Graphical types and constraints

Second-order polymorphism and inference

Who? Boris Yakobowski, under the supervision of Didier Rémy

Where? INRIA Rocquencourt, project Gallium

When? 17th December, 2008

Outline

1 Introduction: polymorphism in programming languages

2 Graphic types and MLF instance

3 Type inference through graphic constraints

4 A Church-style language for MLF

5 Conclusion

Types in programs

Context

- Safety of software
- Expressivity of programming languages

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- Safety of software
- Expressivity of programming languages

A key tool for this: Typing

- Prevents the programmer from writing some forms of erroneous code

  e.g. $1 + \texttt{"I am a string"}$

(Of course, semantically incorrect code is still possible)

Types in programs

Context

- Safety of software
- Expressivity of programming languages

A key tool for this: Typing

- Prevents the programmer from writing some forms of erroneous code

  e.g. $1 +$ "`I am a string`"

  (Of course, semantically incorrect code is still possible)

- Static typing is important

```
if (...) then
   x := x+1;
else // rarely executed code
   print_string(x)
```

Type inference

The compiler infers the types of the expressions of the program

- Removes the need to write (often redundant) type annotations

```
Node n = new Node();
```

- Facilitates rapid prototyping
- Can infer types more general than the ones the programmer had in mind

Type inference issues

- Which type should we give to functions admitting more than one possible type?

Example: finding the length of a list

```
let rec length = function
  | [] -> 0
  | _ :: q -> 1 + length q
```

$$\texttt{length}: \begin{cases} \texttt{int list} \rightarrow \texttt{int} \\ \texttt{float list} \rightarrow \texttt{int} \end{cases}$$

ML-style polymorphism

- Functions no longer receive monomorphic types, but type schemes

  $\texttt{sort}: \quad \forall\alpha.\, \alpha\,\texttt{list} \to \alpha\,\texttt{list}$

- An alternative way of saying

  "for any type α, sort has type $\alpha\,\texttt{list} \to \alpha\,\texttt{list}$"

The symbol $\forall$ introduces universal quantification

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  The symbol $\forall$ introduces universal quantification

ML Polymorphism

- One of the key reasons of the success of ML as a language
- Full type inference
  (annotations are never needed in programs)
- Sometimes a bit limited
  universal quantification only in front of the type

Second-order polymorphism

- Universal quantification under arrows is allowed

$$\lambda(f)\ \ f\ (\lambda(x)\ x)\ \ :\ \forall\alpha.\ ((\forall\beta.\ \beta \to \beta) \to \alpha) \to \alpha$$

- Many uses:
  - Encoding existential types
  - Polymorphic iterators over polymorphic structures
  - State encapsulation $\quad \texttt{runST} :: \forall\alpha.\ (\forall\beta.\ \mathsf{ST}\ \beta\ \alpha) \to \alpha$
  - ...

Second-order polymorphism

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- We want at least the expressivity of System F

  But type inference in System F is undecidable!

System F as a programming language

- System F does not have principal types

Example:

$$\begin{array}{rcll} \mathsf{id} & \triangleq & \lambda(x)\, x & : \quad \forall\beta.\ \beta \to \beta \\ \mathsf{choose} & \triangleq & \lambda(x)\, \lambda(y)\, x & : \quad \forall\alpha.\ \alpha \to \alpha \to \alpha \end{array}$$

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$$\mathsf{choose\ id} : \begin{cases} (\forall\beta.\ \beta \to \beta) \to (\forall\beta.\ \beta \to \beta) & \alpha = \forall\beta.\ \beta \to \beta \\ \forall\gamma.\ (\gamma \to \gamma) \to (\gamma \to \gamma) & \alpha = \gamma \to \gamma \end{cases}$$

No type is more general than the other

This is a fundamental limitation of System-F
(and more generally of System-F types)

Adding flexible quantification to types

Flexible quantification

ML^{F} types extend System F types with an instance-bounded quantification of the form $\forall\,(\alpha \geqslant \tau)\;\tau'$:

- Both τ and τ' can be instantiated inside $\forall\,(\alpha \geqslant \tau)\;\tau'$
- All occurrences of α in τ' must pick the same instance of τ

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- Both τ and τ' can be instantiated inside $\forall\,(\alpha \geqslant \tau)\;\tau'$
- All occurrences of α in τ' must pick the same instance of τ
- **Example:**

$$\begin{array}{rcl} \text{choose id} & : & \forall\,(\alpha \geqslant \forall\beta.\ \beta \to \beta)\ \alpha \to \alpha \\ & \sqsubseteq & (\forall\beta.\ \beta \to \beta) \to (\forall\beta.\ \beta \to \beta) \\ \text{or} & \sqsubseteq & \forall\gamma.\ (\gamma \to \gamma) \to (\gamma \to \gamma) \end{array}$$

