Numerical modeling of anisothermal multi-phase flows in petroleum wellbore and reservoir

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Motivations

Optical fiber sensors

- Increase of subsea wellheads and highly deviated wells
  → Production log are less easy to be performed
- Emerging of new technologies such as optical fiber sensors
- Temperature measurements continuous in time and all along the well
**Motivations**

**Optical fiber sensors**

- Increase of subsea wellheads and highly deviated wells
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**Possible applications**

- Estimate virgin reservoir temperature
- Predict flow profiles and the flow rate of each layer
Motivations

Optical fiber sensors

- Increase of subsea wellheads and highly deviated wells
  - Production log are less easy to be performed
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Interpretation of temperature profiles

→ Need for an EXHAUSTIVE ENERGY EQUATION
Mass conservation law

Single phase flow:

$$\frac{\partial (\phi \rho)}{\partial t} + \nabla \cdot (\rho u) = 0$$
Mass conservation law

Single phase flow:

\[ \frac{\partial (\phi \rho)}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \]

Multi-component multi-phase flow:

\[ \sum_{p=1}^{n_p} \left( \frac{\partial}{\partial t} (\phi_s \rho_p y_{c,p}) + \nabla \cdot (\rho_p y_{c,p} \mathbf{u}_p) \right) = 0, \quad c = 1, \ldots, n_c \]

\( n_c \) is the number of components
\( n_p \) is the number of phases
Mass conservation law

Single phase flow :

\[
\frac{\partial (\phi \rho)}{\partial t} + \nabla \cdot (\rho u) = 0
\]

Darcy’s law : \( u = -\mu^{-1}K(\nabla p - \rho g) \)

Multi-component multi-phase flow :

\[
\sum_{p=1}^{n_p} \left( \frac{\partial}{\partial t} (\phi S_p \rho_p y_c p) + \nabla \cdot (\rho_p y_c p u_p) \right) = 0, \quad c = 1, \ldots, n_c
\]

\( n_c \) is the number of components
\( n_p \) is the number of phases
Mass conservation law

Single phase flow:

\[ \frac{\partial (\varphi \rho)}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \]

Darcy’s law: \[ \mathbf{u} = -\mu^{-1}K(\nabla p - \rho g) \]

Multi-component multi-phase flow:

\[ \sum_{p=1}^{n_p} \left( \frac{\partial}{\partial t} (\varphi S_p \rho_p y_{c,p}) + \nabla \cdot (\rho_p y_{c,p} \mathbf{u}_p) \right) = 0, \quad c = 1, \ldots, n_c \]

\(n_c\) is the number of components
\(n_p\) is the number of phases

Generalized Darcy’s law: \[ \mathbf{u}_p = -k_{rp} \mu_p^{-1} K (\nabla p_p - \rho_p g) \]
Numerical modeling of thermomechanical multi-phase flows, well-porous medium

Mass conservation law

Single phase flow:
\[
\frac{\partial (\phi \rho)}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

Darcy’s law:
\[
\mathbf{u} = -\mu^{-1} \mathbf{K} (\nabla p - \rho \mathbf{g})
\]

Darcy-Forchheimer law:
\[
\mu \mathbf{K}^{-1} \mathbf{u} + F\rho |\mathbf{u}| \mathbf{u} = -(\nabla p - \rho \mathbf{g})
\]

Multi-component multi-phase flow:
\[
\sum_{p=1}^{n_p} \left( \frac{\partial}{\partial t} (\phi S_p \rho_p y_{c,p}) + \nabla \cdot (\rho_p y_{c,p} \mathbf{u}_p) \right) = 0, \quad c = 1, \ldots, n_c
\]

\(n_c\) is the number of components
\(n_p\) is the number of phases

Generalized Darcy’s law:
\[
\mathbf{u}_p = -k_{rp} \mu_p^{-1} \mathbf{K} (\nabla p_p - \rho_p \mathbf{g})
\]
**Energy conservation law**

Single phase flow:

\[
\frac{\partial (\rho E)}{\partial t} = -\nabla \cdot (\rho E \mathbf{u}) + \nabla \cdot (\lambda \nabla T) - \nabla \cdot (p \mathbf{u}) + \nabla \cdot (\tau \mathbf{u})
\]

- **Total energy**: \( E = E_c + E_p + \mathcal{U} \)
Energy conservation law

Single phase flow:

\[ \frac{\partial (\rho E)}{\partial t} = -\nabla \cdot (\rho E \mathbf{u}) + \nabla \cdot (\lambda \nabla T) - \nabla \cdot (p \mathbf{u}) + \nabla \cdot (\tau \mathbf{u}) \]

Kinetic energy:

\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho \mathbf{u}^2 \right) + \nabla \cdot \left( \frac{1}{2} \rho \mathbf{u} \mathbf{u} \right) = \rho g \mathbf{u} - \mathbf{u} \cdot \nabla p + \mathbf{u} \cdot (\nabla : \mathbf{T}) \]
**Energy conservation law**

**Single phase flow :**

\[
\frac{\partial (\rho E)}{\partial t} = -\nabla \cdot (\rho E \mathbf{u}) + \nabla \cdot (\lambda \nabla T) - \nabla \cdot (p \mathbf{u}) + \nabla \cdot (\tau \mathbf{u})
\]

\[
\leftrightarrow \quad \frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \mathbf{u}) + p \nabla \cdot \mathbf{u} - \nabla \cdot (\lambda \nabla T) - \tau : \nabla \cdot \mathbf{u} = 0
\]

- Kinetic energy : \[
\frac{d}{dt} \left( \frac{1}{2} \rho |\mathbf{u}|^2 \right) + \nabla \cdot \left( \frac{1}{2} \rho |\mathbf{u}|^2 \mathbf{u} \right) = \rho g \mathbf{u} - \mathbf{u} \cdot \nabla p + \mathbf{u} \cdot (\nabla : \tau)
\]
**Energy conservation law**

**Single phase flow:**

\[
\frac{\partial (\rho E)}{\partial t} = -\nabla \cdot (\rho E \mathbf{u}) + \nabla \cdot (\lambda \nabla T) - \nabla \cdot (p \mathbf{u}) + \nabla \cdot (\tau \mathbf{u})
\]

\[\leftrightarrow \quad \frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \mathbf{u}) + p \nabla \cdot \mathbf{u} - \nabla \cdot (\lambda \nabla T) - \tau : \nabla \cdot \mathbf{u} = 0\]

- Kinetic energy: \( \frac{d}{dt}\left(\frac{1}{2} \rho \mathbf{u}^2\right) + \nabla \cdot \left(\frac{1}{2} \rho \mathbf{u}^2 \mathbf{u}\right) = \rho g \mathbf{u} - \mathbf{u} \cdot \nabla p + \mathbf{u} \cdot (\nabla \cdot \tau)\)

- Mass conservation: \( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0\)

- Enthalpy expression: \( \rho \mathbf{U} = \rho H - p\)

\[\leftrightarrow \quad \rho \frac{\partial H}{\partial t} - \frac{\partial p}{\partial t} + \rho \mathbf{u} \cdot \nabla H - \mathbf{u} \cdot \nabla p - \nabla \cdot (\lambda \nabla T) - \tau : \nabla \cdot \mathbf{u} = 0\]
Energy conservation law

