Exploration de la valeur de Shapley et des indices d'interaction pour les jeux définis sur des ensembles ordonnés

Fabien Lange

UNIVERSITÉ PARIS 1 PANTHÉON-SORBONNE

THÈSE

pour obtenir le grade de
Docteur de l'Université Paris 1 Panthéon-Sorbonne
Discipline : Informatique

présentée par

Fabien LANGE

*
*   *

EXPLORATION DE LA VALEUR DE SHAPLEY ET DES INDICES D'INTERACTION POUR LES JEUX DÉFINIS SUR DES ENSEMBLES ORDONNÉS

*   *
*

soutenue le 14 décembre 2007 devant le jury composé de

M. Michel GRABISCH	Professeur à l'Université Paris 1	Directeur
M. Ulrich FAIGLE	Professeur à l'Université de Cologne	Rapporteur
M. Stef TIJS	Professeur à l'Université de Tilburg	Rapporteur
M. Joseph ABDOU	Professeur à l'Université Paris 1	Examinateur
M. Christophe LABREUCHE	Chercheur, Thales Research & Technology	Examinateur
M. Jean-François LASLIER	Directeur de recherche au CNRS	Examinateur
M. Bruno LECLERC	Maître de conférences à l'EHESS	Examinateur

Remerciements

Ces années de thèse auront été pour moi l'occasion d'expériences variées dans le monde de la recherche. Pour toutes, sans exception, Michel Grabisch aura été un prodigieux initiateur, présentant conjointement de nombreuses et rares qualités tant sur le plan humain que sur le plan professionnel. Il m'a témoigné une grande patience, m'a soutenu et guidé tout au long de ces années avec une réelle bienveillance. Sa créativité, sa rigueur scientifique, sa détermination et ses talents pédagogiques comptent sans aucun doute pour une grande part dans ma formation et dans mon goût de la recherche. Ces quelques mots sont bien peu de choses en regard de ces quelques lignes qui veulent exprimer ma profonde reconnaissance.

Je remercie grandement les professeurs Ulrich Faigle et Stef Tijs de m'avoir fait l'honneur de rapporter ce travail, et d'y avoir investi une grande partie de leur temps. Je remercie vivement Christophe Labreuche, Bruno Leclerc, Jean-François Laslier et Joseph Abdou d'avoir accepté d'examiner cette thèse.

Cette thèse n'aurait pu être entreprise si dans un premier temps, le laboratoire Marin Mersenne et l'Ecole docorale de philosophie de l'université Paris 1 ne m'avaient pas chaleureusement accueilli les deux premières années. Je remercie ainsi particulièrement Jean-Bernard Baillon et Denis Pennequin pour leur accueil bienveillant et amical en cette première phase de thèse. Un grand merci également aux membres du SAMOS. J'ai également eu la joie, les deux années suivantes, d'avoir pu rejoindre l'équipe du CERMSEM. Merci à son directeur Bernard Cornet, de m'avoir accueilli, ainsi qu'aux membres enseignants et chercheurs d'avoir contribué à de bonnes conditions de travail en son sein. Un non moindre merci à toute une bande de thésards qui ont su amener joie et bonne humeur au quotidien. Chacun d'eux (elles), avec son originalité, a grandement contribué à une vie de laboratoire des plus sympathiques.

Cette expérience n'aurait sans doute pas eu lieu, si je n'avais eu la chance cinq ans plus tôt de suivre la formation du DEA MIASH, avec une fantastique équipe pédagogique à son actif : Olivier Hudry, Marc Demange, Bernard Monjardet, Éric Soutif, ainsi que Michel Grabisch, et Bruno Leclerc, se sont tous révélés être des enseignants de grande qualité, et chacun y est assurément pour quelque chose dans mon choix qui a suivi d'entreprendre une thèse.

La même année, j'ai pu bénéficier de la confiance de Jean-Pierre Leca, qui m'a permis de débuter dans l'enseignement. Qu'il y trouve ici toute ma reconnaissance. À ce sujet, je remercie également Daniel Trécourt, Alain Chateauneuf, et encore Michel Grabisch, qui m'ont permis d'enseigner dans des domaines très variés.

Ces années de thèse ont été l'occasion de nombreux voyages et conférences internationales, qui ont contribué à développer un fort goût du contact scientifique. J'ai ainsi rencontré Ivan Kojadinovic, Christophe Labreuche, Jean-Luc Marichal, Pedro Miranda, entre autres, qui se sont révélés êtres des contacts fort intéressants. Le professeur Endre Pap m'a fait bénéficié par deux fois d'un accueil des plus chaleureux à l'université de Novi Sad. Un grand merci également au professeur H. Naraynan, Fabrice Jammes, Fabrice Mathurin et Julien Reynaud pour leur coopération ou leur aide à des tâches de recherche ou d'enseignement.

Un grand merci à Valentin, Nicolas, Christophe, et Agnieszka pour leur aide précieuse et leur soutien solide, y compris en des circonstances pas toujours favorables ! Merci spécial à Raphaëlle, Damien (dit DD), pour son sens indéfectible de l'amitié, et à ma mère, qui m'a toujours encouragé dans la voie engagée, et sans qui je n'aurais sans doute jamais prolongé mes études au-delà de la maîtrise.

Last but not least, de tout mon cœur, je te remercie, Houa, pour ton soutien sans faille et ton amour. Cette thèse t'est dédiée.

Table des matières

Introduction

La théorie des jeux constitue une approche mathématique de problèmes de stratégie tels qu'on en trouve en recherche opérationnelle et en économie. Elle étudie les situations où les choix de deux ou plusieurs protagonistes ont des conséquences pour les uns comme pour les autres.

Les *jeux coopératifs* constituent une classe de jeux pour lesquels les regroupements de joueurs — les *coalitions* — encouragent les comportements de coopération. Ainsi, dans un tel modèle, un jeu n'est pas tant une compétition entre les individualités qu'entre les différentes coalitions jouables. Dans cette thèse, nous nous intéressons principalement aux jeux coopératifs donnés sous *forme de fonction caractéristique*, c'est-à-dire mathématiquement donnés sous la forme d'une fonction à valeurs réelles définie sur la collection des coalitions d'un ensemble N de joueurs : pour un jeu v donné, et une coalition S de joueurs, le nombre réel $v(S)$ s'interprète selon le cas comme un *gain* dont les membres de S bénéficient dans le cas où la coalition S est jouée (*jeux d'allocation de profit*), ou parfois comme un *coût* (*jeux d'allocation de coût*). Sauf précision contraire, les jeux considérés dans ce document sont des jeux d'allocation de profit.

Un autre outil apparenté aux jeux coopératifs est aussi présenté : les *capacités*. Dans ce contexte, N ne désigne non plus les joueurs mais un ensemble de *critères* ou d'*attributs*. Introduit en 1953 par Choquet, l'outil de capacité [6] permet de modéliser l'importance des groupes de critères, et se révèle donc utile dans de nombreuses situations en *aide multicritère à la décision* (MCDA).

Mathématiquement, une capacité est la donnée d'un jeu *monotone* et normalisé (i.e., $v(N) = 1$). Il s'agit donc d'un objet vérifiant les mêmes axiomes qu'une mesure de probabilité, moins l'additivité. C'est d'ailleurs sous le terme de *mesure non-additive* que Denneberg les conceptualise [8]. Les capacités se révèlent être très utiles, par exemple dans la fusion de données fondées sur les intégrales non additives.

Un concept-clé intervenant dans tout ce document est la *valeur de Shapley*. Dans un papier remarquable écrit en 1953, Shapley [40] émet la possibilité de déterminer le juste partage entre les joueurs, du gain généré par la *grande coalition*, N. Ce qu'il parvient à faire par l'introduction de trois seuls axiomes intuitifs et naturels :

> *Symétrie* : les noms, ou numéros des joueurs, ne jouent aucun rôle dans la détermination de leurs rétributions.
>
> *« Coalition-support » (carrier axiom)* : pour toute coalition P de joueurs dite *support* du jeu v, c'est-à-dire en dehors de laquelle les autres joueurs ont une contribution nulle, les membres de P se partagent $v(P)$.
>
> *Additivité* : si v et w sont deux jeux, alors la rétribution de chacun des joueurs pour le jeu somme $v + w$ est égale à la somme de leurs rétributions pour les jeux v et w.

Noter que des variantes existent dans la présentation de ces axiomes (cf. sous-section 2.2). La première partie de cette thèse (chapitres 1 à 3) a pour objet la généralisation de valeur de Shapley, et une axiomatisation de celle-ci, pour des jeux définis sur des structures très générales de coalitions, ou d'*actions*.

La valeur de Shapley est un bon moyen de mesurer le taux d'implication d'un joueur dans un jeu. Elle ne donne cependant aucune information sur le phénomène de coopération entre les joueurs. Deux joueurs i et j peuvent par exemple afficher une *interaction* positive, ce qui se traduit par le fait que la coalition $\{i, j\}$ « pèse » plus que la somme de ce que produisent séparément i et j ; ou, à l'inverse, ceux-ci peuvent montrer un manque d'intérêt à coopérer, situation dans laquelle la coalition $\{i, j\}$ aura un poids inférieur à la somme des poids individuels. L'*indice d'interaction* de Shapley pour les jeux coopératifs classiques, proposé par Grabisch [17], se présente mathématiquement comme un prolongement de la valeur de Shapley sur l'ensemble des coalitions.

On propose dans la seconde partie de ce document, une exploration de l'indice d'interaction pour les jeux bi-coopératifs et bi-capacités, avec des axiomatisations de celui-ci (chapitre 4), et la mise en œuvre du calcul de l'inverse d'interaction, permettant de retrouver un jeu sous sa forme classique à partir de son indice d'interaction (chapitre 5). Le chapitre 6 considère des jeux coopératifs prenant une forme très générale, et met en évidence l'équivalence de représentations entre un jeu, sa transformée de Möbius, et son indice d'interaction.

1 Différents types de jeux

Dans ce document, on considère un ensemble fini $N := \{1, \ldots, n\}$ dont les éléments seront selon le contexte, des joueurs, des votants (théorie des jeux), des critères, des attributs (décision multicritère, MCDA), etc. Quel que soit ce contexte, notre approche se fait basiquement par l'introduction de fonctions définies sur des structures partiellement ordonnées d'actions. Celles-ci seront selon le cas des parties de N (*jeux coalitionnels*) ou non.

S'il n'y a pas d'ambiguïté, pour éviter des lourdeurs d'écriture, on omettra les accolades pour désigner des singletons ou des paires d'éléments, en écrivant par exemple $N \setminus i$ au lieu de $N \setminus \{i\}$, ou ij au lieu de $\{i, j\}$. D'autre part, le cardinal d'un ensemble $S, T, \ldots$ sera souvent noté par la lettre minuscule correspondante $s, t, \ldots$

On propose pour commencer de donner quelques définitions centrales de théorie des ensembles ordonnés. Soit $(P, \leq)$ un ensemble ordonné. Pour tous éléments x, y de P, on dit que x est *couvert* par y, et on écrit $x \prec y$, si $x \leq y$ et s'il n'existe aucun autre élément $z \in P$ tel que $x \leq z \leq y$. La relation $\prec$ de couverture induite par un ordre partiel quelconque est en fait la plus petite relation binaire dont la fermeture transitive se trouve être cet ordre partiel. Celle-ci est en fait un moyen commode de représenter graphiquement n'importe quel ordre partiel. Cette représentation a pour nom *diagramme de Hasse*. La figure 1 représente par exemple le diagramme de Hasse de l'ensemble des parties de 2^N, avec $n = 3$.

Pour tout couple d'éléments (x, y) de P tel que $x \leq y$, on appelle *chaîne maximale*[1] de x à y sur P, toute suite $(x_0 := x, x_1, \ldots, x_m := y)$ d'éléments de P telle que $x_{i-1} \prec x_i$, $i = 1, \ldots, m$. On appelle *longueur* de la chaîne, l'entier m. Par exemple, $(2^N, \subseteq)$ compte $n!$ chaînes maximales de $\emptyset$ à N, toutes de longueur n, chacune d'entre elles étant associée à une permutation de N.

[1]Si P admet des plus petit et plus grand éléments $\perp$ et $\top$, et que x et y ne sont pas précisés, il sera tacitement question d'une chaîne maximale de $\perp$ à $\top$.

Un partie Q de P est dite *idéal d'ordre* de P si

$$\forall x, y \in P, \quad y \in Q \text{ et } x \leq y \Rightarrow x \in Q.$$

Définition 1 (treillis) *Un* treillis *est un ensemble partiellement ordonné T dans lequel chaque couple d'éléments x, y admet dans T une borne supérieure et une borne inférieure.*

Pour x, y éléments d'un treillis T, on note respectivement $x \vee y$ et $x \wedge y$ les bornes supérieure et inférieure de x et y. En particulier, si T est fini, ces dernières sont définies pour toute partie non vide de T, et dans ce cas, on note respectivement $\top$ et $\bot$, les plus grand et plus petit éléments de T.

On dit que $x \in T$ est un élément *sup-irréductible* de T si x ne peut être exprimé comme borne supérieure d'autres éléments de T. Un résultat classique nous indique que les éléments sup-irréductibles d'un treillis sont précisément ceux couvrant un et un seul élément. On note $\underline{x}$ l'élément couvert par le sup-irréductible x. En particulier, on appelle *atome* tout sup-irréductible x tel que $\underline{x} = \bot$.

1.1 Jeux coopératifs classiques et capacités

On appelle *coalition* tout élément de 2^N, c'est-à-dire de l'ensemble des parties de N. Muni de l'inclusion, 2^N est un ensemble partiellement ordonné, et même un treillis, dont les bornes supérieure et inférieure sont respectivement données par la réunion et l'intersection. Tout ensemble ordonné isomorphe à un tel treillis est dit *treillis booléen*. Dans toute la sous-section, on considère des fonctions d'ensemble définies sur 2^N.

Définition 2 (jeu coopératif classique) *On appelle* jeu coopératif classique *sur N toute application v définie sur 2^N, telle que $v(\emptyset) = 0$.*

On note $\mathcal{G}(2^N)$ l'ensemble des jeux coopératifs classiques sur N.

En économie, de telles applications sont souvent qualifiées de jeux à *utilité transférable* en forme coalitionnelle, de par l'interprétation que l'on fait des valeurs que prennent ces jeux : pour un jeu v, et une coalition S, $v(S)$ peut être généralement

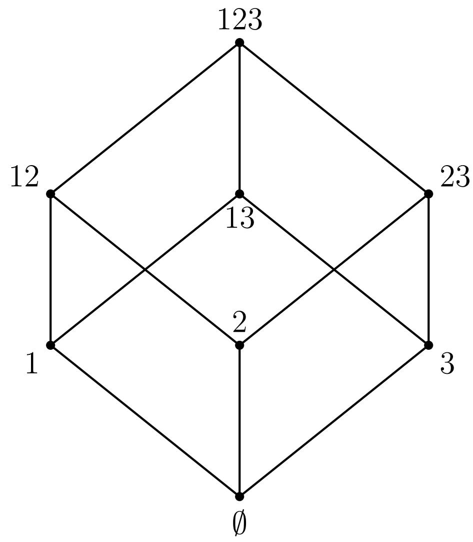


FIG. 1 – Diagramme de Hasse de $(2^N, \subseteq)$ (n=3)

considéré comme la quantité maximale de bien que les membres de S peuvent obtenir s'ils coopèrent.

Un jeu coopératif v est dit *monotone* si

$$\forall S, T \in 2^N \text{ tel que } S \subseteq T, \quad v(S) \leqslant v(T).$$

On distingue parmi les jeux vérifiant cette propriété une catégorie largement utilisée en *théorie des votes* :

Définition 3 (jeu simple) *Un jeu coopératif classique v est dit* simple *s'il est monotone, et si pour toute coalition S, $v(S) \in \{0, 1\}$, avec $v(\emptyset) = 0$, et $v(N) = 1$.*

Dans ce contexte, une coalition S est dite *gagnante* dans le jeu simple v si $v(S) = 1$. On peut employer les jeux simples lorsque les éléments de N sont des partis politiques. À partir d'une règle de décision (adoption d'un projet de loi, par exemple), un jeu simple est modélisé pour déterminer les coalitions votant la règle en majorité. On peut recourir alors dans ce cas à un calcul d'*indice de pouvoir* permettant de déterminer le parti le plus influent (cf. section 2). Les *jeux unanimes* sont des jeux simples particuliers : pour tout $S \in 2^N$, $S \neq \emptyset$,

$$\forall T \in 2^N, \quad u_S(T) := \begin{cases} 1, & \text{si } T \supseteq S, \\ 0, & \text{sinon.} \end{cases}$$

Nous plaçant dans le contexte de la MCDA, on peut utiliser l'outil de capacité, qui peut se révéler très efficace dans l'agrégation par *intégrale de Choquet*, par exemple.

Définition 4 (capacité) *On appelle* capacité *tout jeu coopératif classique monotone ν, tel que $\nu(N) = 1$.*

Selon le contexte et l'historique, d'autres termes désignent ce même concept. Il semblerait ainsi que Vitali ait été le premier à introduire le concept [45] de probabilité non additive en 1925. En 1974, Sugeno les introduit sous le nom de *mesures floues*.

Soit par exemple à évaluer un ensemble d'*alternatives* ou d'*actions*. On dispose pour cela d'un ensemble de critères N sur lesquels chacune de ces alternatives devra être notée. Les *actions binaires* sont des actions représentant une situation particulière dans laquelle un certain sous-ensemble A de critères obtient le score maximal 1, et l'ensemble complémentaire le score minimal 0. On demande à un *décideur* d'attribuer pour chacune de ces actions binaires $(1_A, 0_{N\setminus A})$, une évaluation sur une échelle linéaire. Si cette attribution respecte comme attendu la monotonie (A inclus dans A' implique que l'action binaire $(1_{A'}, 0_{N\setminus A'})$ soit mieux notée que $(1_A, 0_{N\setminus A})$), et si l'action $(1_\emptyset, 0_N)$ (tous les critères sont inacceptables) a une évaluation globale nulle, ceci définit une capacité ν définie sur l'ensemble des parties de N.

Pour une alternative donnée, l'intégrale de Choquet par rapport à la capacité ν est alors un moyen classique en MCDA d'agréger les scores obtenus pour chacun des critères. Cette intégrale de Choquet est en fait une généralisation de la moyenne pondérée dans le sens que si la capacité définie par le décideur se trouve être additive, et donc une probabilité, il en résulte que le résultat obtenu sera la moyenne arithmétique des scores. En outre, et de façon non exhaustive, celle-ci permet ainsi de modéliser non seulement les moyennes pondérées, mais aussi les minimum, maximum, médianes et autres statistiques d'ordre telles que la moyenne dite « olympique »(voir par exemple [48]).

1.2 Jeux bi-coopératifs

Dans d'autres circonstances, il est souhaitable d'établir une *bipolarité* sur les actions. Il a en effet été observé que des décideurs humains intervenant dans une évaluation ont un comportement asymétrique face à des alternatives opposées. Bilbao a ainsi modélisé les *jeux bi-coopératifs* [2], situation dans laquelle chaque action fait l'objet d'une *bi-coalition* (A, B) de joueurs : les joueurs de A collaborent positivement, ceux de B négativement, les joueurs restants s'abstenant de participer. On appelle parfois les joueurs de A coalition *défensive*, et les joueurs de B, coalition *défaitrice*. La valeur du jeu en (A, B) quantifie alors le gain des joueurs de A dans la situation (A, B). Les jeux bi-coopératifs peuvent être vus comme des extensions des jeux de vote ternaires introduits par Felsenthal et Machover [14].

Plus précisément, définissons

$$\mathcal{Q}(N) := \left\{(A, B) \in 2^N \times 2^N \mid A \cap B = \emptyset\right\},$$

que l'on munit de la relation d'ordre

$$(A, B) \sqsubseteq (A', B') \text{ ssi } A \subseteq A' \text{ et } B \supseteq B'.$$

L'ensemble des éléments sup-irréductibles de $\mathcal{Q}(N)$ est

$$\mathcal{J}(\mathcal{Q}(N)) = \{(\emptyset, N \setminus i), (i, N \setminus i) \mid i \in N\}.$$

Le diagramme de Hasse de $(\mathcal{Q}(N), \sqsubseteq)$ est représenté p. 167, où les éléments sup-irréductibles sont représentés par les disques noirs.

En identifiant tout élément (A, B) de $\mathcal{Q}(N)$ à la différence de fonctions indicatrices $1_A - 1_B$, on constate aisément que $(\mathcal{Q}(N), \sqsubseteq)$ est un treillis isomorphe à 3^N. Les bornes supérieure et inférieure de deux éléments sont respectivement données par :

$$\forall (A, B), (A', B') \in \mathcal{Q}(N), \quad \begin{aligned}(A, B) \sqcup (A', B') &= (A \cup A', B \cap B'),\\ (A, B) \sqcap (A', B') &= (A \cap A', B \cup B').\end{aligned}$$

Les plus grand et plus petit éléments de $\mathcal{Q}(N)$ sont

$$\begin{aligned}\top &= (N, \emptyset),\\ \bot &= (\emptyset, N).\end{aligned}$$

Définition 5 (jeu bi-coopératif) *On appelle* jeu bi-coopératif *toute application v définie sur $\mathcal{Q}(N)$ telle que $v(\emptyset, \emptyset) = 0$.*

Grabisch et Labreuche ont adapté ce modèle en définissant à leur tour les *bi-capacités* [18, 20], généralisation des capacités, modélisant les actions *ternaires* que constituent les bi-coalitions de critères : ceux qui sont totalement satisfaits, ceux totalement inacceptables, les autres restant neutres. C'est dans ce contexte de bipolarité que l'on se place aux chapitres 4 et 5.

Définition 6 (bi-capacité) *On appelle* bi-capacité *tout jeu bi-coopératif ν monotone pour l'ordre $\sqsubseteq$, i.e.*

$$\forall A, B \in N, \quad A \subseteq B \textit{ implique } \nu(A, \cdot) \leqslant \nu(B, \cdot) \textit{ et } \nu(\cdot, A) \geqslant \nu(\cdot, B),$$

et vérifiant $\nu(N, \emptyset) = 1 = -\nu(\emptyset, N)$.

Par suite, les mêmes auteurs ont défini et caractérisé l'intégrale de Choquet par rapport à des bi-capacités [21].

1.3 Jeux réguliers et autres jeux coalitionnels

Il arrive parfois que pour diverses raisons (incompatibilité entre les joueurs, contraintes de formation des coalitions, etc.), seul un sous-ensemble de 2^N puisse être accepté comme ensemble de coalitions *réalisables*. Ainsi, on trouve dans la littérature divers types de structures de coalitions. Faigle a par exemple introduit l'idée de *contraintes de précédence* parmi les joueurs [12] (voir sous-section suivante). Bilbao et Edelman définissent des jeux sur des *géométries convexes* [3], ou encore des *anti-matroïdes*. Outre le fait que chacune de ces structures (voir leurs définitions au chapitre 3, p. 99) soit un sous-ensemble $\mathcal{N}$ de 2^N muni de la relation d'inclusion, notons aussi que leur point commun réside dans leur statut de *système de coalitions*, i.e., $\mathcal{N} \ni \emptyset, N$.

Honda et Grabisch ont introduit une classe très générale de jeux coalitionnels, les *jeux réguliers*, qui permettent de choisir avec une grande souplesse les coalitions que l'on désire faire entrer dans le jeu.

Définition 7 (système de coalitions régulier) *On appelle* système de coalitions régulier *tout système de coalitions $\mathcal{N}$ vérifiant la propriété suivante :*

$$\forall S, T \in \mathcal{N} \text{ tels que } S \prec T, \text{ alors } |T \setminus S| = 1,$$

où $\prec$ est la relation de couverture associée à l'ordre d'inclusion dans $\mathcal{N}$.
Une application définie sur un tel ensemble, et s'annulant en $\emptyset$, est dite jeu régulier*.*

Une autre façon de caractériser un système de coalitions régulier, est d'imposer que toutes les chaînes maximales de coalitions de $\emptyset$ à N soient de longueur n (cf. proposition 1, p. 105). Cette contrainte modélise en fait très bien les situations pour lesquelles il est nécessaire pour augmenter une coalition (au sens de l'inclusion), de n'ajouter qu'un seul joueur. L'ensemble 2^N constitue un exemple simple de système de coalitions régulier. Notons que ceux-ci ne sont pas nécessairement des treillis (cf. Chapitre 3, proposition 2).

Par exemple, soit un réseau de connaissance entre n individus, dont l'un d'entre eux, au moins, est dit *individu de contact*. Un tel réseau est représentable, sans perte de généralité, par un graphe connexe non orienté à n sommets (individus), certains étant marqués (individus de contact), et dont les arêtes représentent les liens de connaissance. Alors, si on considère l'ensemble de toutes les coalitions d'individus obtenues par un quelconque parcours dans le graphe à partir d'un individu de contact, on obtient là un système de coalitions régulier. En fait chacune de ces coalitions réalisables est soit un singleton-individu de contact, soit une coalition dans laquelle deux individus quelconques ont entre eux, sinon un lien de connaissance direct, une « chaîne » de connaissance initiée par l'un des individus de contact. Par exemple, le système 2^N est obtenu à partir du graphe complet à n sommets, dont tous sont marqués. La figure 2, p. 34 représente le système de coalitions régulier obtenu à partir du réseau de connaissance à trois individus numérotés 1, 2 et 3, tous trois individus de contact, et avec les liens $1 \leftrightarrow 2$ et $2 \leftrightarrow 3$.

1.4 Jeux définis sur des treillis distributifs

Jeux multi-choix

Introduits en 1993 par Hsiao et Raghavan [27], les *jeux multi-choix* constituaient déjà une étape supplémentaire dans la généralisation des jeux coopératifs. Dans ce modèle, chaque joueur $i \in N$ a à sa disposition un ensemble totalement ordonné $L_i := \{0, \dots, l_i\}$ ($l_i \geqslant 1$) de niveaux de participation[2]. Ainsi le treillis $L := \prod_{i=1}^n L_i$, muni de l'ordre produit, désigne l'ensemble des actions conjointes des n joueurs. Le plus petit élément de L, $\perp = (0_1, \dots, 0_n)$, représente ainsi l'action nulle de L.

Définition 8 (jeu multi-choix) *On appelle* jeu multi-choix *toute application v définie sur un treillis L de la forme ci-dessus, et s'annulant en $\perp$.*

On note $\mathcal{G}(L)$ l'ensemble des jeux multi-choix définis sur L.

> Remarque. Lorsque $L_i = \{0, 1\}$ pour tout i, L est alors booléen, et on reconnaît en $\mathcal{G}(L)$ les jeux coopératifs classiques donnés sous forme de *fonction pseudo-booléenne* [25].

Notons que les *capacités k-aire* représentent l'équivalent dans le domaine de la MCDA [19], des jeux multi-choix à k niveaux de participation non nuls.

Jeux à actions combinées

Tandis que dans le cas des jeux multi-choix, chaque joueur peut agir à un certain niveau d'une échelle linéaire, Grabisch [22] a développé l'idée de jeux définis sur des treillis distributifs.

Définition 9 (treillis distributif) *Un treillis est* distributif *si la loi $\vee$ est distributive sur la loi $\wedge$, ou si la loi $\wedge$ est distributive sur la loi $\vee$ (ces deux conditions étant équivalentes).*

[2] Dans le modèle initial de Hsiao et Raghavan, tous les l_i sont égaux.

Un théorème important en théorie des treillis, est dû à Birkhoff [4] :

Théorème 1 (Birkhoff) *Soit* $(T, \leq)$ *un treillis distributif fini,* J *l'ensemble de ses éléments sup-irréductibles, et* $\mathcal{O}(J)$ *l'ensemble des idéaux de* $(J, \leq)$*. Alors* $(T, \leq)$ *est isomorphe à* $(\mathcal{O}(J), \subseteq)$*.*

Il est à noter que ce résultat caractérise les treillis distributifs finis. D'autre part, chaque ensemble ordonné fini caractérisant un treillis distributif par la donnée de ses idéaux munis de l'inclusion, il existe ainsi une bijection entre la classe des treillis distributifs finis et la classe des ensembles finis partiellement ordonnés. Par exemple, le treillis booléen 2^N, qui est un cas particulier de treillis distributif, a pour ensemble ordonné associé l'antichaîne à n éléments. On trouvera p. 60 (chapitre 1) et p. 200 (chapitre 6), deux autres exemples d'association ensemble ordonné/treillis distributif. Notons que le système de coalitions régulier représenté figure 2 (p. 34) est bien un treillis, mais n'est pas distributif.

Considérons pour chaque joueur $i \in N$ un ensemble fini d'*actions élémentaires* $\mathcal{J}_i$ muni d'un ordre partiel $\leq_i$: $j \leq_i j'$ dans $\mathcal{J}_i$ se traduisant par « l'action j' inclut l'action j »(une illustration p. 60, chapitre 1, donne un exemple d'interprétation). Toute action réalisable pour le joueur i est donc un sous-ensemble d'actions élémentaires donné par un idéal de $\mathcal{J}_i$. Le théorème de Birkhoff nous indique alors que l'ensemble de ces idéaux (actions réalisables) est un treillis distributif noté L_i. Tout élément sup-irréductible de L_i s'écrit alors $\downarrow j := \{x \in \mathcal{J}_i \mid x \leq_i j\}$ $(j \in \mathcal{J}_i)$, idéal qui peut s'identifier à l'action élémentaire j. De manière générale, tout élément de L_i, qui est un idéal J de $\mathcal{J}_i$, peut être identifié à l'ensemble des éléments maximaux de J.

Maintenant, si l'on considère l'ensemble de toutes les actions réalisables « combinées » par l'ensemble des joueurs, celui-ci est donné par le produit cartésien L des L_i. Notons alors que L est encore un treillis distributif dont les éléments sup-irréductibles sont de la forme $(\emptyset, \ldots, \emptyset, \downarrow j_i, \emptyset, \ldots, \emptyset)$, $i \in N$, $j_i \in \mathcal{J}_i$. On notera de manière générale $\mathcal{J}(L)$ l'ensemble des éléments sup-irréductibles de L.

En vertu de l'équivalence des classes d'ensembles ordonnés et treillis distributifs mentionnée plus haut, on donne la définition suivante. Clairement, lorsque tous les L_i sont linéaires, on retrouve alors les jeux multi-choix.

Définition 10 (jeu à actions combinées) *Soit* $(L_i, \leq_i)$ *des treillis distributifs* *finis,* $L := \prod_{i=1}^{n} L_i$*, et* $\leq$ *l'ordre produit des* $\leq_i$*. On appelle* jeu à actions combinées *sur* L *toute application* v *définie sur* L*, telle que* $v(\bot) = 0$*.*

Le jeu à actions combinées $v \in \mathcal{G}(L)$ est dit *monotone* si $v(x) \leq v(y)$, pour tous $x, y \in L$ tels que $x \leq y$.

Jeux sous contraintes de précédence

Un autre interprétation des jeux définis sur treillis distributifs est due à Faigle et Kern [13], où ce sont les joueurs qui sont cette fois partiellement ordonnés ; $P := (N, \leq)$ est un quelconque ensemble ordonné de joueurs où $\leq$ représente une relation de *précédence* : $i \leq j$ signifie que la présence de j impose la présence de i dans la coalition. Dans ce cadre, une coalition réalisable de P est donc un idéal de P. Ainsi, la collection de toutes les coalitions de P munie de l'inclusion, est un treillis distributif, et un jeu sur P, est une application définie sur $\mathcal{O}(P)$, et qui s'annule en $\emptyset$.

Notons que Derks et Peters ont eux aussi introduits des jeux coalitionnels, définis sur des sous-ensembles propres de 2^N. Ces *jeux avec coalitions restreintes* ne sont cependant pas construits sur le même modèle : dans ce contexte, l'ensemble des coalitions réalisables est l'image de 2^N par une projection monotone, ce qui revient à considérer des parties de 2^N stables par réunion ensembliste (voir [10]).

On trouvera chapitre 3, un schéma récapitulatif d'inclusion de toutes ces structures d'actions (figure 1, p. 108). Il est intéressant de noter que les jeux réguliers (sous-section 1.3) y sont représentés comme la classe la plus générale de jeux coopératifs présentés ici[3], lorsque l'on comprend que tout jeu défini sur un treillis distributif peut toujours être « converti » en forme coalitionnelle.

1.5 Autres types de jeux coopératifs

Tous les jeux présentés jusqu'à présent sont, sinon des jeux coalitionnels, modélisables sous une telle forme. Dans le cas des jeux définis sur des treillis distributifs, par exemple, les actions combinées peuvent toujours être représentées par des

[3]Dans [30], Labreuche définit toutefois des jeux sur des structures plus générales que les systèmes de coalitions réguliers.

sous-ensembles d'actions élémentaires ; l'ensemble $\mathcal{Q}(N)$ des bi-coalitions de N peut être mis en bijection avec un sous-ensemble approprié de $2^{N'}$, où $|N'| = 2n$, etc.

Il existe cependant d'autres formes de jeux coopératifs, tels que les *jeux globaux*, introduits par Gilboa et Lehrer [16]. Si N est comme toujours l'ensemble des joueurs (ou pays, partis politiques, etc.), le type de coopération considérée résulte de la manière dont certains « clans » peuvent se former. Sont alors envisagées toutes les partitions possibles de N.

Soit $\mathfrak{P}(N)$ l'ensemble des *partitions* de N. On le munit de l'ordre suivant : pour toutes partitions $\mathcal{P} := \{P_1, \ldots, P_p\}$, $\mathcal{Q} := \{Q_1, \ldots, Q_q\}$ de N, on note $\mathcal{P} \leq \mathcal{Q}$ si pour tout k, P_k est inclus dans l'un des Q_l. $(\mathfrak{P}(N), \leq)$ est alors un treillis géométrique (et non distributif pour tout $n > 2$), dont les plus petit et plus grand éléments $\mathcal{P}_\perp$ et $\mathcal{P}_\top$ sont respectivement donnés par l'ensemble des singletons de N, et l'ensemble $\{N\}$. On considère ainsi dans un tel modèle que le rendement d'une partition de N est facilité lorsque cette dernière est subdivisée en un nombre minimal de coalitions. Un *jeu global* sur N est alors toute application définie sur $\mathfrak{P}(N)$.

On peut alors associer à tout jeu global h sur N un jeu coopératif v_h défini de la manière suivante : pour toute coalition S de N, $v_h(S)$ est l'image par h de la partition $\{S\} \cup \{\{i\} \mid i \in N \setminus S\}$. De même, pour tout jeu $v \in \mathcal{G}(2^N)$, on construit naturellement le jeu global suivant : $h_v(\mathcal{P}) := \sum_{S \in \mathcal{P}} v(S)$, pour toute partition $\mathcal{P}$ de N.

Citons également les *jeux sous forme de fonction de partition (partition function form games)* [43], qui sont définis sur l'ensemble des couples $\big($coalition S, partition $\mathcal{P}\big)$, tels que S soit élément de $\mathcal{P}$. Dans ce modèle, $v(S, \mathcal{P})$ désigne le gain produit par les membres de S en cas de formation de la structure de coalitions $\mathcal{P}$.

2 Concepts de solutions

Un inconvénient propre à la théorie des jeux relève de la complexité de l'information contenue dans un jeu. Par exemple, pour un jeu coopératif sur N, $2^n - 1$ coalitions peuvent prendre des valeurs indépendantes, ce qui induit une information de taille exponentielle par rapport au nombre de joueurs. Un thème de recherche fondamental en théorie des jeux consiste à déterminer quels sont les vecteurs d'attribution *(payoff vectors)* de gain acceptables pour les joueurs participant à un jeu donné.

Dans un livre paru en 1944 qui trace déjà les grandes lignes de la théorie des jeux moderne [46], von Neumann et Morgenstern tentent de réduire l'information contenue dans les jeux coopératifs. Ce sont eux, qui déjà, introduisent la notion de jeu en forme de fonction caractéristique (qui est le modèle utilisé dans ce document), et définissent aussi la notion d'*ensemble stable* pour un jeu coopératif, qui est un ensemble de vecteurs d'attribution correspondant à une « norme de comportement » pour les joueurs (voir [5]).

De multiples travaux pour recueillir ce type d'information essentielle contenue dans un jeu ont depuis été développés sous le nom générique de *concept de solution*. On distingue basiquement deux types d'approche complémentaires : la détermination d'indices d'importance, tels que la valeur de Shapley, et d'autres concepts de solutions offrant généralement plusieurs possibilités de choix de vecteurs d'attribution.

Muni de l'addition usuelle de fonctions et de la multiplication par un scalaire réel, l'ensemble des jeux coopératifs $\mathcal{G}(2^N)$ est clairement un espace vectoriel sur $\mathbb{R}$ de dimension $2^n - 1$ (puisque $v(\emptyset) = 0$).

Pour tout joueur $i \in N$ et pour toute coalition S contenant i, on appelle *contribution marginale* de i dans S la différence $v(S) - v(S \setminus i)$. Pour toute chaîne maximale $\mathcal{C} = (\emptyset, S_1, \ldots, S_n = N)$ de $(2^N, \subseteq)$, il existe un unique élément σ de $\mathcal{S}(N)$ (groupe de permutations de N), tel que $S_1 = \sigma(1)$, $S_2 \setminus S_1 = \sigma(2)$,…,$S_n \setminus S_{n-1} = \sigma(n)$. Pour tout jeu $v \in \mathcal{G}(2^N)$, on définit alors le *vecteur de contributions marginales de v relativement à σ*, et on note $m^\sigma(v)$ (ou indifféremment $m^{\mathcal{C}}(v)$), l'élément de $\mathbb{R}^n$ dont la composante d'ordre $\sigma(i)$ vaut la contribution marginale de $\sigma(i)$ dans S_i, i.e., $v(S_i) - (S_{i-1})$. Par exemple, si $n = 4$, et $\sigma = (2, 4, 3, 1)$, la chaîne maximale associée est alors $(\emptyset, 2, 24, 234, N)$ et pour tout jeu v, on a

$$m^\sigma(v) = \begin{bmatrix} v(N) - v(234) \\ v(2) \\ v(234) - v(24) \\ v(24) - v(2) \end{bmatrix}.$$

2.1 Le cœur et l'ensemble de Weber

Il existe de nombreux travaux proposant la détermination de vecteurs d'attribution convenables. L'un des plus connus de ces procédés est de déterminer un partage entre les joueurs de telle sorte que la somme des attributions de tous les joueurs de chaque coalition soit toujours supérieure ou égale au gain produit par la coalition. L'ensemble de vecteurs d'attributions satisfaisant cette condition est appelé le *cœur* du jeu.

Plus formellement, pour tout jeu coopératif $v \in \mathcal{G}(2^N)$, on appelle *pré-imputation* de v tout vecteur x de $\mathbb{R}^n$ *efficace*, c'est-à-dire tel que $\sum_{i=1}^n x_i = v(N)$. On note $I^*(v)$ l'ensemble des pré-imputations de v. Cet ensemble représente l'ensemble des vecteurs d'attribution réalisables, en terme de partage des gains (on suppose *in fine* que la grande coalition se forme). L'ensemble des *imputations* du jeu est défini par

$$I(v) := \{x \in I^*(v) \mid x_i \geqslant v(i) \text{ pour tout } i \in N\}.$$

Noter que si v est un jeu d'allocation de coût, le sens de l'inégalité ci-dessus doit être inversé. Sans perte de généralité, on suppose dans la suite que v est un jeu d'allocation de profit. Notons que $I(v) \neq \emptyset$ si et seulement si $v(N) \geqslant \sum_{i=1}^n v(i)$.

Définition 11 (cœur) *Pour tout jeu $v \in \mathcal{G}(2^N)$, on appelle* cœur *de v, l'ensemble défini par*

$$C(v) := \big\{x \in I^*(v) \mid \sum_{i \in S} x_i \geqslant v(S),\ \forall S \in 2^N \setminus \{\emptyset\}\big\},$$

On peut caractériser l'ensemble des jeux dont le *cœur* est non vide. Pour toute coalition S non vide de N, on note e^S le vecteur caractéristique de S dans $\mathbb{R}^n$, i.e., $e_i^S := 1$ si $i \in S$, et 0 sinon. On appelle *application balancée* toute fonction λ définie sur $2^N \setminus \{\emptyset\}$ et à valeurs non négatives, telle que $\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)\, e^S = e^N$.

On appelle alors *jeu balancé* tout jeu $v \in \mathcal{G}(2^N)$ tel que pour toute application balancée λ,

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)\, v(S) \leqslant v(N).$$

Il est alors prouvé que le cœur d'un jeu est non vide si et seulement s'il est balancé.

Définition 12 (ensemble de Weber) *Pour tout jeu $v \in \mathcal{G}(2^N)$, on appelle* ensemble de Weber *de v, l'enveloppe convexe des $n!$ vecteurs de contributions marginales de v.*

Un jeu $v \in \mathcal{G}(2^N)$ est dit *convexe* si pour toutes coalitions $S, T \in 2^N$, $v(S \cup T) \geqslant v(S) + v(T) - v(S \cap T)$. On montre alors que tout jeu est convexe si et seulement si son cœur coïncide avec son ensemble de Weber. Les sommets du polytope de $\mathbb{R}^n$ constitué par cet ensemble sont alors les vecteurs de contributions marginales de v. Enfin et dans tous les cas, le cœur d'un jeu est inclus dans son ensemble de Weber.

2.2 La valeur de Shapley

La valeur de Shapley est sans doute le moyen le plus connu permettant de partager d'une manière rationnelle la valeur de la grande coalition entre tous les joueurs. On admet alors dans ce contexte que c'est cette coalition qui est formée.

Définition 13 (valeur de Shapley) *On appelle* valeur de Shapley *pour l'ensemble des jeux coopératifs classiques, l'application à valeurs dans $\mathbb{R}^n$, et définie sur $\mathcal{G}(2^N)$ par*

$$\Phi_{Sh}(v) := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}(N)} m^{\sigma}(v).$$

La valeur de Shapley d'un jeu est donc le centre de l'ensemble de Weber de ce jeu, c'est-à-dire la moyenne arithmétique de tous les vecteurs de contributions marginales de v. Une interprétation probabiliste est la suivante : supposons que l'on tire de manière équiprobable d'une urne, une permutation de N. Les joueurs entrent alors dans une pièce l'un après l'autre dans l'ordre de σ et se voient attribuer la contribution marginale induite par cet ordre. Alors pour tout i, la $i$ème composante $\Phi^i_{Sh}(v)$ du vecteur $\Phi_{Sh}(v)$ représente la « juste contribution » de i allouée par cette procédure aléatoire.

Par définition, la valeur de Shapley est une valeur *d'ordre aléatoire*, c'est-à-dire une valeur donnée comme moyenne pondérée des vecteurs de contributions marginales. Notons que ce n'est pas de cette manière que Shapley a introduit cette valeur dans l'article historique [40], mais bien par le biais des axiomes qu'on donne maintenant précisément :

Linéarité (L) : l'application Φ est linéaire, i.e., pour tous jeux v, w, et tout réel α,

$$\forall i \in N, \quad \Phi^i(v + \alpha \cdot w) = \Phi^i(v) + \alpha\, \Phi^i(w).$$

Un joueur i est dit *nul* pour le jeu v, si pour toute coalition $S \in 2^N$, $v(S \cup i) = v(S)$.

Nullité (N) : pour tout jeu v, pour tout joueur i nul pour v, alors $\Phi^i(v) = 0$.

Symétrie (S) : pour toute permutation $\sigma \in \mathcal{S}(N)$, pour tout jeu v, pour tout $i \in N$,

$$\Phi^i(v \circ \sigma) = \Phi^{\sigma(i)}(v),$$

où $v \circ \sigma$ désigne le jeu $v(\sigma(\cdot))$.

L'efficacité déjà introduite en sous-section précédente est une notion extrêmement naturelle et requise dans la détermination de concepts de solutions fiables.

Efficacité (E) : pour tout jeu v,

$$\sum_{i=1}^{n} \Phi^i(v) = v(N).$$

Notons que l'axiome de coalition-support tel que présenté plus haut, est détourné ici sous la forme des deux axiomes **(N)** et **(E)**. D'autre part, l'additivité mentionnée antérieurement est ici impliquée par l'axiome plus fort de linéarité. L'axiome de nullité est parfois substitué à l'axiome plus fort de *neutralité (dummy axiom)* :

Un joueur i est dit *neutre* pour le jeu v, si pour toute coalition $S \in 2^N$ telle que $i \notin S$, $v(S \cup i) = v(S) + v(i)$.

Neutralité (D) : pour tout jeu v, si le joueur i est neutre pour v, alors $\Phi^i(v) = v(i)$.

On montre que sous les axiomes **(L)** et **(D)**, Φ est une valeur *probabiliste*, c'est-à-dire que pour tout i, Φ^i est donné comme moyenne pondérée des contributions marginales de i :

$$\forall i \in N, \quad \Phi^i(v) = \sum_{S \subseteq N \setminus i} p^i_S \,(v(S \cup i) - v(S)),$$

où $(p^i_S)_{S \subseteq N \setminus i}$ est une distribution de probabilité.

Un résultat intéressant nous indique que toute valeur d'ordre aléatoire est une valeur probabiliste efficace (vérifiant l'axiome d'efficacité), et réciproquement [47]. Donnée sous sa forme probabiliste, Φ_{Sh} vérifie :

$$\forall v \in \mathcal{G}(2^N), \forall i \in N, \quad \Phi^i_{Sh}(v) = \sum_{S \subseteq N \setminus i} \frac{s!\,(n-s-1)!}{n!}\,(v(S \cup i) - v(S)). \qquad (1)$$

La valeur de Shapley a été interprétée de multiples manières, et encore aujourd'hui, elle fait l'objet de nombreux travaux en théorie des jeux. Elle fournit bien sûr une solution au problème du « partage équitable ». En théorie des votes, une telle valeur prend le nom d'*indice de pouvoir*. On rappelle que dans ce cadre, les jeux sont simples. On parle alors de *coalition gagnante* S pour le jeu v lorsque $v(S) = 1$. Un joueur i est dit *pivot* pour v dans une coalition S lorsque la coalition S est gagnante mais $S \setminus i$ ne l'est pas. Ainsi, pour toute chaîne maximale de N, il existe un unique joueur i et une unique coalition S tels que i soit pivot pour v dans S. Alors l'*indice de Shapley-Shubik* [41] du joueur i dans le jeu v est défini comme la proportion de chaînes maximales dans lesquelles i est le joueur pivot. D'après la définition 13, cet indice n'est autre que la valeur de Shapley dans le cas particulier des jeux simples.

2.3 La valeur de Banzhaf

Une autre valeur bien connue est la *valeur de Banzhaf* [1]. Introduite en fait initialement par Penrose [36], celle-ci se définit comme la moyenne artihmétique des contributions marginales de chaque joueur :

$$\forall v \in \mathcal{G}(2^N), \forall i \in N, \quad \Phi^i_B(v) := \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus i} (v(S \cup i) - v(S)).$$

En théorie des votes, l'*indice de Banzhaf* f_B est basé sur la valeur de Banzhaf. Son calcul pour un jeu v fixé est simplement donné pour chaque joueur i, par la proportion de coalitions dans lesquelles i est pivot, sur l'ensemble des occurences où n'importe quel joueur est pivot :

$$\forall v \in \mathcal{G}(2^N), \forall i \in N, \quad f^i_B(v) := \frac{\displaystyle\sum_{S \subseteq N \setminus i} (v(S \cup i) - v(S))}{\displaystyle\sum_{j=1}^{n} \sum_{S \subseteq N \setminus j} (v(S \cup j) - v(S))}.$$

$f_B(v)$ est donc un vecteur proportionnel à $\Phi_B(v)$. Cependant, tandis que l'indice de pouvoir de Banzhaf vérifie l'axiome d'efficacité, la valeur de Banzhaf n'a pas cette propriété. C'est pourquoi on appelle aussi l'indice de Banzhaf, la valeur de Banzhaf normalisée.

2.4 Autres valeurs et indices

Dans [42], Straffin nous donne une comparaison et des exemples d'applications des indices de Shapley-Shubik et Banzhaf. Notons enfin que de nombreux auteurs on proposé d'autres indices de pouvoir, dont la construction est, pour la plupart, inspirée des indices de Shapley-Shubik ou de Banzhaf. On peut citer entre autres les *indice de Johnston* [28] et *indice de Deegan-Packel* [7], basés sur l'indice de Banzhaf, et prenant en considération le nombre de joueurs pivot dans le jeu ; ou encore les *valeurs de Shapley pondérées* (voir [34, 29]), pour lesquelles chaque joueur se voit attribuer un poids.

Tijs a toutefois proposé une valeur tout à fait originale, la τ*-valeur*, définie pour tout *jeu quasi-balancé*[4] v, comme l'unique pré-imputation de v appartenant au segment de $\mathbb{R}^n$ délimité par les *vecteurs supérieur* et *inférieur* du jeu v (voir [44]).

2.5 Valeur de Shapley pour les jeux bi-coopératifs

Soit $\mathcal{G}(3^N)$ l'ensemble des jeux bi-coopératifs sur N. Labreuche et Grabisch ont récemment proposé une généralisation de la valeur de Shapley pour ceux-ci[5]. Cette solution est en fait donnée sous la forme d'une valeur double $\Phi_{i,\emptyset}\big|\Phi_{\emptyset,i}$ définie sur $\mathcal{G}(3^N)$: la première exprime le degré d'importance des joueurs contribuant positivement, relativement à l'absence de prise de position, et la seconde rend compte de l'influence des joueurs quittant l'« opposition » pour la neutralité. On retrouve les axiomes classiques généralisés, ainsi qu'un nouvel axiome, l'*invariance*, prenant en compte les « transferts de bien » effectués entre les deux parties des bi-coalitions.

Linearité ($\mathbf{L}^{\mathcal{B}}$) : pour tout $i \in N$, $\Phi^{i,\emptyset}$ et $\Phi^{\emptyset,i}$ sont linéaires, i.e., pour

[4] Les jeux quasi-balancés sont introduits par l'auteur et incluent la classe des jeux balancés.

[5] Précisons que l'axiomatisation présentée ici n'est plus la même dans le papier soumis à l'heure actuelle [31].

tous jeux $v, w \in \mathcal{G}(3^N)$ et tout réel α,

$$\begin{aligned} \forall i \in N, \quad \Phi^{i,\emptyset}(v + \alpha \cdot w) &= \Phi^{i,\emptyset}(v) + \alpha\, \Phi^{i,\emptyset}(w), \\ \text{et } \Phi^{\emptyset,i}(v + \alpha \cdot w) &= \Phi^{\emptyset,i}(v) + \alpha\, \Phi^{\emptyset,i}(w), \end{aligned}$$

Un joueur $i \in N$ est dit *nul à gauche* (resp. *nul à droite*) pour $v \in \mathcal{G}(3^N)$ si pour toute bi-coalition $(K, L) \in \mathcal{Q}(N \setminus i)$,

$$v(K \cup i, L) \ \big(\text{resp. } v(K, L \cup i)\big) = v(K, L).$$

Nullité à gauche (NG$^\mathcal{B}$) : pour tout jeu $v \in \mathcal{G}(3^N)$ et tout $i \in N$ nul à gauche pour v, $\Phi^{i,\emptyset}(v) = 0$.

Nullité à droite (ND$^\mathcal{B}$) : pour tout jeu $v \in \mathcal{G}(3^N)$ et tout $i \in N$ nul à droite pour v, $\Phi^{\emptyset,i}(v) = 0$.

Symétrie (S$^\mathcal{B}$) : pour toute permutation $\sigma \in \mathcal{S}(N)$, pour tout jeu $v \in \mathcal{G}(3^N)$, pour tout $i \in N$,

$$\begin{aligned} \Phi^{i,\emptyset}(v \circ \sigma) &= \Phi^{\sigma(i),\emptyset}(v), \\ \text{et } \Phi^{\emptyset,i}(v \circ \sigma) &= \Phi^{\emptyset,\sigma(i)}(v). \end{aligned}$$

Invariance (I$^\mathcal{B}$) : pour tous jeux $v, w \in \mathcal{G}(3^N)$, et tout $i \in N$ tel que $\forall (K, L) \in \mathcal{Q}(N \setminus i)$

$$\begin{cases} v(K \cup i, L) = w(K, L), \\ v(K, L) = w(K, L \cup i), \end{cases}$$

alors $\Phi^{i,\emptyset}(v) = \Phi^{\emptyset,i}(w)$.

Efficacité (E$^\mathcal{B}$) : Pour tout jeu $v \in \mathcal{G}(3^N)$,

$$\sum_{i=1}^{n} (\Phi^{i,\emptyset}(v) + \Phi^{\emptyset,i}(v)) = v(N, \emptyset) - v(\emptyset, N).$$

Théorème 2 (Labreuche, Grabisch) *Sous les axiomes **(L$^\mathcal{B}$)**, **(LN$^\mathcal{B}$)**, **(RN$^\mathcal{B}$)**, **(I$^\mathcal{B}$)**, **(S$^\mathcal{B}$)** et **(E$^\mathcal{B}$)**, pour tout jeu bi-coopératif v, pour tout joueur i,*

$$\Phi^{i,\emptyset} = \sum_{S \subseteq N \setminus i} \frac{s!\,(n-s-1)!}{n!} \left[v(S \cup i, N \setminus (S \cup i)) - v(S, N \setminus (S \cup i))\right],$$

$$\Phi^{\emptyset,i} = \sum_{S \subseteq N \setminus i} \frac{s!\,(n-s-1)!}{n!} \left[v(S, N \setminus (S \cup i)) - v(S, N \setminus S)\right].$$

2.6 Valeurs de Shapley généralisées

La valeur de Shapley a fait l'objet de nombreuses généralisations, pour des jeux coopératifs définis sur des structures moins classiques. Dans leur papier introduisant les jeux sous contraintes de précédences (sous-section 1.4), Faigle et Kern axiomatisent une valeur. De même, Hsiao et Raghavan proposent déjà une valeur de Shapley pour les jeux multi-choix qu'ils introduisent dans [27]. On pourra trouver au chapitre 1, un descriptif et une comparaison de ces deux valeurs.

Citons également Peters et Zank, qui dans [37], proposent un concept de solution *(egalitarian solution)* plus convaincant que la valeur de Shapley initialement proposée par Hsiao et Raghavan. On rappelle que dans ce modèle, l'ensemble des actions conjointes est de la forme $L := \prod_{i=1}^{n} L_i$, où pour tout i, L_i est totalement ordonné. L_i privé de l'action nulle 0_i, est appelé ensemble des *actions marginales* à la disposition de i. La solution proposée est alors définie sur le produit cartésien $\mathcal{G}(L)$ et de la réunion de toutes les actions marginales à la disposition des joueurs. Cette valeur est basée sur des axiomes dérivés de ceux de base pour les jeux coopératifs : l'*efficacité*, l'*additivité*, l'*anonymat*, la *contribution zéro*, et la *symétrie*.

Concernant les jeux globaux (sous-section 1.5), Gilboa et Lehrer édifient sur cet ensemble une valeur de Shapley basée sur des axiomes semblables aux axiomes de Shapley [16]. Ils montrent alors que la valeur d'un jeu global h coïncide avec la valeur de Shapley du jeu coalitionnel associé v_h.

Pour toutes ces valeurs généralisées, il est encore possible de définir naturellement le concept de valeur probabiliste. Dans ces modèles, les contributions marginales sont considérées relativement à des joueurs (jeux sous contraintes de précédence), des joueurs jouant positivement/négativement (jeux bi-coopératifs), ou encore des actions marginales (jeux multi-choix). Il est intéressant de noter que toutes les valeurs citées ci-dessus sont, à l'instar de la valeur de Shapley pour les jeux coopératifs, des valeurs probabilistes.

3 L'indice d'interaction de Shapley

Plaçons-nous dans un contexte d'aide à la décision multicritère et imaginons une capacité ν avec $N := \{1, 2\}$. Par définition d'une capacité, on a déjà $\nu(\emptyset) = 0$ et

$\nu(N) = 1$. La valeur de Shapley de ν est alors donnée par

$$\Phi^1_{Sh}(\nu) = \frac{1}{2}\,\nu(1) + \frac{1}{2}\,(1 - \nu(2)),$$
$$\Phi^2_{Sh}(\nu) = \frac{1}{2}\,\nu(2) + \frac{1}{2}\,(1 - \nu(1)).$$

Admettons alors que chacun des deux critères ait la même importance, c'est-à-dire $\Phi^1_{Sh}(v) = \Phi^2_{Sh}(v) = \dfrac{1}{2}$. D'après les formules ci-dessus, ceci équivaut à $\nu(1) = \nu(2)$. Il est clair que selon cette dernière valeur, l'interprétation de l'*interaction* entre les critères change du tout au tout. Dans les cas extrêmes, on a $\nu(1) = \nu(2) = 0$ d'une part, et $\nu(1) = \nu(2) = 1$ d'autre part. La première situation témoigne d'une forte *complémentarité* entre les critères, c'est-à-dire que les actions binaires $(1_1, 0_2)$ et $(1_2, 0_1)$ sont sanctionnées par l'évaluation la plus sévère : zéro. À l'inverse, dans le deuxième cas, les critères sont dits *redondants* puisque les actions binaires citées précédemment sont suffisantes pour garantir le meilleur score possible. En situation intermédiaire, si $\nu(1) = \nu(2) = \dfrac{1}{2}$, ν est alors additive, ce qui dénote d'une certaine indépendance entre les critères.

Une analyse analogue en théorie des jeux conduit à considérer des joueurs complémentaires, dont la réunion favorise leur contribution à un jeu, ou bien des joueurs « substituables », dont la présence des deux ne serait pas, dans une certaine mesure, indispensable.

Afin de décrire ce phénomène, Murofushi et Soneda ont proposé en 1972 un *indice d'interaction* [33], permettant de déterminer le degré d'interaction entre deux critères (notons que c'est Owen qui a initialement introduit ce concept [34]). Pour n'importe quelle paire d'entre eux, une « mesure » de ce degré d'interaction fournit une valeur positive ou négative selon que les critères agissent complémentairement ou de manière redondante. Grabisch est allé jusqu'à généraliser cela en définissant l'indice d'interaction [17] pour toute coalition de joueurs dans des jeux coopératifs classiques.

