# Vorticité dans le modèle de Ginzburg-Landau de la supraconductivité

Hassen Aydi

# THÈSE

présentée par

**Hassen AYDI**

pour obtenir le titre de

## DOCTEUR DE L'UNIVERSITÉ PARIS XII

Spécialité: Mathématiques

Titre: **Vorticité dans le modèle de Ginzburg-Landau de la supraconductivité.**

Soutenue le 17 décembre 2004 devant le jury composé de:

| | |
|---|---|
| M. Fabrice BÉTHUEL | Président de jury |
| Mme Lia BRONSARD | Rapporteur |
| M. Rejeb HADIJI | Examinateur |
| M. Etienne SANDIER | Directeur de thèse |
| Mme Sylvia SERFATY | Rapporteur |

# Remerciements

Je suis sincèrement reconnaissant envers Etienne SANDIER d'avoir accepté de diriger mon travail de thèse, de m'avoir proposé un sujet passionnant et surtout de m'avoir toujours donné d'excellents conseils. J'exprime encore ma profonde gratitude envers lui pour sa rigueur et son enthousiasme. Avec toute mon amitié, je lui adresse ma plus grande estime.

Je suis très heureux que Lia BRONSARD et Sylvia SERFATY aient accepté la lourde tâche de rapporter ma thèse, et particulièrement honoré qu'elles aient toutes les deux accepté d'assister à ma soutenance. Je remercie chaleureusement Fabrice BÉTHUEL et Rejeb HADIJI de m'avoir fait l'honneur et le plaisir de participer à mon jury.

Le bon déroulement de cette thèse doit beaucoup aux excellentes conditions de travail que j'ai pu trouver au laboratoire de mathématiques de Créteil, ainsi qu'à la possibilité qui m'a été offerte de participer à des conférences internationales. C'est l'occasion pour moi de remercier tous les membres du laboratoire, en particulier Frank Pacard.

J'ai apprécié l'esprit d'entraide et de bonne humeur qui règne au laboratoire. En particulier, je remercie Mohamed et Aurélia mes collègues du bureau P3-437 durant ces trois années avec qui j'ai eu des discussions passionnantes et hilarantes. Je garde également un bon souvenir des nombreuses conversations avec Arnaud, alors que nous faisions ensemble le trajet du retour à Cachan ou à Paris. Mes remerciements vont aussi à mes amis Abdellatif, Boris, Fethi, Habib, Jing, Mar, Saida, Sana et Slah, qui ont rendu si agréable ce séjour en France. Je n'oublie pas les innombrables cafés qu'on a pris ensemble. J'en arrive maintenant aux amis de Cachan. Que tous trouvent ici mes remerciements les plus sincères pour l'amitié et les bons moments tout au long de cette dernière année.

J'ai bien sûr une pensée toute particulière pour Dada qui s'est constamment préoccupée de ma scolarité et m'a soutenu pendant toutes mes études universitaires. Encore un grand merci à ma famille qui n'a jamais cessé de m'encourager, surtout à mon oncle Béchir pour l'aide et le soutien qu'il m'a apportés tout au long de ma thèse.

Je voudrais enfin dire un grand merci à tous ceux qui m'entourent et qui se reconnaîtront. Je dédie ce travail à l'âme de mon père qui est toujours là présente dans mon coeur.

**Résumé : Vorticité dans le modèle de Ginzburg-Landau de la supraconductivité**

Prenant $\varepsilon = \frac{1}{\kappa}$ avec $\kappa > 0$ est le paramètre de Ginzburg-Landau, ce mémoire de thèse porte sur l'étude asymptotique dans la limite $\varepsilon \to 0$ des minimiseurs périodiques ainsi que des points critiques de l'énergie de Ginzburg-Landau.
En première partie, on prouve pour des certeins champs magnétiques appliqués $h_{ex}$ à la surface du supraconducteur de l'ordre du premier champ critique $H_{c_1} = \frac{|\log \varepsilon|}{2}$ que pour les minimiseurs périodiques de Ginzburg-Landau, le nombre des vortex par période est de l'ordre de $h_{ex}$ et leur répartition est uniforme. En outre, en prenant des champs $h_{ex}$ proches de $H_{c_1}$ de la forme $h_{ex} = H_{c_1} + f(\varepsilon)$ où $f(\varepsilon) \to +\infty$ et $f(\varepsilon) = o(|\log \varepsilon|)$, on montre que le nombre de vortex des minimiseurs périodiques par période est de l'ordre de $f(\varepsilon)$ et leur répartition est aussi uniforme.
Dans une deuxième partie, toujours dans le modèle périodique, on construit une suite de points critiques ayant des vortex répartis sur un nombre fini de lignes horizontales.
Dans une troisième partie, on construit dans le cas d'un disque une suite de points critiques telle que les vortex sont répartis sur un nombre fini de cercles concentriques de rayon strictement positif et de centre, le centre du disque. Dans le cas où il y a un seul cercle de vorticité, le rayon est bien caractérisé.
Finalement, dans un modèle de Ginzburg-Landau avec "pinning", on s'intéresse à l'étude du signe des degrés des vortex et on donne des résultats partiels indiquant que les degrés ne sont pas toujours positifs.

**Mots clés :** EDP non linéaire ; Équations de Ginzburg-Landau ; Supraconductivité ; Modèle périodique ; Vorticité ; Effets de concentration ; Convergence de mesure ; Comportment asymptotique ; "pinning" de vortex.

**Abstract: Vorticity in the Ginzburg-Landau model of superconductivity**

Taking $\varepsilon = \frac{1}{\kappa}$ where $\kappa > 0$ is the Ginzburg-Landau parameter, this PhD thesis is devoted to the study of the asymptotic behavior in the limit $\epsilon \to 0$ of periodic minimizers and also of critical points of the Ginzburg-Landau energy.
In the first part, we prove for certain applied magnetic fields $h_{ex}$ of the order of the first critical field $H_{c_1} = \frac{|\log \varepsilon|}{2}$ that periodic minimizers of the Ginzburg-Landau energy have a uniform vortex-distribution where their number per period is of the order of $h_{ex}$. Moreover, considering fields $h_{ex}$ close enough to $H_{c_1}$ in the form of $h_{ex} = H_{c_1} + f(\varepsilon)$ where $f(\varepsilon) \to +\infty$ and $f(\varepsilon) = o(|\log \varepsilon|)$, we check that the number of vortices in the periodic minimizers per period is close to $f(\varepsilon)$ and their repartition is uniform too.
In the second part, still in the periodic model, we construct a sequence of critical points such that the vortices are supported on a finite number of horizontal lines.
In the third part, we construct in the case of a disk domain a sequence of critical points such that the vortices are concentrated on a finite number of concentric circles of positive radii and of center, the center of the disk. Also, in the case where there is one circle of vorticity, the radius is well characterized.
Finally, in a Ginzburg-Landau model with pinning, we are interested in the sign of the degrees of the vortices and we give partial results indicating that the degrees may not always be positive.

# Contents

# Chapter 1

# Introduction

Les équations de Ginzburg-Landau sont des équations aux dérivées partielles non linéaires proposées dans les années 50 pour la modélisation de la supraconductivité. Depuis, elles sont devenues un outil très courant dans de nombreux domaines de la physique où des tourbillons et/ou des défauts topologiques interviennent, comme par exemple les super-fluides. De nouveaux problèmes de cette nature apparaissent constamment en physique (par exemple le ferromagnétisme, les condensats de Bose-Einstein,...). Depuis les années 90, des avancées importantes ont eu lieu dans la compréhension mathématique des équations de Ginzburg-Landau. Elles font intervenir des techniques issues de nombreux domaines des mathématiques: EDP non linéaire, théorie géométrique de la mesure, effets de concentration, tourbillons, etc.

## 1 Sur un domaine borné

Dans le modèle de Ginzburg-Landau, l'énergie libre d'un supraconducteur soumis à un champ magnétique extérieur $h_{ex}$ appliqué à sa surface est donnée après renormalisations par

$$J_\Omega(u,A) = \frac{1}{2}\int_\Omega |\nabla u - i\,A\,u|^2 + \frac{1}{2}\int_\Omega |h-h_{ex}|^2 + \frac{\kappa^2}{4}\int_\Omega (1-|u|^2)^2. \tag{1.1}$$

Ici le supraconducteur est assimlié à un cylindre vertical de section $\Omega \subset \mathbb{R}^2$, régulière et simplement connexe. $A$ est le potential-vecteur du champ magnétique induit $h = rotA$ et $u$ est le “paramètre d'ordre” qui indique l'état local du matériau. Là où $|u| \simeq 0$ c'est la phase normale, là où $|u| \simeq 1$ la phase supraconductrice. $\kappa$ est le “paramètre de Ginzburg-Landau”. Le comportement du supraconducteur varie en fonction de $h_{ex}$ et $\kappa$. En effet, si le champ appliqué $h_{ex}$ est assez faible, on observe que le champ magnétique ne pénètre pas dans le matériau (c'est l'effet Meissner). Puis au delà d'un champ critique $H_{c_1}$, il se produit une transition de phase et on observe des filaments de vorticité (ou des vortex) par lesquels le champ pénètre. Plus le champ est grand et plus ils sont nombreux, et comme ils se repoussent, ils tendent à s'organiser en réseau triangulaire dit “réseau d'Abrikosov”. Pour plus de détails concernant l'aspect physique, on renvoit à [Ab], [GL], [SST], [Ti] et [TT].

La fonctionnelle $J_\Omega$, vue dans son aspect mathématique, a suscité beaucoup d'interêt ces dernières années, après les travaux fondateurs de Béthuel, Brezis et

Hélein dans [BBH]. Beaucoup d'auteurs se sont intéressés au cas $\kappa$ grand, qui correspond aux supraconducteurs de type II, la limite $\kappa$ infini étant appelée "limite de London". Le but était de comprendre mathématiquement les mécanismes d'apparition des vortex, les valeurs des champs critiques et d'avoir des descriptions des solutions, de leurs vortex et des estimations de leur énergie dans cette limite. Parmi les résultats obtenus, Serfaty a pu caractériser le premier champ critique $H_{c_1}$ et donner son expression qui est sous la forme $C(\Omega)\ \log\kappa$ (voir [Se1], [Se2]). Il a été prouvé que les minimiseurs de l'énergie sur $H^1(\Omega,\mathbb{C})\times H^1(\Omega,\mathbb{R}^2)$ n'ont pas de vortex en-dessous de $H_{c_1}$, puis en ont au-dessus, et qu'ils se répartissent d'abord près du centre de domaine, en configurations régulières (polygones, etc), puis de manière uniforme dans une sous partie du domaine que l'on peut caractériser par un problème à frontière libre, pour cela voir [ASS], [Se3], [SS2] et [SS3]. Des résultats ont également été prouvés sur la répartition de vortex de solutions non-minimisantes (voir [SS5]).

# 2 Le modèle périodique

Le modèle périodique de Ginzburg-Landau permet d'éviter les effets du bord et de mettre l'accent sur ce qui se passe au coeur du supraconducteur, le "bulk". Ici on prend $\kappa > \frac{1}{\sqrt{2}}$ et $\Omega = \mathbb{R}^2$, ceci correspond à un large supraconducteur infini de type II.

## 2.1 Motivation

Soit $(u, A)$ dans l'espace de Sobolev $H^1_{loc}(\mathbb{R}^2,\mathbb{C})\times H^1_{loc}(\mathbb{R}^2,\mathbb{R}^2)$, alors la densité de l'énergie de Ginzburg-Landau

$$\frac{1}{2}\ |\nabla u - i\ A\ u|^2 + \frac{1}{2}\ |h - h_{ex}|^2 + \frac{\kappa^2}{4}\ (1-|u|^2)^2$$

est dans $L^1_{loc}(\mathbb{R}^2)$. De plus, cette densité est invariante sous une transformation de jauge sous la forme $(v, B) = (u\ e^{i\,g}, B + \nabla g)$ avec $g \in H^2_{loc}(\mathbb{R}^2)$. Toutes les quantités physiques à savoir $|u|$, $h = rotA$ et $(iu, \nabla_A u)$ sont invariantes de jauge. Les équations de Ginzburg-Landau associées sont

$$\begin{cases} -\nabla_A^2 u = \kappa^2\ u\ (1-|u|^2) & \mathbb{R}^2 \\ -\nabla^{\perp} h = (i\ u, \nabla_A u) & \mathbb{R}^2. \end{cases} \tag{1.2}$$

En considérant le cas où le champ appliqué $h_{ex}$ est légèrement inférieur à $H_{c_2} = \kappa^2$ avec $\kappa > \frac{1}{\sqrt{2}}$, Abrikosov a introduit dans [Ab], une modélisation spéciale et a prédit une structure périodique des zéros de $u$ avec $(u, A)$ est une solution de la première équation de Ginzburg-Landau linéarisée en $u$ (en ignorant à droite le terme $u\ |u|^2$) qui . Un tel $(u, A)$ dit solution d'Abrikosov existe si $h_{ex} = H_{c_2}$ et $\kappa > \frac{1}{\sqrt{2}}$. Récemment, dans le cas où $\kappa > \frac{1}{\sqrt{2}}$, Dutour [D] a montré qu'il existe une fonction continue $\kappa \to H_{c_1}(\kappa)$ telle que des solutions de l'équation originale (non linéarisée) existent et correspondent aux solutions d'Abrikosov si $h_{ex}$ est tel que $H_{c_1} < h_{ex} < H_{c_2}$. Notons que $H_{c_1}$ et $H_{c_2}$ sont deux champs critiques et que $H_{c_1}$ se comporte comme $\frac{\log k}{2}$ lorsque $\kappa \to +\infty$. Les solutions d'Abrikosov sont périodiques et leurs zéros forment un réseau et autour de chaque zéro, $u$ a une charge topologique non nulle. Ecrivant $u = |u|\ e^{i\,\varphi}$, et dans les coordonnées polaires $(r, \theta)$ centrées en un zéro de $u$, si $r > 0$ est petit, l'entier

$$\frac{1}{2\,\pi} \int_{\theta=0}^{\theta=2\,\pi} \frac{\partial \varphi}{\partial \theta}(r, \theta)\ d\theta,$$

est non nul. Les points où $u$ s'annule sont appelés vortex et l'entier est le degré du vortex.

On fixe $\kappa > 0$ et $h_{ex} > 0$, et on prend $\mathcal{L}$ un réseau de $\mathbb{R}^2$ dont le domaine fondamental est $\Omega$. Pour définir le modèle périodique, on utilise les conditions au bord de t' Hooft [Th] sous lesquelles le vecteur potentiel $A$ et le paramètre d'ordre $u$ sont périodiques à une transformation de jauge près. Dans ce sens, on dit que $(u, A)$ est $\mathcal{L}$ -périodique si pour tout $v \in \mathcal{L}$, il existe $g^v \in H^2_{loc}(\mathbb{R}^2, \mathbb{R})$ tel que

$$\begin{cases} u(z+v) = u(z)\ e^{i\,g^v(z)} \\ \\ A(z+v) = A(z) + \nabla g^v(z). \end{cases} \tag{1.3}$$

On note $\mathcal{B}_{\mathcal{L}}$ l'espace des configurations $(u, A)$ qui sont $\mathcal{L}$ -périodiques.

**Définition :**

On dit qu'une fonction $T$ définie sur $\mathbb{R}^2$ est périodique si pour tous $z \in \mathbb{R}^2$ et $v \in \mathbb{Z}^2$, $T(z+v) = T(z)$.

Les conditions (1.3) garantissent que les quantités invariantes de jauge sont périodiques. Pour chercher les solutions périodiques des équations (1.2), l'idée naturelle est de trouver la configuration périodique qui minimise l'énergie de Ginzburg-Landau par unité d'aire parmi tous les réseaux possibles de $\mathbb{R}^2$. Ceci revient à étudier

$$\inf_{\mathcal{L}} \inf_{\mathcal{B}_{\mathcal{L}}} \{\frac{J_\Omega(u, A)}{|\Omega|}\}. \tag{1.4}$$

Mais, malheureusement l'étude complète du problème (1.4) est toujours ouverte, donc une analyse rigoureuse des solutions périodiques des équations de Ginzburg-Landau (1.2) reste limitée. Le problème se pose essentiellement au niveau de la recherche de la géométrie du réseau associé à l'énergie minimale.

Le réseau étant fixé, l'analyse de la vorticité des minimiseurs périodiques de l'énergie dans la limite $\kappa \to +\infty$ et pour des champs extérieurs de l'ordre de $\log \kappa$ n'est pas encore étudiée. Pour cela, dans toute la suite de la thèse on fixe le réseau dès le début et on se restreint à un réseau dont le domaine fondamental est un parallélogramme d'aire 1. Par commodité, on prend un réseau carré. Notons que nos résultats décrivent des mesures de vorticité limites et ne semblent pas assez précis pour être influencés par la géométrie du réseau. Ceci explique pourquoi nous nous restreignons à un réseau carré de coté 1.

En reprenant les idées de Sandier et Serfaty et pour des champs appliqués $h_{ex}$ en particulier de l'ordre de $\log\kappa$ avec $\kappa \to +\infty$, ce travail a pour but principal la recherche de la vorticité des minimiseurs de l'énergie de Ginzburg-Landau parmi les configurations périodiques ou parmi celles qui présentent certaines symétries.

## 2.2 Le modèle périodique sur le carré $K$ de coté 1

Les chapitres 2 à 7 traîtent du modèle périodique. Dans le chapitre 2, on introduit le modèle périodique sur le carré $K$ de coté 1. Pour cela, soit $\mathcal{A}$ l'espace des $(u, A)$ dans $H^1_{loc}(\mathbb{R}^2, \mathbb{C}) \times H^1_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ tels que pour tout $v \in \mathbb{Z}^2$, il existe $g^v \in H^2_{loc}(\mathbb{R}^2, \mathbb{R})$ tel que les conditions (1.3) soient vérifiées.

Connaissant que toutes les quantités physiques $|u|$, $h = rotA$ et $(iu, \nabla_A u)$ sont périodiques, il suffit de mesurer l'énergie de Ginzburg-Landau d'une configuration $(u, A)$ $\mathcal{L}$ -périodique sur la période $K$. Ici, on s'intéresse juste à l'étude de

$$\inf_{\mathcal{A}} J_K(u, A).$$

On démontre en particulier les propriétés bien connues de quantification du flux (en chapitre 2) et d'existence de minimiseurs de l'énergie de Ginzburg-Landau ( en chapitre 3).

## 2.3 Résultats sur la vorticité

Dans le chapitre 4, pour des champs appliqués qui sont tels que $h_{ex} \leq C\,|\log\varepsilon|$ avec $\varepsilon = \frac{1}{k}$ et pour des configurations périodiques $(u_\varepsilon, A_\varepsilon)$ d'énergie minimale, on définit des vortex en s'inspirant d'une méthode de Jerrard [J]. De là, on peut associer à $(u_\varepsilon, A_\varepsilon)$ une mesure de vorticité

$$\mu_\varepsilon = \frac{2\pi \sum_i d_i \delta_{a_i}}{h_{ex}}, \tag{1.5}$$

où $(a_i, d_i)_i$ sont les positions et les degrés des vortex de $(u_\varepsilon, A_\varepsilon)$.

Dans le chapitre 5, on prend $h_{ex}$ tel que $\lim_{\varepsilon\to 0} \frac{|\log\varepsilon|}{h_{ex}} = \lambda$, avec la condition additionnelle : si $\lambda = 0$, on impose que $h_{ex} \ll \frac{1}{\varepsilon^2}$. Dans toute la suite de la thèse, on gardera cette définition du paramètre $\lambda$. Alors, en prouvant que

$$\lim_{\varepsilon\to 0} \frac{J_K(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} = \begin{cases} \frac{\lambda}{2}\left(1 - \frac{\lambda}{4}\right) & \text{si } 0 < \lambda < 2 \\ \frac{1}{2} & \text{si } \lambda \geq 2, \end{cases} \tag{1.6}$$

qui est obtenue par une borne supérieure et puis une borne inférieure dans l'esprit de la gamma-convergence (voir [DC]) de l'énergie minimale "normalisée" $\frac{J_K(u_\varepsilon, A_\varepsilon)}{{h_{ex}}^2}$, on peut montrer que dans le cas $\lambda > 0$, on a si $\varepsilon \to 0$,

$$\mu_\varepsilon \rightharpoonup \max\Big(0, 1 - \frac{\lambda}{2}\Big)\, dx,$$

où $dx$ est la mesure de Lebesgue de $\mathbb{R}^2$. Donc, pour $0 < \lambda < 2$, on en déduit que la répartition de la vorticité des minimiseurs périodiques de l'énergie de Ginzburg-Landau est uniforme et que le nombre des vortex sur $K$ est de l'ordre de $h_{ex}$, alors que pour $\lambda \geq 2$, il est petit par rapport à $h_{ex}$. Plus précisement, à partir de l'estimation de l'énergie minimale, ou plutôt de sa borne inférieure, on prouve qu'il n' y a pas de vortex si $\lambda > 2$. Ces derniers résultats sont légèrement différents à ceux de Sandier et Serfaty [SS3]. En effet, ceci est dû à l'absence d'effets de bord dans le modèle périodique.

Dans le chapitre 6, on s'interesse au cas $\lambda = 2$. Ceci correspond à des champs appliqués $h_{ex}$ proches du premier champ critique $H_{c_1} = \frac{|\log \varepsilon|}{2}$ pour $\varepsilon \to 0$, qui sont de la forme

$$h_{ex} = H_{c_1} + f(\varepsilon),$$

avec si $\varepsilon \to 0$, $f(\varepsilon) \to +\infty$ et $f(\varepsilon) = o(|\log \varepsilon|)$. D'après (1.6), on remarque que si $\varepsilon \to 0$, l'énergie minimale $J_K(u_\varepsilon, A_\varepsilon)$ est équivalente à l'énergie sans vorticité sur $K$ égale à $\frac{1}{2}\, h_{ex}^2$. Plus clairement, on montre que

$$\frac{J_K(u_\varepsilon, A_\varepsilon) - \frac{1}{2}\, h_{ex}^2}{(f(\varepsilon))^2} \tag{1.7}$$

est la quantité appropriée à considérer. Dans ce cas, en prouvant une borne inférieure de l'énergie minimale plus fine, qui se base sur la construction des boules précédemment mentionnée, on démontre que si $\varepsilon \to 0$,

$$\frac{2\,\pi\, \sum_i d_i\, \delta_{a_i}}{f(\varepsilon)} \rightharpoonup dx,$$

où $(a_i, d_i)_i$ sont les positions et les degrés des vortex de $(u_\varepsilon, A_\varepsilon)$ définis dans le chapitre 4. Donc la répartition des vortex des minimiseurs périodiques est uniforme et plus précisement leur nombre sur le carré $K$ est de l'ordre de $f(\varepsilon)$. Cela contraste avec [SS1], où il faut un incrément de $|\log|\log \varepsilon||$ pour ajouter un vortex.

# 3 Ligne de vorticité

Toujours dans le cadre périodique, le septième chapitre est consacré à construire une suite de solutions $(u_\varepsilon, A_\varepsilon)$ des équations de Ginzburg-Landau (1.2) telle que dans la limite $\varepsilon \to 0$ et pour des champs appliqués bien précis, les vortex de $(u_\varepsilon, A_\varepsilon)$ sur $K$ se concentrent sur un nombre fini de lignes horizontales. La méthode consiste à minimiser l'énergie de Ginzburg-Landau parmi les configurations périodiques $(u, A)$ ( c'est à dire $(u, A) \in \mathcal{A}$) ayant de plus une symétrie par des translations bien choisies données comme suit

$$\begin{cases} u(x + \frac{1}{p_\varepsilon}, y) = u(x, y)\, e^{i\,\xi(x,y)} \\ A(x + \frac{1}{p_\varepsilon}, y) = A(x, y) + \nabla\, \xi(x, y), \end{cases}$$

avec $\xi \in H^2_{loc}(\mathbb{R}^2, \mathbb{R})$ et $p_\varepsilon \in \mathbb{N}$ est une fonction de $\varepsilon$ telle que la limite suivante existe et ne s'annule pas

$$\alpha = 2\,\pi\, \lim_{\varepsilon \to 0} \frac{p_\varepsilon}{h_{ex}}.$$

Pour $h_{ex} \leq C\, |\log \varepsilon|$, on peut associer à un minimiseur $(u_\varepsilon, A_\varepsilon)$ de l'énergie de Ginzburg-Landau une mesure de vorticité analogue à (1.5). On montre que sur $K$

et dans la limite $\varepsilon \to 0$, elle est portée par un nombre fini de lignes horizontales. En outre, elle attribue à chaque ligne une masse appartenant $\alpha\ \mathbb{Z}$. Ensuite, sous une relation bien précise liant $\lambda$ à $\alpha$, on prouve que la mesure limite de vorticité n'est pas nulle. Ceci implique plutôt qu'il y a au moins une ligne horizontale de vorticité. Malheureusement, on ne connait pas explicitement la mesure de vorticité lorsqu'elle est non nulle. Mais, dans le cas où la restriction de la mesure limite sur $K$ est portée par une seule ligne horizontale, on précise la valeur de sa masse.

# 4 Cercle de vorticité

Le huitième chapitre répond à une question posée par Sandier et Serfaty dans [SS5], et consiste en la construction d'une suite de points critiques $(u_\varepsilon, A_\varepsilon)$ de $J_\Omega$ où $\Omega$ est un disque de rayon $R$ telle que dans la limite $\varepsilon \to 0$, la vorticité de $(u_\varepsilon, A_\varepsilon)$ se concentre sur un nombre fini de cercles concentriques de rayon strictement positif et de centre, le centre de disque. Pour cela, on minimise l'énergie $J_\Omega$ parmi les configurations $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ présentant certaines symétries données par

$$\begin{cases} u\Big(x\ e^{i\ \frac{2\ \pi}{q_\varepsilon}}\Big) = u(x)\ e^{i\ \psi(x)} \\ A\Big(x\ e^{i\ \frac{2\ \pi}{q_\varepsilon}}\Big) = e^{i\frac{2\ \pi}{q_\varepsilon}}\ A(x) + e^{i\ \frac{2\ \pi}{q_\varepsilon}}\ \nabla\psi(x), \end{cases}$$

avec $\psi \in H^2_{loc}(\mathbb{R}^2, \mathbb{C})$ et $q_\varepsilon \in \mathbb{N}$ est une fonction de $\varepsilon$ telle que la limite suivante existe et ne s'annule pas

$$\beta = \lim_{\varepsilon \to 0} \frac{q_\varepsilon}{h_{ex}}.$$

Pour $h_{ex} \leq C\ |\log \varepsilon|$, on peut construire, à partir d'un minimiseur $(u_\varepsilon, A_\varepsilon)$ de $J_\Omega$, une mesure de vorticité notée $\tilde{\mu}_\varepsilon$. Puis, quite à extraire une sous-suite $\varepsilon_n$, on montre que la limite faible de $\frac{h_{\varepsilon_n}}{h_{ex}}$ dans $H^1(\Omega)$ notée $h_\infty$ est radiale et que la limite de $\tilde{\mu}_{\varepsilon_n}$ au sens des mesures de Radon est égale à $-\Delta h_\infty + h_\infty$. Cette mesure limite est portée par un nombre fini de cercles concentriques de rayon strictement positif et de centre, le centre du disque. De plus, elle attribue à chaque cercle une masse appartenant à $2\ \pi\ \beta\ \mathbb{Z}$. Notons que le cas où $-\Delta h_\infty + h_\infty = 0$ n'est pas exclu. Cependant, sous certaines hypothèses, la mesure $-\Delta h_\infty + h_\infty$ n'est pas nulle. En effet, on prouve que pour tout $R > 0$ et pour tout $\beta > 0$ petit, il existe une relation bien choisie liant $\lambda$ à $\beta$ de façon à ce que

$$-\Delta h_\infty + h_\infty \neq 0.$$

Signalons que la preuve nécessite quelques propriétés sur les fonctions de Bessel modifiées du premier ordre. Ceci montre qu' il y a au moins un cercle de vorticité de centre, le centre du disque. L'inconvinient est qu' on ne connait pas explicitement la mesure de vorticité lorsqu'elle est non nulle. Cependant, dans un cas très particulier, si la mesure de vorticité est portée par un seul cercle avec une masse bien donnée ègale à $2\ \pi\ \beta$, on peut caractériser ce cercle par la donnée de son rayon qui sera l'unique solution d'un problème de minimisation.

## 5 "Pinning" des vortex

Enfin dans le neuvième et dernier chapitre, on s'intéresse à l'étude du signe des degrés des vortex qui interviennent dans un modèle de Ginzburg-Landau avec un problème de l'ancrage ("pinning") des vortex, étudié par André, Bauman et Philips dans [APB]. Dans ce cas, l'énergie est

$$J_\varepsilon(u, A) = \frac{1}{2} \int_\Omega |\nabla u - i\, A\, u|^2 + \frac{1}{2} \int_\Omega |h - h_{ex}|^2 + \frac{1}{4\, \varepsilon^2} \int_\Omega (a - |u|^2)^2.$$

Le poids $a(x)$ est positif et s'annule en un nombre fini de points notés $\{x_1, ..., x_n\}$. Pour $\kappa = \frac{1}{\varepsilon}$ et un champ appliqué $h_{ex}$ suffisamment grands, André, Bauman et Philips ont montré que les minimiseurs $(u_\varepsilon, A_\varepsilon)$ de $J_\varepsilon$ sur $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ ont une structure non triviale de vortex près des zéros $x_1, ..., x_n$. Notons $d_i$ le degré de $u_\varepsilon$ autour du point $x_i$. Le $n$-uplet d'entiers $d = (d_1, ..., d_n)$ est un minimum d'une fonctionnelle bien déterminée sur $\mathbb{Z}^n$. On s'intéresse au signe des degrés $d_i$, et on montre que, dans des cas très particuliers, les degrés sont positifs. On donne aussi des indices qui laissent penser que, pour certains choix de poids $a(x)$, ceci pourrait être faux.

## Plan of the thesis

Our interest is to describe the repartition of the vortices in minimizers of the Ginzburg-Landau energy $J$ over appropriate spaces according to the value of the applied field. The plan of the thesis is as follows:

In chapter 1, we have introduced some notations and given some known results on vortices. In addition, we have stated the main results which will be proved in the rest of this work.

In chapter 2, we introduce the periodic Ginzburg-Landau model, and in chapter 3, we give results concerning the minimization of the energy $J$ over a space denoted by $\mathcal{A}$ presenting some periodicities and we consider the Ginzburg-Landau equations, these are a system of partial differential equations that are derived from the model.

In chapter 4, we construct a family of vortex balls in the periodic setting and we give precise lower bound of the energy on these balls.

According to the value of the applied field $h_{ex}$, we give in the chapters 5 and 6, some results concerning the repartition of the vortices and their number of global minimizers of $J$ over the space $\mathcal{A}$ as $\varepsilon \to 0$.
First, in chapter 5, we take the case of applied fields $h_{ex}$ which are of the order of $\mathcal{O}(|\log \varepsilon|)$.
Second, in chapter 6, we take applied fields close enough to the first critical field $H_{c_1} \approx \frac{|\log \varepsilon|}{2}$ defined by

$$h_{ex} = H_{c_1} + f(\varepsilon),$$

where $f(\varepsilon) = o(|\log \varepsilon|)$ and $f(\varepsilon)$ tends to $+\infty$ as $\varepsilon \to 0$.

Moreover, in chapter 7 we construct a sequence of periodic critical points of $J$ such that as $\varepsilon \to 0$, the repartition of the vortices on the square $K$ is supported on a finite number of horizontal lines.

In chapter 8, we show that the distribution of the vortices is scattered on a finite number of concentric circles of positive radius and of center, the center of the disk.

Finally, for bounded applied fields $h_{ex}$, we are concerned in the chapter 9 with the study of the sign of the degrees of the vortices intervening in a Ginzburg-Landau model with pinning.

# Chapter 2

# The periodic Ginzburg-Landau model

In this chapter, we define the periodic Ginzburg-Landau model and we give the space where we minimize the Ginzburg-Landau energy. Moreover, we state some properties of this space. Finally, constructing a gauge transformation, we obtain an equivalent minimization problem.

## 1 Definitions

First, we give some classical properties of Sobolev spaces that we will need later. Letting $n \in \mathbb{N}^*$, $1 \leq p \leq +\infty$ and $m \in \mathbb{Z}$, we say that a distribution $f$ on $\mathbb{R}^n$ belongs to $W^{m,p}(\mathbb{R}^n)$ if

$$D^\alpha f := \frac{\partial^{\alpha_1+\alpha_2+...+\alpha_n} f}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}} \in L^p(\mathbb{R}^n),\ \ \forall\, \alpha_i \in \mathbb{N} \quad \text{such that} \quad |\alpha| = \alpha_1 + ... + \alpha_n \leq m. \tag{2.1}$$

Let $O$ be an open domain of $\mathbb{R}^n$, then we define similarly as the above the spaces $W^{m,p}(O)$ (respectively $W^{m,p}_{loc}(\mathbb{R}^n)$) by imposing that the derivatives until the order $m$ belong to $L^p(O)$ (respectively $L^p_{loc}(\mathbb{R}^n)$). Note that the above derivatives are taken in the sense of distributions. We set

$$H^m_{loc}(\mathbb{R}^n, \mathbb{R}) = W^{m,2}_{loc}(\mathbb{R}^n).$$

Now, let us give from [B] or [GT] or [Ad] some Sobolev's injections which will be useful for the rest. In particular, we state

**Theorem:** For $p$ such that $1 < p < +\infty$, we have with $q = \frac{np}{n-p}$, the following injections

(i) if $n > p$, then $W^{1,p}_{loc}(\mathbb{R}^n) \subset L^q_{loc}(\mathbb{R}^n)$.

(ii) if $n = p$, then $\forall r$ such that $n < r < +\infty$, we have $W^{1,p}_{loc} \subset L^r_{loc}(\mathbb{R}^n)$.

(iii) if $n < p$, then $W^{1,p}_{loc}(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$.

Moreover, if $1 < r < q$ and if $O$ is relatively compact of $\mathbb{R}^n$, then $W^{1,p}(O) \subset L^r(O)$ and the injection is compact.

Before all, let $K$ be any square in $\mathbb{R}^2$. The free Ginzburg-Landau energy of a superconductor given by (1.1) is

$$J_K(u,A) = \frac{1}{2}\int_K |\nabla u - i\,A\,u|^2 + \frac{1}{2}\int_K |h-h_{ex}|^2 + \frac{\kappa^2}{4}\int_K (1-|u|^2)^2. \qquad (2.2)$$

The superconductor is assumed to be an infinite vertical cylinder of section $K$. $A : K \to \mathbb{R}^2$ is the vector potential, and the induced magnetic field in the material is $h = curl A$. The complex-valued function $u$ is called the "order parameter" and $\kappa > 0$ is the Ginzburg-Landau parameter. $\kappa$ is a dimensionless constant and from now on we take $\kappa = \frac{1}{\varepsilon}$, $\varepsilon > 0$. $\varepsilon$ represents the scale of variation for the superconducting order parameters, and in some sense measures the radius of the core region of an isolated vortex. $h_{ex} = h_{ex}(\varepsilon) \geq 0$ is the applied magnetic field on the boundary of the superconductor. One can refer for example to [Ab], [SS1], [SS2], and [SS3] for a discussion of the functional $J = J_K$.

**Definition 2.1.** *A vortex is an isolated zero of $u$ such that restricted to a small ball $\mathcal{C}$ around it, the map $\frac{u}{|u|} = e^{i\,\varphi} : \mathcal{C} \to S^1$ has a nonzero winding number $d$, the degree of the vortex, defined as follows*

$$\int_{\mathcal{C}} \frac{\partial \varphi}{\partial \tau} = 2\,\pi\,d,$$

*where $\tau$ is the unit vector such that if $\nu$ is the inward pointing unit normal vector on $\mathcal{C}$, then $(\tau, \nu)$ is at each point of $\mathcal{C}$ a direct orthonormal frame. Away from vortices, it is expected that $|u| \simeq 1$.*

**Definition 2.2.** *We say that a function $T$ is periodic if it is periodic with respect to the lattice determined by the vectors $\vec{i} = \binom{1}{0}$ and $\vec{j} = \binom{0}{1}$, meaning that*

$$T(x+1,y) = T(x,y) = T(x,y+1) \quad \forall (x,y) \in \mathbb{R}^2. \qquad (2.3)$$

*Here, $T$ may be scalar or vector-valued and may be real or complex-valued. Note that (2.3) implies, for differentiable $T$, that*

$$\frac{\partial T}{\partial x}(x+1,y) = \frac{\partial T}{\partial x}(x,y) = \frac{\partial T}{\partial x}(x,y+1) \quad \forall (x,y) \in \mathbb{R}^2,$$

*and*

$$\frac{\partial T}{\partial y}(x+1,y) = \frac{\partial T}{\partial y}(x,y) = \frac{\partial T}{\partial y}(x,y+1) \quad \forall (x,y) \in \mathbb{R}^2.$$

The subtlety of the periodic Ginzburg-Landau problems is that periodic magnetic fields and currents are generally represented by non-periodic potentials $A$ and order parameter $u$. One setting for such periodic problems is via the t' Hooft boundary conditions [Th], for which one demands that $A$ and $u$ be periodic up to a family of gauge transformations from one period cell the next. This is given as follows

**Definition 2.3.** *Let* $(u, A)$ *be in* $H^1_{loc}(\mathbb{R}^2, \mathbb{C}) \times H^1_{loc}(\mathbb{R}^2, \mathbb{R}^2)$. $(u, A)$ *belongs to the space* $\mathcal{A}$ *if there exists* $(f, g) \in H^2_{loc}(\mathbb{R}^2, \mathbb{R}) \times H^2_{loc}(\mathbb{R}^2, \mathbb{R})$ *such that*

$$\begin{cases} u(x+1, y) = u(x, y)\; e^{i\, f(x,y)} \\ u(x, y+1) = u(x, y)\; e^{i\, g(x,y)}, \end{cases} \tag{2.4}$$

*and*

$$\begin{cases} A(x+1, y) = A(x, y) + \nabla f(x, y) \\ A(x, y+1) = A(x, y) + \nabla g(x, y). \end{cases} \tag{2.5}$$

The conditions (2.4) and (2.5) are called in physics the t'Hooft's boundary conditions. We can refer to [ABB], [ABS], [DGP] to find configurations $(u, A)$ which are given analogously as in the definition 2.3.

# 2 Some properties

As was noted in definition 2.3, the order parameter $u$, or more precisely, the phase of the order parameter, and the magnetic potential $A$ are not periodic (in the sense of definition 2.2). The first of the basic interpretations of the periodic Ginzburg-Landau model for superconductivity concerns the periodic nature of the physical attributes of the superconductor, meaning that the density of superconducting charge carriers $|u|^2$, the magnetic field $h$ and the free energy $J$ are periodic with respect to the lattice vectors $\vec{i} = \binom{1}{0}$ and $\vec{j} = \binom{0}{1}$. Indeed, taking the curl in (2.5), then since $curl \nabla f = curl \nabla g = 0$, hence the induced field $h$ defined by $h = curl A$ satisfies

$$h(x+1, y) = h(x, y) = h(x, y+1).$$

Let locally $u = |u|\; e^{i\,\varphi} = \rho\; e^{i\,\varphi}$ where $\varphi$ denoted the phase of the order parameter $u$. We get from (2.4)

$$\rho(x+1, y) = \rho(x, y) = \rho(x, y+1).$$

Now, again from (2.4)-(2.5), it is obvious that

$$(\nabla\varphi - A)(x+1, y) = (\nabla\varphi - A)(x, y) = (\nabla\varphi - A)(x, y+1).$$

We replace $u$ by $\rho\; e^{i\,\varphi}$ in $(\nabla u - i\; A\; u)$ to write

$$\begin{aligned} \nabla_A u :=& \nabla u - i\; A\; u \\ =& (\nabla\rho + i\; \rho\; \nabla\varphi)\; e^{i\,\varphi} - i\; \rho\; A\; e^{i\,\varphi} \\ =& \Big(\nabla\rho + i\; \rho\; (\nabla\varphi - A)\Big)\; e^{i\,\varphi}. \end{aligned}$$

It follows that

$$|\nabla_A u|^2 = |\nabla\rho|^2 + \rho^2\,|\nabla\varphi - A|^2. \tag{2.6}$$

Let $(u, A) \in \mathcal{A}$. Thanks to the above, it is easy to remark that the fundamental domain of periodicity is any square $K \subset \mathbb{R}^2$ of sidelength 1. Without loss of generality, we take

$$K = [0,1[\times[0,1[.$$

Again from the above, it suffices to compute the energy $J$ given by (2.2) over the period $K = [0,1[\times[0,1[$. Using (2.6), $J$ can be written as follows

$$J(u,A) = J_K(u,A) = \frac{1}{2}\int_K |\nabla\rho|^2 + \rho^2|\nabla\varphi - A|^2 + \frac{1}{2}\int_K |h - h_{ex}|^2 + \frac{1}{4\,\varepsilon^2}\int_K (1-\rho^2)^2. \tag{2.7}$$

For $(u, A) \in \mathcal{A}$, we define the quantity

$$II(x,y) := f(x,y) - f(x,y+1) + g(x+1,y) - g(x,y), \ \ \forall (x,y) \in \mathbb{R}^2. \tag{2.8}$$

Moreover, we define

$$N := \frac{1}{2\,\pi}\int_K h. \tag{2.9}$$

Now, we give a classical property for $(u, A)$ belonging to the space $\mathcal{A}$.

**Lemma 2.4.** *Let $(u, A) \in \mathcal{A}$. Let $h$ be the magnetic field defined by $h = curl A$, then $II(x, y)$ defined by (2.8) is independent of $(x, y)$ and belongs to $2\,\pi\,\mathbb{Z}$ if $u$ is not identically zero. Moreover*

$$N = \frac{1}{2\,\pi}\,II(0,0) \in \mathbb{Z}. \tag{2.10}$$

**Proof**: Decomposing the potential $A(x + 1, y + 1)$ into two different ways, we obtain the following equations

$$\begin{aligned} A(x+1,y+1) =& A(x,y) + \nabla f(x,y+1) + \nabla g(x,y) \\ =& A(x,y) + \nabla f(x,y) + \nabla g(x+1,y). \end{aligned}$$

Then, by identification, we can write for all $(x, y) \in \mathbb{R}^2$

$$\nabla f(x,y) - \nabla f(x,y+1) + \nabla g(x+1,y) - \nabla g(x,y) = 0.$$

Thanks to (2.8), this means

$$\nabla II(x,y) = 0 \quad \forall\ (x,y) \in \mathbb{R}^2.$$

This implies that the function $II$ is independent of $(x, y)$. Also,

$$
\begin{aligned}
u(x+1, y+1) =& u(x, y)\ e^{i\left(f(x,y+1)+g(x,y)\right)} \\
=& u(x, y)\ e^{i\left(f(x,y)+g(x+1,y)\right)}.
\end{aligned}
$$

Thus, if $u$ is not identically zero then for some $(x, y) \in \mathbb{R}^2$

$$\Big(f(x, y) - f(x, y+1) + g(x+1, y) - g(x, y)\Big) \in 2\,\pi\,\mathbb{Z}.$$

Combining the two above properties of the function $II$, we can say that $II$ is a constant in $2\,\pi\,\mathbb{Z}$. Now, integrating the magnetic field strength over the basic unit period cell $K$ and applying Stokes' Theorem, we get

$$
\begin{aligned}
\int_K h = \int_K curl A =& \int_{\partial K} A.\tau \\
=& \int_0^1 (A_1(x,1) - A_1(x,0))\ dx - \int_0^1 (A_2(1,y) - A_2(0,y))\ dy,
\end{aligned} \tag{2.11}
$$

where $A = (A_1, A_2)$ and $\tau = \nu^{\perp}$, $\nu$ is the exterior unit normal on the boundary of $K$, and $A.\tau$ is the component of $A$ in the direction $\tau$. Referring to the definition 2.3, we have

$$
\begin{cases}
A_1(x+1, y) - A_1(x, y) = \partial_x g(x, y) \\
A_2(x, y+1) - A_2(x, y) = \partial_y f(x, y),
\end{cases}
$$

where $\nabla = (\partial_x, \partial_y)$. We insert these equations in (2.11) to get

$$
\begin{aligned}
\int_K h =& f(0,0) - f(0,1) + g(1,0) - g(0,0) \\
=& II(0,0).
\end{aligned} \tag{2.12}
$$

Hence, by definition of $N$ given by (2.9), we find

$$N = \frac{1}{2\,\pi} \int_K h = \frac{1}{2\,\pi}\, II(0,0).$$

Thus, $N \in \mathbb{Z}$. Hence, the total flux per period cell is quantized for any element of $\mathcal{A}$. □

# 3 The minimization of the energy $J$

In this paragraph, our interest is to study the following minimization problem

$$\inf_{(u,A)\in\mathcal{A}} J(u,A). \tag{2.13}$$

First, we remark that the $\mathcal{A}$ given by definition 2.3 is neither a vectorial nor affine space, because of the gauge invariance. Hence, we need to choose a gauge transformation that makes easy the study of (2.13).

## 3.1 Gauge transformation

For $(x,y)\in\mathbb{R}^2$, we introduce the potential vector $\vec{C}=\frac{1}{2}\binom{-y}{x}$. Then, $\vec{C}$ verifies

$$div\vec{C}=0 \quad \text{and} \quad curl\vec{C}=1.$$

First, we need to state the following definition

**Definition 2.5.** *Let $(u_1,Q_1)$ and $(u_2,Q_2)$ be in the space $\mathcal{A}$. We say that $(u_1,Q_1)$ is gauge equivalent to $(u_2,Q_2)$ if there exists $\Phi\in H^2_{loc}(\mathbb{R}^2)$ such that*

$$\begin{cases} Q_1 = Q_2 + \nabla\Phi \quad in \;\; \mathbb{R}^2 \\ u_1 = u_2\, e^{i\,\Phi} \quad in \;\; \mathbb{R}^2. \end{cases}$$

Now, we give a gauge transformation in order to find an equivalent study to the minimization problem (2.13).

**Proposition 2.6.** *Let $(u,A)$ be in $\mathcal{A}$ and $N$ be defined by (2.9). Then, there exists $(v,P)\in H^1_{loc}(\mathbb{R}^2,\mathbb{C})\times H^1_{loc}(\mathbb{R}^2,\mathbb{R}^2)$ such that $(u,A)$ is gauge equivalent to the configuration $(v,2\,\pi\,N\,\vec{C}+P)$ where*

$$v(x+1,y)=e^{i\,\pi\,N\,y}\,v(x,y) \quad \forall\,(x,y)\,\mathbb{R}^2, \tag{2.14}$$

$$v(x,y+1)=e^{-i\,\pi\,N\,x}\,v(x,y) \quad \forall\,(x,y)\,\mathbb{R}^2, \tag{2.15}$$

$$P \quad is\ periodic,$$

$$div\ P=0 \quad in \quad \mathbb{R}^2.$$

*If $N\neq 0$, then we can impose that $\int_K P=0$.*

**Proof**: See [D], Theorem 2.3.2. We can also see [ABB] where there is a similar result on fixing a gauge in the case of the Lawrence-Doniach model. □

**Remark 2.7.** *Thanks to the definition 2.5, the fact that $(u,A)\in\mathcal{A}$ is gauge equivalent to the configuration $(v,2\;\pi\;N\;\vec{C}+P)$ means that there exists a function $\sigma\in H^2_{loc}(\mathbb{R}^2,\mathbb{R})$ such that*

$$\begin{cases} A = 2\,\pi\,N\,\vec{C}+P+\nabla\sigma \quad in \;\; \mathbb{R}^2 \\ u = v\,e^{i\,\sigma} \quad in \;\; \mathbb{R}^2. \end{cases} \tag{2.16}$$

Let us fix $d \in \mathbb{Z}$ and let $v \in H^1_{loc}(\mathbb{R}^2, \mathbb{C})$ such that $\forall\ (x, y) \in \mathbb{R}^2$

$$\begin{cases} v(x+1, y) = e^{i\,\pi\,d\,y}\ v(x, y) \\ v(x, y+1) = e^{-i\,\pi\,d\,x}\ v(x, y), \end{cases} \tag{2.17}$$

then, we are in a situation to introduce the new space

$$\mathcal{B}_d := \left\{ \begin{array}{c} (v, P) \in H^1_{loc}(\mathbb{R}^2) \times H^1_{loc}(\mathbb{R}^2, \mathbb{R}^2) \quad \text{such that} \quad (2.17) \quad \text{is verified,} \\ P \quad \text{is periodic,} \quad div\ P = 0 \quad \text{in} \quad \mathbb{R}^2 \quad \text{and} \quad \int_K P = 0 \quad \text{if} \quad d \neq 0 \end{array} \right\}. \tag{2.18}$$

Observe that $\mathcal{B}_d$ is a vector space. Let us define

$$\mathcal{B} = \cup_{d \in \mathbb{Z}} \mathcal{B}_d. \tag{2.19}$$

By definition of $\mathcal{B}$, we can write

$$(v, P) \in \mathcal{B} \Longleftrightarrow \exists\ \ d \in \mathbb{Z}, \quad (v, P) \in \mathcal{B}_d. \tag{2.20}$$

Now, let $(u, A)$ be in the space $\mathcal{A}$. Then, going back to lemma 2.4, the $N$ given in proposition 2.6 which is such that

$$\int_K h = 2\,\pi\,N,$$

is in $\mathbb{Z}$. Consequently, combining the properties of $(v, P)$ defined by proposition 2.6, we can write referring to (2.18)

$$(v, P) \in \mathcal{B}_N.$$

In particular, this yields $(v, P) \in \mathcal{B}$. In the next paragraph, we will give an equivalent to the minimization problem (2.13).

### 3.2 The equivalent minimization problem

First, we take $d$ to be fixed in $\mathbb{Z}$ and $(v, P) \in \mathcal{B}_d$. Taking

$$u = v \quad \text{and} \quad A = 2\,\pi\,d\,\vec{C} + P,$$

it is clear that $(u, A)$ is in the space $\mathcal{A}$ and is gauge equivalent to the configuration $(v, 2\,\pi\,d\,\vec{C} + P)$. Moreover, in particular for $h = curl A$, we find

$$\int_K h = 2\,\pi\,d,$$

since $curl \vec{C} = 1$ and $\int_K curl P = 0$ which follows from the fact that $P$ is periodic.
Second, reciprocally let $(u, A) \in \mathcal{A}$ and $N$ be such that $N = \frac{1}{2\,\pi} \int_K h$. We know that $N \in \mathbb{Z}$, and then the proposition 2.6 implies that there exists $(v, P)$ in $\mathcal{B}_N$ such that $(u, A)$ is in addition gauge equivalent to the configuration $(v, 2\,\pi\,N\,\vec{C} + P)$.
Obviously, the energy $J$ is invariant under the gauge transformation (2.16), hence we have for $(u, A) \in \mathcal{A}$,

$$J(u,A) = J(v, 2\,\pi\,N\,\vec{C} + P).$$

For $d \in \mathbb{Z}$, let us take the function $G$ defined over $\mathcal{B}_d$ given as follows

$$\begin{aligned} G(v,P) =& J(v, 2\,\pi\,d\,\vec{C} + P) \\ =& \frac{1}{2}\int_K |\nabla v - i\,(2\,\pi\,d\,\vec{C} + P)\,v|^2 + \frac{1}{2}\int_K |2\,\pi\,d + curl\ P - h_{ex}|^2 \\ +& \frac{1}{4\,\varepsilon^2}\int_K (1-|v|^2)^2. \end{aligned} \tag{2.21}$$

As a consequence of all the above, we can deduce that the minimization problem (2.13) of the energy $J$ over the space $\mathcal{A}$ is equivalent to the minimization of $G$ over the space $\mathcal{B}$, i.e.

$$\inf_{\mathcal{A}} J(u,A) = \inf_{\cup_{d\in\mathbb{Z}}\mathcal{B}_d} G(v,P) = \inf_{\mathcal{B}} G(v,P). \tag{2.22}$$

Now, we calculate the quantity

$$\frac{1}{2}\int_K |2\,\pi\,d + curl\ P - h_{ex}|^2. \tag{2.23}$$

Thanks to the periodicity of $P$, we find

$$\int_K (2\,\pi\,d - h_{ex})\ curl\ P = 0. \tag{2.24}$$

Then, we use (2.24) in the decomposition of (2.23) to get

$$\frac{1}{2}\int_K |2\,\pi\,d + curl\ P - h_{ex}|^2 = \frac{1}{2}\int_K |2\,\pi\,d - h_{ex}|^2 + \frac{1}{2}\int_K |curl\ P|^2. \tag{2.25}$$

Consequently, inserting (2.25) in (2.21), we obtain for $(v,P) \in \mathcal{B}_d$

$$\begin{aligned} G(v,P) =& \frac{1}{2}\int_K |\nabla v - i\,(2\,\pi\,d\,\vec{C} + P)\,v|^2 + \frac{1}{4\,\varepsilon^2}\int_K (1-|v|^2)^2 + \frac{1}{2}\int_K |2\,\pi\,d - h_{ex}|^2 \\ &+ \frac{1}{2}\int_K |curl\ P|^2. \end{aligned} \tag{2.26}$$

The next chapter is devoted to study

$$\inf_{\mathcal{B}} G(v,P), \tag{2.27}$$

where $G$ is the functional given by (2.26) and $\mathcal{B}$ is defined by (2.19).

# Chapter 3

# Minimizers and critical points of the Ginzburg-Landau energy in the periodic model

This chapter describes the periodic model introduced at the end of the chapter 2. In the first part, we are concerned with the minimization of the functional $G$ over the space $\mathcal{B}$. In the second part, we give the critical points of $G$ and we give their regularity.

## 1 Existence of minimizing solution for the functional $G$ over the space $\mathcal{B}$

Here, we are concerned with the study of the minimization of the energy $G$ over the space $\mathcal{B}$. More precisely, we will prove that the functional $G$ has a minimizer over the space $\mathcal{B}$. We define the space $\mathcal{B}_d$ by (2.18) and we take $K$ to be any square of sidelength 1.

**Proposition 3.1.** *The minimum of $G$ over the space $\mathcal{B}$ is achieved.*

**Proof**: Let

$$G_{min} = \inf_{(v,P)\in\mathcal{B}} G(v,P). \tag{3.1}$$

Because the functional $G$ is positive, this infimum exists. We consider a minimizing sequence $(v_n, P_n)$ in $\mathcal{B}$. Then, $P_n$ is periodic, divergence free and with zero mean in $K$. There exists $d_n \in \mathbb{Z}$ such that

$$(v_n, P_n) \in \mathcal{B}_{d_n}.$$

First, $(1,0) \in \mathcal{B}$ because $(1,0) \in \mathcal{B}_0$. Then, testing the energy $G$ by the configuration $(1,0)$, we get

$$G(v_n, P_n) \leq G(1,0) = \frac{1}{2}\, h_{ex}^2.$$

If $d_n = 0$, then necessarily $G(v_n, P_n) = \frac{1}{2}\ h_{ex}^2$, and then the infimum of the functional $G$ is obtained for the so-called pure state $v_n = 1$ and $P_n = 0$. The interesting case corresponds to $d_n \neq 0$. From (2.26), since $(G(v_n, P_n))_n$ is bounded, then $(d_n)_n$ also. Taking a subsequence, we may assume $(d_n)_n$ to be constant and equal to some $d \in \mathbb{Z}$. Still from (2.26), $(curl P_n)_n$ is bounded in $L^2(K)$. Since $P_n$ is divergence free and has zero mean in $K$, this implies that $(P_n)_n$ is bounded in $H^1(K)$ and by periodicity in $H^1(U)$ for any bounded open subset $U$ of $\mathbb{R}^2$. In particular, by Sobolev embedding, $(P_n)_n$ is locally bounded in $L^p$ for any $1 < p < +\infty$.
As for $(v_n)_n$, the potential term in (2.26) guaranties that it is locally bounded in $L^4$. Moreover, $\Big(\nabla v_n - i\ (2\ \pi\ d\ \vec{C} + P_n)\ v_n\Big)$ is bounded in $L^2(K)$ and by periodicity is locally bounded in $L^2$. Using the $L^4$ bound on $v_n$ and the $L^p$ bound on $P_n$, we get that $\nabla v_n$ is locally bounded in $L^2$, hence $(v_n)_n$ is locally bounded in $H^1$. Then, passing to a subsequence and using a diagonal argument if necessary, $(v_n)_n$ and $(P_n)_n$ weakly converge in $H^1$ on every bounded open set in $\mathbb{R}^2$. Reasoning as in [BR], the limit minimizes $G$. □

**Remark 3.2.** *We remark that a minimizer $(v, P)$ of the functional $G$ over the space $\mathcal{B}$ depends on the parameter $\varepsilon$, so we will take $(v_\varepsilon, P_\varepsilon)$. But, when it is not necessary to take the $\varepsilon$ and to keep the subscripts, we write $(v, P)$ instead of $(v_\varepsilon, P_\varepsilon)$. Thanks to (2.22), we remark in addition that the minimum of the energy $J$ over the space $\mathcal{A}$ is achieved.*

# 2 Properties of Critical points

Let $(v, P)$ be a minimizer of $G$ over the space $\mathcal{B}$. By definition of $\mathcal{B}$, there exists $d \in \mathbb{Z}$ such that $(v, P) \in \mathcal{B}_d$. If $d = 0$, $G(v, P) = G(1, 0) = \frac{1}{2}$ (the superconducting phase). The interesting case is when $d \neq 0$. In this paragraph, letting $d \neq 0$, we will prove that $(v, P)$ has the regularity $C^\infty$ and verifies a system of partial differential equations. Let $\nabla^\perp$ and $< ., . >$ denote respectively $(-\partial_x, \partial_y)$ and the scalar-product in $\mathbb{C}$ identified with $\mathbb{R}^2$, where $\nabla = (\partial_x, \partial_y)$. First, referring to [D], we can have

**Proposition 3.3.** *For $d \in \mathbb{Z}$, let $(v, P) \in \mathcal{B}_d$ be a critical point of $G$. Then, $(v, P) \in C^\infty_{loc}(\mathbb{R}^2, \mathbb{C}) \times C^\infty_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ and*

$$\begin{cases} -\nabla_B^2 v = \frac{1}{\varepsilon^2}\ v\ (1 - |v|^2) \quad in \quad \mathbb{R}^2 \\ -\nabla^\perp curl B =< i\ v, \nabla_B v > \quad in \quad \mathbb{R}^2, \end{cases} \tag{3.2}$$

*where $B = 2\ \pi\ d\ \vec{C} + P$.*

Now, let $(v, P)$ be a minimizer of $G$ over the space $\mathcal{B}$. By definition of $\mathcal{B}$, there exists $d \in \mathbb{Z}$ such that $(v, P) \in \mathcal{B}_d$. Then, in particular $(v, 2\ \pi\ d\ \vec{C} + P)$ is a minimizer of the energy $J$ over the space $\mathcal{A}$. Consequently, any minimizer of $J$ over $\mathcal{A}$ has the form $(v\ e^{i\,\nu}, 2\ \pi\ d\ \vec{C} + P + \nabla\nu)$ where $\nu \in H^2_{loc}(\mathbb{R}^2, \mathbb{R})$. Now, let $(u, A)$ be a minimizer of the energy $J$ over the space $\mathcal{A}$, so there exists $k \in H^2_{loc}(\mathbb{R}^2, \mathbb{R})$ such that

$$(u, A) = (v\ e^{i\,k}, 2\ \pi\ d\ \vec{C} + P + \nabla k), \tag{3.3}$$

where $(v, P)$ is a minimizer of $G$ over the space $\mathcal{B}$ and $(v, P) \in \mathcal{B}_d$. Since it is easy that (3.2) is invariant under the above gauge transformation, we can state the following corollary

**Corollary 3.4.** *Any minimizer $(u, A)$ of the energy $J$ over the space $\mathcal{A}$ satisfies*

$$\begin{cases} -\nabla_A^2 u = \frac{1}{\varepsilon^2}\, u\,(1-|u|^2) \quad in \quad \mathbb{R}^2 \\ -\nabla^{\perp} curl A =< i\, u, \nabla_A u > \quad in \quad \mathbb{R}^2. \end{cases} \tag{3.4}$$

*Theses equations are termed the Ginzburg-Landau equations.*

# 3 Remarks on the Ginzburg-Landau equations

In this section, we prove a few elementary results concerning solutions of the Ginzburg-Landau equations (3.4), that are going to be useful for the rest and that help understand the idea of the proofs.
First, let $(u, A) \in \mathcal{A}$ be a critical point of $J$ and let $N$ be the corresponding degree. Then, thanks to proposition 2.6, $(u, A)$ is gauge equivalent to $(v, B)$, where

$$B = 2\,\pi\,N\,\vec{C} + P, \tag{3.5}$$

and it follows from the preceding section that $(v, B)$ is smooth. In particular $|u|^2 = |v|^2$ is smooth and similarly are all the gauge-invariant quantities.

Now, we give a standard property for the Ginzburg-Landau equations (3.4).

