in (0, \Lambda]$ *and* Ω' *an open domain such that* $\overline{\Omega'} \subset \Omega$. *Then*

$$\sup_{(\lambda,u)\in\mathcal{S},\lambda\geq\lambda_0} \|u\|_{L^\infty(\Omega')} < +\infty. \tag{4.5}$$

Remark. Lemma 4.3.1 holds in the non symmetric case also.

Proof of Lemma 4.3.1. Let $\tilde{k}(t) \stackrel{\text{def}}{=} f(t)e^{-t}$ and ϕ_0 the positive normalised eigenvector associated to the eigenvalue $\lambda_1(\Omega)$ of $-\Delta$ in Ω with Dirichlet boundary conditions. Then multiplying the equation in (P_λ) by ϕ_0 and integrating by parts we get:

$$\lambda_1(\Omega)\int_\Omega u\phi_0 = \lambda\int_\Omega \tilde{k}(u)e^u\phi_0.$$

From **(H1)** and **(H2)**, there exists $C > 0$ large enough and $0 < \epsilon < \lambda_0$ such that for any $t > 0$, $t \leq C + \epsilon\tilde{k}(t)e^t$. Therefore, for any $(\lambda, u) \in \mathcal{S}$ with $\lambda \geq \lambda_0$ we have

$$(\lambda_0 - \epsilon)\int_\Omega \tilde{k}(u)e^u\phi_0 \leq C$$

which implies that

$$\int_K e^u \leq M(K, \lambda_0, \Omega) < +\infty$$

for any compact subset K of Ω. Using Theorem 4.1.2 with $V(x) \stackrel{\text{def}}{=} \lambda\tilde{k}(u(x))$ and $p = \infty$, we get $u(x)$ is uniformly bounded in Ω' as $(\lambda, u) \in \mathcal{S}$ and $\lambda \geq \lambda_0$. □

Proof of Theorem 4.1.3. From above, $\mathcal{C}_r$ contains radially symmetric and radially decreasing solutions to (P_λ) and is unbounded. Moreover $\Pi_{\mathbb{R}}\mathcal{C}_r \subset [0, \Lambda]$. Then, $\mathcal{C}_r$ admits at least one asymptotic bifurcation point $\tilde{\lambda} \in [0, \Lambda]$. From Lemma 4.3.1 and the fact that u is radially decreasing for $(\lambda, u) \in \mathcal{C}_r$, we deduce that $\lambda = 0$ is the unique asymptotic bifurcation point. Then, Theorem 4.1.3 follows. □

Now, we deal with the case $\alpha > 1$. We will use more carefully the O.D.E. analysis. First, we have

$$(P_\lambda) \qquad \begin{cases} \left.\begin{array}{rcl} -(rw')' &=& \lambda r f(w) \\ w &>& 0 \end{array}\right\} \text{ in } (0,1), \\ \quad w'(0) = w(1) = 0. \end{cases}$$

Therefore, letting $R = \lambda^{\frac{1}{2}}$, (P_λ) can be rewritten as the following ODE boundary value problem via the transformation $w(r) = u(|x| = r)$ for $r \in (0, R)$:

$$(P_R) \qquad \begin{cases} \left.\begin{array}{rcl} -(rw')' & = & rf(w) \\ w & > & 0 \end{array}\right\} \text{ in } (0,R), \\ w'(0) = w(R) = 0. \end{cases}$$

We finally make the following Emden-Fowler transformation

$$y(t) = w(r), \text{ where } r = 2e^{-\frac{t}{2}}, \;\; t \in (2\log(2R^{-1}), \infty).$$

Then, it can be checked that (P_R) is equivalent to the following problem with $T = 2\log(\frac{2}{R})$:

$$\left.\begin{array}{rcl} -y'' & = & e^{-t}f(y) \\ y & > & 0 \end{array}\right\} \text{ in } (T,\infty), \\ y(T) = y'(\infty) = 0.$$

For our purpose, instead of the above boundary value problem, it will be more convenient to consider the following initial-value problem depending upon a parameter $\gamma > 0$:

$$(P_\gamma) \qquad \begin{cases} -y'' = e^{-t}f(y), \\ y(\infty) = \gamma, y'(\infty) = 0. \end{cases}$$