Single phase flow:

\[
\frac{\partial (\rho E)}{\partial t} = -\nabla \cdot (\rho E \mathbf{u}) + \nabla \cdot (\lambda \nabla T) - \nabla \cdot (p \mathbf{u}) + \nabla \cdot (\tau : \mathbf{u})
\]

\[
\leftrightarrow \quad \frac{\partial (\rho U)}{\partial t} + \nabla \cdot (\rho U \mathbf{u}) + p \nabla \cdot \mathbf{u} - \nabla \cdot (\lambda \nabla T) - \tau : \nabla \cdot \mathbf{u} = 0
\]

\[
\rho \frac{\partial H}{\partial t} - \frac{\partial p}{\partial t} + \rho \mathbf{u} \cdot \nabla H - \mathbf{u} \cdot \nabla p - \nabla \cdot (\lambda \nabla T) - \tau : \nabla \cdot \mathbf{u} = 0
\]

\[
\left\{ \begin{array}{l}
\frac{\partial H}{\partial T} = \frac{1-\beta T}{\beta} \\
\frac{\partial H}{\partial p} = \frac{(\rho c)_f}{\rho} \end{array} \right. \quad \beta \text{ coefficient of thermal expansion} \quad (\rho c)_f \text{ specific heat capacity of the fluid}
\]

\[
\leftrightarrow \quad (\rho c)_* \frac{\partial T}{\partial t} + (\rho c)_f \mathbf{u} \cdot \nabla T - \nabla \cdot (\lambda \nabla T) - \phi \beta T \frac{\partial p}{\partial t} - \beta T \mathbf{u} \cdot \nabla p - \tau : \nabla \cdot \mathbf{u} = 0
\]
Energy conservation law

Single phase flow:

\[
\frac{\partial (\rho E)}{\partial t} = -\nabla \cdot (\rho E \mathbf{u}) + \nabla \cdot (\Lambda \nabla T) - \nabla \cdot (p \mathbf{u}) + \nabla \cdot (\tau \mathbf{u})
\]

\[
\frac{\partial (\rho U)}{\partial t} + \nabla \cdot (\rho U \mathbf{u}) + p \nabla \cdot \mathbf{u} - \nabla \cdot (\Lambda \nabla T) - \tau : \nabla \cdot \mathbf{u} = 0
\]

\[
(\rho c)_s \frac{\partial T}{\partial t} + (\rho c)_f \mathbf{u} \cdot \nabla T - \nabla \cdot (\Lambda \nabla T) - \phi \beta T \frac{\partial p}{\partial t} - \beta T \mathbf{u} \cdot \nabla p - \tau : \nabla \cdot \mathbf{u} = 0
\]
Energy conservation law

Single phase flow:

\[
\frac{\partial (\rho E)}{\partial t} = -\nabla \cdot (\rho E \mathbf{u}) + \nabla \cdot (\lambda \nabla T) - \nabla \cdot (p \mathbf{u}) + \nabla \cdot (\tau \mathbf{u})
\]

\[
\leftrightarrow \quad \frac{\partial (\rho U)}{\partial t} + \nabla \cdot (\rho U \mathbf{u}) + p \nabla \cdot \mathbf{u} - \nabla \cdot (\lambda \nabla T) - \Phi \mu = 0
\]

\[
\leftrightarrow \quad (\rho c)^* \frac{\partial T}{\partial t} + (\rho c)_f \mathbf{u} \cdot \nabla T - \nabla \cdot (\lambda \nabla T) - \phi \beta T \frac{\partial p}{\partial t} - \beta \mathbf{T} \cdot \nabla p - \tau : \nabla \mathbf{u} = 0
\]
Energy conservation law

Single phase flow:

\[
\frac{\partial (\rho E)}{\partial t} = - \nabla \cdot (\rho E \mathbf{u}) + \nabla \cdot (\lambda \nabla T) - \nabla \cdot (p \mathbf{u}) + \nabla \cdot (\tau \mathbf{u})
\]

\[
\leftrightarrow \frac{\partial (\rho \mathbf{U})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \mathbf{u}) + p \nabla \cdot \mathbf{u} - \nabla \cdot (\lambda \nabla T) - \Phi_{\mu} = 0
\]

\[
\leftrightarrow (\rho c)_1 \frac{\partial T}{\partial t} + (\rho c)_f \mathbf{u} \cdot \nabla T - \nabla \cdot (\lambda \nabla T) - \phi \beta T \frac{\partial p}{\partial t} - \beta T \mathbf{u} \cdot \nabla p + \mathbf{u} \cdot \nabla p = 0
\]

Equation considered in Denel’s thesis
**Energy conservation law**

**Single phase flow:**

\[
\frac{\partial (\rho E)}{\partial t} = - \nabla \cdot (\rho E \mathbf{u}) + \nabla \cdot (\Lambda \nabla T) - \nabla \cdot (p \mathbf{u}) + \nabla \cdot (\mathbf{I} \mathbf{u})
\]

\[
\leftrightarrow \quad \frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{U} \mathbf{u}) + p \nabla \cdot \mathbf{u} - \nabla \cdot (\Lambda \nabla T) - \Phi_{\mu} = 0
\]

\[
\leftrightarrow \quad (\rho c)_t \frac{\partial T}{\partial t} + (\rho c)_f \mathbf{u} \cdot \nabla T - \nabla \cdot (\Lambda \nabla T) - \phi \beta T \frac{\partial p}{\partial t} - \beta T \mathbf{u} \cdot \nabla p + \mathbf{u} \cdot \nabla p = 0
\]

*Equation considered in Denel’s thesis*

**Multi-component multi-phase flow:**

\[
\sum_p \left( \frac{\partial}{\partial t} \left( \phi S_p \rho_p \mathbf{U}_p \right) + \nabla \cdot \left( \phi S_p \rho_p \mathbf{U}_p \mathbf{u}_p \right) + p_p \nabla \cdot \mathbf{u}_p - \nabla \cdot \left( \Lambda_p \nabla T \right) - \Phi_{\mu,p} \right) = 0
\]

- **Equivalent conductivity:** \( \lambda = (\lambda_s)^{(1 - \phi)} \times (\lambda_w)^{S_w \times \phi} \times (\lambda_o)^{S_o \times \phi} \times (\lambda_g)^{S_g \times \phi} \)

\[
\sum_p \left( \frac{\partial}{\partial t} \left( \phi S_p \rho_p \mathbf{U}_p \right) + \nabla \cdot \left( \phi S_p \rho_p \mathbf{U}_p \mathbf{u}_p \right) + p_p \nabla \cdot \mathbf{u}_p \right) - \nabla \cdot (\Lambda \nabla T) + \sum_p \mathbf{u}_p \cdot \nabla p_p = 0
\]
Outline

* Part I: Coupling of single phase reservoir and wellbore models
  - Coupling of the two models/Transmission conditions
  - Analysis of the continuous global problem
  - Finite element discretization
  - Numerical results

* Part II: Multi-component multi-phase model in reservoir
  - Physical modeling
  - Finite volume discretization
  - Numerical scheme
  - Numerical results
Coupling of single phase reservoir and wellbore models

- Porous media $\Omega$ divided into $N$ geological layers $\Omega_i$
- Layers characterized by their own physical and thermodynamic properties
- Layers saturated with a formation water and a monophasic compressible fluid
- Only the monophasic fluid is mobile
- 2D axisymmetric hypothesis
Porous media $\Omega$ divided into $N$ geological layers $\Omega_i$

- Layers characterized by their own physical and thermodynamic properties
- Layers saturated with a formation water and a monophasic compressible fluid
- Only the monophasic fluid is mobile
- 2D axisymmetric hypothesis

Couple Denel’s reservoir and wellbore models

B. Denel, Simulation numérique et couplage de modèles thermomécaniques puits-milieux poreux, Thèse de doctorat, Université de Pau, 2004
Semi-discretized reservoir model (*Darcy-Forchheimer*):

\[
\begin{align*}
\frac{1}{r} M \mathbf{G}_1 + \nabla p_1 &= -\rho_1^{n-1} \mathbf{g} \\
\frac{1}{r} \mathbf{q}_1 - \nabla T_1 &= 0 \\
rt^p_1 - \frac{b}{\Delta t} T_1 + \text{div} \mathbf{G}_1 &= \frac{a}{\Delta t} p_1^{n-1} - \frac{b}{\Delta t} T_1^{n-1} \\
\frac{a}{\Delta t} T_1 + \kappa \mathbf{G}_1^{n-1} \cdot \nabla T_1 - \frac{f}{\Delta t} p_1 + l \mathbf{G}_1^{n-1} \cdot \nabla p_1 - \text{div} \mathbf{q}_1 &= \frac{a}{\Delta t} T_1^{n-1} - \frac{f}{\Delta t} p_1^{n-1}
\end{align*}
\]

