On the topology and differential geometry of Kahler threefolds

Rares Rasdeaconu

On the topology and differential geometry of Kähler threefolds

A Dissertation, Presented

by

Răsdeaconu Rareş

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2005

Stony Brook University

The Graduate School

Răsdeaconu Rareş

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Claude LeBrun
Professor, Department of Mathematics
Dissertation Director

H. Blaine Lawson Jr.
Professor, Department of Mathematics
Chairman of Dissertation

Sorin Popescu
Associate Professor, Department of Mathematics

Martin Roček
Professor, Department of Physics
Outside Member

This dissertation is accepted by the Graduate School.

Dean of the Graduate School

Abstract of the Dissertation,

On the topology and differential geometry of Kähler threefolds

by

Răsdeaconu Rareş

Doctor of Philosophy

in

Mathematics

Stony Brook University

2005

In the first part of my thesis we provide infinitely many examples of pairs of diffeomorphic, non simply connected Kähler manifolds of complex dimension 3 with different Kodaira dimensions. Also, in any allowed Kodaira dimension we find infinitely many pairs of non deformation equivalent, diffeomorphic Kähler threefolds.

In the second part we study the existence of Kähler metrics of positive total scalar curvature on 3-folds of negative Kodaira dimension. We give a positive answer for rationally connected threefolds. The proof relies on the Mori theory of minimal models, the weak factorization theorem and on a specialization technique.

părinţilor mei

Contents

Acknowledgments

I am indebted to many for their support and encouragement. It is a very pleasant task to thank everyone for their help.

First of all, I would like to thank my advisor Claude LeBrun. He patiently let me choose the areas of geometry that most interested me and helped me become a mathematician. It has been a privilege to learn from him the craft of thinking about Mathematics, and I am deeply grateful to him for being such a supporting and inspiring mentor.

I would like to thank Mark Andrea de Cataldo, Lowell Jones and Sorin Popescu for the time they spent discussing with me, for their teachings, encouragement, and their advice during my graduate years.

This project would not have seen the light without Ioana's support. She was with me in the difficult moments and more importantly, she has been the source of the good times.

I am grateful to my friends who helped me survive graduate school. In no particular order, thank you Vuli, Dan, Ionuţ, Olguţa, and Rodrigo.

To all, my warmest gratitude.

Chapter 1

Kodaira dimension of diffeomorphic threefolds

1.1 Introduction

Let M be a compact complex manifold of complex dimension n. On any such manifold the *canonical line bundle* $K_M = \wedge^{n,0}$ encodes important information about the complex structure. One can define a series of birational invariants of M,

$$P_k(M) := h^0(M, K_M^{\otimes k}),\ \ k \geq 0,$$

called the *plurigenera*. The number of independent holomorphic n-forms on M, $p_g(M) = P_1(M)$ is called the geometric genus. The *Kodaira dimension* $\mathrm{Kod}(M)$, is a birational invariant given by:

$$\mathrm{Kod}(M) = \limsup \frac{\log h^0(M, K_M^{\otimes k})}{\log k}.$$

This can be shown to coincide with the maximal complex dimension of the image of M under the pluri-canonical maps, so that $\mathrm{Kod}(M) \in \{-\infty, 0, 1, \ldots, n\}$. A compact complex n-manifold is said to be of *general type* if $\mathrm{Kod}(M) = n$.

For Riemann surfaces, the classification with respect to the Kodaira dimension, $\mathrm{Kod}(M) = -\infty, 0$ or 1 is equivalent to the one given by the *genus*, $g(M) = 0,\ 1$, and ≥ 2, respectively.

An important question in differential geometry is to understand how the complex structures on a given complex manifold are related to the diffeomorphism type of the underlying smooth manifold or further, to the topological type of the underlying topological manifold. Shedding some light on this question is S. Donaldson's result on the *failure of the h-cobordism conjecture in dimension four.* In this regard, he found a pair of *non-diffeomorphic*, *h*-cobordant, simply connected 4-manifolds. One of them was $\mathbb{CP}_2\#9\overline{\mathbb{CP}}_2$, the blow-up of $\mathbb{CP}_2$ at nine appropriate points, and the other one was a certain properly elliptic surface. For us, an important feature of these two complex surfaces is the fact that they have *different Kodaira dimensions.* Later, R. Friedman and Z. Qin [FrQi94] went further and proved that actually, for complex surfaces of Kähler type, *the Kodaira dimension is invariant under diffeomorphisms.* However, in higher dimensions, C. LeBrun and F. Catanese gave examples [CaLe97] of pairs of diffeomorphic projective manifolds of complex dimensions $2n$ with $n \geq 2$, and Kodaira dimensions $-\infty$ and $2n$.

In this thesis we address the question of the invariance of the Kodaira dimension under diffeomorphisms in complex dimension 3. We obtain the expected negative result:

Theorem A. *For any allowed pair of distinct Kodaira dimensions* (d, d'), *with the exception of* $(-\infty, 0)$ *and* $(0, 3)$, *there exist infinitely many pairs of diffeomorphic Kähler threefolds* (M, M'), *having the same Chern numbers, but* *with* $\mathrm{Kod}(M) = d$ *and* $\mathrm{Kod}(M') = d'$*, respectively.*

Corollary 1.1. *For Kähler threefolds, the Kodaira dimension is not a smooth invariant.*

Our examples also provide negative answers to questions regarding the deformation types of Kähler threefolds.

Recall that two manifolds X_1 and X_2 are called *directly deformation equivalent* if there exists a complex manifold $\mathcal{X}$, and a proper holomorphic submersion $\varpi : \mathcal{X} \to \Delta$ with $\Delta = \{|z| = 1\} \subset \mathbb{C}$, such that X_1 and X_2 occur as fibers of ϖ. The *deformation equivalence* relation is the equivalence relation generated by direct deformation equivalence.

It is known that two deformation equivalent manifolds are orientedly diffeomorphic. For complex surfaces of Kähler type there were strong indications that the converse should also be true. R. Friedman and J. Morgan proved [FrMo97] that, not only the Kodaira dimension is a smooth invariant but the plurigenera, too. However, Manetti [Man01] exhibited examples of diffeomorphic complex surfaces of general type which were *not* deformation equivalent. An easy consequence of our **Theorem A** and of the deformation invariance of plurigenera for 3-folds [KoMo92] is that in complex dimension 3 the situation is similar:

Corollary 1.2. *For Kähler threefolds the deformation type does not coincide with the diffeomorphism type.*

Actually, with a bit more work we can get:

Theorem B. *In any possible Kodaira dimension, there exist infinitely many examples of pairs of diffeomorphic, non-deformation equivalent Kähler threefolds with the same Chern numbers.*

The examples we use are Cartesian products of simply connected, $h-$cobordant complex surfaces with Riemann surfaces of positive genus. The real six-manifolds obtained will therefore be $h-$cobordant. To prove that these six-manifolds are in fact diffeomorphic, we use the $s-$Cobordism Theorem, by showing that the obstruction to the triviality of the corresponding $h-$cobordism, the *Whitehead torsion*, vanishes. Similar examples were previously used by Y. Ruan [Ruan94] to find pairs of diffeomorphic symplectic 6-manifolds which are not symplectic deformation equivalent. However, to show that his examples are diffeomorphic, Ruan uses the classification (up to diffeomorphisms) of *simply-connected*, real 6-manifolds [OkVdV95]. This restricts Ruan's construction to the case of Cartesian products by 2-spheres, a result which would also follow from Smale's h-cobordism theorem.

The examples of pairs complex structures we find are all of Kähler type with the same Chern numbers. This should be contrasted with C. LeBrun's examples [LeB99] of complex structures, mostly non-Kähler, with *different Chern numbers* on a given differentiable *real* manifold.

In our opinion, the novelty of this article is the use of the apparently forgotten *s-Cobordism Theorem.* This theorem is especially useful when combined with a theorem on the vanishing of the *Whitehead group*. For this, there exist nowadays strong results, due to F.T. Farrell and L. Jones [FaJo91].

In the next section, we will review the main tools we use to find our examples: h-cobordisms, the Whitehead group and its vanishing. In section 3 we recall few well-known generalities about complex surfaces. Sections 4 and 5 contain a number of examples and the proofs of **Theorems A** and **B**. In the last section we conclude with few remarks and we raise some natural questions.

1.2 The s-Cobordism Theorem

Definition 1.3. *Let M and M' be two n-dimensional closed, smooth, oriented manifolds. A cobordism between M and M' is a triplet $(W; M, M')$, where W is an $(n+1)$-dimensional compact, oriented manifold with boundary, $\partial W = \overline{\partial W}_- \bigsqcup \partial W_+$ with $\partial W_- = M$ and $\partial W_+ = M'$ (by $\overline{\partial W}_-$ we denoted the orientation-reversed version of ∂W_-).*

We say that the cobordism $(W; M, M')$ is an h-cobordism if the inclusions $i_- : M \to W$ and $i_+ : M' \to W$ are homotopy equivalences between M, M' and W.

The following well-known results [Wall62], [Wall64] allow us to easily check when two simply connected 4-manifolds are h-cobordant:

Theorem 1.4. *Two simply connected smooth manifolds of dimension 4 are h-cobordant if and only if their intersection forms are isomorphic.*

Theorem 1.5. *Any indefinite, unimodular, bilinear form is uniquely determined by its rank, signature and parity.*

In higher dimensions any h-cobordism $(W; M, M')$ is controlled by a complicated torsion invariant $\tau(W; M)$, the *Whitehead torsion,* an element of the so called *Whitehead group* which will be defined below.

Let Π be any group, and $R = \mathbb{Z}(\Pi)$ the integral unitary ring generated by Π. We denote by $GL_n(R)$ the group of all nonsingular $n \times n$ matrices over R. For all n we have a natural inclusion $GL_n(R) \subset GL_{n+1}(R)$ identifying each $A \in GL_n(R)$ with the matrix:

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(R).$$

Let $GL(R) = \bigcup_{n=1}^{\infty} GL_n(R)$. We define the following group:

$$K_1(R) = GL(R)/[GL(R), GL(R)].$$

The *Whitehead group* we are interested in is:

$$Wh(\Pi) = K_1(R)/ < \pm g \mid g \in \Pi > .$$

Theorem 1.6. *Let M be a smooth, closed manifold. For any h-cobordism W of M with $\partial\_W = M$, and with dim $W \geq 6$ there exists an element $\tau(W) \in Wh(\pi_1(M))$, called* the Whitehead torsion, *characterized by the following properties:*

- **s-Cobordism Theorem** $\tau(W) = 0$ *if and only if the h-cobordism is trivial, i.e. W is diffeomorphic to $\partial\_W \times [0, 1]$;*

- **Existence** *Given $\alpha \in Wh(\pi_1(M))$, there exists an h-cobordism W with $\tau(W) = \alpha$;*

- **Uniqueness** $\tau(W) = \tau(W')$ *if and only if there exists a diffeomorphism*

$h : W \to W'$ such that $h_{|M} = id_M$.

For the definition of the *Whitehead torsion* and the above theorem we refer the reader to Milnor's article [Mil66]. However, the above theorem suffices. When M is simply connected, the s-cobordism theorem is nothing but the usual h-cobordism theorem [Mil65], due to Smale.

This theorem will be a stepping stone in finding pairs of diffeomorphic manifolds in dimensions greater than 5, provided knowledge about the vanishing of the *Whitehead groups*. The vanishing theorem that we are going to use here is:

Theorem 1.7 (Farrell, Jones)**.** *Let M be a compact Riemannian manifold of non-positive sectional curvature. Then $Wh(\pi_1(M)) = 0$.*

The uniformization theorem of compact Riemann surfaces yields then the following result which, as it was kindly pointed to us by L. Jones, was also known to F. Waldhausen [Wal78], long before [FaJo91].

Corollary 1.8. *Let Σ be a compact Riemann surface. Then $Wh(\pi_1(\Sigma)) = 0$.*

An important consequence, which will be frequently used is the following:

Corollary 1.9. *Let M and M' be two simply connected, h-cobordant 4-manifolds, and Σ be a Riemann surface of positive genus. Then $M \times \Sigma$ and $M' \times \Sigma$ are diffeomorphic.*

Proof. Let W be an h-cobordism between M and M' such that $\partial_- W = M$ and $\partial_+ W = M'$ and let $\widetilde{W} = W \times \Sigma$. Then $\partial_- \widetilde{W} = M \times \Sigma,\ \partial_+ \widetilde{W} = M' \times \Sigma$, and $\widetilde{W}$ is an h-cobordism between $M \times \Sigma$ and $M' \times \Sigma$. Now, since M is simply connected $\pi_1(M \times \Sigma) = \pi_1(\Sigma)$ and so

$$Wh(\pi_1(M \times \Sigma)) = Wh(\pi_1(\Sigma)).$$

By the uniformization theorem any Riemann surface of positive genus admits a metric of non-positive curvature. Thus, by Theorem 1.7, $Wh(\pi_1(\Sigma)) = 0$, which, by Theorem 1.6, implies that $M \times \Sigma$ and $M' \times \Sigma$ are diffeomorphic. $\square$

1.3 Generalities

To prove **Theorems A** and **B** we will use our Corollary 1.9, by taking for M and M' appropriate h-cobordant, simply connected, complex projective surfaces, and for Σ, Riemann surfaces of genus $g(\Sigma) \geq 1$. To find examples of h-cobordant complex surfaces, we use:

Proposition 1.10. *Let M and M' be two simply connected complex surfaces with the same geometric genus p_g, $c_1^2(M) - c_1^2(M') = m \geq 0$ and let $k > 0$ be any integer. Let X be the blowing-up of M at $k + m$ distinct points and X' be the blowing-up of M' at k distinct points. Then X and X' are h-cobordant,* $\mathrm{Kod}(X) = \mathrm{Kod}(M)$ *and* $\mathrm{Kod}(X') = \mathrm{Kod}(M')$.

Proof. By Noether's formula we immediately see that

$$b_2(M') = b_2(M) + m.$$

Since, by blowing-up we increase each time the second Betti number by one, it follows that

$$b_2(X') = b_2(X).$$

Using the birational invariance of the plurigenera, we have that

$$b_+(X') = 2p_g + 1 = b_+(X).$$

As X and X' are both non-spin, and their intersection forms have the same rank and signature, their intersection forms are isomorphic. Thus, by Theorem 1.4, X and X' are h-cobordant.

The statement about the Kodaira dimension follows immediately from the birational invariance of the plurigenera, too. $\square$

Corollary 1.11. *Let S and S' be two simply connected, h-cobordant complex surfaces. If S_k and S'_k are the blowing-ups of the two surfaces, each at $k \geq 0$ distinct points, then S_k and S'_k are h-cobordant, too. Moreover,* $\mathrm{Kod}(S_k) = \mathrm{Kod}(S)$, *and* $\mathrm{Kod}(S'_k) = \mathrm{Kod}(S')$.

The following proposition will take care of the computation of the Kodaira dimension of our examples. Its proof is standard, and we will omit it.

Proposition 1.12. *Let V and W be two complex manifolds. Then*

$$P_m(V \times W) = P_m(V) \cdot P_m(W).$$

In particular, $\mathrm{Kod}(V \times W) = \mathrm{Kod}(V) + \mathrm{Kod}(W)$.

For the computation of the Chern numbers of the examples involved, we need:

Proposition 1.13. *Let M be a smooth complex surface with $c_1^2(M) = a$, $c_2(M) = b$, and let Σ be a smooth complex curve of genus g, and $X = M \times \Sigma$ their Cartesian product. The Chern numbers $(\mathbf{c_1^3}, \mathbf{c_1 c_2}, \mathbf{c_3})$ of X are*

$$((6-6g)a, (2-2g)(a+b), (2-2g)b).$$

Proof. Let $p : X \to M$, and $q : X \to \Sigma$ be the projections onto the two factors. Then the total Chern class is

$$c(X) = p^* c(M) \cdot q^* c(\Sigma),$$

which allows us to identify the Chern classes. Integrating over X, the result follows immediately. □

1.4 Diffeomorphism Type

In this section we prove **Theorem A**. All we have to do is to exhibit the appropriate examples. Thus, for each of the pairs of Kodaira dimensions stated, we provide infinitely many examples, by taking Cartesian products of appropriate $h-$cobordant Kähler surfaces with Riemann surfaces of positive genus.

Example 1: *Pairs of Kodaira dimensions* $(-\infty, 1)$ *and* $(-\infty, 2)$

Let M be the blowing-up of $\mathbb{CP}_2$ at 9 distinct points given by the intersection of two generic cubics. M is a non-spin, simply connected complex surface with $\mathrm{Kod}(M) = -\infty$ which is also an elliptic fibration, $\pi : M \to \mathbb{CP}_1$. By taking the cubics general enough, we may assume that M has no multiple fibers, and the only singular fibers are irreducible curves with one ordinary double point.

Let M' be obtained from M by performing logarithmic transformations on two of its smooth fibers, with multiplicities p and q, where p and q are two relatively prime positive integers. M' is also an elliptic surface, $\pi' : M' \to \mathbb{CP}_1$, whose fibers can be identified to those in M except for the pair of multiple fibers F_1, and F_2. Let F be homology class of the generic fiber in M'. In homology we have $[F] = p[F_1] = q[F_2]$. By canonical bundle formula, we see that: $K_M = -F$, and

$$K_{M'} = -F + (p-1)F_1 + (q-1)F_2 = \frac{pq - p - q}{pq}F. \tag{1.1}$$

Then $p_g(M) = p_g(M') = 0$, $c_1^2(M) = c_1^2(M') = 0$, and $\mathrm{Kod}(M') = 1$. Moreover, from [FrMo94, Theorem 2.3, page 158] M' is simply connected and non-spin.

For any $k \geq 0$, let M_k and M'_k be the blowing-ups at k distinct points of M and M', respectively, and let Σ be a Riemann surface.

- If $g(\Sigma) = 1$, according to Corollary 1.9 and Proposition 1.12, $(M_k \times \Sigma_1, M'_k \times \Sigma_1),\ k \geq 0$ will provide infinitely many pairs of diffeomorphic Kähler threefolds, whose Kodaira dimensions are $-\infty$ and 1, respectively.

- If $g(\Sigma) \geq 2$, we get infinitely many pairs of diffeomorphic Kähler threefolds of Kodaira dimensions $-\infty$, and 2, respectively.

The statement about the Chern numbers follows from Proposition 1.13. □

Example 2: *Pairs of Kodaira dimensions* $(0,1)$ *and* $(0,2)$

In $\mathbb{CP}_1 \times \mathbb{CP}_2$, let M be the the generic section of line bundle

$$p_1^*\mathcal{O}_{\mathbb{CP}_1}(2) \otimes p_2^*\mathcal{O}_{\mathbb{CP}_2}(3),$$

where p_i, $i = 1, 2$ are the projections onto the two factors. Then M is a $K3$ surface, i.e. a smooth, simply connected complex surface, with trivial canonical bundle. Moreover, using the projection onto the first factor, it fibers over $\mathbb{CP}_1$ with elliptic fibers.

Kodaira [Kod70] produced infinitely many examples of properly elliptic surfaces of Kähler type, homotopically equivalent to a $K3$ surface, by performing two logarithmic transformations on two smooth fibers with relatively prime multiplicities on such elliptic $K3$. Let M' to be any such surface, and let M_k and M'_k be the blowing-ups at k distinct points of M and M', respectively.

As before, let Σ be a Riemann surface.

- If $g(\Sigma) = 1$, the Cartesian products $M_k \times \Sigma$ and $M'_k \times \Sigma$ will provide infinitely many pairs of diffeomorphic Kähler 3-folds of Kodaira dimensions 0 and 1, respectively.

- If $g(\Sigma) \geq 2$, we obtain pairs in Kodaira dimensions 1 and 2, respectively.

Again, the statement about the Chern numbers follows from Proposition 1.13. □

Example 3: *Pairs of Kodaira dimensions* $(-\infty, 2)$ *and* $(-\infty, 3)$

Arguing as before, we present a pair of simply connected, $h-$cobordant projective surfaces, one on Kodaira dimension 2, and the other one of Kodaira dimension $-\infty$.

Let M be the *Barlow surface* [Bar85]. This is a non-spin, simply connected projective surface of general type, with $p_g = 0$ and $c_1^2(M) = 1$. It is therefore h-cobordant to M', the projective plane $\mathbb{CP}_2$ blown-up at 8 points.

By taking the Cartesian product of their blowing-ups by a Riemann surface of genus 1, we obtain diffeomorphic, projective threefolds of Kodaira dimensions 3, and $-\infty$, respectively, while for a Riemann surface of bigger genus, we obtain diffeomorphic, projective threefolds of Kodaira dimensions 2, and $-\infty$, respectively. The invariance of their Chern numbers follows as usual. □

Example 4: *Pairs of Kodaira dimensions* $(0, 2)$ *and* $(1, 3)$

Following [Cat78], we will describe an example of simply connected, minimal surface of general type with $c_1^2 = p_g = 1$.

