Robust Control and Observation of LPV Time-Delay Systems

C. Briat
PhD. defense, November 27th 2008
GIPSA-lab, Control Systems Department, Grenoble, France

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Co-directeurs: Olivier Sename (Professor INPG, GIPSA-lab)
Jean-François Lafay (Professor, Ecole Centrale Nantes)
Administrative Context

3 years PhD thesis

- Advisors:
  - Olivier Sename (GIPSA-Lab)
  - Jean-François Lafay (IRCCyN)

- 6 months journey in GeorgiaTech (Rhone-Alpes Region scholarship)
  - Work with Erik Verriest
  - "Modeling and Control of Disease Epidemics by Vaccination"
Scientific Context

Nonlinear System
Scientific Context

Nonlinear System \( \rightarrow \) LPV System

Approximation

Parameters

C. Briat - PhD. defense [GIPSA-lab / SLR team]
Scientific Context

- Nonlinear System
- LPV System
- Parameters
- Approximation
- Large Scale System
Scientific Context

Nonlinear System \rightarrow \text{Approximation} \rightarrow \text{LPV System} \rightarrow \text{Parameters}

Large Scale System \rightarrow \text{Approximation} \rightarrow \text{Time-Delay System} \rightarrow \text{Delay}
Scientific Context

- Nonlinear System
- Large Scale System
- LPV System
- Time-Delay System
- LPV Time-Delay System

Parameters → Nonlinear System → Approximation → LPV System
Parameters → Large Scale System → Approximation → Time-Delay System
Delay → Time-Delay System
Delay → LPV Time-Delay System
Scientific Context

- Nonlinear System
- Large Scale System
- LPV System
- Time-Delay System
- LPV Time-Delay System
- Stability Analysis
Introduction

Stability Analysis of LPV Time-Delay Systems

Control of LPV Time-Delay Systems

Scientific Context

Nonlinear System

LPV System

Parameters

Large Scale System

Time-Delay System

Parameters

LPV Time-Delay System

Delay

Stability Analysis

Relaxations

Synthesis Tools
Scientific Context

Nonlinear System \rightarrow LPV System \rightarrow LPV Time-Delay System

Parameters

Large Scale System \rightarrow Time-Delay System \rightarrow LPV Time-Delay System

Delay

Stability Analysis

Relaxations

Synthesis Tools

Control \rightarrow Observation \rightarrow Filtering
Scientific Context

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Stability Analysis of LPV Time-Delay Systems

Control of LPV Time-Delay Systems

Nonlinear System

LPV System

Large Scale System

Time-Delay System

LPV Time-Delay System

Parameters

Delay

Approximation

Thesis

Stability Analysis

Relaxations

Synthesis Tools

Control

Observation

Filtering
Contributions of the Thesis

Stability Results

▶ Conference publications [IFAC World Congress ’08], [ECC07]
▶ Journal submissions IEEE TAC, Systems & Control Letters

Design Methods

▶ Conference publications [IFAC World Congress ’08], [ECC07], [IFAC SSSC’07]
▶ Conference submissions [ECC’09]
▶ Journal submissions IEEE TAC, Systems & Control Letters

Modeling and Control of Disease Epidemics

▶ Conference publication [IFAC World Congress ’08]
▶ Journal Submission [Biomedical Signal Processing and Control]
Outline

1. Introduction
2. Stability of LPV Time-Delay Systems
3. Control of LPV Time-Delay Systems
4. Conclusion & Future Works
1. Introduction
   ▶ Presentation of LPV systems
   ▶ Stability Analysis of LPV systems
   ▶ Control of LPV systems
   ▶ Presentation of time-delay systems
   ▶ Stability Analysis of time-delay systems
   ▶ Control of time-delay systems

2. Stability of LPV Time-Delay Systems

3. Control of LPV Time-Delay Systems

4. Conclusion & Future Works
LPV Systems


\[
\begin{align*}
\dot{x}(t) &= A(\rho(t))x(t) + E(\rho(t))w(t) \\
\rho(t) &\in U_\rho \text{ compact} \\
\dot{\rho}(t) &\in \text{co}\{U_\nu\}
\end{align*}
\]

