Calcul d'algèbre de Frobenius sur l'homologie des lacets libres d'une variété.

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A mes parents.
A Delphine ainsi qu'à tous ceux de mon entourage qui m'ont soutenu durant ces trois années.

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Introduction générale.

La *théorie topologique des cordes* (*String topology*) est la description des différentes structures algébriques naturelles supportées par l'homologie (et l'homologie équivariante) de l'espace des lacets libres, LM, d'une variété M compacte connexe orientée de dimension m qui est supposée sans bord.

Précisons que $LM \subset M^{S^1}$ désigne l'espace topologique des applications différentiables par morceaux

$$\gamma : [0,1] \to M \, , \gamma(0) = \gamma(1) \, ,$$

L'espace LM est une variété de Hilbert, [15], [26], [10] et l'application d'évaluation

$$LM \to M \, , \quad \gamma \mapsto \gamma(0)$$

est un fibré localement trivial, [7]-Lemme 5-4-1, de fibre l'espace les lacets pointés, différentiables par morceaux, noté ΩM. Il existe une opération naturelle du groupe $S^1 = \mathbb{R}/\mathbb{Z}$ sur LM :

$$S^1 \times LM \to LM \, , \quad (s, \gamma) \mapsto \gamma_s \, , \gamma_s(t) = \gamma(s+t) \, , t \in [0,1]/0 = 1 = \mathbb{R}/\mathbb{Z} \, .$$

Précisons aussi que pour tout espace topologique X, $H_*(X)$ désigne l'homologie singulière à coefficients dans $\boldsymbol{k}$ où $\boldsymbol{k}$ désigne un anneau commutatif unitaire.

Le premier exemple de ce type de structure a été construit en 1999, par Chas et Sullivan, [8]. Il s'agit du *loop-produit*

$$H_p(LM) \otimes H_q(LM) \to H_{p+q-m}(LM) \, , \quad (x, y) \mapsto x \bullet y$$

qui induit sur la désuspension

$$\mathbb{H}_*(LM) := H_{*+m}(LM)$$

une structure d'algèbre graduée commutative. Dans l'article original, [8], le loop-produit est obtenu par une combinaison de la multiplication des lacets pointés

$$\Omega M \times \Omega M \to \Omega M \, , \quad (\alpha, \beta) \mapsto \alpha \star \beta$$

et du produit d'intersection (= Poincaré dual du cup produit)

$$H_p(M) \otimes H_q(M) \to H_{p+q-m}(M) \quad (x, y) \mapsto x \bullet y \, .$$

Une définition équivalente du loop-produit en termes de bordismes est donnée par Chataur dans [10].

Cette structure d'algèbre commutative sur $\mathbb{H}_*(LM)$ est sous-jacente à deux structures algébriques, celle d'algèbre de Batalin-Vilkovisky et celle d'algèbre de Frobenius. Rappelons que :

- Une *algèbre commutative graduée* est un k-module gradué $A = \{A_i\}_{i\geq 0}$ muni d'une application $\mathbb{k}$-linéaire

$$A_p \otimes A_q \to A_{p+q}\,, \quad x \otimes y \mapsto xy$$

qui est associative, qui admet une unité et qui vérifie la relation $xy = (-1)^{pq} yx$ pour tout $x \in A_p$ et tout $y \in A_q$.

- Une *algèbre de Lie graduée* est un k-module gradué $L = \{L_i\}_{i\geq 1}$ muni d'une application bilinéaire

$$L_p \otimes L_q \to L_{p+q}\,, \quad x \otimes y \mapsto [x, y]$$

et d'un opérateur quadratique

$$L_p \to L_{2p}\,, \quad x \mapsto \phi(x)$$

qui vérifient, pour tout $x \in L_p$, $y \in L_q$ et $z \in L_r$, les axiomes suivants, [5] :
(1) $[x, y] = (-1)^{pq}[y, x]$,
($1\frac{1}{2}$) $[x, x] = 0\,,$ si p est pair,
(2) $(-1)^{pr}[x, [y, z]] + (-1)^{rq}[z, [x, y]] + (-1)^{qp}[y, [z, x]] = 0$,
($2\frac{1}{3}$) $[x, [x, x]] = 0$ si p est impair,
(3) $[x, y] = \phi(x + y) - \phi(x) - \phi(y)$ si p et q sont impairs
(4) $[x, [x, y]] = [\phi(x), y]$ si p est impair.

- Une *algèbre de Gerstenhaber* est une algèbre commutative graduée $A = (\{A_i\}_{i\geq 0}, \cdot)$ munie d'une application bilinéaire

$$A_p \otimes A_q \to A_{p+q+1}\,, \quad x \otimes y \mapsto \{x, y\}$$

qui induit sur A_{*-1} une structure d'algèbre de Lie graduée :

- Une *algèbre de Batalin-Vilkovisky* est algèbre commutative graduée $A = \{A_i\}_{i\geq 0}$ munie d'une application linéaire

$$\Delta : A_* \to A_{*-1}$$

telle que
(a) $\Delta \circ \Delta = 0$
(b) Le crochet $\{\ ,\ \}$ défini par

$$\{x, y\} = (-1)^p \Delta(xy) - (-1)^p \Delta(x)y - x\Delta(y)$$

fait de A une algèbre de Gerstenhaber.

- une *algèbre de Frobenius* est une algèbre commutative graduée $A = \{A_i\}_{i\geq 0}$ munie d'une application A-linéaire

$$\Phi : A_p \to (A \otimes A)_{p-d}$$

qui est coassociative et cocommutative et n'admettant pas nécessairement de co-unité. Ici l'action de A sur A (resp. sur $A \otimes A$) est donnée par la multiplication à gauche (resp. par $a(x \otimes y) = (ax) \otimes y$.

Dans la suite nous nous intéressons uniquement à la structure d'algèbre de Frobenius de $H_*(LM)$ ie au loop-produit

$$H_p(LM) \otimes H_q(LM) \to H_{p+q-m}(LM)\,, \quad (x,y) \mapsto x \bullet y$$

et au loop-coproduit

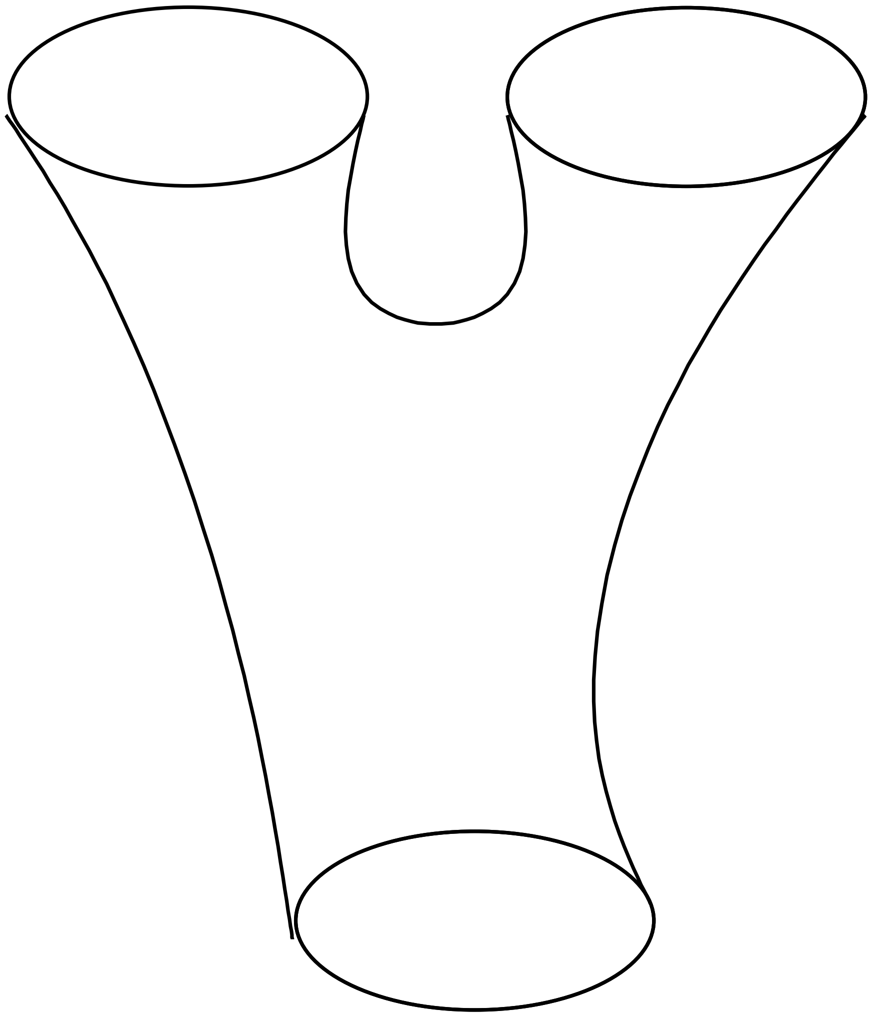
$$H_p(LM) \to (H_*(LM) \otimes H_*(LM))_{p-m}\,, \quad (x) \mapsto \Phi(x)\,.$$

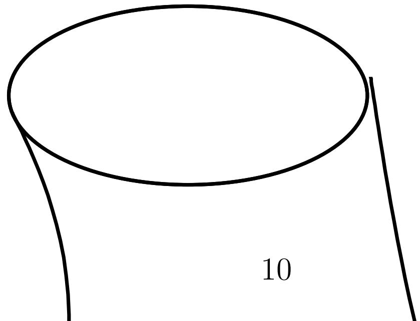
Afin de calculer ce produit et ce coproduit il est préférable d'en donner une définition plus combinatoire.

Le loop-produit et le loop-coproduit sont deux cas particuliers de (p,q)-*loop-opérations* $(p=2, q=1)$ pour le loop-produit et $(p=1, q=2)$ pour le loop-coproduit. Les (p,q)-loop-opérations se définissent en termes de *diagrammes de corde.* Les diagrammes de corde définissent de façon combinatoire les surfaces $\Sigma_{g,p,q}$ topologiques compactes de genre g admettant $p+q$-composantes de bord. Par suite, ces (p,q)-loop-opérations sont reliées à une théorie des champs quantiques topologiques (TQFT) en dimension 2. En particulier, la structure d'algèbre de Frobenius étudiée ici s'interprète en termes de TQFT. (Une interprétation de la la structure d'algèbre de Batalin-Vilkovisky, évoquée ci-dessus, en termes de TFT (Topological Field Theory) est fournie dans [12].)

Précisons ici quelque points.

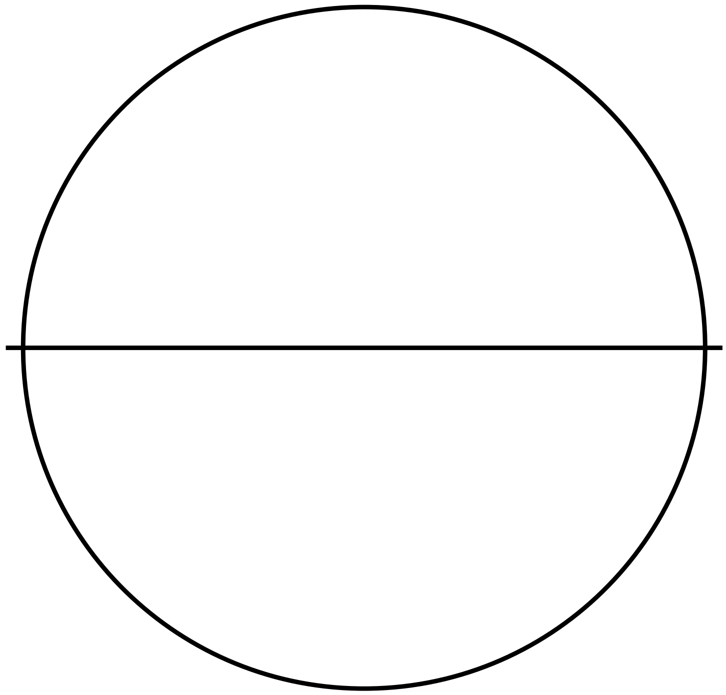
Les TQFT ont été introduites par Atiyah, [4] en 1988. Les structures algébriques induites par ces TQFT ont été beaucoup étudiées pour les dimensions 3 et 4, [18]. En dimension 2, la TQFT est reliée à la théorie des champs conformes en dimension 2, (2-CFT), définie par G. Segal [31] et étudiée par E. Witten [36]. En 1996, L. Abrams, [1], a construit une "équivalence naturelle" entre une TQFT de dimension 2 et une algèbre de Frobenius avec co-unité. En 2003, R. Cohen et V. Godin, [13], introduisent la notion de 2-PBTQFT "positive boundary 2-dimensionnal TQFT" et démontrent que les (p,q)-loop-opérations correspondent aux surfaces de genre 0 admettant p composantes de "bord entrant" et $q \geq 1$ composantes de "bord sortant". Ils en déduisent en particulier que le loop-produit et le loop-coproduit définissent sur $H_*(LM)$ une structure d'algèbre de Frobenius sans co-unité. Par exemple la figure de gauche (resp de droite) correspond au loop-produit (resp. au loop-coproduit).

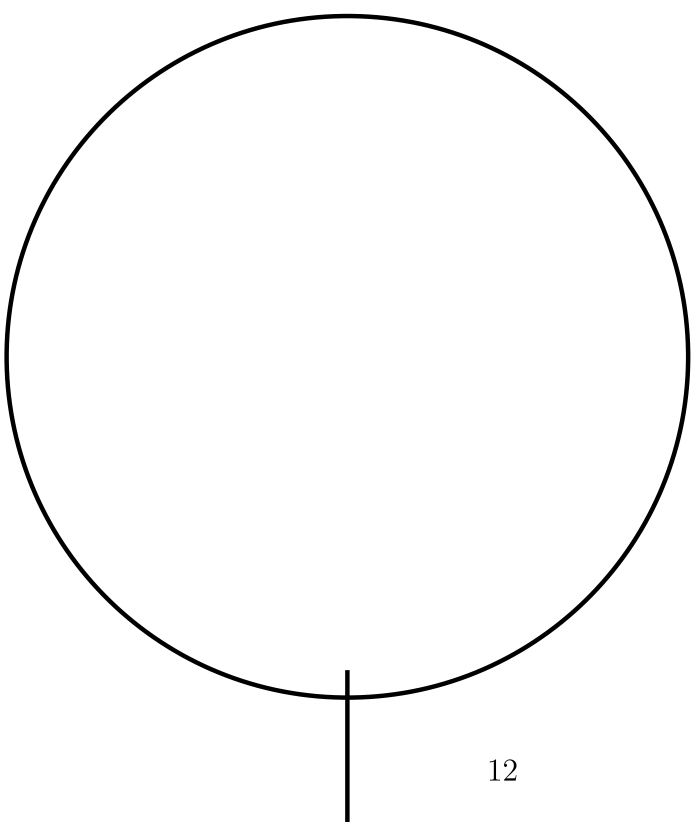




$p = 2, q = 1$ $\qquad\qquad$ $p = 1, q = 2$

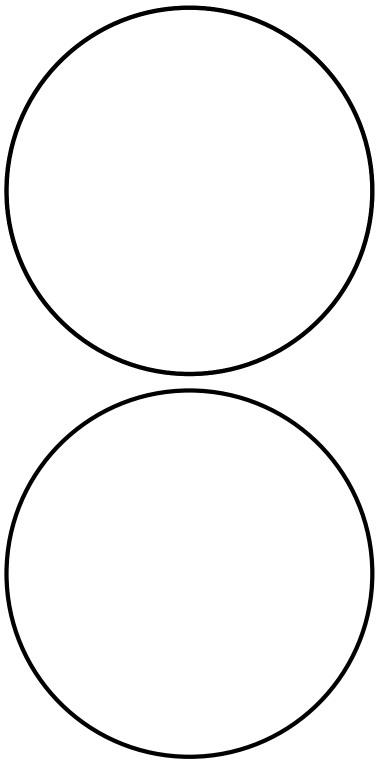
A une surface $\Sigma_{g,p,q}$, compacte connexe de genre g admettant p composantes de “bord entrant” et $q \geq 1$ composantes de “bord sortant” on associe un diagramme de corde $C_{g,p,q}$ qui caractérise topologiquement la surface. Par exemple les surfaces ci-dessus admettent pour diagramme de corde





$p=2, q=1$ $\qquad\qquad$ $p=1, q=2$

Dans le diagramme $C_{0,p,q}$ apparaissent les p "cercles entrants" tandis que les q "cercles sortants" sont décrits commes des "cycles" (cf [34], [13], [35]). De la même manière que l'on enlève les "infinité" dans les diagrammes de Feynman, en écrasant les "arêtes fantômes" nous obtenons le diagramme réduit $SC_{0,p,q}$. Par exemple, les diagrammes de corde ci-dessus se réduisent de la même manière :



Ce procédé de réduction définit de façon purement combinatoire le diagramme commutatif ci-dessous

$$\begin{array}{ccccc} (LM)^{\times q} \simeq \mathrm{map}\,((S^1)\coprod^q, M) & \overset{\rho_{out}}{\longleftarrow} & \mathrm{map}\,(SC_{0,p,q}, M) & \overset{\rho_{in}}{\longrightarrow} & (LM)^{\times p} \simeq \mathrm{map}\,((S^1)\coprod^p, M) \\ ev_{out} \downarrow & & ev_{vertex} \downarrow & & \downarrow ev_{in} \\ M^{\times q} & \overset{\Delta_{out}}{\longleftarrow} & M^{\times p+q} & \overset{\Delta_{in}}{\longrightarrow} & M^{\times p} \end{array}$$

où ev_{-} désigne l'évaluation en certains sommets du diagramme de corde $C_{0,p,q}$ et Δ_{in}, Δ_{out} désignent les "applications diagonales induites"

Le point important dans ce diagramme commutatif est que le carré de gauche qui fait intervenir ρ_{in} est un "pull-back" diagramme de fibrés localement triviaux et que ρ_{in} est un plongement de codimension finie entre deux variétés de Hilbert.

En appliquant la théorie de Thom-Pontryagin, [3], [6], [25], [26], à ρ_{in}, nous obtenons l'application

$$(\rho_{in})_! : H_*(LM)^{\otimes p} \hookrightarrow H_*(LM^{\times p}) \rightarrow H_{*+\chi m}(\mathrm{map}\,(SC_{0,p,q}, M))$$

pour tout anneau de coefficients $\mathbb{k}$ et où $\chi = 2-2g-p-q$ désigne la caractéristique d'Euler de la surface $\Sigma_{g,p,q}$.

La (p,q)-loop-opération $\mu_{C_{0,p,q}}$ est définie comme la composition

$$H_*(LM)^{\otimes p} \overset{(\rho_{in})_!}{\longrightarrow} H_{*+\chi m}(\mathrm{map}\,(SC_{0,p,q}, M)) \overset{H_*(\rho_{out})}{\longrightarrow} H_{*+\chi m}((LM)^q)\,.$$

Par exemple au diagramme de corde $C_{0,2,1}$ correspond le diagramme commutatif

$$\begin{array}{ccccc} LM & \stackrel{\rho_{out}}{\leftarrow} & \mathrm{map}\,(S^1 \vee S^1, M) = LM \times_M LM & \stackrel{\rho_{in}}{\rightarrow} & (LM)^{\times 2} \\ ev_0 \downarrow & & ev_0 \downarrow & & \downarrow (ev_0, ev_0) \\ M & = & M & \stackrel{\Delta}{\rightarrow} & M^{\times 2} \end{array}$$

où ρ_{out} désigne la composition des lacets tandis que ρ_{in} désigne l'inclusion vue comme une application fibrée au-dessus de la diagonale. Le loop-produit est alors défini par

$$H_*(LM)^{\otimes 2} \stackrel{(\rho_{in})_!}{\rightarrow} H_{*-m}(LM \times_M LM) \stackrel{H_*(\rho_{out})}{\rightarrow} H_{*-m}(LM)\,.$$

Au diagramme de corde $C_{0,1,2}$ correspond le diagramme commutatif

$$\begin{array}{ccccc} (LM)^2 & \stackrel{\rho_{out}}{\leftarrow} & LM \times_M LM & \stackrel{\rho_{in}}{\rightarrow} & LM \\ (ev_0)^2 \downarrow & & ev_0 \downarrow & & \downarrow ev_0 \times ev_{\frac{1}{2}} \\ M^{\times 2} & \stackrel{\Delta}{\leftarrow} & M & \stackrel{\Delta}{\rightarrow} & M^2 \end{array}$$

où ρ_{out} désigne l'inclusion et ρ_{in} désigne la composition des lacets vue comme une application fibrée au-dessus de la diagonale. Le loop-coproduit est défini, pour tout anneau de coefficients, par

$$H_*(LM) \stackrel{(\rho_{in})_!}{\rightarrow} H_{*-m}(LM \times_M LM) \stackrel{H_*(\rho_{out})}{\rightarrow} H_{*-m}((LM)^{\times 2}) = (H_*(LM)^{\otimes 2})_{*-m}\,.$$

On imagine facilement que le calcul explicite du loop-produit et du loop-coproduit n'est pas aisé.

En 2002, Cohen et Jones, [15], ont montré que le loop-produit s'interprète en termes de spectres d'anneaux. Cette interprétation théorique avait pour but d'établir un isomorphisme d'algèbres entre $H_*(LM)$ muni du loop-produit et la cohomologie de Hochschild $HH^*(C^*M; C^*M)$ de l'algèbre des cochaînes singulières de M à coefficients dans elle-même. Ce dernier résultat a été établi peu après par S.A. Merkulov, [28] lorsque $\Bbbk = \mathbb{R}$ et ceci à l'aide du formalisme des intégrales itérées. En 2004, Félix, Thomas et Vigué, [23], ont démontré ce même résultat pour $\Bbbk = \mathbb{Q}$ à l'aide de la théorie des modèles de Sullivan, [33], [20]. Ainsi, dans le cas d'un corps de caractéristique zéro, le calcul du loop-produit se ramène à celui du *produit de Gerstenhaber*, [24], sur l'homologie de Hochschild $HH(A, A)$ d'une algèbre différentielle graduée. Cette méthode est explicitée dans [22] et [21]. Peu après, Chataur et Thomas [11] décrivent le loop-coproduit en homologie rationelle en termes du modèle minimal de M, [20], et de la classe diagonale de M, [6].

L'objet principal de cette thèse est le calcul du loop-produit **pour tout anneau** $\Bbbk$ et nous appliquons les mêmes méthodes afin de donner des informations sur le loop-coproduit.

Pour cela nous utilisons les techniques de suites spectrales de Leray-Serre.

Le premier chapitre de cette thèse est consacré au calcul du loop-produit. Dans le second chapitre nous illustrons sur des exemples les techniques développées dans le premier chapitre. En particulier nous calculons une approximation du loop-produit dans le cas où M est une variété de Stiefel : $SO(9)/SO(7)$. Cet exemple original de calcul montre l'efficacité de notre méthode. Dans le troisième chapitre nous donnons des informations sur le loop-coproduit obtenues grâce aux techniques de suites spectrales introduites dans le premier chapitre. Dans le dernier chapitre, nous faisons quelques remarques concernant l'invariance homotopique ainsi que sur les problèmes d'extension.

Chapitre 1

The loop-product spectral sequences.

Abstract

In this paper we prove that the shriek map associated to a finite codimensionnal sub-fiberwise embedding between Hilbert manifolds behaves nicely with respect to the associated Serre Spectral sequences. We apply this result in different settings, in particular we study the Chas-Sullivan loop-product.
AMS Classification : 55P35, 54N45,55N33, 17A65,
81T30, 17B55
Key words : free loop space, loop-homology,
Serre spectral sequence.

1.1 Introduction.

In this paper $\boldsymbol{k}$ is a fixed commutative ring, (co)chain complexes, (co)homology are with coefficients in $\boldsymbol{k}$.

Let M be a connected closed oriented m-manifold and let LM be the space free loops on M. Chas and Sullivan, [8], have constructed a natural product on the desuspension of the homology of the free loop space

$$\mathbb{H}_*(LM) := H_{*+m}(LM)$$

so that $\mathbb{H}_*(LM)$ is a commutative graded algebra. This product is called the *loop-product*. The purpose of this paper is to compute this algebra when the m-manifold M appears as the total space of a fiber bundle, (as for example for Stiefel manifolds).

To be more precise let us recall that if X and Y are two Hilbert connected smooth oriented manifolds without boundary and if $f : X \hookrightarrow Y$ is a for any smooth orientation preserving embedding of codimension $k < \infty$ there is a well defined homomorphism of $H_*(Y)$-comodules

$$f_! : H_*(Y) \rightarrow H_{*-k}(X)\,,$$

called the *homology shriek map* $f_!$ (see section 1.2.1 for details).

Our main result, consists to show that if $f : X \hookrightarrow Y$ is a *sub-fiberwise embedding* (definition below) then $f_!$ behaves really nicely with associated Serre spectral sequences. The remaining results of the paper are consequences of our main result.

Definition 1.1.1 *A sub-fiberwise embedding* (f, f^B) *is a commutative diagram*

$$\begin{array}{ccc} X & \overset{f}{\hookrightarrow} & X' \\ \downarrow p & & \downarrow p' \\ B & \overset{f^B}{\hookrightarrow} & B' \end{array}$$

where

$$(*)\quad \begin{cases} a) & X, X', B \text{ and } B' \text{ are connected Hilbert manifolds without boundary} \\ b) & f \text{ (respectively } f^B\text{) is a smooth embedding of finite codimension } k_X \\ & \text{(respectively } k_B\text{)} \\ c) & p \text{ and } p' \text{ are Serre fibrations} \\ d) & \text{for some } b \in B \text{ the induced map} \\ & \qquad f^F : F := p^{-1}(b) \to {p'}^{-1}(f(b)) := F' \\ & \text{is an embedding of finite codimension } k_F \\ e) & \text{embeddings } f, f^B \text{ and } f^F \text{ admit Thom classes.} \end{cases}$$

Following the terminology of [17] we will also define a fiberwise embedding when $f^B = id_B$ and a pull-back embedding when the above diagram is a pull-back diagram and thus $f^F = id_F$.

We denote by $C_*(X)$ the singular chain complex and by $C^*(X)$ the singular cochain complexe.

Theorem 1.1.2 First part *Let* $f : X \hookrightarrow X'$ *be a sub-fiberwise embedding as above. For each* $n \geq 0$ *there exists filtrations*

$$\{0\} \subset F_0C_n(X) \subset F_1C_n(X) \subset ... \subset F_nC_n(X) = C_n(X)$$
$$\{0\} \subset F_0C_n(X') \subset F_1C_n(X') \subset ... \subset F_nC_n(X') = C_n(X')$$

and a chain representative $f_! : C_*(X') \to C_{*-k_X}(X)$ *of* $f_! : H_*(X') \to H_{*-k_X}(X)$ *satisfying :*

$$f_!\left(F_*C_*(X')\right) \subset F_{*-k_B}C_{*-k_X}(X')\,.$$

Second part

The above filtrations induce the Serre spectral sequences of the fibrations p *and* p'*, denoted by* $\{E^r[p]\}_{r\geq 0}$ *and* $\{E^r[p']\}_{r\geq 0}$*. The chain map* $f_!$ *induces a homomorphism of bidegree* $(-k_B, -k_F)$ *between the associated spectral sequences*

$$\{E^r(f_!)\} : \{E^r[p']\}_{r\geq 0} \to \{E^r[p]\}_{r\geq 0}\,.$$

There exists a chain representative

$$f_!^B : C_*(B') \to C_{*-k_B}(B) \text{ (respectively } f_!^F : C_*(F') \to C_{*-k_F}(F)\text{)}$$

of $f_!^B : H_*(B') \to H_{*-k_B}(B)$ *(respectively of* $f_!^F : H_*(F') \to H_{*-k_F}(F)$ *) such that*

$$\{E^2(f_!)\} = H_*(f_!^B; \mathcal{H}_*(f_!^F)) : E^2_{s,t}[p'] = H_s(B'; \mathcal{H}_t(F')) \to E^2_{s-k_B,t-k_F}[p] = H_{s-k_B}(B; \mathcal{H}_{t-k_F}(F))\,,$$

where $\mathcal{H}(-)$ *denotes the usual system of local coefficients.*

This theorem is proved in section 1.4.5 page 32.

Given a first quadrant homology spectral sequence $\{E_r\}_{r\geq 0}$ it is natural, in our context, to define the (k_B,k_F)-*regraded spectral sequence* $\{\mathbb{E}^r\}_{r\geq 0}$ by :

$$\mathbb{E}^r_{*,*} = E^r_{*+k_B,*+k_F}\,.$$

Hereafter, we apply the main result in four different settings :

1. Classical intersection theory of smooth manifolds.
2. Loop-product on $H_*(LM)$.
3. Relative loop-product on $H_*(LM)$ and intersection morphism.
4. The path space : M^I.

