Uncertainty representation and combination: new results with applications to nuclear safety issues

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Risk analysis → many uncertainties

Example: environmental protection
Overview

Classical situation

- Input variables/parameters → physical model → output variable

Representation → Synthesis → Propagation → Risk Evaluation

Overview

- Representation
- Synthesis
  - Information fusion
  - Reliability assessment
- Propagation
  - Independence assumptions
  - Practical propagation
- Risk evaluation and decision making
Basic setting

Situation

Describe our uncertainty about the value assumed by a variable $X$ on a domain $\mathcal{X}$ (e.g. temperature in a room, state of a sensor, ...). Here, the domain $\mathcal{X}$ is either:

- finite
- the real line $\mathbb{R}$ with associated borel $\sigma$-field

In the latter case, when considering discrete representations, we can come back to a finite domain by taking a suitable partition of $\mathbb{R}$. 

Basic setting
Why imprecise probability frameworks?

Two basic models

- Intervals or sets: no event is more likely to occur than another, complete imprecision (worst-case analysis)
- Probability distributions: precise estimation of the confidence of the occurrence of an event

In practice, often more information than an interval, but not enough to identify a precise probability.
Why imprecise probability frameworks? (Example)

How much grass per day?

Answer: usually around 12 Kg, but can go from 4 to 35 Kg

⇒ interval $[4, 35]$: less information than available

⇒ triangular probability density with mode 12 and support $[4, 35]$ → more information than really available
Coping with imprecision

Three main formal frames (denoted $F$) propose to cope with intermediary states of

- lower/upper probabilities
- random sets
- possibility theory

→ understanding their links, similarities, differences is important to achieve an unified handling of uncertainties.
Generic representation tool

**Capacity**

A capacity on \( \mathcal{X} \) is a function \( \mu \), defined on the power set \( \mathcal{P}(\mathcal{X}) \) of \( \mathcal{X} \), such that:

- \( A \subseteq B \Rightarrow \mu(A) \leq \mu(B) \) (monotonicity)
- \( \mu(\emptyset) = 0, \mu(\mathcal{X}) = 1 \) (boundary conditions)
Generic representation tool

Modeling imprecision and uncertainty

3 state of knowledge → need of two measures $\underline{\mu} \leq \bar{\mu}$:

- Certainty of event $A$: $\underline{\mu}(A) = 1, \bar{\mu}(A) = 1$
- Impossibility of event $A$: $\underline{\mu}(A) = 0, \bar{\mu}(A) = 0$
- Ignorance about event $A$: $\underline{\mu}(A) = 0, \bar{\mu}(A) = 1$

$\underline{\mu} \leq \bar{\mu}$ related by conjugacy relation such that, for any event $E$,

$$\underline{\mu}(E) = 1 - \bar{\mu}(E^c)$$
Generic framework: Lower probabilities (Walley, 91)

Associated set of probabilities

To \( \mu \) correspond a convex set (Credal set) of probabilities \( \mathcal{P}_\mu \) s.t.

\[
\mathcal{P}_\mu := \{ P \in \mathcal{P}_\mathcal{X} | (\forall A \subseteq \mathcal{X})(P(A) \geq \mu(A)) \},
\]

with \( \mathcal{P}_\mathcal{X} \): set of all probability measures on \( \mathcal{X} \).

Consistence/coherence

- \( \mu \) is said consistent if \( \mathcal{P}_\mu \neq \emptyset \)
- \( \mu \) is said coherent if \( \mu(A) = \inf_{P \in \mathcal{P}_\mu} P(A) \)
- If \( \mu \) coherent, \( \mu = P \)
Simple representations

General models: hardly tractable in practice → need for simpler representations, easier to deal with → many of them proposed and still proposed.

Problem

Recent representations (p-boxes, clouds) have not yet been related thoroughly to others.

Why such a study?

Both theoretical and practical issues

- need to know how they settle in existing frameworks
- gain insights about their expressiveness, easiness of use and other features.
A first summary

Credal sets

Coherent Lower/upper probabilities

Probabilities → Point → Sets

A → B

B is particular case of A

..... Classic. Proba.
2-monotone lower probabilities

Definition

A lower probability \( P \) is 2\( - \)monotone if, for every \( A, B \subseteq \mathcal{X} \), the inequality

\[
P(A \cup B) + P(A \cap B) \geq P(A) + P(B)
\]

holds.