3.1 Transformée de Möbius d'un jeu

Introduisons d'abord une notion intervenant largement en théorie des jeux et en théorie de l'utilité multi-attributs. Pour toute fonction f définie sur un ensemble ordonné $(P, \leq)$, la *transformée de Möbius* de f est l'application notée m^f, unique solution [38] de l'équation fonctionnelle

$$f(x) = \sum_{y \leq x} m^f(y), \quad x \in P.$$

Plus précisément, cette application est donnée par

$$m^f(x) := \sum_{y \leq x} \mu(y, x)\, f(y), \quad x \in P,$$

où μ est appelée fonction de Möbius, et est récursivement donnée sur $P \times P$ par

$$\mu(x, y) = \begin{cases} 1, & \text{si } x = y, \\ -\displaystyle\sum_{x \leq t < y} \mu(x, t), & \text{si } x < y, \\ 0, & \text{sinon.} \end{cases}$$

Toute fonction définie sur un ensemble ordonné est ainsi caractérisée par sa transformée de Möbius.

Dans le cas particulier du treillis $(2^N, \subseteq)$, il est bien connu que $\mu(A, B) = (-1)^{|B \setminus A|}$, pour tous sous-ensembles A, B tels que $A \subseteq B$, c'est-à-dire, pour tout jeu coopératif classique $v \in \mathcal{G}(2^N)$,

$$m^v(S) = \sum_{T \subseteq S} (-1)^{s-t}\, v(T).$$

Ainsi, la famille des jeux unanimes $(u_S)_{S \in 2^N \setminus \{\emptyset\}}$ fournit une base de l'espace vectoriel $\mathcal{G}(2^N)$ via

$$v = \sum_{S \in 2^N, S \neq \emptyset} m^v(S) \cdot u_S.$$

En économie, on appelle aussi $m^v(S)$ le *dividende* du jeu v apporté par la coalition S. Il existe une écriture de la valeur de Shapley en terme de ces dividendes [26] :

$$\forall v \in \mathcal{G}(2^N), \forall i \in N, \quad \Phi^i_{Sh}(v) = \sum_{S \in 2^N | i \in S} \frac{1}{s}\, m^v(S).$$

3.2 Dérivées de jeux

Remarquons que la contribution marginale d'un joueur i dans une coalition S n'est pas sans évoquer la dérivation, ceci dans un contexte de mathématiques discrètes. Dans le cas d'une fonction d'ensemble f définie sur le treillis booléen $(2^N, \subseteq)$, on appelle *dérivée* par rapport à $i \in N$ l'opérateur noté Δ_i défini sur $\mathcal{G}(2^N)$ par

$$\forall S \in 2^N, \quad \Delta_i f(S) := f(S \cup i) - f(S).$$

Par récursivité, on définit aussi l'opérateur de dérivation par rapport à n'importe quelle coalition non vide $T := \{i_1, \ldots, i_m\}$ (les i_k étant distincts deux à deux) :

$$\forall S \in 2^N, \quad \Delta_T f(S) := \Delta_{i_m}\big(\ldots \Delta_{i_2}\big(\Delta_{i_1} f(S)\big)\ldots\big).$$

Noter que si $S \cap T \neq \emptyset$, la dérivée est nulle. De manière explicite,

$$\forall S \in 2^N, \quad \Delta_T f(S) = \sum_{K \subseteq T} (-1)^{t-k} f(S \cup K).$$

On pose $\Delta_\emptyset f(S) := f(S)$ pour toute coalition S.

3.3 L'indice d'interaction de Shapley

Grabisch propose la définition suivante pour l'indice d'interaction [17].

Définition 14 (indice d'interaction) *On appelle* indice d'interaction de Shapley *pour l'ensemble des jeux coopératifs classiques, l'application définie sur* $\mathcal{G}(2^N) \times 2^N$ *par*

$$I(v, S) = \sum_{T \subseteq N \setminus S} \frac{(n-s-t)!\, t!}{(n-s+1)!} \Delta_S v(T). \tag{2}$$

Notons que pour tout $i \in N$, $I(v, i)$ coïncide avec la valeur de Shapley $\Phi^i_{Sh}(v)$.

Une axiomatisation de l'indice d'interaction est proposée par Grabisch et Roubens dans [24]. Soit v un jeu coopératif classique et K une partie non vide de N. On appelle *jeu restreint à* K, et on note v^K la restriction de l'application v à 2^K. On appelle *jeu* K*-réduit*, et on note $v_{[K]}$ le jeu défini sur $N_{[K]} := (N \setminus K) \cup \{[K]\}$ par $v_{[K]}(S) := \begin{cases} v(S), & \text{si } [K] \notin S, \\ v((S \setminus [K]) \cup K) & \text{sinon} \end{cases}$, pour toute coalition $S \in 2^{N_{[K]}}$. Interprétation : $[K]$ est comparable à un « macro-joueur ».

L'idée étant de construire un prolongement de la valeur de Shapley, on retrouve les axiomes de Shapley : **linéarité (L)**, **neutralité (D)**, **symétrie (S)**, et **efficacité (E)** (voir sous-section 2.2).

Grabisch et Roubens proposent en sus :

Récursivité 1 (R1) : pour tout jeu v, pour toute coalition $S \subseteq N$ telle que $s > 1$

$$I(v, S) = I(v_{\cup j}^{N \setminus j}, S \setminus j) - I(v^{N \setminus j}, S \setminus j), \quad \forall j \in S,$$

où $v_{\cup B}^{K}$ est le jeu défini par $v_{\cup B}^{K}(S) := v(S \cup B) - v(B),\ S \subseteq K$, où B et K sont des parties disjointes de N.

Récursivité 2 (R2) : pour tout jeu v, pour toute coalition $S \subseteq N$ telle que $s > 1$

$$I(v, S) = I(v_{[S]}, [S]) - \sum_{K \subsetneq S, K \neq \emptyset} I(v^{N \setminus K}, S \setminus K).$$

Notons que **(L)**, **(S)**, **(D)** et **(E)** sont assignés à des applications de $\mathcal{G}(2^N)$ dans $\mathbb{R}^n$. Moyennant une adaptation pour les appliquer à la restriction de I à $\mathcal{G}(2^N) \times \{S \in 2^N \mid s = 1\}$, on peut établir le résultat suivant.

Théorème 3 (Grabisch, Roubens) *Les deux axiomes de récursivité sont équivalents sous **(L)**, **(S)** et **(D)**. Sous **(L)**, **(S)**, **(D)**, **(E)** et l'un des deux axiomes de récursivité, pour tout jeu $v \in \mathcal{G}(2^N)$, pour toute coalition S non vide, $I(v, S)$ est donnée par la formule (2).*

3.4 Représentations équivalentes de fonctions d'ensemble

Considérons un jeu coopératif v, sa transformée de Möbius m^v et son indice d'interaction $I(v, \cdot)$. Il est alors prouvé que chacune de ces applications caractérise les deux autres, ce qui permet entre autres de confirmer la complétude de l'information fournie par l'indice d'interaction d'un jeu. On trouve à ce sujet dans [23], toutes les transformations passant de l'une à l'autre de ces expressions.

Plus récemment, toujours en ce qui concerne les jeux coopératifs classiques, et pour le passage de l'index d'interaction à la transformée de Möbius (opérateur de Bernoulli), une méthode élégante a été développée dans [9] faisant appel au produit de convolution sur $2^N \times 2^N$.

C'est sur ce modèle que l'on parvient à faire de même pour les fonctions de bi-ensembles (chapitre 5) et d'une manière beaucoup plus générale, pour les jeux à actions combinées (chapitre 6).

Résultats principaux

1 Axiomatisation de la valeur de Shapley pour les jeux multi-choix

Les deux premiers chapitres introduisent le concept de jeux coopératifs définis sur des treillis distributifs (voir introduction, sous-section 1.4). On y propose également des axiomatisations de la valeur de Shapley pour les jeux multi-choix.

1.1 Une nouvelle axiomatisation de la valeur de Shapley pour les jeux coopératifs classiques

Commençons par présenter une nouvelle axiomatisation de la valeur de Shapley pour les jeux coopératifs (chapitre 2). On rappelle que Φ est définie sur $\mathcal{G}(2^N)$, à valeurs dans $\mathbb{R}^n$:

Nullité généralisée (NG) : pour tout jeu $v \in \mathcal{G}(2^N)$, pour tout joueur $i \in N$ nul pour v,

$$\begin{cases} \Phi^i(v) = 0, \\ \Phi^j(v) = \Phi^j(v^{-i}), \qquad \text{pour tout joueur } j \text{ autre que } i, \end{cases}$$

où v^{-i} désigne le jeu de $\mathcal{G}(2^{N\setminus i})$, restriction de v à $2^{N\setminus i}$.

L'assignation zéro à $\Phi^i(v)$ n'est autre que l'axiome classique de nullité. L'interprétation de l'assignation $\Phi^j(v)$ est naturelle : si i, joueur nul, quitte le jeu, alors les autres joueurs devraient garder la même valeur dans le jeu restreint.

On rappelle que le jeu unanime u_N de $\mathcal{G}(2^N)$ est défini par $u_N(S) := \begin{cases} 1, & \text{si } S = N, \\ 0, & \text{sinon.} \end{cases}$

Équité (Eq) : pour tout $i \in N$,

$$\Phi^i(u_N) = \frac{1}{n}.$$

Cet axiome stipule que dans le jeu simple où la grande coalition est la seule à produire un gain, tous les joueurs devraient partager la même fraction de ce gain.

Théorème 1.1 *Sous les axiomes de linéarité, de nullité généralisée, et d'équité,* Φ *est la valeur de Shapley sur* $\mathcal{G}(2^N)$.

1.2 Deux axiomatisations de la valeur de Shapley pour les jeux multichoix

Dans l'axiomatisation de la valeur de Shapley pour les jeux multichoix proposée par Peters et Zank, tous les joueurs ont le même nombre de niveaux de contribution. En effet, il est intéressant de remarquer que l'axiome de symétrie qu'ils proposent, n'a aucune portée sur les jeux multi-choix « asymétriques ».

On propose deux nouvelles axiomatisations palliant ce problème. L'une (chapitre 1) fournit un axiome de symétrie généralisée, et l'autre (chapitre 2), basée sur une construction récursive de la valeur de Shapley, généralise l'axiomatique présentée ci-dessus pour les jeux coopératifs classiques.

Notations

On rappelle que $N := \{1, \ldots, n\}$ est un ensemble de joueurs, $L_i := \{0_i, \ldots, l_i\}$ désigne pour chaque joueur i, l'ensemble totalement ordonné des actions mises à sa disposition, et $L := \prod_{i=1}^n L_i$, l'ensemble de toutes les actions conjointes, que l'on munit de l'ordre produit des L_i. Remarquons que l'ensemble $\mathcal{J}(L)$ des éléments sup-irréductibles de L peut être identifié à la réunion de toutes les actions marginales des joueurs : $\bigcup_{i=1}^n \{1_i, \ldots, l_i\}$. Pour éviter des notations trop lourdes, on notera indifféremment k_i pour l'action marginale $k_i \in L_i$ ou l'action conjointe associée de $\mathcal{J}(L)$.

Une valeur sur $\mathcal{G}(L)$ sera notée Φ, et définie sur le produit cartésien de $\mathcal{G}(L)$ par $\mathcal{J}(L)$. On constatera que les solutions caractérisées dans chacun des deux cas coïncident, à ceci près que celle introduite au premier chapitre est donnée sous une forme dite *cumulative*, et dans le deuxième, sous sa forme *différentielle* : $\Phi(v, k_i)$ représente dans le premier cas la valeur de l'action k_i relativement à l'action nulle, tandis que dans le second cas, cette valeur est donnée relativement à l'action précédente $(k-1)_i$. Le passage de l'une à l'autre de ces formes est aisé ; réservant la notation ϕ aux valeurs différentielles, et Φ aux valeurs cumulatives, on a alors les relations suivantes, pour tout jeu $v \in \mathcal{G}(L)$, tout joueur i et toute action marginale $k_i \in L_i \setminus \{0_i\}$:

$$\Phi(v, k_i) = \sum_{l=1}^{k} \phi(v, l_i), \tag{1.1}$$

$$\text{et } \phi(v, k_i) = \begin{cases} \Phi(v, k_i) - \Phi(v, (k-1)_i), & \text{si } k > 1, \\ \Phi(v, k_i), & \text{sinon.} \end{cases} \tag{1.2}$$

Tout élément x de L est noté vectoriellement $(x_1, \ldots, x_n)$. $\top$ désigne le plus grand élément de L. $L_{-i} := \prod_{j \neq i} L_j$. Pour tout $k_i \in L_i$ et tout $y \in L_{-i}$, (y, k_i) dénote l'action combinée x telle que $x_j = y_j$, $j \neq i$ et $x_i = k_i$. En particulier, 0_{-i} désigne l'élément de L_{-i} dont toutes les coordonnées sont nulles. On appelle *sommet* de L tout élément de L dont chacune des coordonnées x_i vaut 0_i ou bien l_i, $i \in N$. On note $\Gamma(L)$ l'ensemble des sommets de L.

Axiomes

Linéarité ($\mathbf{L}^{\mathcal{M}}$) : Φ est linéaire par rapport à la variable v, i.e., pour tous jeux $v, w \in \mathcal{G}(L)$, et tout réel α,

$$\forall k_i \in \mathcal{J}(L), \quad \Phi(v + \alpha \cdot w, k_i) = \Phi(v, k_i) + \alpha\, \Phi(w, k_i).$$

(Idem pour la version différentielle ϕ.)

– Pour tout $k \in L_i, k \neq 0$, le joueur i est dit *cumulativement* (resp. *différentiellement*) k-nul pour $v \in \mathcal{G}(L)$ si $v(x, k_i) = v(x, 0_i)$ (resp. $v(x, k_i) = v(x, (k-1)_i)$), $\forall x \in L_{-i}$.

– Pour tout $k \in L_i, k \neq 0$, le joueur i est dit *cumulativement* (resp. *différentiellement*) *k-neutre* pour $v \in \mathcal{G}(L)$ si $v(x, k_i) = v(x, 0_i) + v(0_{-i}, k_i)$ (resp. $v(x, k_i) = v(x, (k-1)_i) + v(0_{-i}, k_i))$, $\forall x \in L_{-i}$. Si $l_i = 1$, on écrira simplement i est nul (resp. neutre) pour i est cumulativement/différentiellement k-nul (resp. k-neutre).
  Seules les définitions de nullité et neutralité changent selon qu'elles sont cumulatives ou différentielles. Les axiomes sont les mêmes.

Nullité ($\mathbf{N}^{\mathcal{M}}$) : pour tout jeu $v \in \mathcal{G}(L)$, pour tout joueur i cumulativement (resp. différentiellement) k-nul pour v, alors $\Phi(v, k_i) = 0$ (resp. $\phi(v, k_i) = 0$).

Neutralité ($\mathbf{D}^{\mathcal{M}}$) : pour tout jeu $v \in \mathcal{G}(L)$, pour tout joueur i cumulativement (resp. différentiellement) k-neutre pour v, alors $\Phi(v, k_i) = v(k_i)$ (resp. $\phi(v, k_i) = v(k_i)$).

Notons que les axiomes de nullité résultent des axiomes de neutralité.

Monotonie ($\mathbf{M}^{\mathcal{M}}$) : pour tout jeu $v \in \mathcal{G}(L)$, si v est monotone, alors $\Phi(v, k_i) \geqslant 0$, pour toute action marginale k_i.
(Idem pour la version différentielle Φ.)

Invariance ($\mathbf{I}^{\mathcal{M}}$) : soit deux jeux v_1, v_2 de $\mathcal{G}(L)$ tels que pour tout joueur i,

$$\begin{aligned} v_1(x, x_i) &= v_2(x, x_i - 1), \ \forall x \in L_{-i}, \forall x_i > 1 \\ v_1(x, 0_i) &= v_2(x, 0_i), \ \forall x \in L_{-i}. \end{aligned}$$

Alors $\Phi(v_1, k_i) = \Phi(v_2, (k-1)_i), \ 1 < k \leqslant l_i$.

Soit $\mathcal{G}_0(L)$ le sous-espace vectoriel de $\mathcal{G}(L)$ défini par l'ensemble des jeux s'annulant en tout autre point que des sommets de L. On considère la bijection canonique de $\mathcal{G}_0(L)$ dans $\mathcal{G}(2^N)$ définie par $v \mapsto \tilde{v}$, telle que

$$\tilde{v}(S) := v(s), \quad \text{avec } s_i = \begin{cases} l_i, & \text{si } i \in S, \\ 0_i, & \text{sinon,} \end{cases} \forall i \in N.$$

L'axiome suivant inclut la symétrie au sens classique (pour les jeux de $\mathcal{G}(2^N)$), et la complète par l'adjonction d'une condition naturelle que doivent vérifier les actions marginales « maximales ».

Symétrie corrigée ($\mathbf{S}^{\mathcal{M}}$) :

1. Pour toute permutation $\sigma \in \mathcal{S}(N)$, pour tout jeu $v \in \mathcal{G}(2^N)$, pour tout $i \in N$,
$$\Phi^i_{Sh}(v \circ \sigma) = \Phi^{\sigma(i)}_{Sh}(v), \quad \text{(valeur de Shapley sur } \mathcal{G}(2^N)),$$
où $v \circ \sigma$ désigne le jeu $v(\sigma(\cdot))$.
2. Pour tout jeu $v \in \mathcal{G}_0(L)$, et tout joueur i, $\Phi(v, l_i) = \Phi^i(\tilde{v})$.

Joueur nul exclu ($\mathbf{JNE}^{\mathcal{M}}$) : pour tout jeu $v \in \mathcal{G}(L)$, pour tout joueur $i \in N$ tel que $l_i = 1$, si i est nul pour v,
$$\phi(v, l_j) = \phi(v^{-i}, l_j), \qquad \text{pour tout joueur } j \text{ autre que } i,$$
où v^{-i} désigne le jeu de $\mathcal{G}(L_{-i})$, restriction de v à L_{-i}.

Régression de niveau ($\mathbf{RN}^{\mathcal{M}}$) : pour tout jeu $v \in \mathcal{G}(L)$, pour tout joueur $i \in N$ tel que $l_i > 1$, si i est différentiellement l_i-nul pour v,

1. $\phi(v, k_i) = \phi(v^{-l_i}, k_i), \quad$ pour toute action marginale $k_i \neq 0_i, l_i$,
2. $\phi(v, l_j) = \phi(v^{-l_i}, l_j), \quad$ pour tout joueur j autre que i,

où v^{-l_i} désigne la restriction de v à $L_{-i} \times (L_i \setminus \{l_i\})$.

Efficacité ($\mathbf{E}^{\mathcal{M}}$) : pour tout jeu v,
$$\sum_{i=1}^{n} \Phi(v, l_i) = v(\top).$$

Le *jeu unanime* $u_\top$ de $\mathcal{G}(L)$ est défini par $u_\top(x) := \begin{cases} 1, & \text{si } x = \top, \\ 0, & \text{sinon.} \end{cases}$

Équité ($\mathbf{Eq}^{\mathcal{M}}$) : pour tout joueur $i \in N$,
$$\phi(u_\top, l_i) = \frac{1}{n}.$$
Pour tout $x \in L$, on note $h(x) := |\{j \in N \mid x_j = l_j\}|$.

Théorème 1.2 *Sous les axiomes* ***($L^{\mathcal{M}}$),($D^{\mathcal{M}}$),($M^{\mathcal{M}}$),($I^{\mathcal{M}}$),($S^{\mathcal{M}}$)*** *et* ***($E^{\mathcal{M}}$)****, pour tout jeu $v \in \mathcal{G}(L)$, pour toute action marginale k_i,*
$$\Phi(v, k_i) = \sum_{x \in \Gamma(L_{-i})} \frac{h(x)!\,(n - h(x) - 1)!}{n!} [v(x, k_i) - v(x, 0_i)],$$

Théorème 1.3 *Sous les axiomes* ***($L^{\mathcal{M}}$)****,* ***($N^{\mathcal{M}}$)****,* ***($JNE^{\mathcal{M}}$)****,* ***($RN^{\mathcal{M}}$)****, et* ***($Eq^{\mathcal{M}}$)****, pour tout jeu $v \in \mathcal{G}(L)$, pour toute action marginale k_i,*

$$\phi(v, k_i) = \sum_{x \in \Gamma(L_{-i})} \frac{h(x)!\,(n - h(x) - 1)!}{n!} \, [v(x, k_i) - v(x, (k-1)_i)],$$

Via (1.1) et (1.2), on constate que les valeurs produites dans ces deux théorèmes désignent la même valeur. Notons enfin que la valeur obtenue par Peters et Zank (où cependant tous les joueurs ont le même nombre d'actions à leur disposition) est encore la même (voir [37], théorème 4.1).

2 Jeux réguliers, valeur de Shapley et lois de Kirchhoff

On propose au chapitre 3 une valeur pour les jeux réguliers (introduction, sous-section 1.3). Cette valeur généralise la valeur de Shapley pour les jeux coopératifs classiques. Un aspect particulièrement intéressant de son axiomatisation repose sur une forte analogie avec les *lois de Kirchhoff*, qui expriment la conservation de l'énergie et de la charge dans un circuit électrique.

2.1 Valeur de Shapley pour les jeux réguliers

Soit $N := \{1, \ldots, n\}$ l'ensemble des joueurs, et $\mathcal{N}$ un système de coalitions réguliers (cf. définition 7). Une valeur Φ sur $\mathcal{G}(\mathcal{N})$, espace des jeux sur $\mathcal{N}$, est, tout comme la valeur de Shapley sur $\mathcal{G}(2^N)$, une application définie sur $\mathcal{G}(\mathcal{N})$ et à valeurs dans $\mathbb{R}^n$, dont la $i$ème composante sera notée Φ^i.

On retrouve tout d'abord les axiomes habituels :

> **Linéarité ($L^{\mathcal{R}}$)** : l'application Φ est linéaire, i.e., pour tous jeux $v, w \in \mathcal{G}(\mathcal{N})$, et tout réel α,
>
> $$\forall i \in N, \quad \Phi^i(v + \alpha \cdot w) = \Phi^i(v) + \alpha\, \Phi^i(w).$$

Un joueur $i \in N$ est dit nul pour le jeu $v \in \mathcal{G}(\mathcal{N})$, si pour toute coalition $S \in \mathcal{N}$ telle que $S \cup i \in \mathcal{N}$, on a $v(S \cup i) = v(S)$.

Nullité $(\mathbf{N}^{\mathcal{R}})$: pour tout jeu v, pour tout joueur i nul pour v, alors $\Phi^i(v) = 0$.

Sous les axiomes $(\mathbf{L}^{\mathcal{R}})$ et $(\mathbf{N}^{\mathcal{R}})$, Φ est une valeur dite *marginaliste*, à savoir que pour tout joueur i, Φ^i ne dépend que des contributions marginales de i : pour tout $i \in N$, il existe une famille de nombres réels p_S^i telle que

$$\Phi^i(v) = \sum_{S \in \mathcal{N} | i \notin S, S \cup i \in \mathcal{N}} p_S^i \,(v(S \cup i) - v(S)). \tag{2.3}$$

L'axiome suivant est également classique.

Efficacité $(\mathbf{E}^{\mathcal{R}})$: pour tout jeu $v \in \mathcal{G}(\mathcal{N})$,

$$\sum_{i=1}^{n} \Phi^i(v) = v(N).$$

À nouveau, et de manière bien plus marquée que pour les jeux multi-choix, le traitement universel de l'asymétrie des systèmes de coalitions réguliers n'est pas chose aisée, et là encore, l'axiome classique de symétrie est de portée nulle dès lors que l'échange des rôles de deux joueurs altère la structure du système. On propose alors une approche totalement différente, basée sur un axiome naturel que devraient vérifier les jeux *équidistribués*.

Un jeu $v \in \mathcal{G}(\mathcal{N})$ est dit *équidistribué* si v est additif et symétrique, c'est-à-dire que v ne dépend que d'un coefficient de proportionnalité $\alpha \in \mathbb{R}$: $\forall S \in \mathcal{N}$, $v(S) = \alpha \cdot s$. Soit Φ une valeur marginaliste dont les coefficients linéaires des contributions marginales sont notés comme dans (2.3). Pour toute chaîne maximale $\mathcal{C} := (S_0 = \emptyset, S_1, \ldots, S_n = N)$ sur $\mathcal{N}$, on appelle *somme pondérée des contributions marginales des joueurs le long de* $\mathcal{C}$, la somme

$$m_\Phi^{\mathcal{C}}(v) := \sum_{i=1}^{n} p_{S_{i-1}}^{\sigma(i)} \,(v(S_i) - v(S_{i-1})),$$

où σ dénote la permutation associée à $\mathcal{C}$, i.e., $\sigma(i) := S_i \setminus S_{i-1}$, $i = 1, \ldots, n$. Par exemple, si $n = 3$, et $\mathcal{C} := (\emptyset, 3, 31, N = 123)$ est une chaîne maximale sur $\mathcal{N}$, alors relativement aux notation ci-dessus, on a $m_\Phi^{\mathcal{C}}(v) = p_\emptyset^3 \, v(3) + p_3^1 \,(v(13) - v(3)) + p_{13}^2 \,(v(N) - v(13))$.

On introduit alors l'axiome suivant :

Régularité $(\mathbf{R}^{\mathcal{R}})$: si Φ est une valeur marginaliste, pour tout jeu $v \in \mathcal{G}(\mathcal{N})$ équidistribué, pour toutes chaînes maximales $\mathcal{C}_1$ et $\mathcal{C}_2$ sur $\mathcal{N}$,

$$m_\Phi^{\mathcal{C}_1}(v) = m_\Phi^{\mathcal{C}_2}(v).$$

Une valeur marginaliste satisfaisant cet axiome est dite *régulière*.

Théorème 2.1 *Il existe une unique valeur Φ_K marginaliste efficace régulière sur $\mathcal{G}(\mathcal{N})$. Lorsque $\mathcal{N} = 2^N$, Φ_K est la valeur de Shapley pour les jeux coopératifs.*

2.2 Analogie avec la théorie des réseaux électriques

Considérons la représentation par diagramme de Hasse d'un système de coalitions régulier. Par exemple, soit $n = 3$, et $\mathcal{N} := \{1, 2, 3, 12, 23, N\}$.

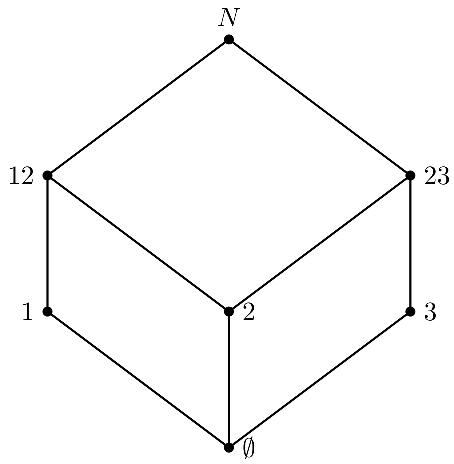


FIG. 2 – Diagramme de Hasse de $(\mathcal{N}, \subseteq)$

Ce diagramme peut être assimilé à un réseau électrique dans lequel les *nœuds* sont représentés par les coalitions réalisables et les *branches* sont représentées par les arêtes du diagramme, c'est-à-dire les couples de coalitions réalisables $(S, S \cup i)$, $S \in \mathcal{N}, S \cup i \in \mathcal{N}, i \notin S$. On notera B_S^i la branche identifiée au couple $(S, S \cup i)$. On appelle *maille* du réseau tout circuit fermé de branches.

Chacune des branches B_S^i d'un tel réseau est dotée d'une certaine *résistance* R_S^i. Si nous supposons qu'un certain *courant électrique* continu I circule du nœud noté $\emptyset$ au nœud noté N, alors chaque branche orientée B_S^i se voit être parcourue par un courant électrique I_S^i, et affectée d'une *tension électrique* (ou *différence de potentiel*) U_S^i. La *loi d'Ohm* énonce alors que pour chacune des branches, $U_S^i = R_S^i \cdot I_S^i$. D'autre part, les *lois de Kirchhoff* déterminent de manière unique le courant et la différence de potentiel de chaque branche :

1. Première loi de Kirchhoff (ou *loi des nœuds*) : la somme algébrique des intensités des courants qui entrent par un nœud est égale à la somme algébrique des intensités des courants qui en sortent.
2. Seconde loi de Kirchhoff (ou *loi des mailles*) : dans une maille quelconque du réseau, la somme algébrique des tensions le long de la maille est nulle.

Si maintenant nous supposons que toutes les branches ont la même résistance, alors par la loi d'Ohm, on peut faire abstraction des tensions électriques. Et si d'autre part le courant électrique I est unitaire, il en résulte que le courant circulant dans la branche B_S^i donné par les lois de Kirchhoff est précisément égal au coefficient p_S^i associé à la contribution marginale de i dans S, de la valeur Φ_K.

En fait, on établit l'analogie suivante : la loi des nœuds se rapporte à l'axiome d'efficacité, tandis que la loi des mailles correspond à l'axiome de régularité.

2.3 Φ_K et la monotonie

Dans [49], Young propose plusieurs axiomes de monotonie pour les jeux coopératifs classiques.

Monotonie ($\mathbf{M}^{\mathcal{R}}$) : pour tout jeu $v \in \mathcal{G}(\mathcal{N})$, si v est monotone, alors Φ est à valeurs dans $\mathbb{R}_+^n$.

Monotonie forte ($\mathbf{MF}^{\mathcal{R}}$) : soit deux jeux $v, w \in \mathcal{G}(\mathcal{N})$ et un joueur $i \in N$ vérifiant pour toute coalition $S \in \mathcal{N}$ telle que $i \notin S$ et $S \cup i \in \mathcal{N}$, $w(S \cup i) - w(S) \geqslant v(S \cup i) - v(S)$. Alors $\Phi^i(w) \geqslant \Phi^i(v)$.

Monotonie coalitionnelle ($\mathbf{MC}^{\mathcal{R}}$) : soit deux jeux $v, w \in \mathcal{G}(\mathcal{N})$ et une coalition $S \in \mathcal{N}$ tels que $w(S) \geqslant v(S)$, et $w(T) = v(T)$, pour toute autre coalition $T \in \mathcal{N}$. Alors pour tout joueur $i \in S$, $\Phi^i(w) \geqslant \Phi^i(v)$.

Monotonie agrégée ($\mathbf{MA}^{\mathcal{R}}$) : soit deux jeux $v, w \in \mathcal{G}(\mathcal{N})$ tels que $w(N) \geqslant v(N)$, et $w(S) = v(S)$, pour toute autre coalition $S \in \mathcal{N}$. Alors pour tout joueur i, $\Phi^i(w) \geqslant \Phi^i(v)$.

Proposition 2.2 *Sous les axiomes ($\boldsymbol{L}^{\mathcal{R}}$) et ($\boldsymbol{N}^{\mathcal{R}}$), i.e., pour toute valeur marginaliste, ($\boldsymbol{M}^{\mathcal{R}}$), ($\boldsymbol{MF}^{\mathcal{R}}$) et ($\boldsymbol{MC}^{\mathcal{R}}$) sont équivalents. Ces axiomes impliquent ($\boldsymbol{MA}^{\mathcal{R}}$).*

Par un contre-exemple assez complexe de système de coalitions régulier (cf. annexe , p. 128), on montre que Φ_K ne vérifie pas en général l'axiome de monotonie. Elle n'est donc pas toujours probabiliste. On a cependant le résultat suivant.

Théorème 2.3 *Pour tout système de coalitions régulier $\mathcal{N}$, Φ_K satisfait la monotonie agrégée.*

3 L'indice d'interaction pour les jeux bi-coopératifs : axiomatisations

Le chapitre 4 introduit des axiomatisations de l'indice d'interaction I pour les bi-capacités. Celles-ci restent bien sûr valides pour l'ensemble des jeux bi-coopératifs (et par extension, aux fonctions de bi-ensembles). De la même manière que l'indice d'interaction sur $\mathcal{G}(2^N)$ est une extension la valeur de Shapley, l'indice d'interaction pour les jeux bi-coopératifs prolonge la valeur de Shapley de ceux-ci.

I est une application définie sur $\mathcal{G}(3^N) \times \mathcal{Q}(N)$. Elle exprime le degré d'interaction de toute bi-coalition de joueurs (ou de critères) dans un jeu. L'interprétation est sujette à discussions selon le contexte. Sommairement, un résultat positif (resp. négatif) pour $I(v, (A, B))$ indique ainsi que globalement, le regroupement des joueurs de A contre ceux de B, est bénéfique (resp. nuisible).

À nouveau, des opérateurs de dérivation interviennent dans la formule caractérisée par les axiomes proposés. Pour tout jeu bi-coopératif v, et tout $i \in N$, on définit

$$\begin{aligned}
\Delta_{i,\emptyset} v(K, L) &:= v(K \cup i, L) - v(K, L), \quad \text{pour tout } (K, L) \in \mathcal{Q}(N \setminus i),\\
\Delta_{\emptyset,i} v(K, L) &:= v(K, L \setminus i) - v(K, L), \quad \text{pour tout } (K, L) \in \mathcal{Q}(N) \text{ avec } i \in L.
\end{aligned}$$

Récursivement, on définit $\Delta_{S,T} v$ pour tout $(K, L) \in \mathcal{Q}(N \setminus S)$ tel que $T \subseteq L$, par

$$\begin{aligned}
\Delta_{S,T} v(K, L) &:= \Delta_{i,\emptyset}(\Delta_{S\setminus i,T} v(K, L))\\
&= \Delta_{\emptyset,j}(\Delta_{S,T\setminus j} v(K, L)),
\end{aligned}$$

pour tout $i \in S$ et tout $j \in T$.

3.1 Une première axiomatisation : l'axiome de récursivité

Une première axiomatisation possible est directement inspirée de l'axiomatisation de I pour les jeux coopératifs classiques et son axiome **(R1)** (voir introduction, sous-section 3.3). Ainsi, on considère à nouveau les axiomes qui caractérisent la valeur de Shapley pour les jeux bi-coopératifs : **linéarité ($\mathbf{L}^{\mathcal{B}}$)**, **nullité à gauche ($\mathbf{NG}^{\mathcal{B}}$)**, **nullité à droite ($\mathbf{ND}^{\mathcal{B}}$)**, **symétrie ($\mathbf{S}^{\mathcal{B}}$)**, et **efficacité ($\mathbf{E}^{\mathcal{B}}$)** (voir introduction, sous-section 2.5), qu'une fois encore on ajustera afin de les rendre valides pour la restriction de I à $\mathcal{G}(3^N) \times \{(i, \emptyset), (\emptyset, i) \mid i \in N\}$.

À partir de tout jeu $v \in \mathcal{G}(3^N)$ et tout joueur i, on définit le jeu $v_+^{N\setminus i}$ (resp. $v_-^{N\setminus i}$), *restriction de v en présence positive* (resp. *négative*) de i, par

$$\begin{aligned}\forall (A, B) \in \mathcal{Q}(N), \quad & v_+^{N\setminus i}(A, B) := v(A \cup i, B) - v(i, \emptyset),\\ (\text{resp.} \quad & v_-^{N\setminus i}(A, B) := v(A, B \cup i) - v(\emptyset, i)).\end{aligned}$$

> **Recursivité ($\mathbf{R}^{\mathcal{B}}$)** : pour tout jeu $v \in \mathcal{G}(3^N)$, pour toute bi-coalition $(S, T) \in \mathcal{Q}(N)$ telle que $s + t \geqslant 2$,
>
> $$\begin{aligned}\forall i \in S, \quad & I(v, (S, T)) = I(v_+^{N\setminus i}, (S \setminus i, T)) - I(v^{N\setminus i}, (S \setminus i, T)), \text{ si } s \geqslant 1,\\ \forall i \in T, \quad & I(v, (S, T)) = I(v^{N\setminus i}, (S, T \setminus i)) - I(v_-^{N\setminus i}, (S, T \setminus i)), \text{ si } t \geqslant 1.\end{aligned}$$

Théorème 3.1 *Sous les axiomes **($\mathbf{L}^{\mathcal{B}}$)**, **($\mathbf{NG}^{\mathcal{B}}$)**, **($\mathbf{ND}^{\mathcal{B}}$)**, **($\mathbf{I}^{\mathcal{B}}$)**, **($\mathbf{S}^{\mathcal{B}}$)**, **($\mathbf{E}^{\mathcal{B}}$)** et **($\mathbf{R}^{\mathcal{B}}$)**, I est donné pour tout jeu $v \in \mathcal{G}(3^N)$ et toute bi-coalition $(S, T) \neq (\emptyset, \emptyset)$, par*

$$I(v, (S, T)) = \sum_{K \subseteq N \setminus (S \cup T)} \frac{k!\,(n - s - t - k)!}{(n - s - t + 1)!} \Delta_{S,T} v(K, N \setminus (K \cup S)). \tag{3.4}$$

3.2 Une seconde axiomatisation : l'axiome de l'association réduite

Récemment, Fujimoto, Kojadinovic et Marichal ont proposé d'élégantes propriétés pour caractériser l'indice d'interaction pour les jeux coopératifs classiques [15]. L'un des intérêts est d'éviter un axiome de récursivité difficilement interprétable. On retrouve les axiomes de linéarité, nullité, symétrie et invariance qu'il est par contre nécessaire d'étendre ici à toutes les bi-coalitions.

Linéarité étendue (LE$^{\mathcal{B}}$) : I est linéaire par rapport à la variable v, i.e., pour tous jeux v, w, et tout réel α,

$$\forall (S,T) \in \mathcal{Q}(N), \quad I(v + \alpha \cdot w, (S,T)) = I(v,(S,T)) + \alpha\, I(w,(S,T)).$$

Nullité à gauche étendue (NGE$^{\mathcal{B}}$) : pour tout jeu $v \in \mathcal{G}(3^N)$ et tout $i \in N$ nul à gauche pour v,

$$I(v,(S \cup i, T)) = 0 \text{ pour toute bi-coalition } (S,T) \in \mathcal{Q}(N \setminus i).$$

Nullité à droite étendue (NDE$^{\mathcal{B}}$) : pour tout jeu $v \in \mathcal{G}(3^N)$ et tout $i \in N$ nul à droite pour v,

$$I(v,(S, T \cup i)) = 0 \text{ pour toute bi-coalition } (S,T) \in \mathcal{Q}(N \setminus i).$$

Symétrie étendue (SE$^{\mathcal{B}}$) : pour toute permutation $\sigma \in \mathcal{S}(N)$, pour tout jeu $v \in \mathcal{G}(3^N)$, pour tout $(S,T) \in \mathcal{Q}$,

$$I(v \circ \sigma, (S,T)) = I(v, (\sigma(S), \sigma(T))).$$

Invariance étendue (IE$^{\mathcal{B}}$) : pour tous jeux bi-coopératifs v, w et tout $i \in N$ tels que

$$\forall (K,L) \in \mathcal{Q}(N \setminus i), \begin{cases} v(K \cup i, L) = w(K,L), \\ v(K,L) = w(K, L \cup i), \end{cases}$$

alors $I(v,(S \cup i, T)) = I(w,(S, T \cup i)), \quad \forall (S,T) \in \mathcal{Q}(N \setminus i)$.

On appelle *association* pour le jeu $v \in \mathcal{G}(3^N)$ toute coalition de joueurs $P \subseteq N$ telle que pour toute bi-coalition de $(S,T) \in \mathcal{Q}(N \setminus P)$, et tous sous-ensembles propres P_+ et P_- de P, on a

$$v(S \cup P_+, T \cup P_-) = v(S,T).$$

L'interprétation est simple : si tous les joueurs d'une association ne coopèrent pas de la *même* manière, i.e., ne sont ni tous dans la partie défensive, ni tous dans la partie défaitrice, alors l'effet du jeu est le même que dans le cas où tous s'abstiennent de participer.

Pour tout jeu $v \in \mathcal{G}(3^N)$, et toute coalition K non vide, on appelle *jeu K-réduit*, et on note $v_{[K]}$ le jeu défini sur $N_{[K]} := (N \setminus K) \cup \{[K]\}$ par

$$v_{[K]}(S,T) := v(S^\star, T^\star),$$

où $A^\star := \begin{cases} A \text{ si } [K] \notin A, \\ (A \setminus [K]) \cup K \text{ sinon.} \end{cases}$

L'axiome suivant est inspiré du travail de Fujimoto, Kojadinovic et Marichal.

> **Association réduite (AR$^{\mathcal{B}}$)** : pour tout jeu bi-coopératif v, et toute association de joueurs $P \subseteq N$ pour v,
>
> $$I(v, (P, \emptyset)) = I(v_{[P]}, ([P], \emptyset)).$$

Théorème 3.2 *Sous les axiomes **(LE$^{\mathcal{B}}$)**, **(NGE$^{\mathcal{B}}$)**, **(NDE$^{\mathcal{B}}$)**, **(IE$^{\mathcal{B}}$)**, **(SE$^{\mathcal{B}}$)** et **(E$^{\mathcal{B}}$)**, les axiomes **(R$^{\mathcal{B}}$)** et **(AR$^{\mathcal{B}}$)** sont équivalents. I est donc donné par la formule (3.4).*

4 L'indice d'interaction généralisé pour les jeux définis sur les treillis distributifs

L'indice d'interaction pour les jeux bi-coopératifs, axiomatisé au chapitre 4, est examiné au chapitre 5. On y établit l'équivalence de représentations de fonctions de bi-ensembles (i.e., définies sur $\mathcal{Q}(N)$). Il est déjà établi que toute fonction de bi-ensembles v est caractérisée par sa transformée de Möbius m^v (cf. introduction, sous-section 3.1). On montre, comme l'a fait Grabisch dans [17] pour les jeux coopératifs classiques, que $I(v, \cdot)$ caractérise de même v. D'autre part, une formule explicitant v en fonction de $I(v, \cdot)$ est fournie.

On élabore au chapitre 6 une formule générale de l'indice d'interaction pour les jeux à actions combinées. On y détermine également une formule explicite de la transformée de Möbius pour ces applications. Les résultats obtenus au chapitre 5 sont généralisés.

Cette section donne les résultats principaux obtenus au chapitre 6. Dans la sous-section 4.3, on donne les résultats obtenus pour les fonctions de bi-ensembles (chapitre 5), qui, bien que soumises à la bipolarité, peuvent malgré tout être perçues comme cas particuliers de fonctions sur treillis distributifs.

4.1 L'indice d'interaction pour les jeux à actions combinées

Soit N l'ensemble des joueurs. Pour tout joueur i, $(L_i, \leq_i)$ est un treillis distributif de plus petit élément (resp. plus grand élément) $\bot_i$ (resp. $\top_i$) construit à partir de l'ensemble partiellement ordonné des actions élémentaires dont il dispose. En vertu du théorème de Birkhoff (p. 11), ces actions élémentaires sont figurées par l'ensemble $\mathcal{J}_i$ des éléments sup-irréductibles de L_i. $L := \prod_{i=1}^{n} L_i$ désigne ainsi l'ensemble de toutes les actions combinées des joueurs (que l'on munit de l'ordre produit $\leq$ des $\leq_i$), et $\mathcal{J}(L)$, l'ensemble des éléments sup-irréductibles de L, qu'on peut identifier à la réunion des $\mathcal{J}_i$. $\mathcal{G}(L)$, l'ensemble des jeux définis sur L, est un sous-espace vectoriel de $\mathbb{R}^L$, ensemble des applications définies sur L.

Par le théorème de Birkhoff, tout élément x de L, est caractérisé par sa *décomposition normale* en éléments sup-irréductibles, qui est l'ensemble des éléments de $\mathcal{J}(L)$ minorant x. On note $\eta(x)$ cet ensemble. De plus, on note $\eta^*(x)$ la *décomposition minimale* en éléments sup-irréductibles de x, qui est l'ensemble des éléments maximaux de $\eta(x)$.

On généralise de cette manière la dérivation dans $\mathbb{R}^L$ (cf. sous-section 3.2) : Soit $f \in \mathbb{R}^L$ et $j \in \mathcal{J}(L)$. La dérivée de *premier ordre* de f par rapport à j au point $x \in L$ est donnée par

$$\Delta_j f(x) := f(x \vee j) - f(x).$$

Via la décomposition minimale $\eta^*(y) = \{j_1, \ldots, j_m\}$ de tout point $y \in L$, on définit récursivement l'opérateur Δ_y sur $\mathbb{R}^L$ par

$$\Delta_y f(x) := \Delta_{j_m}\big(\ldots \Delta_{j_2}\big(\Delta_{j_1} f(x)\big) \ldots\big), \quad x \in L.$$

Cette définition ne dépend pas de l'ordre des j_k et a donc bien un sens. Explicitement, on a

$$\Delta_y f(x) = \sum_{S \subseteq \{1,\ldots,m\}} (-1)^{m-s} f\big(x \vee \bigvee_{k \in S} j_k\big).$$

Un indice d'interaction pour jeux à actions combinées est donné dans [22]. Cependant, dans la formule proposée, toutes les actions combinées ne sont pas traitées. L'indice d'interaction I exprimé au chapitre 6, est défini sur $\mathbb{R}^L \times L$. Cette formule étend ainsi l'indice d'interaction proposé antérieurement.

Pour tout $j \in \mathcal{J}(L)$, on rappelle qu'on note $\underline{j}$ l'unique élément de L couvert par j. On propose d'étendre comme suit cette notation à tout élément de $x \in L$:

$$\underline{x} := \bigvee \{j \in \eta(x) \mid j \notin \eta^*(x)\}.$$

De manière équivalente, on montre que $\underline{x}$ est le plus petit élément de L tel que l'intervalle $[\underline{x}, x]$ soit booléen, ou encore, le plus grand élément de L tel que $[\underline{x}, x]$ contienne tous les éléments couverts par x.

L'indice d'interaction du jeu v évalué en $x \in L$ est exprimé par une moyenne pondérée de dérivées par rapport à x, des sommets de L.

Définition 4.1 *Soit* $v \in \mathbb{R}^L$. *Soit* $x \in L$ *et* $X := \{i \in N \mid x_i \neq \bot_i\}$. *L'*indice d'interaction $I(v, x)$ *est défini par*

$$I(v, x) := \sum_{Y \subseteq N \setminus X} \frac{|Y|!\,(n - |X| - |Y|)!}{(n - |X| + 1)!}\, \Delta_x v(\underline{x} \vee \top_Y),$$

où $\top_Y$ *est le sommet de* L *dont les coordonnées d'ordre* $j \in Y$ *sont* $\top_j$, *et* $\bot_j$ *pour les autres.*

Remarquons à nouveau qu'il n'y a aucun inconvénient à étendre cette définition aux applications définies sur $\mathbb{R}^L$.

4.2 Représentations équivalentes de fonctions définies sur des treillis

Les opérateurs de passage d'une fonction de treillis $f \in \mathbb{R}^L$ à sa transformée de Möbius, ou à son indice d'interaction sont des transformations linéaires. On considère l'ensemble des opérateurs linéaires agissant sur l'ensemble $\mathbb{R}^L$. On peut représenter ces opérateurs comme fonctions définies sur $L \times L$, la composée de deux opérateurs Ψ_1, Ψ_2 se notant

$$(\Psi_1 \star \Psi_2)(x, y) := \sum_{t \in L} \Psi_1(x, t)\, \Psi_2(t, y),$$

et l'action à gauche (resp. à droite) de l'opérateur Ψ sur la fonction f s'écrivant

$$(\Psi \star f)(x) := \sum_{t \in L} \Psi(x, t)\, f(t),$$

$$(f \star \Psi)(x) := \sum_{t \in L} f(t)\, \Psi(t, x).$$

La transformation de Möbius est définie comme l'opérateur inverse de l'opérateur *Zeta*, qui s'écrit

$$\forall x, y \in L, \quad Z(x,y) := \begin{cases} 1, & \text{si } x \leq y, \\ 0, & \text{sinon.} \end{cases}$$

Ainsi, Z agit à droite sur $\mathbb{R}^L$ via $f = m^f \star Z$, pour toute fonction $f \in \mathbb{R}^L$, ce qui par inversion de l'équation, donne $m^f = f \star Z^{-1}$. Via une écriture des dérivées en termes de transformées de Möbius, on a le résultat suivant :

Théorème 4.1 *Soit $f \in \mathbb{R}^L$ et $x \in L$. Alors*

$$I(f,x) = \sum_{z \in [x, \check{x}]} \frac{1}{k(z) - k(x) + 1} \, m^f(z),$$

où $\check{x}_j := \top_j$ si $x_j = \bot_j$, $\check{x}_j := x_j$ sinon, et $k(y)$ est le nombre de coordonnées de $y \in L$ différentes de $\bot_j$, $j = 1, \ldots, n$.

En termes de transformation linéaire, ce résultat s'écrit alors $I(f, \cdot) = \Gamma \star m^f$, pour toute fonction $f \in \mathbb{R}^L$, où l'opérateur Γ est donné par

$$\forall x, y \in L, \quad \Gamma(x,y) := \begin{cases} \dfrac{1}{k(y) - k(x) + 1}, & \text{si } \forall i \in N, x_i = \bot_i \text{ ou } y_i = x_i, \\ 0, & \text{sinon.} \end{cases}$$

Cet opérateur étant triangulaire-supérieur, il est inversible, et on a alors $m^f = \Gamma^{-1} \star I(f, \cdot)$. Ceci établit l'équivalence de représentations de f, m^f et $I(f, \cdot)$. En conséquence de quoi on a les formules suivantes

$$\begin{aligned} I(f, \cdot) &= \Gamma \star (f \star Z^{-1}), \\ \text{et} \qquad f &= (\Gamma^{-1} \star I(f, \cdot)) \star Z. \end{aligned}$$

L'opérateur d'interaction $\mathbb{I}$ qui exprime (par action à droite) la transformation de la fonction f en $I(f, \cdot)$ vérifie alors la relation $\mathbb{I} = Z^{-1} \star {}^t\Gamma$, où ${}^t\Gamma$ désigne l'opérateur transposé de Γ (i.e., ${}^t\Gamma(x,y) := \Gamma(y,x)$ pour tous $x, y \in L$).

Sur le modèle proposé par Denneberg et Grabisch [9] dans le cadre des fonctions d'ensemble, on élabore un procédé calculatoire basé sur la réduction d'*algèbres d'incidence* [11], qui permet le calcul explicite des opérateurs de Möbius Z^{-1} et Γ^{-1} :

Proposition 4.2 *Pour tous $x, y \in L$,*

$$Z^{-1}(x,y) := \begin{cases} (-1)^m, & \textit{si le sous-ensemble } [x,y] \textit{ est un treillis Booléen de cardinal } 2^m, \\ 0, & \textit{sinon,} \end{cases}$$

$$\Gamma^{-1}(x,y) := \begin{cases} B_{k(y-x)}, & \textit{si } \forall i \in N, x_i = \bot_i \textit{ ou } y_i = x_i, \\ 0, & \textit{sinon,} \end{cases}$$

où $(B_k)_{k\in\mathbb{N}}$ désigne la suite de Bernoulli (voir p. 216).

Par la détermination de l'opérateur inverse d'interaction $\mathbb{I}^{-1}$, on a enfin ce dernier résultat :

Théorème 4.3 *Pour toute fonction $f \in \mathbb{R}^L$,*

$$\forall x \in L, \quad f(x) = \sum_{y \in L} b_{k(y_x)}^{k(y)}\, I(f,y),$$

où pour tous $x, y \in L$, y_x est défini par $(y_x)_i := \begin{cases} y_i, & \textit{si } y_i \leq_i x_i, \\ \bot_i, & \textit{sinon,} \end{cases}$ pour tout $i \in N$, et où les nombres b_m^p sont des nombres définis récursivement à partir de la suite de Bernoulli pour tout $p \in \mathbb{N}$ et tout $m \in \{0, \dots, p\}$, par

$$b_m^p := \sum_{j=0}^{m} \binom{m}{j} B_{p-j}.$$

4.3 L'opérateur inverse d'interaction pour les fonctions de bi-ensembles

Notons qu'au chapitre 5, contrairement au procédé utilisé au chapitre 4, la dérivation pour les fonctions de bi-ensembles se construit comme en sous-section 4.1, c'est-à-dire itérativement à partir des dérivées du premier ordre par rapport aux sup-irréductibles de $\mathcal{Q}(N)$.

En quête de l'opérateur inverse d'interaction pour les fonctions de bi-ensembles, nous obtenons au terme d'un développement usant largement de théorie des algèbres d'incidence, le résultat suivant :

Théorème 4.4 *Pour toute fonction de bi-ensembles f*

$$\forall (A,B) \in \mathcal{Q}(N), \quad f(A,B) = \sum_{(C,D)\in\mathcal{Q}(N)} b^{n-|D|}_{n-|B\cup D\cup(A^c\cap C)|} I(f,(C,D)),$$

où les nombres b^p_m *sont définis à la sous-section précédente.*

Références bibliographiques

[1] J.F. Banzhaf. Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, 19:317–343, 1965.

[2] J.M. Bilbao. *Cooperative Games on Combinatorial Structures*. Kluwer Academic Publishers, 2000.

[3] J.M. Bilbao and P.H. Edelman. The Shapley value on convex geometries. *Discrete Applied Mathematics*, 103:33–40, 2000.

[4] G. Birkhoff. *Lattice Theory*. American Mathematical Society, 3d edition, 1967.

[5] R. Branzei, D. Dimitrov, and S. Tijs. *Models in cooperative game theory: crisp, fuzzy and multichoice games*. Springer Verlag, 2005.

[6] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953.

[7] J. Deegan and E.W. Packel. A new index of power for simple n-person games. *Int. J. of Game Theory*, 7:113–123, 1978.

[8] D. Denneberg. *Non-Additive Measure and Integral*. Kluwer Academic, 1994.

[9] D. Denneberg and M. Grabisch. Interaction transform of set functions over a finite set. *Information Sciences*, 48(1):15–27, 1999.

[10] J. Derks and H. Peters. A Shapley value for games with restricted coalitions. *Int. J. of Game Theory*, 21:351–360, 1993.

[11] P. Doubilet, G.C. Rota, and R. Stanley. On the foundations of combinatorial theory (VI): The idea of generating function. In Berkeley University Press, editor, *6th Berkeley Symposium on Mathematical Statistics and Probability*, volume 2, pages 267–318, 1972.

[12] U. Faigle. Cores with restricted cooperation. *ZOR – Methods and Models of Operations Research*, pages 405–422, 1989.

[13] U. Faigle and W. Kern. The Shapley value for cooperative games under precedence constraints. *International Journal of Game Theory*, 21:249–266, 1992.

[14] D. Felsenthal and M. Machover. Ternary voting games. *Int. J. of Game Theory*, 26:335–351, 1997.

[15] K. Fujimoto, I. Kojadinovic, and J.L. Marichal. Axiomatic characterizations of probabilistic and cardinal-probabilistic interaction indices. *Games and Economic Behavior*, 55(1):72–99, 2006.

[16] I. Gilboa and E. Lehrer. Global games. *Int. J. of Game Theory*, 20:129–147, 1991.

[17] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[18] M. Grabisch and Ch. Labreuche. Bi-capacities. Technical Report 2003/002, LIP6, UPMC, http://www.lip6.fr/reports/lip6.2003.002.html, 2003.

[19] M. Grabisch and Ch. Labreuche. Capacities on lattices and k-ary capacities. In *3d Int, Conf. of the European Soc. for Fuzzy Logic and Technology (EUSFLAT 2003)*, pages 304–307, Zittau, Germany, September 2003.

[20] M. Grabisch and Ch. Labreuche. Bi-capacities. Part I: definition, Möbius transform and interaction. *Fuzzy Sets and Systems*, 151:211–236, 2005.

[21] M. Grabisch and Ch. Labreuche. Bi-capacities. Part II: the Choquet integral. *Fuzzy Sets and Systems*, 151:237–259, 2005.

[22] M. Grabisch and Ch. Labreuche. Derivative of functions over lattices as a basis for the notion of interaction between attributes. *Annals of Mathematics and Artificial Intelligence*, 49:151–170, 2007.

[23] M. Grabisch, J.L. Marichal, and M. Roubens. Equivalent representations of set functions. *Mathematics of Operations Research*, 25(2):157–178, 2000.

[24] M. Grabisch and M. Roubens. An axiomatic approach to the concept of interaction among players in cooperative games. *Int. Journal of Game Theory*, 28:547–565, 1999.

[25] P.L. Hammer and S. Rudeanu. *Boolean Methods in Operations Research and Related Areas.* Springer, 1968.

[26] J.C. Harsanyi. A bargaining model for the cooperative n-person game. *Annals of Mathematics Study*, 40:325–355, 1959.

[27] C.R. Hsiao and T.E.S. Raghavan. Shapley value for multichoice cooperative games, I. *Games and Economic Behavior*, 5:240–256, 1993.

[28] R. Johnston. On the measurement of power: some reactions to Laver. *Environment and Planning*, 10:907–914, 1978.

[29] E. Kalai and D. Samet. On weighted Shapley values. *Int. J. of Game Theory*, 16:205–222, 1987.

[30] Ch. Labreuche. Interaction indices for games with forbidden coalitions. In *9th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'2002)*, pages 529–534, Annecy, France, July 2002.

[31] Ch. Labreuche and M. Grabisch. A value for bi-cooperative games. *Int. J. of Game Theory*, submitted.

[32] B. Leclerc. Un cours sur les ensembles ordonnés. Technical Report 134, CAMS, 1997.

[33] T. Murofushi and S. Soneda. Techniques for reading fuzzy measures (III): interaction index. In *9th Fuzzy System Symposium*, pages 693–696, Sapporo, Japan, May 1993. In Japanese.

[34] G. Owen. Multilinear extensions of games. *Management Sci.*, 18:64–79, 1972.

[35] G. Owen. *Game Theory.* Academic Press, 3d edition, 1995.

[36] L. Penrose. The elementary statistics of majority voting. *Journal of the Royal Statistical Society*, 109(1), 1946.

[37] H. Peters and H. Zank. The egalitarian solution for multi-choice games. *Annals of Operations Research*, 137:399–409, 2005.

[38] G.C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 2:340–368, 1964.

[39] A.E. Roth. *Essays in honor of Lloyd S. Shapley*, chapter Introduction to the Shapley value, pages 1–27. Cambridge University Press, 1988.

[40] L.S. Shapley. A value for n-person games. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.

[41] L.S. Shapley and M. Shubik. A method for evaluating the distribution of power in a committee system. *American Political Science Review*, 48:787–792, 1954.

[42] P.D. Straffin. *Handbook of Game Theory with Economic Applications*, volume 2, chapter Power and stability in politics, pages 1127–1151. Elsevier, 1994.

[43] R.M. Thrall and W.F. Lucas. N-person games in partition function form. *Naval Research Logistics Quarterly*, 10:281–293, 1963.

[44] S. Tijs. Bounds for the core and the τ-value. In O. Moeschlin and D. Pallaschke, editors, *Game Theory and Mathematical Economics*, Amsterdam, 1981. North-Holland.

[45] G. Vitali. Sulla definizione di integrale delle funzioni di una variabile. *Ann. Mat. Pura ed Appl.*, (2):111–121, 1925.

