Differences equations and splitting of separatrices

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Equations aux différences et scission de séparatrices.

par

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Je dédie cette thèse à la mémoire de ma Mère.

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Introduction

Dans le cadre de notre travail de thèse, nous nous sommes intéressés aux problèmes de scission de séparatrices pour des systèmes d'équations aux différences à petit pas de la forme

$$\delta_\varepsilon y(t) = F\big(t, \varepsilon, y(t)\big), \tag{0.0.1}$$

où δ_ε est un opérateur de différence, qui se réduisent à des équations différentielles quand $\varepsilon \to 0$ formellement.

Pour deux problèmes particuliers de ce type, nous adaptons la méthode introduite par H. Volkmer et R. Schäfke [10] pour traiter les problèmes de scission de séparatrices pour l'équation logistique et l'équation du pendule. De façon générale, nous suivons les étapes suivantes:

1. Etablir l'existence d'une solution série formelle du type

   $$\widehat{y}(\varepsilon, t) = \sum_{n=0}^{\infty} p_n(g(t))\varepsilon^n \ ,$$

   où p_n sont des polynômes, $g(t)$ est une fonction bien connue avec $g(0) = 0, g(+\infty) = 1$ et où $p_n(0) = p_n(1) = 0$ pour tout $n \geq 2$.

2. Démontrer l'existence d'une famille de solutions analytiques $y(\varepsilon, t)$ pour l'équation (1) et ayant $\widetilde{y}$ pour développement asymptotique.

3. Déterminer une approximation asymptotique des polynômes p_n de la forme:

   $$p_n(u) = \alpha\, n!\, \tau_n(u) + \beta\, (n-2)\,!\tau_{n-2}(u) + O((n-4)\,!)$$

   quand $n \to \infty$, où les τ_n sont des polynômes associés à g.

4. En déduire une quasi-solution de la forme: $\widetilde{y}(\varepsilon,t) = \alpha H_0(\varepsilon,t) + \beta H_2(\varepsilon,t) + R(\varepsilon,t)$, c'est-à-dire une fonction qui satisfait (1) sauf une erreur exponentiellement petite ; la fonction H_0 est encore élémentaire et on montre que sa valeur en $t = +\infty$ est aussi exponentiellement petite, mais plus grande que l'erreur ci-dessus (ainsi que les valeurs $H_2(\varepsilon,+\infty)$, $R(\varepsilon,+\infty)$).

5. On montre que $\alpha \neq 0$.

6. Malgré $p_n(1) = 0$ pour $n \geq 2$, on établit un équivalent exponentiellement petit de la différence $y(\varepsilon,+\infty) - p_0(1) - \varepsilon p_1(1)$ et on montre aussi qu'elle est non nulle pour $\varepsilon > 0$ assez petit.

En bref, nous avons adapté les techniques de [10] pour améliorer les résultats de [3] et pour retrouver les résultats de [6].

0.0.1 Équation logistique

On considère l'équation différentielle

$$y' = 1 - y(t)^2, \tag{0.0.2}$$

dont les solutions sont $y(t) = \tanh(t+c)$. La discrétisation de cette équation par la méthode de Nyström, donne la récurrence :

$$y_{n+1} = y_{n-1} + 2\varepsilon\big(1 - y_n^2\big). \tag{0.0.3}$$

Avec la conditions initiales $y_0 = 0,\ y_1 = \varepsilon$, nous calculons la solution discrète y_n avec 1600 itérations et $\varepsilon = \frac{1}{20}$ (Voir FIG.1). Nous pouvons remarquer que la solution discrète rejoint rapidement le niveau $y = 1$. Elle reste proche de 1 pendant un temps relativement long, puis quitte ce niveau. Les points d'indices pairs suivent une courbe et ceux d'indices impairs en suivent une autre. Tous se rejoignent au niveau $y = -1$, puis la solution discrète recommence un nouveau cycle.

Si on pose $u_n = y_{2n},\ \ v_n = y_{2n+1}$, nous obtenons une récurrence du premier ordre dans le plan

$$(u_{n+1}, v_{n+1}) = \Phi(u_n, v_n) \tag{0.0.4}$$

où le diffeomorphisme $\Phi : \mathbb{R}^2 \mapsto \mathbb{R}^2$, $(u,v) \mapsto (u_1, v_1)$ est défini par

$$\begin{aligned} u_1 &= u + 2\varepsilon\big(1 - v^2\big), \\ v_1 &= v + 2\varepsilon\big(1 - u_1^2\big). \end{aligned}$$

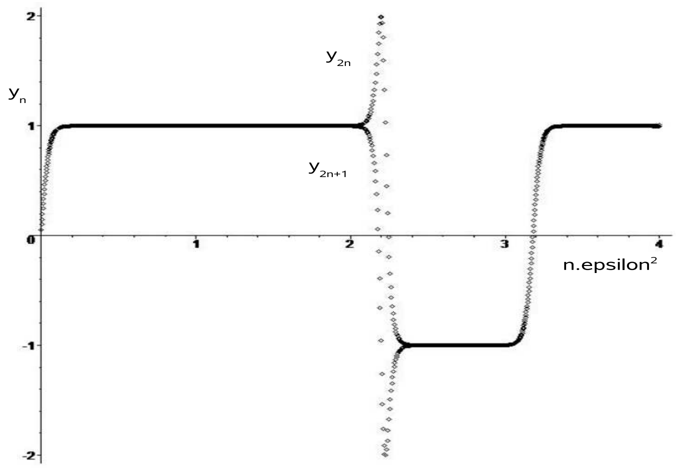

Figure 1: La figure représente y_n en fonction de $n\varepsilon^2$.

Ceci est la discrétisation du système d'équations différentielles suivant

$$\begin{cases} u' = 2\big(1 - v^2\big), \\ v' = 2\big(1 - u^2\big). \end{cases} \tag{0.0.5}$$

L'ensemble $\mathfrak{E} = \big\{(u,v) \in \mathbb{R}^2 \;;\; (u-v)(u^2+uv+v^2-3) = 0\big\}$ est un ensemble invariant pour le système (0.0.5). Ce système possède deux points selles en $A = (1,1)$ et $B = (-1,-1)$. Les variétés stables et instables de A , B sont notées respectivement W_s^{0+}, W_i^{0+}, W_s^{0-} et W_i^{0-} (voir FIG.1). Les point A et B restent des points selles du système discret et ont donc eux aussi des variétés invariantes, ε-proches des variétés $W_{s\,i}^{0\pm}$. Nous allons démontrer que la variété stable W_s^+ de A et la variété instable W_i^- de B ne coïncident plus pour l'équation aux différence, contrairement à celles de l'équation différentielle (Voir FIG.1) et que la distance verticale entre ces deux variétés est exponentiellement petite mais non nulle. Plus précisément, nous donnons une estimation asymptotique de cette distance.

A. Fruchard et R. Schäfke [3] avaient démontré que, pour $0 < \varepsilon < \varepsilon_0$, le point $(0, \varepsilon)$ se situe entre les deux variétés W_s^+ et W_i^- et que la distance de Hausdorff $\Delta(\varepsilon)$ entre les

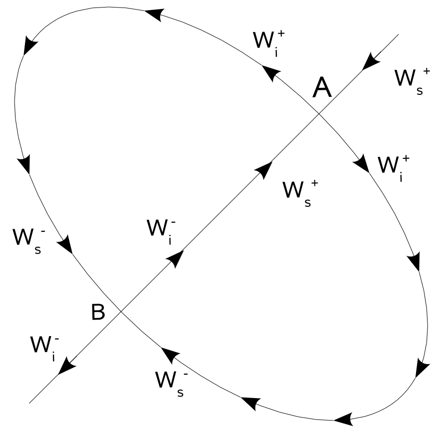

Figure 2: La variété stable et instable pour l'équation différentielle.

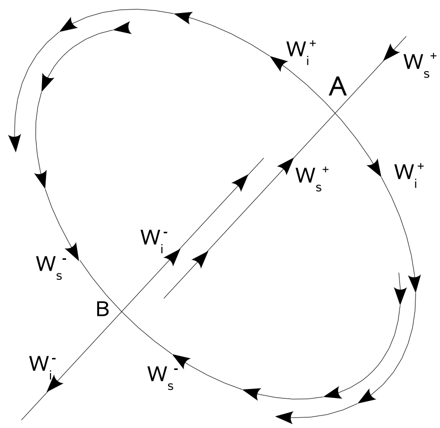

Figure 3: La variété stable et instable pour l'équation aux différences.

deux portions de variétés invariantes $W_s^+ \cap S$ et $W_i^- \cap S$ dans la bande $S = \big\{(u,v) \in \mathbb{R}^2\,;\; -1 \leq u+v \leq 1\big\}$ satisfait

$$\Delta(\varepsilon) \leq \exp\left(\frac{-\pi^2 + o(1)}{2\varepsilon}\right). \tag{0.0.6}$$

Ils ont démontré également l'existence de deux fonctions entières $y_\varepsilon^\pm : \mathbb{C} \longmapsto \mathbb{C}$, solutions de l'équation aux différences

$$y(t+\varepsilon) = y(t-\varepsilon) + 2\varepsilon\big(1 - y(t)^2\big), \tag{0.0.7}$$

et que les fonctions $t \mapsto \big(y_\varepsilon^+(t), y_\varepsilon^+(t+\varepsilon)\big)$, $t \mapsto \big(y_\varepsilon^-(t), y_\varepsilon^-(t+\varepsilon)\big)$ fournissent une paramétrisation $w_s^+(t)$ de W_s^+ pour $t \in [-1, \infty[$, respectivement $w_i^-(t)$ de W_i^- pour $t \in]-\infty, 1]$.

Nous démontrons dans cette partie de la thèse le théorème suivant:

Théorème 1. *Il existe une constante* α *avec* $1.2641497 \le \alpha \le 1.2641509$ *et* $\varepsilon_0 > 0$ *tels que pour* $0 < \varepsilon < \varepsilon_0$

$$\mathrm{Dist}_\varepsilon\big(w_s^+(t), W_i^-\big) = \frac{4\pi\alpha\cos(\frac{\pi}{\varepsilon}t+\pi)}{\varepsilon^3\big(1-\tanh(\frac{d}{\varepsilon}t)^2\big)}\, e^{\frac{-\pi^2}{2\varepsilon}} + O\Big(\frac{1}{\varepsilon^2}\, e^{\frac{-\pi^2}{2\varepsilon}}\Big), \quad \textit{quand } \varepsilon \searrow 0,$$

où $d = \frac{1}{2}$*arcsinh*(2ε) *et* $\mathrm{Dist}_\varepsilon(.,.)$ *est la distance verticale entre le point* $w_s^+(t)$ *et la variété instable* W_i^-.