1. Classical intersection theory revisited.

Recall that if M is a m-dimensionnal oriented closed manifold then the desuspended homology of M :

$$\mathbb{H}_*(M) = H_{*+m}(M)$$

is a commutative graded algebra for the intersection product. The intersection product $x\otimes y \mapsto x\bullet y$, is the Poincaré dual of the cup product, [6]. It is also defined by the composite :

$$H_*(M)\otimes H_*(M) \xrightarrow{\times} H_*(M\times M) \xrightarrow{\Delta_!} H_{*-m}(M)\,,$$

where $\Delta : M\to M\times M$ denotes the diagonal embedding and $\times$ the cross product. We deduce then from our main result :

Proposition 1.1.3 *Let $N\to X\xrightarrow{p} M$ be a Serre fibration such that*

a) N, (respectively M) is a finite dimensionnal smooth closed oriented manifold of dimension n (respectively m),

b) M is arcwise connected.

The (m,n)-regraded Serre spectral sequence $\{\mathbb{E}^r[p]\}_{r\geq 0}$ is a multiplicative spectral sequence which congerges to the algebra $\mathbb{H}_*(X)$. Furthermore, if $\pi_1(M)$ acts trivially on $H_*(N)$, then the tensor product of graded algebras

$$\mathbb{H}_*(M)\otimes\mathbb{H}_*(N)$$

is a subalgebra of $\mathbb{E}^2[p]$.

We prove this proposition in section 1.5.1 page 33. This result is the Poincaré dual of the multiplicative structure of the Serre spectral sequence in cohomology as shown by the following proposition :

Proposition 1.1.4 *Let $N\to X\xrightarrow{p} M$ be a Serre fibration of finite dimensionnal oriented closed manifolds. We denote by $[X]$ the fundamental class of X. Then the Poincaré duality isomorphism $DP : H^*(X)\to\mathbb{H}_{-*}(X)\quad f\mapsto f\cap[X]$ preserves the Serre filtration so that it induces an isomorphism $E^*(DP) : E_n^{*,*}\to\mathbb{E}^n_{-*,-*}$ for $n\geq 2$*

These propositions are proved in section 1.5.1 page 33. The compatibility of the Serre spectral sequence to the intersection product, to the cup product and to the Poincaré duality isomorphism provides immediately the following theorem :

Theorem 1.1.5 *Let $N \to X \xrightarrow{p} M$ be a Serre fibration of finite dimensionnal oriented closed manifolds. Then, the Serre spectral sequence associated to this fibration is a spectral sequence of Frobenius algebra (see [1] for a definition)*

2. Loop-product on $H_*(LM)$.

Chas and Sullivan have defined the loop-product using "transversal geometrical chains". An other description of the loop-product is the following, [16].

We consider the diagram

$$\begin{array}{ccccc} LM \times LM & \xleftarrow{\tilde{\Delta}} & LM \times_M LM & \xrightarrow{Comp} & LM \\ \downarrow{\scriptstyle ev_0 \times ev_0} & & \downarrow{\scriptstyle ev_0} & & \downarrow{\scriptstyle ev_0} \\ M \times M & \xleftarrow{\Delta} & M & \xrightarrow{=} & M \end{array}$$

where

a) $ev(0)(\gamma) = \gamma(0)\,, \gamma \in LM$,

b) the left hand square is a pull-back embedding defined in 1.4.6.

c) *Comp* denotes composition of free loops.

Here we assume that LM is a Hilbert manifold when we restrict to "smooth loops" and thus $(\tilde{\Delta}, \Delta)$ is a sub-fiberwise embedding of codimension m.

The loop-product $x \otimes y \mapsto x \circ y$ is then the composite

$$H_*(LM)^{\otimes 2} \xrightarrow{\times} H_*(LM^{\times 2}) \xrightarrow{\tilde{\Delta}_!} H_{*-m}(LM \times_M LM) \xrightarrow{H_*(Comp)} H_{*-m}(LM)$$

Under this product the desuspended homology of LM,

$$\mathbb{H}_*(LM) = H_{*+m}(LM)$$

is a graded commutative algebra [8].

The first application of our main result for the loop-product is the following slight generalization of the *Cohen-Jones-Yan spectral sequence*, [16].

Theorem 1.1.6 *Let M be a smooth closed oriented m-manifold arcwise connected. The $(m, 0)$-regraded Serre spectral sequence,*

$$\{\mathbb{E}^r[ev(0)]\}_{r \geq 0}$$

of the loop fibration,

$$\Omega M \xrightarrow{i_0} LM \xrightarrow{ev(0)} M$$

is a multiplicative spectral sequence which congerges to the graded algebra $\mathbb{H}_*(LM) = H_{*+m}(LM)$ and whose E^2-term is

$$\mathbb{E}^2[ev(0)] = \mathbb{H}_*(M; \mathcal{H}_*(\Omega M))$$

where $\mathcal{H}_*(\Omega M))$ denotes the usual system of local coefficients.

Furthermore, if we suppose that $\pi_1(M)$ acts trivially on ΩM then the tensor product of the graded algebra $\mathbb{H}_*(M)$ with the Pontryagin algebra $H_*(\Omega M)$ is a graded subalgebra of

$$\mathbb{E}^2[ev(0)] = \mathbb{H}_*(M; H_*(\Omega M))\,.$$

The proof of this theorem is given in section 1.5.2 page 34. **Remark :** In [16], the manifold M is assumed to be 1-connected, we only suppose that M is arcwise connected.

Our second application of the main result for loop-product is about fibered manifolds.

Theorem 1.1.7 *Under hypothesis of Proposition 1.1.3, the (m, n)-regraded Serre spectral sequence*

$$\{\mathbb{E}^r[Lp]\}_{r\geq 0}$$

of the Serre fibration

$$LN \xrightarrow{Li} LX \xrightarrow{Lp} LM$$

is a multiplicative spectral sequence which converges to the algebra $\mathbb{H}_*(LX)$. Moreover, if we assume that $\pi_1(M)$ acts trivially on $H_*(N)$, then the tensor product of graded algebras :

$$\mathbb{H}_*(LM) \otimes \mathbb{H}_*(LN)$$

is a subalgebra of $\mathbb{E}^2[Lp]$.

In particular if $H_*(LM)$ or $H_*(LN)$ is torsion free then

$$\mathbb{E}^2[p] = \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LN)\,.$$

We give a proof of this theorem in section 1.5.3 page 35.

Definition 1.1.8 *This spectral sequence will be called hereafter the loop-product ((m, n)-regraded) Serre spectral sequence.*

It is naturally related to the spectral sequence considered in Proposition 1.1.3 by mean of each of the following diagrams

$$\begin{array}{ccccc} LN & \xrightarrow{Li} & LX & \xrightarrow{Lp} & LM \\ \downarrow{\scriptstyle ev(0)^N} & & \downarrow{\scriptstyle ev(0)^X} & & \downarrow{\scriptstyle ev(0)^M} \\ N & \xrightarrow{i} & X & \xrightarrow{p} & M \end{array} \qquad \begin{array}{ccccc} LN & \xrightarrow{Li} & LX & \xrightarrow{Lp} & LM \\ \uparrow{\scriptstyle \sigma^N} & & \uparrow{\scriptstyle \sigma^X} & & \uparrow{\scriptstyle \sigma^M} \\ N & \xrightarrow{i} & X & \xrightarrow{p} & M \end{array}$$

where σ^X denotes the canonical section of $ev(0)^X$. Indeed both $ev(0)^X$ and σ^X induce homomorphisms of graded algebras between $\mathbb{H}_*(X)$ and $\mathbb{H}_*(LX)$.

3. Relative loop-product on $H_*(LM)$ and intersection morphism.

Let $i : N \hookrightarrow M$ be a smooth finite codimensionnal embedding between closed n-dimensionnal (resp m-dimensionnal) smooth manifolds. Let

$$\tilde{i} : L_N M = \{\gamma \in LM ; \gamma(0) \in N\} \hookrightarrow LM$$

be the natural inclusion. Under convenient hypothesis (see 1.6, page 38).

$$\mathbb{H}_*(L_N M) = H_{*+n}(L_N M)$$

is a graded commutative algebra and Theorems 1.1.6 and 1.1.7 convert into Theorems 1.1.9 and 1.1.10.

Theorem 1.1.9 *The (n,0)-regraded spectral sequence associated to the fibration $\Omega M \longrightarrow L_N M \longrightarrow N$ is multiplicative. Moreover, if $\pi_1(N)$ acts trivially on ΩM, the E^2 − term of the spectral sequence contains $i_!(\mathbb{H}_*(M)) \otimes H_*(\Omega M)$ as subalgebra.*

Now, let state Theorem 1.1.10 the analogous of Theorem 1.1.7 in this case. Let $N \xrightarrow{i} X \xrightarrow{p} M$ be the fibration of Theorem 1.1.7 and assume that the following diagram

$$\begin{array}{ccc} U & \xrightarrow{j_{|U}} & N \\ {\scriptstyle i_{|U}}\downarrow & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{j} & X \\ {\scriptstyle p_{|Y}}\downarrow & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{j_V} & M \end{array}$$

is a sub-fiberwise embedding.

Theorem 1.1.10 *In the above situation there is a morphism of multiplicative spectral sequences*

$$\mathbb{E}^*_{*,*}(Lp) \xrightarrow{E(\tilde{j}_!)} \mathbb{E}^*_{*,*}(Lp_{|Y})$$

given at the E^2-level by

$$E(\tilde{j}_!) = H_*(\tilde{j}_{V!}; \tilde{j}_{|U!})$$

.

These theorems are proved in section 1.6.1 page 38 and page 39.

Definition 1.1.11 *The case when N is a point is particularly interesting since $L_{pt}M = \Omega M$ and the restricted homomorphism*

$$\tilde{i}_! : \mathbb{H}_*(LM) \to H_*(\Omega M)$$

is by definition the intersection morphism denoted by I in [8].

Proposition 1.1.12 *Let $a \in \mathbb{H}_{-m}(LM)$ be the homology class representing a point in $\mathbb{H}_*(LM)$ and denote by $\mu_a : \mathbb{H}_*(LM) \to \mathbb{H}_{*-m}(LM)$ $x \mapsto a \circ x$ the multiplication by a. Then,*

$$\tilde{i}_* \circ \tilde{i}_! = \tilde{i}_* \circ I = \mu_a\,.$$

The proof of this proposition is done in section 1.6.2 page 39.
From this result we deduce our last result concerning the loop-product.

Theorem 1.1.13 *Assume that for each $k \geq 0$, $H_k(\Omega M)$ is finitely generated. Then, the three following propositions are equivalent :*
(a) I is onto
(b) The differentials $\{d_n\}_{n\geq 0}$ of the Cohen-Jones-Yan spectral sequence vanish for $n \geq 2$ so that the spectral sequence collapses.
(c) $\tilde{i}_*$ is injective

We begin to prove this theorem in section 1.6.3 of page 40 and we end this proof in section 1.6.4 of page 1.6.4. Observe that condition (c) is "dual" to the classical Leray-Hirsh result for the cohomology spectral sequence [29].

Explicit computations are given in section 1.6.5 and in section 1.6.6.

4. The path space M^I.

We consider the fibration

$$\Omega M \xrightarrow{i_1} M^I \xrightarrow{ev(0),ev(1)} M \times M$$

First, we observe that the construction of the loop-product extends to M^I so that the desuspended homology $\mathbb{H}_*(M^I)$ is a graded commutative algebra isomorphic to $\mathbb{H}_*(M)$ with the intersection product.

Secondly, there exists on $\mathbf{H}_*(M \times M) = \mathbb{H}_{*-m}(M \times M)$ a natural structure of associative algebra given by the product $\diamond$ defined in 1.7.3 page 44, which is not commutative and without unit.

With these two data, we prove Theorem 1.1.14 in section 1.7.4 page 44 which extends Theorem 1.1.6.

Theorem 1.1.14 *Let M be a smooth closed m-dimensionnal oriented manifold. There is a multiplicative structure on the $(d,0)$-regraded Serre spectral sequence associated to the fibration $\Omega M \longrightarrow M^I \xrightarrow{(ev_1,ev_0)} M \times M$. Furthermore, if we suppose that $\pi_1(M)$ acts trivially on ΩM then we have at the E^2-level :*

$\mathbf{E}^2_{*,*} = \mathbf{H}_*(M \times M; H_*(\Omega M))$ contains $\mathbf{H}_*(M \times M) \otimes H_*(\Omega M)$ as subalgebra. The structure of algebra on $\mathbf{H}_*(M \times M) \otimes H_*(\Omega M)$ is given by $\diamond$ on $\mathbf{H}_*(M \times M)$ and by the Pontryagin product on $H_*(\Omega M)$. Furthermore, the spectral sequence $\mathbf{E}^2_{*,*}$ converges to $\mathbb{H}_*(M)$ as algebra for the intersection product.

This section is a way to link the Pontryagin product of pointed loop spaces to the intersection product by mean of the Serre spectral sequence using the technics develloped in the preceding sections.

The paper is organized as follows. In section 1.2, we recall the definition of the shriek map associated to a smooth embedding. In section 1.3, we recall some results about spectral sequences. In section 1.4 we prove the main result Theorem 1.1.2. In section 1.5 we prove Proposition 1.1.3, Theorem 1.1.6 and Theorem 1.1.7. In section 1.6 we prove Theorem 1.1.9, Theorem 1.1.10, Proposition 1.1.12 Theorem 1.1.13 and we give some examples. In section 1.7 we prove Theorem 1.1.14 and give an example.

1.2 Recollection on Thom-Pontryagin theory.

The purpose of this section is to make precise the definition of the shriek map of an embedding [11].

1.2.1 Thom isomorphism

We denote by E' the sphere bundle associated to the following disk bundle. Let us recall that the Thom class of a disk bundle

$$(p) \qquad D^k \to E \xrightarrow{p} B$$

is a cohomology class

$$\tau_p \in H^k(E, E')$$

whose restriction to each fiber (D^k, S^{k-1}) is a generator of $H^k(D^k, S^{k-1})$. Every disk bundle has a Thom class if $\Bbbk = \mathbb{F}_2$. Every oriented disk bundle has a Thom class with any ring of coefficients $\Bbbk$. If the disk bundle (p) has a Thom class τ_p then the composite

$$H_*(E, E') \stackrel{\tau_p \cap -}{\to} H_{*-k}(E) \stackrel{H_*(p)}{\to} H_{*-k}(B)$$

is an isomorphism of graded modules, called the *homology Thom isomorphism of the fiber bundle* (p).

1.2.2 Shriek map

Let N and M be two smooth closed oriented manifolds and $f : N \hookrightarrow M$ be a smooth orientation preserving embedding. For simplicity, we identify $f(N)$ with N. Assume further, that N is closed subspace of M and that M admits a partition of unity then there is a splitting

$$T(M)_{|_N} = T(N) \oplus \nu_f$$

where ν_f is called the normal fiber bundle of the embedding. The isomorphism class of ν_f is well defined and depends only of the isotopy class of f.

Assume that the rank of $\nu_f = k$ and denote by :

$$(D^k, S^{k-1}) \to (D^k_{p_f}, S^{k-1}_{p_f}) \xrightarrow{p_f} N$$

the associated pair (disk, sphere) bundle. The restriction of the exponential map induces an isomorphism Θ from $(D^k_{p_f}, S^{k-1}_{p_f})$ onto a tubular neighborhood

$$(\text{Tube } f\,, \partial\text{Tube } f)\,.$$

The Thom class of the embedding f is the Thom class, whenever it exists, of the disk bundle p_f.

The definition of $f_! : H_*(M) \longrightarrow H_{*-k}(N)$ is given by the following composition :

$$H_*(M) \xrightarrow{j^M} H_*(M, M-N) \xrightarrow{Exc} H_*(\text{Tube } f, \partial \text{Tube } f) \xrightarrow{\Theta_*} H_*((D^k_{p_f}, S^{k-1}_{p_f}))$$

$$H_*((D^k_{p_f}, S^{k-1}_{p_f})) \xrightarrow{\tau_{p_f} \cap} H_{*-k}(D^k_{p_f}) \xrightarrow{p_*} H_{*-k}(N)$$

where Exc denotes the excision isomorphism, $j^B : H_*(B) \to H_*(B, A)$ the canonical homomorphism induced by the inclusion $A \subset B$.

1.3 The Serre spectral sequence.

1.3.1 Some classical results about spectral sequences.

In this section, we recall some definitions and results about the theory of spectral sequences that are given in [27].

Definition 1.3.1 *(definition 2.3 page 31 of [27]) A filtration F^* on an R-module A is a family of submodules $\{F^pA\}$ for p in $\mathbb{Z}$ so that*

$$... \subset F^{p+1}A \subset F^pA \subset F^{p-1}A \subset ... \subset A, \textit{decreasing filtration}$$

$$... \subset F^{p-1}A \subset F^pA \subset F^{p+1}A \subset ... \subset A, \textit{increasing filtration}$$

Definition 1.3.2 *(definition 2.5 page 33 of [27]) An R-module is a filtered differential graded module if*

(1) A is a direct sum of submodules, $A = \bigoplus_{n=0}^{\infty} A^n$.

(2) There is an R-linear mapping, $d : A \to A$ of degree 1 ($d : A^n \to A^{n+1}$) or degree -1 ($d : A^n \to A^{n-1}$) satisfying $d \circ d = 0$.

(3) A as a filtration F and the differential d respects the filtration, that is, $d : F^pA \to F^pA$

We remark that since the differentials respect the filtrations, $H(A, d) = kerd/imd$ inherits a filtration

$$F^pH(A, d) = image(H(F^pA, d) \overset{H(inclusion)}{\longrightarrow} H(A, d).$$

We can associate to a filtered differential graded module a spectral sequence. More precisely, we have the following theorem of [27] page 33 (rewrite in homology) :

Theorem 1.3.3 *Each filtered differential graded module (A, d, F^*) determines a spectral sequence, $\{E^r_{*,*}, d_r\}$, $r = 1, 2, ...$ with d_r of bidegree $(-r, r-1)$ and*

$$E^1_{p,q} \cong H_{p+q}(F^pA/F^{p-1}A).$$

Suppose further that the filtration is bounded, that is, for each dimension n, there are values $s = s(n)$ and $t = t(n)$, so that

$$A_n = F^tA^n \supset F^{t-1}A^n \supset ... \supset F^sA^n \supset \{0\},$$

then the spectral sequence converges to $H(A, d)$, that is

$$E^\infty_{p,q} \cong F^pH_{p+q}(A, d)/F^{p-1}H_{p+q}(A, d).$$

Now, let recall the definitions of a morphisms of differential graduate filtered modules and of spectral sequences given pages 65 and 66 of [27].

Definition 1.3.4 *A mapping $\phi : (A, d, F) \to (\bar{A}, \bar{d}, \bar{F})$ with $\phi : A \to \bar{A}$ a morphism of graded modules, such that $\phi \circ d = \bar{d} \circ \phi$ and ϕ respects filtrations, that is, $\phi(F^pA) \subset \bar{F}^p\bar{A}$ is the good object to define the notion of morphism of differential graded filtered modules.*

Definition 1.3.5 *Given two spectral sequences, $\{E^r_{*,*}, d_r\}$ and $\{\bar{E}^r_{*,*}, \bar{d}_r\}$, we define a morphism of spectral sequences to be a sequence of homomorphisms of bigraded modules, $f_r : (E^r_{*,*}, d_r) \to (\bar{E}^r_{*,*}, \bar{d}_r)$, for all r of bidegree $(0,0)$, such that f_r commutes with the differentials, that is, $f_r \circ d_r = \bar{d}_r \circ f_r$, and each f_{r+1} is induced by f_r on homology, that is, f_{r+1} is the composite*

$$f_{r+1} : E^{r+1}_{*,*} \cong H(E^r_{*,*}, d_r) \xrightarrow{H(f_r)} H(\bar{E}^r_{*,*}, \bar{d}_r) \cong \bar{E}^{r+1}_{*,*}.$$

1.3.2 Serre spectral sequence for homology.

We recall the theorem 5.1 page 134 of [27] that proves the existence of the Serre spectral sequence of a fibration.

Theorem 1.3.6 *Let G be an abelian group. Suppose $F \hookrightarrow E \xrightarrow{\pi} B$ is a Serre fibration, where B is path-connected and F connected. Then there is a first quadrant spectral sequence, $\{E^r_{*,*}, d_r\}$ converging to $H_*(E, G)$, with*

$$E^2_{p,q} \cong H_p(B; \mathcal{H}_q(F; G)),$$

the homology of the space B with local coefficients in the homology of the fiber of π. Furthermore, this spectral sequence is natural with respect to fiber-preserving maps of fibrations.

Remark : The proof of does'nt need the hypothesis that the fiber is connected. That's the reason why we can slightly extend the result of [16].

As shown in the part 5.3 of [27] (from page 167 to page 174), we can construct this spectral sequence from a filtration of $C_*(E)$ the singular chains of E. As in [27], we denote Δ_p the standard p-simplexe, namely $\Delta_p = \{(x_0, ..., x_p) \in \mathbb{R}^{p+1}; x_i \geq 0, \sum x_i = 1\}$. Let $F^pC_r(E)$ be the subgroup of $C_r(E)$ generated by $T : \Delta^r \to E$ such that $\Pi \circ T = S(i_0, ..., i_r)$, for $i_r \leq p$ and $S : \Delta^p \to B$. This is an increasing filtration with $F^rC_r(E) = C_r(E)$. For short, R.Cohen, J.D.S.Jones and J.Yan deduce of this part 5.3 of [27] the proposition 6 of [16] :

Proposition 1.3.7 *(proposition 6 of [16]) Let $F \hookrightarrow E \to B$ be a filtration as above. We considere the filtration of chain complexes $F^pC_*(E)$ defined above. Then, this filtration induces the Serre spectral sequence converging to $H_*(E)$.*

1.4 Proof of the main result : Theorem 1.1.2.

The aim of this section is to prove our main result. To this end and following [16], we first describe the homology shriek map at the chain level and then we study the compatibility of this chain representative with a filtration which yields a Serre spectral sequence.

1.4.1 Shriek map of an embedding at the chain level.

We will prove the following proposition :

Proposition 1.4.1 *Let $f : X \hookrightarrow X'$ be an embedding of finite codimension k between connected Hilbert manifolds without boundary. We assume that f admits a Thom class τ. Then, there exists a morphism of differential graded modules $\tilde{f}_! : C_*(X') \to C_{*-k}(X)$ that induces $f_! : H_*(X') \to H_{*-k}(X)$ in homology.*

Proof :

We will construct $\tilde{f}_!$ as the composition of six chain maps, the first beginning from $C_*(X')$ and the last ending at $C_{*-k}(X)$.

1) $i_\sharp : C_*(X') \to C_*(X', X' - f(X))$ is induced by the inclusion of pair.

2) The next step is a byproduct of the following proposition :

Proposition 1.4.2 *Let $B \subset A \subset X$ topological spaces such that B is closed and A is open. Then, there exists a quasi-isomorphism $\bar{p} : C_*(X, A) \to C_*(X - B, A - B)$ which inverses $j : C_*(X - B, A - B) \to C_*(X, A)$ in homology.*

This proposition proves the existence at the chain level of a chain map that induces in homology the inverse of the excision isomorphism.

Proof :

Let denote by $\mathcal{U} = \{A, X - B\}$ the open covering of X that is $X = A \cup (X - B)$.

Lemma 1.4.3 *There exists two arrows* $C_*^{\mathcal{U}}(X, A) \rightleftarrows C_*(X, A)$ *that induce identity in homology.*

Proof : From [6] corollary 17.5 chapter IV, there exists $\gamma^k : C_*(X) \to C_*(X)$ such that $\forall \sigma \in C_*(X)$, $\gamma^k(\sigma) \in C_*^{\mathcal{U}}(X)$. The corollary 17.5 of the same chapter proves that there exists $T_k : C_*(X) \to C_{*+1}(X)$ such that $\partial T_k + T_k \partial = \gamma^k - id$.

By naturality of γ^k and T_k (see [6] page 225), the restrictions γ_A^k and T_k^A of γ^k and T_k to $C_*(A)$ are such that $\gamma_A^k - id_{C_*(A)} = \partial T_k^A + T_k^A \partial$. Now, we consider the following commutative diagram with exact rows :

$$\begin{array}{ccccc}
C_*^{\mathcal{U}}(A) & \xrightarrow{inc} & C_*^{\mathcal{U}}(X) & \xrightarrow{pr} & C_*^{\mathcal{U}}(X, A) \\
\gamma_A^k \uparrow \; \downarrow i_A & & \gamma^k \uparrow \; \downarrow i_X & & \bar{\gamma}_A^k \uparrow \; \downarrow i_{(X,A)} \\
C_*(A) & \xrightarrow{inc} & C_*(X) & \xrightarrow{pr} & C_*^{\mathcal{U}}(X, A)
\end{array}$$

where *inc* and *pr* denote the canonical inclusions and the canonical projections. What precedes proves that γ^k and T_k are compatible with quotient proving lemma 1.4.3.

$\square$

Now, we will construct $\bar{p}^{\mathcal{U}} : C_*^{\mathcal{U}}(X, A) \to C_*(X - B, A - B)$

By construction, we have :

$$C_*^{\mathcal{U}}(X) = C_*(A) + C_*(X - B)$$

The sum is not direct, so we have : $C_*(A) \cap C_*(X - B) = C_*(A - B)$. Then, the canonical projection

$$\pi : C_*^{\mathcal{U}}(X) \to C_*^{\mathcal{U}}(X)/C_*(A - B) = C_*(A)/C_*(A - B) \oplus C_*(X - B)/C_*(A - B)$$

ends in a direct sum. Let $pr_2 : C_*(A)/C_*(A - B) \oplus C_*(X - B)/C_*(A - B) \to C_*(X - B)/C_*(A - B)$ be the canonical projection, we define

$$p^{\mathcal{U}} : C_*^{\mathcal{U}}(X) \to C_*(X - B)/C_*(A - B)$$

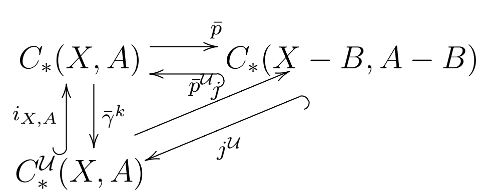
$$p^{\mathcal{U}} = pr_2 \circ \pi.$$

If a lies in $C_*(A)$, then $pr_2(\pi(a)) = 0$ thus $p^{\mathcal{U}}$ is compatible with quotient and we call $\bar{p}^{\mathcal{U}} : C_*^{\mathcal{U}}(X)/C_*(A) = C_*^{\mathcal{U}}(X, A) \to C_*(X - B, A - B)$ the quotient map.

The maps π and pr_2 are chain morphisms and consequently, $p^{\mathcal{U}}$ is a chain map. We define $\bar{p} : C_*(X, A) \to C_*(X - B, A - B)$ with lemma 1.4.3 : $\bar{p} = \bar{p}^{\mathcal{U}} \circ \bar{\gamma}^k$. There only rests to verify that $j \circ \bar{p} \sim id_{C_*(X,A)}$ and $\bar{p} \circ j \sim id_{C_*(X-B,A-B)}$. By construction, we remark that $C_*^{\mathcal{U}}(X - B, A - B) = C_*(X - B, A - B)$ and we denote by

$$j^{\mathcal{U}} = C_*(X - B, A - B) \to C_*^{\mathcal{U}}(X, A)$$

the map induced by (topological) inclusion of pair. We obtain the following non-commutative diagram :



We remark that $\bar{p}^{\mathcal{U}} \circ j^{\mathcal{U}} = id_{C_*(X-B,A-B)}$ and that $j^{\mathcal{U}} \circ \bar{p}^{\mathcal{U}} = id_{C_*(X-B,A-B)}$.

We have $j = i_{(X,A)} \circ j^{\mathcal{U}}$, $\bar{p} = \bar{p}^{\mathcal{U}} \circ \bar{\gamma}^k$, $i_{(X,A)} \circ \bar{\gamma}^k \sim id$ and $\bar{\gamma}^k \circ i_{(X,A)} \sim id$ so that we have :

$$\bar{p} \circ j = \bar{p}^{\mathcal{U}} \circ \bar{\gamma}^k \circ i_{(X,A)} \circ j^{\mathcal{U}} \sim \bar{p}^{\mathcal{U}} \circ j^{\mathcal{U}} \sim id$$

and

$$j \circ \bar{p} = i_{(X,A)} \circ j^{\mathcal{U}} \circ \bar{p}^{\mathcal{U}} \circ \bar{\gamma}^k = i_{(X,A)} \circ \bar{\gamma}^k \sim id.$$

This proves proposition 1.4.2.

$\square$

The next map is $\bar{p} : C_*(X', X' - f(X)) \to C_*(T, T - f(X))$ constructed from proposition 1.4.2 with X' for X in the proposition, T for A, and $X' - T$ for B.

3) Since T is a regular neighbourhood of $f(X)$ in X', then ∂T is a strong deformation retract of $T - f(X)$ that is the inclusion $j : \partial T \hookrightarrow T - f(X)$ admits a homotopy inverse denoted by r. So, there exists a chain map $r_\sharp : C_*(T, T - f(X)) \to C_*(T, \partial T)$.

4) The restriction of the exponential map induces an isomorphism of pair Θ from (η_f, η'_f) onto the tubular neighbourhood $(T, \partial T)$. This map (more precisely the inverse of this map) induces the chain map

$$\Theta_\sharp : C_*(T, \partial T) \to C_*(\eta_f, \eta'_f).$$

5) Let $\tilde{\tau} \in C_k(\eta_f, \eta'_f)$ be a cocycle representing the Thom class of f. Then, the cap product at the chain level provides the morphism $\tilde{\tau} \cap - : C_*(\eta_f, \eta'_f) \to C_{*-k}(\eta_f)$.