In $\mathbb{CP}_2$ we consider two generic smooth cubics F_1 and F_2, which meet transversally at 9 distinct points, $x_1, \cdots, x_9$, and let

$$\sigma : \widetilde{X} \to \mathbb{CP}_2$$

be the blowing-up of $\mathbb{CP}_2$ at $x_1, \cdots, x_9$, with exceptional divisors $\widetilde{E}_i,\ i = 1, ..., 9$. Let $\widetilde{F}_1$ and $\widetilde{F}_2$ be the strict transforms of F_1 and F_2, respectively. Then $\widetilde{F}_1$ and $\widetilde{F}_2$ are two disjoint, smooth divisors, and we can easily see that

$$\mathcal{O}_{\widetilde{X}}(\widetilde{F}_1 + \widetilde{F}_2) = \widetilde{\mathcal{L}}^{\otimes 2},$$

where

$$\widetilde{\mathcal{L}} = \sigma^* \mathcal{O}_{\mathbb{CP}_2}(3) \otimes \mathcal{O}_{\widetilde{X}}(\widetilde{E}_1 + \cdots + \widetilde{E}_9).$$

Let $\pi : \bar{X} \to \widetilde{X}$ to be the double covering of $\widetilde{X}$ branched along the smooth divisor $\widetilde{F}_1 + \widetilde{F}_2$. We denote by

$$p : \bar{X} \to \mathbb{CP}_2$$

the composition $\sigma \circ \pi$, and by $\bar{F}_1,\ \bar{F}_2$ the reduced divisors $\pi^{-1}(\widetilde{F}_1)$, and $\pi^{-1}(\widetilde{F}_2)$, respectively. Since each $\widetilde{E}_i$ intersects the branch locus at 2 distinct points, we can see that for each $i = 1, \ldots, 9,\ \bar{E}_i = \pi^{-1}(\widetilde{E}_i)$ is a smooth (-2)-curve such that

$$\pi_{|\bar{E}_i} : \bar{E}_i \to \widetilde{E}_i$$

is the double covering of $\widetilde{E}_i$ branched at the two intersection points of $\widetilde{E}_1$ with $\widetilde{F}_1 + \widetilde{F}_2$. As the $\widetilde{E}_i$'s are mutually disjoint, the $\bar{E}_i$'s will also be mutually disjoint.

Similarly, if ℓ is a line in $\mathbb{CP}_2$ not passing through any of the intersection points of F_1 with F_2, then

$$L = p^*(\ell) = p^*\mathcal{O}_{\mathbb{CP}_2}(1)$$

is a smooth curve of genus 2, not intersecting any of the $\bar{E}_i$'s. Since

$$p^*\mathcal{O}_{\mathbb{CP}_2}(3) = \mathcal{O}_{\bar{X}}(2\bar{F}_1 + \bar{E}_1 + \cdots + \bar{E}_9),$$

we can write as before

$$\mathcal{O}_{\bar{X}}(L + \bar{E}_1 + \cdots + \bar{E}_9) = \bar{\mathcal{L}}^{\otimes 2},$$

where

$$\bar{\mathcal{L}} = p^*\mathcal{O}_{\mathbb{CP}_2}(2) \otimes \mathcal{O}_{\bar{X}}(-\bar{F}_1).$$

Let now $\phi : \bar{S} \to \bar{X}$ be the double covering of $\bar{X}$ ramified along the smooth divisor

$$L + \bar{E}_1 + \cdots + \bar{E}_9.$$

The surface $\bar{S}$ is non-minimal with exactly 9 disjoint exceptional curves of the first kind, the reduced divisors $\phi^{-1}(\bar{E}_i),\ i = 1, \ldots 9$. The surface S we were looking for is obtained from $\bar{S}$ by blowing down these 9 exceptional curves.

Lemma 1.14. *S is a simply connected, minimal surface with*

$$c_1^2(S) = p_g(S) = 1.$$

Proof. As S is obtained from $\bar{S}$ by blowing-down 9 exceptional curves,

$$c_1^2(S) = c_1^2(\bar{S}) + 9.$$

The canonical line bundle $\mathcal{K}_{\bar{S}}$ of $\bar{S}$ as a double covering of $\bar{X}$ is [BPV84, Lemma 17.1, p. 42]:

$$\mathcal{K}_{\bar{S}} = \phi^* \bar{\mathcal{L}},$$

since the canonical bundle of $\bar{X}$ is trivial. The computation of $c_1^2(\bar{S})$ follows again from [BPV84, Lemma 17.1, p. 42], and we have:

$$\begin{aligned}
c_1^2(\bar{S}) &= (\mathcal{K}_{\bar{S}} \cdot \mathcal{K}_{\bar{S}}) = (\phi^*\bar{\mathcal{L}} \cdot \phi^*\bar{\mathcal{L}}) = 2(\bar{\mathcal{L}} \cdot \bar{\mathcal{L}}) \\
&= 2(p^*\mathcal{O}_{\mathbb{CP}_2}(2) \cdot p^*\mathcal{O}_{\mathbb{CP}_2}(2)) - 4(p^*\mathcal{O}_{\mathbb{CP}_2}(2) \cdot \mathcal{O}_{\bar{X}}(\bar{F}_1)) \\
&\quad + 2(\mathcal{O}_{\bar{X}}(\bar{F}_1) \cdot \mathcal{O}_{\bar{X}}(\bar{F}_1)) \\
&= 4(\sigma^*\mathcal{O}_{\mathbb{CP}_2}(2) \cdot \sigma^*\mathcal{O}_{\mathbb{CP}_2}(2)) - 2(\pi^*\sigma^*\mathcal{O}_{\mathbb{CP}_2}(2) \cdot \pi^*\mathcal{O}_{\widetilde{X}}(\widetilde{F}_1)) \\
&\quad + \frac{1}{2}(\pi^*\mathcal{O}_{\widetilde{X}}(\widetilde{F}_1) \cdot \pi^*\mathcal{O}_{\widetilde{X}}(\widetilde{F}_1)) \\
&= 4(\mathcal{O}_{\mathbb{CP}_2}(2) \cdot \mathcal{O}_{\mathbb{CP}_2}(2)) - 4(\sigma^*\mathcal{O}_{\mathbb{CP}_2}(2) \cdot \mathcal{O}_{\widetilde{X}}(\widetilde{F}_1)) \\
&\quad + (\mathcal{O}_{\widetilde{X}}(\widetilde{F}_1) \cdot \mathcal{O}_{\widetilde{X}}(\widetilde{F}_1)) \\
&= 16 - 4(\mathcal{O}_{\mathbb{CP}_2}(2) \cdot \mathcal{O}_{\mathbb{CP}_2}(3)) \\
&= -8.
\end{aligned}$$

Thus $c_1^2(S) = 1$.

To compute $p_g(S)$ using the birational invariance of the plurigenera, it would be the same to compute

$$p_g(\bar{S}) = h^0(\bar{S}, \mathcal{K}_{\bar{S}}) = h^0(\bar{X}, \phi_* \mathcal{K}_{\bar{S}}).$$

Using the projection formula (cf. [BPV84], p. 182), we have:

$$h^0(\bar{X}, \phi_* \mathcal{K}_{\bar{S}}) = h^0(\bar{X}, \phi_* \phi^* \bar{\mathcal{L}}) = h^0(\bar{X}, \mathcal{O}_{\bar{X}}) + h^0(\bar{X}, \bar{\mathcal{L}}) = 1.$$

For the proof of the simply connectedness, we refer the interested reader to [Cat78]. □

Let S'_k be the blowing-up of a $K3$ surface at k distinct points. Let also S_k denote the blowing-up of S at $k+1$ distinct points, and let Σ be a Riemann surface.

- If $g(\Sigma) = 1$, $(S_k \times \Sigma, S'_k \times \Sigma)$ will provide infinitely many pairs of diffeomorphic Kähler threefolds of Kodaira dimensions 2 and 0, respectively;
- If $g(\Sigma) \geq 2$ we get infinitely many pairs of diffeomorphic Kähler threefolds of Kodaira dimensions 3 and 1, respectively.

The statement about the Chern classes follows as before. □

Example 5: *Pairs of Kodaira dimensions* $(1,2)$ *and* $(2,3)$

In $\mathbb{CP}_1 \times \mathbb{CP}_2$, let M_n be the the generic section of line bundle $p_1^* \mathcal{O}_{\mathbb{CP}_1}(n) \otimes p_2^* \mathcal{O}_{\mathbb{CP}_2}(3)$ for $n \geq 3$, where p_i, $i = 1, 2$ be the projections onto the two factors. Then M_n is a smooth, simply connected projective surface, and using the projection onto the first factor we see that M_n is a properly elliptic surface. By the adjunction formula, the canonical line bundle is:

$$K_{M_n} = p_1^* \mathcal{O}_{\mathbb{CP}_1}(n-2).$$

From this and the projection formula we can find the purigenera:

$$\begin{aligned} P_m(M_n) &= h^0(M_n, K_{M_n}^{\otimes m}) = h^0(M_n, p_1^* \mathcal{O}_{\mathbb{CP}_1}(m(n-2))) \\ &= h^0(\mathbb{CP}_1, p_{1*} p_1^* \mathcal{O}_{\mathbb{CP}_1}(m(n-2))) \\ &= h^0(\mathbb{CP}_1, \mathcal{O}_{\mathbb{CP}_1}(m(n-2))) \\ &= m(n-2)+1. \end{aligned}$$

So, $\mathrm{Kod}(M_n) = 1$, and $p_g(M_n) = n-1$. We can also see that $c_1^2(M_n) = 0$.

Let M' be any smooth sextic in $\mathbb{CP}_3$. M' is a simply connected surface of general type with $p_g(M') = 10$, and $c_1^2(M') = 24$. Let M_k' be the blowing-up of M at $24+k$ distinct points, M_k be the blowing-up of M_{11} at $k+1$ points, and let Σ be a Riemann surface. If $g(\Sigma) = 1$, $(M_k \times \Sigma, M_k' \times \Sigma)$ will provide infinitely many pairs of diffeomorphic Kähler threefolds of Kodaira dimensions 1 and 2, respectively, while if $g(\Sigma) \geq 2$ we get infinitely many pairs of diffeomorphic Kähler threefolds of Kodaira dimensions 2 and 3, respectively. The statement about the Chern classes again follows. □

1.5 Deformation Type

Similar idea can be used to prove **Theorem B**. The proof follows from the examples below.

Example 1: *Kodaira dimension* $-\infty$

Here we use again the Barlow surface M, and M', the blowing-up of $\mathbb{CP}_2$ at 8 points as two h-cobordant complex surfaces. Let S_k and S'_k denote the blowing-ups of M and M', respectively at k distinct points. Then, by the classical h-cobordism theorem, $X_k = S_k \times \mathbb{CP}_1$ and $X'_k = S'_k \times \mathbb{CP}_1$ are two diffeomorphic 3-folds with the same Kodaira dimension $-\infty$. The fact that X_k and X'_k are not deformation equivalent follows as in [Ruan94] from Kodaira's stability theorem [Kod63]. We also see immediately that they have the same Chern numbers. $\square$

Example 2: *Kodaira dimension* 2 *and* 3

We start with a *Horikawa surface*, namely a simply connected surface of general type M with $c_1^2(M) = 16$ and $p_g(M) = 10$. An example of such surface can be obtained as a ramified double cover of $Y = \mathbb{CP}_1 \times \mathbb{CP}_1$ branched at a generic curve of bi-degree $(6, 12)$. If we denote by $p : M \to Y$, its degree 2 morphism onto Y, then the canonical bundle of M is $K_M = \mathcal{O}_Y(1, 4)$, see [BPV84, page 182]. Here by $\mathcal{O}_Y(a, b)$ we denote the line bundle $p_1^*\mathcal{O}_{\mathbb{CP}_1}(a) \otimes p_2^*\mathcal{O}_{\mathbb{CP}_1}(b)$, where $p_i,\ 1 = 1, 2$ are the projections of Y onto the two factors. Notice that the formula for the canonical bundle shows that M is not spin.

Lemma 1.15. *The plurigenera of M are given by:*

$$P_n(M) = \begin{cases} 10 & n = 1 \\ 8n^2 - 8n + 11 & n \geq 2 \end{cases}$$

Proof. Cf. [BPV84] we have $p_*\mathcal{O}_M = \mathcal{O}_Y \oplus \mathcal{O}_Y(-3, -6)$. We have:

$$\begin{aligned} P_n(M) &= h^0(M, p^*\mathcal{O}_Y(n, 4n)) = h^0(Y, p_*p^*\mathcal{O}_Y(n, 4n)) \\ &= h^0(Y, \mathcal{O}_Y(n, 4n) \otimes p_*\mathcal{O}_M) \\ &= h^0(Y, \mathcal{O}_Y(n, 4n)) + h^0(Y, \mathcal{O}_Y(n-3, 4n-6)). \end{aligned}$$

Now, if $n < 3$ we get $P_n(M) = (n+1)(4n+1)$. In particular, $p_g(M) = 10$ and $P_2(M) = 27$. If $n \geq 3$, $P_n(M) = (n+1)(4n+1) + (n-2)(4n-5) = 8n^2 - 8n + 11$. □

Let $M' \subset \mathbb{CP}_3$ be a smooth sextic. The adjunction formula will provide again the the canonical bundle $K_{M'} = \mathcal{O}_{M'}(2)$ and so $c_1^2(M') = 24$.

Lemma 1.16. *The plurigenera of M' are given by:*

$$P_n(M') = \begin{cases} \binom{2n+3}{3} & n = 1, 2 \\ 12n^2 - 12n + 11 & n \geq 3 \end{cases}$$

Proof. From the exact sequence $0 \to \mathcal{O}_{\mathbb{CP}_3}(2n-6) \to \mathcal{O}_{\mathbb{CP}_3}(2n) \to K_{M'}^{\otimes n} \to 0$, we get:

$$0 \to H^0(\mathbb{CP}_3, \mathcal{O}_{\mathbb{CP}_3}(2n-6)) \to H^0(\mathbb{CP}_3, \mathcal{O}_{\mathbb{CP}_3}(2n))$$
$$\to H^0(M', K_{M'}^{\otimes n}) \to H^1(\mathbb{CP}_3, \mathcal{O}_{\mathbb{CP}_3}(2n)) = 0.$$

So, for $n \geq 3$,

$$P_n(M') = \binom{2n+3}{3} - \binom{2n-3}{3} = 12n^2 - 12n + 11,$$

while for $n < 3$, $P_n(M') = \binom{2n+3}{3}$. In particular, $p_g(M') = 10$ and $P_2(M') = 35$. □

Let M_k be the blowing-up of M at k distinct points, M'_k be the blowing-up of M' at $8+k$ distinct points, and let Σ be a Riemann surface. If $g(\Sigma) = 1$, $(M_k \times \Sigma, M'_k \times \Sigma), k \geq 0$ will provide the required examples of Kodaira dimension 2, and if $g(\Sigma) \geq 2$, will provide the required examples of Kodaira dimension 3.

To prove that they are not deformation equivalent we will use the deformation invariance of plurigenera theorem [KoMo92, page 535]. Because of the their multiplicative property cf. Proposition 1.12, it will suffice to look at the plurigenera of M and M'. But, $P_2(M) = 27$ and $P_2(M') = P_2(S) = 35$, and so $M \times \Sigma$ and $M' \times \Sigma$ are not deformation equivalent.

The statement about the Chern numbers of this examples follows immediately. □

Example 3: *Kodaira dimension* 1

Here we use again the elliptic surfaces $\pi : M_{p,q} \to \mathbb{CP}_1$ obtained from the rational elliptic surface by applying logarithmic transformations on two smooth fibers, with relatively prime multiplicities p and q. From (1.1) we get $K_{M_{p,q}}^{\otimes pq} = p^*\mathcal{O}_{\mathbb{CP}_1}((pq-p-q))$. Hence $P_{pq}(M_{p,q}) = pq-p-q+1,$ while if $n \leq pq,\ P_n(M_{p,q}) = 0$, the class of F being a primitive element in $H^2(M_{p,q}, \mathbb{Z})$, cf. [Kod70]. It is easy to see now that, for example, if $(p,q) \neq (2,3),\ P_6(M_{p,q}) \neq P_6(M_{2,3})$. If Σ is any smooth elliptic curve, the $3-$folds $X_{p,q} = M_{p,q} \times \Sigma$ will provide infinitely many diffeomorphic Kähler threefolds of Kodaira dimension 1. Corollary 1.13 shows again that all these threefolds have the same Chern numbers. The above computation of plurigenera shows that, in general, the $X_{p,q}$'s have different plurigenera. Hence, these Kähler threefolds are not deformation equivalent. □

Example 4: *Kodaira dimension* 0

Here we are supposed to start with a simply connected minimal surface of zero Kodaira dimension. But, up to diffeomorphisms there exists only one [BPV84], the $K3$ surface. So our method fails to produce examples in this case. However, M. Gross constructed [Gro97] a pair of diffeomorphic complex threefolds with trivial canonical bundle, which are *not* deformation equivalent. For the sake of completeness we will briefly recall his examples.

Let $E_1 = \mathcal{O}_{\mathbb{CP}_1}^{\oplus 4}$ and $E_2 = \mathcal{O}_{\mathbb{CP}_1}(-1) \oplus \mathcal{O}_{\mathbb{CP}_1}^{\oplus 2}(1) \oplus \mathcal{O}_{\mathbb{CP}_1}$ be the two rank 4 vector bundles over $\mathbb{CP}_1$, and consider $X_1 = \mathbb{P}(E_1)$ and $X_2 = \mathbb{P}(E_2)$ their projectivizations. Note that E_2 deforms to E_1. Let $M_i \in |-K_{X_i}|,\ i = 1,2$ general anticanonical divisors. The adjunction formula immediately shows that $K_{M_i} = 0,\ i = 1,2$, and so M_1 and M_2 have zero Kodaira dimension. While for M_1 is easy to see that can be chosen to be smooth, simply connected and with no torsion in cohomology, Gross shows [Gro97], [Ruan96] that the same holds for M_2. Moreover, the two 3-folds have the same topological invariants, (the second cohomology group, the Euler characteristic, the cubic form, and the first Pontrjaghin class), and so, cf. [OkVdV95], are diffeomorphic. To show that M_1 and M_2, are not deformation equivalent, note that M_2 contains a smooth rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, which is stable under the deformation of the complex structure while M_1, doesn't. Obviously, M_1 and M_2 have the same Chern numbers. By blowing them up simultaneously at k distinct points, we obtain infinitely many pairs of diffeomorphic, projective threefolds of zero Kodaira dimension with the same Chern numbers. □

1.6 Concluding Remarks

1. Let M and M' be any of the pairs of complex surfaces discussed in the previous two sections. A simple inspection shows that they are not spin, and so, their intersection forms will have the form $m\langle 1\rangle \oplus n\langle -1\rangle$. By a result of Wall [Wall62], if $m, n \geq 2$, the intersection form is transitive on the primitive characteristic elements of fixed square. Since, c_1 is characteristic, if it is primitive too, we can assume that the homotopy equivalence $f : M \to M'$ given by an automorphism of such intersection form will carry the first

Chern class of M' into the first Chern class of M. But this implies that the $h-$cobordism constructed between $X = M \times \Sigma$ and $X' = M' \times \Sigma$ also preserves the first Chern classes.

Following Ruan [Ruan94], we can arrange our examples such that c_1 is a primitive class. In the cases when $b_+ > 1$, which is equivalent to $p_g > 0$, it follows that there exists a diffeomorphism $F : X \to X'$ such that $F^* c_1(X') = c_1(X)$, where $F^* : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is the isomorphism induced by F. In these cases our theorems provide either examples of pairs of diffeomorphic Kähler threefolds, with the same Chern classes, but with different Kodaira dimensions, or examples of pairs of non deformation equivalent, diffeomorphic Kähler threefolds, with same Chern classes and of the same Kodaira dimension.

However, in some cases we are forced to consider surfaces with $b_+ = 1$. In these cases it is not clear whether one can arrange the $h-$cobordisms constructed between $X = M \times \Sigma$ and $X' = M' \times \Sigma$ also preserves the first Chern classes.

2. With our method it is impossible to provide examples of diffeomorphic 3-folds of Kodaira dimensions $(0, 3)$ and $(-\infty, 0)$. In the first case, our method fails for obvious reasons. In the second case, the reason is that for a projective surface of Kodaira dimension $-\infty$, the geometric genus p_g is 0, while for a simply connected projective surface of Kodaira dimension 0, $p_g \neq 0$. Thus, any two surfaces of these dimensions will have different b_+, which is preserved under blow-ups. So, no pair of projective surfaces of these Kodaira dimensions can be h-cobordant. However, this raises the following question:

Question 1.17. *Are there examples of pairs of diffeomorphic, projective* 3-

folds (M, M') *of Kodaira dimensions* $(0, 3)$ *or* $(-\infty, 0)$?

Most of the examples exhibited here have the fundamental group of a Riemann surface. Natural questions to ask would be the following:

Question 1.18. *Are there examples of diffeomorphic,* simply connected*, complex, projective* $3-$*folds of different Kodaira dimension?*

Question 1.19. *Are there examples of projective,* simply connected*, diffeomorphic, but not deformation equivalent* 3*-manifolds with the same Kodaira dimension?*

As we showed, the answer is *yes* when the Kodaira dimensions is $-\infty$ or 0, but we are not aware of such examples in the other cases.

Chapter 2

The total scalar curvature of rationally connected threefolds

2.1 Introduction

In the second part of my thesis we address the following question:

Question 2.1. *Let X be a smooth complex n-fold of Kähler type and negative Kodaira dimension. Does X admit Kähler metrics of positive total scalar curvature?*

If we denote by s_g and $d\mu_g$ the scalar curvature and the volume form of g, respectively, this is the same as asking if there is any Kähler metric g on X such that $\int_X s_g d\mu_g > 0$.