Afin de démontrer ce théorème nous suivons les étapes suivantes :

Dans un premier temps nous construisons par récurrence une solution formelle en d avec des coefficients en $u = \tanh\left(\frac{d}{\varepsilon}t\right)$ de la forme

$$\widehat{A}(d,u) = u + \sum_{n=1}^{\infty} A_{2n+1}(u) d^{2n}, \tag{0.0.8}$$

où A_{2n+1} sont des polynômes vérifiant $A_{2n+1}(1) = A_{2n+1}(-1) = 0$ pour tout n. Nous définissons ensuite une norme appropriée sur un espace de polynômes. En utilisant cette norme, nous démontrons que la série formelle $\widehat{A}$ est Gevrey-1. Plus précisément nous obtenons $\|A_n\| = \mathcal{O}\big(n\,!\,\pi^{-n}\big)$. Cela nous permet par la suite de construire une quasi-solution pour l'équation aux différences (0.0.7).

Dans la section suivante, nous introduisons les opérateurs $\mathcal{C}_2, \mathcal{C}, \mathcal{S}_2, \mathcal{S}$ définis par

$$\begin{aligned}
\mathcal{C}(Z)(d,u) &= \tfrac{1}{2}\big(Z(d,T^{+\frac{1}{2}}) + Z(d,T^{-\frac{1}{2}})\big),\\
\mathcal{S}(Z)(d,u) &= \tfrac{1}{2}\big(Z(d,T^{+\frac{1}{2}}) - Z(d,T^{-\frac{1}{2}})\big),\\
\mathcal{C}_2(Z)(d,u) &= \tfrac{1}{2}\Big(Z(d,T^{+}) + Z(d,T^{-})\Big),\\
\mathcal{S}_2(Z)(d,u) &= \tfrac{1}{2}\big(Z(d,T^{+}) - Z(d,T^{-})\big).
\end{aligned} \tag{0.0.9}$$

où

$$\begin{aligned} T^{+}(d,u): &= \frac{u+\tanh(d)}{1+u\tanh(d)} \\ T^{-}(d,u): &= \frac{u-\tanh(d)}{1-u\tanh(d)} \\ T^{+\frac{1}{2}}(d,u) &= T^{+}\big(\tfrac{d}{2},u\big) \\ T^{-\frac{1}{2}}(d,u) &= T^{-}\big(\tfrac{d}{2},u\big) \end{aligned}$$

et $Z(d,u)$ est une série formelle dont les coefficients sont des polynômes en u.

Cette définition nous permet d'écrire l'équation (0.0.7) sous forme

$$\mathcal{S}_2(\widehat{A})(d,u) = \varepsilon\big(1-\widehat{A}(d,u)^2\big). \tag{0.0.10}$$

En faisant quelques changements de variables, nous pouvons réécrire l'équation:

$$\begin{aligned} e_0(d,u)\mathcal{L}(J) &= e_1(d,u)\mathcal{C}(J) + e_2(d,u)F + e_3(d,u)F^2 \\ &\quad + e_4(d,u)\mathcal{C}_2(F) + e_5(d,u) \end{aligned}$$

où

$$\begin{aligned} F(d,u) &= \tfrac{1}{\varepsilon}\Big(U(d,u)-\widehat{A}(d,u)\Big), \\ U(d,u) &= \varepsilon u + (u-u^3)d^3 + \Big(\tfrac{10}{3}u^5 - \tfrac{16}{3}u^3 + 2u\Big)d^5, \end{aligned}$$

$\mathcal{L}$ est un opérateur linéaire défini en fonction des opérateurs $\mathcal{C}$ et $\mathcal{S}$, $e_i(d,u), i=0,...,5$ sont des séries en d et u connues et convergentes, et J est une série formelle qui dépend de la série $F(d,u)$.

Le membre de gauche de cette équation est une série avec un terme principal 1 multiplié par l'opérateur inversible $\mathcal{L}$ appliqué à la série J. Le membre de droite est une expression des séries formelles F et J multipliées par les séries convergentes $e_i(d,u), i = 1,...,5$. Les séries formelles F, J ont été choisies précisément de telle façon que les séries convergentes commencent par un ordre de d assez grand, ce qui rend le second terme plus petit que le premier. Cette propriété sera utile pour construire une récurrence sur n, puis pour inverser les opérateurs $\mathcal{L},\mathcal{C}$. Cette dernière étape nous permettra de trouver l'asymptotique des coefficients de la séries formelle F ; cette asymptotique s'écrit:

$$\Big\| F_n - \alpha n\,!\big(\tfrac{i}{\pi}\big)^{n+1}\tau'_{n+1} - \beta(n-2)\,!\big(\tfrac{i}{\pi}\big)^{n-1}\tau'_{n-1} - \gamma(n-4)\,!\big(\tfrac{i}{\pi}\big)^{n-3}\tau'_{n-3}\Big\|_n =$$

$$\mathcal{O}\big((n-5)!\,\pi^{-n}\log(n+2)\big).$$

où les polynômes $\tau_n(u)$ sont définis par la relation de récurrence:

$$\tau_0(u)=1, \tau_1(u)=u, \tau_{n+1}(u)=\tfrac{1}{n}D\tau_n(u),$$

et où D est l'opérateur

$$D:=(1-u^2)\frac{\partial}{\partial u}.$$

Maintenant que l'on a obtenu une estimation asymptotique des coefficients de la solution formelle, nous allons utiliser la transformée de Laplace pour construire une quasi-solution analytique $\mathcal{A}(d,u)$ dans le disque $|z|<\frac{\pi^2}{2}$ pour l'équation aux différences (0.0.10). L'avantage de cette méthode est d'avoir une solution approchée ayant pour développement asymptotique la séries formelle $\widehat{A}(d,u)$,

$$\mathcal{A}(d,u)\sim\widehat{A}(d,u), \quad d\searrow 0 \text{ pour tout } \; u_0<u<1, -1<u_0<0.$$

Ceci nous permet par la suite de démontrer que la fonction $\mathcal{A}(d,u)$ satisfait l'équation (10) avec une erreur exponentiellement petite. Précisément, nous avons

$$\Big|\mathcal{A}(d,T^+)-\mathcal{A}(d,T^-)-2\varepsilon\big(1-\mathcal{A}(d,u)^2\big)\Big|\leq\tfrac{K}{d}\,e^{-\frac{\pi^2}{2d}}, \quad \text{pour } 0<d<d_0,\ u_0<u<1.$$

La fonction $\mathcal{A}(d,u)$ est une quasi-solution proche de la solution exacte $y(t)$ de l'équation aux différence (0.0.7), et la fonction $\mathcal{A}^-(d,u)=-\mathcal{A}(d,-u)$ est aussi une quasi-solution proche de la solution exacte $y^-(t)$. On note $\widetilde{W_s},\widetilde{W_i}$ les variétés proches de W_s^+, respectivement W_i^- paramétrées par $t\mapsto\big(\xi_+(t),\xi_+(t+\varepsilon)\big)$ respectivement $t\mapsto\big(\xi_-(t),\xi_-(t+\varepsilon)\big)$, où $\xi_+(t)=\mathcal{A}\big(d,u(t)\big)$ et $\xi_-(t)=\mathcal{A}^-\big(d,u(t)\big)$. Nous démontrons que la distance verticale entre un point de la variété stable et la variété $\widetilde{W_s}$ est exponentiellement petite, ainsi que celle entre $\widetilde{W_i}$ et W_i^-. En effet, si (x_0,y_0) est un point sur la variété stable W_s^+, nous obtenons

$$\operatorname{dist}_v\big((x_0,y_0),\widetilde{W_s}\big)=O\Big(\tfrac{1}{\varepsilon^2}\,\exp\big(-\tfrac{\pi^2}{2\varepsilon}\big)\Big), \tag{0.0.11}$$

où dist_v désigne la distance verticale.

En utilisant le théorème des résidus, nous calculons la quantité

$$\begin{aligned} d_\varepsilon\Big(\widetilde{\xi}_+(t),\widetilde{\xi}_-(t)\Big) &= \mathcal{A}(d,u)-\mathcal{A}^-(d,u), \text{ pour } -\tfrac{1}{2}\leq u\leq\tfrac{1}{2} \\ &= \frac{2\pi\alpha\cos(\frac{\pi t}{\varepsilon})}{\varepsilon^3\big(1-\tanh(\frac{d}{\varepsilon}t)^2\big)}\,e^{-\frac{\pi^2}{2\varepsilon}}+O\Big(\tfrac{1}{\varepsilon}\cos(\tfrac{\pi t}{\varepsilon})e^{-\frac{\pi^2}{2d}}\Big). \end{aligned}$$

Finalement, ceci avec (0.0.11) nous permet de d'obtenir

$$\mathrm{Dist}_\varepsilon\big(w_s^+(t), W_i^-\big) = \frac{4\pi\alpha\cos(\frac{\pi}{\varepsilon}t+\pi)}{\varepsilon^3\big(1-\tanh(\frac{d}{\varepsilon}t)\big)}\, e^{\frac{-\pi^2}{2\varepsilon}} + O\Big(\frac{1}{\varepsilon^2}\, e^{\frac{-\pi^2}{2\varepsilon}}\Big), \quad \varepsilon \searrow 0,$$

0.0.2 Équation du pendule (Application standard)

Dans cette partie de la thèse nous nous sommes intéressés à l'équation du pendule. Nous avons utilisé la méthode de la section précédente pour retrouver le résultat de Lazutkin [7][8], Lazutkin et. al. [9], Gelfreich [4] (Voir aussi les références de Gelfreich [4]). La discrétisation de l'équation du pendule $y'' = \sin(y)$ conduit à l'équation aux différences suivante

$$y(t+\varepsilon) + y(t-\varepsilon) - 2y(t) = \varepsilon^2 \sin\big(y(t)\big), \tag{0.0.12}$$

Une discrétisation de l'équation du pendule du premier ordre dans le plan donne le système d'équations aux différences suivant

$$\begin{cases} y(t+\varepsilon) = y(t) + \varepsilon z(t+\varepsilon), \\ z(t+\varepsilon) = z(t) + \varepsilon \sin\big(y(t)\big). \end{cases} \tag{0.0.13}$$

qui est une discrétisation du système différentiel

$$\begin{cases} y' = z, \\ z' = \sin(y). \end{cases} \tag{0.0.14}$$

Ce système possède deux points selles en $A = (0,0)$ et $B = (2\pi, 0)$. Les variétés stable et instable existent pour ce système ainsi que pour le système discrétisé (0.0.13). On note W_s^- la variété instable en A et W_i^+ la variété instable en B pour le système discret. Le système (0.0.14) a une orbite hétérocline connectant les points stationnaires B et A, paramétrée par $\big(q_0(t), q_0'(t)\big)$, où $q_0(t) = 4\arctan\big(e^{-t}\big)$. Il s'agit d'une paramétrisation d'une portion de la courbe $p = -2\sin(q/2)$, égale à la variété stable de (0.0.14) en A ainsi qu'à la variété instable en B. Notre objectif est d'étudier la distance entre les deux variétés invariantes correspondantes du système discret (0.0.13) et de donner un équivalent pour cette distance, ce qui montre en particulier qu'elle est non nulle. Il existe deux familles de fonctions entières $Y^\pm = (y^\pm(t), z^\pm(t))$ solutions du système aux différences (0.0.7) et les fonctions $t \mapsto \big(y_\varepsilon^+(t), z_\varepsilon^+(t)\big)$, $t \mapsto \big(y_\varepsilon^-(t), z_\varepsilon^-(t)\big)$ fournissent une paramétrisation $w_i^+(t)$ de W_i^+, respectivement $w_s^-(t)$ de W_s^-.