Since $f(y(t)) > 0$ as long as $y(t) > 0$, it follows from (P_γ) that y is a strictly concave function as long as it is positive. Therefore, there exists $T_0(\gamma) > -\infty$ such that $y(T_0(\gamma)) = 0$ and $y(t) > 0$ for all $t > T_0(\gamma)$. $T_0(\gamma)$ thus defined, is clearly the first zero of the solution y of (P_γ) as we move left from infinity. Let $y_0 > 0$ be such that g is convex for all $t > y_0$. We also define the point $t_0(\gamma) > T_0(\gamma)$ to be such that $y(t_0(\gamma)) = y_0$ for each $\gamma > 0$. Due to the continuity of the mapping $\gamma \to T_0(\gamma)$, we only have to know the behaviour of $T_0(\gamma)$ in the two limiting cases $\gamma \to 0$ and $\gamma \to +\infty$. The asymptotic of $T_0(\gamma)$ as $\gamma \to 0$ is given by the following Lemma proved in [2] (see also the generalisation to the N-Laplacian case in [26]):

Lemma 4.3.2. *Assume* **(H1)-(H2)**. *Then* $T_0(\gamma) \to +\infty$ *when* $\gamma \to 0$.

The asymptotic for $T_0(\gamma)$ when $\gamma \to +\infty$ is more delicate. Concerning the case $1 < \alpha < 2$, we have the following result which is a straightforward extension of theorem 3 in [6]. Let $g(u) \overset{\text{def}}{=} \log(f(u))$ and let y_0 large enough such that g is convex in $[y_0, +\infty)$. Then define $m(u) \stackrel{\text{def}}{=} \{g(u) - \frac{1}{2}ug'(u)\} - \frac{1}{2}y_0 g'(u)[e^{\frac{(g(u)-g(y_0)}{2}} - 1]^{-1}$

Theorem 4.3.1. *Let $1 < \alpha < 2$. Suppose h satisfies* **(H1)** *and* **(H2)**. *Then $T_0(\gamma)$ satisfies*

$$T_0(\gamma) > m(\gamma) + log\{\frac{1}{2}g'(\gamma)\} - 1, \tag{4.6}$$

and then $T_0(\gamma) \to +\infty$ as $\gamma \to +\infty$.

From the above result, going back to our original variable $x \in B_R$ and defining $u(x) = y(2\log(\frac{2}{|x|}))$ Theorem 4.1.4 follows. Concerning $\alpha = 2$, we can use the results from [26] which extends ealier results from [3]. Precisely, we have the following results. Concerning class **(H3)**, we have

Proposition 4.3.1. *Let f satisfy the hypotheses* **(H1)-(H2)** *and* **(A3)-(A5)**. *Then,* $\limsup_{\gamma\to\infty} T_0(\gamma) < \infty$.

Concerning classes **(H4)** and **(H5)**, we have

Proposition 4.3.2. *Let f satisfy the hypotheses* **(H1)-(H2)** *and either* **(A6)-(A7)** *or* **(A8)**. *Then,* $\lim_{\gamma\to\infty} T_0(\gamma) = \infty$.

From the above propositions, we prove Theorem 4.1.7 and Theorem 4.1.9:

Proof of Theorem 4.1.7. The existence and the unboundednes of $\mathcal{C}_r$ follow from above. From the asymptotic of $T_0(\gamma)$ as $\gamma \to 0^+$ (Lemma 4.3.2) and as $\gamma \to +\infty$ (see Proposition 4.3.2) and the continuity of the map $\gamma \to T_0(\gamma)$, we get that 0 is the unique asymptotic bifurcation point. To get the blow up behaviour of v_λ, as $\lambda \to 0^+$, we use the asymptotics at specific points as in [3] and [26]. Precisely using for $N = 2$ Lemma 5.1, Lemma 5.2, Lemma 5.3, Lemma 5.5 and Lemma 5.7 in [26], we can extend with similar proofs the results in Section 3 and Section 4 in [3] from which we get the blow up analysis at $\lambda = 0$ (in particular Lemma 3.1, Lemma 3.2, Lemma 3.3 in Section 3 and theorem D' in Section 4 based on Lemma 4.1, Lemma 4.2, Lemma 4.3, Lemma 4.4 and Lemma 4.5). This completes the proof of Theorem 4.1.7. □

Now we prove Theorem 4.1.9.