Semi-discretized wellbore model (*Compressible Navier-Stokes*):

\[
\text{div}(r \mathbf{G}_2) = -r \frac{\rho_2 - \rho_2^n}{\Delta t}
\]

\[
\text{div}(r \mathbf{u}_2) = \frac{1}{\rho} (\text{div}(r \mathbf{G}_2) - r \mathbf{G}_2 \cdot \nabla \rho)
\]

\[
\rho \mathbf{u}_2 + r \mathbf{G}_2 \cdot \nabla \mathbf{u}_2 + r \nabla p_2 - \text{div}(r \tau) + \tau_{\theta \theta} \mathbf{e}_r + r F |\mathbf{G}_2| \mathbf{u}_2 = r \rho_2 g + r \rho_2 \frac{u_2^2}{\Delta t}
\]

\[
\frac{1}{\lambda} \mathbf{q}_2 - \nabla T_2 = 0
\]

\[
r c_v \left( \rho \frac{T_2}{\Delta t} + \mathbf{G}_2 \cdot \nabla T_2 \right) - \text{div}(r \mathbf{q}_2) = r c_v \frac{T_2^n}{\Delta t} - \frac{1}{2} \left( \rho \frac{|u_2|^2 - |u_2^n|^2}{\Delta t} + \mathbf{G}_2 \cdot \nabla (|\mathbf{u}_2|^2) \right) - \text{div}(r p_2 \mathbf{u}_2) + \text{div}(r \tau_2 \mathbf{u}_2) + r g \mathbf{G}_2
\]
Mathematical difficulties related to the coupling

- A multiscale problem (stiff coupling):
  1. 2D axisymmetric reservoir model
  2. 1.5D wellbore model
Mathematical difficulties related to the coupling

- A multiscale problem (stiff coupling):
  1. 2D axisymmetric reservoir model
  2. 1.5D wellbore model

Flow has a privileged direction
⇒ Derive a 1.5D model

- Explicit dependency on r:
  \[
  \begin{align*}
  \mathbf{u} &= \frac{r}{R} \mathbf{u}_1(z) + \frac{R - r}{R} \mathbf{u}_0(z) \\
  \mathbf{G} &= \left( \begin{array}{c} 
  \frac{r}{R} \mathbf{G}_1(z) \\
  \mathbf{G}_2(z) \end{array} \right) \\
  \mathbf{q} &= \left( \begin{array}{c} 
  \frac{r}{R} \mathbf{q}_1(z) \\
  \mathbf{q}_2(z) \end{array} \right) \\
  \rho &= \rho(z) \ , \ p = p(z) \ , \ T = T(z)
  \end{align*}
  \]

- Consider only one rectangular mesh in the radial direction
Mathematical difficulties related to the coupling

- A multiscale problem (stiff coupling):
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  2. 1.5D wellbore model

- Fixed point method with respect to $\rho$
### Mathematical difficulties related to the coupling

- **A multiscale problem (stiff coupling):**
  - 2D axisymmetric reservoir model
  - 1.5D wellbore model

- **Fixed point method with respect to $\rho$**

### Evaluation of the specific flux

Use $\text{div}(rG) = -r \frac{\partial \rho}{\partial t}$ and solve:

\[
\begin{align*}
\text{Find } G \in W^* \\
\int_{\Omega_2} \text{div}(rG) \chi dx = - \int_{\Omega_2} r \frac{\rho - \rho^n}{\Delta t} \chi dx, \quad \forall \chi \in M
\end{align*}
\]
Mathematical difficulties related to the coupling

- A multiscale problem (stiff coupling):
  - 2D axisymmetric reservoir model
  - 1.5D wellbore model

Fixed point method with respect to $\rho$

Evaluation of the specific flux

Use $\text{div}(rG) = -r \frac{\rho}{\partial t}$ and solve

\[
\begin{aligned}
\text{Find } G \in W^* \\
\int_{\Omega_2} \text{div}(rG) \chi \, dx = - \int_{\Omega_2} r \frac{\rho - \rho^n}{\Delta t} \chi \, dx, \quad \forall \chi \in M
\end{aligned}
\]

Evaluation of $(u, p)$

Use $\rho u \cdot \nabla u = G \cdot \nabla u$ and solve

\[
\begin{aligned}
\text{Find } u \in V^*, \ p \in M \\
 m(u, v) + n(p, v) &= l_1(v), \quad \forall v \in V^0 \\
 n(q, u) &= l_2(q) \quad \forall q \in M
\end{aligned}
\]
Numerical modeling of thermomechanical multi-phase flows, well-porous medium

Coupling of single phase reservoir and wellbore models

Transmission conditions

Mathematical difficulties related to the coupling

- A multiscale problem (stiff coupling):
  1. 2D axisymmetric reservoir model
  2. 1.5D wellbore model

Fixed point method with respect to $\rho$

Evaluation of the specific flux

Use $\text{div}(rG) = -r \frac{\partial \rho}{\partial t}$ and solve

\[
\begin{cases}
\text{Find } G \in W^* \\
\int_{\Omega_2} \text{div}(rG)\chi dx = -\int_{\Omega_2} r^\rho - \rho^n \frac{\Delta t}{\Delta t} \chi dx, \quad \forall \chi \in M
\end{cases}
\]

Evaluation of $(u, p)$

Use $\rho u \cdot \nabla u = G \cdot \nabla u$ and solve

\[
\begin{cases}
\text{Find } u \in V^*, \ p \in M \\
m(u, v) + n(p, v) = l_1(v), \quad \forall v \in V^0 \\
n(q, u) = l_2(q) \quad \forall q \in M
\end{cases}
\]

Evaluation of $(q, T)$

Use $\rho u \cdot \nabla T = G \cdot \nabla T$ and solve

\[
\begin{cases}
\text{Find } q \in H, \ T \in M \\
a(q, w) + b(T, w) = f_1(w), \quad \forall w \in H \\
b(S, q) - c(T, S) = f_2(S) \quad \forall S \in M
\end{cases}
\]

Evaluation of $(u, p)$
Mathematical difficulties related to the coupling

- A multiscale problem (stiff coupling):
  - 2D axisymmetric reservoir model
  - 1.5D wellbore model

- Additional unknown in the wellbore: Velocity $u_2$

- Density is not constant in the two domains

- Energetic aspect taken into account
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Coupling of single phase reservoir and wellbore models

Transmission conditions

Mathematical difficulties related to the coupling

- A multiscale problem (stiff coupling):
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- Additional unknown in the wellbore: Velocity $u_2$
- Density is not constant in the two domains
- Energetic aspect taken into account

Transmission conditions at the interface

- $[G \cdot n] = 0 \Rightarrow G_1 \cdot n = RG_2 \cdot n$
- $[\sigma n \cdot n] = 0 \Rightarrow -p_1 + \tau_{rr} = -p_2$
- $u_2 \cdot t = 0$
  - or $u_2 \cdot t = -\frac{\sqrt{\alpha}}{\alpha} \sigma_2 n \cdot t$ Beavers-Joseph-Saffman
- $G_1 \cdot n = \rho \ u_2 \cdot n$
- $[q \cdot n] = 0 \Rightarrow q_1 \cdot n = Rq_2 \cdot n$
- $[T] = 0 \Rightarrow T_1 = T_2$
Variational formulation in the reservoir

- We denote: \( x_1 = (G_1, q_1, p_1, T_1) \)
- Functional framework:

\[
X_1 = H(div, \Omega_1) \times H(div, \Omega_1) \times L^2(\Omega_1) \times L^2(\Omega_1)
\]
\[
X_1^0 = \{ x_1 \in X_1 \mid G_1 \cdot n = 0 \text{ on } \Gamma_G, \ q_1 \cdot n = 0 \text{ on } \Gamma_q \}
\]
\[
X_1^* = \{ x_1 \in X_1 \mid G_1 \cdot n = G^* \text{ on } \Gamma_G, \ q_1 \cdot n = q^* \text{ on } \Gamma_q \}
\]