For Kähler metrics, the total scalar curvature has a simpler expression :

$$\int_X s_g d\mu_g = 2\pi n c_1(X) \cup [\omega]^{n-1} \tag{2.1}$$

where $[\omega]$ is the cohomology class of the Kähler form of g. The negativity of the Kodaira dimension is a necessary condition [Yau74] because, arguing by contradiction, if for some $m > 0$, the m^{th} power of the canonical bundle of X is either trivial or has sections one can immediately see that $c_1(X) \cup [\omega]^{n-1}$ is negative, which, by (2.1), would imply that the total scalar curvature of (X, g) was negative.

Question 2.1 has an immediate positive answer in dimension 1. The only smooth complex curve of negative Kodaira dimension is $\mathbb{P}_1$, and the Fubini-Study metric satisfies the required inequality. In complex dimension 2, a positive answer was given by Yau in [Yau74]. His proof is based on the theory of minimal models and the classification of Kähler surfaces of negative Kodaira dimension to find the required metrics on the minimal models. Then he proved that if on a given smooth surface such a metric exists, one can find Kähler metrics of positive total scalar curvature on any of its blowing-ups. Moreover, the metrics he found are Hodge metrics.

Inspired by Yau's approach, we tackle Question 2.1 in the case of projective threefolds of negative Kodaira dimension, where a satisfactory theory of minimal models exists. As in [Yau74], we look for Hodge metrics instead. In this context the natural question to ask is [1]:

Question 2.2. *Let X be a smooth projective 3-fold, with* $\mathrm{Kod}(X) = -\infty$. *Is there any ample line bundle H on X such that $K_X \cdot H^2 < 0$?*

A positive answer to this question can be connected to a deep result of S. Mori and Y. Miyaoka [MiyMo86]. Namely, Question 2.2 can be regarded as a

[1] The same question has also been raised in a different context by F. Campana, J. P. Demailly, T. Peternell and M. Schneider, [DPS96], [CaPe98].

possible effective characterization of the class of smooth, projective threefolds of negative Kodaira dimension.

Also, Question 2.2 can be viewed as extracting some positivity property of the anticanonical bundle. An affirmative answer would yield in dimension three, a weak alternative to the generic semi-positivity theorem of Miyaoka asserting that, for non-uniruled manifolds, the restriction of the cotangent bundle to a general smooth complete intersection curve cut out by elements of $|mH|$ is semi-positive, for any ample divisor H and $m \gg 0$.

As an attempt to answer the original Question 2.1, in this thesis we give a partial positive answer to Question 2.2 in the case of rationally connected threefolds. Recall that cf. [KMM92], a complex projective manifold X of dimension $n \geq 2$ is called *rationally connected* if there is a rational curve passing through any two given points of X. Our result is:

Theorem A. *For every projective, rationally connected manifold X of dimension 3, there exists an ample line bundle H on X such that $K_X \cdot H^2 < 0$.*

One reason to restrict our attention to this important case comes from the observation that for rationally connected manifolds, answering affirmatively to Question 2.1 is equivalent to answering affirmatively to Question 2.2. This is automatically true for surfaces, but is a non-trivial issue in higher dimensions. This follows from their convenient cohomological properties. Namely, if X is such a manifold, then [KMM92]

$$H^i(X, \mathcal{O}_X) = 0, \text{ for } i \geq 1.$$

But in this case, from the Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X)$$

it follows that $H^2(X, \mathbb{C}) \simeq H^{1,1}(X)$. Thus, we can see that any $(1,1)-$form with real coefficients can be approximated [2] by a $(1,1)-$form with rational coefficients. Hence, up to multiplication by positive integers, any Kähler forms can be approximated by first Chern classes of *ample line bundles*. From this the equivalence of our two questions follows easily.

However, this is not the only reason to restrict ourselves to the case of rationally connected threefolds, as it will become apparent from the proof of Theorem A.

In what follows, we outline the proof of Theorem A. To simplify the exposition, for any projective manifold X, We introduce the following definition:

Definition 2.3. *Let X be a smooth projective threefolds. We say that the property $\mathcal{P}_X$ holds true if there exists an ample line bundle H on X such that $K_X \cdot H^2 < 0$.*

Similar to the case of complex surfaces, the idea to prove this theorem is to start with an arbitrary 3-fold X of negative Kodaira dimension, and show that property $\mathcal{P}$ above holds true for its minimal model X_{min}. In the first section, we apply Mori's theory of minimal models on X to get a birational map $f : X \dashrightarrow X_{min}$ to a new three dimensional projective variety X_{min}, with at most $\mathbb{Q}-$factorial terminal singularities, which is either a Mori fiber space,

[2] Here we consider $H^2(X, \mathbb{R})$ as a finite dimensional real vector space, endowed with any metric topology

or has nef canonical bundle.

Deep results Miyaoka [Miy88], [MiPe97] exclude the possibility of X_{min} having nef canonical bundle (nef canonical bundle would imply non-negative Kodaira dimension). Hence, X_{min} has to be a Mori fiber space, i.e X_{min} is either a del Pezzo fibration, a conic bundle or a Fano variety. As in [CaPe98] and [DPS96], it is easy to see that $\mathcal{P}_{X_{min}}$ holds true.

The difficult part of this program is to show that

$$\mathcal{P}_{X_{min}} \Longrightarrow \mathcal{P}_X.$$

As a first step for a better understanding of this problem, in Section 2.3 we prove the following:

Proposition 2.4. *Let $p : X' \to X$ be a resolution of singularities of a projective $\mathbb{Q}-$factorial variety X of dimension three, with terminal singularities. Assume that p is smooth outside the singular locus* $\mathrm{Sing}(X)$. *Then*

$$\mathcal{P}_{X'} \text{ holds true} \Longleftrightarrow \mathcal{P}_X \text{ holds true.}$$

This shows that in order to answer Question 2.2 it is enough to show that $\mathcal{P}$ is a birational property of smooth projective threefolds.

Our approach is to use the weak factorization theorem [AKMW02], which says that any birational map between smooth (projective) manifolds can be decomposed into a finite sequence of blow-ups and blow-downs with nonsingular centers of (projective) manifolds. We prove the following:

Proposition 2.5. *Let $p : Y \to X$ be the blow-up of a smooth, projective 3-fold* *at a point. Then*

$$\mathcal{P}_X \text{ holds true} \Longleftrightarrow \mathcal{P}_Y \text{ holds true.}$$

For the blowing-up along curves the following result is of crucial importance:

Proposition 2.6. *Let $p : Y \to X$ be the blow-up of a smooth, projective 3-fold along a smooth curve C.*

- $\mathcal{P}_X$ holds true $\Longrightarrow \mathcal{P}_Y$ holds true.
- *If $K_X \cdot C < 0$, then*

$$\mathcal{P}_Y \text{ holds true} \Longrightarrow \mathcal{P}_X \text{ holds true.}$$

We should point out that these results do not require the rational connectedness of X.

In the last case to verify, $\mathcal{P}_Y \Longrightarrow \mathcal{P}_X$, where $Y \to X$ is the blowing-up of smooth projective threefolds along smooth curves with $K_X \cdot C \geq 0$, the methods used to prove the previous results do not work anymore. It is the last hurdle, where we use this extra assumption.

The condition $K_X \cdot C < 0$ imposed in the previous proposition can be interpreted, by the Riemann Roch theorem, as saying that the curve C "moves". Our approach is to reduce this case to the case which we have already solved. To be more precise, we are going to find a smooth curve $C' \subset X$ with $K_X \cdot C' < 0$, such that $\mathcal{P}_{Y'}$ holds true, where $Y' \to X$ is the blowing-up of X along C'. What we do in our construction is "forcing C to move", by eventually modifying it, while preserving property $\mathcal{P}$. This is done by a lengthy specialization argument, where we strongly rely on the rational connectedness hypothesis. This argument is inspired from the proof of Noether's theorem of Griffiths and Harris [GH85]. The construction presented in Section 2.4.2, on which all the computations are performed is based on the work of Graber, Harris, Starr [GHS03] and Kollár [ArKo03]. We devote to this specialization argument, the entire Section 2.4.

In Section 2.5 we prove Theorem 1.9. An appendix containing some results used intensively throughout this entire chapter is added for convenience.

Conventions: We work over the field of complex numbers and we use the standard notations and terminology of Hartshorne's Algebraic Geometry book [Har77].

2.2 Minimal models

In this section we introduce the objects which appear in Mori's theory of minimal models and we show that our problem has a positive answer for the "minimal models."

Let X be a variety with $\dim X > 1$, such that K_X is $\mathbb{Q}-$Cartier, i.e. mK_X is Cartier for some positive integer m. If $f : Y \to X$ is a proper birational morphism such that K_Y is a line bundle (e.g. Y is a resolution of X), then mK_Y is linearly equivalent to:

$$f^*(mK_X) + \sum m \cdot a(E_i) \cdot E_i,$$

where the E_i's are the exceptional divisors. Using numerical equivalence, we can divide by m and write:

$$K_Y \equiv_{\mathbb{Q}} f^* K_X + \sum a(E_i) \cdot E_i.$$

Definition 2.7. *We say that X has terminal singularities if for any resolution, and for any i, $a(E_i) > 0$.*

Definition 2.8. *We say that a variety X is $\mathbb{Q}-$factorial if for any Weil divisor D there exist a positive integer m such that mD is a Cartier divisor.*

The minimal model program (MMP) studies the structure of varieties via birational morphisms or birational maps of special types to seemingly simpler varieties. The birational morphisms which appear running the MMP are the following:

Definition 2.9 (divisorial contractions). *Let X be a projective variety with at most $\mathbb{Q}-$factorial singularities. A birational morphism $f : X \to Y$ is called a divisorial contraction if it contracts a divisor, $-K_X$ is $f-$ample and* rank $NS(X) =$ rank $NS(Y) + 1$.

The main difficulty in the higher dimensional minimal model program is the existence of non-divisorial contractions. When the variety X is $\mathbb{Q}-$factorial with only terminal singularities, one may get contractions, called contractions of flipping type $f : X \to Y$, where the exceptional locus E has codimension at least 2, but such that $-K_X$ is $f-$ample and rank $NS(X) =$ rank $NS(Y) + 1$. In this case, K_Y is no longer $\mathbb{Q}-$Cartier. The remedy in dimension 3 is the existence of special birational maps, called flips, which allow to replace X by another $\mathbb{Q}-$factorial variety X^+ with only terminal singularities, but simpler in some sense:

Definition 2.10 (**flips**). *Let $f : X \to Y$ be a flipping contraction as above. A variety X^+ together with a map $f^+ : X^+ \to Y$ is called a flip of f if X^+ is $\mathbb{Q}-$factorial varieties with terminal singularities and K_{X^+} is f^+-ample.*

By abuse of terminology, the birational map $X \dashrightarrow X^+$ will also be called a flip.

The minimal model program starts with an arbitrary projective $\mathbb{Q}$-factorial threefold with at most terminal singularities X on which one applies an suitable sequence of divisorial contractions and flips. To describe the outcome of the MMP, we need to introduce the following definition:

Definition 2.11 (**Mori fiber spaces**). *Let X and Y be two irreducible $\mathbb{Q}-$factorial varieties with terminal singularities, dim $X >$ dim Y, and $f : X \to Y$ a morphism. The triplet (X, Y, f) is called a Mori fiber space if $-K_X$ is $f-$ample, and*

$$\text{rank } NS(X) = \text{rank } NS(Y) + 1.$$

Theorem 2.12 (**Mori**). *Let X be a projective variety with only $\mathbb{Q}-$factorial terminal singularities, and dim $X = 3$. Then there exist a birational map $f : X \dashrightarrow X_{min}$, which is a composition of divisorial contractions and flips, such that either $K_{X_{min}}$ is nef or X_{min} has a Mori fiber space structure.*

In their approach to Question 2.2, Campana and Peternell proved some important cases. Because of the simplicity, we include their proofs.

The following easy lemma will be used frequently throughout the proofs, sometimes without referring to it.

Lemma 2.13. *Let X be a projective $\mathbb{Q}-$factorial variety of dimension 3, with (at most) terminal singularities for which there exists a nef line bundle D with $K_X \cdot D^2 < 0$. Then there exists an ample line bundle H on X such that $K_X \cdot H^2 < 0$.*

Proof. Let L be any ample on X. Then for any positive integer m, $H_m := mD + L$ is an ample line bundle with:

$$K_X \cdot H_m^2 = K_X \cdot (mD + H)^2 = mK_X \cdot D^2 + 2K_X \cdot D \cdot L + K_X \cdot L^2 < 0$$

for $m \gg 0$. □

Proposition 2.14 (Mori fiber spaces). *Let (X, Y, f) be a Mori fiber space, with dim $X = 3$. Then the property $\mathcal{P}_X$ holds true.*

Proof. Since dim $X = 3$ we have 3 cases, according to the dimension of Y :

***Case 1* (dim Y=0)** In this case X is a $\mathbb{Q}-$Fano variety with rank $NS(X) = 1$. In particular, $-mK_X$ is an ample line bundle, for some integer $m > 0$, and the property $\mathcal{P}_X$ follows immediately.

***Case 2* (dim Y=1)** Take L_Y be any ample line bundle on Y. Since $-K_X$ is $f-$ample it follows that $K_X \cdot (f^*L)^2 < 0$, and as f^*L_Y is nef, from Lemma 2.13 we can see that $\mathcal{P}_X$ holds true.

***Case 3* (dim Y=2)** As before we take L_Y be any ample line bundle on Y and H_X be an ample line bundle on X. Then for any positive integer

$H_m := mf^*L + H_X$, is an ample line bundle and we have:

$$K_X \cdot H_m^2 = K_X \cdot (mf^*L + H_X)^2 = K_X \cdot H_X^2 + 2mK_X \cdot H_X \cdot f^*L_Y < 0,$$

for $m \gg 0$, again because $-K_X$ is $f-$ample.

□

Corollary 2.15. *Let X be a $\mathbb{Q}-$factorial, projective variety of dimension three with at most terminal singularities. If* $\mathrm{Kod}(X) = -\infty$ *then $\mathcal{P}_{X_{min}}$ holds true.*

Proof. From Theorem 2.12 we know that X_{min} is either a Mori fiber space, for which $\mathcal{P}_{X_{min}}$ holds true, or $K_{X_{min}}$ is nef. To exclude the second possibility, we note that from Miyaoka's abundance theorem [MiPe97, page 88], in dimension three this would imply that $\mathrm{Kod}(X_{min}) \geq 0$. However, this is impossible since the Kodaira dimension is a birational invariant. □

2.3 Various reductions

Our first reduction takes care of the singularities. Let X be a projective $\mathbb{Q}-$factorial variety X of dimension three, with terminal singularities. By Hironaka's resolution of singularities we can always find resolution $p : X' \to X$ which is an isomorphism outside the singular points of X. We begin by proving the following:

Proposition 2.16. *$\mathcal{P}_{X'}$ holds true if and only if $\mathcal{P}_X$ holds true.*

Proof. Suppose first that $\mathcal{P}_X$ holds true. Hence there exists an ample line bundle L on X such that $K_X \cdot L^2 < 0$, and let $D' = p^*L$. Then D is a nef line bundle on X' and

$$K_{X'} \cdot D'^2 = (p^* K_X + \sum_i a_i E_i) \cdot p^* L \cdot p^* L = K_X \cdot L^2 < 0.$$

Using Lemma 2.13 it follows that $\mathcal{P}_{X'}$ holds also true.

Conversely, suppose now $\mathcal{P}_{X'}$ holds true, and let H' be a an ample line on X' such that $K_{X'} \cdot H'^2 < 0$. Without loss of generality we can assume H' very ample and represented by an irreducible divisor, still denoted by H'. Let $D := p(H')$ its pushforward in X. Following [KoMo98, Lemma 3.39] we have

$$p^* D \equiv_{\mathbb{Q}} H' + \sum_i c_i E_i,$$

where the E_i ‘s are the exceptional divisors of the resolutions and $c_i \geq 0$.

Now, if C is any curve in X, let C' be its strict transform in X' and so $p_* C' = C$. Then, $D \cdot C = D \cdot p_* C' = p^* D \cdot C' = H' \cdot C' + \sum c_i (E_i \cdot C') > 0,\ C'$ not being contained in any of the exceptional divisors. Thus D is a strictly nef divisor. The singularities of X being terminal, $K_{X'} \equiv_{\mathbb{Q}} p^* K_X + \sum a_i E_i$, with $a_i > 0$. Again, since the singularities of X are a finite number of isolated points, $p^* L \cdot E_i = 0$ for any cartier divisor L on X. We immediately obtain:

$$\begin{aligned} K_X \cdot D^2 &= (p^* K_X) \cdot (p^* D)^2 = (p^* K_X) \cdot (H')^2 \\ &= K_{X'} \cdot H'^2 - \sum a_i E_i \cdot H'^2 < 0. \end{aligned}$$

The proposition follows now from Lemma 2.13. □

Proposition 2.16 allows us to interpret the results we proved in the previous section in in the following way. From the minimal models program we obtain a birational map $f : X \dashrightarrow Y$ from a smooth projective threefold X to a singular threefold Y for which $\mathcal{P}_Y$ holds true. We can replace now Y by a *smooth* projective threefold X' for which $\mathcal{P}_{X'}$ holds true. Thus the problem we study reduces to the following:

Question 2.17. *Is $\mathcal{P}$ a birational property of the class of projective threefolds of negative Kodaira dimension?*

This is already a major simplification, because we can use now the weak factorization theorem [AKMW02] of Abramovich, Karu, Matsuki and Włodarczyk:

Theorem 2.18 (Abramovich, Karu, Matsuki, Włodarczyk). *A birational map between projective nonsingular varieties over an algebraically closed field K of characteristic zero is a composite of blowings up and blowings down with smooth centers of smooth projective varieties.*

Therefore, what is left to prove is that the property $\mathcal{P}$ is preserved under blowing-ups and blowing-downs at points and smooth curves, respectively.

2.3.1 Blowing-up at points

Proposition 2.19. *Let $p : Y \to X$ be the blow-up of a smooth, projective 3-fold at a point. Then $\mathcal{P}_X$ holds true if and only if $\mathcal{P}_Y$ holds true.*

Proof. Let E be the exceptional divisor of p. Then by [Har77, Ex. II.8.5], $\mathrm{Pic}(Y) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}[E]$ and $K_Y = p^*K_X + 2E$.

Suppose first that $\mathcal{P}_X$ holds true and let H_X be an ample line bundle on X such that $K_X \cdot H_X^2 < 0$. Then $D_Y \stackrel{\text{def}}{=} p^*H_X$ is a nef line bundle on Y such that $K_Y \cdot D_Y^2 = (p^*K_X + 2E) \cdot p^*H_X \cdot p^*H_X = p^*K_X \cdot p^*H_X \cdot p^*H_X = K_X \cdot H_X^2 < 0$. Using again Lemma 2.13 it follows that $\mathcal{P}_Y$ holds true.

Conversely, suppose that $\mathcal{P}_Y$ holds true, and let H_Y be an ample line bundle on Y such that $K_Y \cdot H_Y^2 < 0$. Then $H_Y = p^*D_X - aE$, for some line bundle $D_X \in \operatorname{Pic}(X)$, and some positive integer a. As in the proof of Proposition 2.16, we can show that D_X is nef and $K_X \cdot D_X^2 < 0$. Let C be any curve in X and let C' be its strict transform in Y. Then $p_*C' = C$, and $D_X \cdot C = D_X \cdot p_*C' = p^*D_X \cdot C' = H_X \cdot C' + aE \cdot C' > 0$, because H_X is ample and C' is not contained in E. Therefore D_X is a nef line bundle and

$$K_X \cdot D_X^2 = p^*K_X \cdot p^*D_X \cdot p^*D_X = p^*K_X \cdot H_Y \cdot H_Y = K_Y \cdot H_Y^2 - 2E \cdot H_Y^2 < 0.$$

Applying again Lemma 2.13 we can conclude the proof of the proposition. □

2.3.2 Blowing-up along curves

In the case of 1-dimensional blowing-up centers, it is easy to prove in one direction:

Proposition 2.20. *Let $p : Y \to X$ be the blow-up of a smooth, projective 3-fold along a smooth curve C. If $\mathcal{P}_X$ holds true then $\mathcal{P}_Y$ holds true.*

Proof. Let H_X be an ample line bundle on X satisfying $K_X \cdot H_X^2 < 0$ and let

$D_Y \stackrel{\text{def}}{=} p^*H_X$. Then D_Y is a nef line bundle on Y and we have

$$K_Y \cdot D_Y^2 = (p^*K_X + E) \cdot p^*H_X \cdot p^*H_X = p^*K_X \cdot p^*H_X \cdot p^*H_X = K_X \cdot H_X^2 < 0.$$

The conclusion follows again from Lemma 2.13. □

For the converse of Proposition 2.20 the following proposition is the key step in our line of argument. Its proof is rather long, but elementary, based on Proposition C.3.

