Affine cluster algebras

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November 6th, 2008

**Motivation**: Framework for a *combinatorial* study of
- total positivity in algebraic groups,
- canonical bases in quantum groups.

**Connections**:
- Combinatorics,
- Lie Theory,
- Poisson Geometry,
- Representation theory...
Problematics

- **Problem**: Find and compute bases in cluster algebras.
- **Canonical Bases**:
  - Sherman-Zelevinsky ($\tilde{A}_2, \tilde{A}_{1,1}$),
  - Cerulli Irelli ($\tilde{A}_{2,1}$).
- **Bases**:
  - Caldero-Keller (finite type),
  - Geiss-Leclerc-Schröer (general, abstract).
- **Strategy**: Give an unified and explicit method to compute bases in cluster algebras using representation theory.
1 Cluster algebras and cluster categories

2 Affine cluster algebras

3 Generalized Chebyshev polynomials

4 Generic variables

5 Further directions
A seed is a pair \((Q, x)\) such that:

- \(Q = (Q_0, Q_1)\) is a quiver without loops and 2-cycles;
- \(x = (x_i : i \in Q_0)\) is a \(Q_0\)-tuple of indeterminates over \(\mathbb{Z}\), called \emph{cluster of the seed} \((Q, x)\).
Mutation of seeds

For every $k \in Q_0$, $\mu_k(Q, x) = (Q', x')$ is the new seed given by:

\[
\begin{array}{|c|c|}
\hline
Q & Q' \\
\hline
i \xrightarrow{r} j & i \xrightarrow{r+st} j \\
\downarrow & \downarrow \\
\downarrow & \downarrow \\
s \xleftarrow{r} k & s \xleftarrow{r-st} k \\
\hline
\end{array}
\]

and

\[
x' = x \setminus \{x_k\} \sqcup \{x'_k\}
\]

where

\[
x_kx'_k = \prod_{i \xrightarrow{k\in Q_1}} x_i + \prod_{k \xrightarrow{i\in Q_1}} x_i.
\]

We denote by $(Q, x) \sim_{\text{mut}} (R, y)$ the generated equivalence relation.
Acyclic cluster algebras

Let \((Q, u)\) be a seed with \(Q\) ayclic.

**Definition**

The cluster algebra \(\mathcal{A}(Q)\) with initial seed \((Q, u)\) is

\[
\mathcal{A}(Q) = \mathbb{Z}[x \mid x \in \mathbf{c} \text{ s.t. } (R, \mathbf{c}) \sim_{\text{mut}} (Q, u)] \subset \mathbb{Q}(u)
\]

The \(\mathbf{c}\) occurring are called the clusters of \(\mathcal{A}(Q)\),

The \(x \in \mathbf{c}\) are called the cluster variables of \(\mathcal{A}(Q)\).

\[
\text{Cl}(Q) = \{\text{cluster variables}\}
\]
The Laurent phenomenon

Theorem, Fomin-Zelevinsky, 2001

\( \mathcal{A}(Q) \subset \mathbb{Z}[u^{\pm 1}] \).

If \( x \in \mathbb{Z}[u^{\pm 1}] \), the denominator vector \( \text{den}(x) \in \mathbb{Z}^Q \) of \( x \) is given by

\[
x = \frac{P(u)}{u^{\text{den}(x)}}
\]

in its irreducible form.
A *cluster monomial* is a monomial in cluster variables belonging to a same cluster. We set

\[ \mathcal{M}(Q) = \{\text{cluster monomials in } A(Q)\} . \]
A cluster algebra $\mathcal{A}(Q)$ is said to be of \textit{finite type} if $|\text{Cl}(Q)| < \infty$. 
Simply-laced Dynkin diagrams

\[ \mathbb{A}_n \quad \mathbb{D}_n \quad \mathbb{E}_6 \quad \mathbb{E}_7 \quad \mathbb{E}_8 \]
\( \mathcal{A}(Q) \) is of finite type if and only if \( Q \) is a Dynkin quiver. In this case, \( \text{den} \) induces a 1-1 correspondence

\[
\text{den} : \text{Cl}(Q) \longrightarrow \Phi_{>0}(Q) \sqcup (-\Pi(Q)).
\]
Theorem, Caldero-Keller, 2005
If $\mathcal{A}(Q)$ is of finite type, then $\mathcal{M}(Q)$ is a $\mathbb{Z}$-basis in $\mathcal{A}(Q)$.