Proof of Theorem 4.1.9. Assertions (i)-(ii) follow from the above arguments (used in the proof of Theorem 4.1.3). Furthermore from the uniqueness of solutions to (P_λ) for $0 < \lambda$ small (given by Proposition 4.3.1), we get that 0 can not be a bifurcation point. Together with the unboundedness of $\mathcal{C}_r$ this proves (iii). Finally, (iv) follows from Theorem

4.4.1 in Section 4.4. $\square$

4.4 Summary

In this section, we recall some results from [16] used in the proofs of the main results in Section 3 for $\alpha = 2$. Let $R > 0$, $B_R = \{x \in \mathbb{R}^2 |\, |x| < R\}$. Consider

$$(\mathrm{P_R}) \qquad \begin{cases} \left.\begin{array}{rcl} -\Delta u & = & f(u) \\ u & > & 0 \end{array}\right\} \text{ in } B_R, \\ u = 0 \ \text{ on } \partial B_R, u \in C^2_{loc}(B_R). \end{cases}$$

in [16], the following result is proved

Theorem 4.4.1. *Let $f : [0, \infty) \to [0, \infty)$ be a C^3 super-exponential type nonlinearity such that $g \stackrel{\text{def}}{=} \log f$ is convex for all large $t > 0$. Suppose there exists a sequence $\{R_n\}$ of positive real numbers with $R_* \stackrel{\text{def}}{=} \liminf_{n\to\infty} R_n > 0$ and a sequence $\{u_n\}$ of solutions to (P_{R_n}) such that $\sup_{B_{R_n}} u_n \to \infty$ as $n \to \infty$. Then the problem (P_0) posed on B_{R_*} admits a solution that blows up only at the origin.*

This result uses the main result in [7] and the following Theorem proved in [16].

Theorem 4.4.2 (removable singularity)**.** *Let $\Omega' = B_R/\{0\}$ (for a fixed R) and*

$$(\mathrm{P}') \qquad \begin{cases} \left.\begin{array}{rcl} -\Delta u & = & f(u) \\ u & \geq & 0 \end{array}\right\} \text{ in } \Omega', \\ u \in L^\infty_{loc}(\Omega'). \end{cases}$$

Then any solution of P' *extends to a distributional solution to* $(\mathrm{P_R})$*.*

Theorems 4.4.1 and 4.4.2 can be extended to $f(t) = \frac{1}{t^\delta} + h(t)e^{t^2}$ with $0 < \delta < 1$ with minor changes in the proof. We refer to [16] for the detailed proof in the regular case.

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Appendix A

Properties of the p-laplacian

We first recall some properties of the p-laplacian operator defined by

$$\Delta_p u = \nabla(|\nabla u|^{p-2}\,\nabla u) \;\; 1 < p < \infty. \tag{A.1}$$

that appears in serveral situations the p-laplacian in the paradigmatic example of degenerated/singular quasilinear elliptic operator. Notice that if $p = 2$ it becomes the classical laplace operator. In this chapter we deal mainly with the Dirichlet problem

$$(P)\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, u > 0 & \text{in } \;\Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $f \in W^{-1,p'}(\Omega)$ $p' = \frac{p}{p-1}$. The boundary condition will be understood as $u \in W_0^{1,p}(\Omega)$.
We have the following elementary result, as application of the classical calculus of variations (see Peral [14] for the detail).