[46] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, New Jersey, 1944.

[47] R.J. Weber. Probabilistic values for games. In A.E. Roth, editor, *The Shapley Value. Essays in Honor of Lloyd S. Shapley*, pages 101–119. Cambridge University Press, 1988.

[48] R.R. Yager. Families of OWA operators. *Fuzzy Sets and Systems*, 55:255–271, 1993.

[49] H.P. Young. Monotonic solutions of cooperative games. *International Journal of Game Theory*, 14:65–72, 1985.

Première partie.
Jeux définis sur des ensembles ordonnés et valeur de Shapley

Chapitre 1.
Jeux sur treillis, jeux multi-choix et valeur de Shapley : Une nouvelle approche

Résumé

La théorie des treillis a un rôle déterminant pour les jeux multi-choix, de même que pour de nombreuses autres généralisations des jeux coopératifs classiques. Nous proposons une définition générale de jeux définis sur des treillis, ainsi qu'une interprétation de ceux-ci. Des définition de la valeur de Shapley pour les jeux multi-choix ont déjà été données, parmi lesquelles celle originelle proposée par Hsiao et Raghavan, et celle de Faigle et Kern. Nous proposons une axiomatisation de valeur dans l'esprit de celle proposée par Shapley, et qui évite une haute complexité de calcul.

Mots clés : jeu coopératif, treillis distributif, jeu multi-choix, valeur de Shapley

Games on lattices, multichoice games and the Shapley value: a new approach

Michel Grabisch
Fabien Lange

Centre d'Economie de la Sorbonne, UMR 8174, CNRS-Université Paris 1

Abstract

Multichoice games, as well as many other recent attempts to generalize the notion of classical cooperative game, can be casted into the framework of lattices. We propose a general definition for games on lattices, together with an interpretation. Some definitions of the Shapley value of multichoice games have already been given, among them the original one due to Hsiao and Raghavan, and the one given by Faigle and Kern. We propose an approach together with its axiomatization, more in the spirit of the original axiomatization of Shapley, and avoiding a high computational complexity.

Keywords: cooperative game, distributive lattice, multichoice game, Shapley value

In *Mathematical Methods of Operations Research*, 65:153–167, 2007.

1 Introduction

The field of cooperative game theory has been enriched these recent years by many new kinds of game, trying to model in a more accurate way the behaviour of players in a real situation. In the classical view of cooperative games, to each coalition of players taking part into the game, an asset or a power (voting games) is associated, and participation is assumed to be of a binary nature, i.e., either a player participates or he does not. From this point, many variations have been introduced, let us cite games with precedence constraints among players (Faigle and Kern [6]) where not all coalitions are valid, ternary voting games (Felsenthal and Machover [7]) where abstention is permitted, bi-cooperative games (Bilbao [2]) where each player can choose to play either in favor, against, or not to play, multichoice games (Hsiao and Raghavan [14]) where each player has a set of m possible ordered actions, fuzzy games (Butnariu and Klement [4], Tijs *et al.* [18]) which can be seen as a continuous generalization of multi-choice games, global games (Gilboa and Lehrer [8]) where coalitions are replaced by partitions of the set of players, etc.

All the above examples of games can be casted into the general framework of games defined on a lattice, i.e., functions $v : (L, \leq) \longrightarrow \mathbb{R}$, where $(L, \leq)$ is a lattice, and such that $v(\perp) = 0$, $\perp$ denoting the bottom element of L. We mention at this point that one can define games on other structures of discrete mathematics, such as matroids and convex geometries; this has been extensively studied by Bilbao [1].

A central question in game theory is to define a *value* or *solution concept* for a game, i.e., how to individually reward players supposing that all players have joined the grand coalition. A famous example for classical games is the *Shapley value*, based on rational axioms for sharing the total worth of the game $v(N)$. A different approach is to consider the *core* of the game, i.e., the set of imputations such that no subcoalition can do better by itself.

The aim of this paper is first to provide a general approach to games on lattices, giving an interpretation in terms of elementary actions, and second to provide a definition for the Shapley value together with an axiomatization. As it will be discussed, other previous definitions of the Shapley value have been given. We will focus on the works of Faigle and Kern [6], and Hsiao and Raghavan [14]. Previous works of the authors around this topic can be found in [12, 11, 10].

2 Mathematical background

We begin by recalling necessary material on lattices (a good introduction on lattices can be found in [5]), in a finite setting. A *lattice* is a set L endowed with a partial order $\leq$ such that for any $x, y \in L$ their least upper bound $x \vee y$ and greatest lower bound $x \wedge y$ always exist. For finite lattices, the greatest element of L (denoted $\top$) and least element $\perp$ always exist. x *covers* y (denoted $x \succ y$) if $x > y$ and there is no z such that $x > z > y$. The lattice is *distributive* if $\vee, \wedge$ obey distributivity. An element $j \in L$ is *join-irreducible* if it cannot be expressed as a supremum of other elements. Equivalently j is join-irreducible if it covers only one element. Join-irreducible elements covering $\perp$ are called *atoms*, and the lattice is *atomistic* if all join-irreducible elements are atoms. The set of all join-irreducible elements of L is denoted $\mathcal{J}(L)$.

An important property is that in a distributive lattice, any element x can be written as an irredundant supremum of join-irreducible elements in a unique way (this is called the *minimal decomposition* of x). We denote by $\eta^*(x)$ the set of join-irreducible elements in the minimal decomposition of x, and we denote by $\eta(x)$ the *normal decomposition* of x, defined as the set of join-irreducible elements smaller or equal to x, i.e., $\eta(x) := \{j \in \mathcal{J}(L) \mid j \leq x\}$. Hence $\eta^*(x) \subseteq \eta(x)$, and

$$x = \bigvee_{j \in \eta^*(x)} j = \bigvee_{j \in \eta(x)} j.$$

Let us rephrase differently the above result. We say that $Q \subseteq L$ is a *downset* of L if $x \in Q$ and $y \leq x$ imply $y \in Q$. For any subset P of L, we denote by $\mathcal{O}(P)$ the set of all downsets of P. Then the mapping η is an isomorphism of L onto $\mathcal{O}(\mathcal{J}(L))$ (Birkhoff's theorem).

In a finite setting, *Boolean lattices* are of the type 2^N for some set N, i.e., they are isomorphic to the lattice of subsets of some set, ordered by inclusion. Boolean lattices are atomistic, and atoms corresponds to singletons. A *linear lattice* is such that $\leq$ is a total order. All elements are join-irreducible, except $\perp$.

Given lattices $(L_1, \leq_1), \ldots, (L_n, \leq_n)$, the product lattice $L = L_1 \times \cdots \times L_n$ is endowed with the product order $\leq$ of $\leq_1, \ldots, \leq_n$ in the usual sense. Elements x of L can be written in their vector form $(x_1, \ldots, x_n)$. We use the notation (x_A, y_{-A}) to indicate a vector z such that $z_i = x_i$ if $i \in A$, and $z_i = y_i$ otherwise. Similarly L_{-i} denotes $\prod_{j \neq i} L_j$. All join-irreducible elements of L are of the form

$(\bot_1, \ldots, \bot_{i-1}, j_i, \bot_{i+1}, \ldots, \bot_n)$, for some i and some join-irreducible element j_i of L_i. A *vertex* of L is any element whose components are either top or bottom. We denote $\Gamma(L)$ the set of vertices of L. Note that $\Gamma(L) = L$ iff L is Boolean, since in this case, denoting the trivial lattice $\{\bot, \top\}$ by 2, we have $L = \underbrace{2 \times \cdots \times 2}_{n \text{ times}} = 2^n$.

3 Games on lattices

We denote by $N := \{1, \ldots, n\}$ the set of players.

Definition 1 *We consider finite distributive lattices $(L_1, \leq_1), \ldots, (L_n, \leq_n)$ and their product $L := L_1 \times \cdots \times L_n$ endowed with the product order $\leq$. A* game on L *is any function $v : L \longrightarrow \mathbb{R}$ such that $v(\bot) = 0$. The set of such games is denoted $\mathcal{G}(L)$. A game is* monotone *if $x \leq x'$ implies $v(x) \leq v(x')$.*

Lattice $(L_i, \leq_i)$ represents the (partially) ordered set of actions, choices, levels of participation of player i to the game. Each lattice may be different.

First, let us examine several particular examples.

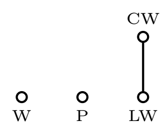
- $(L, \leq) = (2^N, \subseteq)$. This is the classical notion of game. Each player has two possible actions (participate, not participate), hence $L_i = \{0, 1\}$. L is a Boolean lattice.
- $(L, \leq) = (3^N, \leq)$. This case comprises ternary voting games and bi-cooperative games (each L_i can be coded as $L_i = \{-1, 0, 1\}$, where 0 means "no participation", -1 means voting or playing against, and 1 means voting or playing in favor), as well as multi-choice games with $m = 2$, letting $L_i = \{0, 1, 2\}$, with 0 indicating no participation, and 1,2 participation (low and high). In fact, Grabisch distinguishes these two cases, the first one being called *bipolar game* since the L_i's have a symmetric structure around 0 [10].
- $(L, \leq) = (m^N, \leq)$, with $L_i = \{0, 1, \ldots, m\}$. This corresponds to multi-choice games as introduced by Hsiao and Raghavan. In this paper we will call them *m-level games*, and call *multi-choice game* the case where each L_i is a linear lattice $L_i := \{0, 1, \ldots, l_i\}$ (i.e., the number of levels may be different for each player).

- $(L, \leq) = ([0, 1]^n, \leq)$. This corresponds to fuzzy games.

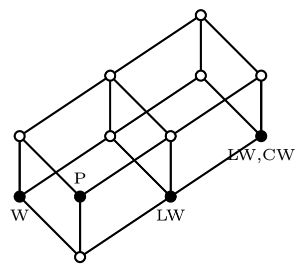
Note that the case of global games cannot be recovered by our definition, since the lattice of partitions is not a product lattice.

Let us turn to the interpretation of our definition. We assume that each player $i \in N$ has at his/her disposal a set of *elementary* or *pure* actions $j_1, \dots, j_{n_i}$. These elementary actions are partially ordered (e.g. in the sense of benefit caused by the action), forming a partially ordered set $(\mathcal{J}_i, \leq)$. Then by virtue of Birkhoff's theorem (see Sec. 2), the set $(\mathcal{O}(\mathcal{J}_i), \subseteq)$ of downsets of $\mathcal{J}_i$ is a distributive lattice denoted L_i, whose join-irreducible elements correspond to the elementary actions. The *bottom action* $\perp$ of L_i is the action which amounts to do nothing. Hence, each action in L_i is either a pure action j_k or a combined action $j_k \vee j_{k'} \vee j_{k''} \vee \cdots$ consisting of doing all actions $j_k, j_{k'}, \dots$ for player i.

For example, assume that players are gardeners who take care of some garden or park. Elementary actions are watering (W), light weeding (LW), careful weeding (CW), and pruning (P). All these actions are benefic for the garden and clearly LW<CW, but otherwise actions seem to be incomparable. They form the following partially ordered set:



which in turn form the following lattice of possible actions:



Let us give another interpretation of our framework, borrowed from Faigle and Kern [6]. Let $P := (N, \leq)$ be a partially ordered set of players, where $\leq$ is a relation of *precedence*: $i \leq j$ if the presence of j enforces the presence of i in any coalition $S \subseteq N$. Hence, a (valid) *coalition* of P is a subset S of N such that

$i \in S$ and $j \leq i$ entails $j \in S$. Hence, the collection $\mathcal{C}(P)$ of all coalitions of P is the collection of all downsets (ideals) of P. A *game* on P is any function $v : \mathcal{C}(P) \to \mathbb{R}$ such that $v(\emptyset) = 0$.

From this definition, it is possible to recover our structure. For each player i in N, let $\mathcal{J}_i := \{j_1, \ldots, j_{n_i}\}$ be the set of elementary actions of player i. Consider the set of virtual players

$$N' := \bigcup_{i \in N} \mathcal{J}_i$$

equipped with the partial order $\leq$ induced by the partial orders on each $\mathcal{J}_i$. Then coalitions of $(N', \leq)$ correspond bijectively to elements of $\prod_{i \in N} \mathcal{O}(\mathcal{J}_i)$.

4 Previous works on the Shapley value

We present in this section the Shapley value defined by Faigle and Kern, and the one defined by Hsiao and Raghavan, together with their axiomatization. A good comparison of these two values can be found in [3]. We present them with our notations, which are rather far from the original ones.

The value introduced by Faigle and Kern is the average of the marginal vectors along all maximal chains in L. A *maximal chain* in a (finite) lattice L is a sequence of elements $C = \{\bot, x, y, z, \ldots, \top\}$ such that $\bot \prec x \prec y \prec z \cdots \prec \top$. We denote by $\mathcal{C}(L)$ the set of all maximal chains on L. Then the Shapley value of Faigle and Kern is defined by:

$$\phi^v_{\mathrm{FK}}(j_i) := \frac{1}{|\mathcal{C}(L)|} \sum_{C \in \mathcal{C}(L)} [v(x_{j_i}) - v(\underline{x_{j_i}})], \tag{1}$$

for any join-irreducible element j_i of L_i, and for any $i \in N$. The element x_{j_i} is the first in the sequence C containing j_i in $\eta(x_{j_i})$, and $\underline{x_{j_i}}$ is its predecessor in the chain C. In the vocabulary of Faigle and Kern, maximal chains correspond to what they call *feasible ranking* of join-irreducible elements (players).

The axiomatic of Faigle and Kern is essentially based on linearity (L) and the unique decomposition of a game on the basis of unanimity games. In this case, a unanimity game u_x is defined by, for any $x \in L$:

$$u_x(y) := \begin{cases} 1, & \text{if } y \geq x \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Then the coordinates of any game v in this basis are given by the Möbius transform (or dividend) of v [17]. It remains then to fix the Shapley value of any unanimity game by some suitable axioms. They are indicated below.

An element $c \in L$ is a *carrier* if $v(x \wedge c) = v(x)$, for all $x \in L$.

Carrier axiom (C): If c is a carrier for v, then $\sum_{j_i \leq c} \phi^v_{\mathrm{FK}}(j_i) = v(c)$.

The *hierarchical strength* of a join-irreducible element $j_i \in L_i$ with respect to some $x \in L$ is defined by the relative number of maximal chains in L where x is the first occurence of j_i in the chain, that is:

$$h_x(j_i) := \frac{1}{|\mathcal{C}(L)|}\big|\big\{C \in \mathcal{C}(L) \mid x_{j_i} = x\big\}\big|.$$

Hierarchical strength axiom (HS): For any $x \in L$ and any join-irreducible elements $j_i, j'_{i'} \in \eta(x)$,

$$h_x(j_i)\phi^{u_x}_{\mathrm{FK}}(j'_{i'}) = h_x(j'_{i'})\phi^{u_x}_{\mathrm{FK}}(j_i).$$

Then, under axioms (L), (C) and (HS), the value of the unanimity game u_x is uniquely determined:

$$\phi^{u_x}_{\mathrm{FK}}(j_i) = \begin{cases} 0, & \text{if } j_i \notin \eta(x) \\ h_x(j_i)/\sum_{k\in\eta(x)} h_x(k), & \text{otherwise.} \end{cases}$$

We turn to the value proposed by Hsiao and Raghavan, which is limited to m-choice games in our terminology. Its construction is similar to the one of Faigle and Kern because it is based also on unanimity games. The main difference is that Hsiao and Raghavan introduced weights for all possible actions of the players, leading to a kind a weighted Shapley value. Let us denote by $w_1, \dots, w_m$ the weights of actions $1, \dots, m$; they are such that $w_1 < \cdots < w_m$. The first axiom is additivity (A) of the value, i.e., $\phi^{v+w}_{\mathrm{HR}} = \phi^v_{\mathrm{HR}} + \phi^w_{\mathrm{HR}}$. The second axiom is the carrier axiom (C), as for Faigle and Kern. The remaining ones are as follows.

Minimal effort axiom (ME): if v is such that $v(x) = 0$ for all $x \not\geq y$, then for all players i, all action $k_i < y_i$, we have $\phi^v_{\mathrm{HR}}(k_i) = 0$.

Weight axiom (W): If $v := \alpha u_x$ for some $\alpha > 0$, then $\phi^v_{\mathrm{HR}}(x_{i,i})$ is proportional to w_{x_i}, for all $i \in N$.

Using these axioms, it can be shown that, j_i denoting action j for player i:

$$\phi^{u_x}_{\mathrm{HR}}(j_i) = \begin{cases} \frac{w_j}{\sum_{i\in N} w_{x_i}}, & \text{if } j = x_i \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Let us discuss these values. As remarked by Faigle and Kern, since the problem of computing the number of maximal chains in a partially ordered set is a ♯P-complete counting problem, it is doubtful whether an efficient algorithm could exist to compute exactly ϕ_{FK}. For multichoice games, the number of maximal chains is, with our notation [3, 6]:

$$|\mathcal{C}(L)| = \frac{\left(\sum_{i\in N} l_i\right)!}{\prod_{i\in N}(l_i!)} = \binom{l}{l_1}\binom{l-l_1}{l_2}\binom{l-l_1-l_2}{l_3}\cdots 1,$$

with $l := \prod_{i\in N} l_i$. For 5 players having each 3 actions (3-level game), this gives already $(15)!/6^5 = 168,168,000$. The same remark applies to ϕ_{HR}, since its explicit expression given in [14] is very complicated. In [3], Branzei *et al.* have shown that ϕ_{FK} and ϕ_{HR} do not coincide in general. Even more, one can find examples where for no system of weights the two values can coincide.

Concerning the axiomatic, the one of Faigle and Kern is very simple, although the meaning of the (HS) axiom is not completely clear, at least in our framework of games on lattices (recall that this axiomatic was primarily intended for games with precedence constraints). The axiomatic of Hsiao and Raghavan is simple and clear, but they need weights on action, which are necessarily all different, so one could ask about what if no weight is wanted, and what do precisely mean these weights (in particular, what is the exact difference between w_j and $v(j_i)$?).

In the next section, we present an alternative view.

5 Axiomatic of the Shapley value for multi-choice games

Our approach will take a different way. We do not use unanimity games, but introduce axioms similar to the original ones of Shapley, adding them one by one as Weber in [19], to see the exact effect of each axiom. Surprisingly, we will come up with a value which is very near the classical Shapley value, and very simple to compute.

5.1 Notations, differential and cumulative values

We recall that for every player i, L_i is a linear lattice denoted $L_i := \{0, 1, 2, \ldots, l_i\}$. The set $\mathcal{J}(L)$ of join-irreducible elements (or virtual players in the framework of Faigle and Kern) of L is $\{(0_1, \ldots, 0_{i-1}, k_i, 0_{i+1}, \ldots, 0_n) \mid i \in N, k \in L_i \setminus \{0\}\}$; hence each join-irreducible element corresponds to a single player playing at a given level. Since we use them constantly in the following, we will often adopt the shorthand $\widetilde{k_i}$ for $(0_1, \ldots, 0_{i-1}, k_i, 0_{i+1}, \ldots, 0_n)$.

Our aim is to define the Shapley value for each join-irreducible element $\widetilde{k_i}$. A first approach would be to define the Shapley value for $\widetilde{k_i}$ as a kind of average contribution of player i playing at level k, compared to the situation where i plays at level $k-1$. We call this a *differential* value, which we denote by $\phi(k_i)$. A differential value obviously satisfies what could be called a *differential null axiom*, saying that $\phi(k_i) = 0$ whenever player i is such that $v(x_{-i}, k_i) = v(x_{-i}, (k-1)_i)$ for all $x_{-i} \in L_{-i}$, using our notation for compound vectors (see Section 2).

A careful look at the previous axiomatizations of Faigle and Kern, and Hsiao and Raghavan, show that their value are differential. This is due to the carrier axiom, which could be implied by the differential null axiom and a suitable efficiency axiom (see also formula (1), which obviously satisfies the differential null axiom).

However, if we stick to the idea that the Shapley value for k_i should be a reward for player i having played at level k, it should express an average of the contribution of player i playing at level k, but compared to the situation where k *does not* participate. Roughly speaking, this amounts to sum all differential values from the first level to the kth level. Hence, such a value could be called a *cumulative* value, and to our opinion, it is the only one of interest, the differential value being merely an intermediate step of computation. We denote by $\Phi(k_i)$ the cumulative value for player i playing at level k.

Our position is to give directly an axiomatization of the cumulative Shapley value, which in the sequel will be called simply “Shapley value”. It is possible however to derive a similar axiomatization for the differential value (see [10] for the case of m-choice games).

5.2 The axiomatic of the (cumulative) Shapley value

Let us give first the following definitions generalizing the ones given for classical games.

- for some $k \in L_i, k \neq 0$, player i is said to be $k-$*null* (or simply k_i is *null*) for $v \in \mathcal{G}(L)$ if $v(x, k_i) = v(x, 0_i)$, $\forall x \in L_{-i}$.
- for some $k \in L_i, k \neq 0$, player i is said to be $k-$*dummy* (or simply k_i is *dummy*) for $v \in \mathcal{G}(L)$ if $v(x, k_i) = v(x, 0_i) + v(0_{-i}, k_i)$, $\forall x \in L_{-i}$.
- $v \in \mathcal{G}(L)$ is said to be *monotone* if $v(x) \leq v(y)$, for all x, y in L such that $x \leq y$.

This enables to introduce the following axioms:

Null axiom (N): $\forall v \in \mathcal{G}(L)$, for all null k_i, $\Phi^v(k_i) = 0$.

Dummy axiom (D): $\forall v \in \mathcal{G}(L)$, for all dummy k_i, $\Phi^v(k_i) = v(k_i)$.

As for classical games, the dummy axiom implies the null axiom. Indeed, assume k_i is null. Then $v(k_i) = v(0) = 0$, so that $v(x, k_i) = v(x, 0_i) + v(k_i)$ holds, i.e., k_i is dummy. Then $\Phi^v(k_i) = v(k_i) = 0$, which proves that (N) holds.

Monotonicity axiom (M): $\forall v \in \mathcal{G}(L)$, if v is monotone, then $\Phi^v(k_i) \geq 0$, for all join-irreducible k_i.

Linear axiom (L): For all join-irreducible k_i, $\Phi(k_i)$ is linear on the set of games $\mathcal{G}(L)$, which directly implies

$$\Phi^v(k_i) = \sum_{x \in L} a_x^{k_i}\, v(x), \quad \text{with } a_x^{k_i} \in \mathbb{R}.$$

Proposition 1 *Under axioms (L) and (N), $\forall v \in \mathcal{G}(L)$, for all join-irreducible k_i,*

$$\Phi^v(k_i) = \sum_{x \in L_{-i}} p_x^{k_i}\,[v(x,k_i) - v(x,0_i)], \quad \textit{with } p_x^{k_i} \in \mathbb{R}.$$

Proof: It is clear that the above formula satisfies the axioms. Conversely, assuming k_i is null,

$$\begin{aligned}
\Phi^v(k_i) &= \sum_{x \in L} a_x^{k_i} v(x) \\
&= \sum_{x \in L_{-i}} \left[a_{(x,0_i)}^{k_i} v(x,0_i) + \cdots + a_{(x,l_i)}^{k_i} v(x,l_i)\right] \\
&= \sum_{x \in L_{-i}} v(x,0_i)[a_{(x,0_i)}^{k_i} + a_{(x,k_i)}^{k_i}] + \sum_{x \in L_{-i}} \sum_{j \neq 0,k} a_{(x,j_i)}^{k_i} v(x,j_i). \qquad (4)
\end{aligned}$$

Consider $v' \in \mathcal{G}(L_{-i})$ and extend it to $\mathcal{G}(L)$:

$$v(x,j_i) = \begin{cases} v'(x), & \text{if } j = k, 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then k_i is null for v, hence (4) applies and reduces to:

$$\Phi^v(k_i) = \sum_{x \in L_{-i}} v'(x)[a_{(x,0_i)}^{k_i} + a_{(x,k_i)}^{k_i}] = 0.$$

This implies $a_{(x,k_i)}^{k_i} = -a_{(x,0_i)}^{k_i}$. Introducing this in (4) we get:

$$\Phi^v(k_i) = 0 = \sum_{x \in L_{-i}} \sum_{j \neq 0,k} a_{(x,j_i)}^{k_i} v(x,j_i).$$

Since this must hold for any game, we deduce that $a_{(x,j_i)}^{k_i} = 0$, $\forall j \neq 0, k$. Letting $p_x^{k_i} := a_{(x,k_i)}^{k_i}$, the result is proven. ■

Proposition 2 *Under axioms (L) and (D), $\forall v \in \mathcal{G}(L)$, for all join-irreducible k_i,*

$$\Phi^v(k_i) = \sum_{x \in L_{-i}} p_x^{k_i}\,[v(x,k_i) - v(x,0_i)], \quad \textit{with } p_x^{k_i} \in \mathbb{R}, \textit{ and } \sum_{x \in L_{-i}} p_x^{k_i} = 1.$$

Proof: We consider the unanimity game u_{k_i} defined by

$$u_{k_i}(x) = \begin{cases} 1, & \text{if } x \geq k_i \\ 0, & \text{otherwise.} \end{cases}$$

k_i is dummy since $u_{k_i}(x, k_i) = 1 = u_{k_i}(x, 0_i) + u_{k_i}(k_i)$. Hence

$$\Phi_{u_{k_i}}(k_i) = u_{k_i}(k_i) = 1 = \sum_{x \in L_{-i}} p_x^{k_i}.1$$

which proves the result. ■

Proposition 3 *Under axioms (L),(N) and (M),* $\forall v \in \mathcal{G}(L)$*, for all join-irreducible* k_i*,*

$$\Phi^v(k_i) = \sum_{x \in L_{-i}} p_x^{k_i} \left[v(x, k_i) - v(x, 0_i)\right], \quad \textit{with } p_x^{k_i} \geq 0.$$

Proof: Let choose some $y \in L$ and define by analogy with classical games

$$\hat{u}_y(x) = \begin{cases} 1, & \text{if } x \geq y, x \neq y \\ 0, & \text{else.} \end{cases}$$

By definition, $\hat{u}_y$ is monotone. Letting $y = (x_0, 0_i)$ for some $x_0 \in L_{-i}$, and applying Prop. 1, we get:

$$\begin{aligned} \phi_{\hat{u}_{(x_0,0_i}}(k_i) &= \sum_{x \in L_{-i}} p_x^{k_i} [\hat{u}_{(x_0,0_i}(x, k_i) - \hat{u}_{(x_0,0_i}(x, 0_i)] \\ &= p_{x_0}^{k_i} \geq 0. \end{aligned}$$

■

As a consequence, one can deduce from Propositions 2 and 3 that under axioms (L),(D) and (M), for every join-irreducible k_i, $(p_x^{k_i})_{x \in L_{-i}}$ will be a probability distribution.

The next axiom enables an easier computation of coefficients $p_x^{k_i}$ while reducing their number:

Invariance axiom (I): Let us consider two games v_1, v_2 of $\mathcal{G}(L)$ such that for some i in N,

$$v_1(x, x_i) = v_2(x, x_i - 1),\ \forall x \in L_{-i}, \forall x_i > 1$$
$$v_1(x, 0_i) = v_2(x, 0_i),\ \forall x \in L_{-i}.$$

Then $\Phi^{v_1}(k_i) = \Phi^{v_2}((k-1)_i),\ 1 < k \leq l_i$.

The axiom says that when a game v_2 is merely a shift of another game v_1 concerning player i, the Shapley values are the same for this player. This implies that the way of computing v does not depend on the level k, as shown in the next proposition.

Proposition 4 *Under axioms (L),(N) and (I), $\forall v \in \mathcal{G}(L)$, for all join-irreducible k_i,*

$$\Phi^v(k_i) = \sum_{x \in L_{-i}} p_x^i \left[v(x, k_i) - v(x, 0_i)\right], \quad \textit{with } p_x^i \in \mathbb{R}.$$

Proof: We have for $k > 1$

$$\begin{aligned}
\Phi_{v_1}(k_i) &= \sum_{x \in L_{-i}} p_x^{k_i} \Big[v_1(x, k_i) - v_1(x, 0_i)\Big] \\
&= \sum_{x \in L_{-i}} p_x^{k_i} \Big[v_2(x, (k-1)_i) - v_2(x, 0_i)\Big] \\
\Phi_{v_2}((k-1)_i) &= \sum_{x \in L_{-i}} p_x^{(k-1)_i} \Big[v_2(x, (k-1)_i) - v_2(x, 0_i)\Big],
\end{aligned}$$

which proves the result. ■

Let us now introduce a symmetry axiom, which is an adaptation of the classical symmetry axiom. The difficulty here is that since the L_i's could be different, applying directly the classical symmetry axiom may lead to meaningless expressions. In this purpose, we introduce a subspace of $\mathcal{G}(L)$:

$$\mathcal{G}_0(L) := \{v \in \mathcal{G}(L) \mid v(x) = 0, \forall x \notin \Gamma(L)\},$$

where we recall that $\Gamma(L) = \{0_1, \top_1\} \times \{0_2, \top_2\} \times \cdots \times \{0_n, \top_n\}$ is the set of vertices of L. For any x in $\Gamma(L)$ and any permutation σ on N, we define $x^\sigma := (x_1^\sigma, \ldots, x_n^\sigma)$ by

$$x_i^\sigma := \begin{cases} 0_i, & \text{if } x_{\sigma(i)} = 0_{\sigma(i)}, \\ \top_i, & \text{if } x_{\sigma(i)} = \top_{\sigma(i)}. \end{cases}$$

Besides, for any $v \in \mathcal{G}_0(L)$, we denote by v^σ the game in $\mathcal{G}_0(L)$ such that $v^\sigma(x) := v(x^\sigma)$, for any x in $\Gamma(L)$. When all l_i's are different, observe that x^σ is a vertice of $\Gamma(L)$, contrary to $\sigma(x) := (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, as well as v^σ is a game in $\mathcal{G}_0(L)$ while $v \circ \sigma$ is not. Let us take for example $L := \{0,1,2\} \times \{0,1,2,3,4\} \times \{0,1,2,3\}$, and

i	1	2	3
$\sigma(i)$	2	3	1

Then $(2,0,0)^\sigma = (0,0,3)$, $(2,0,3)^\sigma = (0,4,3)$.

Symmetry axiom (S): Let σ be a permutation on N. Then for any game v in $\mathcal{G}_0(L)$, and any i in N,

$$\Phi^{v^{\sigma^{-1}}}(\top_i^\sigma) = \Phi^v(\top_i).$$

Thus, as for classical games, this axiom says that the computation of Shapley value should not depend on the labelling of the players. Finally, we give the last axiom:

Efficiency axiom (E): $\forall v \in \mathcal{G}(L)$, $\sum_{i \in N} \Phi^v(\top_i) = v(\top)$.

Theorem 5 *Under axioms (L),(D),(M),(I),(S) and (E), $\forall v \in \mathcal{G}(L)$, for all join-irreducible k_i,*

$$\Phi^v(k_i) = \sum_{x \in \Gamma(L_{-i})} \frac{(n - h(x) - 1)!\, h(x)!}{n!} [v(x, k_i) - v(x, 0_i)],$$

where $h(x) := |\{j \in N \setminus i \mid x_j = \top_j\}|$.

Proof: Let v be a game in $\mathcal{G}_0(L)$ and let σ be a transposition of N, that is to say a permutation which only exchanges two players i and j. This implies $\sigma = \sigma^{-1}$. Then by (S) we have $\Phi^v(\top_i) = \Phi^{v^\sigma}(\top_j)$, which writes, using axioms (L), (D), (M) and (I), and Prop. 3 and 4:

$$\sum_{x\in\Gamma(L_{-i})} p^i_x\,[v(x,\top_i) - v(x,0_i)] = \sum_{x\in\Gamma(L_{-j})} p^j_x\,[v((x,\top_j)^\sigma) - v((x,0_j)^\sigma)],$$

which can be rewritten as

$$\sum_{x\in\Gamma(L_{-i,j})}\sum_{x_j\in\{0,\top_j\}} p^i_{x,x_j}\,[v(x,\top_i,x_j) - v(x,0_i,x_j)] =$$
$$\sum_{x\in\Gamma(L_{-i,j})}\sum_{x_i\in\{0,\top_i\}} p^j_{x,x_i}\,[v((x,x_i,\top_j)^\sigma) - v((x,x_i,0_j)^\sigma)].$$

If $x \in \Gamma(L_{-i,j})$, then $(x, x_i, \top_j)^\sigma = (x, \top_i, x'_j)$, and $(x, x_i, 0_j)^\sigma = (x, 0_i, x'_j)$, where x'_j is of the same nature than x_i, (i.e., $x'_j = 0$ iff $x_i = 0$, and $x'_j = \top_j$ iff $x_i = \top_i$). Consequently, as the above equalities are true for any $v \in \mathcal{G}_0(L)$, we can identify the term of the first member coefficient of which is p^i_{x,x_j} with the term of the second member coefficient of which is p^j_{x,x_i} such that x_i and x_j are of the same nature. This gives equality between these coefficients.

By taking into account all transpositions of N, for any x in $\Gamma(L_{-i,j,l})$, we write

$$\begin{aligned} p^i_{x,x_j,x_l} &= p^j_{x,x'_i,x_l} \text{ where } x'_i \text{ of the same nature than } x_j,\\ &= p^l_{x,x'_i,x'_j} \text{ where } x'_j \text{ of the same nature than } x_l;\\ \text{besides, } p^i_{x,x_j,x_l} &= p^l_{x,x''_i,x_j} \text{ where } x''_i \text{ of the same nature than } x_l \text{ and thus of } x'_j. \end{aligned}$$

As a result, for all $l \neq i, j$ and for all $x \in \Gamma(L_{-i,j,l})$, $p^l_{x,x'_i,x'_j} = p^l_{x,x_i,x_j}$ whenever x_i and x'_j have the same nature, as x'_i and x_j. Consequently, for the computation of p^l_x, $x \in L_{-l}$, any permutation being a composition of transpositions, indices of components "0" and "$\top$" of x have no importance as long as the cardinality $h(x) = |\{i \in N \setminus l \mid x_i = \top_i\}|$ is the same. Therefore, we will use a new notation for p^l_x:

$$p^l_m := p^l_x, \text{ where } m = h(x).$$

Moreover, it is clear that for all $i, j \in N$, for all $m \in \{0, \ldots, n-1\}$, $p^i_m = p^j_m$, due to the effect of the transposition $i \leftrightarrow j$. It follows that one can write p_m instead of p^i_m, $i \in N$.

Now, by efficiency axiom, we have $\sum_{i\in N}\sum_{x\in L_{-i}} p^i_x\,[v(x,\top_i)-v(x,0_i)] = v(\top)$. Assuming v is a game in $\mathcal{G}_0(L)$, this gives the following equation:

$$\sum_{i\in N}\sum_{m=0}^{n-1}\sum_{\substack{x\in L_{-i},\\ h(x)=m}} p_m\,[v(x,\top_i)-v(x,0_i)] = v(\top_1,\dots,\top_n). \tag{5}$$

Le us denote $\mathcal{G}(2^N)$ the set of classical games on N and $v\mapsto\tilde v$ the canonical isomorphism from $\mathcal{G}_0(L)$ to $\mathcal{G}(2^N)$, i.e., for all $S\in 2^N$

$$\tilde v(S) := v(s),\quad \text{with } s_i = \begin{cases}\top_i, & \text{if } i\in S\\ 0_i, & \text{else}\end{cases},\forall i\in N.$$

Observe that, through this mapping, Eq. (5) becomes

$$\sum_{i\in N}\sum_{m=0}^{n-1}\sum_{\substack{S\subseteq N\setminus i,\\ |S|=m}} p_m\,[\tilde v(S\cup i)-\tilde v(S)] = \tilde v(N). \tag{6}$$

We recognize here the classical efficiency axiom, from which we deduce that coefficients p_m's are nothing else that the well-known Shapley coefficients $p_m = \alpha^1_m(n) := \frac{(n-m-1)!\,m!}{n!}$ for all $m\in\{0\dots,n-1\}$.

As a consequence, through inverse of the above isomorphism, we easily obtain the expression of the previous p^i_x when $x\in\Gamma(L_{-i})$:

$$p^i_x = \frac{(n-h(x)-1)!\,h(x)!}{n!}.$$

Finally, as $(p^i_x)_{x\in L_{-i}}$ is a probability distribution, and since we know that $\sum_{S\subseteq N\setminus i}\alpha^1_{|S|} = \sum_{x\in\Gamma(L_{-i})} p^i_x = 1$, it follows that $p^i_x = 0$ for all $x\in L_{-i}\setminus\Gamma(L_{-i})$. ■

Remark. It is possible to give a rather different formulation suggested by the proof of Th. 5 by introducing the following axioms:

Symmetry axiom for classical games (CS): Let σ be a permutation on N. Then for any game ν in $\mathcal{G}(2^N)$, and any i in N, $\Phi^{\nu\circ\sigma^{-1}}(\sigma(i)) = \Phi^{\nu}(i)$.

Full participation axiom (FP): For any game v in $\mathcal{G}_0(L)$, and any i in N, $\Phi^v(\top_i) = \Phi^{\tilde{v}}(i)$.

Consequently, axiom (S) being equivalent to the pair ((CS), (FP)) under axioms (L),(D),(M),(I),(E), the required theorem can also be proven with these axioms and (CS),(FP) instead of (S).

6 Towards the general case

In this section, we present first ideas to define a Shapley value for the general case, where the L_i's are finite distributive, as a basis for future research. Our aim is to obtain $\Phi^v(x_i)$, for any $x_i \in L_i$, $x_i \neq \perp_i$, which should represent the contribution of doing action x_i instead of nothing for player i. We denote as usual the top and bottom elements of each lattice L_i by $\top_i, \perp_i$.

A first approach is to adapt the previous axiomatization for multichoice games to the general case. This can be done under the restriction that in each L_i, the bottom element $\perp_i$ has a unique successor, denoted by 1_i (in other words, 1_i is the unique atom of L_i). Also, for any $x_i \in L_i$, $x_i \neq \perp_i$, $\underline{x}_i := \bigwedge\{y_i \in L_i \mid y_i \prec x_i\}$, i.e., $\underline{x}_i$ is the infimum of all predecessors of x_i. The following axioms and definitions are direct generalizations of the previous ones:

- For some $x_i \in L_i \setminus \perp_i$, player i is x_i-*null* (or simply x_i is *null*) for $v \in \mathcal{G}(L)$ if $v(x, x_i) = v(x, \perp_i)$, $\forall x \in L_{-i}$.

- For some $x_i \in L_i \setminus \perp_i$, player i is x_i-*dummy* (or simply x_i is *dummy*) for $v \in \mathcal{G}(L)$ if $v(x, x_i) = v(x, \perp_i) + v(\perp_{-i}, x_i)$, $\forall x \in L_{-i}$.

- **Null axiom (N):** $\forall v \in \mathcal{G}(L)$, for all null x_i, $\phi^v(x_i) = 0$.

- **Dummy axiom (D):** $\forall v \in \mathcal{G}(L)$, for all dummy x_i, $\phi^v(x_i) = v(\perp_{-i}, x_i)$.

- **Monotonicity axiom (M):** $\forall v \in \mathcal{G}(L)$, if v is monotone, then $\Phi^v(x_i) \geq 0$, for every player i, $x_i \neq \perp_i$.

- **Linearity axiom (L):** For all $x_i \in L_i$, $x_i \neq \perp_i$, $\Phi^v(x_i)$ is linear on $\mathcal{G}(L)$.

- **Invariance axiom (I):** Let us consider two games $v_1, v_2 \in \mathcal{G}(L)$ such that for some $i \in N$,

$$v_1(y, x_i) = v_2(y, \underline{x}_i), \forall y \in L_{-i}, \forall x_i > 1_i$$
$$v_1(y, \perp_i) = v_2(y, \perp_i), \forall y \in L_{-i}.$$

  Then $\Phi^{v_1}(x_i) = \Phi^{v_2}(\underline{x}_i)$, $x_i > 1_i$.

- **Symmetry axiom (S):** Let σ be a permutation on N. Then for any game $v \in \mathcal{G}_0(L)$ and any $i \in N$,

$$\Phi^{v^{\sigma^{-1}}}(\top_i^{\sigma}) = \Phi^v(\top_i),$$

  with same notations as in previous section.

- **Efficiency axiom (E):** $\forall v \in \mathcal{G}(L)$, $\sum_{i \in N} \Phi^v(\top_i) = v(\top)$.

Using the same schemata of proofs as for multichoice games, we come up with the following result:

Theorem 6 *Under axioms (L), (D), (M), (I), (S) and (E), for all $v \in \mathcal{G}(L)$, for all $x_i \in L_i$, $x_i \neq \perp_i$,*

$$\Phi^v(x_i) = \sum_{y \in \Gamma(L_{-i})} \frac{(n - h(y) - 1)!\, h(y)!}{n!} [v(y, x_i) - v(y, \perp_i)],$$

where $h(y) := |\{j \in N \setminus i \mid x_j = \top_j\}|$.

Although the result is appealing by its simplicity, it suffers from the restriction imposed on the L_i's, and by the fact the axiom (I) becomes questionable. Also, the role of join-irreducible elements as a basic element of the construction has disappeared, which is not in accordance with our interpretation of games on lattice, as given in Section 3.

Based on preceding remarks, we suggest an alternative approach, which goes in several steps, and starts from join-irreducible elements.

1. For any join-irreducible element $x_i \in L_i$, we compute the *differential Shapley value* $\phi^v(x_i)$, expressing the contribution of doing action x_i instead of the predecessor of action x_i for player i. Since the predecessor of x_i is unique iff x_i is a join-irreducible element, this makes sense.

2. We compute $\phi^v(x_i)$ for any $x_i \in L_i$, considering its unique irredundant decomposition into join-irreducible elements (see Sec. 2). This unique decomposition always exists since L is distributive.

3. We compute $\Phi^v(x_i)$ by cumulating the differential Shapley values between x_i and $\perp_i$.

To bring this approach to an operational state, first an axiomatization is needed for defining the differential Shapley value for join-irreducible elements. The second problem is how to use the irredundant decomposition of x_i to compute $\phi^v(x_i)$. We suggest the following:

$$\phi^v(x_i) = \sum_{j_i \in \eta(x_i)} \phi^v(j_i) + I^v(\eta(x_i)),$$

where $I^v(S)$ is the *interaction* among elements of $S \subseteq L_i$. The interaction represents the effect of joining elements. For example, for two join-irreducible elements j_i, k_i:

- $I^v(\{j_i, k_i\}) = 0$ if the worth of $j_i \vee k_i$ is the sum of the worths of j_i and k_i
- $I^v(\{j_i, k_i\}) > 0$ (resp. < 0) if the worth of $j_i \vee k_i$ is greater (resp. smaller) than the sum of the worths of j_i and k_i.

The first appearance of the notion of interaction for classical games is due to Owen [16] under the name "co-value". It was rediscovered in a different context by Murofushi and Soneda [15], and generalized by Grabisch [9]. An axiomatization of interaction has been done by Grabisch and Roubens [13], and a general definition for games on lattices has been recently given by Grabisch and Labreuche [12].

We leave the complete setting of this approach for future research.

References

[1] J.M. Bilbao. *Cooperative Games on Combinatorial Structures.* Kluwer Academic Publishers, 2000.

[2] J.M. Bilbao, J.R. Fernandez, A. Jiménez Losada, and E. Lebrón. Bicooperative games. In J.M. Bilbao, editor, *Cooperative games on combinatorial structures*. Kluwer Acad. Publ., 2000.

[3] R. Branzei, D. Dimitrov, and S. Tijs. *Models in cooperative game theory: crisp, fuzzy and multichoice games*. Springer Verlag, 2005.

[4] D. Butnariu and E.P. Klement. *Triangular norm-based measures and games with fuzzy coalitions*. Kluwer Academic Publishers, 1993.

[5] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Orders*. Cambridge University Press, 1990.

[6] U. Faigle and W. Kern. The Shapley value for cooperative games under precedence constraints. *International Journal of Game Theory*, 21:249–266, 1992.

[7] D. Felsenthal and M. Machover. Ternary voting games. *Int. J. of Game Theory*, 26:335–351, 1997.

[8] I. Gilboa and E. Lehrer. Global games. *Int. J. of Game Theory*, 20:129–147, 1991.

[9] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[10] M. Grabisch. An axiomatization of the Shapley value and interaction index for games on lattices. In *SCIS-ISIS 2004, 2nd Int. Conf. on Soft Computing and Intelligent Systems and 5th Int. Symp. on Advanced Intelligent Systems*, Yokohama, Japan, September 2004.

[11] M. Grabisch. The Shapley value for games on lattices (l-fuzzy games). In *Int. Conf. on Fuzzy Sets and Soft Computing in Economics and Finance*, St. Petersburg, Russia, June 2004.

[12] M. Grabisch and Ch. Labreuche. Derivative of functions over lattices as a basis for the notion of interaction between attributes. *Annals of Mathematics and Artificial Intelligence*, 49:151–170, 2007.

[13] M. Grabisch and M. Roubens. An axiomatic approach to the concept of interaction among players in cooperative games. *Int. Journal of Game Theory*, 28:547–565, 1999.

[14] C.R. Hsiao and T.E.S. Raghavan. Shapley value for multichoice cooperative games, I. *Games and Economic Behavior*, 5:240–256, 1993.

[15] T. Murofushi and S. Soneda. Techniques for reading fuzzy measures (III): interaction index. In *9th Fuzzy System Symposium*, pages 693–696, Sapporo, Japan, May 1993. In Japanese.

[16] G. Owen. Multilinear extensions of games. *Management Sci.*, 1972.

[17] G.C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 2:340–368, 1964.

[18] S. Tijs, R. Branzei, S. Ishihara, and S. Muto. On cores and stable sets for fuzzy games. *Fuzzy Sets and Systems*, 146:285–296, 2004.

[19] R.J. Weber. Probabilistic values for games. In A.E. Roth, editor, *The Shapley Value. Essays in Honor of Lloyd S. Shapley*, pages 101–119. Cambridge University Press, 1988.

Chapitre 2.
Un concept de solution récursif pour jeux multi-choix

Résumé

Nous proposons une nouvelle axiomatisation de la valeur de Shapley, où les axiomes de symétrie de d'efficacité sont écartés et remplacés par de nouveaux axiomes naturels. On construit à partir de tout jeu et d'un joueur fixé, un jeu à joueur exclu par le rejet dans son domaine de définition de toutes les coalitions contenant ce joueur. On montre alors que la valeur de Shapley est l'unique solution satisfaisant la linéarité, la nullité, l'axiome du joueur nul exclu, et l'équité. Dans la seconde partie, le matériel ci-dessus est généralisé afin d'axiomatiser la valeur de Shapley pour les jeux multi-choix.

Mots clés : valeur de Shapley, jeu multi-choix, équité, axiome de nullité généralisée

A recursive solution concept for multichoice games

Fabien Lange
Michel Grabisch

Centre d'Economie de la Sorbonne, UMR 8174, CNRS-Université Paris 1

Abstract

We propose a new axiomatization of the Shapley value for cooperative games, where symmetry and efficiency can be discarded and replaced with new natural axioms. From any game, an excluded-player game is built by discarding all coalitions that contain a fixed player. Then it is shown that the Shapley value is the unique value satisfying the linearity axiom, the nullity axiom, the excluded-null-player axiom, and the equity axiom. In the second part, by generalizing the above material, the Shapley value for multichoice games is worked out.

Keywords: Shapley value, multichoice game, equity, generalized nullity axiom

Submitted to *International Journal of Game Theory.*

Short version selected from proceedings of SISY 2007 to the special issue of *Acta Polytechnica Hungarica.*

1 Introduction

The value or solution concept of a game is a key concept in cooperative game theory, since it defines a rational imputation given to the players if they join the grand coalition. In this respect, the Shapley value remains the best known solution concept [11], and it has been axiomatized by many authors in various ways (see especially Weber [13], or the survey by Monderer and Samet [8]).

If the definition and axiomatization of the Shapley value is well established for classical cooperative TU-games, the situation is less clear when considering variants of classical TU-games, like multichoice games [7], games in partition function form [12], etc. In this paper, we focus on multichoice games, where players are allowed to have several (and totally ordered) levels of participation. Hence, a solution for multichoice games assigns a numerical value to each possible participation level and to each player. The original proposal of Hsiao and Raghavan [7] for the Shapley value has been, up to our knowledge, scarcely used due to its complexity. Another proposal is due to Faigle and Kern [5], and compared to the former one by Branzei et al. [3], and also by the authors [6]. The value proposed by Faigle and Kern, although elegant but still with a very high computational complexity, is more rooted in combinatorics than in game theory, and takes as a basis the expression of the Shapley value using maximal chains in the lattice of coalitions. In [6], we proposed an alternative view having the same complexity than the usual Shapley value for classical TU-games. It turned out that our value is identical to the egalitarian value proposed by Peters and Zank [10], although they use different axioms and impose some restrictions (namely, all players should have the same set of participation levels).

Although close to the axiomatization proposed by Weber for classical TU-games, our axiomatization in [6] suffered from a complex symmetry axiom, hard to interpret, the fundamental problem there being that the classical notion of symmetry among players cannot hold since two different players may have a different set of participation levels (note that this difficulty was avoided by Peters and Zank, since they considered multichoice games with all players having the same set of participation levels).

In this paper, we propose a new axiomatization for the so-called egalitarian value, which is based essentially on carriers and on a recursive scheme, and which does not make use of a symmetry axiom. In Section 3, we present the main ideas applied on classical TU-games, and we come up with a very simple and natural axiomatization using linearity, a nullity axiom which uses also carriers, and an equity axiom stating that the sharing should be uniform and efficient for the unanimity game based on the grand coalition (this is in fact a very weak version of the efficiency axiom). In Section 4, the same process is applied to multichoice games. An additional axiom (called decreased level axiom) is used, to take into account the case where a player does not participate at the highest level.

In the sequel, $\mathbb{N}$ refers to the set of positive integers. In order to avoid a heavy notation, we will often omit braces for subsets, by writing i instead of $\{i\}$ or 123 for $\{1, 2, 3\}$. Furthermore, cardinalities of subsets $S, T, \dots$ will be denoted by the corresponding lower case letters $s, t, \dots$

2 Mathematical background

We begin by recalling necessary material on lattices (a good introduction on lattices can be found in [4]), in a finite setting. A *lattice* is a set L endowed with a partial order $\leq$ such that for any $x, y \in L$ their least upper bound $x \vee y$ and greatest lower bound $x \wedge y$ always exist. For finite lattices, the greatest element of L (denoted $\top$) and least element $\bot$ always exist. x *covers* y (denoted $x \succ y$) if $x > y$ and there is no z such that $x > z > y$. A *ranked lattice* is a pair (L, r), where L is a lattice and the rank function $r : L \to \mathbb{N}$ satisfies the property that $r(y) = r(x) + 1$ whenever y covers x in L. The lattice is *distributive* if $\vee, \wedge$ obey distributivity. An element $j \in L$ is *join-irreducible* if it cannot be expressed as a supremum of other elements. Equivalently j is join-irreducible if it covers only one element. The set of all join-irreducible elements of L is denoted $\mathcal{J}(L)$.

An important property is that in a distributive lattice, any element x can be written as an irredundant supremum of join-irreducible elements in a unique way (this is called the *minimal decomposition* of x). We denote by $\eta^*(x)$ the set of join-irreducible elements in the minimal decomposition of x, and we denote by $\eta(x)$ the *normal decomposition* of x, defined as the set of join-irreducible elements smaller or equal to x, i.e., $\eta(x) := \{j \in \mathcal{J}(L) \mid j \leq x\}$. Let us rephrase differently the above result. We say that $Q \subseteq L$ is a *downset* of L if $x \in Q$ and $y \leq x$ imply $y \in Q$. For any subset P of L, we denote by $\mathcal{O}(P)$ the set of all downsets of P. Then, by Birkhoff's theorem [2], the mapping η is an isomorphism of L onto

$\mathcal{O}(\mathcal{J}(L))$.

Given lattices $(L_1, \leq_1), \ldots, (L_n, \leq_n)$, the product lattice $L = L_1 \times \cdots \times L_n$ is endowed with the product order $\leq$ of $\leq_1, \ldots, \leq_n$ in the usual sense. Elements of x can be written in their vector form $(x_1, \ldots, x_n)$. We use the notation (x_S, y_{-S}) to indicate a vector z such that $z_i = x_i$ if $i \in S$, and $z_i = y_i$ otherwise. Similarly L_{-i} denotes $\prod_{k \neq i} L_k$ if $N \setminus i \neq \emptyset$, and the singleton set $\{()\}$ otherwise. By this way, for any vector x, $\big((), x\big)$ simply denotes x. All join-irreducible elements of L are of the form $(\perp_1, \ldots, \perp_{i-1}, j_i, \perp_{i+1}, \ldots, \perp_n)$, for some i and some join-irreducible element j_i of L_i. A *vertex* of L is any element whose components are either top or bottom. We denote $\Gamma(L)$ the set of vertices of L.

3 A new axiomatization of the Shapley value for classical cooperative games

In the whole paper, we consider an infinite denumerable set Ω, the universe of players. As usual, a *game* on Ω is a set function $v : \Omega \to \mathbb{R}$ such that $v(\emptyset) = 0$, which assigns to each *coalition* $S \subseteq \Omega$ its *worth* $v(S)$. We denote by 2^Ω (power set of Ω) the set of coalitions. In this section, we focus on the particular case of *classical cooperative games*, that is to say, each player has the only choice to cooperate or not.

A set $N \subseteq \Omega$ is said to be a *carrier* of a game v when for all $S \subseteq \Omega$, $v(S) = v(N \cap S)$. Thus a game v with carrier $N \subseteq \Omega$ is completely defined by the knowledge of the coefficients $\{v(S)\}_{S \subseteq N}$ and the players outside N have no influence on the game since they do not contribute to any coalition. In this paper, we restrict our attention to finite games, that is to say, games that posses a finite carrier N with n elements. We denote by $\mathcal{G}(N)$ the set of games with the finite carrier N. For the sake of clarity, and to avoid any ambiguity, the domain of $v \in \mathcal{G}(N)$ will be restricted to the elements of 2^N. $\mathcal{G}$ denotes the set of all finite games:

$$\mathcal{G} := \{\mathcal{G}(N) \mid N \subseteq \Omega, n \in \mathbb{N}\}.$$

Identity games of $\mathcal{G}(N)$ are particular games defined by

$$\forall S \subseteq N \setminus \{\emptyset\}, \quad \delta_S(T) := \begin{cases} 1 \text{ if } T = S, \\ 0 \text{ otherwise.} \end{cases}$$

A *value* on $\mathcal{G}(N)$ is a function $\Phi : \mathcal{G}(N) \times N \to \mathbb{R}$ that assigns to every player i in a game $v \in \mathcal{G}(N)$ his prospect $\Phi(v, i)$ for playing the game. For instance, the Shapley value [11] for cooperative games Φ_{Sh} is defined by

$$\forall v \in \mathcal{G}(N), \forall i \in N, \quad \Phi_{Sh}(v, i) := \sum_{S \subseteq N \setminus i} \frac{s!\,(n-s-1)!}{n!}\,(v(S \cup i) - v(S)). \quad (1)$$

The axiomatization is well-known. Φ_{Sh} is the sole value given on $\mathcal{G}(N)$ satisfying (see also Weber [13]):

Linearity (L): for any $i \in N$, $\Phi(v, i)$ is linear w.r.t the variable v.

Player $i \in N$ is said to be *null* for v if $\forall S \subseteq N \setminus i$, $v(S \cup i) = v(S)$.

Nullity (N): for any game $v \in \mathcal{G}(N)$ and any $i \in N$ null for v, $\Phi(v, i) = 0$.

For any permutation σ on N, we denote $v \circ \sigma$ the game defined by $v \circ \sigma(S) := v(\sigma(S)), \ \forall S \in 2^N$.

Symmetry (S): for any permutation σ on N, any game $v \in \mathcal{G}(N)$ and any $i \in N$, $\Phi(v, \sigma(i)) = \Phi(v \circ \sigma, i)$.

This means that Φ must not depend on the labelling of the players.

Efficiency (E): for any game $v \in \mathcal{G}(N)$, $\sum_{i \in N} \Phi(v, i) = v(N)$.

That is to say, the values of the players must be shared in proportion of the overall resources $v(N)$.

We now introduce a new axiomatization of the Shapley value for classical cooperative games. For any game $v \in \mathcal{G}(N)$ and any coalition $S \in 2^N$, we denote by $v^S \in \mathcal{G}(S)$ the *restricted game v to the power set of S*. For $i \in N$, v^{-i} denotes the restricted game $v^{N \setminus i}$. Let us consider the following axioms for values on $\mathcal{G}$.

Excluded-null-player (ENP): for any finite set $N \subseteq \Omega$ and any game $v \in \mathcal{G}(N)$, if $i \in N$ is null for v,

$$\forall j \in N \setminus i, \quad \Phi(v, j) = \Phi(v^{-i}, j).$$

This simply means that if a null player leaves the game, then other players should keep the same value in the associated restricted game. Note that this axiom completes in a certain sense the above axiom **(N)** since the former deals with null players whereas the latter addresses the others. Therefore, one can merge **(N)** and **(ENP)**:

Generalized nullity (GN): for any finite set $N \subseteq \Omega$ and any game $v \in \mathcal{G}(N)$, if $i \in N$ is null for v,

$$\begin{cases} \Phi(v, i) = 0, \\ \Phi(v, j) = \Phi(v^{-i}, j), \text{ for any player } j \in N \setminus i. \end{cases}$$

We define the particular *unanimity game* of $\mathcal{G}(N)$ by $u_N(S) := \begin{cases} 1, \text{ if } S = N, \\ 0, \text{ otherwise.} \end{cases}$

Equity (Eq): for any finite set $N \subseteq \Omega$, for any player $i \in N$,

$$\Phi(u_N, i) = \frac{1}{n}.$$

This natural axiom simply states that in the particular game where the grand coalition is the unique to produce a unitary worth (all others giving nothing), all players should share the same fraction of this unit.

Theorem 1 *Φ_{Sh} is the sole value on $\mathcal{G}$ satisfying axioms* ***(L)**, **(GN)** and **(Eq)**.*

Note that since the result is given over $\mathcal{G}$, axioms **(L)** and **(N)** should be adjusted in accordance with the arbitrariness of the choice of N. Actually, it is sufficient to specify for these axioms "for any finite set $N \subseteq \Omega$, for any game $v \in \mathcal{G}(N)$".