Fact, Sherman-Zelevinsky
In general, $\mathcal{M}(Q)$ does not span $\mathcal{A}(Q)$.

Conjecture, Zelevinsky
In general, $\mathcal{M}(Q)$ is linearly independent over $\mathbb{Z}$. 
The cluster category

Let \( k = \mathbb{C} \), \( Q \) be an acyclic quiver

\[ kQ\text{-mod} \cong \text{rep}(Q). \]

**Definition, BMRRT, 2004**

The *cluster category* of \( Q \) is the orbit category of the auto-functor \( F = \tau^{-1}[1] \) in the bounded derived category \( D^b(kQ) \) of \( kQ\text{-mod} \).

\[ \mathcal{C}_Q = D^b(kQ)/F. \]

**Theorem, K, BMRRT, 2004**

- \( \mathcal{C}_Q \) is a triangulated category;
- \( \text{Ext}^1_{\mathcal{C}_Q}(X, Y) \cong D\text{Ext}^1_{\mathcal{C}_Q}(Y, X) \) (2-Calabi-Yau);
- \( \text{ind}(\mathcal{C}_Q) = \text{ind}(kQ\text{-mod}) \sqcup \{ P_i[1] : i \in Q_0 \} \).
The quiver grassmannian

- Let $M$ be a $kQ$-module and $e \in \mathbb{Z}^Q_0$. We write
  \[ \text{Gr}_e(M) = \{ N \subset M : \dim N = e \} \]
  the quiver grassmannian.
- We denote by $\chi$ the Euler-Poincaré characteristic.
The Caldero-Chapoton map

**Definition, Caldero-Chapoton**

The *Caldero-Chapoton* map is the map \( \mathcal{X} : \text{Ob}(\mathcal{C}_Q) \rightarrow \mathbb{Z}[u^{\pm 1}] \):

- If \( M, N \) are in \( \text{Ob}(\mathcal{C}_Q) \), then \( \mathcal{X}_{M \oplus N} = \mathcal{X}_M \mathcal{X}_N \);
- If \( M \simeq P_i[1] \), then \( \mathcal{X}_{P_i[1]} = u_i \);
- If \( M \) is an indecomposable module, then

\[
\mathcal{X}_M = \sum_e \chi(\text{Gr}_e(M)) \prod_{i \in Q_0} u_i^{-\langle e, \dim S_i \rangle - \langle \dim S_i, \dim M - e \rangle}.
\]

Equality (1) holds for any \( kQ \)-module \( M \).
Theorem, Caldero-Keller

\[ X \] induces a 1-1 correspondence

\[ \{ \text{indecomposable rigid objects in } C_Q \} \xrightarrow{\sim} \text{Cl}(Q). \]

Moreover, the map

\[ T = \bigoplus_{i \in Q_0} T_i \xrightarrow{\sim} \{ \text{clusters in } A(Q) \} \]

is a 1-1 correspondence.

\[ \{ \text{maximal rigid objects in } C_Q \} \]

is a 1-1 correspondence.
Corollary

\( X? \) induces a 1-1 correspondence

\[ \{ \text{rigid objects in } \mathcal{C}_Q \} \xrightarrow{\sim} \mathcal{M}(Q). \]

Corollary

\( \text{den} \) induces a 1-1 correspondence

\[ \text{Cl}(Q) \xrightarrow{\sim} \Phi^{\text{re}, \text{Sc}}(Q) \sqcup (\neg \Pi(Q)) \]
The one-dimensional multiplication formula

**Theorem, CK**

Let $M, N$ be indecomposable objects in $C_Q$ such that $\dim \text{ Ext}^1_{C_Q}(M, N) = 1$. Then

$$X_M X_N = X_B + X_{B'}$$

where $B$ and $B'$ are the unique objects such that there exists non-split triangles

$$M \rightarrow B \rightarrow N \rightarrow M[1],$$

$$N \rightarrow B' \rightarrow M \rightarrow N[1]$$
in $C_Q$. 