- Approximation of nonlinear and LTV systems
- Offer interesting solutions for control \(\rightarrow\) gain scheduling
- Semi-active suspensions [Poussot 2008], robotic systems [Kajiwara 1999], turbo-fan engines [Gilbert 2008], and so on...
- Eigenvalues computation of \(A(\rho)\)

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Stability Analysis of LPV Systems

Time vs. Frequency Domain Methods

- Frequency domain analysis ‘inapplicable’
  - Time Domain analysis → Lyapunov theory for LPV systems

\[ V_q(x) = x^T P x(t) \quad V_r(x) = x^T \rho P(\rho)x(t) \]

Quadratic vs. Robust Stability

- Quadratic stability
  - Unbounded parameter variation rates \( \dot{\rho} \in (-\infty, +\infty) \)
  - Necessary Condition: \( \Re\{\lambda(A(\rho))\} < 0, \rho \in U_{\rho} \)

- Robust stability
  - Bounded parameter variation rates
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Control of LPV Time-Delay Systems

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Control of LPV Systems

Types of Controllers

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| Robust Controller    | \( u(t) = K x(t) \)                                                           | \[
\begin{bmatrix}
\dot{x}_c(t) \\
u(t)
\end{bmatrix} = K \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}
\]                                           |
| Gain-Scheduled Controller | \( u(t) = K(\rho) x(t) \)                                               | \[
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Advantages and Drawbacks of LPV Controllers

+ Flexibility
+ Better performance
− Computation
− Implementation
### Control of LPV Systems

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| Gain-Scheduled       | \( u(t) = K(\rho)x(t) \)                                                      | \[
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Control of LPV Systems

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Advantages and Drawbacks of LPV Controllers

+ Flexibility
+ Better performance

– Computation
– Implementation
Introduction

Stability Analysis of LPV Time-Delay Systems

Control of LPV Time-Delay Systems

Time-Delay Systems

Actuators → System → Sensors

Controller

Measurements

Controls
Time-Delay Systems

![Diagram of a control system with sensors, actuators, network, and delays.]
Introduction

Stability Analysis of LPV Time-Delay Systems

Control of LPV Time-Delay Systems

Linear Time-Delay Systems

General Expression

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + A_h x(t - h(t)) + Ew(t) \\
h(t) &= \text{constant/time - varying} \\
h(t) &= \text{bounded/unbounded} \\
\frac{dh(t)}{dt} &= \text{bounded/unbounded}
\end{align*}
\]

+ Approximation of systems with propagation, diffusion or memory phenomena
  - Networks, combustion processes, population growth, disease propagation, price fluctuations. . .
  - Infinite number of eigenvalues
  - Depend on the delay value
Introduction

Stability Analysis of LPV Time-Delay Systems

Control of LPV Time-Delay Systems

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**Linear Time-Delay Systems**

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Stability Analysis of Time-Delay Systems (1)

Two notions of stability

- Delay-independent stability → unbounded delay
- Delay-dependent stability → bounded delay

Frequency Domain Methods [Niculescu 2001, Gu 2003]

- LTI systems
- Constant delays


- LTV, LPV and Nonlinear systems
- Time-varying delays
Two notions of stability

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Stability Analysis of Time-Delay Systems (2)

Extension of Lyapunov Theory

- Lyapunov-Krasovskii & Lyapunov-Razumikhin

Example


\[ V_i = x(t)^T Px(t) + \int_{t-h}^{t} x(\theta)^T Q x(\theta) d\theta \]

- Delay-dependent stability [Han 2005, Gouaisbaut 2006]

\[ V_d = V_i + \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}(\eta)^T R \dot{x}(\eta) d\eta d\theta \]
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### Advantages & Drawbacks of Memory Controllers

+ Flexibility
+ Better performances
- Needs more memory
- Delay supposed to be exactly known
- Problem of delay measurement/estimation [Belkoura]

Robust controllers with uncertain delay?
## Control of Time-Delay Systems

### Controllers

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Robust controllers with uncertain delay?
Delay-robust control of uncertain LPV time-delay systems

\[ u(t) = K(\rho)x(t) + K_h(\rho)x(t - d(t)) \]