Adding rigid quantification

- Flexible quantification solves the problem of principality
- But not the fact that type inference is undecidable

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Rigid quantification

Instance-bounded quantification, of the form $\forall (\alpha = \tau)\, \tau'$

- τ cannot (really) be instantiated inside $\forall (\alpha = \tau)\, \tau'$
- But $\forall (\alpha = \tau)\, \alpha \to \alpha$ and $\forall (\alpha = \tau)\, \forall (\alpha' = \tau)\, \alpha \to \alpha'$
are different as far as type inference is concerned

ML^{F} as a type system

Extends ML and System F, and combines the benefits of both

Compared to ML

- The expressivity of second-order polymorphism is available
- All ML programs remain typable unchanged

Compared to System F

- ML^{F} has type inference
- Programs have principal types (given their type annotations)

Moreover:

- in practice, programs require very few type annotations
- typable programs are stable under a wide range of program transformations

How to improve ML^{F}

Limitations

- Instance-bounded quantification makes equivalence and instance between types unwieldy
- Meta-theoretical results dense and non-modular
- Algorithmic inefficiency of type inference
- Not suitable for use in a typed compiler, by lack of a language to describe reduction

My work

- Use graphic types and constraints to improve the presentation
- Study efficient type inference
- Define an internal language for ML^{F}

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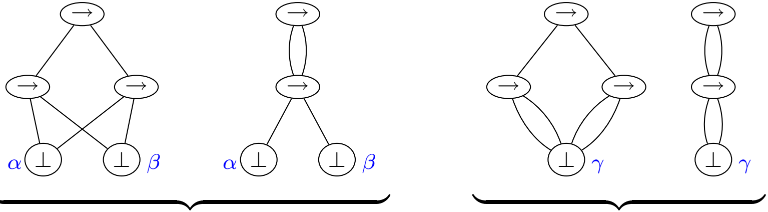
Graphic types: an alternative representation of types

A graphic type

- A term-dag, representing the skeleton of the type
  - Sharing is important, but only for variables
  - Variables are anonymous



$(\alpha\rightarrow\beta)\rightarrow(\alpha\rightarrow\beta)$ $(\gamma\rightarrow\gamma)\rightarrow(\gamma\rightarrow\gamma)$

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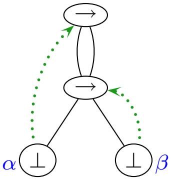

$$\forall\alpha.\ (\forall\beta_1.\ \alpha \rightarrow \beta_1) \rightarrow (\forall\beta_2.\ \alpha \rightarrow \beta_2))$$

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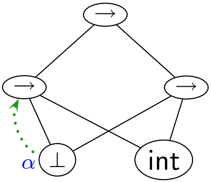
- Some well-scopedness properties



$$(\forall \alpha_1.\ \alpha_1 \rightarrow \text{int}) \rightarrow (\boxed{\alpha_2} \rightarrow \text{int})$$

Ill-scoped!

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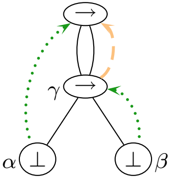
Advantages of graphic types:

- Commutation of binders, no α-conversion, no useless quantification...
- Bring closer theory and implementation
- Same formalism for different systems: ML, System F, ML^{F}, $\mathsf{F}_{\leq}$, ...

Graphic MLF types

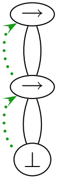
- Two kind of binding edges, for flexible and rigid quantification
- Non-variables nodes can be bound



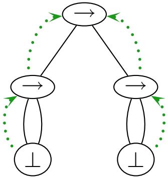
$$\forall\,(\alpha \geqslant \bot)\,\forall\,(\gamma = \forall\,(\beta \geqslant \bot)\;\alpha \to \beta)\;\gamma \to \gamma$$

Graphic MLF types

- Two kind of binding edges, for flexible and rigid quantification
- Non-variables nodes can be bound
- Sharing of non-variable nodes becomes important



$\neq$



$\forall\,(\alpha \geqslant \sigma_{id})\;\alpha \to \alpha$

$\forall\,(\alpha \geqslant \sigma_{id})\,\forall\,(\beta \geqslant \sigma_{id})\;\alpha \to \beta$

Possible type for $\lambda(x)\ x$

Incorrect for $\lambda(x)\ x$

Instance on graphic ML^{F} types

The instance relation $\sqsubseteq$

- Four atomic operations on graphs:

Instance on graphic ML^F types

The instance relation $\sqsubseteq$

Four atomic operations on graphs:

Grafting: replacing a variable by a closed type (variable substitution)

$\sqsubseteq$

Instance on graphic ML^F types

The instance relation $\sqsubseteq$

Four atomic operations on graphs:

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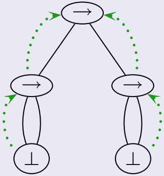
- Merging: fusing two identical subgraphs (correlates the two corresponding subtypes)



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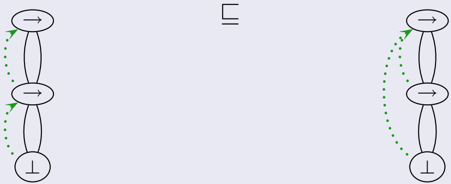

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    (correlates the two corresponding subtypes)
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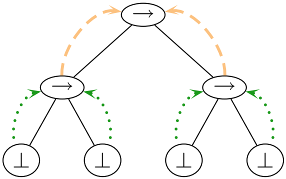
    (removes the possibility to introduce universal quantification)
  - Weakening: turns a flexible edge into a rigid one
    (forbids further instantiation of the corresponding type)
- A control of permissions rejecting some unsafe instances

Permissions on nodes

- Some instances on types would be unsound

Example: $e \quad \triangleq \quad \lambda(x : \forall\alpha.\ \forall\beta.\ \alpha \to \beta)\ x$



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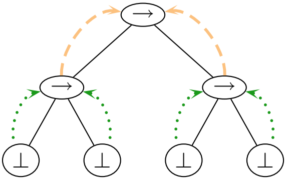
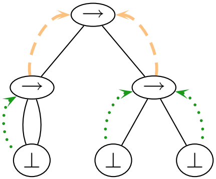


Correct type for e

$\not\sqsubseteq$



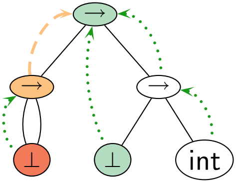
Incorrect type for e:

$e\ (\lambda(y)\ y)$ would have type
$\forall\alpha.\ \forall\beta.\ \alpha \to \beta$

Permissions on nodes

- Some instances on types would be unsound
- Nodes receive permissions according to the binding structure above and below them

  Permissions are represented by colors



- All forms of instance are forbidden on red nodes, as well as grafting on orange ones

  This ensures type soundness

Unification on ML^F graphic types

Unification on graphic types:

- Finds the most general type τ such that $\tau_1 \sqsubseteq \tau$ and $\tau_2 \sqsubseteq \tau$
- *Or* unifies two nodes in a certain type (more general)

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- Unification algorithm
  - First-order unification on the skeleton
  - Minimal raising and weakening so that the binding trees match
  - Control of permissions

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Unification

- is principal on all useful problems
- has linear complexity

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Type inference in graphic ML^F

- Constraints are an elegant way to present type inference
  - Scale better to non-toy languages
  - More general than an algorithm

- Graphic constraints as an extension of graphic types

- Can be used to perform type inference on graphic types

Permit type inference for ML, ML^F, and probably other systems

Graphic constraints

- Graphic types extended with four new constructs
  - Unification edges

    Force two nodes to be equal
  - Existential nodes

    "Floating" nodes, used only to introduce other constraints
  - Generalization nodes G
  - Instantiation edges
- Same instance relation as on graphic types

  Meta-theoretical results can be reused unchanged

Type generalization

- Type generalization is essential in $\mathsf{ML^F}$, just as in ML
- Gen nodes are used to promote types into type schemes

$g : \quad \forall \alpha.\ \alpha \to \alpha$

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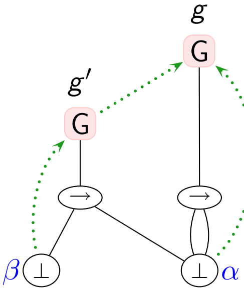

g : $\forall\alpha.\ \alpha \to \alpha$

g' : $\forall\beta.\ \beta \to \alpha$
α is free at the level of g'

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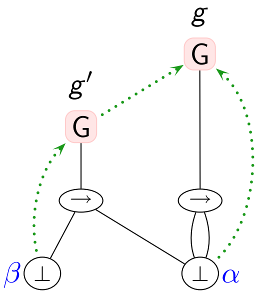

$g: \quad \forall\alpha.\ \alpha \to \alpha$

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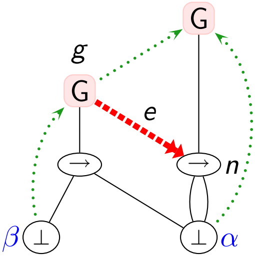
α is free at the level of g'