Proof: First, let us check that Φ_{Sh} satisfies the axioms, which is already known for **(L)** and **(N)**, and obvious for **(Eq)**. It remains to check the validity of

(ENP). We denote by $\alpha_s^1(n) := \dfrac{s!\,(n-s-1)!}{n!}$ the Shapley's coefficient in (1). Let $v \in \mathcal{G}(N)$ and $i \in N$ a player null for v. Then for $j \in N \setminus i$,

$$\begin{aligned}\Phi_{Sh}(v,j) &= \sum_{S \subseteq N \setminus j} \alpha_s^1(n)\,(v(S \cup j) - v(S)) \\ &= \sum_{S \subseteq N \setminus ij} [\alpha_s^1(n) + \alpha_{s+1}^1(n)]\,(v(S \cup j) - v(S)),\end{aligned}$$

since i is null for v. Now

$$\begin{aligned}\alpha_s^1(n) + \alpha_{s+1}^1(n) &= \frac{s!\,(n-s-1)!}{n!} + \frac{(s+1)!\,(n-s-2)!}{n!} \\ &= \frac{s!\,(n-s-2)!}{n!}\,[s+1+n-s-1] \\ &= \frac{s!\,(n-s-2)!}{(n-1)!} = \alpha_s^1(n-1).\end{aligned} \tag{2}$$

Thus

$$\begin{aligned}\Phi_{Sh}(v,j) &= \sum_{S \subseteq (N \setminus i) \setminus j} \alpha_s^1(n-1)\,(v(S \cup j) - v(S)) \\ &= \Phi_{Sh}(v^{-i}, j).\end{aligned}$$

Conversely, let us remind an intermediary result from Weber [13], that gives the general form of values Φ on $\mathcal{G}(N) \times N$, under the nullity axiom and the linearity axiom:

$$\forall v \in \mathcal{G}(N), \forall i \in N, \quad \Phi(v,i) = \sum_{S \subseteq N \setminus i} p_S^i(n)\,(v(S \cup i) - v(S)), \tag{3}$$

where the $p_S^i(n)$'s are some real numbers.

We now show the result by mathematical induction on the cardinality of N. For $N := \{1\}$, (3) gives $\Phi(u_N, 1) = p_\emptyset^1(1)\,u_N(1) = p_\emptyset^1(1)$. Besides, from **(Eq)**, $\Phi(u_N, 1) = 1$. Thus $p_\emptyset^1(1) = 1$ and (1) is satisfied for $n = 1$.

Let us assume that (1) is true for games of $\mathcal{G}(N)$, n fixed in $\mathbb{N}$. We now consider any game $v \in \mathcal{G}(N')$ with $|N'| = n+1$. We suppose that player $i \in N'$ is null for v. Then, for any player $j \in N' \setminus i$, by (3), there are some real coefficients $p_S^j := p_S^j(n+1)$, $S \subseteq N' \setminus j$, such that

$$\Phi(v,j) = \sum_{S \subseteq N' \setminus ij} [p_S^j + p_{S \cup i}^j]\,(v(S \cup j) - v(S)),$$

since i is null for v. Moreover, by **(ENP)**, we also have

$$\Phi(v,j) = \Phi(v^{-i},j) = \sum_{S \subseteq N' \setminus ij} \alpha^1_s(n)\,(v(S \cup j) - v(S)).$$

Since these formula are true for any game $v \in \mathcal{G}(N')$ with player i null for v, they are in particular true for the game $\delta_{S \cup j} + \delta_{S \cup ij}$, $S \subseteq N' \setminus ij$. By identification of the two above formulae, we can deduce that for all $j \in N'$, for all $S \subseteq N' \setminus ij$,

$$p^j_S + p^j_{S \cup i} = \alpha^1_s(n). \tag{4}$$

We now recursively compute the coefficients p^j_S's. Precisely, we show that

$$\forall j \in N', \forall S \subsetneq N' \setminus j, \quad p^j_S = \alpha^1_s(n+1). \tag{5}$$

Let $j \in N$. Considering the unanimity game $u_{N'}$, we have

$$\Phi(u_{N'},j) = p^j_{N' \setminus j} = \frac{1}{n+1} = \alpha^1_n(n+1),$$

where the second equality is due to **(Eq)**. Thus (5) is shown for $S = N' \setminus j$, that is to say $s = n$. Assuming (5) is true $\forall S \subseteq N' \setminus j$ such that s is a fixed cardinality in $\{1, \dots, n\}$, then by (4), for any $i \in S$

$$\begin{aligned} p^j_{S \setminus i} &= \alpha^1_{s-1}(n) - \alpha^1_s(n+1) \\ &= \alpha^1_{s-1}(n+1), \end{aligned}$$

where the second equality comes from (2). Consequently, (5) is also satisfied for any $T \subseteq N' \setminus j$ such that $t = s-1$, thanks to correct choices of S and i. Finally, the result is proved for any subcoalition of $N' \setminus j$. ∎

An important remark is that this new axiomatization has the advantage of characterizing Φ_{Sh} for all games of $\mathcal{G}$, and not only for the games of $\mathcal{G}(N)$, where N is a fixed finite set. This is due to the recursive nature of the axiom **(ENP)**.

We present now another axiomatization of Φ_{Sh}, where the generalized nullity axiom is outlined in another way.

Definition 1 *Let $v \in \mathcal{G}(N)$ be any finite game. We call* support *of v, denoted by $\mathfrak{S}(v)$, the minimal carrier of v, that is,*

$$\mathfrak{S}(v) := \bigcap_{C \text{ is a carrier of } v} \{C \in 2^N\}.$$

Actually, a *carrier axiom* has been introduced for the first time by Myerson [9], saying that, if C is a carrier for the game v, then the worth $v(C)$ should be shared only among the members of the carrier. It is shown that this axiom is equivalent to the conjunction of the above axioms **(N)** and **(E)**. With regard to our work, we focus our attention on the support of the game and give an axiom for players in accordance with their membership of the support of the game. If there is no ambiguity, we denote by $v^{\mathfrak{S}}$ the restricted game $v^{\mathfrak{S}(v)}$.

> **Restricted-support games (RS)**: for any finite set $N \subseteq \Omega$, any game $v \in \mathcal{G}(N)$, and any player $i \in N$,
> $$\Phi(v, i) = \begin{cases} \Phi(v^{\mathfrak{S}}, i) \text{ if } i \in \mathfrak{S}(v), \\ 0 \text{ otherwise.} \end{cases}$$

Corollary 2 *Φ_{Sh} is the sole value on $\mathcal{G}$ satisfying axioms **(L)**, **(RS)** and **(Eq)**.*

To show this result, we propose an alternative characterization of the support of a game:

Lemma 3 *Let $v \in \mathcal{G}(N)$ be any game. Then $\mathfrak{S}(v)$ is the set of players which are not null for v.*

Proof: It suffices to show that a player $i \in N$ is not null for v iff she belongs to every carrier of v. Suppose that player $i \in N$ is null for v and let C be any carrier of v. Then for any $S \in 2^N$, $v(S \cap (C \setminus i)) = v((S \setminus i) \cap C) = v(S \setminus i) = v(S)$. Thus $C \setminus i$ is a carrier of v.
Conversely, if i is not null for v, then $\exists S' \in 2^N$ s.t. $v(S' \cup i) \neq v(S')$. Then for any carrier C, considering the subcoalition $S := S' \cup i$, if $i \notin C$ then $v(S \cap C) = v((S' \cap C) \cup (i \cap C)) = v(S')$ which should also equals $v(S) = v(S' \cup i)$. This contradicts $v(S' \cup i) \neq v(S')$. ■

4 The Shapley value of multichoice games

In previous section, the lattice representing actions of players was $L := \{0, 1\}^{\Omega}$, 0 (resp. 1) denoting absence (resp. presence) of a player. Now, for every player i

belonging to a finite carrier of players N, it is assumed that she may act at a level of participation $k \in L_i$ to the game. Actually, $L_i := \{0, 1, 2, \dots, \top_i\}$ is a linear lattice, where 0 means absence of participation and $\top_i$ represents the maximal participation to the game. Thus $L = L_1 \times \cdots \times L_n$ is the set of all possible joint actions of players of N. We denote by $\mathcal{L}(N)$ the set of all cartesian products of finite linear lattices over N, and by $\mathcal{L}$, the union of all these ones for every finite set N:

$$\mathcal{L}(N) := \{\prod_{i=1}^{n} L_i \mid \top_1, \dots, \top_n \in \mathbb{N}\}; \quad \mathcal{L} := \{\mathcal{L}(N) \mid N \subseteq \Omega, n \in \mathbb{N}\}.$$

Note that it shall be useful for the sequel to introduce the following binary relation over $\mathcal{L}$ defined for all $L \in \mathcal{L}(N), L' \in \mathcal{L}(N')$, by

$$L \; \mathcal{R} \; L' \text{ iff } \begin{cases} n = n', \\ (\top'_1, \dots, \top'_n) \text{ is a permutation of } (\top_1, \dots, \top_n). \end{cases}$$

This relation is obviously an equivalence relation. We denote by $\overline{\mathcal{L}}$ the quotient set $\mathcal{L}/_{\mathcal{R}}$.

Thus, it turns out that $\overline{\mathcal{L}}$ is isomorphic to the set of the partitions of positive integers, where a *partition* of a positive integer m is a finite nonincreasing sequence[1] of positive integers $(\lambda_1, \dots, \lambda_n)$ such that $\sum_{i=1}^{n} \lambda_i = m$ (see [1]). The λ_i's, corresponding to the maximal levels of participation of players, are called the *parts* of the associated partition. With a slight abuse of notation, we may assimilate $\overline{\mathcal{L}}$ to the set of partitions of positive integers. For any $\lambda := (\lambda_1, \dots, \lambda_n) \in \overline{\mathcal{L}}$, $|\lambda|$ is the sum of the λ_i's, i.e., the unique integer whose partition is given by λ. Also, let us endow $\overline{\mathcal{L}}$ with the following order. For all $\lambda := (\lambda_1, \dots, \lambda_n) \in \overline{\mathcal{L}}, \lambda' := (\lambda'_1, \dots, \lambda'_{n'}) \in \overline{\mathcal{L}}$,

$$\lambda' \leq \lambda \text{ iff } \begin{cases} n' \leq n, \\ \forall i \in \{1, \dots, n'\}, \lambda'_i \leq \lambda_i \end{cases} .$$

For instance, we have $(2, 1, 1) \leq (4, 3, 2, 1)$. Note that $\lambda := (1)$ is the bottom of $(\overline{\mathcal{L}}, \leq)$.

Proposition 4 *$(\overline{\mathcal{L}}, \leq)$ is a ranked lattice, whose rank function is given by $r(\lambda) = |\lambda|$, $\forall \lambda \in \overline{\mathcal{L}}$.*

[1]In the sequel, elements of $\overline{\mathcal{L}}$ are assumed to be given under this form.

Proof: We show that supremum and infimum of $(\overline{\mathcal{L}}, \leq)$ are respectively given by

$$(\lambda_1, \ldots, \lambda_n) \vee (\lambda'_1, \ldots, \lambda'_{n'}) = (\lambda_1^{(s)}, \ldots, \lambda_n^{(s)}) := (\lambda_1 \vee \lambda'_1, \ldots, \lambda_{n'} \vee \lambda'_{n'}, \lambda_{n'+1}, \ldots, \lambda_n),$$
$$(\lambda_1, \ldots, \lambda_n) \wedge (\lambda'_1, \ldots, \lambda'_{n'}) = (\lambda_1^{(i)}, \ldots, \lambda_{n'}^{(i)}) := (\lambda_1 \wedge \lambda'_1, \ldots, \lambda_{n'} \wedge \lambda'_{n'}),$$

where it is assumed without loss of generality that $n \geq n'$. Indeed, we easily check that $(\lambda_1^{(i)}, \ldots, \lambda_{n'}^{(i)}) \leq (\lambda_1, \ldots, \lambda_n)$, $(\lambda'_1, \ldots, \lambda'_{n'}) \leq (\lambda_1^{(s)}, \ldots, \lambda_n^{(s)})$. Besides, if $(\lambda_1^{(S)}, \ldots, \lambda_m^{(S)})$ is another partition greater than $(\lambda_1, \ldots, \lambda_n)$ and $(\lambda'_1, \ldots, \lambda'_{n'})$, then $m \geq n$ and $\lambda_j^{(S)} \geq \lambda_j^{(s)}$ for all $j = 1, \ldots, n$. This proves the unicity of $(\lambda_1^{(s)}, \ldots, \lambda_n^{(s)})$. Argument is the same for the infimum.

Now let us define r over $\overline{\mathcal{L}}$ by $r(\lambda) := |\lambda|$, and suppose that λ and λ' satisfy $\lambda' \succ \lambda$ (λ' covers λ, see Section 2). Thus either $n = n'$, and $\exists! j \in \{1, \ldots, n\}$ such that $\lambda'_j = \lambda_j + 1$, or $n' = n + 1$, with $\lambda'_j = \lambda_j$ for all $j = 1, \ldots, n$ and $\lambda_{n'} = 1$. In both cases, we obtain $r(\lambda') = |\lambda'| = |\lambda| + 1 = r(\lambda) + 1$. ■

For $L \in \mathcal{L}$, $\mathcal{G}(L)$ denotes the set of functions defined on L which vanish at $\perp := (0, \ldots, 0)$: this corresponds to *multichoice games* as introduced by Hsiao and Raghavan [7], where each player has a set of possible ordered actions. For the sake of commodity, we will assimilate any element L of $\mathcal{L}$ with its representative element in $\overline{\mathcal{L}}$. In this way, for any $\lambda := (\lambda_1, \ldots, \lambda_n) \in \overline{\mathcal{L}}$, $v \in \mathcal{G}(\lambda)$ means that v is any game with n players such that their maximal participation levels are given *up to the order of players* by $\lambda_1, \ldots, \lambda_n$. We denote by $\mathcal{G}^{\mathcal{M}}$ the set of all multichoice games, that is to say,

$$\mathcal{G}^{\mathcal{M}} := \{\mathcal{G}(L) \mid L \in \mathcal{L}\}.$$

The set $\mathcal{J}(L)$ of join-irreducible elements of L is $\{(0_{-i}, k_i) \mid i \in N, k \in L_i \setminus \{0\}\}$, using our notation for compound vectors (see Section 2); hence each join-irreducible element $(0_{-i}, k_i)$, which we will often denote by k_i if no ambiguity occurs, corresponds to a single player playing at a given level. Thus a value on $\mathcal{G}(L)$ is a function $\Phi : \mathcal{G}(L) \times \mathcal{J}(L) \to \mathbb{R}$ that assigns to every player i playing at the level k in a game $v \in \mathcal{G}(L)$ his prospect $\Phi(v, k_i)$. Our aim is to define the Shapley value $\Phi(v, k_i)$ for each join-irreducible element k_i.

Our approach will take here a similar way, such as the axiomatization given for classical cooperative games. Note that an axiomatization of the Shapley value for multichoice games has already been done in [6] and [10]. The computed formula is the same. However, the former uses a symmetry axiom which is not really natural, whereas the latter is less intuitive and requires more material. Another important difference in [10] is that the extended Shapley value is only given for multichoice games where the number of possible actions is the same for all players. Moreover, none are given in a simple recursive way on the whole set $\mathcal{G}^{\mathcal{M}}$.

Let us first give the following axioms generalizing the ones given for classical games.

Linearity ($\mathbf{L}^{\mathcal{M}}$): for any $L \in \mathcal{L}$, for all join-irreducible $k_i \in \mathcal{J}(L)$, $\Phi(v, k_i)$ is linear on the set of games $\mathcal{G}(L)$, which directly implies

$$\Phi(v, k_i) = \sum_{x \in L} p_x^{k_i}\, v(x), \quad \text{with } p_x^{k_i} \in \mathbb{R}.$$

For some $k \in L_i, k \neq 0$, player i is said to be k-*null* (or simply k_i is *null*) for $v \in \mathcal{G}(L)$ if $v(x, k_i) = v(x, (k-1)_i)$, $\forall x \in L_{-i}$. If $\top_i$ is null for v and $\top_i = 1$, player i is simply said to be null for v.

Nullity ($\mathbf{N}^{\mathcal{M}}$): for any $L \in \mathcal{L}$, for any game $v \in \mathcal{G}(L)$, for any player i who is k-null for v,

$$\Phi(v, k_i) = 0.$$

For some $i \in N$, and $v \in \mathcal{G}(L)$, if $\top_i \neq 1$, we define by $v^{-\top_i}$ the restriction of v to the product $L_{-i} \times (L_i \setminus \top_i)$. Moreover, v^{-i} denotes the mapping defined over $L_{-i} : x \mapsto v(x, 0_i)$.

Excluded-null-player ($\mathbf{ENP}^{\mathcal{M}}$): for any $L \in \mathcal{L}$, for any game $v \in \mathcal{G}(L)$, for any player $i \in N$ such that $\top_i = 1$, if i is null for v,

$$\forall j \in N \setminus i, \quad \Phi(v, \top_j) = \Phi(v^{-i}, \top_j).$$

Decreased-level ($\mathbf{DL}^{\mathcal{M}}$): for any $L \in \mathcal{L}$, for any game $v \in \mathcal{G}(L)$, for any player $i \in N$ such that $\top_i \neq 1$, if $\top_i$ is null for v,

(i) $\forall k \in L_i \setminus \{0, \top_i\}, \quad \Phi(v, k_i) = \Phi(v^{-\top_i}, k_i)$.

(ii) $\forall j \in N \setminus i, \quad \Phi(v, \top_j) = \Phi(v^{-\top_i}, \top_j).$

Likewise the previous section, **(N$^{\mathcal{M}}$)**, **(ENP$^{\mathcal{M}}$)** and **(DL$^{\mathcal{M}}$)** may be merged in the following axiom:

Generalized nullity (GN$^{\mathcal{M}}$): for any $L \in \mathcal{L}$, for any game $v \in \mathcal{G}(L)$, for any player i which is k-null for v, any player $j \in N$ and any level $l \in \{1, \ldots, \top_j\}$,

$$\Phi(v, l_j) = \begin{cases} 0 \text{ if } j = i \text{ and } l = k, \\ \Phi(v^{-i}, l_j) \text{ if } j \neq i \text{ and } k = \top_i = 1, \\ \Phi(v^{-\top_i}, l_j) \text{ if } j \neq i \text{ and } k = \top_i \neq 1. \end{cases}$$

Note that this axiom is stronger than the simple concatenation of **(N$^{\mathcal{M}}$)**, **(ENP$^{\mathcal{M}}$)** and **(DL$^{\mathcal{M}}$)**. Thus its validity is easily verifiable by checking the formulae are true.

For any $L \in \mathcal{L}$, we define the particular *unanimity game* of $\mathcal{G}(L)$ by
$$u_\top(x) := \begin{cases} 1, & \text{if } x = \top, \\ 0, & \text{otherwise.} \end{cases}$$

Equity (Eq$^{\mathcal{M}}$): for any $L \in \mathcal{L}$, for any player $i \in N$,

$$\Phi(u_\top, \top_i) = \frac{1}{n}.$$

Theorem 5 *Under axioms **(L$^{\mathcal{M}}$)**, **(GN$^{\mathcal{M}}$)**, and **(Eq$^{\mathcal{M}}$)**, Φ is given on $\mathcal{G}^{\mathcal{M}}$ by:*

$$\Phi(v, k_i) = \sum_{x \in \Gamma(L_{-i})} \frac{h(x)!\,(n - h(x) - 1)!}{n!} [v(x, k_i) - v(x, (k-1)_i)], \tag{6}$$
$$\text{for any finite set } N \subseteq \Omega, \forall L \in \mathcal{L}(N), \forall v \in \mathcal{G}(L), \forall k_i \in \mathcal{J}(L),$$

where $h(x) := |\{j \in N \setminus i \mid x_j = \top_j\}|$.

Proof: It is clear that the above formula satisfies **(L$^{\mathcal{M}}$)** and **(N$^{\mathcal{M}}$)**. Then we check that Φ satisfies axioms **(ENP$^{\mathcal{M}}$)** and **(DL$^{\mathcal{M}}$)**, **(Eq$^{\mathcal{M}}$)** being easy to verify. Let $v \in \mathcal{G}(L)$ and $i \in N$ a player null for v ($L_i = \{0, 1\}$). Note that the classical Shapley's coefficients appear in the formula, under the form $\alpha^1_{h(x)}(n) = \dfrac{h(x)!\,(n - h(x) - 1)!}{n!}$. Then for $j \in N \setminus i$,

$$\begin{aligned}\Phi(v, \top_j) &= \sum_{x \in \Gamma(L_{-j})} \alpha^1_{h(x)}(n)\,(v(x, \top_j) - v(x, \top_j - 1)) \\ &= \sum_{x \in \Gamma(L_{-ij})} [\alpha^1_{h(x)}(n) + \alpha^1_{h(x)+1}(n)]\,(v(x, 0_i, \top_j) - v(x, 0_i, \top_j - 1)),\end{aligned}$$

since i is null for v and $\top_i = 1$. Then, by refering to (2), we have

$$\alpha^1_{h(x)}(n) + \alpha^1_{h(x)+1}(n) = \alpha^1_{h(x)}(n - 1).$$

Thus

$$\begin{aligned}\Phi(v, \top_j) &= \sum_{x \in \Gamma((L_{-i})_{-j})} \alpha^1_{h(x)}(n - 1)\,\big(v(x, \top_j) - v(x, \top_j - 1)\big) \\ &= \Phi(v^{-i}, \top_j),\end{aligned}$$

and **(ENP$^{\mathcal{M}}$)** is satisfied. If $\top_1 \neq 1$ and $\top_i$ is null for v, **(DL$^{\mathcal{M}}$)-(ii)** is obtained in the same way. Besides, **(DL$^{\mathcal{M}}$)-(i)** is also easy to check since $k_i < \top_i$ and the set of indices $\Gamma(L_{-i})$ under the Sigma symbol of (6) does not depend on L_i.

We now show that the formula is uniquely determined by the axioms. Under **(L$^{\mathcal{M}}$)** and **(N$^{\mathcal{M}}$)**, Φ is given by

$$\Phi(v, k_i) = \sum_{x \in L_{-i}} p^{k_i}_x(L)\,[v(x, k_i) - v(x, (k - 1)_i)], \tag{7}$$
$$\text{for any finite set } N \subseteq \Omega, \forall L \in \mathcal{L}(N), \forall v \in \mathcal{G}(L), \forall k_i \in \mathcal{J}(L),$$

with $p^{k_i}_x(L) \in \mathbb{R}$. Indeed, assuming k_i is null for v,

$$\begin{aligned}\Phi(v, k_i) &= \sum_{x \in L} p^{k_i}_x(L)\,v(x) \\ &= \sum_{x \in L_{-i}} \Big[p^{k_i}_{(x,0_i)}(L)\,v(x, 0_i) + \cdots + p^{k_i}_{(x,\top_i)}(L)\,v(x, \top_i)\Big] \\ &= \sum_{x \in L_{-i}} [p^{k_i}_{(x,(k-1)_i)}(L) + p^{k_i}_{(x,k_i)}(L)]\,v(x, k_i) + \sum_{x \in L_{-i}} \sum_{l \neq k-1,k} p^{k_i}_{(x,l_i)}(L)\,v(x, l_i).\end{aligned} \tag{8}$$

Consider $v' \in \mathcal{G}(L_{-i})$ and extend it to $\mathcal{G}(L)$:

$$v(x, l_i) = \begin{cases} v'(x), & \text{if } l = k-1, k \\ 0, & \text{otherwise.} \end{cases}$$

Then k_i is null for v, hence (8) applies and reduces to:

$$\Phi(v, k_i) = \sum_{x \in L_{-i}} [p^{k_i}_{(x,(k-1)_i)}(L) + p^{k_i}_{(x,k_i)}(L)]\, v'(x) = 0,$$

by axiom $(\mathbf{N}^{\mathcal{M}})$. Since this is true for all $v' \in \mathcal{G}(L_{-i})$, this implies $p^{k_i}_{(x,(k-1)_i)}(L) = -p^{k_i}_{(x,k_i)}(L)$, $\forall x \in L_{-i}$. Introducing this in (8) yields:

$$\Phi(v, k_i) = 0 = \sum_{x \in L_{-i}} \sum_{l \neq k-1,k} p^{k_i}_{(x,l_i)}(L)\, v(x, l_i).$$

Since this must hold for any game v for which k_i is null, we deduce that $p^{k_i}_{(x,l_i)}(L) = 0$, $\forall l \neq k-1, k$. Letting $p^{k_i}_x(L) := p^{k_i}_{(x,k_i)}(L)$, the result is proven.

As for the previous theorem, we compute coefficients of (7) by a basic *transfinite induction*, which is an extension of mathematical induction on sets endowed with a wellfounded relation. A binary relation R is *wellfounded* on a set E if every nonempty subset of E has an R-minimal element; that is, for every nonempty subset X of E, there is an element m of X such that for every element x of X, the pair (x, m) is not in R. Considering the strict order $<$ associated to $\leq$, it is easy to see that $<$ is wellfounded on $\overline{\mathcal{L}}$. Thus, the inductive step rests on showing the formula over $\mathcal{G}(\lambda)$ if it is true for games defined over all predecessors of λ in $(\overline{\mathcal{L}}, \leq)$. Consequently, if the formula is also satisfied on $\mathcal{G}\big((1)\big)$, then the induction hypothesis applies and the result is satisfied for any game of $\mathcal{G}^{\mathcal{M}}$.

The case $\lambda = (1)$ corresponds to classical cooperative games with one player, which has been verified in the proof of Theorem 1. Indeed, in this case, $\mathcal{J}(L)$ has only one element one can denote by 1_1 (which is also one of the only two elements of L), for which (7) under $(\mathbf{Eq}^{\mathcal{M}})$ writes $\Phi(v, 1_1) = v(1_1)$. Besides, (6) writes

$$\begin{aligned}\Phi(v, 1_1) &= \sum_{x \in \{()\}} \frac{h(x)!\,(n - h(x) - 1)!}{n!} [v(x, 1_1) - v(x, 0_1)] \\ &= v(1_1).\end{aligned}$$

For any $\lambda := (\lambda_1, \ldots, \lambda_n) \in \overline{\mathcal{L}} \setminus \{(1)\}$, let us assume that (6) holds for all games of $\mathcal{G}(\lambda')$ such that $\lambda' \prec \lambda$. We now show that under $(\mathbf{ENP}^{\mathcal{M}})$, $(\mathbf{DL}^{\mathcal{M}})$ and

(Eq$^{\mathcal{M}}$), the unicity of all coefficients in (7) is given for any game $v \in \mathcal{G}(\lambda)$. This being done, as it has been checked that (6) satisfies the axioms, the result will be proved. Let N be any set of players of cardinality n, and L be any linear lattice such that maximum levels $\top_1, \dots, \top_n$, in any order, are given by λ.

- We show the unicity of the $\Phi(v, k_i)$'s, for any player $i \in N$ such that $\top_i \neq 1$, and any level $k < \top_i$. Indeed, in this case, if $\top_i$ is null for any $v \in \mathcal{G}(L)$, by **(DL$^{\mathcal{M}}$)-(i)**,

$$\Phi(v, k_i) = \Phi(v^{-\top_i}, k_i) = \sum_{x \in L'_{-i}} p_x^{k_i}(L') \left[v(x, k_i) - v(x, (k-1)_i)\right],$$

where $L' := L_{-i} \times (L_i \setminus \top_i)$. Since associated partition of L' is one of the predecessors of λ, thus all $p_x^{k_i}(L')$ are known by assumption. For any $x \in L \setminus \{\bot\}$, let us denote δ_x the identity game defined by

$$\forall y \in L, \delta_x(y) := \begin{cases} 1 \text{ if } y = x, \\ 0 \text{ otherwise.} \end{cases}$$

Consequently, by identification of the coefficients for the above formula and the one straightforwardly given by (7) for the $\top_i$-null game $v := \begin{cases} \delta_{(x,k_i)} \text{ if } k < \top_i - 1 \\ \delta_{(x,\top_i - 1)} + \delta_{(x,\top_i)} \text{ if } k = \top_i - 1 \end{cases}$, we obtain $p_x^{k_i}(L) = p_x^{k_i}(L')$, for all $x \in L_{-i}$.

- Now, let $i \in N$ be any player and $j \in N \setminus i$ such that $\top_j \neq 1$. Then for any game $v \in \mathcal{G}(L)$ for which $\top_j$ is null,

$$\begin{aligned} \Phi(v, \top_i) &= \sum_{x \in L_{-i}} p_x^{\top_i}(L) \left[v(x, \top_i) - v(x, \top_i - 1)\right] \\ &= \sum_{\substack{x \in L_{-i} \\ x_j \neq \top_j, \top_j - 1}} p_x^{\top_i}(L) \left[v(x, \top_i) - v(x, \top_i - 1)\right] \end{aligned} \tag{9}$$

$$+ \sum_{y \in L_{-ij}} \left(p^{\top_i}_{(y,\top_j - 1)}(L) + p^{\top_i}_{(y,\top_j)}(L)\right) \left[v(y, \top_j - 1, \top_i) - v(y, \top_j - 1, \top_i - 1)\right].$$

Besides, by **(DL$^{\mathcal{M}}$)-(ii)**,

$$\begin{aligned}\Phi(v,\top_i) &= \Phi(v^{-\top_j},\top_i)\\ &= \sum_{\substack{x\in L'_{-i}\\ x_j\neq \top_j-1}} p_x^{\top_i}(L')\,[v(x,\top_i)-v(x,\top_i-1)] \\ &\quad + \sum_{y\in L'_{-ij}} p^{\top_i}_{(y,\top_j-1)}(L')\,[v(y,\top_j-1,\top_i)-v(y,\top_j-1,\top_i-1)],\end{aligned} \tag{10}$$

where $L' := L_{-j}\times(L_j\setminus \top_j)$. Consequently, by identification of coefficients for (9) and (10) in the game $v := \delta_{(x,\top_i)}$ which is $\top_j$-null, we obtain $p_x^{\top_i}(L) = p_x^{\top_i}(L')$ for all $x\in L_{-i}$ such that $x_j\neq \top_j, \top_j-1$. Thus, we have proved the unicity of coefficients $p_x^{\top_i}(L)$ for all $i\in N$, and for all $x\in L_{-i}$ such that $\exists j\in N\setminus i, x_j\neq \top_j,\top_j-1$.

- Lastly, it remains to show the unicity of the $p_x^{\top_i}(L)$'s, where $i\in N$ and $x\in L_{-i}$ such that $\forall j\in N\setminus i$, $x_j\in\{\top_j,\top_j-1\}$. This in view, we consider the partition $\{C_{i,m}\}_{i\in N;0\leq m\leq n-1}$ of these indices, where $C_{i,m}$ denotes the set of elements of L_{-i} whose m coordinates x_j are $\top_j-1$ and the others are $\top_j$. For any $i\in N$, we show the unicity of the $p_x^{\top_i}(L)$'s by induction on m. For $x\in C_{i,0}$, that is to say, $x=\top_{-i} := (\top_1,\dots,\top_{i-1},\top_{i+1},\dots,\top_n)$, $p_x^{\top_i}(L)$ is given by **(Eq$^{\mathcal{M}}$)**:

$$\Phi(u_\top,\top_i) = p^{\top_i}_{\top_{-i}}(L) = \frac{1}{n}.$$

Let us suppose that all $p_x^{\top_i}(L)$'s are given for all elements of $C_{i,m}$, where m is fixed in $\{0,\dots,n-2\}$. We show the unicity of the $p_x^{\top_i}(L)$'s for $x\in C_{i,m+1}$. Indeed, one can associate any $x\in C_{i,m+1}$ to any $j_0\in N\setminus i$ such that $x_{j_0}=\top_{j_0}-1$. We define $x'\in C_{i,m}$ by $x'_j := \begin{cases}\top_{j_0} \text{ if } j=j_0,\\ x_j \text{ otherwise}\end{cases}$. Now, two situations may arise: either $\top_{j_0}\neq 1$ or $\top_{j_0}=1$. In the first case, we refer to (9) and (10) with $j := j_0$: by identification of the coefficients for the $\top_j$-null game $v := \delta_{(x,\top_i)}+\delta_{(x',\top_i)}$, we obtain $p_x^{\top_i}(L)+p_{x'}^{\top_i}(L) = p_x^{\top_i}(L')$, i.e., $p_x^{\top_i}(L) = p_x^{\top_i}(L') - p_{x'}^{\top_i}(L)$, where $p_{x'}^{\top_i}(L)$ is given by hypothesis in the current induction, and $p_x^{\top_i}(L')$ is given by hypothesis in the backward transfinite induction. Finally, if $\top_{j_0}=1$, for any game $v\in\mathcal{G}(L)$ for which

j_0 is null,

$$\begin{aligned}\Phi(v, \top_i) &= \sum_{x \in L_{-i}} p_x^{\top_i}(L)\, [v(x, \top_i) - v(x, \top_i - 1)] \\ &= \sum_{y \in L_{-ij_0}} \left(p^{\top_i}_{(y,0_{j_0})}(L) + p^{\top_i}_{(y,1_{j_0})}(L)\right) [v(y, 0_{j_0}, \top_i) - v(y, 0_{j_0}, \top_i - 1)].\end{aligned} \tag{11}$$

Besides, by **(ENP$^{\mathcal{M}}$)**,

$$\begin{aligned}\Phi(v, \top_i) &= \Phi(v^{-j_0}, \top_i) \\ &= \sum_{y \in L'_{-i}} p_y^{\top_i}(L')\, [v(y, \top_i) - v(y, \top_i - 1)],\end{aligned} \tag{12}$$

where $L' := L_{-j_0}$. Consequently, by identification of coefficients for (11) and (12) for the game $v := \delta_{(x,\top_i)} + \delta_{(x',\top_i)}$ which is j_0-null, we obtain $p_x^{\top_i}(L) + p_{x'}^{\top_i}(L) = p_x^{\top_i}(L')$, i.e., $p_x^{\top_i}(L) = p_x^{\top_i}(L') - p_{x'}^{\top_i}(L)$. Note that even if $m+1$ choices of j_0 are possible, one cannot guarantee the existence of such an index such that $\top_{j_0} = 1$ for all $i \in N$, or such that $\top_{j_0} \neq 1$ for all $i \in N$. As a consequence, axioms **(DL$^{\mathcal{M}}$)-(ii)** and **(ENP$^{\mathcal{M}}$)** are both necessary.

This ends the proof of the current inductive step: $\forall i \in N$, all $p_x^{\top_i}(L)$'s are given for any $x \in L_{-i}$ such that $\forall j \in N \setminus i$, $x_j \in \{\top_j, \top_j - 1\}$. Consequently, for all linear lattice L associated to λ, $\forall k_i \in \mathcal{J}(L)$, $\forall x \in L_{-i}$, all $p_x^{k_i}(L)$'s are given, which also completes the inductive step of the transfinite induction.

■

References

[1] G.E. Andrews. *The Theory of Partitions*. Addison-Wesley, 1976.

[2] G. Birkhoff. *Lattice Theory*. American Mathematical Society, 3d edition, 1967.

[3] R. Branzei, D. Dimitrov, and S. Tijs. *Models in cooperative game theory: crisp, fuzzy and multichoice games*. Springer Verlag, 2005.

[4] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Orders.* Cambridge University Press, 1990.

[5] U. Faigle and W. Kern. The Shapley value for cooperative games under precedence constraints. *International Journal of Game Theory*, 21:249–266, 1992.

[6] M. Grabisch and F. Lange. Games on lattices, multichoice games and the Shapley value: a new approach. *Mathematical Methods of Operations Research*, 65:153–167, 2007.

[7] C.R. Hsiao and T.E.S. Raghavan. Shapley value for multichoice cooperative games, I. *Games and Economic Behavior*, 5:240–256, 1993.

[8] D. Monderer and D. Samet. Variations on the shapley value. In Ed. R.J. Aumann and S. Hart, editors, *Handbook of Game Theory No III.* Elsevier Science, 2002.

[9] R.B. Myerson. Values of games in partition function form. *Int. J. of Game Theory*, 6:23–31, 1977.

[10] H. Peters and H. Zank. The egalitarian solution for multi-choice games. *Annals of Operations Research*, 137:399–409, 2005.

[11] L.S. Shapley. A value for n-person games. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.

[12] R.M. Thrall and W.F. Lucas. N-person games in partition function form. *Naval Research Logistics Quarterly*, 10:281–293, 1963.

[13] R.J. Weber. Probabilistic values for games. In A.E. Roth, editor, *The Shapley Value. Essays in Honor of Lloyd S. Shapley*, pages 101–119. Cambridge University Press, 1988.

Chapitre 3. Valeur sur jeux réguliers et lois de Kirchhoff

Résumé

En théorie des jeux coopératifs, la valeur de Shapley est une notion centrale permettant de définir d'une manière rationnelle le moyen de partager la valeur de la grande coalition entre tous les joueurs. Dans le cadre général de ce papier, l'ensemble des coalitions réalisables où est défini le jeu forme un ensemble ordonné (par l'inclusion) dont toutes les chaînes maximales ont la même longueur. Nous montrons d'abord que certaines définitions et axiomatisations précédemment étudiées par Faigle et Kern de la valeur de Shapley restent valables. Notre principale contribution est de proposer une nouvelle axiomatisation qui évite l'axiome de force hiérarchique de Faigle et Kern, considèrant un nouveau moyen de généraliser l'axiome d'anonymat entre les joueurs. Des idées de la théorie des réseaux électriques sont ensuite empruntées, où nous montrons que notre axiome d'anonymat ainsi que l'axiome bien connu d'efficacité correspondent en fait aux deux lois de Kirchhoff dans un circuit électrique résistif (les noeuds étant données par les coalitions faisables et les branches par les couples de coalitions se précédant). Plus précisément, des analogies sont données entre l'axiome d'efficacité et la loi des noeuds, et entre l'axiome d'anonymat et la loi des mailles. Nous établissons enfin une forme plus faible de l'axiome de monotonie qui est satisfait par la valeur proposée.

Mots clés : système de coalitions régulier, jeu régulier, valeur de Shapley, valeur probabiliste efficace, valeur régulière, lois de Kirchhoff

Values on regular games under Kirchhoff's laws

Fabien Lange
Michel Grabisch

Centre d'Economie de la Sorbonne, UMR 8174, CNRS-Université Paris 1

Abstract

In cooperative game theory, the Shapley value is a central notion defining a rational way to share the total worth of a game among players. In this paper, we address a general framework, namely regular set systems, where the set of feasible coalitions forms a poset where all maximal chains have the same length. We first show that previous definitions and axiomatizations of the Shapley value proprosed by Faigle and Kern, and Bilbao and Edelman still work. Our main contribution is then to propose a new axiomatization avoiding the hierarchical strength axiom of Faigle and Kern, and considering a new way to define the symmetry among players. Borrowing ideas from electric networks theory, we show that our symmetry axiom and the classical efficiency axiom correspond actually to the two Kirchhoff's laws in the resistor circuit associated to the Hasse diagram of feasible coalitions. We finally work out a weak form of the monotonicity axiom which is satisfied by the proposed value.

Keywords: regular set system, regular game, Shapley value, probabilistic efficient value, regular value, Kirchhoff's laws

Submitted to *SIAM Journal on Discrete Mathematics.*

Short version in *Proc. Int. Conf. on Information Processing and Management of Uncertainty (IPMU'06)*, Paris, France, July 2006.

1 Introduction

The value or solution concept of a game is a key concept in cooperative game theory, since it defines a rational imputation given to the players if they join the grand coalition. In this respect, the Shapley value remains the best known solution concept [16, 17] applied also to more general notions of game, like multichoice games [12].

In the above cited classical works, it is assumed that any coalition of players can form. However, this assumption is often irrealistic, for various reasons (incompatibilities between players, precedence constraints, etc.). A great deal of work has been done in order to consider weaker assumptions on the set of *feasible* coalitions. Along this line, we may cite Faigle [10, 11] who introduced the idea of precedence constraints among players, and Bilbao and Edelman, considering that the set of feasible coalitions is a convex geometry. Due to well known results in lattice representation, the construction of Faigle amounts to have a distributive lattice as the set of feasible coalitions, and hence is a particular case of Bilbao and Edelman's construction. We may also cite the recent work of Bilbao, who introduced cooperative games under augmenting systems [2], which are particular structures where the grand coalition is not necessarily feasible.

Despite the mathematical interest of convex geometries, we may argue if they fit or not to the framework of game theory. Specifically, feasible coalitions of a convex geometry should satisfy two conditions: (1) if S is a feasible coalition, then it is possible to find a player i such that $S \cup i$ is still feasible, (2) if S, T are feasible, then their intersection too should be feasible. The first condition is a natural and very weak one in a context where the grand coalition can form, since it says that from a given coalition, it is possible to augment it gradually to reach the grand coalition. On the contrary, it is more difficult to accept the second one.

We propose to consider more general structures, avoiding the closure under intersection, which we call *regular set systems*, which more or less amounts to take condition (1) above and a symmetric one, saying that from a given coalition S, it is possible to withdraw one player while remaining feasible. Regular set systems have been proposed by Honda and Grabisch [14], and have all their maximal chains of same length. One of their main mathematical advantages is that they allow to keep many classical notions defined for games, capacities [5] and other set functions [13], such as the Möbius transform, the core, the Shapley value, the entropy, etc., since all these notions can be defined through maximal chains. A general view of regular set systems, giving connections with more classical ordered structures, is given in Section 2.

Our main aim is the axiomatization of a solution concept for games defined on regular set systems —which we call *regular games*—, close to the Shapley value. In Section 3, we begin by considering probabilistic and marginalist values, and we generalize results obtained by Bilbao and Edelman. In Section 4, we propose a substitute for the classical symmetry axiom, which cannot be straightforwardly generalized in such general coalition structures. Our proposal, called the *regularity axiom*, has a more natural interpretation than the hierarchical strength axiom of Faigle and Kern [11], which is merely a combinatorial axiom. Our main achievement is Theorem 6, which shows that there is a unique marginalist value satisfying the regularity axiom and efficiency. This is done through an analogy with networks and electrical circuits, explained in Section 5. The efficiency and regularity axioms are shown to be respectively equivalent to the first and second Kirchhoff's laws. A last section is devoted to the study of monotonicity. It is shown that our value does not satisfy the monotonicity axiom in general, but a weaker form of monotonicity, which is the aggregate monotonicity.

In the paper, $N := \{1, 2, \ldots, n\}$ refers to the finite set of players. In order to avoid heavy notations, we will often omit braces for subsets, by writing i instead of $\{i\}$ or 123 for $\{1, 2, 3\}$. Furthermore, cardinalities of subsets $S, T, \ldots$ will be denoted by the corresponding lower case letters $s, t, \ldots$

2 Regular games

Let us consider $\mathcal{N}$ a subcollection of the power set 2^N of N. Then we call $(N, \mathcal{N})$ a *set system* on N if $\mathcal{N}$ contains $\emptyset$ and N. In the sequel, $(N, \mathcal{N})$ always denotes a set system.

Elements of $\mathcal{N}$ are called *(feasible) coalitions*. For any two coalitions A, B of $\mathcal{N}$, we say that A is *covered* by B, and write $A \prec B$, if $A \subsetneq B$ and $A \subseteq C \subsetneq B$, with $C \in \mathcal{N}$, implies $C = A$.

Definition 1 *$(N, \mathcal{N})$ is a* regular set system *if it satisfies the following property:*

$$\forall S, T \in \mathcal{N} \text{ such that } S \prec T \text{ in } \mathcal{N}, \text{ then } |T \setminus S| = 1.$$

If in addition, the regular set system has a lattice structure, then we call it a *regular set lattice.*

For any two coalitions S, T in a set system $(N, \mathcal{N})$, we call *maximal chain from S to T* any sequence $(S_0, S_1, \ldots, S_m)$ of elements of $\mathcal{N}$ such that $S_0 = S$, $S_m = T$, and $S_i \prec S_{i+1}$ for every $0 \leq i \leq m-1$. If S and T are not specified, maximal chains are understood to be from $\emptyset$ to N. Note that we find in [14] Definition 1 under the equivalent form **(ii)** below:

Proposition 1 *Let $(N, \mathcal{N})$ be a set system. Then the following assertions are equivalent:*

> ***(i)*** *$(N, \mathcal{N})$ is a regular set system.*
>
> ***(ii)*** *All maximal chains of $(N, \mathcal{N})$ have length n, i.e., all maximal chains have exactly $n+1$ elements.*

Proof: Assuming that $(N, \mathcal{N})$ is regular, let C be a maximal chain of $(N, \mathcal{N})$. Every element of C covers the previous one, and then contains only one extra player. Thus C contains $n+1$ elements. Conversely, if $(N, \mathcal{N})$ is not regular, i.e., there are two elements S, T such that $S \prec T$ and $|T \setminus S| \geq 2$, then any maximal chain going through S and T has necessarily less than $n+1$ elements. ■

Note that regular set systems also satisfy the following properties, which straightforwardly derive from the definition:

(iii) One-point extension: $\forall S \in \mathcal{N}, S \neq N, \quad \exists i \in N \setminus S$ such that $S \cup i \in \mathcal{N}$.

(iv) Accessibility: $\forall T \in \mathcal{N}, T \neq \emptyset, \exists j \in T$ such that $T \setminus j \in \mathcal{N}$.

These properties are not sufficient to characterize regular set systems, and are actually used by Labreuche as an underlying structure of games in [15].

What is interesting for the sequel, the set of regular set systems is a general class embodying some classical structures such as distributive lattices and convex geometries [3]. We now present them.

A *Jordan-Dedekind poset* is any poset such that all its maximal chains between any two elements have the same length. Note that if the Jordan-Dedekind poset has least and greatest elements, it is sufficient to verify that all its maximal chains between them have the same length. Thus we call *Jordan Dedekind set system* any Jordan-Dedekind poset which is a set system. A *convex geometry* is any set system $(N, \mathcal{N})$ satisfying

(C1) One-point extension property.

(C2) Intersection closure: $\forall A, B \in \mathcal{N}, \quad A \cap B \in \mathcal{N}$.

The dual set system of the convex geometry is called *antimatroid*, that is to say any set system satisfying

(A1) Accessibility property.

(A2) Union closure.

A lattice is *distributive* when the infimum and the supremum obey the distributivity law. For any poset $(P, \leqslant)$, a subset $Q \subseteq P$ is called a *downset* of $(P, \leqslant)$ if $x \leqslant y$ and $y \in Q$ imply $x \in Q$. We denote by $\mathcal{O}(P)$ the set of all downsets of P. Besides, a *join-irreducible element* x of a lattice $(L, \leq)$ is an element that is not the least one, and for which $(x = y \vee z)$ implies $(x = y$ or $x = z)$. It is known that the set of all downsets of $(P, \leqslant)$ endowed with the inclusion relation is a distributive lattice. Conversely, a fundamental Theorem due to Birkhoff [4] says that any distributive lattice $(L, \leq)$ is isomorphic to the set $\mathcal{O}(\mathcal{J})$ of all downsets of the set $\mathcal{J}$ of join-irreducible elements of L. Consequently, for any distributive lattice $(L, \leq)$, there is a poset $(P, \leqslant)$ such that $(L, \leq)$ has the isomorphic form $\mathcal{O}(P)$. Moreover, it is also known that distributive lattices which are set systems coincide with the class of set systems closed under intersection and union. Finally, we will call *distributive regular set system* any distributive lattice given under the form $\mathcal{O}(P)$, where P is endowed with the appropriate partial order relation $\leqslant$.

We present now the following inclusion diagram where these set systems structures fit into each other (see Fig. 1).

Proposition 2

(1) *The class of Jordan-Dedekind set systems strictly includes regular set systems.*

(2) *The class of regular set systems strictly includes regular set lattices.*

(3) *The class of regular set lattices strictly includes convex geometries and antimatroids.*

(4) *The intersection of the classes of convex geometries and antimatroids coincides with the class of distributive regular set systems.*

(5) *The class of distributive regular set systems strictly includes distributive regular set systems isomorphic to direct products of linear lattices.*

(6) *The class of direct products linear lattices strictly includes Boolean lattices.*

Proof: We show the successive inclusions **(1)** to **(4)**, **(5)** and **(6)** being well known or evident.

(1) is clear since for any regular set system, for any two coalitions S and T such that $S \subseteq T$, all maximal chains from S to T have clearly $t - s + 1$ elements. However, the reverse property is not true. Indeed, it is self-evident that for any $n \geq 3$, the poset $\{\emptyset, 1, 2, \ldots, n, N\}$ is a Jordan-Dedekind set system but is not regular, since its maximal chains have length 2 and should have length n.

About **(2)**, we have only to show that the inclusion is strict: for $n = 4$, we easily see that the set system $\mathcal{N} := \{\emptyset, 1, 2, 13, 23, 14, 24, 123, 124, 1234\}$ is regular but $\{1\}$ and $\{2\}$ have no supremum thus $(N, \mathcal{N})$ is not a lattice.

By a simple induction using $(C1)$, we show that any maximal chain of a convex geometry $(N, \mathcal{N})$ has necessarily length n. By Proposition 1, $(N, \mathcal{N})$ is a regular set system. This holds for an antimatroid, by the duality principle. Besides, the convex geometry and the antimatroid are lattices $(\mathcal{N}, \subseteq, \vee, \cap, N, \emptyset)$ and $(\mathcal{N}, \subseteq, \cup, \wedge, N, \emptyset)$ where $A \vee B := \cap\{C \in \mathcal{N} \mid A \cup B \subseteq C\}$ and $A \wedge B := \cup\{C \in \mathcal{N} \mid A \cap B \subseteq C\}$, respectively. Conversely, the set system $\{\emptyset, 1, 2, 13, 23, 123\}$ is a regular lattice but is neither a convex geometry nor an antimatroid. Thus, **(3)** is shown.

Now, let $(N, \leqslant)$ be any poset and $L := \mathcal{O}(N)$ be the distributive lattice of all downsets of $(N, \leqslant)$. As said above, it is known that the union and the intersection of any two downsets is also a downset. Furthermore, it is clear that condition **(C1)** holds since from any downset $S \neq N$ of $(N, \leqslant)$, adding a minimal element of the restricted poset $(N \setminus S, \leqslant)$ in S leaves S a downset. By withdrawing a maximal element of $S \neq \emptyset$, the dual condition **(A1)** holds. Conversely, if a set system is a convex geometry and an antimatroid, then supremum and infimum laws are union and intersection, which immediately implies the distributivity law. Thus **(4)** is shown.

Remark that **(5)** and **(6)** are seen as set system inclusions in the sense that for any direct product of linear lattices L, there is a regular set system $(N, \mathcal{N})$ that is isomorphic to L. In addition, for any Boolean lattice $\mathcal{B}$, there is an integer n such that $\mathcal{B}$ is isomorphic to 2^N. ■

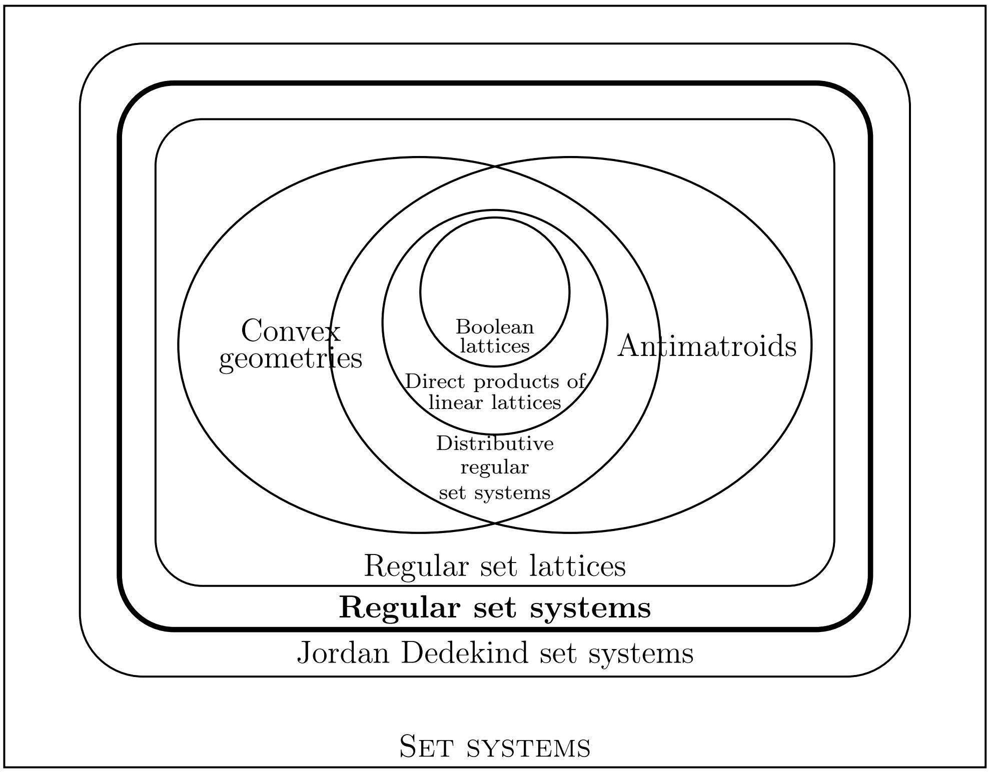


Figure 1: Inclusion diagram of set systems

We call *regular game* any game defined on a regular set system, that is to say, any mapping v defined over a regular set system $(N, \mathcal{N})$ such that $v(\emptyset) = 0$. We denote by $\mathcal{G}(\mathcal{N})$ the $\mathbb{R}$-vector space of games over the set system $(N, \mathcal{N})$. Remark that for the Boolean lattice $\mathcal{N} := 2^N$, $\mathcal{G}(\mathcal{N})$ is the set of classical cooperative games.

Considering a regular set system $(N, \mathcal{N})$, the following mappings

$$\delta_S : \mathcal{N} \to \mathbb{R}$$
$$A \mapsto \begin{cases} 1 \text{ if } A = S, \\ 0 \text{ otherwise,} \end{cases}$$

form a special collection of games in $\mathcal{G}(\mathcal{N})$, that are called *identity games*, for $S \in \mathcal{N}$. Note that the mapping $\delta_\emptyset$ is not a game since $\delta_\emptyset(\emptyset) = 1$.

We also introduce *symmetric games*, whose worths depend only on the cardinality of the coalitions, and *equidistributed games* of $\mathcal{G}(\mathcal{N})$, being regular games v that are both symmetric and additive, that is to say, worths $v(S)$ are proportional to s:

$$\exists \nu \in \mathbb{R} \text{ such that } \forall S \in \mathcal{N},\ v(S) = \nu \cdot s.$$

As a consequence of Proposition 2, the material we propose in what follows, is convenient as well for games on convex geometries [1, chap.7], and thus for games with precedence constraints [11], where feasible coalitions of players are the only ones that respect a given precedence structure on the set of players: let $(N, \leqslant)$ be a partially ordered set of players, where $\leqslant$ is a relation of *precedence* in the sense that $i \leqslant j$ if the presence of j enforces the presence of i in any coalition $S \subseteq N$. Hence, a *coalition* of N is a subset S of N such that $i \in S$ and $j \leqslant i$ entails $j \in S$. Consequently, the collection $\mathcal{C}(N)$ of all coalitions of N is the collection of all downsets of $(N, \leqslant)$, which is a distributive regular set system.

3 Probabilistic and efficient values

From now on, $(N, \mathcal{N})$ refers to a regular set system. A *value* on $\mathcal{G}(\mathcal{N})$ is a mapping $\Phi : \mathcal{G}(\mathcal{N}) \to \mathbb{R}^n$ that associates to each game v a vector $(\Phi^1(v), \dots, \Phi^n(v))$, where the real number $\Phi^i(v)$ represents the payoff to player i in the game v. The Shapley value for cooperative games Φ_{Sh} is well known [16].

$$\forall v \in \mathcal{G}(2^N), \forall i \in N, \quad \Phi^i_{Sh}(v) := \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!}(v(S \cup i) - v(S)). \qquad (1)$$

Following the work of Weber [17], Bilbao has defined and axiomatized a class of values for games defined over convex geometries, the *probabilistic values*. It is possible to define such values for regular games.

First, we denote by $S+i$ the coalition $S \cup i$ whenever $S \not\ni i$. Thus, writing $S+i \in \mathcal{N}$ infers two relations: $i \notin S$ and $S \cup i \in \mathcal{N}$. Similarly, $S-i$ denotes the coalition $S \setminus i$ and infers $S \ni i$.

Definition 2 *A value Φ on $\mathcal{G}(\mathcal{N})$ is a* probabilistic value *if there exists for each player i, a collection of real numbers $\left\{p_S^i \mid S \in \mathcal{N}, S+i \in \mathcal{N}\right\}$ satisfying $p_S^i \geq 0$ and $\sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} p_S^i = 1$ such that*

$$\Phi^i(v) = \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} p_S^i \left(v(S \cup i) - v(S)\right), \tag{2}$$

for every game $v \in \mathcal{G}(\mathcal{N})$.

If no condition is required for real numbers p_S^i, then we call Φ a marginalist value*.*

Observe that for a probabilistic value, the participation of player i is assessed to be a weighted average of his marginal contribution $v(S \cup i) - v(S)$ whenever i joins coalition S (provided that $S \cup i$ is a feasible coalition), p_S^i being the subjective probability that i joins S.

In a cooperative game, it is assumed that all players decide to cooperate among them and form the grand coalition N. This leads to the problem of distributing the amount $v(N)$ among them. In this case, a value Φ is *efficient* if it satisfies:

Efficiency axiom (E): $\forall v \in \mathcal{G}(\mathcal{N}), \quad \sum_{i=1}^n \Phi^i(v) = v(N)$.

We consider also the following axioms.

Linearity axiom (L): $\forall i \in N, \forall v, w \in \mathcal{G}(\mathcal{N}), \forall \alpha \in \mathbb{R}$,
$\Phi^i(\alpha v + w) = \alpha \Phi^i(v) + \Phi^i(w)$.

Player i is a *null player* when his contribution to all coalitions $S \cup i \in \mathcal{N}$ formed with his incorporation to $S \in \mathcal{N}$ has no effect.

Definition 3 *A player $i \in N$ is null for $v \in \mathcal{G}(\mathcal{N})$ if*

$$\forall S \in \mathcal{N} \text{ such that } S+i \in \mathcal{N}, \quad v(S \cup i) = v(S).$$

Player i is dummy for $v \in \mathcal{G}(\mathcal{N})$ if

$$\forall S \in \mathcal{N} \textit{ such that } S + i \in \mathcal{N}, \quad v(S \cup i) - v(S) = \begin{cases} v(i), & \textit{if } i \in \mathcal{N} \\ 0, & \textit{else.} \end{cases}$$

Null axiom (N): If player i is null for v, then $\Phi^i(v) = 0$.