- Choice of Lyapunov-Krasovskii functionals
- Derivation of design results
- Delay uncertainties

Design of delay-scheduled state-feedback controllers

\[ u(t) = K(d(t))x(t) \]

- Delay: parameter vs. operator
- Framework
Delay-robust control of uncertain LPV time-delay systems

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Outline

1. Introduction

2. Stability of LPV Time-Delay Systems
   ▶ Presentation of LPV time-delay systems
   ▶ Choice of Lyapunov-Krasovskii functional
   ▶ Reduction of conservatism

3. Control of LPV Time-Delay Systems

4. Conclusion & Future Works
Example of LPV Time-Delay System

Cutting Process [Zhang 2002]

- Nonlinearities
- Delay: time between two successive passes of the blades

\[
\begin{aligned}
\dot{x}(t) &= A(\rho)x(t) + A_h(\rho)x(t - h(t)) + E(\rho)w(t) \\
\rho &\in U_ho \\
\dot{\rho} &\in co\{U_\nu\} \\
h(t) &\in [0, h_{max}] \\
\left| \frac{dh(t)}{dt} \right| &\leq \mu < 1
\end{aligned}
\]  

Objectives

- Efficient stability tests
- Efficient design tools
- Tackle delay uncertainties
Introduction

Stability Analysis of LPV Time-Delay Systems

Control of LPV Time-Delay Systems

LPV Time-Delay Systems


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\end{align*}
\] (1)

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LPV Time-Delay Systems


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Introduction
Stability Analysis of LPV Time-Delay Systems
Control of LPV Time-Delay Systems
Choice of the Lyapunov-Krasovskii functional

Criteria

- Simple form (few decision matrices, small size of LMIs)
- Avoid model-transformations
- 'Good' results (estimation of delay margin, system norms...)
- Stability over an interval of delay values
- Parameter dependent
Stability Condition

Generalization of [Han 2005, Gouaisbaut 2006] to the LPV case

\[ V = x(t)^T P(\rho)x(t) + \int_{t-h(t)}^{t} x(\theta)^T Q x(\theta) d\theta + h_{\text{max}} \int_{-h_{\text{max}}}^{0} \int_{t+\theta}^{t} \dot{x}(s)^T R \dot{x}(s) ds d\theta \]

- Used along with Jensen's inequality [Han 2005, Gouaisbaut 2006]

Theorem

The LPV Time-delay system (1) is asymptotically stable if there exists \( P(\rho), Q, R > 0 \) such that the LMI

\[
\begin{bmatrix}
A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q - R + \frac{\partial P(\rho)}{\partial \rho} \nu & P(\rho)A_h(\rho) + R & h_{\text{max}} A(\rho)^T R \\
* & -(1-\mu)Q - R & h_{\text{max}} A_h(\rho)^T R \\
* & * & -R
\end{bmatrix} < 0
\]

holds for all \( \rho \in U_\rho \) and \( \nu \in U_\nu \).
Introduction Stability Analysis of LPV Time-Delay Systems

Stability of LPV Time-Delay Systems

Stability Condition

Generalization of [Han 2005, Gouaisbaut 2006] to the LPV case

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V = x(t)^T P(\rho) x(t) + \int_{t-h(t)}^{t} x(\theta)^T Q x(\theta) d\theta + h_{\text{max}} \int_{-h_{\text{max}}}^{0} \int_{t+\theta}^{t} \dot{x}(s)^T R \dot{x}(s) ds d\theta
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- Used along with Jensen's inequality [Han 2005, Gouaisbaut 2006]

Theorem

The LPV Time-delay system (1) is asymptotically stable if there exists \( P(\rho), Q, R > 0 \) such that the LMI

\[
\begin{bmatrix}
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Example - LTI case

LTI system with constant delay

\[
\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} x(t - h)
\]

Comparison with existing results

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+ Computational complexity
+ Competitive

→ Gap → Conservatism
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- Computational complexity
- Competitive
- Gap $\rightarrow$ Conservatism
Origin of Conservatism