Properties

- Always coherent lower probability
- Simplify many mathematical operations
The scheme continued (again)

Credal sets

Coherent Lower/upper probabilities

2-monotone capacities

Probabilities  Sets

Point

A → B

B is particular case of A

..... Classic. Proba.
Expert providing his opinion about the potential value of pH in a given field

Also correspond to: Imprecise histograms, small multinomial samples

Definition

Set \( L = \{ [l(x), u(x)] | x \in \mathcal{X} \} \) of bounds on elements of \( \mathcal{X} \) verifying inducing the credal set

\[
\mathcal{P}_L = \{ P \in \mathbb{P}_\mathcal{X} | \forall x, l(x) \leq p(x) \leq u(x) \}.
\]

We assume bounds \( L \) to be consistent and coherent

Lower (2-monotone) probability s.t.:

\[
P(A) = \max \left( \sum_{x \in A} l(x), 1 - \sum_{x \in A^c} u(x) \right)
\]
The summary continued (once again)

A → B

B is particular case of A
P-box: imprecise cumulative distribution (Ferson, 03, Williamson & Downs, 90)

Definition

Pair of cumulative distribution $[F, \bar{F}]$ on $\mathbb{R}$. Induced Lower probability consistent if $F$ stochastically dominate $\bar{F}$

$$F(x) \leq \bar{F}(x) \ \forall x \in \mathbb{R}$$

Induced credals set

$$\mathcal{P}_{[F, \bar{F}]} = \{P | \forall x, F(x) \leq P((-\infty, x]) \leq \bar{F}(x)\}$$

In practice, discrete p-box induced by a finite set of $n$ constraints

$$i = 1, \ldots, n, \alpha_i \leq P((-\infty, x_i]) \leq \beta_i$$

Expert opinions expressed through percentiles, small interval with confidence band (Kolmogorov-Smirnov distance)
Yet another summary

A \rightarrow B

B is particular case of A

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**Thesis**
- S. Destercke
- Introduction
- Context
- Representation
- Generic tools
- State of art
- Gen. p-boxes
- Clouds
- Multiple sources
- Inf. fusion
- Rel. assess.
- Application
- Propagation
- Independence
- Prac. prop
- Eval and Dec.
- Conclusions & perspectives

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**Diagram**

- Credal sets
- Coherent Lower/upper probabilities
- 2-monotone capacities
- Probability intervals
- P-boxes
- Probabilities
- Point
- Sets

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**Text**

- Yet another summary

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**Notes**

- Classic.
- Proba.
Second framework: random sets (Shafer, 76), (Dempster, 67), (Smets, 94)

Definition

A (discrete) mass distribution is a mapping \( m : \varnothing(\mathcal{X}) \rightarrow [0,1] \) such that \( \sum_{E \subseteq \mathcal{X}} m(E) = 1 \), and a set with masses \( > 0 \) is called focal. \( m(E) \) is a probabilistic mass to allocate to elements of \( E \).

\[
\begin{align*}
\text{Bel}(A) &= m(E_1) + m(E_2) \\
\text{Pl}(A) &= m(E_1) + m(E_2) + m(E_3) + m(E_5)
\end{align*}
\]

\[
\begin{align*}
\text{Bel}(A) &= \sum_{E \subseteq A} m(E) \quad \text{(Masses necessarily } \in A) \\
\text{Pl}(A) &= \sum_{E \cap A \neq \emptyset} m(E) = 1 - \text{Bel}^c(A) \quad \text{(Masses potentially } \in A)
\end{align*}
\]
A belief function \( \text{Bel} \) induce the credal set

\[
\mathcal{P}_{\text{Bel}} := \{ P \in \mathbb{P}_\mathcal{X} | (\forall A \subseteq \mathcal{X})(P(A) \geq \text{Bel}(A)) \},
\]

Practical usefulness: simulating \( \mathcal{P}_{\text{Bel}} \) by sampling \( m \)

P-boxes are special cases of random sets (Kriegler & Held, 05).

Probability Intervals

No particular links between random sets and probability intervals.