Lazutkin et. al. [9], Gelfreich [4], (voir aussi Lazutkin [7][8]) ont donné une estimation asymptotique de l'angle entre les variétés. En partant d'une solution hétérocline de l'équation différentielle, ils ont étudié le comportement des solutions analytiques de l'équation aux différences au voisinage de leurs singularités $t = \pm\frac{\pi}{2}i$.

Nous suivons les même étapes que pour l'équation logistique : nous construisons une solution pour l'équation (0.0.7) sous forme d'une série formelle en ε avec des coefficients polynômes en $u = \tanh(dt/\varepsilon)$ où $d = 2\text{arcsinh}(\varepsilon/2)$, nous cherchons ensuite une approximation asymptotique des coefficients pour construire des quasi-solutions qui vérifient l'équation (0.0.7) avec une erreur exponentiellement petite puis nous démontrons que la distance entre la solution exacte et la quasi-solution est exponentiellement petite et enfin nous obtenons le résultat suivant.

Théorème 2. *Il existe* $\alpha \neq 0$ *et* $\varepsilon_0 > 0$ *tels que pour* $0 < \varepsilon < \varepsilon_0$ *et* $-\frac{4}{3} < t < \frac{4}{3}$

$$\text{dist}_v\big(w^+_{s,\varepsilon}(t), W^-_{u,\varepsilon}\big) = \frac{4\pi\alpha}{\varepsilon^2}\cosh(t)\sin\left(\tfrac{2\pi t}{\varepsilon}\right)e^{-\frac{\pi^2}{\varepsilon}} + O\Big(\frac{1}{\varepsilon}\,e^{-\frac{\pi^2}{\varepsilon}}\Big), \quad \textit{quand } \varepsilon \searrow 0,$$

où $w^+_{s,\varepsilon}(t) = \big(q_0(t), \widetilde{p}^+_{s,\varepsilon}(t)\big)$ *est un point sur la variété stable et où* $\text{dist}_v\big(P, W^-_{i,\varepsilon}\big)$ *désigne la distance verticale entre le point* P *et la variété instable* $W^-_{i,\varepsilon}$.

Ce résultat correspond au résultat de Lazutkin et. al. [9]. En effet l'angle entre les séparatrices à un point d'intersection est asymptotiquement équivalent à

$$\frac{1}{q_0'(t)}\,\frac{d}{dt}\,\text{dist}_v\big(w^+_{s,\varepsilon}(t), W^-_{u,\varepsilon}\big).$$

La preuve de notre théorème principal utilise les étapes suivantes. Nous commençons la construction d'une solution formelle pour l'équation aux différences (0.0.12) sous forme d'une séries de puissance de $d = 2\text{arcsinh}(\varepsilon/2)$, dont les coefficients sont des polynômes en $u = \tanh\left(\frac{d}{\varepsilon}\,t\right)$:

$$A_d(u) = \sum_{n=1}^{+\infty} A_{2n-1}(u)d^{2n}, \tag{0.0.15}$$

où $A_{2n-1}(u)$ sont des polynômes de degrés $\leq 2n-1$. Nous cherchons ensuite une approximation asymptotique des coefficients de la solutions formelle en utilisant une norme appropriée sur l'espace des polynômes. Pour cela nous réécrivons l'équation sous la forme

$$A_d(T^+)\sqrt{\frac{1-(T^+)^2}{1-u^2}} + A_d(T^-)\sqrt{\frac{1-(T^-)^2}{1-u^2}} - 2A_d(u) = f\big(\varepsilon, u, A_d(u)\big). \tag{0.0.16}$$

En faisant un changement de variable précis, nous obtenons

$$V(d,u)\mathcal{S}(Q_1\mathcal{S}(Q_2\,A))(d,u) = g\big(d,u,A(d,u)\big), \tag{0.0.17}$$

où V, Q_1 et Q_2 sont des polynômes connus de d et u et g est une fonction d, u et A, impliquant les opérateurs $\mathcal{S}, \mathcal{C}$ et multipliée par une puissance suffisamment élevée de d. Nous utilisons cette propriété pour inverser l'opérateur $\mathcal{S}$. Si on définit les séries formelles $F(d,u), G(d,u)$ par

$$F(d,u) = A(d,u) - U(d,u)$$
$$G(d,u) = Q(d,u)F(d,u)$$

où

$$\begin{aligned} U(d,u) : &= -\frac{1}{4}ud^2 + \left(\frac{91}{864}u^3 - \frac{47}{576}u\right)d^4 + \left(-\frac{319}{2880}u^5 + \frac{185}{1152}u^3 - \frac{3703}{69120}u\right)d^6 \\ Q(d,u) : &= 1 + \frac{1}{4}(1-u^2)d^2 + \left(\frac{91}{432}u^4 - \frac{13}{48}u^2 + \frac{13}{216}\right)d^4 \\ &+ \left(-\frac{319}{960}u^6 + \frac{1079}{1728}u^4 - \frac{937}{2880}u^2 + \frac{287}{8640}\right)d^6, \end{aligned}$$

on obtient l'approximation asymptotique suivante

$$\begin{aligned} G(d,u) &= \left(\frac{\alpha}{d^2} + \beta + \frac{\alpha}{3}\right)\frac{u\,H_0(d,u)}{\tau_2(u)} - \left(\beta d + \frac{\alpha}{d}\right)H_2(d,u) \\ &+ \delta d H_1(d,u) + S(d,u) \end{aligned} \tag{0.0.18}$$

où $\|S\|_n = \mathcal{O}\Big((n-3)!(2\pi)^n\Big)$ et

$$\begin{aligned} G(d,u) &= \frac{F(d,u)}{Q(d,u)} \\ H_0(d,u) &= \sum_{\substack{n=10 \\ n\ pair}}^{\infty} (n-1)\,!\left(\frac{i}{2\pi}\right)^n \tau_n(u)\,d^n \\ H_1(d,u) &= \sum_{\substack{n=9 \\ n\ impair}}^{\infty} (n-1)\,!\left(\frac{i}{2\pi}\right)^{n+1} \tau_n(u)\,d^n \\ H_2(d,u) &= d\frac{\partial H_1}{\partial d}(d,u) \end{aligned}$$

Nous utilisons cette approximation asymptotique pour construire une quasi-solution $\mathcal{A}(d,u)$ analytique dans $|z| < \pi^2$, dont le développement asymptotique est la solution formelle de l'équation aux différences de départ, cette quasi-solution satisfait l'équation (16) avec une erreur exponentiellement petite. En particulier

$$\begin{aligned} \Bigg| \sqrt{\frac{1-(T^+)^2}{1-u^2}}\mathcal{A}(d,T^+) &+ \sqrt{\frac{1-(T^-)^2}{1-u^2}}\mathcal{A}(d,T^-) - 2\mathcal{A}(d,u) - f\big(\varepsilon, \mathcal{A}(d,u)\big)\Bigg| \\ &\leq Kde^{-\frac{\pi^2}{d}}, \end{aligned} \tag{0.0.19}$$

où K est une constante positive. Si $y_\varepsilon(t)$ est une solution exacte de l'équation aux différences (0.0.12), alors $(y_\varepsilon(t), z_\varepsilon(t))$, où $z_\varepsilon(t) = \frac{1}{\varepsilon}\big(y_\varepsilon(t) - y_\varepsilon(t-\varepsilon)\big)$, est une une solution pour le système (0.0.13). Nous avons mentionné que la variété stable W_s^- de ce système en $A = (0,0)$ est paramétrée par $t \to (y_\varepsilon^-(t), z_\varepsilon^-(t))$ et la variété instable W_i^+ de (0.0.13) en $B = (2\pi, 0)$ est paramétrée par $t \to (y_\varepsilon^+(t), z_\varepsilon^+(t))$, où $(y_\varepsilon^-(t), z_\varepsilon^-(t))$ est une solution exacte de (0.0.13) et $y_\varepsilon^+(t) = 2\pi - y_\varepsilon^-(-t)$, $z_\varepsilon^+(t) = z_\varepsilon^-(-t+\varepsilon)$. Notons par $\widetilde{W}_s$, $\widetilde{W}_i$ les variétés de proches W_s^- respectivement W_i^+ paramétrées par $t \mapsto \big(\xi_-(t), \varphi_-(t)\big)$ respectivement $t \mapsto \big(\xi_+(t), \varphi_+(t)\big)$, où $\xi_-(t) = \sqrt{1-u(t)^2}\mathcal{A}\big(d, u(t)\big) + q_{0d}(t)$ et $\xi_+(t) = 2\pi - \xi_-(-t)$, $\varphi_-(t) = \dfrac{1}{\varepsilon}\big(\xi_-(t) - \xi_-(t-\varepsilon)\big)$, $\varphi_+(t) = \dfrac{1}{\varepsilon}\big(\xi_+(t) - \xi_+(t-\varepsilon)\big)$. Nous calculons la distance entre les points de de la variété stable et la quasi-variété $\widetilde{W}_s$ ainsi que celle entre la variété instable et $\widetilde{W}_i$, ensuite nous calculons la distance entre les quasi-variétés $\widetilde{W}_s$ et $\widetilde{W}_i$, nous obtenons l'asymptotique de la distance verticale entre les deux variétés W_s^-, W_i^+. Plus précisément, pour $0 < \varepsilon < \varepsilon_0$ et $-\frac{4}{3} < t < \frac{4}{3}$

$$\operatorname{dist}_v\big(w_{s,\varepsilon}^+(t), W_{u,\varepsilon}^-\big) = \frac{4\pi\alpha}{\varepsilon^2}\cosh(t)\sin\left(\frac{2\pi t}{\varepsilon}\right)e^{-\frac{\pi^2}{\varepsilon}} + O\left(\frac{1}{\varepsilon}e^{-\frac{\pi^2}{\varepsilon}}\right), \quad \text{as } \varepsilon \searrow 0,$$

Chapter 1

On the distance between separatrices for the discretized logistic differential equation

1.1 Introduction

We consider the logistic equation

$$y' = 1 - y(t)^2, \tag{1.1.1}$$

whose solutions are $y(t) = \tanh(t+c)$. The discretization of this equation by Nystrôm's method, which consists in replacing the derivative by the symmetrical difference, gives the recurrence:

$$y_{n+1} = y_{n-1} + 2\varepsilon\big(1 - y_n^2\big). \tag{1.1.2}$$

With the initial conditions $y_0 = 0,\ y_1 = \varepsilon$, we calculate the discrete solution y_n with 1600 iterates and $\varepsilon = \frac{1}{20}$ (see FIG.1). We observe that the discrete solution rejoins quickly the level $y = 1$, it stays close to this point for a relatively long time before it leaves its neighborhood, then for the little while the even numbered points follow one curve and the odd numbered follow another curve, the two curves meet at the level $y = -1$, the discrete solution stays close to this point for a relatively long time, then it remakes the same cycle.