- Constant matrices $Q, R$

$$V(x_t) = x(t)^TP(\rho)x(t) + \int_{t-h(t)}^{t} x(\theta)^TQx(\theta)d\theta + h_{max}\int_{-h_{max}}^{0} \int_{t+\theta}^{t} \dot{x}(s)^TR\dot{x}(s)dsd\theta$$

- Jensen’s inequality
  - Bound of an integral term over a finite interval
  - For illustration: Conservatism $\equiv$ surface between curves
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Introduction

Stability Analysis of LPV Time-Delay Systems

Control of LPV Time-Delay Systems

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Generalization of the functional [Han 2008]

\[ V = x(t)^T P(\rho)x(t) + \int_{t-h(t)}^{t} x(\theta)^T Q(\theta)x(\theta) d\theta + \int_{-h_{max}}^{0} \int_{t-\theta}^{t} \dot{x}(s)^T R(\theta)\dot{x}(s) ds d\theta \]

Discretization

- \( Q(\cdot), R(\cdot) \): piecewise constant continuous [Gu 2001, Han 2008]

\[ V = x(t)^T P(\rho)x(t) + \sum_{i=0}^{N-1} \int_{t-(i+1)h(t)/N}^{t-ih(t)/N} x(\theta)^T Q_i x(\theta) d\theta \]

\[ + \frac{h_{max}}{N} \sum_{i=0}^{N-1} \int_{-(i+1)h_{max}/N}^{-ih_{max}/N} \int_{t+\theta}^{t} \dot{x}(s)^T R_i \dot{x}(s) ds d\theta \]

Synergy of fragmentation and discretization
Reduction of Conservatism

Generalization of the functional [Han 2008]

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+ \frac{h_{\text{max}}}{N} \sum_{i=0}^{N-1} \int_{-(i+1)h_{\text{max}}/N}^{-ih_{\text{max}}/N} \int_{t+\theta}^{t} \dot{x}(s)^T R_i \dot{x}(s)dsd\theta
\]

Synergy of fragmentation and discretization
**Theorem**

The LPV Time-delay system (1) is asymptotically stable if there exists $P(\rho), Q_1, Q_2, R_1, R_2 > 0$ such that the LMI

$$
\begin{bmatrix}
\mathcal{M}_{11}(\rho, \dot{\rho}) & R_1 & P(\rho)A_h(\rho) & \frac{h_{\text{max}}}{2} A(\rho)^T R_1 \\
\star & U_1 & R_2 & 0 \\
\star & \star & U_2 & \frac{h_{\text{max}}}{2} A_h(\rho)^T R_1 \\
\star & \star & \star & -R_1 \\
\star & \star & \star & \star
\end{bmatrix} \prec 0
$$

holds for all $\rho \in U_\rho$ and $\nu \in U_\nu$ with

$$
\mathcal{M}_{11}(\rho, \dot{\rho}) = A(\rho)^T P(\rho) + P(\rho)A(\rho) + Q_1 - R_1 + \frac{\partial P(\rho)}{\partial \rho} \nu
$$

$$
U_1 = -Q_1 + Q_2 - R_1 - R_2
$$

$$
U_2 = -Q_2 - R_2
$$
### Example - LTI case (cont’d)

#### Comparison with existing method using fragmentation [Peaucelle et al. 2007]

<table>
<thead>
<tr>
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<td>81</td>
<td>–</td>
</tr>
<tr>
<td>$N = 1$</td>
<td>4.4721</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>$N = 2$</td>
<td>5.1775</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>$N = 3$</td>
<td>5.9678</td>
<td>21</td>
<td>27</td>
</tr>
<tr>
<td>$N = 4$</td>
<td>6.0569</td>
<td>27</td>
<td>42</td>
</tr>
<tr>
<td>$N = 9$</td>
<td>6.149</td>
<td>57</td>
<td>177</td>
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<tr>
<td>$N = 30$</td>
<td>6.171</td>
<td>183</td>
<td>1836</td>
</tr>
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<td>Theoretical</td>
<td>6.172</td>
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- Zhang et al. 2000: constant time-delays only
- Approach of Peaucelle et al. based on translation of the state
1. Introduction
2. Stability of LPV Time-Delay Systems
3. Control of LPV Time-Delay Systems
   ▶ Principle of delay-robust stabilization
   ▶ Stabilization test - Relaxations
   ▶ Example
4. Conclusion & Future Works
**Principle of delay-robust stabilization**