Authors have studied mapping of a prob. int. \( L \) into a random set (Lemmer & Kyburg, Denoeux)
This is not a summary

A → B

B is particular case of A
Third framework: possibility theory

**Definition**

A possibility distribution $\pi$ is a mapping $\pi : \mathcal{X} \to [0, 1]$ such that $\exists x, \pi(x) = 1$, and a set with masses $> 0$ is called focal. Given $A \subseteq \mathcal{X}$, two measures are defined:

- $\Pi(A) = \sup_{x \in A} \pi(x)$ (Possibility)
- $N(A) = 1 - \Pi(A^c)$ (Necessity)

And an $\alpha$-cut is defined as $A_\alpha = \{x \in \mathcal{X} | \pi(x) \geq \alpha\}$ (strict if the inequality is strict)
Possibility distribution is a particular case of random sets with nested realisations (Shafer, 76)
Possibilities as Credal sets

A necessity measure: special case of lower probability (Dubois & Prade, 92), (de Cooman & Aeyels, 99) inducing

\[ \mathcal{P}_\pi = \{ P \in \mathcal{P} \mathcal{X} | \forall A \subseteq \mathcal{X}, P(A) \geq N(A) \} \]

Characterization by constraints on \( \alpha \)-cuts (Dubois et al., 04), (Couso et al., 01)

\[ P(A_\alpha) \geq 1 - \alpha \ (\forall \alpha \in [0, 1]) \]

N.B. upper bounds of \( P(A_\alpha) \) always trivial (i.e. =1)
State of the art: summary

- Credal sets
  - Coherent Lower/upper probabilities
  - 2-monotone capacities
  - Random sets ($\infty$-monotone)
  - P-boxes
    - Probability intervals
    - Possibilities
    - Probabilities
    - Sets

A \rightarrow B

B particular case of A

Classic. Proba.
Generalized p-boxes: introduction

Why studying such a model?

- Possibility distributions: nested sets with lower confidence bounds
- (Discrete) P-boxes: lower and upper probabilistic bounds on (nested) sets $(-\infty, x_i]$

Both, even if poorly expressive, are very useful tools in many applications

Basic idea

Extend them both by studying a model where we give lower and upper probabilistic bounds on a collection of nested sets.
Generalized p-boxes: introduction

**Constraints**

Let \( \emptyset \subset A_1 \subset \ldots \subset A_n \subset \mathcal{X} \) be a collection and nested sets. A Generalised p-box represent constraints

\[
\alpha_i \leq P(A_i) \leq \beta_i \quad i = 1, \ldots, n
\]
\[
0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq 1
\]
\[
0 \leq \beta_1 \leq \beta_2 \leq \ldots \leq \beta_n \leq 1
\]

\( \rightarrow \) study the induced lower probability and credal set, and its link to previous representations.
An example

Evaluating impact of radionuclides inhalation on workers (e.g. in Uranium mines) \(\rightarrow\) key parameter: mean diameter of particles (AMAD)

Expert opinion translated in constraints:

- \(0.3 \leq P([4.5, 5.5]) \leq 0.6\)
- \(0.7 \leq P([4, 6]) \leq 0.9\)
- \(1 \leq P([3, 7]) \leq 1\)
Generalised p-boxes enter the picture

A \rightarrow B
B particular case of A
Generalized p-boxes: first results

First links with previous representations

\[ \alpha_i \leq P(A_i) \leq \beta_i \quad i = 1, \ldots, n \]

\[ 0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq 1 \]

\[ 0 \leq \beta_1 \leq \beta_2 \leq \ldots \leq \beta_n \leq 1 \]

- We retrieve possibility distributions when \( \beta_i = 1 \), \( i = 1, \ldots, N \)
- We retrieve p-boxes when \( \mathcal{X} = R \) and \( A_i = (-\infty, x_i) \)
Generalised p-boxes get more involved in the picture

A → B
B particular case of A
Generalized p-boxes: formal definition (Destercke et al., 08)

**Construction**

Nested sets $\emptyset \subset A_1 \subset \ldots \subset A_n = \mathcal{X} \rightarrow$ Sets $A_i \setminus A_{i-1}$ partition of $\mathcal{X}$.