**Lemma 3.5.** *Any solution $(u, A)$ of the Ginzburg-Landau equations (3.4) satisfies*

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq 1. \tag{3.6}$$

**Proof**: Let $(u, A)$ be a solution of the Ginzburg-Landau equations (3.4). Let us adapt the same notations as the above on the configuration $(v, B)$. First, we have $|u| = |v|$, so to prove (3.6), it suffices to show that $|v| \leq 1$, which follows from the maximum principle. Indeed, we will check that at the points where the density of the superconducting electrons which is $|v|^2$ is maximum, the inequality $|v| \leq 1$ holds. From (3.2), we can write under the gauge $divB = divP = 0$ which is known as the Coulomb gauge.

$$\Delta v = \frac{v}{\varepsilon^2}\,(|v|^2 - 1) + |B|^2\, v + 2\, i\, B\, .\nabla v \quad \text{in } \mathbb{R}^2. \tag{3.7}$$

Let us calculate the laplacian of $|v|^2$

$$\begin{aligned} \Delta |v|^2 &= \overline{v}\,\Delta v + v\,\overline{\Delta v} + 2\,|\nabla v|^2 \\ &= 2\,Re(\overline{v}\,\Delta v) + 2\,|\nabla v|^2. \end{aligned}$$

We replace $\Delta v$ with the right-hand side of (3.7) in the above to get

$$
\begin{aligned}
\Delta|v|^2 =&\frac{2}{\varepsilon^2}\ |v|^2\ (|v|^2-1)+2\ |B|^2\ |v|^2+4\ Re(i\ \overline{v}\ B\ .\nabla v)+2\ |i\ \nabla v|^2 \\
=&\frac{2}{\varepsilon^2}\ |v|^2\ (|v|^2-1)+2\ |\nabla v-i\ B\ v|^2.
\end{aligned}
\tag{3.8}
$$

Let $z\in\mathbb{R}^2$ where the maximum of the function $|v|^2$ is achieved, then $\Big(\Delta|v|^2\Big)(z)\leq 0$. We obtain from (3.8) that necessarily $|v|(z)\leq 1$. $z$ is a maximum hence by continuity of $v$ the inequality $|v|\leq 1$ is true everywhere. □

The following two inequalities, in particular the last assertion, will be very useful in the sequel.

**Lemma 3.6.** *Let $(u,A)$ be a solution of the Ginzburg-Landau equations (3.4) and $h=curl\ A$. Then*

$$|\nabla_A u|^2\geq|\nabla h|^2, \tag{3.9}$$

*and*

$$J_K(u,A)\geq\frac{1}{2}\ \|h-h_{ex}\|^2_{H^1(K)}. \tag{3.10}$$

**Proof**: We have noted locally, $u=\rho\ e^{i\ \varphi}$. First, (2.6) is

$$|\nabla_A u|^2=|\nabla\rho|^2+\rho^2\ |\nabla\varphi-A|^2.$$

Moreover, the second Ginzburg-Landau (3.4) gives us

$$|\nabla h|=|<i\ u,\ \nabla_A u>|=\rho^2\ |\nabla\varphi-A|.$$

Hence

$$|\nabla\varphi-A|^2=\frac{|\nabla h|^2}{\rho^4}.$$

This implies that

$$|\nabla_A u|^2=|\nabla\rho|^2+\frac{|\nabla h|^2}{\rho^2}\geq\frac{|\nabla h|^2}{\rho^2}.$$

Using $\rho\leq 1$, we get (3.9). Returning now to the energy given by (2.7), we can write using (3.9)

$$
\begin{aligned}
J_K(u,A) =&\frac{1}{2}\int_K|\nabla_A u|^2+\frac{1}{2}\int_K|h-h_{ex}|^2+\frac{1}{4\ \varepsilon^2}\int_K(1-\rho^2)^2 \\
\geq&\frac{1}{2}\int_K|\nabla h|^2+\frac{1}{2}\int_K|h-h_{ex}|^2+\frac{1}{4\ \varepsilon^2}\int_K(1-\rho^2)^2 \\
\geq&\frac{1}{2}\int_K|\nabla h|^2+\frac{1}{2}\int_K|h-h_{ex}|^2 \\
=&\frac{1}{2}\ \|h-h_{ex}\|^2_{H^1(K)}.
\end{aligned}
\tag{3.11}
$$

□

We state the following a priori estimates of $(u, A)$ solution of the Ginzburg-Landau equations (3.4). Set locally $\rho = |u|$.

**Lemma 3.7.** *Let $(u, A)$ be a solution of the Ginzburg-Landau equations (3.4) and $h = curl A$. Then*

$$\|\nabla \rho\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{\varepsilon}, \tag{3.12}$$

$$\|\nabla_A u\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{\varepsilon}, \tag{3.13}$$

$$\|h - h_{ex}\|_{C^2_{loc}(\mathbb{R}^2)} \leq \frac{C}{\varepsilon^2}, \tag{3.14}$$

$$\|h - h_{ex}\|_{C^1_{loc}(\mathbb{R}^2)} \leq \frac{C}{\varepsilon}. \tag{3.15}$$

**Proof**: These estimates are proved in [HP], proposition 4.2, see also [P] lemma 7.1. They rely on a blow-up at scale $\varepsilon$, which leads to equations at scale 1, for which all the quantities are uniformly bounded. □

**Remark 3.8.** *Let $(u, A)$ be a minimizer of the energy $J$ over the space $\mathcal{A}$. In particular, it is a solution of the Ginzburg-Landau equations (3.4), and then the results of lemmas 3.5, 3.6 and 3.7 remain true.*
*From now on, we will only consider the energy $J$ and take the configurations $(u, A)$ which are in $\mathcal{A}$.*

# Chapter 4

# Construction of vortex balls

In this chapter, keeping the same notations as in chapter 2, we define the vortices of $(u, A) \in \mathcal{A}$ with their degrees, by defining balls $\Big(B_i\Big)_{i\in I}$, such that $|u| \geq \frac{3}{4}$ on $K \setminus \cup_{i\in I} B_i$. We also give a suitable lower bound of the energy $J$ on the balls $B_i$ which will be very useful for the rest.

## 1 The main result

In this chapter, we take applied fields $h_{ex}$ satisfying the a priori bound

$$h_{ex} \leq C\ |\log \varepsilon|.$$

We take $K$ to be any square of sidelength 1. Now, we define some quantities that will be useful in the sequel. For $(u_\varepsilon, A_\varepsilon) \in \mathcal{A}$ —where $\mathcal{A}$ is given by definition 2.3 — we set for any compact $O \subset \mathbb{R}^2$

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, O) := \frac{1}{2}\int_O |\nabla u_\varepsilon - i\ A_\varepsilon\ u_\varepsilon|^2 + \frac{1}{4\ \varepsilon^2}\int_O (1 - |u_\varepsilon|^2)^2. \tag{4.1}$$

We know that the quantities $|\nabla u_\varepsilon - i\ A_\varepsilon\ u_\varepsilon|^2$, $(1 - |u_\varepsilon|^2)^2$ and $h_\varepsilon$ are periodic, thus there exists $C > 0$ depending on $O$ such that

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, O) \leq C\ F_\varepsilon(u_\varepsilon, A_\varepsilon, K), \quad \|h_\varepsilon\|_{L^2(O)} \leq C\ \|h_\varepsilon\|_{L^2(K)}. \tag{4.2}$$

Let us define

$$\gamma_\varepsilon := \|h_\varepsilon\|_{L^2(K)}. \tag{4.3}$$

Now, we construct a family of disjoint balls $(B_i)_i$ containing the set $\{|u| < \frac{3}{4}\}$. The main result is

**Proposition 4.1.** *Let $K$ be any square of sidelength 1. If $h_{ex} \leq C\ |\log \varepsilon|$, there exists $\varepsilon_0$ such that if $\varepsilon < \varepsilon_0$ and $(u_\varepsilon, A_\varepsilon) \in \mathcal{A}$ satisfies $|\nabla u_\varepsilon - i\ A_\varepsilon\ u_\varepsilon| < \frac{C}{\varepsilon}$, $\gamma_\varepsilon = \|h_\varepsilon\|_{L^2(K)} \leq C\ h_{ex}$, and*

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, K) \leq C\ \alpha_\varepsilon\ h_{ex},$$

*where* $1 \ll \alpha_\varepsilon \leq C\,|\log\varepsilon|$ *and* $m_\varepsilon = o(1)$ *satisfy*

$$\frac{\log \alpha_\varepsilon}{\alpha_\varepsilon} \ll m_\varepsilon, \tag{4.4}$$

*then there exist a square of sidelength 1 still denoted* $K$ *and a family of disjoint balls* $\Big(B_i = B_i(a_i, r_i)\Big)_{i\in I_\varepsilon}$ *of center* $a_i$ *and of radii* $r_i$ *satisfying*

$$\{x \in K,\ |u_\varepsilon(x)| < \frac{3}{4}\} \subset \cup_{i\in I_\varepsilon} B_i, \tag{4.5}$$

$$\overline{\cup_{i\in I_\varepsilon} B_i(a_i, r_i)} \subset K, \tag{4.6}$$

$$\sum_{i\in I_\varepsilon} r_i \leq C\ \alpha_\varepsilon\ e^{-m_\varepsilon\ \alpha_\varepsilon}, \tag{4.7}$$

$$card(I_\varepsilon) \leq C\ \alpha_\varepsilon\ h_{ex}, \tag{4.8}$$

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i) \geq \pi\ |d_i|\ \Big(|\log\varepsilon| - \log\gamma_\varepsilon - m_\varepsilon\ \alpha_\varepsilon\Big), \tag{4.9}$$

*where* $d_i$ *is the degree of the map* $\frac{u_\varepsilon}{|u_\varepsilon|}$ *restricted to* $\partial B_i$*.*

**Remark 4.2.** *Thanks to the construction of the vortex balls that we recall in the above proposition, the fundamental domain of periodicity of* $(u_\varepsilon, A_\varepsilon)$ *in* $\mathcal{A}$ *will be from now on the square* $K = [0,1[\times[0,1[$*. We will refer to* $(a_i, d_i)_{i\in I_\varepsilon}$ *as the family of the vortices defined on* $K$ *associated to* $(u_\varepsilon, A_\varepsilon)$ *and to* $(B_i)_{i\in I_\varepsilon}$ *as the family of the vortex balls. In particular, the balls* $(B_i)_{i\in I_\varepsilon}$ *can be extended by periodicity to* $\mathbb{R}^2$*. For this, for any* $i \in I_\varepsilon$*, denote by* $B_i = B_{i,0,0}$*, then we let* $B_{i,n,m}$ *be the ball image of* $B_i = B_{i,0,0}$ *by translation of vector* $(n\,\vec{i} + m\,\vec{j})$ *where* $n$ *and* $m$ *are in* $\mathbb{Z}$ *. Going back to (4.5)-(4.6), then by periodicity of* $|u_\varepsilon|$*, we can write*

$$|u_\varepsilon| \geq \frac{3}{4} \quad on\ \ \mathbb{R}^2\backslash\Big(\cup_{(i\in I_\varepsilon,\ n,m\in\mathbb{Z})} B_{i,n,m}\Big).$$

Let us give the meaning of the different inequalities given in the proposition 4.1. First, (4.5) locates the set where $|u_\varepsilon|$ is less than $\frac{3}{4}$, which is contained in a union of the disjoint balls $(B_i)_{i\in I_\varepsilon}$. Second, from (4.6), there is no intersection between the balls and the boundary of K. Finally, (4.7) gives us a control on the size of the balls and (4.9) states a lower bound of the energy. Note that $d_i$ be the degree of $\frac{u_\varepsilon}{|u_\varepsilon|}$ restricted to $\partial B_i = \partial B_i(a_i, r_i)$. Writing locally $\frac{u_\varepsilon}{|u_\varepsilon|} = e^{i\,\varphi_\varepsilon}$, then by definition of $d_i$, we have

$$d_i = deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_i) = \frac{1}{2\,\pi}\int_{\partial B_i} \nabla\varphi_\varepsilon\ .\tau.$$

Now, taking $B_{i,1,0}$ which is the ball image of $B_{i,0,0}$ by translation of vector $\vec{i}$, we have

$$deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_{i,1,0}) = \frac{1}{2\,\pi}\int_{\partial B_{i,1,0}} \nabla\varphi_\varepsilon\ .\tau. \tag{4.10}$$

Thanks to (2.4), there exist $l \in \mathbb{Z}$ and $f \in H^2_{loc}(\mathbb{R}^2)$ such that $\forall\ (x,y) \in \mathbb{R}^2$

$$\varphi_\varepsilon(x,y) = \varphi_\varepsilon(x-1,y) + f(x-1,y) + 2\,\pi\, l.$$

We take the gradient

$$\nabla\varphi_\varepsilon(x,y) = \nabla\varphi_\varepsilon(x-1,y) + \nabla f(x-1,y).$$

Obviously,

$$\frac{1}{2\,\pi}\int_{\partial B_{i,1,0}} \frac{\partial\varphi_\varepsilon}{\partial\tau} = \frac{1}{2\,\pi}\int_{\partial B_{i,0,0}} \frac{\partial\varphi_\varepsilon}{\partial\tau} + \frac{\partial f}{\partial\tau}.$$

$f \in H^2_{loc}(\mathbb{R}^2)$, hence it is continuous on $\mathbb{R}^2$. Then, in view of $curl\nabla f = 0$, we obtain

$$\int_{\partial B_{i,0,0}} \frac{\partial f}{\partial\tau} = \int_{B_{i,0,0}} curl\nabla f = 0.$$

Thus, for any $i \in I_\varepsilon$

$$deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_{i,1,0}) = \frac{1}{2\,\pi}\int_{\partial B_{i,1,0}} \frac{\partial\varphi_\varepsilon}{\partial\tau} = \frac{1}{2\,\pi}\int_{\partial B_{i,0,0}} \frac{\partial\varphi_\varepsilon}{\partial\tau} = d_i = deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_{i,0,0}). \tag{4.11}$$

Similar to (7.51), we can prove for any $i \in I_\varepsilon$ and $n, m \in \mathbb{Z}$

$$d_i = deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_{i,0,0}) = deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_{i,n,m}). \tag{4.12}$$

This means that the degree is invariant under periodicity.

## 2 Corollaries from Proposition 4.1

**Corollary 4.3.** *Under the hypotheses of proposition 4.1 and using the notations there, we have*

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i) \geq \pi\, |d_i|\, |\log\varepsilon|\, \Big(1 - o(1)\Big), \tag{4.13}$$

*and if* $\gamma_\varepsilon = \mathcal{O}(\alpha_\varepsilon)$,

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i) \geq \pi\, |d_i|\, \Big(|\log\varepsilon| - C\, m_\varepsilon\, \alpha_\varepsilon\Big). \tag{4.14}$$

*Proof.* Combining the assumption $\alpha_\varepsilon \leq C\ |\log\varepsilon|$ together with $\gamma_\varepsilon \leq C\ |\log\varepsilon|$ in (4.9), the lower bound (4.9) rewrites as

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i) \geq \pi\, |d_i|\, |\log\varepsilon|\, \Big(1 - o(1)\Big).$$

In the case where $\gamma_\varepsilon = \mathcal{O}(\alpha_\varepsilon)$, we have from (4.4)

$$\log\gamma_\varepsilon + m_\varepsilon\, \alpha_\varepsilon \leq \log C + \log\alpha_\varepsilon + m_\varepsilon\, \alpha_\varepsilon \leq C\, m_\varepsilon\, \alpha_\varepsilon.$$

Inserting this in (4.9), the proof of (4.14) is completed. □

The second corollary is

**Corollary 4.4.** *Let $(u_\varepsilon, A_\varepsilon) \in \mathcal{A}$ satisfying the hypotheses of proposition 4.1 and $h_\varepsilon = curl A_\varepsilon$ be the induced magnetic field, then*

$$N_\varepsilon = \sum_{i \in I_\varepsilon} d_i, \tag{4.15}$$

*where the family $(d_i)_{i \in I_\varepsilon}$ is given by the proposition 4.1 and $N_\varepsilon = \frac{1}{2\,\pi} \int_K h_\varepsilon$.*

*Proof.* Let

$$w_\varepsilon = \frac{u_\varepsilon}{|u_\varepsilon|} = e^{i\,\varphi_\varepsilon}. \tag{4.16}$$

Then

$$(i\, w_\varepsilon, \nabla w_\varepsilon) = Re(i\, w_\varepsilon\, \nabla \bar{w}_\varepsilon) = \nabla \varphi_\varepsilon. \tag{4.17}$$

It follows that $curl(i\, w_\varepsilon, \nabla w_\varepsilon) = 0$ in $K \setminus \cup_{i \in I_\varepsilon} B_i$. In particular,

$$\int_{K \setminus \cup_{i \in I_\varepsilon} B_i} curl(i\, w_\varepsilon, \nabla w_\varepsilon) = 0. \tag{4.18}$$

Thanks to (4.6), which is $\overline{\cup_{i \in I_\varepsilon} B_i(a_i, r_i)} \subset K$, (4.18) implies

$$\int_{\partial K} (i\, w_\varepsilon, \frac{\partial w_\varepsilon}{\partial \tau}) = \sum_{i \in I_\varepsilon} \int_{\partial B_i} (i\, w_\varepsilon, \frac{\partial w_\varepsilon}{\partial \tau}). \tag{4.19}$$

Hence from (4.17), (4.19) reads

$$\int_{\partial K} \frac{\partial \varphi_\varepsilon}{\partial \tau} = \sum_{i \in I_\varepsilon} \int_{\partial B_i} \frac{\partial \varphi_\varepsilon}{\partial \tau}. \tag{4.20}$$

On the one hand, by definition of the degree $d_i$ of $\frac{u_\varepsilon}{|u_\varepsilon|}$ restricted to $\partial B_i$, we have

$$\sum_{i \in I_\varepsilon} \int_{\partial B_i} \frac{\partial \varphi_\varepsilon}{\partial \tau} = 2\,\pi \sum_{i \in I_\varepsilon} d_i. \tag{4.21}$$

On the other hand, using definition 2.3, a simple calculation gives

$$\begin{aligned} \int_{\partial K} \frac{\partial \varphi_\varepsilon}{\partial \tau} &= \int_0^1 (A_1(x,1) - A_1(x,0))dx - \int_0^1 (A_2(1,y) - A_2(0,y))dy \\ &= f_\varepsilon(0,0) - f_\varepsilon(0,1) + g_\varepsilon(1,0) - g_\varepsilon(0,0) \\ &= II_\varepsilon(0,0) = 2\,\pi\, N_\varepsilon. \end{aligned} \tag{4.22}$$

Combining now (4.21) together with (4.22) in (4.20) proves (4.15). □

# 3 Proof of Proposition 4.1

Here, let $\Omega \subset \mathbb{R}^2$ be the smooth, bounded and connected section of the superconductor. We consider $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ and $h = curl A$ the induced magnetic field. We use the method of R. Jerrard introduced in [J] in order to construct balls containing all the zeroes of $u$, on which we have a suitable lower bound of the energy (see [Sa] for a similar construction). The size of the balls has to be large enough, so that most of the energy is concentrated in these balls.
We follow closely the proofs of [J] and we adopt the presentation of [SS1].
First, we include the set $\left\{x,\ |u(x)| < \frac{3}{4}\right\}$ in well-chosen disjoint small balls $B_i$ of radii $r_i > \varepsilon$ such that

$$F_\varepsilon(u, A, B_i) \geq \frac{C\, r_i}{\varepsilon},$$

where $F_\varepsilon(u, A, B_i)$ is defined in (4.1). This is possible according to the following.

**Lemma 4.5.** *Let $u : \Omega \to \mathbb{C}$, $A : \Omega \to \mathbb{R}^2$ be such that $|\nabla u - i\ A\ u| < \frac{C}{\varepsilon}$. Then, there exist disjoint balls $B_1, ..., B_k$ of radii $r_i$ such that*

$$(1)\ \forall\, 1 \leq i \leq k,\ r_i > \varepsilon,$$

$$(2)\ \{x \in \Omega,\ |u(x)| < \frac{3}{4}\} \subset \cup_{1 \leq i \leq k} B_i,$$

$$(3)\ \forall\, 1 \leq i \leq k,\ F_\varepsilon(u, A, B_i \cap \Omega) \geq \frac{C\, r_i}{\varepsilon}.$$

## 3.1 Proof of lemma 4.5

We use the notation $S(x, r)$ for the circle in $\mathbb{R}^2$ of center $x$ and radius $r$. Let $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ and $h = curl\ A$. Let us take

$$\gamma = \|h\|_{L^2(\Omega)},$$

then we will take $\gamma$ to be fixed and establish lower bounds for $F_\varepsilon$ on circles in which $\gamma$ appears as a parameter. For $y \in \mathbb{R}$, set $y^+ = \max\{y, 0\}$.
We have

**Lemma 4.6.** *If $u : \Omega \to \mathbb{C}$ and $A : \Omega \to \mathbb{R}^2$, there exist $r(\Omega)$, $C(\Omega)$ such that $\forall\, x \in \Omega$, $\forall\, \varepsilon < r < r(\Omega)$, letting $m = \min_{S_r \cap \Omega} |u|$,*

$$F_\varepsilon(u, A, S_r) \geq \frac{(1 - m)^C}{C\, \varepsilon}. \tag{4.23}$$

**Proof**: We write locally $u = \rho\, e^{i\,\varphi}$. For $y = (y_1, y_2) \in S_r$, let $\tau(y) := \frac{1}{|y|}\,(-y_2, y_1)$ denote the oriented tangent at $y$. We start by

$$F_\varepsilon(u, A, S_r) = \int_{S_r} \frac{1}{2}\,|\nabla\rho|^2 + \frac{1}{2}\,\rho^2\,|\nabla\varphi - A|^2 + \frac{1}{4\,\varepsilon^2}\,\int_K (1 - |\rho|^2)^2$$

$$\geq \int_{S_r} \frac{1}{2}\,|\nabla_\tau\rho|^2 + \frac{1}{4\,\varepsilon^2}\,\int_K (1 - |\rho|^2)^2,$$

then, using lemma 2.5 of [J] completes the proof of lemma 4.6. □

Using (4.23) and replacing in the proof of lemma 3.1 of [SS1] the quantity $\int_{S_r} |\nabla u|^2 + \frac{1}{2\,\varepsilon^2}\,\int_{S_r} (1 - |u|^2)^2$ with $F_\varepsilon(u, A, S_r)$, the lemma 4.5 is proved.

## 3.2 Estimation on circle

From [J], lemma 6.1, we have the following

**Lemma 4.7.** *There exist constants $C, p > 0$ such that if $u : S_t \to \mathbb{C}$ and $A : S_t \to \mathbb{R}^2$, where $S_t$ is a circle of radius $t$ in $\mathbb{R}^2$ such that $t > \varepsilon$, then*

$$F_\varepsilon(u, A, S_t) \geq \lambda_\varepsilon(t, |d|),$$

*where*

$$\lambda_\varepsilon(r, d) = \min_{m\in[0,1]} \left\{ \frac{m^2}{r}\,\left( (\sqrt{\pi}\,d - \frac{r\,\gamma}{2})^+ \right)^2 + \frac{1}{C\,\varepsilon}\,|1 - m|^p \right\}. \tag{4.24}$$

*Moreover*

$$\lambda_\varepsilon(r, d) \geq \frac{\pi}{r}\,\left[ \left( d - \frac{r\,\gamma}{2\,\sqrt{\pi}} \right)^+ \right]^2 \left[ 1 - C\,\frac{\varepsilon^q}{r^q} \right], \tag{4.25}$$

*where $q = \frac{1}{p-1} > 0$.*

Let us define a function $\Lambda_\varepsilon$, which provides a convenient way of keeping track of lower bounds on balls, and we record several properties of $\Lambda_\varepsilon$. First, denote by $a \wedge b = \min(a, b)$ for any $a, b \in \mathbb{R}$, and then we set for $r > 0$

$$\Lambda_\varepsilon(r) := \int_0^r \lambda_\varepsilon(s, 1) \wedge \frac{c_0}{\varepsilon} ds, \tag{4.26}$$

where $c_0$ is a constant to be selected below.

**Remark 4.8.** *In [J], proposition 6.1, Jerrard has assumed that $\gamma$ is bounded independently of $\varepsilon$ and has found that*

$$\Lambda_\varepsilon(r) \geq \pi\,\log\frac{1}{\varepsilon} + \log(r \wedge \gamma^{-1}) - C, \quad \forall\, r > 0.$$

*However, this assertion remains true under the weaker assumption*

$$\gamma\,\varepsilon \leq C.$$

*For the reader 's convenience, we will give a proof of this in the next paragraph.*

### 3.3 Properties of $\Lambda_\varepsilon$

The function $\Lambda_\varepsilon$ satisfies the following properties.

**Lemma 4.9.** *The function $\Lambda_\varepsilon$ is increasing. Moreover*

$$(1)\ \Lambda_\varepsilon(r+s) \le \Lambda_\varepsilon(r) + \Lambda_\varepsilon(s) \quad \forall\, r, s \ge 0.$$

$$(2)\ \frac{\Lambda_\varepsilon(s)}{s} \quad \textit{is nonincreasing on } \mathbb{R}_+.$$

*Assume that $\gamma \le \frac{c}{\varepsilon}$, then $\forall\, r > 0$*

$$(3)\ \Lambda_\varepsilon(r) \ge \pi\ \log\Big(\frac{r \wedge \frac{1}{\gamma}}{\varepsilon}\Big) - C.$$

**Proof**: From the definition (4.24) of $\lambda_\varepsilon$, it is clear that $\lambda_\varepsilon > 0$ and that $r \to \lambda_\varepsilon(r,1)$ is nonincreasing. The first of these facts implies that $\Lambda_\varepsilon$ is increasing; from the two facts together it is easy to see that the assertion (1) holds.
Next, the assertion (2) is clear, since $\frac{1}{s}\ \Lambda_\varepsilon(s)$ is just the average over the interval $[0,s]$ of the nonincreasing function $r \longmapsto \lambda_\varepsilon(r,1) \wedge \frac{c_0}{\varepsilon}$.
Now, we provide a proof of the assertion (3). Recall from (4.25) that

$$\lambda_\varepsilon(r,1) \ge \pi\ \Big((1 - \frac{r\,\gamma}{2\,\sqrt{\pi}})^+\Big)^2\ \Big(\frac{1}{r} - C\ \frac{\varepsilon^q}{r^{q+1}}\Big). \tag{4.27}$$

We need to find a condition on $r$ in order to obtain

$$\pi\ \Big((1 - \frac{r\,\gamma}{2\,\sqrt{\pi}})^+\Big)^2\ \Big(\frac{1}{r} - C\ \frac{\varepsilon^q}{r^{q+1}}\Big) \le \frac{c_0}{\varepsilon}. \tag{4.28}$$

First, $\Big(\frac{1}{r} - C\ \frac{\varepsilon^q}{r^{q+1}}\Big) \le \frac{c_1}{\varepsilon}$ whenever $r \ge c_2\ \varepsilon$ for some $c_2 > 0$. Second, the quantity $\Big((1 - \frac{r\,\gamma}{2\,\sqrt{\pi}})^+\Big)^2 \le 1$ even when $\gamma$ is unbounded. Then, there exists $c_0 > 0$, which is $c_0 = \pi\ c_2$, such that for $c_2\ \varepsilon \le r$, the inequality (4.28) holds. As a result from (4.27)-(4.28), we can write for $c_2\ \varepsilon \le r$

$$\lambda_\varepsilon(r,1) \wedge \frac{c_0}{\varepsilon} \ge \pi\ \Big((1 - \frac{r\,\gamma}{2\,\sqrt{\pi}})^+\Big)^2\ \Big(\frac{1}{r} - C\ \frac{\varepsilon^q}{r^{q+1}}\Big). \tag{4.29}$$

Now, we assume that $\gamma\,\varepsilon \le c^0$. We distinguish the two following cases.
**Case 1:** $\quad r \le \frac{2\,\sqrt{\pi}}{\gamma}$
Here, we can write

$$(1 - \frac{r\,\gamma}{2\,\sqrt{\pi}})^+ = 1 - \frac{r\,\gamma}{2\,\sqrt{\pi}}.$$

For $c_2\ \varepsilon \le s \le r$, we get $s \le \frac{2\,\sqrt{\pi}}{\gamma}$, so

$$(1-\frac{s\,\gamma}{2\,\sqrt{\pi}})^+ = 1-\frac{s\,\gamma}{2\,\sqrt{\pi}}.$$

Consequently, we insert (4.29) in (4.26) to get

$$\begin{aligned}\Lambda_\varepsilon(r) &\geq \int_{c_2\,\varepsilon}^{r} \pi\,\Big((1-\frac{s\,\gamma}{2\,\sqrt{\pi}})^+\Big)^2\,\Big(\frac{1}{s}-C\,\frac{\varepsilon^q}{s^{q+1}}\Big)\\ &\geq \int_{c_2\,\varepsilon}^{r} \pi\,\Big(1-\frac{s\,\gamma}{\sqrt{\pi}}\Big)\,\Big(\frac{1}{s}-C\,\frac{\varepsilon^q}{s^{q+1}}\Big).\end{aligned}$$

Using the assumption $\gamma\,\varepsilon \leq c^0$ and thanks to $r \leq \frac{2\,\sqrt{\pi}}{\gamma}$, we obtain

$$\begin{aligned}\Lambda_\varepsilon(r) \geq& \pi\,\int_{c_2\,\varepsilon}^{r}\Big(\frac{1}{s}-C\,\frac{\varepsilon^q}{s^{q+1}}\Big)-\pi\,\int_{c_2\,\varepsilon}^{r}\frac{\gamma}{\sqrt{\pi}}\\ \geq& \pi\,\int_{c_2\,\varepsilon}^{r}\Big(\frac{1}{s}-C\,\frac{\varepsilon^q}{s^{q+1}}\Big)-\pi\,\int_{c_2\,\varepsilon}^{\frac{2\,\sqrt{\pi}}{\gamma}}\frac{\gamma}{\sqrt{\pi}}.\end{aligned}$$

A simple calculation gives us

$$\begin{aligned}\Lambda_\varepsilon(r) &\geq \pi\,\log(\frac{r}{\varepsilon})-C\,\varepsilon^\alpha\int_{c_2\,\varepsilon}^{\frac{2\,\sqrt{\pi}}{\gamma}} s^{-1-q}ds-(\frac{2\,\sqrt{\pi}}{\gamma}-c_2\,\varepsilon)\,\frac{\gamma}{\sqrt{\pi}}\\ &\geq \pi\,\log(\frac{r}{\varepsilon})+C\,(\varepsilon\,\gamma)^q-C.\end{aligned}$$

We find

$$\Lambda_\varepsilon(r) \geq \pi\,\log(\frac{r}{\varepsilon})-C. \tag{4.30}$$

**Case 2:** $\quad r > \frac{2\,\sqrt{\pi}}{\gamma}$

Here, we can write $(1-\frac{r\,\gamma}{2\,\sqrt{\pi}})^+ = 0$. First, using the assumption $\gamma\,\varepsilon \leq c^0$, we can state

$$\begin{aligned}\Lambda_\varepsilon(r) &\geq \int_{c_2\,\varepsilon}^{r} \pi\,\Big((1-\frac{s\,\gamma}{2\,\sqrt{\pi}})^+\Big)^2\,\Big(\frac{1}{s}-C\,\frac{\varepsilon^q}{s^{q+1}}\Big)\\ &\geq \int_{c_2\,\varepsilon}^{\frac{2\,\sqrt{\pi}}{\gamma}} \pi\,\Big((1-\frac{s\,\gamma}{2\,\sqrt{\pi}})^+\Big)^2\,\Big(\frac{1}{s}-C\,\frac{\varepsilon^q}{s^{q+1}}\Big)\end{aligned}$$

Since $c_2\,\varepsilon \leq s \leq \frac{2\,\sqrt{\pi}}{\gamma}$,

$$(1-\frac{s\,\gamma}{2\,\sqrt{\pi}})^+ = 1-\frac{s\,\gamma}{2\,\sqrt{\pi}}.$$

Obviously as in the case 1,

$$\Lambda_\varepsilon(r) \geq \int_{c_2\ \varepsilon}^{\frac{2\sqrt{\pi}}{\gamma}} \pi\ \Big(1-\frac{s\ \gamma}{2\ \sqrt{\pi}}\Big)^2\ \Big(\frac{1}{s}-C\ \frac{\varepsilon^q}{s^{q+1}}\Big)$$

$$\geq \pi\ \log\Big(\frac{2\ \sqrt{\pi}\ \gamma^{-1}}{\varepsilon}\Big)+C\ (\varepsilon\ \gamma)^q - C.$$

We obtain

$$\Lambda_\varepsilon(r) \geq \pi\ \log(\frac{\gamma^{-1}}{\varepsilon}) - C. \tag{4.31}$$

From (4.30)-(4.31), we can deduce

$$\Lambda_\varepsilon(r) \geq \pi\ \log\Big(\frac{r \wedge \frac{1}{\gamma}}{\varepsilon}\Big) - C.$$

This completes the proof of lemma 4.9. □

## 3.4 Estimation on an annulus

The proof of proposition 4.1 involves dilating the balls $B_i$ ( which are defined by lemma 4.5) into balls $B'_i$. A lower bound for $F_\varepsilon(u, A, B'_i)$ is obtained by combining the lower bound for $F_\varepsilon(u, A, B_i)$ and a lower bound of the energy on the annulus $B'_i \backslash \overline{B_i}$. In particular, referring to proposition 6.2 of [J], we have the following

**Lemma 4.10.** *$\forall r > s > \varepsilon$, if $B_r$ and $B_s$ are two concentric balls of respective radii $r$ and $s$, and if $u : B_r\backslash\overline{B_s} \to \mathbb{C}$ is such that $|u| > \frac{3}{4}$, $d = deg(u, \partial B_r)$, and $A : B_r\backslash\overline{B_s} \to \mathbb{R}^2$, then*

$$F_\varepsilon(u, A, B_r\backslash\overline{B_s}) \geq |d|\ \Big(\Lambda_\varepsilon(\frac{r}{|d|}) - \Lambda_\varepsilon\ (\frac{s}{|d|})\Big). \tag{4.32}$$

*Also,*

$$F_\varepsilon(u, A, B_r\backslash\overline{B_s}) \geq \Lambda_\varepsilon(r) - \Lambda_\varepsilon\ (s). \tag{4.33}$$

## 3.5 Growing and merging

The method consists in starting from $\{|u| < \frac{3}{4}\}$, and when this set is not too big, including it the balls $B_i$, $1 \leq i \leq k$ that shall grow progressively. The energy on each ball is controlled during the growth process thanks to lemma 4.10. Then, it may happen that some balls intersect. We then merge them into a larger ball of a radius equal to the sum of the merged balls, and check that we still have a suitable lower bound on the energy over the new ball. We proceed with the growing and merging until the balls have the desired size. For more details of this phenomena, we can refer to [Sa]. The following lemma sums up the whole growth process.

**Lemma 4.11.** *Let $u : \Omega \to \mathbb{C}$, $A : \Omega \to \mathbb{R}^2$ be such that $|\nabla u - iA\, u| < \frac{C}{\varepsilon}$ and $\{B_i\}_i$ be a family of balls of radii satisfying the results of lemma 4.5.*
*Let*

$$s_0 = \min_{\{d_i \neq 0\}} (\frac{r_i}{|d_i|}),$$

*where $d_i = deg(u_\varepsilon, \partial B_i)$ if $\overline{B_i} \subset \Omega$ and 0 otherwise. Then, for every $s \geq s_0$, there exists a family $\mathcal{B}(s)$ of disjoint balls $B_1(s), ..., B_{k(s)}(s)$ of radii $r_i(s)$ such that*

*(1) the family of balls is monotone,*

*(2) for every $i$, $F_\varepsilon(u, A, B_i(s)) \geq \frac{r_i(s)\, \Lambda_\varepsilon(s)}{s}$, where $\Lambda_\varepsilon$ is defined by lemma 4.10,*

*(3) if $d_i(s) = deg(u, \partial B_i(s))$ with $\overline{B_i(s)} \subset \Omega$, then $r_i(s) \geq s\, |d_i(s)|$.*

**Proof**: The proof is as in [SS1], proposition 3.1, replacing the quantity $\frac{1}{2}\int_{B_i(s)} |\nabla u|^2 + \frac{1}{4\,\varepsilon^2}\int_{B_i(s)} (1 - |u|^2)^2$ with $F_\varepsilon(u, A, B_i(s))$. □

### 3.6 The final balls

Now, we get as a consequence of the above, the following proposition which gives us the final balls that we need.

**Proposition 4.12.** *Let $u : \Omega \to \mathbb{C}$, $A : \Omega \to \mathbb{R}^2$ be such that $|\nabla u - i\, A\, u| < \frac{C}{\varepsilon}$ and $F_\varepsilon(u, A, \Omega) \leq C\, \alpha_\varepsilon\, |\log \varepsilon|$ where $1 \ll \alpha_\varepsilon \leq C\, |\log \varepsilon|$ and $\gamma_\varepsilon \leq C\, |\log \varepsilon|$, then letting $m_\varepsilon = o(1)$ be any sequence verifying*

$$\frac{\log \alpha_\varepsilon}{\alpha_\varepsilon} \ll m_\varepsilon,$$

*there is an $\varepsilon_0$ such that $\forall\ \varepsilon < \varepsilon_0$, there exists a finite family of disjoint balls $\Big(B_i = B(a_i, r_i)\Big)_{i \in \urcorner}$ of center $a_i$ and of radii $r_i$ such that*

$$(1)\ \{x \in \Omega,\ |u(x)| < \frac{3}{4}\} \subset \cup_{i \in \urcorner} B_i,$$

$$(2)\ \sum_{i \in \urcorner} r_i \leq C\, \alpha_\varepsilon\, e^{-m_\varepsilon\, \alpha_\varepsilon},$$

$$(3)\ card\ \urcorner \leq C\, \alpha_\varepsilon\, |\log \varepsilon|.$$

*In addition*

$$F_\varepsilon(u, A, B_i \cap \Omega) \geq \pi\, |d_i|\, \Big(|\log \varepsilon| + \log\Big(e^{-m_\varepsilon\, \alpha_\varepsilon} \wedge \gamma_\varepsilon^{-1}\Big)\Big), \tag{4.34}$$

*where $d_i$ is the degree of the map $\frac{u}{|u|}$ restricted to $\partial B_i$ if $\overline{B_i} \subset \Omega$ and is equal to 0 otherwise.*

**Proof of proposition 4.12**

First, consider the balls given by lemma 4.5, then apply the lemma 4.11 to get bigger balls. If $s_0 = \min_{\{d_i \neq 0\}}(\frac{r_i}{|d_i|})$, we must then check that $s_0$ is small enough to be able to apply the lemma 4.11 for $s$ large enough. By the assertion (3) of lemma 4.5,

$$C\ r_i < \varepsilon\ F_\varepsilon(u, A, B_i \cap \Omega) \leq C\ \varepsilon\ |\log \varepsilon|\ \alpha_\varepsilon, \tag{4.35}$$

so that $s_0 \leq C\ \varepsilon\ |\log\varepsilon|\ \alpha_\varepsilon$. We can apply the lemma 4.11 for all $s \geq C\ \varepsilon\ |\log\varepsilon|\ \alpha_\varepsilon$. We take $m_\varepsilon = o(1)$ any sequence verifying

$$\frac{\log \alpha_\varepsilon}{\alpha_\varepsilon} \ll m_\varepsilon.$$

Note that $m_\varepsilon$ is positive. Now, we choose in particular

$$s_1 = e^{-m_\varepsilon\ \alpha_\varepsilon}.$$

In other words, the lemma 4.11 yields the final balls $\mathcal{B}(s_1)$ such that

$$\forall\ i \quad \text{if} \quad \overline{B_i(s_1)} \subset \Omega, \quad F_\varepsilon(u, A, B_i) \geq \frac{r_i(s_1)}{s_1} \Lambda_\varepsilon(s_1),$$

with

$$r_i(s_1) \geq s_1\ |d_i(s_1)|.$$

Furthermore

$$F_\varepsilon(u, A, B_i) \geq |d_i(s_1)|\ \Lambda_\varepsilon(s_1).$$

Then, from the assertion (3) of the lemma 4.10, we have under $\gamma_\varepsilon\ \varepsilon \leq c$

$$F_\varepsilon(u, A, B_i) \geq |d_i(s_1)|\ \Big(\pi\ \log \frac{s_1 \wedge \gamma_\varepsilon^{-1}}{\varepsilon} - C\Big), \tag{4.36}$$

which holds in our case because we have assumed that $\gamma_\varepsilon \leq C\ |\log\varepsilon|$, so for $\varepsilon$ a small enough, $\gamma_\varepsilon\ \varepsilon \leq c$. We thus get the lower bound (4.34) on $F_\varepsilon$. Now, we will prove the assertion (2). From the assertion (2) of the lemma 4.11,

$$\sum_{i \in \urcorner} r_i\ \frac{\Lambda_\varepsilon(s_1)}{s_1} \leq \sum_{i \in \urcorner} F_\varepsilon(u, A, B_i(s_1)).$$

Since the balls $\Big(B_i(s_1)\Big)_{i \in \urcorner}$ are disjoint,

$$\sum_{i \in \urcorner} F_\varepsilon(u, A, B_i(s_1)) = F_\varepsilon(u, A, \cup_{i \in \urcorner} B_i(s_1)) \leq F_\varepsilon(u, A, \Omega) \leq C\ |\log\varepsilon|\ \alpha_\varepsilon.$$

It follows that

$$\sum_{i \in \urcorner} r_i(s_1) \leq C\ \frac{s_1}{\Lambda_\varepsilon(s_1)}\ |\log\varepsilon|\ \alpha_\varepsilon. \tag{4.37}$$

Moreover, thanks to fact that $\gamma_\varepsilon$ and $\alpha_\varepsilon$ are less than $C\ h_{ex}$, together with the fact that $m_\varepsilon \gg \frac{\log \alpha_\varepsilon}{\alpha_\varepsilon}$, we have for $\varepsilon$ sufficiently small

$$
\begin{aligned}
\Lambda_\varepsilon(s_1) &\geq \pi \left( |\log \varepsilon| + \log(e^{-m_\varepsilon\ \alpha_\varepsilon} \wedge \gamma_\varepsilon^{-1}) \right) \\
&\geq \pi \left( |\log \varepsilon| - m_\varepsilon\ \alpha_\varepsilon - \log \gamma_\varepsilon \right) \\
&\geq C\ |\log \varepsilon|.
\end{aligned} \tag{4.38}
$$

We insert (4.38) in (4.37) to get

$$
\begin{aligned}
\sum_{i \in \daleth} r_i(s_1) \leq & C\ \frac{s_1}{\Lambda_\varepsilon(s_1)}\ |\log \varepsilon|\ \alpha_\varepsilon \\
\leq & C\ s_1\ \alpha_\varepsilon \simeq C\ \alpha_\varepsilon\ e^{-m_\varepsilon\ \alpha_\varepsilon}.
\end{aligned} \tag{4.39}
$$

Thanks to $m_\varepsilon \gg \frac{\log \alpha_\varepsilon}{\alpha_\varepsilon}$, we remark for a sufficiently small $\varepsilon$

$$
\sum_{i \in \daleth} r_i(s_1) = o(1).
$$

There only remains to show that the assertion (3) holds. This is easy, since in lemma 4.5 each ball satisfies $F_\varepsilon(u, A, B_i \cap \Omega) \geq C\ \frac{r_i}{\varepsilon}$, with $r_i > \varepsilon$, hence carries an energy that is bounded from below by a constant independent from $\varepsilon$. As $F_\varepsilon \leq C\ \alpha_\varepsilon\ |\log \varepsilon|$, we see that the number of these balls has to be bounded by $C\ \alpha_\varepsilon\ |\log \varepsilon|$. Then, the procedure of lemma 4.11 does not increase the number of balls, so that property (3) is true. This completes the proof of proposition 4.12.

## 3.7 Completing the proof of proposition 4.1

Here, we apply proposition 4.12 in $\Omega = [0, 2[ \times [0, 2[$, taking $u = u_\varepsilon$ and $A = A_\varepsilon$ such that $|\nabla u_\varepsilon - i\ A_\varepsilon\ u_\varepsilon| \leq \frac{C}{\varepsilon}$ and $F_\varepsilon(u_\varepsilon, A_\varepsilon, \Omega) \leq C\ \alpha_\varepsilon\ |\log \varepsilon|$ with

$$
1 \ll \alpha_\varepsilon \leq C\ |\log \varepsilon|, \quad \gamma_\varepsilon \leq C\ |\log \varepsilon| \quad \text{and} \quad \frac{\log \alpha_\varepsilon}{\alpha_\varepsilon} \ll m_\varepsilon = o(1).
$$

The conclusion is that there exist balls (keeping the same notations) $\Big(B_i(a_i, r_i)\Big)_{i \in \daleth_\varepsilon}$ such that the assertions (1), (2), (3) and the lower bound (4.34) hold.

**Lemma 4.13.** *If $\varepsilon$ is sufficiently small, there exist $0 < x_0 < 1$ and $0 < y_0 < 1$ such that there is no intersection between the boundary of the square $K^0 = [x_0, x_0 + 1[ \times [y_0, y_0 + 1[$ and any ball of the family $\Big(B_i(a_i, r_i)\Big)_{i \in \daleth_\varepsilon}$.*

**Proof**: We project the balls $\Big(B_i(a_i, r_i)\Big)_{i \in \daleth_\varepsilon}$ on the horizontal line of equation $y = 1$ contained in $\Omega = [0, 2[ \times [0, 2[$. Then, since $\frac{\log \alpha_\varepsilon}{\alpha_\varepsilon} \ll m_\varepsilon = o(1)$, the assertion (2) in proposition 4.12 gives us

$$
\sum_{i \in \daleth_\varepsilon} r_i \leq C\ \alpha_\varepsilon\ e^{-m_\varepsilon\ \alpha_\varepsilon} = o(1).
$$

From the above identity and if $\varepsilon$ is sufficiently small, there must exist $0 < x_0 < 1$ such that the two lines contained in $\Omega$ of equations $x = x_0$ and $x = x_0 + 1$ don't intersect any ball of the family $\Big(B_i(a_i, r_i)\Big)_{i\in\daleth_\varepsilon}$. Similarly, using the same argument, then if $\varepsilon$ is sufficiently small there exists $0 < y_0 < 1$ such that there is no intersection between the two lines contained in $\Omega$ of equations $y = y_0$ and $y = y_0 + 1$, and the balls $\Big(B_i(a_i, r_i)\Big)_{i\in\daleth_\varepsilon}$.
Consequently, for an $\varepsilon$ small enough, it is clear that the boundary of the square $K^0 = [x_0, x_0+1[\times[y_0, y_0+1[$ does not intersect any ball of the family $\Big(B_i(a_i, r_i)\Big)_{i\in\daleth_\varepsilon}$. □

Now, let $card(I_\varepsilon)$ be the number of the balls from $\Big(B_i(a_i, r_i)\Big)_{i\in\daleth_\varepsilon}$ which are contained in the square $K^0$. It is obvious from the lemma 4.13 that $\Big(B_i(a_i, r_i)\Big)_{i\in I_\varepsilon}$ is the new family of balls verifying, thanks to lemma 4.13,

$$\overline{\cup_{i\in I_\varepsilon} B_i(a_i, r_i)} \subset K^0. \tag{4.40}$$

Remark that $K^0$ can be considered as the fundamental domain of periodicity for $(u_\varepsilon, A_\varepsilon) \in \mathcal{A}$.
The balls $\Big(B_i(a_i, r_i)\Big)_{i\in\daleth_\varepsilon}$ are disjoint, then by definition of $I_\varepsilon$, the balls $\Big(B_i(a_i, r_i)\Big)_{i\in I_\varepsilon}$ are disjoint too. Moreover, it is immediate that the assertions (1), (2), (3) and the lower bound (4.34) in proposition 4.12 hold.

Finally, let $(u_\varepsilon, A_\varepsilon)$ verify the hypotheses of proposition 4.1, then referring to (4.2), we can find that the hypotheses of proposition 4.12 remain true. Hence, using the above completes the proof of the proposition 4.1 ( Here $K^0$ is the square of sidelength 1 where the family of balls $\Big(B_i(a_i, r_i)\Big)_{i\in I_\varepsilon}$ is defined). Proposition 4.1 is then proved with $K = K^0$.

**Remark 4.14.** *Without loss of generality, we will assume that the square $K$ above is simply* $[0, 1[\times[0, 1[$.

# Chapter 5

# Applied magnetic fields of the order of $H_{c_1}$

In this chapter, we assume that $h_{ex}$ is of the order of $H_{c_1}$ where $H_{c_1}$ behaves for $\varepsilon \to 0$ as $\frac{|\log \varepsilon|}{2}$. We will study in the limit $\varepsilon \to 0$ the asymptotic behavior of global minimizers $(u_\varepsilon, A_\varepsilon)$ of the Ginzburg-Landau energy $J_K$ over the space $\mathcal{A}$ and we will explore the vortex-structure of $(u_\varepsilon, A_\varepsilon)$. In particular, our interest is to describe the repartition and the number of the vortices. Our work will be based on the construction of the vortex balls summarized in proposition 4.1.

## 1 Statement of results

Consider $h_{ex}$ a function of $\varepsilon$ and assume

$$\lambda = \lim_{\varepsilon \to 0} \frac{|\log \varepsilon|}{h_{ex}}, \tag{5.1}$$

exists and is finite. If $\lambda = 0$, we require in addition that $h_{ex} \ll \frac{1}{\varepsilon^2}$. We take $K$ to be any square of sidelength 1. Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J_K$ over the space $\mathcal{A}$ and $h_\varepsilon = curl A_\varepsilon$ be the induced magnetic field. From now on, we will write $J$ instead of $J_K$.

In the case $\lambda = 0$, we have the following.

**Proposition 5.1.** *Assume $|\log \varepsilon| \ll h_{ex} \ll \frac{1}{\varepsilon^2}$. Let $h_\varepsilon$ be the induced magnetic field of a minimizing configuration $(u_\varepsilon, A_\varepsilon)$. Then, $\frac{h_\varepsilon}{h_{ex}}$ tends strongly to 1 locally in $H^1$ and*

$$\lim_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} = 0. \tag{5.2}$$

Now, we restrict to the case $\lambda > 0$, i.e. $h_{ex} \le C\,|\log \varepsilon|$. We define the space $V$

$$V := \Big\{ f \in H^1_{loc}(\mathbb{R}^2, \mathbb{R}) \ \text{ such that } \ f \ \text{ is periodic and } \ \mu = -\Delta f + f \ \text{ is a Radon measure} \Big\}. \tag{5.3}$$

We define for $f \in V$

$$E(f) = \frac{\lambda}{2} \int_K |-\Delta f + f| + \frac{1}{2} \int_K |\nabla f|^2 + \frac{1}{2} \int_K |f-1|^2, \tag{5.4}$$

and

$$P(f) = \frac{1}{2} \int_K |\nabla f|^2 + \frac{1}{2} \int_K |f|^2.$$

We have

**Proposition 5.2.** *There exists a unique $h_*$ such that*

$$E(h_*) = \min_{f \in V} E(f) = \frac{1}{2} - \min_{f \in W_0} P(f) = \frac{1}{2} - P(h_*),$$

*where*

$$W_0 =: \Big\{ f \in H^1_{loc}(\mathbb{R}^2, \mathbb{R}), \quad f \;\; \textit{is periodic}, \;\; \textit{and} \, \|f-1\|_{L^\infty(\mathbb{R}^2)} \le \frac{\lambda}{2} \Big\}. \tag{5.5}$$

## 1.1 Notations

We recall some facts about the weak convergence of general Radon measures.
**Theorem 1([EG])**
Let $\nu$, $\nu_k$ $(k = 1, 2, ...)$ be Radon measures defined on $\mathbb{R}^2$. The following three statements are equivalent:
(i) $\lim_{k\to\infty} \int_{\mathbb{R}^2} f \, d\nu_k = \int_{\mathbb{R}^2} f \, d\nu$ for all $f \in C_c(\mathbb{R}^2)$.
(ii) $\limsup_{k\to\infty} \nu_k(O) \le \nu(O)$ for each compact set $O \subset \mathbb{R}^2$ and $\nu(U) \le \liminf_{k\to\infty} \nu_k(U)$ for each open set $U \subset \mathbb{R}^2$.
(iii) $\lim_{k\to\infty} \nu_k(B) = \nu(B)$ for each bounded Borel set $B \subset \mathbb{R}^2$ with $\nu(\partial B) = 0$.

**Definition 5.3.** *If $(i)$ through $(iii)$ hold, then we say that the measures $\nu_k$ converge weakly to the measure $\nu$, it is written*

$$\nu_k \rightharpoonup \nu. \tag{5.6}$$

We also have
**Theorem 2 ([EG])**
Let $\{\nu_k\}_{k=1}^\infty$ be a sequence of Radon measures on $\mathbb{R}^2$ satisfying

$$\sup_k \nu_k(O) < \infty \quad \text{for each compact set } \; O \subset \mathbb{R}^2.$$

Then, there exists a subsequence $\{\nu_{k_j}\}_{j=1}^\infty$ and a Radon measure $\nu$ on $\mathbb{R}^2$ such that (in the sense of (5.6))

$$\nu_{k_j} \rightharpoonup \nu.$$

## 1.2 Main Theorem

Let $J^0$ be the energy of the test configuration $(u \equiv 1, A \equiv 0)$, also called the vortex-less energy. Then

$$J^0 = J(u \equiv 1, A \equiv 0) = \frac{1}{2}\, h_{ex}^2.$$

Once we restrict to the case $\lambda > 0$, we use the construction of the vortex balls that we recalled in proposition 4.1. Indeed, letting $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over the space $\mathcal{A}$, we will prove after (precisely, in the section of the lower bound of the energy $J$) that the hypotheses of the proposition 4.1 remain true, so we take the "vortices" $(a_i, d_i)_{i \in I_\varepsilon}$ defined by that proposition on the square $K = [0,1[\times[0,1[$. The main result is the following

**Theorem 5.4.** *Assume $\lambda > 0$. Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over $\mathcal{A}$ and $h_\varepsilon = curl A_\varepsilon$ be the associated magnetic field. Then, as $\varepsilon \to 0$*

$$\frac{h_\varepsilon}{h_{ex}} \rightharpoonup h_* = \max\Big(0, 1 - \frac{\lambda}{2}\Big) \quad \textit{weakly locally in } H^1.$$

*In addition,*

$$\lim_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} = \begin{cases} \frac{\lambda}{2}\,(1 - \frac{\lambda}{4}) & \textit{if } 0 < \lambda < 2 \\ \frac{1}{2} & \textit{if } \lambda \geq 2. \end{cases} \tag{5.7}$$

*Moreover, letting $\mu_\varepsilon$ be the extended measure by periodicity to $\mathbb{R}^2$ of the measure $\frac{\sum_{i \in I_\varepsilon} 2\,\pi\, d_i\, \delta_{a_i}}{h_{ex}}$, we have as $\varepsilon \to 0$*

$$\mu_\varepsilon \rightharpoonup \max\Big(0, 1 - \frac{\lambda}{2}\Big)\, dx \tag{5.8}$$

*where $dx$ is the Lebesgue measure on $\mathbb{R}^2$.*

Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over $\mathcal{A}$ and $h_\varepsilon = curl A_\varepsilon$. Recall that, if $N_\varepsilon$ is defined by

$$N_\varepsilon := \frac{1}{2\,\pi} \int_K h_\varepsilon, \tag{5.9}$$

then from corollary 4.4, we have $N_\varepsilon = \sum_{i \in I_\varepsilon} d_i$, and then $N_\varepsilon$ represents the number of the vortices per period. In particular, Theorem 5.4 gives us the order of $N_\varepsilon$ when $\varepsilon$ tends to 0.

**Corollary 5.5.** *Let $\lambda \geq 0$, then*

$$\lim_{\varepsilon \to 0} \frac{N_\varepsilon}{h_{ex}} = \max\Big(0, \frac{1}{2\,\pi}(1 - \frac{\lambda}{2})\Big). \tag{5.10}$$

*Proof.* Since $N_\varepsilon = \frac{1}{2\,\pi} \int_K h_\varepsilon$, we have

$$\lim_{\varepsilon \to 0} \frac{N_\varepsilon}{h_{ex}} = \frac{1}{2\,\pi} \lim_{\varepsilon \to 0} \frac{h_\varepsilon}{h_{ex}}. \tag{5.11}$$

If $\lambda > 0$, Theorem 5.4 implies that $\frac{h_\varepsilon}{h_{ex}}$ tends weakly in $H^1(K)$ to $h_*$, where $h_* = \max(0, 1-\frac{\lambda}{2})$, and modulo a subsequence, the convergence is strong in $L^1$. Therefore

$$2\,\pi \lim_{n\to+\infty} \frac{N_{\varepsilon_n}}{h_{ex}} = \int_K h_* = \max\left(0, 1 - \frac{\lambda}{2}\right). \tag{5.12}$$

If $\lambda = 0$, then $\frac{h_\varepsilon}{h_{ex}}$ tends to 1 strongly in $H^1$ from corollary 5.1, and therefore

$$2\,\pi \lim_{n\to+\infty} \frac{N_{\varepsilon_n}}{h_{ex}} = 1. \tag{5.13}$$

This completes the proof. □

**Proposition 5.6.** *If $\lambda > 2$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$*

$$\sum_{i\in I_\varepsilon} |d_i| = 0,$$

*where $(a_i, d_i)_{i\in I_\varepsilon}$ is the family of vortices defined on $K$ and associated to minimizers of $J$ over the space $\mathcal{A}$.*

### 1.3 Interpretations and commentaries

The results of Theorem 5.4 first indicate that $(h_{ex}\ \max\big(0, 1-\frac{\lambda}{2}\big))$ can be seen as a good approximation of $h_\varepsilon$ as $\varepsilon \to 0$, and provide the asymptotic expansion of the energy. Also, the convergence (5.8) gives us an idea on the limit measure of vortices, so it describes the repartition of the vortices for global minimizers $(u_\varepsilon, A_\varepsilon)$ of the energy $J$ over the space $\mathcal{A}$. While, (5.10) gives us an estimate on the number of the vortices per period. In particular, we have with respect to $\lambda$ (and then the applied field $h_{ex}$) the following

- If $0 < \lambda < 2$, (5.8) implies that there is a uniform-vortex distribution, and from corollary 5.5, we remark that the number of the vortices per period is expected to be proportional to the applied magnetic field $h_{ex}$.

- If $\lambda \geq 2$, the number of vortices in the material is negligible compared to $h_{ex}$, but the Theorem 6.1 does not give us an affirmative answer on the number and the repartition of the vortices. However, from proposition 5.6, the minimizers of $J$ have no vortices when $\lambda > 2$. While, the case $\lambda = 2$ will be addressed in chapter 6.

The above gives us a meaning to the value of the first critical field $H_{c_1}$ which behaves for $\varepsilon \to 0$ as $\frac{|\log\varepsilon|}{2}$. Our results are slightly different to [SS3]. Indeed, in [SS3] the vortices of minimizers of the Ginzburg-Landau energy $J_\Omega$ over $H^1(\Omega,\mathbb{C})\times H^1(\Omega,\mathbb{R}^2)$ are scattered uniformly in an inner region denoted $\omega_\lambda$ with a vortex-density equal to $h_{ex} - \frac{|\log\varepsilon|}{2}$ (note that $\Omega$ is the section of the superconductor) . In the outer region $\Omega\backslash\overline{\omega_\lambda}$, there are no vortices. Moreover, taking $\psi$ solution to

$$\begin{cases} -\Delta\psi + \psi = -1 \quad \text{in } \Omega \\ \psi = 0 \quad \text{on } \partial\Omega, \end{cases}$$

then as $\lambda$ decreases, the vortex-region first appears at the minimum of $\psi$. More clearly, a necessary and sufficient condition for $\omega_\lambda$ to be nonempty is

$$\omega_\lambda \neq \emptyset \Longleftrightarrow \lim_{\varepsilon\to 0} \frac{h_{ex}}{|\log\varepsilon|} \geq \frac{1}{2\ \max|\psi|} \Longleftrightarrow \lambda \leq 2\ \max|\psi|.$$

In addition, there exists $C > 0$ such that

$$dist(\omega_\lambda, \partial\Omega) \geq C\ \lambda.$$

Note that the difference between our results and those of [SS3] is due to the fact that the periodic model removes the boundary effects.

The proof of Theorem 5.4 will be obtained by getting first an upper bound on the energy by construction of approximate solutions, and then a lower bound based on energy estimates and convergence of measures, in the spirit of gamma-convergence (see [DC]) of the "normalized" energy $\frac{J}{{h_{ex}}^2}$ defined on $\mathcal{A}$ to the functional $E$ over the space $V$.

# 2 Upper bound of the energy

In this section, we bound from above $\frac{J(u_\varepsilon,A_\varepsilon)}{h_{ex}^2}$ where $(u_\varepsilon, A_\varepsilon)$ is a minimizer of $J$ over the space $\mathcal{A}$.

## 2.1 Preliminaries

We take $a \in \mathbb{R}_+$. Let $f \in V$ — where $V$ is defined by (5.3) — satisfy

$$\mu = a = -\Delta f + f \quad \text{in } \mathbb{R}^2, \tag{5.14}$$

where $d\mu = a\ dx$. The fact that $f$ is bounded in (5.14) leads to

$$f(x) = a \quad \forall\, x \in \mathbb{R}^2. \tag{5.15}$$

Inserting (5.15) in (5.4), we obtain

$$E(f) = \frac{\lambda\ a}{2} + \frac{(a-1)^2}{2}. \tag{5.16}$$

We define $G$ to be the solution to

$$-\Delta_x G(x,y) + G(x,y) = \delta_y \quad \text{in } \mathbb{R}^2. \tag{5.17}$$

Remark that $G$ exists and it is unique. We state some well known properties of $G$ (see [Ti] for instance).