Now letting $u_n = y_{2n},\ \ v_n = y_{2n+1}$, we obtain a recurrence of first order in the plane

$$(u_{n+1}, v_{n+1}) = \Phi(u_n, v_n) \tag{1.1.3}$$

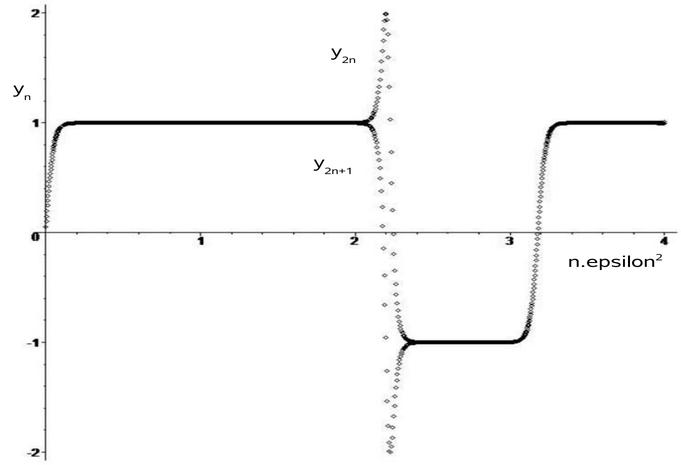

Figure 1.1: This figure represents y_n as a function of $n\varepsilon^2$.

where the diffeomorphism $\Phi : \mathrm{IR}^2 \mapsto \mathrm{IR}^2$ is defined by

$$\begin{aligned} u_1 &= u + 2\varepsilon\big(1 - v^2\big), \\ v_1 &= v + 2\varepsilon\big(1 - u_1^2\big). \end{aligned}$$

This is a discretization of the following system of differential equations

$$\begin{cases} u' = 2\big(1 - v^2\big), \\ v' = 2\big(1 - u^2\big). \end{cases} \tag{1.1.4}$$

We notice easily that the set $\mathfrak{E} = \big\{(u, v)\backslash(u - v)(u^2 + uv + v^2 - 3) = 0\big\}$ is an invariant set for this system. The system (1.1.4) has two saddle points in $A = (1, 1)$ and $B = (-1, -1)$ and corresponding stable and unstable manifolds lie on its invariant set $\mathfrak{E}$.

The stable manifold at A and unstable manifolds at B are part of set $\big\{(u, v)\backslash u = v\big\}$, where as the unstable manifold at A and stable manifolds at B are part of set $\big\{(u, v)\backslash u^2 + uv + v^2 = 3\big\}$. The stable manifold of A coincides with the unstable manifold of B (See FIG.2).

For the discretized equation (1.1.3), these manifolds still exist [3], let W_s^+ , W_i^- denote the stable manifold at A and the unstable manifold at B respectively, W_i^+ and W_s^- unstable manifold at A and stable manifold at B respectively, W_s^+ and W_i^- do not coincide an more (See FIG.3) as we want to show.

In the paper [3], after introducing the notion of length of the first level $l_1(\varepsilon) = 2\varepsilon n_1(\varepsilon)$, where $n_1(\varepsilon) = \inf\big\{n \in \mathrm{IN}\backslash y_{2n}(\varepsilon) + y_{2n+1}(\varepsilon) < 0\big\}$, Fruchard-Schäfke had shown that there exist a postive constant K such that

$$\begin{aligned} \Delta(\varepsilon) &\leq \exp\left(-\frac{\pi^2 + o(1)}{2\varepsilon}\right), \quad \text{as } \varepsilon \searrow 0, \\ l_1(\varepsilon) &\geq \frac{\pi^2}{4\varepsilon} + o(1), \end{aligned}$$

where $\Delta(\varepsilon)$ denote the distance between the sets $W_s^+ \cap S$ and $W_i^- \cap S$ in the sense of Hausdorff and $S = \big\{(u,v)\backslash -1 \leq u+v \leq 1\big\}$.

With the initial condition $y_0 = 0,\ y_0 = \varepsilon$, the length of the first level satisfies $l_1(\varepsilon) = -\frac{1}{2}\big(1+o(1)\big)\log(\Delta(\varepsilon))$ as $\varepsilon \searrow 0$[3]. They also showed that there are two families of entire functions $y_\varepsilon^\pm : \mathbb{C} \longmapsto \mathbb{C}$, solutions of the difference equation

$$y(t+\varepsilon) = y(t-\varepsilon) + 2\varepsilon\big(1 - y(t)^2\big), \tag{1.1.5}$$

and the functions $t \mapsto \big(y_\varepsilon^+(t), y_\varepsilon^+(t+\varepsilon)\big)$, $t \mapsto \big(y_\varepsilon^-(t), y_\varepsilon^-(t+\varepsilon)\big)$ provide parametrization $w_s^+(t)$ of W_s^+ for $t \in [-1,\infty[$ respectively $w_i^-(t)$ of W_i^- for $t \in]-\infty, 1]$.

In this work we will prove

Theorem 1.1.1. *There exist a constant* α *with* $1.2641497 \leq \alpha \leq 1.2641509$ *and* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$

$$Dist_\varepsilon\big(w_s^+(t), W_i^-\big) = \frac{4\pi\alpha\cos(\frac{\pi}{\varepsilon}t + \pi)}{\varepsilon^3\big(1-\tanh(t)^2\big)} e^{-\frac{\pi^2}{2\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon^2} e^{-\frac{\pi^2}{2\varepsilon}}\right), \quad \textit{as } \varepsilon \searrow 0,$$

where $Dist_\varepsilon$ *is defined as the vertical distance between the stable and unstable manifolds.*

In order to show the theorem 1.1.1, we start with the construction of a formal power series solution in d whose coefficients are polynomials in $u = \tanh(d\,t/\varepsilon)$, afterwards we will give asymptotic approximations of these coefficients using appropriate norms on the spaces of polynomials. The next step is to construct a quasi-solution i.e. a function that satisfies the equation (1.1.5) except for an exponentially small error, then we show that this quasi-solution and the exact solution of equation (1.1.5) are exponentially close.

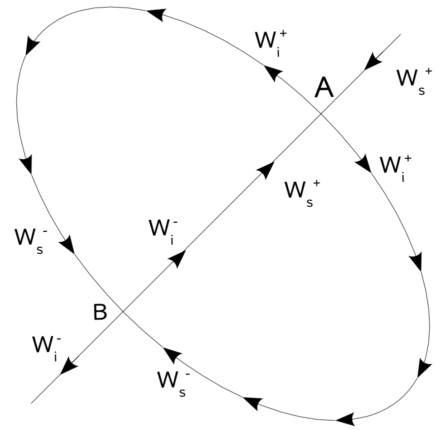

Figure 1.2: The stable and unstable manifolds for the logistic differential equation.

Finally we give an asymptotic estimate of the distance between the stable manifold of A and the unstable manifold of B and we show that this distance is exponentially small but not zero, thus completing the proof of the theorem 1.1.1.

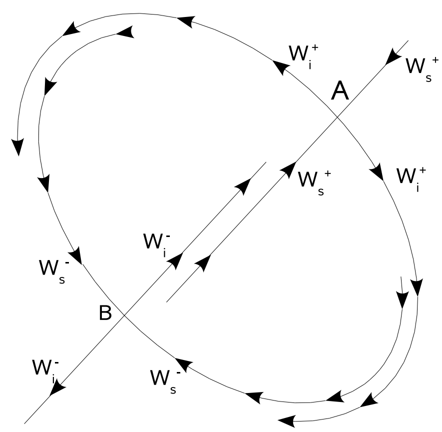

Figure 1.3: The stable and unstable manifolds for the difference equation.

1.2 Formal solutions

We consider the difference equation

$$y(t+\varepsilon) = y(t-\varepsilon) + 2\varepsilon\big(1 - y(t)^2\big), \tag{1.2.1}$$

where $\varepsilon > 0$ is the discretization step and $y(t) \to 1$ as $t \to +\infty$. Our first aim is to transform (1.2.1) in such a way that the new equation admits a formal solutions whose coefficients are polynomials. We define u and $A(d,u)$ such that

$$u := \tanh(\frac{d}{\varepsilon}t), \tag{1.2.2}$$

$$y_\varepsilon(t) = A(d,u), \tag{1.2.3}$$

where $d := \sum_{n=1}^{+\infty} d_n \varepsilon^n$ is a series to be determined. When these variables are substituted in $(1.2.1)$, the following equation satisfied by $A(d,u)$ is obtained:

$$A\big(d, T^+(d,u)\big) - A\big(d, T^-(d,u)\big) = f\big(\varepsilon, A(d,u)\big), \tag{1.2.4}$$

where

$$f(\varepsilon, x) := 2\varepsilon(1 - x^2), \tag{1.2.5}$$

$$T^+ := T^+(d,u) = \frac{u + \tanh(d)}{1 + u\tanh(d)}, \tag{1.2.6}$$

$$T^- := T^-(d,u) = \frac{u - \tanh(d)}{1 - u\tanh(d)}. \tag{1.2.7}$$

For small ε one can construct a formal expansion in powers of ε^2 of the form $\sum_{n=0}^{\infty} A_{2n+1}(u) d^{2n}$, where A_{2n+1} are polynomials which satisfy $A_1(u) = u$ and $A_{2n+1}(\pm 1) = 0$, for $n \geq 1$.

The existence of a such formal solution is only possible if d and ε are coupled in a very special way. Indeed, suppose that there exists a such formal power series solution. We differentiate (1.2.4) with respect to u and obtain at $u = 1$

$$\frac{\partial A}{\partial u}(d,1) \cdot \lim_{u \to 1} \frac{T^+(d,u) - 1}{u - 1} - \frac{\partial A}{\partial u}(d,1) \cdot \lim_{u \to 1} \frac{T^-(d,u) - 1}{u - 1} = \frac{\partial f}{\partial x}(\varepsilon, 1) \cdot \frac{\partial A}{\partial u}(d,1).$$

Because $A_{2n+1}(1) = 0$ for $n \geq 1$ implies $A(d,1) = 1$. Thus with $\frac{\partial f}{\partial x}(\varepsilon, 1) = -4\varepsilon$ and $\frac{\partial A}{\partial u}(d,1) = 1 + O(d) \neq 0$, the above equation implies

$$\varepsilon = \frac{\tanh(d)}{1 - \tanh(d)^2} = \frac{1}{2}\sinh(2d). \tag{1.2.8}$$

Theorem 1.2.1. *Let ε and d be coupled by $\varepsilon = \frac{1}{2}\sinh(2d)$. Then (1.2.4) has a formal power series solution that can be written in the form*

$$A(d,u) = u + \sum_{n=1}^{\infty} A_{2n+1}(u) d^{2n}, \tag{1.2.9}$$

where $A_{2n+1}(u)$ *are odd polynomial and* $A_{2n+1}(1) = A_{2n+1}(-1) = 0$ *for all* n*.*

Proof. We will use the Induction Principle for even n to show that there exist unique odd polynomials $A_1, A_3, A_5 ... A_{n+1}$ such that

$$Z(d,u) = \sum_{\substack{k=0 \\ k \ even}}^{n} A_{k+1} d^k, \tag{1.2.10}$$

satisfy

$$Z\big(d, T^+(d,u)\big) - Z\big(d, T^-(d,u)\big) = f\big(\varepsilon, Z(d,u)\big) \ mod \ d^{n+2}. \tag{1.2.11}$$