- Variational formulation:

\[
\begin{aligned}
\text{Find } x_1 & \in X_1^* \\
\mathcal{A}_1(x_1, x'_1) = F_1(x'_1) & \quad \forall x'_1 \in X_1^0
\end{aligned}
\]

Weak formulation in the wellbore

- We denote: \( x_2 = (G_2, u_2, p_2, q_2, T_2) \)
- Functional framework:

\[
X_2 = W \times V \times M \times H \times M
\]
\[
X_2^* = W^* \times V^* \times M \times H \times M
\]
\[
Y_2 = M \times V^0 \times M \times H \times M
\]

- Weak formulation:

\[
\begin{aligned}
\text{Find } x_2 & \in X_2^* \\
\mathcal{A}_2(x_2, x'_2) = F_2(x'_2) & \quad \forall x'_2 \in Y_2
\end{aligned}
\]
Numerical modeling of thermomechanical multi-phase flows, well-porous medium

Coupling of single phase reservoir and wellbore models

Analysis of the continuous global problem

**An integration by part in the reservoir yields the terms:**

\[
\int_{\Sigma} p_1 G_1' \cdot n \ d\sigma - \int_{\Sigma} T_1 q_1' \cdot n \ d\sigma
\]

**An integration by part in the wellbore yields the terms:**

\[
\int_{\Sigma} R(p_2 - \tau_2 n \cdot n) u_2' \cdot n \ d\sigma - \int_{\Sigma} R T_2 q_2' \cdot n \ d\sigma
\]

**Dualization by Lagrange multipliers** \( \Lambda = (\theta, \mu) \):

\[
\theta = p_1 = p_2 - \tau_2 n \cdot n \quad \mu = T_1 = T_2
\]

**Multipliers’ spaces:**

\[
\mathbb{L} = L^2(\Sigma) \times L^2(\Sigma) \quad \mathbb{K} = L^2(\Sigma) \times L^2(\Sigma) \times L^2(\Sigma)
\]

**Bilinear forms:**

\[
I(\Lambda, x') = \int_{\Sigma} (G_1' \cdot n - Ru_2' \cdot n) \theta \ d\sigma - \int_{\Sigma} (q_1' \cdot n - Rq_2' \cdot n) \mu \ d\sigma
\]

\[
J(\Lambda', x) = \int_{\Sigma} (G_1 \cdot n - R \rho_2 u_2 \cdot n) \theta' \ d\sigma + \int_{\Sigma} (G_1 \cdot n - RG_2 \cdot n) \zeta' \ d\sigma - \int_{\Sigma} (q_1 \cdot n - Rq_2 \cdot n) \mu' \ d\sigma
\]
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Coupling of single phase reservoir and wellbore models

Analysis of the continuous global problem

- An integration by part in the reservoir yields the terms:

\[ \int_{\Sigma} p_1 G'_1 \cdot n \, d\sigma - \int_{\Sigma} T_1 q'_1 \cdot n \, d\sigma \]

- An integration by part in the wellbore yields the terms:

\[ \int_{\Sigma} R (p_2 - \tau_2 \cdot n) u'_2 \cdot n \, d\sigma - \int_{\Sigma} RT_2 q'_2 \cdot n \, d\sigma \]

- Dualization by Lagrange multipliers \( \Lambda = (\theta, \mu) \):

\[ \theta = p_1 = p_2 - \tau_2 \cdot n \quad \mu = T_1 = T_2 \]

- Multipliers’ spaces:

\[ \mathcal{L} = L^2(\Sigma) \times L^2(\Sigma) \quad \mathcal{K} = L^2(\Sigma) \times L^2(\Sigma) \times L^2(\Sigma) \]

- Bilinear forms:

\[ I(\Lambda, x') = \int_{\Sigma} (G'_1 \cdot n - Ru'_2 \cdot n) \theta \, d\sigma - \int_{\Sigma} (q'_1 \cdot n - Rq'_2 \cdot n) \mu \, d\sigma \]

\[ J(\Lambda', x) = \int_{\Sigma} (G_1 \cdot n - R\rho u_2 \cdot n) \theta' \, d\sigma + \int_{\Sigma} (G_1 \cdot n - RG_2 \cdot n) \zeta' \, d\sigma - \int_{\Sigma} (q_1 \cdot n - Rq_2 \cdot n) \mu' \, d\sigma \]
An integration by part in the reservoir yields the terms:

\[ \int_{\Sigma} p_1 G'_1 \cdot n \, d\sigma - \int_{\Sigma} T_1 q'_1 \cdot n \, d\sigma \]

An integration by part in the wellbore yields the terms:

\[ \int_{\Sigma} R \left( p_2 - \tau_2 n \cdot n \right) u'_2 \cdot n \, d\sigma - \int_{\Sigma} RT_2 q'_2 \cdot n \, d\sigma \]

Dualization by Lagrange multipliers \( \Lambda = (\theta, \mu) \):

\[ \theta = p_1 = p_2 - \tau_2 n \cdot n \quad \mu = T_1 = T_2 \]

Multipliers’ spaces:

\[ \mathcal{L} = L^2(\Sigma) \times L^2(\Sigma) \quad \mathcal{K} = L^2(\Sigma) \times L^2(\Sigma) \times L^2(\Sigma) \]

Bilinear forms:

\[ I(\Lambda, x') = \int_{\Sigma} \left( G'_1 \cdot n - Ru'_2 \cdot n \right) \theta \, d\sigma - \int_{\Sigma} (q'_1 \cdot n - Rq'_2 \cdot n) \mu \, d\sigma \]

\[ J(\Lambda', x) = \int_{\Sigma} (G_1 \cdot n - R\rho_2 u_2 \cdot n) \theta' \, d\sigma + \int_{\Sigma} (G_1 \cdot n - RG_2 \cdot n) \zeta' \, d\sigma - \int_{\Sigma} (q_1 \cdot n - Rq_2 \cdot n) \mu' \, d\sigma \]
Weak formulation for the coupled problem

- **Functional framework:**
  \[
  \mathcal{X} = \{ x = (x_1, x_2) \in X_1 \times X_2 ; \ G_1 \cdot n, \ q_1 \cdot n \in L^2(\Sigma) \} \\
  \mathcal{Y} = \{ x' = (x'_1, x'_2) \in X_1 \times Y_2 \Sigma ; \ G_1 \cdot n, \ q_1 \cdot n \in L^2(\Sigma) \} \\
  \mathcal{X}^* = \{ (x_1, x_2) \in \mathcal{X} ; \ G_1 \cdot n = G^* \text{ on } \gamma_G \setminus \Sigma, \ q_1 \cdot n = q^* \text{ on } \gamma_q \setminus \Sigma, \ u_2 \cdot n = Q \text{ on } \Gamma_s \} \\
  \mathcal{Y}^0 = \{ x' \in \mathcal{Y} ; \ G_1' \cdot n = 0 \text{ on } \gamma_G \setminus \Sigma, \ q_1' \cdot n = 0 \text{ on } \gamma_q \setminus \Sigma, \ u_2' \cdot n = 0 \text{ on } \Gamma_s \} \\
  \]

- **Weak formulation:**
  \[
  \begin{cases}
  \text{Find } x \in \mathcal{X}^*, \Lambda \in \mathcal{L} \\
  \mathcal{A}(x, x') + \mathcal{I}(\Lambda, x') = \mathcal{F}(x') \quad \forall x' \in \mathcal{Y}^0 \\
  \mathcal{J}(\Lambda', x) = 0 \quad \forall \Lambda' \in \mathcal{K} 
  \end{cases}
  \]

  Where:
  \[
  \mathcal{A}(x, x') = \mathcal{A}_1(x_1, x'_1) + \mathcal{A}_2(x_2, x'_2) \quad \forall x \in \mathcal{X}, \ \forall x' \in \mathcal{Y} \\
  \mathcal{F}(x') = \mathcal{F}_1(x'_1) + \mathcal{F}_2(x'_2) \quad \forall x' \in \mathcal{Y} 
  \]
Analysis of the continuous global problem

Babuška’s theorem:

- $\mathcal{I}$ and $\mathcal{J}$ satisfy an inf-sup condition
- The coupled problem reduces to:

\[
\begin{cases}
\text{Find } x \in \mathcal{J}^* \\
\mathcal{A}(x, x') = \mathcal{F}(x') \quad \forall x' \in \mathcal{I}
\end{cases}
\]

Where:

\[
\begin{aligned}
\mathcal{J}^* &= \{x \in \mathcal{X}^* ; \mathcal{J}(\Lambda', x) = 0 \quad \forall \Lambda' \in \mathcal{K}\} \\
\mathcal{I} &= \{x' \in \mathcal{Y}^0 ; \mathcal{I}(\Lambda, x') = 0 \quad \forall \Lambda \in \mathcal{L}\}
\end{aligned}
\]

Theorem: \( \forall x \in \mathcal{J}^0 \setminus \{0\}, \sup_{x' \in \mathcal{I}} \frac{\mathcal{A}(x, x')}{{\|x'\|}_Y} > 0 \)

\( \rightarrow \) **UNIQUENESS OF THE SOLUTION**

- Existence: Galerkin’s method (via finite element spaces)
Analysis of the continuous global problem

Babuška’s theorem:

- $I$ and $J$ satisfy an inf-sup condition

The coupled problem reduces to:

$$\begin{cases}
\text{Find } x \in J^* \\
A(x, x') = F(x') \quad \forall x' \in I
\end{cases}$$

Where:

$$J^* = \{ x \in X^* ; J(\Lambda', x) = 0 \quad \forall \Lambda' \in K \}$$

$$I = \{ x' \in Y^0 ; I(\Lambda, x') = 0 \quad \forall \Lambda \in L \}$$

Theorem: \( \forall x \in J^0 \setminus \{0\}, \sup_{x' \in I} \frac{A(x, x')}{\|x'\|_Y} > 0 \)

\( \implies \text{UNIQUENESS OF THE SOLUTION} \)

Existence: Galerkin’s method (via finite element spaces)
## Proof of the uniqueness

- **By putting**: \( U = (G_1, q_1, G_2, u_2, q_2) \quad U' = (G_1', q_1', \chi, u_2', q_2') \quad s = (p_1, T_1, p_2, T_2) \)

- **Non-standard Mixed formulation**:

\[
\begin{aligned}
\text{Find } (U, s) \in \mathbb{U}^* \times S \\
A(U, U') + B(s, U') = F_1(U') \quad \forall U' \in T^0 \\
-B(s', U) + C(s, s') = F_2(s') \quad \forall s' \in S
\end{aligned}
\]

Where:

\[
\begin{aligned}
A(U, U') &= \int_{\Omega_1} \frac{1}{r} MG_1 \cdot G_1' dx + \int_{\Omega_1} \frac{1}{r \lambda_1} q_1 \cdot q_1' dx + \int_{\Omega_2} \chi \text{div}(r G_2) dx + \int_{\Omega_2} r \frac{1}{r \lambda_1} q_2 \cdot q_2' dx + a(u_2, u_2') \text{ non symmetric} \\
B(s, U') &= -\int_{\Omega_1} p_1 \text{div} G_1' dx + \int_{\Omega_1} T_1 \text{div} q_1' dx - \int_{\Omega_2} p_2 \text{div}(ru_2') dx + \int_{\Omega_2} T_2 \text{div}(r q_2') \\
C(s, s') &= \int_{\Omega_1} r \frac{\alpha}{\Delta t} p_1 p_1' dx - \int_{\Omega_1} r \frac{b}{\Delta t} T_1 p_1' dx + \int_{\Omega_1} r \frac{d}{\Delta t} T_1 T_1' dx - \int_{\Omega_1} r \frac{f}{\Delta t} p_1 T_1' dx + \int_{\Omega_2} r \frac{c p_2}{\Delta t} T_2 T_2' dx \text{ non symmetric}
\end{aligned}
\]
Numerical modeling of thermomechanical multi-phase flows, well-porous medium

- Coupling of single phase reservoir and wellbore models
- Analysis of the continuous global problem

**The homogeneous problem admits only the trivial solution ??**

Let \((U, s)\) a solution of:

\[
\begin{align*}
A(U, U') + B(s, U') &= 0 \quad \forall U' \in T^0 \\
-B(s', U) + C(s, s') &= 0 \quad \forall s' \in S
\end{align*}
\]
The homogeneous problem admits only the trivial solution ??

Let \((U, s)\) a solution of :

\[
\begin{align*}
A(U, U') + B(s, U') &= 0 \quad \forall U' \in T^0 \\
-B(s', U) + C(s, s') &= 0 \quad \forall s' \in S
\end{align*}
\]

There exists \(R : U^0 \to T^0\) linear, continuous and satisfying :

\[
A(U, RU) > 0, \quad U - U' \in \text{Ker}B
\]
The homogeneous problem admits only the trivial solution ??

Let \((U,s)\) a solution of :

\[
\begin{align*}
A(U,U') + B(s,U') &= 0 \quad \forall U' \in T^0 \\
-B(s',U) + C(s,s') &= 0 \quad \forall s' \in S
\end{align*}
\]

* There exists \(R : U^0 \rightarrow T^0\) linear, continuous and satisfying :

\[A(U,RU) > 0, \quad U - U' \in \text{Ker}B\]

\[U = (G_1,q_1,G_2,u_2,q_2) \in U^0\) define \(RU = U' = (G'_1,q_1,\chi,u_2,q_2)\) satisfying :

\[G'_1 \cdot n = \frac{1}{\rho_2}G_1 \cdot n \text{ on } \Sigma, \quad \text{div}G'_1 = \text{div}G_1 \text{ in } \Omega_1, \quad \|G'_1\|_{0,\Omega_1} + \|\chi\|_{0,\Omega_2} \leq c \|U\|.
\]

\[
\begin{align*}
U' \in T^0, \quad \|U'\| \leq c \|U\|, \quad B(s,U) = B(s,U') \\
A(U,U') \geq c \left(\|q_1\|^2_{0,\Omega_1} + \|q_2\|^2_{0,\Omega_2}\right) + m(u_2,u_2) + \int_{\Omega_1} \frac{1}{2}MG_1 \cdot G'_1 dx + \int_{\Omega_2} \chi \text{div}(rG_2) dx
\end{align*}
\]

Bound \(m(u_2,u_2)\) by means of Young’s inequality
The homogeneous problem admits only the trivial solution ??

Let \((U, s)\) a solution of:

\[
\begin{cases}
A(U, U') + B(s, U') = 0 & \forall U' \in T^0 \\
-B(s', U) + C(s, s') = 0 & \forall s' \in S
\end{cases}
\]

- There exists \(\mathcal{R} : U^0 \rightarrow T^0\) linear, continuous and satisfying:
  \[A(U, \mathcal{R}U) > 0, \quad U - U' \in \text{Ker}B\]

- \(\forall s \in S, \quad C(s, s) \geq \gamma' \left( \|p_1\|_{0, \Omega_1}^2 + \|T_1\|_{0, \Omega_1}^2 + \|T_2\|_{0, \Omega_2}^2 \right)\)
The homogeneous problem admits only the trivial solution ??