The dummy axiom of Bilbao introduced to axiomatize games on convex geometries, is:

Dummy axiom (D): If player i is dummy for v, then $\Phi^i(v) = v(i)$, whenever $i \in \mathcal{N}$ and 0 otherwise.

Monotonicity axiom (M): If the game $v \in \mathcal{G}(\mathcal{N})$ is monotonic, that is to say, $S \subseteq T$ implies $v(S) \leq v(T)$ for all $S, T \in \mathcal{N}$, then the values Φ^i are nonnegative.

Let us present the axiomatization of probabilistic values for games on regular set systems, as already seen in [1].

Proposition 3 *Let Φ a value on $\mathcal{G}(\mathcal{N})$. Under axioms **(L)** and **(N)**, Φ is a marginalist value.*

Proof: First, under **(L)**, for all $i \in N$, there is a unique collection of real numbers $\{c^i_S \mid S \in \mathcal{N}, S \neq \emptyset\}$ such that

$$\Phi^i(v) = \sum_{S \in \mathcal{N}, S \neq \emptyset} c^i_S \, v(S),$$

for every game $v \in \mathcal{G}(\mathcal{N})$. Indeed, the collection of identity games is clearly a basis of $\mathcal{G}(\mathcal{N})$ since every game v can be written as

$$v = \sum_{S \in \mathcal{N}, S \neq \emptyset} v(S) \, \delta_S,$$

in a unique way. By axiom **(L)**,

$$\Phi^i(v) = \sum_{S \in \mathcal{N}, S \neq \emptyset} v(S)\, \Phi^i(\delta_S).$$

Now, this formula can also write

$$\Phi^i(v) = \sum_{\substack{S \in \mathcal{N} \mid \\ S-i \in \mathcal{N} \text{ or } S+i \in \mathcal{N}}} v(S)\, \Phi^i(\delta_S) + \sum_{\substack{S \in \mathcal{N} \mid \\ S-i \notin \mathcal{N} \text{ or } S+i \notin \mathcal{N}}} v(S)\, \Phi^i(\delta_S).$$

Next, assume that $S \in \mathcal{N}$, $i \notin S$ and $S + i \notin \mathcal{N}$ (resp. $S \in \mathcal{N}$, $i \in S$ and $S - i \notin \mathcal{N}$). Thus i is null for δ_S and, by **(N)**, $\Phi^i(\delta_S) = 0$. Then, the second part of the above sum vanishes. Therefore

$$\begin{aligned}
\Phi^i(v) &= \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} [v(S)\, \Phi^i(\delta_S) + v(S \cup i)\, \Phi^i(\delta_{S \cup i})] \\
&= \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} [v(S \cup i) - v(S)]\, \Phi^i(\delta_{S \cup i}) + \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} v(S)\, [\Phi^i(\delta_S) + \Phi^i(\delta_{S \cup i})] \\
&= \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} \Phi^i(\delta_{S \cup i})\, [v(S \cup i) - v(S)] + \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} v(S)\, \Phi^i(\delta_S + \delta_{S \cup i}).
\end{aligned}$$

Since i is null for $\delta_S + \delta_{S \cup i}$ whenever $S \in \mathcal{N}$ and $S + i \in \mathcal{N}$, we conclude that $\Phi^i(\delta_S + \delta_{S \cup i}) = 0$ by **(N)**. Thus

$$\Phi^i(v) = \sum_{S \in \mathcal{N} \mid S+i \in \mathcal{N}} \Phi^i(\delta_{S \cup i})\, [v(S \cup i) - v(S)]. \tag{3}$$

■

Observe now that the dummy axiom implies the null axiom since a null player i is a particular dummy player satisfying $v(i) = 0$. Bilbao has shown that values for games over convex geometries (that are particular regular games) which satisfy axioms **(L), (D), (M)** and **(E)**, are precisely the efficient probabilistic values [1, chap.7].

We improve now this result by weakening the set of axioms and considering more general structures.

Theorem 4 *Let Φ be a value on $\mathcal{G}(\mathcal{N})$. Under axioms **(L)**, **(N)**, **(M)** and **(E)**, Φ is a probabilistic and an efficient value.*

Proof: Whenever $i \notin S$, we denote by p_S^i the coefficient $\Phi^i(\delta_{S \cup i})$ of formula (3) above.

Let choose some $T \in \mathcal{N}$ and define the game of $\mathcal{G}(\mathcal{N})$

$$\hat{u}_T(S) = \begin{cases} 1, & \text{if } S \supsetneq T \\ 0, & \text{else.} \end{cases}$$

By definition, $\hat{u}_T$ is monotonic. Letting $i \in N$ and $T \in \mathcal{N}$ such that $T + i \in \mathcal{N}$, under **(L), (N)** and **(M)**, we get:

$$\begin{aligned} \Phi^i(\hat{u}_T) &= \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p_S^i \left(\hat{u}_T(S \cup i) - \hat{u}_T(S)\right) \\ &= p_T^i \geq 0. \end{aligned}$$

Lastly, it remains to show under the efficiency axiom that for all $i \in N$, the collections of number $\{p_S^i \mid S \in \mathcal{N} : S + i \in \mathcal{N}\}$ form probability distributions, so that we could conclude to the result. For any $i \in N$, let us consider the game

$$u_i(S) = \begin{cases} 1, & \text{if } S \ni i \\ 0, & \text{else.} \end{cases}$$

Then, on the one hand, we have under **(E)**,

$$\sum_{j=1}^{n} \Phi^j(u_i) = u_i(N) = 1.$$

And on the other hand ,

$$\begin{aligned} \sum_{j=1}^{n} \Phi^j(u_i) &= \sum_{j=1}^{n} \sum_{S \in \mathcal{N} | S+j \in \mathcal{N}} p_S^j \left(u_i(S \cup j) - u_i(S)\right) \\ &= \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p_S^i \left(u_i(S \cup i) - u_i(S)\right) + \sum_{j \in N \setminus i} \sum_{S \in \mathcal{N} | S+j \in \mathcal{N}} p_S^j \left(u_i(S \cup j) - u_i(S)\right) \\ &= \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p_S^i, \end{aligned}$$

since differences of the first sum always worth 1 whereas differences of the second one vanish. This achieves the proof. ■

We present now an important result about marginalist values, already known for convex geometries [1, chap.7].

Proposition 5 *Let Φ be a marginalist value on $\mathcal{G}(\mathcal{N})$, defined by*

$$\Phi^i(v) = \sum_{S\in\mathcal{N}|S+i\in\mathcal{N}} p^i_S\,(v(S\cup i) - v(S)),$$

for every game v and for all $i \in N$, where p^i_S are real numbers. Then Φ satisfies the efficiency axiom if and only if

$$\sum_{i\in N|i\in\mathcal{N}} p^i_\emptyset = \sum_{i\in N|N\setminus i\in\mathcal{N}} p^i_{N\setminus i} = 1, \tag{4}$$

$$\sum_{i\in N|S-i\in\mathcal{N}} p^i_{S\setminus i} = \sum_{i\in N|S+i\in\mathcal{N}} p^i_S, \tag{5}$$

for all $S \in \mathcal{N} \setminus \{\emptyset, N\}$.

Proof: For every $v \in \mathcal{G}(\mathcal{N})$, we compute the sum of the values $\Phi^i(v)$.

$$\begin{aligned}
\sum_{i=1}^n \Phi^i(v) &= \sum_{i=1}^n \sum_{S\in\mathcal{N}|S+i\in\mathcal{N}} p^i_S\,(v(S\cup i) - v(S))\\
&= \sum_{S\in\mathcal{N}} \Big(\sum_{i\in N|S-i\in\mathcal{N}} p^i_{S\setminus i} - \sum_{i\in N|S+i\in\mathcal{N}} p^i_S \Big)\, v(S)\\
&= \sum_{\substack{S\in\mathcal{N}|\\ S\neq\emptyset,N}} \Big(\sum_{i\in N|S-i\in\mathcal{N}} p^i_{S\setminus i} - \sum_{i\in N|S+i\in\mathcal{N}} p^i_S \Big)\, v(S) + \Big(\sum_{i\in N|N-i\in\mathcal{N}} p^i_{N\setminus i} \Big)\, v(N).
\end{aligned}$$

If the coefficients satisfy (4) and (5), then it is clear that Φ satisfies the efficiency axiom.

Conversely, fix $T \in \mathcal{N}$ such that $T \neq \emptyset, N$ and consider the identity game δ_T. The efficiency axiom straightforwardly implies that $\sum_{i=1}^n \Phi^i(\delta_T) = 0$. Applying the above equality to δ_T, we have

$$\sum_{i=1}^n \Phi^i(\delta_T) = 0 = \sum_{i\in N|T-i\in\mathcal{N}} p^i_{T\setminus i} - \sum_{i\in N|T+i\in\mathcal{N}} p^i_T,$$

that is to say (5) is proven. If $T = N$, then the equality becomes

$$\sum_{i=1}^n \Phi^i(\delta_N) = 1 = \sum_{i\in N|N-i\in\mathcal{N}} p^i_{N\setminus i},$$

which partially proves (4). Finally, consider the game $\hat{u}_\emptyset := \sum_{T\in\mathcal{N}, T\neq\emptyset} \delta_T$. Then

$$\sum_{i=1}^{n} \Phi^i(\hat{u}_\emptyset) = 1 = \sum_{i=1}^{n} \sum_{S\in\mathcal{N}|S+i\in\mathcal{N}} p_S^i \left(\hat{u}_\emptyset(S\cup i) - \hat{u}_\emptyset(S)\right)$$
$$= \sum_{i\in N|i\in\mathcal{N}} p_\emptyset^i\, \hat{u}_\emptyset(i) = \sum_{i\in N|i\in\mathcal{N}} p_\emptyset^i,$$

which achieves the proof. ■

4 The Shapley value for regular games

If we focus now on the particular case of classical cooperative games, we know that Weber has characterized the Shapley value on $\mathcal{G}(2^N)$ as the unique probabilistic value satisfying the well known *symmetry axiom*, assuming that the coefficients of the value should not depend on the labelling of the elements of N, that is a very natural property.

The fundamental idea of the symmetry axiom rests on permutations of players. Symmetry could be naturally defined in regular games. Unfortunately, this generalization has a very limited interest: apart from particular cases of regular set systems, players generally cannot be permuted, which leaves this axiom ineffective.

Faigle and Kern attempted to generalize the Shapley value for their *games under precedence constraints* [11], which are games over distributive regular set systems (cf. Proposition 2), and thus particular regular games: in their framework, maximal chains correspond to what they call feasible ranking of players. For that, the *hierarchical strength axiom* is introduced, which is actually difficult to interpret.

We propose a different approach for the axiomatization of the Shapley value on $\mathcal{G}(\mathcal{N})$. Consider first the classical Shapley value on $\mathcal{G}(2^N)$ where $p_S^i := \frac{s!(n-s-1)!}{n!}$ (cf. (1)).

Observing that for any subsets $A, A+i+j \subseteq N$, the equality $p_A^i + p_{A\cup i}^j = p_A^j + p_{A\cup j}^i$ holds, one may wonder if this property is sufficient to form the Shapley value from a probabilistic and efficient value. Actually, the answer is positive. Furthermore, one can generalize it to any case of game defined on a regular set system.

In this perspective, we introduce the following material. Let A, B be any two coalitions of $\mathcal{N}$ such that $A \subseteq B$, and $\mathcal{C} := (S_a, S_{a+1}, \ldots, S_b)$ be a maximal chain from A to B. Thus we denote by $\sigma^{\mathcal{C}}$ the mapping defined over $\{a+1, \ldots, b\}$ by $\sigma^{\mathcal{C}}(i) := S_i \setminus S_{i-1}$. Remark that σ is a permutation of N if $A = \emptyset$ and $B = N$. Besides, for any marginalist value Φ whose coefficients are given by (2) and any game $v \in \mathcal{G}(\mathcal{N})$, we denote the cumulative sum of marginal contributions of players of $B \setminus A$ along $\mathcal{C}$ by:

$$m_{\Phi}^{\mathcal{C}}(v) := \sum_{i=a+1}^{b} p_{S_{i-1}}^{\sigma^{\mathcal{C}}(i)} \left(v(S_i) - v(S_{i-1})\right).$$

Let us consider any equidistributed game (Section 2). Thus, *for such a game, the cumulative sum of expected marginal contributions of players of $B \setminus A$ involved should not depend on the considered maximal chain from A to B*, since the path taken from A to B has no effect on the successive increasing worth $v(C)$, $A \subseteq C \subseteq B$.

Indeed, if we assume that $\mathcal{N}$ contains the coalitions $1, 12, 2, 23$ and 123, for instance, then the following equalities should hold for every equidistributed game v:

$$\begin{aligned} & p_{\emptyset}^{1}\, v(1) + p_{1}^{2}\,(v(12) - v(1)) + p_{12}^{3}\,(v(123) - v(12)) \\ & \quad = p_{\emptyset}^{2}\, v(2) + p_{2}^{3}\,(v(23) - v(2)) + p_{23}^{1}\,(v(123) - v(23)), \\ & p_{2}^{1}\,(v(12) - v(2)) + p_{12}^{3}\,(v(123) - v(12)) \\ & \quad = p_{2}^{3}\,(v(23) - v(2)) + p_{23}^{1}\,(v(123) - v(23)), \end{aligned}$$

that are respectively equivalent to

$$\begin{aligned} p_{\emptyset}^{1} + p_{1}^{2} + p_{12}^{3} &= p_{\emptyset}^{2} + p_{2}^{3} + p_{23}^{1}, \\ p_{2}^{1} + p_{12}^{3} &= p_{2}^{3} + p_{23}^{1}, \end{aligned}$$

since v is equidistributed.

In this spirit, we propose the following axiom.

Regularity axiom (R): For any equidistributed game $v \in \mathcal{G}(\mathcal{N})$, for any couple of maximal chains $\mathcal{C}_1, \mathcal{C}_2$ of $(N, \mathcal{N})$, then $m_{\Phi}^{\mathcal{C}_1}(v) = m_{\Phi}^{\mathcal{C}_2}(v)$.

We call *regular value* any marginalist value satisfying the regularity axiom.

Actually, the regularity axiom may be seen as a generalization of the Shapley's symmetry axiom. The next result confirms this view, and gives a generalization of the Shapley's result asserting the unicity of the value Φ under the linearity axiom, the null axiom, the efficiency axiom and the regularity axiom.

Theorem 6 *Let $(N, \mathcal{N})$ be a regular set system. Then there is a unique efficient regular value Φ_K on $\mathcal{G}(\mathcal{N})$.*

To show this result, we need to introduce new material and definitions, that is done in the next section and in Appendix .

5 The Shapley value in the framework of network theory

Considering results of previous section, let us fix a regular set system $(N, \mathcal{N})$ and a marginalist value Φ on $\mathcal{G}(\mathcal{N})$, and let $\{p_S^i \mid i \in N, S \in \mathcal{N} \text{ s.t. } S + i \in \mathcal{N}\}$ be the set of associated coefficients. We associate to any regular set system $(N, \mathcal{N})$ an *electrical network*, that is, an interconnection of electrical components such as *resistors*. Precisely, we consider the mapping

$$(N, \mathcal{N}) \mapsto \mathcal{E}(N, \mathcal{N}),$$

where $\mathcal{E}(N, \mathcal{N})$ is built in this way: *nodes* of $\mathcal{E}(N, \mathcal{N})$ are simply the elements of $\mathcal{N}$, whereas its *branches* are directed wires given by the couples $b_S^i := (S, S + i)$ of $(N, \mathcal{N})$. Note that at this stage, $\mathcal{E}(N, \mathcal{N})$ may be seen as the Hasse diagram of $(N, \mathcal{N})$, since the b_S^i's are given by the covering relation $\prec$ of $(N, \mathcal{N})$. We complete the building by adding another branch $b_{N,\emptyset}$ by connecting the node N with the node $\emptyset$.

In $\mathcal{E}(N, \mathcal{N})$, we call *circuit*[1], any sequence $(b_1, \dots, b_m)$ of branches such that the b_j's are different and two consecutive edges are incident, as well as b_1 and b_m. For convenience, we also may write a circuit as a sequence of m nodes $(S_0, S_1, \dots, S_m = S_0)$.

Now, one can attribute to the branches b_S^i (resp. $b_{N,\emptyset}$) some weights $I_{S \to S \cup i}$ and $V_{S \to S \cup i}$ (resp. $I_{N \to \emptyset}$ and $V_{N \to \emptyset}$), where I_b's are the worths of a directed

[1]This terminology shall also be used for $(N, \mathcal{N})$.

commodity flowing in the branches called *electrical current*, and where V_b's are worths proportional to currents I_b's: V_b's are called *potential drops* and satisfy the well-known *Ohm's law*: $V_b = R_b \cdot I_b$, where R_b is said to be the *resistance* of the non-oriented branch b. In our framework, we assign for any electrical network a unitary resistance to every branch b_S^i, so that $V_b = I_b$. Precisely, the necessary and sufficient following conditions for the electrical current and potential drops must be satisfied:

> **First Kirchhoff's law:** The sum of all currents entering a node is equal to the sum of all currents leaving the node.

> **Second Kirchhoff's law:** The directed sum of the electrical potential drops around a circuit must be zero.

Therefore, let us now assign to any branch b_S^i the coefficient p_S^i of the marginalist value Φ, with $I_{N\to\emptyset} := 1$. Thus Proposition 5 asserts that Φ satisfies the efficiency axiom if and only if the first Kirchhoff's law is satisfied in $\mathcal{E}(N, \mathcal{N})$.

We also establish a straight link between the second Kirchhoff's law and the regularity axiom in Corollary 8. In this respect, for any circuit $\mathcal{M} := (b_1, \dots, b_m)$ on $(N, \mathcal{N})$, for $j = 1, \dots, m$, if b_j represents the couple $(S, S+i)$, we denote by $\bar{p}_j = \bar{p}_S^i$ the signed coefficient of Φ associated to b_j relatively to its orientation in $\mathcal{M}$:

$$\bar{p}_S^i := \begin{cases} \ \ p_S^i & \text{if in } \mathcal{M},\ b_j \text{ is directed in accordance with } \subseteq, \\ -\, p_S^i & \text{otherwise.} \end{cases}$$

As a consequence, the second Kirchhoff's law fitted for the marginalist value Φ may be expressed by

> **Circuit property:** For any circuit $(b_1, \dots, b_m)$ of $(N, \mathcal{N})$, $\sum_{j=1}^m \bar{p}_j = 0$.

We call *potential* over $(N, \mathcal{N})$ any real-valued mapping $\mathbf{V}$ defined on $\mathcal{N}$ satisfying for any coefficient p_S^i

$$p_S^i = \mathbf{V}(S+i) - \mathbf{V}(S). \tag{6}$$

Proposition 7 *Let Φ be a marginalist value on $\mathcal{G}(\mathcal{N})$ and $\{p^i_S \mid i \in N, S \in \mathcal{N} \mid S+i \in \mathcal{N}\}$ be the set of associated coefficients. Then the circuit property holds if and only if there exists a potential over $(N, \mathcal{N})$.*

Moreover, for a given instance of coefficients $p := (p^i_S)_{(S,S+i)\in\mathcal{N}^2}$ satisfying the circuit property, there is a unique potential $\mathbf{V}$ vanishing at $\emptyset$. We call it the potential associated to p and grounded on $\emptyset$, *and denote it by $\mathbf{V}^p_{\mathbf{0}}$.*

Proof: The sufficiency condition is clear. Indeed, let $(b_1, \ldots, b_m)$ be a circuit of $(N, \mathcal{N})$, where the b_j's belong to E and $(S_0, S_1, \ldots, S_m = S_0)$ be the same circuit expressed in terms of coalitions. Then

$$\begin{aligned}\sum_{j=1}^{m} V_{b_j} &= \sum_{j=1}^{m} (\mathbf{V}(S_j) - \mathbf{V}(S_{j-1})) \\ &= \mathbf{V}(S_m) - \mathbf{V}(S_0) \\ &= 0.\end{aligned}$$

Conversely, assume that the circuit property holds. Let us define by induction on the cardinality of the coalitions, the following correspondance:

$$\begin{gathered}\mathbf{V}(\emptyset) := 0, \\ \forall S \in \mathcal{N} \setminus \{\emptyset\}, \text{ and } i \in S,\ \mathbf{V}(S) := \mathbf{V}(S \setminus i) + p^i_{S\setminus i}.\end{gathered}$$

Then by definition, $\mathbf{V}$ has the required property of a potential. Thus it remains to show that this mapping is properly defined. Indeed, let $S \in \mathcal{N}$ and i, k two distinct players in S. By an inductive argument, we make the assumption that $\mathbf{V}$ is properly defined on $\{T \in \mathcal{N} \mid t < s\}$. Thus we must show that $\mathbf{V}(S \setminus i) + p^i_{S\setminus i} = \mathbf{V}(S \setminus k) + p^k_{S\setminus k}$. Let S_0 be any maximal element of the set $\{T \in \mathcal{N} \mid T \subseteq S \setminus i, T \subseteq S \setminus k\}$ and $m := s - s_0$ $(m \geq 2)$. Thus there is a circuit $\mathcal{M} = (b_1, \ldots, b_{2m})$ such that b_1 is directed from S_0, $b_m = b^i_{S\setminus i}$ and $b_{m+1} = b^k_{S\setminus k}$. By induction, we easily compute $\mathbf{V}(S\setminus i) = \mathbf{V}(S_0) + \sum_{j=1}^{m-1} p(b_j)$ and $\mathbf{V}(S\setminus k) = \mathbf{V}(S_0) + \sum_{j=m+2}^{2m} p(b_j)$. Besides, if we denote by $(S_0, S_1, \ldots, S_{2m} = S_0)$

the same circuit in terms of coalitions, we get by the circuit property

$$\sum_{j=1}^{2m} V_{b_j} = 0$$
$$\text{iff} \ \sum_{j=1}^{m} p(b_j) = \sum_{j=m+1}^{2m} p(b_j)$$
$$\text{iff} \ \mathbf{V}(S_0) + \sum_{j=1}^{m-1} p(b_j) + p^i_{S\setminus i} = \mathbf{V}(S_0) + \sum_{j=m+2}^{2m} p(b_j) + p^k_{S\setminus k}$$
$$\text{iff} \ \mathbf{V}(S \setminus i) + p^i_{S\setminus i} = \mathbf{V}(S \setminus k) + p^k_{S\setminus k}.$$

Now, we show that if there are two mappings $\mathbf{V}_1$ and $\mathbf{V}_2$ satisfying (6), then they are the same up to an additive constant. Let us denote by c the real number $\mathbf{V}_2(\emptyset) - \mathbf{V}_1(\emptyset)$. For any $S \in \mathcal{N}$, there is a maximal chain $(T_0 = \emptyset, T_1, \ldots, T_s = S)$ from $\emptyset$ to S. Thus, by denoting i_j the singleton $T_j \setminus T_{j-1}$, we get by (6)

$$\mathbf{V}_k(S) = \mathbf{V}_k(T_s) = \mathbf{V}_k(T_{s-1}) + p^{i_s}_{T_{s-1}} = \mathbf{V}_k(T_{s-2}) + p^{i_{s-1}}_{T_{s-2}} + p^{i_s}_{T_{s-1}} = \cdots = \mathbf{V}_k(\emptyset) + \sum_{j=1}^{s} p^{i_j}_{T_{j-1}},$$

for $k = 1, 2$. Thus, $\mathbf{V}_2(S) - \mathbf{V}_1(S) = \mathbf{V}_2(\emptyset) - \mathbf{V}_1(\emptyset) = c$. As a consequence, by fixing $\mathbf{V}$ on any vertex, we have a unique possibility for the potential, which gives $\mathbf{V}^p_{\mathbf{0}}$ if $\emptyset$ is assigned to 0. ■

Lastly, we deduce the following result, asserting that Φ satisfies the regularity axiom if and only if the second Kirchhoff's law is satisfied in $\mathcal{E}(N, \mathcal{N})$.

Corollary 8 *Let Φ be a marginalist value on $\mathcal{G}(\mathcal{N})$:*

$$\Phi^i(v) = \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p^i_S \left(v(S \cup i) - v(S)\right),$$

where p^i_S are real numbers, for all $i \in N$. Then Φ satisfies the regularity axiom if and only if the circuit property holds.

Proof: The regularity axiom means that along any maximal chain of $(N, \mathcal{N})$, for an equidistributed game v, the cumulative sum of marginal contributions of players do not depend on the considered maximal chain. Since v is equidistributed, worths $v(S)$ only depend on a multiplicative constant, thus it is equivalent to only consider coefficients along the chain.

Assuming firstly that the circuit property holds, we consider any potential $\mathbf{V}$ associated to the coefficients p_S^i's. Therefore, if $\mathcal{C} := (S_0 = \emptyset, S_1, \ldots, S_n = N)$ is any maximal chain of $(N, \mathcal{N})$ and v is any equidistributed game defined by $v(S) := \nu \cdot s,\ \forall S \in \mathcal{N}$, we obtain

$$\begin{aligned} m_\Phi^{\mathcal{C}}(v) &= \nu \sum_{j=1}^{n} (\mathbf{V}(S_j) - \mathbf{V}(S_{j-1})) \\ &= \nu \cdot (\mathbf{V}(N) - \mathbf{V}(\emptyset)), \end{aligned}$$

which does not depend on $\mathcal{C}$, and thus the regularity axiom holds.

Conversely, if the regularity axiom holds, we build the same correspondance than in the proof of Proposition 7, that is to say

$$\begin{aligned} \mathbf{V}(\emptyset) &:= 0, \\ \forall S \in \mathcal{N} \setminus \{\emptyset\},\ \text{and } i \in S,\ \mathbf{V}(S) &:= \mathbf{V}(S \setminus i) + p^i_{S \setminus i}. \end{aligned}$$

Then by a similar argument than in the proof of Proposition 7, we show that $\mathbf{V}$ is a potential, which will implies that the circuit property holds. Once again, we just have to show that $\mathbf{V}$ is properly defined, that is to say, by making again the assumption that $\mathbf{V}$ is properly defined on $\{T \in \mathcal{N} \mid t < |S|\}$, where $S \in \mathcal{N}$ and $i, k \in S$ $(i \neq k)$, we must show that $\mathbf{V}(S \setminus i) + p^i_{S \setminus i} = \mathbf{V}(S \setminus k) + p^k_{S \setminus k}$. Let $\mathcal{C}^1 := (b^1_1, \ldots, b^1_n)$ and $\mathcal{C}^2 := (b^2_1, \ldots, b^2_n)$ be any two maximal chains such that $b^1_s = (S \setminus i, S)$, $b^2_s = (S \setminus k, S)$ and $b^1_j = b^2_j$ for every $j > s$. We easily verify that $\mathbf{V}(S \setminus i) = \sum_{j=1}^{s-1} p(b^1_j)$ and $\mathbf{V}(S \setminus k) = \sum_{j=1}^{s-1} p(b^2_j)$. Besides, by the regularity axiom applied on $\mathcal{C}^1$ and $\mathcal{C}^2$

$$\begin{aligned} & \sum_{j=1}^{n} p(b^1_j) = \sum_{j=1}^{n} p(b^2_j) \\ \text{iff } & \mathbf{V}(S \setminus i) + p^i_{S \setminus i} + \sum_{j=s+1}^{n} p(b^1_j) = \mathbf{V}(S \setminus k) + p^k_{S \setminus k} + \sum_{j=s+1}^{n} p(b^2_j) \\ \text{iff } & \mathbf{V}(S \setminus i) + p^i_{S \setminus i} = \mathbf{V}(S \setminus k) + p^k_{S \setminus k}. \end{aligned}$$

■

To sum up, we could say with a slight abuse of language that the unique efficient regular value on $\mathcal{G}(\mathcal{N})$, is also the unique marginalist value satisfying the Kirchhoff's laws in the sense that it satisfies (4), (5) and the circuit property. That is why we may call Φ_K the *Kirchhoff's value* or also the *Shapley-Kirchhoff value.*

This being introduced, we have now a sufficient material to show Theorem 6, that is made in Appendix .

6 Shapley-Kirchhoff value and monotonicity axioms

At this point of the work, a natural question arises about a last property of the Shapley-Kirchhoff value. Indeed, Φ_K satisfies linearity axiom, null axiom, efficiency axiom and regularity axiom. However, we have no information about the monotonicity axiom, that would make of Φ_K a probabilistic value. With this in mind, we present the following handy short result.

Lemma 9 *Let Φ be a marginalist value on $\mathcal{G}(\mathcal{N})$, defined by*

$$\Phi^i(v) = \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p_S^i \, (v(S \cup i) - v(S)),$$

for every game v and for all $i \in N$, where p_S^i are real numbers. Then Φ satisfies the monotonicity axiom if and only if all p_S^i's are nonnegative.

Proof: Let v be a game on $\mathcal{G}(\mathcal{N})$. If all p_S^i's are nonnegative and v monotonic, $\Phi^i(v)$ is clearly nonnegative for any $i \in N$.

Conversely, let $i \in N$ and $S \in \mathcal{N}$ with $S + i \in \mathcal{N}$, such that $p_S^i < 0$. Let $\hat{u}_S \in \mathcal{G}(\mathcal{N})$ be the *unanimity game* defined by

$$\forall T \in \mathcal{N}, \quad \hat{u}_S(T) = \begin{cases} 1 \text{ if } T \supsetneq S \\ 0 \text{ otherwise.} \end{cases}$$

Thus, u_S is clearly monotonic, nevertheless $\Phi^i(v) = p_S^i < 0$. ■

In the light of this result, for Φ_K to satisfy monotonicity axiom, we should show that coefficients p^i_S's are nonnegative. Actually, it turns out that there are regular set systems for which the Shapley-Kirchhoff value does not satisfy the monotonicity axiom, as the counterexample given in Appendix proves it.

Nevertheless, it appears that there are other kinds of monotonicity axioms in the framework of cooperative games, as Young worked out in [18]. Indeed, monotonicity is a general principle of fair division which states that as the underlying data of a problem change, the solution should change in parallel fashion. We give the following monotonicity axioms adapted to our framework (cf. p.111).

A frequently encountered form of monotonicity is *aggregate monotonicity.* This principle states that if the worth of the coalition of the whole increases, while the worth of all other coalitions remains fixed, then no player should get less than before.

> **Aggregate monotonicity axiom (AM):** Let v, w two games in $\mathcal{G}(\mathcal{N})$ such that $w(N) \geq v(N)$ and $w(S) = v(S)$ for other coalitions $S \in \mathcal{N}$. Then, $\forall i \in N, \quad \Phi^i(w) \geq \Phi^i(v)$.

Coalitional monotonicity is satisfied if an increase in the worth of a particular coalition implies no decrease in the allocation to any member of that coalition. Thus the following axiom is stronger than the previous one.

> **Coalitional monotonicity axiom (CM):** Let $S \in \mathcal{N}$ and v, w two games in $\mathcal{G}(\mathcal{N})$ such that $w(S) \geq v(S)$ and $w(T) = v(T)$ for other coalitions $T \in \mathcal{N}$. Then, $\forall i \in S, \quad \Phi^i(w) \geq \Phi^i(v)$.

Coalitional monotonicity refers to monotonic changes in the *absolute* worth of the coalitions a given player. There are also situations where the worth of coalitions containing a given player i increase *relatively* to the worth of coalitions not containing i:

> **Strong monotonicity axiom (SM):** Let v, w two games in $\mathcal{G}(\mathcal{N})$ and a player i such that for every $S \in \mathcal{N}$ satisfying $S + i \in \mathcal{N}$, $w(S \cup i) - w(S) \geq v(S \cup i) - v(S)$. Then $\Phi^i(w) \geq \Phi^i(v)$.

Having described these monotonicity axioms, we give results in context of games over regular set systems.

Proposition 10 *Under linearity and null axioms,* ***(M)****,* ***(SM)*** *and* ***(CM)*** *are equivalent.*
In addition, these axioms are strictly stronger than ***(AM)****, whatever the regular set system is.*

Proof: We successively show that **(M)** $\Leftrightarrow$ **(SM)** and **(M)** $\Leftrightarrow$ **(CM)**.

- Due to Lemma 9, if **(M)** holds, then **(SM)** also holds since $\Phi^i(v)$ depends only on the marginal contributions of player i. Conversely, let $i \in N$ and $S \in \mathcal{N}$ with $S + i \in \mathcal{N}$, such that $p^i_S < 0$, so that **(M)** is not true. Let v, w be any two games such that $w(S \cup i) - w(S) = v(S \cup i) - v(S) + 1$, and for any other coalition $T \in \mathcal{N}$ satisfying $T + i \in \mathcal{N}$, $w(T \cup i) - w(T) = v(T \cup i) - v(T)$. Then assumption of **(SM)** is satisfied but $\Phi^i(w) - \Phi^i(v) = p^i_S < 0$, that is to say **(SM)** does not hold.

- Due to Lemma 9, if **(M)** holds then **(CM)** clearly holds. Indeed, if only the worth of a coalition S increases in the game v, then for any player i of the coalition, in $\Phi^i(v)$, the associated marginal contribution $p^i_{S \setminus i}(v(S) - v(S \setminus i))$ increases, and other marginal contribution remain the same. Conversely, let us assume that **(M)** is not satisfied, that is to say there is a player $i \in N$ and a coalition $S \in \mathcal{N}$ with $S + i \in \mathcal{N}$, such that $p^i_S < 0$. Let v, w be any two games such that $w(S \cup i) = v(S \cup i) + 1$, and $w(T) = v(T)$ for any other coalition $T \in \mathcal{N}$. Then $\Phi^i(w) - \Phi^i(v) = p^i_S < 0$, that is to say **(SM)** does not hold.

Second part of the result is clear and is an immediate consequence of Lemma 11.
■

Thus, as the aggregate monotonicity axiom is weaker than the classical one, one may wonder if the Shapley-Kirchhoff values satisfies it. The answer is actually positive and rests on the following characterization.

Lemma 11 *Let Φ be a marginalist value on $\mathcal{G}(\mathcal{N})$, defined by*

$$\Phi^i(v) = \sum_{S \in \mathcal{N} | S+i \in \mathcal{N}} p_S^i \left(v(S \cup i) - v(S)\right),$$

for every game v and for all $i \in N$, where p_S^i are real numbers. Then Φ satisfies the aggregate monotonicity axiom if and only if for every i in N such that $N \setminus i \in \mathcal{N}$, $p_{N \setminus i}^i$ is nonnegative.

Proof: The sufficient condition derives from Lemma 9. Indeed, if all $p_{N\setminus i}^i$'s are nonnegative and v, w any two games having the same worths, except for N with $w(N) \geq v(N)$, then $\Phi^j(w) - \Phi^j(v) = p_{N \setminus j}^j \left(w(N) - v(N)\right)$ or vanishes, depending on whether $N \setminus j$ is a coalition of $\mathcal{N}$ or not. Conversely, let us assume that there is a player $i \in N$ satisfying $N \setminus i \in \mathcal{N}$, such that $p_{N\setminus i}^i < 0$. Let v, w be any two games such that $w(N) = v(N) + 1$ and $w(T) = v(T)$ for any other coalition. Then $\Phi^i(w) - \Phi^i(v) = p_{N\setminus i}^i < 0$. Thus the necessary condition is satisfied. ■

Finally, we show this final result, whose proof uses the *Maximum Principle* applied to a valued graph (see Appendix).

Theorem 12 *For any regular set system $(N, \mathcal{N})$, the Shapley-Kirchhoff value Φ_K satisfies the aggregate monotonicity axiom.*

Appendix A. Dirichlet problem on a graph and potentials

The problem of finding the currents on the branches of a resistor network, an entering current being given, is easy to solve if seen as the solution of a Dirichlet problem associated to that network [8, 6]. Thus, we introduce now a few rudimentary notions of discrete *potential theory*, in order to prove Theorems 6 and 12.

Let $G = (V, E)$ be a non-oriented connected graph where the set of vertices V is randomly divided into two distinct categories, a non-empty set of *boundary points*

V_0 and the set of *interior points* $V_1 = V \setminus V_0$. For any interior point x, we denote by $d(x)$ the degree of vertex x in G, that is to say $d(x) := \big|\{y \in V \mid \{x,y\} \in E)\}\big|$. A function f defined on V is said to be *harmonic* on G if, for points x in V_1, it has the averaging property

$$f(x) = \frac{\sum_{\{x,y\}\in E} f(y)}{d(x)}, \tag{7}$$

with no restriction on the values of f at the boundary points.

Now the problem of finding a harmonic function given its boundary values is called the *Dirichlet problem*, and the *Uniqueness Principle* for the Dirichlet problem asserts that there cannot be two different harmonic functions having the same boundary values. We approach the Uniqueness Principle by way of the *Maximum Principle* for harmonic functions.

> **Maximum Principle.** A harmonic function f defined on V takes on its maximum value M and its minimum value m on the boundary.

Proof: If M is the maximum value of f and if $f(x) = M$ for x an interior point, then since $f(x)$ is the average of the values of f at its neighbors, these values must all equal M also. By working our way, repeating this argument at every step, we eventually reach a boundary point x_0 for which we can conclude that $f(x_0) = M$. That same argument works for the minimum value m. ■

> **Uniqueness Principle.** If f and g are harmonic on V such that $f = g$ on V_0, then $f(x) = g(x)$ for all $x \in V$.

Proof: Let $h := f - g$. Then if x is any interior point,

$$\frac{\sum_{\{x,y\}\in E} h(y)}{d(x)} = \frac{\sum_{\{x,y\}\in E} f(y)}{d(x)} - \frac{\sum_{\{x,y\}\in E} g(y)}{d(x)} = f(x) - g(x) = h(x).$$

Therefore h is a harmonic function which vanishes on V_0, and hence, by the Maximum Principle, the maximum and minimum values of h are 0. Thus h is identically null, and $f = g$. ■

Once again, let us consider graph $G = ((V_0, V_1), E)$ as an electrical network, where $V = V_0 \cup V_1$ is the set of nodes, with $|V_0| = 2$, and E is the set of branches, each of them having a unit resistance. If a voltage $\mathbf{V}$ is given on V_0, and satisfies the first Kirchhoff's law for interior points:

$$\forall x \in V_1, \sum_{\{x,y\}\in E} (\mathbf{V}(y) - \mathbf{V}(x)) = 0, \tag{8}$$

then $\mathbf{V}$ is precisely a harmonic function. Indeed, since all resistances of the branches are unitary, the current in the oriented branch (x, y) is expressed by the potential drop $\mathbf{V}(y) - \mathbf{V}(x)$.

Let x be an interior node, then solving (8) for $\mathbf{V}(x)$ straightforwardly gives the averaging property (7), and by the Uniqueness Principle, we conclude that $f = \mathbf{V}$. It remains to show that such a function exists, which rests on basic linear algebra. Indeed, by expressing (8) (or (7)) for each interior point, we easily obtain a linear system $\Gamma(\mathbf{V_1}) = B$, Γ being a $v_1 \times v_1$ morphism, $\mathbf{V_1}$ being the vector of unknown variables $(\mathbf{V}(x))_{x\in V_1}$, and where the right-hand member B depends only on the values taken by $\mathbf{V}$ on the boundary points. Γ being an injective endomorphism, then it is also bijective. Thus by

$$\mathbf{V_1} = \Gamma^{-1}(B), \tag{9}$$

the existence of $\mathbf{V}$ follows.

Proof: [Theorem 6]
Let $V_0 := \{\emptyset, N\}$ and $V_1 := \mathcal{N} \setminus V_0$. By the Uniqueness Principle, given a real number R, there is a unique potential $\mathbf{V}$ defined on $\mathcal{N}$, such that $\mathbf{V}(\emptyset) = 0$, $\mathbf{V}(N) = R$, and satisfying the first Kirchhoff's law on V_1. Moreover, we know by (9) that $\mathbf{V}$ is linearly dependent on R, that is to say, proportional to R:

$$\forall S \in \mathcal{N},\ \mathbf{V}(S) = \mathbf{V_0^1}(S) \cdot R,$$

where $\mathbf{V_0^1}$ is the required potential for $R = 1$. Thus one can adjust the value of R so that

$$\sum_{S\succ\emptyset} \mathbf{V}(S) = 1. \tag{10}$$

Indeed, let $C_{\mathcal{N}} := \sum_{S\succ\emptyset} \mathbf{V_0^1}(S)$ and $R_{\mathcal{N}} := C_{\mathcal{N}}^{-1}$ (note that $C_{\mathcal{N}}$ is non null since by the argument used for Maximum Principle, all $\mathbf{V_0^1}(S)$'s, $S \neq \emptyset$, are necessarily strictly positive). Thus for $R = R_{\mathcal{N}}$, (10) holds. We denote by $\mathbf{V_0}$ the associated potential, that is to say, satisfying $\mathbf{V_0}(\emptyset) = 0$ and $\mathbf{V_0}(N) = R_{\mathcal{N}}$.

Now, for any $(S, S+i) \in \mathcal{N}$, define $p_S^i := \mathbf{V_0}(S+i) - \mathbf{V_0}(S)$. Therefore, the marginalist value associated to the coefficients p_S^i's is an efficient regular value. Indeed, by Proposition 7, the circuit property holds, and thus by Corollary 8, the regularity axiom also holds. Besides, by Proposition 5, the efficency axiom holds, (5) being equivalent to (8), and (4) being expressed by (10) on the one hand, and by conservation of the flow pattern $(p_S^i)_{(S,S+i)\in\mathcal{N}^2}$ on the other hand.

Lastly, coefficients p_S^i's being determined by the unique potential grounded on $\emptyset$ and satisfying (10), the unicity of Φ_K is also shown. ■

Proof: [Theorem 12]
Let $\mathbf{V}$ be a potential associated to the coefficients p_S^i's of Φ_K. The source node of the electrical network being $\emptyset$, $\mathbf{V}$ is necessarily greater on bound N than bound $\emptyset$. Then by the Maximum Principle, $\forall i \in N$ such that $N \setminus i \in \mathcal{N}$, $p_{N\setminus i}^i = \mathbf{V}(N) - \mathbf{V}(N \setminus i)$ is nonnegative. By Lemma 11, the result follows. ■

Appendix B. Example of regular set system for which Φ_K does not satisfy the monotonicity axiom

Let $N := \{\alpha, \beta, \gamma, 1, 2, \dots, n'\}$ be the set of players, with $N' := \{1, \dots, n'\}$ and $n' \geq 1$. We make the assumption that the set of coalitions $\mathcal{N}$ is defined by

$$\mathcal{N} := \Big\{\emptyset, N, N \setminus \beta, N \setminus \gamma\Big\} \cup \bigcup_{i=0}^{n'} \Big\{\{\alpha, 1, 2, \dots, i\}\Big\} \cup \bigcup_{S \in 2^{N'}} \Big\{S \cup \beta\Big\}.$$

Then $(N, \mathcal{N})$ is a regular set system. The associated Shapley-Kirchhoff value defined over $\mathcal{G}(\mathcal{N})$ satisfies the monotonicity axiom if and only if $n' < 5$. Indeed, we show that if it is not the case, coefficient $p_{N'\cup\alpha}^{\beta}$ is negative, which corresponds to a negative current in the associated directed branch $(N' \cup \alpha, N' \cup \alpha\beta)$ (cf. Section 5). Fig. 2 is given with $n' = 5$, where the bold line represents the above mentionned branch.

Proof: The first point is to verify that $(N, \mathcal{N})$ is a regular set system. By checking that each coalition S of $\mathcal{N}$ has all its successors (for inclusion order) T satisfying $|T \setminus S| = 1$, we have the result.

We compute now the different coefficients p_S^i of the Shapley-Kirchhoff Φ_K value associated to the regular set system $(N, \mathcal{N})$. Let us denote by x the coefficient $p_\emptyset^\beta$ and y the coefficient $p_{N'\cup\alpha}^\beta$. On the one hand, we consider the sub-order $\mathcal{N}'_\beta$ induced by all vertices associated to coalitions including β and included in $N' \cup \beta$. $\mathcal{N}'_\beta$ being isomorphic to the Boolean lattice $2^{N'}$, any permutation of N' leaves unchanged $\mathcal{N}'_\beta$, and since the circuit property is a symmetric rule (in the sense that labels of players have no importance for it), the coefficients computed for the edges of $\mathcal{N}'_\beta$ are proportional to the coefficients of the classical Shapley value over $\mathcal{G}(\mathcal{N})$ whenever $\mathcal{N} = 2^{N'}$. Indeed, for any subset S of N' and any $i \in N' \setminus S$, let us denote by x_s^1 the coefficient $p_{S\cup\beta}^i$. By Proposition 5, we get

$$\begin{aligned}
&x_0^1 = \frac{x}{n'},\\
&s \cdot x_{s-1}^1 = (n' - s) \cdot x_s^1, \quad \forall s \in \{1, \ldots, n' - 1\},
\end{aligned}$$

which immediately conducts by induction on s to

$$x_s^1 = \frac{s!(n' - s - 1)!}{n'!} \cdot x, \quad \forall s \in \{0, \ldots, n' - 1\}.$$

Remark that the left coefficient also known under the form $p_s^1(n')$, is well-known since being a coefficient of Shapley for games with n' players (cf. (1)).
Naturally, we also get $p_{N'\cup\beta}^\alpha = \sum_{i=1}^{n'} x_{n'-1}^1 = x$.

On the other hand, also by Proposition 5, we successively deduce:

- On node $\emptyset$, $p_\emptyset^\alpha = 1 - x$.
- On nodes $\alpha 1 \ldots i,\ i \in N'$, $p_\alpha^1 = p_{\alpha 1}^2 = \cdots = p_{\alpha 12\ldots(n'-1)}^{n'} = 1 - x$.
- On nodes $N' \cup \alpha, N \setminus \beta, N \setminus \gamma$, $p_{N'\cup\alpha}^\gamma = p_{N\setminus\beta}^\beta = 1 - x - y$, and $p_{N\setminus\gamma}^\gamma = x + y$.

Now, let $\mathcal{M}_1$ be the circuit $(\emptyset, \alpha, \alpha 1, \alpha 12, \ldots, N'\cup\alpha, N\setminus\gamma, N'\cup\beta, \ldots, \beta 12, \beta 1, \beta, \emptyset)$ and $\mathcal{M}_2$ be the circuit $(N, N \setminus \gamma, N' \cup \alpha, N \setminus \beta, N)$. Then by the circuit property applied to these two circuits, we have the system:

$$\begin{cases}
(n' + 1)(1 - x) + y - x - \sum_{s=0}^{n'-1} x_s^1 - x & = 0\\
-(x + y) - y + 2(1 - x - y) & = 0
\end{cases}.$$

By denoting $\varsigma(n')$ the sum of the coefficients of Shapley $\sum_{s=0}^{n'-1} p_s^1(n')$, the above system writes

$$\begin{cases} (n'+3+\varsigma(n'))x - y & = n'+1 \\ 3x+4y & = 2 \end{cases},$$

and has for determinant the strictly positive number $\Delta := 4(n'+\varsigma(n'))+15$. Thus, we obtain

$$x = \frac{4n'+6}{\Delta} \quad \text{and}$$
$$y = \frac{2\varsigma(n')-n'+3}{\Delta}.$$

For all the above p_S^i's given in terms of x and y, except for $p_{N'\cup\alpha}^{\beta} = y$, we straightforwardly verify that $p_S^i > 0$, for all $n' \geq 1$. Thus it remains to find condition on n' so that y is nonnegative. For $n' \leq 4$, it can be checked that $2\varsigma(n') - n' + 3$ is positive. Besides, $\forall n' \geq 1, \forall s \in \{0, \ldots, n'-1\}$, $p_s^1(n') = \frac{s!(n'-s-1)!}{n'!} \leq \frac{(s+(n'-s-1))!}{n'!} = \frac{1}{n'}$, the inequality being strict whenever $s \neq 0, n'-1$. Thus, $\forall n' > 2$, $\varsigma(n') = \sum_{s=0}^{n'-1} p_s^1(n') < 1$. Moreover,

$$\Delta \cdot y/2 = \varsigma(n') - (n'-3)/2,$$

where $(n'-3)/2 \geq 1, \forall n' \geq 5$. As a consequence, $\varsigma(n') - (n'-3)/2 < 0, \forall n' \geq 5$, and so is y. The result finally follows, by Lemma 9.

In particular, whenever $n' = 5$, which corresponds to the regular system of Fig. 2, Φ_K is not monotonic, with $p_{\alpha 12345}^{\beta} = -\frac{14}{557}$. ■

Acknowledgment

The first author is very grateful to Professor H. Narayanan for his kind and helpful attitude, giving much of his time in the investigation of the counterexample of Appendix .

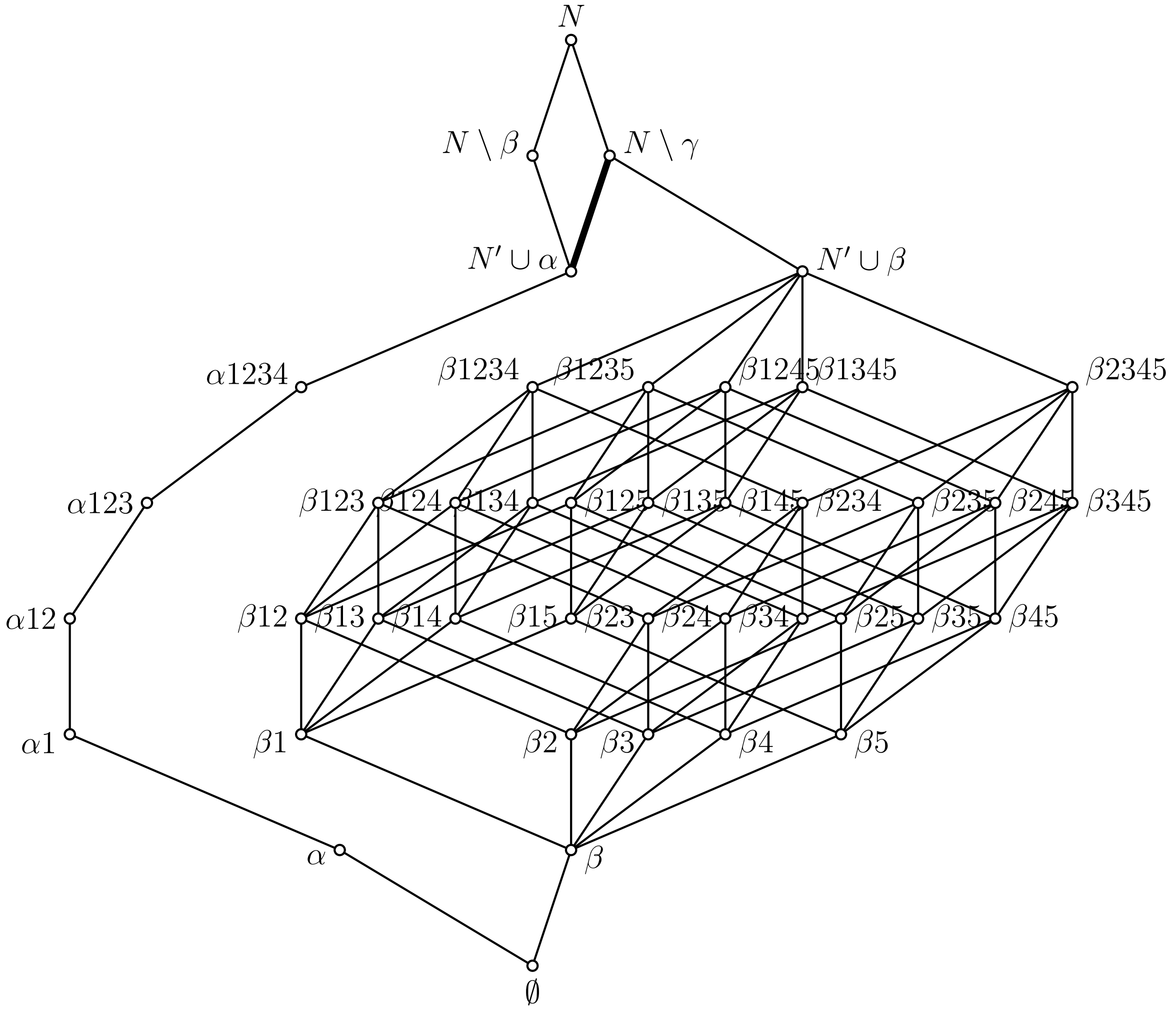


Figure 2: A regular set system where **(M)** does not hold for Φ_K

References

[1] J.M. Bilbao. *Cooperative Games on Combinatorial Structures*. Kluwer Academic Publishers, 2000.

[2] J.M. Bilbao. Cooperative games under augmenting systems. *SIAM Journal on Discrete Mathematics*, 17(1):122–133, 2003.

[3] J.M. Bilbao and P.H. Edelman. The Shapley value on convex geometries. *Discrete Applied Mathematics*, 103:33–40, 2000.

[4] G. Birkhoff. *Lattice Theory*. American Mathematical Society, 3d edition, 1967.

[5] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953.

[6] Y. Colin de Verdière. *Spectres de graphes*. Collection S.M.F., 1998.

[7] N. Deo. *Graph Theory with Applications to Engineering and Computer Science*. Prentice-Hall, 1974.

[8] P.G. Doyle and J.L. Snell. *Random walks and electric networks*. The mathematical association of America, 1984.

[9] P. Dubey, A. Neyman, and R.J. Weber. Value theory without efficiency. *Mathematics of Operations Research*, 6(1):122–128, 1981.

[10] U. Faigle. Cores with restricted cooperation. *ZOR – Methods and Models of Operations Research*, pages 405–422, 1989.

[11] U. Faigle and W. Kern. The Shapley value for cooperative games under precedence constraints. *International Journal of Game Theory*, 21:249–266, 1992.

[12] M. Grabisch and F. Lange. Games on lattices, multichoice games and the Shapley value: a new approach. *Mathematical Methods of Operations Research*, 65:153–167, 2007.

[13] M. Grabisch, J.L. Marichal, and M. Roubens. Equivalent representations of set functions. *Mathematics of Operations Research*, 25(2):157–178, 2000.

[14] A. Honda and M. Grabisch. Entropy of capacities on lattices and set systems. *Information Sciences*, 176(23):3472–3489.

[15] Ch. Labreuche. Interaction indices for games with forbidden coalitions. In *9th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'2002)*, pages 529–534, Annecy, France, July 2002.

[16] L.S. Shapley. A value for n-person games. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.

[17] R.J. Weber. Probabilistic values for games. In A.E. Roth, editor, *The Shapley Value. Essays in Honor of Lloyd S. Shapley*, pages 101–119. Cambridge University Press, 1988.

[18] H.P. Young. Monotonic solutions of cooperative games. *International Journal of Game Theory*, 14:65–72, 1985.

Seconde partie.
L'indice d'interaction de Shapley : Axiomatisations et calculs

Chapitre 4. Axiomatisations de l'indice d'interaction pour les bi-capacités

Résumé

Les bi-capacités se révèlent être une généralisation naturelle des capacités (ou mesures floues) dans un contexe de prise de décision, où les échelles sous-jacentes sont bipolaires. Elles sont capables de prendre en compte une large variété de comportements décisionnels. Après une courte présentation de la structure de base, on introduit la valeur de Shapley et l'indice d'interaction pour les capacités. On étudie ensuite le cas des bi-capacités, et on fournit deux axiomatisations de leur indice d'interaction.

Mots clés : bi-capacité, valeur de Shapley, indice d'interaction, association de critères

Axiomatizations of the Shapley interaction index for bi-capacities

Fabien Lange
Michel Grabisch

Centre d'Economie de la Sorbonne, UMR 8174, CNRS-Université Paris 1

Abstract

Bi-capacities are a natural generalization of capacities (or fuzzy measures) in a context of decision making where underlying scales are bipolar. They are able to capture a wide variety of decision behaviours. After a short presentation of the basis structure, we introduce the Shapley value and the interaction index for capacities. Afterwards, the case of bicapacities is studied with axiomatizations of the interaction index.

Keywords: bi-capacity, Shapley value, interaction index, partnership of criteria

Short version in proceedings of *Joint 4th EUSFLAT and 11th LFA Conference*, Barcelona, Spain, September 2005.

1 Introduction

Real-valued set functions are widely used in operations research [10], while capacities [2] have become a fundamental tool in decision making. There has been some attempts to define more general concepts, among which can be cited *bi-cooperative games* [1], in game theory, which generalize the idea of *ternary voting games* [3]. In the field of multicriteria decision making, there has been a recent proposal of more general functions, motivated by multicriteria decision making, leading to *bi-capacities*, which have been introduced by Grabisch and Labreuche [6]. Specifically, let us consider a set N of criteria and a set X of alternatives in a multicriteria decision making problem, where each alternative x is described by a vector of real valued score $(x_1, \ldots, x_n)$. A *decision maker* may provide a capacity ν defined over 2^N, where $\nu(A)$ for any $A \subseteq N$ is the score of every *binary alternative* $(1_A, 0_{A^c})$: all criteria of A have score 1 and others, 0. Then it is well known that the *Choquet integral* enables to compute an overall score of the alternative x by interpolation between binary alternatives. Motivated with perceptible limitations of such a model, the decision maker may score alternatives of X on a bipolar scale in this way: to each *bi-coalition* (A, B) of criteria — positive vs. negative ones — a *ternary alternative* $(1_A, -1_B, 0_{(A\cup B)^c})$ is associated: every criterion of A (the *positive part*) has a score equal to 1 (total satisfaction), every one in B (the *negative part*) has a score equal to -1 (total unsatisfaction) and the others have a score equal to 0 (neutrality). Scores are given to each ternary alternative, which defines a bi-capacity.

The concept of interaction index, can be seen as an extension of the notion of *value* or power index [13]. It is fundamental for it enables to measure the interaction phenomena modeled by a capacity on a set of criteria; such phenomena can be for instance substitution or complementarity effects between some criteria [7]. Our aim is to provide axiomatizations of the *Shapley interaction index* of a bi-capacity. Two of them are proposed: at first a *recursive axiom* is used by extension of the one of Grabisch and Roubens [9], and subsequently we work out the *reduced-partnership-consistency axiom* using the concept of partnership [4].

2 Capacities and bi-capacities

Throughout the paper, $N := \{1, \dots, n\}$ denotes the finite referential set. Furthermore, cardinalities of subsets $S, T, \dots$ are denoted by the corresponding lower case letters $s, t, \dots$.

We begin by recalling basic notion about capacities for finite sets [2]. A *cooperative game* $\nu : 2^N \to \mathbb{R}^+$ is a set function such that $\nu(\emptyset) = 0$, and ν is said to be a *capacity* if $A \subseteq B \subseteq N$ implies $\nu(A) \leq \nu(B)$ (monotonicity condition). If in addition $\nu(N) = 1$, the capacity is said to be *normalized*.