Theorem A.0.3. *Assume $\Omega \subset \mathbb{R}^N$ a bounded domain $f \in W^{-1,p'}(\Omega)$, then the problem (P) has a solution $u \in W_0^{1,p}(\Omega)$ in the weak sense, namely*

$$\int_\Omega \{\langle |\nabla u|^{p-2}\nabla u, \nabla\Phi\rangle - f\Phi\}\,\mathrm{d}x = 0, \;\; \forall\, \Phi \in W_0^{1,p}(\Omega). \tag{A.2}$$

We are interested in the properties of the inverse operator

$$-(\Delta_p)^{-1} : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$$

and we need the following inequalities (see [10]).

Lemma A.0.1. *let $x, y \in \mathbb{R}^N$ and $\langle .,. \rangle$ the standard product in $\mathbb{R}^N$. Then*

$$\left\langle |x|^{p-2}x - |y|^{p-2}y, x-y \right\rangle \geq \begin{cases} c_p\,|x-y|^p\,, & \text{if } p \geq 2 \\ c_p\frac{|x-y|^2}{(|x|+|y|)^{2-p}}, & \text{if } 1 < p < 2. \end{cases}$$

The main properties of $(-\Delta_p)$ and $(-\Delta_p)^{-1}$ are summarized in the following theorem (see Peral [14] for the detail):

Theorem A.0.4. *Let Ω a bounded domain for $\mathbb{R}^N$.*

i) $\Delta_p : W_0^{1,p}(\Omega) \longrightarrow W_0^{-1,p'}(\Omega)$ *is uniformly continuous on bounded sets.*

ii) $(\Delta_p)^{-1} : W_0^{-1,p'}(\Omega) \longrightarrow W_0^{1,p}(\Omega)$*,and is continuous.*

iii) the composed operator

$$(\Delta_p)^{-1} : W_0^{-1,p'}(\Omega) \longrightarrow W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact if $1 \leq q < \frac{pN}{N-p}$.

Now consider the Dirichlet problems

$$-\nabla \cdot (\mathbf{a}(x, \nabla u)) = f(x) \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega \tag{A.3}$$

$$-\nabla \cdot (\mathbf{a}(x, \nabla v)) = g(x) \text{ in } \Omega; \quad v = 0 \text{ on } \partial\Omega \tag{A.4}$$

We assume that $\mathbf{a}$ satisfies the following *ellipticity* and *growth conditions*:

(H) There exist some constants $\kappa \in [0,1]$, $\gamma, \Gamma \in (0,\infty)$, and $\alpha \in (0,1)$, such that

$$a_i(x,0) = 0; \quad i = 1, \dots, N, \tag{A.5}$$

$$\sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \eta_j}(x,\eta)\cdot \xi_i\xi_j \geq \gamma \cdot (\kappa + |\eta|)^{p-2}\cdot |\xi|^2, \tag{A.6}$$

$$\sum_{i,j=1}^{N} \left|\frac{\partial a_i}{\partial \eta_j}(x,\eta)\right| \leq \Gamma \cdot (\kappa + |\eta|)^{p-2}, \tag{A.7}$$

$$\sum_{i=1}^{N} |a_i(x,\eta) - a_i(y,\eta)| \leq \Gamma \cdot (1 + |\eta|)^{p}\cdot |x-y|^{\alpha}, \tag{A.8}$$

for all $x, y \in \Omega$, all $\eta \in \mathbb{R}^N \setminus \{0\}$, and all $\xi \in \mathbb{R}^N$.

We have the following weak comparison principle for nonnegative weak solutions $u, v \in W_0^{1,p}(\Omega)$ are any weak solutions of the Dirichlet problems (A.3) and (A.4), respectively (see [5], [3] and [6] for the detail).

Lemma A.0.2. *(Weak comparison principle) Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. Assume that $f \leq g$ in $L^{\frac{p}{(p-1)}}(\Omega)$, and $u, v \in W_0^{1,p}(\Omega)$ are any weak solutions of the Dirichlet problems* (A.3) *and* (A.4)*, respectively and assume that a satisfies condition* **(H)***. Then $u \leq v$ holds everywhere in Ω.*

In the following Theorem we recall the result of the strong comparison principle for nonnegative weak solutions $u, v \in W_0^{1,p}(\Omega)$ are any weak solutions of the Dirichlet problems (A.3) and (A.4), respectively (see [5], [3], [6] and [15] for the detail).