Proposition 2.21. *Let $p : Y \to X$ be the blowing-up of a smooth, projective 3-fold along a smooth curve C such that $K_X \cdot C < 0$. If $\mathcal{P}_Y$ holds true then $\mathcal{P}_X$ holds true.*

Proof. Let H_Y be an ample line bundle on X such that $\mathcal{P}_Y$ holds true. Without loss of generality, we can assume that H_X is very ample. Since $p : Y \to X$ is the blowing-up of X along $C \subset X$, the exceptional divisor $E = \mathbb{P}_C(N^\vee_{C/X})$ will be a ruled surface over C. Let $d = \deg_C(N_{C/X})$, and let g be the genus of C. Let f be a fiber of $p_{|_E} : E \to C$. We denote by a the intersection number $(H_Y \cdot f)$ in the Chow ring $A(Y)$. We can write:

$$H_Y = p^*L_X - aE \tag{2.2}$$

for some line bundle $L_X \in \operatorname{Pic}(X)$. As in the proof of Proposition 2.19, we can check that $L_X \cdot C' > 0$ for any irreducible curve $C' \subset X$, different than C. Let $\tilde{C}$ be the strict transform of C'. Then:

$$L_X \cdot C' = (H_Y + aE) \cdot C' = H_Y \cdot \tilde{C} + aE \cdot \tilde{C} > 0,$$

because $E \cdot \tilde{C} \geq 0$, the curve $\tilde{C}$ is an irreducible curve, obviously not contained in E. Thus, in order to show that L_X is nef we only have to check that $L_X \cdot C \geq 0$. A straightforward application of the projection formula and of Proposition B.2 gives:

$$\begin{aligned}
K_Y \cdot H_Y^2 &= (p^*K_X + E) \cdot (p^*L_X - aE) \cdot (p^*L_X - aE) \\
&= p^*K_X \cdot p^*L_Y \cdot p^*E - 2ap^*K_X \cdot p^*L_X \cdot E + a^2 p^*K_X \cdot E^2 \\
&\quad + E \cdot p^*L_X \cdot p^*L_X - 2aE^2 \cdot p^*L_X + a^2E^3 \\
&= K_X \cdot L_X^2 - 2aE^2 \cdot p^*L_X + a^2 p^*K_X \cdot E^2 + a^2E^3 \\
&= K_X \cdot L_X^2 + 2a(L_X \cdot C) - a^2(K_X \cdot C) - a^2 d \\
&= K_X \cdot L_X^2 + 2a(L_X \cdot C) - a^2(2g-2).
\end{aligned}$$

Observation 2.22. Since $K_Y \cdot H_Y^2 < 0$, to conclude the proof of the Proposition 2.21 it will suffice to prove that

$$2a(L_X \cdot C) - a^2(2g-2) \geq 0, \tag{2.3}$$

because we would obtain:

- $K_X \cdot L_X^2 < 0$,
- $L_X \cdot C \geq 0$, *if* $g \geq 1$.

If $g = 0$, we still have to check that $L_X \cdot C \geq 0$.

For a better understanding of (2.3) the following considerations are necessary.

On $E = \mathbb{P}_C(N^{\vee}_{C/X})$, let C_0 be the section of minimal self-intersection $C_0^2 = -e$. We use $\{C_0, f\}$ as a basis for $\mathrm{Num}_{\mathbb{Z}}(E)$. With respect to this basis:

$$H_{X|_E} \equiv aC_0 + bf, \text{ for } \text{ some } b \in \mathbb{Z} \ ;$$
$$E_{|_E} \equiv xC_0 + yf,$$

where x and y can be determined as follows:

$$-1 = E \cdot f = E_{|_E} \underset{E}{\cdot} f = (xC_0 + yf) \underset{E}{\cdot} f = x;$$
$$-d = E^3 = E_{|_E} \underset{E}{\cdot} E_{|_E} = (-C_0 + yf)^2 = -e - 2y,$$

so $y = \frac{d-e}{2}$. Here we denoted by " $\underset{E}{\cdot}$ " the intersection product on the exceptional smooth divisor E.

Remark 2.23. Note that the two invariants, d and e, of $E = \mathbb{P}_C(N^{\vee}_{C/X})$, have the same parity.

Lemma 2.24. *In the above notations, we have:*

$$2a(L_X \cdot C) - a^2(2g-2) = 2ab - a^2e - a^2(K_X \cdot C). \tag{2.4}$$

Proof. Computing $H_Y \cdot E \cdot p^*L_X$ in two ways, we obtain:

$$H_Y \cdot E \cdot p^*L_X = (L_X \cdot C)L_X \cdot f = a(L_X \cdot C);$$
$$H_Y \cdot E \cdot p^*L_X = H_Y \cdot E \cdot (H_Y + aE) = H_Y^2 \cdot E + aH_Y \cdot E^2.$$

Thus

$$
\begin{aligned}
2a(L_X \cdot C) - a^2(2g-2) &= 2(H_Y^2 \cdot E + aH_Y \cdot E^2) - a^2(2g-2) \\
&= 2(H_Y^2 \cdot E + aH_Y \cdot E^2) - a^2 d - a^2(K_X \cdot C).
\end{aligned}
$$

Furthermore:

$$
\begin{aligned}
2(H_Y^2 \cdot E + aH_Y \cdot E^2) &= 2[(H_{X|_E} \underset{E}{\cdot} H_{X|_E}) + a(H_{X|_E} \underset{E}{\cdot} E_{|_E})] \\
&= 2(aC_0 + bf)^2 + a(aC_0 + bf) \underset{E}{\cdot} [-2C_0 + (d-e)f] \\
&= 2a^2C_0^2 + 4ab - 2a^2C_0^2 - 2ab + a^2(d-e) \\
&= 2ab + a^2(d-e).
\end{aligned}
$$

Therefore $2a(L_X \cdot C) - a^2(2g-2) = 2ab - a^2e - a^2(K_X \cdot C)$. □

We can finish now the proof of Proposition 2.21:

- If $g \geq 0$, by Proposition C.3, we have two subcases:

  i) ***Case* $\mathbf{e} \geq \mathbf{0}$** : Since H_Y is ample, $H_{Y|_E}$ is ample, and so, by Proposition C.3, $a > 0$ and $b > ae$. Remembering that $K_X \cdot C < 0$, from (2.4) we can see that:

$$
\begin{aligned}
2a(L_X \cdot C) - a^2(2g-2) &= 2ab - a^2e - a^2(K_X \cdot C) \\
&> a^2e - a^2(K_X \cdot C) > 0.
\end{aligned}
$$

  By the crucial Observation 2.22 and by Proposition 2.13 we are done.

ii) ***Case* $\mathbf{e < 0}$:** Similarly, since H_Y is ample, $H_{Y|_E}$ is ample, too. Thus, by Proposition C.3, $a > 0$ and $b > \frac{1}{2}ae$. Then:

$$\begin{aligned} 2a(L_X \cdot C) - a^2(2g-2) =& 2ab - a^2e - a^2(K_X \cdot C) \\ >& -a^2(K_X \cdot C) > 0, \end{aligned}$$

and we are again done.

- If $g = 0$ then $e \geq 0$, and in this case it suffices to show that $2ab + a^2(d - e) \geq 0$. Since $H_{Y|_E}$ is ample, $a > 0$ and $b > ae$. We have:

$$2ab + a^2(d-e) > a^2e + a^2d = a^2(e - 2 - K_X \cdot C).$$

So, if $K_X \cdot C \leq -2$ it follows immediately that $L_X \cdot C > 0$, and with the help of Observation 2.22 and Lemma 2.13 we are done again. If $K_X \cdot C = -1$, then $d = -1$ and since d and e have the same parity, $e \geq 1$ and we obtain again $L_X \cdot C > 0$, and we can conclude as above.

With this Proposition 2.21 is completely proved. □

Remark 2.25. The proof of Proposition 2.21 also works when $K_X \cdot C = 0$ and $g > 0$. However, when C is a rational curve $d = \deg N_{C/X} = -2$, and the above arguments show that a possible exception occurs only when $e = 0$, and $0 < b < a$. In this case, $N_{C/X} \simeq \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$, and what fails is only the nefness of L_X.

When $K_X \cdot C > 0$, nothing can be said with the above approach.

This remark inspires the following conjecture:

Conjecture 2.26. *Let Y be the blowing up of of a smooth, projective threefold X along a curve $C \simeq \mathbb{P}_1$ with $N_{C/X} \cong \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$. If the contraction of the exceptional divisor of Y along the "other direction" is projective, then:*

$$\mathcal{P}_Y \text{ holds true} \Longrightarrow \mathcal{P}_X \text{ holds true.}$$

In the next section we will give a proof of a special case of this conjecture as a part of our argument.

2.4 Specialization argument

From what we proved so far, to show that $\mathcal{P}$ is a birational property in the class of smooth, projective threefolds it would be enough to answer affirmatively to the following question:

Question 2.27. *Let $p : X_C \to X$ be the blowing up of smooth projective threefold X along a smooth curve $C \subset X$ with $K_X \cdot C \geq 0$. Suppose that $\mathcal{P}_{X_C}$ holds true. Does $\mathcal{P}_X$ also hold true?*

Proposition 2.21 is inspirational, suggesting that a positive answer is possible if we can replace the blowing-up $p : X_C \to X$ of X along the curve C by the blowing-up $p' : X_{C'} \to X$ of X along a smooth curve $C' \subset X$, but such that $K_X \cdot C' < 0$, as long as we are able to show that $\mathcal{P}_{X_{C'}}$ also holds true. We will show that such an approach works in the case of *rationally connected* projective threefolds.

A more precise description of our strategy to answer Question 2.27, and the outline of the structure of this section is the following. In the next subsection we introduce the results from the theory of rationally connected manifolds we need. Then using the outcome of Theorem 2.31, in subsection 2.4.2, we construct a smooth family over the unit disk $\mathcal{X} \to \Delta$, whose general fiber X_{C_t} is the blowing-up of X along a smooth curve C_t with $K_X \cdot C_t < 0$. The central fiber of this family will be a normal crossing divisor whose irreducible components are smooth rationally connected threefolds. In subsection 2.4.3 we show that any line bundle on the central fiber of $\mathcal{X} \to \Delta$ extends to $\mathcal{X}$. Moreover, if the line bundle on the central fiber is chosen to be ample, its extension restricted to X_{C_t} will also be ample, by eventually shrinking Δ. In subsection 2.4.5, we apply the results obtained in the previous subsection, to show how to construct ample line bundles on one of the components of the central fiber of $\mathcal{X} \to \Delta$. In the next subsection, we show how to use the result of the previous subsection to construct an ample line bundles on the whole central fiber of $\mathcal{X} \to \Delta$. Finally, in the last subsection, we set up the intersection theory of the central fiber, and show that on the central fiber of $\mathcal{X} \to \Delta$ satisfies property $\mathcal{P}$ holds true, which will imply that $\mathcal{P}_{X_{C_t}}$ holds true, too.

2.4.1 Rationally connected manifolds

In this section we collect the necessary information from the theory of rationally connected manifolds. For the definitions and the main results presented we refer the interested reader to [KMM92], [Kol96] and especially to [ArKo03].

Let X denote a complex projective manifold with $\dim X \geq 2$.

Definition 2.28. *A nonsingular, complex, projective variety X will be called* rationally connected *if any pair of points in X can be connected by a rational curve.*

The main properties and characterizations of rationally connected manifolds are summarized in the following:

Theorem 2.29. 1) *Rationally connectedness is a birational property and is invariant under smooth deformations.*

2) *Rationally connected manifolds are simply connected and satisfy*

$$H^0(X, \Omega_X^{\otimes m}) = 0 \text{ for } m > 0 \text{ and } H^i(X, \mathcal{O}_X) = 0 \text{ for } i > 0.$$

3) *X is rationally connected if and only if for any point $x \in X$ there exists a smooth rational curve $L \subset X$ passing through x, with arbitrarily prescribed tangent direction and such that its normal bundle $N_{L|X}$ is ample.*

We should point out that the statement in 3) is not one of the usual characterizations of rationally connectedness. However, it easily follows from [Deb01, page 110].

Definition 2.30. *A* comb *with* n teeth *is a projective curve with $n+1$ irreducible components $C, L_1, \dots, L_n$ such that:*

- *The curves $L_1, \cdots, L_n$ are mutually disjoint, smooth rational curves.*
- *Each L_i, $i \neq 0$ meets C transversely in a single smooth point of C.*

The curve C is called the handle *of the comb, and $L_1, \dots, L_n$ are called the* teeth.

The key result we use is the following theorem of Graber, Harris and Starr [GHS03], which we present in the shape given by J. Kollár [ArKo03]:

Theorem 2.31. *Let X be a smooth, complex, projective variety of dimension at least 3. Let $C \subset X$ be a smooth irreducible curve. Let $L \subset X$ be a rational curve with ample normal bundle intersecting C and let $\mathcal{L}$ be a family of rational curves on X parametrized by a neighborhood of $[L]$ in* Hilb(X). *Then there are curves $L_1, \ldots, L_n \in \mathcal{C}$ such that $C_0 = C \cup L_1 \cup \cdots \cup L_n$ is a comb and satisfies the following conditions:*

1) *The sheaf $N_{C_0/X}$ is generated by the global sections.*

2) $H^1(C_0, N_{C_0/X}) = 0$.

Obviously the hypotheses are fulfilled in the case of rationally connected manifolds.

For a better understanding of this theorem the following corollary [ArKo03] is very useful. Since we consider that its proof gives some useful information about our construction, we include for convenience Kollár's proof.

Corollary 2.32. Hilb(X) *has a unique irreducible component containing $[C_0]$. This component is smooth at $[C_0]$ and a non-empty subset of it parametrizes smooth, irreducible curves in X.*

Proof. Since the curve C_0 is locally complete intersection, its normal sheaf $N_{C_0/X}$ is locally free. We have an exact sequence

$$0 \to N_{C/X} \to N_{C_0/X_{|C}} \to Q \to 0$$

where Q is a torsion sheaf supported at the points $P_i = C \cap L_i$, for $i = 1, \dots n$. Since $N_{C_0/X}$ is globally generated, we can find a global section $s \in H^0(C_0, N_{C_0/X})$ such that, for each i, the restriction of s to a neighborhood of P_i is not in the image of $N_{C/X}$. This means that s corresponds to a first-order deformation of C_0 that smoothes the nodes P_i of C_0. From the vanishing of $H^1(C_0, N_{C_0/X})$ we see that there are no obstructions finding a global deformation of C_0 that smoothes its nodes P_i.

To be more explicit, we choose local holomorphic coordinates, so that near one of its nodes P, C_0 is given by:

$$z_1 z_2 = z_3 = \cdots = z_n = 0.$$

Consider now a general $1-$parameter deformation corresponding to a section of $N_{C_0/X}$ which does not belong to the subspace of $N_{C_0/X,P}$ generated by $z_3, \cdots, z_n$. This deformation will be given by the equations:

$$z_1 z_2 + tf(t, \mathbf{z}) = z_3 + tf_3(t, \mathbf{z}) = \cdots = z_n + tf_n(t, \mathbf{z}),$$

and $f(t, \mathbf{z}) \neq 0$, by assumption. We can change new coordinates $z_1{}' := z_1$, $z_2{}' := z_2$ and $z_i{}' := z_i + tf_i(t, \mathbf{z})$ for $i = 3, \dots n$, to get new, simpler equations:

$$z_1{}' z_2{}' + t(a + F(t, \mathbf{z})) = z_3{}' = \cdots = z_n{}' = 0, \tag{2.5}$$

where $a \neq 0$ and $F(0,0) = 0$. The singular points are given by the equations:

$$z_1{}' + t\frac{\partial F}{\partial z_2{}'} = z_2{}' + t\frac{\partial F}{\partial z_1{}'} = \cdots = z_n{}' = 0.$$

Substituting back these equations into $z_1'z_2' + t(a + F(t, \mathbf{z}) = 0$ we get a new equation for the supposed singular point:

$$ta = -tF(t, \mathbf{z}) - t^2 \frac{\partial F}{\partial z_1'} \frac{\partial F}{\partial z_2'}$$

The latter has no solution for $t \neq 0$ and $t, z_1', z_2', \dots z_n'$ small since $a \neq 0$ and $F(0, 0) = 0$. □

Remark 2.33. Using the implicit function theorem we can change one more time the coordinates in (2.5) such that near the node P_i, C_0 is given by:

$$z_1 z_2 + t = z_3 = \cdots = z_n = 0. \tag{2.6}$$

This change of coordinates is given by $z_i := z_i'$ for $i = 1, \dots, n$, and $t := t(a + F(t, \mathbf{z}))$.

2.4.2 Construction of the specialization

We start with our blowing-up $p : X_C \to X$ of a projective, rationally connected threefold X along a smooth curve $C \subset X$. Let E be the exceptional divisor. We will construct a degeneration having an appropriate blowing-up of X_C as one of the components of the central fiber.

Since X is rationally connected, we can always attach [ArKo03] to the curve $C \subset X$ a finite number of disjoint, smooth rational curves $L_1, \dots L_n \subset X$, with ample normal bundle, meeting C at transversely at exactly one point $P_i = C \cap L_i,\ i = 1, \dots, n$. Using Theorem 2.31 and Corollary 2.32, the comb $C_0 = C \cup L_1 \cup \cdots \cup L_n$ is smoothable for $n \gg 0$. As in the proof of Corollary

2.32 this means that we can find a small deformation of C_0 parametrized by a one dimensional disk $\Delta \subset \mathrm{Hilb}(X)$ centered in $[C_0]$. That is there exists a smooth submanifold $\mathcal{C} \subset X \times \Delta$, such that its projection $\pi : \mathcal{C} \to \Delta$ is flat, and

$$\pi^{-1}(t) = \begin{cases} C_0, & \text{if } t = 0 \\ C_t, & \text{if } t \neq 0, \end{cases}$$

where C_t is a smooth irreducible curve. From Corollary 2.32 and Remark 2.33, in local coordinates chosen w.r.t a neighborhood of the node P_i, π is the projection

$$(z_1, z_2, z_3, t) \mapsto t$$

and $\mathcal{C}$ is given by $z_1 z_2 + t = z_3 = 0$. In these local coordinates, $C \subset \mathcal{C}$ is given by $z_1 = z_3 = t = 0$, and L_i by $z_2 = z_3 = t = 0$.

Let $\varpi : \mathcal{X}_{\mathcal{C}} \to X \times \Delta$ be the blow-up of $X \times \Delta$ along $\mathcal{C}$, and let

$$\Pi : \mathcal{X}_{\mathcal{C}} \to \Delta$$

be the projection onto Δ.

Lemma 2.34. (Structure of $\Pi : \mathcal{X}_{\mathcal{C}} \to \Delta$)

i) *$\mathcal{X}_{\mathcal{C}}$ is a smooth variety, and $\Pi : \mathcal{X}_{\mathcal{C}} \to \Delta$ is a flat, proper family of projective varieties.*

ii) *For $t \neq 0$, $X_{\mathcal{C},t} = \Pi^{-1}(t)$ is the blowing-up of X along C_t, while $X_{\mathcal{C},0} = \Pi^{-1}(0)$ is the blowing-up of X along the ideal sheaf of $C_0 \subset X$.*

Proof. i) This are standard facts about blowing-up, see sections II. 7 and II. 8 of [Har77].

ii) For the proof we can either quote the universal property of the blowing-up, Corollary II.7.15 of [Har77] or use local equations as in the proof of Corollary 2.32. We adopt the latter. The results we want to prove here are of local nature. In a neighborhood of a node of $C_0 \subset U \subset X \times \Delta$, $\mathcal{C}$ is given by the equations:

$$z_1 z_2 + t = z_3 = 0.$$

$\mathcal{X}_{\mathcal{C}|_U} \subset U \times \mathbb{P}_1$ will therefore given by the equations:

$$(z_1 z_2 + t)v = z_3 u, \tag{2.7}$$

where $[u : v]$ are the homogeneous coordinates on $\mathbb{P}_1$, and the conclusion follows now immediately. □

Let $\Pi_0 : X_{\mathcal{C},0} \to X$ denote the blowing-up map of X along the ideal sheaf of C_0.

Lemma 2.35. (Structure of the central fiber $X_{\mathcal{C},0}$)

i) $X_{\mathcal{C},0}$ *has exactly* n *distinct ordinary double points as singularities.*

ii) *The exceptional divisor of* Π_0, *denoted by* E^* *is a union of smooth Weil divisors* $E^*_C,\ E^*_1, \ldots, E^*_n$.

Proof. i) From the arguments used in the above Lemma we can see that the singular points of $X_{\mathcal{C},0}$ can occur only over the singular points of C_0. The type of this singularities can be seen from (2.7) for $t = 0$. It follows that for node of C_0, in the above coordinates, there is exactly one singular point of $X_{\mathcal{C},0}$, which appears in the chart where $v \neq 0$ and is given by the local equation

$$z_1 z_2 = z_3 u', \tag{2.8}$$

where $u' = \frac{u}{v}$.

ii) Since the center of the blowing-up has exactly $n+1$ components, it follows the exceptional divisor of $X_{\mathcal{C},0}$ has $n+1$ components too, one over each of the components of C_0. Using (2.8) the other claims easily follow. $\square$

Let $Q_i \in \mathcal{X}_{\mathcal{C}},\ 1 = 1, \ldots, n$ denote the singular points of $X_{\mathcal{C},0}$.