**Nominal stabilization of LPV time-delay systems**

- Gain-scheduled memoryless controller:
  \[ u(t) = K_0(\rho)x(t) \]

- Gain-scheduled exact memory controller:
  \[ u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t)) \]

**Delay-robust stabilization of LPV time-delay systems**

- Gain-scheduled approximate memory controller:
  \[ u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - d(t)) \quad \text{with} \quad |h(t) - d(t)| \leq \delta \]

Few studied in the literature
Principle of delay-robust stabilization

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Few studied in the literature
Recall of stability test for $N = 1$

**Theorem**

System (1) is asymptotically stable if there exists $P(\rho), Q, R \succ 0$ such that the LMI

$$
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holds for all $\rho \in U_\rho$ and $\nu \in U_\nu$.

**Goal**

- Derive efficient design results
- Tackle delay uncertainty
Stabilization test for $N = 1$

Stability Analysis of LPV Time-Delay Systems

System and Controller

$$\dot{x}(t) = A(\rho)x(t) + A_h(\rho)x(t - h(t)) + B(\rho)u(t)$$
$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t))$$

Theorem

The system is asymptotically stabilizable by a control law with exact memory if there exists $P(\rho), Q, R > 0$ and $K_0(\rho), K_h(\rho)$ such that the LMI

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holds for all $\rho \in \mathcal{U}_\rho$ and $\nu \in \mathcal{U}_\nu$ with $A_{cl} = A + BK_0$ and $A_{hcl} = A_h + BK_h$.

Convexity

- Bilinear matrix inequality $\rightarrow$ non-convex
- Single terms in $R$ and multiple products $\rightarrow$ Linearization not possible !!
Stabilization test for $N = 1$

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\[
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A_{cl}^T R & \star \\
\star & -R
\end{bmatrix}
\]

\[
\preceq 0
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Stabilization test for $N = 1$

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Preliminary Relaxations

Common Relaxations

- Remove single terms in $R$
  - Avoid Jensen’s inequality
    - High increase of conservatism
- Set $P(\rho) = \varepsilon(\rho)R$
  - Difficult choice of $\varepsilon(\rho)$
  - High increase of conservatism
  - Increase of computational complexity

Relaxation of [Briat. IFAC World Congress 2008]

- Use of adjoint system and projection lemma
  - Non conservative
  - Nonlinear optimization problem (expensive, local convergence)

High increase of conservatism and/or computational complexity
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High increase of conservatism and/or computational complexity
Proposed Relaxation

Origin of the Problem

- Substitution of the closed-loop but convexity not preserved
- Relaxation done after substitution

Proposed Methodology

- Test modification → ‘convexity preserving’ form
  - Relaxation done before substitution
  - Orientation of the relaxation

Relaxation features

- Decoupling multiple matrix products
  - Introduction of a new variable
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Relaxed stability test for $N = 1$

**Theorem**

System (1) is asymptotically stable if there exists $P(\rho), Q, R \succ 0, X(\rho)$ and $K_0(\rho), K_h(\rho)$ such that the LMI

$$
\begin{bmatrix}
-X(\rho) - X(\rho)^T & X(\rho)^T A(\rho) + P(\rho) & X(\rho)^T A_h(\rho) & X(\rho)^T & h_{\max} R \\
* & -P(\rho) + Q - R + \dot{P} & R & 0 & 0 \\
* & * & -(1 - \mu)Q - R & 0 & 0 \\
* & * & * & -P(\rho) & -h_{\max} R \\
* & * & * & * & -R
\end{bmatrix} \prec 0
$$

holds for all $\rho \in U_\rho$ and $\nu \in U_\nu$.

- Additional variable $X(\rho)$
- No multiple products anymore
Relaxed stabilization test for $N = 1$

- Stabilization of system (1) by an exact memory control law:

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t))$$ (2)

- After some manipulations...