For $n = 0$, we put $A_1(u) = u$ and $Z(d,u) = u$ and obtain

$$Z\big(d, T^+(d,u)\big) - Z\big(d, T^-(d,u)\big) = T^+(d,u) - T^-(d,u) = (2 - 2u^2)d + \mathcal{O}(d^3).$$

and

$$f\big(\varepsilon, Z(d,u)\big) = 2\varepsilon(1-u^2) = (2-2u^2)d + \mathcal{O}(d^3).$$

This gives

$$Z\big(d, T^+(d,u)\big) - Z\big(d, T^-(d,u)\big) = f\big(\varepsilon, Z(d,u)\big) \ mod \ d^2.$$

Now suppose that for some even n already $A_1, A_3, A_5 ... A_{n+1}$ have been found with the above properties. We have to construct A_{n+3}. First, we show that $Z(d,u)$ satisfies (1.2.11) even modulo d^{n+3}. To this purpose, let

$$Z\big(d, T^+\big) - Z\big(d, T^-\big) = f\big(\varepsilon, Z(d,u)\big) + R_{n+2}(u) d^{n+2} + \mathcal{O}(d^{n+3}) \tag{1.2.12}$$

we replace d by $-d$. Using that Z and ε are even in d and n, we obtain

$$Z\big(d, T^-\big) - Z\big(d, T^+\big) = -2\varepsilon\big(1 - Z(d,u)^2\big) + R_{n+2}(u) d^{n+2} + \mathcal{O}(d^{n+3}),$$

this gives

$$Z\big(d, T^+\big) - Z\big(d, T^-\big) = 2\varepsilon(d)\big(1 - Z(d,u)^2\big) - R_{n+2}(u)(d)^{n+2} + \mathcal{O}(d^{n+3}).$$

With (1.2.12) this implies $R_{n+2}(u) = 0$ and consequently

$$Z\big(d, T^+\big) - Z\big(d, T^-\big) = f\big(\varepsilon, Z(d,u)\big) + R_{n+3}(u)d^{n+3} + \mathcal{O}(d^{n+4}). \tag{1.2.13}$$

We want to construct $A_{n+3}(u)$ such that

$$\widetilde{Z}\big(d, T^+\big) - \widetilde{Z}\big(d, T^-\big) = f\big(\varepsilon, \widetilde{Z}(d,u)\big) + \mathcal{O}(d^{n+4}), \tag{1.2.14}$$

if we put

$$\widetilde{Z}(d,u) = Z(d,u) + A_{n+3}(u)d^{n+2}. \tag{1.2.15}$$

To this purpose we use again Taylor expansion

$$\begin{aligned} f\big(\varepsilon, \widetilde{Z}(d,u)\big) &= f\big(\varepsilon, Z(d,u)\big) - 4u\, A_{n+3}(u)d^{n+3} + \mathcal{O}(d^{n+4}), \\ \widetilde{Z}\big(d, T^+(d,u)\big) &= Z(d,u) + A_{n+3}(u)d^{n+2} + (1-u^2)\frac{\partial A_{n+3}}{\partial u}(u)d^{n+3} + \mathcal{O}(d^{n+4}), \\ \widetilde{Z}\big(d, T^-(d,u)\big) &= Z(d,u) + A_{n+3}(u)d^{n+2} - (1-u^2)\frac{\partial A_{n+3}}{\partial u}(u)d^{n+3} + \mathcal{O}(d^{n+4}). \end{aligned}$$

With (1.2.13), this gives

$$\widetilde{Z}\big(d, T^+\big) - \widetilde{Z}\big(d, T^-\big) = f\big(\varepsilon, \widetilde{Z}(d,u)\big) + \widetilde{R}_{n+3}(u)d^{n+3} + \mathcal{O}(d^{n+4}), \tag{1.2.16}$$

where

$$\widetilde{R}_{n+3}(u) = 2(1-u^2)\frac{\partial A_{n+3}}{\partial u}(u) + 4u\, A_{n+3}(u) + R_{n+3}(u).$$

We see that (1.2.14) is satisfied if only if

$$2(1-u^2)\frac{\partial A_{n+3}}{\partial u}(u) + 4u\, A_{n+3}(u) + R_{n+3}(u) = 0. \tag{1.2.17}$$

This equation has a unique odd solution which is given by

$$A_{n+3}(u) = -(1-u^2)\int_0^u \frac{R_{n+3}(t)}{2(1-t^2)^2}dt.$$

We can prove that this solution is polynomial. For that, it is necessary that $R_{n+3}(u)$ and $R'_{n+3}(u)$ vanish into 1 and -1. Indeed, if we take (1.2.13) with $u = 1$, we obtain

$$R_{n+3}(1)d^{n+3} = Z\big(d, T^+(d,1)\big) - Z\big(d, T^-(d,1)\big) - f\big(\varepsilon, Z(d,1)\big) + \mathcal{O}(d^{n+4}). \tag{1.2.18}$$

Since, $T^+(d,1) = T^-(d,1) = Z(d,1) = 1$ and $f\big(\varepsilon, 1\big) = 0$, we obtain

$$R_{n+3}(1)d^{n+3} = \mathcal{O}(d^{n+4}), \tag{1.2.19}$$

therefore $R_{n+3}(1) = 0$. In order to show that $R'_{n+3}(1) = 0$, we derive formally (1.2.13) and replace u by 1. Using $T^+(d,1) = T^-(d,1) = Z(d,1) = 1$ and $\frac{\partial f}{\partial u}\big(\varepsilon, 1\big) = -4\varepsilon$, we obtain

$$R'_{n+3}(1)d^{n+3} = \frac{\partial Z}{\partial u}(d,1)\left(\frac{\partial T^+}{\partial u}(d,1) - \frac{\partial T^-}{\partial u}(d,1) + 4\varepsilon\right) + \mathcal{O}(d^{n+4}). \qquad (1.2.20)$$

By choice of d, cf (1.2.8). Thus we have $\frac{\partial T^+}{\partial u}(d,1) - \frac{\partial T^-}{\partial u}(d,1) = -2\sinh(2d)$. with (2,8) we obtain

$$R'_{n+3}(1)d^{n+3} = \mathcal{O}(d^{n+4}), \qquad (1.2.21)$$

therefore $R'_{n+3}(1) = 0$. As formula (1.2.13) shows that $R_{n+3}(u)$ is odd, then $R_{n+3}(-1) = R'_{n+3}(-1) = 0$.

The first polynomials $A_{n+1}(u)$ with even n are given by

n	0	2	4	6
$A_{n+1}(u)$	u	$u - u^3$	$(1-u^2)\Big(\frac{4}{3}u - \frac{10}{3}u^3\Big)$	$(1-u^2)\Big(\frac{62}{3}u^5 - \frac{190}{9}u^3 + \frac{182}{45}u\Big)$

We introduce the operators $\mathcal{C}_2, \mathcal{C}, \mathcal{S}_2, \mathcal{S}$ defined by

$$\begin{aligned}
\mathcal{C}(Z)(d,u) &= \tfrac{1}{2}\big(Z(d, T^{+\frac{1}{2}}) + Z(d, T^{-\frac{1}{2}})\big),\\
\mathcal{S}(Z)(d,u) &= \tfrac{1}{2}\big(Z(d, T^{+\frac{1}{2}}) - Z(d, T^{-\frac{1}{2}})\big),\\
\mathcal{C}_2(Z)(d,u) &= \tfrac{1}{2}\big(Z(d, T^{+}) + Z(d, T^{-})\big),\\
\mathcal{S}_2(Z)(d,u) &= \tfrac{1}{2}\big(Z(d, T^{+}) - Z(d, T^{-})\big).
\end{aligned} \qquad (1.2.22)$$

where $T^{+\frac{1}{2}} = T^+(\frac{d}{2}, u)$, $T^{-\frac{1}{2}} = T^-(\frac{d}{2}, u)$ and $Z(d,u)$ is a formal power series of d whose coefficients are polynomials. We rewrite equation (1.2.4) as

$$\mathcal{S}_2(A)(d,u) = \varepsilon\big(1 - A(d,u)^2\big). \qquad (1.2.23)$$

1.3 Norms for polynomials and basis

In the sequel we denote:

- $\mathcal{P}$ the set of all polynomial whose coefficents are complex,
- $\mathcal{P}_n$ the spaces of all polynomials of degree less than or equal to n,

- $\mathcal{Q} := \{Q(d,u) = \sum_{n=0}^{\infty} Q_n(u)d^n,$ where $Q_n(u) \in \mathcal{P}_n,$ for all $n \in \mathbb{N}\}$.

Proposition 1.3.1. *[10] If we define the sequence of the polynomial functions $\tau_n(u)$ by $\tau_0(u) = 1, \tau_1(u) = u, \tau_{n+1}(u) = \frac{1}{n}D\tau_n(u)$, where the operator D is defined by $D := (1-u^2)\frac{\partial}{\partial u}$, we have*

1. $T^+(d,u) = \sum_{n=0}^{\infty} \tau_{n+1}(u)d^n$,
2. $\tau_n(u)$ *has exactly degree* n,
3. $\tau_n(\tanh(z)) = \frac{1}{(n-1)!}\big(\frac{d}{dz}\big)^{n-1}\big(\tanh(z)\big)$.

Definition 1.3.2. *Let $p \in \mathcal{P}_n$. As $\tau_0(u), \tau_1(u), ..., \tau_n(u)$ form a basis of $\mathcal{P}_n$, we can write p as*

$$p = \sum_{k=0}^{n} a_k \tau_k(u).$$

Then we define the norm

$$\|p\|_n = \sum_{i=0}^{n} a_i \left(\frac{\pi}{2}\right)^{n-i}. \tag{1.3.1}$$

Theorem 1.3.3. *[10] Let n,m be positive integers and $p \in \mathcal{P}_n,\ q \in \mathcal{P}_m$. The norms $\| \, \|_n$ of the above definition have the following property:*

1. $\|Dp\|_{n+1} \leq n\|p\|_n$.
2. *If p odd we have $\|p\|_n \leq \|Dp\|_{n+1}$.*
3. *There exists a constant M_1 such that $\|pq\|_{n+m} \leq M_1\|p\|_n|q\|_m$.*
4. *There is a constant M_2 such that $|p(u)| \leq M_2\left(\frac{2}{\pi}\right)^n \|p\|_n \ (-1 \leq u \leq 1)$.*
5. *There is a constant M_3 such that for all $n > 1$ with $p(-1) = p(-1) = 0$,*

$$\left\|\frac{p}{\tau_2}\right\|_{n-2} \leq M_3\|p\|_n.$$

1.4 Operators

In this section we will use definitions and results adapted from [10] by replacing $\frac{\pi}{2}$ by π.

Definition 1.4.1. *Let f be formal power series of z whose coefficients are complex. We define a linear operator $f(dD)$ on $\mathcal{Q}$ by*

$$f(dD)Q = \sum_{n=0}^{\infty} \Big(\sum_{i=0}^{n} f_i Q_{n-i}(u) \Big) d^n, \tag{1.4.1}$$

Where $f(z) = \sum_{i=0}^{\infty} f_i z^i$ and $Q = \sum_{i=0}^{\infty} Q_n(u) d^n \in \mathcal{Q}$.