- Gen nodes also delimit generalization scopes

Instantiation edges

- Constrain a node to be an instance of a type scheme



- e constrains n to be an instance of g

Instantiation edges

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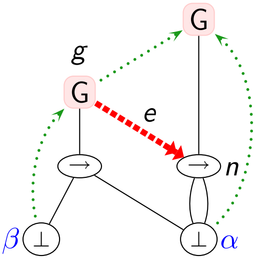

g : $\forall\beta.\ \beta \to \alpha$
n : $\alpha \to \alpha$

e is solved (take $\beta = \alpha$)

- e constrains n to be an instance of g

Instantiation edges

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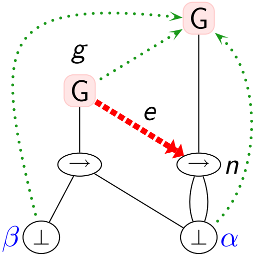


$g: \quad \beta \to \alpha$
$n: \quad \alpha \to \alpha$

e is not solved $\quad (\beta \neq \alpha)$

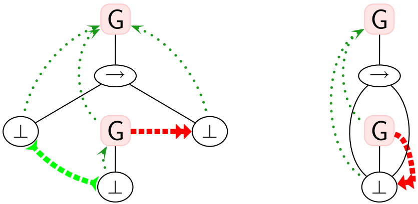
- *e* constrains *n* to be an instance of *g*

Semantics of constraints

Presolutions

A presolution of a constraint χ is an instance of χ in which all the instantiation and unification edges are solved.

Presolutions correspond to typing derivations, and are in correspondance with Church-style λ-terms



Semantics of constraints

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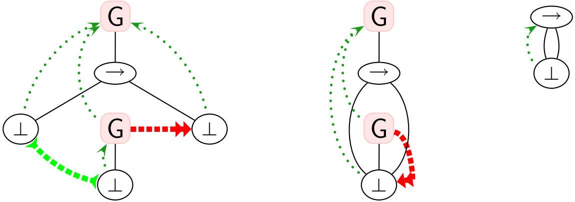
- ## Solutions

  A solution of a constraint is the type scheme represented by a presolutions of a constraint.



Typing constraints

- Source language: (ML^F only)

$$a ::= x \mid \lambda(x)\ a \mid a\ a \mid \text{let } x = a \text{ in } a \ \mid (a : \tau) \mid \lambda(x : \tau)\ a$$

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- λ-terms are translated into constraints compositionnally

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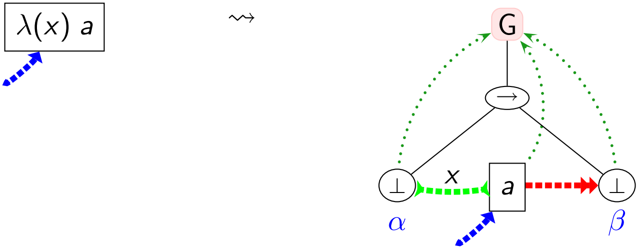
- One generalization scope by subexpression

  in ML, only needed for let; in ML^F, needed everywhere

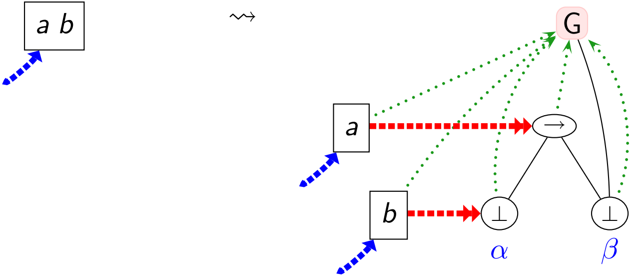
- Same typing constraints for ML and ML^F
  - the superfluous gen nodes can be removed in ML
  - ML^F constraints can be instantiated by the more general types of ML^F

Typing constraint for an abstraction



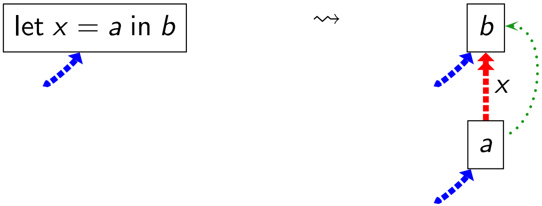
- $\lambda(x)\ a$ can receive type $\alpha \rightarrow \beta$, provided
  - α is the (common) type of all the occurrences of x in a
  - β is an instance of the type of a.