6) Let $\pi_f : \eta_f \to X$ be the canonical projection. The last map in the definition of $\tilde{f_!}$ is the map induced by π_f :

$$\pi_{f\sharp} : C_{*-k}(\eta_f) \to C_{*-k}(X).$$

1.4.2 Beginning of the proof

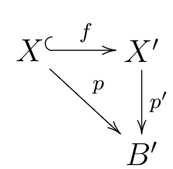
We begin by filtering $C_*(X)$ and $C_*(X')$ in the way of Serre (see 1.3.2). Then, we need to check that $\tilde{f_!}$ preserves the regraded filtration that is : $\tilde{f_!}(F_pC_{p+q}(X')) \subset F_{p-k_B}C_{p+q-k_x}(X)$.

The naturality of the Serre filtration according to continuous maps proves that $i_\sharp$, $r_\sharp$, $\Theta_\sharp$ and $\pi_{f\sharp}$ preserve the induced filtrations. The map $\bar{p}$ is a projection of a free module on another. Let σ be an element of $F_pC_{p+q}(X', X' - f(X))$. If $\bar{p}(\sigma) = 0$ then $\bar{p}(\sigma)$ lies in $F_pC_{p+q}(T, T - f(X))$. If $\bar{p}(\sigma) = \sigma$, then $\bar{p}(\sigma)$ lies in $F_pC_{p+q}(T, T - f(X))$. This proves that $\bar{p}$ preserves the filtration. The more technical point is to prove that the cap product with $\tilde{\tau}$ preserves filtrations. That's what we will prove now.

1.4.3 Fiberwise embedding

We begin by proving Theorem 1.1.2 for what we call a fiberwise embedding wich is a more restrictive case that a sub-fiberwise embedding of definition 1.1.1 as it doesn't suggest. This terminology refers to the point of view of [17].

Definition 1.4.4 *A fiberwise embedding is a sub-fiberwise embedding such that $f^B = id$. It corresponds to the following commutative diagram over B'.*



We begin by proving the first part of the main result for fiberwise embeddings :

Lemma 1.4.5 *Denote by ν_f the normal bundle of the embedding $f : X \hookrightarrow X'$ of the fiberwise embedding 1.4.4 and by $k := k_X$ the codimension of this embedding. If we choose $\tilde{\tau} \in C^k(D^k(\nu_f), S^{k-1}(\nu_f))$ representing the Thom class of f such that $\tilde{\tau}$ vanishes on degenerate simplices, then*

$$F_pC_{p+q}(D^k(\nu_f), S^{k-1}(\nu_f)) \xrightarrow{\tilde{\tau}\cap -} F_pC_{p+q-k}(D^k(\nu_f))$$

for p,q, integers, is a morphism of differential graded filtered modules (dgfm for short).

Let $\omega \in F_pC_{p+q}(X')$. Then there exists $\Omega \in C_p(B)$ such that $p'_\sharp(\omega) = \Omega(i_0, ..., i_{p+q})$ (see section 1.3.2 for the definition of $(i_0, ..., i_{p+q})$). Another way to explain that is that the diagram

$$\begin{array}{ccc} \Delta^{p+q} & \xrightarrow{\omega} & X' \\ {\scriptstyle (i_0,\dots,i_{p+q})}\downarrow & & \downarrow{\scriptstyle p'} \\ \Delta^p & \xrightarrow{\Omega} & B \end{array}$$

commutes. Thus $\tau_f \cap \omega$ is a chain of $p + q - k$-faces of simplices of ω. Then, the diagram

$$\begin{array}{ccc} \Delta^{p+q-k} & & \\ {\scriptstyle (i_0,\dots,i_{p+q-k})}\downarrow & \searrow^{\tau\cap\omega} & \\ \Delta^{p+q} & \xrightarrow{\omega} & X' \\ {\scriptstyle (i_0,\dots,i_{p+q})}\downarrow & & \downarrow{\scriptstyle p'} \\ \Delta^p & \xrightarrow{\Omega} & B \end{array}$$

commutes proving that $\tilde{\tau} \cap \omega)$ lies in $F_pC_{p+q-k}(X)$.

It remains to show that $\tilde{\tau} \cap -$ commutes to the differentials (the context being sufficiently clear to know about which differential we refer, all differentials are denoted by d). Let $c \in C_*(X')$, since $d(\tilde{\tau}) = 0$ we have :

$$\begin{aligned} d(\tilde{\tau} \cap c) &= d(\tilde{\tau}) \cap c + (-1)^k \tilde{\tau} \cap dc \\ &= (-1)^{k_B} \tilde{\tau} \cap dc \end{aligned}$$

This ends the proof of Lemma 1.4.5.

□

The second part of the main result follows by classical theory of spectral sequences more exavtly by Theorem 1.3.3 of section 1.3.

It remains to explain what happen at the E^2-level. Using the notation of the definition of sub-fiberwise embeddings 1.1.1 and fiberwise embeddings 1.4.4, we have the following commutative diagram :

$$\begin{array}{ccc} F & \xrightarrow{f_{|F}} & F' \\ {\scriptstyle j'|_F}\downarrow & & \downarrow{\scriptstyle j'} \\ X & \xrightarrow{f} & X' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & B & \end{array}$$

This allows us to have a fiberwise description of the tubular neighborhood of the embedding f. We denote by T_f this tubular neighbourhood and by $T_{f_{|F}}$ the tubular neighbourhood of $f_{|F}$. Then the diagram

$$\begin{array}{ccccc} F & \hookrightarrow & T_{f_{|F}} & \hookrightarrow & F' \\ {\scriptstyle j'|_F}\downarrow & & \downarrow & & \downarrow{\scriptstyle j'} \\ X & \hookrightarrow & T_f & \hookrightarrow & X' \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle p'_{|T}} & \swarrow{\scriptstyle p'} & \\ & & B & & \end{array}$$

commutes by construction.

Furthermore, we have immediately that the upper square of the first diagram is a pull-back diagram then $\tau_{f_{|F}} = j'^*(\tau_f)$. Then, for some $x \in H_n(F')$, $n \in \mathbb{N}$, $j'_*(\tau_{f_{|F}} \cap x) = j'_*(j'^*(\tau_f \cap x)) = \tau_f \cap j'_*(x)$. Since the other maps in the definition of the shriek map are compatible with j'_*, we have the following commutative square :

$$\begin{array}{ccc} H_{*-k}(F) & \xleftarrow{f_{|F!}} & H_*(F') \\ \downarrow{\scriptstyle j'_*} & & \downarrow{\scriptstyle j'_*} \\ H_{*-k}(E) & \xleftarrow[f_!]{} & H_*(E') \end{array} .$$

On the spectral sequences, this proves that $E^2_{*,*}(f_!) = H_*(id, \mathcal{H}(f_{|F!}))$.

□

1.4.4 Pull-back embedding

The second case for which we prove the main result is the following.

Definition 1.4.6 *Consider a sub-fiberwise embedding of definition 1.1.1 such that for all $b \in B$, f^F is identity.*

$$\begin{array}{ccc} F & \xrightarrow{id_F} & F \\ \downarrow & & \downarrow \\ X & \hookrightarrow & X' \\ \downarrow & & \downarrow \\ B & \hookrightarrow & B' \end{array}$$

In this case, the diagram of the definition 1.1.1 is a pull-back diagram that we call a pull-back embedding.

Now we prove the first part of the main result for pull-back embeddings :

Lemma 1.4.7 *Denote by ν_f the normal bundle of the embedding $f : X \hookrightarrow X'$ of the pull-back diagram and by $k := k_B$ the codimension of this embedding. If we choose*

$\tilde{\tau} \in C^k(D^k(\nu_f), S^{k-1}(\nu_f))$ representing the Thom class of f such that $\tilde{\tau}$ vanishes on degenerated simplices, then

$$F_pC_{p+q}(D^k(\nu_f), S^{k-1}(\nu_f)) \xrightarrow{\tilde{\tau}\cap -} F_{p-k}C_{p+q-k}(D^k(\nu_f))$$

for p,q, integers, is a morphism of differential graded filtered modules.

Proof of Lemma 1.4.7 :

Let τ_{f^B} be the Thom class of f^B. Since in this case the diagram of 1.1.1 is a pull back-diagram, we have $\tilde{\tau} = p'^*(\tau_{f^B})$. At the chain level we can choose a cocycle representing the Thom classes, also called τ_{f^B} and $\tilde{\tau}$, such that

$$\tilde{\tau} = p'^\sharp(\tau_{f^B}).$$

For some $\sigma \in F_pC_{p+q}(X')$, by definition of the Serre filtration, there exists $\Sigma \in C_p(B')$ and $(i_0, ..., i_{p+q})$ such that $p'(\sigma) = \Sigma(i_0, ..., i_{p+q})$. Thus

$$\begin{aligned} p'_\sharp(\tilde{\tau} \cap \sigma) &= p'_\sharp(p'^\sharp(\tau_{f^B}) \cap \sigma) \\ &= \tau_{f^B} \cap \Sigma(i_0, ..., i_{p+q}) \\ &= \tau_{f^B}(\sigma(i_{p+q-k}, ..., i_{p+q}))\Sigma(i_0, ..., i_{p+q-k}). \end{aligned}$$

Since $\tau_{f^B}(\sigma(i_{p+q-k}, ..., i_{p+q})) \neq 0$ by definition of the Thom class, $i_{p+q-k}, ..., i_{p+q}$ are pairwise distinct then we can write $\Sigma(i_0, ..., i_{p+q-b})$ as an element of $C_{p-k}(B')$ so that $\tilde{\tau} \cap (\sigma)$ lies in $F_{p-k}C_{p+q-k}(X)$.

It remains to show that $\tilde{\tau} \cap -$ commutes to the differentials (the context being sufficiently clear to know about wich differential we refer, all differentials are denoted by d). Let $c \in C_*(X')$, since $d(\tilde{\tau}) = 0$, we have : $d(\tilde{\tau} \cap c) = d(\tilde{\tau}) \cap c + (-1)^k\tilde{\tau} \cap dc$ $= (-1)^k\tilde{\tau} \cap dc$.

□

The second part of the proof of the main result follows from 1.4.7 in the same way as in the case of fiberwise embedding in the preceding subsection. Observe that at the E^2-level, $E^2(f_!) = H_*(f_!^B, \mathcal{H}_*(id))$.

□

1.4.5 Proof of Theorem 1.1.2 for a sub-fiberwise embedding

The proof is a direct consequence of 1.4.3 and 1.4.4 since :

Lemma 1.4.8 *A sub-fiberwise embedding (f, f^B) (definition 1.1.1) decomposes as the commutative diagram :*

$$\begin{array}{ccccc} X & \overset{f_2}{\hookrightarrow} & X' \times_{B'} B & \overset{f_1}{\hookrightarrow} & X' \\ & \searrow^{p} & \downarrow{p'} & & \downarrow{p'} \\ & & B & \overset{f^B}{\hookrightarrow} & B' \end{array}$$

where the right part of the diagram is a pull-back diagram and the left part a fiberwise embedding.

Proof of lemma 1.5.2 : We begin by constructing the pull-back diagram from p' and f^B. Then, we complete by the left part of the diagram. The lemma follows immediately by construction of the diagram.

□

1.5 Main applications : classical Serre spectral sequence for finite dimensionnal manifolds, Cohen-Jones-Yan spectral sequence and string Serre spectral sequence. Proof of Proposition 1.1.3, 1.1.12 and Theorems 1.1.6, 1.1.7.

1.5.1 Classical Serre spectral sequence for finite dimensionnal manifolds. Proof of Proposition 1.1.3

For a topological space Y, we denote by Δ_Y the diagonal embedding of Y in $Y \times Y$. First, it is easy to check that the commutative diagram

$$\begin{array}{ccc} N & \overset{\Delta_N}{\hookrightarrow} & N \times N \\ \downarrow & & \downarrow \\ X & \overset{\Delta_X}{\hookrightarrow} & X \times X \\ \downarrow & & \downarrow \\ M & \overset{\Delta_M}{\hookrightarrow} & M \times M \end{array}$$

is a sub-fiberwise embedding 1.1.1.

Applying the naturality of the Serre spectral sequence according to the cross product (denoted by $\times$) and the main result, we immediately prove that the composite map

$$H_*(X) \otimes H_*(X) \xrightarrow{\times} H_*(X \times X) \xrightarrow{\Delta_{X!}} H_{*-(m+n)}(X)$$

induces on the (m, n)-regraded Serre spectral sequence associated to the fibration $N \to X \to M$ a multiplicative structure. This spectral sequence converges as algebra to $\mathbb{H}_*(X)$ with the intersection product. Furthermore, if we assume that $\pi_1(M)$ acts trivially on $H_*(N)$, then the local coefficients are constant, so the multiplicative structure $\mathbb{E}^2_{*,*} = \mathbb{H}_*(M; \mathbb{H}_*(N))$ contains $\mathbb{H}_*(M) \otimes \mathbb{H}_*(N)$ as subalgebra.

□

Proof of Proposition 1.1.4. Recall the classical theorem (see for instance theorem 3.5 p66 of [27]).

Theorem 1.5.1 *A morphism of differential graded filtered modules $\Phi : (A, d, F) \to (A', d', F')$ determines a morphism of the associated spectral sequence. If for some $n \in \mathbb{N}$, $\Phi_n : E_n \to E'_n$ is an isomorphism of bigraded modules, then $\Phi_r : E_r \to E'_r$ is an isomorphism for all r, $n \leq r \leq \infty$. If the filtration are bounded, then Φ induces an isomorphism $H(\Phi) : H(A, d) \to H(A', d')$.*

Let $x := dim(X)$ and $[X] \in C_x(X)$ be a cycle representing the fundamental class of X. We define

$$DP : C^{x-*}(X) \to C_*(X)$$

$$t \mapsto t \cap [X]$$

the linear quasi isomorphism which induces in homology the Poincaré duality isomorphism. Recall the definition of the Serre filtration of $C^*(X)$ is defined in chapter 5 of [27] :

$$F^pC^{p+q}(X) := \{f \in C^{p+q}(X); f(T) = 0 \quad \forall T \in F_{p-1}C_{p+q}(X)\}.$$

We need to prove that DP induces a morphism $F^{m-p}C^{x-*}(X) \to F_pC_*(X)$ (recall that $m = dimM$) of differential graded modules that preserves filtrations. For reasons of degree, the fundamental class $[X]$ is a $\mathbb{Z}$-linear sum of simplices $\{\sigma_i\}$ of $F_mC_x(X)$ that we write $[X] = \sum_i \alpha_i\sigma_i \quad \alpha_i \in \mathbb{Z}$. Then for $f \in F^{m-p}C^{x-(p+q)}(X)$,

$$DP(f) = f \cap [M] = \sum_i \alpha_i f \cap \Delta_i.$$

Now we need the following notations of [6]. For a singular simplex $\sigma : \Delta_r \to X$ and for $r + t = n$, $0 \leq r$, $t \leq n$, we denote by

$$\sigma\rfloor_r : \Delta_r \to \Delta_n \to X$$

the "front r-face " of σ, which is the composition of σ with the inclusion $(1, ..., r)$ of Δ_r in Δ_n (see 1.3.2 page 26). Similarely, we denote by

$$\lfloor_t\sigma : \Delta_t \to \Delta_n \to X$$

the "back t-face" of σ which is the inclusion $(n-t, ..., n)$ of Δ_t in Δ_n (see 1.3.2 page 26). Now, let compute $f \cap \sigma_i \quad i \in \mathbb{N}$. Using the classical formulas for the cap product (see for example [6]), we have :

$$f \cap \sigma_i = (-1)^{(x-(p+q))(p+q)} f(\lfloor_{(x-(p+q))}\sigma).\sigma\rfloor_{p+q}$$

If $\lfloor_{(x-(p+q))}\sigma$ is in $F_{m-p-1}C_{x-(p+q)}(X)$, then $f(\lfloor_{(x-(p+q))}\sigma) = 0$ by definition of $F^{m-p}C^{x-(p+q)}(X)$. If $\lfloor_{(x-(p+q))}\sigma$ is not in $F_{m-p-1}C_{x-(p+q)}(X)$, then for reason of degree since σ_i lies in $F_mC_x(X)$, $\sigma\rfloor_{p+q}$ lies in $F_pC_{p+q}(X)$. This proves that DP preserves the filtration.

Since $E^2(DP)$ is an isomorphism induced by Poincaré duality on the basis and on the fiber, we can conclude the proof of Proposition 1.1.4 by applying Theorem 1.5.1.

□

1.5.2 The Cohen-Jones-Yan spectral sequence. Proof of Theorem 1.1.6.

We construct the following commutative diagram :

$$\begin{array}{ccccc}
\Omega M \times \Omega M & \xrightarrow{i \times i} & LM \times LM & \xrightarrow{ev(0) \times ev(0)} & M \times M \\
\uparrow{\scriptstyle id} & & \uparrow{\scriptstyle \tilde{\Delta}} & & \uparrow{\scriptstyle \Delta} \\
\Omega M \times \Omega M & \xrightarrow{i \times i} & LM \times_M LM & \xrightarrow{ev_\infty} & M \\
\downarrow{\scriptstyle Comp} & & \downarrow{\scriptstyle Comp} & & \downarrow{\scriptstyle id} \\
\Omega M & \xrightarrow{i} & LM & \xrightarrow{ev(0)} & M
\end{array}$$

where the maps are defined as in the introduction. We need to prove that $(\tilde{\Delta}, \Delta)$ is a pull-back embedding of codimension m. By adapting a proof of [7] lemma 5.4.1 p 220, we prove that $ev : LM \to M$ is locally trivial. We also have that LM and $LM \times_M LM$ are Hilbert manifolds (see [11]). The Thom class of $\tilde{\Delta}$ is obtained from this of Δ (see [10]). Now, composition of the maps

$$C_{*+d}(LM)\otimes C_{*+d}(LM) \stackrel{\times}{\to} C_{*+2d}(LM\times LM) \stackrel{\tilde{\Delta}_!}{\to} C_{*+d}(LM\times_M LM) \stackrel{Comp_\sharp}{\to} C_{*+d}(LM)$$

induces at the homology level Chas and Sullivan product denoted by μ. The Serre spectral sequence associated to the fibration

$$\Omega M \xrightarrow{i} LM \xrightarrow{ev(0)} M$$

satisfys $\mathbb{E}^2_{*,*} = \mathbb{H}_*(M; H_*(\Omega M))$ (here $\pi_1(M)$ acts trivially on ΩM since M is arcwise connected).

We considere the Serre filtration for each of the horizontal fibrations. By using the main result for $(\tilde{\Delta}, \Delta)$ and the naturality of the Serre spectral sequence for the Eilenberg-Zilbert morphism and for $\gamma_\sharp$, we extend the result of [16].Namely, we prove that there is a multiplicative structure on this Serre spectral sequence containing at the E^2-level the tensor product of $\mathbb{H}_*(M)$ with intersection product and $H_*(\Omega M)$ with Pontryagin product. This spectral sequence of algebra converges to $\mathbb{H}_*(LM)$.

1.5.3 String Serre spectral sequence. Proof of Theorem 1.1.7.

For any topological space T, the map $map(T, E) \to map(T, B)$ is a fibration with fiber $map(T, F)$. Since $LT = map(S^1, T)$ and $LT \times_T LT = map(S^1 \vee S^1, T)$, we have the following fibrations :

$$LN \xrightarrow{Li} LX \xrightarrow{Lp} LM \quad \text{and} \quad LN \times_N LF \xrightarrow{Li} LX \times_X LX \xrightarrow{Lp} LM \times_M LM$$

and by looping the sub-fiberwise embedding

$$\begin{array}{ccc} X & \xhookrightarrow{\Delta_X} & X \times X \\ \downarrow p & & \downarrow p\times p \\ M & \xhookrightarrow{\Delta_M} & M \times M \end{array}$$

we obtain the sub-fiberwise embedding

$$\begin{array}{ccc} LXp & \xhookrightarrow{\tilde{\Delta_X}} & LX \times_X LX \\ \downarrow L & & \downarrow Lp\times Lp \\ LM & \xhookrightarrow{\tilde{\Delta_M}} & LM \times_M LM. \end{array}$$

where $\tilde{\Delta_X}$ denotes the embedding obtained by pull-back of free loop spaces over diagonal embedding. Applying our main result to this sub-fiberwise embedding, we proves that $\tilde{\Delta_X}$ induces a morphism of spectral sequences between $E^*(LX \times LX)$ the Serre spectral sequence associated to the fibration $LX \times LX \to LM \times LM$ and

$E^*(LX \times_X LX)$ the Serre spectral sequence associated to the fibration $LX \times_x LX \to LM \times_M LM$ of bidegree $(-m, -n)$.

Finally, by naturality, the composition of loops : $comp_X : LX \times_X LX \to LX$ induces a morphism $E^*(comp_{X*}) : E^*_{*,*}(LX \times_X LX) \to E^*_{*,*}(LX)$. Since the loop product on LX namely μ_X is by definition $comp_{X*} \circ \tilde{\Delta}_{X!}$, we have shown the compatibility of this Serre spectral sequence (m, n)-regraded to this product.

The following diagram resumes the situation and allows to see what is induced on the base and on the fiber.

$$\begin{array}{ccccc}
LN \times LN & \longrightarrow & LX \times LX & \longrightarrow & LM \times LM \\
\uparrow{\scriptstyle \tilde{\Delta_N}} & & \uparrow{\scriptstyle \tilde{\Delta_X}} & & \uparrow{\scriptstyle \tilde{\Delta_M}} \\
LN \times_N LN & \longrightarrow & LX \times_X LX & \longrightarrow & LM \times_M LM \\
\downarrow{\scriptstyle Comp_N} & & \downarrow{\scriptstyle Comp_X} & & \downarrow{\scriptstyle Comp_M} \\
LN & \longrightarrow & LX & \longrightarrow & LM
\end{array}.$$

We remark that the morphism induced on the base (resp. the fiber) is the loop product for LM (resp. LN). With this remark, we give a description of the multiplicative structure of the spectral sequence at the E^2-level :

$$\mathbb{E}^2(\mu_X) = \mathbb{H}(\mu_M; \mathcal{H}(\mu_N)) :$$

$$\mathbb{E}^2_{*,*}(LX)^{\otimes 2} = \mathbb{H}_*(LM; \mathcal{H}_{*+n}(LN))^{\otimes 2} \to \mathbb{H}_*(LM; \mathcal{H}_{*+n}(LN)) = \mathbb{E}^2_{*,*}(LX).$$

In many cases, the local coefficients are constant. Indeed :

Lemma 1.5.2 *Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration such that $\pi_1(B)$ acts trivially on $H_*(F)$, then $\pi_1(LB)$ acts trivially on $H_*(LF)$.*

Proof of lemma 1.5.2 :

For a fixed loop $\gamma \in \Omega B$, the holonomy operation defines maps

$$\Psi_\gamma : F \longrightarrow F \qquad x \mapsto \gamma.x.$$

Since $\pi_1(B)$ acts trivially on $H_*(F)$, there exists a homotopy $H_\gamma : F \times I \longrightarrow F \quad (x,t) \mapsto H_\gamma(x,t)$ such that $H_\gamma(x,0) = x$, and $H_\gamma(x,1) = \gamma.x$. For a fixed $\Gamma \in \Omega LB$, the holonomy action of ΩLB on LF (associated to the fibration $LF \xrightarrow{Lj} LE \xrightarrow{Lp} LB$) yields

$$\Phi_\gamma : LF \longrightarrow LF \qquad f \mapsto (s \mapsto \Psi_{\Gamma(-,s)}(f(s)))$$

Now the homotopy

$$\mathcal{H}_\Gamma : LF \times I \longrightarrow LF \qquad f(-), t \mapsto \mathcal{H}_\Gamma(f(-), t) = (s \mapsto \mathcal{H}_\Gamma(-,s)(f(s), t))$$

satisfies

$$\mathcal{H}_\Gamma(f, 0) = (s \mapsto H_{\Gamma(-,s)}(f(s), 0)) = f(-))$$

$$\mathcal{H}_\Gamma(f, 1) = (s \mapsto H_{\Gamma(-,s)}(f(s), 1)) = \Gamma(-,-).f(-) = \Phi_{\Gamma.f}$$

$s \in S^1$.

$\square$

Since with the hypothesis of this Theorem 1.1.7, the lemma holds, the local coefficients are constant then $\mathbb{E}^2_{*,*} = H_{*+m}(LM; H_{*+n}(LN))$ contains $\mathbb{H}_*(LM) \otimes \mathbb{H}(LN)$ as subalgebra.

$\square$

1.6 Restricted Chas and Sullivan algebra and intersection morphism. Proof of theorems 1.1.9, 1.1.10, 1.1.13 and proposition 1.1.12.

In this section, we apply the theory to the case where the base points of the free loops are contained in a given submanifold. We study particularly the case where the submanifold is a point to give some results about the intersection morphism of Chas and Sullivan [8].

1.6.1 The restricted Chas and Sullivan loop-product.

Let $i : N \hookrightarrow M$ be a smooth finite codimensionnal embedding of Hilbert closed manifold admitting a Thom class. Define $L_N M$ as the right corner of the pull-back diagram :

$$\begin{array}{ccc} L_N M & \overset{\tilde{i}}{\hookrightarrow} & LM \\ \downarrow{\scriptstyle ev(0)} & & \downarrow{\scriptstyle ev(0)} \\ N & \overset{i}{\hookrightarrow} & M \end{array}$$

i.e $L_N M$ is the space of free loops LM based on N. This commutative diagram is a pull-back embedding. We denote $\mathbb{H}_*(L_N M) := H_{*+n}(L_N M)$. We define the restricted loop product

$$\mu_N : \mathbb{H}_*(L_N M) \otimes \mathbb{H}_*(L_N M) \to \mathbb{H}_*(L_N M)$$

$$\alpha \otimes \beta \mapsto \mu_N(\alpha \otimes \beta) := \alpha \circ_N \beta$$

by putting $\mu_N = Comp_* \circ \Delta_{N!} \circ \times$ where we consider the commutative diagram :

$$\begin{array}{ccccc} L_N M & \overset{Comp}{\longleftarrow} & L_N M \times_N L_N M & \overset{\tilde{\Delta_N}}{\longrightarrow} & L_N M \times_N L_N M \\ \downarrow{\scriptstyle ev(0)} & & \downarrow{\scriptstyle ev_\infty} & & \downarrow{\scriptstyle ev(0) \times ev(0)} \\ N & \underset{=}{\longleftarrow} & N & \overset{\Delta_N}{\longrightarrow} & N \times N \end{array}$$

Proof of Theorem 1.1.9. Starting with the following commutative diagram :

$$\begin{array}{ccccc} \Omega M \times \Omega M & \longrightarrow & L_N M \times L_N M & \overset{ev(0) \times ev(0)}{\longrightarrow} & N \times N \\ \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle \tilde{\Delta_N}} & & \uparrow{\scriptstyle \Delta_N} \\ \Omega M \times \Omega M & \longrightarrow & L_N M \times_N L_N M & \overset{ev_\infty}{\longrightarrow} & N \\ \downarrow{\scriptstyle Comp} & & \downarrow{\scriptstyle Comp} & & \downarrow{\scriptstyle id} \\ \Omega M & \longrightarrow & L_N M & \overset{ev(0)}{\longrightarrow} & N \end{array}$$

the proof of Theorem 1.1.6 works as well to prove Theorem 1.1.9.

□

Proposition 1.6.1 *$\tilde{i}_!$ is a morphism of algebras and induces a morphism of multiplicative spectral sequences : $E^*(\tilde{i}_!) : \mathbb{E}^*(LM) \longrightarrow \mathbb{E}^*(L_N M)$.*

Proof : Applying the main result Theorem 1.1.2 to the pull-back embedding $(i, \tilde{i})$ provides a morphism of bidegree $(m-n, 0)$. The fact that this morphism is multiplicative comes from the following commutative diagram :

$$\begin{array}{ccc} L_N M \times L_N M & \xrightarrow{\tilde{i}\times\tilde{i}} & LM \times LM \\ \uparrow & & \uparrow \\ L_N M \times_N L_N M & \xrightarrow{\tilde{i}\times_N\tilde{i}} & LM \times_M LM \\ \downarrow{\scriptstyle Comp} & & \downarrow{\scriptstyle Comp} \\ L_N M & \xrightarrow{\tilde{i}} & LM \end{array} .$$

Indeed $\tilde{i}_! \circ \mu = \mu_N \circ \tilde{i}_! \otimes \tilde{i}_!$.

□

Now, we prove Theorem 1.1.10.

By pulling-back on the loops the preceding sub-fiberwise embedding, we show that the following commutative diagram

$$\begin{array}{ccc} L_U N & \xrightarrow{\tilde{j}_{|U}} & LN \\ \downarrow{\scriptstyle Li|_U} & & \downarrow{\scriptstyle Li} \\ L_Y N & \xrightarrow{\tilde{j}} & LX \\ \downarrow{\scriptstyle Lp_{|Y}} & & \downarrow{\scriptstyle Lp} \\ L_V N & \xrightarrow{\tilde{j}_V} & LM \end{array}$$

is a sub-fiberwise embedding. The theorem comes directly from the main result.