Let us denote $\mathcal{Q}(N) := \{(A, B) \in 2^N \times 2^N \mid A \cap B = \emptyset\}$.

Definition 1 *A function $v : \mathcal{Q}(N) \to \mathbb{R}$ is a* bi-capacity *if it satisfies:*

(i) $v(\emptyset, \emptyset) = 0$.

(ii) $A \subseteq B$ *implies* $v(A, \cdot) \leq v(B, \cdot)$ *and* $v(\cdot, A) \geq v(\cdot, B)$.

In addition, v is normalized *if* $v(N, \emptyset) = 1 = -v(\emptyset, N)$.

In a multicriteria decision making framework, $v(A, B)$ represents the score of the ternary alternative $(1_A, -1_B, 0_{(A\cup B)^c})$. Note that the definition implies that $v(\cdot, \emptyset) \geq 0$ and $v(\emptyset, \cdot) \leq 0$. Actually, bi-capacities are particular *bi-cooperative games* [1], that is, functions defined over $\mathcal{Q}(N)$ with only condition *(i)* holding.

From its definition, $\mathcal{Q}(N)$ is isomorphic to the set of mappings from N to $\{-1, 0, 1\}$, hence $|\mathcal{Q}(N)| = 3^n$. Also, it is easy to see that $\mathcal{Q}(N)$ is a lattice, when equipped with the order:

$$(A, B) \sqsubseteq (C, D) \text{ if } A \subseteq C \text{ and } B \supseteq D.$$

Supremum and infimum are respectively

$$(A, B) \sqcup (C, D) = (A \cup C, B \cap D)$$
$$(A, B) \sqcap (C, D) = (A \cap C, B \cup D),$$

and top and bottom are respectively $(N, \emptyset)$ and $(\emptyset, N)$. We give in Fig. 1 the Hasse diagram of $(\mathcal{Q}(N), \sqsubseteq)$ for $n = 3$ (where top, bottom and the central point $(\emptyset, \emptyset)$ are represented by black circles).

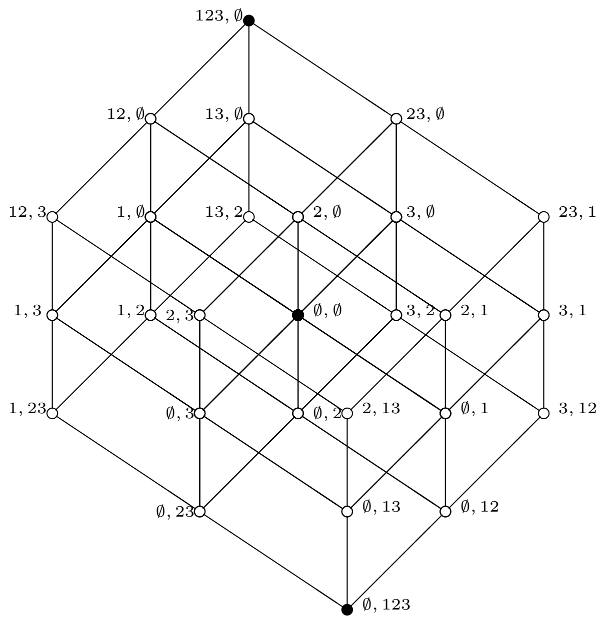


Figure 1: The lattice $\mathcal{Q}(N)$ for $n = 3$

Derivatives of bi-capacities play a central role in the definition of interaction [6] and are defined in this way: if v is a bi-capacity, and $i \in N$,

$$\begin{aligned} \Delta_{i,\emptyset} v(K,L) &:= v(K \cup i, L) - v(K,L), \quad \text{for any } (K,L) \in \mathcal{Q}(N \setminus i); \\ \Delta_{\emptyset,i} v(K,L) &:= v(K, L \setminus i) - v(K,L), \quad \text{for any } (K,L) \in \mathcal{Q}(N) \text{ with } i \in L. \end{aligned}$$

Recursively, we define $\Delta_{S,T} v$ for any $(K,L) \in \mathcal{Q}(N \setminus S)$ with $T \subseteq L$, for any $i \in S$ and any $j \in T$, by

$$\begin{aligned} \Delta_{S,T} v(K,L) &:= \Delta_{i,\emptyset}(\Delta_{S\setminus i,T} v(K,L)) \\ &= \Delta_{\emptyset,j}(\Delta_{S,T\setminus j} v(K,L)), \end{aligned}$$

so that these values are always non-negative. This generalizes the notion of derivative for a capacity ν, that is $\Delta_i \nu(A) := \nu(A \cup i) - \nu(A)$ if $i \in N, A \subseteq N \setminus i$ and $\Delta_S \nu(A) := \Delta_i(\Delta_{S\setminus i} \nu(A))$ if $A \subseteq N \setminus S$. The general expression for the (S,T)-derivative is given by, for any $(S,T) \in \mathcal{Q}(N)$, $(S,T) \neq (\emptyset, \emptyset)$ (cf. [8]):

$$\Delta_{S,T} v(K,L) = \sum_{\substack{S' \subseteq S \\ T' \subseteq T}} (-1)^{(s-s')+(t-t')} v(K \cup S', L \setminus T'),$$

$$\text{for all } (K,L) \in \mathcal{Q}(N \setminus S), L \supseteq T. \quad (1)$$

Although we develop our results for bi-capacities, we emphasize the fact that all subsequent results remain valid for bi-cooperative games.

3 Previous work on interaction index for capacities

We recall in this section two main ways which have been conducted to axiomatize the interaction index for capacities. Since the following axioms extend the ones of the Shapley value, we may adopt the terminology of *Shapley interaction index*.

In this section, ν denotes a capacity on N. Let us recall its *Shapley value*: for any element $i \in N$,

$$\phi^{\nu}(i) := \sum_{S \subseteq N \setminus i} p_s^1(n)\, (\nu(S \cup i) - \nu(S)),$$

where the coefficients $p_s^1(n) := \dfrac{(n-s-1)!\, s!}{n!}$ define a probability distribution over $\{S \subseteq N \setminus i\}$.

The classical axioms introduced by Shapley [13] (see also Weber [14]) are the following

- Linearity: for any $i \in N$, $\phi(i)$ is linear on the set of capacities on N.
- $i \in N$ is said to be *dummy* for ν if $\forall S \subseteq N \setminus i$, $\nu(S \cup i) = \nu(S) + \nu(i)$.
- Dummy axiom: For any capacity ν and any $i \in N$ dummy for ν, $\phi^{\nu}(i) = \nu(i)$.
- Symmetry axiom: for any permutation σ on N, any capacity ν and any $i \in N$, $\phi^{\nu \circ \sigma^{-1}}(\sigma(i)) = \phi^{\nu}(i)$. This means that ϕ^{ν} must not depend on the labelling of the criteria.
- Efficiency axiom (E^c): for any capacity ν, $\sum_{i \in N} \phi^{\nu}(i) = \nu(N)$; that is to say the values of the criteria must be divided in proportion of the overall score $\nu(N)$.

By generalizing Murofushi and Soneda [12], Grabisch has defined the *interaction index* of capacities [5]. A first axiomatization have been proposed by Grabisch and Roubens and rests on a recursivity axiom [9]. For this, they introduce the following definitions:

Let K a non-empty subset of N and $B \subseteq N \setminus K$. The *restricted capacity* ν^K is the capacity ν restricted to 2^K. The *restriction of* ν *to* K *in the presence of* B is the capacity defined by

$$\nu^K_{\cup B}(S) := \nu(S \cup B) - \nu(B)$$

for any $S \subseteq K$. Lastly, the *reduced capacity* $\nu^{[K]}$ is the capacity defined on $N_{[K]} := (N \setminus K) \cup \{[K]\}$ by

$$\nu^{[K]}(A) := \nu(A^\star)$$

where $A^\star := \Big\{ A \text{ if } [K] \notin A(A \setminus [K]) \cup K \text{ else }$; $[K]$ actually indicates a single hypothetical player, which is the representative of the players in K.

> **Recursivity axiom 1** (R1^c): For any capacity ν, $\forall S \subseteq N$, $s > 1$, $\forall i \in S$,
>
> $$I^\nu(S) = I^{\nu^{N\setminus i}_{\cup i}}(S \setminus i) - I^{\nu^{N\setminus i}}(S \setminus i).$$

> **Recursivity axiom 2** (R2^c): For any capacity ν, $\forall S \subseteq N$, $s > 1$,
>
> $$I^\nu(S) = I^{\nu^{[S]}}([S]) - \sum_{\substack{K \subsetneq S \\ K \neq \emptyset}} I^{\nu^{N\setminus K}}(S \setminus K).$$

Theorem 1 (Grabisch, Roubens [9]) *Under linear axiom, dummy axiom, symmetry axiom, efficiency axiom (E*^{c}*) and* $\big($*(R1*^{c}*) or (R2*^{c}*)*$\big)$*, for any capacity* ν*,* $\forall S \subseteq N$*,* $S \neq \emptyset$*,*

$$I^\nu(S) = \sum_{T \subseteq N \setminus S} p^s_t(n)\, \Delta_S \nu(T),$$

where $p^s_t(n) := p^1_t(n-s+1) = \dfrac{(n-s-t)!\,t!}{(n-s+1)!}$.

Actually, the authors have shown that (R1^c) and (R2^c) are equivalent under the first axioms [9].

Now we present an axiomatization of Fujimoto, Kojadinovic and Marichal based on the concept of *partnership coalition* [4]; we use for this the following generalized axioms:

Linear axiom (L^c): For any $S \subseteq N$, $I(S)$ is linear on the set of capacities on N.

Dummy axiom (D^c): For any capacity ν and any $i \in N$ dummy for ν,

$$\begin{cases} I^\nu(i) = \nu(i), \\ I^\nu(S \cup i) = 0, \quad \forall S \subseteq N \setminus i, S \neq \emptyset. \end{cases}$$

Symmetry axiom (S^c): For any permutation σ on N, any capacity ν and any $S \subseteq N$,

$$I^{\nu \circ \sigma^{-1}}(\sigma(S)) = I^\nu(S).$$

For any $P \subseteq N$, P is said to be a *partnership* for ν if

$$\forall S \subsetneq P,\ \forall T \subseteq N \setminus P,\ \nu(S \cup T) = \nu(T).$$

In other words, as long as the elements of P are not present, the worth of any coalition outside P is left unchanged.

Reduced-partnership-consistency axiom (RPC^c): For any capacity ν and $P \subseteq N$ partnership for ν,

$$I^\nu(P) = I^{\nu^{[P]}}([P]).$$

Theorem 2 (Fujimoto, Kojadinovic, Marichal, [4]) *Under (L^c), (D^c), (S^c), (E^c) and (RPC^c), for any capacity ν, $\forall S \subseteq N$, $S \neq \emptyset$,*

$$I^\nu(S) = \sum_{T \subseteq N \setminus S} p_t^s(n)\, \Delta_S \nu(T),$$

As in Theorem 1, I^ν is again the Shapley interaction index of ν.

Let us point out that I is *cardinal-probabilistic*, that is to say, $(p_t^s(n))_{T \subseteq N \setminus S}$ is a probability distribution, for any $S \subseteq N$, $S \neq \emptyset$ (see [4]).

4 Axiomatizations of the interaction for bi-capacities

In the sequel, v is a bi-capacity. Since criterion i has two possible situations (either being in the positive part or in the negative part of the bi-coalition), the effects of which being not necessarily symmetric on v, we should define a value $\Phi_{i,\emptyset}$ representing the contribution of i "joining the positive part" and a value $\Phi_{\emptyset,i}$ representing the contribution of i "leaving the negative part". Indeed, Labreuche and Grabisch have already axiomatized a Shapley value for bi-capacities [11], which is done by introducing axioms similar to the original ones of Shapley that we recalled above:

Linearity (L): For any $i \in N$, $\Phi_{i,\emptyset}$ and $\Phi_{\emptyset,i}$ are linear on the set of bi-capacities on N.

$i \in N$ is said to be *left-null* (resp. *right-null*) for v if $\forall (K,L) \in \mathcal{Q}(N \setminus i)$,

$$v(K \cup i, L) \ \big(\text{resp. } v(K, L \cup i)\big) = v(K,L).$$

Left-null axiom (LN): For any bi-capacity v and any $i \in N$ left-null for v,

$$\Phi^v_{i,\emptyset} = 0.$$

Right-null axiom (RN): For any bi-capacity v and any $i \in N$ right-null for v,

$$\Phi^v_{\emptyset,i} = 0.$$

Invariance axiom (I): For any two bi-capacities v, w, and any $i \in N$ such that $\forall (K,L) \in \mathcal{Q}(N \setminus i)$

$$\begin{cases} v(K \cup i, L) = w(K,L), \\ v(K,L) = w(K, L \cup i), \end{cases}$$

then $\Phi^v_{i,\emptyset} = \Phi^w_{\emptyset,i}$.

This axiom which, has no equivalent in the case of capacities, says that when a game w behaves symmetrically with v, then the Shapley values are the same.

Symmetry axiom (S): For any permutation σ on N, any bi-capacity v and any $i \in N$,

$$\Phi^{v\circ\sigma^{-1}}_{\sigma(i),\emptyset} = \Phi^{v}_{i,\emptyset} \text{ and } \Phi^{v\circ\sigma^{-1}}_{\emptyset,\sigma(i)} = \Phi^{v}_{\emptyset,i}.$$

Efficiency axiom (E): For any bi-capacity v,

$$\sum_{i\in N}(\phi^{v}_{i,\emptyset} + \phi^{v}_{\emptyset,i}) = v(N,\emptyset) - v(\emptyset,N).$$

Theorem 3 (Labreuche, Grabisch [11]) *Under (L), (LN), (RN), (I), (S) and (E), for any bi-capacity v, $\forall i \in N$,*

$$\Phi^{v}_{i,\emptyset} = \sum_{S\subseteq N\setminus i} p^1_s(n)\,[v(S\cup i, N\setminus(S\cup i)) - v(S, N\setminus(S\cup i))],$$
$$\Phi^{v}_{\emptyset,i} = \sum_{S\subseteq N\setminus i} p^1_s(n)\,[v(S, N\setminus(S\cup i)) - v(S, N\setminus S)].$$

Now, since Grabisch and Labreuche have also defined an interaction index I^v over $\mathcal{Q}(N)$ for bi-capacities [8], it is necessary to give satisfactory properties to characterize it.

In the first place, as the interaction index for capacities can be obtained from the Shapley value by a recursion formula, we give here a similar approach to build $I^v_{S,T}$ from $\Phi^v_{i,\emptyset} =: I^v_{i,\emptyset}$ and $\Phi^v_{\emptyset,i} =: I^v_{\emptyset,i}$. Practically, $I^v_{S,T}$ denotes the interaction index when S is added to the positive part, and T is withdrawn from the negative part (i.e., the elements of T become neutral).

For any non-empty subset K, the *restricted* bi-capacity v^K is the restriction of v to $\mathcal{Q}(K)$. Besides, $v^{N\setminus i}_+$ and $v^{N\setminus i}_-$ are particular restricted bi-capacities defined by

$$v^{N\setminus i}_+(A,B) := v(A\cup i, B) - v(i,\emptyset)$$
$$v^{N\setminus i}_-(A,B) := v(A, B\cup i) - v(\emptyset,i),$$

for any $(A,B) \in \mathcal{Q}(N\setminus i)$. We respectively call $v^{N\setminus i}_+$ and $v^{N\setminus i}_-$ the *restrictions* of v in *positive* and *negative presence* of i. Note that the substractions of $v(i,\emptyset)$ and $v(\emptyset,i)$ are necessary to constraint the nullity in $(\emptyset,\emptyset)$. The following axiom generalizes (R1^c).

Recursivity axiom (R): For any bi-capacity v, $\forall (S,T) \in \mathcal{Q}(N)$, $s+t \geq 2$;

$$\forall i \in S, \quad I^v_{S,T} = I^{v_+^{N\setminus i}}_{S\setminus i,T} - I^{v^{N\setminus i}}_{S\setminus i,T}, \text{ if } s \geq 1,$$
$$\forall i \in T, \quad I^v_{S,T} = I^{v^{N\setminus i}}_{S,T\setminus i} - I^{v_-^{N\setminus i}}_{S,T\setminus i}, \text{ if } t \geq 1.$$

Theorem 4 *Under (L), (LN), (RN), (I), (S), (E) and (R), for any bi-capacity v, for any bi-coalition (S,T), $(S,T) \neq (\emptyset,\emptyset)$,*

$$I^v_{S,T} = \sum_{K \subseteq N\setminus(S\cup T)} p_k^{s+t}(n)\, \Delta_{S,T} v(K, N \setminus (K \cup S)). \tag{2}$$

Proof: By Theorem 3, $I^v_{i,\emptyset}$ and $I^v_{\emptyset,i}$ write for all $i \in N$:

$$I^v_{i,\emptyset} = \sum_{S\subseteq N\setminus i} \frac{(n-s-1)!s!}{n!} [v(S\cup i, N\setminus(S\cup i)) - v(S, N\setminus(S\cup i))],$$
$$I^v_{\emptyset,i} = \sum_{S\subseteq N\setminus i} \frac{(n-s-1)!s!}{n!} [v(S, N\setminus(S\cup i)) - v(S, N\setminus S)].$$

We show the result by induction on $m = s+t$.

- For $m = 1$, it is immediate.
- Assume that (2) is shown for $m \in \{1, \ldots, n-1\}$. Let $(S,T) \in \mathcal{Q}(N)$ and $s+t = m+1$. If $s \geq 1$ and $i \in S$ then

$$\begin{aligned}
I^v_{S,T} &= I^{v_+^{N\setminus i}}_{S\setminus i,T} - I^{v^{N\setminus i}}_{S\setminus i,T} \\
&= \sum_{K\subseteq (N\setminus i)\setminus((S\setminus i)\cup T)} \frac{((n-1)-(s-1)-t-k)!\,k!}{((n-1)-(s-1)-t+1)!} \\
&\qquad \Delta_{S\setminus i,T}[v_+^{N\setminus i} - v^{N\setminus i}](K, (N\setminus i)\setminus(K\cup(S\setminus i))) \\
&= \sum_{K\subseteq N\setminus(S\cup T)} \frac{(n-s-t-k)!\,k!}{(n-s-t+1)!} \\
&\qquad \Delta_{S\setminus i,T}[v(\cdot\cup i,\cdot) - v(i,\emptyset) - v + v(i,\emptyset)](K, N\setminus(K\cup S)) \\
&= \sum_{K\subseteq N\setminus(S\cup T)} \frac{(n-s-t-k)!\,k!}{(n-s-t+1)!} \\
&\qquad \Delta_{S\setminus i,T}\Delta_{i,\emptyset} v(K, N\setminus(K\cup S)).
\end{aligned}$$

If $t \geq 1$ and $i \in T$ then

$$\begin{aligned}
I^v_{S,T} &= I^{v^{N\setminus i}}_{S,T\setminus i} - I^{v_-^{N\setminus i}}_{S,T\setminus i} \\
&= \sum_{K\subseteq (N\setminus i)\setminus((S\cup(T\setminus i))} \frac{((n-1)-s-(t-1)-k)!\,k!}{((n-1)-s-(t-1)+1)!} \\
&\qquad\qquad \Delta_{S,T\setminus i}[v^{N\setminus i} - v_-^{N\setminus i}](K,(N\setminus i)\setminus(K\cup S)) \\
&= \sum_{K\subseteq N\setminus(S\cup T)} \frac{(n-s-t-k)!\,k!}{(n-s-t+1)!} \\
&\qquad\qquad \Delta_{S,T\setminus i}[v - v(\emptyset,i) - v(\cdot,\cdot\cup i) + v(\emptyset,i)](K, N\setminus(K\cup S\cup i)) \\
&= \sum_{K\subseteq N\setminus(S\cup T)} \frac{(n-s-t-k)!\,k!}{(n-s-t+1)!} \\
&\qquad\qquad \Delta_{S,T\setminus i}\Delta_{\emptyset,i}v(K, N\setminus(K\cup S)).
\end{aligned}$$

Since operators $\Delta_{S\setminus i,T}\Delta_{i,\emptyset}$ and $\Delta_{S,T\setminus i}\Delta_{\emptyset,i}$ are by definition $\Delta_{S,T}$, the result is shown for $s+t=m+1$.

■

Let us remark that a such result has also been derived from a generalization of (R2^c) (see [8]).

In the second place, one can take inspiration from the Fujimoto, Kojadinovic and Marichal's work [4] in working out an equivalent axiom of the above (RPC) axiom for capacities. Let us start by defining the concepts of partnership and reduced bi-capacity.

For any $P \subseteq N$, P is said a *partnership* for v if

$$\begin{aligned}
&\forall (S,T) \in \mathcal{Q}(N\setminus P), \quad \forall P_+, P_- \subsetneq P \text{ such that } P_+ \cap P_- = \emptyset, \\
&\qquad v(S\cup P_+, T\cup P_-) = v(S,T).
\end{aligned}$$

The meaning is the same as for capacities, that is to say, if all elements of P are not joined together then they have a null effect on the worth of v.

For any non-empty subset K, the *reduced* bi-capacity $v^{[K]}$ is the bi-capacity defined on $N_{[K]} := (N\setminus K)\cup\{[K]\}$ by

$$v^{[K]}(S,T) := v(S^\star, T^\star),$$

where $A^\star := \begin{cases} A \text{ if } [K] \notin A \\ (A \setminus [K]) \cup K \text{ else} \end{cases}$, and $[K]$ is still comparable to a single macro player.

Reduced-partnership-consistency axiom (RPC): For any bi-capacity v and any partnership $P \subseteq N$ for v,

$$I^v_{P,\emptyset} = I^{v^{[P]}}_{[P],\emptyset}.$$

A first remark is that one could replace this axiom with its symmetric, that is, $I^v_{\emptyset,P} = I^{v^{[P]}}_{\emptyset,[P]}$, when P is still a partnership for v, one or the other being sufficient. On the other hand, from this axiom and the above ones (N), (LN), (RN), (I), (S) and (E), it is impossible to compute every $I^v_{S,T}$ whenever $T \neq \emptyset$. Consequently, we do it by generalizing these axioms:

Generalized linearity (GL): For any $(S,T) \in \mathcal{Q}(N)$, $I_{S,T}$ is linear on the set of bi-capacities on N.

Generalized left-null axiom (GLN): For any bi-capacity v and any $i \in N$ left-null for v,

$$I^v_{S \cup i, T} = 0, \quad \forall (S,T) \in \mathcal{Q}(N \setminus i).$$

Generalized right-null axiom (GRN): For any bi-capacity v and any $i \in N$ right-null for v,

$$I^v_{S, T \cup i} = 0, \quad \forall (S,T) \in \mathcal{Q}(N \setminus i).$$

Generalized invariance axiom (GI): For any two bi-capacities v, w and any $i \in N$ such that $\forall (K,L) \in \mathcal{Q}(N \setminus i)$, $\begin{cases} v(K \cup i, L) = w(K,L), \\ v(K,L) = w(K, L \cup i), \end{cases}$ we have

$$I^v_{S \cup i, T} = I^w_{S, T \cup i}, \quad \forall (S,T) \in \mathcal{Q}(N \setminus i).$$

Generalized symmetry axiom (GS): For any permutation σ on N, any bi-capacity v and any $(S,T) \in \mathcal{Q}(N)$,

$$I^{v \circ \sigma^{-1}}_{\sigma(S), \sigma(T)} = I^v_{S,T}.$$

Proposition 5 *Under (GL), (GLN), (GRN), (GI) and (GS), for any bi-capacity v, and any $(S,T) \in \mathcal{Q}(N) \setminus (\emptyset, \emptyset)$, $I^v_{S,T}$ is given by*

$$I^v_{S,T} = \sum_{(K,L) \in \mathcal{Q}(N \setminus (S \cup T))} p^{s+t}_{k,l}(n)\, \Delta_{S,T} v(K, L \cup T), \tag{3}$$

where $(p^u_{k,l}(n))_{(K,L) \in \mathcal{Q}(N \setminus U)}$, $U := S \cup T$, is a probability distribution.

Proof: We straightforwardly derive from (GL) that for any bi-capacity v

$$I^v_{S,T} = \sum_{(K,L) \in \mathcal{Q}(N)} p^{(S,T)}_{(K,L)}\, v(K,L) \quad \forall (S,T) \in \mathcal{Q}(N),$$

where the $p^{(S,T)}_{(K,L)}$'s are real numbers.

1. For all $i \in N$ and all $(S,T) \in \mathcal{Q}(N \setminus i)$,

   $$I^v_{S \cup i,T} = \sum_{(K,L) \in \mathcal{Q}(N \setminus i)} \Big[p^{(S\cup i,T)}_{(K \cup i,L)}\, v(K \cup i, L) + p^{(S \cup i,T)}_{(K,L)}\, v(K,L) + p^{(S\cup i,T)}_{(K,L \cup i)}\, v(K, L \cup i) \Big].$$

   Then if i is left-null:

   $$I^v_{S \cup i,T} = \sum_{(K,L) \in \mathcal{Q}(N \setminus i)} \Big[\Big(p^{(S\cup i,T)}_{(K \cup i,L)} + p^{(S \cup i,T)}_{(K,L)} \Big) v(K,L) + p^{(S\cup i,T)}_{(K,L \cup i)}\, v(K, L \cup i) \Big].$$

   From (GLN), $I^v_{S\cup i,T}$ vanishes for all v, which leads to equalities $\begin{cases} p^{(S\cup i,T)}_{(K,L)} = -p^{(S\cup i,T)}_{(K\cup i,L)} \\ p^{(S\cup i,T)}_{(K,L\cup i)} = 0 \end{cases}$.
   Let $p^{S\cup i,T}_{K,L} := p^{(S\cup i,T)}_{(K\cup i,L)}$.

2. Similarly, if i is right-null, we have

   $$I^v_{S,T \cup i} = \sum_{(K,L) \in \mathcal{Q}(N \setminus i)} \Big[p^{(S,T\cup i)}_{(K \cup i,L)}\, v(K \cup i, L) + \Big(p^{(S,T\cup i)}_{(K,L)} + p^{(S,T\cup i)}_{(K,L\cup i)} \Big) v(K,L) \Big],$$

   which implies from (GRN): $\begin{cases} p^{(S,T\cup i)}_{(K\cup i,L)} = 0 \\ -p^{(S,T\cup i)}_{(K,L\cup i)} = p^{(S,T\cup i)}_{(K,L)} =: p^{S,T\cup i}_{K,L} \end{cases}$,
   $\forall i \in N, \forall (S,T) \in \mathcal{Q}(N \setminus i), \forall (K,L) \in \mathcal{Q}(N \setminus i)$.

3. Let$v$ w two bi-capacities, and $i \in N$; thus

   $$I^v_{S\cup i,T} = \sum_{(K,L) \in \mathcal{Q}(N \setminus i)} p^{S\cup i,T}_{K,L}\, (v(K \cup i, L) - v(K,L)),$$

   $$I^w_{S,T\cup i} = \sum_{(K,L) \in \mathcal{Q}(N \setminus i)} p^{S,T\cup i}_{K,L}\, (w(K,L) - w(K, L \cup i)).$$

If we assume that $\begin{cases} v(K \cup i, L) = w(K, L) \\ v(K, L) = w(K, L \cup i), \end{cases}$ $\forall (K, L) \in \mathcal{Q}(N \setminus i)$ then the second equality above writes

$$I^w_{S,T\cup i} = \sum_{(K,L)\in\mathcal{Q}(N\setminus i)} p^{S,T\cup i}_{K,L} \left(v(K \cup i, L) - v(K, L)\right).$$

since $I^v_{S\cup i,T} = I^w_{S,T\cup i}$ for all v, by (GI), we have $p^{S\cup i,T}_{K,L} = p^{S,T\cup i}_{K,L}, \forall (K, L) \in \mathcal{Q}(N \setminus i)$. Note that we get

$$\begin{aligned} I^v_{S\cup i,T} &= \sum_{(K,L)\in\mathcal{Q}(N\setminus i)} p^{S\cup i,T}_{K,L}\, \Delta_{i,\emptyset} v(K, L), \\ I^w_{S,T\cup i} &= \sum_{(K,L)\in\mathcal{Q}(N\setminus i)} p^{S,T\cup i}_{K,L}\, \Delta_{\emptyset,i} w(K, L \cup i) \text{ où } p^{S,T\cup i}_{K,L} = p^{S\cup i,T}_{K,L}. \end{aligned}$$

By applying (GI) for another criterion $j \neq i$ de N, for all $(S, T) \in \mathcal{Q}(N\setminus ij)$, we have

$$\begin{aligned} I^v_{S\cup ij,T} &= \sum_{(K,L)\in\mathcal{Q}(N\setminus ij)} p^{S\cup ij,T}_{K,L}\, \Delta_{i,\emptyset}\Delta_{j,\emptyset} v(K, L), \\ I^v_{S\cup i,T\cup j} &= \sum_{(K,L)\in\mathcal{Q}(N\setminus ij)} p^{S\cup i,T\cup j}_{K,L}\, \Delta_{i,\emptyset}\Delta_{\emptyset,j} v(K, L \cup j), \\ I^v_{S,T\cup ij} &= \sum_{(K,L)\in\mathcal{Q}(N\setminus ij)} p^{S,T\cup ij}_{K,L}\, \Delta_{\emptyset,i}\Delta_{\emptyset,j} v(K, L \cup ij), \end{aligned}$$

where $p^{S,T\cup ij}_{K,L} = p^{S\cup i,T\cup j}_{K,L} = p^{S\cup ij,T}_{K,L}$, $\forall (K, L) \in \mathcal{Q}(N \setminus ij)$. Thus, by successively applying (GI), we deduce that $(S, T) \in \mathcal{Q}(N) \setminus \{(\emptyset, \emptyset)\}$ and $(K, L) \in \mathcal{Q}(N \setminus (S \cup T))$, $p^{S,T}_{K,L}$ only depend on $S \cup T$ and (K, L). Let $U := S \cup T$ and $p^U_{K,L} := p^{U,\emptyset}_{K,L}$. We have

$$I^v_{S,T} = \sum_{(K,L)\in\mathcal{Q}(N\setminus(S\cup T))} p^{S\cup T}_{K,L}\, \Delta_{S,T} v(K, L \cup T).$$

4. Finally, let σ be any permutation of N. From (GS), we get $I^{v\circ\sigma^{-1}}_{\sigma(S),\sigma(T)} = I^v_{S,T}$ for all (S, T) of $\mathcal{Q}(N) \setminus \{(\emptyset, \emptyset)\}$. Besides,

$$\begin{aligned} I^{v\circ\sigma^{-1}}_{\sigma(S),\sigma(T)} &= \sum_{(K,L)\in\mathcal{Q}(N\setminus\sigma(S\cup T))} p^{\sigma(S\cup T)}_{K,L}\, \underbrace{\Delta_{\sigma(S),\sigma(T)} v\circ\sigma^{-1}\,(K, L \cup T)}_{\Delta_{S,T} v(\sigma^{-1}(K),\sigma^{-1}(L,\cup T))} \\ &= \sum_{(K,L)\in\mathcal{Q}(N\setminus(S\cup T)} p^{\sigma(S\cup T)}_{\sigma(K),\sigma(L)}\, \Delta_{S,T} v(K, L \cup T). \end{aligned}$$

Thus $p^{\sigma(S\cup T)}_{\sigma(K),\sigma(L)} = p^{S\cup T}_{K,L}$, $\forall (K,L) \in \mathcal{Q}(N \setminus (S \cup T))$, that is to say, $p^U_{K,L}$ depend only on the cardinals of U, K, L. Let $p^u_{k,l} := p^U_{K,L}$, then (3) is shown.

■

Under this form, the mapping I is said to be *cardinal-probabilistic*, as a generalization of cardinal-probabilistic indices defined for capacities.

Finally, we have the following result:

Theorem 6 *Under (GL), (GLN), (GRN), (GI), (GS) and (E), axioms (R) and (RPC) are equivalent, thus for any bi-capacity v, for any bi-coalition (S,T), $(S,T) \neq (\emptyset,\emptyset)$,*

$$I^v_{S,T} = \sum_{K\subseteq N\setminus(S\cup T)} p^{s+t}_k(n)\,\Delta_{S,T}v(K, N \setminus (K \cup S)).$$

Proof: Note that (GL), (GLN), (GRN), (GI) and (GS) respectively imply (L), (LN), (RN), (I) and (S). Thus by Theorem 2, it is sufficient to prove that the formula holds with the first axioms and (RPC).

1. Clearly, for all $i \in N$,

$$I^v_{i,\emptyset} = \sum_{S\subseteq N\setminus i} \frac{(n-s-1)!s!}{n!}\,[v(S\cup i, N\setminus(S\cup i)) - v(S, N\setminus(S\cup i))], \quad (4)$$

$$I^v_{\emptyset,i} = \sum_{S\subseteq N\setminus i} \frac{(n-s-1)!s!}{n!}\,[v(S, N\setminus(S\cup i)) - v(S, N\setminus S)].$$

2. Let us compute $I^v_{S,\emptyset}$, $s \geq 2$.
   By proposition 5, there are some real numbers $p^s_{k,l}(n)$, $k+l \leq n-s$ such that

$$\begin{aligned} I^v_{S,\emptyset} &= \sum_{(K,L)\in\mathcal{Q}(N\setminus S)} p^s_{k,l}(n)\,\Delta_{S,\emptyset}v(K,L) \\ &= \sum_{(K,L)\in\mathcal{Q}(N\setminus S)} p^s_{k,l}(n)\,\big(v(K\cup S, L) + \sum_{S'\subsetneq S}(-1)^{s-s'}v(K\cup S', L)\big), \end{aligned}$$

from the explicit expression (1) of $\Delta_{S,T}v$. Now, let S be a partnership, then for all $S' \subsetneq S$, $v(K \cup S', L) = v(K, L)$. Also, since

$$\begin{aligned}\sum_{S' \subsetneq S} (-1)^{s-s'} &= \sum_{S' \subseteq S} (-1)^{s-s'} - 1 \\ &= \sum_{s'=0}^{s} \binom{s}{s'} (-1)^{s-s'} - 1 \\ &= (1-1)^s - 1 \\ &= -1,\end{aligned}$$

$$\text{then} \quad I^v_{S,\emptyset} = \sum_{(K,L) \in \mathcal{Q}(N \setminus S)} p^s_{k,l}(n) \big(v(K \cup S, L) - v(K, L) \big). \tag{5}$$

Moreover, by (RPC) then (4), we have also

$$\begin{aligned}I^v_{S,\emptyset} &= I^{v^{[S]}}_{[S],\emptyset} \\ &= \sum_{K \subseteq (N \setminus S \cup [S]) \setminus [S]} \frac{((n-s+1)-k-1)!\,k!}{(n-s+1)!} \\ &\qquad \Delta_{[S],\emptyset} v^{[S]}(K, (N \setminus S \cup [S]) \setminus (K \cup [S])) \\ &= \sum_{K \subseteq N \setminus S} \frac{(n-s-k)!\,k!}{(n-s+1)!} \\ &\qquad \big(v(K \cup S, N \setminus (K \cup S)) - v(K, N \setminus (K \cup S)) \big).\end{aligned} \tag{6}$$

Let $U := S$. By identifying coefficients of (5) in (6) (formulae are ture for all v), we get $\forall u \in \{1, \ldots, n\}$, $\forall k \in \{0, \ldots, n-u\}$, $\forall l \in \{0 \ldots, n-u-k\}$:

- For the terms of (5) that arise in (6): let $K \subseteq N \setminus U$ and $L = N \setminus (U \cup K)$. Note that $k + l = n - u$.

  $$\begin{aligned}p^u_{k,l}(n) &= p^u_{k,n-u-k}(n) \\ &= \frac{(n-u-k)!\,k!}{(n-u+1)!}.\end{aligned}$$

  Note that these coefficients are identical to those given in Theorem 1, i.e., $p^u_{k,l}(n) = p^u_k(n)$.
- For all other coefficients, i.e., if $k + l \leq n - u$, then

  $$p^u_{k,l}(n) = 0.$$

This ends the proof in this case.

3. The computation of the $I^v_{S,T}$'s with $s+t \geq 2,\ t \geq 1$ is already given above. Indeed, all the $p^{s+t}_{k,l}(n)$'s of (3) are given with $s+t=u$.

4. Finally, for all $u \in \{1,\dots,n\}$,

$$\begin{aligned}\sum_{(K,L)\in\mathcal{Q}(N\setminus U)} p^u_{k,l}(n) &= \sum_{K\subseteq N\setminus U} p^u_k(n)\\ &= 1,\end{aligned}$$

since the Shapley interaction index for capacities is cardinal-probabilistic (see Section 3, p. 146). Thus I for bi-capacities is also cardinal-probabilistic.

∎

Conclusion

Axiomatic characterizations of the interaction index of bi-capacities have been proposed. The presented description is based on generalizations of the recursivity axiom and the reduced-partnership-consinstency axiom. According to the choice of one or the other, more or less powerful linearity, invariance and symmetry are required.

References

[1] J.M. Bilbao, J.R. Fernandez, A. Jiménez Losada, and E. Lebrón. Bicooperative games. In J.M. Bilbao, editor, *Cooperative games on combinatorial structures*. Kluwer Acad. Publ., 2000.

[2] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953.

[3] D. Felsenthal and M. Machover. Ternary voting games. *Int. J. of Game Theory*, 26:335–351, 1997.

[4] K. Fujimoto, I. Kojadinovic, and J.L. Marichal. Axiomatic characterizations of probabilistic and cardinal-probabilistic interaction indices. *Games and Economic Behavior*, 55(1):72–99, 2006.

[5] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[6] M. Grabisch and Ch. Labreuche. Bi-capacities for decision making on bipolar scales. In *EUROFUSE Workshop on Informations Systems*, pages 185–190, Varenna, Italy, September 2002.

[7] M. Grabisch and Ch. Labreuche. Interaction between attributes in a general setting for knowledge discovery. In *4th Int. JIM Conf. (Journées de l'Informatique Messine) on Knowledge Discovery and Discrete Mathematics*, pages 215–222, Metz, France, September 2003.

[8] M. Grabisch and Ch. Labreuche. Bi-capacities. Part I: definition, Möbius transform and interaction. *Fuzzy Sets and Systems*, 151:211–236, 2005.

[9] M. Grabisch and M. Roubens. An axiomatic approach to the concept of interaction among players in cooperative games. *Int. Journal of Game Theory*, 28:547–565, 1999.

[10] P.L. Hammer and S. Rudeanu. *Boolean Methods in Operations Research and Related Areas*. Springer, 1968.

[11] Ch. Labreuche and M. Grabisch. A value for bi-cooperative games. *Int. J. of Game Theory*, submitted.

[12] T. Murofushi and S. Soneda. Techniques for reading fuzzy measures (III): interaction index. In *9th Fuzzy System Symposium*, pages 693–696, Sapporo, Japan, May 1993. In Japanese.

[13] L.S. Shapley. A value for n-person games. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.

[14] R.J. Weber. Probabilistic values for games. In A.E. Roth, editor, *The Shapley Value. Essays in Honor of Lloyd S. Shapley*, pages 101–119. Cambridge University Press, 1988.

Chapitre 5.
La transformée d'interaction pour fonctions de bi-ensembles sur un ensemble fini

Résumé

Les fonctions d'ensemble apparaissent comme un outil très pratique dans de nombreux champs de la recherche opérationnelle. Plusieurs transformations linéaires inversibles ont été introduites pour ces fonctions, telles que la transformée de Möbius, et la transformée d'interaction. Ce papier établit des résultats similaires pour les fonctions de bi-ensembles. Ces dernières ont récemment été introduites en aide à la décision (bi-capacités) et en théorie des jeux (jeux bi-coopératifs), et apparaissent ouvrir de nouvelles appplications dans ces champs.

Mots clés : fonction d'ensemble, fonction de bi-ensembles, transformée de Möbius, transformée d'interaction

Interaction transform for bi-set functions over a finite set

Fabien Lange
Michel Grabisch

Centre d'Economie de la Sorbonne, UMR 8174, CNRS-Université Paris 1

Abstract

Set functions appear as a useful tool in many areas of decision making and operations research, and several linear invertible transformations have been introduced for set functions, such as the Möbius transform and the interaction transform. The present paper establish similar transforms and their relationships for bi-set functions, i.e. functions of two disjoint subsets. Bi-set functions have been recently introduced in decision making (bi-capacities) and game theory (bi-cooperative games), and appear to open new areas in these fields.

Keywords: set function, bi-set function, Möbius transform, interaction transform

In *Information Sciences*, 176:2279–2303, 2006.

1 Introduction

In the field of decision theory and operations research, set functions vanishing on the empty set are an important mathematical tool. In cooperative game theory, they are called *games in characteristic form* (see e.g. Owen [19]), while in operations research they correspond to *pseudo-Boolean functions* [17]. If in addition we require monotonicity with respect to inclusion, we get *capacities* as defined by Choquet [4], or *fuzzy measures* (Sugeno [24]), which happened to be very useful in decision under risk and uncertainty [21], and multicriteria decision making [11]. Well-known particular cases of capacities are belief functions (Shafer [22]), possibility measures (Dubois and Prade [9]), etc.

In the case where the underlying set is finite, there exist close connections with combinatorics. The first one, known since Rota [20], is the *Möbius transform*, which has been widely used in the field of belief functions (under the name *probabilistic mass assignment*), capacities [3], and game theory since the Möbius transform of a game v is the coordinates of v in the basis of unanimity games [23]. The second one, which has been developed in [7] by Denneberg and Grabisch, is the *interaction transform*. It can be viewed as a generalization of the Shapley value [23], and brings very useful tools to multicriteria decision making [12].

Recently, set functions with two arguments have begun to play an important rôle in decision theory, leading to the concepts of *bi-cooperative games* [1], ternary voting games [10], and bi-capacities [14, 15]. Let us describe first the motivation behind bi-capacities, as given in [15] in the framework of multicriteria decision making. We consider a set X of alternatives in a multicriteria decision problem, where each alternative is described by a set of n real-valued scores $(a_1, \dots, a_n)$. Suppose one wants to compute a global score of this alternative by the Choquet integral w.r.t. a capacity μ, namely $\mathcal{C}_\mu(a_1, \dots, a_n)$. Then it is well known that the correspondence between the capacity and the Choquet integral is $\mu(A) = \mathcal{C}_\mu(1_A, 0_{A^c})$, $\forall A \subseteq N$, where $(1_A, 0_{A^c})$ is an alternative having 1 as score on all criteria in A, and 0 otherwise. Such an alternative is called *binary alternative*, and the above result says that the capacity represents the overall score of all binary alternatives.

However, in many practical situations, it is suitable to score alternatives on a bipolar scale, i.e., with a central value 0 having the meaning of a borderline between positive scores, considered as good, and negative scores, considered as bad. It has been observed that most often human decision makers have a different behaviour when faced with alternatives having positive or negative scores, which means that a decision model based solely on the classical Choquet integral, hence on binary alternatives, is no more sufficient. One should, in the general case, consider all *ternary alternatives*, i.e., alternatives of the form $(1_A, -1_B, 0_{(A\cup B)^c})$. Clearly, we need two arguments to denote the overall score of ternary alternatives, namely $v(A, B)$, with $A, B \subseteq N$ being disjoint. This defines bi-capacities, by analogy with capacities.

Motivations in game theory are similar. In classical voting games, $v(A)$ represents the result of a vote concerning some bill, if all voters in A vote in favor of the bill, the remaining voters being against. In ternary voting games, each voter has three alternatives: voting in favor, against, or abstain. Then $v(A, B)$ depicts the result of the vote when voters in A vote in favor, voters in B vote against, and the other ones abstain.

Hence, such "bi-set" functions enable a richer modelling of situations in decision making. The question is then to recover the usual tools associated with set functions, namely the Möbius and interaction transforms. The aim of this paper is precisely to fill this gap. We provide a construction of these two transforms, in the same spirit as the one done by Denneberg and Grabisch in [7], so that the present paper can be seen as a natural continuation of the former. For this reason, we will remain at a general level and deal with *bi-set functions*, instead of more specific cases, as bi-capacities, bi-cooperative games, etc. We will see that analogous results are obtained, despite the fact the the underlying structure is very different.

The organization of the paper is as follows. Section 2 provides necessary background on set functions and bi-set functions, Section 3 introduces the incidence algebras mathematical tool, Section 4 recalls the construction of the interaction transform for set functions as done in [7], Section 5 introduces operators on $\mathcal{Q}\times\mathcal{Q}$, where $\mathcal{Q}$ is the set of pairs of disjoint subsets of N, while Sections 6 and 7 introduce particular cases of such operators, called level operators and cardinality operators. Section 8 gives the expression of the inverse interaction transform, which enables to have a commutative diagram between bi-set functions and their Möbius and interaction transform.

To simplify notations, cardinality of sets $S, T, \ldots$ will be denoted by the corresponding lower case letters $s, t, \ldots$.

2 Set functions and bi-set functions

We introduce necessary concepts for the sequel. We consider a finite set $N := \{1, \ldots, n\}$ which can be thought as the set of criteria, states of nature, voters, etc., depending on the application. We set $\mathcal{P} := \mathcal{P}(N)$. We know that $(\mathcal{P}, \subseteq)$ is the Boolean lattice 2^n, and any $A \in \mathcal{P}$ can be written as a *binary tuple* $x = (x_1, \ldots, x_n) \in \{0,1\}^n$, where $x_i = 1$ iff $i \in A$.

A *set function* v on N is a real-valued mapping on $\mathcal{P}$. Several particular cases are of interest. A set function vanishing on the empty set is called a *game*, while a game satisfying monotonicity, i.e., $v(A)v(B)$ whenever $A \subseteq B$, is called a *capacity* [4], or *non-additive measure* [6], or *fuzzy measure* [24]. Note that when subsets are considered as binary tuples, set functions are called *pseudo-Boolean functions* [17].

For any $C \subseteq N$, the *unanimity game* u_C is defined as:

$$u_C(A) := \begin{cases} 1, & \text{if } A \supseteq C \\ 0, & \text{otherwise} \end{cases}, \quad A \subseteq N.$$

Remark that $u_\emptyset$ is not a game since $u_\emptyset(\emptyset) = 1$.

The *Möbius transform* $m^v : \mathcal{P} \longrightarrow N$ [20] of a set function v is the unique solution of the equation

$$v(A) = \sum_{B \subseteq A} m^v(B), \quad \forall A \subseteq N, \tag{1}$$

and is given by

$$m^v(A) := \sum_{C \subseteq A} (-1)^{a-c}\, v(C), \quad \forall A \subseteq N. \tag{2}$$

Eq. (1) can be rewritten as, using unanimity games:

$$v(A) = \sum_{C \in \mathcal{P}} m^v(C)\, u_C(A), \quad A \subseteq N. \tag{3}$$

Hence, the set of unanimity games forms a 2^n-dimensional basis of set functions, and the Möbius transform represents the coordinates of v in that basis.

For any S belonging to $\mathcal{P} \setminus \{\emptyset\}$, the *derivative* of v with respect to S at point $K \in \mathcal{P}(N \setminus S)$ is given by [16]:

$$\Delta_S v(K) := \sum_{S' \subseteq S} (-1)^{s-s'} v(K \cup S').$$

We set $\Delta_\emptyset v(K) := v(K)$, for any $K \subseteq N$.

The *interaction index* has been proposed by Grabisch [13] and expresses the interaction among a coalition (group) $S \subseteq N$ of elements:

$$I^v(S) := \sum_{K \subseteq N \setminus S} \frac{(n-k-s)!k!}{(n-s+1)!} \Delta_S v(K). \tag{4}$$

This definition extends in fact the *Shapley value* [23] ϕ^v and the interaction index I_{ij} for a pair of elements i, j in N, introduced by Murofushi and Soneda [18]. In particular, the Shapley value is defined by

$$\phi_i^v := \sum_{K \subseteq N \setminus i} \frac{(n-k-1)!k!}{n!} \Delta_i v(K), \quad i \in N.$$

We have $I^v(\{i\}) = \phi_i^v$. As it will be explained in the next section, I^v can be seen as a transform of v, like the Möbius transform.

Let us denote by $\mathcal{Q}(N)$ or simply $\mathcal{Q}$ if there is no fear of ambiguity the set of all pairs of disjoint subsets:

$$\mathcal{Q} := \{(A, B) \in \mathcal{P} \times \mathcal{P} \mid A \cap B = \emptyset\}.$$

We endow $\mathcal{Q}$ with the following partial order:

$$(A, B) \sqsubseteq (C, D) \Leftrightarrow A \subseteq C \text{ and } B \supseteq D.$$

It is easy to see that $(\mathcal{Q}, \sqsubseteq)$ is the lattice 3^n, noting that any element (A, B) of $\mathcal{Q}$ can be written as a *ternary tuple* $x = (x_1, \dots, x_n) \in \{-1, 0, 1\}^n$, where $x_i = 1$ iff $i \in A$ and $x_i = -1$ iff $i \in B$) [15][1]. Supremum and infimum are respectively

$$\begin{aligned} &(A, B) \sqcup (C, D) = (A \cup C, B \cap D), \\ &(A, B) \sqcap (C, D) = (A \cap C, B \cup D), \quad (A, B), (C, D) \in \mathcal{Q}. \end{aligned}$$

Top and bottom of $\mathcal{Q}$ are respectively denoted by $\top := (N, \emptyset)$ and $\bot := (\emptyset, N)$. We give as an illustration $(\mathcal{Q}, \sqsubseteq)$ for $n = 3$ in Fig. 1.

[1]Equivalently, one could have chosen a different coding, as 0, 1 and 2 instead of -1, 0 and 1. Our choice, which is unimportant in this paper, is just suited to the original motivation in decision making explained in the introduction.

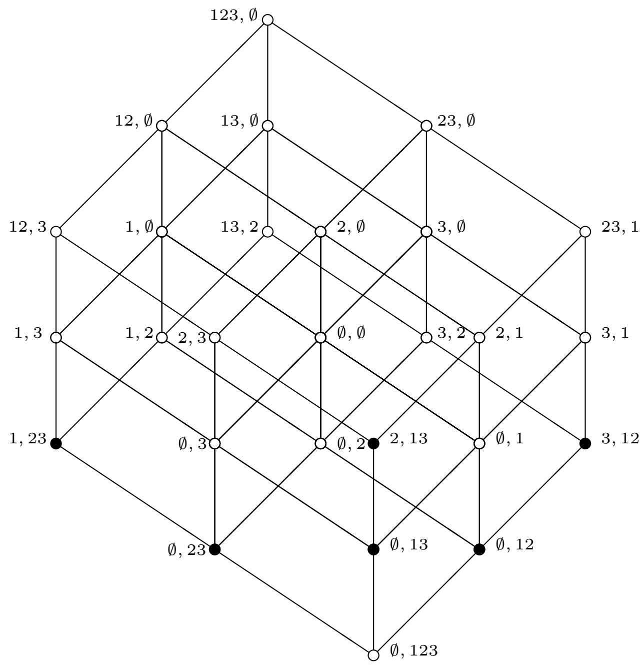


Figure 1: The lattice $\mathcal{Q}$ for $n = 3$

A *bi-set function* v on N is a real-valued mapping on $\mathcal{Q}$. As explained in the introduction, particular cases of interest are bi-cooperative games, where it is required that $v(\emptyset, \emptyset) = 0$, and bi-capacities which require in addition monotonicity, i.e., $(A, B) \sqsubseteq (C, D)$ implies $v(A, B) \leq v(C, D)$.

The lattice $(\mathcal{Q}, \sqsubseteq)$ being distributive, by Birkhoff's theorem [2], any element of the lattice can be written as a unique irredundant supremum over a set of *join-irreducible elements* (elements having only one predecessor). In the case of $(\mathcal{Q}, \sqsubseteq)$ the set of all join-irreducible elements (which are represented by black circles in Fig. 1) is

$$\mathcal{J}(\mathcal{Q}) = \{(\emptyset, i^c), (i, i^c), i \in N\},$$

and the unique irredundant decomposition writes [15]:

$$(A, B) = \bigsqcup_{i \in A} (i, i^c) \sqcup \bigsqcup_{j \in N \setminus (A \cup B)} (\emptyset, j^c).$$

This permits to define *layers* in $\mathcal{Q}$ as follows: for k in $N_0 := \{0, \ldots, n\}$, layer k contains all elements (A, B) whose decomposition has exactly k join-irreducible elements, which is equivalent to say that $|B| = n - k$. We denote by $\| \cdot \|$ the function which maps to every element of $\mathcal{Q}$ the layer to which it belongs.

It is convenient for the sequel to define the following linear order $\leq$ on $\mathcal{Q}$. Recalling that any element of $\mathcal{Q}$ can be written as a ternary tuple $x \in \{-1,0,1\}^n$ or equivalently in $\{0,1,2\}^n$, we can assign to each element (A,B) of $\mathcal{Q}$ an integer $n_{(A,B)}$ whose coding in basis $\{0,1,2\}$ is precisely the ternary tuple x associated to (A,B). For example, taking $n=4$ and the element $(\{1\},\{3\})$, the corresponding tuple is $(2,1,0,1)$, which gives the number $2\times 3^0 + 1\times 3^1 + 0\times 3^2 + 1\times 3^3 = 32$. Obviously, the correspondence between integers and elements of $\mathcal{Q}$ is unique. Hence, we say that $(A,B)\leq(C,D)$ iff $n_{(A,B)}\leq n_{(C,D)}$. This leads to the following order:

$$\cdots(2,3)\,(12,3)\,\Big[(\emptyset,12)\,(\emptyset,2)\,(1,2)\,\Big[(\emptyset,1)\,\Big[(\emptyset,\emptyset)\Big](1,\emptyset)\Big]\,(2,1)\,(2,\emptyset)\,(12,\emptyset)\Big]\,(3,12)\,(3,2)\cdots \tag{5}$$

The brackets are there to enhance the fact that this order is in some sense "recursive", since it can be built by an initial pattern (which is $(\emptyset,\emptyset)$) and a systematic way of augmenting the current pattern, which is to add a new element of N either to the left part or to the right part of any element of the current pattern.

The Möbius transform m^v of v is the unique solution of the equation

$$v(A,B) = \sum_{(C,D)\sqsubseteq(A,B)} m^v(C,D), \quad (A,B)\in\mathcal{Q}, \tag{6}$$

and is given by [15]:

$$m^v(A,B) := \sum_{\substack{(C,D)\sqsubseteq(A,B)\\ D\cap A=\emptyset}} (-1)^{a-c+d-b}\, v(C,D), \quad (A,B)\in\mathcal{Q}. \tag{7}$$

We extend the notion of derivative of a set function to bi-set functions. As bi-set functions are defined on $\mathcal{Q}$, so should be the variables used in the derivation. For any $i\in N$, the derivatives with respect to any join-irreducible elements (i,i^c) and $(\emptyset,i^c)$ of v at point (K,L) are given by [15]:

$$\begin{array}{ll} \forall(K,L)\in\mathcal{Q}(N\setminus i), & \Delta_{(i,i^c)}v(K,L) := v(K\cup i,L) - v(K,L), \quad \text{and}\\ \forall(K,L)\in\mathcal{Q} \text{ with } i\in L, & \Delta_{(\emptyset,i^c)}v(K,L) := v(K,L\setminus i) - v(K,L). \end{array}$$

These derivatives are non negative whenever v is monotonic. Higher order derivatives can be defined recursively for any $(S,T)\in\mathcal{Q}\setminus\{(\emptyset,N)\}$ by:

$$\Delta_{(S,T)}v(K,L) := \Delta_{(i,i^c)}(\Delta_{(S\setminus i,T\cup i)}v(K,L))$$

if (i,i^c) belongs to the irredundant decomposition of (S,T), or

$$\Delta_{(S,T)}v(K,L) := \Delta_{(\emptyset,i^c)}(\Delta_{(S,T\cup i)}v(K,L)),$$

if $(\emptyset, i^c)$ belongs to the irredundant decomposition of (S,T).
We set $\Delta_{(\emptyset,N)}v(K,L) = v(K,L)$, for any $(K,L) \in \mathcal{Q}$.

In [15], the following definition of the interaction index for bi-set functions has been given, as a natural generalization of the definition for set functions:

$$I^v(S,T) := \sum_{K \subseteq T} \frac{(t-k)!k!}{(t+1)!} \Delta_{(S,T)} v(K,(K \cup S)^c), \quad \forall (S,T) \in \mathcal{Q}. \tag{8}$$

3 Incidence algebras

In what follows, we have to take into account *incidence algebras*, which is a usefull structure proposed by Doubilet, Rota and Stanley [8]. Specifically, let $(P, \leq)$ be a partially ordered set. A *segment* $[x,y]$, for x and y in P such that $x \leq y$, is the set of all elements z which satisfy $x \leq z \leq y$. We denote by $S(P)$ the set of all segments of P. A poset is *locally finite* if every segment is finite. Then we define the *incidence algebra* $\mathbf{I}(P,K)$ of a locally finite poset P, over a field K as follows:

$$\mathbf{I}(P,K) := \{\text{mappings } f : P \times P \to K \mid f(x,y) = 0 \text{ if } x \not\leq y\}.$$

Remark that $\mathbf{I}(P,K)$ can also be written as the set of K-valued mappings f defined over $S(P) \cup \{\emptyset\}$, such that $f(\emptyset) = 0$. We will write $\mathbf{I}(P)$ for $\mathbf{I}(P,K)$ if there is no ambiguity about the field K. In $\mathbf{I}(P,K)$, the sum $(+)$ of two functions and multiplication $(\cdot)$ by scalars are defined as usual, and the product $f \star g = h$ is defined as follows:

$$h(x,y) := \sum_{z \in P} f(x,z)\, g(z,y).$$

Endowed with $+, \cdot$ and $\star$, $\mathbf{I}(P,K)$ is an associative K-algebra with identity, where a *K-algebra* $\mathbb{A}$ is a K-vector space together with a bilinear map $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$, whose possible associativity and presence of identity refer to the mentionned bilinear map.

The identity of $\mathbf{I}(P,K)$ is the function $\Delta(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$

Definition 1 *Let $\sim$ be an equivalence relation on the set $S(P)$ of segments of a locally finite poset P. $\sim$ is said to be* order compatible *(or* compatible*) when it* *satisfies the following condition:*

$$\forall f, g \in \mathbf{I}(P),\ \ \textit{if} \begin{cases} f(x,y) = f(u,v) \\ g(x,y) = g(u,v) \end{cases} \textit{for all } [x,y],[u,v] \in S(P)$$
$$\textit{such that } [x,y] \sim [u,v],\ \ \textit{then } (f \star g)(x,y) = (f \star g)(u,v).$$

We call *type* (relative to $\sim$) any equivalence class of segments of P relative to $\sim$. Now, consider the set $\mathbf{R}(P,\sim)$ of all functions defined on the set of types, with addition and external multiplication defined as usual, and convolution $f \star g = h$ defined as follows:

$$h(\alpha) := \sum \left[{}^{\alpha}_{\beta,\gamma}\right] f(\beta)\, g(\gamma), \tag{9}$$

where the sum ranges over all pairs β, γ of types, and where we define the *incidence coefficients* as follows: $\left[{}^{\alpha}_{\beta,\gamma}\right]$ stands for the number of distinct elements z in a segment $[x,y]$ of type α, such that $[x,z]$ is of type β and $[z,y]$ is of type γ.