**Lemma 5.7.** *The function $G(x, y)$, solution of (5.17), has the following properties*

*(1) $G(x, y)$ is symmetric and positive.*

*(2) $G(x, y) + \frac{1}{2\,\pi}\ \log|x - y|$ has a $C^1$ extension on $\mathbb{R}^2$.*

*(3) As $|x - y| \to +\infty$ we have that $G(x, y)$, $\nabla_x G(x, y)$ are $\mathcal{O}(e^{-|x-y|})$.*

*Finally*

$$\int_{\mathbb{R}^2} G(x, y)\ dx = 1. \tag{5.18}$$

**Proof**: The first property is well known, and so is the third. The second property follows by noting that $U(x) = G(x, y) + \frac{1}{2\,\pi}\ \log|x - y|$ satisfies the equation

$$-\Delta U + U = \frac{1}{2\,\pi}\ \log|x - y|.$$

The right hand side is in $L^q$ locally for any $q$, hence locally by elliptic regularity, the function $U$ is locally in $W^{2,q}$ , and therefore $C^1$.
Finally, letting $B(y, R)$ (resp. $B(y, r)$) be the ball of center $y$ and of radius $R > 0$ (resp. $r > 0$), (5.18) follows by integrating the equation $-\Delta_x G(x, y) + G(x, y) = 0$ in $B(y, R) \setminus B(y, r)$, letting $R \to +\infty$ and $r \to 0$, and using the asymptotics of $G$ to estimate the boundary terms. □

## 2.2 Main result

The upper bound of the energy $J$ we prove is

**Proposition 5.8.** *Let $h_{ex}$ be such that $\lim_{\varepsilon\to 0} \frac{|\log \varepsilon|}{h_{ex}} = \lambda$, with the additional condition; if $\lambda = 0$, that $h_{ex} \ll \frac{1}{\varepsilon^2}$. Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J$ over the space $\mathcal{A}$, then for any $a \geq 0$*

$$\limsup_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \leq \frac{\lambda\ a}{2} + \frac{(a-1)^2}{2}. \tag{5.19}$$

Minimizing the right-hand side with respect to $a \in \mathbb{R}_+$ yields

**Corollary 5.9.** *Under the same assumptions of proposition 5.8, we have*

- *If $0 < \lambda < 2$, we have*

$$\limsup_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \leq \frac{\lambda}{2}\ (1 - \frac{\lambda}{4}). \tag{5.20}$$

- *If $\lambda = 0$, we have*

$$\limsup_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} = 0. \tag{5.21}$$

**Remark 5.10.** *Let $h_\varepsilon$ be the induced magnetic field of a minimizing configuration $(u_\varepsilon, A_\varepsilon)$. If $\lambda = 0$, (5.21) holds, then going back to (3.11) to write*

$$\frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \geq \frac{1}{2} \left\| \frac{h_\varepsilon}{h_{ex}} - 1 \right\|^2_{H^1(K)}. \tag{5.22}$$

*Using (5.2) in (5.22) implies that $\frac{h_\varepsilon}{h_{ex}}$ tends to 1 strongly in $H^1(K)$. Now, thanks to the periodicity of $h_\varepsilon$, there exists $C > 0$ such that for each compact $O \subset \mathbb{R}^2$*

$$\left\| \frac{h_\varepsilon}{h_{ex}} - 1 \right\|_{H^1(O)} \leq C \left\| \frac{h_\varepsilon}{h_{ex}} - 1 \right\|_{H^1(K)}.$$

*It follows that $\frac{h_\varepsilon}{h_{ex}}$ tends to 1 strongly in $H^1(O)$ for each compact $O \subset \mathbb{R}^2$. The proposition 5.1 is then proved.*

### 2.3 Proof of proposition 5.8

Proposition 5.8 is proved by constructing a test configuration having approximately $\frac{a\,h_{ex}}{2\,\pi}$ vortices of degree one regularly spread in $K = [0,1[\times[0,1[$. We follow closely [SS3], proposition 2.2.

**Step 1**

Let

$$p = \left[ \sqrt{\frac{a\,h_{ex}}{2\,\pi}} \right], \tag{5.23}$$

where $[x]$ denotes the greatest integer less than or equal to $x$. We place a point at the center of each square $[\frac{k}{p}, \frac{k+1}{p}[\times[\frac{l}{p}, \frac{l+1}{p}[$, where $0 \leq k, l < p$. We call $(a_i^\varepsilon)_{1 \leq i \leq n(\varepsilon)}$ the resulting family of points in the square $K$. The total number of points $n(\varepsilon)$ is

$$n(\varepsilon) = p^2 \simeq \frac{a\,h_{ex}}{2\,\pi}.$$

Letting $\mu_\varepsilon^i$ be the uniform measure on $\partial(B_i(a_i^\varepsilon, \varepsilon))$ of mass $2\,\pi$, we define $\mu_\varepsilon^K$ to be $\frac{\sum_{1 \leq i \leq n(\varepsilon)} \mu_\varepsilon^i}{h_{ex}}$ and $\mu_\varepsilon$ to be the extension of $\mu_\varepsilon^K$ to $\mathbb{R}^2$ by periodicity. In other words, $\mu_\varepsilon = \sum_K \mu_\varepsilon^K$ where the sum runs over a tiling of $\mathbb{R}^2$ by squares of sidelength 1. It is clear from the above

$$\mu_\varepsilon \rightharpoonup a\,dx. \tag{5.24}$$

**Step 2**

Let $\beta > 0$ and set $\Delta_\beta$ to be a $\beta$ neighborhood of the diagonal in $K \times K$. Namely,

$$\Delta_\beta = \{(x, y) \in K \times K, \quad |x - y| < \beta\}.$$

Since $\mu_\varepsilon \rightharpoonup a\,dx$, we have

$$\mu_\varepsilon \otimes \mu_\varepsilon \rightharpoonup a^2\,dx \otimes dx \quad \text{as} \quad \varepsilon \to 0.$$

In view of the continuity of $G$ on $\Big(K \times K\Big) \backslash \Delta_\beta$, we are led to

$$\limsup_{\varepsilon \to 0} \frac{1}{2} \int\int_{(K \times K) \backslash \Delta_\beta} G\,d\mu_\varepsilon\,d\mu_\varepsilon = \frac{a^2}{2} \int_{(K \times K) \backslash \Delta_\beta} G\,dx\,dy. \tag{5.25}$$

Moreover, since $G(.,.)$ is continuous on $K \times (\mathbb{R}^2 \backslash K)$, it follows that

$$\lim_{\varepsilon \to 0} \int_{K \times (\mathbb{R}^2 \backslash K)} G \, d\mu_\varepsilon \, d\mu_\varepsilon = a^2 \int_{K \times (\mathbb{R}^2 \backslash K)} G \, dx \, dy. \tag{5.26}$$

Now, we treat the integral on $\Delta_\beta$. Referring to [SS3], proposition 2.2, there exists a constant $c(\beta) \to 0$ as $\beta \to 0$ such that

$$\frac{1}{2} h_{ex}^2 \int\int_{\Delta_\beta} G(x,y) \, d\mu_\varepsilon(x) \, d\mu_\varepsilon(y) \le C \, \beta \, \pi \, n(\varepsilon) \Big( |\log \varepsilon| + 1 \Big) + n(\varepsilon) \Big( \pi \, |\log \varepsilon| + c(\beta) \Big)$$
$$+ C \, a^2 \, h_{ex}^2 \, |\Delta_{2\,\beta}|. \tag{5.27}$$

We divide by $h_{ex}^2$ and we replace $n(\varepsilon)$ with $\frac{a\, h_{ex}}{2\,\pi}$ to have for a small enough $\varepsilon$

$$\frac{1}{2} \int\int_{\Delta_\beta} G(x,y) \, d\mu_\varepsilon(x) \, d\mu_\varepsilon(y) \le \frac{a \, |\log \varepsilon|}{2 \, h_{ex}} + C \, \beta \, \frac{1}{h_{ex}} \Big( |\log \varepsilon| + 1 \Big)$$
$$+ \frac{c(\beta)}{h_{ex}} + C \, |\Delta_{2\beta}|.$$

We use $\lambda \simeq \frac{|\log \varepsilon|}{h_{ex}}$ as $\varepsilon \to 0$ to get

$$\limsup_{\varepsilon \to 0} \frac{1}{2} \int\int_{\Delta_\beta} G(x,y) \, d\mu_\varepsilon(x) \, d\mu_\varepsilon(y) \le \frac{\lambda}{2} \, a + C \, |\Delta_{2\beta}| + \mathcal{M}(\beta),$$

where $\mathcal{M}(\beta)$ is a constant of $\beta$ tending to 0 as $\beta \to 0$. Thanks to (5.25) and to the fact that $\lim_{\beta \to 0} |\Delta_{2\beta}| = 0$, we get

$$\limsup_{\varepsilon \to 0} \frac{1}{2} \int_{K \times K} G(x,y) d\mu_\varepsilon(x) \, d\mu_\varepsilon(y) \le \frac{\lambda}{2} \, a + \frac{a^2}{2} \int_{K \times K} G(x,y) \, dx \, dy. \tag{5.28}$$

Now, combining (5.26) together with (5.28), we can deduce

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \frac{1}{2} \int_{K \times \mathbb{R}^2} G(x,y) d\mu_\varepsilon(y) \, d\mu_\varepsilon(x) \\
&\qquad \le \limsup_{\varepsilon \to 0} \frac{1}{2} \int_{K \times K} G(x,y) d\mu_\varepsilon(y) \, d\mu_\varepsilon(x) + \frac{1}{2} \lim_{\varepsilon \to 0} \int_{K \times (\mathbb{R}^2 \backslash K)} G \, d\mu_\varepsilon \, d\mu_\varepsilon \\
&\qquad \le \frac{\lambda}{2} \, a + \frac{1}{2} \int_{K \times K} G(x,y) \, dx \, dy + \frac{a^2}{2} \int_{K \times (\mathbb{R}^2 \backslash K)} G \, dx \, dy \\
&\qquad \le \frac{\lambda}{2} a \, + \frac{a^2}{2} \int_{K \, \times \mathbb{R}^2} G(x,y) \, dx \, dy \\
&\qquad = \frac{\lambda}{2} \, a + \frac{a^2}{2}.
\end{aligned}$$

Consequently, we find

$$\limsup_{\varepsilon\to 0} \frac{1}{2} \int_{K\ \times \mathbb{R}^2} G(x,y)\ d(\nu_\varepsilon - 1)(x)\ d(\nu_\varepsilon - 1)(y) \le \frac{\lambda}{2}\ a + \frac{1}{2}\ (a-1)^2. \qquad (5.29)$$

**Step 3**

Now, our interest is to prove (5.19), so for this we construct an appropriate test configuration $(v_\varepsilon, B_\varepsilon)$ in $\mathcal{A}$. First, we define $h_\varepsilon$ to be the periodic solution of

$$-\Delta h_\varepsilon + h_\varepsilon = h_{ex}\ \mu_\varepsilon \quad \text{in } \mathbb{R}^2, \qquad (5.30)$$

where $\mu_\varepsilon$ was defined in step 1. Alternatively, $h_\varepsilon(x) = h_{ex} \int_{\mathbb{R}^2} G(x,y)\, d\mu_\varepsilon(y)$. Now, let $B_\varepsilon$ be a solution of

$$curl\ B_\varepsilon = h_\varepsilon.$$

$B_\varepsilon$ is taken to be the magnetic potential.

We then need to define $v_\varepsilon$ on $\mathbb{R}^2$. Writing $v_\varepsilon = \rho_\varepsilon e^{i\phi_\varepsilon}$, we define $\rho_\varepsilon$ to be periodic and in the square $K$

$$\rho_\varepsilon(x) = \begin{cases} 0 & \text{in } \cup_{1\le i\le n(\varepsilon)} \Big(B_i(a_i^\varepsilon, \varepsilon)\Big) \\ 1 & \text{in } K\backslash \cup_{1\le i\le n(\varepsilon)} \Big(B_i(a_i^\varepsilon, 2\varepsilon)\Big) \end{cases} \qquad (5.31)$$

and such that $0 \le \rho_\varepsilon \le 1$, and for each $1 \le i \le n(\varepsilon)$,

$$\int_{B_i(a_i^\varepsilon,\varepsilon)} |\nabla \rho_\varepsilon|^2 + \frac{1}{2\ \varepsilon^2}(1-\rho_\varepsilon^2)^2 \le C. \qquad (5.32)$$

To define $\phi_\varepsilon$, let first

$$B_i = B_i(a_i^\varepsilon, \varepsilon) \quad \text{for } 1 \le i \le n(\varepsilon).$$

For any $1 \le i \le n(\varepsilon)$, let $B_{i,n,m}$ be the image of $B_i$ by translation of vector $n\ \vec{i} + m\ \vec{j}$ where $n$ and $m$ are in $\mathbb{Z}$. We need only to define the function $\phi_\varepsilon$ only modulo $2\ \pi$, and where $\rho_\varepsilon \ne 0$. The fact that $\phi_\varepsilon$ is not defined on $\cup_{(1\le i\le n(\varepsilon),\ n,m\in\mathbb{Z})} B_{i,n,m}$ is not important, since $\rho_\varepsilon = 0$ there. Choosing a point $x_0 \in \mathbb{R}^2 \backslash \cup_{(1\le i\le n(\varepsilon),\ n,m\in\mathbb{Z})} B_{i,n,m}$, we define for any $x \in \mathbb{R}^2 \backslash \cup_{(1\le i\le n(\varepsilon),\ n,m\in\mathbb{Z})} B_{i,n,m}$, the function

$$\phi_\varepsilon(x) = \oint_{(x_0,x)} B_\varepsilon.\tau - \nabla h_\varepsilon.\nu, \qquad (5.33)$$

where $(x_0, x)$ is any curve joining $x_0$ to $x$ in $\mathbb{R}^2 \backslash \cup_{(1\le i\le n(\varepsilon),\ n,m\in\mathbb{Z})} B_{i,n,m}$ and $(\tau, \nu)$ is the Frenet frame on the curve. Note that the function $\phi_\varepsilon$ is well defined modulo $2\ \pi$. Indeed, if $\omega \subset \mathbb{R}^2$ is such that $\partial\omega \subset \mathbb{R}^2 \backslash \cup_{(1\le i\le n(\varepsilon),\ n,m\in\mathbb{Z})} B_{i,n,m}$, then by (5.33)

$$\int_{\partial\omega} B_\varepsilon.\tau - \nabla h_\varepsilon.\nu = \int_\omega -\Delta h_\varepsilon + h_\varepsilon = h_{ex}\ \mu_\varepsilon(\omega).$$

This quantity is in turn equal to $2\ \pi\ k$, where $k$ is the number of points $a_i^\varepsilon$ in $\omega$. Thus, $e^{i\ \phi_\varepsilon}$ defined by (5.33) does not depend on the choice of the particular curve $(x_0, x)$. Now, let us take $v_\varepsilon = \rho_\varepsilon\ e^{i\ \phi_\varepsilon}$.

**Step 4**

We begin with

**Lemma 5.11.** *The test configuration $(v_\varepsilon, B_\varepsilon)$ belongs to the space $\mathcal{A}$ where $\mathcal{A}$ is given by the definition 2.3.*

**Proof**: The periodicity of the magnetic field $h_\varepsilon$ yields for $(x,y) \in \mathbb{R}^2$ that

$$h_\varepsilon(x+1,y) = h_\varepsilon(x,y) = h_\varepsilon(x,y+1).$$

The magnetic potential $B_\varepsilon$ is taken to solve $curl\ B_\varepsilon = h_\varepsilon$, hence there exist $R_1$ and $R_2$ in $H^1_{loc}(\mathbb{R}^2)$ such that

$$\begin{cases} B_\varepsilon(x+1,y) = B_\varepsilon(x,y) + R_1(x,y) \\ \\ B_\varepsilon(x,y+1) = B_\varepsilon(x,y) + R_2(x,y), \end{cases} \tag{5.34}$$

where

$$\begin{cases} curl\ R_1(x,y) = 0 \\ curl\ R_2(x,y) = 0. \end{cases}$$

For $1 \le i \le 2$, $curl R_i = 0$ implies the existence of $(f_0, g_0) \in H^2_{loc}(\mathbb{R}^2) \times H^2_{loc}(\mathbb{R}^2)$ such that

$$\begin{cases} R_1(x,y) = \nabla f_0(x,y) \\ R_2(x,y) = \nabla g_0(x,y). \end{cases} \tag{5.35}$$

We insert (5.35) in (5.34) to get

$$\begin{cases} B_\varepsilon(x+1,y) = B_\varepsilon(x,y) + \nabla f_0(x,y) \\ \\ B_\varepsilon(x,y+1) = B_\varepsilon(x,y) + \nabla g_0(x,y). \end{cases} \tag{5.36}$$

Now, from the construction of $\phi_\varepsilon$, we have in $\mathbb{R}^2 \setminus \cup_{(1 \le i \le n(\varepsilon),\, n,m \in \mathbb{Z})} B_{i,n,m}$

$$\nabla \phi_\varepsilon = B_\varepsilon - \nabla^\perp h_\varepsilon. \tag{5.37}$$

On the one hand, we use again the periodicity of $h_\varepsilon$ with (5.36) in (5.37) to write

$$\begin{aligned} \nabla \phi_\varepsilon(x+1,y) =& B_\varepsilon(x+1,y) - \nabla^\perp h_\varepsilon(x+1,y) \\ =& B_\varepsilon(x,y) + \nabla f_0(x,y) - \nabla^\perp h_\varepsilon(x,y) \\ =& \nabla \phi_\varepsilon(x,y) + \nabla f_0(x,y). \end{aligned} \tag{5.38}$$

By integration, there exists $c \in \mathbb{R}$ such that

$$\phi_\varepsilon(x+1,y) = \phi_\varepsilon(x,y) + f_0(x,y) + c.$$

Let us set $f_\varepsilon(x,y) = f_0(x,y) + c$, hence

$$\phi_\varepsilon(x+1,y) = \phi_\varepsilon(x,y) + f_\varepsilon(x,y). \tag{5.39}$$

On the other hand, proceeding similarly as (5.39), there exists $g_\varepsilon \in H^2_{loc}(\mathbb{R}^2)$ such that we have in $\mathbb{R}^2 \setminus \cup_{(1\leq i\leq n(\varepsilon),\, n,m\in\mathbb{Z})} B_{i,n,m}$

$$\phi_\varepsilon(x, y+1) = \phi_\varepsilon(x, y) + g_\varepsilon(x, y). \tag{5.40}$$

Recall that $v_\varepsilon = \rho_\varepsilon \, e^{i\,\phi_\varepsilon}$, then combining (5.39)- (5.40) together with the periodicity of $\rho_\varepsilon$, we get the two following equations in $\mathbb{R}^2$

$$\begin{cases} v_\varepsilon(x+1, y) = v_\varepsilon(x, y)\; e^{i\, f_\varepsilon(x,y)} \\ v_\varepsilon(x, y+1) = v_\varepsilon(x, y)\; e^{i\, g_\varepsilon(x,y)}, \end{cases} \tag{5.41}$$

since $\rho_\varepsilon$ is equal to 0 in $\cup_{(1\leq i\leq n(\varepsilon),\, n,m\in\mathbb{Z})} B_{i,n,m}$. We replace $f_0$ and $g_0$ respectively with $(f_\varepsilon - c)$ and $(g_\varepsilon - c')$ in (5.36)

$$\begin{cases} B_\varepsilon(x+1, y) = B_\varepsilon(x, y) + \nabla f_\varepsilon(x, y) \\ B_\varepsilon(x, y+1) = B_\varepsilon(x, y) + \nabla g_\varepsilon(x, y). \end{cases} \tag{5.42}$$

A Combination of (5.41) together with (5.42) gives us that the configuration $(v_\varepsilon, B_\varepsilon) \in \mathcal{A}$. This completes the proof of lemma 5.11. □

**Step 5: Completing the proof of proposition 5.8**

From the equation (5.30), the induced magnetic field $h_\varepsilon$ satisfies

$$-\Delta h_\varepsilon + h_\varepsilon - h_{ex} = h_{ex}\,(\mu_\varepsilon - 1) \quad \text{in} \quad \mathbb{R}^2. \tag{5.43}$$

Hence, from (5.18) we can write

$$h_\varepsilon(y) = h_{ex} \int_{\mathbb{R}^2} G(y, x)\, d\mu_\varepsilon(x), \qquad \forall\; y \in K. \tag{5.44}$$

Now, multiplying (5.43) by $(h_\varepsilon - h_{ex})$, integrating on $K$, and using (5.44) with the periodicity of $h_\varepsilon$, it follows that

$$\begin{aligned} \int_K |\nabla h_\varepsilon|^2 + \int_K |h_\varepsilon - h_{ex}|^2 &= \int_K (-\Delta h_\varepsilon + h_\varepsilon - h_{ex})\,(h_\varepsilon - h_{ex}) \\ &= \int_K h_{ex}\,(h_\varepsilon - h_{ex})(y)\; d(\mu_\varepsilon - 1)(y) \\ &= h_{ex}^2 \int_K \int_{\mathbb{R}^2} G(y, x)\; d(\mu_\varepsilon - 1)(x)\; d(\mu_\varepsilon - 1)(y), \end{aligned}$$

where $(\mu_\varepsilon - 1)$ denotes the difference between of the measure $\mu_\varepsilon$ and the Lebesgue measure on $\mathbb{R}^2$. We divide by $2\,h_{ex}^2$ to get

$$\limsup_{\varepsilon\to 0} \frac{\frac{1}{2}\int_K |\nabla h_\varepsilon|^2 + \frac{1}{2}\int_K |h_\varepsilon - h_{ex}|^2}{h_{ex}^2} = \limsup_{\varepsilon\to 0} \frac{1}{2} \int_{K\times\mathbb{R}^2} G(x, y)\; d(\mu_\varepsilon - 1)(y)\; d(\mu_\varepsilon - 1)(x).$$

We use (5.29) to have

$$\limsup_{\varepsilon\to 0} \frac{\frac{1}{2}\int_K |\nabla h_\varepsilon|^2 + \frac{1}{2}\int_K |h_\varepsilon - h_{ex}|^2}{h_{ex}^2} \le \frac{\lambda\, a}{2} + \frac{(a-1)^2}{2}. \tag{5.45}$$

In addition, by definition of $\rho_\varepsilon$ and the fact that $n(\varepsilon) = \mathcal{O}(h_{ex})$, it is clear

$$\limsup_{\varepsilon\to 0} \frac{\frac{1}{2}\int_K |\nabla \rho_\varepsilon|^2 + \frac{1}{4\,\varepsilon^2}\int_K (1-\rho_\varepsilon^2)^2}{h_{ex}^2} = 0. \tag{5.46}$$

Moreover, from (5.37)

$$\rho_\varepsilon^2\,|\nabla\phi_\varepsilon - B_\varepsilon|^2 \le |\nabla h_\varepsilon|^2. \tag{5.47}$$

In particular, (5.47) leads to

$$\begin{aligned}\limsup_{\varepsilon\to 0} \frac{J_K(v_\varepsilon, B_\varepsilon)}{h_{ex}^2} \le & \limsup_{\varepsilon\to 0}\Big(\frac{\frac{1}{2}\int_K |\nabla h_\varepsilon|^2 + \frac{1}{2}\int_K |h_\varepsilon - h_{ex}|^2}{h_{ex}^2}\Big) \\ & + \limsup_{\varepsilon\to 0}\Big(\frac{\frac{1}{2}\int_K |\nabla \rho_\varepsilon|^2 + \frac{1}{4\,\varepsilon^2}\int_K (1-\rho_\varepsilon^2)^2}{h_{ex}^2}\Big).\end{aligned} \tag{5.48}$$

A combination of (5.45) together with (5.46) in (5.48) allows to write

$$\limsup_{\varepsilon\to 0} \frac{J_K(v_\varepsilon, B_\varepsilon)}{h_{ex}^2} \le \frac{\lambda\, a}{2} + \frac{(a-1)^2}{2}, \quad \forall\, a \ge 0. \tag{5.49}$$

This inequality is true for the test configuration $(v_\varepsilon, B_\varepsilon)$, so it is true in particular for any minimizer of $J$ over $\mathcal{A}$. This completes the proof of proposition 5.8.

## 3 Lower bound

Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J$ over the space $\mathcal{A}$ and $h_\varepsilon = curl A_\varepsilon$ be the induced magnetic magnetic field. We take $K$ to be any square of sidelength 1. From now on, we assume that $\lambda > 0$, i.e. the applied field satisfies $h_{ex} \le C\,|\log\varepsilon|$. First, it is clear by testing $J$ with the configuration $(u \equiv 1, A \equiv 0)$, that the minimum of $J$ is less than $J^0 = \frac{1}{2}\; h_{ex}^2$. Then, from the expression of $J$, and by the definition (4.3), we have

$$\gamma_\varepsilon = \|h_\varepsilon\|_{L^2(K)} \le C\; h_{ex}.$$

On the other hand, from the expressions of the energy $J$ and the functional $F_\varepsilon$, we have

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, K) \le J(u_\varepsilon, A_\varepsilon). \tag{5.50}$$

Knowing $J(u_\varepsilon, A_\varepsilon) \le C\; h_{ex}^2$, hence $F_\varepsilon(u_\varepsilon, A_\varepsilon, K) \le C\; h_{ex}^2$. Let

$$\alpha_\varepsilon = h_{ex},$$

then $\alpha_\varepsilon \to +\infty$ as $\varepsilon \to 0$ and $\alpha_\varepsilon \le c\, h_{ex}$ $(c > 1)$. Moreover, we can write

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, K) \le C\, \alpha_\varepsilon\, h_{ex}.$$

We define $m_\varepsilon = o(1)$ to be any sequence verifying

$$\frac{\log h_{ex}}{h_{ex}} \ll m_\varepsilon. \tag{5.51}$$

Note that $m_\varepsilon$ is positive. Combining all the above, we can say that the hypotheses of proposition 4.1 hold. Hence, there exist a square of sidelength 1, (without loss of generality the square is $K = [0,1[\times[0,1[$), and a family of disjoint balls $\Big(B_i = B_i(a_i, r_i)\Big)_{i\in I_\varepsilon}$ such that

$$\overline{\cup_{i\in I_\varepsilon} B_i(a_i, r_i)} \subset K, \tag{5.52}$$

where the sum of radii $r_i$ verifies

$$\sum_{i\in I_\varepsilon} r_i \le C\, h_{ex}\;\; e^{-m_\varepsilon\, h_{ex}}. \tag{5.53}$$

Note that thanks to (5.51), we have as $\varepsilon \to 0$

$$\sum_{i\in I_\varepsilon} r_i = o(1). \tag{5.54}$$

For any such set of balls, we can associate to $u_\varepsilon$ the vorticity measure $\frac{2\,\pi\,\sum_{i\in I_\varepsilon} d_i\, \delta_{a_i}}{h_{ex}}$, where $d_i$ is the degree of $\frac{u_\varepsilon}{|u_\varepsilon|}$ restricted to $\partial B_i(a_i, r_i)$. Now, let $\mu_\varepsilon$ be the periodic measure on $\mathbb{R}^2$ whose restriction to $K$ is $\frac{\sum_{i\in I_\varepsilon} 2\,\pi\, d_i\, \delta_{a_i}}{h_{ex}}$. We begin with the following proposition.

**Proposition 5.12.** *For all $\varepsilon_n \to 0$, we can extract a subsequence such that there exist a periodic $h_0$ in $H^1_{loc}(\mathbb{R}^2)$ and a Radon measure $\mu_0$ on $\mathbb{R}^2$ satisfying*

$$\frac{h_{\varepsilon_n}}{h_{ex}} \rightharpoonup h_0 \quad \textit{weakly in} \quad H^1_{loc}(\mathbb{R}^2), \tag{5.55}$$

*and*

$$\mu_{\varepsilon_n} \rightharpoonup \mu_0. \tag{5.56}$$

*Also, we have*

$$-\Delta h_0 + h_0 = \mu_0 \quad \textit{in} \quad \mathbb{R}^2. \tag{5.57}$$

*Hence, $\mu_0 \in H^{-1}$ and $h_0 \in V$ where $V$ is defined by (5.3).*

## 3.1 Proof of proposition 5.12

We split the proof into five steps
**Step 1**
We start with the inequality (3.10) which is

$$\frac{1}{2}\, \|h_\varepsilon - h_{ex}\|^2_{H^1(K)} \leq J(u_\varepsilon, A_\varepsilon).$$

Using then $J(u_\varepsilon, A_\varepsilon) \leq \frac{1}{2}\, h_{ex}^2$ allows to say that $\frac{h_\varepsilon}{h_{ex}}$ is bounded in $H^1(K)$. Then, by periodicity of $h_\varepsilon$, we can say that $\frac{h_\varepsilon}{h_{ex}}$ is bounded in $H^1(O)$ for any compact $O \subset \mathbb{R}^2$, so in particular it is bounded in $H^1_{loc}(\mathbb{R}^2)$. Hence, for a subsequence $\varepsilon_n$, there exists $h_0$ in $H^1_{loc}(\mathbb{R}^2)$ such that $\frac{h_{\varepsilon_n}}{h_{ex}}$ tends to $h_0$ weakly in $H^1_{loc}(\mathbb{R}^2)$ as $n \to +\infty$. Again, the periodicity of $h_\varepsilon$ implies that the weak limit $h_0$ is periodic.
**Step 2**
The lower bound of the energy on the vortex balls $\Big(B_i(a_i, r_i)\Big)_{i\in I_\varepsilon}$ defined by (4.13) is

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i) \geq \pi\, |d_i|\, |\log\varepsilon|\, \Big(1 - o(1)\Big). \tag{5.58}$$

The balls are disjoint, then a summation yields

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, K) \geq F_\varepsilon(u_\varepsilon, A_\varepsilon, \cup_{i\in I_\varepsilon} B_i) \geq \pi \sum_{i\in I_\varepsilon} |d_i|\, |\log\varepsilon|\, \Big(1 - o(1)\Big).$$

We use (5.50) to get

$$J_K(u_\varepsilon, A_\varepsilon) \geq \pi \sum_{i\in I_\varepsilon} |d_i|\, |\log\varepsilon|\, \Big(1 - o(1)\Big). \tag{5.59}$$

Inserting $h_{ex} \leq C\, |\log\varepsilon|$ in the fact that $J_K(u_\varepsilon, A_\varepsilon) \leq \frac{1}{2}\, h_{ex}^2$, we have

$$J_K(u_\varepsilon, A_\varepsilon) \leq C\, h_{ex}\, |\log\varepsilon|.$$

We divide (5.59) by $h_{ex}\, |\log\varepsilon|$ and we use the above to deduce from the definition of the measure $\mu_\varepsilon$

$$\frac{1}{2} \int_K |\mu_\varepsilon| - o(1) = \frac{\pi\, \sum_{i\in I_\varepsilon} |d_i|}{h_{ex}} - o(1) \leq \frac{J_K(u_\varepsilon, A_\varepsilon)}{h_{ex}\, |\log\varepsilon|} \leq C. \tag{5.60}$$

Thus, by periodicity

$$\sup_\varepsilon \mu_\varepsilon(O) < \infty \quad \text{for each compact set} \quad O \subset \mathbb{R}^2.$$

Thus, thanks to the Theorem 2 [EG] given in the beginning of the chapter, there exists a Radon measure $\mu_0$ on $\mathbb{R}^2$ such that

$$\mu_{\varepsilon_n} \rightharpoonup \mu_0 \quad \text{as} \;\; n \to +\infty.$$

Now, we pass to the proof of the third assertion giving us the relation between $\mu_0$ and $h_0$.

**Step 3**
Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over $\mathcal{A}$. Recall that

$$N_\varepsilon = \frac{1}{2\,\pi}\int_K h_\varepsilon \in \mathbb{Z}.$$

Thanks to the proposition 2.6 given in chapter 2, there exists $(v_\varepsilon, P_\varepsilon) \in H^1_{loc}(\mathbb{R}^2,\ \mathbb{C})\times H^1_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ such that $(u_\varepsilon, A_\varepsilon)$ is gauge equivalent to $(v_\varepsilon, 2\ \pi\ N_\varepsilon\ \vec{C} + P_\varepsilon)$ where $(v_\varepsilon, P_\varepsilon) \in \mathcal{B}_{N_\varepsilon}$ and

$$\vec{C} = \frac{1}{2}\begin{pmatrix} -y \\ x \end{pmatrix} \text{ in } \mathbb{R}^2.$$

As mentioned in (3.5), we take

$$B_\varepsilon = P_\varepsilon + 2\ \pi\ N_\varepsilon\ \vec{C}.$$

Having $curl\vec{C} = 0$, hence the induced field $h_\varepsilon$ defined by $h_\varepsilon = curl\ A_\varepsilon$ necessarily satisfies

$$h_\varepsilon = curl\ B_\varepsilon.$$

Clearly, $(v_\varepsilon, B_\varepsilon)$ is again a minimizer of the energy $J$ over the space $\mathcal{A}$. In particular, it is a solution of the two Ginzburg-Landau equations defined by (3.4). Recall that the second Ginzburg-Landau equation holds

$$-\nabla^\perp h_\varepsilon = -\nabla^\perp curl B_\varepsilon =< i\ v_\varepsilon, \nabla_{B_\varepsilon} v_\varepsilon > \quad \text{in } \mathbb{R}^2,$$

where by definition $\nabla_{B_\varepsilon} v_\varepsilon = \nabla v_\varepsilon - i\ B_\varepsilon\ v_\varepsilon$. Taking the curl and dividing by $h_{ex}$, we find

$$-\Delta\frac{h_\varepsilon}{h_{ex}} + \frac{h_\varepsilon}{h_{ex}} = \frac{curl\Big((1-|v_\varepsilon|^2)\ B_\varepsilon\Big)}{h_{ex}} + \frac{curl(i\ v_\varepsilon, \nabla\ v_\varepsilon)}{h_{ex}} \quad \text{in } \mathbb{R}^2. \tag{5.61}$$

Now, since $N_\varepsilon \le C\ h_{ex}$ and reasoning as in the proof of proposition 3.1, we get that $B_\varepsilon$ and $v_\varepsilon$ are locally bounded in $H^1$ by $C\ h_{ex}$.
For $q \in \mathbb{N}^*$, we let

$$K^q = [-q, q[\times[-q, q[. \tag{5.62}$$

Let us fix $q$ in $\mathbb{N}^*$. For any $\xi \in W^{1,r}_{0,r>2}(K^q)$, we have

$$\int_{K^q} \xi\ curl\Big((1-|v_\varepsilon|^2)\ B_\varepsilon\Big) = \int_{K^q} B_\varepsilon\ (1-|v_\varepsilon|^2)\ \nabla^\perp\ \xi.$$

Then, using the Cauchy-Schwartz and the a priori bound on $B_\varepsilon$,

$$\begin{aligned}\Big|\int_{K^q} \xi\ curl\Big((1-|v_\varepsilon|^2)\ B_\varepsilon\Big)\Big| &\le C\ \|\nabla\xi\|_{L^r(K^q)}\ \|B_\varepsilon\|_{L^p(K^q)}\ \|1-|v_\varepsilon|^2\|_{L^2(K^q)} \\ &\le C\ h^2_{ex}\ \varepsilon\ \|\nabla\xi\|_{L^2(K^q)},\end{aligned}$$

for some $p < 2$. The right-hand side tends to 0 as $\varepsilon \to 0$, therefore

$$curl\Big((1 - |v_\varepsilon|^2)\ B_\varepsilon\Big) \to 0 \tag{5.63}$$

in $W^{-1,p}(K^q)$. The family of the vortex balls contained in the square $K^q$ is $\{B_{i,m,m'},\ i \in I_\varepsilon, m, m' \in [-q, q-1]\}$, where $B_{i,m,m'}$ is the image of $B_i$ by translation of vector $m\ \vec{i} + m'\ \vec{j}$ with $m, m' \in \mathbb{Z}$. Thanks to (5.52), this family satisfies

$$\overline{\cup_{(i \in I_\varepsilon,\ m,m' \in [-q,q-1])} B_{i,m,m'}} \subset K^q.$$

Referring to (5.51) and (5.53), we get as $\varepsilon \to 0$

$$(4\ q^2)\ |\log \varepsilon| \sum_{i \in I_\varepsilon} r_i = o(1), \tag{5.64}$$

and therefore the sum of the radii of the vortex balls $\{B_{i,m,m'}\}$ tends to zero as $\varepsilon \to 0$.

**Step 4**

Using the above we can deduce thanks to [ASS], lemma 2.2,

$$\Big|\frac{curl(i\ v_\varepsilon, \nabla\ v_\varepsilon)}{h_{ex}} - \mu_\varepsilon\Big|_{W^{-1,p}_{p<2}(K^q)} \to 0, \tag{5.65}$$

where $\mu_\varepsilon$ is the extended measure by periodicity to the square $K^q$ of the measure $\frac{\sum_{i \in I_\varepsilon} 2\ \pi\ d_i\ \delta_{a_i}}{h_{ex}}$.

**Step 5**

Combining (5.63) together with (5.65) in the identity (5.61), we obtain

$$\Big| - \Delta \frac{h_\varepsilon}{h_{ex}} + \frac{h_\varepsilon}{h_{ex}} - \mu_\varepsilon\Big|_{W^{-1,p}_{p<2}(K^q)} \to 0. \tag{5.66}$$

Finally, having (5.66), then using the same procedure as in [SS3], lemma 3.1, one can check

$$\Big| - \Delta \frac{h_\varepsilon}{h_{ex}} + \frac{h_\varepsilon}{h_{ex}} - \mu_0\Big|_{W^{-1,p}_{p<2}(K^q)} \to 0. \tag{5.67}$$

The convergence (5.67) holds independently of $q$ in $\mathbb{N}^*$, then $(-\Delta \frac{h_\varepsilon}{h_{ex}} + \frac{h_\varepsilon}{h_{ex}} - \mu_0)$ converges to 0 locally in $W^{-1,p}_{p<2}$. We know that $\frac{h_{\varepsilon_n}}{h_{ex}} \rightharpoonup h_0$ weakly in $H^1_{loc}(\mathbb{R}^2)$, hence again up to subsequence

$$\frac{h_{\varepsilon_n}}{h_{ex}} \to h_0 \quad \text{strongly in} \quad W^{1,p}_{p<2,loc}(\mathbb{R}^2).$$

Thus, passing to the limit in $(-\Delta \frac{h_\varepsilon}{h_{ex}} + \frac{h_\varepsilon}{h_{ex}})$, $h_0$ satisfies

$$\mu_0 = -\Delta h_0 + h_0 \quad \text{in} \quad \mathbb{R}^2.$$

The properties which we have found on $h_0$ are $h_0 \in H^1_{loc}(\mathbb{R}^2)$ and $h_0$ is periodic such that $(-\Delta h_0 + h_0)$ is a Radon measure on $\mathbb{R}^2$. Then, by definition of the space $V$ defined by (5.3), $h_0 \in V$. Finally, the fact that $h_0 \in H^1_{loc}(\mathbb{R}^2)$ in $\mu_0 = -\Delta h_0 + h_0$ gives us

$$\mu_0 \in H^{-1}.$$

This completes the proof of the proposition 5.12. We also have the following result, proved in [SS3], lemma 3.2.

**Lemma 5.13.** *We have for the* $(h_0, \mu_0)$ *defined by proposition 5.12 that*

$$\liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \geq \frac{\lambda}{2} \int_K |-\Delta h_0 + h_0| + \frac{1}{2} \int_K |\nabla h_0|^2 + \frac{1}{2} \int_K |h_0 - 1|^2 = E(h_0). \tag{5.68}$$

## 3.2 Minimization of $E$ over $V$

Let us minimize the functional $E$ defined by (5.4) over the space $V$. Having that $V$ is convex, closed, not empty and $E$ is strictly convex, hence $\inf_{f\in V} E(f)$ is uniquely achieved. We denote by $h_*$ the minimum. First, let us split $E$

$$\begin{aligned} E(u) =& \frac{\lambda}{2} \int_K |-\Delta u + u| + \frac{1}{2} \int_K |\nabla u|^2 + \frac{1}{2} \int_K |u-1|^2 \\ =& \frac{1}{2} + \frac{1}{2} \|u\|^2_{H^1(K)} + \frac{\lambda}{2} \int_K |-\Delta u + u| - \int_K u. \end{aligned}$$

For $u \in V$, let

$$\Phi(u) = \frac{\lambda}{2} \int_K |-\Delta u + u| - \int_K u,$$

and $\Phi(u) = +\infty$ if $u \notin V$. It follows that

$$\forall\, u \in V, \;\; E(u) = \frac{1}{2} + \frac{1}{2} \|u\|^2_{H^1(K)} + \Phi(u). \tag{5.69}$$

Now, we use the following Lemma (see [BS]).

**Lemma 5.14.** *Let* $\Phi$ *be convex lower semi-continuous from a Hilbert space* $H$ *to* $(-\infty, +\infty]$*, then*

$$\min_{h\in H}\Big(\frac{1}{2}\|h\|_H^2 + \Phi(h)\Big) = -\min_{f\in H}\Big(\frac{1}{2}\|f\|_H^2 + \Phi^*(-f)\Big), \tag{5.70}$$

*and minimizers coincide, where* $\Phi^*$ *is the convex conjugate of* $\Phi$ *defined by*

$$\Phi^*(f) = \sup_{u\in Dom(\Phi)} \Big( < f, u > -\Phi(u)\Big), \tag{5.71}$$

*where* $Dom(\Phi)$ *is the domain of* $\Phi$ *and* $< .,. >$ *is the scalar product in* $H$.

For duality problem, we can refer to [ET]. Let us take $H$ to be

$$H := \Big\{ f \in H^1_{loc}(\mathbb{R}^2, \mathbb{R}) \ \text{ such that } \ f \ \text{ is periodic } \Big\}.$$

Observe that $H$ is Hilbert and the $\Phi$ defined above from $H$ to $(-\infty, +\infty]$ is convex and lower semi- continuous. Let us calculate its conjugate. By the definition (5.71), the conjugate of $G$ for all $f \in H$ is the following

$$\Phi^*(f) = \sup_{u \in Dom(G)} \Big( \int_K \nabla f \, \nabla u + \int_K f \, u - \frac{\lambda}{2} \int_K |-\Delta u + u| + \int_K u \Big).$$

Using $Dom(G) = V$ and the fact that $f$ is periodic, we can write

$$\begin{aligned}
\Phi^*(f) &= \sup_{u \in V} \Big( -\frac{\lambda}{2} \int_K |-\Delta u + u| + \int_K (-\Delta u + u) f + \int_K (-\Delta u + u) \Big) \\
&= \sup_{\{\mu \in H^{-1} \text{ and } \mu \text{ is a Radon measure}\}} \Big( -\frac{\lambda}{2} \int_K |\mu| + \int_K \mu \, (f+1) \Big) \\
&= \sup_{t \in \mathbb{R}_+} \Big[ \sup_{\{\mu \text{ is a Radon measure}, \ \|\mu\| = t\}} \Big( -\frac{\lambda}{2} ||\mu||(K) + \int_K \mu \, (f+1) \Big) \Big].
\end{aligned}$$

On the one hand, for $f \in L^\infty(\mathbb{R}^2, \mathbb{R})$, we have

$$\begin{aligned}
\Phi^*(f) &= \sup_{t \in \mathbb{R}_+} \Big( -\frac{\lambda}{2} \, t + t \, ||f+1||_{L^\infty} \Big) \\
&= \sup_{t \in \mathbb{R}_+} \Big( (||f+1||_{L^\infty} - \frac{\lambda}{2}) \, t \Big) \\
&= \begin{cases} +\infty & \text{if } \ \|f+1\|_{L^\infty(\mathbb{R}^2)} > \frac{\lambda}{2} \\ 0 & \text{if } \ \|f+1\|_{L^\infty(\mathbb{R}^2)} \le \frac{\lambda}{2}. \end{cases}
\end{aligned} \tag{5.72}$$

On the other hand, if we take $f \notin L^\infty(\mathbb{R}^2, \mathbb{R})$, we get $\Phi^*(f) = +\infty$. From lemma 5.14, equation (5.69), we then deduce

$$E(h_*) = \min_{f \in W_0} \Big( \frac{1}{2} \, \|f\|^2_{H^1(K)} \Big) + \frac{1}{2}, \tag{5.73}$$

where

$$W_0 = \{ f \in H^1_{loc}(\mathbb{R}^2) \ \text{ such that } \ f \ \text{ is periodic and } \ \|f - 1\|_{L^\infty(\mathbb{R}^2)} \le \frac{\lambda}{2} \}.$$

Moreover $h_*$ is the unique minimizer of this problem. It immediately follows that

**Lemma 5.15.** *We have*

$$h_* = \max(1 - \frac{\lambda}{2}, 0). \tag{5.74}$$

Combining all the above shows us that the proposition 5.2 is proved.

# 4 Completing the proof of Theorem 5.4

We can now complete the convergence results. From lemmas 5.13 and 5.15, we deduce

$$\liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \geq E(h_0) \geq E(h_*). \tag{5.75}$$

We distinguish the two following cases:
**Case 1: If** $0 < \lambda < 2$. In this case, from lemma 5.15, we have $h_* = 1 - \frac{\lambda}{2}$. Therefore,

$$E(h_*) = \frac{\lambda}{2}\,(1 - \frac{\lambda}{4}).$$

We insert this in (5.75) to get

$$\liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \geq E(h_0) \geq \frac{\lambda}{2}\,(1 - \frac{\lambda}{4}).$$

On the other hand, proposition (5.8) gives us

$$\limsup_{n\to+\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \leq \frac{\lambda}{2}\,(1 - \frac{\lambda}{4}).$$

We compare the two above inequalities to get

$$\lim_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} = E(h_0) = \frac{\lambda}{2}\,(1 - \frac{\lambda}{4}) = E(h_*).$$

Now, thanks to the fact that $h_* = 1 - \frac{\lambda}{2}$ is the unique minimizer of $E$ over $V$, we conclude

$$h_0 = h_* = 1 - \frac{\lambda}{2} \quad \text{in } \mathbb{R}^2.$$

Hence, $\frac{h_{\varepsilon_n}}{h_{ex}} \to h_0 = h_* = 1 - \frac{\lambda}{2}$ weakly in $H^1_{loc}(\mathbb{R}^2)$. In addition, knowing that $\mu_0 = -\Delta h_0 + h_0$, hence in view of lemma 5.12

$$\mu_{\varepsilon_n} \rightharpoonup \mu_0 = (1 - \frac{\lambda}{2})\,dx, \tag{5.76}$$

where $dx$ is the Lebesgue measure on $\mathbb{R}^2$. This explains the uniform-vortex distribution.

**Case 2: If** $\lambda \geq 2$. In this case, from lemma 5.15, we have $h_* = 0$, thus $E(h_*) = \frac{1}{2}$. We insert this in (5.75) to have

$$\liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \geq E(h_0) \geq \frac{1}{2}.$$

Second, since $J(u_\varepsilon, A_\varepsilon) \leq \frac{1}{2}\,h_{ex}^2$, we find

$$\limsup_{n\to+\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \leq \frac{1}{2}.$$

Comparing the two above inequalities, we get

$$\lim_{n\to+\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} = E(h_0) = \frac{1}{2} = E(h_*).$$

Since $h_* = 0$ is the unique minimizer of $E$ over $V$, we conclude that $h_0 = h_* = 0$. It follows from lemma 5.12 that $\frac{h_{\varepsilon_n}}{h_{ex}} \to 0$ weakly in $H^1_{loc}(\mathbb{R}^2)$. Consequently, thanks to lemma 5.12,

$$\mu_{\varepsilon_n} \rightharpoonup \mu_0 = 0. \tag{5.77}$$

Now, the limits in the above two cases are independent of the chosen subsequence, therefore the whole sequence converges. This completes the proof of Theorem 5.4.

**Remark 5.16.** *In the case $\lambda \geq 2$, the convergence (5.77) does not give us an idea on the number of the vortices and their repartition. Indeed, taking $\lambda \geq 2$ in the corollary 5.5, we can deduce*

$$N_\varepsilon = \sum_{i\in I_\varepsilon} d_i \ll h_{ex},$$

*so we just find that the number of vortices is negligible to $h_{ex}$. We start with the study of the case $\lambda > 2$ in the next paragraph.*

# 5 The case $\lambda > 2$

From the definition of the parameter $\lambda$, we can write in the limit $\varepsilon \to 0$,

$$h_{ex} = \frac{1}{\lambda}\, |\log \varepsilon|. \tag{5.78}$$

Here, assume that $\lambda > 2$. Splitting the energy $J_K$ of a minimizer $(u_\varepsilon, A_\varepsilon)$ between the contribution inside the vortex-balls and the contribution outside, we get using (5.59)

$$\pi \sum_{i\in I_\varepsilon} |d_i|\, |\log \varepsilon|\, (1 - o(1)) + \frac{1}{2} \int_K |h_\varepsilon - h_{ex}|^2 \leq J(u_\varepsilon, A_\varepsilon). \tag{5.79}$$

But, since $\int_K |h_\varepsilon - h_{ex}|^2 = \int_K |h_\varepsilon|^2 + h_{ex}^2 - 4\,\pi\, N_\varepsilon\, h_{ex}$ and $J(u_\varepsilon, A_\varepsilon) \leq \frac{1}{2}\, h_{ex}^2$, we find

$$\pi \sum_{i\in I_\varepsilon} |d_i|\, |\log \varepsilon|\, (1 - o(1)) - 2\,\pi\, N_\varepsilon\, h_{ex} \leq 0. \tag{5.80}$$

Thanks to (5.78) and $N_\varepsilon \leq \sum_{i\in I_\varepsilon} |d_i|$,

$$\pi \sum_{i\in I_\varepsilon} |d_i| \left((1 - \frac{2}{\lambda})\, |\log \varepsilon| \, - o(1)\, |\log \varepsilon|\right) \leq 0. \tag{5.81}$$

The fact that $\lambda > 2$ yields $(1 - \frac{2}{\lambda}) > 0$, and therefore there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$

$$\sum_{i \in I_\varepsilon} |d_i| = 0.$$

Thus, the minimizers of the energy $J$ have no vortices when $\lambda > 2$. This proves the proposition 5.6. The above gives a meaning to the value of the first critical field $H_{c_1}$ which behaves for $\varepsilon \to 0$ as $\frac{|\log \varepsilon|}{2}$.

**Remark 5.17.** *Thanks to Theorem 5.4 and proposition 5.6, the study of the vortex structure of minimizers of $J$ over $\mathcal{A}$ is well known for applied field $h_{ex}$ which are such that $\lambda > 0$ with $\lambda \neq 2$. The case $\lambda = 2$ will be treated separately in the next chapter (chapter 6).*

# Chapter 6

# The case of applied fields close to $H_{c_1}$

In this chapter, we will be concerned with the case of applied fields $h_{ex}$ which are close to the first critical field $H_{c_1}$. More precisely, we assume

$$h_{ex} = H_{c_1} + f(\varepsilon), \tag{6.1}$$

where $H_{c_1}$ behaves for $\varepsilon \to 0$ as $\frac{|\log \varepsilon|}{2}$ and $f(\varepsilon)$ is any sequence tending to $+\infty$ such that

$$f(\varepsilon) = o(|\log \varepsilon|).$$

We will study, in the limit $\varepsilon \to 0$, the vortex-structure of global minimizers $(u_\varepsilon, A_\varepsilon)$ of the Ginzburg-Landau energy $J$ over the space $\mathcal{A}$, in a more precise way than in the preceding chapter.

## 1 Statement of the result

We take $K$ to be any square of sidelengeth 1. The first critical field $H_{c_1}$ behaves for $\varepsilon \to 0$ as $\frac{|\log \varepsilon|}{2}$, i.e. $H_{c_1} \approx \frac{|\log \varepsilon|}{2}$. Since, the parameter $\varepsilon$ is taken usually to tend to 0, we will write from now on $H_{c_1} = \frac{|\log \varepsilon|}{2}$ instead of $H_{c_1} \approx \frac{|\log \varepsilon|}{2}$. We consider applied fields defined by $h_{ex} = H_{c_1} + f(\varepsilon)$ where $f(\varepsilon) \to +\infty$ as $\varepsilon \to 0$ and $f(\varepsilon) = o(|\log \varepsilon|)$, and then $\lambda = \lim_{\varepsilon \to 0} \frac{|\log \varepsilon|}{h_{ex}} = 2$. Thus, letting $\lambda = 2$ in (5.7) gives us

$$\lim_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} = \frac{1}{2},$$

where $(u_\varepsilon, A_\varepsilon)$ is a minimizer of the energy $J$ over the space $\mathcal{A}$. The above limit implies that $J(u_\varepsilon, A_\varepsilon)$ is equivalent to $J^0$ as $\varepsilon \to 0$, where $J^0 = \frac{h_{ex}^2}{2}$. From now on, we will be concerned with the estimate of the energy

$$\frac{J(u_\varepsilon, A_\varepsilon) - J^0}{(f(\varepsilon))^2}$$

as $\varepsilon \to 0$ and we show that it is the appropriately normalized quantity to consider. From the chapter 5, we know that for $(u_\varepsilon, A_\varepsilon)$ a minimizer of $J$ over $\mathcal{A}$, the hypotheses of the proposition 4.1 hold, and applying it yields vortices $(a_i, d_i)_{i \in I_\varepsilon}$. The main result we prove in this chapter is

**Theorem 6.1.** *We take applied magnetic fields defined by $h_{ex} = H_{c_1} + f(\varepsilon)$ such that $f(\varepsilon)$ tends to $+\infty$ and $f(\varepsilon) = o(|\log \varepsilon|)$ when $\varepsilon \to 0$. Consider, for every $\varepsilon$, $(u_\varepsilon, A_\varepsilon)$ minimizing the energy $J$ over $\mathcal{A}$, and $h_\varepsilon = curl A_\varepsilon$. Then, as $\varepsilon \to 0$,*

$$\frac{h_\varepsilon}{f(\varepsilon)} \to 1 \quad \text{strongly in } W^{1,p}_{loc,p<2}(\mathbb{R}^2). \tag{6.2}$$

*In addition,*

$$\lim_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon) - J^0}{f(\varepsilon)^2} = -\frac{1}{2}. \tag{6.3}$$

*Finally, letting $\nu_\varepsilon$ be the extended measure by periodicity to $\mathbb{R}^2$ of $\frac{2\,\pi\,\sum_{i\in I_\varepsilon} d_i\,\delta_{a_i}}{f(\varepsilon)}$, we have as $\varepsilon \to 0$,*

$$\nu_\varepsilon \rightharpoonup dx, \tag{6.4}$$

*where $dx$ is the Lebesgue measure on $\mathbb{R}^2$.*

**Remark 6.2.** *The Theorem 6.1 gives us an idea on the repartition of the vortices and their number par period. Indeed, we obtain from (6.4) that the minimizers of $J$ over the space $\mathcal{A}$ have a uniform scattering of vortices. Moreover, the number of the vortices per period is close to $f(\varepsilon)$. This contrasts the result of [SS1]. Indeed, Sandier and Serfaty have found for minimizers of the energy $J_\Omega$ over $H^1(\Omega,\mathbb{C}) \times H^1(\Omega,\mathbb{R}^2)$ that we need to have an increment of $|\log|\log\varepsilon||$ to add a vortex (where $\Omega$ is the section of the domain occupied by the superconductor). Note that the difference between our results and those of [SS1] is due to the fact that the periodic model removes the boundary effects.*

In order to prove the Theorem 6.1, we give first an upper bound on the energy, through the proposition 6.3, and then a lower bound in proposition 6.12. Note that the upper bound will be obtained by construction of a test configuration in the space $\mathcal{A}$, while the lower bound of the energy will follow essentially from a combination of the suitable lower bound of the energy $J$ on the vortex balls that we recalled in proposition 4.1 and the property

$$N_\varepsilon = \sum_{i\in I_\varepsilon} d_i.$$

# 2 Upper bound of the energy

The main result we prove here is

**Proposition 6.3.** *Set $h_{ex} = \frac{|\log\varepsilon|}{2} + f(\varepsilon)$ with $f(\varepsilon)$ tends to $+\infty$ and $f(\varepsilon) = o(|\log\varepsilon|)$. Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J$ over $\mathcal{A}$, then*

$$\limsup_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon) - J^0}{f(\varepsilon)^2} \le -\frac{1}{2}. \tag{6.5}$$

We split the proof into four steps:

**Step 1**

Let $h_{ex} = \frac{|\log \varepsilon|}{2} + f(\varepsilon)$ where $f(\varepsilon) = o(|\log \varepsilon|)$ and tends to $+\infty$. Arguing as in the proof of proposition 5.8, step 1, we construct points $(a_i^\varepsilon)_i$, $1 \leq i \leq p(\varepsilon)^2$ in $K$ and equally spaced, with

$$n(\varepsilon) = p(\varepsilon)^2, \quad p(\varepsilon) = \left[\sqrt{\frac{f(\varepsilon)}{2\,\pi}}\right].$$

Then, $2\,\pi\; n(\varepsilon) \approx f(\varepsilon)$ as $\varepsilon \to 0$ and $2\,\pi\; n(\varepsilon) \leq f(\varepsilon)$. Letting $\mu_\varepsilon$ be the extended measure by periodicity to $\mathbb{R}^2$ of $\frac{\sum_{1\leq i\leq n(\varepsilon)} \mu_\varepsilon^i}{2\,\pi\; n(\varepsilon)}$, where $\mu_\varepsilon^i$ is the uniform measure on $\partial(B_i(a_i^\varepsilon, \varepsilon))$ of mass $2\,\pi$, we also have

$$\mu_\varepsilon \rightharpoonup dx \quad \text{as} \ \ \varepsilon \to 0, \tag{6.6}$$

where $dx$ is the Lebesgue measure on $\mathbb{R}^2$.