Theorem A.0.5. *(Strong comparison principle) Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. Assume that $f \leq g$ in $L^\infty(\Omega)$ be such that $0 \leq f \leq g$ and $f \not\equiv g$ in Ω and $u, v \in W_0^{1,p}(\Omega)$ are any weak solutions of the Dirichlet problems* (A.3) *and* (A.4)*, respectively and assume that a satisfies condition* **(H)***. Then*

$$0 \leq u < v \quad \text{in } \Omega \ \text{ and } \ \frac{\partial u}{\partial \nu} < \frac{\partial v}{\partial \nu} \leq 0 \quad \text{on } \partial\Omega.$$

Appendix B

Boundedness of solutions

The boundedness of the solutions in the case $p \geq N$ is a consequence of Morrey's theorem and the classical bootstrapping.

One of the few general results for problems with critical growth is the following one on $C^{1,\alpha}$ regularity. We will concentrate on the following problem

$$(P)\begin{cases} -\Delta_p u = -div(|\nabla u|^{p-2}\nabla u) = f(x,u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, u > 0 & \text{in } \Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $1 < p < N$ and f is a Carathéodory function and satisfies

$$|f(x,u)| \leq C(1 + |u|^r), \text{ with } r + 1 \leq p^* = \frac{Np}{N-p}, \tag{B.1}$$

that is the critical Sobolev exponent. The regularity of the solution of the problem (P) for $r < p^* - 1$ is a consequence of the results in the paper by Serrin [9] about the L^∞ estimates and the results by Di Benedetto [7] and Tolksdorf [12], for the $C^{1,\alpha}$ regularity. The case $r = p^* - 1$ is much more delicate and will be obtained below, while for the supercritical case the result is not true: in general in the supercritical case a weak solution is not bounded.

The idea of the L^∞ estimate can be found in the argument used by Trudinger in [13] for Yamabe's problem. The main point is to use some nonlinear test functions in the line of the classical Moser method.

If we have that solution of the problem (P) are boundeds then the regularity $C^{1,\alpha}$ is a consequence of the results in [7] or [12]. I would like to remark that a different (Schauder) approach to the regularity of problem

(P), can be seen in Guedda-Véron, [8].

The result of this chapter is the following adaptation of the announced result by TRUDINGER [13] (see PERAL [14] for the detail).

Theorem B.0.6. *Let $u \in W_0^{1,p}(\Omega)$ be a solution of the problem (P). If f verifies* (B.1)*, then $u \in L^\infty(\Omega)$.*

We give the proof of the theorem B.0.6 in the critical case and $f(x,u) = \lambda b(x)|u|^{p-2}u$ given in DRABEK-KUFNER-NICOLOSI [4] based on Moser iterations. Precisely we consider the homogeneous eigenvalue problem

$$-\nabla \cdot (|\nabla u|^{p-2}\nabla u) = \lambda b(x)|u|^{p-2}u \ \text{ in } \ \Omega; \ \ u = 0 \ \text{ on } \ \partial\Omega \tag{B.2}$$

where $b(x)$ is measurable function satisfying

$$b(x) \geq 0 \tag{B.3}$$

for a.e $x \in \Omega$. We assume that either $b \in L^{\frac{q}{q-p}}(\Omega)$ with some q satisfying $p < q < p^*$ or $b \in L^\infty(\Omega)$.

Proof. For $M > 0$ define $v_M(x) = \min\{u(x), M\}$. Let us choose $\varphi = v_M^{\kappa p+1}$ $(\kappa \geq 0)$ in

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi \mathrm{d}x = \lambda_1 \int_\Omega b(x)|u|^{p-2}u\varphi \mathrm{d}x. \tag{B.4}$$

Obviously $\varphi \in X \cap L^\infty(\Omega)$. It follows from (B.4) that

$$(\kappa p + 1)\int_\Omega v_M^{\kappa p}|\nabla v_M|^p \mathrm{d}x = \lambda_1 \int_\Omega b(x)u^{p-1}v_M^{\kappa p+1}\mathrm{d}x. \tag{B.5}$$