Typing constraint for an application



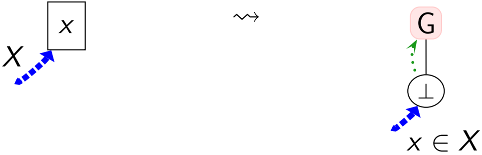
- *a b* can receive type β, provided there exists α such that
  - $a \to \beta$ is an instance of the type of *a*
  - α is an instance of the type of *b*

Typing constraint for a let



- As in ML
- Each occurrence of x in b must have a (possibly different) instance of the type of a

Typing constraint for variables

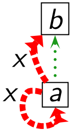


- the variable node is constrained by the appropriate edge from the typing environment

Acyclic constraints

- Constraints can encode problems with polymorphic recursion

$\boxed{\text{let rec } x = a \text{ in } b}$ $\rightsquigarrow$



- Restriction to constraints with an acyclic dependency relation

Dependency relation

g depends on g' if g' is in the scope of g, or if $g' \dashrightarrow n$ with n in the scope of g

- All typing constraints are acyclic

Solving acyclic constraints

Demo

Solving acyclic constraints

Demo

- Principal presolutions and solutions

Complexity of type inference

- ML : type inference is $\mathrm{DExp\text{-}Time}$ complete
  (if types are not printed)

- [McAllester 2003]: type inference in $O(kn(d + \alpha(kn)))$
  - k is the maximal size of type schemes
  - d is the maximal nesting of type schemes

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  - k is the maximal size of type schemes
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- In ML, d is the maximal left-nesting of let
  (*i.e.* let $x =$ (let $y = \ldots$ in $\ldots$) in $\ldots$)

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- In ML^F, unification has the same complexity as in ML, but we introduce more type schemes

  Still, d is invariant by right-nesting of let

Complexity of ML^F type inference

Under the hypothesis that programs are composed of a cascade of toplevel let declarations, type inference in ML^F has linear complexity.

Outline

1 Introduction: polymorphism in programming languages

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4 A Church-style language for MLF

5 Conclusion

An explicit langage for ML^{F}

- Study subject reduction in ML^{F}
- To be used inside a typed compiler

  ML^{F} types are more expressive than F ones

  System F cannot be used as a target langage
- Need for a core, Church-style, langage for ML^{F}, called $x\mathrm{ML}^{\mathsf{F}}$

From System F to $x\mathrm{ML}^\mathsf{F}$

$x\mathrm{ML}^\mathsf{F}$ generalizes System F

- Types: $\sigma \ ::= \ \bot \ \mid \ \forall\,(\alpha \geqslant \sigma)\,\sigma \ \mid \ \alpha \ \mid \ \sigma \to \sigma$

Rigid quantification is only needed for type inference, and is inlined in $x\mathrm{ML}^\mathsf{F}$

- Terms : $a \ ::= \ x \ \mid \ \lambda(x : \sigma)\ a \ \mid a\ a \ \mid \ \mathsf{let}\ x = a\ \mathsf{in}\ a$
  $\qquad \mid \ \Lambda(\alpha \geqslant \sigma)\ a \ \mid \ a[\varphi]$

- Typing rules are the same as in System F, except for type application

$$\textsc{TApp}\quad \frac{\Gamma \vdash a : \sigma \qquad \Gamma \vdash \varphi : \sigma \leq \sigma'}{\Gamma \vdash a[\varphi] : \sigma'}$$

Type computations

Instance is explicitely witnessed through the use of type computations

$$\varphi \ ::= \ \varepsilon \ \mid \ \varphi ; \varphi \ \mid \ \triangleright \sigma \ \mid \ \alpha \triangleleft \ \mid \ \forall (\geqslant \varphi) \mid \ \forall (\alpha \geqslant) \, \varphi \ \mid \ \& \ \mid \ \invamp$$

Type computations

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$$\varphi \;::=\; \varepsilon \;\mid\; \varphi ; \varphi \;\mid\; \triangleright \sigma \;\mid\; \alpha \triangleleft \;\mid\; \forall (\geqslant \varphi) \mid \forall (\alpha \geqslant) \, \varphi \;\mid\; \& \;\mid\; \aleph$$

Inst-Reflex

$$\frac{}{\Gamma \vdash \varepsilon : \sigma \leq \sigma}$$

Inst-Trans

$$\frac{\Gamma \vdash \varphi_1 : \sigma_1 \leq \sigma_2 \qquad \Gamma \vdash \varphi_2 : \sigma_2 \leq \sigma_3}{\Gamma \vdash \varphi_1 ; \varphi_2 : \sigma_1 \leq \sigma_3}$$

Inst-Bot

$$\frac{}{\Gamma \vdash \triangleright \sigma : \bot \leq \sigma}$$

Inst-Hyp

$$\frac{\alpha \geqslant \sigma \in \Gamma}{\Gamma \vdash \alpha \triangleleft : \sigma \leq \alpha}$$