By the above definition and (1) of the proposition 1.3.1, we can show that

$$Q\big(d, T^+(\theta d, u)\big) = \big(\exp(\theta dD)Q\big)(d, u), \quad \text{for } Q \in \mathcal{Q} \text{ and all } \theta \in \mathbb{C}. \tag{1.4.2}$$

With (1.2.22) this implies

$$\begin{aligned} \mathcal{C}(Q)(d,u) &= \cosh(\tfrac{d}{2}D)Q, \ \mathcal{S}(Q)(d,u) = \sinh(\tfrac{d}{2}D)Q, \\ \mathcal{C}_2(Q)(d,u) &= \cosh(dD)Q, \ \mathcal{S}_2(Q)(d,u) = \sinh(dD)Q, \end{aligned}$$

for polynomial series Q in $\mathcal{Q}$. We denote $\big\|Q\big\|_n = \big\|Q_n\big\|_n$.

Theorem 1.4.2. . *Let $f(z)$ be formal power series having a radius of convergence greater than π and let k be a positve integer. There is a constant K such that: If Q is a polynomial series having the following property*

$$\|Q\|_n \leq \begin{cases} 0 & \text{for } n < k \\ M(n-k)!\pi^{-n} & \text{for } n \geq k \end{cases}$$

where M independent of n and $Q \in \mathcal{Q}$ then the polynomial series $f(dD)Q$ satisfies

$$\|f(dD)Q\|_n \leq \begin{cases} 0 & \text{for } n < k \\ MK(n-k)!\pi^{-n} & \text{for } n \geq k \end{cases}$$

Theorem 1.4.3. *There exists a positive constant K such that, if Q is a polynomial series such that Q_n odd, $\|Q\|_n = 0$ for $n < k$ for some positive integer k and*

$$\|dDQ\|_n \leq M(n-k)\,!\pi^{-n} \quad \text{for } n \geq k$$

where M independent of n and $Q \in \mathcal{Q}$, the polynomial series $\mathcal{C}^{-1}(Q)$ satisfies

$$\left\|\mathcal{C}^{-1}(Q)\right\|_n \leq MK\pi^{-n} \begin{cases} n\,! & \text{for } k = 1 \\ (n-1)\,!\log(n) & \text{for } k = 2 \\ (n-1)\,! & \text{for } k \geq 3 \end{cases}$$

Theorem 1.4.4. *We consider a polynomial series*

$$Q_\alpha(d,u) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \alpha_n (n-1)\,!\Big(\frac{i}{\pi}\Big)^{n-1} \tau_n(u) d^n$$

where $\alpha_n = \mathcal{O}(n^{-k})$ as $n \to \infty$ with some integer $k \geq 2$. Let $\alpha := \frac{1}{\pi}\sum_{n=1}^{\infty} \alpha_n$. then the coefficients $\{\mathcal{C}^{-1}(Q_\alpha)\}_n$ of $\mathcal{C}^{-1}(Q_\alpha)$ satisfy

$$\left\| \left\{\mathcal{C}^{-1}(Q_\alpha)\right\}_n - \alpha(n-1)\,!\Big(\frac{i}{\pi}\Big)^{n-1}\tau_n \right\|_n = \mathcal{O}\Big((n-k)\,!\pi^{-n}\Big),$$

as $n \to \infty$ for odd n.

Theorem 1.4.5. *Let k, l, p, q be positve integer with $p \geq k$ and $q \geq l$. Define m as the minimum of $k+q$ and $l+p$. Then there is a constant K with the following property:*

If P and Q are polynomial series such that $\|P\|_n = 0$ for $n < p$, $\|Q\|_n = 0$ for $n < q$ and

$$\begin{aligned} \|P\|_n &\leq M_1(n-k)\,!\,\pi^{-n}, \quad \text{for } n \geq k, \\ \|Q\|_n &\leq M_2(n-l)\,!\,\pi^{-n}, \quad \text{for } n \geq l. \end{aligned}$$

Then

$$\|PQ\|_n \leq KM_1M_2(n-m)\,!\,\pi^{-n}, \quad \text{for } n \geq p+q.$$

Theorem 1.4.6. *Let $Q_1(d,u)$ be a convergent polynomial series which is even with respect to both variables and has constant term 1.*
Let $Q_2(d,u) = d^2(1-u^2)Q_1(d,u)$ and $P(d,u) = \mathcal{S}(Q_2)/\mathcal{C}(Q_2)$. Consider the linear operator defined by

$$\mathcal{L}(Q) = \mathcal{S}(Q) - P(d,u)\cdot\mathcal{C}(Q), \quad Q \in \mathcal{Q}. \tag{1.4.3}$$

Then, there exist a constant K with the following property. If Q is an odd polynomial series with odd coefficients $Q_n(u)$ satisfying $Q_n(1) = 0$ for all n, $\|\mathcal{L}(Q)\| = 0$ for $n < 6$ and

$$\|\mathcal{L}(Q)\|_n \le M(n-6)\,!\pi^{-n} \text{ for } n \ge 6,$$

then also

$$\|dDQ\|_n \le KM(n-6)\,!\pi^{-n} \text{ for } n \ge 6.$$

1.5 Asymptotic approximation of the coeffi cients of the formal solution

The objective in this section is to construct an asymptotic aproximation of the coefficents of the formal solution (1.2.9). It will turn out to be convenient to consider the new series $B(d,u) = \varepsilon A(d,u)$,

$$B(d,u) = \varepsilon A(d,u) = \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} A_{n+1}(u)\varepsilon d^n + \varepsilon u,$$

this gives

$$B(d,u) = \sum_{\substack{n=3 \\ n \text{ odd}}}^{\infty} B_n(u) d^n + \varepsilon u,$$

where $B_n(u)$ are odd polynomials. Furthermore we have $B_n(u) \in \mathcal{P}_n$ and $B_n(1) = B_n(-1) = 0$ for all n. The new equation for $B(d,u)$ is

$$\mathcal{S}_2(B)(d,u) = \varepsilon^2 - B(d,u)^2. \tag{1.5.1}$$

We saw in section 2 that the series A was a formal solution of the starting equation. For the moment, nothing is known about the norms of its coefficients, but we will show that this series is Gevrey-1, more precisely $\|A\|_n = O\big(n\,!\,\pi^{-n}\big)$. This enables us thereafter to construct a quasi-solution. To this purpose, we will prepare the equation (1.5.1) so that we can construct a recurrence.

We start with the decompostion of series $B(d,u)$ in the form:

$$B = U + F, \text{ where } U = \varepsilon u + (u - u^3)d^3 + \Big(\frac{10}{3}u^5 - \frac{16}{3}u^3 + 2u\Big)d^5. \quad (1.5.2)$$

Next, we define

$$\begin{aligned} G: &= Q\,F, \\ J: &= \frac{\mathcal{C}(G)}{Q_1}. \end{aligned} \quad (1.5.3)$$

where

$$\begin{aligned} Q(d,u): &= (1-u^2)d^2 + (u^4 - u^2)d^4 - \Big(\frac{13}{6}u^6 - \frac{7}{2}u^4 + \frac{4}{3}u^2\Big)d^6 \\ &+ \Big(\frac{47}{6}u^8 - \frac{31}{6}u^6 + \frac{58}{15}u^4 - \frac{104}{45}u^2 + 1\Big)d^8, \\ Q_1(d,u) &= (1-u^2)d^2 + \frac{3}{2}(u^2 - u^4)d^4. \end{aligned} \quad (1.5.4)$$

The choice of Q and Q_1 and by using the properties of the operators $\mathcal{S}$, $\mathcal{C}$, we will be able to rewrite the equation (1.5.1) in the form:

$$\begin{aligned} e_0(d,u)\mathcal{L}(J) &= e_1(d,u)\mathcal{C}(J) + e_2(d,u)F + e_3(d,u)F^2 \\ &+ e_4(d,u)\mathcal{C}_2(F) + e_5(d,u) \end{aligned}$$

where $\mathcal{L}$ is the operator defined in (1.4.3) and $e_i(d,u), i = 0, ..., 5$ are convenient known convergent series in d and u which will be thereafter given.

The left hand side of this equation is series with leading term 1 multiplied by the invertible operator $\mathcal{L}$ applied to the series J. The right hand side is an expression of F and J multiplied by known convergent series $e_i(d,u), i = 1, ..., 5$.

U, Q, Q_1, V, W were chosen so that the series $e_i(d,u), i = 0..5$ are of a rather large order in d, this makes the second term smaller than the right hand side. This property will be useful to construct a recurrence on n and to reverse then the operators $\mathcal{L}, \mathcal{C}$, which makes possible to estimate the coefficients of series $\mathcal{C}(G)$ and finally the coefficients of the formal solution of the equation (1.5.1).

More precisely, we insert (1.5.2) into the equation (1.5.1) and find

$$\mathcal{S}_2(F) = -F^2 - 2U\,F + \varepsilon^2 - U^2 - \mathcal{S}_2(U). \tag{1.5.5}$$

We define $\mathcal{X}$ by

$$\mathcal{X} := V(d,u)\cdot\mathcal{SC}(G) + W(d,u)\cdot\mathcal{C}^2(G) \tag{1.5.6}$$

where

$$\begin{aligned} V(d,u): &= 1 + 2u^2d^2 - \big(\frac{7}{3}u^4 - 3u^2\big)d^4 - \Big(\frac{31}{3}u^6 - \frac{221}{9}u^4 + \frac{253}{15}u^2 - \frac{41}{15}\Big)d^6, \\ W(d,u): &= 2\varepsilon u. \end{aligned}$$

Then, because of $G = Q\,F$, we have $\mathcal{X}(G) = V\cdot\mathcal{SC}(F\,Q) + W\cdot\mathcal{C}^2(F\,Q)$. Using the formulas

$$\begin{aligned} &\mathcal{C}_2 = 2\mathcal{C}^2 - Id, \\ &\mathcal{S}_2 = 2\mathcal{SC}, \\ &\mathcal{S}_2(F\,Q) = \mathcal{S}_2(F)\mathcal{C}_2(Q) + \mathcal{S}_2(Q)\mathcal{C}_2(F), \\ &\mathcal{C}_2(F\,Q) = \mathcal{C}_2(F)\mathcal{C}_2(Q) + \mathcal{S}_2(Q)\mathcal{S}_2(F). \end{aligned} \tag{1.5.7}$$

We obtain

$$\begin{aligned} \mathcal{X} &= \frac{1}{2}V\cdot\mathcal{S}_2(F\,Q) + \frac{1}{2}W\cdot\Big(\mathcal{C}_2(F\,Q) + Q\,F\Big) \\ &= \frac{1}{2}\Big(V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q)\Big)\mathcal{S}_2(F) + \frac{1}{2}\Big(V\,\mathcal{S}_2(Q) + W\,\mathcal{C}_2(Q)\Big)\mathcal{C}_2(F) \\ &\quad + \frac{1}{2}W\,Q\cdot F. \end{aligned}$$