□

1.6.2 Proof of Proposition 1.1.12.

We consider the following commutative diagram :

$$\begin{array}{ccc} pt \times LM & \xrightarrow{j} & LM \times LM \\ \uparrow{\scriptstyle id\times\tilde{i}} & & \uparrow{\scriptstyle \tilde{\Delta}} \\ pt \times \Omega M & \longrightarrow & LM \times_M LM \\ \downarrow{\scriptstyle Comp} & & \downarrow{\scriptstyle Comp} \\ \Omega M & \xrightarrow{\tilde{i}} & LM \end{array}$$

We observe that $\tilde{\Delta}_{|pt\times\Omega M} = id \times \tilde{i}$ and that $Comp : pt \times \Omega M \to \Omega M$ is homotopic to the identity. Denote by EZ the cross product. The map $\mathbb{H}_*(LM) \to \mathbb{H}_{*-m}(LM)$ $x \mapsto Comp_* \circ \tilde{\Delta}_! \circ EZ(pt, x)$ is in fact multiplication by a, with a the homology class of pt in $\mathbb{H}_{*-m}(LM)$. The other map $\tilde{i}_* \circ Comp_* \circ (id \times \tilde{i}_!) \circ EZ(pt, -)$ is equal to $\tilde{i}_* \circ \tilde{i}_!$.

□

1.6.3 Intersection morphism and Cohen-Jones-Yan spectral sequence. Beginning of the proof of Theorem 1.1.13.

(1) Assume that for $n \geq 2$, all the differentials of the Cohen-Jones-Yan spectral sequence vanish, then the homomorphism $E^*_{*,*}(\tilde{i}_!)$ induced by the pull-back embedding

$$\begin{array}{ccc}
\Omega M & \xrightarrow{id} & \Omega M \\
\downarrow{\scriptstyle id} & & \downarrow \\
\Omega M & \xrightarrow{\tilde{i}} & LM \\
\downarrow{\scriptstyle ev(0)} & & \downarrow{\scriptstyle ev(0)} \\
pt & \xrightarrow{i} & M
\end{array}$$

is clearly onto from the E^2-level. We denote by $E^*_{*,*}(1)$ the spectral sequence associated to the left fibration and by $\mathbb{E}^*_{*,*}(2)$ the $(d,0)$-regraded spectral sequence associated to the right fibration. Then, $E^\infty(\tilde{i}_!)$ is onto on the graded space of $H_*(\Omega M)$ then I is onto.

(2) We begin by proving that all the differentials starting from $\mathbb{E}^*_{0,*}(2)$ vanish. We write the naturality of the Serre spectral sequence to the shriek map of the fiber embedding $(\tilde{i}, i)$ shown in the main result. We have the following commutative diagram :

$$\begin{array}{ccccccc}
H_*(\Omega M) \simeq \mathbb{E}^2_{0,*}(2) & \supseteq \mathbb{E}^3_{0,*}(2) & \supseteq \ldots & \supseteq \mathbb{E}^\infty_{0,*}(2) & \twoheadleftarrow & \mathbb{H}_*(LM) \\
\downarrow{\scriptstyle E^2(\tilde{i}_!)=i_!\otimes id} & & & \downarrow{\scriptstyle E^\infty(\tilde{i}_!)} & \swarrow{\scriptstyle \tilde{i}_!} & \\
H_*(\Omega M) \simeq E^2_{0,*}(1) & = E^3_{0,*}(1) & = \ldots & = E^\infty_{0,*}(1) = H_*(\Omega M) & &
\end{array}$$

If $I = \tilde{i}_!$ is onto and since each $H_k(\Omega M)$ is finitely generated, we have

$$\mathbb{E}^2_{0,*}(2) = \mathbb{E}^3_{0,*}(2) = \ldots = \mathbb{E}^\infty_{0,*}(2).$$

Thus the differentials starting from $\mathbb{E}^*_{0,*}(2)$ vanish. We remark that in this case, the morphism at the top right of the above diagram is in fact I. The existence of the canonical section $M \to LM$ implies that the differentials starting from $\mathbb{E}^*_{*,0}$ vanish. The multiplicative structure of $\mathbb{E}^2_{*,*}(2)$ implies that all the differentials of the Cohen-Jones-Yan spectral sequence vanish.

To end the proof of Theorem 1.1.13 we need :

Lemma 1.6.2 *If one of the differential of the Cohen-Jones-Yan spectral sequence is non zero, then there exists a non-zero differential arriving on* $\mathbb{E}^*_{-d,*}(2)$.

Proof :

Denote by $[M] \in \mathbb{H}_*(M)$ the fundamental class of M and by 1_ω the unit of $H_*(\Omega M)$. Assume that there exists a non-zero differential in the Cohen-Jones-Yan spectral sequence. We consider the first page of the spectral sequence where there is a non-zero differential. This page is isomorphic to $\mathbb{E}^2_{*,*}$ as an algebra. Since the Cohen-Jones-Yan spectral sequence is multiplicative, there exists a non-zero differential starting from a generator of $\mathbb{E}^2_{*,*}$ namely an element $[M]\otimes\omega$ of $\mathbb{E}^2_{0,*}$. Let $x\otimes\omega' = d([M]\otimes\omega)$ and $y\in\mathbb{H}_*(M)$ such that $x\bullet y = *$ with $*\in\mathbb{H}_{-d}(M)$ representing a fixed point. Then, $d(y\otimes\omega) = \pm d(y\otimes 1_\omega\circ[M]\otimes\omega) = \pm y\otimes 1_\omega\circ d([M]\otimes\omega) = \pm y\otimes 1_\omega\circ x\otimes\omega' = \pm x\bullet y\otimes\omega' = \pm *\otimes\omega'$. We use that $d(y\otimes 1_\omega) = 0$ because of the existence of a section.

□

1.6.4 End of proof of Theorem 1.1.13.

Observation

If $f: M\to N$ is a map between Poincaré duality manifolds, then $f_!$ is onto iff f_* injective. We prove that this result is true for the embedding $\tilde{i}:\Omega M\hookrightarrow LM$.

Proof :

(1) Assume that for $n\geq 2$, all the differentials of the Cohen-Jones-Yan spectral sequence vanish, then the homomorphism $E^*_{*,*}(\tilde{i}_*)$ is clearly injective. Furthermore, we have the injective map $\mathbb{E}^\infty_{-d,*}(2)\hookrightarrow\mathbb{H}_*(LM)$. The composition of these two maps is $\tilde{i}_*$.

(2) Assume $\tilde{i}_*$ is injective. From the naturality of the Serre spectral sequence for $\tilde{i}_*$, we deduce the following commutative diagram :

$$\begin{array}{ccccccccc}
H_*(\Omega M)\simeq E^2_{0,*}(1) & = & E^3_{0,*}(1) & = & \dots & = & E^\infty_{0,*}(1) & = & H_*(\Omega M)\,. \\
\downarrow{\scriptstyle E^2(\tilde{i}_*)} & & & & & & \downarrow{\scriptstyle E^\infty(\tilde{i})} & & \downarrow{\scriptstyle \tilde{i}} \\
H_*(\Omega M)\simeq \mathbb{E}^2_{-d,*}(2) & \twoheadrightarrow & \mathbb{E}^3_{0,*}(2) & \twoheadrightarrow & \cdots & \twoheadrightarrow & \mathbb{E}^\infty_{0,*}(2) & \hookrightarrow & \mathbb{H}_*(LM)
\end{array}$$

Since $\tilde{i}_*$ is injective and $H_k(\Omega M)$, $H_k(LM)$ are of finite type, the surjective maps at the bottom of the diagram are in fact equalities. This proves that there is no non-zero differentials arriving on $\mathbb{E}^*_{-d,*}(2)$. We conclude with the above lemma 1.6.2 that all the differentials of the Cohen-Jones-Yan spectral sequence are zero.

□

1.6.5 Intersection morphism of the spheres.

Assume $n\geq 2$.

$I:\mathbb{H}_k(LS^{2n-1})\to H_k(\Omega S^{2n-1})$ *is an isomorphism for* $k = 2ni$ $i\geq 0$, 0 *otherwise.*

$I:\mathbb{H}_k(LS^{2n})\to H_k(\Omega S^{2n})$ *is an isomorphism for* $k = 2i(2n-1)$ $i\geq 0$, 0 *otherwise.*

Proof :

In [16], Cohen Jones and Yan have shown that all the differentials of their spectral sequence are zero for odd spheres. Then Theorem 1.1.13 proves that $im(I) = H_*(\Omega S^{2n+1})$.

For the case of even dimensionnal spheres (except the 2-sphere), we need the results of [22] wich proves the following result with rational coefficients : $im(I) =$

$H_k(\Omega S^{2n};\mathbb{Q})$ $k = 2i(n-1)$, 0 elsewhere. This result gives the image of the torsion free part of I. For reasons of degree, the image of the torsion part is zero.

□

1.6.6 Intersection morphism of a Stiefel manifold.

Consider the fibration $S^7 \to SO(9)/SO(7) \to S^8$. Using together Theorems 1.1.6 and 1.1.7, we prove that the differentials of the Cohen-Jones-Yan spectral sequence are zero from level 2. Then we have :

$$E^{\infty}_{*,*}(\mathbb{H}_*(L(SO(9)/SO(7))) = \mathbb{H}_*(L(SO(9)/SO(7)) \otimes H_*(\Omega(SO(9)/SO(7))$$
$$= (\lambda(u) \otimes \mathbb{Z}/2\mathbb{Z}[v]/v^2) \otimes (\mathbb{Z}[a] \otimes \mathbb{Z}_2[b])$$

avec $deg(u) = -15$, $deg(v) = -8$, $deg(a) = 14$ and $deg(b) = 6$. Applying theorem 1.6.3, we obtain that

$$im(I) = H_*(\Omega(SO(9)/SO(7))) = \mathbb{Z}[a] \otimes \mathbb{Z}_2[b]$$

with $deg(a) = 2(8-1) = 14$ and $deg(b) = 7 - 1 = 6$.

This computation is done in chapter 3 page 66.

1.7 Application of the main result to the space of free paths.

Our goal here is to apply the theory of section 1.4 to find some structure of algebra on the spectral sequences associated to spaces of free paths.

1.7.1 Product of composition of paths

Our last application uses the composition product on the space of free paths of M, denoted by M^I. More explicitely, for two paths $\gamma_1, \gamma_2 \in M^I$ such that $\gamma_1(1) = \gamma_2(0)$, we denote by $\gamma_1 * \gamma_2$ the composed path.

Definition 1.7.1 *On homology, we define the path product*

$$\tilde{\mu} = Comp_* \circ \hat{i}_! : H_*(M^I) \otimes H_*(M^I) \longrightarrow H_{*-m}(M^I)$$

from the following diagram :

$$M^I \times M^I \xleftarrow{\hat{i}} M^I \times_M M^I \xrightarrow{Comp} M^I$$

where $\hat{i}$ is the canonical inclusion.

1.7.2 Intersection product and product of composition of paths

Proposition 1.7.2 *The path product on $\mathbb{H}_*(M^I)$ is identified, via the isomorphism $H_*(M^I) \simeq H_*(M)$ to the intersection product on $\mathbb{H}_*(M)$.*

Proof :
The homotopy

$$\mathcal{H} : LM \times I \longrightarrow M^I$$
$$\gamma, s \longmapsto (t \mapsto \gamma(st))$$

proves that the inclusion $j : LM \hookrightarrow M^I$ is homotopic to the projection $ev(0) : LM \longrightarrow M$. Since composition of paths on M^I restricted to LM is the composition of loops, the following diagram is commutative :

$$\begin{array}{ccccc}
\mathbb{H}_*(M^I) \otimes \mathbb{H}_*(M^I) & \longleftrightarrow & \mathbb{H}_*(M) \otimes \mathbb{H}_*(M) & \xleftarrow{ev(0)\otimes ev(0)} & \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LM) \\
\downarrow \tilde{\mu} & & \bullet \downarrow & & \downarrow \circ \\
\mathbb{H}_*(M^I) & \longleftrightarrow & \mathbb{H}_*(M) & \xleftarrow{ev(0)} & \mathbb{H}_*(M^I)
\end{array}$$

where $\circ$ is the Chas and Sullivan loop product, $\bullet$ is the intersection product and $\tilde{\mu}$ is the path product on $\mathbb{H}_*(M^I)$. This ends the proof of Proposition 1.7.2.

□

1.7.3 Diamond product

We denote $\mathbf{H}_*(M \times M) = H_{*+d}(M \times M) = \mathbb{H}_{*-d}(M \times M)$.

Definition 1.7.3 *We define the new product*

$$\diamond : \mathbf{H}_*(M \times M) \otimes \mathbf{H}_*(M \times M) \longrightarrow \mathbf{H}_*(M \times M)$$

$$(a \times b) \otimes (c \times d) \longmapsto p_{2*}(a \times (b \bullet c) \times d)$$

where $p_2 : M \times M \times M \longrightarrow M \times M$ is the projection on the first and the third factor of $M \times M \times M$.

This product is associative, not commutative without unit.

1.7.4 Proof of Theorem 1.1.14

We consider the following commutative diagram :

$$\begin{array}{ccccc}
\Omega M \times \Omega M & \longrightarrow & M^I \times M^I & \xrightarrow{(ev(0),ev(1))\times(ev(0),ev(1))} & M \times M \times M \times M \\
\uparrow{\scriptstyle id} & & \uparrow{\scriptstyle \tilde{D}} & & \uparrow{\scriptstyle D} \\
\Omega M \times \Omega M & \longrightarrow & M^I \times_M M^I & \longrightarrow & M \times M \times M \\
\downarrow{\scriptstyle Comp} & & \downarrow{\scriptstyle Comp} & & \downarrow{\scriptstyle p_2} \\
\Omega M & \longrightarrow & M^I & \xrightarrow{(ev(0),ev(1))} & M \times M
\end{array}$$

where $D : M^3 \longrightarrow M^4 \quad , \quad (x, y, z) \longmapsto (x, y, y, z)$ and $\tilde{D}$ is defined from D by pull-back so that $(D, \tilde{D})$ is a pull-back embedding. Let us denote γ the composition of paths or pointed loops. The upper part of the diagram is a sub-fiberwise embedding, so that, applying the main result, $\tilde{D}_!$ induces the morphism of Serre spectral sequences $E^*_{*,*}(\tilde{D}_!)$. The lower part of the diagram is a morphism of fibrations. Then $E^*_{*,*}(Comp_* \circ \tilde{D}_!) : \mathbf{E}^*_{*,*}(M^I \times M^I) \longrightarrow \mathbf{E}^*_{*,*}(M^I)$ is a morphism of spectral sequences which provides the announced multiplicativity of the Serre spectral sequence associated to the fibration $\Omega M \longrightarrow M^I \longrightarrow M \times M$.

$\square$

1.7.5 Remark

In the case of a fibred space, we could state the Theorem 1.1.10' analogous of Theorems 1.1.7 and 1.1.10 , but it is not necessary since, by Proposition 1.7.2, this product is the intersection product then we recover Proposition 1.1.3.

1.7.6 Example : the $\diamond$ product on $\mathbf{H}_*(S^3 \times S^3)$.

We apply the above result to the fibration $\Omega S^3 \longrightarrow S^{3^I} \xrightarrow{(ev(1),ev(0))} S^3 \times S^3$. Denote by $[S^3] \in H_3(S^3)$ the fundamental class and by 1 a generator of $H_0(S^3)$.

Then, we obtain the following "table of multiplication" for $\diamond$:

$\diamond$	1×1	$[S^3]\times 1$	$1\times [S^3]$	$[S^3]\times [S^3]$
1×1	0	1×1	0	$1\times [S^3]$
$[S^3]\times 1$	0	$[S^3]\times 1$	0	$[S^3]\times [S^3]$
$1\times [S^3]$	1×1	0	$1\times [S^3]$	0
$[S^3]\times [S^3]$	$1\times [S^3]$	0	$[S^3]\times [S^3]$	0

Here we have : $\mathbf{E}^2_{*,*} = \mathbf{E}^2_{-3,*} \oplus \mathbf{E}^2_{0,*} \oplus \mathbf{E}^2_{3,*}$. We denote by 1_Ω a generator of $H_0(\Omega S^3)$ and by u a generator of $H_2(\Omega S^3)$. We put

$$a = (1\times [S^3] + [S^3]\times 1)\otimes 1_\Omega$$

$$b = (1\times [S^3] - [S^3]\times 1)\otimes 1_\Omega$$

and

$$c = ([S^3]\times [S^3])\otimes 1_\Omega.$$

The only non zero differential is d_3 and we have $d_3(a) = 0$, $d_3(b) = (1\times 1)\otimes 1_\Omega$ and $d_3(c) \neq 0$ lies in $\mathbf{E}^3_{0,2}$. At the E^∞-level, it remains only $(1\times 1)\otimes 1_\Omega$ representing 1 in $\mathbb{H}_{-3}(S^3)$ and a representing $[S^3]$ in $\mathbb{H}_0(S^3)$. Let us denote by $\circ$ the induced product on the shifted spectral sequence. We check that

$$\begin{aligned} a\circ a &= (1\times [S^3] + [S^3]\times 1)\otimes 1_\Omega \quad \circ \quad (1\times [S^3] + [S^3]\times 1)\otimes 1_\Omega \\ &= (1\times [S^3] + [S^3]\times 1)\diamond (1\times [S^3] + [S^3]\times 1)\otimes (1_\Omega * 1_\Omega) \\ &= (1\times [S^3] + [S^3]\times 1)\otimes 1_\Omega = a \end{aligned}$$

which corresponds to $[S^3]\bullet [S^3] = [S^3]$ in $\mathbb{H}_*(S^3)$. In the same way, we check that

$$(1\times 1)\otimes 1_\Omega \quad \circ \quad (1\times 1)\otimes 1_\Omega = 0$$

and that

$$(1\times 1)\otimes 1_\Omega \quad \circ \quad (1\times [S^3] + [S^3]\times 1)\otimes 1_\Omega = (1\times [S^3] + [S^3]\times 1)\otimes 1_\Omega$$

and

$$(1\times [S^3] + [S^3]\times 1)\otimes 1_\Omega \quad \circ \quad (1\times 1)\otimes 1_\Omega = (1\times [S^3] + [S^3]\times 1)\otimes 1_\Omega.$$

Thus we recover the intersection product on $\mathbb{H}_*(S^3)$.

1.7.7 Remark

As a final remark, let us consider the pull-back embedding (see definition 1.4.6) :

$$\begin{array}{ccc}
\Omega M & \xrightarrow{id} & \Omega M \\
\downarrow & & \downarrow \\
LM & \overset{\tilde{\Delta}}{\hookrightarrow} & M^I \\
\downarrow{\scriptstyle ev(0)} & & \downarrow{\scriptstyle ev(0)\times ev(1)} \\
M & \overset{\Delta}{\hookrightarrow} & M \times M
\end{array}$$

(Δ is the diagonal embedding). Applying the main result, there is a morphism of spectral sequences : $E(\tilde{\Delta}_!) : \mathbf{E}^*_{*,*} \longrightarrow \mathbb{E}^*_{*-d,*}$ given at the E^2-level by : $E^2(\tilde{\Delta}_!) : \mathbf{E}^2_{*,*} = \mathbf{H}_*(M \times M; \mathcal{H}_*(\Omega M)) \longrightarrow \mathbb{H}_{*-d}(M; \mathcal{H}_*(\Omega M))\,, \quad (x \times y; \omega) \longmapsto (x \bullet y; \omega)$. This morphism of spectral sequences is not multiplicative.

Chapitre 2

Exemples de calculs.

Résumé

Dans cette partie, nous donnons quelques exemples de calculs du loop-produit et du morphisme d'intersection pour illustrer les techniques de la première partie.

Tous les calculs d'homologie sont donnés à coefficients entiers. Nous désignons par $\Lambda(x,y)$ l'algèbre graduée commutative libre engendrée par x et y.

2.1 L'algèbre de Chas et Sullivan des sphères

Nous rappelons les résultats sur l'homologie des sphères obtenus dans [16]. Si $n \in \mathbb{N}$ est impair, $\mathbb{H}_*(LS^n) = \Lambda(u) \otimes \mathbb{Z}[v]$ avec $deg(u) = -n$ et $deg(v) = n-1$. Si $n \in \mathbb{N}$ est pair, $\mathbb{H}_*(LS^n) = \Lambda(b) \otimes \mathbb{Z}[a,v]/(a^2, ab, 2av)$ avec $deg(a) = -n$ et $deg(b) = -1$ et $deg(v) = 2n-1$.

2.2 Espaces projectifs quaternioniques

Le cas des espaces projectifs quaternioniques $\mathbb{H}P^n$ se traite exactement comme le font R.Cohen, J.D.S.Jones et J.Yan [16] pour les espaces projectifs complexes $\mathbb{C}P^n$. La suite spectrale de Serre associée à la fibration homotopique $\Omega S^{4n+3} \to \Omega\mathbb{H}P^n \to S^3$ permet de calculer la structure d'anneau de Pontryagin : $H_*(\Omega\mathbb{H}P^n) = \Lambda(t) \otimes \mathbb{Z}[x]$ avec $deg(t) = 3$ et $deg(x) = 4n+2$. De plus, $H^*(\mathbb{H}P^n) = \mathbb{Z}[\alpha]/\alpha^{n+1}$ avec $deg(\alpha) = 4$ [29]. L'homologie de $L\mathbb{H}P^n$ est calculée dans [37].

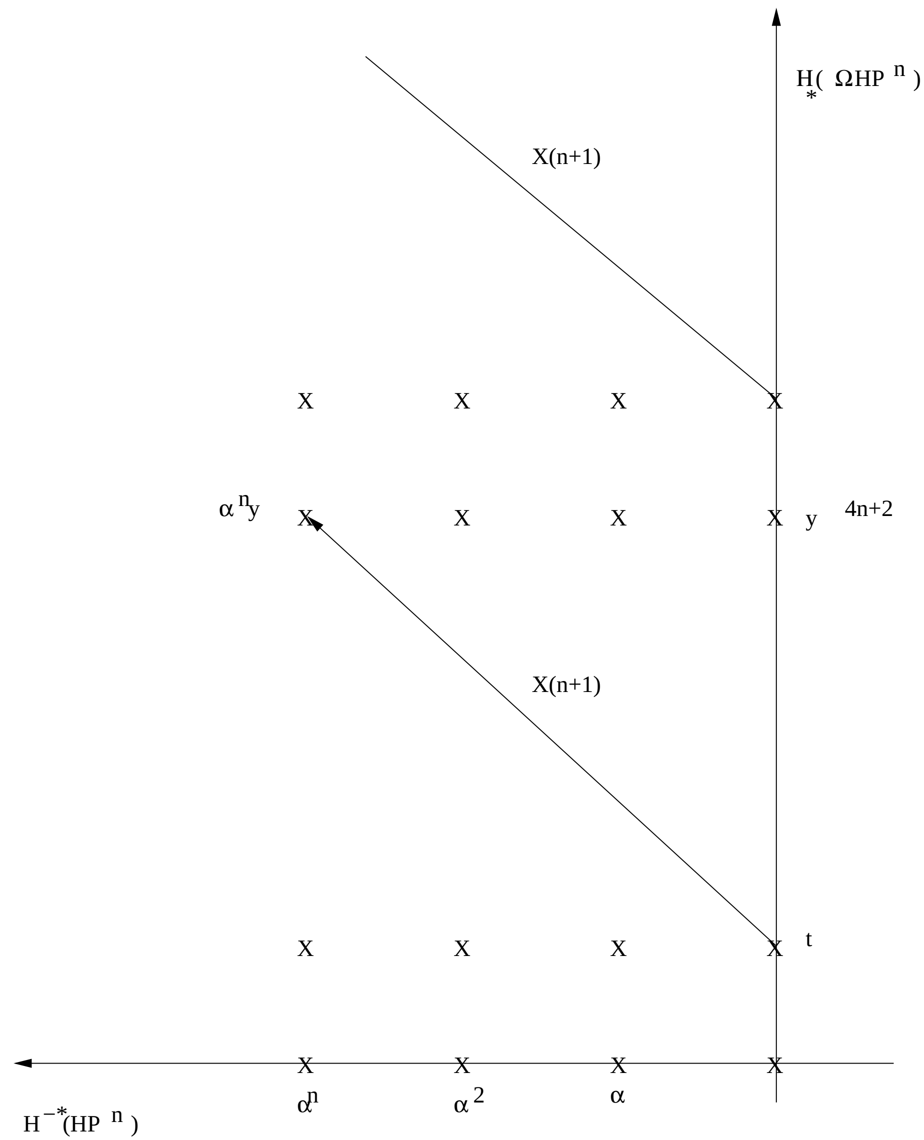
$$H_k(L\mathbb{H}P^n) = \begin{cases} \mathbb{Z} & \text{si } k = 0,\ 2m(2n+1)+4l\ m \geq 0,\ l = 1,...,n \\ \mathbb{Z} & \text{si } k = 2m(2n+1)+4l-4n+1,\ m \geq 1,\ l = 0,...,n-1 \\ \mathbb{Z}/(n+1)\mathbb{Z} & \text{si } k = 2m(2n+1),\ m \geq 1\,. \end{cases}$$

Nous pouvons en déduire les différentielles de la suite spectrale de Cohen-Jones-Yan (voir page 50). Pour des raisons de degré, il n'y a pas de problèmes d'extension d'algèbre (voir Chapitre 4 et [27] au sujet de ces problèmes d'extension d'algèbre). Nous avons alors :

$$\mathbb{H}_*(L\mathbb{H}P^n) = \mathbb{Z}[a,b,y]/(a^{n+1}, a^n b, (n+1)a^n y, b^2)$$

avec $deg(a) = -4$, $deg(b) = -1$, $deg(y) = 4n+2$.

Suite spectrale de Cohen-Jones-Yan des espaces projectifs quaternioniques


H_*(ΩHP^n)
X(n+1)
α^n y
y 4n+2
X(n+1)
t
α^n
α^2
α
H^{-*}(HP^n)

2.3 Les fibrations de Hopf

2.3.1 La fibration $LS^7 \to LS^{15} \to LS^8$

But :

Le but est d'illustrer la suite spectrale d'homologie des lacets libres $(8,7)$-regraduée associée à la fibration de Hopf $LS^7 \to LS^{15} \to LS^8$ connaissant la structure d'algèbre de Chas et Sullivan sur la base, la fibre et l'aboutissant.