1. Cluster algebras and cluster categories

2. **Affine cluster algebras**

3. Generalized Chebyshev polynomials

4. Generic variables

5. Further directions
Motivation

- Finite-tame-wild classification theorem
- Affine quivers are minimal among representation-infinite quivers
- Representation theory of affine quivers is well-known
Simply laced affine diagrams

\[ \tilde{A}_n \]

\[ \tilde{D}_n \]

\[ \tilde{E}_6 \]

\[ \tilde{E}_7 \]

\[ \tilde{E}_8 \]
Definition
A quiver $Q$ is called affine if it is acyclic and if its underlying diagram is an affine diagram.

Definition
A cluster algebra $\mathcal{A}(Q)$ is called affine if $Q$ is an affine quiver.
Affine root systems

\[ \Phi_{>0}(Q) = \Phi_{>0}^{\text{re}}(Q) \sqcup \mathbb{N}^* \delta \]

\[ \Phi^{\text{Sc}}(Q) = \Phi^{\text{re,Sc}}(Q) \sqcup \{ \delta \} \]

Kac’s theorem

Let \( d \in \mathbb{N}^{Q_0} \). Then

- \( \exists M \) indecomposable in \( \text{rep}(Q, d) \) iff \( d \in \Phi_{>0}(Q) \);
- \( \exists! M \) indecomposable in \( \text{rep}(Q, d) \) iff \( d \in \Phi_{>0}^{\text{re}}(Q) \);
- There exists a 1-parameter family of pairwise non-isomorphic indecomposable representations in \( \text{rep}(Q, n\delta) \) for every \( n \geq 1 \).
The Auslander-Reiten quiver of \( kQ\)-mod

\[
\begin{align*}
\mathcal{P} & \quad \mathcal{R} & \quad \mathcal{I} \\
{\text{real Schur}} & \quad & \quad {\text{real Schur}}
\end{align*}
\]
Tubes in \( \Gamma(kQ) \)
Tubes in $\Gamma(kQ)$
Tubes in $\Gamma(kQ)$
Contents

1. Cluster algebras and cluster categories
2. Affine cluster algebras
3. Generalized Chebyshev polynomials
4. Generic variables
5. Further directions
**Problem:** Understand $X_?$ on regular components.

**Strategy:** Use the combinatorial description of regular components in order to have a *combinatorial* description of the behaviour of $X_?$. 
Let $x_i, i \geq 1$ be indeterminates over $\mathbb{Z}$.

**Definition**

The *n*-th generalized Chebyshev polynomial $P_n$ is given by

$$P_n(x_1, \ldots, x_n) = \det \begin{bmatrix} x_n & 1 & \cdots & (0) \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ (0) & \cdots & \cdots & x_1 \end{bmatrix} \in \mathbb{Z}[x_1, \ldots, x_n]$$
A tube

\[
\begin{array}{cccccc}
R_2 & R_0 & R_1 & R_2 & R_0 \\
\end{array}
\]
A tube

\[ R_2 \rightarrow R_0 \rightarrow R_1 \rightarrow R_2 \rightarrow R_0 \]

\[ R_0^{(2)} \rightarrow R_0^{(3)} \rightarrow R_1^{(2)} \]
A tube
Example in type $\tilde{A}_{3,1}$

Let $Q$ be an affine quiver of type $\tilde{A}_{3,1}$.

\[
Q : 1 \rightarrow 2 \rightarrow 3 \rightarrow 4
\]