With (1.5.5) this implies

$$\mathcal{X} = \frac{1}{2}W_1\,F + \frac{1}{2}W_2\,\mathcal{C}_2(F) - \frac{1}{2}\big(V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q)\big)F^2 + \mu(\varepsilon,u), \tag{1.5.8}$$

where

$$\begin{aligned} &W_1 = -2U\big(V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q)\big) + W\,Q, \\ &W_2 = V\,\mathcal{S}_2(Q) + W\,\mathcal{C}_2(Q), \\ &\mu(\varepsilon,u) = \tfrac{1}{2}\big(V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q)\big)\big(\varepsilon^2 - U^2 - \mathcal{S}_2(U)\big). \end{aligned}$$

The lowest power of d in W_1 and W_2 is the eleventh $W_1 = \mathcal{O}(d^{11}) = W_2$, $\mu(\varepsilon, u)$ is analytic. On the other hand $\mathcal{C}(G) = Q_1\, J$, this implies

$$\mathcal{X}(G) = V \cdot \mathcal{S}(Q_1\, J) + W \cdot \mathcal{C}(Q_1\, J). \tag{1.5.9}$$

If we use the product formulas

$$\begin{aligned} \mathcal{S}(Q_1\, J) &= \mathcal{S}(J)\mathcal{C}(Q_1) + \mathcal{S}(Q_1)\mathcal{C}(J), \\ \mathcal{C}(Q_1\, J) &= \mathcal{C}(J)\mathcal{C}(Q_1) + \mathcal{S}(Q_1)\mathcal{S}(J). \end{aligned} \tag{1.5.10}$$

We have

$$\mathcal{X}(G) = \Big(V\mathcal{C}(Q_1) + W\mathcal{S}(Q_1)\Big)\mathcal{S}(J) + \Big(V\mathcal{S}(Q_1) + W\mathcal{C}(Q_1)\Big)\mathcal{C}(J). \tag{1.5.11}$$

With (1.5.8) and (1.4.3), this implies

$$\begin{aligned} V_1\mathcal{L}(J) &= -(W_3 + PV_1)\mathcal{C}(J) + \frac{1}{2}W_1F - \frac{1}{2}\big(V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q)\big)F^2 \\ &\quad + \frac{1}{2}W_2\,\mathcal{C}_2(F) + \mu(\varepsilon, u), \end{aligned} \tag{1.5.12}$$

$$\begin{aligned} V_1 &= V\mathcal{C}(Q_1) + W\mathcal{S}(Q_1), \\ W_3 &= V\mathcal{S}(Q_1) + W\mathcal{C}(Q_1), \\ P &= \mathcal{S}\big((1-u^2)d^2\big)/\mathcal{C}\big((1-u^2)d^2\big) = -ud + \mathcal{O}(d^2), \\ \mathcal{L}(J) &= \mathcal{S}(J) - P \cdot \mathcal{C}(J). \end{aligned} \tag{1.5.13}$$

We divide (1.5.12) by $(1-u^2)d^2$, this implies

$$\begin{aligned} \frac{V_1}{(1-u^2)d^2}\mathcal{L}(J) &= -\frac{W_3 + PV_1}{(1-u^2)d^2}\mathcal{C}(J) + \frac{W_1}{2(1-u^2)d^2}F - \frac{V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q)}{2(1-u^2)d^2}F^2 \\ &\quad + \frac{W_2}{2(1-u^2)d^2}\mathcal{C}_2(F) + \frac{\mu(\varepsilon, u)}{(1-u^2)d^2}, \end{aligned} \tag{1.5.14}$$

where F and J are coupled by the equation $Q_1 J = \mathcal{C}(Q\,F) = \mathcal{C}(G)$.

Theorem 1.5.1. *With the above notations, we have the following estimation*

$$\big\|dD\mathcal{C}(G)\big\|_{n+1} = \mathcal{O}\Big((n-7)\,!\pi^{-n}\Big) \quad \textit{as } n \to \infty.$$

Proof. We set

$$e_n := \frac{\left\|dD\mathcal{C}(G)\right\|_{n+1}}{(n-6)\,!\pi^{-n}} \text{ for } n \geq 7. \tag{1.5.15}$$

We must show that $e_n = \mathcal{O}(n^{-1})$. In the sequel we will use the convention: If $a_n, n = 0, 1, ...$ is any sequence of positive real numbers, then

$$a_n^+ := \max(a_0, a_1, ...a_n), \text{ for all } n \geq 0. \tag{1.5.16}$$

In this proof $K_1, K_2, ...$ will always denote constants independent of n. Theorem 1.4.3 gives

$$\left\|G\right\|_n \leq K_1 e_n^+ (n-1)\,!\pi^{-n}, \tag{1.5.17}$$

$$\left\|G^2\right\|_n \leq K_2 f_n (n-10)\,!\pi^{-n}, \quad \text{for } n \geq 18, \tag{1.5.18}$$

where

$$f_n = \sum_{i=9}^{n-9} e_i^+ e_{n-i}^+ \frac{(i-1)\,!\,(n-i-1)\,!}{(n-10)\,!}, \quad \text{for } n \geq 18.$$

Using theorem 1.3.3, (5), we find

$$\begin{aligned} \left\|F\right\|_n &\leq K_4 e_{n+1}^+ (n+1)\,!\pi^{-n}, \\ \left\|F^2\right\|_n &\leq K_3 f_{n+4} (n-6)\,!\pi^{-n}. \end{aligned} \tag{1.5.19}$$

Using theorem 1.4.2, we obtain

$$\left\|\mathcal{C}_2(F)\right\|_n \leq K_5 e_{n+1}^+ (n+1)\,!\pi^{-n}. \tag{1.5.20}$$

Observe that the convergent polynomials series $W_1/2(1-u^2)d^2$, $W_2/2(1-u^2)d^2$ begins with d^9 and $\big(V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q)\big)/2(1-u^2)d^2$ begin with 1. Using theorem 1.4.5 we thus obtain

$$\left\|\frac{W_2}{2(1-u^2)d^2}\mathcal{C}_2(F)\right\|_n \leq K_6 e_{n-7}^+ (n-7)\,!\pi^{-n}, \tag{1.5.21}$$

$$\left\|\frac{W_1}{2(1-u^2)d^2} F\right\|_n \leq K_7 e_{n-7}^+ (n-7)\,!\pi^{-n}, \tag{1.5.22}$$

$$\left\|\frac{\big(V\,\mathcal{C}_2(Q) + W\,\mathcal{S}_2(Q)\big)}{2(1-u^2)d^2} F^2\right\|_n \leq K_8 f_{n+4}^+ (n-6)\,!\pi^{-n}. \tag{1.5.23}$$

On the other hand

$$\left\|Q_1 J\right\|_n = \left\|\mathcal{C}(G)\right\|_n \leq e_n (n-6)\,!\pi^{-n}.$$

We apply the theorems 1.3.3, (5) and 1.4.2 and find

$$\left\|\mathcal{C}(J)\right\|_n \leq K_9 e_{n+2} (n-4)\,!\pi^{-n}.$$

Because the convergent polynomial series $(W_3 + PV_1)/(1-u^2)d^2 = \mathcal{O}(d^3)$, then

$$\left\|\frac{(W_3 + PV_1)}{(1-u^2)d^2}\mathcal{C}(J)\right\|_n \leq K_{10} e^+_{n-1} (n-9)\,!\pi^{-n}. \tag{1.5.24}$$

With a crude estimate of the convergent terms $\mu(\varepsilon, u)$ the inequalities (1.5.21)- (1.5.24) gives

$$\left\|\frac{V_1}{2(1-u^2)d^2}\mathcal{L}(J)\right\|_n \leq K_{11}(1 + e^+_{n-1} + f^+_{n+4})(n-6)\,!\pi^{-n}.$$

The multiplication by the convergent terme $2(1-u^2)d^2/\,V_1$ only changes the constant K_{11}. Since theorem 4.5 applies to the operator $\mathcal{L}$ defined in (1.4.3), we obtain

$$\left\|dDJ\right\|_n \leq K_{12}(1 + e^+_{n-1} + f^+_{n+4})(n-6)\,!\pi^{-n}. \tag{1.5.25}$$

This with theorem 1.4.5 gives via

$$dD\mathcal{C}(G) = dD(Q_1 J) = J \cdot (dDQ_1) + Q_1 \cdot (dDJ),$$

the estimate

$$\left\|dD\mathcal{C}(G)\right\|_n \leq K_{13}(1 + e^+_{n-3} + f^+_{n+2})(n-8)\,!\pi^{-n}. \tag{1.5.26}$$

By equation (1.5.15), we obtain

$$e_{n-1} \leq \frac{K}{n}(1 + e^+_{n-3} + f^+_{n+2}). \tag{1.5.27}$$

Lemma 1.5.2. *Under the condition (1.5.27), we have* $e_n = \mathcal{O}(n^{-1})$ *as* $n \to \infty$.

Proof. Let $K_1 \geq 9\,!e^+_{10}$ an arbitrary number. We assume that

$$e_n \leq \frac{K_1 (n+p)\,!}{(n-1)\,!(p+10)\,!}, \qquad \text{for } 10 \leq n \leq N-3, \tag{1.5.28}$$

with $p \geq -1,\ N \geq 14$. This gives for $13 \leq n \leq N$

$$(n-8)\,!f_{n+2} \leq 2e_9^+ K_1 \frac{8\,!(n+p-7)\,!}{!(p+10)\,!} + K_1^2 \sum_{i=10}^{n-8} \frac{(i+p)\,!(n+p-i+2)\,!}{\big((p+10)\,!\big)^2}.$$

Using the inequality,

$$\sum_{i=10}^{n-8} (i+p)\,!(n+p-i+2)\,! \leq (p+10)\,!(n+p-7)\,!,$$

we obtain

$$f_{n+2} \leq K_2 \frac{(n+p-7)\,!}{(p+10)\,!(n-8)\,!} \leq K_2 \frac{(n+p-3)\,!}{(p+10)\,!(n-4)\,!} \qquad \text{for } 13 \leq n \leq N,$$

with a constant K_2 depends on k_1, independent of p. The asumption (1.5.27) of the lemma yields

$$e_{n-1} \leq \frac{K}{n}(1+K_1+K_2)\frac{(n+p-3)\,!}{(p+10)\,!(n-4)\,!} \qquad \text{for } 13 \leq n \leq N,$$

this implies

$$e_n \leq \frac{K_3}{n+1}\frac{(n+p)\,!}{(p+10)\,!(n-1)\,!}, \qquad \text{for } 12 \leq n \leq N-1,$$

with a constant K_3 only depending upon K_1. Now we choose $N_0 \geq 12$ so large that $\frac{K_3}{N_0} \leq K_1$ and then p so large that (1.5.28) holds. Then

$$e_n \leq \frac{K_1}{n+1}\frac{(n+p)\,!}{(p+10)\,!(n-1)\,!} \qquad \text{for } 12 \leq n \leq N-1, \text{ with } N \geq N_0.$$