Due to the imbedding $X \hookrightarrow L^{p^*}(\Omega)$ we have

$$(\kappa p + 1)\int_\Omega v_M^{\kappa p}|\nabla v_M|^p \mathrm{d}x \geq (\kappa p + 1)\int_\Omega v_M^{\kappa p}|\nabla v_M|^p \mathrm{d}x \tag{B.6}$$

$$= \frac{\kappa p + 1}{(\kappa+1)^p}\int_\Omega v_M^{\kappa p}|\nabla v_M|^p \mathrm{d}x \geq c_3 \frac{\kappa p + 1}{(\kappa+1)^p}\left(\int_\Omega (v_M^{\kappa+1})^{p^*}\mathrm{d}x\right)^{\frac{p}{p^*}}.$$

Hence it follows from (B.3), (B.4), (B.5), (B.6) and the Hölder inequality that

$$\left(\int_\Omega (v_M^{\kappa+1})^{p^*}\mathrm{d}x\right)^{\frac{p}{p^*}} \leq \frac{1}{c_3}\frac{(\kappa+1)^p}{\kappa p + 1}\int_\Omega b(x)u^{p-1}v_M^{\kappa p+1}\mathrm{d}x \tag{B.7}$$

$$\leq \frac{1}{c_3}\frac{(\kappa+1)^p}{\kappa p+1}\left(\int_\Omega b(x)^{\frac{q}{q-p}}\mathrm{d}x\right)^{\frac{q-p}{q}}\left(\int_\Omega u^{(\kappa+1)q}\mathrm{d}x\right)^{\frac{p}{q}}.$$

Due to the assumptions on $b(x)$ we obtain formally

$$\left(\int_\Omega (v_M^{\kappa+1})^{p^*}\mathrm{d}x\right)^{\frac{p}{p^*}} \leq \frac{1}{c_4}\frac{(\kappa+1)^p}{\kappa p+1}\left(\int_\Omega u^{(\kappa+1)q}\mathrm{d}x\right)^{\frac{p}{q}}. \tag{B.8}$$

$$\|v_M\|_{(\kappa+1)p^*} \leq c_5^{\frac{1}{\kappa+1}}\left(\frac{\kappa+1}{(\kappa p+1)^{\frac{1}{p}}}\right)^{\frac{1}{\kappa+1}}\|u\|_{(\kappa+1)q} \tag{B.9}$$

with $c_5 = c_4^{\frac{1}{p}}$. Since $u \in L^{p^*}(\Omega)$, we can choose κ_1 in (B.9) such that $(\kappa_1+1)q = p^*$, i.e. $\kappa = 1 = \frac{p_s^*}{q} - 1$. Then we have

$$\|v_M\|_{(\kappa_1+1)p^*} \leq c_5^{\frac{1}{\kappa_1+1}}\left(\frac{\kappa_1+1}{(\kappa_1 p+1)^{\frac{1}{p}}}\right)^{\frac{1}{\kappa_1+1}}\|u\|_{p^*} \tag{B.10}$$

for any $M > 0$. Due to $u(x) = \lim\limits_{M\to\infty} v_M(x)$, the Fatou Lemma and (B.10) imply

$$\|u\|_{(\kappa_1+1)p^*} \leq c_5^{\frac{1}{\kappa_1+1}}\left(\frac{\kappa_1+1}{(\kappa_1 p+1)^{\frac{1}{p}}}\right)^{\frac{1}{\kappa_1+1}}\|u\|_{p^*}. \tag{B.11}$$

Hence, we can choose κ_2 in (B.9) such that $(\kappa_2+1)q = (\kappa_1+1)p^* = \frac{(p^*)^2}{q}$ and repeating the same argument we get

$$\|u\|_{(\kappa_2+1)p^*} \leq c_5^{\frac{1}{\kappa_2+1}}\left(\frac{\kappa_2+1}{(\kappa_2 p+1)^{\frac{1}{p}}}\right)^{\frac{1}{\kappa_2+1}}\|u\|_{(\kappa_1+1)p^*}. \tag{B.12}$$