Inst-Inner

$$\frac{\Gamma \vdash \varphi : \sigma_1 \leq \sigma_2}{\Gamma \vdash \forall (\geqslant \varphi) : \forall (\alpha \geqslant \sigma_1) \, \sigma \leq \forall (\alpha \geqslant \sigma_2) \, \sigma}$$

Inst-Outer

$$\frac{\Gamma, \varphi : \alpha \geqslant \sigma \vdash \varphi : \sigma_1 \leq \sigma_2}{\Gamma \vdash \forall (\alpha \geqslant) \, \varphi : \forall (\alpha \geqslant \sigma) \, \sigma_1 \leq \forall (\alpha \geqslant \sigma) \, \sigma_2}$$

Inst-Quant-Elim

$$\frac{}{\Gamma \vdash \& : \forall (\alpha \geqslant \sigma) \, \sigma' \leq \sigma' \{\alpha \leftarrow \sigma\}}$$

Inst-Quant-Intro

$$\frac{\alpha \notin \mathsf{ftv}(\sigma)}{\Gamma \vdash \aleph : \sigma \leq \forall (\alpha \geqslant \bot) \, \sigma}$$

Example: back to choose id

$$\begin{array}{rcll} \text{choose} & \triangleq & \Lambda(\alpha \geqslant \bot)\ \lambda(x : \alpha)\ \lambda(y : \alpha)\ x & : \ \forall\,(\alpha \geqslant \bot)\ \alpha \to \alpha \to \alpha \\ \text{id} & \triangleq & \Lambda(\beta \geqslant \bot)\ \lambda(x : \beta)\ x & : \ \forall\,(\beta \geqslant \bot)\ \beta \to \beta \end{array}$$

- To make choose id well-typed, we must choose a type into which α must be instantiated

Example: back to choose id

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- To make choose id well-typed, we must choose a type into which α must be instantiated
- $e \triangleq \Lambda(\gamma \geqslant \sigma_{id})\ \underbrace{(\text{choose}[\forall\,(\geqslant \triangleright \gamma); \&])}_{\gamma\to\gamma}\ \underbrace{(\text{id}[\gamma \triangleleft])}_{\gamma} : \forall\,(\gamma \geqslant \sigma_{id})\ \gamma \to \gamma$

Example: back to choose id

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- $\begin{cases} e[\&] & : \sigma_{id} \to \sigma_{id} \\ e[\otimes; \forall(\delta \geqslant)\ (\forall(\geqslant \forall(\geqslant \triangleright \delta); \&); \&)] & : \forall(\delta \geqslant \bot)\ (\delta \to \delta) \to (\delta \to \delta) \end{cases}$

Reducing expressions

- Usual β-reduction

$$
\begin{array}{rcl}
(\lambda(x:\tau)\ a_1)\ a_2 & \longrightarrow & a_1\{x \leftarrow a_2\} \\
\text{let } x = a_2 \text{ in } a_1 & \longrightarrow & a_1\{x \leftarrow a_2\}
\end{array}
$$

Reducing expressions

- Usual β-reduction
- 6 specific rules to reduce type applications

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\\
a[\varepsilon] & \longrightarrow & a \\
a[\varphi; \varphi'] & \longrightarrow & a[\varphi][\varphi'] \\
a[\otimes] & \longrightarrow & \Lambda(\alpha \geqslant \bot)\ a \\
& & \text{if } \alpha \notin \mathrm{ftv}(a) \\
\\
(\Lambda(\alpha \geqslant \tau)\ a)[\&] & \longrightarrow & a\{\alpha\triangleleft \leftarrow \varepsilon\}\{\alpha \leftarrow \tau\} \\
(\Lambda(\alpha \geqslant \tau)\ a)[\forall\,(\geqslant \varphi)] & \longrightarrow & \Lambda(\alpha \geqslant \tau[\varphi])\ a\{\alpha\triangleleft \leftarrow \varphi; \alpha\triangleleft\} \\
(\Lambda(\alpha \geqslant \tau)\ a)[\forall\,(\alpha \geqslant)\ \varphi] & \longrightarrow & \Lambda(\alpha \geqslant \tau)\ (a[\varphi])
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Reducing expressions

- Usual β-reduction
- 6 specific rules to reduce type applications
- Context rule

$$
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(\Lambda(\alpha \geqslant \tau)\ a)[\forall\,(\alpha \geqslant)\ \varphi] & \longrightarrow & \Lambda(\alpha \geqslant \tau)\ (a[\varphi]) \\
\\
E\{a\} & \longrightarrow & E\{a'\} \\
 & & \text{if } a \longrightarrow a'
\end{array}
$$