Pour des raisons de degré, les premières différentielles non nulles se trouvent au niveau E^7 (voir page 54) :

$$d_7(y) = b.x.$$

La structure multiplicative de la suite spectrale permet de déduire toutes les autres différentielles au niveau E^7. De même, aux autres niveaux, nous avons (voir pages 55 et 56) :

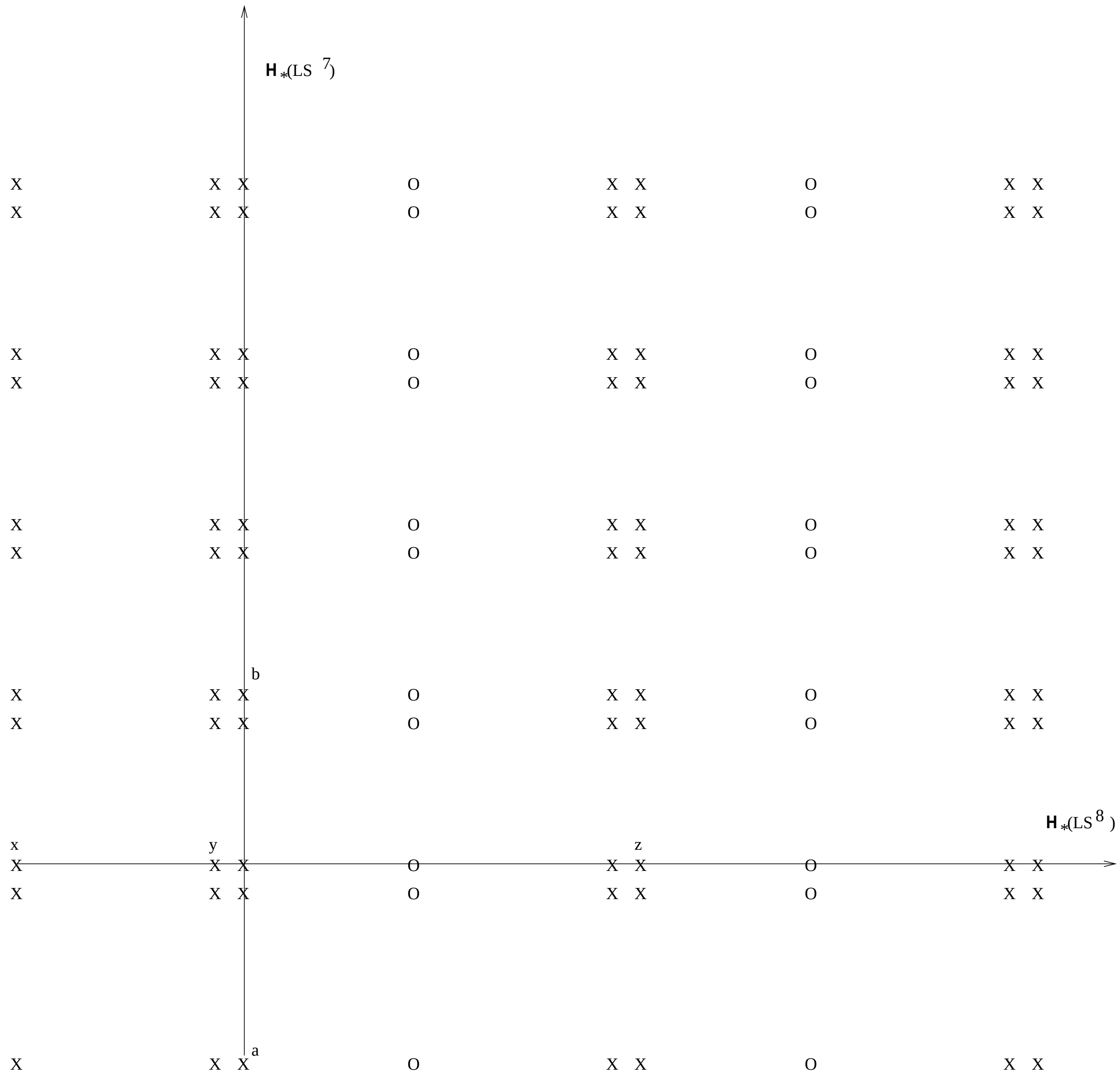
$$d_8(a) = x$$

$$d_{13}(yz) = b^2$$

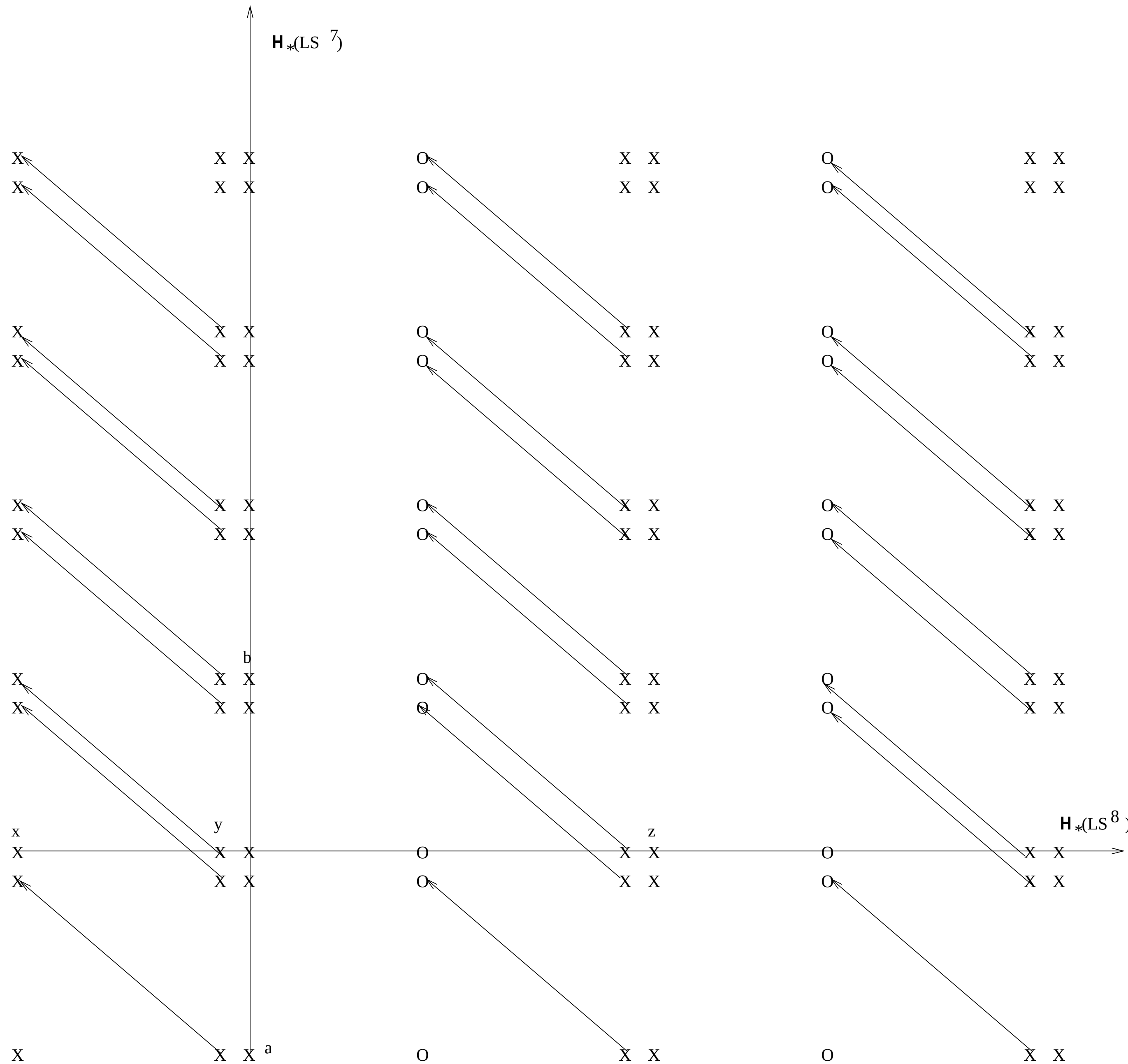
$$d_{14}(az) = b.$$

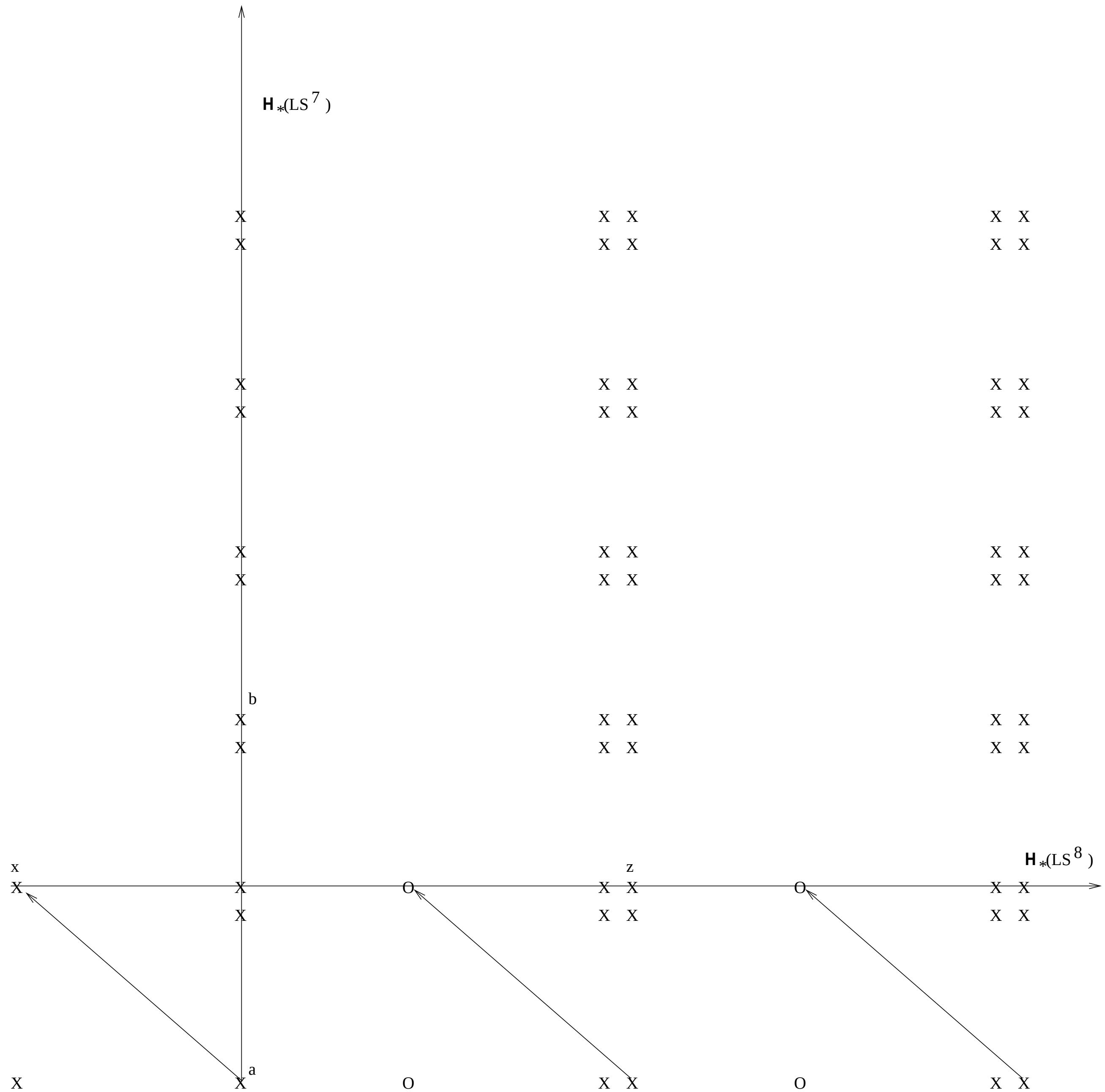
La suite spectrale stationne au niveau E^{15}. Les problèmes d'extension sont non triviaux (voir page 57). Dans $F_8H_{14}(LS^{15})$, nous avons les entiers pairs c'est-à-dire $2.H_{14}(LS^{15})$ et $F_{14}H_{14}(LS^{15}) = H_{14}(LS^{15}) = \mathbb{Z}$ c'est pourquoi $E^{\infty}_{14,0}(H_*(LS^{15})) = \mathbb{Z}/2\mathbb{Z}$. De même, chaque fois que l'on est en degré total multiple de 14 dans la suite spectrale de Serre non regraduée, nous levons les problèmes d'extension de la même manière.

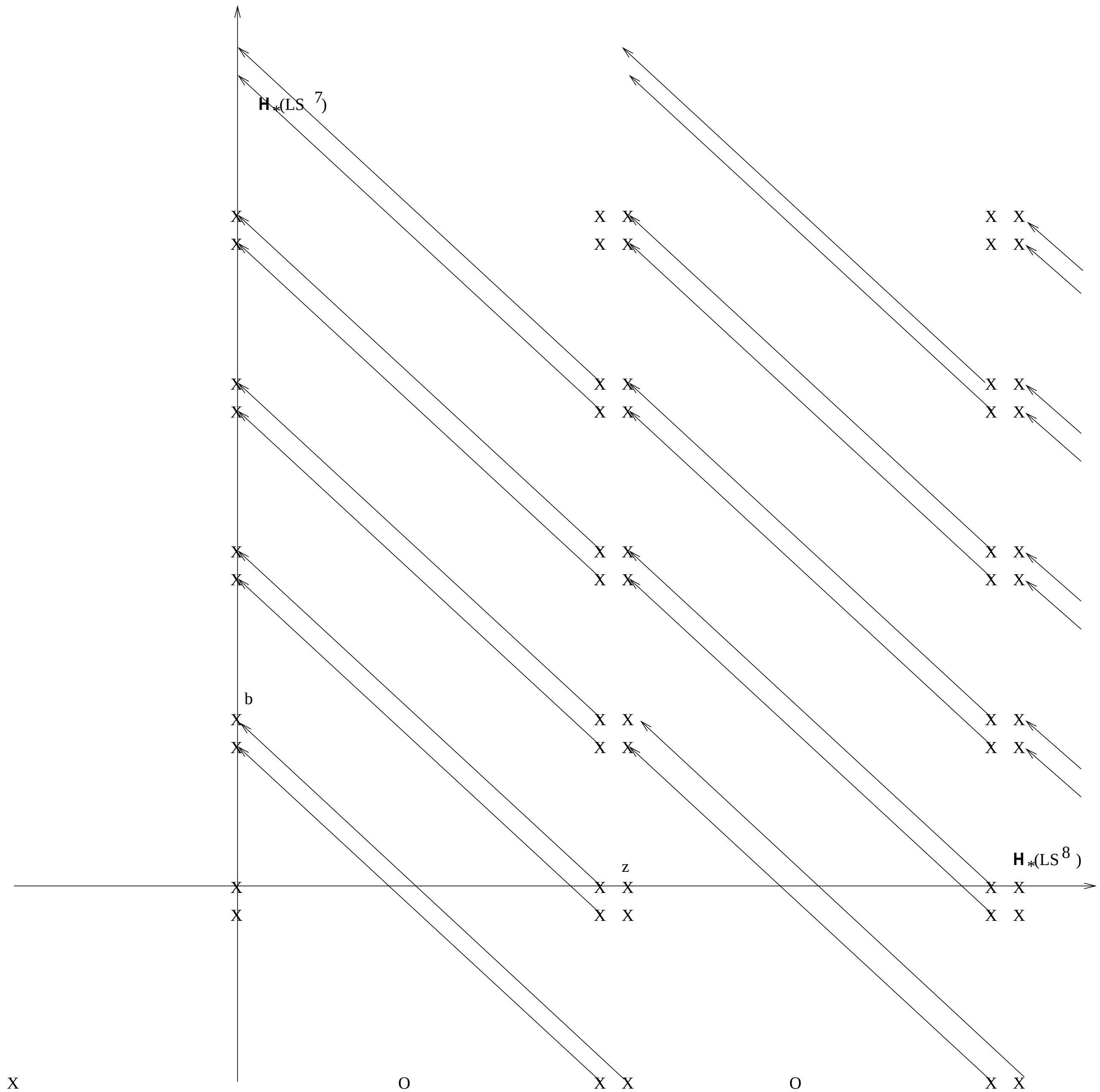
Suite spectrale de la fibration de Hopf : $LS^7 \to LS^{15} \to LS^8$



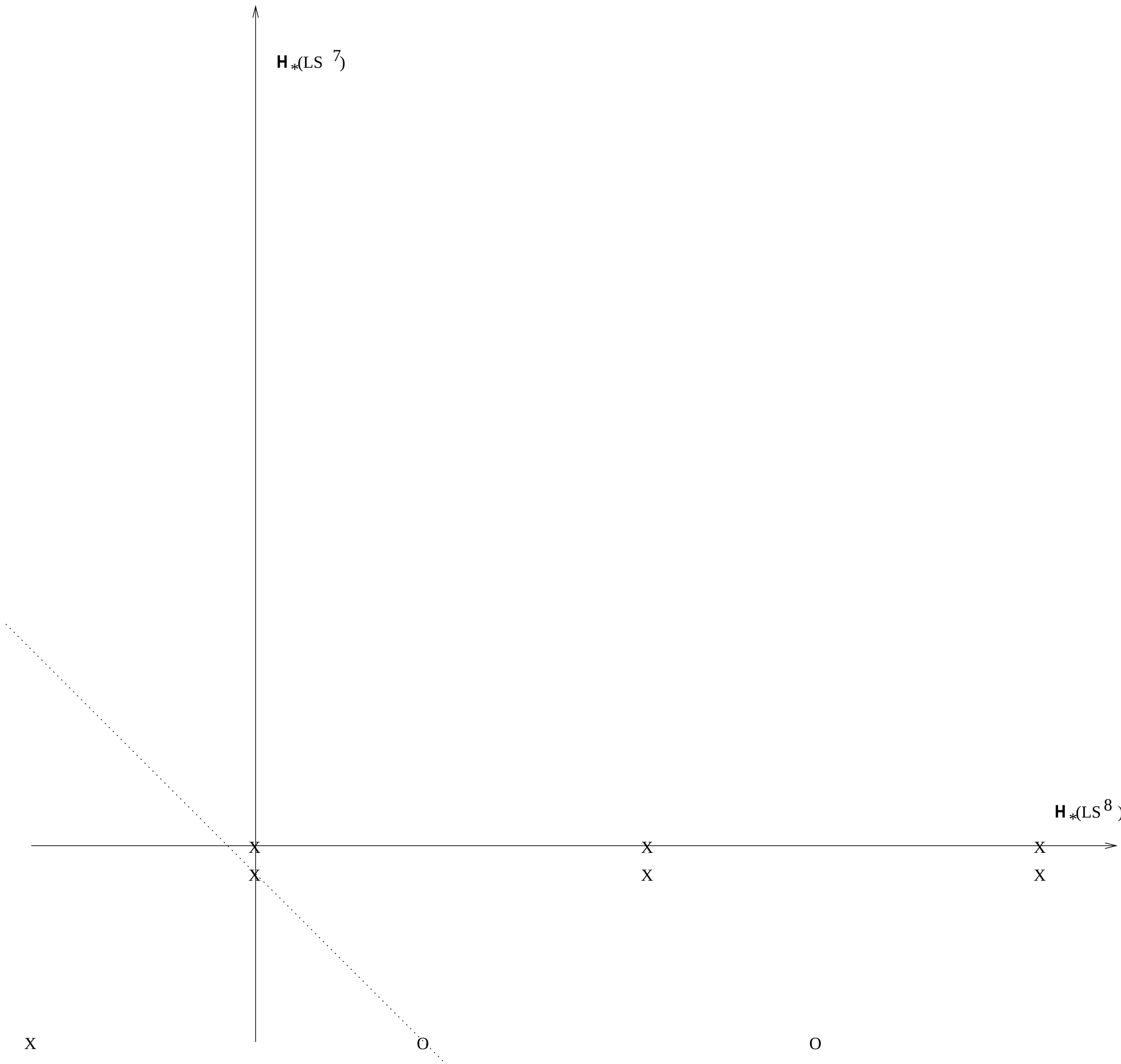
H*(LS 7)
H*(LS 8)
x
y
z
a
b


H*(LS 7)
H*(LS 8)
b
x
y
z
a


H*(LS7)
H*(LS8)
a
b
x
z



H*(LS 7)
b
z
H*(LS 8)
X
O


$\mathbf{H}_*(LS^7)$
$\mathbf{H}_*(LS^8)$
X
X
X
X
X
X
X
O
O

2.3.2 La fibration $LS^3 \to LS^{11} \to L\mathbb{H}P^2$, calcul du morphisme d'intersection $I : \mathbb{H}_*(L\mathbb{H}P^2) \to H_*(\Omega\mathbb{H}P^2)$.

Calcul des différentielles dans la suite spectrale des lacets libres $(8,3)$-regraduée.

Le but est le même que dans la section précédente, nous connaissons la structure d'algèbre de Chas et Sullivan sur la base, la fibre et l'aboutissant, nous en déduisons les différentielles de la suite spectrale des lacets libres $(8,3)$-regraduée (voir page 60). Pour des raisons de degré, les premières différentielles non nulles se trouvent au niveau E^3 (voir page 61) :

$$d_3(b) = a.v$$

La structure multiplicative de la suite spectrale permet de déduire toute les autres différentielles au niveau E^3. De même, aux autres niveaux, nous avons (voir pages 62, 63 et 64) :

$$d_4(u) = a$$

$$d_5(aby) = v^2$$

$$d_6(ayu) = v.$$

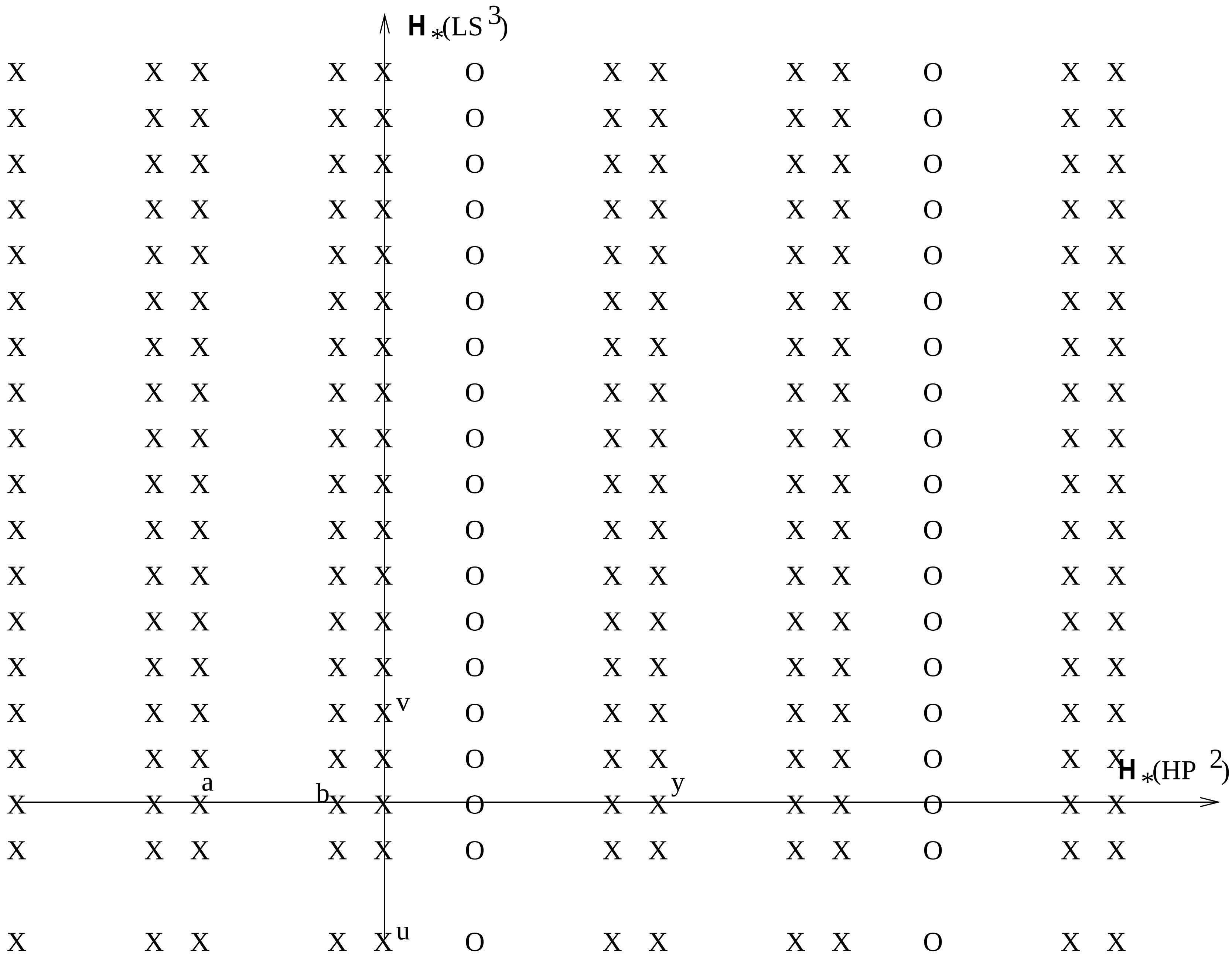
La suite spectrale stationne au niveau E^7. Les problèmes d'extension sont non triviaux (voir page 65). Dans $F_8H_{10}(LS^{11})$, nous avons les entiers multiples de 3 c'est-à-dire $3.H_{10}(LS^{11})$ et $E^\infty_{10,0}(H_*(LS^{11})) = \mathbb{Z}/3\mathbb{Z}$. De même, chaque fois que l'on est en degré total multiple de 10 dans la suite spectrale de Serre non regraduée, nous levons les problèmes d'extension de la même manière.

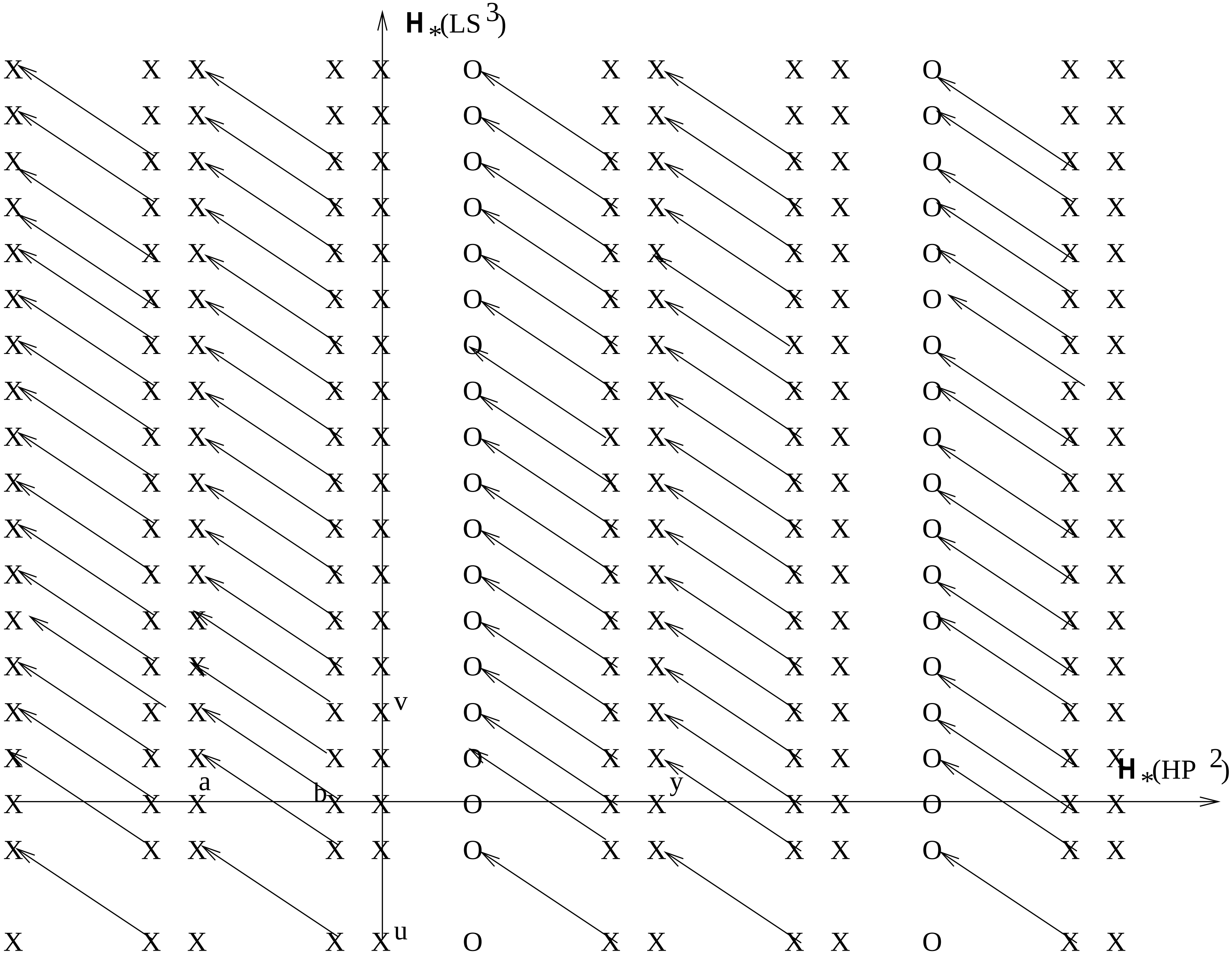
Calcul de I

Soit $\mathbb{E}^*_{*,*}(1)$ (resp. $\mathbb{E}^*_{*,*}(2)$) la suite spectrale de Serre $(8,3)$-regraduée (resp. $(0,0)$-regraduée) associée à la fibration $LS^3 \to LS^{11} \to L\mathbb{H}P^2$ (resp. $\Omega S^3 \to \Omega S^{11} \to \Omega\mathbb{H}P^2$) (voir pages 60 et suivantes). Nous calculons $\mathbb{E}^*_{*,*}(2)$. Au niveau E^3, $d_3(\mu) = t$ et la structure multiplicative de cette suite spectrale donne toutes les autres différentielles. La suite spectrale stationne au niveau E^4. Pour des raisons de degré, il n'y a pas de problèmes d'extension. Comme prouvé dans le chapitre 1 (Theorem 1.1.2), I induit un morphisme de suites spectrales : $\mathbb{E}^*_{*,*}(I) : \mathbb{E}^*_{*,*}(1) \to E^*_{*,*}(2)$ donné au niveau E^2 par $\mathbb{E}^2_{*,*}(I) = I_{\mathbb{H}P^2} \otimes I_{S^3}$ où $I_{\mathbb{H}P^2}$ et I_{S^3} sont les morphismes d'intersection associés à $\mathbb{H}P^2$ et à S^3. L'image de $E^\infty_{*,*}(I)$ est $E^\infty_{*,*}(1)$ car $I_{S^{11}}$ est surjectif. Nous en déduisons puisque $y \in \mathbb{E}^\infty_{10,0}(1)$ survit à l'aboutissant que $E^\infty(I)(y) = \nu$ et par multiplicativité nous avons $E^\infty(I)(y^n) = \nu^n$. Nous en déduisons $Im(I_{\mathbb{H}P^2}) = vect_\mathbb{Z}(\nu^n, n \geq 0)$.