**Step 2**

We define $G$ to be the Green function solution of $-\Delta_x G(x,y) + G(x,y) = \delta_y$ in $\mathbb{R}^2$. We prove

**Lemma 6.4.**

$$\limsup_{\varepsilon\to 0}\Big(\frac{1}{2}\int_{K\times\mathbb{R}^2} G(x,y)\; d\mu_\varepsilon(y)\; d\mu_\varepsilon(x) - \frac{|\log \varepsilon|}{4\,\pi\; n(\varepsilon)}\Big) \leq \frac{1}{2}. \tag{6.7}$$

**Proof**: Let $\beta > 0$ and take

$$\Delta_\beta = \{(x,y) \in K \times K, \quad |x-y| < \beta\}.$$

From (6.6), recall that $\mu_\varepsilon \rightharpoonup dx$. Hence, it follows that

$$\mu_\varepsilon \otimes \mu_\varepsilon \rightharpoonup dx \otimes dx \quad \text{as} \quad \varepsilon \to 0.$$

In view of the continuity of $G$ in $(K\times K)\backslash\Delta_\beta$, we are led to

$$\lim_{\varepsilon\to 0} \frac{1}{2} \int\int_{(K\times K)\backslash\Delta_\beta} G\; d\mu_\varepsilon\; d\mu_\varepsilon = \frac{1}{2}\int_{(K\times K)\backslash\Delta_\beta} G\; dx\; dy. \tag{6.8}$$

Now, we treat the integral on $\Delta_\beta$. From the definition of $\mu_\varepsilon$,

$$\iint_{\Delta_\beta} G(x,y)\; d\mu_\varepsilon(x)\; d\mu_\varepsilon(y) \leq \frac{1}{f(\varepsilon)^2}\Big(\sum_{\substack{1\leq i\neq j\leq n(\varepsilon)\\ |a_i^\varepsilon - a_j^\varepsilon| < 2\beta}} \iint G\; d\mu_\varepsilon^j\; d\mu_\varepsilon^i + \sum_{i=1}^{n(\varepsilon)} \iint G\; d\mu_\varepsilon^i\; d\mu_\varepsilon^i\Big). \tag{6.9}$$

The analogous of (5.27) is

$$\frac{1}{2}\sum_{i=1}^{n(\varepsilon)} \int\int G\; d\mu_\varepsilon^i\; d\mu_\varepsilon^i \leq n(\varepsilon)(\pi\;|\log \varepsilon| + C). \tag{6.10}$$

Moreover,

$$\frac{1}{2} \sum_{\substack{1\le i\ne j\le n(\varepsilon)\\ |a_i^\varepsilon - a_j^\varepsilon|<2\beta}} \iint G\, d\mu_\varepsilon^j\, d\mu_\varepsilon^i \le C\, n(\varepsilon)^2\, |\Delta_{2\,\beta}|. \tag{6.11}$$

A combination of (6.10) together with (6.11) in (6.9) leads to

$$\frac{4\,\pi^2\, n(\varepsilon)^2}{2} \iint_{\Delta_\beta} G(x,y)\, d\mu_\varepsilon(x)\, d\mu_\varepsilon(y) \le n(\varepsilon)\, (\pi\, |\log\varepsilon| + C) + C\, n(\varepsilon)^2\, |\Delta_{2\beta}|.$$

Then

$$\frac{1}{2} \int_{K\times K} G\, d\mu_\varepsilon\, d\mu_\varepsilon \le \frac{|\log\varepsilon|}{4\,\pi\, n(\varepsilon)} + \int_{(K\times K)\setminus\Delta_\beta} G\, d\mu_\varepsilon\, d\mu_\varepsilon + o_\varepsilon(1) + C|\Delta_{2\beta}|,$$

where $o_\varepsilon(1)$ denotes a function of $\varepsilon$ which goes to zero as $\varepsilon \to 0$. Passing to the limit as $\varepsilon \to 0$ and using (6.8) imply

$$\limsup_{\varepsilon\to 0}\Big(\frac{1}{2} \int_{K\times K} G\, d\mu_\varepsilon\, d\mu_\varepsilon - \frac{|\log\varepsilon|}{4\,\pi\, n(\varepsilon)}\Big) \le \frac{1}{2} \int_{(K\times K)\setminus\Delta_\beta} G\, dx\, dy. \tag{6.12}$$

Letting ($\beta \to 0$) in (6.12) yields

$$\limsup_{\varepsilon\to 0}\Big(\frac{1}{2} \int_{K\times K} G\, d\mu_\varepsilon\, d\mu_\varepsilon - \frac{|\log\varepsilon|}{4\,\pi\, n(\varepsilon)}\Big) \le \frac{1}{2} \int_{K\times K} G\, dx\, dy. \tag{6.13}$$

We go back to (5.26) to write

$$\lim_{\varepsilon\to 0} \int_{K\times(\mathbb{R}^2\setminus K)} G\, d\mu_\varepsilon\, d\mu_\varepsilon = \int_{K\times(\mathbb{R}^2\setminus K)} G\, dx\, dy.$$

Combining this together with (6.13) gives us

$$\begin{aligned}\limsup_{\varepsilon\to 0}\Big(\frac{1}{2} \int_{K\times \mathbb{R}^2} G\, d\mu_\varepsilon\, d\mu_\varepsilon - \frac{|\log\varepsilon|}{4\,\pi\, n(\varepsilon)}\Big) &\le \frac{1}{2} \int_{K\times K} G\, dx\, dy + \frac{1}{2} \int_{K\times(\mathbb{R}^2\setminus K)} G\, dx\, dy \\ &= \frac{1}{2} \int_{K\times \mathbb{R}^2} G\, dx\, dy = \frac{1}{2}.\end{aligned}$$

This completes the proof of (6.7). □

**Step 3**

The proof of (6.5) needs a construction of an appropriate test configuration $(v_\varepsilon, B_\varepsilon)$ in $\mathcal{A}$. First, we define $h_\varepsilon$ by

$$h_\varepsilon(x) = 2\,\pi\, n(\varepsilon) \int_{\mathbb{R}^2} G(x,y)\, d\mu_\varepsilon(y).$$

Then, $h_\varepsilon$ is periodic, continuous and in $H^1_{loc}(\mathbb{R}^2)$. It satisfies

$$-\Delta h_\varepsilon + h_\varepsilon = 2\,\pi\, n(\varepsilon)\, \mu_\varepsilon \quad \text{in} \quad \mathbb{R}^2. \tag{6.14}$$

We let $B_\varepsilon$ be a solution of *curl* $B_\varepsilon = h_\varepsilon$. Then, we define $v_\varepsilon$ as in the proof of proposition 5.8 in a way such that $(v_\varepsilon, B_\varepsilon) \in \mathcal{A}$ and

$$J_K(v_\varepsilon, B_\varepsilon) \leq \frac{1}{2} \int_K |\nabla h_\varepsilon|^2 + \frac{1}{2} \int_K |h_\varepsilon - h_{ex}|^2 + C\ n(\varepsilon). \tag{6.15}$$

**Step 4: Proof of proposition 6.3, completed**
From (6.14),

$$-\Delta h_\varepsilon + h_\varepsilon - h_{ex} = \Big(2\ \pi\ n(\varepsilon)\ \mu_\varepsilon - h_{ex}\Big) \quad \text{in} \quad \mathbb{R}^2. \tag{6.16}$$

Now, we multiply (6.16) by $(h_\varepsilon - h_{ex})$, we integrate on $K$, and we use the periodicity of $h_\varepsilon$ to obtain

$$\int_K |\nabla h_\varepsilon|^2 + \int_K |h_\varepsilon - h_{ex}|^2 = \int_K (h_\varepsilon - h_{ex})(x)\ \Big(2\ \pi\ n(\varepsilon)\ \mu_\varepsilon - h_{ex}\Big)(x). \tag{6.17}$$

It follows that

$$\int_K |\nabla h_\varepsilon|^2 + \int_K |h_\varepsilon - h_{ex}|^2 = \int_K (2\ \pi\ n(\varepsilon))^2\ \Big(\int_{\mathbb{R}^2} G(x,y)\ d\mu_\varepsilon(y)\Big)\ d\mu_\varepsilon(x) - 4\ \pi\ n(\varepsilon)\ h_{ex} \int_K \mu_\varepsilon + h_{ex}^2. \tag{6.18}$$

Thus

$$\int_K |\nabla h_\varepsilon|^2 + \int_K |h_\varepsilon - h_{ex}|^2 - h_{ex}^2 = (2\ \pi\ n(\varepsilon))^2 \int_{K\times\mathbb{R}^2} G(x,y)\ d\mu_\varepsilon(y)\ d\mu_\varepsilon(x) - 4\ \pi\ n(\varepsilon)\ h_{ex}.$$

Now, we divide by $(2\ \pi\ n(\varepsilon))^2$ to get

$$\frac{\int_K |\nabla h_\varepsilon|^2 + \int_K |h_\varepsilon - h_{ex}|^2 - h_{ex}^2}{(2\ \pi\ n(\varepsilon))^2} = \int_{K\times\mathbb{R}^2} G(x,y)\ d\mu_\varepsilon(y)\ d\mu_\varepsilon(y) - \frac{2\ h_{ex}}{2\ \pi\ n(\varepsilon)}.$$

Then, replacing the applied field $h_{ex}$ with $\Big(\frac{|\log \varepsilon|}{2} + f(\varepsilon)\Big)$ and recalling that $2\ \pi\ n(\varepsilon) \leq f(\varepsilon)$, we find

$$\frac{\int_K |\nabla h_\varepsilon|^2 + \int_K |h_\varepsilon - h_{ex}|^2 - h_{ex}^2}{(f(\varepsilon))^2} \leq \int_{K\times\mathbb{R}^2} G(x,y)\ d\mu_\varepsilon(y)\ d\mu_\varepsilon(x) - \frac{|\log \varepsilon| + 2\ f(\varepsilon)}{2\ \pi\ n(\varepsilon)}. \tag{6.19}$$

We refer to (6.7) and the fact that $2\ \pi\ n(\varepsilon) \approx f(\varepsilon)$ as $\varepsilon \to 0$ to deduce

$$\limsup_{\varepsilon \to 0} \frac{1}{2\ (f(\varepsilon))^2}\ \Big(\int_K |\nabla h_\varepsilon|^2 + \int_K |h_\varepsilon - h_{ex}|^2 - \ h_{ex}^2\Big) \leq -\frac{1}{2}. \tag{6.20}$$

Going back to (6.15), we obtain from (6.20)

$$\limsup_{\varepsilon \to 0} \frac{J(v_\varepsilon, B_\varepsilon) - J^0}{f(\varepsilon)^2} \leq -\frac{1}{2}.$$

This inequality is true for the test configuration $(v_\varepsilon, B_\varepsilon)$, so it is true in particular for any minimizer of $J$ over the space $\mathcal{A}$. This completes the proof of (6.5). The proposition 6.3 is then proved.

## 3 Lower bound

We start with

### 3.1 Preliminary estimates

We consider $(u_\varepsilon, A_\varepsilon)$ a family of minimizers of the energy $J$ over $\mathcal{A}$ and $h_\varepsilon = curl\ A_\varepsilon$ the associated induced magnetic field. Let $K$ be any square of sidelengeth 1.
First, in the following lemma, we give for an $\varepsilon$ small enough a preliminary idea on the estimate of the order of the quantities $\gamma_\varepsilon = \|h_\varepsilon\|_{L^2(K)}$, $N_\varepsilon = \frac{1}{2\,\pi}\ \int_K h_\varepsilon$ and $F_\varepsilon(u_\varepsilon, A_\varepsilon, K)$.

**Lemma 6.5.** *Set $h_{ex} = \frac{|\log\varepsilon|}{2} + f(\varepsilon)$ with $f(\varepsilon) = o(|\log\varepsilon|)$ and tends to $+\infty$ when $\varepsilon \to 0$. Then, for $\varepsilon$ sufficiently small*

$$N_\varepsilon = o\,(|\log\varepsilon|), \tag{6.21}$$

$$\gamma_\varepsilon \le C\,\sqrt{N_\varepsilon\,|\log\varepsilon|}, \tag{6.22}$$

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, K) \le C\ N_\varepsilon\ |\log\varepsilon|. \tag{6.23}$$

**Proof**: First, $h_{ex} = \frac{|\log\varepsilon|}{2} + f(\varepsilon)$, hence letting $\lambda = \lim_{\varepsilon\to 0}\frac{|\log\varepsilon|}{h_{ex}} = 2$ in the convergence (5.10),

$$\lim_{\varepsilon\to 0}\frac{N_\varepsilon}{h_{ex}} = \max\Big(0, \frac{1}{2\,\pi}(1-\frac{\lambda}{2})\Big) = 0.$$

It means that $N_\varepsilon = o\,(h_{ex})$, so in particular $N_\varepsilon = o\,(|\log\varepsilon|)$ as $\varepsilon \to 0$. Second, by definition of the functional $F_\varepsilon$,

$$J(u_\varepsilon, A_\varepsilon) = F_\varepsilon(u_\varepsilon, A_\varepsilon, K) + \frac{1}{2}\ \int_K\ |h_\varepsilon - h_{ex}|^2.$$

We split $\frac{1}{2}\ \int_K |h_\varepsilon - h_{ex}|^2$ to get

$$\begin{aligned}\frac{1}{2}\ \int_K |h_\varepsilon - h_{ex}|^2 =& \frac{1}{2}\ \int_K |h_\varepsilon|^2 + \frac{1}{2}\ h_{ex}^2 - h_{ex}\ \int_K h_\varepsilon\\ =&\frac{1}{2}\ \int_K |h_\varepsilon|^2 + \frac{1}{2}\ h_{ex}^2 - 2\,\pi\ N_\varepsilon\ h_{ex}.\end{aligned}$$

Replacing $\int_K |h_\varepsilon|^2$ with $\gamma_\varepsilon^2$ leads to

$$J(u_\varepsilon, A_\varepsilon) = F_\varepsilon(u_\varepsilon, A_\varepsilon, K) + \frac{1}{2}\ \gamma_\varepsilon^2 + \frac{1}{2}\ h_{ex}^2 - 2\,\pi\ N_\varepsilon\ h_{ex}. \tag{6.24}$$

Since $(u_\varepsilon, A_\varepsilon)$ is a minimizer, in particular

$$J(u_\varepsilon, A_\varepsilon) \le J(1,0) = J^0 = \frac{1}{2}\ h_{ex}^2.$$

Using this in (6.24),

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, K) + \frac{1}{2}\,\gamma_\varepsilon^2 \le 2\,\pi\,N_\varepsilon\,h_{ex} \le C\,N_\varepsilon\,|\log\varepsilon|. \tag{6.25}$$

For any $\varepsilon > 0$, we obtain

$$\gamma_\varepsilon \le C\,\sqrt{N_\varepsilon\,|\log\varepsilon|},$$

and

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, K) \le 2\,\pi\,N_\varepsilon\,h_{ex} \le C\,N_\varepsilon\,|\log\varepsilon|.$$

□

## 3.2 Vortices have mostly positive degrees

**Lemma 6.6.** *Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer and $h_\varepsilon = curl A_\varepsilon$. For a sufficiently small $\varepsilon$,*

$$N_\varepsilon \ne 0. \tag{6.26}$$

**Proof**: We argue by contradiction. Assume that $N_\varepsilon = 0$. On the one hand, from (6.22), it is immediate that $\gamma_\varepsilon = 0$, consequently $h_\varepsilon = 0$. Letting $N_\varepsilon = 0$ in (6.23), we get

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, K) = 0.$$

By definition of $F_\varepsilon$, this implies that $|u_\varepsilon| = 1$. This means that the material is in its superconducting phase, so the energy of $(u_\varepsilon, A_\varepsilon)$ is

$$J(u_\varepsilon, A_\varepsilon) = J^0 = \frac{1}{2}\,h_{ex}^2.$$

In particular,

$$\lim_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon) - J^0}{f(\varepsilon)^2} = 0. \tag{6.27}$$

On the other hand, proposition 6.3 gives us

$$\limsup_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon) - J^0}{f(\varepsilon)^2} \le -\frac{1}{2}. \tag{6.28}$$

Comparing (6.27) to (6.28), we get a contradiction. This means that $N_\varepsilon \ne 0$, and thanks again to (6.23), $N_\varepsilon$ is necessarily positive.

□

### 3.3 The vortex balls

Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over $\mathcal{A}$. Now, let

$$\alpha_\varepsilon = \max(N_\varepsilon, f(\varepsilon)). \tag{6.29}$$

Having $h_{ex} = \frac{|\log \varepsilon|}{2} + f(\varepsilon)$ with $1 \ll f(\varepsilon) = o(|\log \varepsilon|)$, hence in particular from (6.29) and lemma 6.5, $1 \ll \alpha_\varepsilon \ll h_{ex}$. Using the fact that $N_\varepsilon \leq \alpha_\varepsilon$ in (6.23), we get

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, K) \leq C\ \alpha_\varepsilon\ |\log \varepsilon|.$$

Thanks to (6.21) and (6.22), we have $\gamma_\varepsilon \leq C\ |\log \varepsilon|$. Moreover, let $m_\varepsilon$ be any sequence verifying

$$\frac{\log \alpha_\varepsilon}{\alpha_\varepsilon} \ll m_\varepsilon = o(1). \tag{6.30}$$

Of course, $m_\varepsilon$ is positive. A combination of all the above yields that the hypotheses of proposition 4.1 hold, so there exists $\varepsilon_0$ such that if $\varepsilon < \varepsilon_0$, there exist a square of sidelength 1, (without loss of generality the square $K = [0,1[\times[0,1[$), and a family of disjoint balls still denoted $\Big(B_i = B_i(a_i, r_i)\Big)_{i \in I_\varepsilon}$ verifying

$$\overline{\cup_{i\in I_\varepsilon} B_i} \subset K,$$

such that

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i) \geq \pi\ |d_i|\ \Big(|\log \varepsilon| - \log \gamma_\varepsilon - m_\varepsilon\ \alpha_\varepsilon\Big), \tag{6.31}$$

where $d_i$ is the degree of $\frac{u_\varepsilon}{|u_\varepsilon|}$ on $\partial(B_i)$. Let us define for $\varepsilon < \varepsilon_0$,

$$\mathbf{D}_\varepsilon = \sum_{i \in I_\varepsilon} |d_i|. \tag{6.32}$$

Recall that, (4.15) holds

$$N_\varepsilon = \sum_{i \in I_\varepsilon} d_i.$$

We compare this to (6.32) to get

$$N_\varepsilon \leq \mathbf{D}_\varepsilon \quad \forall\ \varepsilon < \varepsilon_0. \tag{6.33}$$

**Lemma 6.7.** *Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over $\mathcal{A}$ and $h_\varepsilon = curl A_\varepsilon$ be the induced magnetic field. Then, for $\alpha_\varepsilon$ and $m_\varepsilon$ defined respectively by (6.29) and (6.30), there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$*

$$\pi\ \mathbf{D}_\varepsilon\ \Big(|\log \varepsilon| - \log \gamma_\varepsilon - m_\varepsilon\ \alpha_\varepsilon\Big) + \frac{1}{2}\int_{K \setminus \bigcup_{i\in I_\varepsilon} B_i} |\nabla h_\varepsilon|^2 + \frac{1}{2}\int_K |h_\varepsilon|^2 \leq 2\ \pi\ N_\varepsilon\ h_{ex}. \tag{6.34}$$

**Proof**: We split the energy $J$ between the contribution inside the vortex-balls and the contribution outside as follows

$$J(u_\varepsilon, A_\varepsilon) \geq \int_{\cup_{i\in I_\varepsilon} B_i} |\nabla u_\varepsilon - i\ A_\varepsilon\ u_\varepsilon|^2 + \int_{K\setminus \cup_{i\in I_\varepsilon} B_i} |\nabla u_\varepsilon - i\ A_\varepsilon\ u_\varepsilon|^2 + \frac{1}{4\ \varepsilon^2}\int_{\cup_{i\in I_\varepsilon} B_i} (1-|u_\varepsilon|^2)^2$$
$$+\frac{1}{2}\int_K |h_\varepsilon - h_{ex}|^2.$$

From the expression of $F_\varepsilon$,

$$J(u_\varepsilon, A_\varepsilon) \geq F_\varepsilon(u, A, \cup_{i\in I_\varepsilon} B_i) + \int_{K\setminus \cup_{i\in I_\varepsilon} B_i} |\nabla u_\varepsilon - i\ A_\varepsilon\ u_\varepsilon|^2 + \frac{1}{2}\int_K |h_\varepsilon - h_{ex}|^2.$$

Using $|\nabla h_\varepsilon|^2 \leq |\nabla u_\varepsilon - i\ A_\varepsilon\ u_\varepsilon|^2$, we get

$$J(u_\varepsilon, A_\varepsilon) \geq F_\varepsilon(u_\varepsilon, A_\varepsilon, \cup_{i\in I_\varepsilon} B_i) + \int_{K\setminus \cup_{i\in I_\varepsilon} B_i} |\nabla h_\varepsilon|^2 + \frac{1}{2}\int_K |h_\varepsilon - h_{ex}|^2. \quad (6.35)$$

We know

$$\frac{1}{2}\int_K |h_\varepsilon - h_{ex}|^2 = \frac{1}{2}\int_K |h_\varepsilon|^2 + \frac{1}{2}\ h_{ex}^2 - 2\ \pi\ N_\varepsilon\ h_{ex}.$$

The lower bound of the energy on the balls $\Big(B_i(a_i, r_i)\Big)_{i\in I_\varepsilon}$ defined by the (6.31) is

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i) \geq \pi\ |d_i|\ \Big(|\log\varepsilon| - m_\varepsilon\ \alpha_\varepsilon - \log\gamma_\varepsilon\Big).$$

Now, thanks to the fact that the balls $(B_i)_{i\in I_\varepsilon}$ are disjoint, then using the above inequality and the identity $\mathbf{D}_\varepsilon = \sum_{i\in I_\varepsilon} |d_i|$, it is clear

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, \cup_{i\in I_\varepsilon} B_i) = \sum_{i\in I_\varepsilon} F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i)$$
$$\geq \pi\ \mathbf{D}_\varepsilon\ \Big(|\log\varepsilon| - m_\varepsilon\ \alpha_\varepsilon - \log\gamma_\varepsilon\Big).$$

Combining all the above in (6.35), we find

$$\pi\ \mathbf{D}_\varepsilon\ \Big(|\log\varepsilon| - \log\gamma_\varepsilon - m_\varepsilon\ \alpha_\varepsilon\Big) - 2\ \pi\ N_\varepsilon\ h_{ex} + \frac{1}{2}\int_{K\setminus \bigcup_{i\in I_\varepsilon} B_i} |\nabla h_\varepsilon|^2 + \frac{1}{2}\int_K |h_\varepsilon|^2 + \frac{1}{2}\ h_{ex}^2 \leq J(u_\varepsilon, A_\varepsilon). \quad (6.36)$$

Using again the inequality $J(u_\varepsilon, A_\varepsilon) \leq J^0 = \frac{1}{2}\ h_{ex}^2$ in (6.36) gives us (6.34). The lemma is then proved. □

From (6.33), we know that $N_\varepsilon \leq \mathbf{D}_\varepsilon$, hence thanks to lemma 6.6, we have for $\varepsilon$ small enough ($\varepsilon < \varepsilon_0$)

$$\mathbf{D}_\varepsilon \neq 0. \tag{6.37}$$

Then, (6.37) explains the presence of vortices in the superconductor. The rest is devoted to find the exact order of $N_\varepsilon \geq 1$. We start with

**Corollary 6.8.** *For an applied field $h_{ex} = H_{c_1} + f(\varepsilon)$ where $f(\varepsilon) = o(|\log\varepsilon|)$ and tends to $+\infty$, we have*

$$\frac{N_\varepsilon}{\mathbf{D}_\varepsilon} \quad \textit{tends to} \quad 1 \quad \textit{as} \quad \varepsilon \quad \textit{tends to} \quad 0. \tag{6.38}$$

**Proof**: From (6.34),

$$\pi\, \mathbf{D}_\varepsilon \left(|\log\varepsilon| - \log\gamma_\varepsilon - m_\varepsilon\ \alpha_\varepsilon\right) \leq 2\ \pi\ N_\varepsilon\ h_{ex}. \tag{6.39}$$

We insert now the applied field $h_{ex} = \frac{|\log\varepsilon|}{2} + f(\varepsilon)$ in (6.39) to get

$$\pi\left(\mathbf{D}_\varepsilon - N_\varepsilon\right)|\log\varepsilon| - \pi\ \mathbf{D}_\varepsilon\ \log\gamma_\varepsilon + 2\ \pi\ N_\varepsilon\ f(\varepsilon) \leq \pi\ \mathbf{D}_\varepsilon\ m_\varepsilon\ \alpha_\varepsilon. \tag{6.40}$$

Dividing now (6.40) by $\pi\ \mathbf{D}_\varepsilon\ |\log\varepsilon|$, $\left(\mathbf{D}_\varepsilon \neq 0\right)$,

$$(1 - \frac{N_\varepsilon}{\mathbf{D}_\varepsilon}) \leq 2\ \frac{N_\varepsilon}{\mathbf{D}_\varepsilon}\ \frac{f(\varepsilon)}{|\log\varepsilon|} + m_\varepsilon\ \frac{\alpha_\varepsilon}{|\log\varepsilon|} + \frac{\log\gamma_\varepsilon}{|\log\varepsilon|}.$$

Then, using $N_\varepsilon \leq \mathbf{D}_\varepsilon$, one finds

$$0 \leq (1 - \frac{N_\varepsilon}{\mathbf{D}_\varepsilon}) \leq 2\ \frac{f(\varepsilon)}{|\log\varepsilon|} + m_\varepsilon\ \frac{\alpha_\varepsilon}{|\log\varepsilon|} + \frac{\log\gamma_\varepsilon}{|\log\varepsilon|}.$$

Thanks to (6.22)-(6.29) together with the fact that $m_\varepsilon \to 0$ and $f(\varepsilon) = o\ (|\log\varepsilon|)$, the right-hand side of the above inequality tends to 0 as $\varepsilon \to 0$. This implies for $\varepsilon \to 0$,

$$\frac{N_\varepsilon}{\mathbf{D}_\varepsilon} \to 1.$$

□

## 3.4 Estimate of $N_\varepsilon$ and $\gamma_\varepsilon$

Now, using (6.38), we prove the optimal bound on $\gamma_\varepsilon$ and $N_\varepsilon$.

**Proposition 6.9.** *For any $\varepsilon < \varepsilon_0$, there exists $C > 0$ such that*

$$\gamma_\varepsilon \leq C\ f(\varepsilon), \quad N_\varepsilon \leq C\ f(\varepsilon).$$

**Proof**: **Step 1:** $\gamma_\varepsilon^2 \leq C\ \alpha_\varepsilon\ \max\Big(f(\varepsilon), \alpha_\varepsilon\Big)$
First, by Cauchy-Schwartz inequality

$$\Big(\int_K h_\varepsilon\Big)^2 \leq \int_K |h_\varepsilon|^2.$$

Then, we replace $\int_K h_\varepsilon$ with $2\ \pi\ N_\varepsilon$ and $\|h_\varepsilon\|_{L^2(K)}$ with $\gamma_\varepsilon$ to obtain

$$4\ \pi^2\ N_\varepsilon^2 \leq \gamma_\varepsilon^2. \tag{6.41}$$

From (6.34),

$$\frac{1}{2}\ \int_K |h_\varepsilon|^2 \leq 2\ \pi\ N_\varepsilon\ h_{ex} - \pi\ \mathbf{D}_\varepsilon\ \Big(|\log\varepsilon| - \log\gamma_\varepsilon - m_\varepsilon\ \alpha_\varepsilon\Big).$$

Replacing $h_{ex}$ with $(\frac{|\log\varepsilon|}{2} + f(\varepsilon))$ and $\frac{1}{2}\ \int_K |h_\varepsilon|^2$ with $\frac{1}{2}\ \gamma_\varepsilon^2$, we get

$$\frac{1}{2}\ \gamma_\varepsilon^2 \leq \pi\ (N_\varepsilon - \mathbf{D}_\varepsilon)\ |\log\varepsilon| + 2\ \pi\ N_\varepsilon\ f(\varepsilon) + \pi\ \mathbf{D}_\varepsilon\ \log\gamma_\varepsilon + \pi\ \mathbf{D}_\varepsilon\ m_\varepsilon\ \alpha_\varepsilon.$$

$\lim_{\varepsilon\to 0} \frac{\mathbf{D}_\varepsilon}{N_\varepsilon} = 1$, hence for $\varepsilon$ small enough, $\mathbf{D}_\varepsilon \leq C\ N_\varepsilon$. We use this with the fact that $N_\varepsilon \leq \mathbf{D}_\varepsilon$ to find

$$\frac{1}{2}\ \gamma_\varepsilon^2 \leq 2\ \pi\ N_\varepsilon\ f(\varepsilon) + C\ N_\varepsilon\ \log\gamma_\varepsilon + C\ m_\varepsilon\ N_\varepsilon\ \alpha_\varepsilon.$$

Then, by definition of $\alpha_\varepsilon$, we have $N_\varepsilon \leq \alpha_\varepsilon$, so using this and $m_\varepsilon \to 0$ together with the fact that $\log\gamma_\varepsilon \leq \gamma_\varepsilon$,

$$\gamma_\varepsilon^2 \leq C\ \alpha_\varepsilon\ \gamma_\varepsilon + C\ \alpha_\varepsilon\ f(\varepsilon) + C\ \alpha_\varepsilon^2.$$

Thus, we are led to

$$\gamma_\varepsilon^2 - C\ \alpha_\varepsilon\ \gamma_\varepsilon - C\ \alpha_\varepsilon\ \Big(f(\varepsilon) + \alpha_\varepsilon\Big) \leq 0.$$

It is obvious that

$$\gamma_\varepsilon^2 \leq C\ \alpha_\varepsilon\ \max\Big(f(\varepsilon), \alpha_\varepsilon\Big). \tag{6.42}$$

By definition of $\alpha_\varepsilon$, we get $\gamma_\varepsilon \leq C\ \alpha_\varepsilon$.
**Step 2:** $N_\varepsilon \leq C\ f(\varepsilon)$
Now, we argue by contradiction. Assume that $N_\varepsilon \gg f(\varepsilon)$.
In particular $\alpha_\varepsilon = N_\varepsilon$, and then $\gamma_\varepsilon \leq C\ N_\varepsilon$. Thanks again to (6.34),

$$\pi\ \mathbf{D}_\varepsilon\ \Big(|\log\varepsilon| - \log\gamma_\varepsilon - m_\varepsilon\ N_\varepsilon\Big) - \pi\ N_\varepsilon\ |\log\varepsilon| + \frac{1}{2}\ \int_K |h_\varepsilon|^2 \leq 2\ \pi\ N_\varepsilon\ f(\varepsilon).$$

Rearranging the terms, we get

$$\frac{1}{2}{\gamma_\varepsilon}^2 \leq \pi\ (N_\varepsilon - \mathbf{D}_\varepsilon)\ |\log\varepsilon| + 2\ \pi\ N_\varepsilon f(\varepsilon) + \pi\ \mathbf{D}_\varepsilon\ (\log\gamma_\varepsilon + m_\varepsilon\ N_\varepsilon).$$

Using the fact $\mathbf{D}_\varepsilon \geq N_\varepsilon$ and that $\mathbf{D}_\varepsilon$ is equivalent to $N_\varepsilon$ as $\varepsilon \to 0$ together with $\gamma_\varepsilon \geq N_\varepsilon$, we rewrite the above as

$$N_\varepsilon{}^2 \leq C\, N_\varepsilon \Big( f(\varepsilon) + \log \gamma_\varepsilon + m_\varepsilon\, N_\varepsilon \Big).$$

Now we use $\gamma_\varepsilon \leq C\, N_\varepsilon$ and $m_\varepsilon = o(1)$ to deduce from the above that $N_\varepsilon \leq C\, f(\varepsilon)$. The proposition follows inserting this in the inequality proved in step 1. □

## 3.5 Improved lower bound on the vortex balls

Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J$ over the space $\mathcal{A}$ and $h_\varepsilon$ be the induced magnetic field. The fact that $N_\varepsilon \leq C\, f(\varepsilon)$ and (6.23) yield

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, K) \leq C\, f(\varepsilon)\, |\log \varepsilon|. \tag{6.43}$$

Thus, we may choose

$$\alpha_\varepsilon = f(\varepsilon) \tag{6.44}$$

in the vortex-ball construction. Indeed, since $1 \ll f(\varepsilon) \ll |\log \varepsilon|$, the same holds for $\alpha_\varepsilon$, and $F_\varepsilon(u_\varepsilon, A_\varepsilon, K) \leq C\, \alpha_\varepsilon\, |\log \varepsilon|$. Then, we choose $m_\varepsilon$ satisfying

$$\frac{\log f(\varepsilon)}{f(\varepsilon)} = \frac{\log \alpha_\varepsilon}{\alpha_\varepsilon} \ll m_\varepsilon = o(1).$$

Therefore, the results of proposition 4.1 become the following. First, the sum of the radii of the balls $(B_i)_{i\in I_\varepsilon}$ satisfies $\sum_{i\in I_\varepsilon} r_i \leq C\, f(\varepsilon)\, e^{-m_\varepsilon\, f(\varepsilon)}$ and thanks to the above

$$\sum_{i\in I_\varepsilon} r_i = o(1) \quad \text{as} \ \ \varepsilon \to 0.$$

In addition

$$card(I_\varepsilon) \leq C\, f(\varepsilon)\, |\log \varepsilon|.$$

Second, combining $\alpha_\varepsilon = f(\varepsilon)$ and $\gamma_\varepsilon \leq C\, f(\varepsilon)$ together with $\frac{\log f(\varepsilon)}{f(\varepsilon)} \ll m_\varepsilon$ in (6.31), the lower bound of $J$ on the ball $B_i$ becomes

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i) \geq \pi\, |d_i|\, \Big( |\log \varepsilon| - m_\varepsilon\, f(\varepsilon) \Big). \tag{6.45}$$

Thanks to the boundedness of $\frac{N_\varepsilon}{f(\varepsilon)}$ and the positivity of $N_\varepsilon$, then up to extraction of $\varepsilon_n$ from $\varepsilon$, the following limit exists and it is finite

$$0 \leq L = \lim_{\varepsilon_n \to 0} \frac{N_{\varepsilon_n}}{f(\varepsilon_n)} = \lim_{\varepsilon_n \to 0} \frac{\mathbf{D}_{\varepsilon_\mathbf{n}}}{f(\varepsilon_n)} < +\infty. \tag{6.46}$$

Now, to get a better suited normalization of the induced magnetic field $h_\varepsilon$ and the vorticity-measure associated to minimizers, we define

$$T_\varepsilon = \frac{h_\varepsilon}{f(\varepsilon)}, \tag{6.47}$$

and $\nu_\varepsilon$ to be the extended measure by periodicity to $\mathbb{R}^2$ of $\frac{2\,\pi\, \sum_{i\in I_\varepsilon} d_i\, \delta_{a_i}}{f(\varepsilon)}$.

### 3.6 Convergence of $T_\varepsilon$ and $\nu_\varepsilon$

From Proposition 6.9 and the periodicity of $h_\varepsilon$ and $\nu_\varepsilon$, we immediately deduce

**Lemma 6.10.** *From any sequence $\varepsilon_n \to 0$, we can extract a subsequence such that there exist a periodic $T_0 \in L^2_{loc}(\mathbb{R}^2)$ and a Radon measure $\nu_0$ on $\mathbb{R}^2$ such that*

$$T_{\varepsilon_n} \rightharpoonup T_0 \quad \textit{weakly in} \quad L^2_{loc}(\mathbb{R}^2), \tag{6.48}$$

*and*

$$\nu_{\varepsilon_n} \rightharpoonup \nu_0. \tag{6.49}$$

### 3.7 Relation between of $T_0$ and $\nu_0$

In this paragraph, our interest is to find a relation between the two limits $T_0$ and $\nu_0$. Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J$ over $\mathcal{A}$. Recall that

$$N_\varepsilon = \frac{1}{2\,\pi} \int_K h_\varepsilon \in \mathbb{Z}.$$

From (6.43)

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, K) \leq C\ f(\varepsilon)\ |\log \varepsilon|. \tag{6.50}$$

Working in a Coulomb gauge, from the fact that $h_\varepsilon$ is bounded locally in $L^2$ by $C\ f(\varepsilon)$, we find that $A_\varepsilon$ is bounded locally in $H^1$ by $C\ f(\varepsilon)$. The second Ginzburg-Landau equation yields

$$-\Delta \frac{h_\varepsilon}{f(\varepsilon)} + \frac{h_\varepsilon}{f(\varepsilon)} = \frac{curl\Big((1-|u_\varepsilon|^2)\ A_\varepsilon\Big)}{f(\varepsilon)} + \frac{curl(i\ u_\varepsilon, \nabla\ u_\varepsilon)}{f(\varepsilon)} \quad \text{in } \mathbb{R}^2. \tag{6.51}$$

It is clear that

$$\frac{curl\Big((1-|u_\varepsilon|^2)\ A_\varepsilon\Big)}{f(\varepsilon)} \to 0 \tag{6.52}$$

at least in the sense of distributions, since $\frac{A_\varepsilon}{f(\varepsilon)}$ is bounded in $H^1$ and $(1-|u_\varepsilon|^2)$ goes to zero in $L^2$, then the product goes to 0 in the sense of distributions and so does the curl. However, the second term is more difficult to treat and the reason is that

$$\Big(|\log \varepsilon| \sum_{i \in I_\varepsilon} r_i\Big) = o(1),$$

does not need be true because that the balls $\Big(B_i = B_i(a_i, r_i)\Big)_{i \in I_\varepsilon}$ we work with are too large. Consequently, examining the proof of proposition 5.12, step 4, then as a consequence of the fact that $\Big(|\log \varepsilon|\ \sum_{i \in I_\varepsilon} r_i\Big) = o(1)$ does not hold, we are not able to obtain locally the convergence

$$\Big|\frac{curl(i\ u_\varepsilon, \nabla\ u_\varepsilon)}{f(\varepsilon)} - \nu_\varepsilon\Big|_{W^{-1,p}_{p<2}} \to 0. \tag{6.53}$$

The solution to this, following [SS6] is to go back to the proof of proposition 4.12 and to choose a parameter $\tilde{s}$ between the two reals $s_0$ and $s_1$, in order to obtain new balls denoted by $\mathcal{B}(\tilde{s})$ smaller than the balls $\Big(B_i(a_i, r_i)\Big)_{i \in I_\varepsilon} = \mathcal{B}(s_1)$ and of course greater than the balls of the family $\mathcal{B}(s_0)$. The next proposition explains the method.

**Proposition 6.11.** *The limit configuration $(T_0, \nu_0)$ defined by lemma 6.10 satisfies*

$$-\Delta T_0 + T_0 = \nu_0 \quad \text{in } \mathbb{R}^2, \tag{6.54}$$

$$T_{\varepsilon_n} \to T_0 \quad \text{strongly in} \quad W^{1,p}_{loc, p<2}(\mathbb{R}^2). \tag{6.55}$$

*Moreover,*

$$L = \frac{1}{2\,\pi} \int_K \nu_0 = \frac{1}{2\,\pi} \int_K T_0, \tag{6.56}$$

*where $L$ is defined by (6.46).*

**Proof**: We split the proof into four steps.
**Step1**
We go back to the proof of proposition 4.12 and we choose un parameter $\tilde{s}$ between $s_0$ and $s_1$ given there, then using the same arguments taken as for the construction of the balls $\mathcal{B}(s_1) = \Big(B_i(a_i, r_i)\Big)_{i \in I_\varepsilon}$, there exists a family of disjoint balls denoted by $\mathcal{B}(\tilde{s}) = \Big(\tilde{B}_i(\tilde{a}_j, \tilde{r}_j)\Big)_{j \in \tilde{I}_\varepsilon}$, that covers the region $\{x \in K,\ |u_\varepsilon| \le \frac{3}{4}\}$. In particular, let us take

$$\tilde{s} = \frac{1}{f(\varepsilon)\ |\log \varepsilon|^6}.$$

Note that $s_0 < \tilde{s} < s_1$ where $s_0$ and $s_1$ are given in the proof of proposition 4.12. Let $\tilde{d}_j$ be the degree of $\frac{u_\varepsilon}{|u_\varepsilon|}$ restricted on $\partial(\tilde{B}_j)$. For $i \in I_\varepsilon$, let $m_i$ be the number of balls of the family $(\tilde{B}_j)_{j \in \tilde{I}_\varepsilon}$ contained in the ball $B_i$. Then, we can write

$$\cup_{j=1}^{m_i} \tilde{B}_i(\tilde{a}_j, \tilde{d}_j) \subset B_i, \quad \forall\, i \in I_\varepsilon.$$

Therefore, it is obvious that the degree of $u_\varepsilon$ restricted to $\partial B_i$ is written as

$$d_i = \sum_{j=1}^{m_i} \tilde{d}_j \quad \forall\, i \in I_\varepsilon. \tag{6.57}$$

Consequently, since $\gamma_\varepsilon$ and $\alpha_\varepsilon$ are less than $C\ f(\varepsilon)$, we get from (4.39)

$$\sum_{j \in \tilde{I}_\varepsilon} \tilde{r}_j \le C\ \tilde{s}\ f(\varepsilon) \le C \frac{1}{|\log \varepsilon|^6}.$$

Moreover, referring to (4.36), the lower bound of $J$ on $\tilde{B}_j$ is

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, \tilde{B}_j) \ge \pi\ |\tilde{d}_j|\ \Big(|\log \varepsilon| - C\ |\log|\log \varepsilon|| - C \log f(\varepsilon)\Big).$$

Since $f(\varepsilon) = o(|\log\varepsilon|)$, it follows that

$$\sum_{j\in\tilde{I}_\varepsilon} F_\varepsilon(u_\varepsilon, A_\varepsilon, \tilde{B}_j) \geq \pi \sum_{j\in\tilde{I}_\varepsilon} |\tilde{d}_j|\, |\log\varepsilon|\, (1 - o(1)). \tag{6.58}$$

In other terms, recall that (6.23) holds

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, K) \leq C\; N_\varepsilon\; |\log\varepsilon|.$$

We need to give an upper bound of the energy on the balls $(B_i)_{i\in I_\varepsilon}$, so using the fact that $N_\varepsilon \leq C\; f(\varepsilon)$ in the above inequality, we obtain

$$\begin{aligned}\sum_{i\in I_\varepsilon} F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i) &\leq F_\varepsilon(u_\varepsilon, A_\varepsilon, K)\\ &\leq C\; f(\varepsilon)\; |\log\varepsilon|.\end{aligned} \tag{6.59}$$

Now, since the $m_i$ balls of the family $(\tilde{B}_j)_{j\in\tilde{I}_\varepsilon}$ contained in the ball $B_i$ are disjoint,

$$\sum_{j=1}^{m_i} F_\varepsilon(u_\varepsilon, A_\varepsilon, \tilde{B}_j) \leq F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i).$$

We compare the energy $J$ on the families $(B_i)_{i\in I_\varepsilon}$ and $(\tilde{B}_j)_{j\in\tilde{I}_\varepsilon}$ to get using the above inequality

$$\sum_{j\in\tilde{I}_\varepsilon} F_\varepsilon(u_\varepsilon, A_\varepsilon, \tilde{B}_i) = \sum_{i\in I_\varepsilon}\sum_{j=1}^{m_i} F_\varepsilon(u_\varepsilon, A_\varepsilon, \tilde{B}_j) \leq \sum_{i\in I_\varepsilon} F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i).$$

Inserting (6.58) and (6.59) in the above, we find

$$C\,\pi \sum_{j\in\tilde{I}_\varepsilon} |\tilde{d}_j|\, |\log\varepsilon|\, (1 - o(1)) \leq C\; f(\varepsilon)\; |\log\varepsilon|.$$

We deduce

$$\frac{\sum_{j\in\tilde{I}_\varepsilon} |\tilde{d}_j|}{f(\varepsilon)} \leq C. \tag{6.60}$$

Let $\tilde{\nu}_\varepsilon$ be the extended measure by periodicity to $\mathbb{R}^2$ of the measure $\frac{2\,\pi\, \sum_{j\in\tilde{I}_\varepsilon} \tilde{d}_j\, \delta_{\tilde{a}_j}}{f(\varepsilon)}$. Thanks to (6.60), we can say that $(\tilde{\nu}_{\varepsilon_n})$ is a bounded sequence of measures, and extracting again if necessary, we can assume that there exists a measure $\tilde{\nu}_0$ on $\mathbb{R}^2$ such that

$$\tilde{\nu}_{\varepsilon_n} \rightharpoonup \tilde{\nu}_0. \tag{6.61}$$

**Step 2:** $-\Delta T_0 + T_0 = \tilde{\nu}_0$

The balls $\left(\tilde{B}_i(\tilde{a}_j, \tilde{r}_j)\right)_{j\in\tilde{I}_\varepsilon}$ are so much small as that we have $\sum_{j\in\tilde{I}_\varepsilon} \tilde{r}_j \leq \frac{1}{|\log\varepsilon|^6}$, hence

$$|\log\varepsilon| \sum_{j\in\tilde I_\varepsilon} \tilde r_j = o(1) \quad \text{as} \ \ \varepsilon \to 0. \tag{6.62}$$

Using (6.62) and referring to the proof of proposition 5.12, steps 3,4, we find similarly as (5.66),

$$-\Delta T_\varepsilon + T_\varepsilon - \tilde\nu_\varepsilon \to 0 \tag{6.63}$$

locally in $W^{-1,p}_{p<2}$. Now, having (6.63) and using the same procedure as in [SS4], lemma 3.1, one can check

$$-\Delta T_\varepsilon + T_\varepsilon - \tilde\nu_0 \to 0 \tag{6.64}$$

locally in $W^{-1,p}_{p<2}$. $T_0$ is the weak limit of $T_{\varepsilon_n}$ locally in $L^2$, hence by uniqueness of the limit, $T_{\varepsilon_n} \to T_0$ locally in $W^{1,p}_{p<2}$ and

$$-\Delta T_0 + T_0 = \tilde\nu_0 \quad \text{in} \quad \mathbb{R}^2. \tag{6.65}$$

**Step 3:** $\nu_0 = \tilde\nu_0$

Now, having $\sum_{i\in I_\varepsilon} r_i = o(1)$, we can claim

$$\tilde\nu_{\varepsilon_n} - \nu_{\varepsilon_n} \rightharpoonup 0. \tag{6.66}$$

Indeed, first for any $f \in C^\infty_c(K)$

$$\int_K f\ (\nu_\varepsilon - \tilde\nu_\varepsilon) = \frac{2\,\pi}{f(\varepsilon)} \Big( \sum_{i\in I_\varepsilon} d_i f(a_i) - \sum_{i\in\tilde I_\varepsilon} \tilde d_i\ f(\tilde a_i)\Big). \tag{6.67}$$

Referring to (6.57),

$$\begin{aligned}\int_K f\ (\nu_\varepsilon - \tilde\nu_\varepsilon) &= \frac{2\,\pi}{f(\varepsilon)} \Big( \sum_{i\in I_\varepsilon}\sum_{j=1}^{m_i} \tilde d_j\ f(a_i) - \sum_{i\in I_\varepsilon}\sum_{j=1}^{m_i} \tilde d_j\ f(\tilde a_j)\Big)\\ &= \frac{2\,\pi}{f(\varepsilon)}\ \sum_{i\in I_\varepsilon}\Big(\sum_{j=1}^{m_i} \tilde d_j\ [f(a_i) - f(\tilde a_j)]\Big).\end{aligned} \tag{6.68}$$

Therefore, the function $f$ satisfies

$$|f(a_i) - f(\tilde a_j)| \le C\ |a_i - \tilde a_j|.$$

We know that $a_i$ is the center of $B_i$ and $\tilde a_j \in B_i(a_i, r_i)$, hence $|a_i - \tilde a_j| \le r_i$. Using this in (6.68), we get

$$\begin{aligned}\Big|\int_K f\ (\nu_\varepsilon - \tilde\nu_\varepsilon)\Big| &\le \frac{2\,\pi}{f(\varepsilon)}\ \sum_{i\in I_\varepsilon}\Big(\sum_{j=1}^{m_i} C\ |\tilde d_j|\ r_i\Big)\\ &\le \frac{C}{f(\varepsilon)}\ \sum_{i\in I_\varepsilon} r_i\ \sum_{j=1}^{m_i} |\tilde d_j|.\end{aligned}$$

Obviously, it is immediate that

$$\Big| \int_K f\ (\nu_\varepsilon - \tilde\nu_\varepsilon)\Big| \leq \frac{C}{f(\varepsilon)} \max_{i\in I_\varepsilon} r_i \ \sum_{i\in I_\varepsilon}\sum_{j=1}^{m_i} |\tilde d_j|$$

$$\leq \frac{C}{f(\varepsilon)} \Big(\sum_{i\in I_\varepsilon} r_i\Big) \Big(\sum_{i\in \tilde I_\varepsilon} |\tilde d_i|\Big).$$

We go back to (6.60) to get

$$\Big| \int_K f\ (\nu_\varepsilon - \tilde\nu_\varepsilon)\Big| \leq C \ \sum_{i\in I_\varepsilon} r_i.$$

Then, thanks to $\sum_{i\in I_\varepsilon} r_i = o(1)$, we deduce

$$\int_K f\ (\nu_\varepsilon - \tilde\nu_\varepsilon) \to 0 \quad \text{as } \ \varepsilon \to 0.$$

This means that the measure $\Big(\frac{2\,\pi\ \sum_{j\in \tilde I_\varepsilon} \tilde d_j\ \delta_{\tilde a_j}}{f(\varepsilon)} - \frac{2\,\pi\ \sum_{i\in I_\varepsilon} d_i\ \delta_{a_i}}{f(\varepsilon)}\Big)$ converges to 0 in the sense of distributions, i.e. in $\Big(C_c^\infty(K)\Big)'$. But, due to the fact that $\Big(\frac{2\,\pi\ \sum_{j\in \tilde I_\varepsilon} \tilde d_j\ \delta_{\tilde a_j}}{f(\varepsilon)} - \frac{2\,\pi\ \sum_{i\in I_\varepsilon} d_i\ \delta_{a_i}}{f(\varepsilon)}\Big)$ is a bounded sequence of measures, and extracting again if necessary, hence by uniqueness of the limit, $\Big(\frac{2\,\pi\ \sum_{j\in \tilde I_{\varepsilon_n}} \tilde d_j\ \delta_{\tilde a_j}}{f(\varepsilon_n)} - \frac{2\,\pi\ \sum_{i\in I_{\varepsilon_n}} d_i\ \delta_{a_i}}{f(\varepsilon_n)}\Big)$ converges to 0 in the sense of measures.

Now, let $q$ be fixed in $\mathbb{N}^*$ and $f \in C_c^\infty(K^q)$. Proceeding similar to the above, we can prove

$$\int_{K^q} f\ (\nu_\varepsilon - \tilde\nu_\varepsilon) \to 0 \quad \text{as } \ \varepsilon \to 0,$$

where $\nu_\varepsilon$ and $\tilde\nu_\varepsilon$ are respectively the extended measure to $K^q$ of $\frac{2\,\pi\ \sum_{i\in I_\varepsilon} d_i\ \delta_{a_i}}{f(\varepsilon)}$ and $\frac{2\,\pi\ \sum_{j\in \tilde I_\varepsilon} \tilde d_j\ \delta_{\tilde a_j}}{f(\varepsilon)}$. This means that $(\nu_\varepsilon - \tilde\nu_\varepsilon)$ converges to 0 in the sense of distributions and therefore $\nu_0 = \tilde\nu_0$, from which it follows that $-\Delta T_0 + T_0 = \nu_0$.

**Step 4**

The fact that

$$L = \frac{1}{2\,\pi} \int_K \nu_0 = \frac{1}{2\,\pi} \int_K T_0$$

follows from the above and the strong convergence of $T_{\varepsilon_n}$ to $T_0$ in $L^1$. □

## 3.8 The lower bound

In this paragraph, we give the lower bound of the minimal energy $\frac{J(u_\varepsilon, A_\varepsilon) - J^0}{(f(\varepsilon))^2}$ which complements the upper bound given by (6.5).

**Proposition 6.12.** *Let $f(\varepsilon)$ tend to $+\infty$ such that $f(\varepsilon) = o(|\log \varepsilon|)$. Then, for $T_0$ defined by lemma 6.10, we have*

$$\liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n}) - J^0}{(f(\varepsilon_n))^2} \geq \frac{1}{2}\, \|T_0 - 1\|^2_{H^1(K)} - \frac{1}{2}. \tag{6.69}$$

**Proof**: First, (6.36) gives us

$$J_K(u_\varepsilon, A_\varepsilon) - J^0 \geq \pi\, \mathbf{D}_\varepsilon\, \Big(|\log \varepsilon| - m_\varepsilon\, f(\varepsilon)\Big) - 2\,\pi\, N_\varepsilon\, h_{ex} + \frac{1}{2} \int_{K\setminus \bigcup_{i\in I_\varepsilon} B_i(a_i, r_i)} |\nabla h_\varepsilon|^2 + \frac{1}{2}\int_K |h_\varepsilon|^2.$$

We multiply the above by $\frac{1}{(f(\varepsilon))^2}$, we replace $h_{ex}$ with $(\frac{|\log \varepsilon|}{2} + f(\varepsilon))$ and we pass to the limit to get

$$\liminf_{n\to\infty} \frac{J_K(u_{\varepsilon_n}, A_{\varepsilon_n}) - J^0}{(f(\varepsilon_n))^2} \geq \frac{1}{2}\, \liminf_{n\to\infty} \int_{K\setminus\cup_{i\in I_{\varepsilon_n}} B_i(a_i,r_i)} \frac{|\nabla h_{\varepsilon_n}|^2}{(f(\varepsilon_n))^2} + \frac{1}{2}\, \liminf_{n\to\infty} \int_K \frac{|h_{\varepsilon_n}|^2}{(f(\varepsilon_n))^2}$$
$$+ \lim_{n\to\infty} \Big(\pi\, (\mathbf{D}_{\varepsilon_\mathbf{n}} - N_{\varepsilon_n}) \frac{|\log \varepsilon_n|}{(f(\varepsilon_n))^2} - \pi\, \frac{m_{\varepsilon_n}\, \mathbf{D}_{\varepsilon_\mathbf{n}}}{f(\varepsilon_n)} - 2\,\pi\, \frac{N_{\varepsilon_n}}{f(\varepsilon_n)}\Big). \tag{6.70}$$

A combination of $m_\varepsilon \to 0$ and $N_\varepsilon \leq \mathbf{D}_\varepsilon$ together with the fact that $\frac{N_\varepsilon}{f(\varepsilon)}$ tends to $L$ in (6.70) yield

$$\liminf_{n\to\infty} \frac{J_K(u_{\varepsilon_n}, A_{\varepsilon_n}) - J^0}{(f(\varepsilon_n))^2} \geq \frac{1}{2}\, \liminf_{n\to\infty} \int_{K\setminus\cup_{i\in I_{\varepsilon_n}} B_i(a_i,r_i)} \frac{|\nabla h_{\varepsilon_n}|^2}{(f(\varepsilon_n))^2} + \frac{1}{2}\, \lim_{n\to\infty} \int_K \frac{|h_{\varepsilon_n}|^2}{(f(\varepsilon_n))^2} - 2\,\pi\, L. \tag{6.71}$$

By definition of the function $T_\varepsilon$ which is $\frac{h_\varepsilon}{f(\varepsilon)}$,

$$\liminf_{n\to\infty} \frac{J_K(u_{\varepsilon_n}, A_{\varepsilon_n}) - J^0}{(f(\varepsilon_n))^2} \geq \frac{1}{2}\, \liminf_{n\to\infty} \int_{K\setminus\cup_{i\in I_{\varepsilon_n}} B_i(a_i,r_i)} |\nabla T_{\varepsilon_n}|^2 + \frac{1}{2}\, \lim_{n\to\infty} \int_K |T_{\varepsilon_n}|^2 - 2\,\pi\, L. \tag{6.72}$$

From proposition 6.11,

$$\begin{cases} T_{\varepsilon_n} \to T_0 \quad \text{strongly in } \; W^{1,p}_{p<2}(K) \\ \nabla T_{\varepsilon_n} \to \nabla T_0 \;\; a.e. \end{cases}$$

Let $X_\varepsilon = \nabla T_\varepsilon$ in $K\setminus \cup_{i\in I_{\varepsilon_n}} B_i(a_i, r_i)$ and 0 otherwise, so thanks to [SS6]

$$X_{\varepsilon_n} \to \nabla T_0 \quad a.e.$$

In particular, using Fatou lemma, we have

$$\liminf_{n\to\infty} \int_{K\setminus\cup_{i\in I_{\varepsilon_n}} B_i(a_i,r_i)} |X_{\varepsilon_n}|^2 \geq \int_K |\nabla T_0|^2.$$

Consequently, by definition of the function $X_\varepsilon$, this implies

$$\liminf_{n\to\infty} \int_{K\setminus \cup_{i\in I_{\varepsilon_n}} B_i(a_i,r_i)} |\nabla T_{\varepsilon_n}|^2 \geq \int_K |\nabla T_0|^2. \tag{6.73}$$

Again, since $T_{\varepsilon_n} \to T_0$ weakly in $L^2(K)$,

$$\liminf_{n\to\infty} \int_K |T_{\varepsilon_n}|^2 \geq \int_K |T_0|^2. \tag{6.74}$$

Inserting (6.73) and (6.74) in (6.72),

$$\liminf_{n\to\infty} \frac{J_K(u_{\varepsilon_n}, A_{\varepsilon_n}) - J^0}{(f(\varepsilon_n))^2} \geq \frac{1}{2} \int_K |T_0|^2 + \frac{1}{2} \int_K |\nabla T_0|^2 - 2\,\pi\, L. \tag{6.75}$$

Referring to (6.56), we know

$$\int_K T_0 = 2\,\pi\, L.$$

Consequently,

$$\liminf_{n\to\infty} \frac{J_K(u_{\varepsilon_n}, A_{\varepsilon_n}) - J^0}{(f(\varepsilon_n))^2} \geq \frac{1}{2} \int_K |T_0|^2 + \frac{1}{2} \int_K |\nabla T_0|^2 - \int_K T_0. \tag{6.76}$$

More precisely, this means

$$\liminf_{n\to\infty} \frac{J_K(u_{\varepsilon_n}, A_{\varepsilon_n}) - J^0}{(f(\varepsilon_n))^2} \geq \frac{1}{2}\, \|T_0 - 1\|^2_{H^1(K)} - \frac{1}{2}. \tag{6.77}$$

□

# 4 Proof of Theorem 6.1, completed

First, combining the properties of the limit configuration $T_0$, we can say that $T_0 \in V$. From a comparison of the upper and the lower bound of the quantity $\frac{J-J^0}{(f(\varepsilon))^2}$, we present the values taken by the limiting configuration of vortices $(T_0, \nu_0)$ in the following lemma.

**Lemma 6.13.** *The $(T_0, \nu_0)$ defined by lemma 6.10 satisfies*

$$\begin{cases} T_0 = 1 \\ \nu_0 = dx \end{cases} \tag{6.78}$$

*where $dx$ is the Lebesgue measure on $\mathbb{R}^2$. Moreover,*

$$\lim_{n\to+\infty} \frac{\mathbf{D}_{\varepsilon_n}}{f(\varepsilon_n)} = \frac{1}{2\,\pi}.$$

**Proof**: Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$, then propositions 6.3 and 6.12 both give us

$$\frac{1}{2}\,\|T_0 - 1\|^2_{H^1(K)} - \frac{1}{2} \leq \liminf_{n\to\infty} \frac{J_K(u_{\varepsilon_n}, A_{\varepsilon_n}) - J^0}{(f(\varepsilon_n))^2} \leq \limsup_{n\to\infty} \frac{J_K(u_{\varepsilon_n}, A_{\varepsilon_n}) - J^0}{(f(\varepsilon_n))^2} \leq -\frac{1}{2}.$$

Examining the left and right-side of the above, we find

$$\lim_{n\to\infty} \frac{J_K(u_{\varepsilon_n}, A_{\varepsilon_n}) - J^0}{\Big(f(\varepsilon_n)\Big)^2} = -\frac{1}{2}, \tag{6.79}$$

and

$$T_0 = 1.$$

Thanks to proposition 6.11,

$$T_{\varepsilon_n} \to T_0 = 1 \quad \text{strongly in} \quad W^{1,p}_{loc, p<2}(\mathbb{R}^2).$$

Moreover, the limit measure, which is $\nu_0 = -\Delta T_0 + T_0$, is written as

$$\nu_0 = dx \quad \text{in} \quad \mathbb{R}^2, \tag{6.80}$$

where $dx$ is the Lebesgue measure on $\mathbb{R}^2$. Consequently, thanks to (6.49)

$$\nu_{\varepsilon_n} \rightharpoonup \nu_0 = dx,$$

where $\nu_\varepsilon$ is the extended measure by periodicity to $\mathbb{R}^2$ whose restriction on $K$ is $\frac{2\,\pi\,\sum_{i\in I_\varepsilon} d_i\,\delta_{a_i}}{f(\varepsilon)}$. This means that the vortex-repartition is uniform. In addition, from (6.80),

$$\int_K \nu_0 = 1,$$

and in view of (6.56),

$$\int_K \nu_0 = 2\,\pi\,L,$$

then by identification, we get thanks to the fact that $\lim_{\varepsilon\to 0} \frac{N_\varepsilon}{\mathbf{D}_\varepsilon} = 1$

$$L = \lim_{n\to+\infty} \frac{N_{\varepsilon_n}}{f(\varepsilon_n)} = \lim_{n\to+\infty} \frac{\mathbf{D}_{\varepsilon_\mathbf{n}}}{f(\varepsilon_n)} = \frac{1}{2\,\pi}. \tag{6.81}$$

This allows to ensure that the number of vortices per period is of the order of $f(\varepsilon)$. The above limits don't depend on the chosen of the subsequence and since it is true for any $\varepsilon_n \to 0$, the whole sequence converges. Combining all the above completes the proof of Theorem 6.1. □

**Remark 6.14.** *In the chapters 5 and 6, we have studied in the limit $\varepsilon \to 0$ the vortex-structure of minimizers of the Ginzburg-Landau energy over the space $\mathcal{A}$ with respect to the all possible values of $\lambda > 0$. In particular, we have given the value of $\lambda$ for which the vortices appear, and when there are vortices we have successfully stated their repartition in the superconductor and their number per period.*

# Chapter 7

# Vortices's concentration along one line

In this chapter, we construct a periodic critical point of the energy $J$, i.e. a solution of the Ginzburg-Landau equations (3.4), such that as $\varepsilon \to 0$, the vortices contained in the square $K$ of $(u_\varepsilon, A_\varepsilon)$ minimizer of the energy $J$ over an appropriate space, are concentrated along a finite number of horizontal lines.

## 1 Introduction

Here, we deal with applied fields $h_{ex}$ given by the following limit

$$\lambda = \lim_{\varepsilon \to 0} \frac{|\log \varepsilon|}{h_{ex}}. \tag{7.1}$$

We assume that the limit exists. From now on, we consider fields such that

$$0 < \lambda < 2. \tag{7.2}$$

Note that $\lambda > 0$, i.e. $h_{ex} \leq C \ |\log \varepsilon|$. We take $K$ any square of sidelength 1. Our goal is to find a sequence of solutions of the Ginzburg-Landau equations (3.4) such that as $\varepsilon \to 0$, the vortices contained in $K$ concentrate on a finite number of horizontal lines. The sequence is constructed by minimizing the energy $J$ over an appropriate space. First, let $p_\varepsilon \in \mathbb{N}$ be a function of $\varepsilon$ such that the following limit exists and does not vanish

$$\alpha = 2\,\pi \lim_{\varepsilon \to 0} \frac{p_\varepsilon}{h_{ex}}. \tag{7.3}$$

Now, we define the space where we perform the minimization of the Ginzburg-Landau energy $J$.