Let \((\mathbf{U}, s)\) a solution of:

\[
\begin{cases}
    A(\mathbf{U}, \mathbf{U}') + B(s, \mathbf{U}') = 0 & \forall \mathbf{U}' \in T^0 \\
    -B(s', \mathbf{U}) + C(s, s') = 0 & \forall s' \in S
\end{cases}
\]

* There exists \(\mathcal{R} : \mathbb{U}^0 \to T^0\) linear, continuous and satisfying:

\[
A(\mathbf{U}, \mathcal{R}\mathbf{U}) > 0, \quad \mathbf{U} - \mathbf{U}' \in \text{Ker} B
\]

* \(\forall s \in S, \quad C(s, s) \geq \gamma(\|p_1\|^2_{0, \Omega_1} + \|T_1\|^2_{0, \Omega_1} + \|T_2\|^2_{0, \Omega_2})\)

\[
C(s, s') = \int_{\Omega_1} r \frac{a}{\Delta t} p_1 p'_1 \, dx - \int_{\Omega_1} r \frac{b}{\Delta t} T_1 p'_1 \, dx + \int_{\Omega_1} r \frac{d}{\Delta t} T_1 T'_1 \, dx - \int_{\Omega_1} r \frac{f}{\Delta t} p_1 T'_1 \, dx + \int_{\Omega_2} r \frac{c \rho_2}{\Delta t} T_2 T'_2 \, dx
\]

If \(4ad - (b + f)^2 \geq c\) a.e. in \(\Omega_1\), we have:

\[
\int_{\Omega_1} r \frac{a}{\Delta t} p_1 p_1 \, dx - \int_{\Omega_1} r \frac{b}{\Delta t} T_1 p_1 \, dx + \int_{\Omega_1} r \frac{d}{\Delta t} T_1 T_1 \, dx - \int_{\Omega_1} r \frac{f}{\Delta t} p_1 T_1 \, dx \geq \frac{c}{\Delta t} \left(\|p_1\|^2_{0, \Omega_1} + \|T_1\|^2_{0, \Omega_1}\right)
\]
The homogeneous problem admits only the trivial solution ??

Let \((U, s)\) a solution of :

\[
\begin{align*}
A(U, U') + B(s, U') &= 0 \quad \forall U' \in T^0 \\
-B(s', U) + C(s, s') &= 0 \quad \forall s' \in S
\end{align*}
\]

* There exists \(R : U^0 \rightarrow T^0\) linear, continuous and satisfying :

\[
A(U, RU) > 0, \quad U - U' \in \text{Ker}B
\]

* \(\forall s \in S, \quad C(s, s) \geq \gamma (\|p_1\|_{0,\Omega_1}^2 + \|T_1\|_{0,\Omega_1}^2 + \|T_2\|_{0,\Omega_2}^2)
\]

then \(U = 0\) and \((p_1, T_1, T_2) = 0\)
The homogeneous problem admits only the trivial solution ??

Let \((U, s)\) a solution of:
\[
\begin{align*}
A(U, U') + B(s, U') &= 0 \quad \forall U' \in T^0 \\
-B(s', U) + C(s, s') &= 0 \quad \forall s' \in S
\end{align*}
\]

* There exists \(\mathcal{R} : U^0 \rightarrow T^0\) linear, continuous and satisfying:
\[
A(U, \mathcal{R}U) > 0, \quad U - U' \in \text{Ker}B
\]

* \(\forall s \in S, \quad C(s, s) \geq \gamma (\|p_1\|_{0, \Omega_1}^2 + \|T_1\|_{0, \Omega_1}^2 + \|T_2\|_{0, \Omega_2}^2)\)

then \(U = 0\) and \((p_1, T_1, T_2) = 0\)

* There exists \(\beta > 0\) such that:
\[
\beta \|s\| \leq \sup_{U' \in T^0} \frac{B(s, U')}{\|U'\|}
\]

Fortin’s trick:
\[
s = (p_1, T_1, p_2, T_2) \in S \quad \mapsto \quad U' = (G'_1, q_1, \chi, u_2, q_2) \in T^0 \quad \text{satisfying} \quad \begin{cases} 
B(s, U') \geq c_1\|s\|^2 \\
\|U'\| \leq c_2\|s\|
\end{cases}
\]
The homogeneous problem admits only the trivial solution ??

Let \((U,s)\) a solution of :

\[
\begin{align*}
    A(U, U') + B(s, U') &= 0 \quad \forall U' \in T^0 \\
    -B(s', U) + C(s, s') &= 0 \quad \forall s' \in S
\end{align*}
\]

* There exists \(\mathcal{R} : U^0 \rightarrow T^0\) linear, continuous and satisfying :

\[
A(U, \mathcal{R}U) > 0, \quad U - U' \in \text{Ker}B
\]

* \(\forall s \in S, \quad C(s, s) \geq \gamma(\|p_1\|_{0, \Omega_1}^2 + \|T_1\|_{0, \Omega_1}^2 + \|T_2\|_{0, \Omega_2}^2)
\]

then \(U = 0\) and \((p_1, T_1, T_2) = 0\)

* There exists \(\beta > 0\) such that :

\[
\beta_2 \|s\| \leq \sup_{U \in T^0} \frac{B(s, U')}{\|U'\|}
\]

then \(p_2 = 0\)
Discrete problem

- Suppose that the two meshes match on the perforations
- Denote by $\mathcal{E}_h$ the set of edges situated on the interface $\Sigma$

Finite dimensional spaces

- Conservative variables (specific flux, heat flux):
  \[ RT_0 = \left\{ \begin{pmatrix} ar + b \\ az + c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\} \]
  \[ V_h = \{ G \in H(div, \Omega) ; G/K \in RT_0 \quad \forall K \in \mathcal{T}_h \} \]
- Scalar variables (pressure, temperature, density):
  \[ L_h = \{ p \in L^2(\Omega) ; p/K \in P_0 \quad \forall K \in \mathcal{T}_h \} \]
- Fluid’s velocity:
  \[ X_h = \{ u \in H^1(\Omega) ; v/K \in Q_1 \quad \forall K \in \mathcal{T}_h \} \]
- Lagrange multipliers on the interface:
  \[ K_h = \{ \mu \in L^2(\Sigma) ; \mu \in P_0(e) \quad \forall e \in \mathcal{E}_h \} \]
**Numerical modeling of thermomechanical multi-phase flows, well-porous medium**

**Coupling of single phase reservoir and wellbore models**

**Finite element discretization**

---

### Discrete formulation

Find $x_h \in X_h^*, \Lambda_h \in \mathbb{L}_h$

\[
\begin{aligned}
\mathcal{A}_h(x_h, x') + \mathcal{I}(\Lambda_h, x') &= \mathcal{F}_h(x') \quad \forall x' \in \mathcal{Y}_h \\
\mathcal{J}(\Lambda', x_h) &= 0 \quad \forall \Lambda' \in \mathcal{K}_h
\end{aligned}
\]

---

#### Upwind scheme for convective terms

- $\partial K^- = \{ e \in \partial K / G_{h}^{n-1} \cdot n < 0 \}$
- For any $P_0$ functions $T$:
  \[
  \int_K \kappa G_{h}^{n-1} \cdot \nabla T \, dx = \sum_{e \in \partial K^-} \kappa (T_e - T_{\partial K}) \int_e G_{h}^{n-1} \cdot n \, d\sigma \quad \forall T \in L_h
  \]
- For any $Q_1$-continuous functions $\phi$ and $v$:
  \[
  \int_K r G_{h}^{n-1} \cdot \nabla \phi v \, dx = \sum_{e \in \partial K^-} (\phi^e - P_K(\phi)) \int_e r G_{h}^{n-1} \cdot n \, v \, d\sigma \quad \forall K \in T_h
  \]
Discrete formulation

\[
\begin{align*}
\text{Find } x_h &\in X_h^*, \Lambda_h \in L_h \\
A_h(x_h, x') + I(\Lambda_h, x') &= F_h(x') \quad \forall x' \in Y_h \\
J(\Lambda', x_h) &= 0 \quad \forall \Lambda' \in K_h
\end{align*}
\]

Upwind scheme for convective terms

- \( \partial K^- = \{ e \in \partial K / \mathbf{G}_h^{n-1} \cdot n < 0 \} \)
- For any \( P_0 \) functions \( T \):
  \[
  \int_K \kappa \mathbf{G}_h^{n-1} \cdot \nabla T \, dx = \sum_{e \in \partial K^-} \kappa (T^* - T_K) \int_e \mathbf{G}_h^{n-1} \cdot n \, d\sigma \quad \forall T \in L_h
  \]
- For any \( Q_1 \)-continuous functions \( \phi \) and \( v \):
  \[
  \int_K r \mathbf{G}_h^{n-1} \cdot \nabla \phi \, v \, dx = \sum_{e \in \partial K^-} (\phi^* - P_K(\phi)) \int_e r \mathbf{G}_h^{n-1} \cdot n \, v \, d\sigma \quad \forall K \in T_h
  \]
Theorem: The discrete problem has a unique solution for $\Delta t$ small enough