Define $[F, \overline{F}]$ such that, if $x \in A_i \setminus A_{i-1}$, $\overline{F} = \beta_i, F = \alpha_i$

Two mappings $f, f'$ from $\mathcal{X} \rightarrow \mathbb{R}$ are comonotone iff $\forall x, y \in \mathcal{X}, f(x) < f(y) \rightarrow f'(x) \leq f'(y)$

**Definition**

A generalized p-box is a pair of comonotone mappings $\overline{F} : \mathcal{X} \rightarrow [0,1]$ and $\underline{F} : \mathcal{X} \rightarrow [0,1]$ s.t. $\exists x, \overline{F}(x) = \underline{F}(x) = 1$
Generalized p-boxes: formal definition (Destercke et al., 08)

Induced credal set

The credal set $\mathcal{P}_{[E,F]}$ induced by a gen. p-box $[E,F]$ is defined as

$$\mathcal{P}_{[E,F]} = \{P \in \mathcal{P}_X | \forall A_i, \alpha_i \leq P(A_i) \leq \beta_i\}$$
Generalized p-boxes: links with other representations

Theorem (Destercke et al., 08)

From any generalized p-box \([F, \overline{F}]\), we can define two possibility distributions \(\pi_F, \pi_{\overline{F}}\) on \(X\) such that

\[
P[F, \overline{F}] = P_{\pi_F} \cap P_{\pi_{\overline{F}}}
\]

holds

\(\Rightarrow\) Generalized p-boxes are representable by pairs of possibility distributions.
Generalised p-boxes get comfortable in the picture

A \rightarrow B
B particular case of A
A represent B

\begin{itemize}
\item Credal sets
\item Coherent Lower/upper probabilities
\item 2-monotone capacities
\item Random sets (\(\infty\)-monotone)
\item Generalized p-boxes
\item Probability intervals
\item P-boxes
\item Probabilities
\item Point
\item Sets
\item Possibilities
\end{itemize}
Generalized p-boxes: links with other representations

**Theorem (Destercke et al., 08)**

Any generalized p-box $[\underline{F}, \overline{F}]$ can be represented as a particular random set for which, to every level $\alpha \in [0, 1]$, we associate the focal element

$$E_\alpha \setminus F_\alpha$$

with $E_\alpha$: $\alpha$-cut of $\pi_{\overline{F}}$ and $F_\alpha$: $\alpha$-cut of $1 - \pi_{\overline{F}}$

$\Rightarrow$ Calculus used for generic random sets can be directly applied to generalized p-boxes
Generalised p-boxes and links: illustration
Theorem (Destercke et al., 08)

From a probability interval $L$, it is possible to build $|\mathcal{X}|/2$ generalized p-boxes $[F_1, \overline{F}_1], \ldots, [F_n, \overline{F}_n]|\mathcal{X}|/2$ such that

$$\mathcal{P}_L = \bigcap_{i=1}^{n} \mathcal{P} [F_i, \overline{F}_i]$$

⇒ Probability intervals representable by generalized p-boxes
Generalised p-boxes in the picture: the end

- Credal sets
- Coherent Lower/upper probabilities
- 2-monotone capacities
- Random sets ($\infty$-monotone)
- Generalised p-boxes
- Probability intervals
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- Point
- Sets
- Possibilities

A → B
B particular case of A
A → B
A represent B

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Classic. Proba.
Clouds: introduction and definition

Introduced (Neumaier, 04) to deal with imprecision in high dimensions.

**Definition**

Cloud \([\pi, \delta]\): pair of mappings \(\delta : \mathcal{X} \rightarrow [0,1]\), \(\pi : \mathcal{X} \rightarrow [0,1]\), with \(\delta \leq \pi\), \(\pi(x) = 1\) for at least one element \(x\) in \(\mathcal{X}\), and \(\delta(y) = 0\) for at least one element \(y\) in \(\mathcal{X}\).

**Induced credal set (Neumaier, 04)**

\[
P[\pi,\delta] = \{ P \in \mathbb{P} \mathcal{X} | P(\delta_x) \leq 1 - \alpha \leq P(\pi_x) \}
\]
Now clouds want to get in

Credal sets

Coherent Lower/upper probabilities

2-monotone capacities

Random sets (∞-monotone)

Generalised p-boxes

Probability intervals

P-boxes

Probabilities

Point

Sets

Possibilities

Pair of Poss. Dist.