**Definition 7.1.** *Let $(u, A) \in H^1_{loc}(\mathbb{R}^2, \mathbb{C}) \times H^1_{loc}(\mathbb{R}^2, \mathbb{R}^2)$. Then, $(u, A)$ belongs to $\mathcal{G}_\varepsilon$ if there exists $(k_1, k_2) \in H^2_{loc}(\mathbb{R}^2, \mathbb{R}) \times H^2_{loc}(\mathbb{R}^2, \mathbb{R})$ such that $\forall (x, y) \in \mathbb{R}^2$*

$$\begin{cases} u\Big(x+\frac{1}{p_\varepsilon},y\Big) = u(x,y)\ e^{i\,k_1(x,y)} \\ u(x,y+1) = u(x,y)\ e^{i\,k_2(x,y)} \end{cases} \tag{7.4}$$

*and*

$$\begin{cases} A\Big(x+\frac{1}{p_\varepsilon},y\Big) = A(x,y) + \nabla\ k_1(x,y) \\ A(x,y+1) = A(x,y) + \nabla\ k_2(x,y). \end{cases} \tag{7.5}$$

Proceeding similarly as in the chapter 3, the infimum of $J$ over $\mathcal{G}_\varepsilon$ is achieved. We denote by $(u_\varepsilon, A_\varepsilon)$ a sequence of minimizers and $h_\varepsilon = curl A_\varepsilon$ its associated magnetic field. Then, it is a solution of the Ginzburg-Landau equations, namely

$$\begin{cases} \nabla^2_{A_\varepsilon} u_\varepsilon = \frac{1}{\varepsilon^2}\ u_\varepsilon\ (1-|u_\varepsilon|^2) & \text{in } \mathbb{R}^2 \\ -\nabla^\perp h_\varepsilon =< i\ u_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon > & \text{in } \mathbb{R}^2. \end{cases}$$

We restrict our attention to the asymptotic behavior of the minimizers $(u_\varepsilon, A_\varepsilon)$ over $\mathcal{G}_\varepsilon$ when $\varepsilon$ tends to 0 and their vortices. From now ow, let $K = [0,1[\times[0,1[$. First, we state some notations and definitions.

**Notations**

Let $f$ be a function on $\mathbb{R}^2$.
i) We mean by the $K$ -periodicity of $f$ that there is a periodicity with respect to the square $K$, i.e. $\forall\ (x,y) \in \mathbb{R}^2$

$$f(x+1,y) = f(x,y) = f(x,y+1).$$

ii) We say that $f$ is $R$ -periodic if

$$f(x+\frac{1}{p_\varepsilon},y) = f(x,y) \quad \forall\ (x,y) \in \mathbb{R}^2.$$

iii) We say that $f$ is $KR$ -periodic if for any $(x,y) \in \mathbb{R}^2$,

$$f(x+\frac{1}{p_\varepsilon},y) = f(x,y) = f(x,y+1).$$

Now, let $(u,A)$ be in the space $\mathcal{G}_\varepsilon$ and $h = curl A$. Then, in particular from (7.4) and (7.5), a simple calculation gives us that the physical quantities like $h$, $|u|$ and $< i\ u,\ \nabla_A u >$ are $KR$ -periodic (in the sense of (iii)).

## 2 An upper bound of the energy

Recall that $\alpha$ is defined by (7.3). First, we give the space

$$
\mathcal{U} := \left\{ \begin{array}{l} f \in H^1_{loc}(\mathbb{R}^2, \mathbb{R}) \quad \text{such that } f \text{ is } K \text{ }-\text{periodic, } x \to f(x,y) \text{ is constant and the} \\ \text{restriction of the measure } \ \nu = -\Delta f + f \ \text{ on } \ K \ \text{ is supported on a finite number} \\ \text{of horizontal lines such that the mass of } \ \nu \ \text{ on each one belongs to } \ \alpha \ \mathbb{Z} \end{array} \right\}. \tag{7.6}
$$

We take for any $f \in \mathcal{U}$ the measure

$$
\nu = -\Delta f + f \quad \text{in} \quad \mathbb{R}^2. \tag{7.7}
$$

For $y \in K$, we define $G$ to be the Green function solution of

$$
-\Delta_x G(x,y) + G(x,y) = \delta_y \quad \text{in } \mathbb{R}^2. \tag{7.8}
$$

Remark that $G$ exists, is unique and symmetric, i.e. $G(x,y) = G(y,x)$. Let $I$ be the functional

$$
I(\nu) = \frac{\lambda}{2} \int_K |\nu| + \frac{1}{2} \int_{K \times \mathbb{R}^2} G(x,y) \, d(\nu - 1)(y) \, d(\nu - 1)(x). \tag{7.9}
$$

**Lemma 7.2.** *We have for any $f \in \mathcal{U}$,*

$$
E(f) = \frac{\lambda}{2} \int_K |-\Delta f + f| + \frac{1}{2} \int_K |\nabla f|^2 + \frac{1}{2} \int_K |f - 1|^2 = I(\nu) \qquad \forall \ \nu = -\Delta f + f. \tag{7.10}
$$

**Proof**: First, we multiply the equation (7.8) by $x \to f(x)$ and we integrate over $\mathbb{R}^2$ to get for $y \in K$

$$
\int_{\mathbb{R}^2} \Big( -\Delta_x G(x,y) + G(x,y) \Big) \, f(x) \, dx = \int_{\mathbb{R}^2} \delta_y(x) \, f(x) \, dx = f(y). \tag{7.11}
$$

Second, thanks to lemma 5.7, it is easy to find

$$
\int_{\mathbb{R}^2} -\Delta_x G(x,y) \, f(x) \, dx = \int_{\mathbb{R}^2} -\Delta_x f(x) \, G(x,y) \, dx.
$$

We insert this in (7.11) to have

$$
\int_{\mathbb{R}^2} \Big( -\Delta_x f(x) + f(x) \Big) \, G(x,y) \, dx = f(y). \tag{7.12}
$$

Going back to (7.7) and the symmetry of $G$, we obtain

$$
f(y) = \int_{\mathbb{R}^2} G(y,x) \, d\nu(x), \quad y \in K. \tag{7.13}
$$

Now, set for $f \in \mathcal{U}$

$$F(f) = \int_K |\nabla f|^2 + \int_K |f-1|^2.$$

It follows that

$$E(f) = \frac{\lambda}{2} \int_K |-\Delta f + f| + \frac{1}{2} F(f).$$

We need to give the explicit form of the functional $F$ only in function of the measure $\nu$. Using the $K$ -periodicity of $f$

$$\int_K \Delta f = 0 \quad \text{and} \quad \int_K -f\, \Delta f = \int_K |\nabla f|^2.$$

We insert this in the functional $F$ to obtain

$$\begin{aligned} F(f) &= \int_K (f-1) \Big( -\Delta f + f - 1 \Big) \\ &= \int_K (f-1)(y)\, d(\nu - 1)(y). \end{aligned} \tag{7.14}$$

The equation (5.18) gives us

$$\int_{\mathbb{R}^2} G(y,x)\, dx = 1.$$

Using this in (7.13),

$$(f-1)(y) = \int_{\mathbb{R}^2} G(y,x)\, d(\nu-1)(x). \tag{7.15}$$

The measure $(\nu - 1)$ denotes the difference between of the measure $\nu$ and the Lebesgue measure on $\mathbb{R}^2$. Inserting (7.15) in (7.14) leads to

$$F(f) = \int_K \Big( \int_{\mathbb{R}^2} G(y,x)\, d(\nu-1)(x) \Big) d(\mu - 1)(y).$$

We obtain for $\nu = -\Delta f + f$,

$$\begin{aligned} E(f) =& \frac{\lambda}{2} \int_K |-\Delta f + f| + \frac{1}{2} F(f) \\ =& \frac{\lambda}{2} \int_K |\nu| + \frac{1}{2} \int_{K \times \mathbb{R}^2} G(x,y)\, d(\nu-1)(y)\, d(\nu-1)(x) \\ =& I(\nu). \end{aligned}$$

□

## 2.1 Main result

The upper bound on the minimal energy is stated in the following

**Proposition 7.3.** *Set $h_{ex}$ be such that $\lim_{\varepsilon\to 0}\frac{|\log\varepsilon|}{h_{ex}}=\lambda$ exists, is finite and does not vanish. Let $\nu$ be any $K$ -periodic Radon measure on $\mathbb{R}^2$ constant on horizontal lines such that the restriction of $\nu$ on $K$ is supported on a finite number of horizontal lines and its mass on each line belongs to $\alpha\,\mathbb{N}$, and $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J$ over the space $\mathcal{G}_\varepsilon$, then*

$$\limsup_{\varepsilon\to 0}\frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2}\le I(\nu). \tag{7.16}$$

Thanks to lemma 7.2, the proposition 7.3 can be stated differently

**Corollary 7.4.** *If $\lambda>0$, then for any $f\in\mathcal{U}$ with $(-\Delta f+f)$ is positive, we have for a minimizer $(u_\varepsilon, A_\varepsilon)$ of the energy $J$ over $\mathcal{G}_\varepsilon$*

$$\limsup_{\varepsilon\to 0}\frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2}\le E(f).$$

*Proof.* Let $f\in\mathcal{U}$, then by definition of the space $\mathcal{U}$, $f$ is $K$ -periodic, so the measure $\nu=-\Delta f+f$ is in particular $K$ -periodic. Again, $\nu$ is constant on horizontal lines and its restriction on $K$ is concentrated on a finite number of horizontal lines. Moreover, $\nu$ is taken to be positive, so the mass of $\nu$ on each line belongs to $\alpha\,\mathbb{N}$. Combining all the above, the proposition 7.3 implies

$$\limsup_{\varepsilon\to 0}\frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2}\le I(\nu).$$

Therefore, for $\nu=-\Delta f+f$, the lemma 7.2 leads to

$$\limsup_{\varepsilon\to 0}\frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2}\le E(f).$$

□

## 2.2 The proof of proposition 7.3

Suppose that the assumptions of proposition 7.3 hold, then without loss of generality, we assume that the restriction of the measure $\nu$ on $K$ is supported on $m$ horizontal lines denoted by $\{\Sigma_i,\ 1\le i\le m\}$. Since the mass of $\nu$ on each horizontal line belongs to $\alpha\,\mathbb{N}$, there exist $(y_i)_{1\le i\le m}$ with $0<y_1<y_2<...<y_m<1$ and $(n_i)_{1\le i\le m}$ with $n_i\in\mathbb{N}$ such that the restriction of $\nu$ on $K$ is equal to $\alpha\,\sum_{i=1}^m n_i\,\delta_{\Sigma_i}$ where $\delta_{\Sigma_i}$ is the measure of arclength along $\Sigma_i$ and the equation of $\Sigma_i$ is $y=y_i$.
The upper bound (7.16) is obtained by a construction of a test configuration $(v_\varepsilon, B_\varepsilon)$ in the space $\mathcal{G}_\varepsilon$. For this, we need to describe the vortices of $(v_\varepsilon, B_\varepsilon)$. We split the proof into three steps.
**Step1**
We consider the sequence $p_\varepsilon$ defined by (7.3). Let $R_j$ be the rectangle

$$R_j = [\frac{j-1}{p_\varepsilon}, \frac{j}{p_\varepsilon}[\times[0,1[, \quad 1 \le j \le p_\varepsilon.$$

We place in the rectangle $R_j$ the points

$$(a_j^k)_{1\le k\le m} = \Big(\frac{j-1/2}{p_\varepsilon}, y_k\Big)_{1\le k\le m}. \tag{7.17}$$

The extended points on $K$ are $\Big(a_j^k\Big)_{1\le k\le m,\ 1\le j\le p_\varepsilon}$. We deduce that there are $(m\, p_\varepsilon)$ points in the square $K$. Now, we define $\nu_\varepsilon$ to be the extended measure to $\mathbb{R}^2$ by $K$ -periodicity of $\frac{2\,\pi}{h_{ex}}\ \sum_{k=1}^m \Big(n_k\ \sum_{i=1}^{p_\varepsilon} \delta_{a_i^k}\Big)$. Let $1 \le k \le m$ be fixed, then as $\varepsilon \to 0$

$$\frac{\sum_{i=1}^{p_\varepsilon} \delta_{a_i^k}}{p_\varepsilon} \to \delta_{\Sigma_k} \quad \text{in the sense of measures,}$$

where $\Sigma_k$ is the horizontal line of equation $y = y_k$. Consequently, using the fact that $\alpha\, h_{ex} \simeq 2\,\pi\, p_\varepsilon$ as $\varepsilon \to 0$, we find

$$\nu_\varepsilon \rightharpoonup \nu \quad \text{as} \quad \varepsilon \to 0.$$

**Step2**

We refer to the proof of proposition 5.8 to have

$$\limsup_{\varepsilon\to 0} \frac{1}{2}\ \int_{K\ \times\mathbb{R}^2} G(x,y)\ d(\nu_\varepsilon - 1)(x)\ d(\nu_\varepsilon - 1)(y) \le \frac{\lambda}{2}\ \nu(K) + \int_{K\times\mathbb{R}^2} G(x,y)\ d(\nu-1)(x)\ d(\nu-1)(y). \tag{7.18}$$

**Step3**

Now, we construct a test configuration $(v_\varepsilon, B_\varepsilon)$ to be in the space $\mathcal{G}_\varepsilon$. First, we construct a function $h_\varepsilon$ $KR$ -periodic by letting

$$h_\varepsilon(x) = h_{ex}\ \int_{\mathbb{R}^2} G(x,y)\, d\nu_\varepsilon(y),$$

so that

$$-\Delta h_\varepsilon + h_\varepsilon = h_{ex}\ \nu_\varepsilon \quad \text{in}\ \mathbb{R}^2. \tag{7.19}$$

$h_\varepsilon$ is taken as the magnetic field. Then, $v_\varepsilon$ and $B_\varepsilon$ are defined as in the proof of proposition 5.8 in a way such that $h_\varepsilon = curl B_\varepsilon$ and $(v_\varepsilon, B_\varepsilon) \in \mathcal{G}_\varepsilon$ with

$$\frac{J(v_\varepsilon, B_\varepsilon)}{h_{ex}^2} \le \frac{\frac{1}{2}\ \int_K |\nabla h_\varepsilon|^2 + \frac{1}{2}\ \int_K |h_\varepsilon - h_{ex}|^2}{h_{ex}^2} + o_\varepsilon(1), \tag{7.20}$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. Then, following again the proof of proposition 5.8 and using (7.18) yield

$$\limsup_{\varepsilon\to 0} \frac{\frac{1}{2}\ \int_K |\nabla h_\varepsilon|^2 + \frac{1}{2}\ \int_K |h_\varepsilon - h_{ex}|^2}{h_{ex}^2} \le I(\nu).$$

Combining with (7.20) allows to conclude

$$\limsup_{\varepsilon \to 0} \frac{J(v_\varepsilon, B_\varepsilon)}{h_{ex}^2} \leq I(\nu). \tag{7.21}$$

This inequality is true for the test configuration $(v_\varepsilon, B_\varepsilon)$, so it is true in particular for any minimizer of the energy $J$ over the space $\mathcal{G}_\varepsilon$ and (7.16) is proved.

**Assumption**

Let $f \in \mathcal{U}$, then without loss of generality we assume from now on that the restriction of the measure $\nu = -\Delta f + f$ on the square $K$ is supported on $m$ horizontal lines. We take $(y_i)_{1 \leq i \leq m}$ such that $0 < y_1 < y_2 < ... < y_m < 1$ and we define $\Sigma_i$ be the horizontal line contained in $K$ and of equation $y = y_i$. As a consequence of the above,

$$\{\Sigma_i, \quad 1 \leq i \leq m\}$$

is the family of the $m$ disjoint horizontal lines where the restriction of the measure $\nu$ on $K$ concentrates. Since the mass of $\nu$ on each line belongs to $\alpha\,\mathbb{Z}$, there exists $n_i \in \mathbb{Z}$ such that the mass of the measure $\nu$ on $\Sigma_i$ is equal to $\alpha\, n_i$. It means that the restriction of the measure $\nu$ on $K$ can be written as

$$\nu = \sum_{i=1}^{m} \alpha \;\; n_i \;\; \delta_{\Sigma_i}. \tag{7.22}$$

Now, for $f \in \mathcal{U}$ and under the above assumptions, our interest is to rewrite the energy $E$, which is given by lemma 7.2, only in function of the family $(y_i, n_i)_{1 \leq i \leq m}$. This will be the subject of the next paragraph.

# 3 New formulation of the energy $E$

Here, we take $K = [0,1[\times[0,1[$. Let $f \in \mathcal{U}$, then in particular, the restriction of the measure $(-\Delta f + f)$ on $K$ is concentrated on a finite number of horizontal lines. We start with the case where the measure $(-\Delta f + f)$ is not concentrated on any horizontal line.

## 3.1 Energy without horizontal lines of concentration

Let $f \in \mathcal{U}$. In the case of absence of concentration's horizontal lines, $f$ verifies in particular

$$-\Delta f + f = 0 \quad \text{in } \mathbb{R}^2. \tag{7.23}$$

Since $f$ is bounded, it is easy that

$$f = 0 \quad \text{in } \mathbb{R}^2.$$

Consequently, letting $f = 0$ in the energy without line of concentration, we deduce

$$E(f) = \frac{1}{2}.$$

### 3.2 The energy in the presence of horizontal line (or lines) of concentration

In this case, let $m \geq 1$ and $(y_i, n_i)_{1 \leq i \leq m}$ such that $0 < y_1 < ... < y_m < 1$ and $n_i \in \mathbb{Z}$ for $1 \leq i \leq m$. Let $f \in \mathcal{U}$, then taking the assumption given in the above paragraph and thanks to (7.22), the restriction of the measure $(-\Delta f + f)$ on $K$ is

$$-\Delta f + f = \sum_{i=1}^{m} \alpha \ \ n_i \ \ \delta_{\Sigma_i}, \tag{7.24}$$

where $\Sigma_i$ is the horizontal line contained in $K$ and of equation $y = y_i$. Due to the parameter $m$, then for the family $(y_i, n_i)_{1 \leq i \leq m}$ defined in the above, let us set the space

$$\mathcal{U}_m := \left\{ \begin{array}{l} f \in H^1_{loc}(\mathbb{R}^2, \mathbb{R}) \ \text{ such that } \ f \ \text{ is } \ K-\text{periodic}, \ x \to f(x,y) \text{ is constant and the} \\ \text{restriction of the measure } \ (-\Delta f + f) \ \text{ on } \ K \ \text{ is of the form } \ \sum_{i=1}^{m} \ \alpha \ \ n_i \ \ \delta_{\Sigma_i} \\ \text{where } \ n_i \in \mathbb{Z} \ \ \text{for } 1 \leq i \leq m \end{array} \right\}.$$

Let $f \in \mathcal{U}_m$, then there exist $(n_i)_{1 \leq i \leq m}$ with $n_i \in \mathbb{Z}$ for any $1 \leq i \leq m$ such that the restriction of the measure $(-\Delta f + f)$ on $K$ can be written as $\sum_{i=1}^{m} \alpha \, n_i \, \delta_{\Sigma_i}$. Again by definition of the space $\mathcal{U}_m$, $x \to f(x,y)$ is constant in $[0,1]$, hence to drop the subscripts, we set for $y \in [0,1]$, $g(y) = f(x,y)$. In particular, we deduce

$$\int_K |-\Delta f + f| = \alpha \sum_{i=1}^{m} n_i \quad \text{and} \quad \int_K (-\Delta f + f)(f-1) = \alpha \sum_{i=1}^{m} n_i \ (g-1)(y_i).$$

Using $g(y) = f(x,y)$ together with these two identities and the fact that $f$ is $K$ -periodic, the energy $E$ corresponding to $m$ horizontal lines can be written as

$$\begin{aligned} E(f) =& \frac{\lambda}{2} \, \alpha \sum_{i=1}^{m} n_i + \frac{1}{2} \int_K (-\Delta f + f - 1)(f-1) \\ =& \frac{\lambda}{2} \, \alpha \sum_{i=1}^{m} n_i + \frac{\alpha}{2} \sum_{i=1}^{m} n_i \ (g-1)(y_i) - \frac{1}{2} \int_K (f-1). \end{aligned}$$

We can write

$$E(f) = \frac{1}{2} + \sum_{i=1}^{m} \alpha \ \ n_i \ \ \Big( \frac{g(y_i)}{2} - (\frac{1}{2} - \frac{\lambda}{2}) \Big) - \frac{1}{2} \int_K f. \tag{7.25}$$

We need to calculate $\int_K f$. For this, denote by $g'_l$ (resp. $g'_r$) the left (resp. right) derivative of $g$, so it is clear

$$\alpha \; n_i = g_l'(y_i) - g_r'(y_i) \quad \forall \, 1 \le i \le m. \tag{7.26}$$

By definition of the function $g$,

$$\begin{aligned}\int_K f = \int_0^1 g(y) \; dy = \int_0^{y_1} g(y) + \int_{y_m}^1 g(y) \; dy + \sum_{i=1}^{m-1} \int_{y_i}^{y_{i+1}} g(y) \\ = \int_0^{y_1} g''(y) \; dy + \int_{y_m}^1 g''(y) \; dy + \sum_{i=1}^{m-1} \int_{y_i}^{y_{i+1}} g''(y) \; dy.\end{aligned} \tag{7.27}$$

Using (7.26) and $g'(0) = g'(1)$ (which follows from $f \in \mathcal{U}_m$),

$$\int_K f = \sum_{i=1}^{m} \alpha \;\; n_i. \tag{7.28}$$

Inserting (7.28) in (7.25),

$$E(f) = \frac{1}{2} + \sum_{i=1}^{m} \alpha \;\; n_i \;\; \Big( \frac{g(y_i)}{2} - (1 - \frac{\lambda}{2}) \Big). \tag{7.29}$$

From now on, we restrict to the case $m = 1$. In particular, we have

**Lemma 7.5.** *If $f \in \mathcal{U}$ is such that $-\Delta f + f$ is equal on $K$ to $\alpha \; n_1 \; \delta_{\Sigma_1}$ where $n_1 \in \mathbb{Z}$, then*

$$E(f) = E_1(n_1, y_1) = \frac{1}{2} + (1 - \frac{\lambda}{2}) \; \alpha \;\; n_1 + \frac{e+1}{4 \; (e-1)} \;\; \alpha^2 \;\; n_1^2, \tag{7.30}$$

*where $\Sigma_1$ is the horizontal line contained in $K$ and of equation $y = y_1$.*

**Proof**: The restriction of the measure $(-\Delta f + f)$ on $K$ is

$$-\Delta f + f = \;\alpha \;\; n_1 \;\; \delta_{\Sigma_1}.$$

Taking $g(y) = f(x, y)$, we have thanks to (7.26),

$$\alpha \; n_1 = g_l'(y_1) - g_r'(y_1).$$

By definition of $g$, $g(0) = g(1)$ and $g'(0) = g'(1)$. Now, combining the above together with the continuity of $g$ at $y_1$, a simple calculation gives us

$$g(y_1) = \frac{e+1}{2 \; (e-1)} \;\; \alpha \;\; n_1. \tag{7.31}$$

However, letting $m = 1$ in (7.29),

$$E(f) = E_1(n_1, y_1) = \frac{1}{2} + \alpha \;\; n_1 \;\; \Big( \frac{g(y_1)}{2} - (1 - \frac{\lambda}{2}) \Big). \tag{7.32}$$

We insert (7.31) in (7.32) to deduce

$$E(f) = E_1(n_1, y_1) = \frac{1}{2} + (1 - \frac{\lambda}{2})\ \alpha\ \ n_1 + \frac{e+1}{4\,(e-1)}\ \ \alpha^2\ \ n_1^2.$$

This completes the proof of the lemma 7.5. □

The expression given by (7.32) does not depend on $y_1$, hence for simplification, we take for $n_1 \in \mathbb{Z}$

$$F(n_1) = E_1(n_1, y_1) = \frac{1}{2} + (1 - \frac{\lambda}{2})\ \alpha\ \ n_1 + \frac{e+1}{4\,(e-1)}\ \ \alpha^2\ \ n_1^2. \tag{7.33}$$

Now, let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over the space $\mathcal{G}_\varepsilon$, then going back to the upper bound given by corollary 7.4 and using the definition of $F$, we can deduce for any $n_1 \in \mathbb{N}$

$$\limsup_{\varepsilon \to 0} \frac{J_K(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \leq F(n_1). \tag{7.34}$$

## 3.3 The finer upper bound of the minimal energy

The fundamental result of this section which will be very useful for the rest is stated in the following lemma

**Lemma 7.6.** *Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over the space $\mathcal{G}_\varepsilon$. Then, if*

$$1 - \frac{\lambda}{2} > \alpha\ \frac{e+1}{4\,(e-1)}, \tag{7.35}$$

*we have*

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} < \frac{1}{2}. \tag{7.36}$$

**Remark 7.7.** *Thanks to the assumption (7.2), we remark that the left-hand side of (7.35) is positive. Then, for a sufficiently small $\alpha > 0$, the condition (7.35) has a sense. Note that the right-hand side of (7.36) which is $\frac{1}{2}$ corresponds to the energy without horizontal line (or lines). Moreover, the inequality given by (7.36) will be very essential at the end of the chapter.*

**Proof**: We take particularly $n_1 = 1$ in (7.34) to get

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \leq F(1).$$

By definition of the functional $F$ given by (7.33),

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \leq \frac{1}{2} - (1 - \frac{\lambda}{2})\ \alpha + \frac{e+1}{4\,(e-1)}\ \ \alpha^2.$$

Now, we choose

$$1 - \frac{\lambda}{2} > \alpha \, \frac{e+1}{4\,(e-1)},$$

to conclude from the above

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} < \frac{1}{2}.$$

The proof of lemma 7.6 is then completed. □

# 4 Lower bound

Here, we assume that the applied field $h_{ex}$ is such that

$$0 < \lambda < 2.$$

Consider $(u_\varepsilon, A_\varepsilon)$ a family of minimizers of the energy $J$ over the space $\mathcal{G}_\varepsilon$, thus a family of critical points of $J$ and let $h_\varepsilon = curl A_\varepsilon$ be the induced field. Similar to the proposition 4.1, we can state

**Proposition 7.8.** *For $h_{ex} \leq C\,|\log\varepsilon|$, there exists $\varepsilon_0$ such that if $\varepsilon < \varepsilon_0$ and $(u_\varepsilon, A_\varepsilon)$ a minimizer of $J$ over $\mathcal{G}_\varepsilon$, then there exist a rectangle $R^1$ of the form $[x, x+\frac{1}{p_\varepsilon}[\times[y, y+1[$ $x, y \in \mathbb{R}$, (without loss of generality the rectangle is $R^1 = [0, \frac{1}{p_\varepsilon}[\times[0,1[)$, and a family of disjoint balls $\Big(B_i = B_i(a_i, r_i)\Big)_{i \in \mathcal{L}_\varepsilon}$ of center $a_i$ and of radii $r_i$ satisfying*

$$\{x \in R^1,\ |u_\varepsilon(x)| < \frac{3}{4}\} \subset \cup_{i \in \mathcal{L}_\varepsilon} B_i, \tag{7.37}$$

$$\overline{\cup_{i \in \mathcal{L}_\varepsilon} B_i(a_i, r_i)} \subset R^1, \tag{7.38}$$

$$\sum_{i \in \mathcal{L}_\varepsilon} r_i \leq C\,|\log\varepsilon|\, e^{-\sqrt{|\log\varepsilon|}}, \tag{7.39}$$

$$card(\mathcal{L}_\varepsilon) \leq C\,|\log\varepsilon|\, h_{ex}, \tag{7.40}$$

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i) = \frac{1}{2}\int_{B_i} |\nabla u_\varepsilon - i\,A_\varepsilon\,u_\varepsilon|^2 + \frac{1}{4\,\varepsilon^2}\int_{B_i}(1-|u_\varepsilon|^2)^2 \geq \pi\,|d_i|\,|\log\varepsilon|\,(1-o(1)), \tag{7.41}$$

*where $d_i$ is the degree of the map $\frac{u_\varepsilon}{|u_\varepsilon|}$ restricted to $\partial B_i$.*

## 4.1 Proof of proposition 7.8

First, letting $\Omega = [0,1[\times[0,2[$, $m_\varepsilon = \frac{1}{\sqrt{|\log\varepsilon|}}$ and $\alpha_\varepsilon = |\log\varepsilon|$ in the proposition 4.12, we have

**Lemma 7.9.** *For $h_{ex} \le C\ |\log\varepsilon|$, there exists $\varepsilon_0$ such that if $\varepsilon < \varepsilon_0$ and $(u_\varepsilon, A_\varepsilon)$ satisfies $|\nabla u_\varepsilon - i\ A_\varepsilon\ u_\varepsilon| < \frac{C}{\varepsilon}$ and $F_\varepsilon(u_\varepsilon, A_\varepsilon, \Omega) \le C\ |\log\varepsilon|^2$, then there exists a family of disjoint balls $\Big(B_i = B_i(a_i, r_i)\Big)_{i\in I_\varepsilon}$ of center $a_i$ and of radii $r_i$ such that*

$$\{x \in \Omega,\ |u_\varepsilon(x)| < \frac{3}{4}\} \subset \cup_{i\in I_\varepsilon} B_i, \tag{7.42}$$

$$\sum_{i\in I_\varepsilon} r_i \le C\ |\log\varepsilon|\ e^{-\sqrt{|\log\varepsilon|}}, \tag{7.43}$$

$$card(I_\varepsilon) \le C\ |\log\varepsilon|\ h_{ex}, \tag{7.44}$$

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i) \ge \pi\ |d_i|\ |\log\varepsilon|\ (1 - o(1)), \tag{7.45}$$

*where $d_i$ is the degree of the map $\frac{u_\varepsilon}{|u_\varepsilon|}$ restricted to $\partial B_i$ if $\overline{B_i} \subset \Omega$ and $d_i = 0$ otherwise.*

Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over $\mathcal{G}_\varepsilon$, then it is a critical point, so going back to (3.13), we have

$$|\nabla u_\varepsilon - i\ A_\varepsilon\ u_\varepsilon| \le \frac{C}{\varepsilon}. \tag{7.46}$$

We test the energy $J$ by the configuration $(1, 0)$, since it belongs to $\mathcal{G}_\varepsilon$. The minimum of the energy $J_\Omega$ is then less than $J_\Omega(1, 0) \le C\ h_{ex}^2 \le C\ |\log\varepsilon|\ h_{ex}$. Hence

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, \Omega) \le J_\Omega(u_\varepsilon, A_\varepsilon) \le C\ |\log\varepsilon|\ h_{ex}.$$

So, the hypotheses of lemma 7.9 are verified. Then, applying it, there exists a family of balls in $\Omega$ depending on $\varepsilon$ denoted by $(B_i)_{i\in I_\varepsilon} = \Big(B_i(a_i, r_i)\Big)_{i\in I_\varepsilon}$ such that the three assertions (7.43)-(7.44) and (7.45) hold.
We start by the proof of the assertion (7.38). Thanks to (7.43), we have

$$\sum_{i\in I_\varepsilon} r_i \le C\ |\log\varepsilon|\ e^{-\sqrt{|\log\varepsilon|}}.$$

Then, since $p_\varepsilon = \mathcal{O}(|\log\varepsilon|)$,

$$\sum_{i\in I_\varepsilon} r_i = o(\frac{1}{p_\varepsilon}).$$

Consequently, projecting the balls $\Big(B_i(a_i, r_i)\Big)_{i\in I_\varepsilon}$ on the horizontal line of equation $y = \frac{1}{2}$, then if $\varepsilon$ is sufficiently small there exists $0 < x_1 < 1$ such that the two lines of equations $x = x_1$ and $x = x_1 + \frac{1}{p_\varepsilon}$ don't intersect any ball of the family $\Big(B_i(a_i, r_i)\Big)_{i\in I_\varepsilon}$. Using the same argument, then if $\varepsilon$ is sufficiently small there exists $0 < y_1 < 1$ such that there is no intersection between the two lines of equation $y = y_1$ and $y = y_1 + 1$, and the balls $\Big(B_i(a_i, r_i)\Big)_{i\in I_\varepsilon}$. We define

$$\mathcal{L}_\varepsilon = \Big\{ i \in I_\varepsilon,\quad B_i(a_i, r_i) \subset R^1 = [x_1, x_1 + \frac{1}{p_\varepsilon}[\times[y, y_1 + 1[ \Big\}.$$

Note that the balls $\Big(B_i(a_i, r_i)\Big)_{i\in\mathcal{L}_\varepsilon}$ defined on $R^1$ are disjoint, since the initial balls $\Big(B_i = B(a_i, r_i)\Big)_{i\in I_\varepsilon}$ are disjoint. In addition, the lemma 7.9 implies in particular that the other assertions of proposition 7.8 hold. Without loss of generality, the rectangle $R^1$ is $[0, \frac{1}{p_\varepsilon}[\times[0,1[$.
Combining all the above completes the proof of proposition 7.8.

**Notation**

In the above proposition, the rectangle $R^1$ is $[0, \frac{1}{p_\varepsilon}[\times[0,1[$, then from now on we will take $K = [0,1[\times[0,1[$. Now, let us extend the balls $\Big(B_i(a_i, r_i)\Big)_{i\in\mathcal{L}_\varepsilon}$ by $R$ -periodicity to $K$. For simplification, the ball $B_i(a_i, r_i)$ defined on $R^1$ will be denoted

$$B_i(a_i, r_i) = B_i^1(a_i^1, r_i), \quad \forall\, i \in \mathcal{L}_\varepsilon.$$

Note that the rectangle $R^1$ can be taken as the fundamental domain of periodicity for $(u_\varepsilon, A_\varepsilon) \in \mathcal{G}_\varepsilon$. Then, for $i \in \mathcal{L}_\varepsilon$, we let $B_i^j(a_i^j, r_i)$, $1 \le j \le q_\varepsilon$ be the extended ball of $B_i^1(a_i^1, r_i)$ by $R$ -periodicity to the rectangle $R^j = [j-1, j[\times[0,1[$. Consequently, $\Big(B_i^j(a_i^j, r_i)\Big)_{(1\le j\le p_\varepsilon,\ i\in\mathcal{L}_\varepsilon)}$ is the family of the vortex balls defined on the square $K$.

## 4.2 Preliminaries

Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over $\mathcal{G}_\varepsilon$. Then, proposition 7.8 gives us the existence of the balls $\Big(B_i^1(a_i^j, r_i)\Big)_{i\in\mathcal{L}_\varepsilon}$ defined on $R^1$. Let us take

$$\mathbf{D}_\varepsilon := \sum_{i\in\mathcal{L}_\varepsilon} |d_i|. \tag{7.47}$$

Our interest now is to estimate, for an $\varepsilon$ small enough, the order of $\mathbf{D}_\varepsilon$. In particular, we give an upper bound for $\mathbf{D}_\varepsilon$.

**Lemma 7.10.** *For a sufficiently small $\varepsilon$, there exists $C > 0$ independently of $\varepsilon$ such that*

$$\mathbf{D}_\varepsilon \le C. \tag{7.48}$$

**Proof**: First, knowing $\overline{\cup_{i\in\mathcal{L}_\varepsilon} B_i^1} \subset R^1$, we have

$$J_{R^1}(u_\varepsilon, A_\varepsilon) \ge J_{\cup_{i\in\mathcal{L}_\varepsilon} B_i^1}(u_\varepsilon, A_\varepsilon). \tag{7.49}$$

It is immediate by $R$ -periodicity that

$$J_{R^1}(u_\varepsilon, A_\varepsilon) = \frac{J_K(u_\varepsilon, A_\varepsilon)}{p_\varepsilon}.$$

Then, inserting this together with (7.41) in (7.49), we obtain thanks to the fact $2\,\pi\, p_\varepsilon \simeq \alpha\, h_{ex}$ as $\varepsilon \to 0$

$$\pi\, \mathbf{D}_\varepsilon\, |\log\varepsilon|\, (1 - o(1)) \le \frac{h_{ex}^2}{2\, p_\varepsilon} \le C\, h_{ex}. \tag{7.50}$$

Since $h_{ex} \leq C \, |\log \varepsilon|$, (7.50) leads to

$$\mathbf{D}_\varepsilon \leq C.$$

$\square$

The inequality (7.48) gives us a bounded vorticity in each rectangle $R^j$, $1 \leq j \leq p_\varepsilon$. Now, knowing that $B_i^{j+1}$ is the ball image of $B_i^j$ by translation of vector $\frac{1}{p_\varepsilon}\vec{i}$, then we can easily prove for any $i \in \mathcal{L}_\varepsilon$

$$deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_i^j) = deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_i^{j+1}). \tag{7.51}$$

It is then clear that the degree of $\frac{u_\varepsilon}{|u_\varepsilon|}$ restricted to $\partial B_i^1$ where $i \in \mathcal{L}_\varepsilon$, is invariant under the $KR$ -periodicity to $\mathbb{R}^2$. In particular, (7.51) implies that the number of the vortices of $u_\varepsilon$ in $K$ is a multiple of $p_\varepsilon$.

**Remark 7.11.** *Let $(u_\varepsilon, A_\varepsilon) \in \mathcal{G}_\varepsilon$, then by definition of the spaces $\mathcal{G}_\varepsilon$ and $\mathcal{A}$, it is clear that $(u_\varepsilon, A_\varepsilon)$ belongs to the space $\mathcal{A}$. If $(u_\varepsilon, A_\varepsilon)$ is in addition a minimizer of the energy $J$ over $\mathcal{G}_\varepsilon$, then using (7.51) together with the fact that*

$$\overline{\cup_{\left(i\in\mathcal{L}_\varepsilon,\, 1\leq j\leq p_\varepsilon\right)} B_i^j(a_i^j, r_i)} \subset K,$$

*and proceeding similar to the corollary 4.4, we can deduce for a sufficiently small $\varepsilon$*

$$\int_K h_\varepsilon = 2\,\pi\,p_\varepsilon \sum_{i\in\mathcal{L}_\varepsilon} d_i.$$

## 4.3 Convergence of minimizing sequence

Consider $h_{ex}$ satisfying (7.1) and (7.2). Let $(u_\varepsilon, A_\varepsilon)$ be a sequence of minimizers of the Ginzburg-Landau energy $J$ over $\mathcal{G}_\varepsilon$ and $h_\varepsilon = curl A_\varepsilon$ be the induced magnetic field. For any such set of the balls $\left(B_i^j(a_i^j, r_i)\right)_{(i\in\mathcal{L}_\varepsilon,\, 1\leq j\leq p_\varepsilon)}$ defined on $K$ by proposition 7.8, we can associate to $u_\varepsilon$ the extended measure by $K$ -periodicity to $\mathbb{R}^2$ of $\frac{2\,\pi\,\sum_{i\in\mathcal{L}_\varepsilon} d_i \left(\sum_{j=1}^{p_\varepsilon} \delta_{a_i^j}\right)}{h_{ex}}$, denoted by $\nu_\varepsilon$. Using the fact that $J_K(u_\varepsilon, A_\varepsilon) \leq C\,h_{ex}^2$ in (3.10), we have

$$\frac{1}{2}\,\|h_\varepsilon - h_{ex}\|^2_{H^1(K)} \leq J_K(u_\varepsilon, A_\varepsilon) \leq C\,h_{ex}^2.$$

Then, $\frac{h_\varepsilon}{h_{ex}}$ is bounded in $H^1(K)$, so thanks to the $K$ -periodicity of $h_\varepsilon$, $\frac{h_\varepsilon}{h_{ex}}$ is bounded in $H^1(O)$ for each compact $O \subset \mathbb{R}^2$. In particular, it is bounded in $H^1_{loc}(\mathbb{R}^2)$. So, up an extraction we can find a subsequence $\varepsilon_n \to 0$ and there exists $f_0 \in H^1_{loc}(\mathbb{R}^2)$ such that

$$\frac{h_{\varepsilon_n}}{h_{ex}} \rightharpoonup f_0 \quad \text{weakly in } H^1_{loc}(\mathbb{R}^2). \tag{7.52}$$

Moreover, using again the $K$ -periodicity of $\frac{h_\varepsilon}{h_{ex}}$ implies that

$$f_0 \text{ is } K \text{ -- periodic.}$$

Now, by definition of the restriction of the Radon measure $\nu_\varepsilon$ on $K$, it is clear

$$\int_K |\nu_{\varepsilon_n}| = 2\,\pi\, p_{\varepsilon_n}\, \frac{\mathbf{D}_{\varepsilon_n}}{h_{ex}}.$$

Thanks to (7.48), $\left(\frac{p_{\varepsilon_n}\mathbf{D}_{\varepsilon_n}}{h_{ex}}\right)_n$ is bounded, hence by $K$ -periodicity to $\mathbb{R}^2$, we can write for any compact $O \subset \mathbb{R}^2$

$$\int_O |\nu_{\varepsilon_n}| \leq C.$$

Thus, $(\nu_{\varepsilon_n})_n$ is a bounded sequence of measures, and extracting again if necessary, we can assume that there exists a Radon measure $\nu_0$ on $\mathbb{R}^2$ such that

$$\nu_{\varepsilon_n} \rightharpoonup \nu_0.$$

Finally, proceeding similarly as in the proof of proposition 5.12, the relation between $\nu_0$ and $f_0$ is

$$\nu_0 = -\Delta f_0 + f_0 \quad \text{in } \mathbb{R}^2. \tag{7.53}$$

Now, let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J_K$ over the space $\mathcal{G}_\varepsilon$. Knowing that $\mathcal{G}_\varepsilon$ belongs to the space $\mathcal{A}$, hence in particular the lemma 5.13 yields

$$\liminf_{n\to\infty} \frac{J_K(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \geq \frac{\lambda}{2} \int_K |-\Delta f_0 + f_0| + \frac{1}{2} \int_K |\nabla f_0|^2 + \frac{1}{2} \int_K |f_0 - 1|^2 = E(f_0). \tag{7.54}$$

## 4.4 Properties of $f_0$ and $\nu_0$

Let us start with

**Lemma 7.12.** *$f_0$ is continuous on $\mathbb{R}^2$.*

**Proof**: First, referring to [SS5], lemma 4.1, we have

$$|\nabla f_0| \in W^{1,p}_{loc}(\mathbb{R}^2), \quad 1 \leq p < +\infty.$$

Thus, in particular

$$f_0 \in W^{1,p}_{loc}(\mathbb{R}^2), \quad 1 \leq p < +\infty.$$

By Sobolev injection, we conclude

$$f_0 \in C^{0,\alpha}_{loc}(\mathbb{R}^2), \quad 0 \leq \alpha < 1,$$

which completes the proof of lemma. □

**Proposition 7.13.** *The limit configuration $(f_0, \nu_0)$ verifies*

- *$x \to f_0(x,y)$ is constant.*
- *The restriction of $\nu_0$ on $K$ is concentrated on a finite number of horizontal lines .*
- *The mass of $\nu_0$ on each horizontal line belongs to $\alpha\,\mathbb{Z}$.*

**Remark 7.14.** *The case where $\nu_0 = 0$ is included in the result of the above proposition; it corresponds to the case where the measure $\nu_0$ is not concentrated on any horizontal line. Moreover, combining the fact that $\nu_0 = -\Delta f_0 + f_0$ in $\mathbb{R}^2$ together with the above proposition, it is clear from the definition of the space $\mathcal{U}$ defined by (7.6),*

$$f_0 \in \mathcal{U}.$$

**Proof**: **Step1:** $x \to f_0(x,y)$ **is constant**
We know that $f_\varepsilon(x+\frac{k}{p_\varepsilon}, y) = f_\varepsilon(x)$ for any integer $k$. Now, taking any real number $a$, there exists a sequence of integers $k_\varepsilon$ such that $\frac{k_\varepsilon}{p_\varepsilon} \to a$. We denote $t_\varepsilon$ the translation $(x,y) \to (x+\frac{k_\varepsilon}{p_\varepsilon}, y)$ and $t_a$ the translation $(x,y) \to (x+a,y)$. Then, taking $g$ any smooth compactly supported function and using change of variables

$$\int f_\varepsilon\, g = \int (f_\varepsilon \circ t_\varepsilon)\, g = \int f_\varepsilon\, (g \circ {t_\varepsilon}^{-1}).$$

$f_0$ is the limit of $f_\varepsilon$, hence passing to the limit we find

$$\int f_0\, g = \int f_0\, (g \circ {t_a}^{-1}) = \int (f_0 \circ t_a)\, g,$$

and therefore $f_0 = f_0 \circ t_a$. Step 1 is then proved.

**Step2: The restriction of $\nu_0$ on $K$ is concentrated on a finite number of horizontal lines such that the mass of $\nu_0$ on each line belongs to $\alpha\,\mathbb{Z}$**
The vortex balls $\left(B_i^j(a_i^j, r_i)\right)_{(i\in\mathcal{L}_\varepsilon,\, 1\le j\le p_\varepsilon)}$ defined on $K$ depends on $\varepsilon$, hence from now on, we write

$$d_i(\varepsilon) = d_i \quad \text{and} \quad a_i^j(\varepsilon) = a_i^j \quad \text{for} \quad i \in \mathcal{L}_\varepsilon \;\text{ and }\; 1 \le j \le p_\varepsilon,$$

where $d_i = deg\left(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_i^j(a_i^j, r_i)\right)$. First, for a sufficiently small $\varepsilon$, lemma 7.10 gives us $\mathbf{D}_\varepsilon = \sum_{i\in\mathcal{L}_\varepsilon} |d_i(\varepsilon)| \le C$. Thus, the cardinal of $\{i \in \mathcal{L}_\varepsilon,\;\; d_i(\varepsilon) \ne 0\}$ is bounded independently of $\varepsilon$. First, if for any $\varepsilon < \varepsilon_0$, $d_i(\varepsilon) = 0$, $\forall\, i \in \mathcal{L}_\varepsilon$. This means that for any $\varepsilon < \varepsilon_0$, $\mathbf{D}_\varepsilon = 0$ , so by definition of the measure $\nu_\varepsilon$, we have $\nu_\varepsilon = 0$. Then, the limit measure

$$\nu_0 = 0. \tag{7.55}$$

Second, if for a sufficiently small $\varepsilon$, there exist points with non zero degrees, then without loss of generality, there exists $m \in \mathbb{N}^*$ such that these points are denoted

$\{a_i^j(\varepsilon),\ \ 1 \le i \le m,\ \ 1 \le j \le p_\varepsilon\}$. Now, up to extraction from $\varepsilon \to 0$, we can get from the above that $\forall\ 1 \le i \le m$ there exist $q_i \in \mathbb{Z}$ and $b_i^1 \in R^1$ such that

$$d_i(\varepsilon_n) \to q_i, \quad \text{and} \;\; a_i^1(\varepsilon_n) \to b_i^1.$$

For simplification, let

$$\forall\, 1 \le i \le m, \quad b_i^1 = (x_i, y_i) \quad \text{where} \quad 0 < y_1 < ... < y_m < 1.$$

Note that $y_i$ is constant and does not depend on $\varepsilon$. The extended points of $(b_i^1)_{1\le i \le m}$ by $R$ -periodicity to $K$ are $\{b_i^j = (x_i + \frac{(j-1)}{p_\varepsilon}, y_i),\ \ 1 \le i \le m, \quad 1 \le j \le p_\varepsilon\}$. It is easy that $\forall\ 1 \le i \le m$, as $n \to +\infty$

$$\frac{\sum_{j=1}^{p_{\varepsilon_n}} \delta_{a_i^j(\varepsilon_n)}}{p_{\varepsilon_n}} \to \delta_{([0,1]\times\{y_i\})} \quad \text{in the sense of measures.} \tag{7.56}$$

Consequently, using $d_i(\varepsilon_n) \to q_i$ together with $\alpha\ h_{ex} \simeq 2\ \pi\ p_{\varepsilon_n}$ as $n \to +\infty$ in (7.56), we find for any $1 \le i \le m$

$$2\ \pi\ d_i(\varepsilon_n)\ \frac{\sum_{j=1}^{p_{\varepsilon_n}} \delta_{a_i^j}}{h_{ex}} \to \alpha\ q_i\ \delta_{([0,1]\times\{y_i\})} \quad \text{in the sense of measures.} \tag{7.57}$$

Finally, as $n \to +\infty$

$$2\ \pi \sum_{i=1}^{m} d_i(\varepsilon_n)\ \frac{\sum_{j=1}^{p_{\varepsilon_n}} \delta_{a_i^j}(\varepsilon_n)}{h_{ex}} \to \alpha \sum_{i=1}^{m} q_i\ \delta_{([0,1]\times\{y_i\})} \quad \text{in the sense of measures.} \tag{7.58}$$

We define $\Sigma^i$ to be the horizontal line contained in $K$ and of equation $y = y_i$. Hence, $\{\Sigma^i,\ 1 \le i \le m,\}$ is the family of the $m$ horizontal disjoint lines where the restriction of the limit measure $\nu_0$ on $K$ concentrates. The left hand-side of (7.58) is the restriction of the measure $\nu_\varepsilon$ on $K$, hence we can conclude that the restriction of the measure $\nu_0$ on $K$ is equal to $\sum_{i=1}^m \alpha\ \ q_i\ \delta_{\Sigma^i}$. The mass of the limit measure $\nu_0$ on the line $\Sigma^i$ is equal to $(\alpha\ q_i)$. The conclusion from this and (7.55) is that the mass of $\nu_0$ on the horizontal lines which are contained on $K$ belongs to $\alpha\ \mathbb{Z}$. This completes the proof of proposition 7.13. □

Now, under some condition relaying $\lambda$ ( and then the applied field) to the parameter $\alpha$, we state a fundamental property for the limit measure $\nu_0$.

**Lemma 7.15.** *If* $1 - \frac{\lambda}{2} > \alpha\ \frac{e+1}{4\ (e-1)}$*, we have* $\nu_0 \ne 0$.

**Proof**: We argue by contradiction and we suppose that $\nu_0 = 0$. First, let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over the space $\mathcal{G}_\varepsilon$, then going back to (7.54) and using the fact that $\nu_0 = -\Delta f_0 + f_0$, we have

$$\liminf_{n\to\infty} \frac{J_K(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \ge \frac{1}{2} \int_K |\nabla f_0|^2 + \frac{1}{2} \int_K |f_0 - 1|^2, \tag{7.59}$$

where $\Delta f_0 + f_0 = 0$ in $\mathbb{R}^2$. In this case, we have $f_0 = 0$. Inserting this in (7.59), we get

$$\liminf_{n\to\infty} \frac{J_K(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \ge \frac{1}{2}. \tag{7.60}$$

However, if we choose $1 - \frac{\lambda}{2} > \alpha\ \frac{e+1}{4\,(e-1)}$, then the lemma 7.6 gives us

$$\limsup_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} < \frac{1}{2}.$$

Comparing this to (7.60), we get a contradiction. Consequently, $\nu_0 \neq 0$. □

Now, under the hypotheses of lemma 7.15, we have obtained that the limit measure of vorticity $\nu_0$ verifies $\nu_0 \neq 0$, which allows to say that the restriction of $\nu_0$ on $K$ concentrates on at least one horizontal line. From now on, we restrict to the case where the restriction of the limit measure $\nu_0$ on $K$ is supported exactly on one horizontal line.

# 5 Vortices's concentration on one horizontal line

In this paragraph, we assume that $1 - \frac{\lambda}{2} > \alpha\ \frac{e+1}{4\,(e-1)}$. If the restriction of the limit measure $\nu_0$ on $K$ is supported exactly on one horizontal line, then $\nu_0$ is written on $K$ as

$$\nu_0 = \alpha\ d\ \ \delta_\Sigma, \tag{7.61}$$

where $d \in \mathbb{Z}^*$ and $y = y_1 \in ]0,1[$ is the equation of the horizontal line $\Sigma$. Note that the mass of $\nu_0$ on $\Sigma$ is $\alpha\ d$. My interest is then to give for certain applied fields, the value of $d$.

**Lemma 7.16.** *If in addition*

$$1 - \frac{\lambda}{2} \leq \alpha\ \frac{e+1}{2\,(e-1)}, \tag{7.62}$$

*then the $d$ defined by (7.61) is equal to 1. Moreover, letting $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over the space $\mathcal{G}_\varepsilon$, we get*

$$\lim_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} = \frac{1}{2} - (1 - \frac{\lambda}{2})\ \ \alpha + \frac{e+1}{4\,(e-1)}\ \ \alpha^2.$$

**Proof**: From remark 7.14, $f_0 \in \mathcal{U}$, then from (7.61) which is

$$\nu_0 = -\Delta f_0 + f_0 = \alpha\ d\ \ \delta_\Sigma \quad \text{on} \quad K,$$

it is clear, $f_0 \in \mathcal{U}_1$. In particular, from lemma 7.5, we have by definition of the functional $F$ given by (7.33)

$$E(f_0) = F(d) = \frac{1}{2} - \alpha\ (1 - \frac{\lambda}{2})\ d + \alpha^2\ \frac{1+e}{4\,(e-1)}\ d^2. \tag{7.63}$$

Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J$ over the space $\mathcal{G}_\varepsilon$. Going back to (7.54),

$$\liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \geq E(f_0) = F(d). \tag{7.64}$$

Now, let us take in particular $n_1 = 1$ in (7.34) to find

$$\limsup_{\varepsilon\to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \leq F(n_1) = F(1). \tag{7.65}$$

Combining (7.64) together with (7.65), we have for $d \in \mathbb{Z}^*$

$$F(d) \leq \liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \leq \limsup_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \leq F(1). \tag{7.66}$$

Let $d_*$ be the minimum of $F$ over $\mathbb{Z}$. Since $x \to F(x)$ is convex in $\mathbb{R}$, the minimum of $F$ over $\mathbb{R}$ is achieved at

$$x_{min} = \frac{e-1}{e+1}\ \frac{2-\lambda}{\alpha}. \tag{7.67}$$

Having $1 - \frac{\lambda}{2} > \alpha\ \frac{e+1}{4\,(e-1)}$, hence $x_{min} > \frac{1}{2}$. Now, let us take in addition

$$1 - \frac{\lambda}{2} \leq \alpha\ \frac{e+1}{2\,(e-1)}.$$

Inserting this in (7.67), we get $x_{min} \leq 1$. Consequently, under the assumptions (7.35) and (7.62), $\frac{1}{2} < x_{min} \leq 1$. This implies that the unique minimum of $F$ over $\mathbb{Z}$ is $d_* = 1$. Inserting this in (7.66) to find

$$F(1) = F(d_*) \leq F(d) \leq \liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \leq \limsup_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \leq F(1) = F(d_*).$$

In particular, we get $F(d) = F(1) = F(d_*)$. In view of the uniqueness of the minimum $d_* = 1$ of the functional $F$ over $\mathbb{Z}$, we obtain $d = 1$. We deduce using (7.63),

$$\lim_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} = F(1) = \frac{1}{2} - (1 - \frac{\lambda}{2})\ \alpha + \frac{e+1}{4\,(e-1)}\ \alpha^2.$$

Finally, inserting $d = 1$ in (7.61), the restriction of the limit measure $\nu_0 = -\Delta f_0 + f_0$ on $K$ is written as $\alpha\ \ \delta_\Sigma$ where $\Sigma$ is an arbitrary horizontal line. Note that the mass of the measure $\nu_0$ on $\Sigma$ is equal to $\alpha$. □

As a consequence, combining all the above, the main result that we have proved along the chapter is stated in the following.

**Theorem 7.17.** *A- Convergence:*
*Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J$ over the space $\mathcal{G}_\varepsilon$ and $h_\varepsilon = curl A_\varepsilon$ be the associated magnetic field. Then, letting $\nu_\varepsilon$ be the extended measure by $K$ -periodicity to $\mathbb{R}^2$ of $\frac{2\,\pi\,\sum_{i\in\mathcal{L}_\varepsilon} d_i \left(\sum_{j=1}^{p_\varepsilon} \delta_{a_i^j}\right)}{h_{ex}}$, there exist a $K$ -periodic $f_0 \in H^1_{loc}(\mathbb{R}^2)$ and a Radon measure $\nu_0$ on $\mathbb{R}^2$ such that up an extraction of $\varepsilon_n$ from $\varepsilon$*

$$\frac{h_{\varepsilon_n}}{h_{ex}} \to f_0 \quad \text{weakly in } \ H^1_{loc}(\mathbb{R}^2). \tag{7.68}$$

$$\nu_{\varepsilon_n} \rightharpoonup \nu_0 = -\Delta f_0 + f_0. \tag{7.69}$$

*B-Properties of $(f_0, \nu_0)$:*
*We have $f_o \in \mathcal{U}$. Moreover, if*

$$1 - \frac{\lambda}{2} > \alpha\ \frac{e+1}{4\,(e-1)}$$

*we have* $\nu_0 \neq 0$. *In addition, when the restriction of* $\nu_0$ *on* $K$ *concentrates on one horizontal line, then if*

$$1 - \frac{\lambda}{2} \leq \alpha \, \frac{e+1}{2\,(e-1)}$$

*we have*

$$\lim_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} = \frac{1}{2} - (1 - \frac{\lambda}{2}) \;\; \alpha + \frac{e+1}{4\,(e-1)} \;\; \alpha^2,$$

*and the restriction of* $\nu_0$ *on* $K$ *is equal to* $(\alpha \; \delta_\Sigma)$ *where* $\Sigma$ *is any arbitrary horizontal line.*

**Remark 7.18.** *In the above Theorem, we don't study the case where the vortices contained in* $K$ *are concentrated on more than one horizontal line. It is rather difficult to obtain a concentration of vortices on* $m \geq 2$ *horizontal lines. The first step is to determine the expression of the functional defined by (7.29) only in function of the family* $(n_i, y_i)_{1 \leq i \leq m}$ *where* $n_i \in \mathbb{Z}$ *for any* $1 \leq i \leq m$ *and* $0 < y_1 < ... < y_m < 1$. *The second step consists in the minimization of this expression among all the configurations* $(n_i, y_i)_{1 \leq i \leq m}$ *defined above. Unfortunately, this minimization is not easy to study, so we can not give explicitly the limit measure of vorticity. This explains the fact that we only consider the case* $m = 1$, *which corresponds to one horizontal line of vortices in* $K$.

# Chapter 8

# Vortices's concentration along one circle

In this chapter, the domain is taken to be the disk $B_R$ of center O and of radius $R > 0$. We then construct a sequence $(u_\varepsilon, A_\varepsilon)_{\varepsilon>0}$ of critical points of the energy $J_{B_R}$, such that in the limit $\varepsilon \to 0$ and under some condition on the applied magnetic field $h_{ex}$, the vortices of $(u_\varepsilon, A_\varepsilon)$ are supported on a finite number of concentric circles of center O and of strict positive radii (at least, there is concentration on one circle). In particular, if the limit measure of vorticity is concentrated exactly on one circle such that its mass is known, we will characterize this circle of vorticity by giving its radius which will be the solution of a minimization problem.

## 1 Statement of the problem

### 1.1 Purpose of the chapter

Let $\Omega$ be a bounded, regular and simply connected domain in $\mathbb{R}^2$. Let $(u, A)$ denote a critical point of the energy $J_\Omega$ and $h$ the magnetic field will denote $curlA$. Then from [SS5], $(u, A)$ is a solution of the Ginzburg-Landau equations, namely

$$\begin{cases} \nabla_A^2 u = \frac{1}{\varepsilon^2}\, u(1-|u|^2) & \text{in } \Omega \\ -\nabla^\perp h = < i\, u, \nabla_A u > & \text{in } \Omega \end{cases} \tag{8.1}$$

with the boundary conditions on $\partial\Omega$

$$\begin{cases} h = h_{ex} \\ \nabla_A u.\nu = 0, \end{cases} \tag{8.2}$$

where $\nu$ is the unit outward normal to the boundary $\partial\Omega$. We take $H_1^1(\Omega)$ to be the space of functions $f$ in $H^1(\Omega)$ such that $f = 1$ on the boundary $\partial\Omega$. Let again $BV(\Omega)$ be the space of functions with bounded variations on $\Omega$. In the sequel, $\mathcal{M}(\Omega)$ will be the set of Radon measures on $\Omega$.
Now, we give a result of Sandier and Serfaty that describes the asymptotic behavior

of critical points of $J$ when $\varepsilon \to 0$.
**Theorem [SS5]:** Let $\varepsilon_n \to 0$ and $(u_n, A_n)$ be critical points of $J_\Omega$ with $J_\Omega(u_n, A_n) \leq C\ h_{ex}^2$ and $h_{ex} \leq C\ |\log \varepsilon|$. Then, up to extraction of a subsequence, there exist $h_\infty \in H_1^1(\Omega)$ and $\mu_\infty \in \mathcal{M}(\Omega)$ such that

$$\frac{h_n}{h_{ex}} \to h_\infty \quad \text{weakly in } H_1^1(\Omega),$$

and

$$\mu_n = \frac{2\ \pi\ \sum_{i\in I} d_i\ \delta_{a_i}}{h_{ex}} \to \mu_\infty = -\Delta h_\infty + h_\infty \quad \text{in the sense of measures,}$$

where $\{(a_i, d_i)_{i\in I}\}$ is the family of vortices defined by proposition 4.12. Moreover, $h_\infty$ is stationary with respect to inner variations for the functional

$$L(f) = \frac{1}{2} \int_\Omega |\nabla f|^2 + |f|^2,$$

defined over $H_1^1(\Omega)$. If $\nabla h_\infty$ is continuous on $\Omega$ and $|\nabla h_\infty| \in BV(\Omega)$, then

$$\begin{cases} h_\infty \in C^{1,\alpha}(\Omega, \mathbb{R}) \\ h_\infty = 1 \qquad \text{on } \partial\Omega \\ 0 \leq h_\infty \leq 1 \\ \mu_\infty = h_\infty\ \mathbf{1}_{|\nabla h_\infty|=0}. \end{cases} \tag{8.3}$$

Thus, $\mu_\infty$ is a nonnegative $L^\infty$ function and $\mu_\infty \ll dx$ holds.