**Theorem**

System (1) is stabilizable using (2) if there exists $P(\rho), Q, R \succ 0, X$ and $Y_0(\rho), Y_h(\rho)$ such that the LMI

$$
\begin{bmatrix}
-X - X^T & A(\rho)X + B(\rho)Y_0(\rho) + P(\rho) & A_h(\rho)X + B(\rho)Y_h(\rho) & X^T & \tilde{R} \\
* & -P(\rho) + Q - R + \dot{P}(\rho) & 0 & 0 & 0 \\
* & * & -(1 - \mu)Q - R & 0 & 0 \\
* & * & * & \dot{R} & -R \\
* & * & * & * & -R
\end{bmatrix} \preceq 0
$$

holds for all $\rho \in U_\rho$ and $\nu \in U_\nu$ with $\tilde{R} = h_{max}R$.

Suitable controller gains are given by $K_0(\rho) = Y_0(\rho)X^{-1}$ and $K_h(\rho) = Y_h(\rho)X^{-1}$. 

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Relaxed stabilization test for $N = 1$

- Stabilization of system (1) by an exact memory control law:

$$u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t))$$  \hspace{1cm} (2)

- After some manipulations...

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$$\begin{bmatrix}
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* & -P(\rho) + Q - R + \hat{P}(\rho) & R & 0 & 0 \\
* & * & -(1 - \mu)Q - R & 0 & 0 \\
* & * & * & P(\rho) & -\tilde{R} \\
* & * & * & * & -R
\end{bmatrix} \prec 0$$

holds for all $\rho \in U_{\rho}$ and $\nu \in U_{\nu}$ with $\tilde{R} = h_{max}R$.

Suitable controller gains are given by $K_0(\rho) = Y_0(\rho)X^{-1}$ and $K_h(\rho) = Y_h(\rho)X^{-1}$. 
Example (1)

LPV time-delay system [Zhang et al., 2005]

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 + 0.1\rho(t) \\ -2 & -3 + 0.2\rho(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.2\rho(t) \\ 0.1 + 0.1\rho(t) \end{bmatrix} u(t) \\
&\quad + \begin{bmatrix} 0.2\rho(t) \\ -0.2 + 0.1\rho(t) \end{bmatrix} x(t - h(t)) + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w(t) \\
z(t) &= \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) \\
\rho(t) &= \sin(t)
\end{align*}
\]

Goal

Find a controller such that such that the closed-loop system

1. is asymptotically stable for all \( h(t) \in [0, h_{\text{max}}] \) with \( |\dot{h}(t)| \leq \mu < 1 \) and
2. satisfies

\[ ||z||_{\mathcal{L}_2} \leq \gamma ||w||_{\mathcal{L}_2} \]
Example (1)

- LPV time-delay system [Zhang et al., 2005]

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix}
0 & 1 + 0.1 \rho(t) \\
-2 & -3 + 0.2 \rho(t)
\end{bmatrix} x(t) + \begin{bmatrix}
0.2 \rho(t) \\
0.1 + 0.1 \rho(t)
\end{bmatrix} u(t) \\
&\quad + \begin{bmatrix}
0.2 \rho(t) \\
-0.2 + 0.1 \rho(t)
\end{bmatrix} x(t - h(t)) + \begin{bmatrix}
-0.2 \\
-0.2
\end{bmatrix} w(t) \\
\end{align*}
\]

\[
\begin{align*}
z(t) &= \begin{bmatrix}
0 & 10 \\
0 & 0
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
0.1
\end{bmatrix} u(t) \\
\end{align*}
\]

\[
\rho(t) = \sin(t)
\]

Goal

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\[
||z||_2 \leq \gamma ||w||_2
\]
Example (2)

Case 1: \( \dot{h}(t) \leq 0.5, \ h(t) \in [0, 0.5] \)

- Design of a memoryless state-feedback control law

\[
u(t) = K_0(\rho)x(t)
\]

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\[
K_0(\rho) = \begin{bmatrix}
-1.0535 - 2.9459\rho + 1.9889\rho^2 \\
-1.1378 - 2.6403\rho + 1.9260\rho^2
\end{bmatrix}^T
\]