General clouds

A ——→ B

B particular case of A

A ——→ B

A represent B

Classic. Proba.
Clouds: links with other representation

Theorem (Destercke et al., 08)

The two following statements are equivalent:

(i) The cloud \([\pi, \delta]\) can be encoded as a generalised p-box \([E, F]\) such that \(\mathcal{P}[\pi, \delta] = \mathcal{P}[E, F]\)

(ii) \(\delta\) and \(\pi\) are comonotonic \((\delta(x) < \delta(y) \Rightarrow \pi(x) \leq \pi(y))\)

and a cloud is said comonotonic if \(\delta\) and \(\pi\) are comonotonic.

\(\Rightarrow\) comonotonic clouds and generalised p-boxes: equivalent representations
They already fit in quite well
Clouds: links with other representations

**Theorem (Destercke et al., 08)**

A cloud \([\pi, \delta]\) is representable by the pair of possibility distributions \(1 - \delta\) and \(\pi\), in the following sense:

\[
P[\pi, \delta] = P\pi \cap P_{1-\delta}
\]
Theorem (Destercke et al., 08)

There are families of non-comonotonic clouds \([\pi, \delta]\) such that the lower probability induced by the credal set \(\mathcal{P}[\pi, \delta]\) is not even 2-monotone

\(\Rightarrow\) clouds not special cases of random sets, and non-comonotonic clouds appears of less practical interest.
Finally

Credal sets

Coherent Lower/upper probabilities

2-monotone capacities

Random sets ($\infty$-monotone)

Comonotonic clouds

Generalised p-boxes

General clouds

Pair of Poss. Dist.

Probability intervals

P-boxes

Probabilities

Point

Sets

Possibilities

A → B

B particular case of A

A → B

A represent B

Classic. Proba.
Overview

- Representation
- **Synthesis**
  - Information fusion
  - Reliability assessment
- Propagation
  - Independence assumptions
  - Practical propagation
- Risk evaluation and decision making
Information fusion: setting

Receiving and representing Information from multiple sources (e.g., experts, physical models) → summarise this information into a single representation

*Example: expert opinions on the same variable (e.g., AMAD)*

- Expert opinion 1
- Expert opinion 2
- Expert opinion N

Rep. 1 \( (R_1) \)  \( \rightarrow \) Rep. 2 \( (R_2) \)  \( \rightarrow \) Rep. N \( (R_N) \)

\[ \varphi(R_1, R_2, \ldots, R_N) \]

? Final representation
Behaviours of $\varphi$

**Choice of $\varphi$**

Can be guided by the presence/absence of conflict between sources

$\varphi$ can follow three main kinds of behaviour:

- **Conjunctive ($\cap$):** $\varphi(R_1,\ldots,R_N) \subseteq R_i$ for $i = 1,\ldots,N$. Result is more informative than each source. Assume reliability of all sources and no conflict between them.

- **Disjunctive ($\cup$):** $\varphi(R_1,\ldots,R_N) \supseteq R_i$ for $i = 1,\ldots,N$. Result is not more informative than each source. Assume reliability of at least one source.

- **Compromise:** result between conjunctive and disjunctive behaviours.
Conjunction/disjunction: illustration

Conjunction result: \( \emptyset \)

Disjunction result:

\[ \Rightarrow \text{Conjunction not reliable.} \]
\[ \Rightarrow \text{Disjunction too imprecise.} \]
\[ \Rightarrow \text{Inadequate to cope with partial conflict.} \]
Adaptive fusion rules

Goes from conjunction when there is no conflict towards disjunction when conflict increase

use of maximal coherent subsets as a general approach (Walley, 82), (Dubois & Prade, 90)
maximal coherent subsets: principles

Original idea from logic (Rescher & Manor, 70)

Resolve inconsistencies in knowledge bases:
- extract maximal subsets of consistent formulas (conjunction)
- proposition true if true in every subsets (disjunction)

Application to uncertainty representations
- extract $k$ maximal subsets $K_i \subseteq \{R_1, \ldots, R_N\}$ of representations having non-empty conjunction
- take the disjunction of all conjunctions.
Maximal coherent subsets: illustration (Dubois, Fargier, Prade, 00)

Maximal coherent subsets: $K_1 = \{I_1, I_2\}$ and $K_2 = \{I_2, I_3, I_4\}$

Final result: $(I_1 \cap I_2) \cup (I_2 \cap I_3 \cap I_4)$
MCS: practical issue

Problem
Maximal coherent subsets theoretically and conceptually attractive, but

Extracting MCS $\rightarrow$ NP-complete problem in boolean logic: computational intractability!