Sketch of the proof

- Follow the proof of the continuous case, with constants independent of $h$
- Use Fortin’s trick and interpolate continuous functions
- Need an auxiliary result:

For any $(p, \theta) \in M_h \times K_h$, there exists $G \in V_h$ satisfying:

$$
\begin{align*}
    G \cdot n &= \theta \text{ on } \Sigma, \\
    G \cdot n &= 0 \text{ on } \Gamma_G \setminus \Sigma \\
    \text{div} G &= p \text{ in } \Omega_1 \\
    \|G\|_{H(\text{div}, \Omega_1)} + \|G \cdot n\|_{0, \Sigma} &\leq c(\|p\|_{0, \Omega_1} + \|\theta\|_{0, \Sigma})
\end{align*}
$$
Mesh convergence

Test 1: two-layered reservoir

- Two layers with the same properties
- Only the lower one is perforated
- Production of light oil during seven days

\[ Q = 500 \text{ m}^3/\text{j} \]

\[ k_h = 1000 mD \quad k_v = 350 mD \quad \phi = 0.28 \quad s_w = 0.15 \]

\[ p = 400 \text{ bars} \]

\[ \dot{q}.\vec{n} = 0 \]
Convergence rate for the pressure at $t=7$ days

(a) order of the error in the reservoir

(b) order of the error in the well

Numerically, $\|p - p_h\|_{0, \Omega} \leq C|h|^\alpha$, with $\alpha \approx 1.39$ in the reservoir and $\alpha \approx 1.61$ in the well
Test 2: realistic reservoir

- Seven layers with different properties
- Production of oil during 28 days

Conditions of the simulation for the coupled problem
Specific flux at the end of the production

(c) Same scale in the 2 domains

(d) Different scales (ratio equal to 10)
Behaviour of the pressure during one month production

(e) Pressure at t=0 day

(f) Pressure at t=2 days

(g) Pressure at t=7 days

(h) Pressure at t=28 days
Behaviour of the temperature during one month production

(i) Temperature at t=0 day

(j) Temperature at t=2 days

(k) Temperature at t=7 days

(l) Temperature at t=28 days
Comparison with the separate reservoir and wellbore codes

- Previous simulation conditions for the coupled code
- Conditions of the simulation for the sole reservoir model:

<table>
<thead>
<tr>
<th>Condition</th>
<th>$k_h$ (mD)</th>
<th>$k_v$ (mD)</th>
<th>$\phi$</th>
<th>$s_w$</th>
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</thead>
<tbody>
<tr>
<td>$P=390$ bars, $q.n=0$</td>
<td>$2000$</td>
<td>$350$</td>
<td>$0.20$</td>
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<tr>
<td></td>
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<tr>
<td></td>
<td>$1000$</td>
<td>$15$</td>
<td>$0.24$</td>
<td>$0.40$</td>
</tr>
</tbody>
</table>

$P=400$ bars, $q.n=0$
Pressure maps in the reservoir at $t = 28$ days

(m) Pressure given by reservoir code

(n) Pressure given by coupled code
Temperature maps in the reservoir at $t = 28$ days

(o) Temperature given by reservoir code

(p) Temperature given by coupled code
Vertical mass fluxes in the wellbore at $t = 28$ days

(q) $G_z$ given by wellbore code

(r) $G_z$ given by coupled code
Multi-component multi-phase model in reservoir

- Three phases ($p$): water ($w$), oil ($o$) and gas ($g$)
- $n_c$ components: water, heavy hydrocarbons, light hydrocarbons, methan....
- $n_h$ hydrocarbon components ($n_h = n_c - 1$)

<table>
<thead>
<tr>
<th></th>
<th>$\bar{w}$</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>...</th>
<th>...</th>
<th>$n_h$</th>
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<tr>
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<td>$g$</td>
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</tr>
</tbody>
</table>

- 3D / Porous medium $\Omega$ with $n_W$ wells
Multi-component multi-phase model in reservoir

- Three phases \( (p) \): water \((w)\), oil \((o)\) and gas \((g)\)

- \( n_c \) components: water, heavy hydrocarbons, light hydrocarbons, methane...

- \( n_h \) hydrocarbon components \((n_h = n_c - 1)\)

<table>
<thead>
<tr>
<th></th>
<th>( \bar{\bar{w}} )</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( \ldots )</th>
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<th>( n_h )</th>
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<tr>
<td>( w )</td>
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- 3D / Porous medium \( \Omega \) with \( n_W \) wells
Three phases \((p)\): water \((w)\), oil \((o)\) and gas \((g)\)

\(n_c\) components: water, heavy hydrocarbons, light hydrocarbons, methane...

\(n_h\) hydrocarbon components \((n_h = n_c - 1)\)

\[\bar{w} \quad n_1 \quad n_2 \quad \ldots \quad \ldots \quad n_h\]

\[
\begin{array}{cccccc}
\checkmark & \checkmark & \checkmark & \checkmark & \checkmark & \checkmark \\
\end{array}
\]

3D / Porous medium \(\Omega\) with \(n_W\) wells
**Governing equations**

- **Mass conservation equation for each component** \( c \): 
  \[
  \mathcal{F}_c = \sum_{p=\{o,g,w\}} \left( \frac{\partial}{\partial t} (\phi S_p \rho_p y_{c,p}) + \nabla \cdot (\rho_p u_p y_{c,p}) \right) = 0
  \]

  \( u_p \) is given by the generalized Darcy law: 
  \[
  u_p = -k_{rp} \mu_p^{-1} K(\nabla p_p - \rho_p g)
  \]

- **Energy equation**: 
  \[
  \mathcal{F}_T = \frac{\partial}{\partial t} \left[ \sum_{p=\{o,g,w\}} (\phi S_p \rho_p H_p - p_p) + (1 - \phi) \rho_s H_s \right] + \sum_{p=\{o,w,g\}} \nabla \cdot (\phi S_p \rho_p H_p u_p) - \nabla \cdot (\lambda \nabla T) + \sum_{p=\{o,g,w\}} u_p \cdot \nabla p_p = 0
  \]

- **Capillary pressure constraints**: 
  \[
  \begin{align*}
  p_{c,ow} &= p_o - p_w \quad \text{(oil-water capillary pressure)} \\
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- **Saturation constraint**: 
  \[
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  \]

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  \[
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### Primary and secondary variables

- According to Gibb’s phase rule, the number of primary variables is equal to:

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- Use linear constraint equations to remove two pressures, one saturation and two component mole fractions

  $\rightarrow 2n_h + 2$ non-linear equations and variables left

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- **Primary equations** are the \( n_c + 1 \) mass and energy balance equations \((F_p = \{F_c, F_T\})\)
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- **Primary variables** \( X_p \) are:
  - \( p, T, S_o, S_g, \ldots \) when both oil and gas phases are present
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Flash calculations are used to check the state of hydrocarbon phases in gridblocks

- **Phase disappearance for a gridblock with two hydrocarbon phases**
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Finite volume discretization

- Equations are integrated over each gridblock $V$
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- For diffusive terms:

$$\int_K \text{div}(K \nabla v) dx = \sum_{\sigma \in \mathcal{K}} \int_{\sigma} \nabla v \cdot n_{K,\sigma} d\sigma$$

$$= \sum_{\sigma \in \mathcal{K}} k_{K,\sigma} \frac{v_\sigma - v_K}{d_{K,\sigma}} \text{mes}(\sigma)$$

$$= - \sum_{\sigma \in \mathcal{K}} F_{K,\sigma}$$
Finite volume discretization

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$$
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$$

$$
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$$

$$
= - \sum_{\sigma \in \mathcal{E}_K} F^v_{K,\sigma} = \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma (v_L - v_K)
$$