Remark 2.36. It can be seen that $X_{\mathcal{C},0}$ is a Gorenstein non $\mathbb{Q}-$factorial variety. Hence push-forward arguments, as the ones we used in the previous section cannot be applied.

In order to perform the computation to follow, we need a better understanding of the components E_i^*'s of E^*.

Proposition 2.37. (The components $E_i^*,\ i = 1, \ldots n$)

i) $E_i^* = \mathbb{P}_{L_i}(N^{\vee}_{C_0/\mathcal{X}_{|L_i}})$.

ii) *The conormal bundle of E_i^* is given by the extension:*

$$0 \longrightarrow \mathcal{O}_{E_i^*} \longrightarrow N^{\vee}_{E_i^*/\mathcal{X}_{\mathcal{C}}} \longrightarrow \mathcal{J}_{Q_i} \otimes \mathcal{O}_{E_i^*}(1) \otimes \mathcal{O}_{E_i^*}(f) \longrightarrow 0,$$

where, by $\mathcal{J}_{Q_i}$ we denoted the ideal sheaf of Q_i on E_i^*, $\mathcal{O}_{E_i^*}(1)$ is the dual of the tautological bundle of the ruled surface E_i^*, and f is its fiber.

Proof. i) This is a well-known fact. We include a short proof for convenience. The general theory of blowing-ups tells us that, since $L_i \subset \mathcal{C}$, $E_i^* = \mathbb{P}_{L_i}(N^\vee_{\mathcal{C}/X\times\Delta_{|L_i}})$. To compute $N^\vee_{\mathcal{C}/X\times\Delta_{|L_i}}$ we use the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{L_i} & = & \mathcal{O}_{L_i} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^\vee_{\mathcal{C}/X\times\Delta_{|L_i}} & \longrightarrow & N^\vee_{L_{i/X}} \oplus \mathcal{O}_{L_i} & \longrightarrow & \mathcal{O}_{L_i}(P_i) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^\vee_{C_0/X_{|L_i}} & \longrightarrow & N^\vee_{L_i/X} & \longrightarrow & \mathcal{O}_{P_i}(P_i) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2.9}$$

The first row is given by the exact sequence of conormal bundles of the inclusions $L_i \subset \mathcal{C} \subset X \times \Delta$. We have the obvious isomorphism $N^\vee_{L_{i/X\times\Delta}} \simeq N^\vee_{L_{i/X}} \oplus \mathcal{O}_{L_i}$.

On the smooth surface $\mathcal{C}$, since the L_i's are mutually disjoint *rational curves* and meet C transversally at exactly one point, we have:

$$0 = L_i \cdot C_t = L_i \cdot C_0 = L_i \cdot (C + L_1 + \cdots L_n) = 1 + L_i^2.$$

Therefore, the L_i's are actually $(-1)-$curves and $N^\vee_{L_{i|\mathcal{C}}} \simeq \mathcal{O}_{L_i}(P_i)$.

The second row is the exact sequence of Andreatta-Wisniewski [AnWi98, page 265]. From the snake lemma, we can see now that

$$N^{\vee}_{\mathcal{C}/X\times\Delta_{|L_i}} \simeq N^{\vee}_{C_0/X_{|L_i}}.$$

ii) Let $\bar{F} : \bar{\mathcal{X}} \to \mathcal{X}_{\mathcal{C}}$ be the blowing-up of $\mathcal{X}_{\mathcal{C}}$ at the points Q_i, for $i = 1, \dots, n$, and $\bar{\Pi} : \bar{\mathcal{X}} \to \Delta$ the projection onto Δ. We denote by $\bar{X}$ the strict transform of $X_{\mathcal{C},0}$, and by $Z_i,\ i = 1, \dots, n$, the exceptional divisors of $\bar{F}$. $\bar{\Pi}$ has the following fibers:

$$\bar{\Pi}^{-1}(t) = \begin{cases} X_{C_t}, & \text{if } t = 0 \\ \bar{X} + 2Z_1 + \dots 2Z_n, & \text{if } t \neq 0. \end{cases}$$

Since $\mathcal{X}_{\mathcal{C}}$ is smooth, we have $Z_i \simeq \mathbb{P}_3$, and $N_{Z_i/\bar{\mathcal{X}}} \simeq \mathcal{O}_{\mathbb{P}_3}(-1)$. The multiplicities of the Z_i's in the central fiber are caused by the ordinary double point singularities of $X_{\mathcal{C},0}$. The reduced component $\bar{X}$ is a big resolution of $X_{\mathcal{C},0}$. The induced map $\bar{X} \to X_{\mathcal{C},0}$ has n exceptional divisors, $T_i = \bar{X} \cap Z_i,\ i = 1, \dots, n$, each of them isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$, with $N_{T_i/\bar{X}} \simeq \mathcal{O}(-1,-1)$. Moreover, $N_{T_i/Z_i} \simeq \mathcal{O}(1,1)$, for all $i = 1, \dots, n$.

To compute the conormal bundle $N^{\vee}_{E_i^*/\mathcal{X}_{\mathcal{C}}}$, we need a good understanding of the main component $\bar{X}$ of $\bar{\Pi}^{-1}(0)$. Let $\bar{p} : \bar{X} \to X$ be the natural morphism onto X. This has the following alternative description :

- Consider $p_L : X_L \to X$, the blowing up of X, along the disjoint union of curves $L_1, \dots, L_n$. Let $E_1, \dots, E_n$ denote the exceptional divisors and $\bar{C}$ denote the strict transform of C and $\{x_i\} = \bar{C} \cap E_i$. The E_i's are rational ruled surfaces over L_i, $E_i = \mathbb{P}_{L_i}(N^{\vee}_{L_i/X})$, with $N_{E_i/X_L} \simeq \mathcal{O}_{E_i}(-1)$. Consider $f_i \in E_i$, the fiber of E_i through x_i, for all $i = 1, \dots, n$.

- Let $p_{\bar{C}} : X_{L,\bar{C}} \rightarrow X_L$ be the blowing-up of X_L along $\bar{C}$. We denote by $E_{\bar{C}}$ the exceptional divisor, and by $\bar{E}_i$ the strict transforms of E_i, for all $i = 1, \ldots, n$. Each of the $\bar{E}_i{}'$s is the blowing-up of E_i at x_i. Let ℓ_i denote the exceptional divisor of these blowing-up. $E_{\bar{C}}$ and $\bar{E}_i$ meet transversally along ℓ_i, and ℓ_i sits in $E_{\bar{C}}$ as fiber. Moreover, we have $N_{\bar{E}_i/X_{L,\bar{C}}} = p_{\bar{C}}^* N_{E_i/X_L}$ (see [Ful98]). In each $\bar{E}_i$, we denote by $\bar{f}_i$ the strict transform of f_i.

  We can immediately see that $N_{\bar{E}_i/X_{L,\bar{C}|\bar{f}_i}} \simeq \mathcal{O}_{\mathbb{P}_1}(-1)$, and the exact sequence:

  $$0 \rightarrow N_{\bar{f}_i/\bar{E}_i} \rightarrow N_{\bar{f}_i/X_{L,\bar{C}}} \rightarrow N_{\bar{E}_i/X_{L,\bar{C}|\bar{f}_i}} \rightarrow 0$$

  yields

  $$N_{\bar{f}_i/X_{L,\bar{C}}} \simeq \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1).$$

- We blow-up now $X_{L,\bar{C}}$ along $\bar{f}_i$, for all $i = 1, \ldots, n$. The resulting $3-$fold is isomorphic to $\bar{X}$, where the exceptional divisors of the last blowing-up coincide with T_i, $i = 1, \ldots, n$. Let $p_{\bar{F}} : \bar{X} \rightarrow X_{L,\bar{C}}$ be the blowing-up map. The map $\bar{p}$ is the composition:

  $$\bar{p} = p_L \circ p_{\bar{C}} \circ p_{\bar{F}}.$$

  Denote by $R_i = p_{\bar{F}}^* \bar{E}_i - T_i$ the strict transforms of $\bar{E}_i$. Since $\bar{f}_i \subset \bar{E}_i$, R_i is isomorphic to $\bar{E}_i$, the blowing-up of of E_i at x_i. Let $\bar{E}$ be the strict transform of $E_{\bar{C}}$, and $\bar{\ell}_i$ be the strict transform of ℓ_i, for all $i = 1, \ldots, n$. $\bar{E}$ is isomorphic to $E_{\bar{C}}$ blown-up at the intersection points of $E_{\bar{C}}$ with the curves $\bar{f}_i$, and intersects R_i transversally along $\bar{\ell}_i$ for all $i = 1, \ldots, n$.

First we need to determine $N_{R_i/\bar{X}}$. To do this, we have to analyze more closely the position of the exceptional divisors of the map $\bar{p}$.

- R_i and T_i meet transversally along h_i, one of the rulings of T_i which coincides with $\bar{f}_i$, under the identification of R_i with $\bar{E}_i$;
- $\bar{E}$ and T_i meet transversally along k_i, the other ruling of T_i, for all $i = 1, \ldots, n$;
- $\bar{E} \cap R_i \cap T_i = \{\text{point}\}$, for all $i = 1, \ldots, n$;
- $R_i \cap R_j = \emptyset$, for $i \neq j$.

Let $p_i : R_i \to E_i$ be the blowing up of E_i at x_i, where h_i is the strict transform of the fiber through x_i, and $\bar{\ell}_i$ denotes the exceptional divisor. Using i) we can see that E_i^* is actually the elementary transform of E_i centered at x_i. Consequently, we denote by $q_i : R_i \to E_i^*$, the blowing-down of $\bar{f}_i$, for every $i = 1, \ldots, n$.

Claim 2.37.1. $N_{R_i/\bar{X}} \simeq \mathcal{O}_{R_i}(-h_i) \otimes p_i^* \mathcal{O}_{E_i}(-1)$.

Proof of Claim. Since R_i is the blowing-up of E_i, we can write $N_{R_i/\bar{X}}$ as

$$\mathcal{O}_{R_i}(ah_i) \otimes p_i^* \mathcal{O}_{E_i}(b) \otimes p_i^* \mathcal{O}_{E_i}(cf),$$

where f is the generic fiber of the ruled surface E_i. Let $d_i = \deg N_{L_i/X}$. Let also denote by $\bar{f}$ the strict transform in $\bar{\mathcal{X}}$ of the generic fiber of E_i. From the fact that $R_i = p_{\bar F}^* \bar E_i - T_i$ and the projection formula, we compute:

$$\begin{aligned}
R_i \cdot \bar f &= (p_{\bar F}^* \bar E_i - T_i) \cdot \bar f = E_i \cdot f = -1;\\
R_i \cdot \bar\ell_i &= (p_{\bar F}^* \bar E_i - T_i) \cdot \bar\ell_i = p_{\bar C}^* E_i \cdot \ell_i - T_i \cdot \bar\ell_i = -1;\\
R_i^3 &= (p_{\bar F}^* \bar E_i - T_i)^3 = \bar{E_i}^3 - 3p_{\bar F}^* \bar E_i \cdot p_{\bar F}^* \bar E_i \cdot T_i + 3p_{\bar F}^* \bar E_i \cdot T_i^2 - T_i^3\\
&= E_i^3 + 3(\bar E_i \cdot \bar f_i) \cdot (T_i \cdot k_i) + 2 = -d_i + 3 - 2 = -d_i + 1.
\end{aligned}$$

On the other hand, computing on the surface R_i, we have:

- $R_i \cdot \bar f = (ah_i + p_i^* \mathcal{O}_{E_i}(b) + cp_i^* f) \cdot p_i^* f = b$, and so $b = -1$.
- $R_i \cdot \bar\ell_i = (ah_i + p_i^* \mathcal{O}_{E_i}(b) + cp_i^* f) \cdot \bar\ell_i = a(p_i^* f - \bar\ell_i) \cdot \bar\ell_i = a$, and so $a = -1$;
- $R_i^3 = (-h_i - p_i^* \mathcal{O}_{E_i}(1) + cp_i^* f)^2 = -1 - d_i + 2 - 2c = -d_i + 1 - 2c$, and so $c = 0$.

□

We compute now $N^\vee_{R_i/\bar{\mathcal{X}}}$ from the conormal sequence of the inclusions $R_i \subset \bar X \subset \bar{\mathcal{X}}$:

$$0 \to N^\vee_{\bar X/\bar{\mathcal{X}}_{|R_i}} \to N^\vee_{R_i/\bar{\mathcal{X}}} \to N^\vee_{R_i/\bar X} \to 0. \tag{2.10}$$

In $\bar{\mathcal{X}}$, we have $\bar X + 2Z_1 + \cdots 2Z_n \sim 0$, (linearly equivalence) and so

$$N^\vee_{\bar X/\bar{\mathcal{X}}} \simeq \mathcal{O}_{\bar X}(2T_1 + \cdots 2T_n).$$

Tensoring by $\mathcal{O}_{R_i}$, we get $N^\vee_{\bar X/\bar{\mathcal{X}}_{|R_i}} \simeq \mathcal{O}_{R_i}(2h_i)$. Hence we obtained:

$$0 \to \mathcal{O}_{R_i}(2h_i) \to N^\vee_{R_i/\bar{\mathcal{X}}} \to \mathcal{O}_{R_i}(h_i) \otimes p_i^* \mathcal{O}_{E_i}(1) \to 0. \tag{2.11}$$

On the other hand, since $\mathcal{X}_{\mathcal{C}}$ and E_i^* are smooth, R_i is the strict transform of E_i^* in $\bar{\mathcal{X}}$. Moreover, the restriction of blowing-up map $\bar{F}$ to R_i coincides with the blowing-up q_i with h_i as exceptional divisor. Therefore, by [Ful98, page 437],

$$N_{R_i/\bar{\mathcal{X}}} \simeq q_i^* N_{E_i^*/\mathcal{X}_{\mathcal{C}}} \otimes \mathcal{O}_{R_i}(-h_i).$$

From (2.11) we obtain:

$$0 \to \mathcal{O}_{R_i}(h_i) \to q_i^* N^{\vee}_{E_i^*/\mathcal{X}_{\mathcal{C}}} \to p_i^* \mathcal{O}_{E_i}(1) \to 0. \tag{2.12}$$

Lemma 2.38. *On the surface R_i, we have:*

$$p_i^* \mathcal{O}_{E_i}(1) = q_i^* \mathcal{O}_{E_i^*}(1) \otimes q_i^* \mathcal{O}_{E_i^*}(f) \otimes \mathcal{O}_{R_i}(-h_i),$$

where here f denotes the generic fiber of E_i^*.

Proof of Lemma. Computing the canonical line bundle of R_i in two ways, we get:

$$p_i^* \mathcal{O}_{E_i}(K_{E_i}) \otimes \mathcal{O}_{R_i}(\bar{\ell}_i) = q_i^* \mathcal{O}_{\bar{E}_i}(K_{\bar{E}_i}) \otimes \mathcal{O}_{R_i}(h_i). \tag{2.13}$$

Using the canonical bundle formula for ruled surfaces, and the fact that E_i^* is the elementary transform of E_i centered at x_i, from (2.13) we have:

$$\begin{aligned} &p_i^* \mathcal{O}_{E_i}(-2) \otimes p_i^* \mathcal{O}_{E_i}(-d_i f) \otimes \mathcal{O}_{R_i}(\bar{\ell}_i) = \\ &q_i^* \mathcal{O}_{E_i^*}(-2) \otimes q_i^* \mathcal{O}_{E_i^*}((-d_i - 1) f) \otimes \mathcal{O}_{R_i}(h_i). \end{aligned} \tag{2.14}$$

But, $\mathcal{O}_{R_i}(\bar{\ell}_i) = q_i^* \mathcal{O}_{E_i}(f) \otimes \mathcal{O}_{R_i}(-h_i)$, and $p_i^* \mathcal{O}_{E_i}(f) = q_i^* \mathcal{O}_{E_i^*}(f)$. Simplifying

(2.14) we get:

$$p_i^*\mathcal{O}_{E_i}(-2) = q_i^*\mathcal{O}_{E_i^*}(-2) \otimes q_i^*\mathcal{O}_{E_i^*}(-2f) \otimes \mathcal{O}_{R_i}(2h_i),$$

and the proof of the lemma follows. $\square$

To finish the proof of the proposition, notice that we obtained the following exact sequence:

$$0 \to \mathcal{O}_{R_i}(h_i) \to q_i^* N^\vee_{E_i^*/\mathcal{X}_\mathcal{C}} \to q_i^*\mathcal{O}_{E_i^*}(1) \otimes q_i^*\mathcal{O}_{E_i^*}(f) \otimes \mathcal{O}_{R_i}(-h_i) \to 0.$$

By pushing forward on E_i^*, since $R^1 q_{i*}\mathcal{O}_{R_i}(h_i) = 0$ and $q_{i*}\mathcal{O}_{R_i}(h_i) = \mathcal{O}_{E_i^*}$, the projection formula yields:

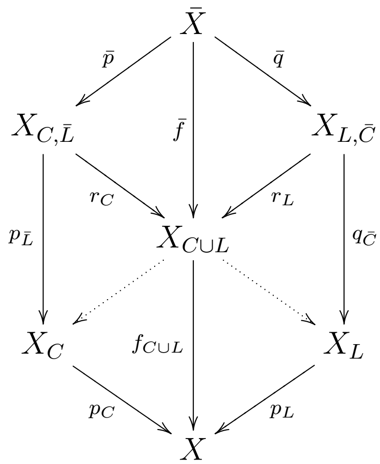
$$0 \longrightarrow \mathcal{O}_{E_i^*} \longrightarrow N^\vee_{E_i^*/\mathcal{X}_\mathcal{C}} \longrightarrow \mathcal{J}_{Q_i} \otimes \mathcal{O}_{E_i^*}(1) \otimes \mathcal{O}_{E_i^*}(f) \longrightarrow 0,$$

where, $\mathcal{J}_{Q_i}$ is the ideal sheaf of the point Q_i, and we are done. $\square$

Corollary 2.39. *The Chern classes of* $N^\vee_{E_i^*/\mathcal{X}_\mathcal{C}}$ *are:*

- $\det(N^\vee_{E_i^*/\mathcal{X}_\mathcal{C}}) = \mathcal{O}_{E_i^*}(1) \otimes \mathcal{O}_{E_i^*}(f)$;
- $c_2(N^\vee_{E_i^*/\mathcal{X}_\mathcal{C}}) = 1$.

Remark 2.40. The description of $\bar{X}$ in the proof of the above proposition is of local nature and it comes from the well-known diagram below [EiHa00, pages 178-179]. This *commutative diagram* exhibits the relation between the two small resolutions and the natural big resolutions of a three-dimensional ordinary double points, as those appearing as singularities of $X_{\mathcal{C},0}$.



(2.15)

To simplify the notation and explain the diagram (2.15), let X be an arbitrary threefold, and $C,\ L \subset X$ be two smooth curves intersecting transversally at exactly one point $\{x\} = C \cap L$.

- $f_{C\cup L} : X_{C\cup L} \to X$ is the blowing-up of X along the ideal sheaf of $C \cup L$;
- $\bar{f} : \bar{X} \to X_{C\cup L}$ is the big resolution of $\bar{X}$ obtained by blowing-up the singular point.
- $p_C : X_C \to X$ is the blowing-up of X along C. Let f_C be the fiber of the exceptional divisor over x.
- $q_L : X_L \to X$ is the blowing-up of X along L. Let f_L be the fiber of the exceptional divisor over x.
- $p_{\bar{L}} : X_{C,\bar{L}} \to X$ is the blowing-up of X_C along $\bar{L}$, the proper transform of L in X_C. Let $\bar{f}_C$ denote the proper transform of f_C.
- $q_{\bar{C}} : X_{L,\bar{C}} \to X$ is the blowing-up of X_L along $\bar{C}$, the proper transform of C in X_L; Let $\bar{f}_L$ denote the proper transform of f_L.

- $\tilde{p} : \tilde{X} \to X_{C,\bar{L}}$ is the blowing-up of $X_{C,\bar{L}}$ along $\bar{f}_C$.
- $\tilde{q} : \tilde{X} \to X_{L,\bar{C}}$ is the blowing-up of $X_{L,\bar{C}}$ along $\bar{f}_L$.
- $r_L : X_{L,\bar{C}} \to \bar{X}$ and $r_C : X_{C,\bar{L}} \to \bar{X}$ are the two *small resolutions* of the singular point of $\bar{X}$.

We will modify the family $\Pi : \mathcal{X}_{\mathcal{C}} \to \Delta$ to produce a flat, proper map

$$\Phi : \mathcal{X} \to \Delta$$

with normal crossing central fiber, and with X_{C_t} as the general fiber. Of course, such a map can be viewed as a degeneration of X_{C_t}.

The map Φ is obtained as the composition

$$\mathcal{X} \xrightarrow{F} \mathcal{X}_{\mathcal{C}} \xrightarrow{\Pi} \Delta,$$

where $F : \mathcal{X} \to \mathcal{X}_{\mathcal{C}}$ is the blowing-up $X_{\mathcal{C}}$ along E_i^*, for $i = 1, \dots, n$.

It is easy to see that the generic fiber of Φ is X_{C_t}, the blowing-up of X along the smooth curve C_t. The central fiber of Φ is a normal crossing thereefold

$$X_0 = X_p \cup X_1 \cup \cdots \cup X_n,$$

with exactly $n+1$ irreducible, smooth components.