$\Gamma(kQ)$ contains an unique exceptional tube $\mathcal{T}_0$ and $\text{rg}(\mathcal{T}_0) = 3$. We denote by $E_0, E_1, E_2$ the quasi-simple modules in $\mathcal{T}_0$. 
Example: Quasi-simples in the exceptional tube of $\tilde{A}_{3,1}$

$E_0 : \begin{array}{ccc}
0 & \rightarrow & k \\
\uparrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}$

$E_1 : \begin{array}{ccc}
k & \rightarrow & 0 \\
\uparrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}$

$E_2 : \begin{array}{ccc}
0 & \rightarrow & 0 \\
\uparrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}$
The exceptional tube of $\tilde{A}_{3,1}$
Variables in the exceptional tube of $\tilde{A}_{3,1}$

\[ x_0 = X_{E_0} = \frac{u_2 + u_4}{u_3}, \quad x_1 = X_{E_1} = \frac{u_1 + u_3}{u_2}, \]

\[ x_2 = X_{E_2} = \frac{1 + u_1 u_3 + u_2 u_4}{u_1 u_4}. \]
Variables in the exceptional tube of $\tilde{A}_{3,1}$

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\[ X_{E_0^{(2)}} = \frac{u_1 u_2 + u_1 u_4 + u_3 u_4}{u_2 u_3}. \]
Variables in the exceptional tube of $\tilde{\mathbb{A}}_{3,1}$

\[ x_0 = X_{E_0} = \frac{u_2 + u_4}{u_3}, \quad x_1 = X_{E_1} = \frac{u_1 + u_3}{u_2}, \]

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\[ X_{E_0^{(2)}} = \frac{u_1 u_2 + u_1 u_4 + u_3 u_4}{u_2 u_3}. \]

\[ X_{M_0} = \frac{u_1^2 u_3 u_4 + u_1^2 u_2 u_3 + u_1 u_3^2 u_4 + u_1 u_4 + u_1 u_2 + u_3 u_4 + u_2 u_3 u_4^2}{u_1 u_2 u_3 u_4}. \]
1. Cluster algebras and cluster categories
2. Affine cluster algebras
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Motivations

- "Generalizing" cluster monomials,
- Analogue of the dual semicanonical basis.
Lemma, D. 2008

Let $Q$ be an acyclic quiver and $d \in \mathbb{N}^{Q_0}$. Then, there exists an open dense subset $U_d \subset \text{rep}(Q, d)$ such that $X_?$ is constant over $U_d$. We denote by $X_d$ the value of $X_?$ on this open subset.
Definition of generic variables

Let $d \in \mathbb{Z}^{Q_0}$. We set

$$X_d = X_{[d]} + \prod_{d_i < 0} u_i^{-d_i}$$

the generic variable of dimension $d$.

$$\mathcal{B}'(Q) = \left\{ X_d : d \in \mathbb{Z}^{Q_0} \right\}$$
Generic variables and cluster monomials

**Proposition, D. 2008**

Let $Q$ be an acyclic quiver. Then

$$\mathcal{M}(Q) \subset \mathcal{B}'(Q).$$

Moreover, if $Q$ is Dynkin, then

$$\mathcal{M}(Q) = \mathcal{B}'(Q).$$
Let $Q$ be an acyclic quiver, $d \in \mathbb{N}^{Q_0}$ and $d = d_1 \oplus \cdots \oplus d_n$ its canonical decomposition. Then,

$$X_d = \prod_{i=1}^{n} X_{d_i}.$$ 

It thus suffices to compute $X_d$ for $d \in \Phi^{Sc}$. 
Proposition, D. 2008

Let $Q$ be an affine quiver and $d \in \Phi^{Sc}(Q)$.

- If $d \in \Phi^{Sc, re}(Q)$, then $X_d \in \mathcal{M}(Q)$;
- Otherwise, $d = \delta$ and $X_\delta = X_{\mathcal{M}_\lambda}$ for any $\lambda \in \mathbb{P}^1_0$.