In the above Theorem, there is unfortunately no way that ensures $\mu_\infty \ll dx$ is true, unless we know that $\nabla h_\infty$ is continuous and $|\nabla h_\infty| \in BV(\Omega)$. $\mu_\infty$ could be a measure that concentrates on lines (since it has to belong to $H^{-1}$). Yet, the above Theorem only asserts that $|\nabla h_\infty|$ is continuous, but not necessarily $\nabla h_\infty$. There are counter-examples of $(h_\infty, \mu_\infty)$ satisfying these conclusions with $\nabla h_\infty$ discontinuous, thus without $\mu_\infty \ll dx$.

We state now a counter example. We restrict ourselves to the case of the disk domain $\Omega = B(\mathrm{O}, R)$ such that $R > 0$ is the radius and O is the center. Taking $R_1 < R$, let us solve

$$\begin{cases} -\Delta h_1 + h_1 = 0 \quad \text{in } B(\mathrm{O}, R_1) \\ \\ h_1 = 1 \quad \text{on } \partial B(\mathrm{O}, R_1), \end{cases} \tag{8.4}$$

and

$$\begin{cases} -\Delta h_2 + h_2 = 0 \quad \text{in } B(\mathrm{O}, R)\backslash\overline{B(\mathrm{O}, R_1)} \\ \\ h_2 = 1 \quad \text{on } \partial B(\mathrm{O}, R) \cup \partial B(\mathrm{O}, R_1). \end{cases} \tag{8.5}$$

The two functions $h_1$ and $h_2$ are radial, and we can adjust $R_1$ and $R$ (see remark 8.9) in such a way that

$$\frac{\partial h_1}{\partial r}(R_1) = -\frac{\partial h_2}{\partial r}(R_1). \tag{8.6}$$

We define $h$ as $h_1$ in $B(\mathrm{O}, R_1)$ and $h_2$ in $B(\mathrm{O}, R)\backslash B(\mathrm{O}, R_1)$, then $h$ is in $H^1(B(\mathrm{O}, R))$. Moreover, $\nabla h$ is discontinuous on $\partial B(\mathrm{O}, R_1)$, while $|\nabla h|$ remains continuous. The measure $\mu = -\Delta h + h$ is positive and it is supported on $\partial B(\mathrm{O}, R_1)$, thus $\mu \ll dx$ does not hold. Nothing allows us to exclude that there are sequences of critical points converging to such limiting configurations. They would correspond to solutions with vortices of positive degrees concentrated along the circle $\partial B(\mathrm{O}, R_1)$. These sequences of critical points are constructed by minimizing the energy $J$ over an appropriate space.

## 1.2 Definitions

In this chapter, the domain is taken to be the disk $B_R = B(\mathrm{O}, R)$ where $R > 0$ is its radius and O is its center. Assume that $\lambda = \lim_{\varepsilon\to 0} \frac{|\log \varepsilon|}{h_{ex}}$ exists, is finite and does not vanish. We define $q_\varepsilon \in \mathbb{N}$ to be a function of $\varepsilon$ such that the following limit exists, is finite and does not vanish too

$$\beta = \lim_{\varepsilon\to 0} \frac{q_\varepsilon}{h_{ex}}. \tag{8.7}$$

Note that when it is not necessary, we will write $J$ instead of $J_{B_R}$. The natural space where we perform the minimization of the energy $J_{B_R}$ is denoted by $G_\varepsilon$ and it is defined as follows.

**Definition 8.1.** *Let $(u, A) \in H^1(B_R, \mathbb{C}) \times H^1(B_R, \mathbb{R}^2)$, then $(u, A)$ belongs to the space $G_\varepsilon$ if there exists $f \in H^2(B_R, \mathbb{C})$ such that for any $x \in B_R$*

$$u\Big(x\, e^{i\,\frac{2\,\pi}{q_\varepsilon}}\Big) = u(x)\, e^{i\, f(x)}, \tag{8.8}$$

*and*

$$A\Big(x\, e^{i\,\frac{2\,\pi}{q_\varepsilon}}\Big) = e^{i\frac{2\,\pi}{q_\varepsilon}}\, A(x) + e^{i\,\frac{2\,\pi}{q_\varepsilon}}\, \nabla f(x). \tag{8.9}$$

Now, let us choose the following gauge named the Coulomb gauge

$$\begin{cases} divA = 0 & \text{in } B_R \\ A.\nu = 0 & \text{on } \partial B_R. \end{cases} \tag{8.10}$$

In the presence of this gauge, we can check that the infimum of $J$ over the space $G_\varepsilon$ is achieved. Without loss of generality, we denote by $(u_\varepsilon, A_\varepsilon)_{\varepsilon>0}$ a sequence of minimizers. Then, it is a critical point, hence a solution of the Ginzburg-Landau equations (8.1) and (8.2). Let $h_\varepsilon = curlA_\varepsilon$ be the induced field. We restrict our attention to the asymptotic behavior of minimizers $(u_\varepsilon, A_\varepsilon)$ as $\varepsilon \to 0$ and we explore the vortex-structure of minimizers which will be obtained by getting first an upper bound on the minimal energy $J(u_\varepsilon, A_\varepsilon)$, and then a lower bound.

# 2 Construction of an upper bound

In this section, letting $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over the space $G_\varepsilon$, we are interested in giving an upper bound for the "renormalized" energy $\frac{J(u_\varepsilon,A_\varepsilon)}{h_{ex}^2}$.

## 2.1 Preliminaries

Recall that the parameter $\beta$ is defined by (8.7). Then, we start by giving the space

$$Y := \left\{ \begin{array}{l} f \in H_1^1(B_R,\mathbb{R}) \quad \text{such that } f \text{ is radial and } \mu = -\Delta f + f \text{ is supported on} \\ \text{a finite number of concentric circles of center O and the mass of } \mu \text{ on each} \\ \text{one belongs to } 2\,\pi\,\beta\,\mathbb{Z} \end{array} \right\} \tag{8.11}$$

**Remark 8.2.** *we have for any $f \in Y$, $f \in H_1^1(B_R,\mathbb{R})$. Then, the measure $\mu = -\Delta f + f$ belongs to $H^{-1}$, so it does not concentrate on isolated points (in particular the point O, the center of the disk $B_R$). Hence, the finite number of concentric circles where $\mu$ concentrates have strict positive radii. In addition, for $f \in Y$, we remark that $f$ is continuous, but $\nabla f$ is not continuous.*

First, any $f \in Y$ is solution of

$$\begin{cases} -\Delta f + f = \mu \ \text{ in } \ B_R \\ f = 1 \quad \text{on } \ \partial B_R. \end{cases} \tag{8.12}$$

Then, $\forall\, x \in B_R$

$$(f-1)(x) = \int_{B_R} G(x,y)\; d(\mu - 1)(y),$$

where $G$ is the Green solution of

$$\begin{cases} -\Delta_x G(x,y) + G(x,y) = \delta_y \quad \text{in } \ B_R \\ G(x,y) = 0 \quad x \in \partial B_R. \end{cases} \tag{8.13}$$

Recall that the functional $E$, defined over $Y$, is

$$E(f) = \frac{\lambda}{2}\int_{B_R} |-\Delta f + f| + \frac{1}{2}\int_{B_R} |\nabla f|^2 + \frac{1}{2}\int_{B_R} |f-1|^2. \tag{8.14}$$

We refer to [SS3], proposition 2.1 to get for any $f \in Y$

$$E(f) = I(\mu) = \frac{\lambda}{2}\int_{B_R} |\mu| + \frac{1}{2}\int_{B_R\times B_R} G(x,y)\; d(\mu-1)(x)\; d(\mu-1)(y), \quad \mu = -\Delta f + f. \tag{8.15}$$

## 2.2 The upper bound

**Proposition 8.3.** *Consider $h_{ex} \leq C\,|\log \varepsilon|$. Let $\mu$ be any Radon measure invariant by rotation and concentrated on a finite number of concentric circles of center O and of strict positive radii such that the mass of $\mu$ on each one belongs to $2\,\pi\,\beta\,\mathbb{N}$. Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J$ over the space $G_\varepsilon$, then*

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \leq I(\mu) = \frac{\lambda}{2} \int_{B_R} |\mu| + \frac{1}{2} \int_{B_R \times B_R} G(x,y)\, d(\mu - 1)(x)\, d(\mu - 1)(y). \tag{8.16}$$

Thanks to (8.15), proposition 8.3 can be stated differently

**Corollary 8.4.** *If $\lambda > 0$, then for any $f \in Y$ with $(-\Delta f + f)$ is positive, we have*

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \leq E(f). \tag{8.17}$$

*Proof.* First, the fact that $\lambda > 0$ means that $h_{ex} \leq C\,|\log \varepsilon|$. Let $f \in Y$, then by definition of the space $Y$, the measure $\mu = -\Delta f + f$ is invariant by rotation and it is concentrated on a finite number of concentric circles of center O. Now, thanks to the remark 8.2, these concentric circles have strict positive radii. Moreover, $\mu = -\Delta f + f$ is taken to be positive, so the mass of $\mu$ on each concentric circle belongs to $2\,\pi\,\beta\,\mathbb{N}$. Combining all the above, the proposition 8.3 implies

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \leq I(\mu).$$

Therefore, (8.15) leads to

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \leq E(f).$$

□

## 2.3 Proof of proposition 8.3

Suppose that the assumptions of proposition 8.3 hold, then without loss of generality, we assume that the measure $\mu$ is supported on $m$ concentric circles denoted by $(\Gamma_i)_{1 \leq i \leq m}$, of center O and of strict positive radii. The mass of $\mu$ on each circle belongs to $2\,\pi\,\beta\,\mathbb{N}$, hence there exist $(r_i)_{1 \leq i \leq m}$ with $0 < r_1 < r_2 < \ldots < r_m < R$ and $(m_i)_{1 \leq i \leq m}$ with $m_i \in \mathbb{N}$ for $1 \leq i \leq m$ such that

$$\int_{\Gamma_i} \mu = 2\,\pi\;\beta\;m_i.$$

Note that $r_i$ is taken to be the radius of the circle $\Gamma_i$. From the concentration of the measure $\mu$ on the $m$ concentric circles $\Big(\Gamma_i\Big)_{1 \leq i \leq m}$, $\int_{B_R} \mu$ can be written as

$$\int_{B_R} \mu = \int_{\cup_{i=1}^m \Gamma_i} \mu = \sum_{i=1}^m \int_{\Gamma_i} \mu.$$

It follows that

$$\int_{B_R} \mu = 2\ \pi\ \ \beta\ \sum_{i=1}^m\ m_i.$$

Then, it is clear that the measure $\mu$ is given as

$$\mu = \beta\ \sum_{i=1}^m \frac{m_i}{r_i}\ \delta_{\Gamma_i}, \tag{8.18}$$

where $\delta_{\Gamma_i}$ is the measure of arclength along $\Gamma_i$.
The upper bound (8.16) is obtained by a construction of a test configuration $(v_\varepsilon, B_\varepsilon)$ in the the space $G_\varepsilon$. For this, we need to describe the vortices of $(v_\varepsilon, B_\varepsilon)$. We decompose the proof of the proposition 8.3 into five steps.
**Step1**
We consider the sequence $q_\varepsilon$ defined by (8.7). Let $S_j$ be the sector

$$S_j = \{r\ e^{i\,\theta}, \quad 0 \le r < R, \quad \theta \in [\frac{2\ \pi\ (j-1)}{q_\varepsilon}, \frac{2\ \pi\ j}{q_\varepsilon}[, \ \ 1 \le j \le q_\varepsilon\}.$$

First, we place in the sector $S_1$ the points

$$(a_1^k)_{1 \le k \le m} = \left(r_k\ e^{i\ \frac{\pi}{q_\varepsilon}}\right)_{1 \le k \le m},$$

where $\{r_k,\ 1 \le k \le m\}$ are the radii of the circles where the measure $\mu$ concentrates. Then, by rotation of center O and of angle $\frac{2\,\pi}{q_\varepsilon}$, we extend these points to the ball $B_R$. In particular, the extended points of $(a_1^k)_{1 \le k \le m}$ to the sector $S_j$, $1 \le j \le q_\varepsilon$, are denoted

$$(a_j^k)_{1 \le k \le m} = \left(r_k\ e^{i\ 2\ \pi\ \frac{j-\frac{1}{2}}{q_\varepsilon}}\right)_{1 \le k \le m}.$$

We deduce that there are $(m\ q_\varepsilon)$ points in the ball $B_R$ which are

$$\left(a_j^k\right)_{1 \le k \le m,\ 1 \le j \le q_\varepsilon} = \left(r_k\ e^{i\ 2\ \pi\ \frac{j-\frac{1}{2}}{q_\varepsilon}}\right)_{1 \le k \le m,\ \ 1 \le j \le q_\varepsilon}. \tag{8.19}$$

We define for any $0\ \le j\ \le q_\varepsilon$

$$\Sigma_j = \{r\ e^{i\ \frac{2\,\pi\,j}{q_\varepsilon}}, \quad 0 \le r < R\}.$$

Remark that the boundary of $S_j$ is

$$\partial S_j = (\partial B_R \cap \overline{S_j}) \cup \Sigma_{j-1} \cup \Sigma_j, \ \ 1\ \le j\ \le q_\varepsilon.$$

From now on, we say that a function $T$ is $S$ -periodic means

$$T(x\ e^{i\ \frac{2\,\pi}{q_\varepsilon}}) = T(x), \quad x \in B_R.$$

Note that we pass from the sector $S_j$ to $S_{j+1}$ by a rotation of center O and of angle $\frac{2\,\pi}{q_\varepsilon}$. Now, we define the measure $\mu_\varepsilon$

$$\mu_\varepsilon = \frac{2\,\pi}{h_{ex}} \sum_{k=1}^{m} \Big( m_k \sum_{i=1}^{q_\varepsilon} \delta_{a_i^k} \Big), \tag{8.20}$$

where $m_k$ and $a_i^k$ are defined respectively by (8.18) and (8.19). Now, let $1 \leq k \leq m$ be fixed, then it is clear that as $\varepsilon \to 0$,

$$\frac{\sum_{i=1}^{q_\varepsilon} \delta_{a_i^k}}{q_\varepsilon} \to \frac{1}{2\,\pi\, r_k} \delta_{\Gamma_k} \quad \text{in the sense of measures,}$$

where $\Gamma_k$ is the circle of center O and of radius $r_k$. Consequently, using the fact that $\beta\, h_{ex} \simeq q_\varepsilon$ as $\varepsilon \to 0$,

$$2\,\pi\, m_k \frac{\sum_{p=1}^{q_\varepsilon} \delta_{a_i^k}}{h_{ex}} \to \beta\, \frac{m_k}{r_k}\, \delta_{\Gamma_k}. \tag{8.21}$$

It follows that as $\varepsilon \to 0$,

$$\sum_{k=1}^{m} 2\,\pi\, m_k \Big( \frac{\sum_{p=1}^{q_\varepsilon} \delta_{a_i^k}}{h_{ex}} \Big) \to \beta \sum_{k=1}^{m} \frac{m_k}{r_k}\, \delta_{\Gamma_k} \quad \text{in the sense of measures.} \tag{8.22}$$

Then, by definition of the measures $\mu_\varepsilon$ and $\mu$, we deduce from (8.22), as $\varepsilon \to 0$,

$$\mu_\varepsilon \to \mu \quad \text{in the sense of measures.}$$

**Step2**

Here, thanks to [SS3], proposition 2.2, we can state

$$\limsup_{\varepsilon\to 0} \frac{1}{2} \int\int_{B_R \times B_R} G(x,y)\, d(\mu_\varepsilon - 1)(x)\, d(\mu_\varepsilon - 1)(y) \leq \frac{\lambda}{2}\, \mu(B_R) + \int_{B_R}\int_{B_R} G(x,y)\, d(\mu-1)(x)\, d(\mu-1)(y). \tag{8.23}$$

**Step3**

Now, we construct a test configuration $(v_\varepsilon, B_\varepsilon)$ to be in $G_\varepsilon$. First, we construct a function $h_\varepsilon$ $S$ -periodic. Indeed, let $h_\varepsilon$ be the unique solution of

$$\begin{cases} -\Delta h_\varepsilon + h_\varepsilon = \sum_{k=1}^{m}\ 2\,\pi\, m_k\, \delta_{a_1^k} & \text{in } S_1 \\ h_\varepsilon = h_{ex} & \text{on } \overline{S_1} \cap \partial B_R \\ \frac{\partial h_\varepsilon}{\partial \nu} = 0 & \text{on } \Sigma_0 \cup \Sigma_1, \end{cases}$$

where the points $\left(a_1^k\right)_{1\leq k\leq m}$ are defined by (8.19). Because, we have set $\frac{\partial h_\varepsilon}{\partial \nu} = 0$ on $\Sigma_0 \cup \Sigma_1$, and thanks to the fact that $h_\varepsilon$ has the symmetry of the sector $S_1$, the extended $h_\varepsilon$ by $S$ -periodicity to the ball $B_R$ necessarily verifies

$$\begin{cases} -\Delta h_\varepsilon + h_\varepsilon = h_{ex}\, \mu_\varepsilon & \text{in } B_R \\ h_\varepsilon = h_{ex} & \text{on } \partial B_R, \end{cases} \tag{8.24}$$

where $\mu_\varepsilon$ is defined by (8.20). In particular, we obtain

$$h_\varepsilon\Big(x\, e^{i\frac{2\,\pi}{q_\varepsilon}}\Big) = h_\varepsilon(x), \quad x \in B_R.$$

$h_\varepsilon$ is taken as the magnetic field. Having defined $h_\varepsilon$ on $B_R$, we let $B_\varepsilon$ be a solution of $curl B_\varepsilon = h_\varepsilon$. $B_\varepsilon$ is taken to be the magnetic potential. Furthermore, we define the function $\phi_\varepsilon$ only modulo $2\,\pi$ where $\rho_\varepsilon \neq 0$. Set $x_0 \in B_R \backslash \Big[\bigcup_{(1\le k\le m,\ 1\le j\le q_\varepsilon)}\Big(B(a_j^k, \varepsilon)\Big)\Big]$ and the function

$$\phi_\varepsilon(x) = \oint_{(x_0,x)} e^{-i\,\frac{4\,\pi}{q_\varepsilon}}\ B_\varepsilon.\tau - \nabla h_\varepsilon.\nu, \tag{8.25}$$

where $(x_0, x)$ is any curve joining $x_0$ to $x$ in $B_R \backslash \Big[\bigcup_{(1\le k\le m,\ 1\le j\le q_\varepsilon)}\Big(B(a_j^k, \varepsilon)\Big)\Big]$. Let us then choose $\rho_\varepsilon$ such that $0 \le \rho_\varepsilon \le 1$, $\rho_\varepsilon = 0$ in $\bigcup_{1\le k\le m}\Big(B(a_1^k, \varepsilon)\Big)$, $\rho_\varepsilon = 1$ in $S_1 \backslash \Big(\cup_{1\le k\le m} B(a_1^k, 2\,\varepsilon)\Big)$, and $\rho_\varepsilon = \frac{|x - a_1^k|}{\varepsilon} - 1$ otherwise. We may extend $\rho_\varepsilon$ by $S$ -periodicity to $B_R$, hence

$$\rho_\varepsilon\Big(x\, e^{i\frac{2\,\pi}{q_\varepsilon}}\Big) = \rho_\varepsilon(x) \quad \forall\, x \in B_R.$$

Similar to the proof of proposition 5.8, step 3, $e^{i\,\phi}$ is well defined, so let us take

$$v_\varepsilon = \rho_\varepsilon\, e^{i\,\phi_\varepsilon}.$$

**Step4**
Here, we prove

**Lemma 8.5.** *The test configuration* $\Big(v_\varepsilon, B_\varepsilon\Big)$ *belongs to the space* $G_\varepsilon$.

**Proof**: First, thanks to (8.24)

$$h_\varepsilon\Big(x\, e^{i\,\frac{2\,\pi}{q_\varepsilon}}\Big) = h_\varepsilon(x) \quad \forall\, x \in B_R. \tag{8.26}$$

The magnetic potential $B_\varepsilon \in H^1(B_R, \mathbb{R}^2)$ is taken to solve $curl B_\varepsilon = h_\varepsilon$. For simplification, set

$$b_\varepsilon = e^{i\,\frac{2\,\pi}{q_\varepsilon}}.$$

Then, (8.26) becomes $h_\varepsilon(b_\varepsilon\ x) = h_\varepsilon(x)$. We replace $h_\varepsilon$ with $curl B_\varepsilon$ in (8.26) to obtain

$$(curl B_\varepsilon)(b_\varepsilon\ x) = (curl B_\varepsilon)(x).$$

But,

$$(curl B_\varepsilon)(b_\varepsilon\ x) = \frac{1}{b_\varepsilon}\ curl\Big(B_\varepsilon(b_\varepsilon\ x)\Big),$$

then by identification, we get

$$curl\Big(\frac{1}{b_\varepsilon}\ B_\varepsilon(b_\varepsilon\ x)-B_\varepsilon(x)\Big)=0.$$

In view of this and to the fact that the quantity $\Big(\frac{1}{b_\varepsilon}\ B_\varepsilon(b_\varepsilon\ x)-B_\varepsilon(x)\Big)$ is a complex-vector potential, there exists a function $g_\varepsilon\in H^2(B_R,\mathbb{C})$ such that

$$\frac{1}{b_\varepsilon}\ B_\varepsilon(b_\varepsilon\ x)-B_\varepsilon(x)=\nabla g_\varepsilon(x).$$

It follows that

$$B_\varepsilon(b_\varepsilon\ x)=b_\varepsilon\ B_\varepsilon(x)+b_\varepsilon\ \nabla g_\varepsilon(x)\quad\forall\ x\in B_R.$$

Consequently, the potential vector $B_\varepsilon$ satisfies for any $x\in B_R$

$$B_\varepsilon\Big(x\ e^{i\ \frac{2\ \pi}{q_\varepsilon}}\Big)=e^{\frac{2\ i\ \pi}{q_\varepsilon}}\ B_\varepsilon(x)+e^{\frac{2\ i\ \pi}{q_\varepsilon}}\ \nabla g_\varepsilon(x).\tag{8.27}$$

Now, from the construction of the function $\phi_\varepsilon$, it is obvious that on $B_R\backslash\bigcup_{(1\le k\le m,\ 1\le j\le q_\varepsilon)}\Big(B(a_j^k,\varepsilon)\Big)$

$$\nabla\phi_\varepsilon=e^{-i\ \frac{4\ \pi}{q_\varepsilon}}\ B_\varepsilon-\nabla^\perp h_\varepsilon.$$

It means, using $b_\varepsilon=e^{i\ \frac{2\ \pi}{q_\varepsilon}}$

$$\nabla\phi_\varepsilon=\frac{1}{b_\varepsilon^2}\ B_\varepsilon-\nabla^\perp h_\varepsilon.$$

In particular, $\forall x\in B_R\backslash\bigcup_{(1\le k\le m,\ 1\le j\le q_\varepsilon)}\Big(B(a_j^k,\varepsilon)\Big)$

$$(\nabla\phi_\varepsilon)(b_\varepsilon\ x)=\frac{1}{b_\varepsilon^2}\ B_\varepsilon(b_\varepsilon\ x)-(\nabla^\perp h_\varepsilon)(b_\varepsilon\ x).\tag{8.28}$$

On the one hand, the left-hand side of (8.28) is

$$(\nabla\phi_\varepsilon)(b_\varepsilon\ x)=\frac{1}{b_\varepsilon}\ \nabla\Big(\phi_\varepsilon(b_\varepsilon\ x)\Big).\tag{8.29}$$

On the other hand, using (8.26)-(8.27) in the right-hand side of (8.28), we have for $x\in B_R\backslash\bigcup_{(1\le k\le m,\ 1\le j\le q_\varepsilon)}\Big(B(a_j^k,\varepsilon)\Big)$

$$\begin{aligned}\frac{1}{b_\varepsilon^2}\ B_\varepsilon(b_\varepsilon\ x)-(\nabla^\perp h_\varepsilon)(b_\varepsilon\ x)=&\frac{1}{b_\varepsilon}B_\varepsilon(x)+\frac{1}{b_\varepsilon}\ \nabla g_\varepsilon(x)-\frac{1}{b_\varepsilon}\ \nabla^\perp h_\varepsilon(x)\\=&\frac{1}{b_\varepsilon}\ \nabla\phi_\varepsilon(x)+\frac{1}{b_\varepsilon}\ \nabla g_\varepsilon(x).\end{aligned}\tag{8.30}$$

Comparing (8.28)-(8.29) to (8.30), we get by identification

$$\nabla\Big(\phi_\varepsilon(b_\varepsilon\ x)\Big)=\frac{1}{b_\varepsilon}\ B_\varepsilon(b_\varepsilon\ x)-b_\varepsilon\ (\nabla^\perp h_\varepsilon)(b_\varepsilon\ x)=\nabla\phi_\varepsilon(x)+\nabla g_\varepsilon(x).\tag{8.31}$$

By integration of (8.31), there exists a constant $c\in\mathbb{C}$ such that

$$\phi_\varepsilon(b_\varepsilon\ x) = \phi_\varepsilon(x) + g_\varepsilon(x) + c.$$

Set $f_\varepsilon(x) = g_\varepsilon(x) + c$ and replace $b_\varepsilon$ with $e^{i\ \frac{2\,\pi}{q_\varepsilon}}$ to get

$$\phi_\varepsilon\Big(x\ e^{i\ \frac{2\,\pi}{q_\varepsilon}}\Big) = \phi_\varepsilon(x) + f_\varepsilon(x). \tag{8.32}$$

Using (8.32) together with the fact that $\rho_\varepsilon\Big(x\ e^{i\ \frac{2\,\pi}{q_\varepsilon}}\Big) = \rho_\varepsilon(x)$ in $v_\varepsilon = \rho_\varepsilon\ e^{i\,\phi_\varepsilon}$, we obtain in $B_R\backslash \bigcup_{(1\le k\le m,\ 1\le j\le q_\varepsilon)}\Big(B(a_j^k,\varepsilon)\Big)$,

$$\begin{aligned} v_\varepsilon\Big(x\ e^{i\frac{2\,\pi}{q_\varepsilon}}\Big) &= \rho_\varepsilon\Big(x\ e^{i\frac{2\,\pi}{q_\varepsilon}}\Big)\ e^{i\ \phi_\varepsilon\Big(x\ e^{i\ \frac{2\,\pi}{q_\varepsilon}}\Big)} \\ &= \rho_\varepsilon(x)\ e^{i\ \phi_\varepsilon(x)}\ e^{i\ f_\varepsilon(x)} \\ &= v_\varepsilon(x)\ e^{i\ f_\varepsilon(x)}. \end{aligned} \tag{8.33}$$

Thanks to the fact that $\rho_\varepsilon$ is equal to 0 in $\bigcup_{(1\le k\le m,\ 1\le j\le q_\varepsilon)}\Big(B(a_j^k,\varepsilon)\Big)$, we find for any $x\in B_R$

$$v_\varepsilon\Big(x\ e^{i\frac{2\,\pi}{q_\varepsilon}}\Big) = v_\varepsilon(x)\ e^{i\ f_\varepsilon(x)}. \tag{8.34}$$

Finally, we replace $g_\varepsilon$ with $(f_\varepsilon - c)$ in (8.27) to have

$$B_\varepsilon\Big(x\ e^{i\ \frac{2\,\pi}{q_\varepsilon}}\Big) = e^{\frac{2\ i\ \pi}{q_\varepsilon}}\ B_\varepsilon(x) + e^{\frac{2\ i\ \pi}{q_\varepsilon}}\ \nabla f_\varepsilon(x). \tag{8.35}$$

Combining (8.34) together with (8.35) completes the proof of the lemma 8.5. □

**Step5**

From the equation (8.24), the induced magnetic field $h_\varepsilon$ satisfies

$$\begin{cases} -\Delta h_\varepsilon + h_\varepsilon - h_{ex} = h_{ex}\ (\mu_\varepsilon - 1) & \text{in} \quad B_R \\ h_\varepsilon = h_{ex} \quad \text{on}\ \ \partial B_R. \end{cases}$$

Hence, in particular

$$(h_\varepsilon - h_{ex})(y) = h_{ex}\ \int_{B_R} G(y,x)\ d(\mu_\varepsilon - 1)(x), \qquad \forall\ \ y\in B_R. \tag{8.36}$$

Now, multiplying $-\Delta h_\varepsilon + h_\varepsilon - h_{ex} = h_{ex}\ (\mu_\varepsilon - 1)$ by $(h_\varepsilon - h_{ex})$, integrating on $B_R$, and using (8.36), it follows that

$$\begin{aligned} \int_{B_R} |\nabla h_\varepsilon|^2 + \int_{B_R} |h_\varepsilon - h_{ex}|^2 &= \int_{B_R} (-\Delta h_\varepsilon + h_\varepsilon - h_{ex})\ (h_\varepsilon - h_{ex}) \\ &= \int_{B_R} h_{ex}\ (h_\varepsilon - h_{ex})(y)\ d(\mu_\varepsilon - 1)(y) \\ &= h_{ex}^2\ \int_{B_R}\int_{B_R} G(y,x)\ d(\mu_\varepsilon - 1)(x)\ d(\mu_\varepsilon - 1)(y), \end{aligned}$$

where $(\mu_\varepsilon - 1)$ denotes the difference between of the measure $\mu_\varepsilon$ and the Lebesgue measure on $B_R$. We divide by $2\, h_{ex}^2$ to get

$$\limsup_{\varepsilon\to 0} \frac{\frac{1}{2}\int_{B_R}|\nabla h_\varepsilon|^2 + \frac{1}{2}\int_{B_R}|h_\varepsilon - h_{ex}|^2}{h_{ex}^2} = \limsup_{\varepsilon\to 0} \frac{1}{2}\int_{B_R\times B_R} G(x,y)\, d(\mu_\varepsilon - 1)(y)\, d(\mu_\varepsilon - 1)(x).$$

Using (8.23),

$$\limsup_{\varepsilon\to 0} \frac{\frac{1}{2}\int_{B_R}|\nabla h_\varepsilon|^2 + \frac{1}{2}\int_{B_R}|h_\varepsilon - h_{ex}|^2}{h_{ex}^2} \le \frac{\lambda}{2}\int_{B_R}|\mu| + \frac{1}{2}\int_{B_R\times B_R} G(x,y)\, d(\mu - 1)(y)\, d(\mu - 1)(x).$$

We remark that the right-hand side is the functional $I$ defined by (8.15), so that

$$\limsup_{\varepsilon\to 0} \frac{\frac{1}{2}\int_{B_R}|\nabla h_\varepsilon|^2 + \frac{1}{2}\int_{B_R}|h_\varepsilon - h_{ex}|^2}{h_{ex}^2} \le I(\mu). \tag{8.37}$$

In addition, thanks to the fact that there are $(m\ q_\varepsilon)$ points $(a_i^k)_{(1\le i\le q_\varepsilon,\ 1\le k\le m)}$ in $B_R$, hence by definition of $\rho_\varepsilon$, it is clear

$$\limsup_{\varepsilon\to 0} \frac{\frac{1}{2}\int_{B_R}|\nabla \rho_\varepsilon|^2 + \frac{1}{4\,\varepsilon^2}\int_{B_R}(1-\rho_\varepsilon^2)^2}{h_{ex}^2} = 0. \tag{8.38}$$

Here, it is easy that (5.47) holds, so in particular

$$\begin{aligned}\limsup_{\varepsilon\to 0} \frac{J_{B_R}(v_\varepsilon, B_\varepsilon)}{h_{ex}^2} \le & \limsup_{\varepsilon\to 0}\Big(\frac{\frac{1}{2}\int_{B_R}|\nabla h_\varepsilon|^2 + \frac{1}{2}\int_{B_R}|h_\varepsilon - h_{ex}|^2}{h_{ex}^2}\Big) \\ & + \limsup_{\varepsilon\to 0}\Big(\frac{\frac{1}{2}\int_{B_R}|\nabla \rho_\varepsilon|^2 + \frac{1}{4\,\varepsilon^2}\int_{K}(1-\rho_\varepsilon^2)^2}{h_{ex}^2}\Big).\end{aligned} \tag{8.39}$$

A combination of (8.37) together with (8.38) in (8.39) allows to write

$$\limsup_{\varepsilon\to 0} \frac{J_{B_R}(v_\varepsilon, B_\varepsilon)}{h_{ex}^2} \le I(\mu). \tag{8.40}$$

This inequality is true for the test configuration $(v_\varepsilon, B_\varepsilon) \in G_\varepsilon$, so it is true in particular for any minimizer of $J$ over the space $G_\varepsilon$. This completes the result of the proposition 8.3.

## 3 New formulation of the energy $E$

Let $f \in Y$. In particular, the measure $(-\Delta f + f)$ is concentrated on a finite number of concentric circles of center O and of strict positive radii. Recall that

$$E(f) = \frac{\lambda}{2}\int_{B_R}|-\Delta f + f| + \frac{1}{2}\int_{B_R}|\nabla f|^2 + \frac{1}{2}\int_{B_R}|f-1|^2.$$

Let us start with the case where the measure $(-\Delta f + f)$ is not concentrated on any circle.

### 3.1 Energy without circle of concentration

In the case where the measure $(-\Delta f + f)$ is not concentrated on any circle, each $f \in Y$ is solution of

$$\begin{cases} -\Delta f + f = 0 \quad \text{in } B_R \\ f = 1 \quad \text{on } \partial B_R. \end{cases} \tag{8.41}$$

Consequently, for the $f$ unique solution of (8.41), $E(f)$ is the energy which corresponds to the case of the absence of circle of concentration. Using (8.41), we deduce

$$\begin{aligned} E(f) =& \frac{1}{2} \int_{B_R} |\nabla f|^2 + \frac{1}{2} \int_{B_R} |f-1|^2 \\ =& \frac{1}{2} \int_{B_R} (-\Delta f + f - 1)\,(f-1) \\ =& -\frac{1}{2} \int_{B_R} (f-1) \\ =& \frac{\pi\, R^2}{2} - \frac{1}{2} \int_{B_R} f. \end{aligned} \tag{8.42}$$

Now, our interest is to calculate $\int_{B_R} f$. In polar coordinates, remember that the Laplacian reads

$$\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2.$$

Let us take $f(r\, e^{i\,\theta}) = f(r,\theta)$. The scalar function $f \in Y$ is radial, hence using the above in (8.41), it solves

$$-\partial_{rr} f(r,\theta) - \frac{\partial_r f(r,\theta)}{r} + f(r,\theta) = 0 \quad \text{in } [0,R] \times [0, 2\,\pi] \quad \text{and} \quad f(R,\theta) = 1. \tag{8.43}$$

Again $f$ is radial, hence there exists $g : [0,R] \to \mathbb{R}$ such that

$$f(r\, e^{i\,\theta}) = g(r) \quad \text{for any } \ \theta \in [0, 2\,\pi].$$

In particular, (8.43) becomes

$$-g'' - \frac{g'}{r} + g = 0 \quad \text{in } [0,R] \quad \text{and} \quad g(R) = 1. \tag{8.44}$$

Note that the continuity of $f$ yields that $g$ is continuous on $[0,R]$ too.

**Modified Bessel functions**

Now, let us resolve the following ordinary differential equation

$$-y'' - \frac{y'}{x} + y = 0 \quad \text{in} \ [a, b], \ \ 0 \leq a < b \leq +\infty \tag{8.45}$$

such that $y$ is continuous on $[a, b]$. We define $I_0$ and $K_0$ to be respectively the modified Bessel function of the first kind and of the second kind. We need to give some properties of the Bessel functions $I_0$ and $K_0$. For this, we can refer to the literature [W]. First,

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2 \, 2^{2\,n}}. \tag{8.46}$$

Note that $I_0$ increases and $I_0(0) = 1$. We define $I_1$ to be the derivative of $I_0$, so it is positive. Second, $K_0$ is given as follows

$$K_0(x) = -\Big( \log(\frac{x}{2}) + \gamma \Big) \ I_0(x) + \sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2 \, 2^{2\,n}} \ \Phi(n), \tag{8.47}$$

where $\Phi(n) = 1 + \frac{1}{2} + ... + \frac{1}{n}$ for $n \neq 0$, $\Phi(0) = 0$, and $\gamma = (\lim_{n \to +\infty} \Phi(n) - \log n)$. We note that $K_0$ is positive, decreases and tends to $+\infty$ as $x \to 0$. Let $K_1$ be the derivative of the function $(-K_0)$, then it is positive and thanks to (8.47), $K_1$ tends to $+\infty$ as $x \to 0$.

Let $y$ be a solution of (8.45). We distinguish two cases:

**Case 1: If** $a > 0$

Here, there exist $C_1, \ C_2 \in \ \mathbb{R}$ such that for any $x \in [a, b]$, $y(x)$ can be written as

$$y(x) = C_1 \ \ I_0(x) + C_2 \ \ K_0(x).$$

**Case 2: If** $a = 0$

In this case, knowing that $y$ is continuous on $[0, b]$ (especially at 0), hence necessarily the constant $C_2 = 0$ (given in the case 1) because $K_0$ is not well defined at 0, so there exists only $C_1 \in \mathbb{R}$ such that

$$y(x) = C_1 \ \ I_0(x).$$

**Lemma 8.6.** *Let $f$ be the solution of (8.41), then*

$$E(f) = \pi \ \left( \frac{R^2}{2} - R \ \frac{I_1(R)}{I_0(R)} \right).$$

**Proof**: First, we go back to resolve (8.44). Using the fact that $g$ is continuous in $[0, R]$ implies that there exists $C_1 \in \mathbb{R}$ such that $g(r) = C_1 \ I_0(r)$ in $[0, R]$. Knowing $g(R) = 1$, then in particular $C_1 = \frac{1}{I_0(R)}$. It follows that

$$g(r) = \frac{I_0(r)}{I_0(R)} \quad \text{in} \ [0, R]. \tag{8.48}$$

Using (8.48),

$$g'(R) = \frac{I_1(R)}{I_0(R)}.$$

Let again $f(r,\theta) = g(r)$, then having $f = \Delta f$ in $B_R$ gives us

$$\int_{B_R} f = \int_{B_R} \Delta f = \int_{\partial B_R} \frac{\partial f}{\partial \nu} = 2\,\pi\,R\,g'(R) = 2\,\pi\,R\,\frac{I_1(R)}{I_0(R)},$$

where $\nu$ is the outward normal at the boundary of $B_R$. Inserting the above in (8.42), we have

$$E(f) = \frac{\pi\,R^2}{2} - \pi\,R\,\frac{I_1(R)}{I_0(R)}.$$

For simplification, we take

$$\hat{J}_0 = E(f) = \pi\,\left(\frac{R^2}{2} - R\,\frac{I_1(R)}{I_0(R)}\right). \tag{8.49}$$

□

## 3.2 The energy in the presence of circle (or circles) of concentration

In this section, we take $f \in Y$ such that $(-\Delta f + f)$ is positive. Let us consider the measure

$$\mu = -\Delta f + f.$$

We take the assumption on the measure $\mu$ given by the proposition 8.3, so that $\mu$ is supported on $m \geq 1$ concentric circles of center O. Of course, it is known that these circles have positive radii. For $(r_i)_{1\leq i\leq m}$ with $0 < r_1 < r_2 < ... < r_m < R$, we take $\Gamma_i(r_i)$ to be the circle of center O and of radius $r_i$. As a consequence, $\{\Gamma_i(r_i),\ \ 1 \leq i \leq m\}$ is taken to be the family of the $m$ disjoint concentric circles where the measure $\mu$ concentrates. Again, by definition of the space $Y$ and using the fact that the measure $\mu$ is taken to be positive, so the mass of $\mu$ on each circle belongs to $2\,\pi\,\beta\,\mathbb{N}$. Hence, there exist $(m_i)_{1\leq i\leq m}$ with $m_i \in \mathbb{N}$ for any $1 \leq i \leq m$ such that the mass of the measure $\mu$ on the circle $\Gamma_i(r_i)$ is equal to $2\,\pi\,\beta\,m_i$.
From now on, when we write $(r_i, m_i)_{1\leq i\leq m}$, it means that this family verifies the above assumptions.
Thanks to the concentration of the measure $\mu$ on the $m$ disjoint concentric circles $\Big(\Gamma_i(r_i)\Big)_{1\leq i\leq m}$, we get as (8.18)

$$\mu = -\Delta f + f = \sum_{i=1}^{m} \frac{\beta\,m_i}{r_i}\,\delta_{\Gamma_i(r_i)} \quad \text{in } \ B_R.$$

Letting $f(r,\theta) = g(r)$, then proceeding as (8.44), we can write

$$-g''(r) - \frac{g'(r)}{r} + g(r) = \sum_{i=1}^{m} \frac{\beta\,m_i}{r_i}\,\delta_{r_i} \quad \text{in } [0,R], \quad g(R) = 1. \tag{8.50}$$

Let us denote $g'_l$ (resp. $g'_r$) the left (resp. right) derivative of $g$. We have for $1 \le i \le m$

$$\beta \ \frac{m_i}{r_i} = g'_l(r_i) - g'_r(r_i) \quad (r_i, m_i) \in ]0, R[\times \mathbb{N}. \tag{8.51}$$

The $g$ solution of (8.50) on the intervals $[0, r_1[, ]r_i, r_{i+1}[$ for $1 \le i \le m-1$ and $]r_m, R]$ is in particular solution of the following ordinary differential equation

$$-g''(r) - \frac{g'(r)}{r} + g(r) = 0. \tag{8.52}$$

Due to the parameter of $m$ and under the above, let us take the space

$$Y_m := \left\{ \begin{array}{l} f \in H^1_1(B_R, \mathbb{R}) \ \text{ such that } \ f \text{ is radial and } \ (-\Delta f + f) \ \text{ is of the form} \\ \sum_{i=1}^m \frac{\beta\, m_i}{r_i} \, \delta_{\Gamma_i(r_i)} \ \text{ in } \ B_R \ \text{ where } \ 0 < r_1 < ... < r_m < R \ \text{ and } \ m_i \in \mathbb{N} \\ \text{for } 1 \le i \le m \end{array} \right\}. \tag{8.53}$$

Taking $f \in Y_m$, our interest now is to determine the energy $E$ defined by (8.14) only in function of the family $(r_i, m_i)_{1 \le i \le m}$. The following lemma presents a preliminary expression of $E$.

**Lemma 8.7.** *Let $m \ge 1$. If $f \in Y$ is such that $-\Delta f + f = \beta \ \sum_{i=1}^m \frac{m_i}{r_i} \ \delta_{\Gamma_i(r_i)}$ where $(r_i, m_i)_{1 \le i \le m}$ are such that $0 < r_1 < ... < r_m < R$ and $m_i \in \mathbb{N}$ for $1 \le i \le m$, then letting $g(r) = f(r, \theta)$, we have*

$$\frac{E(f)}{\pi} = \frac{R^2}{2} - R\, g'(R) + \sum_{i=1}^m \beta \ m_i \ \Big(g(r_i) - (2 - \lambda)\Big). \tag{8.54}$$

**Proof**: Letting $f \in Y$,

$$E(f) = \frac{\lambda}{2} \ \int_{B_R} | - \Delta f + f| + \frac{1}{2} \ \int_{B_R} \Big( - \Delta f + f - 1\Big) \ (f - 1), \tag{8.55}$$

since $f = 1$ on $\partial B_R$. We use the fact that $f$ is radial and verifies

$$-\Delta f + f = \beta \ \sum_{i=1}^m \frac{m_i}{r_i} \ \delta_{\Gamma_i(r_i)},$$

to obtain

$$\int_{B_R} | - \Delta f + f| = 2\, \pi \ \beta \ \sum_{i=1}^m m_i, \tag{8.56}$$

and by definition of the function $g$ which is $g(r) = f(r, \theta)$,

$$\int_{B_R} (-\Delta f + f) \ (f - 1) = 2\, \pi\, \beta \ \sum_{i=1}^m m_i \ \ (g - 1)(r_i). \tag{8.57}$$

We insert (8.56)-(8.57) in (8.55) to have

$$E(f) = \lambda\, \pi\, \beta \sum_{i=1}^{m} m_i + \pi\, \beta \sum_{i=1}^{m} m_i\ (g-1)(r_i) + \frac{\pi\, R^2}{2} - \frac{1}{2} \int_{B_R} f. \tag{8.58}$$

Now, we need to calculate $\int_{B_R} f$. First, the fact that $f$ is radial leads to

$$\int_{B_R} f(r,\theta)\ r\ dr\ d\theta = 2\,\pi \int_0^R g(r)\ r\ dr. \tag{8.59}$$

Second, we decompose the interval $[0, R]$ and we use (8.50) to deduce

$$\begin{aligned}\int_0^R g(r)\ r\ dr &= \sum_{i=1}^{m}\Big( \int_0^{r_i} g(r)\ r\ dr + \int_{r_i}^{R} g(r)\ r\ dr\Big)\\ &= \sum_{i=1}^{m}\Big( \int_0^{r_i} (g''(r)\ r + g'(r))\ dr + \int_{r_i}^{r} (g''(r)\ r + g'(r))\ dr\Big) \qquad (8.60)\\ &= \sum_{i=1}^{m}\Big[r\ g'(r)\Big]_0^{r_i} + \sum_{i=1}^{m}\Big[r\ g'(r)\Big]_{r_i}^{R},\end{aligned}$$

where $[S(x)]_a^b = S(b) - S(a)$ for any $a$, $b \in \mathbb{R}$ and $S$ any function defined on $\mathbb{R}$. Referring to the fact that $\beta\ \frac{m_i}{r_i} = g_l'(r_i) - g_r'(r_i)$ for $1 \leq i \leq m$,

$$\int_0^R g(r)\ r\ dr = R\ \ g'(R) + \sum_{i=1}^{m} r_i\ \Big(g_l'(r_i) - g_r'(r_i)\Big) = R\ \ g'(R) + \sum_{i=1}^{m} \beta\ \ m_i.$$

Inserting this (8.59), we find

$$\int_{B_R} f = 2\,\pi\,\Big(R\, g'(R) + \sum_{i=1}^{m} \beta\ \ m_i\Big).$$

We insert again this in (8.58) to complete the proof of the lemma. □

In the next paragraph, we will be interested in giving the expression of the energy $E$ defined by (8.54) only in function of $(r_i, m_i)_{1\leq i\leq m}$. For this, it suffices to determine the quantities $g'(R)$ and $g(r_i)$, $1 \leq i \leq m$. First, let us define the function $X$ on $]0, R]$ as follows

$$\forall\ x \in ]0, R], \quad X(x) := I_0(R)\ K_0(x) - K_0(R)\ I_0(x).$$

We mention that $X(R) = 0$. Moreover, since $I_1 = I_0' \geq 0$ and $K_1 = -K_0' \geq 0$, it is clear that the function $X$ is decreasing in $]0, R]$. Using $I_0(0) = 1$ and the fact that $K_0$ tends to $+\infty$ as $x \to 0$, then $X$ tends to $+\infty$ as $x \to 0$. As a consequence, the function $X$ is positive on $]0, R[$. In addition, by definition of the Bessel functions $I_0$ and $K_0$ as solutions of (8.45), the function $X$ satisfies for $0 < x \leq R$

$$X'' + \frac{X'}{x} = X.$$

**Remark 8.8.** *Let $f \in Y$ be such that $-\Delta f + f = \beta \sum_{i=1}^{m} \frac{m_i}{r_i} \delta_{\Gamma_i(r_i)}$ where $(r_i, m_i)_{1 \leq i \leq m}$ are such that $0 < r_1 < ... < r_m < R$ and $m_i \in \mathbb{N}$ for $1 \leq i \leq m$. Let us take*

$$E(f) = E_m(r_1, ..., r_m, m_1, ..., m_m). \tag{8.61}$$

*In this case, (8.17) can be rewritten to be*

$$\limsup_{n \to \infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \leq E_m(r_1, ..., r_m, m_1, ..., m_m). \tag{8.62}$$

*It seems be not easy to give the expression of the energy $E_m$ for large $m$, even for $m \geq 2$. From now on, we just restrict to the case $m = 1$.*

**Remark 8.9.** *Here, we show how $R_1$ and $R$, which are given in (8.4) and (8.5), are adjusted in order to prove (8.6). The solutions $h_1$ and $h_2$ respectively of (8.4) and (8.5) are radial. Using the Bessel functions, we can find*

$$h_1(r) = \frac{I_0(r)}{I_0(R)} \quad \forall\, 0 \leq r \leq R_1, \tag{8.63}$$

*and*

$$h_2(r) = \frac{K_0(R_1) - K_0(R)}{X(R_1)} \; I_0(r) + \frac{I_0(R) - I_0(R_1)}{X(R_1)} \; K_0(r) \quad \forall\, R_1 \leq r \leq R. \tag{8.64}$$

*In particular*

$$h_1'(R_1) = \frac{I_1(R_1)}{I_0(R)}, \tag{8.65}$$

*and*

$$h_2'(R_1) = \frac{K_0(R_1) - K_0(R)}{X(R_1)} \; I_1(R_1) + \frac{I_0(R_1) - I_0(R)}{X(R_1)} \; K_1(R_1). \tag{8.66}$$

*Now, we adjust $R_1$ and $R$ in order to get*

$$I_1(R_1) \left( \frac{1}{I_0(R)} + \frac{K_0(R_1) - K_0(R)}{X(R_1)} \right) = \frac{I_0(R) - I_0(R_1)}{X(R_1)} \; K_1(R_1). \tag{8.67}$$

*Thus, $h_1'(R_1) = -h_2'(R_1)$. (8.6) is then proved.*

### 3.3 The case $m = 1$

**Lemma 8.10.** *If $f \in Y$ is such that $-\Delta f + f = \beta \frac{m_1}{r_1} \delta_{\Gamma_1(r_1)}$ where $0 < r_1 < R$ and $m_1 \in \mathbb{N}$, then letting $g(r) = f(r, \theta)$, we have*

$$g(r_1) = \frac{I_0(r_1)}{I_0(R)} \left( 1 + \frac{\beta\, m_1 \;\, X(r_1)}{R \Big( I_0(R)\, K_1(R) + I_1(R)\, K_0(R) \Big)} \right), \tag{8.68}$$

*and*

$$g'(R) = \frac{I_1(R)}{I_0(R)} - \frac{\beta}{R\, I_0(R)} \; I_0(r_1) \;\, m_1. \tag{8.69}$$

**Proof**: Under the assumption on $f$, then by definition of the function $g$, it is clear

$$-g''(r) - \frac{g'(r)}{r} + g(r) = \beta \, \frac{m_1}{r_1} \, \delta_{r_1} \quad \text{in} \quad [0, R] \ \text{ and } \ g(R) = 1.$$

Using the continuity of $g$ at 0, there exist real constants $\sigma_0$, $\sigma_1$, and $\omega_1$ such that $g(r)$ is written as

$$g(r) = \begin{cases} \sigma_0 \ I_0(r) & \text{in } [0, r_1] \\ \sigma_1 \ I_0(r) + \omega_1 \ K_0(r) & \text{in } [r_1, R]. \end{cases}$$

Our aim is to find $g(r_1)$ and $g'(R)$, so we need to find the parameters $\sigma_0$, $\sigma_1$ and $\omega_1$. First, the boundary condition $g(R) = 1$ yields

$$g(R) = \sigma_1 \ I_0(R) + \omega_1 \ K_0(R) = 1. \tag{8.70}$$

The continuity of $g$ at $r_1$ reads $g_g(r_1) = g_d(r_2)$, so that

$$\sigma_0 \ I_0(r_1) = \sigma_1 \ I_0(r_1) + \omega_1 \ K_0(r_1). \tag{8.71}$$

(8.71) gives us

$$\sigma_0 - \sigma_1 = \omega_1 \ \frac{K_0(r_1)}{I_0(r_1)}. \tag{8.72}$$

Now, we use (8.51) to get

$$\beta \ \frac{m_1}{r_1} = g_l'(r_1) - g_r'(r_1).$$

In particular, we have

$$\beta \ \frac{m_1}{r_1} = \sigma_0 \ I_1(r_1) - \sigma_1 \ I_1(r_1) + \omega_1 \ K_1(r_1). \tag{8.73}$$

Therefore,

$$\sigma_0 - \sigma_1 = \beta \ \frac{m_1}{r_1 \ I_1(r_1)} - \omega_1 \ \frac{K_1(r_1)}{I_1(r_1)}. \tag{8.74}$$

We compare (8.72) to (8.74) to have

$$\omega_1 = \frac{\beta \ m_1 \ I_0(r_1)}{r_1} \ \frac{1}{I_0(r_1) \ K_1(r_1) + I_1(r_1) \ K_0(r_1)}. \tag{8.75}$$

Let us define the function

$$b(x) = I_0(x) \ K_1(x) + I_1(x) \ K_0(x), \quad x \in ]0, R]. \tag{8.76}$$

The derivative of the function $b$ is $b'(x) = -\frac{b(x)}{x^2}$, so by integration there exists $a \in \mathbb{R}$ such that

$$b(x) = \frac{a}{x} \quad \forall x \in ]0, R].$$

In particular, we have for $x = R$, $b(R) = \frac{a}{R} = I_0(R)\ K_1(R) + I_1(R)\ K_0(R)$. It is then clear that $b(x) = \frac{a}{x}$ for any $x \in ]0, R]$, where

$$a = R\ \Big(I_0(R)\ K_1(R) + I_1(R)\ K_0(R)\Big). \tag{8.77}$$

Inserting $b(r_1) = \frac{a}{r_1}$ in (8.75),

$$\omega_1 = \frac{\beta\, m_1\ I_0(r_1)}{r_1\ b(r_1)} = \frac{\beta}{a}\ m_1\ I_0(r_1). \tag{8.78}$$

Replacing $\omega_1$ with (8.78) in (8.70), we have

$$\sigma_1 = \frac{1}{I_0(R)} - \frac{\beta}{a}\ \ m_1\ \frac{I_0(r_1)\ K_0(R)}{I_0(R)}. \tag{8.79}$$

Consequently, inserting the two quantities (8.78) and (8.79) in $g(r_1) = \sigma_1\ I_0(r_1) + \omega_1\ K_0(r_1)$ and referring to the function $X$,

$$g(r_1) = \frac{I_0(r_1)}{I_0(R)}\ \ \Big(1 + \frac{\beta}{a}\ \ m_1\ \ X(r_1)\Big).$$

Moreover, we have

$$\begin{aligned} g'(R) =& \sigma_1\ I_1(R) - w_1\ K_1(R) \\ =& \frac{I_1(R)}{I_0(R)} - \frac{\beta\, m_1}{a}\ I_0(r_1)\ K_0(R)\ \frac{I_1(R)}{I_0(R)} - \frac{\beta}{a}\ m_1\ I_0(r_1)\ K_1(R) \\ =& \frac{I_1(R)}{I_0(R)} - \frac{\beta}{a}\ m_1\ \frac{I_0(r_1)\ b(R)}{I_0(R)} \\ =& \frac{I_1(R)}{I_0(R)} - \frac{I_0(r_1)}{R\ I_0(R)}\ \beta\ \ m_1. \end{aligned} \tag{8.80}$$

The lemma 8.10 is then proved.

□

**Corollary 8.11.** *If $f \in Y$ is such that $-\Delta f + f = \beta\ \ \frac{m_1}{r_1}\ \ \delta_{\Gamma_1(r_1)}$ where $0 < r_1 < R$ and $m_1 \in \mathbb{N}$, then we have*

$$\frac{E(f)}{\pi} = \frac{E_1(r_1, m_1)}{\pi} = \frac{\hat{J}_0}{\pi} + \Big(\lambda - (2 - 2\frac{I_0(r_1)}{I_0(R)})\Big)\ \beta\, m_1 + \frac{\beta^2\ m_1^2}{a}\ \frac{I_0(r_1)\ X(r_1)}{I_0(R)}, \tag{8.81}$$

*where $a$ is defined by (8.77).*

*Proof.* Let $g(r) = f(r, \theta)$. Inserting (8.68)-(8.80) in (8.54), we get

$$\begin{aligned} \frac{E_1(r_1, m_1)}{\pi} =& \frac{R^2}{2} - R\ g'(R) + \beta\ \ m_1\ \ \Big(g(r_1) - (2 - \lambda)\Big) \\ =& \frac{R^2}{2} - R\ \ \frac{I_1(R)}{I_0(R)} + \frac{I_0(r_1)}{I_0(R)}\ \ \beta\ \ m_1 - (2 - \lambda)\ \ \beta\ \ m_1 + \beta\ \ \frac{I_0(r_1)}{I_0(R)}\ \ \Big(\frac{\beta}{a}\ \ m_1\ \ X(r_1) + 1\Big)\ \ m_1. \end{aligned}$$

From (8.49), we have $\left(\frac{R^2}{2} - R\ \frac{I_1(R)}{I_0(R)}\right) = \frac{\hat{J}_0}{\pi}$. It follows that

$$\frac{E_1(r_1, m_1)}{\pi} = \frac{\hat{J}_0}{\pi} + \beta\ \ m_1\ \ (\lambda - 2) + \frac{2\ \beta\ \ m_1\ \ I_0(r_1)}{I_0(R)} + \frac{\beta^2}{a}\ \ \frac{m_1^2\ \ I_0(r_1)\ \ X(r_1)}{I_0(R)}.$$

□

**Remark 8.12.** *Now, let us take* $m_1 = 1$*. We show at the end of the chapter the reason which allows me to choose* $m_1 = 1$*. In particular, thanks to (8.81),*

$$\frac{E_1(r_1, 1)}{\pi} = \frac{\hat{J}_0}{\pi} + \Big(\lambda - (2 - 2\frac{I_0(r_1)}{I_0(R)})\Big)\ \beta\ + \frac{I_0(r_1)\ X(r_1)}{a\ I_0(R)}\ \beta^2. \tag{8.82}$$

*To simplify, we set*

$$F(x) = \frac{E_1(x, 1)}{\pi} \quad x \in ]0, R[. \tag{8.83}$$

*Let* $(u_\varepsilon, A_\varepsilon)$ *be a minimizer of* $J_{B_R}$ *over the space* $G_\varepsilon$*. In particular, letting* $m = 1$ *with* $r_1 = x \in ]0, R[$ *and* $m_1 = 1$ *in (8.62),*

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \leq E_1(x, 1) \quad \forall\ x \in ]0, R[. \tag{8.84}$$

*The next paragraph is devoted to minimize the right-hand side of (8.84), which is the functional* $x\ \to E_1(x, 1)$*, over the interval* $]0, R[$*. By (8.83), we will be interested in minimizing the functional* $F$ *over* $]0, R[$*.*

# 4 Minimization of $F$ over $]0, R[$

First, we state some properties of the Bessel functions $I_i$ and $K_i$, $0 \leq i \leq 1$ which will be very useful for the rest.

## 4.1 Some properties of the modified Bessel functions

**Lemma 8.13.** $I_1$ *is increasing in* $[0, +\infty[$ *and* $K_1$ *is decreasing in* $]0, +\infty[$*. Moreover for any* $x > 0$

$$I_0 - \frac{2}{x}\ I_1 \geq 0 \ \text{ and }\ K_0 + \frac{2}{x}\ K_1 \geq 0. \tag{8.85}$$

*In addition, for any* $0 \leq i \leq 1$ *and* $x > 0$

$$I_i(x) \simeq \frac{e^x}{\sqrt{2\ \pi\ x}} \quad \text{when } x \text{ is large enough,} \tag{8.86}$$

$$K_i(x) \simeq \frac{e^{-x}}{\sqrt{2\ \pi\ x}} \quad \text{when } x \text{ is large enough.} \tag{8.87}$$

*Finally,*

$$I_0 \geq I_1 \quad \text{and} \quad K_0 < K_1. \tag{8.88}$$

**Proof**: First, referring to the expressions of $I_0$ and $K_0$ given respectively by (8.46) and (8.47), it is obvious that the assertion (8.85) is immediate. Second, we can refer to [W] to find the assertions (8.86) and (8.87). Now, we will prove (8.88). Let us take for $x \geq 0$

$$N_1(x) = (I_0(x))^2 - (I_1(x))^2.$$

We have

$$I_1' = I_0 - \frac{I_1}{x},$$

since $I_0'' + \frac{I_0'}{x} = I_0$ and $I_1 = I_0'$. Using the fact that $I_0 \geq \frac{2\, I_1}{x}$ yields that $I_1' \geq 0$, so $I_1$ is increasing. A derivation of $N_1$ gives us

$$N_1'(x) = 2\; I_0\; I_1 - 2\; I_1\; (I_0 - \frac{I_1}{x}) = 2\; \frac{(I_1)^2}{x} > 0.$$

In particular, we deduce that $N_1(x) \geq N_1(0) = 1$, which proves $I_0 \geq I_1$ in $[0, +\infty[$. Now, let us take for $x > 0$

$$N_2(x) = (K_1(x))^2 - (K_0(x))^2.$$

From $K_0'' + \frac{K_0'}{x} = K_0$ and the definition of $K_I$, we have

$$K_1' = -K_0 - \frac{K_1}{x}.$$

Using the fact that $K_0 + \frac{2\, K_1}{x} \geq 0$ yields that $K_1$ is decreasing. The derivative of $N_2$ is

$$N_2'(x) = -2\; \frac{(K_1)^2}{x} < 0.$$

Thanks to lemma 8.13, $N_2$ tends to 0 as $x \to +\infty$, hence with the fact that $N_2$ is decreasing, we get $N_2(x) > 0$ for any $x \in ]0, +\infty[$. □

## 4.2 Critical point (or points) of the functional $F$

Recall that for $a = R\, \Big(I_0(R)\; K_1(R) + I_1(R)\; K_0(R)\Big)$ and for any $r \in ]0, R[$

$$F(r) = \frac{\hat{J}_0}{\pi} + \;\beta\;\; (\lambda - 2) + 2\;\beta\; \frac{I_0(r)}{I_0(R)} + \frac{\beta^2}{a}\; \frac{I_0(r)\;\; X(r)}{I_0(R)}.$$

Now, my interest is to determine the critical points of the functional $F$. First, let $B$ be the derivative of $(-X)$, so that

$$B(x) = I_0(R)\; K_1(x) + K_0(R)\; I_1(x), \quad \forall\, x \in ]0, R].$$

Note that $B(x) > 0$ for any $x \in ]0, R]$ and in particular from (8.76), we have $B(R) = b(R) = \frac{a}{R}$. The derivative of $F$ is

$$F'(r) = 2\,\beta\,\frac{I_1(r)}{I_0(R)} + \frac{\beta^2}{a\,I_0(R)}\,\Big(I_1(r)\,X(r) - I_0(r)\,B(r)\Big) \quad \forall\, r \in ]0, R[.$$

Let us define

$$T(x) = I_0(x)\,B(x) - I_1(x)\,X(x) \quad \forall\, x \in ]0, R]. \tag{8.89}$$

We know from lemma 8.13 that $K_1 \geq K_0$, then it is immediate that $B \geq X$ in $]0, R]$. Moreover, since $I_0 \geq I_1$, it is clear from (8.89)

$$T(x) \geq 0 \quad \text{for any} \quad x \in ]0, R].$$

We replace $(I_0\,B - I_1\,X)$ with $T$ in $F'(r)$ to get for any $r \in ]0, R[$

$$F'(r) = \frac{\beta}{I_0(R)}\Big(2\,I_1(r) - \frac{\beta}{a}\,T(r)\Big). \tag{8.90}$$

Letting $F'(r) = 0$, we get $\frac{\beta}{a} = 2\,\frac{I_1(r)}{T(r)}$. Hence, if we take the function

$$G(x) = 2\,\frac{I_1(x)}{T(x)}, \quad x \in ]0, R], \tag{8.91}$$

it follows that any critical point $r$ in $]0, R[$ of the functional $F$ satisfies the following identity

$$\frac{\beta}{a} = G(r).$$

Consequently, the critical points of $x \to F(x)$ in the plane $(x, y)$ are the intersection between the graph of $x \to G(x)$ and the horizontal line of equation $y = \frac{\beta}{a}$. Thus, to determine this intersection, we need to know the sense of variation of the function $G$.