To describe the main component, denoted by X_p, as before, let $X_L \to X$ denote the blowing-up of X along the curves $L_i,\ i = 1, \dots, n$. Then X_p is obtained by blowing-up X_L along $\bar{C}$, the strict transforms of the C. It actually coincides with the $3-$fold $X_{L,\bar{C}}$, described in the above proposition. We denote by

$$f_p : X_p \to X,$$

the projection onto X.

The other components,

$$X_i = \mathbb{P}_{E_i^*}(N^{\vee}_{E_i^*|\mathcal{X}_{\mathcal{C}}}) \xrightarrow{f_i} E_i^*,$$

are the exceptional divisors of the of the blowing-up of $\mathcal{X}_{\mathcal{C}}$ along E_i^*, for $i = 1, \ldots, n$.

Moreover, the intersections $S_i := X_p \cap X_i$ are smooth surfaces, the blowing-up of E_i^* at Q_i. Again, the S_i actually coincides with the surfaces $\bar{E}_i$ described in the above proposition.

It is easy to notice that all of the components of the central fiber and their intersections are in fact rationally connected manifolds.

2.4.3 Extensions of line bundles

In this section we will show that any line bundle on the central fiber of Φ extends to a line bundle on an open neighborhood of the central fiber. The results are probably standard and well-known [Per77, Theorem Q, page 50]. However, since we couldn't find any reference for the proof, we include it for completeness.

First, we must describe the line bundles on X_0. To do this we follow [Fri83].

Since the components X_p and X_i of X_0 intersect transversally along S_i, and $X_i \cap X_j = \emptyset$, for $i \neq j$, a line bundle H_{X_0} on X_0 consists of a $(n+1)-uple$

of line bundles $(H_p, H_1, \dots H_n)$ on X_p and X_i, for $i = 1, \dots, n$, respectively, such that

$$H_{p|_{S_i}} \simeq H_{i|_{S_i}},$$

for any $i = 1, \dots, n$.

As an example, we consider [Fri83] the case of dualizing sheaf of ω_{X_0}, an example which will be needed later. We have:

$$\omega_p = \omega_{X_0|_{X_p}} = \mathcal{O}_{X_p}(K_{X_p} + S_1 + \cdots S_n),$$
$$\omega_i = \omega_{X_0|_{X_i}} = \mathcal{O}_{X_i}(K_{X_i} + S_i).$$

By adjunction, we have the canonical isomorphisms:

$$\omega_{p|_{S_i}} = \mathcal{O}_{X_p}(K_{X_p} + S_i)_{|_{S_i}} = \mathcal{O}_{R_i}(K_{S_i}) = \mathcal{O}_{X_i}(K_{X_i} + S_i)_{|_{S_i}} = \omega_{i|_{S_i}},$$

showing that ω_p agrees with ω_i on S_i, and so

$$\omega_{X_0} = (\mathcal{O}_{X_p}(K_{X_p} + R_1 + \cdots R_n), \mathcal{O}_{X_1}(K_{X_i} + R_1), \dots \mathcal{O}_{X_n}(K_{X_n} + R_n)). \quad (2.16)$$

Lemma 2.41. $H^1(X_0, \mathcal{O}_{X_0}) = H^2(X_0, \mathcal{O}_{X_0}) = 0.$

Proof. Recall that the central fiber X_0 has $(n+1)$ smooth components

$$X_0 = X_p \cup X_1 \cup \cdots \cup X_n,$$

where $S_i = X_p \cap X_s$, is the blowing up of a rational ruled surface, and $X_i \cap X_j =$

$\emptyset$. for $i \neq j$. Their structure sheaves are related by the Mayer-Vietoris sequence:

$$0 \to \mathcal{O}_{X_0} \to \mathcal{O}_{X_p} \oplus \mathcal{O}_{X_1} \oplus \cdots \oplus \mathcal{O}_{X_n} \to \mathcal{O}_S \to 0, \tag{2.17}$$

where S is the reduced, reducible surface $S_1 + \cdots + S_n$.

The relevant part of the cohomology sequence is:

$$\begin{aligned} \cdots \to H^0(X_p, \mathcal{O}_{X_p}) \oplus H^0(X_1, \mathcal{O}_{X_1}) \oplus \cdots \oplus H^0(X_n, \mathcal{O}_{X_n}) \xrightarrow{s} H^0(S, \mathcal{O}_S) \\ \to H^1(X_0, \mathcal{O}_{X_0}) \to H^1(X_p, \mathcal{O}_{X_p}) \oplus H^1(X_1, \mathcal{O}_{X_1}) \oplus \cdots \oplus H^1(X_n, \mathcal{O}_{X_n}) \\ \to H^1(S, \mathcal{O}_S) \to H^2(X_0, \mathcal{O}_{X_0}) \to H^2(X_p, \mathcal{O}_{X_p}) \oplus H^2(X_1, \mathcal{O}_{X_1}) \oplus \cdots \\ \oplus H^2(X_n, \mathcal{O}_{X_n}) \to \cdots . \end{aligned} \tag{2.18}$$

Under the identification

$$H^k(S, \mathcal{O}_S) \simeq H^k(S_1, \mathcal{O}_{S_1}) \oplus \cdots \oplus H^k(S_n, \mathcal{O}_{S_n}),\ \forall k \geq 0,$$

the map:

$$s : H^0(X_p, \mathcal{O}_{X_p}) \oplus H^0(X_1, \mathcal{O}_{X_1}) \oplus \cdots \oplus H^0(X_n, \mathcal{O}_{X_n}) \to H^0(S, \mathcal{O}_S)$$

is given by $(s_p, s_1, \ldots, s_n) \mapsto (s_p - s_1, \ldots, s_p - s_n)$, and is obviously surjective.

Since, as discussed in Theorem 2.29, the structural sheaf of a rationally connected manifold has no higher cohomology, it follows that:

$$H^k(X_p, \mathcal{O}_{X_p}) = H^k(X_i, \mathcal{O}_{X_i}) = H^k(S_i, \mathcal{O}_{S_i}) = 0,$$

for $k = 1, 2$ and $i = 1, \ldots, n$ and the lemma follows now from a simple inspection of (2.18). □

Corollary 2.42. $R^1\Phi_*\mathcal{O}_{\mathcal{X}} = R^2\Phi_*\mathcal{O}_{\mathcal{X}} = 0.$

Proof. Since X is rationally connected, X_{C_t} will also be rationally connected for any $t \in \Delta,\ t \neq 0$. Then [ArKo03] we have:

$$H^1(X_{C_t}, \mathcal{O}_{X_{C_t}}) = H^2(X_{C_t}, \mathcal{O}_{X_{C_t}}) = 0.$$

But, since Φ is proper and $H^1(X_0, \mathcal{O}_{X_0}) = H^2(X_0, \mathcal{O}_{X_0}) = 0$, from the semi-continuity theorem it follows that

$$R^1\Phi_*\mathcal{O}_{\mathcal{X}} = R^2\Phi_*\mathcal{O}_{\mathcal{X}} = 0.$$

Of course, we shrink Δ whenever necessary. □

Theorem 2.43. *Any line bundle $\mathcal{L}_{X_0} \in \mathrm{Pic}(X_0)$ extends to a line bundle over $\mathcal{X}$.*

Proof. Applying Φ_* over the exponential sequence

$$0 \to \mathbb{Z}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}^*_{\mathcal{X}} \to 0$$

we obtain

$$0 = R^1\Phi_*\mathcal{O}_{\mathcal{X}} \to R^1\Phi_*\mathcal{O}^*_{\mathcal{X}} \to R^2\Phi_*\mathbb{Z}_{\mathcal{X}} \to R^2\Phi_*\mathcal{O}_{\mathcal{X}} = 0.$$

Hence $R^1\Phi_*\mathcal{O}^*_{\mathcal{X}} \simeq R^2\Phi_*\mathbb{Z}_{\mathcal{X}}$, and passing to global sections we get

$$\mathrm{Pic}(\mathcal{X}) \simeq H^2(\mathcal{X}, \mathbb{Z}).$$

However, X_0 is a deformation retract of $\mathcal{X}$, and therefore

$$H^2(X_0, \mathbb{Z}) \simeq H^2(\mathcal{X}, \mathbb{Z}).$$

But, from Lemma 2.41 and the exponential sequence we can see that

$$\mathrm{Pic}(X_0) \simeq H^2(X_0, \mathbb{Z}),$$

and the proposition follows. □

The following proposition is well-known. It is the key that allows us to solve the ampleness issue discussed in Remark 2.25. For its proof we refer the interested reader to [KoMo98, Proposition 1.41].

Proposition 2.44. *Let $f : X \to Y$ be a proper morphism, and D a Cartier divisor on X. Let $y \in Y$ be a point and X_y the fiber of f over y. If $\mathcal{O}_{X_y}(D)$ is ample, then D is ample over some open set $U \ni y$ of Y.*

2.4.4 Intersection Theory I

In this section we set up the intersection theory of the central fiber X_0 of $\Phi : \mathcal{X} \to \Delta$. To do this, we will adopt a cohomological approach, as discussed in Appendix A. In our situation we need the following:

Proposition 2.45. *If $L_0^i = (L_p^i, L_1^i, \dots L_n^i),\ i = 1, 2, 3$ are three line bundles on X_0, then*

$$L_0^1 \cdot L_0^2 \cdot L_0^3 = L_p^1 \cdot L_p^2 \cdot L_p^3 + \sum_{k=1}^{n} L_k^1 \cdot L_k^2 \cdot L_k^3.$$

Proof. Let $L_0 = (L_p, L_1, \dots L_n)$ be an arbitrary line bundle on X_0. Of course, $L_p \in \mathrm{Pic}(X_p),\ L_i \in \mathrm{Pic}(X_i)$ for $i = 1, \dots, n$, with the property that:

$$L_{S_i} \overset{\text{def}}{=} L_{p|S_i} \simeq L_{i|S_i}.$$

Tensoring the Mayer-Vietoris sequence (2.18) by $L_0^\vee$ we get:

$$0 \to L_0^\vee \to L_p^\vee \oplus L_1^\vee \oplus \dots \oplus L_n^\vee \to L_{S_1}^\vee \oplus \dots \oplus L_{S_n}^\vee \to 0 \tag{2.19}$$

From (2.19) we immediately obtain:

$$\chi(-L_0) = \chi(-L_p) + \sum_{k=1}^{n} \chi(-L_k) - \sum_{k=1}^{n} \chi(-L_{S_k}). \tag{2.20}$$

Now using (2.20) in (2.47) we see that

$$L_0^1 \cdot L_0^2 \cdot L_0^3 = L_p^1 \cdot L_p^2 \cdot L_p^3 + \sum_{k=1}^{n} L_k^1 \cdot L_k^2 \cdot L_k^3 - \sum_{k=1}^{n} L_{S_k}^1 \cdot L_{S_k}^2 \cdot L_{S_k}^3.$$

But, since S_k is a smooth surface $L_{S_k}^1 \cdot L_{S_k}^2 \cdot L_{S_k}^3 = 0$, and the conclusion follows. □

Corollary 2.46. *Let $\mathcal{L}^i,\ i = 1, 2, 3$ be three line bundles on $\mathcal{X}$, and denote by L_t^i their restriction to the fiber $\Phi^{-1}(t)$. If $L_0^i = (L_p^i, L_1^i, \dots L_n^i),\ i = 1, 2, 3,$*

then:

$$L_t^1 \cdot L_t^2 \cdot L_t^3 = L_p^1 \cdot L_p^2 \cdot L_p^3 + \sum_{k=1}^{n} L_k^1 \cdot L_k^2 \cdot L_k^3, \quad \forall t \in \Delta.$$

Proof. Let $\mathcal{L}$ be an arbitrary line bundle on $\mathcal{X}$. We will denote by L_t its restriction to $\Phi^{-1}(t)$ an arbitrary fiber of Φ. Since the Euler characteristic is constant in flat families, we have:

$$\chi(L_0) = \chi(L_t), \quad \forall t \in \Delta. \tag{2.21}$$

From (2.47), (2.21) and the above proposition, the result follows. □

Remark 2.47. The proposition and its corollary are nothing but the three dimensional version of [Per77, Proposition 2.4.1]. There, U. Persson sets up the intersection theory on the central fiber of degenerations of surfaces for extendable line bundles. However, our cohomological approach is different than his topological approach.

Remark 2.48. It is worth mentioning that Proposition 2.45 and Corollary 2.46 clearly hold in a more general settings. In the next section, we are going to apply these results on a deformation to the normal cone construction.

2.4.5 Deformation to the normal cone

In this section we will provide an application of the previously obtained results that will help to construct an appropriate *ample* line bundle on the main component of the central fiber of Φ. This will be done by answering affirmatively to our Conjecture 2.26 in the case of rationally connected manifolds. The method we adopt is based on the classical *deformation to the normal cone construction* [Ful98].

We will show how an ample line bundle on the central fiber of the deformation to the normal cone can be constructed. Then with the help of the results in subsections 2.4.3 and 2.4.4 we will show that on the central fiber of this degeneration property $\mathcal{P}$ holds true. This can be seen not only as an important part of our argument, but also as a prelude to the next section.

We start with a smooth, projective threefold X containing a smooth rational curve $C \subset X$, with $N_{C/X} \simeq \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$. Let $p : X_C \to X$ be the blowing-up of X along C, and suppose there exists an ample line bundle H_{X_C} on X_C such that $K_{X_C} \cdot H^2_{X_C} < 0$, i.e. $\mathcal{P}_{X_C}$ holds true. We will prove the following:

Proposition 2.49. *If X is a rationally connected threefold, then*

$$\mathcal{P}_{X_C} \textit{ holds true} \Longrightarrow \mathcal{P}_X \textit{ holds true.}$$

The deformation to the normal cone construction

Let $\Delta \subset \mathbb{C}$ denote the unit disk. Also, let $p : \mathcal{X} \to X \times \Delta$ be the blowing-up of $X \times \Delta$ along $C \times \{0\}$ and $q : \mathcal{X} \to \Delta$ the projection onto Δ.

$$\begin{array}{ccc} & \mathcal{X} & \\ & \downarrow{\scriptstyle p} & \\ C \times \{0\} \subset & X \times \Delta & {\scriptstyle q} \\ & \downarrow{\scriptstyle pr_2} & \\ & \Delta & \end{array} \tag{2.22}$$

The fiber $q^{-1}(t)$ is clearly isomorphic to X for $t \neq 0$, while the central fiber $X_0 \overset{def}{=} q^{-1}(0)$ is a normal crossing divisor in $\mathcal{X}$ with two smooth irreducible components. One of them is X_C as the strict transform of $X \times \{0\}$ in $\mathcal{X}$, while the other one is $P \simeq \mathbb{P}_{\mathbb{P}_1}(\mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}) \xrightarrow{f} C$, the exceptional divisor of q. These two components intersect transversally along E, the exceptional divisor of q. Since $N_{C/X} \simeq \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$, we have

$$E \simeq \mathbb{P}_C(N^{\vee}_{C/X}) \simeq \ \mathbb{P}_{\mathbb{P}_1}(\mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}(1)) \simeq \mathbb{P}_1 \times \mathbb{P}_1,$$

where $f_E : E \to C$ is the projection onto to first factor identified with C. E sits naturally in P and the restriction of f to E is f_E.

Construction of the line bundle

A line bundle on X_0 is given by a pair (L_{X_C}, L_P) of isomorphism classes of line bundles $L_{X_C} \in \mathrm{Pic}(X_C)$ and $L_P \in \mathrm{Pic}(P)$, such that their restriction to E coincide. From our starting hypothesis we have a clear choice for L_{X_C}, namely the ample line bundle H_{X_C}. Its restriction to E can be written as

$$L_{X_C|E} \simeq f_E^* \mathcal{O}_C(a) \otimes \mathcal{O}_{\mathbb{P}_1}(b), \tag{2.23}$$

with $a,\ b > 0$. Here we identify $\mathcal{O}_{\mathbb{P}_1}(1)$ with $\mathcal{O}_E(1)$.

Since $f_{|E} = f_E,\ \mathcal{O}_P(1) \otimes \mathcal{O}_E$ will be isomorphic to $\mathcal{O}_E(1)$, and this forces the choice of the line bundle L_P on P. Namely,

$$L_P = f^* \mathcal{O}_C(a) \otimes \mathcal{O}_P(b). \tag{2.24}$$

With this choice, we obviously have

$$L_{X_{C|E}} \simeq L_{P|E}.$$

Thus, the pair (L_{X_C}, L_P) defines a line bundle $\mathcal{L}_0$ on X_0.

Proof of Proposition 2.49. First we show that $\mathcal{P}_{X_0}$ holds true. To do this, we consider on X_0 the line bundle $\mathcal{L}_{X_0}$ constructed above. To simplify the notations, we will denote by ξ the line bundle $\mathcal{O}_P(1) \in \mathrm{Pic}(P)$, and by ξ_E the line bundle $\mathcal{O}_E(1) \in \mathrm{Pic}(E)$. Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}(1) \oplus \mathcal{O}_{\mathbb{P}_1}$, and L_E be the line bundle $L_{X_{C|E}} \simeq L_{P|E}$. Also, for simplicity we will abusively use additive notations for line bundles.

Lemma 2.50. *$\mathcal{L}_0$ is ample.*

Proof. Since P is the projectivization of a globally generated line bundle, it is not hard to see that ξ is nef. The 4-dimensional linear system $|\xi|$ is also base-point free [Deb01, page 26], and contains E as a member. The induced map

$$\phi_{|\xi|} : P \to \mathbb{P}_4$$

is an immersion outside the smooth rational curve, $D = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1}) \subset P$. The curve D has $N_{D/P} = \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$, and is contracted to an ordinary double point. The image $\phi_{|\xi|}(P)$ is the cone over the Segre embedding of $\mathbb{P}_1 \times \mathbb{P}_1 \subset \mathbb{P}_3$.

From this, by Nakai-Moishezon we can easily infer that $L_P = f^*\mathcal{O}_C(a) + b\xi$ is ample. Since L_{X_C} was chosen to be ample, $\mathcal{L}_0 = (L_{X_C}, L_P)$ is ample, too. □

As in Section 2.4.4, we have $K_{X_0} = (K_{X_C} + E, K_P + E)$, where K_P is given by

$$K_P = f^*(K_C + \det \mathcal{E}) - 3\xi = -3\xi.$$

Using Proposition 2.45 and Proposition C.2, we have:

$$\begin{aligned}
K_{X_0} \cdot \mathcal{L}_0^2 =& (K_{X_C} + E) \cdot L_{X_C}^2 + (K_P + E) \cdot L_P^2 \\
=& K_{X_C} \cdot H_{X_C}^2 + K_P \cdot L_P^2 + 2(L_E \underset{E}{\cdot} L_E) \\
=& K_{X_C} \cdot H_{X_C}^2 - 3\xi \cdot (f^*\mathcal{O}_C(a) + b\xi)^2 + 2(f_E^*\mathcal{O}_C(a) + b\xi_E)^2 \\
=& K_{X_C} \cdot H_{X_C}^2 - 3\xi(b^2\xi^2 + 2b\xi f^*\mathcal{O}_C(a)) + 4ab + 4b^2 \\
=& K_{X_C} \cdot H_{X_C}^2 - 6b^2 - 6ab + 4ab + 4b^2 \\
=& K_{X_C} \cdot H_{X_C}^2 - 2b^2 - 2ab \\
<& \ K_{X_C} \cdot H_{X_C}^2 < 0. \qquad (2.25)
\end{aligned}$$

But this implies that $\mathcal{P}_{X_0}$ holds true.

Now, since X_C, P and E are all rationally connected, the vanishing results from Section 2.4.3 hold and any line bundle on X_0 extends to $\mathcal{X}$. In particular, the line bundle $\mathcal{L}_0$ will extend to a line $\mathcal{L}$ bundle on $\mathcal{X}$. But, since $\mathcal{L}_0$ is ample on X_0, by eventually shrinking Δ, we can see that $L_t = \mathcal{L}_{|X_t}$ is ample on X and with the help Corollary 2.46, from (2.25) it follows that $\mathcal{P}_X$ holds true. □

Recall that in Section 2.4.2, we described the components of X_0. The one called "the main component", denoted by X_p will play an important rôle in our computations. As it will become apparent from future computations, X_p must be endowed with an ample line bundle H_p, such that $K_{X_p} \cdot H_p^2 < 0$. As we do not have to many options, this should be done by starting with the God-given line bundle H_{X_C} on X_C. A natural choice for the line bundle H_p would be the proper transform of H_{X_C} on X_p, as X_p and X_C are obviously birational. However, since H_{X_C} is ample, it is not hard to see that its proper transform will have negative degree exactly on the rational curves $\bar{f}_i$ introduced in Section 2.4.2.