Corollary, D. 2008

Let $Q$ be an affine quiver. Then,

$$B'(Q) = \mathcal{M}(Q) \sqcup \{X_\delta^n X_E : n \geq 1, E \in \mathcal{E}_R\}$$
The difference property

Definition

Let $Q$ be an affine quiver. We say that $Q$ satisfies the difference property if for every indecomposable $kQ$-modules $M, M_\lambda$ in $\text{rep}(Q, \delta)$ belonging respectively to an exceptional and an homogeneous tube, we have:

$$X_{M_\lambda} = X_M - X_{q.\text{rad}M/q.\text{soc}M}.$$
The difference property

\[ \frac{\text{q.rad}(M)}{\text{q.soc}(M)} \]

\( M \)
Let $Q$ be an affine quiver of type $\tilde{A}$. Then $Q$ satisfies the difference property.

Conjecture

Every affine quiver satisfies the difference property.
Lemma, D. 2008
Let $Q$ be an affine quiver satisfying the difference property. Then,

$$\mathbb{Z}[X_M : M \in \text{Ob}(C_Q)] = \mathcal{A}(Q).$$

Corollary, D. 2008
Let $Q$ be an affine quiver satisfying the difference property. Then,

$$\mathcal{B}'(Q) \subset \mathcal{A}(Q).$$
The semicanonical basis

**Theorem, D. 2008**

Let $Q$ be an affine quiver such that every quiver reflection-equivalent to $Q$ satisfies the difference property. Then, $\mathcal{B}'(Q)$ is a $\mathbb{Z}$-basis in $\mathcal{A}(Q)$.

**Corollary, D. 2008**

Let $Q$ be an affine quiver of type $\tilde{\mathbb{A}}$. Then, $\mathcal{B}'(Q)$ is a $\mathbb{Z}$-basis in $\mathcal{A}(Q)$.

**Conjecture**

Let $Q$ be an affine quiver. Then, $\mathcal{B}'(Q)$ is a $\mathbb{Z}$-basis in $\mathcal{A}(Q)$. 
If $\lambda \neq 0$, $M_\lambda$ is a quasi-simple in an homogeneous tube. $M_0$ is in $\mathcal{T}_0$ and

$$q.\text{soc} \ M_0 \simeq E_0, \quad q.\text{rad} \ M_0 \simeq E_0^{(2)}.$$
The exceptional tube of $\tilde{A}_{3,1}$
Example of difference property for type $\tilde{A}_{3,1}$

<table>
<thead>
<tr>
<th>e</th>
<th>$0$</th>
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$$X_{M_0} = \frac{u_1^2 u_3 u_4 + u_1^2 u_2 u_3 + u_1 u_3^2 u_4 + u_1 u_4 + u_1 u_2 + u_3 u_4 + u_2 u_3 u_4^2}{u_1 u_2 u_3 u_4}$$

$$X_{M_\lambda} = \frac{u_1^2 u_2 u_3 + u_1 u_2 + u_1 u_4 + u_3 u_4 + u_2 u_3 u_4^2}{u_1 u_2 u_3 u_4}$$

$$X_{M_0} = X_{M_\lambda} + \frac{u_2 + u_4}{u_3} = X_{M_\lambda} + X_{E_0}$$
The semicanonical basis of $\mathcal{A}(\tilde{\mathbb{A}}_{3,1})$

\[ x_0 = X_{E_0}, x_1 = X_{E_1}, x_2 = X_{E_2}, \]
\[ y_0 = X_{E_0^{(2)}}, \quad y_1 = X_{E_1^{(2)}}, \quad y_2 = X_{E_2^{(2)}}, \]
\[ z = X_{M_{\lambda}} \]

Alors,

\[ B'(Q) = \mathcal{M}(Q) \sqcup \{ z^n x_i^r y_i^s : n > 0, r, s \geq 0, i = 0, 1, 2 \} \]

est une $\mathbb{Z}$-base de $\mathcal{A}(Q)$. 
Further directions

- Canonical bases for affine quivers,
- Cluster algebras with coefficients,
- Semicanonical bases for wild quivers,
- Connections with the dual semicanonical basis.