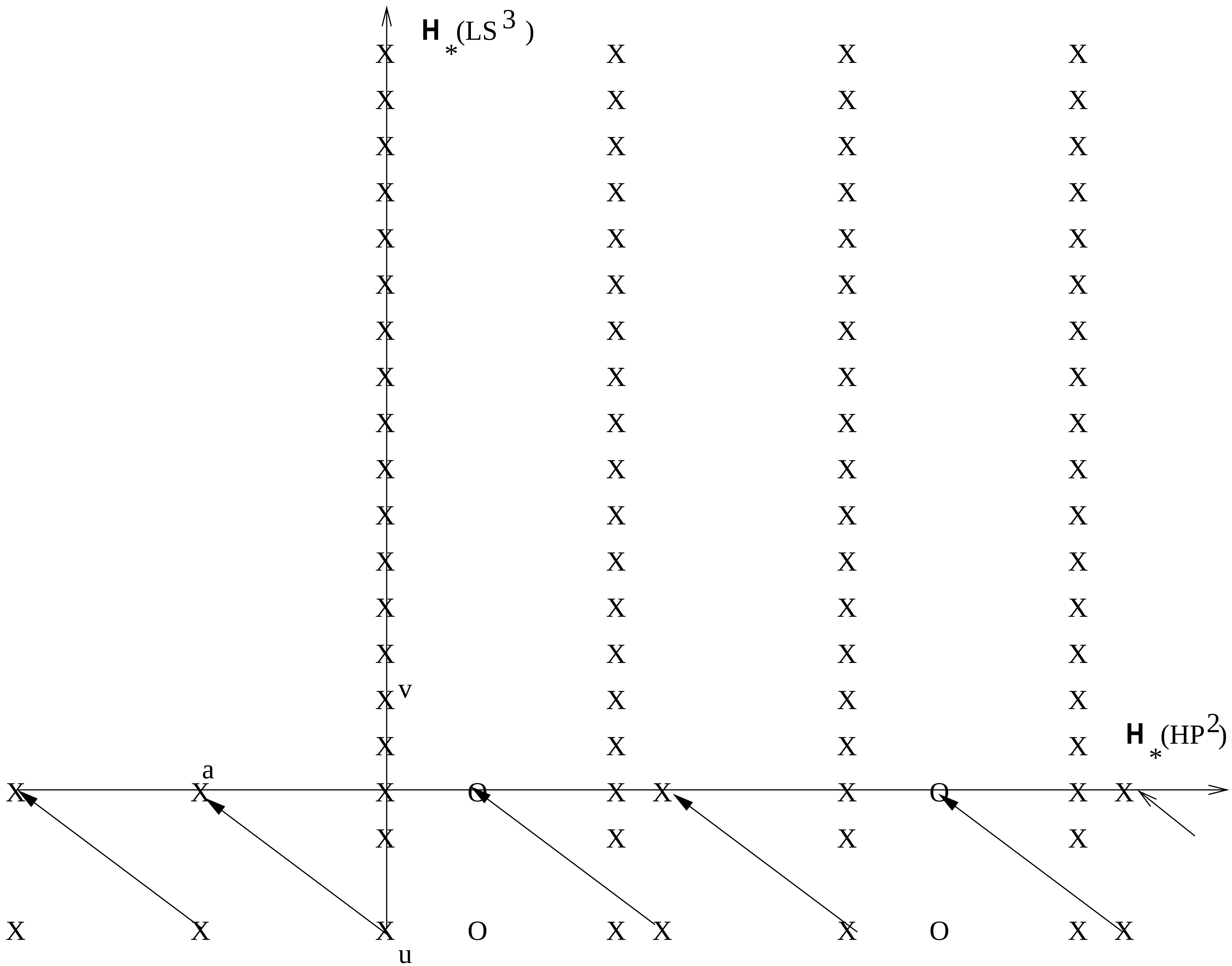
Ces résultats se généralisent aisément aux fibrations $LS^3 \to LS^{4(n+1)-1} \to L\mathbb{H}P^n$.

Suite spectrale associée à la fibration de Hopf
$LS^3 \to LS^{11} \to L\mathbb{H}P^2$

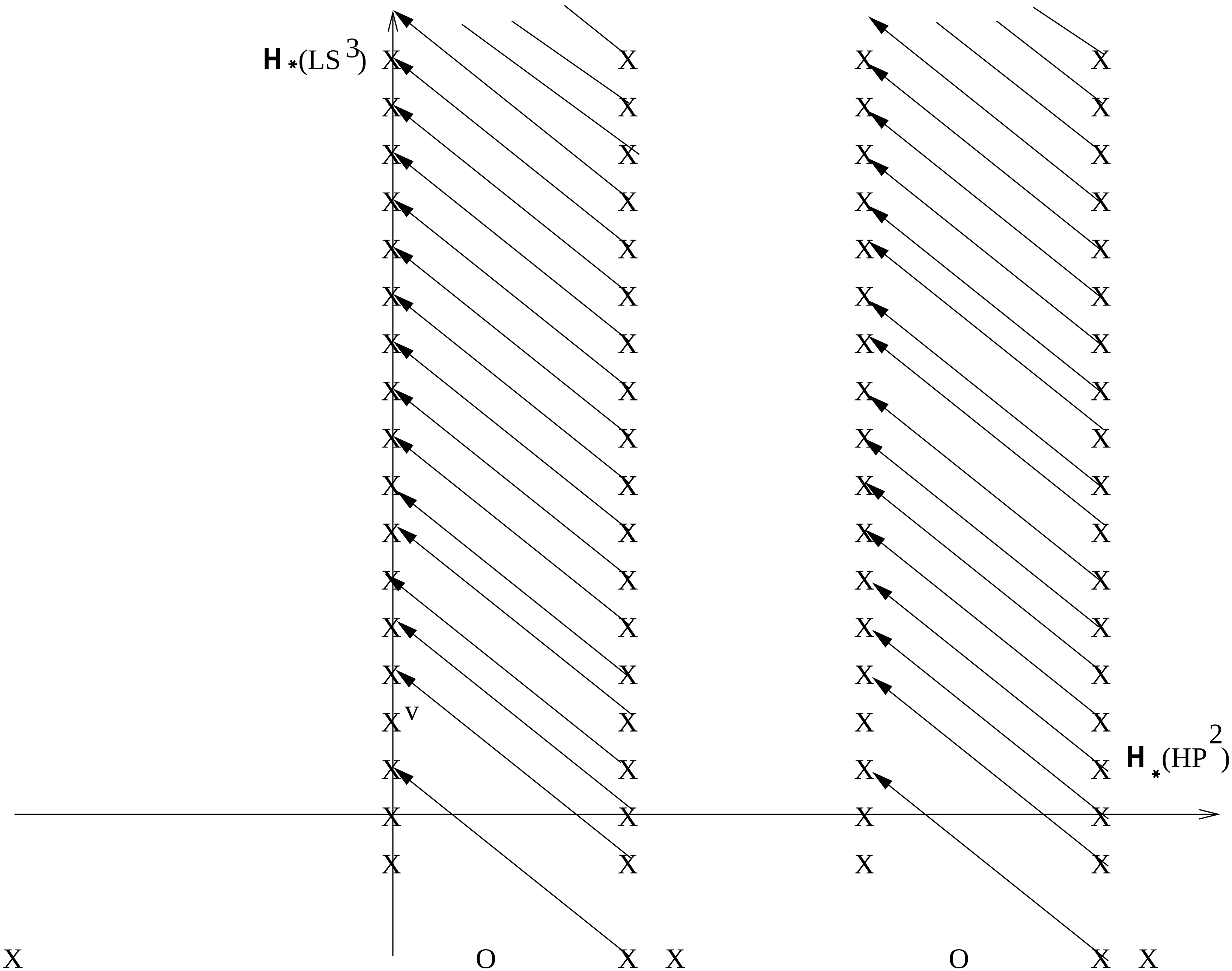

$\mathbf{H}_*(LS^3)$
$\mathbf{H}_*(HP^2)$
a
b
y
v
u

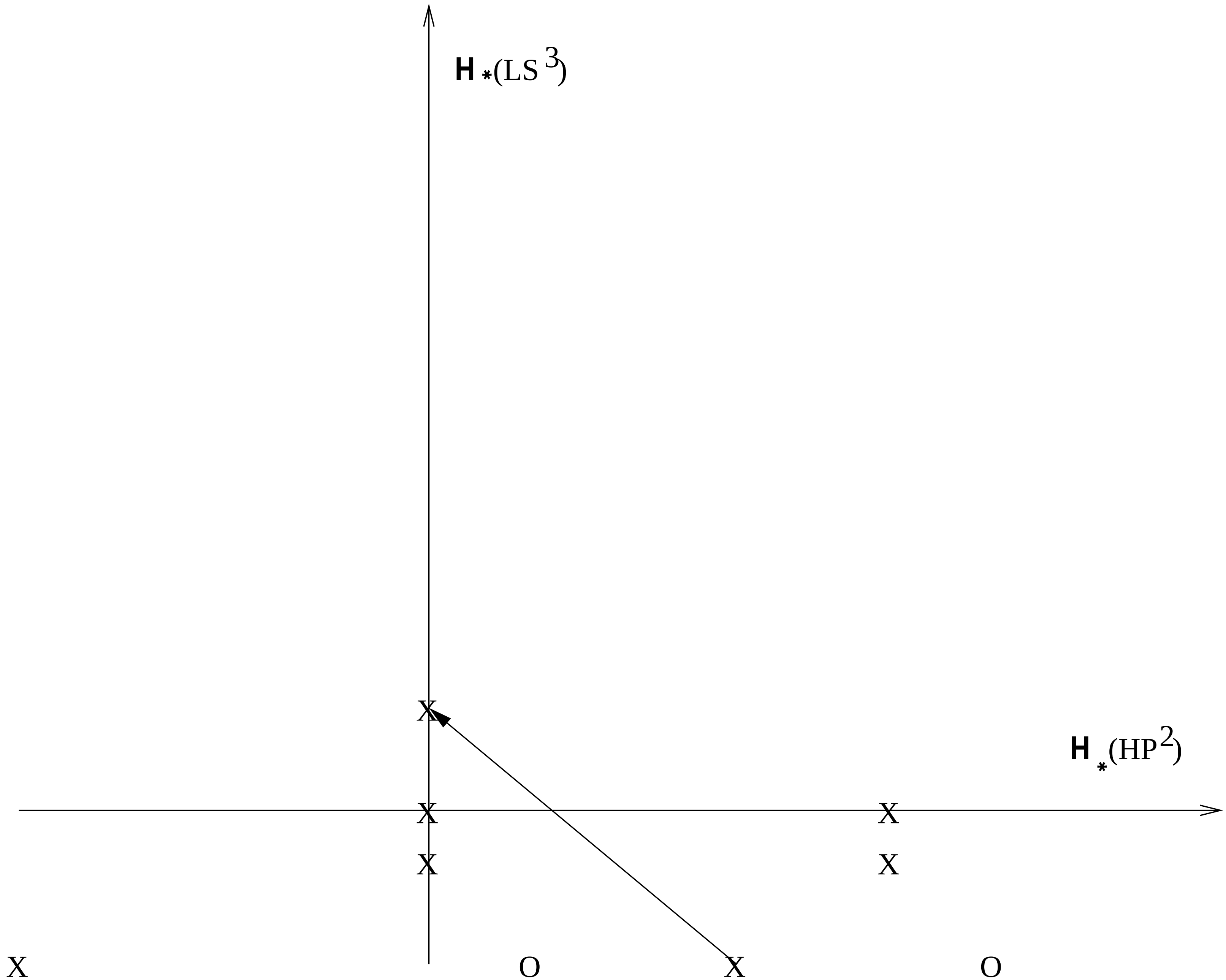


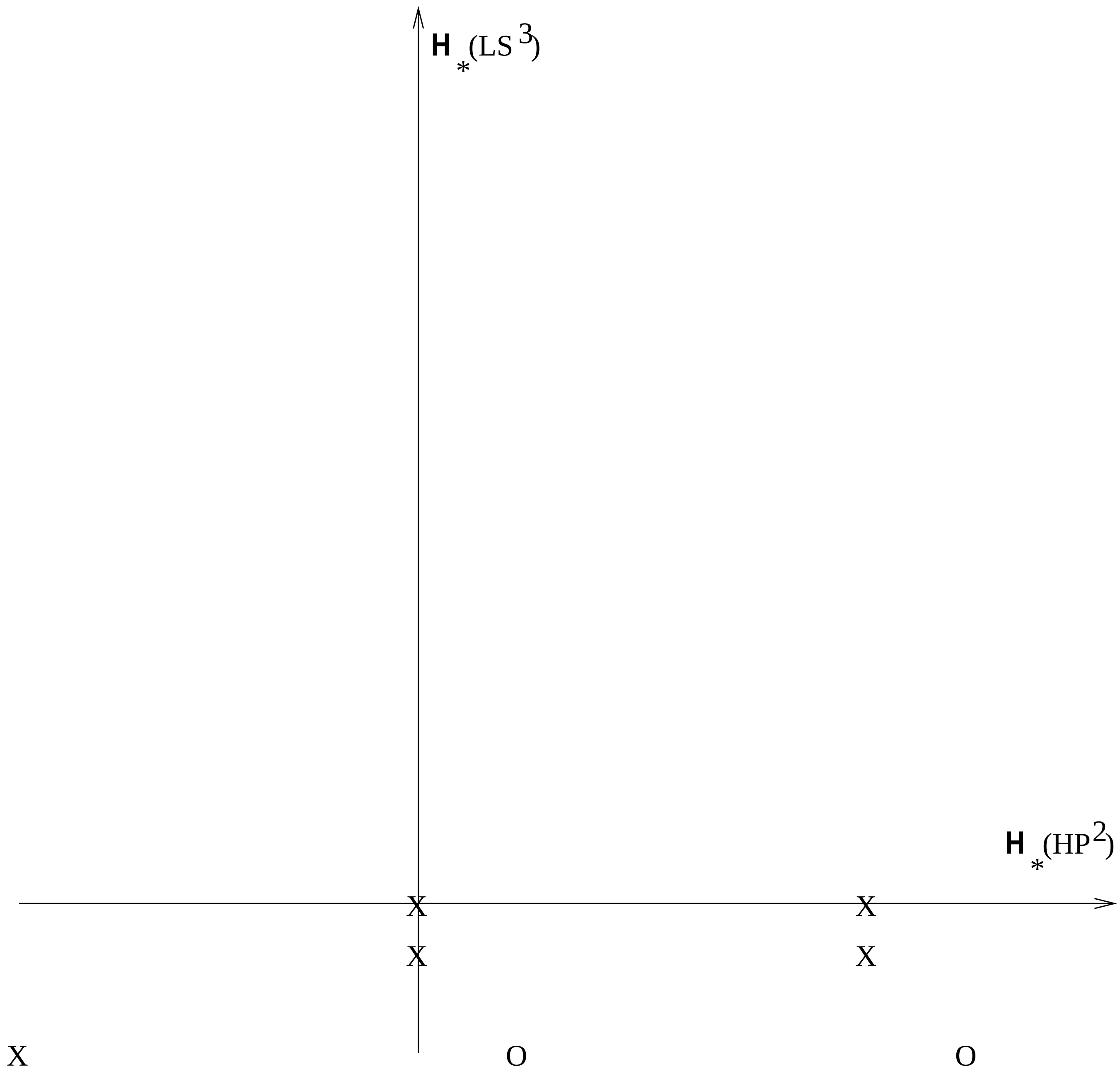
H_*(LS^3)
H_*(HP^2)
a
b
v
u
y



$\mathbf{H}_*(LS^3)$
$\mathbf{H}_*(HP^2)$
v
a
u


H*(LS3)
v
H*(HP2)
O
O


H∗(LS³)
H∗(HP²)


H_*(LS^3)
H_*(HP^2)
X
X
X
X
X
O
O

2.4 Calcul de $E^\infty(\mathbb{H}_*(L(SO(9)/SO(7))))$ et du morphisme d'intersection.

Méthode :

Nous écrivons deux suites spectrales, l'une est la suite spectrale de Cohen-Jones-Yan associée à la fibration $\Omega(SO(9)/SO(7)) \to L(SO(9)/SO(7)) \to SO(9)/SO(7)$: $\mathbb{E}^*_{*,*}(1)$ (voir page 69), l'autre est la suite spectrale des lacets libres $(8,7)$-regraduée associée à la fibration $LS^7 \to L(SO(9)/SO(7)) \to LS^8$: $\mathbb{E}^*_{*,*}(2)$ (voir page 73). En raisonnant successivement sur chacune de ces suites spectrales, nous parvenons à montrer que les différentielles de la suite spectrale de Cohen-Jones-Yan sont toutes nulles ce qui permet d'appliquer le théorème 1.1.13 page 21 du premier chapitre pour conclure que le morphisme d'intersection est surjectif. Malheureusement, nous ne savons lever les problèmes d'extension afin de déduire $\mathbb{H}_*(L(SO(9)/SO(7)))$ de $E^\infty(\mathbb{H}_*(L(SO(9)/SO(7))))$.

2.4.1 La suite spectrale de Cohen-Jones-Yan

Nous partons de $H^*(SO(9)/SO(7)) = \Lambda(\delta) \otimes \mathbb{Z}/2\mathbb{Z}[\gamma]$ avec $deg(\delta) = -15$ et $deg(\gamma) = -8$ [29]. En comparant la suite spectrale de Serre associée à la fibration des chemins basés $\Omega(SO(9)/SO(7)) \to P(SO(9)/SO(7)) \to SO(9)/SO(7)$ et la suite spectrale de Serre associée à la fibration $\Omega S^7 \to \Omega(SO(9)/SO(7)) \to \Omega S^8$ (voir page 76), on en déduit la structure d'anneau de Pontryagin : $H_*(\Omega(SO(9)/SO(7))) = \Lambda(\beta) \otimes \mathbb{Z}/2\mathbb{Z}[\alpha]$ avec $deg(\beta) = 14$ et $deg(\alpha) = 6$. Nous pouvons alors écrire le terme E^2 de la suite spectrale de Cohen-Jones-Yan de $\mathbb{H}_*(L(SO(9)/SO(7)))$ (voir page 69).

$$E^2_{*,*}(\mathbb{H}_*(L(SO(9)/SO(7)))) =$$

$$(\Lambda(\delta)\otimes\mathbb{Z}/2\mathbb{Z}[\gamma]) \bigotimes (\Lambda(\beta)\otimes\mathbb{Z}/2\mathbb{Z}[\alpha]) \bigoplus tor(\mathbb{H}_{-8}(SO(9)/SO(7)); H_*(\Omega(SO(9)/SO(7))))$$

2.4.2 Calculs

Le terme E^2 de la suite spectrale des lacets libres est : $\mathbb{E}^2_{*,*}(2) = \mathbb{H}_*(LS^7) \otimes \mathbb{H}_*(LS^8)$ (voir page 73).

1) Pour des raisons de degré, la première différentielle non nulle de $\mathbb{E}^*_{*,*}(2)$ se trouve au niveau 7 (voir page 74).

$$d_7(y) = 2xv$$

et on complète cette page en utilisant la structure multiplicative.

2) (voir page 75)

$$d_8(u) = 2x$$

3) Puisque d'après $E^*(1)$, α ne peut être un bord, il faut déterminer si α est un cycle dans $E^*(1)$ pour savoir si il survit à l'aboutissant. Nous remarquons que

α correspond au niveau de $E^2(1)$ au produit tensoriel $[SO(9)/SO(7)] \otimes \omega_6$ de la classe fondamentale de $SO(9)/SO(7)$ avec un générateur de $H_6(\Omega(SO(9)/SO(7)))$. Or cet élément en degré 6 de $H_*(\Omega(SO(9)/SO(7)))$ provient dans la suite spectrale associée à la fibration $\Omega S^7 \to \Omega(SO(9)/SO(7)) \to \Omega S^8$ d'un générateur de $H_6(\Omega S^7)$. Considérons maintenant v un générateur de $\mathbb{E}^2_{0,6}(2)$. Cet élément est construit comme $[S^8] \otimes e_6$ où e_6 est un générateur de $\mathbb{H}_6(LS^7)$. Or $\mathbb{H}_*(LS^7) = \mathbb{H}_*(S^7) \otimes H_*(\Omega S^7)$ c'est pourquoi e_6 se décompose comme suit : $e_6 = [S^7] \otimes \omega'_6$ avec ω'_6 un générateur de $H_6(\Omega S^7)$. Il apparaît ainsi que v et α correspondent géométriquement au même élément or pour des raisons de degré, v est forcément un cycle si bien que α aussi. Nous en concluons que toutes les différentielles partant de α sont nulles.

4) Pour des raisons de degré (voir page 75), z survit à l'aboutissant dans $\mathbb{E}^*_{*,*}(2)$ donc il y a un terme contenant du $\mathbb{Z}$ en degré 14 ce qui prouve (voir page 69) que $d_{15}(\beta) = 0$. Puisque pour tout $n \geq 2$, $d_n(\alpha) = d_n(\beta) = 0$ et puisque la fibration $\Omega(SO(9)/SO(7)) \to (LSO(9)/SO(7)) \to SO(9)/SO(7)$ admet une section (ce qui entraîne que pour tout $n \geq 2$, $d_n(\gamma) = d_n(\delta) = 0$), les différentielles s'annulent sur tous les générateurs d'algèbre ce qui prouve qu'elles sont toutes nulles et prouve ainsi les deux points énoncés au début de cette partie.

Nous complétons (voir pages 76, 77 et 78) les différentielles de $\mathbb{E}(2)$.

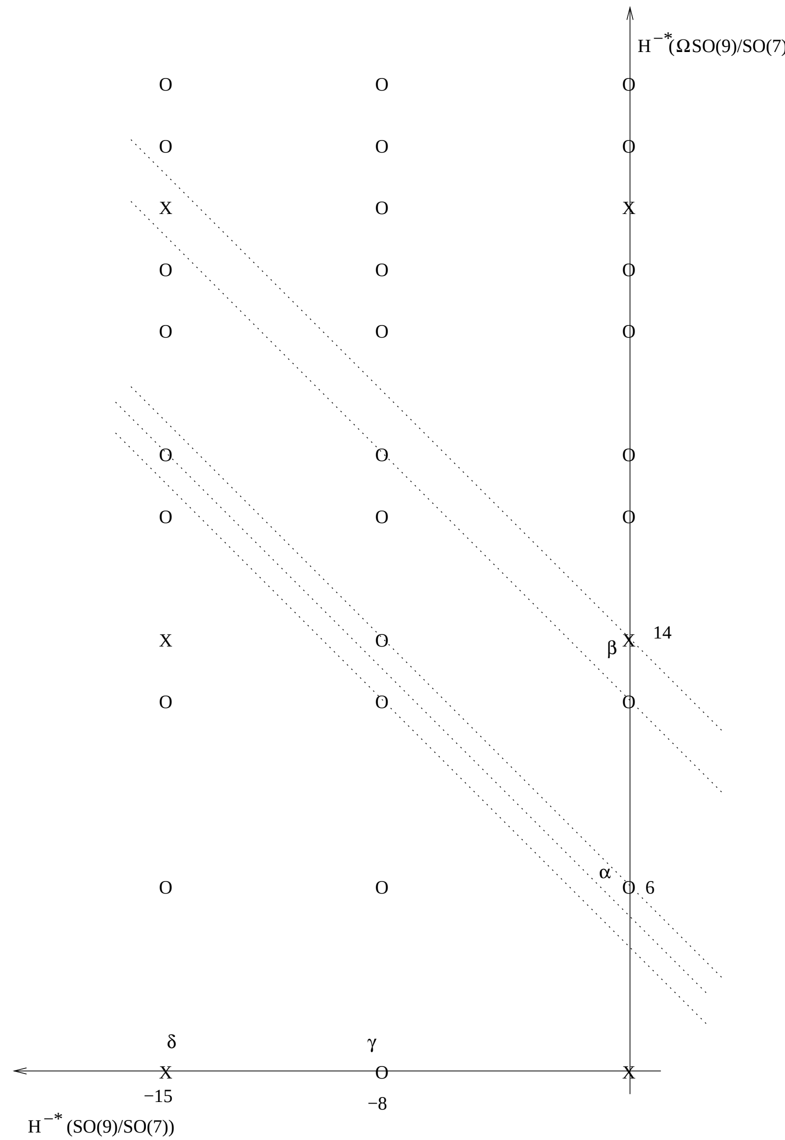
2.4.3 Remarque

La méthode précédente ne se généralise pas facilement pour des calculs sur les variétés de Stiefel

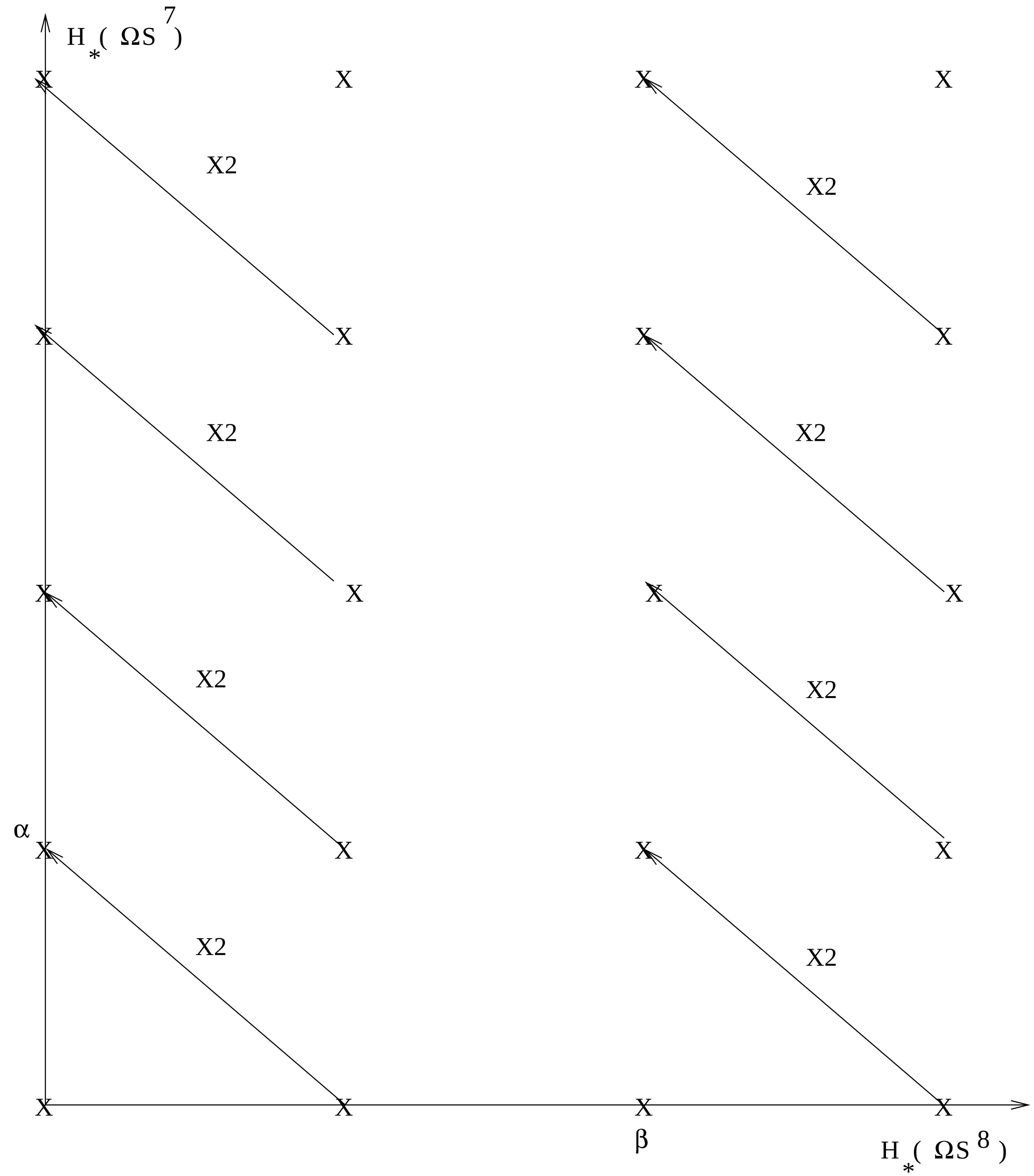
$$V_2(\mathbb{R}^n) = SO(n+1)/SO(n-1).$$

Néammoins, nous pouvons utiliser les identifications données dans [19] pour calculer le terme E^2 de la suite spectrale de Cohen-Jones-Yan : $V_2(\mathbb{R}^n)$ est homotopiquement équivalent à $\tau(S^n)$ le fibré en sphères du fibré normal de S^n lui même homotopiquement équivalent à $F_3(S^n)$ l'espace de configuration de trois points sur S^n. Nous reviendrons sur cette remarque dans le chapitre 4 pour évoquer le calcul de $\mathbb{H}_*(L(SO(4)/SO(2)))$ et $\mathbb{H}_*(L(SO(8)/SO(6)))$.