Solutions
- use heuristics and approximations
- work in a restricted but tractable framework: intervals on the real line $\rightarrow$ polynomial complexity (Dubois, Fargier, Prade, 00)
Level-wise MCS with possibility distributions

Our proposition

$N$ distributions $\pi_i$: apply MCS to each level $\alpha \in [0,1]$.

Results for $\neq$ levels $\rightarrow$ not necessarily nested
Level-wise MCS with possibility distributions

Finite set of values $\beta_i \ i = 0, 1, \ldots, n$ such that sets $E_\alpha$ resulting from MCS for $\alpha \in (\beta_i, \beta_{i+1}]$ are nested

Result: $n$ possibility distributions with weights ($\sum m(F_i) = 1$)
Level-wise MCS with possibility distributions

Summarizing the information

$m(F_i)$ Complex structure $\rightarrow$ compute contour function $\pi_c$ as an interpretable summary (weighted average of $F_i$)

\[
m(F_1) = \beta_1 - 0(\beta_0) \\
m(F_2) = \beta_2 - \beta_1 \\
m(F_3) = 1(\beta_3) - \beta_2
\]
Fusion rules for clouds?

Definition

Let \([\pi, \delta]_1, \ldots, [\pi, \delta]_N\) be \(N\) clouds, we propose the following fusion rules:

- **Conjunction:** \([\pi, \delta]_\cap = [\pi_\cap, \delta_\cap] = [\min_{i=1}^N (\pi_i), \max_{i=1}^N (\delta_i)]\).
- **Disjunction:** \([\pi, \delta]_\cup = [\pi_\cup, \delta_\cup] = [\max_{i=1}^N (\pi_i), \min_{i=1}^N (\delta_i)]\).

\(\rightarrow\) Conjunction and disjunction defined, maximal coherent subsets follow.
Overview

- **Representation**
- **Synthesis**
  - Information fusion
  - Reliability assessment
- **Propagation**
  - Independence assumptions
  - Practical propagation
- **Risk evaluation and decision making**
Evaluation of source reliability

Principle (Cooke, 91), (Sandri et al., 95)

Evaluate sources from past performance. Two quantitative values:

- **Precision** of information delivered by source. The more precise the information, the more useful it is $\Rightarrow$ proposition of a general criteria based on cardinality

- **Accuracy**: consistency between delivered information and observed (experimental) values $\Rightarrow$ proposition of a general criteria based on inclusion index

- **Global**: global score $= \text{precision} \times \text{accuracy}$
Ten different institutes use their own models and experts to reproduce a simulated accident → use fusion rules and information evaluation technics to analyse information, with the help of SUNSET software.
Le cas BEMUSE : incertitudes cibles

Problème générique : chaque participant a fourni, pour chaque variable d'intérêt, la réponse obtenue suite à son analyse d'incertitude. La comparaison de ces résultats, relativement différents, n'est pas aisée (au-delà de quelques constatations évidentes)

Dans le cas BEMUSE : utilisation d'écarts maximaux et d'indicateurs statistiques simples pour tenter d'extraire de l'information.

Experiment to reproduce

10 ≠ mod-elling of this experiment

10 ≠ Results to compare

CATHARE
ATHLET
RELAP5
MARS
Application to result of OCDE project BEMUSE

Result

- Detection of participants overestimating (bad precision, good accuracy) or underestimating (good precision, bad accuracy) their uncertainty
- Quantified evaluation of conflict between subgroups of sources
- Generic tool to validate computer codes

Interest of non-experts

- Results added to final report
- Price at $\lambda \mu$ conference (high number of participants from industry)
Overview

- Representation
- Synthesis
  - Information fusion
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- Risk evaluation and decision making
Propagate uncertainty through a model $f(X_1, \ldots, X_N) = Y$ to evaluate uncertainty on $Y$.