- Better performances
- Lower controller gains
- Lower numerical complexity
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Example (3)

Case 2: \( \dot{h}(t) \leq 0.9, h(t) \in [0, 10] \)

- Synthesis of both memoryless and exact memory controllers

\[
\begin{align*}
  u(t) &= K_0(\rho)x(t) \\
  u(t) &= K_0(\rho)x(t) + K_h(\rho)x(t - h(t))
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- Delayed term important
- Needs the exact value of the delay at any time
- Problem of delay estimation [Belkoura et al. 2008]

Robust synthesis w.r.t. delay uncertainty on implemented delay
Case 2: $\dot{h}(t) \leq 0.9, \, h(t) \in [0, 10]$

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$$u(t) = K_0(\rho)x(t) \quad u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - h(t))$$

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Robust synthesis w.r.t. delay uncertainty on implemented delay
Delay-Robust Controllers (1)

System and Controller

\[ \dot{x}(t) = A(\rho)x(t) + A_h(\rho)x(t - h(t)) + B(\rho)u(t) \]
\[ u(t) = K_0(\rho)x(t) + K_h(\rho)x(t - d(t)) \]

with \( |d(t) - h(t)| \leq \delta \).

Objectives

▶ Given maximal error \( \delta \) on the delay knowledge, find a controller such that the closed-loop system

1. is asymptotically stable for all \( h(t) \in [0, h_{\text{max}}] \) with \( |\dot{h}(t)| \leq \mu < 1 \), \( |d(t) - h(t)| \leq \delta \) and
2. satisfies the input/output relationship

\[ ||z||_{L_2} \leq \gamma ||w||_{L_2} \]
Introduction Stability Analysis of LPV Time-Delay Systems

Control of LPV Time-Delay Systems

Delay-Robust Controllers (1)

System and Controller

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- Given maximal error \(\delta\) on the delay knowledge, find a controller such that the closed-loop system
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\[
\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2}
\]
Delay-Robust Controllers (2)

Closed-loop system

- System with two constrained delays

\[
\dot{x}(t) = A_{cl}(\rho)x(t) + A_h(\rho)x(t - h(t)) + B(\rho)K_h(\rho)x(t - d(t))
\]

\[
A_{cl}(\rho) = A(\rho) + B(\rho)K_0(\rho)
\]

with \(|d(t) - h(t)| \leq \delta\)

Model Transformation

\[
\nabla(\eta) := \frac{1}{\delta} \int_{t-h(t)}^{t-d(t)} \eta(s) ds
\]

- Linear dynamical time-varying operator \(\|\nabla\|_{\mathcal{L}_2 - \mathcal{L}_2} \leq 1\)

\[
\nabla(\dot{x}) = \frac{1}{\delta}(x(t - d(t)) - x(t - h(t))) \Rightarrow x(t - h(t)) = x(t - d(t)) + \delta \nabla(\dot{x})
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How to consider the relation between the delays?

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with $|d(t) - h(t)| \leq \delta$

How to consider the relation between the delays?

Model Transformation

$$\nabla(\eta) := \frac{1}{\delta} \int_{t-h(t)}^{t-d(t)} \eta(s)ds$$

- Linear dynamical time-varying operator $||\nabla||_{\mathcal{L}_2-\mathcal{L}_2} \leq 1$

$$\nabla(\dot{x}) = \frac{1}{\delta}(x(t - d(t)) - x(t - h(t))) \Rightarrow x(t - h(t)) = x(t - d(t)) + \delta \nabla(\dot{x})$$
Transformed Closed-Loop System

\[ \dot{x}(t) = A_{cl}(\rho)x(t) + A_{hcl}(\rho)x(t - d(t)) + \delta A_h w_0(t) \]

\[ z_0(t) = \dot{x}(t) \]

\[ w_0(t) = \nabla(z_0(t)) \]

- Uncertain system with one delay
- System stable for if
  - nominal system stable ($\delta = 0$)
  - $\|z_0\|_{L_2} < \|w_0\|_{L_2}$ for $\delta \neq 0$ (small gain)
Example (1)

- Previous results: $\gamma = 12.8799$ (Memoryless), $\gamma = 4.1641$ (Exact Memory)