Then, Proposition 4.3. of [8] (p. 278) tell us that the set $\mathbf{R}(P,\sim)$ of functions on types, called the *reduced incidence algebra modulo* $\sim$, forms an associative algebra with identity, which is isomorphic to the following subalgebra of $\mathbf{I}(P)$:

$$\big\{f \in \mathbf{I}(P) \mid \forall [x,y],[u,v] \in S(P) \text{ having the same type, } f(x,y) = f(u,v)\big\}.$$

We denote by $\hat{f}$ the unique element of the above subalgebra, associated to some function f on types.

In the case of partially ordered set $(\mathcal{P}, \subseteq)$, the following holds.

Proposition 1 *Let $\sim_l$ be the equivalence relation on the set $S(\mathcal{P})$ defined by:*

$$[A,B] \sim_l [C,D] \ \textit{iff}\ B \setminus A = D \setminus C,$$

and let $\overline{\mathcal{P}}$ be the set of types. Then $\sim_l$ is compatible and $\overline{\mathcal{P}}$ is isomorphic to $(\mathcal{P}, \subseteq)$.

Particular reduced incidence algebras are given by specific compatible equivalence relations:

Definition 2 *Let P be a locally finite poset with a least element and $\sim$ be a compatible equivalence relation.* $\mathbf{R}(P,\sim)$ *is said to be an* algebra of full binomial type *if the types are in one to one correspondence with a subset* N' *of* $\mathbb{N}$*, the type of a segment* $[x, y]$ *being denoted* $O(x, y)$*, satisfying:*

(A) for any $z \in [x, y]$, $O(x, y) = O(x, z) + O(z, y)$;

(B) let $\left[{m \atop k}\right] := |\{z \in [x, y] \mid O(x, z) = k \text{ and } O(z, y) = m - k\}|$,
where $[x, y]$ *is of type* m*. Then for all* $m \in N'$ *and* $k \leq m$, $\left[{m \atop k}\right] \neq 0$.

Proposition 2 *Let* $\sim_c$ *be the equivalence relation on the set* $S(\mathcal{P})$ *defined by:*

$$[A, B] \sim_c [C, D] \text{ iff } |B \setminus A| = |D \setminus C|.$$

Then $\sim_c$ *is compatible and* $\mathbf{R}(\mathcal{P}, \sim_c)$ *is an algebra of full binomial type, where types are in correspondence with* $N_0 = \{0, \ldots, n\}$*.*
Moreover, $\mathbf{R}(\mathcal{P}, \sim_c)$ *is the* maximally reduced incidence algebra *on* $\mathcal{P}$*, that is to say,* $\sim_c$ *is the coarsest compatible equivalence relation: every partition of* $S(\mathcal{P})$ *that is the quotient set of some compatible equivalence relation, is a refinement of* $S(\mathcal{P})/\sim_c$*. In this way,* $\mathbf{R}(\mathcal{P}, \sim_c)$ *is also denoted* $\overline{\mathbf{R}}(\mathcal{P})$*.*

4 Interaction transform for set functions

We recall in this section main results given in [7], where a new invertible transform of set functions is introduced, called the *interaction transform*. The authors lay down a general framework of transformations of set functions by introducing an algebraic structure on set functions and operators (set functions of two variables), which enable the writing of the formulae given in the previous section under a simplified algebraic form. Links with incidence algebras can be examined in Appendix .

In the first place, we recall main definitions. We call *operator* on $\mathcal{P}$ a real-valued function on $\mathcal{P} \times \mathcal{P}$, and introduce a multiplication $\star$ between operators, and between operators and set functions as follows. Let v be a set function and Φ, Ψ some operators; for A_1, A_2 belonging to $\mathcal{P}$, we have:

$$(\Phi \star \Psi)(A_1, A_2) := \sum_{C \in \mathcal{P}} \Phi(A_1, C)\, \Psi(C, A_2),$$

$$(\Phi \star v)(A_1) := \sum_{C \in \mathcal{P}} \Phi(A_1, C)\, v(C),$$

$$(v \star \Phi)(A_2) := \sum_{C \in \mathcal{P}} v(C)\, \Phi(C, A_2).$$

Let us now consider a subset $\mathcal{G}_{\mathcal{P}}$ of these operators[2], defined by the operators Φ which have the property

$$\Phi(A_1, A_2) = \begin{cases} 1, & \text{if } A_1 = A_2 \\ 0, & \text{if } A_1 \not\subseteq A_2, \end{cases}$$

for any $A_1, A_2 \in \mathcal{P}$. The family $\mathcal{G}_{\mathcal{P}}$ endowed with the operation $\star$ is a group (see [7]).Into, we found the so-called operators Zeta ($Z_{\mathcal{P}}$) and Möbius ($Z_{\mathcal{P}}^{-1}$) defined by:

$$\begin{aligned} Z_{\mathcal{P}}(A_1, A_2) &:= \begin{cases} 1, & \text{if } A_1 \subseteq A_2 \\ 0, & \text{otherwise} \end{cases}, \quad A_1, A_2 \in \mathcal{P}, \\ Z_{\mathcal{P}}^{-1}(A_1, A_2) &= \begin{cases} (-1)^{a_2 - a_1}, & \text{if } A_1 \subseteq A_2 \\ 0, & \text{otherwise} \end{cases}, \quad A_1, A_2 \in \mathcal{P}. \end{aligned}$$

> **Remark.** Describing the domain of $Z_{\mathcal{P}}$ in terms of elements of $S(\mathcal{P})$ justifies the terminologies of Zeta and Möbius function: $Z_{\mathcal{P}}(A_1, A_2) = 1$ if $[A_1, A_2] \neq \emptyset$ and 0 otherwise.

Then, Equations (3) and (2) can be written as:

$$v = m^v \star Z_{\mathcal{P}} \text{ and } m^v = v \star Z_{\mathcal{P}}^{-1},$$

> **Remark.** The symbol $\star$ has not here the same meaning than the convolution operation in $\mathbf{I}(\mathcal{P})$ (see previous Section) since v and m^v are not elements of $\mathbf{I}(\mathcal{P})$.

A central role is also played by the operator $\Gamma_{\mathcal{P}} \in \mathcal{G}_{\mathcal{P}}$,

$$\Gamma_{\mathcal{P}}(A_1, A_2) := \begin{cases} \dfrac{1}{a_2 - a_1 + 1}, & \text{if } A_1 \subseteq A_2 \\ 0, & \text{else} \end{cases}, \tag{10}$$

which is called the *inverse Bernoulli operator*. This name will be justified in Section 8. Actually, we have the relation

$$I^v = \Gamma_{\mathcal{P}} \star m^v.$$

[2]The sets and functions denoted with the suffix $_{\mathcal{P}}$ are sets and functions defined on $\mathcal{P}$ refering to [7].

We call *level operator* an operator Φ satisfying

$$\Phi(A_1, A_2) = \begin{cases} \Phi(\emptyset, A_2 \setminus A_1), & \text{if } A_1 \subseteq A_2 \\ 0, & \text{otherwise} \end{cases},$$

and the set of all level operators is denoted by $\mathcal{G}'_{\mathcal{P}}$. Endowed with $\star$, $\mathcal{G}'_{\mathcal{P}}$ is a subgroup of $\mathcal{G}_{\mathcal{P}}$. Let us introduce

$$\mathbf{g}_{\mathcal{P}} := \{\varphi : \mathcal{P} \to \mathbb{R} \mid \varphi(\emptyset) = 1\},$$

and associate with any level operator Φ the function φ_Φ of $\mathbf{g}_{\mathcal{P}}$ defined by $\varphi_\Phi(\cdot) := \Phi(\emptyset, \cdot)$. Indeed, it is easy to see that φ_Φ determines Φ uniquely: let φ be in $\mathbf{g}_{\mathcal{P}}$; if we define

$$\Phi_\varphi(A_1, A_2) := \begin{cases} \varphi(A_2 \setminus A_1), & \text{for } A_1 \subseteq A_2, \\ 0, & \text{else} \end{cases}, \quad A_1, A_2 \in \mathcal{P},$$

we have $\Phi_\varphi = \Phi$ iff $\varphi := \varphi_\Phi$. Now, if we define the operation $\star$ between two elements φ, ψ of $\mathbf{g}_{\mathcal{P}}$ by

$$\varphi \star \psi(A) := \Phi_\varphi \star \Phi_\psi(\emptyset, A), \quad A \in \mathcal{P}, \tag{11}$$

then $(\mathcal{G}'_{\mathcal{P}}, \star)$ and $(\mathbf{g}_{\mathcal{P}}, \star)$ are isomorphic. $\varphi \star \psi$ is the *convolution* of $\varphi, \psi \in \mathbf{g}_{\mathcal{P}}$,

$$\varphi \star \psi(A) = \sum_{C \subseteq A} \varphi(C)\, \psi(A \setminus C), \quad A \in \mathcal{P}.$$

Since the inverse Bernoulli operator is a level operator, its corresponding function $\gamma_{\mathcal{P}} := \varphi_{\Gamma_{\mathcal{P}}}$ is:

$$\gamma_{\mathcal{P}}(A) = \frac{1}{a+1}, \quad A \in \mathcal{P}.$$

A *cardinality function* on $\mathcal{P}$ is an element of $\mathbf{g}_{\mathcal{P}}$ that only depends on the cardinality of the variable. The above inverse Bernoulli function is an example of cardinality function. We denote by $\mathbf{c}_{\mathcal{P}}$ the set of all cardinality functions, and endowed with $\star$, $\mathbf{c}_{\mathcal{P}}$ is subgroup of $\mathbf{g}_{\mathcal{P}}$. To each cardinality function φ we associate its *cardinal representation* f_φ in the set

$$\mathfrak{r} := \{f : N_0 \to \mathbb{R} \mid f(0) = 1\}$$

in a bijective way; for any $A \in \mathcal{P}$

$$f_\varphi(|A|) = \varphi(A).$$

Conversely, for $f \in \mathfrak{r}$, we put

$$\varphi_{f,\mathcal{P}}(A) := f(a), \quad A \in \mathcal{P}.$$

Once more, $\mathfrak{r}$ is an Abelian group with $f_\delta := 1_{\{0\}}$ as neutral element, and where the convolution operation is, for any $f, g \in \mathfrak{r}$, $\quad A \in \mathcal{P}$

$$f \star g(|A|) := \varphi_{f,\mathcal{P}} \star \varphi_{g,\mathcal{P}}(A),$$

that is to say, for any $m \in N_0$:

$$f \star g(m) = \sum_{k=0}^{m} \binom{m}{k} f(k)\, g(m-k). \tag{12}$$

Hence, $(\mathfrak{r}, \star)$ is isomorphic to $(\mathbf{c}_{\mathcal{P}}, \star)$. We will denote $f^{\star -1}$ the inverse of an element f of $\mathfrak{r}$.

There is still a way to consider the sets of cardinality functions and cardinal representations as a reduced incidence algebra (see Appendix).

5 Operators on $\mathcal{Q} \times \mathcal{Q}$

We will proceed in the same way for bi-set functions, the basis working set being $\mathcal{Q}$. We consider real-valued functions on $\mathcal{Q}$ in one and two variables, the latter ones being called operators, and we introduce a multiplication $\star$ between operators, and between a bi-set function and an operator. Let v be a bi-set function and Φ, Ψ some operators on $\mathcal{Q} \times \mathcal{Q}$; for $(A_1, B_1), (A_2, B_2)$ belonging to $\mathcal{Q}$, we define:

$$(\Phi \star \Psi)((A_1, B_1), (A_2, B_2)) := \sum_{(C,D) \in \mathcal{Q}} \Phi((A_1, B_1), (C, D))\, \Psi((C, D), (A_2, B_2)),$$

$$(\Phi \star v)(A_1, B_1) := \sum_{(C,D) \in \mathcal{Q}} \Phi((A_1, B_1), (C, D))\, v(C, D),$$

$$(v \star \Psi)(A_2, B_2) := \sum_{(C,D) \in \mathcal{Q}} v(C, D)\, \Phi((C, D), (A_2, B_2)).$$

Endowed with $\star$, the set of these operators contains the neutral element Δ defined by

$$\Delta((A_1,B_1),(A_2,B_2)) := \begin{cases} 1, & \text{if } (A_1,B_1)=(A_2,B_2) \\ 0, & \text{else} \end{cases}, \quad (A_1,B_1),(A_2,B_2) \in \mathcal{Q},$$

and satisfies associativity. When it exists, we will denote by Φ^{-1} the inverse of an operator Φ, that is to say the operator verifying $\Phi \star \Phi^{-1} = \Phi^{-1} \star \Phi = \Delta$.

The following proposition deals with an important subset of operators.

Proposition 3 *The family* $\mathcal{G}$ *of operators defined by:*

$$\Phi \in \mathcal{G} \iff \Phi((A_1,B_1),(A_2,B_2)) = \begin{cases} 1, & \text{if } (A_1,B_1)=(A_2,B_2) \\ 0, & \text{if } (A_1,B_1) \not\sqsubseteq (A_2,B_2) \end{cases},$$
$$(A_1,B_1),(A_2,B_2) \in \mathcal{Q},$$

endowed with the operation $\star$ *is a group. The inverse* Φ^{-1} *of* Φ *in* $\mathcal{G}$ *computes recursively through*

$$\Phi^{-1}((A_1,B_1),(A_1,B_1)) = 1,$$
$$\Phi^{-1}((A_1,B_1),(A_2,B_2)) = -\sum_{\substack{(C,D)\in\\ [(A_1,B_1),(A_2,B_2)[}} \Phi^{-1}((A_1,B_1),(C,D))\,\Phi((C,D),(A_2,B_2)).$$

We can naturally find inside the set $\mathcal{G}$ the *Zeta operator* Z:

$$Z((A_1,B_1),(A_2,B_2)) := \begin{cases} 1, & \text{if } (A_1,B_1) \sqsubseteq (A_2,B_2) \\ 0, & \text{otherwise} \end{cases}, \quad (A_1,B_1),(A_2,B_2) \in \mathcal{Q},$$

which allows us to rewrite Equation (6) as:

$$v = m^v \star Z.$$

Similarly, the *Möbius operator*, defined as the inverse of Z, permits to rewrite Equation (7) as:

$$m^v = v \star Z^{-1},$$

and Proposition 3 gives, for any $(A_1,B_1),(A_2,B_2) \in \mathcal{Q}$

$$Z^{-1}((A_1,B_1),(A_2,B_2)) = \begin{cases} (-1)^{a_2-a_1+b_1-b_2}, & \text{if } \begin{cases} (A_1,B_1) \sqsubseteq (A_2,B_2) \\ B_1 \cap A_2 = \emptyset \end{cases} \\ 0, & \text{otherwise} \end{cases},$$

as expected (see (7)).

In the previous section, the interaction index of a set function was expressed through the $\mathcal{G}_{\mathcal{P}}$ operator $\Gamma_{\mathcal{P}}$ (see (10)), which facilitated the inversion of (4). We shall undertake to do the same thing for bi-set functions. From (8) and according to an expression of the derivatives based on Möbius transform [15], we have for every $(S,T) \in \mathcal{Q}$:

$$I^v(S,T) = \sum_{\substack{(S',T')\in\\ [(S,T),(S\cup T,\emptyset)]}} \frac{1}{t-t'+1}\, m^v(S',T'). \tag{13}$$

As a result, if we set down:

$$\Gamma((A_1,B_1),(A_2,B_2)) := \begin{cases} \dfrac{1}{b_1-b_2+1}, & \text{if } (A_1,B_1) \sqsubseteq (A_2,B_2) \sqsubseteq (A_1\cup B_1,\emptyset) \\ 0, & \text{otherwise} \end{cases}, \tag{14}$$

we can write from (13) the relation:

$$I^v = \Gamma \star m^v. \tag{15}$$

Let us notice that Γ is an operator in $\mathcal{G}$. As $\Gamma_{\mathcal{P}}$, we call it the *inverse Bernoulli operator.*

Γ has a similar expression to that of $\Gamma_{\mathcal{P}}$ (see (10)):

$$\Gamma_{\mathcal{P}}(A_1,A_2) := \begin{cases} \dfrac{1}{a_2-a_1}, & \text{if } A_1 \subseteq A_2 \\ 0, & \text{otherwise} \end{cases}, \quad A_1,A_2 \in \mathcal{P},$$

with however a rather unexpected inequality $(A_2,B_2) \sqsubseteq (A_1\cup B_1,\emptyset)$ which will complicate the continuation of the work. Nevertheless, at this point, we can set the following fundamental result, already known in the case of set functions (see Fig. 2).

Theorem 4 *For any bi-set function v, the triangular diagram where appear the functions v, m^v, I^v and the operators of transition Z, Γ is commutative.*

Proof: Commutativity between v and m^v is clear according to Equation (6). The one between m^v and I^v is known due to (15). By transitivity, the result follows.

■

$$\begin{tikzcd}
v \arrow[rr, shift left, "\Gamma \star (\bullet \star Z^{-1})"] \arrow[dr, shift right, "\bullet \star Z^{-1}"'] & & I^v \arrow[ll, shift left, "(\Gamma^{-1} \star \bullet) \star Z"] \arrow[dl, shift right, "\Gamma^{-1} \star \bullet"'] \\
& m^v \arrow[ul, shift right, "\bullet \star Z"'] \arrow[ur, shift right, "\Gamma \star \bullet"'] &
\end{tikzcd}$$

Figure 2: Three ways of representing bi-set functions

6 Level operators

Our aim being now the inversion of Γ, a few results about lattice theory need to be brought in. First, the double inequality in (14) suggests us to introduce a new binary relation on $\mathcal{Q}$, denoted by $\trianglelefteq$:

$$(A_1, B_1) \trianglelefteq (A_2, B_2) \text{ if and only if}$$
$$\begin{cases} (A_1, B_1) \sqsubseteq (A_2, B_2) \\ A_2 \subseteq A_1 \cup B_1 \end{cases} \text{or equivalently} \begin{cases} A_1 \subseteq A_2 \subseteq A_1 \cup B_1 \\ B_1 \supseteq B_2. \end{cases}$$

It is easy to see that $\trianglelefteq$ is an ordered relation included in $\sqsubseteq$.

Moreover, as we use the notation $[(A_1, B_1), (A_2, B_2)]$ to denote the segment of $(\mathcal{Q}, \sqsubseteq)$ delimited by (A_1, B_1) and (A_2, B_2), we will use the notation $\lfloor (A_1, B_1), (A_2, B_2) \rceil$ for the same of $(\mathcal{Q}, \trianglelefteq)$ — by replacing, if needed, $\lfloor$ by $\rfloor$ or $\rceil$ by $\lceil$ if we deprive the segment of the associated bound. We denote $S(\mathcal{Q})$ the set of segments of $(\mathcal{Q}, \trianglelefteq)$ and we also introduce

$$\mathcal{Q}_{(A,B)} := \lfloor \perp, (A, B) \rceil.$$

We have the following proposition which is useful for the sequel:

Proposition 5 *For any (A, B) of $\mathcal{Q}$, the ordered subset $(\mathcal{Q}_{(A,B)}, \trianglelefteq)$ of $(\mathcal{Q}, \trianglelefteq)$ is a Boolean lattice isomorphic to $(\mathcal{P}(B^c), \subseteq)$ by the mapping:*

$$\begin{aligned} q_{(A,B)} :& \mathcal{Q}_{(A,B)} \to \mathcal{P}(B^c) \\ & (C, D) \mapsto D^c. \end{aligned} \tag{16}$$

In particular, $(\mathcal{Q}_\top, \trianglelefteq)$ is a Boolean lattice isomorphic to $(\mathcal{P}, \subseteq)$.

Endowed with this new order relation, we can define the following operation in $\mathcal{Q}$:

Definition 3 *The* strict difference *operation in $\mathcal{Q}$ is defined for every $((A_1, B_1), (A_2, B_2)) \in \mathcal{Q} \times \mathcal{Q}$ such that $(A_1, B_1) \trianglelefteq (A_2, B_2)$ by:*

$$(A_2, B_2) \setminus\!\!\setminus (A_1, B_1) := (A_2 \setminus A_1, (B_1 \setminus B_2)^c).$$

Note that $((A_2, B_2) \setminus\!\!\setminus (A_1, B_1)) \sqcup (A_1, B_1) = (A_2, B_2)$.

Remark. One can give a graphic interpretation of the $\trianglelefteq$ order and the $\setminus\!\!\setminus$ operation: we call *vertices* of $\mathcal{Q}$ any element (A, B) such that $A \cup B = N$, since they coincide with the vertices of $[-1, 1]^n$. In the same way, we define the vertices of any sub-lattice of $\mathcal{Q}$. So, for any $(A, B) \in \mathcal{Q}$, $\mathcal{Q}_{(A,B)}$ is the set of vertices of the sub-lattice $[\perp, (A, B)]$. Moreover, two elements $(C_1, D_1), (C_2, D_2)$ of $\mathcal{Q}$ are said *complementary* w.r.t. an element (A, B) of $\mathcal{Q}$ if $(C_1, D_1), (C_2, D_2) \in \mathcal{Q}_{(A,B)}$ and:

$$\begin{aligned} &(A, B) \setminus\!\!\setminus (C_1, D_1) = (C_2, D_2), \text{ which is equivalent to} \\ &(A, B) \setminus\!\!\setminus (C_2, D_2) = (C_1, D_1). \end{aligned}$$

In particular, the pairs of elements which are complementary w.r.t. $\top$ are the pairs $\{(A, A^c), (A^c, A)\}$, for every $A \subseteq N$. As a consequence, for $(A, B) \in \mathcal{Q}$, complementarity w.r.t. an element of $\mathcal{Q}_{(A,B)}$ entails the same property than complementarity w.r.t. an element of $\mathcal{P}$: if (C, D) belongs to $\mathcal{Q}_{(A,B)}$, (C, D) and its complement w.r.t. (A, B) are opposite vertices in the sub-lattice $\mathcal{Q}_{(A,B)}$.

Like the set difference in $\mathcal{P}$ (cf. [7]), the strict difference operation in $\mathcal{Q}$ will allow us to transform some $\mathcal{G}$ operators into operators of a single variable. In addition, the $\star$ operation will be transformed into a convolution operation.

Now, let us derive results for $\mathcal{Q} \times \mathcal{Q}$ operators.

Proposition 6 *Let $\sim_l$ be the equivalence relation on the set $S(\mathcal{Q})$ defined by:*

$$\lfloor (A_1, B_1), (A_2, B_2) \rceil \sim_l \lfloor (C_1, D_1), (C_2, D_2) \rceil \quad \textit{iff} \quad (A_2, B_2) \mathbin{\backslash\!\backslash} (A_1, B_1) = (C_2, D_2) \mathbin{\backslash\!\backslash} (C_1, D_1),$$

and let $\overline{\mathcal{Q}}$ be the set of types. Then $\sim_l$ is compatible and $\overline{\mathcal{Q}}$ is isomorphic to $\mathcal{Q}$.

Let us introduce $\mathbf{g} := \{\varphi : \mathcal{Q} \to \mathbb{R} \mid \varphi(\bot) = 1\}$, which is the subset of $\mathbf{R}(\mathcal{Q}, \sim_l)$ of functions of value 1 at $\bot$, and let $\mathcal{G}'$ be the image of $\mathbf{g}$ by the following mapping:

$$\mathbf{R}(\mathcal{Q}, \sim_l) \to \mathbf{I}(\mathcal{Q})$$

$$\varphi \mapsto \Phi_\varphi, \text{ where } \Phi_\varphi : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$$

$$\Phi_\varphi((A_1, B_1), (A_2, B_2)) := \begin{cases} \varphi\big(\text{type}(\lfloor (A_1, B_1), (A_2, B_2) \rceil)\big) & \text{if } (A_1, B_1) \trianglelefteq (A_2, B_2), \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{G}'$ is called the set of *level operators*. We deduce the following result:

Proposition 7 *Let $\star$ be the convolution operation on $\mathbf{R}(\mathcal{Q}, \sim_l)$ given by (9). Then $(\mathcal{G}', \star)$ is isomorphic to $(\mathbf{g}, \star)$.*

Next, we can give the inverse mapping $\Lambda : \mathcal{G}' \longrightarrow \mathbf{g}$, which associates to any level operator Φ the function $\Lambda(\Phi)$, also denoted φ_Φ for convenience:

$$\varphi_\Phi(A, B) := \Phi(\bot, (A, B)), \quad (A, B) \in \mathcal{Q}.$$

> **Remark.** We can notice that Γ is a level operator, contrary to Z and Z^{-1}, even if in the case of set functions, we can find in the $\mathcal{G}'_{\mathcal{P}}$ set the $\Gamma_{\mathcal{P}}$ operator but also the Zeta and Möbius $\mathcal{P} \times \mathcal{P}$ operators.

As a consequence, we can express the inverse Bernoulli function $\gamma := \varphi_\Gamma$. Thanks to what we have seen before, we can directly write

$$\gamma(A, B) = \frac{1}{n - b + 1}, \quad (A, B) \in \mathcal{Q}. \tag{17}$$

$\gamma^{\star -1}$ is called the *Bernoulli function*.

7 Cardinality operators

A real-valued function on $\mathcal{Q}$ is called a *cardinality function* if it only depends on the layer of the variable, and is equal to 1 at $\bot$. We denote by $\mathbf{c}$ the set of these functions. We recall that $\mathfrak{r} := \{f : N_0 \longrightarrow \mathbb{R} \mid f(0) = 1\}$, and introduce the mapping $\lambda : \mathbf{c} \longrightarrow \mathfrak{r}$, which associates to each cardinality function φ its *cardinal representation* $\lambda(\varphi)$, also denoted by f_φ for convenience, defined by:

$$f_\varphi(\|(A, B)\|) = \varphi(A, B), \quad (A, B) \in \mathcal{Q}.$$

Conversely, for any $f \in \mathfrak{r}$ we define $\varphi_f((A, B)) := f(\|(A, B)\|)$, $(A, B) \in \mathcal{Q}$. Thus, λ is bijective.

Proposition 8 *Let $\sim_c$ be the equivalence relation on the set $S(\mathcal{Q})$ defined by:*

$$\lfloor (A_1, B_1), (A_2, B_2) \rceil \sim_c \lfloor (C_1, D_1), (C_2, D_2) \rceil \text{ iff } \\ \|(A_2, B_2) \setminus\!\!\setminus (A_1, B_1)\| = \|(C_2, D_2) \setminus\!\!\setminus (C_1, D_1)\|.$$

Then $\sim_c$ is compatible and $\mathbf{R}(\mathcal{Q}, \sim_c)$ is an algebra of full binomial type, where types are in correspondence with $N_0 = \{0, \ldots, n\}$.
As a consequence, $\mathbf{R}(\mathcal{Q}, \sim_c)$ is the maximally reduced incidence algebra on $\mathcal{Q}$: $\mathbf{R}(\mathcal{Q}, \sim_c) = \overline{\mathbf{R}}(\mathcal{Q})$.

Furthermore, we call *cardinality operator* of $\mathcal{Q} \times \mathcal{Q}$ (resp. $\mathcal{P} \times \mathcal{P}$) any level operator Φ whose associated function φ_Φ of $\mathbf{g}$ (resp. $\mathbf{g}_\mathcal{P}$) belongs to $\mathbf{c}$ (resp. $\mathbf{c}_\mathcal{P}$). We denote by $\mathcal{G}''$ (resp. $\mathcal{G}''_\mathcal{P}$) the set of cardinality operators. As shown by the following Lemma, $(\mathbf{c}, \star)$ is a subgroup of $(\mathbf{g}, \star)$.

Lemma 9 (fundamental) $(\mathbf{c}, \star)$ *is an Abelian group isomorphic to* $(\mathfrak{r}, \star)$*, and the triangular diagram representing* $\mathbf{c}_\mathcal{P}$*,* $\mathbf{c}$ *and* $\mathfrak{r}$ *is commutative.*

Therefore, by (17) the inverse Bernoulli function for bi-set functions has cardinal representation $f_\gamma(m) = \frac{1}{m+1}, \quad m \in N_0$. In fact, it appears that $f_\gamma = f_{\gamma_\mathcal{P}}$. This link with the previous result is fundamental in our work.

As a conclusion to these three sections, we can give the following recapitulative result, illustrated by Figure 3.

Proposition 10 *The diagram successively representing* $\mathcal{G}''_{\mathcal{P}}$ *(cardinality operators for set functions),* $\mathbf{c}_{\mathcal{P}}$ *(cardinality functions for set functions),* $\mathfrak{r}$ *(functions defined on* N_0*, equal to* 1 *in* 0*),* $\mathbf{c}$ *(cardinality functions for bi-set functions) and* $\mathcal{G}''$ *(cardinality operators for bi-set functions) sets, is commutative.*

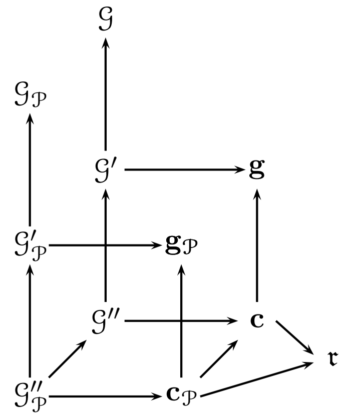


Figure 3: Summary diagram

- In the foreground, we have the set functions whereas in the background, the bi-set functions are represented.
- On the left, the operators; on the right, the functions of a single variable.
- At the top, the triangular operators of $\mathcal{G}$ and $\mathcal{G}_{\mathcal{P}}$; in the middle layer, the level operators and the level functions; at the bottom, the cardinality operators and the cardinality functions.
- The horizontal arrows correspond to group isomorphisms whereas the vertical ones stand for subgroups relations.

8 The inverse interaction transform

We will now examine the automorphism *inv* on $\mathfrak{r}$ defined by

$$\begin{aligned} inv : \mathfrak{r} &\to \mathfrak{r} \\ f &\mapsto f^{\star -1}. \end{aligned}$$

Proposition 3.1 of [7] explicitely gives us the expression of $f^{\star-1}$. In particular, the inverse of the function $f_{\gamma_{\mathcal{P}}}$ is given by Proposition 3.3:

$$\begin{aligned} f^{\star-1}_{\gamma_{\mathcal{P}}}(0) &= 1, \\ f^{\star-1}_{\gamma_{\mathcal{P}}}(m) &= -\frac{1}{m+1}\sum_{k=0}^{m-1}\binom{m+1}{k} f^{\star-1}_{\gamma_{\mathcal{P}}}(k), \quad m \in N. \end{aligned}$$

This last formula extended to natural numbers is named the *Bernoulli sequence*, which explains our former name inverse Bernoulli function for γ and $\gamma_{\mathcal{P}}$. The sequence $(\mathfrak{b}_m)_{m\in\mathbb{N}}$ of Bernoulli numbers starts with $1, -1/2, 1/6, 0,$ $-1/30, \ldots$, and it is well known that $\mathfrak{b}_m = 0$ for $m \geq 3$ odd.

Yet, since it is the inversion in $\mathcal{G}$ that interests us, we use Proposition 10 which provides us the required inversion automorphism of $\mathcal{G}''$. On the other hand, since $f_\gamma = f_{\gamma_{\mathcal{P}}}$, it is easy to find step by step, the inverse of Γ:

$$\Gamma \mapsto \gamma \mapsto \underline{f_\gamma = f_{\gamma_{\mathcal{P}}}} \mapsto \underline{f^{\star-1}_\gamma = f^{\star-1}_{\gamma_{\mathcal{P}}} = (\mathfrak{b}_m)_{m\in N_0}} \mapsto \gamma^{\star-1} \mapsto \Gamma^{-1}$$

This immediately implies:

Proposition 11 *The Bernoulli operator* Γ^{-1} *is given by:*

$$\Gamma^{-1}((A_1,B_1),(A_2,B_2)) = \begin{cases} \mathfrak{b}_{b_1-b_2} \text{ if } (A_1,B_1) \trianglelefteq (A_2,B_2) \\ 0 \text{ otherwise} \end{cases},$$
$$(A_1,B_1),(A_2,B_2) \in \mathcal{Q},$$

where $(\mathfrak{b}_m)_{m\in\mathbb{N}}$ *is the sequence of Bernoulli numbers.*

As a consequence, thanks to the inversion of (15), we can write:

$$\begin{aligned} m^v(A,B) &= \sum_{(C,D)\trianglerighteq(A,B)} \mathfrak{b}_{b-d}\, I^v(C,D) \\ &= \sum_{m=0}^{b} \mathfrak{b}_m \sum_{\substack{(C,D)\trianglerighteq(A,B) \\ b-d=m}} I^v(C,D), \qquad (A,B) \in \mathcal{Q}. \end{aligned}$$

Finally, we obtain:

Theorem 12 *For any bi-set function v, we have:*

$$v(A,B) = \sum_{(C,D)\in \mathcal{Q}} b^{n-|D|}_{n-|B\cup D\cup(A^c\cap C)|}\, I^v(C,D), \quad (A,B)\in\mathcal{Q},$$

where

$$b^p_m := \sum_{k=0}^{m} \binom{m}{k} \mathfrak{b}_{p-k}$$

for $0 \le m \le p$, and $(\mathfrak{b}_m)_{m\in\mathbb{N}}$ is the sequence of Bernoulli numbers.

Remark. Let us notice that numbers b^p_m have been introduced in [7] to express a set function v from its interaction index I^v:

$$v(A) = \sum_{C\in\mathcal{P}} b^{|C|}_{|C\cap A|}\, I^v(C), \quad A\in\mathcal{P}.$$

It is easy to compute them from the sequence of Bernoulli ($b^p_0 = \mathfrak{b}_p$ for any $p\in\mathbb{N}$), and thanks to the recursion of the "Pascal's triangle":

$$b^{p+1}_{m+1} = b^{p+1}_m + b^p_m, \quad 0\le m\le p.$$

Furthermore, the coefficients b^p_m show the following symmetry:

$$b^p_m = (-1)^p\, b^p_{p-m}, \quad 0\le m\le p.$$

Appendix A. Incidence algebras within the framework of the operators

Whenever the poset P is the set $\mathcal{P}$ of subsets of N and the field K is $\mathbb{R}$, the obtained incidence algebra $\mathbf{I}(\mathcal{P},\mathbb{R}) = \mathbf{I}(\mathcal{P})$ includes $\mathcal{G}_{\mathcal{P}}$, whose operators necessarily has value 1 on one point segments of $S(P)$. Therefore, even if $\mathcal{G}_{\mathcal{P}}$ endowed with $\star$ is a group, it is neither closed under the sum nor under the scalar multiplication, thus it cannot be an algebra. As a consequence, to take advantage of the Doubilet, Rota and Stanley's results, we shall reduce these to the case where the sets are only endowed with the convolution operation.

As a consequence of Proposition 1, we can identify the set of level functions $\mathbf{g}_{\mathcal{P}}$ with the subset of all functions f of $\mathbf{R}(\mathcal{P},\sim_l)$ verifying $f(\overline{\emptyset}) = 1$ ($\overline{\emptyset}$ being the class of all one point segments of $\mathcal{P}$, that is singletons of $\mathcal{P}$), both endowed with their respective operation $\star$ (11) and (9). Indeed, the incidence coefficient $[{}_{\beta,\gamma}^{\alpha}]$ of (9) equals 1 whenever there exists $B \subseteq A$ such that $\beta = \overline{B}$ and $\gamma = \overline{A \setminus B}$, and 0 otherwise, where A is any point of $\mathcal{P}$ such that $\overline{A} = \alpha$. Furthermore, the mapping $\varphi \mapsto \Phi_\varphi$ given in Section 4 is exactly the one given just before Prop 1, $f \mapsto \hat{f}$, so that we can retrieve $\mathcal{G}'_{\mathcal{P}} = \widehat{\mathbf{g}_{\mathcal{P}}}$.

Under Proposition 2 and according to Definition 2 (algebras of full binomial type), we see that the set $\mathfrak{r}$ of cardinal representations is identified with the subset of all functions f of $\overline{\mathbf{R}}(\mathcal{P})$ verifying $f(\overline{0}) = 1$ ($\overline{0}$ being the type in correspondence with 0, i.e., the subset of one point segments). Furthermore, the incidence coefficients of (9) are easily retrieved: if $\alpha = \overline{m}$ and $k \leq m$, there are exactly $[{}_{k,m-k}^{\ m}] = \binom{m}{k}$ elements of type $\overline{k}$ (their complements being of type $\overline{m-k}$) in a segment of type α, and 0 otherwise. This enables to retrieve (12).

Similarly, whenever $P = \mathcal{Q}$, there is again an underlying structure of incidence algebra in our framework: clearly, if we consider the locally finite poset $(\mathcal{Q}, \sqsubseteq)$, we obtain the incidence algebra $\mathbf{I}(\mathcal{Q}^{\sqsubseteq})$ that includes the set $\mathcal{G}$ of operators of value 1 on one point segments of $S(\mathcal{Q})$. See also Sections 6 and 7.

Appendix B. Proofs of the results

Proof (Proposition 1).
First, it is clear that $\sim_l$ is an equivalence relation. Then, let $[A, B]$ and $[C, D]$ in $S(\mathcal{P})$ such that $[A, B] \sim_l [C, D]$, and $\mathbf{b}$ the mapping $[A, B] \to [C, D]$ defined by $\mathbf{b}(E) := C \cup (E \setminus A)$. It is easy to see that $\mathbf{b}$ is a bijection. Besides, for any segment $[A_1, B_1]$ included in $[A, B]$, we have easily $[A_1, B_1] \sim_l [\mathbf{b}(A_1), \mathbf{b}(B_1)]$ so that we can apply Proposition 4.1. of [8], which concludes to the compatibility of $\sim_l$.

Let us now show the second assertion. For any A in $\mathcal{P}$, let $\overline{A}$ be the set of all segments $[A_1, A_2]$ verifying $A_2 \setminus A_1 = A$, i.e.

$$\overline{A} = \{[A', A' \cup A] \mid A' \subseteq A^c\}.$$

Consequently, the set of types $\overline{\mathcal{P}} = \{\overline{A} \mid A \in \mathcal{P}\}$ ordered by its natural induced order is isomorphic to $(\mathcal{P}, \subseteq)$. ∎

Proof (Proposition 2).
It is clear that $\sim_c$ is an equivalence relation. Moreover, it is exactly the one given in Example 4.1. p. 276 ([8]), that is, fixing two segments equivalent when they are isomorphic makes the relation compatible. Consider that $[A', A' \cup A] \sim_c [B', B' \cup B]$ with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$. This implies $a = b$. Then if we consider any bijection $\mathbf{b}$ from A to B, it is easy to see that $A' \cup A'' \mapsto B' \cup \mathbf{b}(A'')$, where $A'' \subseteq A$, is an order isomorphism from $[A', A' \cup A]$ to $[B', B' \cup B]$.
Let us denote the types relative to $\sim_c$ by $\overline{k}$ for any $k \in N_0$ with

$$\overline{k} = \{[A', A' \cup A] \mid A \cap A' = \emptyset,\ |A| = k\}.$$

Then the set N' of Definition 2 is exactly N_0 and for every $[A, B] \in S(\mathcal{P})$, we have $O(A, B) = |B \setminus A| = b - a$. Thus (A) is immediate. Concerning (B), let $M \in \mathcal{P}$ of cardinal $m \in N_0$ and $k \leq m$. Then formula of (B) writes:

$$\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right] := |\{Z \in [\emptyset, M] \mid |Z| = k \text{ and } m - |Z| = m - k\}|,$$

that is to say $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right] = \binom{m}{k}$.
For the second assertion, let $\sim$ defining a coarser partition of $S(\mathcal{P})$ than $\sim_c$: a partition π is said to be *coarser* than a partition σ if every block of σ is contained in a block of π. Then Lemma 4.1. p.278 ([8]) implies that two equivalent (for $\sim$) segments have the same number of elements in $\mathcal{P}$. Thus, as types of $\sim_c$ are exactly the sets of segments of the same cardinality, $\sim$ coincides with $\sim_c$. ■

Proof (Proposition 3).
$(\mathcal{G}, \star)$ is a group if:

(i) $\mathcal{G}$ is stable under $\star$.

(ii) $\star$ is an associative operation.

(iii) Δ is the neutral element.

(iv) If $\Phi \in \mathcal{G}$, there is an inverse Φ^{-1} in $\mathcal{G}$: $\Phi \star \Phi^{-1} = \Phi^{-1} \star \Phi = \Delta$.

The first property is evident, and according to incidence algebras theory, **(ii)** and **(iii)** hold.

Concerning the fourth property, it is sufficient to show that there is an element Φ^{-1} belonging to $\mathcal{G}$ verifying $\Phi^{-1} \star \Phi = \Delta$; under this condition, $\Phi \star \Phi^{-1} = \Phi \star \Delta \star \Phi^{-1} = \Phi \star \Phi^{-1} \star \Phi \star \Phi^{-1} = \Delta \star \Delta = \Delta$. Let us construct Φ^{-1}. If Φ^{-1} exists (and belongs to $\mathcal{G}$):

$$\Phi^{-1}\star\Phi((A_1,B_1),(A_2,B_2)) = \sum_{\substack{(C,D)\in\\ [(A_1,B_1),(A_2,B_2)]}} \Phi^{-1}((A_1,B_1),(C,D))\,\Phi((C,D),(A_2,B_2)).$$

If $(A_1,B_1) \sqsubseteq (A_2,B_2)$ with $(A_1,B_1) \neq (A_2,B_2)$, then:

$$\begin{aligned}\Phi^{-1}&((A_1,B_1),(A_2,B_2))\,\Phi((A_2,B_2),(A_2,B_2))+\\ &\sum_{\substack{(C,D)\in\\ [(A_1,B_1),(A_2,B_2)[}} \Phi^{-1}((A_1,B_1),(C,D))\,\Phi((C,D),(A_2,B_2)) =\\ &\qquad\qquad\qquad\qquad\Delta((A_1,B_1),(A_2,B_2)) = 0.\end{aligned}$$

That is to say:

$$\Phi^{-1}((A_1,B_1),(A_2,B_2)) = -\sum_{\substack{(C,D)\in\\ [(A_1,B_1),(A_2,B_2)[}} \Phi^{-1}((A_1,B_1),(C,D))\,\Phi((C,D),(A_2,B_2)).$$

Conversely, the operator given above, taking value 1 for every $((A,B),(A,B))$ of $\mathcal{Q}\times\mathcal{Q}$ and 0 for every $(A_1,B_1),(A_2,B_2)$ such that $(A_1,B_1) \not\sqsubseteq (A_2,B_2)$, satisfies $\Phi^{-1}\star\Phi = \Delta$. ■

Proof (Proposition 5).
We know that $(\mathcal{P}(B^c),\subseteq)$ is a Boolean lattice. To show that $(\mathcal{Q}_{(A,B)},\trianglelefteq)$ is a lattice isomorphic to $(\mathcal{P}(B^c),\subseteq)$, it suffices to show that $q_{(A,B)}$ as defined above is an order-isomorphism, i.e., that $q_{(A,B)}$ is a bijection from $(\mathcal{Q}_{(A,B)},\trianglelefteq)$ to $(\mathcal{P}(B^c),\subseteq)$, and that for any pair of elements $(C,D),(C',D')$ of $(\mathcal{Q}_{(A,B)},\trianglelefteq)$ we have $(C,D)\trianglelefteq(C',D')$ iff $q_{(A,B)}(C,D)\subseteq q_{(A,B)}(C',D')$ [5].

First, observe the following equivalences:

$$(C,D)\in\mathcal{Q}_{(A,B)} \text{ iff } \begin{cases} C\subseteq A\subseteq C\cup D\\ B\subseteq D\\ C\cap D=\emptyset\end{cases} \text{ iff } \begin{cases} A\subseteq C\cup D\\ B\subseteq D\\ A^c\subseteq C^c\\ C\cap D=\emptyset\end{cases} \text{ iff } \begin{cases} A=C\cup(A\cap D)\\ B\subseteq D\\ C\cap D=\emptyset\end{cases}.$$

Let us show that $q_{(A,B)}$ is a bijection. Obviously $q_{(A,B)}$ is onto $(\mathcal{P}(B^c), \subseteq)$. Moreover, $D^c \subseteq B^c$ has a unique antecedent by $q_{(A,B)}$, which is (C, D), with $C := A \setminus (A \cap D)$, by the above equivalence.

Secondly, consider $(C, D) \trianglelefteq (C', D')$. Then $D \supseteq D'$, which means that $q_{(A,B)}(C, D) = D^c \subseteq D'^c = q_{(A,B)}(C', D')$. Conversely, if $D^c \subseteq D'^c$, the inverse images are $(A \setminus (A \cap D), D)$ and $(A \setminus (A \cap D'), D')$. Since $A \setminus (A \cap D) \subseteq A \setminus (A \cap D') \subseteq A \cup D$, $q^{-1}_{(A,B)}(D) \trianglelefteq q^{-1}_{(A,B)}(D')$, and $q_{(A,B)}$ is order-isomorphic. ■

Proof (Proposition 6).
First, $\sim_l$ is clearly an equivalence relation. If $\lfloor (A_1, B_1), (A_2, B_2) \rceil$ and $\lfloor (C_1, D_1), (C_2, D_2) \rceil$ are equivalent, let $\mathbf{b}$ the mapping defined over $\lfloor (A_1, B_1), (A_2, B_2) \rceil$ by

$$\mathbf{b}(E, F) := (C_1, D_1) \sqcup ((E, F) \setminus\!\!\setminus (A_1, B_1)).$$

It is easy to verify that the image of $\mathbf{b}$ is included in $\lfloor (C_1, D_1), (C_2, D_2) \rceil$ and for any (E, F) in $\lfloor (C_1, D_1), (C_2, D_2) \rceil$, we have

$$(E, F) = (C_1, D_1) \sqcup ((E, F) \setminus\!\!\setminus (C_1, D_1)),$$

so that we can deduce the only possible antecedent of (E, F) by $\mathbf{b}$ being $(A_1, B_1) \sqcup ((E, F) \setminus\!\!\setminus (C_1, D_1))$, which makes $\mathbf{b}$ a bijection. Furthermore, it is elementary to show that any segment $\lfloor (A'_1, B'_1), (A'_2, B'_2) \rceil$ included in $\lfloor (A_1, B_1), (A_2, B_2) \rceil$ is equivalent to its image. Thus, following Proposition 4.1. of [8], this implies that $\sim_l$ is compatible.

Now, if we endow $\overline{\mathcal{Q}}$ with the natural order denoted by:

$$\overline{(A, B)} \trianglelefteq \overline{(C, D)} \text{ whenever } (A, B) \trianglelefteq (C, D),$$

then it is self-evident that $(\overline{\mathcal{Q}}, \trianglelefteq)$ is isomorphic to $(\mathcal{Q}, \trianglelefteq)$.

Then for more convenience, we may use $\mathcal{Q}$ instead of $\overline{\mathcal{Q}}$. ■

Proof (Proposition 7).
Let f, g be in $\mathbf{R}(\mathcal{Q}, \sim_l)$. Then by considering (9), for any $(A, B) \in \mathcal{Q}$, we have

$$f \star g(A, B) = \sum \begin{bmatrix} (A,B) \\ (C,D),(C',D') \end{bmatrix} f(C, D)\, g(C', D'), \text{ where}$$

$$\begin{aligned}\left[{}^{(A,B)}_{(C,D),(C',D')}\right] = \big|\{(E,F) \in \lfloor \perp, (A,B)\rceil \mid \\ \mathrm{type}(\lfloor \perp, (E,F)\rceil) = (C,D),\ \mathrm{type}(\lfloor (E,F),(A,B)\rceil) = (C',D')\}\big|.\end{aligned}$$

Then it is easy to see that this incidence coefficient equals 1 iff $(C,D) \trianglelefteq (A,B)$, $(C',D') \trianglelefteq (A,B)$, such that $(C,D) \sqcup (C',D') = (A,B)$, and 0 otherwise. This implies:

$$f \star g(A,B) = \sum_{(C,D) \trianglelefteq (A,B)} f(C,D)\, g((A,B) \setminus\!\!\setminus (C,D)).$$

Proposition 4.3. of [8] tells us that $\mathbf{R}(\mathcal{Q}, \sim_l)$ is isomorphic to its image in $\mathbf{I}(\mathcal{Q})$ (as $\mathbb{R}$-algebras), by the mapping $\varphi \mapsto \Phi_\varphi$. Moreover, we find that $\mathbf{g}$ endowed with $\star$ is a group: the inverse $\varphi^{\star -1}$ of an element φ of $\mathbf{g}$ is in $\mathbf{R}(\mathcal{Q}, \sim_l)$, which necessarily takes value 1 at $\perp$ (clear). Then as algebra structure induces group structure, this implies that $(\mathcal{G}', \star)$ is isomorphic to $(\mathbf{g}, \star)$. The convolution being a commutative operation, $(\mathbf{g}, \star)$ is an Abelian group and so is $(\mathcal{G}', \star)$. ■

Proof (Proposition 8).
To show that $\sim_c$ is compatible, the method is the same than the one of Proposition 2. Consider the extended mapping of the one above:

$$\begin{aligned}\overline{\mathbf{R}}(\mathcal{Q}) &\to \mathbf{R}(\mathcal{Q}, \sim_l) \\ f &\mapsto \varphi_f,\end{aligned}$$

we obtain that according to Proposition 4.3 ([8],) its image is a subalgebra of $\mathbf{R}(\mathcal{Q}, \sim_l)$. And as $(\mathfrak{r}, \star)$ is a group, this implies that $(\mathbf{c}, \star)$ is a group. The remainder follows. ■

Proof (Lemma 9 fundamental).

- We already know that $\mathbf{c}_{\mathcal{P}}$ is isomorphic to $\mathfrak{r}$ (cf. section 4).
- According the above proposition, $\overline{\mathbf{R}}(\mathcal{P})$ and $\overline{\mathbf{R}}(\mathcal{Q})$ are identicals. So are $\mathbf{c}_{\mathcal{P}}$ and $\mathbf{c}$.

■

Proof (Proposition 10).
We have shown commutativity of the triangle $\mathbf{c}_{\mathcal{P}}, \mathbf{c}, \mathfrak{r}$ in Lemma 9 (in particular, $\mathbf{c}_{\mathcal{P}}$ and $\mathbf{c}$ isomorphic through $\lambda^{-1} \circ \lambda_{\mathcal{P}}$). Commutativity between $\mathcal{G}''_{\mathcal{P}}, \mathbf{c}_{\mathcal{P}}$ and $\mathcal{G}'', \mathbf{c}$ are given through isomorphisms $\Lambda_{\mathcal{P}}$ and Λ restricted to $\mathcal{G}''_{\mathcal{P}}$ and $\mathcal{G}''$. ■

Proof (Theorem 12).
For all $(A, B) \in \mathcal{Q}$, according to (6) and Proposition 11 we have

$$\begin{aligned} v(A,B) &= \sum_{(C,D) \sqsubseteq (A,B)} m^v(C,D) \\ &= \sum_{(C,D) \sqsubseteq (A,B)} \sum_{(E,F) \trianglerighteq (C,D)} \mathfrak{b}_{d-f}\, I^v(E,F) \\ &= \sum_{(E,F) \in \mathcal{Q}} \Bigg(\sum_{\substack{(C,D) \trianglelefteq (E,F) \\ (C,D) \sqsubseteq (A,B)}} \mathfrak{b}_{d-f} \Bigg) I^v(E,F). \end{aligned}$$

Let us show that $\begin{cases} (C,D) \trianglelefteq (E,F) \\ (C,D) \sqsubseteq (A,B) \end{cases}$ iff $(C,D) \trianglelefteq (A \cap E, B \cup F \cup (E \cap A^c))$:
First consider the "if" part. If we assume that $\begin{cases} C \subseteq A \cap E \subseteq C \cup D \\ B \cup F \cup (E \cap A^c) \subseteq D \end{cases}$, we have easily $C \subseteq E$, $F \subseteq D$, $C \subseteq A$ and $B \subseteq D$. Moreover, $A \cap E \subseteq C \cup D$ and $A^c \cap E \subseteq D \subseteq C \cup D$ thus $E \subseteq C \cup D$. For the "only if" part, if $\begin{cases} C \subseteq E \subseteq C \cup D \\ F \subseteq D \\ C \subseteq A \\ B \subseteq D \end{cases}$, then $C \subseteq A \cap E$ and $A \cap E \subseteq C \cup D$ are obvious. Next, $E \subseteq C \cup D$ and $A^c \subseteq C^c$ thus $E \cap A^c \subseteq (C \cup D) \cap C^c = D$ since $C \cap D = \emptyset$. Finally, $B \cup F \cup (E \cap A^c) \subseteq D$ is verified.

Therefore, we can write thanks to Proposition 5

$$\begin{aligned}
\sum_{\substack{(C,D)\trianglelefteq(E,F)\\(C,D)\sqsubseteq(A,B)}} \mathfrak{b}_{d-f} &= \sum_{(C,D)\trianglelefteq(A\cap E, B\cup F\cup(E\cap A^c))} \mathfrak{b}_{d-f}\\
&= \sum_{D^c\subseteq(B\cup F\cup(E\cap A^c))^c} \mathfrak{b}_{d-f}\\
&= \sum_{k=0}^{n-|B\cup F\cup(E\cap A^c)|} \binom{n-|B\cup F\cup(E\cap A^c)|}{k} \mathfrak{b}_{n-f-k}\\
&= b^{n-f}_{n-|B\cup F\cup(E\cap A^c)|}.
\end{aligned}$$

The result follows. ∎

References

[1] J.M. Bilbao. *Cooperative Games on Combinatorial Structures*. Kluwer Academic Publishers, 2000.

[2] G. Birkhoff. *Lattice Theory*. American Mathematical Society, 3d edition, 1967.

[3] A. Chateauneuf and J.Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences*, 17:263–283, 1989.

[4] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953.

[5] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Orders*. Cambridge University Press, 1990.

[6] D. Denneberg. *Non-Additive Measure and Integral*. Kluwer Academic, 1994.

[7] D. Denneberg and M. Grabisch. Interaction transform of set functions over a finite set. *Information Sciences*, 48(1):15–27, 1999.

[8] P. Doubilet, G.C. Rota, and R. Stanley. On the foundations of combinatorial theory (VI): The idea of generating function. In Berkeley University Press,

editor, *6th Berkeley Symposium on Mathematical Statistics and Probability*, volume 2, pages 267–318, 1972.

[9] D. Dubois and H. Prade. *Possibility Theory.* Plenum Press, 1988.

[10] D. Felsenthal and M. Machover. Ternary voting games. *Int. J. of Game Theory*, 26:335–351, 1997.

[11] M. Grabisch. The application of fuzzy integrals in multicriteria decision making. *European J. of Operational Research*, 89:445–456, 1996.

[12] M. Grabisch. Alternative representations of discrete fuzzy measures for decision making. *Int. J. of Uncertainty, Fuzziness, and Knowledge Based Systems*, 5:587–607, 1997.

[13] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[14] M. Grabisch and Ch. Labreuche. Bi-capacities. In *Joint Int. Conf. on Soft Computing and Intelligent Systems and 3d Int. Symp. on Advanced Intelligent Systems*, Tsukuba, Japan, October 2002.

[15] M. Grabisch and Ch. Labreuche. Bi-capacities. Part I: definition, Möbius transform and interaction. *Fuzzy Sets and Systems*, 151:211–236, 2005.

[16] M. Grabisch, J.L. Marichal, and M. Roubens. Equivalent representations of set functions. *Mathematics of Operations Research*, 25(2):157–178, 2000.

[17] P.L. Hammer and S. Rudeanu. *Boolean Methods in Operations Research and Related Areas.* Springer, 1968.

[18] T. Murofushi and S. Soneda. Techniques for reading fuzzy measures (III): interaction index. In *9th Fuzzy System Symposium*, pages 693–696, Sapporo, Japan, May 1993. In Japanese.

[19] G. Owen. *Game Theory.* Academic Press, 3d edition, 1995.

[20] G.C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 2:340–368, 1964.

[21] D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, 57(3):571–587, 1989.

[22] G. Shafer. *A Mathematical Theory of Evidence*. Princeton Univ. Press, 1976.

[23] L.S. Shapley. A value for n-person games. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.

[24] M. Sugeno. *Theory of fuzzy integrals and its applications*. PhD thesis, Tokyo Institute of Technology, 1974.

Chapitre 6.
Jeux sur treillis distributifs et indice d'interaction

Résumé

Cet article propose une approche générale de l'interaction entre les joueurs ou attributs. On y examine la notion de jeu à actions combinées, qui sont des applications définies sur des treillis distributifs. Une définition généralisée de l'indice d'interaction pour ces jeux est fournie, ainsi que la mise en évidence d'équivalence de représentations entre un jeu, sa transformée de Möbius et son indice d'interaction. Des opérateurs linéaires sur l'ensemble des jeux sont construits, permettant le passage de l'une à l'autre de ces formes.

Mots clés : fonction de treillis, transformée de Möbius, dérivée booléenne, indice d'interaction, action de groupe

Games on distributive lattices and the Shapley interaction transform

Fabien Lange
Michel Grabisch

Centre d'Economie de la Sorbonne, UMR 8174, CNRS-Université Paris 1

Abstract

The paper proposes a general approach of interaction between players or attributes. It generalizes the notion of interaction defined for players modeled by games, by considering functions defined on distributive lattices. A general definition of the interaction index is provided, as well as the construction of operators establishing transforms between games, their Möbius transforms and their interaction indices.

Keywords: lattice game, Möbius transform, interaction index, equivalent representations

1 Introduction

Interaction index has been developped in [8] by Grabisch, and can be seen as a generalization of the Shapley value. Roughly speaking, the interaction index quantifies the genuine contribution of a coalition with reference to all its sub-coalitions, where a positive (resp. negative) interaction corresponds to a positive (resp. negative) correlation. In game theory, it describes the synergy between players or voters, the interest in forming or not forming certain coalitions. In multicriteria decision, it tells which criteria play a key role (and how), and which criteria are redundant (with which ones) in the decision process.