**Proposition 8.14.** *If*

$$\beta < \frac{2\,R\,I_1(R)}{I_0(R)},$$

*then the functional $F$ has one critical point in $]0, R[$. Precisely, this critical point is the minimizer of $F$ over $]0, R[$ and it is in $]0, R[$.*

**Remark 8.15.** *Note that $T(R) = \frac{a\,I_0(R)}{R}$, since $X(R) = 0$ and $B(R) = \frac{a}{R}$. Thus, (8.91) gives us*

$$G(R) = \frac{2\,R\,I_1(R)}{a\,I_0(R)}.$$

*In particular, the assumption on the parameter $\beta$ given by the above proposition becomes*

$$\frac{\beta}{a} < G(R).$$

### 4.3 Proof of proposition 8.14

To prove the proposition 8.14, we are concerned firstly with the determination of the sense of variation of the function $G$ in the interval $]0, R]$. To drop the subscripts, when it is not necessary, we omit the variable $x$. Let us give the derivatives of the functions $B$ and $T$. We have for $x \in ]0, R]$,

$$B' = -X - \frac{B}{x}, \qquad T' = 2\ I_1\ B - 2\ I_0\ X - \frac{T}{x}.$$

The functions $B$ and $X$ are respectively positive and nonnegative on $]0, R]$, hence $B' < 0$, meaning that $B$ is decreasing in $]0, R]$. Using the above derivatives, we have for $x \in ]0, R]$

$$G'(x) = \frac{2}{T^2}\ \Big(I_0\ T + 2\ I_1\ (I_0\ X - I_1\ B)\Big). \tag{8.92}$$

We replace $T$ by the right-hand side of (8.89) in (8.92) to find

$$\frac{T^2}{2}\ G' = (I_0^2 - 2\ I_1^2)\ B + I_0\ I_1\ X. \tag{8.93}$$

In view of the fact that $X(R) = 0$, $B(R) = \frac{a}{R}$ and $T(R) = \frac{a\ I_0(R)}{R}$, hence again from (8.93),

$$\frac{a\ I_0(R)^2}{2\ R}\ G'(R) = \Big(I_0(R)\Big)^2 - 2\ \Big(I_1(R)\Big)^2.$$

We remark that the sign of $G'(R)$ depends on the sign of the quantity

$$\Big(I_0(R)\Big)^2 - 2\ \Big(I_1(R)\Big)^2,$$

consequently it depends on $\Big(I_0(R) - \sqrt{2}\ I_1(R)\Big)$.

Now, let us take for $x \in [0, +\infty[$

$$M(x) = \Big(I_0(x)\Big)^2 - 2\ \Big(I_1(x)\Big)^2.$$

For $x > 0$, the derivative of $M$ is

$$M'(x) = 2\ I_1\ (-I_0 + \frac{2}{x}\ I_1).$$

Thanks to lemma 8.13, we have $(-I_0 + \frac{2}{x}\ I_1) < 0$. Then, $M'(x) < 0$, so the function $M$ is decreasing in $[0, +\infty[$. Again, from lemma 8.13, $M(x)$ tends to $-\infty$ as $x \to +\infty$, which with the fact that $M$ decreases and $M(0) = 1 > 0$ imply that there exists a unique $0 < R^* < +\infty$ such that $M(R^*) = 0$, meaning that

$$I_0(R^*) = \sqrt{2}\ I_1(R^*).$$

Note that $R^* \simeq 2$, and from (8.93) $G'(R^*) > 0$.

**Step 1: The sense of variation of $G$**

To determine the cardinal of the set $\{r \in ]0, R[, \quad \frac{\beta}{a} = G(r)\}$, we need to know the sense of variation of the function $G$. For this, we distinguish with respect to $R$ the radius of the disk $B_R$ the two following cases. We start with

**Case1:** $I_0(R) \geq \sqrt{2}\, I_1(R)\Big( \Longleftrightarrow R \leq R^*\Big)$
In this case, we have $G'(R) \geq 0$. The function $M$ is decreasing in $[0, +\infty[$, hence in particular $\forall\, x \in ]0, R]$,

$$(I_0(x))^2 - 2\,(I_1(x))^2 \geq (I_0(R))^2 - 2\,(I_1(R))^2.$$

We insert this in (8.93) to have

$$\frac{T^2(x)}{2}\ G'(x) \geq \Big((I_0(R))^2 - 2\,(I_1(R))^2\Big)\ B(x). \tag{8.94}$$

Thanks to the fact that $I_0(R) \geq \sqrt{2}\, I_1(R)$, we get

$$\forall\, 0 < x < R, \quad G'(x) > 0.$$

This implies that $G$ is increasing in $]0, R[$, so

$$\forall\, x \in ]0, R[, \quad G(x) < G(R).$$

Remember that any critical point $r$ of $F$ satisfies $\frac{\beta}{a} = G(r)$, so the intersection between the graph of $x \to G(x)$ and the horizontal line of equation $y = \frac{\beta}{a}$ is restricted to one point ( even without a condition on $\beta$). Consequently, there is a unique critical point of $F$ in $]0, R[$.

**Case2:** $I_0(R) < \sqrt{2}\, I_1(R)\Big( \Longleftrightarrow R > R^*\Big)$
First, it is clear that $G'(r) > 0$ for any $r \in ]0, R^*]$. But, unfortunately we have no idea on the sign of $G'$ on the interval $[R^*, R]$. Then, from now on we will be concerned with the study of the behavior of $G$ on the interval $[R^*, R]$. Knowing $R > R^*$, we have $G'(R) < 0$, then combining this with the fact that $G'(R_*) > 0$, there exists at least $r_+$, $R^* < r_+ < R$ such that

$$G'(r_+) = 0.$$

We will prove that $r_+$ is the unique point in $[R^*, R]$ where the function $G'$ vanishes. Firstly, after a simple calculation, the second derivative of the function $G$ for $r$ in $[R^*, R]$ is

$$\begin{aligned}\frac{T^4\ G''(r)}{2} =& I_1\ T^3 + I_0\ T^2\ (2\ I_1\ B - 2\ I_0\ X - \frac{T}{r}) + 2\ (I_0 - \frac{I_1}{r})\ (I_0\ X - I_1\ B)\ T^2 + 2\ I_1\ T^2\ (I_1\ X - I_0\ B)\\ & -2\ I_1\ T^2\ B\ (I_0 - \frac{I_1}{r}) + 2\ I_1^2\ T^2\ (X + \frac{B}{r}) - 2\ T\ T'\ \Big(I_0\ T + 2\ I_1\ (I_0\ X - I_1\ B)\Big).\end{aligned} \tag{8.95}$$

Because we know that $G'(r_+) = 0$ and $r_+ \in [R^*, R]$, so the set of the critical points of $G$ in $[R^*, R]$ is not empty. Let $r$ be an arbitrary critical point of $G$ in $[R^*, R]$, then in particular thanks to (8.92),

$$I_1(r)\ B(r) - I_0(r)\ X(r) = \frac{I_0\ T(r)}{2\ I_1(r)}. \tag{8.96}$$

Replacing $T$ with $(I_0\ B - I_1\ X)$ in (8.96), we can write

$$X(r) = \frac{\Big(2\ (I_1(r))^2 - (I_0(r))^2\Big)\ B(r)}{I_0(r)\ I_1(r)}. \tag{8.97}$$

Now, replacing $(I_0\ B - I_1\ X)$ with $T$ and Inserting (8.96) in (8.95), it is easy to find

$$\frac{T^2\ G''(r)}{2\ I_1} = 4\ I_1\ \frac{B}{r} - 3\ T = (\frac{I_1}{r} - 3\ I_0)\ B - 3\ I_1\ X. \tag{8.98}$$

Let us replace the $X$ given in (8.98) with the right-hand side of (8.97), then any critical point $r$ of $G$ in $[R^*, R]$ satisfies

$$\frac{r\ I_0\ T^2\ G''(r)}{4\ I_1\ B} = 2\ I_1\ I_0 + 3\ r\ (I_1)^2 - 3\ r\ (I_0)^2. \tag{8.99}$$

Let us study the right-hand side of (8.99). Its derivative is equal to the quantity $\Big(-(I_0)^2 - (I_1)^2 - 2\ \frac{I_1\ I_0}{r}\Big)$, which is negative. Then, the right-hand side of (8.99) is decreasing in the interval $[R^*, R]$, so by the definition of $R^*$ which is such that $I_0(R^*) = \sqrt{2}\ I_1(R^*)$ and then $R^* \simeq 2$, we have for any $x \in [R^*, R]$

$$\begin{aligned} 2\ I_1(x)\ I_0(x) + 3\ x\ (I_1(x))^2 - 3\ x\ (I_0(x))^2 &\leq 2\ I_1(R^*)\ I_0(R^*) + 3\ R^*\ (I_1(R^*))^2 - 3\ R^*\ (I_0(R^*))^2 \\ &= (2\ \sqrt{2}\ - 3\ R^*)\ (I_1(R^*))^2 \leq 0. \end{aligned}$$

Thus, going back to (8.99), we conclude that any critical point $r$ of the function $G$ in $[R^*, R]$ satisfies

$$G''(r) \leq 0. \tag{8.100}$$

Thus, $r$ is necessarily a maximum of $G$ in $[R^*, R]$, so that by continuity of $G$, there is a unique critical point of $G$ in $[R^*, R]$. But, knowing that $G'(r_+) = 0$, hence $r_+ = r$ is the unique critical point of $G$, and it is then the maximum of $G$. So, in particular $G'(r) > 0$ for any $r \in [R^*, r_+[$. We know that $G'(r) > 0$ for any $r \in ]0, R^*]$, then as a consequence of the above, we have $G'(r_+) = 0$ and $G'(r) > 0$ for $r \in ]0, r_+[$ with $G'(r) < 0$ for $r \in ]r_+, R]$. It means that $G$ is increasing in $]0, r_+[$ and is decreasing in $]r_+, R]$. Thus, we must have $G(r_+) > G(R)$. Now, assume that the parameter $\beta$ satisfies

$$\beta < \frac{2\ R\ I_1(R)}{I_0(R)}.$$

Then, we can write

$$\frac{\beta}{a} < \frac{2\ R\ I_1(R)}{a\ I_0(R)}.$$

Note that the right-hand side of the above inequality is $G(R)$, so this means that $\frac{\beta}{a} < G(R)$. The set $\{r \in ]0, R[, \ \ G(r) = \frac{\beta}{a}\}$ is then restricted to one point, so there is a unique critical point of $F$ in $]0, R[$.

**Step 2: The nature of the critical point of $F$**

Finally, let us determine the nature of the critical point of $F$ in each case. On the one hand, going back to (8.90) and using the fact that $T(R) = \frac{a\ I_0(R)}{R}$, we get

$$F'(R) = \frac{\beta}{I_0(R)} \Big(2\ I_1(R) - \frac{\beta}{a}\ T(R)\Big) = \frac{\beta}{I_0(R)} \Big(2\ I_1(R) - \beta\ \frac{I_0(R)}{R}\Big).$$

Then, under the fact that $\beta < \frac{2\ R\ I_1(R)}{I_0(R)}$, we get $F'(R) > 0$. On the other hand, by definition of the functions $I_i$ and $K_i$ where $0 \leq i \leq 1$, we can deduce $T(x) \to +\infty$ as $x \to 0$. Inserting this in (8.90),

$$F'(x) \to -\infty \quad \text{as} \ \ x \to 0.$$

Consequently, combining this together with $F'(R) > 0$ implies that the unique critical point of the functional $F$ in the interval $]0, R[$ is necessarily a minimizer and in particular, it is in $]0, R[$. The proposition 8.14 is then proved.

## 4.4 The finer upper bound of the minimal energy

From now on, the applied magnetic field is taken to satisfy

$$\lambda < 2 - \frac{2}{I_0(R)}. \tag{8.101}$$

Moreover, assume that

$$\beta < \frac{2\ R\ I_1(R)}{I_0(R)},$$

then, let us define $R_0$ by

$$F(R_0) = \inf_{r \in ]0,R[} F(r). \tag{8.102}$$

From the proposition 8.14, the minimum $R_0$ exists, is unique and it is in $]0, R[$. In addition, it satisfies

$$G(R_0) = \frac{2\ I_1(R_0)}{T(R_0)} = \frac{\beta}{a}, \tag{8.103}$$

where $a = R\ \Big(I_0(R)\ K_1(R) + I_1(R)\ K_0(R)\Big)$. We finish the section with the following fundamental result which will be essential for the rest.

**Lemma 8.16.** *Let $(u_\varepsilon, A_\varepsilon)$ a minimizer of $J$ over $G_\varepsilon$ and $a = R\ \Big(I_0(R)\ K_1(R) + I_1(R)\ K_0(R)\Big)$. Assume that*

$$\beta < \frac{2\ R\ I_1(R)}{I_0(R)},$$

*then if the applied field is such that*

$$\frac{2\ I_0(R) - 2}{I_0(R)} - \lambda > \frac{2\ I_0(R_0) - 2}{I_0(R)} + \beta\ \frac{I_0(R_0)\ X(R_0)}{a\ I_0(R)}, \tag{8.104}$$

*we have*

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} < \hat{J}_0. \tag{8.105}$$

**Remark 8.17.** *Thanks to the assumption (8.101), we remark that the left-hand side of (8.104) is positive. Then, for a small enough $\beta > 0$, the condition (8.104) has a sense. Moreover thanks to the inequality (8.105), we obtain in the limit $\varepsilon \to 0$ the presence of concentric circles of vortices (at least one circle) of center $O$, the center of the disk.*

**Proof**: Let us evaluate the energy $F(R_0)$

$$F(R_0) = \frac{\hat{J}_0}{\pi} + \beta\ \ \lambda - \beta\ \left(2 - \frac{2\ I_0(R_0)}{I_0(R)}\right) + \beta^2\ \ \frac{I_0(R_0)\ X(R_0)}{a\ I_0(R)}. \tag{8.106}$$

If the applied field is such that

$$\frac{2\ I_0(R) - 2}{I_0(R)} - \lambda > \frac{2\ I_0(R_0) - 2}{I_0(R)} + \beta\ \frac{I_0(R_0)\ X(R_0)}{a\ I_0(R)},$$

we obtain from (8.106)

$$F(R_0) < \frac{\hat{J}_0}{\pi}.$$

Referring to the definition of $F$ given by (8.83), we deduce

$$E_1(R_0, 1) < \hat{J}_0. \tag{8.107}$$

Now, going back to (8.84), we can write

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \leq E_1(R_0, 1). \tag{8.108}$$

Thanks to (8.107), we conclude

$$\limsup_{\varepsilon \to 0} \frac{J(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} < \hat{J}_0.$$

The lemma is then proved. □

# 5 Lower bound

Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J$ over the space $G_\varepsilon$ and $h_\varepsilon = curl A_\varepsilon$. Here, the applied field is taken to verify

$$0 < \lambda < 2\left(1 - \frac{1}{I_0(R)}\right).$$

## 5.1 The vortex balls

Similar to the proposition 4.1, we can state

**Proposition 8.18.** *For $h_{ex} \le C\ |\log\varepsilon|$, there exists $\varepsilon_0$ such that if $\varepsilon < \varepsilon_0$ and $(u_\varepsilon, A_\varepsilon)$ a minimizer of $J$ over $G_\varepsilon$, then there exist $r_\varepsilon \in ]\frac{1}{|\log\varepsilon|}, \frac{2}{|\log\varepsilon|}[$, $\theta_1 \in [0, 2\pi]$ and a family of disjoint balls $\Big(B_i = B(a_i, r_i)\Big)_{i\in\mathcal{L}_\varepsilon\cup\mathcal{T}_\varepsilon}$ of center $a_i$ and of radii $r_i$ such that*

$$\overline{\cup_{i\in\mathcal{L}_\varepsilon} B_i(a_i, r_i)} \subset B(0, r_\varepsilon), \tag{8.109}$$

$$\overline{\cup_{i\in\mathcal{T}_\varepsilon} B_i(a_i, r_i)} \subset \{r\, e^{i\,\theta},\ \ r_\varepsilon < r \le R,\ \ \ \theta_1 < \theta < \theta_1 + \frac{2\,\pi}{q_\varepsilon}\}, \tag{8.110}$$

$$\sum_{i\in\mathcal{L}_\varepsilon\cup\mathcal{T}_\varepsilon} r_i \le C\ |\log\varepsilon|\ e^{-\sqrt{|\log\varepsilon|}}, \tag{8.111}$$

$$card(\mathcal{L}_\varepsilon \cup \mathcal{T}_\varepsilon) \le C\ |\log\varepsilon|\ h_{ex}, \tag{8.112}$$

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i) \ge \pi\ |d_i|\ |\log\varepsilon|\ (1 - o(1)), \tag{8.113}$$

*where $d_i$ is the degree of the map $\frac{u_\varepsilon}{|u_\varepsilon|}$ restricted to $\partial B_i$ if $\overline{B_i} \subset B_R$ and $d_i = 0$ otherwise.*

### Notation

Taking the radius $r_\varepsilon$ and the parameter $\theta_1$ given by the above proposition, we take $S_{r_\varepsilon,\theta_1}$ the sector

$$S_{r_\varepsilon,\theta_1} = \{r\, e^{i\,\theta},\ \ \ r_\varepsilon < r \le R,\ \ \ \theta_1 < \theta < \theta_1 + \frac{2\,\pi}{q_\varepsilon}\}. \tag{8.114}$$

Note that the angle of $S_{r_\varepsilon,\theta_1}$ is $\frac{2\,\pi}{q_\varepsilon}$.

## 5.2 Proof of proposition 8.18

Before all, letting $\Omega = B_R$, $m_\varepsilon = \frac{1}{\sqrt{|\log\varepsilon|}}$, and $\alpha_\varepsilon = |\log\varepsilon|$ in the proposition 4.12, we have

**Lemma 8.19.** *If $h_{ex} \le C\ |\log\varepsilon|$, there exists $\varepsilon_0$ such that if $\varepsilon < \varepsilon_0$ and $(u_\varepsilon, A_\varepsilon)$ satisfies $|\nabla u_\varepsilon - i\ A_\varepsilon\ u_\varepsilon| < \frac{C}{\varepsilon}$ and $F_\varepsilon(u_\varepsilon, A_\varepsilon, B_R) \le C\ |\log\varepsilon|\ h_{ex}$, then there exists a family of disjoint balls $\Big(B_i = B(a_i, r_i)\Big)_{i\in\urcorner_\varepsilon}$ of center $a_i$ and of radii $r_i$ such that*

$$\{x \in B_R,\ |u_\varepsilon| < \frac{3}{4}\} \subset \cup_{i\in\urcorner_\varepsilon}, B_i, \tag{8.115}$$

$$\sum_{i \in \daleth_\varepsilon} r_i \leq C\ |\log \varepsilon|\ e^{-\sqrt{|\log \varepsilon|}}, \tag{8.116}$$

$$card(\daleth_\varepsilon) \leq C\ |\log \varepsilon|\ h_{ex}, \tag{8.117}$$

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B_i) \geq \pi\ |d_i|\ |\log \varepsilon|\ (1 - o(1)), \tag{8.118}$$

*where* $d_i$ *is the degree of the map* $\frac{u_\varepsilon}{|u_\varepsilon|}$ *restricted to* $\partial B_i$ *if* $\overline{B_i} \subset B_R$ *and* $d_i = 0$ *otherwise.*

Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over $G_\varepsilon$, then it is solution of the Ginzburg-Landau equations (8.1)-(8.2), so going back to (3.13),

$$|\nabla u_\varepsilon - i\ A_\varepsilon\ u_\varepsilon| \leq \frac{C}{\varepsilon}. \tag{8.119}$$

$(1, 0) \in G_\varepsilon$, then testing the energy $J$ by the configuration $(1, 0)$, the minimum of the energy $J_{B_R}$ is less than $J_{B_R}(1, 0) = \frac{\pi\, R^2}{2}\ h_{ex} \leq C\ |\log \varepsilon|\ h_{ex}$. By definition of the functional $F_\varepsilon$, it follows that

$$F_\varepsilon(u_\varepsilon, A_\varepsilon, B_R) \leq J_{B_R}(u_\varepsilon, A_\varepsilon) \leq C\ |\log \varepsilon|\ h_{ex}.$$

So combining all the above, the hypotheses of lemma 8.19 are verified. Then applying it there exists a family of balls in $B_R$ depending on $\varepsilon$ denoted by $(B_i)_{i \in \daleth_\varepsilon} = \Big(B(a_i, r_i)\Big)_{i \in \daleth_\varepsilon}$ such that the assertions (8.115)-(8.116)-(8.117) and (8.118) hold. We start by the proof of the assertions (8.109)-(8.110). First,

$$\sum_{i \in \daleth_\varepsilon} r_i \leq C\ |\log \varepsilon|\ e^{-\sqrt{|\log \varepsilon|}}.$$

Therefore,

$$\sum_{i \in \daleth_\varepsilon} 2\ r_i = o(\frac{1}{|\log \varepsilon|}).$$

Hence, if $\varepsilon$ is small enough, there exists $c$, $1 < c < 2$ such that when we take $r_\varepsilon = \frac{c}{|\log \varepsilon|}$, the boundary of the ball of center O and of radius $r_\varepsilon$ does not intersect any ball of the family $\Big(B_i(a_i, r_i)\Big)_{i \in \daleth_\varepsilon}$. We define

$$\mathcal{L}_\varepsilon = \{i \in \daleth_\varepsilon, \quad B_i(a_i, r_i) \subset B(0, r_\varepsilon)\},$$

then (8.109) is satisfied. Now, in view of the fact that $q_\varepsilon \simeq \beta\ |\log \varepsilon|$ as $\varepsilon \to 0$ and $r_\varepsilon = \frac{c}{|\log \varepsilon|}$, we can write

$$\sum_{i \in \daleth_\varepsilon} \frac{2\ r_i}{r_\varepsilon} = o(\frac{2\ \pi}{q_\varepsilon}). \tag{8.120}$$

Consequently, projecting the balls $\Big(B_i(a_i, r_i)\Big)_{i \in \daleth_\varepsilon}$ on the curve

$$\{r\ e^{i\,\theta}, \ \ r = r_\varepsilon, \ \ \theta \ \text{ belongs to an intervall of length } \ \frac{2\ \pi}{q_\varepsilon}\},$$

then thanks to (8.120) which gives us a comparison of the angles, and if $\varepsilon$ is small enough, there exists necessarily $0 \leq \theta_1 \leq 2\,\pi$ such that the two lines $\{r\ e^{i\theta},\ \ r \in [r_\varepsilon, R]$ and $\theta = \theta_1\}$ and $\{r\ e^{i\theta},\ \ r \in [r_\varepsilon, R]$ and $\theta = \theta_1 + \frac{2\,\pi}{q_\varepsilon}\}$ don't intersect any ball of the family $\{\Big(B_i(a_i, r_i)\Big)_{i \in \daleth_\varepsilon \setminus \mathcal{L}_\varepsilon}\}$. These two lines together with $\{(r, \theta),\ \ r = r_\varepsilon$ and $\theta_1 < \theta < \theta_1 + \frac{2\,\pi}{q_\varepsilon}\}$ form in the disk $B_R$ the boundary of the sector $S_{r_\varepsilon, \theta_1}$ which is defined by (8.114). Now, let us define

$$\mathcal{T}_\varepsilon = \Big\{ i \in \daleth_\varepsilon,\ \ B_i(a_i, r_i) \subset \{r\ e^{i\,\theta},\ \ r_\varepsilon < r \leq R,\quad \theta_1 < \theta < \theta_1 + \frac{2\,\pi}{q_\varepsilon}\} = S_{r_\varepsilon, \theta_1} \Big\}.$$

Thanks to the fact that the balls $\Big(B_i(a_i, r_i)\Big)_{i \in \daleth_\varepsilon}$ are disjoint, hence in particular by definition of $\mathcal{L}_\varepsilon$ and $\mathcal{T}_\varepsilon$, the balls $\Big(B_i = B(a_i, r_i)\Big)_{i \in \mathcal{L}_\varepsilon \cup \mathcal{T}_\varepsilon}$ are disjoint too. Moreover, it is clear that the three assertions (8.111)-(8.112) and (8.113) hold. Combining all the above completes the proof of the proposition 8.18.

**Notation**

Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over the space $G_\varepsilon$. As defined by proposition 8.18, note that $\{(a_i, d_i)_{i \in \mathcal{L}_\varepsilon}\}$ is the associated family of vortices in the ball $B_{r_\varepsilon} = B(0, r_\varepsilon)$, while $\{(a_i, d_i)_{i \in \mathcal{T}_\varepsilon}\}$ is the associated family of vortices in the sector $S_{r_\varepsilon, \theta_1}$. Now, let us extend the family $\{B_i(a_i, r_i)_{i \in \mathcal{T}_\varepsilon}\}$ by S -periodicity to $\overline{B_R} \backslash B_{r_\varepsilon}$. For simplification, let

$$S^1_{r_\varepsilon, \theta_1} = S_{r_\varepsilon, \theta_1}.$$

For any $i \in \mathcal{T}_\varepsilon$, the ball $B_i(a_i, r_i)$ defined on $S^1_{r_\varepsilon, \theta_1}$ will be denoted

$$B_i(a_i, r_i) = B^1_i(a^1_i, r_i),\ \ \forall\, i \in \mathcal{T}_\varepsilon.$$

Then, for $i \in \mathcal{T}_\varepsilon$, we let $B^j_i(a^j_i, r_i)$, $1 \leq j \leq q_\varepsilon$ be the extended of the ball $B^1_i(a^1_i, r_i)$ by $S$ -periodicity to $S^j_{r_\varepsilon, \theta_j}$, $1 \leq j \leq q_\varepsilon$ where $\theta_j = \theta_1 + \frac{2\,\pi\,(j-1)}{q_\varepsilon}$. So that the extended balls defined on $S^j_{r_\varepsilon, \theta_j}$ are $\Big(B^j_i(a^j_i, r_i)\Big)_{i \in \mathcal{T}_\varepsilon}$. Consequently, we get in the annulus $\overline{B_R} \backslash B_{r_\varepsilon}$

$$\Big(B^j_i(a^j_i, r_i)\Big)_{(1 \leq j \leq q_\varepsilon,\ i \in \mathcal{T}_\varepsilon)}.$$

Note that the sector $S^1_{r_\varepsilon, \theta_1}$ can be taken as the fundamental domain of periodicity for $(u_\varepsilon, A_\varepsilon) \in G_\varepsilon$ in the annulus $\overline{B_R} \backslash B_{r_\varepsilon}$. Moreover, thanks to (8.115) and by definition of $\mathcal{T}_\varepsilon$ and $\mathcal{L}_\varepsilon$, it is easy to get from the fact that $|u_\varepsilon|(x\ e^{i\,\frac{2\,\pi}{q_\varepsilon}}) = |u_\varepsilon|$, the following property for the vortex balls

$$\{x \in B_R,\ |u_\varepsilon| < \frac{3}{4}\} \subset \Big( [\cup_{(1 \leq j \leq q_\varepsilon,\ i \in \mathcal{T}_\varepsilon)} B^j_i] \cup [\cup_{\, i \in \mathcal{L}_\varepsilon} B_i] \Big). \tag{8.121}$$

## 5.3 Preliminaries

Recall that for $1 \leq j \leq q_\varepsilon$, we have $\theta_j = \theta_1 + \frac{2\,\pi\,(j-1)}{q_\varepsilon}$. Then

$$S^j_{r_\varepsilon,\theta_j} = \{r\, e^{i\,\theta}, \quad r_\varepsilon < r \leq R, \quad \theta_j < \theta < \theta_j + \frac{2\,\pi}{q_\varepsilon}\}.$$

For $i \in \mathcal{T}_\varepsilon$, let $B^j_i(a^j_i, r_i)$ be contained strictly in the sector $S^j_{r_\varepsilon,\theta_j}$, then writing locally $\frac{u_\varepsilon}{|u_\varepsilon|} = e^{i\,\varphi_\varepsilon}$ and taking $B^{j+1}_i$ which is the image of $B^j_i$ by rotation of angle $\frac{2\,\pi}{q_\varepsilon}$ and center O, we have

$$deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B^j_i) = \frac{1}{2\,\pi} \int_{\partial B^j_i} \nabla\varphi_\varepsilon\ .\tau. \tag{8.122}$$

Using (8.8), there exists $l \in \mathbb{Z}$ such that $\forall\ x \in B_R$

$$\varphi_\varepsilon(x) = \varphi_\varepsilon(x\ e^{-i\frac{2\,\pi}{q_\varepsilon}}) + f(x\ e^{-i\frac{2\,\pi}{q_\varepsilon}}) + 2\,\pi\ l.$$

We take the gradient

$$\nabla\varphi_\varepsilon(x) = e^{-i\frac{2\,\pi}{q_\varepsilon}}\ \Big(\nabla\varphi_\varepsilon\Big)(x\ e^{-i\frac{2\,\pi}{q_\varepsilon}}) + e^{-i\frac{2\,\pi}{q_\varepsilon}}\Big(\nabla f\Big)(x\ e^{-i\frac{2\,\pi}{q_\varepsilon}}).$$

We insert this in (8.122)

$$\begin{aligned} deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B^{j+1}_i) =& \frac{1}{2\,\pi} \int_{\partial B^j_i} \frac{\partial\varphi_\varepsilon}{\partial\tau} + \frac{\partial f}{\partial\tau} \\ =& \frac{1}{2\,\pi} \int_{\partial B^j_i} \frac{\partial\varphi_\varepsilon}{\partial\tau} = deg(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B^j_i). \end{aligned} \tag{8.123}$$

Set

$$D_\varepsilon := \sum_{i\in\mathcal{T}_\varepsilon} |d_i|.$$

Now, our interest is to give the order of $D_\varepsilon$. $h_{ex} \leq C\ |\log\varepsilon|$, hence thanks to (8.113) and (8.123),

$$\pi\ \Big(q_\varepsilon\ D_\varepsilon + \sum_{i\in\mathcal{L}_\varepsilon} |d_i|\Big)\ |\log\varepsilon|\ (1 - o(1)) \leq J_{B_R}(u_\varepsilon, A_\varepsilon) \leq C\ h^2_{ex} \leq C\ |\log\varepsilon|\ h_{ex}. \tag{8.124}$$

We deduce from (8.124) that

$$\sum_{i\in\mathcal{L}_\varepsilon} |d_i| \leq C\ h_{ex}, \tag{8.125}$$

and

$$q_\varepsilon\ D_\varepsilon \leq C\ h_{ex}. \tag{8.126}$$

Inserting $\frac{h_{ex}}{q_\varepsilon}$ tends to $\frac{1}{\beta}$ as $\varepsilon$ tends to 0 in (8.126), we conclude

$$D_\varepsilon \leq C. \tag{8.127}$$

### 5.4 Expansion of the energy

Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over $G_\varepsilon$. We have noted that the resulting family of vortices in $B_R$ is

$$\Big\{(a_i, d_i), \ \ i \in \mathcal{L}_\varepsilon\Big\} \cup \Big\{(a_i^j, d_i), \quad i \in \mathcal{T}_\varepsilon, \quad 1 \le j \le q_\varepsilon\Big\}. \tag{8.128}$$

For any such of the family of vortices defined by (8.128), we take the measure

$$\mu_\varepsilon = \frac{2\,\pi\,\Big(\sum_{i\in\mathcal{L}_\varepsilon} d_i\,\delta_{a_i} + \sum_{i\in\mathcal{T}_\varepsilon} d_i\,(\sum_{j=1}^{q_\varepsilon}\delta_{a_i^j})\Big)}{h_{ex}}. \tag{8.129}$$

In such the Coulomb gauge (8.10), the $H^1$ norms of $u_\varepsilon$ and $A_\varepsilon$ are controlled by $\sqrt{J_{B_R}(u_\varepsilon, A_\varepsilon)}$, consequently by $C\ h_{ex}$. Then, applying the Theorem [SS5] for a minimizer $(u_\varepsilon, A_\varepsilon)$ over $G_\varepsilon$ (because it is a critical point of $J$), there exist $h_\infty \in H_1^1(B_R, \mathbb{R})$ and a Radon measure $\mu_\infty$ such that up an extraction of $\varepsilon_n$ from $\varepsilon$

$$\frac{h_{\varepsilon_n}}{h_{ex}} \to h_\infty \quad \text{weakly in } H_1^1(B_R), \tag{8.130}$$

and

$$\mu_{\varepsilon_n} \to \mu_\infty \ \ \text{in the sense of measures.} \tag{8.131}$$

Moreover, we have

$$\mu_\infty = -\Delta h_\infty + h_\infty. \tag{8.132}$$

**Lemma 8.20.** *Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over $G_\varepsilon$, then*

$$\liminf_{n\to+\infty}\frac{J_{B_R}(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \ge E(h_\infty) = \frac{\lambda}{2}\int_{B_R}|-\Delta h_\infty + h_\infty| + \frac{1}{2}\int_{B_R}|\nabla h_\infty|^2 + \frac{1}{2}\int_{B_R}|h_\infty - 1|^2. \tag{8.133}$$

**Proof**: Splitting the energy $J_{B_R}$ between the contribution inside the vortex-balls $\Big([\cup_{(1\le j\le q_\varepsilon,\ i\in\mathcal{T}_\varepsilon)} B_i^j] \cup [\cup_{\ i\in\mathcal{L}_\varepsilon} B_i]\Big)$ and the contribution outside, we get

$$J_{B_R}(u_\varepsilon, A_\varepsilon) \ge \pi\,\Big(q_\varepsilon\ D_\varepsilon + \sum_{i\in\mathcal{L}_\varepsilon}|d_i|\Big)\,|\log\varepsilon|\,(1 - o(1)) + \frac{1}{2}\int_{B_R\setminus\Big((\cup_{i\in\mathcal{L}_\varepsilon}B_i)\cup(\cup_{(i\in\mathcal{T}_\varepsilon,\ 1\le j\le q_\varepsilon)}B_i^j)\Big)}|\nabla h_\varepsilon|^2$$
$$+\frac{1}{2}\int_{B_R\setminus\Big((\cup_{i\in\mathcal{L}_\varepsilon}B_i)\cup(\cup_{(i\in\mathcal{T}_\varepsilon,\ 1\le j\le q_\varepsilon)}B_i^j)\Big)}|h_\varepsilon - h_{ex}|^2 - o(1). \tag{8.134}$$

Now, we divide (8.134) by $h_{ex}^2$ and we proceed similarly as in [SS3], lemma 2.2 to obtain

$$\liminf_{n\to+\infty}\frac{J_{B_R}(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \ge \frac{\lambda}{2}\int_{B_R}|-\Delta h_\infty + h_\infty| + \frac{1}{2}\int_{B_R}|\nabla h_\infty|^2 + \frac{1}{2}\int_{B_R}|h_\infty - 1|^2.$$

□

From (8.130) and (8.131), we can mention that $\mu_\infty \in H^{-1}$, so in particular no concentration of the vorticity on isolated points. From now on, our interest is to determine the support of the limit measure of vorticity.

## 5.5 Properties of $h_\infty$ and $\mu_\infty$

First, we begin with

**Lemma 8.21.** *$h_\infty$ is continuous on $B_R$.*

**Proof**: Referring to [SS5], lemma 4.1, we have

$$|\nabla h_\infty| \in W^{1,p}(B_R), \quad 1 \leq p < +\infty.$$

In particular

$$h_\infty \in W^{1,p}(B_R), \quad 1 \leq p < +\infty.$$

By Sobolev injection, we conclude

$$h_\infty \in C^{0,\alpha}(B_R), \quad 0 \leq \alpha < 1,$$

which completes the proof of lemma. □

The following proposition gives us other properties of the limiting configuration of vortices $(h_\infty, \mu_\infty)$.

**Proposition 8.22.** *We have*

$$h_\infty \in Y,$$

*where $Y$ is defined by (8.11).*

We split the proof of proposition 8.22 into two steps.

**Step 1: $h_\infty$ is radial**

First, we take for any $x \in B_R$

$$x = r\, e^{i\,\theta}, \quad 0 \leq r < R, \quad 0 \leq \theta \leq 2\,\pi.$$

For $\theta \in [0, 2\,\pi]$, let $\varepsilon_n \to 0$ and $k_n$ an integer such that

$$\frac{2\,\pi\,k_n}{q_{\varepsilon_n}} \to \theta \quad \text{as} \quad n \to +\infty.$$

We take $R_n$ to be the rotation of center O and of angle $\frac{2\,\pi\,k_n}{q_{\varepsilon_n}}$. Taking the curl in (8.9), we get for any $n \in \mathbb{N}$

$$h_{\varepsilon_n} \circ R_n = h_{\varepsilon_n}. \tag{8.135}$$

Since, $\{\frac{h_{\varepsilon_n}}{h_{ex}}\}_n$ is bounded in $H^1(\Omega)$, there exists a subsequence still denoted $n$ such that $\{\frac{h_{\varepsilon_n}}{h_{ex}}\}_n$ and $\{\frac{h_{\varepsilon_n} \circ R_n}{h_{ex}}\}_n$ converge weakly in $H^1$ to the same limit which thanks to (8.130) is $h_\infty$. In addition, for any $\Phi \in C_0^\infty(B_R)$, and by change of variables we obtain

$$\int_{B_R} \frac{h_{\varepsilon_n} \circ R_n}{h_{ex}}\ \Phi = \int_{B_R} \frac{h_{\varepsilon_n}}{h_{ex}}\ (\Phi \circ R_n^{-1}), \tag{8.136}$$

where $R_n^{-1}$ is the rotation of center O and of angle $-\frac{2\,\pi\,k_n}{q_{\varepsilon_n}}$. Inserting (8.135) in (8.136) to have

$$\int_{B_R} \frac{h_{\varepsilon_n}}{h_{ex}}\ \Phi = \int_{B_R} \frac{h_{\varepsilon_n}}{h_{ex}}\ (\Phi \circ R_n^{-1}), \tag{8.137}$$

But, as $n \to +\infty$

$$\Phi \circ R_n^{-1} \to \Phi \circ R_{-\theta} \quad \text{in} \quad C^k(B_R)\ \forall\, k, \tag{8.138}$$

where $R_{-\theta}$ is the rotation of center O and of angle $-\theta$. Thus, we pass to the limit in (8.137) and we use (8.138) to find

$$\int_{B_R} h_\infty\ \Phi = \int_{B_R} h_\infty\ (\Phi \circ R_{-\theta}), \tag{8.139}$$

Now, again by change of variables, it is easy that

$$\int_{B_R} h_\infty\ (\Phi \circ R_{-\theta}) = \int_{B_R} (h_\infty \circ R_\theta)\ \Phi. \tag{8.140}$$

Comparing (8.139) to (8.142), we get for any $\Phi \in C_0^\infty(B_R)$

$$\int_{B_R} h_\infty\ \Phi = \int_{B_R} (h_\infty \circ R_\theta)\ \Phi. \tag{8.141}$$

We deduce for any $\theta \in [0, 2\,\pi]$

$$h_\infty = h_\infty \circ R_\theta. \tag{8.142}$$

It means that $h_\infty$ is radial. The step 1 is then proved.

**Step2: $\mu_\infty$ is supported on a finite number of concentric circles of center $O$ and of strict positive radii such that the mass of $\mu_\infty$ on each one belongs to $2\,\pi\,\beta\,\mathbb{Z}$**

The balls $\left(B_i^j(a_i^j, r_i)\right)_{(i \in \mathcal{T}_\varepsilon,\ 1 \le j \le q_\varepsilon)}$ defined in $\overline{B_R} \backslash B_{r_\varepsilon}$ by proposition 8.18 depends on $\varepsilon$, hence from now on we write

$$d_i(\varepsilon) = d_i \quad \text{and} \quad a_i^j(\varepsilon) = a_i^j \quad \text{for} \quad i \in \mathcal{L}_\varepsilon,\ 1 \le j \le q_\varepsilon,$$

where $d_i = deg\Big(\frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_i^j(a_i^j, r_i)\Big)$. First, for any $\varepsilon < \varepsilon_0$, (8.127) gives us

$$D_\varepsilon = \sum_{i \in \mathcal{T}_\varepsilon} |d_i| \le C.$$

Thus, the cardinal of $\{i \in \mathcal{T}_\varepsilon,\ \ d_i(\varepsilon) \neq 0\}$ is bounded independently of $\varepsilon$. First, if for any $\varepsilon < \varepsilon_0$, $d_i(\varepsilon) = 0$, $\forall\, i \in \mathcal{L}_\varepsilon$. This means that for any $\varepsilon < \varepsilon_0$, $D_\varepsilon = 0$, so the measure $\mu_\varepsilon$ defined by (8.129) is written as

$$\mu_\varepsilon = \frac{2\,\pi\,\sum_{i\in\mathcal{L}_\varepsilon} d_i\,\delta_{a_i}}{h_{ex}}.$$

The points $(a_i)_{i\in\mathcal{L}_\varepsilon}$ are in the ball $B_{r_\varepsilon}$, then using $r_\varepsilon \to 0$ together with the fact that the limit measure $\mu_\infty$ is not concentrated on isolated points ( in particular on the center of the disk $B_R$), we find

$$\mu_\infty = 0. \tag{8.143}$$

Second, if for sufficiently small $\varepsilon$, there exist points with non zero degrees, then without loss of generality there exists $m \in \mathbb{N}^*$ such that these points are denoted

$$\{a_i^j(\varepsilon),\ \ 1 \le i \le m,\ \ 1 \le j \le q_\varepsilon\}.$$

Then, up to extraction from $\varepsilon \to 0$, we can get for any $1 \le i \le m$

$$d_i(\varepsilon_n) \to p_i, \quad \text{and} \quad a_i^1(\varepsilon_n) \to b_i^1 \quad \text{as} \quad n \to +\infty, \tag{8.144}$$

where $p_i \in \mathbb{Z}$ and $b_i^1$ is contained strictly in the sector $S^1_{r_\varepsilon,\theta_1}$. To simplify, we take

$$\forall\, 1 \le k \le m, \quad b_k^1 = r_k\, e^{i\,\theta_k} \quad \text{where} \quad 0 < r_1 < ... < r_m < R.$$

Note that $r_k$ is constant and does not depend on $\varepsilon$. The extended points of $(b_k^1)_{1\le k\le m}$ by $S$ -periodicity to $B_R \backslash B_{r_\varepsilon}$ are

$$\{b_k^j = (r_k\, e^{i\frac{2\,\pi\,(j-1)}{q_\varepsilon}}\ e^{i\,\theta_k}),\ \ 1 \le k \le m, \quad 1 \le j \le q_\varepsilon\}.$$

Let $\Gamma^k(r_k)$ be the circle of center 0 and of radius $r_k$. It is clear for $1 \le k \le m$ and $n \to \infty$,

$$\frac{\sum_{j=1}^{q_{\varepsilon_n}} \delta_{a_k^j(\varepsilon_n)}}{q_{\varepsilon_n}} \to \frac{1}{2\,\pi\, r_k}\ \delta_{\Gamma^k(r_k)} \quad \text{in the sense of measures.} \tag{8.145}$$

Consequently, using $d_k(\varepsilon_n) \to p_k$ together with $\beta\, h_{ex} \simeq q_{\varepsilon_n}$ as $n \to +\infty$ in (8.145),

$$2\,\pi\, d_k(\varepsilon_n)\ \frac{\sum_{j=1}^{q_{\varepsilon_n}} \delta_{a_k^j(\varepsilon_n)}}{h_{ex}} \to \beta\, \frac{p_k}{r_k}\ \delta_{\Gamma^k(r_k)}\ \ \forall\, 1 \le k \le m. \tag{8.146}$$

Finally,

$$2\,\pi \sum_{k=1}^{m} d_k(\varepsilon_n)\ \frac{\sum_{j=1}^{q_{\varepsilon_n}} \delta_{a_k^j(\varepsilon_n)}}{h_{ex}} \to \sum_{k=1}^{m} \beta\, \frac{p_k}{r_k}\ \delta_{\Gamma^k(r_k)} \quad \text{in the sense of measures.} \tag{8.147}$$

However, for any $i \in \mathcal{L}_\varepsilon$, $a_i \in B_{r_\varepsilon}$ and $r_\varepsilon \to 0$ as $\varepsilon \to 0$, then using the fact that $\mu_\infty \in H^{-1}$, we can find

$$\frac{\sum_{i\in\mathcal{L}_\varepsilon} 2\,\pi\ \ d_i\,\delta_{a_i}}{h_{ex}} \to 0 \quad \text{in the sense of measures.}$$

Combining this together with (8.147) in the definition of the measure $\mu_\varepsilon$, which is given by (8.129), implies

$$\mu_{\varepsilon_n} \to \beta \sum_{k=1}^{m} \frac{p_k}{r_k} \, \delta_{\Gamma^k(r_k)} \quad \text{in the sense of measures.} \tag{8.148}$$

Note that $\{\Gamma^k(r_k),\ 1 \leq k \leq m,\quad 0 < r_1 < ... < r_m < R\}$ is the family of the $m$ concentric circles of strict positive radii where the limit measure $\mu_\infty$ concentrates. We can conclude

$$\mu_\infty = \sum_{k=1}^{m} \beta \ \frac{p_k}{r_k} \, \delta_{\Gamma^k(r_k)}. \tag{8.149}$$

The mass of $\mu_\infty$ on the circle $\Gamma^k(r_k)$ is $(2\ \pi\ \beta\ p_k)$. The conclusion from this and (8.143) is that the mass of the measure $\mu_\infty$ on each circle of vorticity belongs to $2\ \pi\ \beta\ \mathbb{Z}$. We combine the properties of $(h_\infty, \mu_\infty)$ given in the two above steps to conclude from the definition of the space $Y$

$$h_\infty \in Y.$$

This completes the proof of proposition 8.22.
Now, under some conditions on the parameters $\beta$ and $\lambda$, we will give the fundamental property on the limit measure of vorticity $\mu_\infty$.

**Lemma 8.23.** *We assume that* $\beta < \frac{2\,R\,I_1(R)}{I_0(R)}$. *Let* $R_0$ *be given by (8.102) and* $a = R\left(I_0(R)\ K_1(R) + I_1(R)\ K_0(R)\right)$, *then if*

$$\frac{2\ I_0(R) - 2}{I_0(R)} - \lambda > \frac{2\ I_0(R_0) - 2}{I_0(R)} + \beta\ \frac{I_0(R_0)\ X(R_0)}{a\ I_0(R)},$$

*we have* $\mu_\infty \neq 0$.

**Proof**: We argue by contradiction. Suppose that $\mu_\infty = -\Delta h_\infty + h_\infty = 0$, then in particular (8.133) gives us

$$\liminf_{n\to+\infty} \frac{J_{B_R}(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \geq \frac{1}{2} \int_{B_R} |\nabla h_\infty|^2 + \frac{1}{2} \int_{B_R} |h_\infty - 1|^2, \tag{8.150}$$

where $(u_\varepsilon, A_\varepsilon)$ is a minimizer of the energy $J$ over the space $G_\varepsilon$. Note that $h_\infty$ satisfies

$$\begin{cases} -\Delta h_\infty + h_\infty = 0 \quad \text{in} \ \ B_R \\ \\ h_\infty = 1 \quad \text{on} \ \ \partial B_R. \end{cases} \tag{8.151}$$

$h_\infty$ is radial, then referring to (8.49), we have

$$\frac{1}{2} \int_{B_R} |\nabla h_\infty|^2 + \frac{1}{2} \int_{B_R} |h_\infty - 1|^2 = \hat{J}_0 = \pi \left(\frac{R^2}{2} - R\ \frac{I_1(R)}{I_0(R)}\right).$$

Replacing this in the right-hand side of (8.150), we find

$$\liminf_{\varepsilon\to 0} \frac{J_{B_R}(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} \geq \hat{J}_0. \tag{8.152}$$

Now, going back to lemma 8.16, then under the hypotheses of this proposition we have

$$\limsup_{\varepsilon \to 0} \frac{J_{B_R}(u_\varepsilon, A_\varepsilon)}{h_{ex}^2} < \hat{J}_0.$$

A comparison of this to (8.152) yields a contradiction. □

Now, under the hypotheses of lemma 8.23, we have obtained that the limit measure of vorticity verifies $\mu_\infty \neq 0$, which allows to say that $\mu_\infty$ concentrates on at least one circle of center O, the center of the disk $B_R$. From now on, we restrict to the case where $\mu_\infty$ is supported exactly on one circle of center O.

## 5.6 Vortices's concentration along one circle

In this paragraph, we assume that

$$\beta < \frac{2\ R\ I_1(R)}{I_0(R)}, \tag{8.153}$$

and

$$\frac{2\ I_0(R) - 2}{I_0(R)} - \lambda > \frac{2\ I_0(R_0) - 2}{I_0(R)} + \beta\ \frac{I_0(R_0)\ X(R_0)}{a\ I_0(R)}, \tag{8.154}$$

where $a = R\ \Big(I_0(R)\ \ K_1(R) + I_1(R)\ \ K_0(R)\Big)$. In the case where the vortices's concentration is exactly along one circle, the limit measure $\mu_\infty$ can be written as

$$\mu_\infty = \beta\ \frac{d}{r}\ \ \delta_\Gamma, \tag{8.155}$$

where $d \in \mathbb{Z}^*$ and $\Gamma$ is the circle of center O and of radius $r$ such that $0 < r < R$. The mass of $\mu_\infty$ on $\Gamma$ is then $2\ \pi\ \beta\ d$.

**Lemma 8.24.** *The $d$ defined by (8.155) is in $\mathbb{N}^*$.*

**Proof**: Let $h_0$ be the solution of

$$\begin{cases} -\Delta h_0 + h_0 = 0 \quad \text{in} \ \ B_R \\ \\ h_0 = 1 \quad \text{on} \ \ \partial B_R. \end{cases} \tag{8.156}$$

By definition of the measure $\mu_\varepsilon$ and thanks to the convergence of $\mu_{\varepsilon_n}$ to $\mu_\infty$,

$$\begin{aligned} \int_{B_R} (h_0 - 1)\ \mu_\infty &= \lim_{n \to +\infty} \int_{B_R} (h_0 - 1)\ \mu_{\varepsilon_n} \\ &= \lim_{n \to +\infty} \Big( \frac{2\ \pi\ \sum_{i \in \mathcal{L}_{\varepsilon_n}} d_i\ (h_0 - 1)(a_i)}{h_{ex}} + \frac{2\ \pi\ q_{\varepsilon_n}\ \sum_{i \in \mathcal{T}_{\varepsilon_n}} d_i(\varepsilon_n)\ (h_0 - 1)(a_i^1(\varepsilon_n))}{h_{ex}} \Big). \end{aligned} \tag{8.157}$$

Now, referring to [SS4], proposition 2, we can find an expansion of the energy $J_{B_R}$ of a minimizer $(u_\varepsilon, A_\varepsilon)$ over the space $G_\varepsilon$ in terms of the positions and degrees of its vortices. It is slightly different to (8.134) and it is given as follows

$$\begin{aligned} J_{B_R}(u_\varepsilon, A_\varepsilon) \geq h_{ex}^2\ \hat{J}_0 + \pi \Big( \sum_{i \in \mathcal{L}_\varepsilon} |d_i| + q_\varepsilon\ D_\varepsilon \Big)\ |\log \varepsilon|\ (1 - o(1)) + 2\ \pi\ h_{ex} \sum_{i \in \mathcal{L}_\varepsilon} d_i\ (h_0 - 1)(a_i) \\ + 2\ \pi\ h_{ex}\ q_\varepsilon \sum_{i \in \mathcal{T}_\varepsilon} d_i\ (h_0 - 1)(a_i^1(\varepsilon)) + \frac{1}{2} \int_{B_R \setminus \Big( (\cup_{i \in \mathcal{L}_\varepsilon} B_i) \cup (\cup_{(i \in \mathcal{T}_\varepsilon,\ 1 \leq j \leq q_\varepsilon)} B_i^j) \Big)} |\nabla (h_\varepsilon - h_{ex}\ h_0)|^2 \\ + \frac{1}{2} \int_{B_R} |h_\varepsilon - h_{ex}\ h_0|^2 - o(1). \end{aligned} \tag{8.158}$$

From (8.158), we can write

$$\begin{aligned} \pi \Big( \sum_{i \in \mathcal{L}_\varepsilon} |d_i| + q_\varepsilon \sum_{i \in \mathcal{T}_\varepsilon} |d_i| \Big)\ |\log \varepsilon|\ (1 - o(1)) \\ + 2\ \pi\ h_{ex} \Big( \sum_{i \in \mathcal{L}_\varepsilon} d_i\ (h_0 - 1)(a_i) + q_\varepsilon \sum_{i \in \mathcal{T}_\varepsilon} d_i(\varepsilon)\ (h_0 - 1)(a_i^1(\varepsilon)) \Big) \leq o(1). \end{aligned} \tag{8.159}$$

Using (8.125) and (8.126),

$$\lim_{n \to +\infty} \frac{(\sum_{i \in \mathcal{L}_{\varepsilon_n}} |d_i| + q_{\varepsilon_n}\ \sum_{i \in \mathcal{T}_{\varepsilon_n}} |d_i|)\ |\log \varepsilon_n|\ o(1)}{h_{ex}^2} = 0.$$

We divide (8.159) by $h_{ex}^2$ and we use the above convergence to obtain

$$\begin{aligned} \lim_{n \to +\infty} \frac{\pi\ |\log \varepsilon_n|\ \Big( \sum_{i \in \mathcal{L}_{\varepsilon_n}} |d_i| + q_{\varepsilon_n}\ \sum_{i \in \mathcal{T}_{\varepsilon_n}} |d_i| \Big)}{h_{ex}^2} \\ \leq -\ 2\ \pi\ \lim_{n \to \infty} \frac{\Big( \sum_{i \in \mathcal{L}_{\varepsilon_n}} d_i\ (h_0 - 1)(a_i) + q_{\varepsilon_n} \sum_{i \in \mathcal{T}_{\varepsilon_n}} d_i(\varepsilon_n)\ (h_0 - 1)(a_i^1(\varepsilon_n)) \Big)}{h_{ex}}. \end{aligned}$$

Inserting this in (8.157),

$$\int_{B_R} (h_0 - 1)\ \mu_\infty \leq -\lim_{n \to +\infty} \frac{\pi\ |\log \varepsilon_n|\ \Big( \sum_{i \in \mathcal{L}_{\varepsilon_n}} |d_i| + q_{\varepsilon_n}\ \sum_{i \in \mathcal{T}_{\varepsilon_n}} |d_i(\varepsilon_n)| \Big)}{h_{ex}^2}. \tag{8.160}$$

By definition of the measure $\mu_\varepsilon$,

$$\int_{B_R} (h_0 - 1)\ \mu_\infty \leq -\liminf_{n \to +\infty} \frac{|\log \varepsilon_n|}{2\ h_{ex}} \int_{B_R} |\mu_{\varepsilon_n}|. \tag{8.161}$$

We know that $\lambda = \lim_{\varepsilon\to 0} \frac{|\log\varepsilon|}{h_{ex}} > 0$ as $\varepsilon \to 0$. Moreover, using the fact that $\int_{B_R} |\mu_\infty| \le \liminf_{n\to+\infty} \int_{B_R} |\mu_{\varepsilon_n}|$ together with $\mu_\infty \neq 0$, the inequality (8.161) becomes as

$$\int_{B_R} (h_0 - 1)\,\mu_\infty \le -\frac{\lambda}{2} \int_{B_R} |\mu_\infty| < 0. \tag{8.162}$$

Now, using (8.155) and the fact that $h_0$ is radial leads to

$$\int_{B_R} (h_0 - 1)\,\mu_\infty = 2\,\pi\,d\,\beta\,(h_0 - 1)(r). \tag{8.163}$$

Comparing (8.162) to (8.163) gives us

$$d\,(h_0 - 1)(r) \le 0.$$

From (8.156), we can check that $0 < h_0 < 1$ in $B_R$, then in particular we get $(h_0 - 1)(r) < 0$, since $0 < r < R$ . Thus, it is clear that $d \ge 0$ which with $d \neq 0$ yield $d \in \mathbb{N}^*$. □

The above lemma implies that the mass of $\mu_\infty$ on the circle $\Gamma$ belongs to $2\,\pi\,\beta\,\mathbb{N}^*$. Now, assuming that the mass of $\mu_\infty$ on $\Gamma$ is equal to $2\,\pi\,\beta$, we have

**Lemma 8.25.** *Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over the space $G_\varepsilon$. If $\mu_\infty = \frac{\beta}{r}\,\delta_\Gamma$, then $r = R_0$. Moreover,*

$$\lim_{n\to+\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} = E_1(R_0, 1) = \hat{J}_0 + \beta\,\pi\,\lambda - \beta\,\pi\,\Big(2 - \frac{2\,I_0(R_0}{I_0(R)}\Big) + \beta^2\,\pi\,\frac{I_0(R_0)\,X(R_0)}{a\,I_0(R)}.$$

**Proof**: The mass of $\mu_\infty$ on the circle $\Gamma$ is equal to $2\,\pi\,\beta$, then $\mu_\infty$ is written as

$$\mu_\infty = -\Delta h_\infty + h_\infty = \frac{\beta}{r}\,\delta_\Gamma, \tag{8.164}$$

where $0 < r < R$ is the radius of $\Gamma$. Consequently, it is clear that $h_\infty \in Y_1$ where $Y_1$ is the space defined by (8.53). In particular, in the sense of the definition (8.61), we have thanks to (8.164),

$$E(h_\infty) = E_1(r, 1). \tag{8.165}$$

Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of $J$ over the space $G_\varepsilon$. Going back to (8.133), we can write using (8.165)

$$\liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \ge E_1(r, 1). \tag{8.166}$$

Now, returning to (8.84) we have for any $0 < x < R$

$$\limsup_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \le E_1(x, 1). \tag{8.167}$$

Combining (8.166) together with (8.167), we get for $r \in ]0, R[$ (the radius $r$ is defined by (8.164))

$$E_1(r,1) \leq \liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \leq \limsup_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \leq E_1(x,1) \quad \forall\, x \in ]0,R[. \tag{8.168}$$

By definition of the functional $F$ given by (8.83) as $F(x) = \frac{E_1(x,1)}{\pi}$ for $x$ in $]0,R[$. Then, (8.168) gives us

$$\pi\ F(r) \leq \liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \leq \limsup_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \leq \pi\ F(x) \quad \forall\, x \in ]0,R[. \tag{8.169}$$

We know from (8.102)

$$\inf_{x\in]0,R[} F(x) = F(R_0).$$

We can then write for $r \in ]0,R[$, thanks to (8.169),

$$\pi\ F(R_0) \leq \pi\ F(r) \leq \liminf_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \leq \limsup_{n\to\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} \leq \pi\ F(R_0). \tag{8.170}$$

The uniqueness of $R_0$ $(0 < R_0 < R)$ minimum of the functional $x \to F(x)$ over $]0,R[$ in (8.170) implies

$$r = R_0.$$

Consequently, the radius of the circle of vortices is $R_0$ where $0 < R_0 < R$. So that, the limit measure of vorticity is

$$\mu_\infty = -\Delta h_\infty + h_\infty = \frac{\beta}{R_0}\ \delta_\Gamma,$$

where $\Gamma$ is the circle of radius $R_0$ and of center O. Finally, using the expression of $E_1(R_0,1) = \pi\ F(R_0)$ given by corollary 8.11, it follows from (8.170)

$$\lim_{n\to+\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} = E_1(R_0,1) = \pi\ F(R_0) = \hat{J}_0 + \beta\ \pi\ \lambda - \beta\ \pi\ \left(2 - \frac{2\ I_0(R_0}{I_0(R)}\right) + \beta^2\ \pi\ \frac{I_0(R_0)\ X(R_0)}{a\ I_0(R)}.$$

□

As a consequence of all the above, the main Theorem that we have proved is the following

**Theorem 8.26.** *Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J$ over the space $G_\varepsilon$ and $h_\varepsilon = curl A_\varepsilon$ be the induced magnetic field. Then, up to extraction of $\varepsilon_n$ from $\varepsilon$, there exist $h_\infty \in H_1^1(B_R)$ and $\mu_\infty \in \mathcal{M}(B_R)$ such that*

$$\frac{h_{\varepsilon_n}}{h_{ex}} \to h_\infty \quad \text{weakly in} \;\; H_1^1(B_R),$$

*and*

$$\mu_{\varepsilon_n} \to \mu_\infty = -\Delta h_\infty + h_\infty \quad \textit{in the sense of measures,}$$

*where* $\mu_\varepsilon$ *is defined by (8.129). Again,* $h_\infty$ *is radial and* $\mu_\infty$ *is supported on a finite number of concentric circles with strict positive radii such that the mass of* $\mu_\infty$ *on each circle belongs to* $2\,\pi\,\beta\,\mathbb{Z}$*. In addition, taking* $a = R\,\Big(I_0(R)\,K_1(R) + I_1(R)\,K_0(R)\Big)$*, then if*

$$\beta < \frac{2\,R\,I_1(R)}{I_0(R)},$$

*there exists a unique* $0 < R_0 < R$ *defined by (8.107) such that if*

$$\frac{2\,I_0(R)-2}{I_0(R)} - \lambda > \frac{2\,I_0(R_0)-2}{I_0(R)} + \beta\,\frac{I_0(R_0)\,X(R_0)}{a\,I_0(R)},$$

*we have* $\mu_\infty \neq 0$*. Moreover, if* $\mu_\infty$ *concentrates on one circle with a mass equal to* $2\,\pi\,\beta$*, the radius of this circle of vortices is* $R_0$*. Finally*

$$\lim_{n\to+\infty} \frac{J(u_{\varepsilon_n}, A_{\varepsilon_n})}{h_{ex}^2} = \pi\,\left(\frac{R^2}{2} - R\,\frac{I_1(R)}{I_0(R)}\right) + \beta\,\pi\,\lambda - \beta\,\pi\,\left(2 - \frac{2\,I_0(R_0}{I_0(R)}\right)$$
$$+\beta^2\,\pi\,\frac{I_0(R_0)\,X(R_0)}{a\,I_0(R)}.$$

**Remark 8.27.** *Analogously to the remark 7.18, we don't give explicitly the limit measure of vorticity.*

# Chapter 9

# Vortex pinning with bounded fields

In [APB], N. André, P. Bauman and D. Philips investigate vortex "pinning" in solutions to the Ginzburg-Landau energy. The coefficient $a(x)$ in the free Ginzburg-Landau energy modelling non-uniform superconductivity is nonnegative and is allowed to vanish at a finite number of points. For a sufficiently large applied field $h_{ex}$ and for all sufficiently large values of the Ginzburg-Landau parameter $\kappa = \frac{1}{\varepsilon}$, they show that the minimizers $(u_\varepsilon, A_\varepsilon)$ have nontrivial vortex structures around the zeroes of $a(x)$. Denote $d_i$ be the degree of $u_\varepsilon$ around the zero $x_i$ of the function $a$, and then $d = (d_1, ..., d_n)$ minimizes a precise functional defined on $\mathbb{Z}^n$. In this chapter, we are interested in the sign of the degrees $(d_i)_{1 \leq i \leq n}$. We give partial results indicating that the degrees may not always be positive.