**Delay-robust synthesis**

**Fig.:** Best $\mathcal{L}_2$ performance $\gamma$ vs. maximal error uncertainty $\delta$
Example (1)

- Previous results: $\gamma = 12.8799$ (Memoryless), $\gamma = 4.1641$ (Exact Memory)

Delay-robust synthesis

**Fig.**: Best $\mathcal{L}_2$ performance $\gamma$ vs. maximal error uncertainty $\delta$

- Characterization of intermediate performances
- Direct generalization of the previous approach
Towards Delay-Scheduled Controllers (1)

Drawbacks of memory controllers

- Memory size (store past values)
- Implementing time-varying delays

Delay-scheduled controllers

\[ u(t) = K(\rho, h(t))x(t) \]

- Still using delay information
- Less memory
- Difficult synthesis
Towards Delay-Scheduled Controllers (1)

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Towards Delay-Scheduled Controllers (2)

Model Transformations

\[
\nabla_1(\eta) = \int_{t-h(t)}^{t} \frac{1}{h(s) + h_{\text{max}} + h_{\text{min}}} \eta(s) ds
\]

Comparison Models

\[
\begin{align*}
\dot{x}(t) &= (A + A_h)x(t) - A_h w_0(t) \\
z_0(t) &= (h(t) + h_{\text{max}} - h_{\text{min}}) \dot{x}(t) \\
w_0(t) &= \nabla_1(x(t))
\end{align*}
\]
Towards Delay-Scheduled Controllers (2)

Model Transformations

\[ \nabla_2(\eta) = \sqrt{\frac{1}{h(t)h_{\text{max}}}} \int_{t-h(t)}^{t} \eta(s)ds \]

Comparison Models

\[
\begin{align*}
\dot{x}(t) &= (A + A_h)x(t) - A_h \sqrt{h(t)h_{\text{max}}} w_0(t) \\
z_0(t) &= \dot{x}(t) \\
w_0(t) &= \nabla_2(x(t))
\end{align*}
\]
Outline

1. Introduction
2. Stability of LPV Time-Delay Systems
3. Control of LPV Time-Delay Systems
4. Conclusion & Future Works
Conclusion

- Methodology to derive stabilization results from stability results
- Based on LMI relaxation
- Generalizes to discretized versions of Lyapunov-Krasovskii functionals
- Synthesis of memoryless and memory controllers
- Synthesis of delay-robust controllers using either a adapted functional or (scaled) small gain results.
Future works

- Improve the results for system with time-varying delays
- Generalize to system with non small-delays \((h_{\text{min}} > 0)\)
- Develop new model transformations for delay-scheduled controller synthesis
- Enhance results on delay-scheduled controllers
Thank you for your attention

Vi ringrazio per l’attenzione

Merci de votre attention
\[ \int_0^{+\infty} \int_{t-h(t)}^{t} \phi(t) \eta(s) d\sigma dt = \int_0^{+\infty} \int_{q(s)}^{s} \phi(t) \eta(s) d\sigma ds \]

with \( q := p^{-1} \).
Existence and Unicity of controller/observers (1)

Synthesis problem

Find $Z(\rho), \mathcal{X}(\rho)$ such that

$$\Psi(\rho, \dot{\rho}, \mathcal{X}(\rho)) + \mathcal{U}(\rho)Z(\rho)\mathcal{V}(\rho) + (\star)^T \prec 0$$

holds for all $(\rho, \dot{\rho}) \in I_\rho \times \text{co}\{U_\nu\}$.

Controller existence - Projection Lemma

$$\text{Ker}[\mathcal{U}(\rho)]\Psi(\rho, \dot{\rho})\text{Ker}[\mathcal{U}(\rho)]^T \prec 0 \quad \text{Ker}[\mathcal{V}(\rho)]^T\Psi(\rho, \dot{\rho})\text{Ker}[\mathcal{V}(\rho)] \prec 0$$

Controller construction

- Implicit
- Explicit [Iwasaki] : $Z = f(\mathcal{U}, \mathcal{V}, \Psi, M)$ for every matrix $M \in \mathcal{M}$ to be chosen