By induction we obtain

$$\|u\|_{(\kappa_n+1)p^*} \leq c_5^{\frac{1}{\kappa_n+1}}\left(\frac{\kappa_n+1}{(\kappa_n p+1)^{\frac{1}{p}}}\right)^{\frac{1}{\kappa_n+1}}\|u\|_{(\kappa_{n-1}+1)p^*}. \tag{B.13}$$

for any $n \in \mathbb{N}$, where $(\kappa_n+1) = \left(\frac{p^*}{q}\right)^n$. It follows from (B.11), (B.13) that

$$\|u\|_{(\kappa_n+1)p^*} \leq c_5^{\sum_{k=1}^n \frac{1}{\kappa_k+1}}\left[\left(\frac{\kappa_1+1}{(\kappa_1 p+1)^{\frac{1}{p}}}\right)^{\frac{1}{\sqrt{\kappa_1+1}}}\right]^{\frac{1}{\sqrt{\kappa_1+1}}}. \tag{B.14}$$

$$\left[\left(\frac{\kappa_2+1}{(\kappa_2 p+1)^{\frac{1}{p}}}\right)^{\frac{1}{\sqrt{\kappa_2+1}}}\right]^{\frac{1}{\sqrt{\kappa_2+1}}} \cdots \left[\left(\frac{\kappa_n+1}{(\kappa_n p+1)^{\frac{1}{p}}}\right)^{\frac{1}{\sqrt{\kappa_n+1}}}\right]^{\frac{1}{\sqrt{\kappa_n+1}}} \|u\|_{p^*}.$$

Since $\left(\frac{y+1}{(yp+1)^{\frac{1}{p}}}\right)^{\frac{1}{\sqrt{y+1}}} > 1$ and $\lim_{y\to\infty}\left(\frac{y+1}{(yp+1)^{\frac{1}{p}}}\right)^{\frac{1}{\sqrt{y+1}}} = 1$, there exists $c_6 > 1$ (independent of κ_n) such that

$$\|u\|_{(\kappa_n+1)p^*} \leq c_5^{\sum_{k=1}^n \frac{1}{\kappa_k+1}} c_6^{\sum_{k=1}^n \frac{1}{\sqrt{\kappa_k+1}}} \|u\|_{p^*}. \tag{B.15}$$

However, $\sum_{k=1}^n \frac{1}{\kappa_k+1} = \sum_{k=1}^n \left(\frac{q}{p^*}\right)^k$, $\sum_{k=1}^n \frac{1}{\sqrt{\kappa_k+1}} = \sum_{k=1}^n \left(\sqrt{\frac{q}{p^*}}\right)^k$ and $\frac{q}{p^*} < \sqrt{\frac{q}{p^*}} < 1$. Hence it follows from (B.15) that there exists a constant $c_7 > 0$ such that we get

$$\|u\|_{r_n} \leq c_7\|u\|_{p^*} \tag{B.16}$$

with $r_n = (\kappa_n+1)p^* \to \infty$ when $n \to \infty$. Let us assume that $\|u\|_\infty > c_7\|u\|_{p^*}$. Then there exists $\eta > 0$ and a set $\mathcal{A}$ of positive measure in Ω such that $u(x) \geq c_7\|u\|_{p^*} + \eta$ for $x \in \mathcal{A}$. Applying the Fatou Lemma we get

$$\lim_{r_n\to\infty}\inf\left(\int_\Omega |u(x)|^{r_n}\mathrm{d}x\right)^{\frac{1}{r_n}} \geq \lim_{r_n\to\infty}\inf\left(\int_{\mathcal{A}} |u(x)|^{r_n}\mathrm{d}x\right)^{\frac{1}{r_n}}$$

$$\geq \lim_{r_n\to\infty}\inf(c_7\|u\|_{p^*}+\eta)(meas\mathcal{A})^{\frac{1}{r_n}} = c_7\|u\|_{p^*}+\eta,$$

which contradicts (B.16). Hence

$$\|u\|_\infty \leq c_7\|u\|_{p^*}$$

and the proof is complete $\square$

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