## 1 Notations

In this chapter, we consider the Ginzburg-Landau energy of superconductivity with a pinning coefficient $a(x)$ given by

$$J_\varepsilon(u, A) = \frac{1}{2} \int_\Omega |\nabla u - i\, A\, u|^2 + \frac{1}{2} \int_\Omega |h - h_{ex}|^2 + \frac{1}{4\, \varepsilon^2} \int_\Omega (a - |u|^2)^2. \tag{9.1}$$

$\Omega \subset \mathbb{R}^2$ is a bounded regular simply connected domain and $a : \Omega \to \mathbb{R}$. We require that the function $a(x)$ satisfies the following:
Assume that $a \in C^1(\Omega \backslash \{x_1, x_2, ..., x_n\}) \cap C^\beta(\Omega)$ for some $\beta > 0$, $\sqrt{a} \in H^1(\Omega)$, $a(x) \geq 0$ for all $x$ in $\Omega$, and $a(x) = 0$ if and only if $x \in \{x_1, x_2, ..., x_n\}$ where $x_1, .., x_n$ are distinct points in $\Omega$ and $n \in \mathbb{N}^*$. Moreover, assume that there are positive constants $m_i$, $M_i$ and $\alpha_i$ such that

$$m_i\, |x - x_i|^{\alpha_i} \leq a(x) \leq M_i\, |x - x_i|^{\alpha_i} \quad \text{for} \quad 1 \leq i \leq n$$

in some neighborhood of $x_i$. Let $\varepsilon > 0$, say that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ is a critical point of $J_\varepsilon$ if it is solution of the Ginzburg-Landau equations, namely

$$\begin{cases} \nabla^2_{A_\varepsilon} u_\varepsilon = \frac{1}{\varepsilon^2}\, u_\varepsilon (a - |u_\varepsilon|^2) & \text{in } \Omega \\ -\nabla^\perp h_\varepsilon = < i\, u_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon > & \text{in } \Omega, \end{cases}$$

with the following boundary conditions

$$\begin{cases} h_\varepsilon = h_{ex} & \text{on } \partial\Omega \\ < \nabla_{A_\varepsilon} u_\varepsilon, \nu >= 0 & \text{on } \partial\Omega. \end{cases}$$

In this chapter, we take the Coulomb gauge (8.10). Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J_\varepsilon$ over the space $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ and $h_\varepsilon = curl A_\varepsilon$ be the associated induced magnetic field.

## 1.1 Absence of vortices

Here, we prove

**Lemma 9.1.** *Let $(u_\varepsilon, A_\varepsilon)$ be a minimizer of the energy $J_\varepsilon$ over the space $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$. Then, if $\forall\, x \in \Omega,\ |u_\varepsilon(x)| > 0$, we have*

$$h_\varepsilon \geq 0 \quad in \quad \Omega.$$

**Proof**: First, writing locally $u_\varepsilon = \rho_\varepsilon\, e^{i\,\varphi}$, the second Ginzburg-Landau equation holds

$$-\nabla^\perp h_\varepsilon =< i\, u_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon >= \rho_\varepsilon^2\, (\nabla\varphi - A_\varepsilon) \quad \text{in } \Omega. \tag{9.2}$$

We take the curl to have

$$-\Delta h_\varepsilon + \rho_\varepsilon^2\, h_\varepsilon = 2\, \rho_\varepsilon\, \nabla\rho_\varepsilon\, (A_\varepsilon^\perp - \nabla^\perp \varphi_\varepsilon). \tag{9.3}$$

But, in view of (9.2),

$$\rho_\varepsilon^2\, \nabla\rho_\varepsilon\, (A_\varepsilon^\perp - \nabla^\perp \varphi_\varepsilon) = -\nabla\rho_\varepsilon.\, \nabla h_\varepsilon.$$

Multiplying (9.3) by $\rho_\varepsilon$ and using the above identity, we find

$$-\rho_\varepsilon\, \Delta h_\varepsilon + 2\, \rho_\varepsilon\, \nabla\rho_\varepsilon\, \nabla h_\varepsilon + \rho_\varepsilon^3\, h_\varepsilon = 0 \quad \text{in } \Omega. \tag{9.4}$$

Let $z_0$ be a minimizer of the function $h_\varepsilon$, so in particular $\nabla h_\varepsilon(z_0) = 0$. It follows from (9.4)

$$-\rho_\varepsilon(z_0)\, \Delta h_\varepsilon(z_0) + \rho_\varepsilon^3(z_0)\, h_\varepsilon(z_0) = 0. \tag{9.5}$$

Knowing that $\forall\, x \in \Omega\ \ |u_\varepsilon(x)| > 0$, hence $\rho_\varepsilon(z_0) > 0$. Then, thanks to $\Delta h_\varepsilon(z_0) \geq 0$, (9.5) leads to $h_\varepsilon(z_0) \geq 0$, which gives us the nonnegativity of $h_\varepsilon$ in $\Omega$ since $z_0$ is a minimizer of $h_\varepsilon$.

□

When $h_{ex} = 0$, every minimizer $(u_\varepsilon, A_\varepsilon)$ of $J_\varepsilon$ in $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ satisfies $A_\varepsilon = 0$ and $\alpha\ u_\varepsilon > 0$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, so in particular $h_\varepsilon = 0$. In this chapter, let $h_{ex} \geq 0$ be bounded independently of $\varepsilon$ and take the function $a(x)$ vanish at a finite number of points denoted $\{x_1, ..., x_n\}$, hence unfortunately nothing allows us to say that the result of the above lemma remains true.

## 1.2 Presence of vortices

Before all, we set the space

$$\mathbf{U} = \Big\{ g \in H^1(\Omega), \ \int_\Omega a^{-1} \, |\nabla g|^2 < \infty \Big\}.$$

Then, $\mathbf{U}$ is a Hilbert space with the norm

$$||g||_{\mathbf{U}} = \Big( \int_\Omega a^{-1} \, |\nabla g|^2 + |g|^2 \Big)^{\frac{1}{2}}.$$

Let us consider the $(n+1)$ functions in $\mathbf{U}$, $\{\eta_0, ..., \eta_n\}$ solving that

$$\begin{cases} -div(a^{-1} \, \nabla \eta_0) + \eta_0 = -1 \quad \text{in} \quad \Omega \\ \eta_0 = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{9.6}$$

and for $1 \leq i \leq n$

$$\begin{cases} -div\Big(a^{-1} \, \nabla \eta_i\Big) + \eta_i = 2 \, \pi \, \delta_{x_i} \quad \text{in} \quad \Omega \\ \eta_i = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{9.7}$$

Note that thanks to [APB], lemma 2.1, we have $\delta_{x_i} \in \mathbf{U}'$, the dual space of $\mathbf{U}$, and clearly $1 \in \mathbf{U}'$. Thus, the Lax-Milgram lemma gives us the existence and the uniqueness of $\eta_0$ and $\eta_i$, $1 \leq i \leq n$, solutions respectively of (9.6) and (9.7). Now, we define the quantities $a_{ij}$ for $1 \leq i, j \leq n$ and $X_i$ for $0 \leq i \leq n$ to be given as follows

$$a_{ij} = \int_\Omega (a^{-1} \, \nabla \eta_i \, \nabla \eta_j + \eta_i \, \eta_j) \quad \text{for } 1 \leq i, j \leq n, \tag{9.8}$$

and

$$X_0 = \int_\Omega a^{-1} \, |\nabla \eta_0|^2 + |\eta_0|^2 \quad \text{and} \quad X_i = -\int_\Omega (a^{-1} \, \nabla \eta_0 \, \nabla \eta_i + \eta_0 \, \eta_i) \quad \text{for } 1 \leq i \leq n. \tag{9.9}$$

Fix $h_{ex} \geq 0$. Let $(u_{\varepsilon_k}, A_{\varepsilon_k})$ be a sequence of minimizers of $J_{\varepsilon_k}$ in $H^1 \times H^1$ with $\varepsilon_k \to 0$ as $k \to +\infty$. Then, thanks to [APB], Theorem 3, there exists a subsequence $\varepsilon_{k_l}$ such that as $l \to +\infty$

$$(u_{\varepsilon_{k_l}}, A_{\varepsilon_{k_l}}) \to (u, A) \quad \text{weakly in} \quad H^1 \times H^1 \quad \text{as} \quad l \to +\infty, \tag{9.10}$$

where $|u| = \sqrt{a}$. Moreover,

$$J_{\varepsilon_{k_l}}(u_{\varepsilon_{k_l}}, A_{\varepsilon_{k_l}}) \to \frac{1}{2} \, X_0 \, h_{ex}^2 + \frac{1}{2} \int_\Omega |\nabla \sqrt{a}|^2 + \frac{1}{2} \, S(d), \tag{9.11}$$

where

$$S(d) = \inf_{c \in \mathbb{Z}^n} S(c) = \inf_{c \in \mathbb{Z}^n} \Big( (\mathbf{A} \, c, c) - 2 \, h_{ex} \, (X, c) \Big), \tag{9.12}$$

such that $X$ is the vector $(X_1, ..., X_n)$ and $\mathbf{A}$ is the matrix defined by

$$\mathbf{A} = [a_{ij}]_{1 \leq i,j \leq n}.$$

Note that, for $c = (c_1, ..., c_n) \in \mathbb{Z}^n$

$$S(c) = \sum_{i,j=1}^{n} a_{ij}\ c_i\ c_j\ - 2\ h_{ex}\ \sum_{i=1}^{n} X_i\ c_i. \tag{9.13}$$

If $r > 0$ and $\overline{B_r(x_i)}$ are disjoint balls of $\Omega$ for $i = 1, ..., n$, then in addition from [APB], for all $l$ sufficiently large, $|u_{\varepsilon_{kl}}|$ is uniformly positive outside $\cup_{i=1}^{n} B_r(x_i)$ and the degree of $u_{\varepsilon_{kl}}$ in $\overline{B_r(x_i)}$ is $d_i$ where $d = (d_1, ..., d_n)$ minimizes the functional $S$ over $\mathbb{Z}^n$. Thus, for $\varepsilon$ sufficiently small, we remark that minimizers $(u_\varepsilon, A_\varepsilon)$ of $J_\varepsilon$ over the space $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ have "pinned" vortices near the zeroes $x_1, ..., x_n$ of the function $a(x)$.

# 2 Goal of the chapter

In this chapter, our interest is to study the minimization problem (9.12), and then to give some properties of $d = (d_1, ..., d_n)$ minimum of $S$ over $\mathbb{Z}^n$. Recall that $d_i$ is the degree of $u_{\varepsilon_{kl}}$ around the zero $x_i$ of the function $a$. In particular, we will be concerned with the sign of the degrees $d_i$ for $1 \leq i \leq n$.

# 3 Some properties of the functions $\eta_i$

In this paragraph, we give some properties of the functions $\eta_i$ for $0 \leq i \leq n$. Before all, we note that the functions $\eta_i$ for $0 \leq i \leq n$ are continuous in $\Omega$. We can state in addition the following

**Lemma 9.2.** *We have*

$$-1 < \eta_0 < 0 \quad \textit{and} \ \ \eta_i > 0 \ \ \forall\, 1 \leq i \leq n. \tag{9.14}$$

*Moreover,*

$$X_0 > 0 \ \ \textit{and} \ \forall\, 1 \leq i \leq n, \quad X_i = -2\ \pi\ \eta_0(x_i) = \int_\Omega \eta_i > 0. \tag{9.15}$$

*In addition, the matrix $\mathbf{A} = [a_{ij}]_{1 \leq i,j \leq n}$ is positive definite and*

$$a_{ji} = a_{ij} = 2\ \pi\ \eta_i(x_j) = 2\ \pi\ \eta_j(x_i) > 0 \quad \textit{for}\ 1 \leq i, j \leq n. \tag{9.16}$$

**Proof**: First, using the maximum principle in (9.6), it is immediate to show

$$-1 < \eta_0 < 0 \quad \text{in} \quad \Omega.$$

Second, $\eta_i^- = min(\eta_i, 0) \in \mathbf{U} \cap H_0^1(\Omega)$, hence we use it as a test function in (9.7) to find

$$\int_{\{\eta_i \le 0\}} a^{-1} \, |\nabla \eta_i|^2 + |\eta_i|^2 = 2\,\pi\,\eta_i^-(x_i) \le 0.$$

We deduce

$$\forall\, 1 \le i \le n, \quad \eta_i > 0 \quad \text{in} \quad \Omega.$$

(9.14) is then proved. Now, we restrict to prove (9.15) and (9.16). Recall that

$$X_0 = \int_\Omega a^{-1} \, |\nabla \eta_0|^2 + |\eta_0|^2. \tag{9.17}$$

Then, $X_0 > 0$ because that $\eta_0 \neq 0$. Multiplying (9.6) by $\eta_i$ for $1 \le i \le n$ and integrating over $\Omega$, we have

$$\int_\Omega a^{-1} \, \nabla \eta_0 \, \eta_i + \eta_0 \, \eta_i = -\int_\Omega \eta_i \quad \text{for } 1 \le i \le n.$$

Again, we multiply (9.7) by $\eta_0$ and we integrate over $\Omega$ to get

$$\int_\Omega a^{-1} \, \nabla \eta_0 \, \eta_i + \eta_0 \, \eta_i = 2\,\pi\,\eta_0(x_i) \quad \text{for } 1 \le i \le n.$$

Comparing the two above identities to the definition (9.9) of $X_i$ for $1 \le i \le n$, we deduce

$$X_i = -2\,\pi\,\eta_0(x_i) = \int_\Omega \eta_i.$$

The fact that $\eta_i > 0$ for $1 \le i \le n$ implies that $X_i > 0$ for $1 \le i \le n$. By definition of $a_{ij}$, we remark that $a_{ji} = a_{ij}$. Moreover, we multiply (9.7) by $\eta_j$ and we integrate over $\Omega$ to have

$$\int_\Omega a^{-1} \, \nabla \eta_i \, \eta_j + \eta_i \, \eta_j = 2\,\pi\,\eta_i(x_j) \quad \text{for } 1 \le i, j \le n.$$

By (9.8), we find

$$a_{ji} = a_{ij} = 2\,\pi\,\eta_i(x_j) = 2\,\pi\,\eta_j(x_i) > 0 \quad \text{for } 1 \le i, j \le n.$$

Finally, thanks to [APB], lemma 3.3, the matrix $\mathbf{A}$ is positive definite. The lemma is then proved. □

**Proposition 9.3.** *We have*

$$\eta_i(x_i) > \eta_i(x_j) \quad \forall\, 1 \le i \neq j \le n. \tag{9.18}$$

**Proof**: To prove (9.18), it suffices to show that $x_i$ is the global maximum of $\eta_i$, $1 \le i \le n$. Let us split the demonstration into two steps.

**Step 1**

We define on the space $\mathbf{U} \cap H_0^1(\Omega)$ the functional

$$T(f) = \frac{1}{2} \int_\Omega a^{-1} |\nabla f|^2 + \frac{1}{2} \int_\Omega |f|^2 - 2\, \pi\, f(x_i).$$

Obviously, note that each critical point $f$ of the functional $T$ on the space $\mathbf{U} \cap H_0^1(\Omega)$ is solution of

$$\begin{cases} -div\Big(a^{-1}\, \nabla f\Big) + f = 2\, \pi\, \delta_{x_i} \quad \text{in} \quad \Omega \\ f = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

But, by uniqueness of solution of the above system and referring to (9.7), then we can say that $\eta_i$ is the unique critical point of the functional $T$ on the space $\mathbf{U} \cap H_0^1(\Omega)$. Let us determine the nature of the above critical point. For this, it suffices to compare $T(\eta_i)$ to a test configuration in the space $\mathbf{U} \cap H_0^1(\Omega)$. From (9.16),

$$\int_\Omega a^{-1}\, |\nabla \eta_i|^2 + \int_\Omega |\eta_i|^2 = 2\, \pi\, \eta_i(x_i).$$

Then, we find

$$\begin{aligned} T(\eta_i) =& \frac{1}{2} \int_\Omega a^{-1}\, |\nabla \eta_i|^2 + \frac{1}{2} \int_\Omega |\eta_i|^2 - 2\, \pi\, \eta_i(x_i) \\ =& -\pi\, \eta_i(x_i). \end{aligned}$$

Since $\eta_i(x_i) > 0$, we get $T(\eta_i) < T(0) = 0$ ( note that $0 \in \mathbf{U} \cap H_0^1(\Omega)$). As a consequence of the above, we can say that $\eta_i$ is necessarily the unique minimum of the functional $T$ over the space $\mathbf{U} \cap H_0^1(\Omega)$.

**Step 2**

Here, let us take the function

$$v(x) = \begin{cases} \eta_i(x) & \text{if } \eta_i(x) \leq \eta_i(x_i) \\ \eta_i(x_i) & \text{if } \eta_i(x) \geq \eta_i(x_i). \end{cases} \tag{9.19}$$

By definition, it is clear that $v$ belongs to the space $\mathbf{U} \cap H_0^1(\Omega)$. Let us evaluate the quantity $T(v)$. Indeed,

$$\begin{aligned} T(v) =& \frac{1}{2} \int_\Omega a^{-1}\, |\nabla v|^2 + \frac{1}{2} \int_\Omega |v|^2 - 2\, \pi\, v(x_i) \\ =& \frac{1}{2} \int_{\{\eta_i \leq \eta_i(x_i)\}} a^{-1}\, |\nabla \eta_i|^2 + \frac{1}{2} \int_{\{\eta_i \leq \eta_i(x_i)\}} |\eta_i|^2 + \frac{1}{2} \int_{\{\eta_i > \eta_i(x_i)\}} |\eta_i(x_i)|^2 - 2\, \pi\, \eta_i(x_i). \end{aligned} \tag{9.20}$$

Remark that $\int_{\{\eta_i > \eta_i(x_i)\}} |\eta_i(x_i)|^2 \leq \int_{\{\eta_i > \eta_i(x_i)\}} |\eta_i|^2$. Then, inserting it in (9.21), it follows that

$$
\begin{aligned}
T(v) \leq & \frac{1}{2} \int_\Omega a^{-1} \, |\nabla \eta_i|^2 + \frac{1}{2} \int_{\{\eta_i \leq \eta_i(x_i)\}} |\eta_i|^2 + \frac{1}{2} \int_{\{\eta_i > \eta_i(x_i)\}} |\eta_i|^2 - 2 \, \pi \, \eta_i(x_i) \\
\leq & \frac{1}{2} \int_\Omega a^{-1} \, |\nabla \eta_i|^2 + \frac{1}{2} \int_\Omega |\eta_i|^2 - 2 \, \pi \, \eta_i(x_i) \\
= & T(\eta_i).
\end{aligned}
\tag{9.21}
$$

Using the fact that $\eta_i$ is the unique minimum of $T$ over the space $\mathbf{U} \cap H_0^1(\Omega)$ allows to deduce

$$
v(x) = \eta_i(x) \quad \forall \, x \in \Omega. \tag{9.22}
$$

By definition of the function $v$, (9.22) implies for $1 \leq i \leq n$

$$
\eta_i(x) \leq \eta_i(x_i) \quad \forall \, x \in \Omega. \tag{9.23}
$$

This means that $x_i$ is the global maximum of the function $\eta_i$. The proposition 9.3 is then proved. □

We summarize the previous information in the following

**Proposition 9.4.** *The function $S$ defined in* (9.12) *satisfies*

$$
a_{ij} > 0, \quad X_i > 0, \quad a_{ii} \geq a_{ij} \quad \forall \, 1 \leq i, j \leq n.
$$

*Moreover, the matrix $(a_{ij})$ is positive definite.*

Then, let us give

**Definition 9.5.** *We say a function $S : \mathbb{Z}^n \to \mathbb{R}$ is admissible if it is of the form*

$$
S(d) = (\mathbf{A} \, d, d) - 2 \, h_{ex} \, \sum_{i=1}^{n} d_i \, X_i,
$$

*where $h_{ex}, X_i$ are positive, and $\mathbf{A} = (a_{ij})$ is a symmetric positive definite matrix such that $a_{ii} \geq a_{ij}$ for every $1 \leq i, j \leq n$.*

## 4 Some properties of $d$

Remember that $X_i > 0$ for $1 \leq i \leq n$. We start with

**Lemma 9.6.** *Set*

$$
\overline{h_{ex}} = \min\{\frac{a_{ii}}{2 \, X_i}, \;\; i = 1, 2, ..., n\}.
$$

*If $h_{ex} > \overline{h_{ex}}$, then $d \neq 0$ where $d$ minimizes $S$ over $\mathbb{Z}^n$.*

**Proof**: Set $j \in \{1,...,n\}$ satisfy $\overline{h_{ex}} = \frac{a_{jj}}{2\,X_j}$. Let $\vec{e_j}$ be the vector in $\mathbb{Z}^n$ whose $i^{th}$ component is $\delta_{ij}$ for $i = 1,...,n$ ($\delta_{ij} = 1$ if $i = j$ and 0 if not). When $h_{ex} > \overline{h_{ex}}$, then from (9.13),

$$\begin{aligned} S(\vec{e_j}) =& a_{jj} - 2\ h_{ex}\ X_j \\ =& 2\ X_j\ (\overline{h_{ex}} - h_{ex}) < 0 = S(0). \end{aligned}$$

By definition of $d$ minimum of $S$ over $\mathbb{Z}^n$, we must have $d \neq 0$. □

From now on, $h_{ex}$ is taken to satisfy the fact that $h_{ex} > \overline{h_{ex}}$. Now, we give a preliminary idea on the sign of the degrees $(d_i)_{1\leq i\leq n}$.

**Lemma 9.7.** *Let $d = (d_1,...,d_n)$ minimize $S$ over $\mathbb{Z}^n$, then there exists $i_0$ where $1 \leq i_0 \leq n$ such that $d_{i_0} \geq 0$.*

**Proof**: We argue by contradiction by assuming that $d_i < 0,\ \forall\ \ 1 \leq i \leq n$. We know that $\forall\ \ 1 \leq i \leq n,\ \ X_i > 0$, hence

$$\sum_{i=1}^{n} X_i\ d_i < 0.$$

Recall that the matrix $\mathbf{A}$ is positive definite, so in particular we have $(\mathbf{A}\ d, d) > 0$, since $d \neq 0$. Combining the above in $S(d)$ to get

$$S(d) = (\mathbf{A}\ d, d) - 2\ h_{ex}\ \sum_{i=1}^{n} X_i\ d_i > 0 = S(0).$$

This contradicts the fact that $d$ minimizes $S$ over $\mathbb{Z}^n$. □

### Notation

Recall that $d = (d_1,...,d_n)$ minimizes $S$ over $\mathbb{Z}^n$. Without loss of generality, we mean by the fact that $d$ is positive if $d_i$ is nonnegative for each $1 \leq i \leq n$, and by $d$ is not positive if there exists $1 \leq j \leq n$ such that $d_j < 0$.
The study of the degrees $(d_i)_{1\leq i\leq n}$ seems be not easy. Then, when it is necessary to make our study easier we need to make extra hypotheses on the domain $\Omega$, on the function $a(x)$ and on the location of its zeroes $x_1$,...,$x_n$.
In this case, assume that the domain is the disk $B_R$ of center O and of radius $R > 0$. The zeroes $\{x_1, x_2, ..., x_n\}$ of the function $a$ are such that $x_1\ x_2 ... x_n$ is a polygon with the same sidelength and of center O, the center of the disk $B_R$. Moreover, letting $\Delta_i$ be the mediatrix of the line $[x_i,\ x_{i+1}]$ and $S_{\Delta_i}$ be the axial symmetry with respect to $\Delta_i$, the function $a(x)$ is taken to be invariant under each $S_{\Delta_i}$, $1 \leq i \leq n$, meaning that the weight $a(x)$ verifies

$$a(x) = a(S_{\Delta_1}(x)) = ... = a(S_{\Delta_n}(x)) \quad \forall\ x \in B_R. \tag{9.24}$$

Note that the above is called symmetric hypotheses. It is clear in this case that $\eta_i(x_j)$ depends only on the value modulo $n$ of $|i - j|$. Therefore, in this case, from the definition of the coefficients $(a_{ij})$, we see that $a_{ij}$ depends only on the value of $|i-j|$ modulo $n$ and that $X_i = X_j$ for any $i, j$. Motivated by this example, we give the following.

**Definition 9.8.** *We will say an admissible function*

$$S(d) = (\mathbf{A}\; d, d) - 2\; h_{ex} \sum_{i=1}^{n} d_i\; X_i,$$

*is symmetric if* $\mathbf{A} = (a_{ij})$*, and* $a_{ij}$ *depends only on the value of* $|i-j|$ *modulo* $n$ *and if* $X_i = X_j$ *for all* $i, j$*.*

# 5 The main result

## 5.1 Setting of the Theorem

The main result here is

**Theorem 9.9.** *For* $h_{ex} > \overline{h_{ex}}$*, then*
*A)The non symmetric case*
*1) If* $n = 1$*, and* $d$ *is the minimum of an admissible* $S$*, then* $d \geq 1$*.*
*2) If* $n = 2$*, there exists an admissible* $S$ *such that every minimum of* $S$ *is not positive.*
*B) The symmetric case*
*1) If* $S$ *is admissible and symmetric, then for* $n = 2$ *or* $n = 3$ *any minimum of* $S$ *is positive.*
*2) If* $n = 4$*, there exist an admissible symmetric* $S$ *such that every minimum of* $S$ *is not positive.*

## 5.2 Remark

In the cases (A-2) and (B-2), it is open whether there exists a pinning coefficient giving rise to the function $S$ we have found.

# 6 The case A: The non symmetric case

## 6.1 The case $n = 1$

Here, $\mathbf{A} = a_{11} > 0$ and $X = X_1 > 0$. Then, for $c \in \mathbb{Z}$ the functional $S$ is $S(c) = a_{11}\, c^2 - 2\, h_{ex}\, X\, c$. Remark that the minimum of $c \to S(c)$ over $\mathbb{R}$ is achieved at $c_0 = \frac{2\, h_{ex}\, X_1}{a_{11}}$. Now, if $h_{ex}$ is such that $h_{ex} > \frac{a_{11}}{2\, X_1}$, we have $c_0 > 1$. Then, the minimum $d$ of $S$ over $\mathbb{Z}$ does not vanish and verifies $d \geq 1$.

## 6.2 The case $n = 2$

Here, we have

$$a_{11} > a_{12} = a_{21}, \quad a_{22} > a_{12}, \quad X_1, X_2 > 0.$$

Let $d = (d_1, d_2)$ be a minimum of $S$ over $\mathbb{Z}^2$. Our aim is to determine the sign of the two degrees $d_1$ and $d_2$. From the lemma 9.7, either $d_1 \geq 0$ or $d_2 \geq 0$. We start with

**Lemma 9.10.** *Let $h_{ex} > \overline{h_{ex}}$. If $X_1 = X_2$, then $d_1$ and $d_2$ are nonnegative (and at least one of them is positive).*

**Proof**: We argue by contradiction. First, suppose that $d_1\, d_2 = 0$ and $d_1$ or $d_2$ is negative. Without loss of generality, we assume that $d_1 = 0$ and $d_2 \leq -1$, then

$$S(d_1, d_2) = S(0, d_2) = a_{22}\, d_2^2 - 2\, h_{ex}\, X_1\, d_2 > 0 = S(0, 0),$$

which contradicts the fact that $(d_1, d_2)$ is a minimum of $S$.
Second, we suppose that $d_1\, d_2 < 0$. On the one hand, comparing $S(d)$ to $S(d_1+d_2, 0)$ and using $X_1 = X_2$,

$$\begin{aligned} S(d_1, d_2) - S(d_1 + d_2, 0) =& (a_{22} - a_{11})\, d_1^2 + 2\, (a_{12} - a_{11})\, d_1\, d_2 - 2\, h_{ex}\, (X_2 - X_1)\, d_2 \\ =& (a_{22} - a_{11})\, d_1^2 + 2\, (a_{12} - a_{11})\, d_1\, d_2. \end{aligned} \tag{9.25}$$

Since $a_{12} < a_{11}$ and $d_1\, d_2 < 0$, it follows that

$$(a_{12} - a_{11})\, d_1\, d_2 > 0,$$

Now, if $(a_{22} - a_{11}) \geq 0$, then combining the two above inequalities in (9.25),

$$S(d_1, d_2) > S(d_1 + d_2, 0). \tag{9.26}$$

On the other hand,

$$\begin{aligned} S(d_1, d_2) - S(0, d_1 + d_2) =& (a_{11} - a_{22})\, d_2^2 + 2\, (a_{12} - a_{22})\, d_1\, d_2 - 2\, h_{ex}\, (X_1 - X_2)\, d_1 \\ =& (a_{11} - a_{22})\, d_2^2 + 2\, (a_{12} - a_{22})\, d_1\, d_2. \end{aligned}$$

Using $a_{12} < a_{22}$, $(a_{12} - a_{22})\, d_1\, d_2 > 0$. In addition, if $(a_{11} - a_{22}) \geq 0$, we get

$$S(d_1, d_2) > S(0, d_1 + d_2). \tag{9.27}$$

A combination of (9.26) together with (9.27) contradicts the fact that $d$ is a minimum of $S$ over $\mathbb{Z}^2$. This leads to $d_1\, d_2 \geq 0$ independently of the sign of $(a_{11} - a_{22})$. But, going back to lemma 9.7, there exists $1 \leq i_0 \leq 2$ such that $d_{i_0} \geq 0$. Then, necessarily the two components are nonnegative, meaning that $d$ is positive. Moreover, when $h_{ex} > \overline{h_{ex}}$, we note that if one component is equal to 0, then thanks to lemma 9.6, the other component must be positive. □

**Lemma 9.11.** *If $X_2 < X_1$, then $d_1$ is nonnegative.*

**Proof**: We do the proof by contradiction. Assume that $d_1$ is strictly negative, i. e. $d_1 \leq -1$. From lemma 9.7, we have necessarily $i_0 = 2$, meaning that $d_2 \geq 0$. We exclude the case of $d_2 = 0$ because

$$S(d_1, 0) = a_{11}\, d_1^2 - 2\, h_{ex}\, d_1 > 0 = S(0, 0).$$

Then, $d_2 \geq 1$ which with $d_1 < 0$ yield that $d_1\, d_2 < 0$. On the one hand

$$S(d_1, d_2) - S(d_1 + d_2, 0) = (a_{22} - a_{11})\, d_1^2 + 2\, (a_{12} - a_{11})\, d_1\, d_2 - 2\, h_{ex}\, (X_2 - X_1)\, d_2.$$

We insert $X_2 < X_1$, $d_2 \geq 1$, $d_1\, d_2 \leq -1 < 0$ and $a_{12} < a_{11}$ in the above identity to find

$$S(d_1, d_2) > S(d_1 + d_2, 0) \quad \text{if} \quad a_{22} \geq a_{11}.$$

Proceeding similarly as in the above, we can get

$$S(d_1, d_2) > S(0, d_1 + d_2) \quad \text{if} \quad a_{11} \geq a_{22}.$$

The two above inequalities contradict the fact that $(d_1, d_2)$ is a minimum. We must have $d_1 \geq 0$, i.e. $i_0 = 1$. □

We know prove assertion A) 2) of the Theorem. More precisely

**Proposition 9.12.** *If $X_1 > X_2$ and if*

$$a_{11}X_2 < (a_{22} + 2a_{12})X_1 < 3a_{11}X_2, \tag{9.28}$$

$$a_{22}(2X_2 - X_1) + 3a_{11}X_2 - 2a_{12}(X_1 + X_2) < 0, \tag{9.29}$$

$$\max\Big( \min(\frac{a_{11}}{2\, X_1}, \frac{a_{22}}{2\, X_2}), \frac{3a_{11} + a_{22} - 4a_{12}}{2(X_1 - X_2)} \Big) < h_{ex} < \frac{a_{22} + 2a_{12}}{2\, X_2}, \tag{9.30}$$

*then $d_2 < 0$.*

**Remark 9.13.** *Remark that the first two assumptions ensure that the set of $h_{ex}$ satisfying the third assumption is not empty.*

## 6.3 Proof of proposition 9.12

First, thanks to the lemma 9.6, each minimum $d$ of $S$ over $\mathbb{Z}^2$ does not vanish if the applied magnetic field satisfies

$$h_{ex} > \overline{h_{ex}} = \min(\frac{a_{11}}{2\, X_1}, \frac{a_{22}}{2\, X_2}). \tag{9.31}$$

Since $X_1 > X_2$, we have from lemma 9.11 $d_1 \geq 0$. First, let us start by minimizing the functional $S$ over $\mathbb{N}^2$.

**Step1**

We define $(n_1, n_2)$ to be a minimum of $S$ over $\mathbb{N}^2$. If $a_{11}\, X_2 < (a_{22} + 2\, a_{12})\, X_1$ and

$$h_{ex} < \frac{a_{22} + 2\, a_{12}}{2\, X_2}, \tag{9.32}$$

and $n_1, n_2 \geq 1$, then

$$
\begin{aligned}
S(n_1,n_2) =& a_{11}\ n_1^2 - 2\ h_{ex}\ X_1\ n_1 + a_{22}\ n_2^2 + 2\ a_{12}\ n_1\ n_2 - 2\ h_{ex}\ X_2\ n_2 \\
\geq & a_{11}\ n_1^2 - 2\ h_{ex}\ X_1\ n_1 + n_2\ (a_{22} + 2\ a_{12} - 2\ h_{ex}\ X_2) \\
> & S(n_1, 0),
\end{aligned}
$$

which contradicts the fact that $(n_1, n_2)$ is a minimum of $S$ over $\mathbb{N}^2$. Moreover, $(n_1, n_2) \neq (0,0)$ because $h_{ex} > \frac{a_{11}}{2\ X_1}$, hence the two only possibilities for $(n_1, n_2)$ are the following

$$
(n_1, n_2) = (n_1, 0) \quad \text{with } \ n_1 \geq 1 \quad \text{or} \quad (n_1, n_2) = (0, n_2) \ \text{ with } \ \ n_2 \geq 1.
$$

In the first case, $n_1 \geq 2$ and

$$
S(n_1, 0) - S(1,0) = (n_1 - 1)\ \Big(a_{11}\ (n_1 + 1) - 2\ h_{ex}\ X_1\Big). \tag{9.33}
$$

Then, using $n_1 \geq 2$, we get

$$
a_{11}\ (n_1 + 1) - 2\ h_{ex}\ X_1 \geq 3\ a_{11}\ - 2\ h_{ex}\ X_1.
$$

Moreover, assume that

$$
\frac{a_{22} + 2\ a_{12}}{2\ X_2} < \frac{3\ a_{11}}{2\ X_1}. \tag{9.34}
$$

Inserting this in (9.32), we have $(3\ a_{11}\ - 2\ h_{ex}\ X_1) > 0$. This yields that the quantity $(n_1 - 1)\ \Big(a_{11}\ (n_1 + 1) - 2\ h_{ex}\ X_1\Big)$ is strictly positive for $n_2 \geq 2$. Hence, (9.33) gives us

$$
S(n_1, 0) > S(1, 0).
$$

So that if $(n_1, 0)$ is a minimum of $S$ over $\mathbb{N}^2$, then necessarily $n_1 = 1$. Now, we study the second possibility which is $(n_1, n_2) = (0, n_2)$ with $n_2 \geq 1$. For this, assume that $n_2 \geq 2$, hence using the same argument as for the first possibility,

$$
S(0, n_2) - S(0,1) \geq 3\ a_{22}\ - 2\ h_{ex}\ X_2.
$$

Knowing $a_{12} < a_{22}$, then from (9.32, $(3\ a_{22}\ - 2\ h_{ex}\ X_2) > 0$, so that $S(0, n_2) > S(0, 1)$. Then, we must have $n_2 = 1$. Consequently, if $(n_1, n_2)$ is a minimum of $S$ over $\mathbb{N}^2$, then necessarily we get

$$
(n_1, n_2) = (1, 0) \quad \text{or} \quad (n_1, n_2) = (0, 1).
$$

Let us compare $S(1,0)$ to $S(0,1)$. In fact,

$$
S(1,0) - S(0,1) = a_{11} - 2\ h_{ex}\ X_1 - a_{22} + 2\ h_{ex}\ X_2 = a_{11} - a_{22} - 2\ h_{ex}\ (X_1 - X_2). \tag{9.35}
$$

We assume that

$$a_{22}\,(2\,X_2 -\; X_1) + 3\,a_{11}\,X_2 - 2\,a_{12}\,(X_1 + X_2) < 0. \tag{9.36}$$

Using the fact that $2\,a_{12} < a_{22} + a_{11}$ with $X_1 > X_2$, we can write

$$\frac{3\,a_{11} + a_{22} - 4\,a_{12}}{2\,X_1 - 2\,X_2} > \frac{a_{11} - a_{22}}{2\,X_1 - 2\,X_2}.$$

Hence, if $h_{ex}$ is such that

$$h_{ex} > \frac{3\,a_{11} + a_{22} - 4\,a_{12}}{2\,X_1 - 2\,X_2}, \tag{9.37}$$

we obtain

$$h_{ex} > \frac{a_{11} - a_{22}}{2\,X_1 - 2\,X_2}. \tag{9.38}$$

Note that the assumption (9.36) gives a sense to the inequalities (9.32)-(9.37). Then, inserting (9.38) in (9.35) implies that $S(1,0) < S(0,1)$. Finally, the minimum of $S$ over $\mathbb{N}^2$ is achieved at $(1,0)$.

Our interest now is to find a point in the region $\mathbb{Z}^2 \cap (x \geq 0,\; y < 0)$ such that its image by $S$ is less than $S(1,0)$.

**Step 2**

Let us take the point $(2,-1)$, then comparing $S(2,-1)$ to $S(1,0)$, we obtain

$$S(2,-1) - S(1,0) = 3\,a_{11} + a_{22} - 4\,a_{12} - 2\,h_{ex}\,X_1 + 2\,h_{ex}\,X_2.$$

Note that (9.37) gives us

$$S(2,-1) < S(1,0).$$

This implies that $(1,0)$ is not a minimum of $S$ over $\mathbb{Z}^2$, and there exists a non positive minimum of $S$ over $\mathbb{Z}^2$. The proposition 9.12 is then proved.

# 7 The case B: the symmetric case

## 7.1 The case $n = 2$

In this case we deduce from $S(d_1, d_2) = S(d_2, d_1)$ that $X_1 = X_2$. In particular, going back to lemma 9.10, we can say that any minimum $d$ of $S$ over $\mathbb{Z}^2$ is positive. Moreover, if $h_{ex} > \overline{h_{ex}}$, the lemma 9.6 leads to $d \neq 0$. Thus, if one component is equal to 0, then the second component must be positive.

## 7.2 The case $n = 3$

In this case,

$$a_{11} = a_{22} = a_{33}, \quad a_{12} = a_{23} = a_{31}, \quad X_1 = X_2 = X_3.$$

Let $d = (d_1, d_2, d_3)$ be a minimum of the functional $S$ over $\mathbb{Z}^3$. We start with a preliminary result on the sign of the degrees $(d_i)_{1 \leq i \leq 3}$.

**Lemma 9.14.** *For any* $1 \le i, j \le 3$,

$$d_i \, d_j \ge 0.$$

**Proof**: We compare $S(d)$ to $S(d_1 + d_2, d_3, 0)$ to get

$$S(d_1, d_2, d_3) - S(d_1 + d_2, d_3, 0) = 2\,(a_{12} - a_{11})\, d_1 \, d_2.$$

Since, $a_{12} < a_{11}$, we can write by definition of $(d_1, d_2, d_3)$ minimum of $S$ over $\mathbb{Z}^3$,

$$d_1 \, d_2 \ge 0.$$

Similarly as in the above, we find again $d_3 \, d_2 \ge 0$ and $d_1 \, d_3 \ge 0$. The lemma 9.14 is then proved. □

Our result is the following

**Lemma 9.15.** *Let* $h_{ex} > \overline{h_{ex}}$*, then for the functions* $\eta_0$ *and* $\eta_i$*,* $1 \le i \le 3$*, solutions respectively of (9.6) and (9.7), the three degrees* $d_1$*,* $d_2$ *and* $d_3$ *are nonnegative and at least one of them is positive.*

**Proof**: We argue by contradiction. Without loss of generality, we assume for example that $d_1 \le -1$. We refer to the fact that $d_1 \, d_2 \ge 0$ and $d_1 \, d_3 \ge 0$ to find necessarily $d_2 \le 0$ and $d_3 \le 0$. We insert these in the expression of $S(d)$ to get

$$S(d_1, d_3, d_2) > 0 = S(0).$$

It is the contradiction. Thus, the three degrees $d_1$, $d_2$ and $d_3$ are nonnegative. In addition, if $h_{ex} > \overline{h_{ex}}$, the lemma 9.6 gives us that $d \ne 0$, so in particular there is at least one positive degree from $(d_i)_{1 \le i \le 3}$. □

## 7.3 The case $n = 4$

In this case, we define

$$\gamma_1 = a_{11} = a_{22} = a_{33} = a_{44}, \quad \gamma_2 = a_{12} = a_{23} = a_{34} = a_{14}, \quad \gamma_3 = a_{31} = a_{24}.$$

Let $d = (d_1, d_2, d_3, d_4)$ be a minimum of $S$ over $\mathbb{Z}^4$. We start by giving a preliminary idea on the sign of the components $(d_i)_{1 \le i \le 4}$.

**Lemma 9.16.** *We have,*

$$d_1 \, d_3 \ge 0 \quad \textit{and} \quad d_2 \, d_4 \ge 0.$$

**Proof**: Testing the minimal functional $S(d_1, d_2, d_3, d_4)$ by the configuration $(d_1 + d_3, d_2, 0, d_4)$, we have

$$S(d_1, d_2, d_3, d_4) \leq S(d_1 + d_3, d_2, 0, d_4).$$

A simple calculation gives us

$$(\gamma_3 - \gamma_1)\ d_1\ d_3 \leq 0.$$

$d_1\ d_3 \geq 0$, since $\gamma_3 < \gamma_1$. Moreover, writing

$$S(d_1, d_2, d_3, d_4) - S(d_1, d_2 + d_4, 0, d_3) = 2\ (\gamma_3 - \gamma_1)\ d_2\ d_4 \leq 0,$$

and using again $\gamma_3 < \gamma_1$, we obtain $d_2\ d_4 \geq 0$. □

By symmetry, remark that

$$S(d_1, d_2, d_3, d_4) = S(d_1, d_4, d_3, d_2) = S(d_3, d_2, d_1, d_4) = S(d_3, d_4, d_1, d_2). \tag{9.39}$$

Then, if $(d_1, d_2, d_3, d_4)$ is a minimum, the above identities give us three other minimums. Here, nothing allows us that all the degrees $d_i$, $1 \leq i \leq 4$, are nonnegative. For this, we state a condition on the applied magnetic field $h_{ex}$ and on the functions $\eta_i$, $0 \leq i \leq 4$, giving us different signs on the degrees $d_i$, $1 \leq i \leq 4$. This will finish the proof of the Theorem.

**Proposition 9.17.** *If $\gamma_1$, $\gamma_2$ and $\gamma_3$ are such that*

$$\gamma_1 < 2\ \gamma_3, \tag{9.40}$$

$$\gamma_1 - 2\ \gamma_2\ + \gamma_3 < 0, \tag{9.41}$$

*then if in addition $h_{ex}$ is such that*

$$\max\Big(\gamma_1,\ 9\ \gamma_1 - 16\ \gamma_2 + 10\ \gamma_3\Big) < 2\ h_{ex}\ X_1 < \gamma_1 + 2\ \gamma_3, \tag{9.42}$$

*there exists a non positive minimum of $S$ over $\mathbb{Z}^4$.*

**Remark 9.18.** *We note that the assumptions (9.40) and (9.41) are taken in order to ensure that the condition (9.42) is not empty. Indeed, under those assumptions, we have*

$$\begin{aligned} 9\ \gamma_1 - 16\ \gamma_2 + 10\ \gamma_3 - (\gamma_1 + 2\ \gamma_3) =& 8\ \gamma_1 - 16\ \gamma_2 + 8\ \gamma_3 \\ =& 8\ (\gamma_1 - 2\ \gamma_2 +\ \gamma_3) < 0, \end{aligned}$$

*which is the condition (9.41).*

### 7.4 Proof of proposition 9.17

We begin with the minimization of the functional $S$ over $\mathbb{N}^4$. In particular, we have

**Lemma 9.19.** *Assume that $\gamma_1 < 2\,\gamma_3$, then if $h_{ex}$ is such that*

$$\gamma_1 < 2\,h_{ex}\,X_1 < \gamma_1 + 2\,\gamma_3,$$

*the minimum of $S$ over $\mathbb{N}^4$ is achieved at $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$.*

**Proof**: Let $n = (n_1, n_2, n_3, n_4)$ be a minimum of $S$ over $\mathbb{N}^4$. First, we suppose that $n_i \geq 1$, $\forall\, 1 \leq i \leq 4$, then

$$\begin{aligned}
S(n_1,n_2,n_3,n_4) =& \Big(\mathbf{A}\,(n_1,n_2,n_3,n_4),\ (n_1,n_2,n_3,n_4)\Big) - 2\,h_{ex}\,X_1\,(n_1+n_2+n_3+n_4)\\
=&\gamma_1\,(n_1^2+n_2^2+n_3^2+n_4^2) + 2\,\gamma_2\,(n_1\,n_2+n_2\,n_3+n_3\,n_4+n_4\,n_1) + 2\,\gamma_3\,(n_1\,n_3+n_2\,n_4)\\
&-2\,h_{ex}\,X_1\,(n_1+n_2+n_3+n_4)\\
=&S(n_1,n_2,n_3,0) + \gamma_1\,n_4^4 + 2\,\gamma_2\,(n_3\,n_4+n_4\,n_1) + 2\,\gamma_3\,n_2\,n_4 - 2\,h_{ex}\,X_1\,n_4\\
\geq&S(n_1,n_2,n_3,0) + n_4\,(\gamma_1 + 2\,\gamma_3 - 2\,h_{ex}\,X_1).
\end{aligned} \tag{9.43}$$

If the applied field is such that

$$h_{ex} < \frac{\gamma_1 + 2\,\gamma_3}{2\,X_1}, \tag{9.44}$$

then (9.43) gives us

$$S(n_1,n_2,n_3,n_4) > S(n_1,n_2,n_3,0),$$

which contradicts the fact that $(n_1,n_2,n_3,n_4)$ is a minimum. This implies that there exists $1 \leq i_0 \leq 4$ such that $n_{i_0} = 0$. Without loss of generality, we suppose that $n_1 = 0$ and $n_i \geq 1$ for $2 \leq i \leq 4$. From now on, $h_{ex}$ is taken to verify (9.44). The case $n_2 \geq 1$ is excluded, indeed if it is true, we have

$$\begin{aligned}
S(0,n_2,n_3,n_4) =&S(0,0,n_3,n_4) + \gamma_1\,n_2^4 + 2\,\gamma_2\,n_2\,n_3 + 2\,\gamma_3\,n_2\,n_4 - 2\,h_{ex}\,X_1\,n_2\\
\geq&S(0,0,n_3,n_4) + n_2\,(\gamma_1 + 2\,\gamma_2 - 2\,h_{ex}\,X_1).
\end{aligned} \tag{9.45}$$

$\gamma_1 + 2\,\gamma_3 < \gamma_1 + 2\,\gamma_2$, since $\gamma_3 < \gamma_2$. Then, using this together with (9.44), we find

$$\gamma_1 + 2\,\gamma_2 - 2\,h_{ex}\,X_1 > 0.$$

We insert this in (9.45) to deduce for $n_2 \geq 1$

$$S(0,n_2,n_3,n_4) > S(0,0,n_3,n_4),$$

which contradicts the fact that $(0,n_2,n_3,n_4)$ is a minimum. Consequently, we can obtain $n_2 = 0$. Third, we suppose that $n_3 \geq 1$ and $n_4 \geq 1$, then

$$\begin{aligned} S(0,0,n_3,n_4) =& S(0,0,0,n_4) + \gamma_1\ n_3^4 + 2\ \gamma_2\ n_3\ n_4 - 2\ h_{ex}\ X_1\ n_3 \\ \geq& S(0,0,0,n_4) + n_3\ (\gamma_1 + 2\ \gamma_2 - 2\ h_{ex}\ X_1) \\ >& S(0,0,0,n_4). \end{aligned} \tag{9.46}$$

The same argument implies that $n_3 = 0$. Now, combining all the above, we have under (9.44) that at this stage the only possibility of $n$ minimum of $S$ over $\mathbb{N}^4$ is $n = (0,0,0,n_4)$ with $n_4 \geq 0$. But, since

$$h_{ex} > \frac{\gamma_1}{2\ X_1},$$

we have $n_4 \geq 1$. Moreover,

$$\begin{aligned} S(0,0,0,n_4) - S(0,0,0,1) &= \gamma_1\ (n_4^2 - 1) - 2\ h_{ex}\ X_1\ (n_4 - 1) \\ &= (n_4 - 1)\ \Big(\gamma_1\ (n_4 + 1) - 2\ h_{ex}\ X_1\Big). \end{aligned} \tag{9.47}$$

Suppose that $n_4 \geq 2$, then using the fact that $\gamma_3 < \gamma_1$, we have from (9.44)

$$\gamma_1\ (n_4 + 1) - 2\ h_{ex}\ X_1 > 3\ \gamma_1\ - 2\ h_{ex}\ X_1 > \gamma_1 + 2\ \gamma_3\ - 2\ h_{ex}\ X_1 > 0.$$

Thanks to this, (9.47) implies for $n_4 \geq 2$

$$S(0,0,0,n_4) > S(0,0,0,1).$$

Finally, we must have $n_4 = 1$, so $(0,0,0,0,1)$ minimizes $S$ over $\mathbb{N}^4$. Referring to (9.39), the proof of lemma 9.19 is completed. □

**Completing the proof of proposition 9.17**

Here, our interest is to find a minimum of $S$ over $\mathbb{Z}^4$ such that it does not belong to $\mathbb{N}^4$. For this, it suffices to find a point in the part $\mathbb{Z}^4 \cap (x \geq 0, y < 0, z \geq 0, t < 0)$ such that its image by $S$ is strictly less than $S(1,0,0,0)$, since $(1,0,0,0)$ is a minimum of $S$ over $\mathbb{N}^4$. Now, we assume in addition to (9.40) that

$$\gamma_1 - 2\ \gamma_2 + \gamma_3 < 0,$$

then we can find an applied field $h_{ex}$ satisfying (9.42). In particular,

$$9\ \gamma_1 - 16\ \gamma_2 + 10\ \gamma_3 < 2\ h_{ex}\ X_1. \tag{9.48}$$

Note that

$$S(2,-1,2,-1) = 10\,\gamma_1 - 16\,\gamma_2 + 10\,\gamma_3 - 4\,h_{ex}\,X_1.$$

Comparing $S(2,-1,2,-1)$ to $S(1,0,0,0)$, we get

$$S(2,-1,2,-1) - S(1,0,0,0) = 9\,\gamma_1 - 16\,\gamma_2 + 10\,\gamma_3 - 2\,h_{ex}\,X_1. \tag{9.49}$$

We deduce thanks to (9.48),

$$S(2,-1,2,-1) < S(1,0,0,0).$$

Consequently, there exists a minimum $d$ of $S$ over $\mathbb{Z}^4$ in the region $(x \geq 0, y < 0, z \geq 0, t < 0)$, meaning that there exists a non positive minimum of $S$ over $\mathbb{Z}^4$. This completes the proof of proposition 9.17.
Finally, combining all the above, the Theorem 9.9 is then proved.

# Bibliography


[Ab] Abrikosov A , Magnetic properties of group $II$ superconductor, J. Exptl. Theoret. Phys. (USSR), 32, 1174-1182, V 5, (1957).

[Ad] Adams R. D, Sobolev spaces, Birkhaüsser, (1975).

[ABB] Alama S, Berlinsky A. J, Bronsard L, Periodic vortex lattices for the Lawrance-Doniach model of layered superconductors in a parallel field, Commun. Contemp. Math, 3, N 3, 457-494, (2001).

[ABS] Alama S, Bronsard L, Sandier E, On the shape of interlayer vortices in the Lawrence-Doniach model, (2003).

[APB] André A, Bauman P, Philips D, Vortex pinning with bounded fields for the Ginzburg-Landau equation, Annales de l'institut Henri Poincaré (C), Non linear analysis, 20, N 4, 705-729, (25), July (2003).

[ASS] Aftalion A, Sandier E, Serfaty S, Pinning phenomena in the Ginzburg-Landau model of superconductivty, J. Math. Pures. Appl, 80, N 3, (2001), 339-372.

[B] Brezis H, Analyse fonctionnelle, théories et applications, collection mathématique appliquée pour la maîtrise. Masson, (1983).

[BBH] Béthuel F, Brezis H, Hélein F, Ginzburg-Landau vortices, Progress in non linear differential equations and their Applications, 13, Birkhaüsser, (1994).

[BR] Béthuel F, Rivière T, Vortices for a variational problem related to superconductivty, Annales de l'institut Henri Poincaré (Analyse Non Linéaire), 12, (1995), N 3 , 243-303.

[BS] Brezis H, Serfaty S, A variational for the two-sided obstacle problem with measure data, Comm. Contemp. Math, 4, N 2, 17-28, (2003).

[D] Dutour M, (PhD thesis), Bifurcation vers l'état d'Abrikosov et diagramme de phase, University of Orsay, (France), (1999).

[DC] Donato P, Cioranescu D, Introduction to homegenization. Astérisque

[DGP] Du Q, Gunzbuger M . D, Peterson J. S, Modeling and analysis of a periodic Ginzburg-Landau model for type II superconductors, SIAM. J. applied. math, 53, N 3, 689-717, (1993).

[EG] Evans L. C, Gariepy R. F, Measure theory and fine properties of functions, Studies in Advanced Mathematics.

[ET] Ekeland I, Temam R, Analyse convexe et problèmes variationnels, Études mathématiques, Paris-Bruxelles-Montreal; Dunod; Gauthier-Villars, (1974).

[GL] Ginzburg V. L, Landau L.D, On the theory of superconductivity, Zh. Ekesperim.i theor. Fiz, 20 , 1064- 1082, (1950).

[GT] Gilbarg. D, Trudinger. N. S, Elliptic partial differential equations of second order, 2nd Edition, Springer-Verlag, (1983).

[HP] Helffer B, Pan X. B, Upper critical field and location of surface nucleation of superconductivity, Ann. Inst. H. Poincaré, Analyse non linéaire, Feb (2001) .

[J] Jerrard R, Lower bounds for generalized Ginzburg-Landau functionals, SIAM. J. Math. Anal, 30, N 3, 721-746, (1999).

[P] Pan X. B, Surface superconductivity in applied magnetic fields above $H_{c_2}$, Comm. Math. Phys, 228, 327-377, (2000).

[Sa] Sandier E, Lower bounds for the energy of unit vector fields and applications, J. Func. Ana, 152, N 2, (1998), 379-403.

[SS1] Sandier E, Serfaty S, Global minimizers for the Ginzburg-Landau functional below the first critical magnetic field, Annales IHP, Analyse non linéaire, 17, N 1, 119-145, (2000).

[SS2] Sandier E, Serfaty S, On the energy of type-II superconductors in the mixed phase, Reviews in Math. Phys, 12 , N 9, 1219-1257, (2000).

[SS3] Sandier E, Serfaty S, A rigorous derivation of a free-boundary problem arising in superconductivty, Annales scientifiques de l'école normale supérieure, 4e série, 33, (2000), 561-592.

[SS4] Sandier E, Serfaty S, Ginzburg-Landau minimisers near first critical field have bounded vorticity, Calc. Var. PDE, V 17, N 1, May (2003).

[SS5] Sandier E, Serfaty S, Limiting vorticities for the Ginzburg-Landau equation, Duke. Math, J, 117, N 3, 403-446 (2003).

[SS6] Sandier E, Serfaty S, Vortices in the magnetic Ginzburg-Landau model, monograph in preparation, to be published by Birkhaüsser.

[Se1] Serfaty S, Local minimizers for the Ginzburg-Landau energy near critical magnetic field, Part I, Comm. Contemp. Math, 1, N 2, (1999), 213-254.

[Se2] Serfaty S, Local minimizers for the Ginzburg-Landau energy near critical magnetic field, Part II, Comm. Contemp. Math, 1, V 3, (1999), 298-333.

[Se3] Serfaty S, Stable configurations in superconductivity: uniqueness, multiplicity and vortex-nucleation, Arch. Rat. Mech. Anal, 149, (1999), 329-365.

[SST] Siant-James D, Sarma G, Thomas E. J, Type-II supercondivity, Pergamon Press, (1969).

[Th] T'Hooft G, A property of electric and magnetic flux in non-Abelian gauge theories, Nucl. Phy, V B153, (1979), 141-160.

[Ti] Tinkham M, Introduction to superconductivity, 2nd edition. Mc Graw-Hill, (1996).

[TT] Tilly D, Tiley J, Superfluidity and superconductivity, 2nd edition, Adam Hilger Ltd, Bristol, (1986).

[W] Watson, Treatise on the theory of Bessel functions, Cambridge University Press, Cambridge, England, The Macmillan Campany, NY, (1994).