Suite spectrale de Cohen-Jones-Yan de la variété de Stiefel $SO(9)/SO(7)$

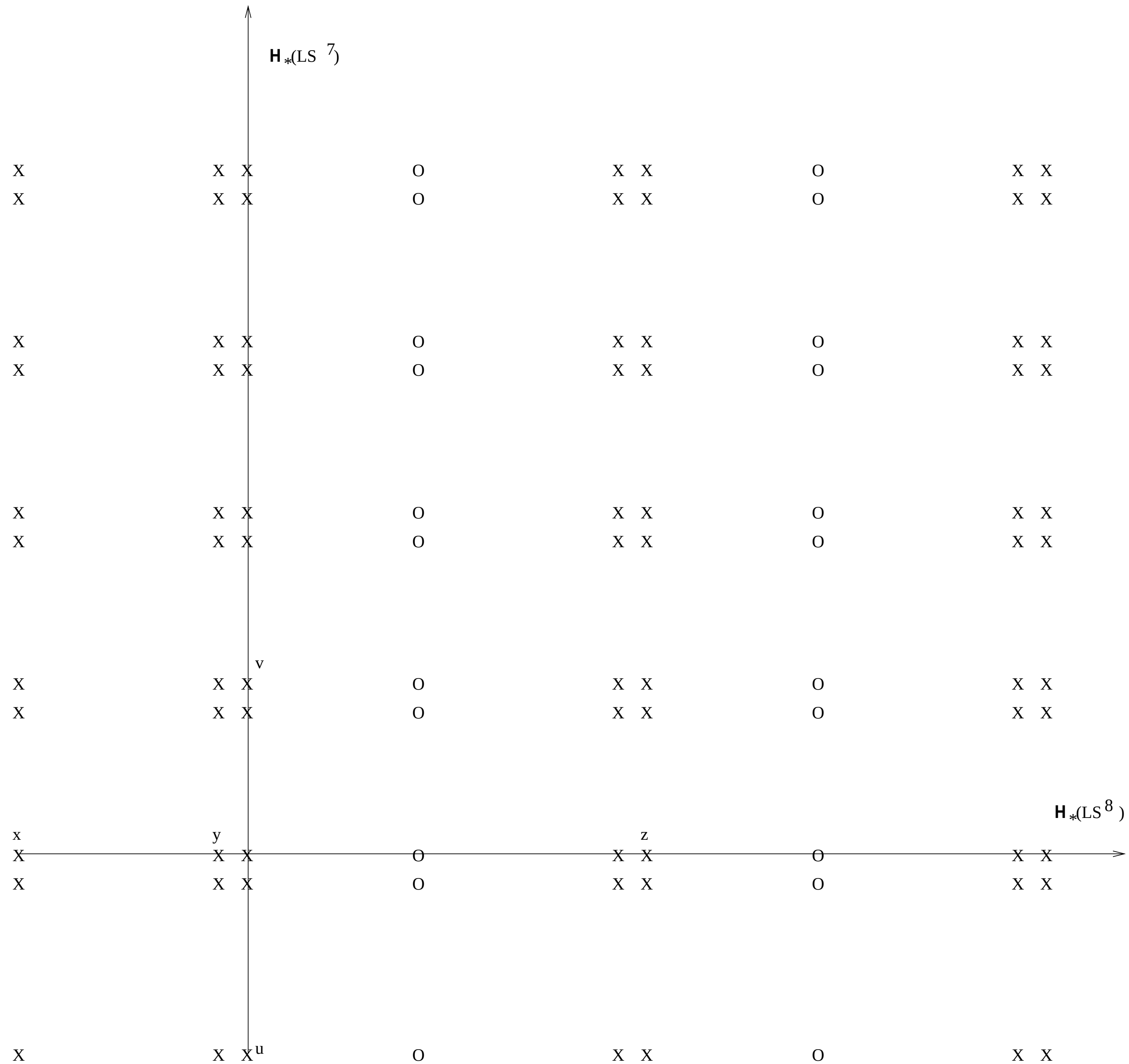

H^{-*}(ΩSO(9)/SO(7))
H^{-*} (SO(9)/SO(7))
14
6
−15
−8
α
β
γ
δ

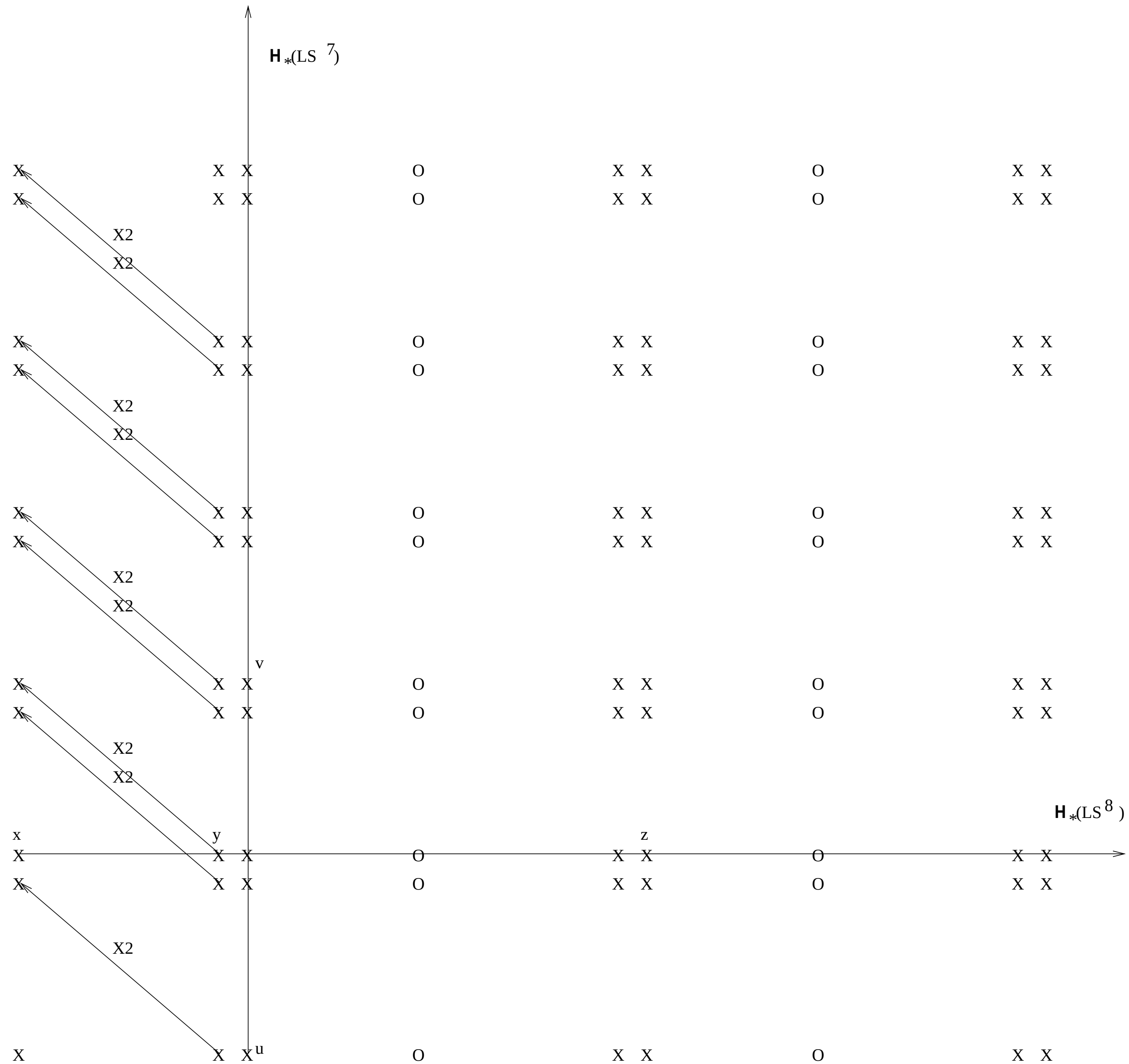
Suite spectrale associée à la fibration des lacets pointés
$\Omega S^7 \to \Omega(SO(9)/SO(7)) \to \Omega S^8$

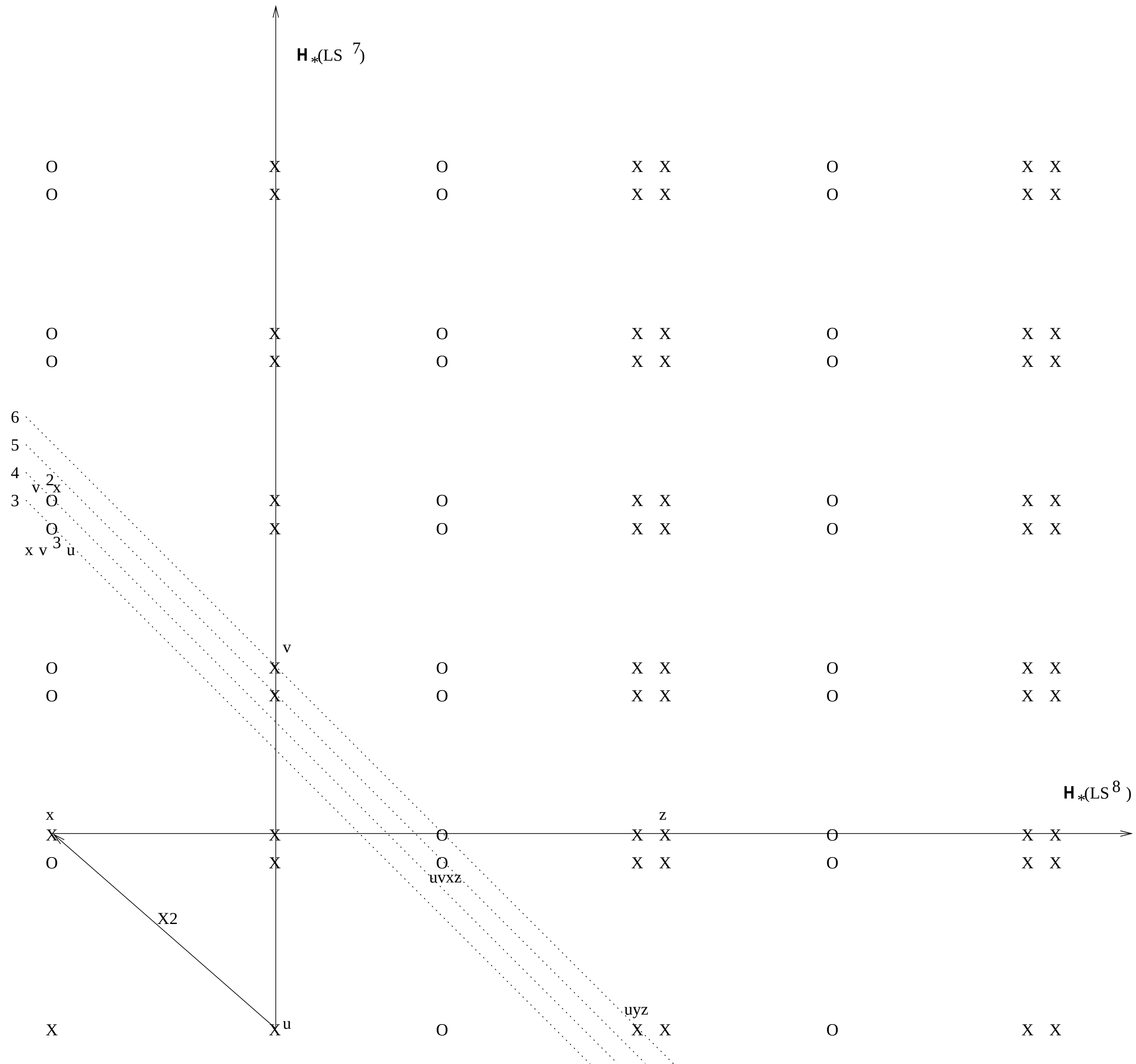


H_*(ΩS 7)
H_*(ΩS 8)
α
β
X2

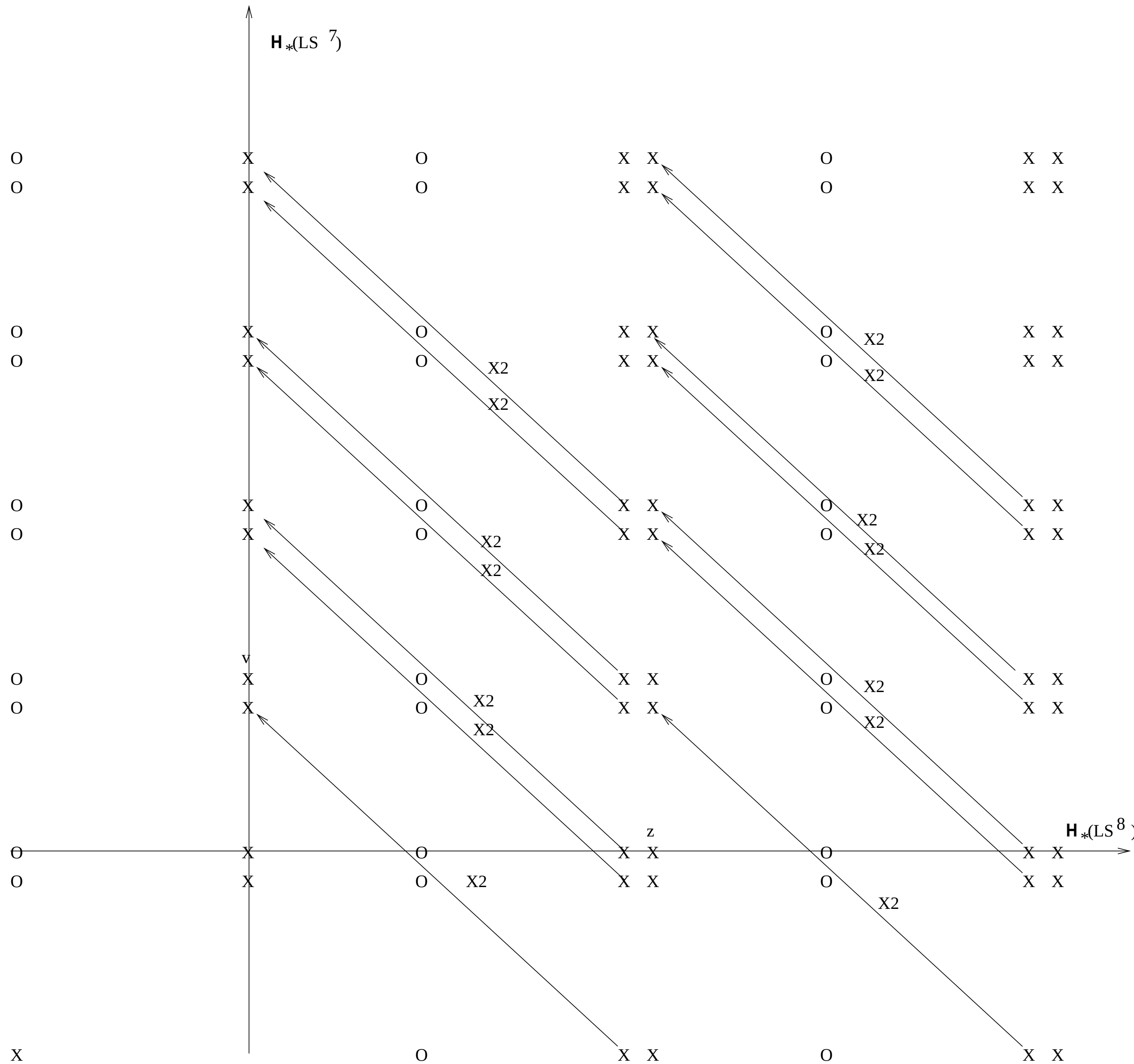
Suite spectrale associée à la fibration $LS^7 \to L(SO(9)/SO(7)) \to LS^8$

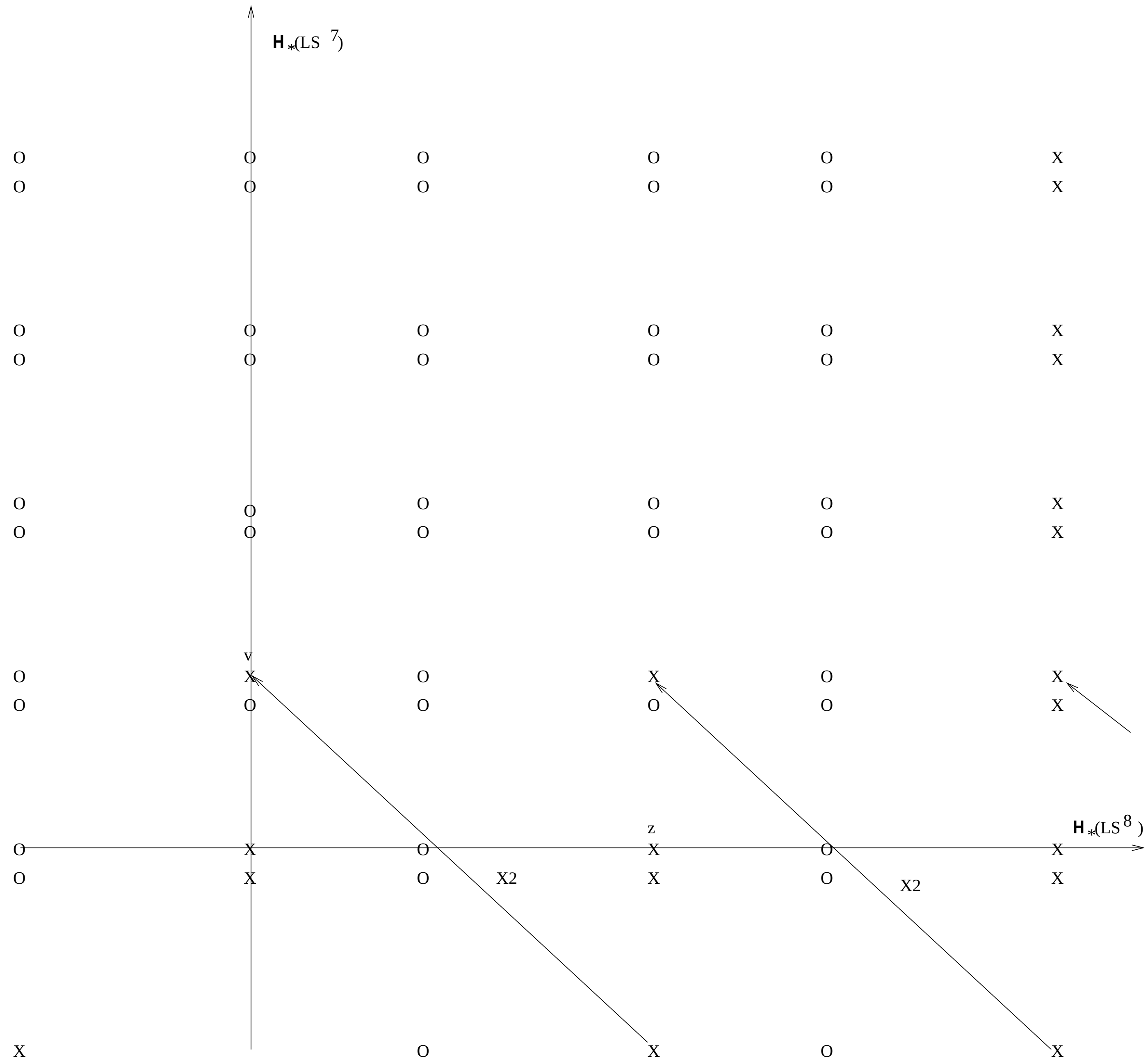

H*(LS 7)
H*(LS 8)
x
y
z
u
v


H*(LS 7)
H*(LS 8)
X2
x
y
z
u
v

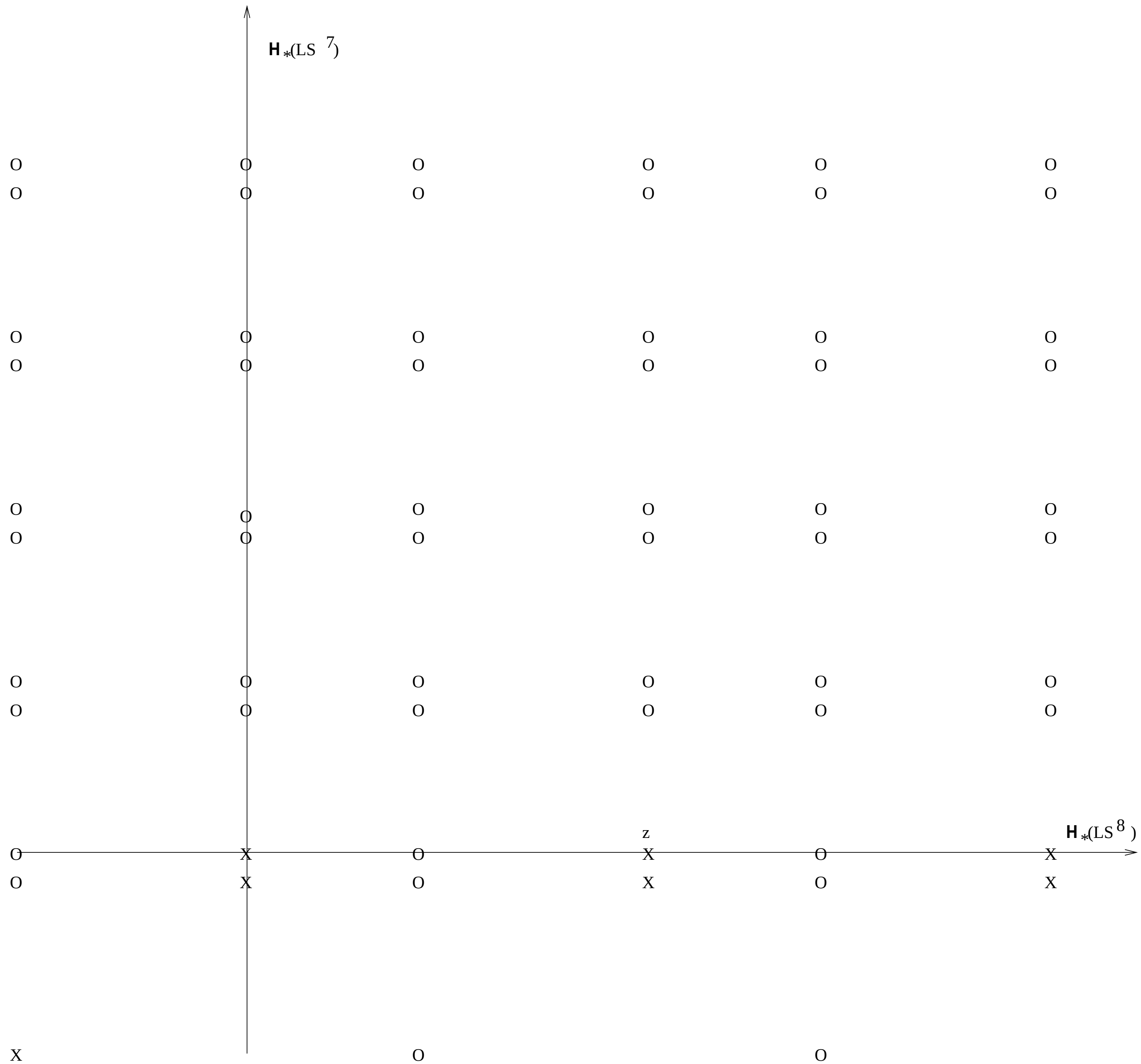


H*(LS 7)
H*(LS 8)
6
5
4
3
v 2 x
x v 3 u
v
x
z
X2
uvxz
uyz
u


H*(LS 7)
H*(LS 8)
v
z
X2



$\mathbf{H}_*(LS^7)$
$\mathbf{H}_*(LS^8)$
v
z
X2
X2



H*(LS 7)
H*(LS 8)
z

Chapitre 3

The loop-coproduct spectral sequences.

Abstract

Let M be a closed oriented d-dimensionnal manifold and let LM be the space of free loops on M. In this paper, we give a geometrical interpretation of the loop-coproduct and we study it's compatibility with the Serre spectral sequence associated to the fibration $\Omega M \to LM \stackrel{ev(0)}{\to} M$. Then, we show that the spectral sequence associated to the free loop fibration $LN \to LX \to LM$ of some Serre fibration $N \to X \to M$ is a spectral sequence of Frobenius algebra.

AMS Classification : 55P35, 54N45,55N33, 17A65,
81T30, 17B55

Key words : free loop space, loop-homology, Serre spectral sequence, loop-coproduct.

3.1 Introduction.

Let M be a arcwise connected closed oriented d-manifold and let LM be the Hilbert manifold homotopically equivalent to M^{S^1} the space of free loops of M [10]. We denote by $ev(0)$ (respectively ev(1/2)) the evaluation map at 0 (respectively 1/2).

$$ev(0) : LM \to M \qquad \gamma \mapsto \gamma(0)$$

$$ev(1/2) : LM \to M \qquad \gamma \mapsto \gamma(1/2)$$

We put

$$L_{1/2}M =: \{\gamma \in LM \quad / \quad ev(1/2)(\gamma) = ev(0)(\gamma)\}$$

and

$$\Omega_{1/2}M =: \{\gamma \in \Omega M \quad / \quad ev(1/2) = *\}$$

where $*$ denotes the base point of M. Then we have the pull-back diagrams :

$$\begin{array}{ccc} L_{1/2}M & \xhookrightarrow{\tilde{j}} & LM \\ \downarrow{\scriptstyle ev(1/2)} & & \downarrow{\scriptstyle ev(0)\times ev(1/2)} \\ M & \xhookrightarrow{\Delta} & M \times M \end{array} \qquad \begin{array}{ccc} \Omega_{1/2}M & \xhookrightarrow{\tilde{i}} & \Omega M \\ \downarrow{\scriptstyle ev(1/2)} & & \downarrow{\scriptstyle ev(1/2)} \\ * & \xhookrightarrow{i} & M \end{array}$$

where Δ is the diagonal embedding and i the canonical inclusion of the base point of M. These diagrams define the d-codimensionnal embeddings $\tilde{j}$ and $\tilde{i}$. We need again the following notation :

$$LM \times_M LM = \{(\alpha, \beta) \in LM \times LM \quad / \quad \alpha(0) = \beta(0)\}.$$

We have the following commutative pull-back diagram [16] :

$$\begin{array}{ccc} LM \times_M LM & \xhookrightarrow{\tilde{\Delta}} & LM \times LM \\ \downarrow{\scriptstyle ev_\infty} & & \downarrow{\scriptstyle ev(0)\times ev(0)} \\ M & \xhookrightarrow{\Delta} & M \times M \end{array}$$

where ev_∞ denotes the evaluation at 0 :

$$LM \times_M LM \to M \qquad (\alpha, \beta) \mapsto \alpha(0) = \beta(0)$$

.

We define

$$\gamma : LM \times_M LM \to L_{1/2}M$$

$$(\alpha, \beta) \mapsto \alpha * \beta$$

with $\alpha * \beta = \alpha(2t)$ if $t \in [0, 1/2]$ and $\beta(2t - 1)$ if $t \in [1/2, 1]$ the reparametrisation of a couple of same basepoint loops. We remark that γ is an homeomorphism and we have

$$\gamma^{-1} : L_{1/2}M \to LM \times_M LM$$

$$\eta \mapsto (t \mapsto (\eta(t/2), \eta(1/2 + t/2)))$$

We denote γ_ω (resp. γ_ω^{-1}) the restriction of γ (resp. γ^{-1}) to $\Omega M \times \Omega M$ (resp. $\Omega_{1/2} M$). We denote by $comp : LM \times_M LM \to LM$ the composition of composable loops and $comp_\omega$ its restriction to pointed loops. Remark that $comp = \tilde{j} \circ \gamma$ for free loops and $comp_\omega = \tilde{i} \circ \gamma_\omega$ for pointed loops.

Assume that $\Bbbk$ is a fixed ring. The following homology groups are supposed having their coefficients in $\Bbbk$. M.Chas and D.Sullivan have constructed a product

$$P : H_*(LM \times LM) \to H_{*-d}(LM)$$

called the loop product such that the desuspended homology of LM namely $\mathbb{H}_*(LM) =: H_{*+d}(LM)$ is a commutative graded algebra. With our notations, $P =: comp_* \circ \tilde{\Delta}_!$.

In [13], R.Cohen and V.Godin have constructed a coproduct

$$\Phi : H_*(LM) \to H_{*-d}(LM \times LM).$$

Since $\tilde{j} : L_{1/2}M \hookrightarrow LM$ is a smooth finite codimensionnal embedding of Hilbert manifolds, we can define $\tilde{j}_! : H_*(LM) \to H_{*-d}(L_{1/2}M)$ (see [16] or [30] for details). Furthermore, considering γ as a 0-codimensionnal smooth embedding, we can also consider

$$\gamma_! : H_*(L_{1/2}M) \to H_*(LM \times_M LM).$$

Remark : The coproduct on based loop space $comp_{\omega !}$ is the coproduct defined by Sullivan in [34].

The remaining of this paper consists of proving the following results :

Proposition 3.1.1 *Let $h : X \to Y$ be a diffeomorphism between smooth Hilbert closed connected manifolds without boundary. We have : $h_! = h_*^{-1}$*

This proposition is proved in section 3.4 page 85.