Results on xMLF

Correctness:

- Subject reduction, for all contexts (including under λ and Λ)

- Progress for call-by-value with or without the value restriction, and for call-by-name

  This is the first time that MLF is proven sound for call-by-name

- Mechanized proof of a previous version of the system

Results on xML^F

Correctness:

- Subject reduction, for all contexts (including under λ and Λ)
- Progress for call-by-value with or without the value restriction, and for call-by-name

  This is the first time that ML^F is proven sound for call-by-name
- Mechanized proof of a previous version of the system
- Confluence of strong reduction
- The reduction rule of System F for type applications is derivable

$$(\Lambda(\alpha)\ a)[\sigma] \longrightarrow a\{\alpha \leftarrow \sigma\}$$

(when a is a System F term, and σ a System F type)

From presolutions to xML^F terms

- ML^F presolutions can be algorithmically translated into well-typed xML^F terms

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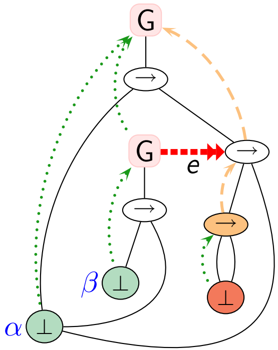
- Nodes flexibly bound on gen nodes are translated into xML^F type abstractions

- The fact that an instantiation edge is solved is translated into a type computation

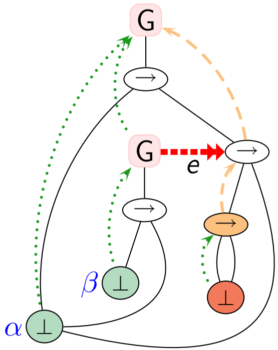
From presolutions to xML$^{\mathsf{F}}$ terms: example



A presolution for $K \triangleq \lambda(x)\ \lambda(y)\ x$

$$K : \forall\,(\alpha)\ \alpha \rightarrow \sigma_{id} \rightarrow \alpha$$

From presolutions to xMLF terms: example



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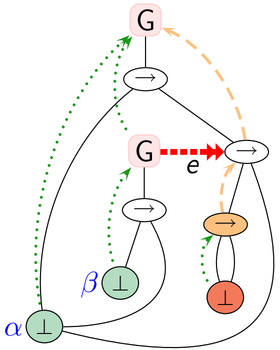

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Related works

- Bringing System F and ML closer
  - restriction to predicative fragment
  - higher-order unification
  - local type inference
  - boxy types
  - FPH, HML
- Typing constraints for ML
- Encoding ML^{F} into System F

Contributions

- Graphic types and constraints are the good way to study ML^{F}
- Presentation of ML^{F} well-understood, and modular
- Generic type inference framework: works indifferently for ML or ML^{F}
- Optimal theoretical complexity, and excellent practical complexity for type inference

Graphs can be used to explain type inference in a simple way, and not only for ML^{F}

- $x\mathrm{ML}^{\mathsf{F}}$ makes ML^{F} suitable for use in a typed compiler

Perspectives

- Extensions to advanced typing features
  - qualified types
  - GADTs, recursive types
  - dependent types
  - F^ω
- Revisit HML and FPH using our inference framework

Thanks

Equivalence and instance on types

- $\sqsubseteq$ permits only more sharing/raising/weakening
  - exactly corresponds to implementation
  - simpler to reason about

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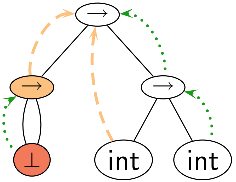
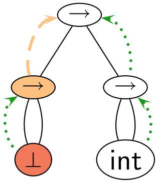


$\approx$



Equivalence and instance on types

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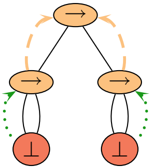

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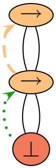

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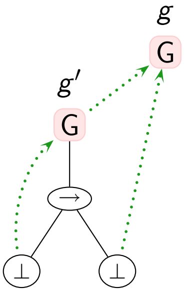
- $\boxminus$ views types up to rigid quantification and $\approx$
- $\sqsubseteq^{\boxminus}$ is $\sqsubseteq$ modulo $\boxminus$
  - most expressive system
  - undecidable type inference
  - terms typable for $\sqsubseteq^{\boxminus}$ are typable for $\sqsubseteq$ through type annotations

Expansion

Expansion takes a fresh instance of a type scheme



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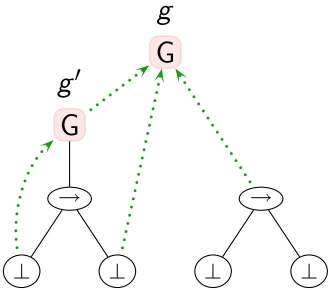