To circumvent this ampleness issue, will we work on the normal cone construction. As an application of Proposition 2.49, we obtain the following:

Corollary 2.51. *The main component X_p of the central fiber of Φ admits an ample line bundle H_p with $K_{X_p} \cdot H_p^2 < 0$.*

Proof. By induction, the Corollary follows from Proposition 2.49 and the description of X_p. Since $\mathcal{P}_{X_C}$ holds true, $\mathcal{P}_{X_{C,\bar{L}}}$ will also hold true. Applying repeatedly Proposition 2.49 we can see that $\mathcal{P}_{X_{L,\bar{C}}}$ holds true, i.e. $\mathcal{P}_{X_p}$ holds true. □

2.4.6 Construction of the line bundle

Recall now the initial package. We have a smooth threefold X containing a smooth curve C, with $K_X \cdot C \geq 0$, and let $p : X_C \to X$ be the blowing-up of X along C. On X_C we have an ample line bundle H_{X_C} such that $K_{X_C} \cdot H_{X_C}^2 < 0$. We want to find an ample line bundle H_0 on X_0 such that $K_{X_0} \cdot H_{X_0}^2 < 0$.

The construction of a line bundle on X_0 is guidelined by the method exhibited in the previous section. What we are going to do is to start with the line bundle on the main component X_p given by Corollary 2.51, and by studying its restriction to S_i we will extend it across X_0.

Remember that $f_p : X_p \to X$ is gotten by first blowing-up X along the curves $L_1, \ldots, L_n$, followed by blowing-up $\bar{C}$, the strict transform of C. We denoted the exceptional divisors by $S_1, \ldots, S_n$, and $E_{\bar{C}}$, respectively. Moreover, the S_i's can be either viewed as blowing-ups of E_i, the exceptional divisors of X along the L_i's, at one point or as blowing-ups of E_i^* at one point. The components X_i of X_0 are pairwise disjoint and meet X_p transversally along S_i.

To simplify the exposition, we introduce the following notations:

- $d_i = \deg_{L_i}(N_{L_i/X}) > 2$, since L_i is a very free rational curve;
- $p_i : S_i \longrightarrow E_i$ is the blowing-up of E_i at $P_i = \bar{C} \cap E_i$ with e_i its exceptional divisor.
- k_i be the strict transform in S_i of the fiber of the ruled surface E_i that passes through P_i;
- $q_i : S_i \longrightarrow E_i^*$ is the contraction of k_i;
- $h_i = \mathcal{O}_{E_i}(1),\ h_i^* = \mathcal{O}_{E_i^*}(1)$;
- f denotes the generic fiber of either E_i or E_i^*. We will still abusively denote by f its strict transform in S_i.
- $c_1 = \det(N^{\vee}_{E_i^*/\mathcal{X}_\mathcal{C}}) = h_i^* + f$;
- $\xi_i = \mathcal{O}_{X_i}(1)$.

What we do next is to find the restriction of H_p to S_i, for each $i = 1, \ldots, n$. For convenience we will denote the restriction of H_p to S_i by H_{S_i}.

Since $\operatorname{Pic}(S_i) \simeq q_i^* \operatorname{Pic}(E_i^*) \oplus \mathbb{Z}[k_i]$, we can write:

$$H_{S_i} = q_i^* M_i - x_i k_i, \tag{2.26}$$

for some line bundle $M_i \in \operatorname{Pic}(E_i^*)$, and some integer x_i.

To construct the line bundle on X_i, recall that

$$X_i = \mathbb{P}(N^\vee_{E_i^*|\mathcal{X}_C}) \xrightarrow{f_i} E_i^*$$

is the exceptional divisor of the blowing-up of $\mathcal{X}$ along E_i^*, and X_i intersects transversally X_p. This transversality can be used, as in [Per77, page 41], to see that

$$N_{X_i/\mathcal{X}_{|S_i}} = N_{S_i/X_p}.$$

In our notations and using Lemma 2.38 this translates into:

$$\xi_{i|S_i} = p_i^* h_i = q_i^* h_i^* + q_i^* f - k_i. \tag{2.27}$$

We can write now:

$$H_{S_i} = q_i^*(M_i - x_i h_i^* - x_i f) + x_i \xi_{i|S_i}. \tag{2.28}$$

Thus, if we let $N_i = M_i - x_i h_i^* - x_i f = M_i - x_i c_1$ we can see that we can write H_{S_i} as

$$H_{S_i} = q_i^* N_i + x_i \xi_{i|S_i}. \tag{2.29}$$

With this is now clear that, we have a unique choice for H_i, namely:

$$H_i = f_i^* N_i + x_i \xi_i. \tag{2.30}$$

As can be easily seen from the above consideration, the choice of the line bundle $H_p \in \mathrm{Pic}(X_p)$ is not necessary to extend it across X_0. Thus, we have just proved:

Proposition 2.52. *Let L_p be an arbitrary line bundle on X_p. Then L_p extends to a unique line bundle H_0 on X_0.*

For furher considerations, we need to show the ampleness of the line bundles H_i. Before we proceed we need the following:

Lemma 2.53. $\mathcal{O}_{X_i}(S_i) = \xi_i$.

Proof. In additive notations, we can write $\mathcal{O}_{X_i}(S_i) = a\xi_i + bf_i^* h_i^* + cf_i^* \mathcal{O}_{E_i^*}(f)$, with $a,\ b,\ c \in \mathbb{Z}$. Restricting to S_i, and using (2.27) we obtain:

$$\begin{aligned} \mathcal{O}_{S_i}(S_i) &= a\xi_{i|S_i} + bq_i^* h_i^* + cq_i^* \mathcal{O}_{E_i^*}(f) \\ &= (a+b)q_i^* h_i^* + (a+c)q_i^* \mathcal{O}_{E_i^*}(f) - ak_i. \end{aligned} \tag{2.31}$$

On the other hand, the restriction of $\mathcal{O}_{X_i}(S_i)$ to S_i is the normal bundle N_{S_i/X_i}. But, from the transversality of the intersection of X_i with X_p we have [Per77, page 41], $N_{S_i/X_i} \simeq N^\vee_{S_i/X_p} = p_i^* h_i$, and so:

$$\mathcal{O}_{S_i}(S_i) = p_i^* h_i = q_i^* h_i^* + q_i^* \mathcal{O}_{E_i^*}(f) - k_i$$

Comparing with (2.31) in $\mathrm{Pic}(S_i)$, we can immediately see that $a = 1$ and $b = c = 0$. $\square$

For the computations involved in proving the ampleness of the H_i's, we will work with the more traditional basis of $\mathrm{Pic}(E_i^*)$ described in Appendix C. $\mathrm{Pic}(E_i^*)$ is generated by $\{\mathcal{O}_{E_i^*}(C_0),\ \mathcal{O}_{E_i^*}(f)\}$, where $\mathcal{O}_{E_i^*}(C_0)$ is the line bundle associated to the section of E_i^* of minimal self-intersection, and f is the class of a fiber. Since E_i^* is rational, $C_0^2 = -e \leq 0$. It is easy to see that $h_i^* = C_0 - \frac{d_i+1-e}{2}f$. Using this basis, any line bundle $L \in \mathrm{Pic}(X_i)$ can be written as $L = a\xi_i + bf_i^*\mathcal{O}_{E_i^*}(C_0) + cf_i^*\mathcal{O}_{E_i^*}(f)$, where $a,\ b,\ c \in \mathbb{Z}$. We have:

Proposition 2.54. *Any line bundle $L \in \mathrm{Pic}(X_i)$ which admits sections, can be written as*

$$L = \lambda(\xi_i - f_i^*c_1) + \beta f_i^*\mathcal{O}_{E_i^*}(C_0) + (\gamma + \beta e)f_i^*\mathcal{O}_{E_i^*}(f),$$

with $\lambda,\ \beta,\ \gamma \geq 0$.

Proof. Let $L = a\xi_i + bf_i^*\mathcal{O}_{E_i^*}(C_0) + cf_i^*\mathcal{O}_{E_i^*}(f)$ be an arbitrary line bundle on X_i. For simplicity, we will abusively denote $\mathcal{O}_{E_i^*}(C_0)$ by C_0 and $\mathcal{O}_{E_i^*}(f)$ by f. If L has sections, its restriction to an arbitrary fiber f of f_i has sections. But $L_{|f} \simeq \mathcal{O}_{\mathbb{P}_1}(a)$, and so $a \geq 0$.

Next we restrict our line bundle to S_i. Using again (2.27) have:

$$\begin{aligned} L_{|S_i} &= a\xi_{i|S_i} + bf_i^*C_{0|S_i} + cf_i^*f_{|S_i} \\ &= a(q_i^*h_i^* + q_i^*f - k_i) + bq_i^*C_0 + cq_i^*f \\ &= (a+b)q_i^*C_0 + (c + a - a\frac{d_i+1-e}{2})q_i^*f - ak_i. \end{aligned} \tag{2.32}$$

Now, if L has sections, $L_{|S_i}$ restricted to f, the strict transform in S_i of a generic fiber of E_i^*, still has sections. But this can happen only if $a+b \geq 0$. We restrict now to the strict transform $\bar{C}_0$ of C_0. Since E_i^* is the elementary transform of E_i, $\bar{C}_0$ will intersect k_i transversally at one point. If we assume that L has sections, it will follow as before that $c - a\frac{d_i+1-e}{2} - (a+b)e \geq 0$. By a change of notation, if we let $\lambda = a \geq 0$, $\beta = a+b \geq 0$ and $\gamma = c - a\frac{d_i+1-e}{2} - (a+b)e \geq 0$, a simple calculation will show that we can write any line bundle $L \in \mathrm{Pic}(X_i)$ with sections as:

$$L = \lambda(\xi - f_i^* h_i^*) + \beta f_i^* C_0 + (\gamma + \beta e) f_i^* f,$$

where $\lambda, \ \beta, \ \gamma \geq 0$. □

Remark 2.55. One could study if the necessary conditions we found are also sufficient for line bundles on X_i to admit sections. However, the result we proved is enough for our purpose.

Proposition 2.56. *The line bundles H_i constructed above are ample for any $i = 1, \dots n$.*

Proof. **Step 1:** We show that for any $i = 1, \dots n$, the line bundle $N_i = M_i - x_i(h_i^* + f) \in \mathrm{Pic}(E_i^*)$ is ample.

Writing the line bundle $M_i = ah_i^* + bf \in \mathrm{Pic}(E_i^*)$ $a, \ b \in \mathbb{Z}$ in our preferred basis $\{C_0, \ f\}$, we see that

$$M_i - x_i(h_i^* + f) = (a - x_i)C_0 + [(b - x_i) - (a - x_i)\frac{d_i + 1 - e}{2}]f.$$

To check the ampleness of this this line bundle, we use the conditions of

Proposition C.3. First, if in (2.26) we take the intersection with the fiber of E_i^* passing through the center of the blowing-up, we immediately see that $a - x_i > 0$. The other condition to check is:

$$(b-x_i)-(a-x_i)\frac{d_i+1-e}{2} > (a-x_i)e \iff (b-x_i) > (a-x_i)\frac{d_i+1+e}{2} \quad (2.33)$$

Now, using the Nakai-Moishezon criterion is easy to check that since $H_{S_i} = q_i^* M_i - x_i k_i$ is ample, then $M_i \in \mathrm{Pic}(E_i^*)$ is ample, too. Proposition C.3 yields that:

$$b > a\frac{d_i+1+e}{2}, \quad (2.34)$$

which at its turn implies (2.33).

Step 2: We show that H_i is strictly nef. Let $C \subset X_i$ be an arbitrary irreducible curves. Depending on the position of the curve C with respect to the surface $S_i \subset X_i$ we distinguish three cases.

1. $C \subseteq S_i$. In this case the intersection pairing can be computed on S_i, where

$$H_i \cdot C = H_{S_i} \underset{S_i}{\cdot} C > 0,$$

since H_{S_i} is ample.

2. $C = f$ is a fiber of $X_i \to E_i^*$. Then

$$H_i \cdot C = (f_i^* N_i + x_i \xi_i) \cdot f = x_i(\xi_i \cdot f) = x_i > 0.$$

3. When $C \nsubseteq S_i$, using Lemma 2.31, we have:

$$H_i \cdot C = (f_i^* N_i + x_i \xi_i) \cdot C = f_i^* N_i \cdot C + x_i S_i \cdot C \geq f_i^* N_i \cdot C > 0,$$

as N_i is ample and C is not contained in a fiber of f_i.

Step 3: We show that H_i is ample. Using the Nakai-Moishezon, all we need to prove are the following:

1. $H_i^2 \cdot S > 0$ for any irreducible surface $S \subseteq X_i$. Using Lemma 2.54, we can write the line bundle associated to S as $\lambda(\xi - f_i^* h_i^*) + \beta f_i^* C_0 + (\gamma + \beta e) f_i^* f$, where $\lambda,\ \beta,\ \gamma \geq 0$, are not simultaneously zero. We have:

$$\begin{aligned} H_i^2 \cdot S &= (f_i^* N_i + x_i \xi_i)^2 \cdot (\lambda(\xi - f_i^* h_i^*) + \beta f_i^* C_0 + (\gamma + \beta e) f_i^* f) \\ &= \lambda (f_i^* N_i + x_i \xi_i)^2 \cdot (\xi - f_i^* h_i^*) + \beta (f_i^* N_i + x_i \xi_i)^2 \cdot f_i^* C_0 \\ &\quad + (\gamma + \beta e)(f_i^* N_i + x_i \xi_i)^2 \cdot f_i^* f. \end{aligned} \tag{2.35}$$

We compute now separately the terms involved.

$$\begin{aligned} (f_i^* N_i + x_i \xi_i)^2 \cdot f_i^* f &= (f_i^* N_i \cdot f_i^* N_i + 2 x_i \xi_i \cdot f_i^* N_i + x_i^2 \xi_i^2) \cdot f_i^* f \\ &= 2 x_i \xi_i \cdot f_i^* N_i \cdot f_i^* f + x_i^2 \xi_i^2 \cdot f_i^* f \\ &= 2 x_i (N_i \cdot f) + x_i^2 (f \cdot c_1) \\ &= 2 x_i [(M_i - x_i c_1) \cdot f] + x_i^2 [f \cdot (h_i^* + f)] \\ &= 2 a x_i - x_i^2 > 0, \end{aligned} \tag{2.36}$$

since $a > x_i$.

$$(f_i^* N_i + x_i\xi_i)^2 \cdot f_i^* C_0 = (f_i^* N_i \cdot f_i^* N_i + 2x_i\xi_i \cdot f_i^* N_i + x_i^2\xi_i^2) \cdot f_i^* C_0$$
$$= 2x_i\xi_i \cdot f_i^* N_i \cdot f_i^* C_0 + x_i^2\xi_i^2 \cdot f_i^* C_0$$
$$= 2x_i(N_i \cdot C_0) + x_i^2(C_0 \cdot c_1)$$
$$= x_i(N_i \cdot C_0) + x_i(M_i \cdot C_0), \tag{2.37}$$

as N_i and M_i ar both ample line bundles on E_i^*. Finally,

$$(f_i^* N_i + x_i\xi_i)^2 \cdot (\xi - f_i^* h_i^*) = (f_i^* N_i + x_i\xi_i)^2 \cdot \xi - (f_i^* N_i + x_i\xi_i)^2 f_i^* h_i^*$$
$$= H_{S_i}^2 - 2x_i(N_i \cdot h_i^*) - x_i^2(h_i^*)^2$$
$$= M_i^2 - 2x_i(M_i \cdot h_i^*) + x_i^2(h_i^*)^2 + x_i^2. \tag{2.38}$$

To show that this last term is positive we use (2.34) to get:

$$M_i^2 - 2x_i(M_i \cdot h_i^*) + x_i^2(h_i^*)^2 + x_i^2$$
$$= (ah_i^* + bf)^2 - 2x_i[(ah_i^* + bf) \cdot h_i^*] + x_i^2(h_i^*)^2 + x_i^2$$
$$= 2b(a - x_i) - a^2(d_i + 1) + 2ax_i(d_i + 1) - d_i x_i^2$$
$$> a(a - x_i)(d_i + 1) - a^2(d_i + 1) + 2ax_i(d_i + 1) - d_i x_i^2$$
$$= ax_i(d_i + 1) - d_i x_i^2 > 0, \tag{2.39}$$

since $a > x_i$.

From (2.35),(2.36),(2.37), (2.38), (2.39), and since λ, β and γ cannot simultaneously vanish, this step is concluded.

2. $H_i^3 > 0$. This follows again by a direct computation and Step 1.

$$\begin{aligned} H_i^3 &= (f_i^* N_i + x_i \xi_i)^3 \\ &= [(f_i^* N_i + x_i\xi_i)^2 \cdot f_i^* N_i] + x_i[\xi_i \cdot (f_i^* N_i + x_i \xi_i)^2] \\ &= 2x_i[\xi_i \cdot (f_i^* N_i)^2] + x_i^2(\xi_i^2 \cdot f_i^* N_i) + x_i H_{S_i}^2 \\ &= 2x_i(N_i \cdot N_i) + x_i^2(N_i \cdot c_1) + x_i H_{S_i}^2 \\ &= x_i(N_i \cdot N_i) + x_i(N_i \cdot M_i) + x_i H_{S_i}^2 > 0, \end{aligned} \tag{2.40}$$

since $x_i > 0$, and the line bundles $M_i,\ N_i \in \mathrm{Pic}(E_i^*)$ are ample.

□

2.4.7 Intersection Theory II

In this subsection we conclude our arguments of this entire section, by proving the the following:

Proposition 2.57. *If $H_{X_0} = (H_p, H_1, \ldots, H_n)$ is the line bundle constructed above,*

$$K_{X_0} \cdot H_{X_0}^2 < 0.$$

Proof. Using Proposition A.3 and Corollary 2.45 we compute now $K_{X_0} \cdot H_{X_0}^2$:

$$\begin{aligned} K_{X_0} \cdot H_{X_0}^2 =& (K_{X_p} + S_1 + \cdots + S_n) \cdot H_p^2 + \sum_{i=1}^{n} (K_{X_i} + S_i) \cdot H_i^2 \\ =& K_{X_p} \cdot H_p^2 + \sum_{i=1}^{n} (K_{X_i} \cdot H_i^2 + 2H_{S_i}^2). \end{aligned} \tag{2.41}$$

We compute separately $K_{X_i} \cdot H_i^2$. From Proposition C.1 we have that $K_{X_i} =$

$f_i^* L_i - 2\xi_i$, where

$$\begin{aligned} L_i =& K_{E_i^*} + c_1 = -2h_i^* - (d_i+3)f + h_i^* + f \\ =& - h_i^* - (d_i+2)f = -c_1 - (d_i+1)f. \end{aligned} \tag{2.42}$$

A straightforward computation using Proposition C.1 again yields:

$$\begin{aligned} K_{X_i} \cdot H_i^2 =& (f_i^* L_i - 2\xi_i) \cdot (f_i^* N_i + x_i \xi_i)^2 \\ =& (f_i^* L_i - 2\xi_i) \cdot (f_i^* N_i \cdot f_i^* N_i + 2x_i f_i^* N_i \cdot \xi_i + x_i^2 \xi_i^2) \\ =& 2x_i f_i^* L_i \cdot f_i^* N_i \cdot \xi_i + x_i^2 f_i^* L_i \cdot \xi_i^2 \\ & - 2 f_i^* N_i \cdot f_i^* N_i \cdot \xi_i - 4x_i f_i^* N_i \cdot \xi_i^2 - 2x_i^2 \xi_i^3 \\ =& 2x_i (L_i \cdot N_i) + x_i^2 (L_i \cdot c_1) - 2(N_i \cdot N_i) \\ & - 4x_i (N_i \cdot c_1) - 2x_i^2 (c_1^2 - c_2). \end{aligned} \tag{2.43}$$

Making use of (2.42), we can complete this computation as follows:

$$\begin{aligned} (L_i \cdot c_1) &= [(-c_1 - (d_i+1)f) \cdot c_1] = -c_1^2 - (d_i+1)(c_1 \cdot f) \\ &= d_i - 1 - d_i - 1 = -2; \\ (L_i \cdot N_i) &= [L_i \cdot (M_i - x_i c_1)] = (L_i \cdot M_i) - x_i (L_i \cdot c_1) = (L_i \cdot M_i) + 2x_i \\ (N_i \cdot N_i) &= [(M_i - x_i c_1) \cdot (M_i - x_i c_1)] = (M_i \cdot M_i) - 2x_i (M_i \cdot c_1) + x_i^2 c_1^2; \\ &= H_{S_i}^2 + x_i^2 - 2x_i (M_i \cdot c_1) + x_i^2 c_1^2; \\ (N_i \cdot c_1) &= [(M_i - x_i c_1) \cdot c_1] = (M_i \cdot c_1) - x_i c_1^2. \end{aligned} \tag{2.44}$$

From (2.43), we get:

$$\begin{aligned}K_{X_i} \cdot H_i^2 + 2H_{S_i}^2 = & \ 2H_{S_i}^2 + 2x_i(L_i \cdot M_i) + 2x_i^2 - 2x_i^2 - 2H_{S_i}^2 \\ & - 2x_i^2 + 4x_i(M_i \cdot c_1) - 2x_i^2 c_1^2 - 4x_i(M_i \cdot c_1) \\ & + 4x_i^2 c_1^2 - 2x_i^2 c_1^2 + 2x_i^2 \\ = & 2x_i(L_i \cdot M_i). \end{aligned} \tag{2.45}$$

It is not hard to see, from [Har77] that on the ruled surface E_i^*, the line bundle $-L_i = h_i^* + (d_i + 2)f$ has a section which is also a section of E_i^*, (actually we can say much more, namely that $-L_i$ is ample) and since M_i is ample it follows immediately that

$$K_{X_i} \cdot H_i^2 + 2H_{S_i}^2 < 0.$$

Finally, from (2.41) we obtain:

$$K_{X_0} \cdot H_{X_0}^2 = K_{X_p} \cdot H_p^2 + \sum_{i=1}^{n} (K_{X_i} \cdot H_i^2 + 2H_{S_i}^2) < K_{X_p} \cdot H_p^2 < 0. \tag{2.46}$$

□

Corollary 2.58. *$\mathcal{P}_{X_0}$ holds true.*

Proof. The line bundle $H_{X_0} = (H_p, H_1, \ldots, H_n)$ has H_p ample by Corollary 2.51 and, by Proposition 2.56, the H_i 's are also ample. Thus H_{X_0} is ample. With this, Proposition 2.57 actually says that $\mathcal{P}_{X_0}$ holds true. □

Remark 2.59. It would seem that the choice of the line bundle H_p is not important, as by attaching more curves to C we could arbitrarily decrease $K_{X_0} \cdot H_{X_0}^2$. However, this is not correct, since H_p depends on the number of curves we attach, which makes the intersection number $K_{X_0} \cdot H_{X_0}^2$ hard to control. This is the main reason we made a choice for the line bundle H_p in Section 2.4.6.