- Conservation of fluxes over the face $\sigma$:

$$
F^v_{K,\sigma} = -F^v_{L,\sigma}
$$

- $\tau_\sigma$ is the transmissibility:

$$
\tau_\sigma = \text{mes}(\sigma) \frac{k_{K,\sigma} k_{L,\sigma}}{k_{K,\sigma} d_{L,\sigma} + k_{L,\sigma} d_{K,\sigma}}.
$$
Finite volume discretization

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$$\int_K \text{div}(K \nabla v) dx = - \sum_{\sigma \in \varepsilon_K} \Gamma_{K,\sigma}^e = \sum_{\sigma \in \varepsilon_K} \tau_{\sigma} (v_L - v_K)$$
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\]

For convective terms:

\[
\int_K u \cdot \nabla v \ dx = \int_{\partial K} v \ u \cdot n \ d\sigma - \int_K v_K \ div(u) dx
\]

\[
\approx \int_{\partial K^+} v_K \ u \cdot n \ d\sigma + \int_{\partial K^-} v_{ext} \ u \cdot n \ d\sigma - \int_{\partial K} v_K \ u \cdot n \ d\sigma
\]

\[
= \sum_{\sigma \in \partial K^-} (v_{ext} - v_K) \int_{\sigma} u \cdot n_{K,\sigma} \ d\sigma
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  \]

- For convective terms:
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  \int_K u \cdot \nabla v \, dx = \int_{\partial K} v u \cdot n \, d\sigma - \int_K v K \text{div}(u) \, dx
  \approx \int_{\partial K^+} v_K u \cdot n \, d\sigma + \int_{\partial K^-} v_{ext} u \cdot n \, d\sigma - \int_{\partial K} v_K u \cdot n \, d\sigma
  = \sum_{\sigma \in \partial K^-} (v_{ext} - v_K) \int_{\sigma} u \cdot n_{K,\sigma} \, d\sigma
  \]

- In our equation, $u$ is equal to $K \nabla p$:
  \[
  \int_K K \nabla p \cdot \nabla v \, dx = - \sum_{\sigma \in \partial K^-} (v_{ext} - v_K) F_{K,\sigma}^p
  \]
Discretized system

\[
\forall K \in \mathcal{T}_h,
\]

\[
E^c_K = V_K \frac{(\phi \sum_p S_p \rho_p y_{c,p})_K^{n-1} - (\phi \sum_p S_p \rho_p y_{c,p})_K}{\Delta t} + Q^c_{lim,K}
\]

\[
- \sum_p (\sum_{\sigma = K \cap L} \tau_\sigma (\Lambda_p y_{c,p})_K/(p_{p,L} - p_{p,K} - \gamma(Z_L - Z_K))) = 0, \quad c = 1, \ldots, n_c
\]

\[
E^T_K = V_K \frac{(\sum_p (\phi \sum_p S_p \rho_p y_{p,p} H_p - p_p) + (1 - \phi) \rho_s H_s)_K^{n-1} - (\sum_p (\phi \sum_p S_p \rho_p y_{p,p} H_p - p_p) + (1 - \phi) \rho_s H_s)_K}{\Delta t}
\]

\[
- \sum_p (\sum_{\sigma = K \cap L} \tau_\sigma (\Lambda_p H_p)_K/(p_{p,L} - p_{p,K} - \gamma(Z_L - Z_K))) + Q^T_{lim,K}
\]

\[
- \sum_p \sum_{\{\sigma\} = K \cap \bar{L}} \tau_\sigma (\phi^{-1} \sum_p \rho_p^{-1} \mu_{pp}^{-1} k_{rp})_K/(p_{p,L} - p_{p,K} - \gamma(Z_L - Z_K))(p_{p,L} - p_{p,K})
\]

\[
- \sum_{\{\sigma\} = K \cap L} \tau'_\sigma \lambda_{K/L} (T_L - T_K) + Q^\lambda_{lim,K} = 0
\]
Numerical scheme

- Extend an existing isothermal simulator GPRS (General Purpose Reservoir Simulator), developed at the University of Stanford

- Iterative Newton Raphson method:

\[ J \Delta X = -F(X'') \] with \[ J = \frac{\partial F}{\partial X}(X'') \]

![Diagram showing Jacobian and Extended Jacobian](image-url)
Numerical scheme

- Extend an existing isothermal simulator GPRS (General Purpose Reservoir Simulator), developed at the University of Stanford

- The jacobian matrix of the full system can be written as:

  \[ J = \begin{bmatrix} \frac{\partial F_p}{\partial X_p} & \frac{\partial F_p}{\partial X_s} \\ \frac{\partial F_s}{\partial X_p} & \frac{\partial F_s}{\partial X_s} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad -F = \begin{bmatrix} -F_p \\ -F_s \end{bmatrix} \]
Numerical scheme

- Extend an existing isothermal simulator GPRS (General Purpose Reservoir Simulator), developed at the University of Stanford

- The jacobian matrix of the full system can be written as:

\[
J = \begin{bmatrix}
\frac{\partial F_p}{\partial X_p} & \frac{\partial F_p}{\partial X_s} \\
\frac{\partial F_s}{\partial X_p} & \frac{\partial F_s}{\partial X_s}
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\text{ and } -F = \begin{bmatrix}
-F_p \\
-F_s
\end{bmatrix}
\]

- First, extract primary variables:

\[(A - BD^{-1}C) \Delta X_p = (-F_p + BD^{-1}F_s)\]

- Next, update secondary ones:

\[\Delta X_s = -D^{-1}F_s - (D^{-1}C) \Delta X_p\]
Comparison with isothermal GPRS

- Reservoir of dimensions $5000 \text{ft} \times 5000 \text{ft} \times 50 \text{ft}$
- Three components: methan $CH_4$, butan $C_4H_{10}$ and heptan $C_7H_{16}$
- Production during 50 days by imposing a bottom hole pressure of 300 psi
Behaviour of the pressure during 50 days production

(a) Pressure at $t=0$ day
(b) Pressure at $t=2$ days
(c) Pressure at $t=10$ days
(d) Pressure at $t=20$ days
(e) Pressure at $t=35$ days
(f) Pressure at $t=50$ days
Behaviour of the temperature during 50 days production

(g) Temperature at t=0 day
(h) Temperature at t=2 days
(i) Temperature at t=10 days

(j) Temperature at t=20 days
(k) Temperature at t=35 days
(l) Temperature at t=50 days
Behaviour of the gas saturation during 50 days production

- (m) Saturation of gas phase at \( t=0 \) day
- (n) Saturation of gas phase at \( t=2 \) days
- (o) Saturation of gas phase at \( t=10 \) days
- (p) Saturation of gas phase at \( t=20 \) days
- (q) Saturation of gas phase at \( t=35 \) days
- (r) Saturation of gas phase at \( t=50 \) days
Comparison of production rates

(s) Gas production rate (MSCF/DAY)

(t) Oil production rate (STB/DAY)
Comparison of pressure and saturations in the well block

(u) Pressure in \textit{psia}

(v) Saturations of oil and gas phases
Sensibility vs. boundary conditions

- Reservoir of dimensions $9000 \text{ ft} \times 9000 \text{ ft} \times 30 \text{ ft}$
- Two components: methan $\text{CH}_4$ and butan $\text{C}_4\text{H}_{10}$
- Production of gas during 90 days by imposing a constant flowrate
Numerical modeling of thermomechanical multi-phase flows, well-porous medium

Multi-component multi-phase model in reservoir

Numerical results

Behaviour of the pressure: constant pressure on the exterior boundary

(a) Pressure at t=5 days  
(b) Pressure at t=60 days  
(c) Pressure at t=90 days

Behaviour of the pressure: no flow on the exterior boundary

(d) Pressure at t=5 days  
(e) Pressure at t=60 days  
(f) Pressure at t=90 days
Comparison of pressures in the well block and in the well

(g) Pressures in the well block

(h) Pressures in the well
**Perspectives**

1. Extend multi-phase simulator in order to treat steam injection
2. Develop a multi-phase anisothermal wellbore model and couple it with the reservoir
3. Solve inverse problems
   - Determine initial temperature
   - Determine flow profiles
4. Consider deviated wells
Thank you for your attention