Corollary 3.1.2 $\gamma_! = \gamma_*^{-1}$ *and* $\gamma_{\omega !} = \gamma_{\omega *}^{-1}$

Definition 3.1.3 *We define $comp_! =: \gamma_! \circ \tilde{j}_! = \gamma_*^{-1} \circ \tilde{j}_!$ (by Proposition 3.1.1).*

Definition 3.1.4 *As the same, we define $comp_{\omega !} =: \gamma_{\omega !} \circ \tilde{i}_! = {\gamma_\omega^{-1}}_* \circ \tilde{i}_!$.*

From the definition of Φ, we obtain immediately :

Theorem 3.1.5 *According to the preceding notations, $\Phi = \tilde{\Delta}_* \circ comp_!$.*

We remark that the loop-coproduct $\Phi = \tilde{\Delta}_* \circ comp_!$ can be thought as "Poincaré-dual" of the loop-product $P = comp_* \circ \tilde{\Delta}_!$.

Now, let us consider $\{E^*_{*,*}[ev(0)]\}$ the Serre spectral sequence associated to the fibration $\Omega M \to LM \stackrel{ev(0)}{\to} M$.

Theorem 3.1.6 *The spectral sequence $\{E^*_{*,*}[ev(0)]\}$ is comultiplicative and converges to the coalgebra $(H_*(LM), \Phi)$. At the E^2-level, $E^2(\Phi) = \Delta_* \otimes comp_{\omega!}$.*

This Theorem is proved in section 3.5 page 85. Theorem 3.1.6 explains the interest to compute $comp_{\omega!}$ in order to do some computations with Φ. So we have :

Theorem 3.1.7 *The pointed loop-coproduct $comp_{\omega!}$ is zero.*

This Theorem is proved in section 3.6 page 87.

Remark : Theorem 3.5 and Theorem 3.6 proves that the loop-coproduct induced on the Serre spectral sequence associated to the fibration $\Omega M \to LM \stackrel{ev(0)}{\to} M$ vanishes at the E^∞-level. In [11], the computation of this loop-coproduct is done for $H_*(L\mathbb{C}P^n; \mathbb{Q})$, using rationnal homotopy theory. The two authors prove that in this case, the loop-coproduct is non-zero. This indicates that the extension issues in our spectral sequence are not trivial.

Now, let $N \to X \stackrel{p}{\to} M$ be a locally trivial fibration satisfying hypothesis of proposition 1.1.3 page 18 of chapitre 1 namely :

a) N, (respectively M) is a finite dimensionnal smooth closed oriented manifold of dimension n (respectively m),

b) M is a connected space.

Then, we state the following theorem proved in section 3.7 page 88. :

Theorem 3.1.8 *Under the above hypothesis, the loop-coproduct Φ_X induces on the Serre spectral sequence associated to the fibration $LN \to LX \stackrel{Lp}{\to} LM$ a structure of coalgebra with a coproduct of degree $-(m+n) = dim X$.Furthermore, if we assume that $\pi_1(M)$ acts trivially on $H_*(N)$, then the tensor product of coalgebra $(H_*(N), \Phi_N) \otimes (H_*(B), \Phi_B)$ is a sub-coalgebra of $E^2[p]$.*

Let us recall the following theorem 1.1.7 of chapitre 1 :

Theorem 1.1.7. *Under hypothesis of Proposition 1.1.3 of chapitre 1, the (m,n)-regraded Serre spectral sequence $\{\mathbb{E}^r[Lp]\}_{r\geq 0}$ of the Serre fibration $LN \to LX \stackrel{Lp}{\to} LM$ is a multiplicative spectral sequence which converges to the algebra $\mathbb{H}_*(LX)$. Moreover the tensor product of graded algebras : $\mathbb{H}_*(LM) \otimes \mathbb{H}_*(LN)$ is a subalgebra of $\mathbb{E}^2[Lp]$. In particular if $H_*(LM)$ is torsion free then $\mathbb{E}^2[p] = \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LN)$.*

As a immediate byproduct of theorem 3.1.8 and of this theorem 1.1.7, we get :

Corollary 3.1.9 *The regraded Serre spectral sequence associated to the fibration $LN \to LX \stackrel{Lp}{\to} LM$ is a spectral sequence of Frobenius algebra. Moreover the tensor product of graded Frobenius algebras : $\mathbb{H}_*(LM) \otimes \mathbb{H}_*(LN)$, is a sub-Frobenius algebra of $\mathbb{E}^2[Lp]$.*

Acknowledgement We would like to thank David Chataur and Jean-Claude Thomas for their precious comments.

The paper is organized as follows :

1) Definition of the loop coproduct

2) Geometrical interpretation of the loop coproduct. Proof of Theorem 3.1.5

3) Shriek map of a diffeomorphism. Proof of Proposition 3.1.1

4) The loop coproduct and the Cohen-Jones-Yan spectral sequence. Proof of Theorem 3.1.6

5) The pointed loop coproduct. Proof of Theorem 3.1.7

6) The loop coproduct and the string Serre spectral sequence. Proof of Theorem 3.1.8

3.2 Definition of the loop coproduct.

Following [13] and [34] the loop product (respectively the loop coproduct) is a particular case of an "operation"

$$\mu_\Sigma : H_*(LM)^{\otimes p} \to H_*(LM)^{\otimes q}$$

with $p = 2$, $q = 1$ (respectively $p = 1$, $q = 2$) that will define a positive boundary TQFT. Here Σ denotes an oriented surface of genus 0 with a fixed parametrization of the $p + q$ boundary components :

$$(S^1)^{\coprod p} \overset{in}{\to} \partial_{in}\Sigma \hookrightarrow \Sigma \hookleftarrow \partial_{out}\Sigma \overset{out}{\leftarrow} (S^1)^{\coprod q}$$

Applying the functor $Map(-, M)$ on gets the diagram :

$$(1) \qquad (LM)^{\times p} \overset{in^\sharp}{\leftarrow} Map(\Sigma, M) \overset{out^\sharp}{\to} (LM)^{\times q}$$

Diagram (1) is homotopically equivalent to the diagram

$$(2) \qquad (LM)^{\times p} \overset{\rho_{in}}{\leftarrow} Map(c, M) \overset{\rho_{out}}{\to} (LM)^{\times q}$$

where c denotes a reduced Sullivan chord diagram with marking which is associated to Σ [13]-§2.

In order to define the loop coproduct we restrict to the case $p = 1$ and $q = 2$. Then Σ is the oriented surface of genus 0 with one entry and two exits components of the boundary. The associated Sullivan chord diagram with markings is determined by the pushout diagram

$$\begin{array}{ccc} * \coprod * & \longrightarrow & * \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{in} & c \end{array}$$

In particular, $c \simeq S^1 \vee S^1$, $S^1 \overset{in}{\to} c$ is homotopic to the pinching map $\nabla : S^1 \to S^1 \vee S^1$ and $S^1 \coprod S^1 \overset{out}{\to} c$ is homotopic to the natural projection $S^1 \coprod S^1 \to S^1 \vee S^1$. Diagram (2) then reduces to a commutative diagram of fibrations :

$$\begin{array}{ccccc} LM \times LM & \xleftarrow{\delta_{out}} & LM \times_M LM & \xrightarrow{\delta_{in}} & LM \\ \downarrow{\scriptstyle ev(0)\times ev(0)} & & \downarrow{\scriptstyle ev_\infty} & & \downarrow{\scriptstyle ev(0)\times ev(1/2)} \\ M \times M & \xleftarrow[\Delta]{} & M & \xrightarrow{\Delta} & M \times M \end{array}$$

By definition (see [13]), $\Phi = \delta_{out*} \circ \delta_{in!}$.

3.3 Geometrical interpretation of the loop coproduct. Proof of Theorem 3.1.5

Geometrically, δ_{out} is the inclusion of composable loops in $LM \times LM$, namely $\tilde{\Delta}$. The other map, δ_{in}, can be thought as the inclusion of $Map(8, M)$ in LM. Then this coproduct Φ can be understood as follows : the space $L_{1/2}M$ is the space of decomposable loops which embeds in LM. This embedding is d-codimensionnal so that we can define the shriek map of the embedding. Then, we decompose the loops of the space of decomposable loops.

The top line of the preceding diagram can be decomposed as follows :

$$(*) \qquad LM \times LM \overset{\tilde{\Delta}=\delta_{out}}{\longleftarrow} LM \times_M LM \overset{\gamma}{\to} L_{1/2}M \overset{\tilde{j}}{\hookrightarrow} LM.$$

Observe that $\delta_{in} = \tilde{j} \circ \gamma = comp$ then $\Phi = \delta_{out*} \circ \delta_{in!} = \tilde{\Delta}_* \circ comp_!$. This proves Theorem 3.3.

□

3.4 Shriek map of a diffeomorphism. Proof of Proposition 3.1.1

By definition, $h_!$ is given by the following composition (see chapitre 1 section 1.2.1) :

$$H_*(Y) \overset{inc_*}{\to} H_*(Y, Y - h(Y)) \overset{exc}{\to} H_*(\mathrm{Tube}\, h/\partial \mathrm{Tube}\, h) \overset{exp_*}{\to} H_*(\nu_Y, \partial\nu_Y) \overset{\pi_*(\tau\cap -)}{\to} C_*(X)$$

where inc_* denotes the inclusion of pair, exc is the excision isomorphism, exp is the homeomorphism given by the exponential between the tubular neighbourhood of the embedding and it's normal bunddle and $\pi_*(\tau \cap -)$, where π denotes the projection of the normal bundle and $\tau \in H^*(\nu_Y, \partial\nu_Y)$ is the Thom class of the embedding, is the Thom isomorphism. In the case we consider, the normal bundle of h is $\nu_Y =: * \to Y \overset{h^{-1}}{\to} X$. Then, $\mathrm{Tube}\, h = Y = E(\nu_Y)$ and $\partial \mathrm{Tube}\, h = \emptyset = \partial E(\nu_Y)$ so that the three first maps of the definition of $h_!$ are identity. We remark that the projection of the normal bundle ν_Y is only h^{-1} and that τ lies in $H^0(\nu_Y, \partial\nu_Y) = H^0(Y)$. Then, the Thom isomorphism is only h_*^{-1}, this achieves the proof of Proposition 3.1.1.

□

3.5 The loop coproduct and the Cohen-Jones-Yan spectral sequence. Proof of Theorem 3.1.6

For the reader convenience, we recall here the definition 1.1.1 of a sub-fiberwise embedding of chapitre 1 and the main result (Theorem 1.1.2) of chapitre 1.

Definition We define a sub-fiberwise embedding (f, f^B) as a commutative diagram

$$\begin{array}{ccc} X & \overset{f}{\to} & X' \\ p\downarrow & & \downarrow p' \\ B & \overset{f^B}{\to} & B' \end{array}$$

where

$$(*)\quad \begin{cases} a) & X, X', B \text{ and } B' \text{ are connected Hilbert manifolds without boundary} \\ b) & f \text{ (respectively } f^B) \text{ is a smooth embedding of finite codimension } k_X \\ & \text{(respectively } k_B) \\ c) & p \text{ and } p' \text{ are Serre fibrations} \\ d) & \text{for some } b \in B \text{ the induced map} \\ & \qquad f^F : F := p^{-1}(b) \to {p'}^{-1}(f(b)) := F' \\ & \text{is an embedding of finite codimension } k_F \\ e) & \text{embeddings } f, f^B \text{ and } f^F \text{admit Thom classes.} \end{cases}$$

First part of the main result. *Let $f : X \to X'$ be a fiber embedding as above. For each $n \geq 0$ there exist filtrations*

$$\begin{cases} \{0\} \subset F_0 C_n(X) \subset F_1 C_n(X) \subset ... \subset F_n C_n(X) = C_n(X) \\ \{0\} \subset F_0 C_n(X') \subset F_1 C_n(X') \subset ... \subset F_n C_n(X') = C_n(X') \end{cases}$$

and a chain representative $f_! : C_*(X') \to C_{*-k_X}(X)$ of $f_! : H_*(X') \to H_{*-k_X}(X)$ satisfying : $f_!\,(F_* C_*(X')) \subset F_{*-k_B} C_{*-k_X}(X')$.

Let $\{E^r[p]\}_{r\geq 0}$ and $\{E^r[p']\}_{r\geq 0}$ be the spectral sequences induced by the above filtrations.

Second part of the main result. *The chain map $f_!$ induces a homomorphism of bidegree $(-k_B, -k_F)$ between the associated spectral sequences $\{E^r(f_!)\} : \{E^r[p']\}_{r\geq 0} \to \{E^r[p]\}_{r\geq 0}$. There exists a chain representative $f_!^B : C_*(B') \to C_{*-k_B}(B)$ (respectively $f_!^F : C_*(F') \to C_{*-k_F}(F)$) of $f_!^B : H_*(B') \to H_{*-k_B}(B)$ (respectively of $f_!^F : H_*(F') \to H_{*-k_F}(F)$) such that*

$$E^2(f_!) = H_*(f_!^B; \mathcal{H}_*(f_!^F)) : E^2_{s,t}[p'] =$$

$$H_s(B'; \mathcal{H}_t(F')) \to E^2_{s-k_B, t-k_F}[p] = H_{s-k_B}(B; \mathcal{H}_{t-k_F}(F)),$$

where $\mathcal{H}(-)$ denote the usual system of local coefficients. These spectral sequences are the Serre spectral sequences of the fibration.

We use the description of the loop coproduct Φ given in the proof of Theorem 3.1.5. Thus we obtain the following commutative diagram of fibrations, where the central column is the composite $(*)$:

$$\begin{array}{ccccc} \Omega M & \longrightarrow & LM & \xrightarrow{ev(0)} & M \\ \uparrow{\scriptstyle \tilde{i}} & & \uparrow{\scriptstyle \tilde{j}} & & \uparrow{\scriptstyle =} \\ \Omega_{1/2} M & \longrightarrow & L_{1/2} M & \xrightarrow{ev(0)} & M \\ \downarrow{\scriptstyle \gamma_\omega^{-1}} & & \downarrow{\scriptstyle \gamma^{-1}} & & \downarrow{\scriptstyle =} \\ \Omega M \times \Omega M & \longrightarrow & LM \times_M LM & \xrightarrow{ev_\infty} & M \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \tilde{\Delta}} & & \downarrow{\scriptstyle \Delta} \\ \Omega M \times \Omega M & \longrightarrow & LM \times LM & \xrightarrow{ev(0)\times ev(0)} & M \times M \end{array}$$

We remark that $(\tilde{j}, id)$ is a sub-fiberwise embedding in the sense of definition 1.1.1, more precisely :

a)LM, $L_{1/2}M$, M are connected Hilbert manifolds without boundary.

b)$\tilde{j}$ (resp. id) is a smooth embedding of finite codimension d (resp. 0).

c) $ev(0)$ and $ev(1/2)$ are locally trivial fibrations ($ev(1/2)$ is locally trivial since it is homeomorphic to ev_∞ which is locally trivial).

d) $\tilde{i}$ is an embedding of finite codimension d.

e) $\tilde{j}$, id and $\tilde{i}$ admit Thom classes. If we denote τ the Thom class of $\tilde{j}$ and $u : \Omega M \to LM$ the canonical embedding, then, the Thom class of $\tilde{i}$ is $u^*(\tau)$, [11].

Then, we apply Theorem 1.1.2 to prove that $\tilde{j}_!$ induces a morphism of spectral sequences :

$$E^*_{*,*}(\tilde{j}_!) : E^*_{*,*}[ev(0)] \to: E^*_{*-d,*}[ev(1/2)].$$

By naturality of the Serre spectral sequences, γ^{-1} and $\tilde{\Delta}$ induce a morphism of spectral sequences :

$$E^*_{*,*}(\gamma_*^{-1}) : E^*_{*-d,*}[ev(1/2)] \to E^*_{*-d,*}[ev_\infty]$$

and :

$$E^*_{*,*}(\tilde{\Delta}_*) : E^*_{*-d,*}[ev_\infty] \to E^*_{*-d,*}[ev(0) \times ev(0)].$$

Then, composing this morphisms, we define a coproduct

$$E^*(\Phi) : E^*_{*,*}[ev(0)] \to E^*_{*-d,*}[ev(0) \times ev(0)]$$

induced by Φ. On the base of the fibration $ev(0)$, Φ induces Δ_*. On the fiber, Φ induces $comp_{\omega !}$ such that at the E^2-level, $E^2(\Phi) = \Delta_* \otimes comp_{\omega !}$.

$\square$

3.6 The pointed loop coproduct. Proof of Theorem 3.1.7

We consider the following pull-back diagram :

$$\begin{array}{ccc}
\Omega_{1/2}M & \xrightarrow{=} & \Omega_{1/2}M \\
\downarrow & & \downarrow \\
\Omega_{1/2}M & \xhookrightarrow{\tilde{i}} & \Omega M \\
\downarrow & & \downarrow {\scriptstyle ev(1/2)} \\
* & \xhookrightarrow{i} & M
\end{array}$$

where i and $\tilde{i}$ are defined in the introduction. This diagram is a pull-back embedding (see definition 1.4.6 page 31) thus $\tilde{i}_!$ induces a morphism of Serre spectral sequences : $E^*(\tilde{i}_!) : E^*_{*,*}(2) \to E^*_{*-d,*}(1)$ where $E^*_{*,*}(1)$ (resp. $E^*_{*,*}(2)$) denotes the Serre spectral sequence associated to the left fibration of the above diagram (resp. the Serre spectral sequence associated to the right fibration of the above diagram). Moreover, $\tilde{i}_*$ induces a morphism of Serre spectral sequences [27] : $E^*(\tilde{i}_*) : E^*_{*,*}(1) \to E^*_{*,*}(2)$. We know that $\tilde{i}_* \circ \gamma_* = comp_*$ is onto ($comp_*$ has a unit). Since γ_* is an isomorphism, this proves that $\tilde{i}$ is onto. The spectral sequence $E^*_{*,*}(1)$ collapses, it has only one column at this level :

$$E^n_{0,*}(1) \simeq H_*(\Omega_{1/2}M)\,,\; n \geq 2.$$

Then, since $\tilde{i}_*$ is onto, $E^\infty(\tilde{i}_*)$ is onto. Moreover, $E^2(\tilde{i}_*) = i_* \otimes id$ thus $Im(E^2(\tilde{i}_*)) = E^2_{0,*}(2)$ so that $Im(E^2(\tilde{i}_*)) \subset E^\infty_{0,*}(2)$ that is why $E^\infty_{*,*}(2) = Im(E^\infty(\tilde{i}_*)) \subset E^\infty_{0,*}(2)$.

We have proved that

$$E^\infty_{*,*}(2) = E^\infty_{0,*}(2).$$

But $E^2(\tilde{i}_!) = i_! \otimes id$ and $\tilde{i}_!$ is non zero only in degree d with values in degree 0 ($i_!$ sends the fundamental class of $H_d(M)$ on a generator of $H_0(M)$ and is zero elsewhere). Then, at the E^2-level of $E^*_{*,*}(2)$, only the column of abscissa d has a non zero image by $E^2(\tilde{i}_!)$. Consequently, at the E^∞-level, only $E^\infty_{d,*}(2)$ has a non zero image by $E^\infty(\tilde{i}_!)$. We have shown that $E^\infty_{d,*}(2) = 0$ thus $Im(E^\infty(\tilde{i}_!)) = 0$.

□

3.7 The loop coproduct and the string Serre spectral sequence. Proof of Theorem 3.1.8

Notations. In this section, we use the same notations as in the introduction but we add a subscript to indicate which manifold we refer (for example, Φ_X denotes the loop-coproduct on $H_*(LX)$).

We check easily that the commutative diagram

$$\begin{array}{ccc}
L_{1/2}N & \hookrightarrow & LN \\
\downarrow & & \downarrow \\
L_{1/2}X & \hookrightarrow & LX \\
\downarrow & & \downarrow \\
L_{1/2}M & \hookrightarrow & LM
\end{array}$$

is a sub fiberwise embedding (see definition 1.1.1 page 17).

Proof of Theorem 3.1.8 : we consider the following commutative diagram :

$$\begin{array}{ccccc}
LN & \longrightarrow & LX & \longrightarrow & LM \\
\uparrow{\scriptstyle \tilde{j}_N} & & \uparrow{\scriptstyle \tilde{j}_X} & & \uparrow{\scriptstyle \tilde{j}_M} \\
L_{1/2}N & \longrightarrow & L_{1/2}X & \longrightarrow & L_{1/2}M \\
\uparrow{\scriptstyle h_N} & & \uparrow{\scriptstyle h_X} & & \uparrow{\scriptstyle h_X} \\
LN \times_N LN & \longrightarrow & LX \times_X LX & \longrightarrow & LM \times_M LM \\
\downarrow{\scriptstyle \tilde{\Delta}_N} & & \downarrow{\scriptstyle \tilde{\Delta}_X} & & \downarrow{\scriptstyle \tilde{\Delta}_M} \\
LN \times LN & \longrightarrow & LX \times_X LX & \longrightarrow & LM \times_M LM
\end{array}\quad .$$

It comes from Theorem 1.1.2 page 17 that $\tilde{j}_{X!}$ induces a morphism of spectral sequences of degree $-x$ so that $\Phi_X = \tilde{\Delta}_{X*} \circ h_{X*} \circ i_{X!}$ induces a morphism of spectral sequences $E^*[\Phi_X]$. This morphism induces on the homology of M the coproduct $\Phi_M = \tilde{\Delta}_{M*} \circ h_{M*} \circ \tilde{j}_{M!}$ and on the homology of N the coproduct $\Phi_N = \tilde{\Delta}_{N*} \circ h_{N*} \circ \tilde{j}_{N!}$. This achieves the proof of Theorem 3.7.

□

Chapitre 4

About extension issues and homotopy invariance in string topology.

The paper [14] is an attempt to prove the homotopy invariance of the loop product. That is, given a homotopy equivalence $f : M_1 \to M_2$ between smooth closed finite dimensionnal manifolds, are $\mathbb{H}_*(LM_1)$ and $\mathbb{H}_*(LM_2)$ isomorphic as algebras ?

As a byproduct of this result, we could prove the following results :

$$\mathbb{H}_*(L(SO(4)/SO(2))) \simeq \mathbb{H}_*(LS^3) \otimes \mathbb{H}_*(LS^2)$$

and

$$\mathbb{H}_*(L(SO(8)/SO(6))) \simeq \mathbb{H}_*(LS^7) \otimes \mathbb{H}_*(LS^6).$$

For instance the second isomorphism is obtained as follows : $SO(8)/SO(6) \simeq \tau(S^7)$ the sphere bundle of the tangent bundle of S^7 (see [19]) and $\tau(S^7) \simeq F_3(S^7)$, the configuration space of three points on S^7. Since S^7 is a Lie group,

$$F_3(S^7) \simeq S^7 \times F_2(S^7 - pt) \simeq S^7 \times S^6.$$

□

A good way to explain some extension issues of our spectral sequences of algebras (see [27]) is to exhibit the failure of our method to prove the homotopy invariance of the loop product.

In one hand, the homotopy invariance of the shriek map in the case of finite dimensionnal closed manifolds is immediately given by Poincaré duality. More precisely, let $g : N_1 \to N_2$ be another homotopy equivalence between smooth closed finite dimensionnal manifolds and k_1, k_2 some continuous maps such that

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ {\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle k_2} \\ N_1 & \xrightarrow{g} & N_2 \end{array}$$

commutes, then

$$\begin{array}{ccc} H_{*-(n-m)}(M_1) & \xrightarrow{H_*(f)} & H_{*-(n-m)}(M_2) \\ \uparrow{\scriptstyle k_{1!}} & & \uparrow{\scriptstyle k_{2!}} \\ H_*(N_1) & \xrightarrow{H_*(g)} & H_*(N_2) \end{array}$$

commutes.

In the other hand, the homotopy invariance of the Pontryagin product of pointed loop spaces $H_*(\Omega M_1) \to H_*(\Omega M_2)$ is given by the naturality of this product to continuous maps. The idea of using the Cohen-Jones-Yan spectral sequence to prove the homotopy invariance of the loop product fails because of the extension issues. More precisely, we only prove :

Proposition 4.0.1 *For M_1 and M_2 defined as above, the Cohen-Jones-Yan spectral sequences $\mathbb{E}^*_{*,*}(LM_1)$ and $\mathbb{E}^*_{*,*}(LM_2)$ are isomorphic as algebra from the E^2-level.*

Proof :

As we state in Theorem 1.1.2 page 17, the loop product induces a morphism of differential graduate filtered module $F_pC_{p+q}(LM_i \times LM_i) \to F_{p-m_i}C_{p+q-m_i}(LM_i)$ $(i = 1, 2)$. By naturality, f induces a morphism of differential graduate module that preserves filtration

$$C_*(f) : F_pC_*(LM_1) \to F_pC_*(LM_2)$$

and

$$C_*(f \times f) : F_pC_*(LM_1 \times LM_1) \to F_pC_*(LM_2 \times LM_2)$$

At the E^2-level, the homotopy invariance of the intersection product and of the Pontryagin product provides the following commutative diagram :

$$\begin{array}{ccc} \mathbb{E}^2(LM_1 \times LM_1) & \xrightarrow{\mathbb{E}^2(Lf \times Lf)} & \mathbb{E}^2(LM_2 \times LM_2) \\ {\scriptstyle \mathbb{E}^2(\mu_1)}\downarrow & & \downarrow{\scriptstyle \mathbb{E}^2(\mu_2)} \\ \mathbb{E}^2(LM_1) & \xrightarrow{\mathbb{E}^2(Lf)} & \mathbb{E}^2(LM_2) \end{array}$$

Where μ_i is the loop product on $\mathbb{H}_*(LM_i)$, $i = 1, 2$. Applying Theorem 3.5 of [27], we conclude that

$$\begin{array}{ccc} \mathbb{E}^\infty(LM_1 \times LM_1) & \xrightarrow{\mathbb{E}^\infty(Lf \times Lf)} & \mathbb{E}^\infty(LM_2 \times LM_2) \\ {\scriptstyle \mathbb{E}^\infty(\mu_1)}\downarrow & & \downarrow{\scriptstyle \mathbb{E}^\infty(\mu_2)} \\ \mathbb{E}^\infty(LM_1) & \xrightarrow{\mathbb{E}^\infty(Lf)} & \mathbb{E}^\infty(LM_2) \end{array}$$

commutes. This proves the result.

$\square$

Why shouldn't we conclude that $\mathbb{H}_*(LM_1)$ and $\mathbb{H}_*(LM_2)$ are isomorphic ? Recall as in [27] that for $p, q \in \mathbb{N}$,

$$E^\infty_{p,q}(H_*(LM_1)) = F_pH_{p+q}(LM_1)/F_{p-1}H_{p+q}(LM_1)$$

Assume that there exists $x_i \in E^{\infty}_{p,q}(H_*(LM_i))$ and $y_i \in E^{\infty}_{p',q'}(H_*(LM_i))$, $i = 1, 2$, such that $\mu_i(x_i, y_i) = 0 \in E^{\infty}_{p+p'-m,q+q'}(LM_i)$. This doesn't proves that $\mu_i(\tilde{x}_i, \tilde{y}_i) = 0$ in $\mathbb{H}_*(LM_i)$ with $\tilde{x}_i$ and $\tilde{y}_i$ the elements that lift x_i and y_i in $\mathbb{H}_*(LM_i)$. This only proves that $\mu_i(x_i, y_i)$ lies in $F_{p+p'-m-1}H_{p+p'+q+q'-m}(LM_i)$ and if

$$F_{p+p'-m-1}H_{p+p'+q+q'-m}(LM_i) \neq 0$$

there are ambiguities on this product. That's precisely this extension issues of algebra which prevent us to conclude.

Nonetheless, we can obtain a result by applying this method to the intersection morphism (definition 1.1.11 page 21)

$$I_i : \mathbb{H}_*(LM_i) \to H_*(\Omega M_i) \quad i = 1, 2.$$

The following commutative diagram

$$\begin{array}{ccc} LM_1 & \xrightarrow{Lf} & LM_2 \\ \uparrow & & \uparrow \\ \Omega M_1 & \xrightarrow{\Omega f} & \Omega M_2 \end{array}$$

induces four morphisms of dgfm at the chain level and at the E^2-level,

$$\begin{array}{ccc} \mathbb{E}^2(LM_1) & \longrightarrow & \mathbb{E}^2(LM_2) \\ \downarrow & & \downarrow \\ E^2(\Omega M_1) & \longrightarrow & E^2(\Omega M_2) \end{array}$$

where the spectral sequences $E^*(\Omega M_i)$ are the Serre spectral sequences associated to the fibration $\Omega M \to \Omega M \to pt$. Then as before,

$$\begin{array}{ccc} \mathbb{E}^{\infty}(LM_1) & \longrightarrow & \mathbb{E}^{\infty}(LM_2) \\ \downarrow & & \downarrow \\ E^{\infty}(\Omega M_1) & \longrightarrow & E^{\infty}(\Omega M_2) \end{array}$$

commutes. There is no extension issues because $E^{\infty}(\Omega M_i) = H_*(\Omega M_i)$. This proves the following result :

Proposition 4.0.2 *The intersection morphism is homotopy invariant.*

We can also applied this method to the coproduct. This provides the following result :

Proposition 4.0.3 $\mathbb{E}^{\infty}_{*,*}(LM_1)$ *is isomorphic to* $\mathbb{E}^{\infty}_{*,*}(LM_2)$ *as coalgebra.*

Remark : by Theorem 3.1.7 page 82, this structure of coalgebra is zero.

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