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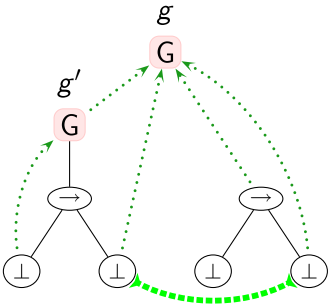

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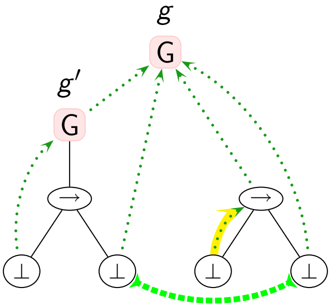

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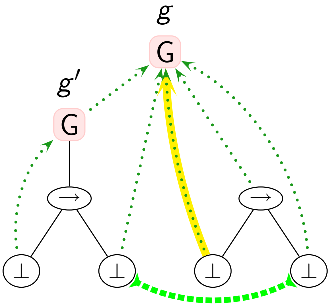

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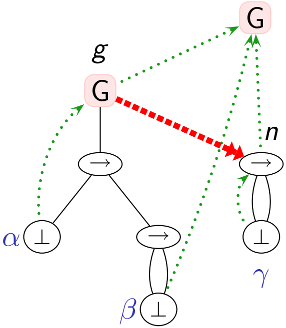
- Where to bind nodes?
  - in ML^{F}, inner polymorphism
  - in ML, to the gen node at which the copy is bound (less general)

Propagation

- Used to enforce the constraints imposed by an instantiation edge



g : $\forall\alpha.\ \alpha \to (\beta \to \beta)$
n : $\forall\gamma.\ \gamma \to \gamma$

Propagation

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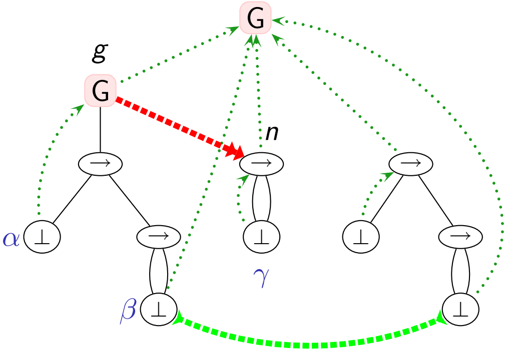

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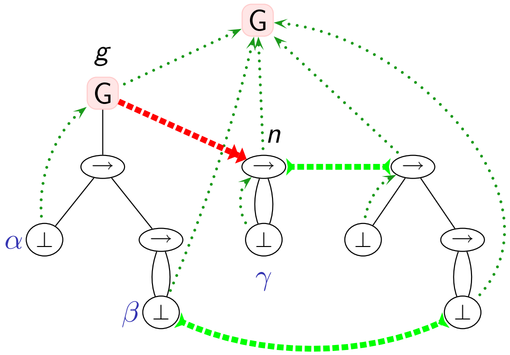

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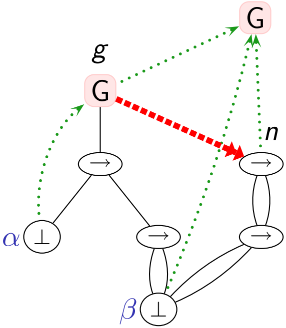


$$g : \quad \forall \alpha.\ \alpha \to (\beta \to \beta)$$
$$n : \quad (\beta \to \beta) \to (\beta \to \beta)$$

- Solving the unification edges enforces the constraint

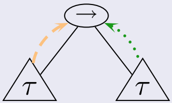
Coercions

- Annotated terms are not primitive, but syntactic sugar
  - $(a : \tau) \quad \triangleq \quad c_\tau \; a$
  - $\lambda(x : \tau)\; a \quad \triangleq \quad \lambda(x)\; \text{let } x = (x : \tau) \text{ in } a$

- **Coercion functions**

Primitives of the typing environment

c_τ :



- The domain of the arrow is frozen
- The codomain can be freely instantiated

Solving acyclic constraints

Solving an acyclic constraint χ

1. Solve the initial unification edges (by unification)
2. Order the instantiation edges according to the dependency relation
3. Propagate the first unsolved instantiation edge *e*, then solve the unification edges created

   This solves *e*, and does not break the already solved instantiation edges
4. Iterate step 3 until all the instantiation edge are solved

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Correctness

This algorithm computes a principal presolution of χ