2.5 Proof of the Main Theorem

Proposition 2.60. *Let X be a rationally connected, projective threefold, and let X_C be the blowing-up of X along a smooth curve $C \subset X$. Then $\mathcal{P}_{X_C}$ holds true if and only if $\mathcal{P}_X$ holds true.*

Proof. The implication $\mathcal{P}_X \Longrightarrow \mathcal{P}_{X_C}$ is the conclusion of Proposition 2.20.

Conversely, we have two cases.

- If $K_X \cdot C < 0$ the result follows from Proposition 2.21.
- If $K_X \cdot C \geq 0$ applying the results of the previous section we obtain a flat, proper family $\Phi : \mathcal{X} \to \Delta$ over the unit disk, such that:

1) $\mathcal{X}$ is smooth;

2) For any $t \in \Delta$, $t \neq 0$, $\Phi^{-1}(t) = X_{C_t}$, the blowing-up of X along a smooth curve with $K_X \cdot C_t < 0$;

3) The central fiber $X_0 = \Phi^{-1}(0)$ is a normal crossing divisor in $\mathcal{X}$ with smooth rationally connected components. Moreover, $\mathcal{P}_{X_0}$ holds true.

By eventually shrinking Δ, we know from Theorem 2.43 and Proposition 2.44 that H_{X_0} can be extended to a line bundle on $\mathcal{X}$, whose restriction $H_{X_{C_t}}$ to the fiber X_{C_t} is ample. Moreover, since $K_{X_0} \cdot H_{X_0}^2 < 0$, from Corollary 2.46 it follows that $K_{X_t} \cdot H_{X_{C_t}}^2 < 0$. Thus, $\mathcal{P}_{X_{C_t}}$ holds true.

Since for all $i = 1, \ldots, n,\ K_X \cdot L_i < 0$, by attaching sufficiently many such curves, $K_X \cdot C_0 < 0$ and by the conservation law we will have $K_X \cdot C_t < 0$, for any $t \in \Delta$. Thus the blowing-up $X_{C_t} \to X$ is along the curve C_t for which $K_X \cdot C_t < 0$. Applying again Proposition 2.21 it follows that $\mathcal{P}_{X_{C_t}} \Longrightarrow \mathcal{P}_X$ and we are done. □

From the weak factorization theorem and Propositions 2.19 and 2.60 we easily get:

Theorem 2.61. *$\mathcal{P}$ is a birational property of rationally connected, projective threefolds.*

We can turn now to the main result of this paper:

Proof of Theorem A. If X is a rationally connected threefold, its Kodaira dimension is negative, and X is birational to a Mori fiber space, which we denote by X_{min}. Then, by Proposition 2.14, $\mathcal{P}_{X_{min}}$ holds true. Moreover, if X' is an appropriate desingularization of X_{min}, by Proposition 2.16 $\mathcal{P}_{X'}$ also holds true. Since X and X' ar smooth and birational, the conclusion of the Theorem A follows now from Theorem 2.61. □

Appendix

A Intersection Theory

Throughout this thesis we used the intersection theory algebraic varieties, from a cohomological viewpoint. This is introduced via the the following theorem due to Snapper [Kle66].

Theorem A.1. *Let X be a proper scheme over a field k, $\mathcal{F}$ a coherent sheaf on X, and let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be r Cartier divisors on X. Then the Euler-Poincaré function*

$$(m_1, \ldots, m_r) \longmapsto \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{L}_r^{\otimes m_r})$$

takes the same values on $\mathbb{Z}^r$ as a polynomial with rational coefficients of degree at most the dimension of the support of $\mathcal{F}$.

Definition A.2. *Let $L_1, \ldots, L_r$ be Cartier divisors on a proper scheme X, $\mathcal{F}$ a coherent sheaf on X. Suppose $r \geq$ dim Supp Z. The intersection number of $L_1, \ldots, L_r$ with $\mathcal{F}$, denoted by*

$$(L_1 \cdots L_r \cdot \mathcal{F})$$

is the coefficient of $m_1, \cdots, m_r$ *in the polynomial*

$$\chi(X, \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \cdots \otimes \mathcal{L}_r^{\otimes m_r})$$

Notations: Let X be a proper scheme over a field $\mathbf{k}$. If $Z \subset X$ a closed subscheme of dimension s and $L_1, \ldots, L_r$ Cartier divisors on X, for any $r \geq s$ we will denote by $(L_1 \cdots L_r \cdot Z)$ the intersection number $(L_1 \cdots L_r \cdot \mathcal{O}_Z)$. If $L_1 = \cdots = L_d$ then we also use the notation $(L^d \cdot Z)$. When $Z = X$ we use $(L_1 \cdots L_d)$ if no confusion is likely.

The basic properties of intersection numbers [Kle66] are summarized in the following:

Proposition A.3. *The intersection numbers* $(L_1 \cdots L_r \cdot \mathcal{F})$ *are uniquely determined by the following properties:*

1) *The intersection number is an integer.*

2) $(L_1 \cdots L_r \cdot Z)$ *is symmetric and multilinear in the* L_i.

3) $(L_1 \cdots L_r \cdot Z) = 0$ *if dim* $Z < r$.

4) *For any exact sequence of coherent sheaves*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

we have

$$(L_1 \cdots L_r \cdot \mathcal{F}) = (L_1 \cdots L_r \cdot \mathcal{F}') + (L_1 \cdots L_r \cdot \mathcal{F}'').$$

5) *If j denotes the the inclusion $j : Z \to X$, we have*

$$(j^*L_1 \cdots j^*L_r \cdot Z)_Z = (L_1 \cdots L_r \cdot Z)_X.$$

6) *If D is an effective Cartier divisor and $L_r = \mathcal{O}_X(D)$, we have*

$$(L_1 \cdots L_r) = (L_1 \cdots L_{r-1} \cdot \mathcal{O}_D) = (L_1 \cdots L_{r-1} \cdot D).$$

7) *If $f : X' \to X$ is a morphism of finite degree, and $r \geq$ dim $X =$ dim X', then*

$$(L_1 \cdots L_r) = \deg(f)(f^*L_1 \cdots f^*L_r).$$

Corollary A.4. *We have:*

1) *If $L_1 \equiv 0$ then $(L_1 \cdots L_d \cdot Z) = 0$.*

2) *If $X \subset \mathbb{P}_N$ and $\mathcal{O}_X(L) = \mathcal{O}_X(1)$ then $(L^d \cdot Z) = \deg\ Z$.*

3) *If $f : X' \to X$ is a birational morphism, and $r \geq$ dim $X =$ dim X', then*

$$(L_1 \cdots L_r \cdot X) = (f^*L_1 \cdots f^*L_r \cdot X').$$

For our computations we use the following identity (see [Deb01, page 8]), which follows easily from Proposition A.3.4:

$$(L_1 \cdots L_r) = \sum_{I \subset \{1,\ldots,r\}} \varepsilon_I \chi \left(X, -\sum_{i \in I} L_i \right) \tag{2.47}$$

where $\varepsilon_I = (-1)^{\mathrm{Card}(I)}$.

B The Blowing Up

Of particular importance for the computations we performed was the multiplication table in the Chow ring of the blowing-up of threefolds. The results [GH78] we used are:

Proposition B.1. *Let X be the a smooth projective threefold, and let $p : X' \to X$ be the blow-up at a point. Let $E \simeq \mathbb{P}_2$ be the exceptional divisor, and f the class of a line of E in the Chow ring $A^*(X')$. Then*

$$A^*(X') = p^*A^*(X) \oplus \mathbb{Z} \cdot E \oplus \mathbb{Z} \cdot f$$

as an additive group. Moreover $p_*E = p_*f = 0$, and $p_*p^*A^*(X) = A^*(X)$. The multiplicative structure of $A^*(X')$ is determined by:

$$E^2 = -f,\ E^3 = -E \cdot f = 1,\ E \cdot p^*Z = f \cdot p^*Z = 0$$

for any cycle $Z \in A^*(X)$.

Proposition B.2. *Let X be the a smooth projective threefold, and let $p : X' \to X$ be the blow-up along a smooth curve $C \subset X$. Let $E \simeq \mathbb{P}(N^\vee_{C/X})$ be the exceptional divisor, and f the class of a fiber of E in the Chow ring $A^*(X')$. Then*

$$A^*(X') = p^*A^*(X) \oplus \mathbb{Z} \cdot E \oplus \mathbb{Z} \cdot f$$

as an additive group. Moreover $p_*E = p_*f = 0$, and $p_*p^*A^*(X) = A^*(X)$. The *multiplicative structure of* $A^*(X')$ *is determined by:*

$$E^2 = -p^*C + \deg_C(N_{C/X}) \cdot f,\ E^3 = -\deg_C(N_{C/X}),\ E \cdot f = -1,$$
$$E \cdot p^*D = (C \cdot D)f,\ f \cdot p^*D = 0,\ \forall D \in A^1(X),$$
$$E \cdot p^*Z = f \cdot p^*Z = 0,\ \forall Z \in A^2(X).$$

In addition, the usual relation holds:

$$\deg_C(N_{C/X}) = 2g(C) - 2 - K_X \cdot C,$$

where $g(C)$ *is the genus of the curve* C, *and* $N_{C/X}$ *is the normal bundle of* C *in* X.

C Ruled Manifolds

The computations in Proposition 2.57 are based on the following:

Proposition C.1. *Let* $X = \mathbb{P}(\mathcal{E}) \xrightarrow{f} S$ *be the projectivization of a rank 2 vector bundle* $\mathcal{E}$ *over a smooth surface* S. *Let* $\xi = \mathcal{O}_X(1),\ c_1 = \det \mathcal{E}$, *and* $c_2 = c_2(\mathcal{E})$. *We have:*

i) $f^*L \cdot f^*K \cdot f^*M = 0,\ \forall L, K, M \in \mathrm{Pic}(S)$;

ii) $f^*L \cdot f^*K \cdot \xi = (L \cdot K),\ \forall L, K \in \mathrm{Pic}(S)$;

iii) $f^*L \cdot \xi^2 = (L \cdot c_1),\ \forall L \in \mathrm{Pic}(S)$;

iv) $\xi^3 = c_1^2 - c_2$;

v) $K_X = f^*L - 2\xi$, *where* $L = K_S + c_1$.

Proof. *i*) and *ii*) are obvious, while for *iii*) and *iv*) we use the identity [GH85]:

$$\xi^2 - f^*c_1 \cdot \xi + f^*c_2 = 0.$$

Thus:

$$f^*L \cdot \xi^2 = f^*L \cdot f^*c_1 \cdot \xi - f^*L \cdot f^*c_2 = (L \cdot c_1);$$
$$\xi^3 = f^*c_1 \cdot \xi^2 - f^*c_2 \cdot \xi = c_1^2 - c_2.$$

v) is also well-known. □

Similarly, for projectivizations of rank three vector bundles we needed the following:

Proposition C.2. *Let* $X = \mathbb{P}(\mathcal{E}) \xrightarrow{f} C$ *be the projectivization of a rank 3 vector bundle* $\mathcal{E}$ *over a smooth curve* C. *Let* $\xi = \mathcal{O}_X(1)$, $c_1 = \det \mathcal{E}$, *and* $c_2 = c_2(\mathcal{E})$. *We have:*

i) $f^*L \cdot f^*K \cdot M = 0, \ \forall L, K \in \mathrm{Pic}(C), M \in \mathrm{Pic}(X)$;

ii) $f^*L \cdot \xi^2 = \deg L, \ \forall L \in \mathrm{Pic}(C)$;

iii) $\xi^3 = \deg \mathcal{E}$;

iv) $K_X = f^*L - 3\xi$, *where* $L = K_S + c_1$.

Proof. As in the proof of the previous proposition, *i*) and *ii*) are obvious and

iv) is well known. For two we use again the fundamental identity [GH85]:

$$\xi^2 - f^* c_1 \cdot \xi = 0.$$

Multiplying by ξ, from ii) we immediately get $\xi^3 = \deg \mathcal{E}$. □

Also, of particular importance are the classical ampleness conditions on ruled surfaces. For the following results, the interested reader is referred to [Har77].

Recall that if $\pi : X \to C$ is a ruled surface, one can write $X \simeq \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a normalized, locally free, rank two sheaf on C. The integer $e = -\deg(\mathcal{E})$ is an invariant of X, and there is a section $\pi_0 : C \to X$ with image C_0, such that $\mathcal{L}(C_0) \simeq \mathcal{O}_X(1)$. Any element of $\mathrm{Pic}(X)$ can be written as $aC_0 + \pi^*\ell$ with $a \in \mathbb{Z}$ and $\ell \in \mathrm{Pic}(C)$, and any element of the $\mathrm{Num}(X)$ can be written $aC_0 + bf$ with $a, b \in \mathbb{Z}$ and where f is the class of a fiber of X. We denote by g the genus of the curve C.

Proposition C.3. *Let $\pi : X \to C$ be a ruled surface, and a divisor $D \equiv aC_0 + bf$.*

(a) If $g = 0$, then $e \geq 0$, and D is ample if and only if $a > 0$ and $b > ae$.

(b) If $g > 0$, according to the the sign of e we have:

i) If $e \geq 0$, D is ample if and only if $a > 0$, $b > ae$.

ii) If $e < 0$, D is ample if and only if $a > 0$, $b > \frac{1}{2}ae$.

Bibliography

[AKMW02] D. Abramovich, K. Karu, K. Matsuki, J. Włodarczyk, *Torification and factorization of birational maps*, J. of Amer. Math. Soc., 15 (2002), 531–572.

[AnWi98] M. Andreatta, J. Wiśniewski, *On contractions of smooth varieties*, J. of Algebraic Geom., 7 (1998), no. 2, 253–312.

[ArKo03] C. Araujo, J. Kollár, *Rational curves on varieties*, in Higher Dimensional Varieties and Rational Points, Bolyai Society Mathematical Studies, 12 (2003), 13–69.

[Bar85] R. Barlow, *A simply connected surface of general type with* $p_g = 0$, Invent. Math., 79 (1985), 293–301.

[BPV84] W. Barth, C. Peters, A. van de Ven, *Compact Complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 4, Springer Verlag, Berlin, 1984.

[Bes87] A. Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10, Springer Verlag, Berlin, 1987.

[CaPe98] F. Campana, T. Peternell, *Rational curves and ampleness properties of the tangent bundle of algebraic varieties*, Manuscripta Math. 97 (1998), 59–74.

[Cat78] F. Catanese, *Surfaces with $K^2 = p_g = 1$ and their period mapping*, in Algebraic Geometry (Proc. of Summer Meeting, Univ. of Copenhagen, Copenhagen, 1978), Lecture Notes in Math., 732, Springer, Berlin, 1979, 1–29.

[CaLe97] F. Catanese, C. LeBrun, *On the scalar curvature of Einstein manifolds*, Math. Res. Lett., 4 (1997), 843–854.

[Deb01] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001.

[DPS96] J. P. Demailly, T. Peternell, M. Schneider, *Holomorphic line bundles with partially vanishing cohomology*, in Proceedings of Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Israel Math. Conf. Proc., Vol. 9, Bar-Ilan Univ. Ramat Gan, 1996, 165–198.

[Don87] S.K. Donaldson, *Irrationality and the h-cobordism conjecture*, J. of Diff. Geom., 26 (1987), 141–168.

[EiHa00] D. Eisenbud, J. Harris, *The geometry of schemes* Graduate Texts in Mathematics, 197. Springer-Verlag, New York, 2000.

[FaJo91] F.T. Farrell, L.E. Jones, *Rigidity in geometry and topology*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo 1991, 653–663.

[Fre82] M. Freedman, *The topology of four manifolds*, J. of Diff. Geom., 17 (1982), 357–454.

[FrMo97] R. Friedman, J.W. Morgan *Algebraic surfaces and Seiberg-Witten invariants*, J. Alg. Geom., 6 (1997), 445–479.

[Fri83] R. Friedman, *Global smoothings of varieties with normal crossings*, Ann. of Math. (2) 118 (1983), no. 1, 75–114.

[FrMo94] R. Friedman, J.W. Morgan *Smooth four-manifolds and complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 27, Springer Verlag, Berlin, 1994.

[FrQi94] R. Friedman, Z. Qin, *The smooth invariance of the Kodaira dimension of a complex surface*, Math. Res. Lett., 1 (1994), 369–376.

[Ful98] Fulton, William, *Intersection theory. Second edition*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 2, Springer Verlag, Berlin, 1998.

[GHS03] T. Graber, J. Harris, J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. 16 (2003), no. 1, 57–67

[GH78] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley, New York, 1978.

[GH85] P. Griffiths, J. Harris, *On the Noether-Lefschetz theorem and some remarks on codimension - two cycles*, Math. Ann. 271 (1985), no. 1, 31–51.

[Gro97] M. Gross, *The deformation space of Calabi-Yau n-folds with canonical singularities can be obstructed*, in Mirror Symmetry Vol. II, AMS/IP. Stud. Adv. Math. 1, Amer. Math. Soc., Providence, RI, 1997, 401–411.

[Har77] R. Hartshorne, *Algebraic Geometry*, Graduate Text in Mathematics, 52, Springer Verlag, New-York-Heidelberg, 1977.

[Kle66] S. L. Kleiman, *Toward a numerical theory of ampleness*, Ann. of Math. (2) 84, 1966, 293–344.

[Kod70] K. Kodaira, *On homotopy K3 surfaces*, Essay on Topology and Related Topics (dedicated to G. de Rham), Springer, New York, 1970, 58–69.

[Kod63] K. Kodaira, *On stability of compact submanifolds of complex manifolds*, Amer. J. Math. 85, 1963, 79–94.

[Kol96] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 32, Springer Verlag, Berlin, 1996.

[KMM92] J. Kollár,Y. Miyaoka, S. Mori, *Rationally connected varieties*, J. Algebraic Geom., 1 (1992), 429–448.

[KoMo98] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, 134, Cambridge University Press, Cambridge, 1998.

[KoMo92] J. Kollár, S. Mori, *Classification of three-dimensional flips*, J. Amer. Math. Soc., 5 (1992), 533–703.

[LeB99] C. LeBrun, *Topology versus Chern numbers for complex* $3-$*folds*, Pacific J. Math., 191 (1999), 123–131.

[Man01] M. Manetti, *On the moduli space of diffeomorphic surfaces*, Invent. Math., 143 (2001), 29–76.

[Mil65] J. Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, Princeton, NJ, 1965.

[Mil66] J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc., 72 (1966), 356–426.

[Miy88] Y. Miyaoka, *On the Kodaira dimension of minimal threefolds*, Math. Ann. 281 (1988), no. 2, 325–332.

[MiyMo86] Y. Miyaoka, S. Mori, *A numerical criterion for uniruledness*, Ann. of Math. (2) 124 (1986), 65–69.

[MiPe97] Y. Miyaoka, Th. Peternell, *Geometry of higher-dimensional algebraic varieties*, DMV Seminar, 26. Birkhuser Verlag, Basel, 1997.

[OkVdV95] Ch. Okonek, A. Van de Ven, *Cubic forms and complex* 3-*folds*, Enseign. Math. 41 (1995), 297–333.

[Per77] U. Persson, *On degenerations of algebraic surfaces*, Mem. Amer. Math. Soc. 11, no. 189, 1977.

[Ruan94] Y. Ruan, *Symplectic topology on algebraic* 3-*folds*, J. of Diff. Geom. 39 (1994), 215–227.

[Ruan96] Y. Ruan, *Topological sigma model and Donaldson-type invariants in Gromov theory.*, Duke Math. J. 83 (1996), 461–500.

[Wal78] F. Waldhausen, *Algebraic K-theory of generalized free products III, IV.*, Ann. of Math., (2) 108 (1978), 205–256.

[Wall62] C.T.C. Wall, *On the orthogonal groups of unimodular quadratic forms*, Math. Ann. 147 (1962), 328-338.

[Wall64] C.T.C. Wall, *On simply connected 4-manifolds*, J. of London Math. Soc. 39 (1964), 141–149.

[Yau74] S. T. Yau, *On the curvature of compact Hermitian manifolds*, Invent. Math. 25 (1974), 213–239