Games defined over *distributive lattices* are very general objects which enable to capture a large variety of behaviors, since every playable action can be expressed in terms of pure or elementary actions. An alternative use of these games has been proposed by Faigle and Kern [7]: from a partially ordered set of players taking part into the game (the relation of precedence), a game is built over the set of all feasible coalitions. An axiomatization of the Shapley value has been proposed in [11], as well as in [7]. In this paper, we aim at generalizing the interaction index concept for games over distributive lattices, which is based on our Shapley value, and which encompasses the interaction index for classical cooperative games, as axiomatized in [12].

In [5], the authors have worked out a framework in order to underline linear and bijective correspondences between a classical cooperative game, its Möbius transform, and its interaction index. In [16], we generalized this construction for *bi-set functions*, which are functions defined over the set of couples of subsets (A, B) (bi-coalitions) of a basis finite set N, such that $A \cap B = \emptyset$, A representing the coalition of defensive players, and B, the defeaters players. We provided a framework to express any game in TU-form from its interaction index by means of the incidence algebras [6]. The objective in this paper is now to extend this framework to games defined over distributive lattices.

In Section 2, we propose a short introduction to distributive lattices, and provide a general definition of lattice games, with some examples. Section 3 gives definition of the Möbius transform, and brings some mathematical background about derivative of lattice functions. In Section 4, we introduce the interaction index for lattice games. *Group actions* are a useful algebraic tool which enable any bijective linear transformation (isomorphism) to operate over the set of lattice functions. Thanks to them, we set up in Section 5 a commutative diagram in the set of lattice functions, which proves that any lattice function, its Möbius transform and its interaction index characterize the same object. We work out in Section 6 an explicit formula for the Möbius transform of distributive lattice functions, as well as the inversion of a fundamental formula of Section 4 which expresses the interaction index of any game in terms of its Möbius transform. Finally, we provide in Section 7 the inverse interaction operator, and the straight expression of any lattice game from its interaction index.

$\mathbb{N}$ denotes the set of nonnegative integers $\{0, 1, 2, \ldots\}$. If no ambiguity occurs, we denote by the lower case letters $s, t, \ldots$ the cardinals of sets $S, T, \ldots$ and we will often omit braces for singletons.

2 Lattice functions and games

We introduce some basic notions about lattices and distributive lattices. A *lattice* L is any partially ordered set (poset) $(L, \leq)$ in which every pair of elements x, y has a supremum $x \vee y$ and an infimum $x \wedge y$. Note that whenever L is finite[1], L is a *complete lattice*, that is, for any nonempty subset, their supremum and infimum always exist. The greatest element of a lattice (denoted $\top$) and least element $\bot$ always exist. In the sequel, it shall be convenient to lay down the convention $\bigvee \emptyset = \bigwedge \emptyset = \bot$.

A lattice is *distributive* if $\vee, \wedge$ obey distributivity. An element $j \in L$ is *join-irreducible* if it cannot be expressed as a supremum of other elements. Equivalently j is join-irreducible if it covers only one element, where x *covers* y (we also say that y is a *predecessor* of x, and denote $x \succ y$) means that $x > y$ and there is no z such that $x > z > y$. The set of all join-irreducible elements of L is denoted by $\mathcal{J}(L)$.

An important property is that in a distributive lattice, any element x can be written as an irredundant supremum of join-irreducible elements in a unique way (this is called the *minimal decomposition* of x). We denote by $\eta^*(x)$ the set of join-irreducible elements in the minimal decomposition of x, and we denote by $\eta(x)$ the *normal decomposition* of x, defined as the set of join-irreducible elements

[1] In the context of the paper, all considered lattices are finite.

smaller or equal to x, i.e., $\eta(x) := \{j \in \mathcal{J}(L) \mid j \leq x\}$. Hence $\eta^*(x) \subseteq \eta(x)$, and

$$x = \bigvee_{j \in \eta^*(x)} j = \bigvee_{j \in \eta(x)} j.$$

For any poset $(P, \leq)$, $Q \subseteq P$ is said to be a *downset* of P if $x \in Q$ and $y \leq x$ imply $y \in Q$. We denote by $\mathcal{O}(P)$ the set of all downsets of P. One can associate to any poset $(P, \leq)$ a distributive lattice which is $\mathcal{O}(P)$ endowed with inclusion. As a consequence of the above results, the mapping η is an isomorphism of L onto $\mathcal{O}(\mathcal{J}(L))$ (Birkhoff's theorem, [1]).

In the whole paper, $N := \{1, \dots, n\}$ is a finite set which can be thought as the set of players or also voters, criteria, states of nature, depending on the application. We consider finite distributive lattices $(L_1, \leq_1), \dots, (L_n, \leq_n)$ and their product $L := L_1 \times \cdots \times L_n$ endowed with the product order $\leq$. Elements x of L can be written in their vector form $(x_1, \dots, x_n)$. L is also a distributive lattice whose join-irreducible elements are of the form $(\bot_1, \dots, \bot_{i-1}, j_i, \bot_{i+1}, \dots, \bot_n)$, for some i and some join-irreducible element j_i of L_i. In the sequel, with some abuse of language, we shall also call j_i this element of L. We denote by $\mathcal{J}(L)$ the set of join-irreducible elements of L (Section 4). A *vertex* of L is any element whose components are either top or bottom. Vertices of L will be denoted by $\top_X$, $X \subseteq N$, whose coordinates are $\top_k$ if $k \in X$, $\bot_k$ else.

Lattice functions are real-valued mappings defined over product lattices of the above form. Lattice functions which vanishes at $\bot$ are called *lattice games* (or games) on $(L, \leq)$. We denote by $\mathbb{R}^L$ the set of lattice functions over L, and by $\mathcal{G}(L)$ the subset of games. Each lattice $(L_i, \leq_i)$ may be different, and represents the (partially) ordered set of actions, choices, levels of participation of player i to the game. A game v is *monotone* if $x \leq y$ implies $v(x) \leq v(y)$ for all $x, y \in L$. Several particular cases of lattice games are of interest.

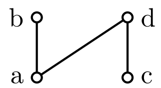
- $L = \{0,1\}^n$. This is the classical notion of cooperative game in *pseudo-Boolean functions* form [14]. Indeed, $(L, \leq)$ is isomorphic to the Boolean lattice[2] $(2^N, \subseteq)$ of the subsets of N, also called *coalitions* of N. Monotone games of $\mathcal{G}(2^N)$ are called *capacities* [3], or *non-additive measures* [4], or *fuzzy measures* [19].

[2]To avoid heavy notations, we will sometimes denote by 2^m any Boolean lattice isomorphic to 2^M, $|M| = m$.

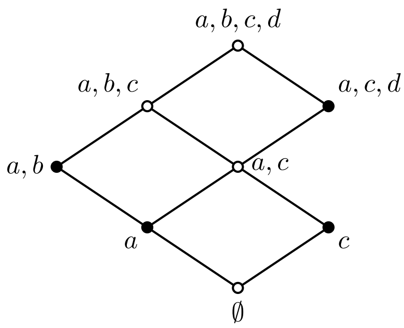
- L is the direct product of some linear lattices: $\forall i \in N,\ L_i = \{0, 1, \ldots, l_i\}$. This corresponds to multichoice games as introduced by Hsiao and Raghavan [15].

- We propose the following interpretation for games on L in the general case, i.e., L is any direct product of n distributive lattices. We assume that each player $i \in N$ has at her/his disposal a set of *elementary* or *pure* actions $j_1, \ldots, j_{n_i}$. These elementary actions are partially ordered (e.g. in the sense of benefit caused by the action), forming a partially ordered set $(\mathcal{J}_i, \leq_i)$. Then by Birkhoff's theorem (see above), the set $(\mathcal{O}(\mathcal{J}_i), \subseteq)$ of downsets of $\mathcal{J}_i$ is a distributive lattice denoted L_i, whose join-irreducible elements correspond to the elementary actions. The *bottom action* $\perp$ of L_i is the action which amounts to do nothing. Hence, each action in L_i is either a pure action j_k or a combined action $j_k \vee j_{k'} \vee j_{k''} \vee \ldots$ consisting of doing all pure actions $j_k, j_{k'}, \ldots$ for player i.

For example, let us suppose that for a given player i, elementary actions are a, b, c, d endowed with the order $\leq_i := \{(a, b), (a, d), (c, d)\}$. They form the following poset:



which in turn form the following lattice L_i of possible actions (black circles represent join-irreducible elements of L_i):



Another interpretation of our framework is borrowed from Faigle and Kern [7]. Let $P := (N, \leq)$ be a partially ordered set of players, where $\leq$ is a relation of *precedence*: $i \leq j$ if the presence of j enforces the presence of i in any coalition $S \subseteq N$. Hence, a (valid) *coalition* of P is a subset S of N such that $i \in S$ and $j \leq i$ entails $j \in S$. Hence, the collection $\mathcal{C}(P)$ of all coalitions of P is the collection of all downsets of P.

It is possible to combine both structures. For each player i in N, let $\mathcal{J}_i := \{j_1, \ldots, j_{n_i}\}$ be the set of elementary actions of player i. Consider the set of all elementary actions $N' := \bigcup_{i \in N} \mathcal{J}_i$ equipped with the partial order $\leq$ induced by the partial orders on each $\mathcal{J}_i$. Then N' might be seen as a set of virtual players whose valid coalitions bijectively correspond to elements of $\prod_{i \in N} \mathcal{O}(\mathcal{J}_i)$.

3 The Möbius transform and derivatives of lattice functions

We introduce in this section some useful material for lattice functions. The *Möbius transform* initially takes its name from number theory[3], and is a key concept in decision analysis (see e.g. [2]). Let $(P, \leq)$ be any poset. The Möbius transform m^f of a mapping $f : P \to \mathbb{R}$ is the unique solution [17] of the equation

$$f(x) = \sum_{y \leq x} m^f(y), \quad x \in P, \tag{1}$$

given by

$$m^f(x) := \sum_{y \leq x} \mu(y, x)\, f(y), \quad x \in P, \tag{2}$$

where $\mu : P \times P \to \mathbb{Z}$ is recursively given by

$$\mu(x, y) = \begin{cases} 1, & \text{if } x = y, \\ -\displaystyle\sum_{x \leq t < y} \mu(x, t), & \text{if } x < y, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

For instance, whenever P is the Boolean lattice 2^N endowed with inclusion, it is well-known that $\mu(A, B) = (-1)^{|B \setminus A|}$, for all subsets A, B such that $A \subseteq B$.

[3] Underlying lattice is in this case the set of all divisors of any positive integer endowed with divisibility relation.

As it will be seen in the next section, *derivatives* of lattice functions are a very useful tool, and have been generalized (in particular) for distributive lattice functions in [10]. Let $(L, \leq)$ be a distributive lattice and $j \in \mathcal{J}(L)$. The *first-order derivative* of the lattice function f w.r.t. j at element $x \in L$ is given by

$$\Delta_j f(x) := f(x \vee j) - f(x).$$

Using the minimal irredundant decomposition $\eta^*(y) = \{j_1, \dots, j_m\}$ of some $y \in L$, we iteratively define the derivative of f w.r.t. y at x by

$$\Delta_y f(x) := \Delta_{j_m}\big(\dots \Delta_{j_2}\big(\Delta_{j_1} f(x)\big)\dots\big), \quad x \in L.$$

Note that if for some k, $j_k \leq x$, the derivative is null. Also, this definition does not depend on the order of the j_k's and thus is well defined. Actually, we easily show by induction that

$$\Delta_y f(x) = \sum_{S \subseteq \{1,\dots,m\}} (-1)^{m-s} f\big(x \vee \bigvee_{k \in S} j_k\big).$$

In particular, whenever $(L, \leq)$ is the Boolean lattice $(2^N, \subseteq)$, for any nonempty $S \subseteq N$,

$$\Delta_S f(A) := \sum_{T \subseteq S} (-1)^{s-t} f(A \cup T), \quad A \subseteq N.$$

We set $\Delta_\perp f(x) := f(x)$, for any $x \in L$.

Note that the derivative w.r.t. y at x takes values at points of $[x, x \vee y]$. If this interval is isomorphic to $2^{\eta^*(y)}$, the derivative is said to be *Boolean*. Equivalently, the derivative is Boolean if $j_k \not\leq x$, $\forall k = 1, \dots, m$, and $[x, x \vee y]$ is Boolean. The reader is invited to refer to [10] for more details about Boolean derivatives. In the same paper, the authors provide a close link between any Boolean derivative and the Möbius transform of a lattice function:

Proposition 1 *Let $x, y \in L$, such that $\Delta_y f(x)$ is Boolean. Then*

$$\Delta_y f(x) = \sum_{z \in [y, x \vee y]} m^f(z).$$

4 The interaction index for lattice functions

From now on, L is a direct product of n finite distributive lattices. Let $v \in \mathcal{G}(L)$. We propose a general definition of interaction[4]. We begin by defining the *importance index*, introduced in [10], as a *power index* of the game defined for elementary actions of every player (that is to say, w.r.t. each join-irreducible element of each lattice L_i). This means that we try to provide an equitable way to share the worth $v(\top)$ between all elementary actions. For any game v, we denote by ϕ^v this power index. We consider some player i and some j_i join-irreducible element of L, and we try to characterize $\phi^v(j_i)$. In this purpose, we first propose the following axioms.

> **Linearity (L)**: ϕ is linear on the set $\mathcal{G}(L)$, i.e., for any $j_i \in \mathcal{J}(L)$
> $$\phi^v(j_i) = \sum_{y \in L} \alpha_y^{j_i}\, v(y),$$
> with $\alpha_y^{j_i} \in \mathbb{R}$.

We say that j_i is *null* for v if $v(y \vee j_i) = v(y \vee \underline{j_i})$ for every $y \in L$ s.t. $y_i = \bot_i$, where $\underline{j_i}$ is the (unique) predecessor of j_i in L.

> **Nullity (N)**: $\forall v \in \mathcal{G}(L)$, and any j_i null for v, then $\phi^v(j_i) = 0$.

Proposition 2 *Under **(L)** and **(N)**, ϕ is given by*

$$\phi^v(j_i) = \sum_{y \in L \mid y_i = \bot_i} \alpha_y^{j_i}\, \Delta_{j_i} v(y \vee \underline{j_i}),$$

where the $\alpha_y^{j_i}$'s are some real numbers.

Proof: It is easy to check that the above formula satisfies the two axioms. Conversely, by **(L)**, we have:

$$\begin{aligned}\phi^v(j_i) &= \sum_{y \in L} \alpha_y^{j_i}\, v(y) \\ &= \sum_{y \in L \mid y_i = \bot_i} \left[\alpha_{y \vee j_i}^{j_i}\, v(y \vee j_i) + \alpha_{y \vee \underline{j_i}}^{j_i}\, v(y \vee \underline{j_i})\right] + \sum_{\substack{y \in L \\ y_i \neq j_i, \underline{j_i}}} \alpha_y^{j_i}\, v(y). \qquad (4)\end{aligned}$$

[4]Although the interaction transform is proposed for games, all subsequent results are also mathematically valid for any lattice function.

Assume j_i is null for v. Then (4) becomes

$$\sum_{y\in L|y_i=\perp_i}\Big[(\alpha^{j_i}_{y\vee j_i}+\alpha^{j_i}_{y\vee \underline{j_i}})\,v(y\vee \underline{j_i})\Big]+\sum_{\substack{y\in L\\ y_i\neq j_i,\underline{j_i}}}\alpha^{j_i}_y\,v(y). \tag{5}$$

Let v be such that

$$\begin{aligned} v(y'\vee j_i) &= v(y'\vee \underline{j_i})\neq 0 \text{ for some } y'\in L \text{ s.t. } y'_i=\perp_i,\\ v(y) &= 0 \text{ otherwise.}\end{aligned}$$

Then j_i is null for v, and (5) yields $\alpha^{j_i}_{y'\vee j_i}+\alpha^{j_i}_{y'\vee \underline{j_i}}=0$. Since y' was arbitrary, this holds for any $y'\in L$ s.t. $y'_i=\perp_i$. Hence (5) becomes for any v such that j_i is null:

$$\sum_{\substack{y\in L\\ y_i\neq j_i,\underline{j_i}}}\alpha^{j_i}_y\,v(y)=0. \tag{6}$$

Let us take v such that

$$\begin{aligned} v(y')&\neq 0 \text{ for some } y'\in L, y'_i\neq j_i,\underline{j_i},\\ v(y)&=0,\quad \forall y\in L, y\neq y', y_i\neq j_i,\underline{j_i},\\ v(y\vee j_i)&=v(y\vee \underline{j_i}),\quad \forall y\in L \text{ s.t. } y_i=\perp_i.\end{aligned}$$

Hence j_i is null for v, and (6) yields $\alpha^{j_i}_y=0$, where $y:=y'$. Since y' is arbitrary, only coefficients $\alpha^{j_i}_y$ with $y_i=j_i$ or $\underline{j_i}$ are non zero. ■

For a given elementary action j_i, the importance index writes under a particular form of the above formula, as a weighted average of the marginal contributions of j_i, taken at vertices of L.

Definition 1 *Let $i\in N$ and j_i any join-irreducible element of L. Let $v\in\mathcal{G}(L)$. The* importance index *w.r.t. j_i of v is defined by*

$$\phi^v(j_i)=\sum_{Y\subseteq N\setminus i}\alpha^1_{|Y|}\big(v(\top_Y\vee j_i)-v(\top_Y\vee \underline{j_i})\big),$$

where $\alpha^1_k:=\dfrac{k!\,(n-k-1)!}{n!}$, *for all* $k\in\{0,\dots,n-1\}$.

Note that if $L = 2^N$, we retrieve the definition of the *Shapley value* [18]. In [11], we proposed an axiomatization of the Shapley value for multichoice games, where the obtained formula is also the one given above (all the L_i's are completely ordered).

As an extension of the importance index for every element of L, we propose a definition for the interaction index. For any $x \in L$, $I^v(x)$ expresses the interaction in the game among all elementary actions j of the minimal decomposition $x = \bigvee_{j \in \eta^*(x)} j$.

An interaction index has been proposed in [10]. However, the formula was only defined for a certain subset of L (containing join-irreducible elements of L but not the whole lattice). We present here I^v as a mapping defined over L. For that, we give the following generalized definition of $\underline{x}$ for any $x \in L$.

Definition 2 *Let $x \in L$. We call* antecessor *of x the unique element of L defined by $\underline{x} := \bigvee\{j \in \eta(x) \mid j \notin \eta^*(x)\}$.*

If $x \in \mathcal{J}(L)$, the antecessor of x is obviously its predecessor, in accordance with the notation $\underline{x}$. By the convention of Section 2, the antecessor of $\perp$ is itself. Note also that the definition of $\underline{x} \in L$ is consistent with the structure of direct product of distributive lattices of L. Indeed, we easily check that $\underline{x} = (\underline{x_1}, \ldots, \underline{x_n})$.

Lemma 3 *Let $x \in L$. For any $J \subseteq \eta^*(x)$, $\exists!\, y_J \in [\underline{x}, x]$ such that $x = y_J \vee \bigvee_{j \in J} j$. Moreover, the mapping $\mathrm{pred}_x : 2^{\eta^*(x)} \to [\underline{x}, x]$ which associates to any J the element y_J, is a bijection.*

Proof: Let $J \subseteq \eta^*(x)$. Since all j's in J are some maximal elements of $\eta^*(x)$, $\eta(x) \setminus J$ is a downset of $\mathcal{J}(L)$ and thus the normal decomposition of some element $y_J \leq x$. Besides, $y_J \geq \underline{x}$ since $\eta(x) \setminus J \supseteq \eta(x) \setminus \eta^*(x)$, which is the normal decomposition of $\underline{x}$, by definition. This defines the mapping pred_x, which is injective, since $y_J = y_{J'} \Rightarrow \eta(x) \setminus J = \eta(x) \setminus J' \Rightarrow J = J'$. Moreover, surjectivity of pred_x is clear since for any element y of $[\underline{x}, x]$, $\eta(x) \setminus \eta^*(\underline{x}) \subseteq \eta(y) \subseteq \eta(x)$, i.e., there is a subset J of $\eta^*(x)$ such that $\eta(y) = \eta(x) \setminus J$. ∎

The following proposition provides three characterizations and an important property of the antecessor.

Proposition 4 *Let $x \in L$, and $p(x) := \{y \in L \mid y \prec x\}$. Then the following assertions hold.*

(i) $\underline{x} = \bigwedge p(x)$.
(ii) $\underline{x}$ *is the greatest element s.t.* $[\underline{x}, x]$ *contains* $p(x)$.
(iii) $\underline{x}$ *is the least element s.t.* $[\underline{x}, x]$ *is Boolean.*
(iv) $[\underline{x}, x] \cong 2^{\eta^*(x)}$.

Proof: For any predecessor y of $x \neq \bot$, there is a unique element $j \in \eta^*(x)$ such that $\eta(y) = \eta(x) \setminus j$. Indeed, $y \prec x \Rightarrow \eta(y) \subseteq \eta(x)$, and at least one element of $\eta(x) \setminus \eta(y)$ belongs to $\eta^*(x)$, otherwise $x = y$. If two elements of $\eta^*(x)$ are removed, say j and j', then clearly $y \prec y \vee j \prec x$, which contradicts $y \prec x$. Conversely, for any $j \in \eta^*(x)$, $\eta(x) \setminus j$ is the decomposition into join-irreducible elements of a predecessor of x. Hence $\eta(\bigwedge p(x)) = \bigcap_{j \in \eta^*(x)} \eta(x) \setminus \{j\} = \eta(x) \setminus \eta^*(x)$, which proves (i).

We straightforwardly derive (iv) from Lemma 3. If $[x', x]$ is an interval containing $p(x)$, x' must be a lower bound of any element of $p(x)$, hence by (i), $x' = \underline{x}$ is the greatest possible element, and (ii) is shown. Besides, by Lemma 3, for all $y \in [\underline{x}, x]$, $[y, x]$ is Boolean. At last, for any $z < x$ s.t. $z \notin [\underline{x}, x]$, we have $z < y < x$, where $y \in p(x)$. Hence $[z, x]$ is clearly not Boolean, which proves (iii). As a result, $\underline{x}$ is the sole element such that $[\underline{x}, x]$ is Boolean and contains $p(x)$. ■

The interaction index $I^v(x)$ is expressed as a weighted average of the derivatives w.r.t. x, taken at vertices of L.

Definition 3 *Let $v \in \mathcal{G}(L)$. Let $x \in L$ and $X := \{i \in N \mid x_i \neq \bot_i\}$. The* interaction index *w.r.t. x of v is defined by*

$$I^v(x) := \sum_{Y \subseteq N \setminus X} \alpha^{|X|}_{|Y|} \, \Delta_x v(\underline{x} \vee \top_Y), \tag{7}$$

where $\alpha^j_k := \dfrac{k!\,(n-j-k)!}{(n-j+1)!}$, *for all* $j = 0, \dots, n$ *and* $k = 0, \dots, n-j$.

In fact, this extends Definition 1. Besides, the formula overlaps previous definitions of the interaction introduced and axiomatized in [5, 13] for classical cooperative games, and also in [10] for multichoice games whose all L_i's are identical.

We now express the interaction index in terms of the Möbius transform by means of the following result.

Lemma 5 *For any $x \in L$, $\Delta_x v(y)$ is Boolean for any y such that for all k, $y_k = \bot_k$ or $\top_k$ if $x_k = \bot_k$, and $y_k = \underline{x_k}$ otherwise.*

Proof: We have to prove that $[y, x \vee y]$ is isomorphic to $2^{\eta^*(x)}$. It suffices to prove that $[y_k, (x \vee y)_k]$ is isomorphic to $2^{\eta^*(x_k)}$ for each coordinate k. If $x_k = \bot_k$, then $[y_k, x_k \vee y_k] = \{y_k\} \cong 2^{\emptyset}$. If $x_k \neq \bot_k$, then $[y_k, x_k \vee y_k] = [\underline{x_k}, x_k] \cong 2^{\eta^*(x_k)}$, by Proposition 4. ■

The following result generalizes one given in [10].

Theorem 6 *Let $v \in \mathcal{G}(L)$ and $x \in L$. Then*

$$I^v(x) = \sum_{z \in [x, \check{x}]} \frac{1}{k(z) - k(x) + 1} m^v(z), \tag{8}$$

where $\check{x}_j := \top_j$ if $x_j = \bot_j$, $\check{x}_j := x_j$ else, and $k(y)$ is the number of coordinates of $y \in L$ not equal to $\bot_j$, $j = 1, \dots, n$.

Proof: Since the derivative in (7) is Boolean by Lemma 5, we can apply Proposition 1, and we get:

$$I^v(x) = \sum_{Y \subseteq N \setminus X} \alpha^{|X|}_{|Y|} \sum_{z \in [x, x \vee \top_Y]} m^v(z).$$

Consequently, $I^v(x)$ can be linearly expressed in terms of $m^v(z)$, where the z's may belong to any $[x, x \vee \top_Y]$, $Y \subseteq N \setminus X$, i.e., $z \in \bigcup_{Y \subseteq N \setminus X} [x, x \vee \top_Y] = [x, x \vee \top_{N \setminus X}]$, that is to say:

$$I^v(x) = \sum_{z \in [x, \check{x}]} \beta_z m^v(z).$$

To compute β_z for a given $z \in [x, \check{x}]$, we have to examine for which Y's of $N \setminus X$, z belongs to $[x, x \vee \top_Y]$. Note that $z_j = x_j$ for all $j \in X$. If $j \in N \setminus X$, and $z_j \neq \bot_j$, then Y must contain j. As a result:

$$\beta_z = \sum_{Z \subseteq Y \subseteq N \setminus X} \alpha^{|X|}_{|Y|},$$

where $Z := \{j \in N \setminus X \mid z_j \neq \perp_j\}$. Observing that $|X| = k(x)$ and $|Z| = k(z) - k(x)$, we get

$$\begin{aligned}
\beta_z &= \sum_{j=k(z)-k(x)}^{n-k(x)} \binom{n-k(z)}{j-k(z)+k(x)} \alpha_j^{k(x)} \\
&= \sum_{j=0}^{n-k(z)} \binom{n-k(z)}{j} \alpha_{j+k(z)-k(x)}^{k(x)} \\
&= \sum_{j=0}^{n-k(z)} \frac{(n-k(z))!\,(j+k(z)-k(x))!}{j!\,(n-k(x)+1)!}.
\end{aligned}$$

It has been proved in [9] that

$$\sum_{i=0}^{l} \frac{(m-i-1)!\,l!}{m!\,(l-i)!} = \frac{1}{m-l}, \qquad m \in \mathbb{N} \setminus \{0\}, l \in \{0, \dots, m-1\}.$$

Applying the above formula with $m = n-k(x)+1$, $l = n-k(z)$ and $i = n-k(z)-j$, we obtain

$$\beta_z = \frac{1}{k(z)-k(x)+1},$$

which is the desired result. ■

5 Linear transformations on sets of lattice functions

In [5], the authors laid down a general framework of transformations of set functions by introducing an algebraic structure on set functions and operators (set functions of two variables), which enable the writing of the formulae given in the previous section under a simplified algebraic form. Then in [16], the same has been done for bi-set functions, by introducing *incidence algebras* [6]. Although this tool may be useful in combinatorics of order theory, we do not now proceed in the same way for lattice functions, making the choice to use a more suitable algebraic strucure, namely the group actions.

We call *operator* on L a real-valued function on $L \times L$. A binary operation $\star$ (multiplication or convolution) between operators is introduced as follows:

$$(\Phi \star \Psi)(x,y) := \sum_{t \in L} \Phi(x,t)\, \Psi(t,y).$$

Endowed with $\star$, the set of operators contains the identity element

$$\Delta(x,y) := \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise,} \end{cases} \quad x, y \in L,$$

and also satisfies associativity, which makes it a monoid. When it exists, we denote by Φ^{-1} the inverse of an operator Φ, that is to say the operator satisfying $\Phi \star \Phi^{-1} = \Phi^{-1} \star \Phi = \Delta$. Consequently, the set of all inversible operators is a group. We denote it by $\mathbb{G}$. We denote by ${}^t\Phi$ the *transpose* of the operator Φ, i.e., ${}^t\Phi(x,y) := \Phi(y,x)$ for all $x, y \in L$.

Let $\leqq$ be any partial order on L included in the usual order $\leq$, and $\lneqq$ the associated strict order. We denote by $I(L, \leqq)$ the set of intervals of L w.r.t. the order $\leqq$, i.e., the family of subsets $[x,y]_{\leqq} := \{t \in L \mid x \leqq t \leqq y\}$, with $x \leqq y$. An operator Φ is said to be *unit upper-triangular* (resp. *unit lower-triangular*) relatively to $\leqq$, or shortly $\mathrm{UUT}_{\leqq}$ (resp. $\mathrm{ULT}_{\leqq}$), if it equals 1 on the diagonal of L^2, and vanishes at all pairs (x,y) s.t. $[x,y]_{\leqq} = \emptyset$ (resp. $[y,x]_{\leqq} = \emptyset$):

$$\Phi(x,y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \nleqq y, \end{cases} \quad x, y \in L.$$

Note that the transpose of any $\mathrm{UUT}_{\leqq}$ operator is $\mathrm{ULT}_{\leqq}$.

Proposition 7 *The subset $G(\leqq)$ of all $UUT_{\leqq}$ operators endowed with $\star$, is a subgroup of $\mathbb{G}$. The inverse Φ^{-1} of $\Phi \in G(\leqq)$ computes recursively through*

$$\begin{aligned} &\Phi^{-1}(x,x) = 1, \\ &\Phi^{-1}(x,y) = -\sum_{x \leqq t \lneqq y} \Phi^{-1}(x,t)\, \Phi(t,y), \quad x \lneqq y. \end{aligned}$$

Proof: $G(\leqq)$ being nonempty, it suffices to check that it is closed under multiplication and inversion. For any $\Phi, \Psi \in G(\leqq)$, $\Psi \star \Phi$ clearly belongs to $G(\leqq)$. Besides, let us examine $\Phi^{-1}(x,y)$ for $x \lneqq y$.

$$\Phi^{-1} \star \Phi(x,y) = \sum_{t \leqq y} \Phi^{-1}(x,t)\, \Phi(t,y).$$

Then

$$\Phi^{-1}(x,y)\,\Phi(y,y) + \sum_{t \lneqq y} \Phi^{-1}(x,t)\,\Phi(t,y) = \Delta(x,y) = 0.$$

Thus:

$$\Phi^{-1}(x,y) = -\sum_{t \lneqq y} \Phi^{-1}(x,t)\,\Phi(t,y).$$

In addition, we easily verify that the unit upper-triangular operator satisfying the above formula (which implies that the sum is over $x \leqq t \lneqq y$), suits as the inverse of Φ. ■

Applying this result for the *Zeta operator* $Z \in G(\leq)$:

$$Z(x,y) := \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise,} \end{cases} \qquad x, y \in L, \tag{9}$$

we recognize the recursive formula (3) (Section 3) of the Möbius operator, i.e., $Z^{-1} = \mu$.

In order to rewrite formulae (1), (2) and also (8) in a reduced form, we introduce some group actions of $\mathbb{G}$ on the set of lattice functions: a *left* (resp. *right*) *group action* of a group $(\mathscr{G}, *)$ on a set $\mathcal{S}$ is a binary function

$$\mathscr{G} \times \mathcal{S} \to \mathcal{S} \quad (\text{resp. } \mathcal{S} \times \mathscr{G} \to \mathcal{S})$$

denoted

$$(\Phi, f) \mapsto \Phi * f \quad (\text{resp. } (f, \Phi) \mapsto f * \Phi),$$

satisfying the following axioms:

GA1. $(\Psi * \Phi) * f = \Psi * (\Phi * f)$ (resp. $f * (\Phi * \Psi) = (f * \Phi) * \Psi$), for all Φ, Ψ in $\mathscr{G}$ and $f \in \mathcal{S}$.

GA2. $E * f = f$ (resp. $f * E = f$), for every $f \in \mathcal{S}$, where E is the identity element of $(\mathscr{G}, *)$.

Let $\Phi \in \mathbb{G}$, and f be a lattice function over L. For x belonging to L, we define:

$$(\Phi \star f)(x) := \sum_{t \in L} \Phi(x,t)\, f(t), \tag{10}$$

$$(f \star \Phi)(x) := \sum_{t \in L} f(t)\, \Phi(t,x). \tag{11}$$

It is easy to verify that (10) and (11) respectively define a left and a right group action of $\mathbb{G}$ on $\mathbb{R}^L$. Note that the subgroup $G(\leq)$ is not stable under the transpose operation.

Now, (1) and (2) respectively rewrites as

$$f = m^f \star Z, \quad \text{and} \quad m^f = f \star Z^{-1}, \quad f \in \mathbb{R}^L.$$

Similarly, if we set down:

$$\Gamma(x,y) := \begin{cases} \dfrac{1}{k(y)-k(x)+1}, & \text{if } \forall i \in N, x_i = \bot_i \text{ or } y_i = x_i, \\ 0, & \text{otherwise,} \end{cases} \qquad x, y \in L, \quad (12)$$

we notice that $\Gamma \in G(\leq)$, and we can write from (8) the relation:

$$I^f = \Gamma \star m^f, \quad f \in \mathbb{R}^L,$$

which in turns gives by inversion

$$m^f = \Gamma^{-1} \star I^f, \quad f \in \mathbb{R}^L. \qquad (13)$$

It is also possible to do without left group actions. Indeed, we easily show that the left action $\mathbb{G} \times \mathbb{R}^L \to \mathbb{R}^L$ can be converted into the right action $\mathbb{R}^L \times \mathbb{G} \to \mathbb{R}^L$ by $(\Phi, f) \mapsto (f, {}^t\Phi)$. Consequently,

$$I^f = m^f \star {}^t\Gamma, \quad f \in \mathbb{R}^L.$$

Note that ${}^t\Gamma$ and ${}^t\Gamma^{-1}$ are unit lower-triangular.

As a conclusion of these results, any lattice function may be seen as the interaction index or the Möbius transform of some lattice function. This actually generalizes a result *(equivalent representations)* of [8] by the result below (see Figure 1).

Theorem 8 *Operators Z and Γ generate a commutative diagram in $\mathbb{R}^L$.*

Proof: From axioms **GA1** and **GA2**, it follows that for every Φ in $G(\leq)$, the function which maps f in $\mathbb{R}^L$ to $f \star \Phi$ (or $\Phi \star f$) is a bijective map from $\mathbb{R}^L$ to $\mathbb{R}^L$. Applying the result for $\Phi = Z$ and $\Phi = \Gamma$, the result follows. ■

We call *interaction operator*, the operator $\mathbb{I} := Z^{-1} \star {}^t\Gamma$. Hence, the interaction index of $f \in \mathbb{R}^L$ is given by $I^f = f \star \mathbb{I}$. Note that $\mathbb{I}$ is neither UUT nor ULT.

$$\begin{tikzcd}
f \arrow[rr, shift left, "\mathbb{I}"] \arrow[dr, shift right, "Z^{-1}"'] & & I^f \arrow[ll, shift left, "\mathbb{I}^{-1}"] \arrow[dl, shift right, "{}^t\Gamma^{-1}"'] \\
& m^f \arrow[ul, shift right, "Z"'] \arrow[ur, shift right, "{}^t\Gamma"'] &
\end{tikzcd}$$

Figure 1: Lattice functions and their representations (operators act on the right)

6 The Möbius and Bernoulli operators

We now aim at giving an explicit formula for the Möbius operator and the Bernoulli operator[5] Γ^{-1}. Let $\sim$ be an equivalence relation on the set $I(L, \leqq)$. We denote by $\overline{[x,y]_{\leqq}}$ the class of any interval $[x,y]_{\leqq}$ by the relation $\sim$. We consider the following property for operators of $G(\leqq)$ relatively to this relation:

$$\begin{aligned}&\Phi \text{ is constant on each equivalence class of } \sim, \text{ i.e.,} \\ &\forall [x,y],[x',y'] \in I(L,\leqq), \quad \overline{[x,y]_{\leqq}} = \overline{[x',y']_{\leqq}}, \text{ then } \Phi(x,y) = \Phi(x',y').\end{aligned} \tag{14}$$

The relation $\sim$ is said to be *compatible*, if the set of operators satisfying (14) is stable under multiplication.

We now consider the particular equivalence relation $\cong$ (order isomorphism) on $I(L, \leqq)$. Then it is a compatible equivalence relation (see [6]). One can notice that relatively to $\cong$ and the usual order, Z satisfies (14). However, it is not the case of Γ in the general case; for instance, if $L := L_1 = \{0,1,2\}$, $\dfrac{1}{2} = \Gamma(0,1) \neq \Gamma(1,2) = 0$ although $[0,1] \cong [1,2]$.

We denote by $\tilde{G}(\leqq)$ the subset of $G(\leqq)$ of operators satisfying property (14) relatively to the compatible relation $\cong$. It is possible to reduce the algebra structure of operators when dealing with the elements of $\tilde{G}(\leqq)$: to any $\Phi \in \tilde{G}(\leqq)$, we associate the following function φ defined on $\tilde{I}(L,\leqq)$, quotient set of $I(L,\leqq)$ by $\cong$:

$$\varphi(\overline{[x,y]_{\leqq}}) := \Phi(x,y), \quad [x,y]_{\leqq} \in I(L,\leqq). \tag{15}$$

[5]This name will be justified in Corollary 12.

The identity operator Δ clearly belongs to $\tilde{G}(\leqq)$, and has for associated function

$$\delta(\overline{[x,y]_{\leqq}}) := \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise,} \end{cases} \quad [x,y]_{\leqq} \in I(L,\leqq).$$

Let $\tilde{g}(\leqq) := \{\varphi : \tilde{I}(L,\leqq) \to \mathbb{R} \mid \forall x \in L, \varphi(\overline{\{x\}}) = 1\}$. Clearly, (15) being reversible, we see that any real-valued mapping φ on $\tilde{I}(L,\leqq)$ such that $\varphi(\overline{\{x\}}) = 1$, $x \in L$, determines uniquely an operator of $\tilde{G}(\leqq)$. For $\varphi, \psi \in \tilde{g}(\leqq)$, we define

$$\varphi \star \psi(\overline{[x,y]_{\leqq}}) := \Phi \star \Psi(x,y), \quad [x,y]_{\leqq} \in I(L,\leqq), \tag{16}$$

where Φ and Ψ are the operators of $\tilde{G}(\leqq)$ respectively induced by φ and ψ.

Proposition 9 *$(\tilde{G}(\leqq),\star)$ and $(\tilde{g}(\leqq),\star)$ are isomorphic groups. δ is the identity element of $(\tilde{g}(\leqq),\star)$.*

Proof: We successively show that $(\tilde{G}(\leqq),\star)$ is a subgroup of $(G(\leqq),\star)$, then $(\tilde{G}(\leqq),\star)$ and $\cong$ $(\tilde{g}(\leqq),\star)$ are isomorphic. By definition of a compatible equivalence relation, the closure of $\tilde{G}(\leqq)$ under convolution follows, and the closure under inversion is straightforwardly derived from Proposition 7. Considering the bijection given by (15), for $\Phi, \Psi \in \tilde{G}(\leqq)$, we denote by φ, ψ and $f_{\Phi\star\Psi}$ the respective images of Φ, Ψ and $\Phi \star \Psi$. Applying (15) for $f_{\Phi\star\Psi}$ and (16) for $\varphi \star \psi$, we get $f_{\Phi\star\Psi} = \varphi \star \psi$. ■

We now address the particular order relation $\leqq$ that enables the writing of operation $\star$ in $\tilde{g}(\leqq)$ in terms of binomial coefficients, which makes brighter the terminology "convolution". From the description of (12) of Γ, we define the following binary relation in L:

$$x \trianglelefteq y \quad \text{iff} \quad \forall i \in N, x_i = \perp_i \text{ or } x_i = y_i.$$

One can easily check that $\trianglelefteq$ in an order relation. Besides, for all x, y s.t. $x \trianglelefteq y$, we naturally define the element $y - x$ of L by

$$(y-x)_i := \begin{cases} y_i, & \text{if } x_i = \perp_i, \\ \perp_i, & \text{if } x_i = y_i, \end{cases} \quad i \in N.$$

Note that if $x \trianglelefteq y$, $k(y-x) = k(y) - k(x)$.

By the following result, one can easily check that $\Gamma \in \tilde{G}(\trianglelefteq)$.

Lemma 10 *Let $x, y \in L$ such that $x \trianglelefteq y$. Then*

$$[x, y]_{\trianglelefteq} \cong 2^{k(y-x)}.$$

As a consequence, the elements of $\tilde{I}(L, \trianglelefteq)$ are given by the classes of intervals of $I(L, \trianglelefteq)$ which are isomorphic to a certain Boolean lattice.

Proof: Let $z \in [x, y]_{\trianglelefteq}$. For any $i \in N$, either $x_i = \bot_i$ and $y_i \neq x_i$, or $x_i = y_i$. The first case implies $z_i = \bot_i$ or $z_i = y_i$ (with $y_i \neq \bot_i$), and the second case implies $z_i = x_i = y_i$. As a result,

$$[x, y]_{\trianglelefteq} = \prod_{i \in N | x_i \neq y_i} \{\bot_i, y_i\} \times \prod_{i \in N | x_i = y_i} \{y_i\} \cong 2^{k(y-x)}.$$

■

Let $w(\mathcal{J}(L))$ be the *width* of $\mathcal{J}(L)$, that is to say the cardinal of a maximal antichain of $\mathcal{J}(L)$, that is also the sum of the cardinals of maximal antichains of the $\mathcal{J}(L_i)$'s. As a result, the greatest intervals of L isomorphic to a Boolean lattice, are isomorphic to $2^{w(\mathcal{J}(L))}$. Note that $n \leq w(\mathcal{J}(L)) \leq |\mathcal{J}(L)|$.

Considering the elements of $\tilde{I}(L, \leq)$, we denote by $\overline{m}$ the class of all Boolean intervals isomorphic to 2^m, $m = 0, \ldots, w(\mathcal{J}(L))$. In the same way, $\overline{\overline{m}}$ denotes the element of $\tilde{I}(L, \trianglelefteq)$ representing all intervals $[x, y]_{\trianglelefteq}$ s.t. $k(y - x) = m$, $m = 0, \ldots, n$. Clearly, all these classes are nonempty. In particular, $\overline{0}$ and $\overline{\overline{0}}$ are the unique elements of $\tilde{g}(\leq)$ and $\tilde{g}(\trianglelefteq)$ containing singletons of L: $\overline{0} = \overline{\overline{0}} = \{\{x\} \mid x \in L\}$. Consequently, the identity element of $\tilde{g}(\leq)$ (resp. $\tilde{g}(\trianglelefteq)$) simply writes as the function which is 1 onto $\overline{0}$ (resp. $\overline{\overline{0}}$), and 0 elsewhere. One can note that in the general case, $\tilde{I}(L, \trianglelefteq) = \{\overline{\overline{0}}, \ldots, \overline{\overline{n}}\}$, but $\tilde{I}(L, \leq) \supsetneq \{\overline{0}, \ldots, \overline{w(\mathcal{J}(L))}\}$ (there are some classes having not a "Boolean type").

By (9) and (12), the associated functions $\zeta \in \tilde{g}(\leq)$ of Z and $\gamma \in \tilde{g}(\trianglelefteq)$ of Γ respectively write

$$\zeta(\alpha) = 1, \quad \alpha \in \tilde{I}(L, \leq),$$
$$\text{and} \quad \gamma(\overline{\overline{m}}) = \frac{1}{m+1}, \quad m = 0, \ldots, n.$$

Theorem 11 *For all $\varphi, \psi \in \tilde{g}(\leq)$, and any $m \in \{0, \dots, w(\mathcal{J}(L))\}$,*

$$\varphi \star \psi(\overline{m}) = \sum_{j=0}^{m} \binom{m}{j} \varphi(\overline{j})\, \psi(\overline{m-j}).$$

Besides, the inverse of φ computes recursively through

$$\begin{aligned}\varphi^{-1}(\overline{0}) &= 1,\\ \varphi^{-1}(\overline{m}) &= -\sum_{j=0}^{m-1} \binom{m}{j} \varphi^{-1}(\overline{j})\, \varphi(\overline{m-j}).\end{aligned}$$

The same formulae hold for $\varphi \star \psi(\overline{\overline{m}})$ and $\varphi^{-1}(\overline{\overline{m}})$, $\varphi, \psi \in \tilde{g}(\trianglelefteq)$ and $m \in \{0, \dots, n\}$.

Proof: Let $\varphi, \psi \in \tilde{g}(\leq)$, $m \in \{0, \dots, w(\mathcal{J}(L))\}$, and any interval $[x, y]$ of L such that $[x, y] \cong 2^m$. Note that $\forall t \in [x, y]$, $[x, t]$ and $[t, y]$ are also Boolean, with $[x, t] \cong 2^j$, $[t, y] \cong 2^{j'}$ s.t. $j + j' = m$. Then by (16),

$$\begin{aligned}\varphi \star \psi(\overline{m}) &= \Phi \star \Psi(x, y)\\ &= \sum_{x \leq t \leq y} \Phi(x, t)\, \Psi(t, y)\\ &= \sum_{j=0}^{m} \sum_{t \in [x,y] \mid \overline{[x,t]} = \overline{j}} \varphi(\overline{j})\, \psi(\overline{m-j})\\ &= \sum_{j=0}^{m} \binom{m}{j} \varphi(\overline{j})\, \psi(\overline{m-j}).\end{aligned}$$

By definition of φ^{-1}, then by Proposition 7, we have also

$$\begin{aligned}\varphi^{-1}(\overline{0}) &= \Phi^{-1}(x, x) = 1,\\ \text{and for } m \neq 0,\ \varphi^{-1}(\overline{m}) &= \Phi^{-1}(x, y)\\ &= -\sum_{x \leq t < y} \Phi^{-1}(x, t)\, \Phi(t, y)\\ &= -\sum_{j=0}^{m-1} \sum_{t \in [x,y] \mid \overline{[x,t]} = \overline{j}} \varphi^{-1}(\overline{j})\, \varphi(\overline{m-j})\\ &= -\sum_{j=0}^{m-1} \binom{m}{j} \varphi^{-1}(\overline{j})\, \varphi(\overline{m-j}).\end{aligned}$$

Now, by Lemma 10, any interval of $I(L, \trianglelefteq)$ is Boolean. Consequently, for $\varphi, \psi \in \tilde{g}(\trianglelefteq)$, and $m \in \{0, \ldots, n\}$, we obtain the same formulae for $\varphi \star \psi(\overline{\overline{m}})$ and $\varphi^{-1}(\overline{\overline{m}})$. ∎

Note that the above result is not general and does not apply for any $\tilde{g}(\leqq)$. Actually, $\tilde{G}(\leq)$ and $\tilde{G}(\trianglelefteq)$ are very particular subgroups of $G(\leq)$, which refer to particular algebras, namely of *binomial type* in the framework of incidence algebras.

Let $(B_m)_{m \in \mathbb{N}}$ be the *sequence of Bernoulli numbers*, computing recursively through

$$\begin{aligned} B_0 &= 1, \\ B_m &= -\frac{1}{m+1} \sum_{j=0}^{m-1} \binom{m+1}{j} B_j, \quad m \in \mathbb{N} \setminus \{0\}. \end{aligned}$$

$(B_m)_m$ starts with $1, -1/2, 1/6, 0, -1/30, 0, 1/42 \ldots$, and it is well-known that $B_m = 0$ for $m \geq 3$ odd. From Theorem 11, we derive the following result.

Corollary 12 *The inverses of ζ in $\tilde{g}(\leq)$ and γ in $\tilde{g}(\trianglelefteq)$ are given by*

$$\zeta^{-1}(\alpha) = \begin{cases} (-1)^m, & \alpha = \overline{m} \quad (m = 0, \ldots, w(\mathcal{J}(L))), \\ 0, & \textit{otherwise}, \end{cases}$$

$$\textit{and} \quad \gamma^{-1}(\overline{\overline{m}}) = B_m, \quad m = 0, \ldots, n.$$

Proof: By applying Theorem 11, we get $\zeta^{-1}(\overline{0}) = 1 = (-1)^0$, and for any $m \in \{1, \ldots, w(\mathcal{J}(L))\}$, by induction on m, we get

$$\begin{aligned} \zeta^{-1}(\overline{m}) &= -\sum_{j=0}^{m-1} \zeta^{-1}(\overline{j})\, \zeta(\overline{m-j}), \\ &= -\big((-1+1)^m - (-1)^m\big) \\ &= (-1)^m. \end{aligned}$$

Then we check that $\zeta^{-1}(\alpha) = 0$ for all $\alpha \in \tilde{I}(L, \leq) \setminus \{\overline{j} \mid j = 0, \ldots, w(\mathcal{J}(L))\}$, suits as the inverse of ζ. Indeed, let α be such an element, and $[x, y]$ be any interval s.t. $\overline{[x,y]} = \alpha$ ($[x,y]$ is not Boolean). Note that $\underline{y} \in [x,y]$. Then

$$\begin{aligned}
\zeta \star \zeta^{-1}(\alpha) &= Z \star Z^{-1}(x,y) \\
&= \sum_{x \le t \le y} Z(x,t)\, Z^{-1}(t,y) \\
&= \sum_{\underline{y} \le t \le y} Z(x,t)\, Z^{-1}(t,y) + \sum_{x \le |t \not\ge \underline{y}} Z(x,t)\,.
\end{aligned}$$

By Proposition 4–(iv), the first sum is $(1+(-1))^{\eta^*(y)}$. Besides, by Proposition 4–(iii), all the $[t,y]$'s in the second sum are not Boolean, and thus the $Z^{-1}(t,y)$'s vanish.

Now, $\gamma^{-1}(\overline{\overline{0}}) = 1 = B_0$, and for any $m \in \{1,\dots,n\}$, we get by Theorem 11

$$\begin{aligned}
\gamma^{-1}(\overline{\overline{m}}) &= -\sum_{j=0}^{m-1} \binom{m}{j} \gamma^{-1}(\overline{\overline{j}})\, \frac{1}{m-j+1} \\
&= -\frac{1}{m+1} \sum_{j=0}^{m-1} \binom{m+1}{j} \gamma^{-1}(\overline{\overline{j}}),
\end{aligned}$$

which is precisely the definition of the Bernoulli sequence. ■

By the bijection (15), we finally deduce from ζ^{-1} and γ^{-1} some explicit formulae for the Möbius operator and the Bernoulli operator.

$$Z^{-1}(x,y) := \begin{cases} (-1)^m, & \text{if } [x,y] \cong 2^m, \\ 0, & \text{otherwise,} \end{cases} \qquad x,y \in L,$$

$$\text{and} \quad \Gamma^{-1}(x,y) := \begin{cases} B_{k(y-x)}, & \text{if } x \trianglelefteq y, \\ 0, & \text{otherwise,} \end{cases} \qquad x,y \in L.$$

7 The interaction operator and its inverse

By means of the expression of the Bernoulli operator and Eq. (13), for any lattice function f, we get

$$m^f(x) = \sum_{y \trianglerighteq x} B_{k(y-x)}\, I^f(y). \tag{17}$$

For any $p \in \mathbb{N}$, and $m = 0, \dots, p$, we define

$$b^p_m := \sum_{j=0}^{m} \binom{m}{j} B_{p-j},$$

These numbers have been introduced in [5] to express a lattice function v from its interaction index I^v. It is easy to compute them from the sequence of Bernoulli: $b^p_0 = B_p$, $p \in \mathbb{N}$, and by applying the recursion of the Pascal's triangle:

$$b^{p+1}_{m+1} = b^{p+1}_m + b^p_m, \quad 0 \le m \le p.$$

The coefficients also satisfy the following symmetry:

$$b^p_m = (-1)^p \, b^p_{p-m}, \quad 0 \le m \le p.$$

The values of b^p_m, $p \le 6$, are

		m						
		0	1	2	3	4	5	6
p	0	1						
	1	$-\frac{1}{2}$	$\frac{1}{2}$					
	2	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$				
	3	0	$\frac{1}{6}$	$-\frac{1}{6}$	0			
	4	$-\frac{1}{30}$	$-\frac{1}{30}$	$\frac{2}{15}$	$-\frac{1}{30}$	$-\frac{1}{30}$		
	5	0	$-\frac{1}{30}$	$-\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{30}$	0	
	6	$\frac{1}{42}$	$\frac{1}{42}$	$-\frac{1}{105}$	$-\frac{8}{105}$	$-\frac{1}{105}$	$\frac{1}{42}$	$\frac{1}{42}$

We now give an explicit formula for the inverse interaction operator $\mathbb{I}^{-1} = Z\star{}^t\Gamma^{-1}$ (cf. Section 5).

Theorem 13 *For all $x, y \in L$,*

$$\mathbb{I}^{-1}(x, y) = b^{k(x)}_{k(x_y)},$$

where $(x_y)_i := \begin{cases} x_i, & \text{if } x_i \le y_i \\ \perp_i, & \text{otherwise} \end{cases}$, $i \in N$. *Consequently, for any lattice function f,*

$$f(x) = \sum_{z \in L} b^{k(z)}_{k(z_x)} \, I^f(z), \quad x \in L.$$

Proof: For all $x \in L$, according to (1) and (17), we have

$$\begin{aligned} f(x) &= \sum_{y \leq x} m^f(y) \\ &= \sum_{y \leq x} \sum_{z \trianglerighteq y} B_{k(z-y)}\, I^f(z) \\ &= \sum_{z \in L} \Big(\sum_{\substack{y \trianglelefteq z \\ y \leq x}} B_{k(z-y)} \Big)\, I^f(z). \end{aligned}$$

Note that $y \trianglelefteq z$ and $y \leq x$ iff $y_i \leq x_i$ and ($y_i = \perp_i$ or $y_i = z_i$), $i \in N$, which is equivalent to $y \trianglelefteq z_x$. Let $K(z_x) := \{i \in N \mid (z_x)_i \neq \perp_i\}$. Then

$$\begin{aligned} \sum_{\substack{y \trianglelefteq z \\ y \leq x}} B_{k(z-y)} &= \sum_{y \trianglelefteq z_x} B_{k(z-y)} \\ &= \sum_{Y \subseteq K(z_x)} B_{k(z)-|Y|} \\ &= \sum_{j=0}^{k(z_x)} \binom{k(z_x)}{j} B_{k(z)-j} \\ &= b_{k(z_x)}^{k(z)}, \end{aligned}$$

where the second equality is due to Lemma 10. ■

References

[1] G. Birkhoff. *Lattice Theory.* American Mathematical Society, 3d edition, 1967.

[2] A. Chateauneuf and J.Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences*, 17:263–283, 1989.

[3] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953.

[4] D. Denneberg. *Non-Additive Measure and Integral.* Kluwer Academic, 1994.

[5] D. Denneberg and M. Grabisch. Interaction transform of set functions over a finite set. *Information Sciences*, 48(1):15–27, 1999.

[6] P. Doubilet, G.C. Rota, and R. Stanley. On the foundations of combinatorial theory (VI): The idea of generating function. In Berkeley University Press, editor, *6th Berkeley Symposium on Mathematical Statistics and Probability*, volume 2, pages 267–318, 1972.

[7] U. Faigle and W. Kern. The Shapley value for cooperative games under precedence constraints. *International Journal of Game Theory*, 21:249–266, 1992.

[8] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[9] M. Grabisch and Ch. Labreuche. Bi-capacities. Part I: definition, Möbius transform and interaction. *Fuzzy Sets and Systems*, 151:211–236, 2005.

[10] M. Grabisch and Ch. Labreuche. Derivative of functions over lattices as a basis for the notion of interaction between attributes. *Annals of Mathematics and Artificial Intelligence*, 49:151–170, 2007.

[11] M. Grabisch and F. Lange. Games on lattices, multichoice games and the Shapley value: a new approach. *Mathematical Methods of Operations Research*, 65:153–167, 2007.

[12] M. Grabisch and M. Roubens. An axiomatic approach of interaction in multicriteria decision making. In *5th Eur. Congr. on Intelligent Techniques and Soft Computing (EUFIT'97)*, pages 81–85, Aachen, Germany, september 1997.

[13] M. Grabisch and M. Roubens. An axiomatic approach to the concept of interaction among players in cooperative games. *Int. Journal of Game Theory*, 28:547–565, 1999.

[14] P.L. Hammer and S. Rudeanu. *Boolean Methods in Operations Research and Related Areas*. Springer, 1968.

[15] C.R. Hsiao and T.E.S. Raghavan. Shapley value for multichoice cooperative games, I. *Games and Economic Behavior*, 5:240–256, 1993.

[16] F. Lange and M. Grabisch. Interaction transform for bi-set functions over a finite set. *Information Sciences*, 176:2279–2303, 2006.

[17] G.C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 2:340–368, 1964.

[18] L.S. Shapley. A value for n-person games. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.

[19] M. Sugeno. *Theory of fuzzy integrals and its applications*. PhD thesis, Tokyo Institute of Technology, 1974.

Résumé

Les fonctions de treillis, apparaissent être des outils essentiels en recherche opérationnelle. Elles ouvrent en effet de nouveaux champs d'application en théorie des jeux coopératifs, et en aide à la décision (les jeux sont dans ce cas des capacités, ou mesures floues).

Cette thèse a pour objet l'investigation de concepts de solutions pour les jeux définis sur des structures générales de coalitions. À cette fin, nous proposons plusieurs généralisations et axiomatisations de la valeur de Shapley pour les jeux multi-choix, les jeux à actions combinées, et les jeux réguliers.

L'indice d'interaction quantifie la véritable contribution d'une coalition par rapport à toutes ses sous-coalitions. Mathématiquement, il s'agit d'un prolongement de la valeur de Shapley. Nous proposons des axiomatisations de l'indice d'interaction de Shapley pour les jeux bi-coopératifs, ainsi que des procédés calculatoires permettant de déterminer l'opérateur d'interaction et son inverse.

Mots clés : jeu coopératif, ensemble ordonné, valeur de Shapley, transformée de Möbius, indice d'interaction

Abstract

Lattice functions appear to be an essential tool in operations research, opening new areas in the fields of cooperative game theory (players or agents form coalitions in games), and decision making (capacities or fuzzy measures are defined over some coalitions structures of criteria).

The thesis aims at investigating solution concepts for games defined on general coalitions structures. In this purpose, we propose several generalizations of the Shapley value with axiomatizations for multichoice games, games over distributive lattices, and regular games.

The interaction index quantifies the genuine contribution of a coalition with reference to all its subcoalitions. Mathematically, it is an extension of the Shapley value, and it involves the derivatives of the game. We propose some axiomatizations of the Shapley interaction index for bi-cooperative games, and some means for computing it from games in transferable utility form, and vice versa.

Keywords: cooperative game, partially ordered set, Shapley value, Möbius transform, interaction index