- Often, information given separately for $X_1, \ldots, X_N$
- Then propagate through $f$ with independence assumptions between
- Many different notions of independence when using imprecise probabilistic frameworks

→ need to make some sense of them, to relate them and to understand their respective usefulness
Our contribution

Preliminary work

First classification of independence notions based on:

- Informative vs non-informative
- Symmetric vs Asymmetric
- Objective vs Subjective

Practical results:

- using more tractable independence notions as conservative approximation of less tractable ones
- relating notions of independence to imprecise probabilistic trees (work with G. de Cooman)
Overview

- Representation
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Starting point

**Hybrid propagation**

Propagate by differentiating aleatory uncertainty (probabilistic calculus) from epistemic uncertainty (possibilistic calculus)

High computational cost to concentrate on specific summary → sometimes unaffordable
"RaFu" method (implemented in SUNSET software)

Use hybrid propagation → sample from distributions only values needed to compute desired result.

Reduce number of computations (∼ 10 to 20 times less) by concentrating on desired result ⇒ currently applied in BEMUSE propagation
Overview

- Representation
- Synthesis
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  - Independence assumption
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- Risk evaluation and decision making
Computing expectations

With probabilities

Decision making based on the computation of expected value $\mathbb{E}_P(u)$ of a function $u : \mathcal{X} \rightarrow \mathbb{R}$, given a probability measure $P$:

$\mathbb{E}_P(u) = \sum_{x \in \mathcal{X}} u(x) P(\{x\})$ if $\mathcal{X}$ finite

$\mathbb{E}_P(u) = \int_{\mathbb{R}} u(x) \, dP$ if $\mathcal{X} = \text{real line}$
Computing expectations

With imprecise probabilities

Expected values become imprecise → compute $[\underline{E}(u), \overline{E}(u)]$

- When $\mathcal{X}$ finite → efficient algorithms to compute them (Utkin & Augustin, 05)
- When $\mathcal{X} = \mathbb{R}$ → hard problem in general

→ start from simple representations → p-boxes
Computing expectations

With P-boxes (work with L. Utkin)

Given a (cont.) function $u$ on $\mathbb{R}$ and a (classical) P-box $[F, \overline{F}]$, find

$$E_{[F, \overline{F}]}(u) = \inf_{F \in [F, \overline{F}]} \int_{\mathbb{R}} u(x) dF(x),$$

$$\overline{E}_{[F, \overline{F}]}(u) = \sup_{F \in [F, \overline{F}]} \int_{\mathbb{R}} u(x) dF(x).$$

→ Find $F$ inside $[F, \overline{F}]$ reaching $[E_{[F, \overline{F}]}(u), \overline{E}_{[F, \overline{F}]}(u)]$

$F$ for which **lower expectation** is reached with $a_i$: local maxima, $b_i$: local minima
Conclusions

New results and new methodologies regarding the problems of

- Representing uncertainty: **Gen. P-boxes, relations with clouds.**
- Dealing with multiple sources: **MCS method on possibilities**
- Propagating uncertainties: **improving IRSN algorithm**
- Making decision under uncertainty: **computation of expectations on p-boxes**

Keeping in mind the three frameworks we chose to work in and that successful applications need:

1. Theoretically sound methods
2. Tractable methods
Next challenges and perspectives

Theoretical

As we have done for uncertainty representations, there is a need to provide a unified framework for the problems of

- Information fusion (e.g., study idempotent rules in random set theory)
- Independence modelling (e.g., how to model both source dependencies and variable dependencies)
- Conditioning our knowledge on some event (e.g., compare the notions of focusing on a particular subfamily, revising my information and learning from new information)
Next challenges and perspectives

**Practical**

- uncertainty representations:
  - build sound elicitation methods
- multiple sources treatment:
  - propose efficient algorithm to fuse information using maximal coherent subsets approach in general frames
- propagation
  - algorithmic work on the combined use of MC simulation + interval analysis + heuristic approaches
  - design efficient methods to simulate credal sets
- decision making
  - explore the computation of lower/upper expectations for other representations and for multiple variables
Next challenges and perspectives

Applications

With the help of SUNSET software, applications in perspective encompass:

- Evaluation of environmental impacts of radioactive wastes on river populations (few data available)
- Similar study as the one in BEMUSE programme to study/validate the results provided by computer codes simulating fires
- Expert system using MCS approach in dosimetry (monitoring of exposed workers)