Since $K_1 \geq 9\,!e_{10}$ is an arbitrary number, we have shown for any $p \geq -1$, taht

$$e_n = \mathcal{O}\left(\frac{(n+p)\,!}{(n-1)\,!}\right) \text{ as } n \to \infty,$$

implies that

$$e_n = \mathcal{O}\left(\frac{(n+p-1)\,!}{(n-1)\,!}\right) \text{ as } n \to \infty.$$

Consequently

$$e_n = \mathcal{O}\big(n^{-1}\big), \text{ as } n \to \infty$$

Thus we have shown that

$$\big\|dD\mathcal{C}(G)\big\|_{n+1} = \mathcal{O}\big((n-7)\,!\pi^{-n}\big) \text{ as } n \to \infty \,.$$

Let $E := \mathcal{C}(G)$. Like G, the polynomial series E is odd in d and the coefficients are odd in u. We partition them

$$\begin{aligned} E_n &= \alpha_n(n-1)\,!\Big(\frac{i}{\pi}\Big)^{n-1}\tau_n(u) + \beta_{n-2}(n-3)\,!\Big(\frac{i}{\pi}\Big)^{n-3}\tau_{n-2}(u) \\ &\quad + \gamma_{n-4}(n-5)\,!\Big(\frac{i}{\pi}\Big)^{n-5}\tau_{n-4}(u) + \overline{E}_{n-6}, \end{aligned} \tag{1.5.29}$$

for odd $n \geq 7$, where α_n and β_n are real number and also $\overline{E}$ have at most degree n for all n. For the whole series E this is equivalent to

$$\mathcal{C}(G) = E = E_1 + d^2E_2 + d^4E_4 + d^6\overline{E}, \tag{1.5.30}$$

where

$$\begin{aligned} E_1 &= \sum_{n=7}^{+\infty} \alpha_n(n-1)\,!\Big(\frac{i}{\pi}\Big)^{n-1}\tau_n(u)d^n, \\ E_2 &= \sum_{n=5}^{+\infty} \beta_n(n-1)\,!\Big(\frac{i}{\pi}\Big)^{n-1}\tau_n(u)d^n, \\ E_3 &= \sum_{n=3}^{+\infty} \gamma_n(n-1)\,!\Big(\frac{i}{\pi}\Big)^{n-1}\tau_n(u)d^n, \\ \overline{E} &= \sum_{n=1}^{+\infty} \overline{E}_n(u)d^n. \end{aligned}$$

Theorem 1.5.1 and the definition of the norms yields

$$\alpha_n = \mathcal{O}(n^{-7}),\ \beta_n = \mathcal{O}(n^{-5}),\ \gamma_n = \mathcal{O}(n^{-3}) \text{ and } \|D\overline{E}_n\|_{n+1} = \mathcal{O}\big((n-1)\,!\pi^{-n}\big).$$

We apply $\mathcal{C}^{-1}$ to (1.5.29) and obtain

$$G = \mathcal{C}^{-1}(E_1) + d^2\mathcal{C}^{-1}(E_2) + d^4\mathcal{C}^{-1}(E_4) + d^6\mathcal{C}^{-1}(\overline{E}). \tag{1.5.31}$$

To the first two summands, theorem 1.4.4 applies and yields

$$\begin{aligned}
\left\|\{\mathcal{C}^{-1}(E_1)\}_n - \alpha(n-1)\,!\Big(\frac{i}{\pi}\Big)^{n-1}\tau_n(u)\right\|_n &= \mathcal{O}\big((n-7)\,!\pi^{-n}\big), \\
\left\|\{\mathcal{C}^{-1}(E_2)\}_n - \beta(n-1)\,!\Big(\frac{i}{\pi}\Big)^{n-1}\tau_n(u)\right\|_n &= \mathcal{O}\big((n-5)\,!\pi^{-n}\big), \\
\left\|\{\mathcal{C}^{-1}(E_3)\}_n - \gamma(n-1)\,!\Big(\frac{i}{\pi}\Big)^{n-1}\tau_n(u)\right\|_n &= \mathcal{O}\big((n-3)\,!\pi^{-n}\big),
\end{aligned}$$

where

$$\alpha = \frac{1}{\pi}\sum_{n=7}^{\infty}\alpha_n,\ \beta = \frac{1}{\pi}\sum_{n=5}^{\infty}\beta_n,\ \gamma = \frac{1}{\pi}\sum_{n=3}^{\infty}\gamma_n.$$

To the last part of (1.5.29), we apply theorem 1.4.3 and obtain

$$\left\|\{\mathcal{C}^{-1}(\overline{E})\}_n\right\|_n = \mathcal{O}\big((n-1)\,!\pi^{-n}\log(n)\big).$$

Using $G = \mathcal{C}^{-1}(E)$, we have shown

$$\begin{aligned}
\left\|G_n - \alpha(n-1)\,!\Big(\frac{i}{\pi}\Big)^{n-1}\tau_n - \beta(n-3)\,!\Big(\frac{i}{\pi}\Big)^{n-3}\tau_{n-2} - \gamma(n-5)\,!\Big(\frac{i}{\pi}\Big)^{n-5}\tau_{n-4}\right\|_n \\
= \mathcal{O}\Big((n-7)\,!\pi^{-n}\log(n)\Big).
\end{aligned}$$

If we use (1.5.3) and the relation $\tau_{n+1} = \frac{1}{n}D\tau_n$ and we apply theorem 1.3.3, (5), we obtain our final result of this section

$$\begin{aligned}
\left\|F_n - \alpha n\,!\Big(\frac{i}{\pi}\Big)^{n+1}\tau'_{n+1} - \beta(n-2)\,!\Big(\frac{i}{\pi}\Big)^{n-1}\tau'_{n-1} - \gamma(n-4)\,!\Big(\frac{i}{\pi}\Big)^{n-3}\tau'_{n-3}\right\|_n = \\
\mathcal{O}\Big((n-5)\,!\pi^{-n}\log(n+2)\Big).
\end{aligned}$$

and

$$\begin{aligned}
\left\|F_n - \alpha(n+1)\,!\Big(\frac{i}{\pi}\Big)^{n+1}\frac{\tau_{n+2}}{\tau_2} - \beta(n-1)\,!\Big(\frac{i}{\pi}\Big)^{n-1}\frac{\tau_n}{\tau_2} - \gamma(n-3)\,!\Big(\frac{i}{\pi}\Big)^{n-3}\frac{\tau_{n-2}}{\tau_2}\right\|_n \\
= \mathcal{O}\Big((n-5)\,!\pi^{-n}\log(n+2)\Big).
\end{aligned}$$

1.6 Preparation of the functions to construct a quasi-solutions

In the previous section , we have shown that equation (1.2.4) has a formal solution and we found an asymptotic approximation of the coefficients of this formal solution. We will use this to construct the quasi-solution. To that purpose, we define the functions

$$H_n(u) := (n+1)\,!\Big(\frac{i}{\pi}\Big)^{n+1}\tau_{n+2}(u) \tag{1.6.1}$$

$$h(t,u) := \sum_{\substack{n=7\\ n\ odd}}^{\infty} H_n(u)\frac{t^{n-1}}{(n-1)\,!}. \tag{1.6.2}$$

We rewrite

$$h(t,u) = \Big(\frac{i}{\pi}\Big)^2 \sum_{\substack{n=6\\ n\ even}}^{\infty} (n+1)(n+2)\Big(\frac{it}{\pi}\Big)^n \tau_{n+3}(u). \tag{1.6.3}$$

Using proposition (1.3.1)-(4), we obtain

$$h(t,u) = \frac{-1}{\pi^2} \sum_{\substack{n=6\\ n\ even}}^{\infty} \frac{1}{n\,!}\Big(\frac{it}{\pi}\Big)^n \frac{d^n}{d^n\xi}\Big(\frac{d^2}{d^2\xi}\big(\tanh(\xi)\big)\Big),$$

or equivalently

$$h(t,u) = \frac{-1}{\pi^2} \sum_{\substack{n=6\\ n\ even}}^{\infty} \frac{1}{n\,!}\Big(\frac{it}{\pi}\Big)^n \frac{d^n}{d^n\xi}\Big(g(\xi)\Big),$$

where $g(\xi) = -2\big(\tanh(\xi) - \tanh(\xi)^3\big)$ and $\xi = \xi(u) = artanh(u)$.
and hence that

$$h(t,u) = \frac{-1}{\pi^2} \sum_{\substack{n=0\\ n\ even}}^{\infty} \frac{1}{n\,!}\Big(\frac{it}{\pi}\Big)^n \frac{d^n}{d^n\xi}\Big(g(\xi)\Big) + \mu(t,u),$$

where

$$\mu(t,u) := -\frac{2\,u}{\pi^2}\tau_2(u) + \frac{4}{\pi^4}(3u^3-2u)\tau_2(u)\,t^2 - \frac{2}{3\pi^6}(45u^5-60u^3+17u)\tau_2(u)\,t^4.$$

Thus we can write

$$h(t,u) = \frac{-1}{2\pi^2}\Big[g\big(\xi + \frac{it}{\pi}\big) + g\big(\xi - \frac{it}{\pi}\big)\Big] + \mu(t,u). \qquad (1.6.4)$$

This gives

$$\begin{aligned} h(t,u) &= (1-u^2)\left[\frac{(4u^3-4u)\sin\left(\frac{t}{\pi}\right)^4 - (6u^3-2u)\sin\left(\frac{t}{\pi}\right)^2 + 2u}{2\pi^2\Big(\cos\left(\frac{t}{\pi}\right)^2 + u^2\sin\left(\frac{t}{\pi}\right)^2\Big)^3}\right] \\ &+ \mu(t,u). \qquad (1.6.5)\end{aligned}$$

For fixed real u the function $h(.,u)$ is analytic in $|t| < \rho$, where $\rho = \frac{\pi^2}{2}$. In the subsequent definition, we consider real values of u, $0 < u \leq 1$, here $h(.,u)$ is also analytic with respect to t on the positive real axis.

Now we define the function $\mathcal{H}(d,u)$ by

$$\mathcal{H}(d,u) := \int_0^{+\infty} e^{-\frac{t}{d}} h(t,u)dt, \quad \text{for} \quad (0 < u \leq 1). \qquad (1.6.6)$$

The function $\mathcal{H}(d,.)$ is real analytic; they can be continued analytically to the interval $-1 < u \leq 1$ in the following way. Choose some positive number M and let Γ_1 the path consisting of the segment from 0 to Mi and of the ray $t \mapsto t + Mi, t \geq 0$. Let Γ_2 the symmetric path that could also be obtained using $-M$ instead of M. Recalling (1.6.4), we can also define

$$\begin{aligned} \mathcal{H}(d,u) : &= -\frac{1}{\pi^2}\left[\int_{\Gamma_2} e^{-\frac{t}{d}} g\big(\xi + \frac{it}{\pi}\big)\,dt + \int_{\Gamma_1} e^{-\frac{t}{d}} g\big(\xi - \frac{it}{\pi}\big)\,dt\right] \\ &+ \mu_1(d,u),\end{aligned}$$

where

$$\mu_1(d,u) := \int_0^{\infty} e^{-\frac{t}{d}} \mu(t,u)dt,$$

for $-\tanh(\frac{2}{\pi}M) < u \leq 1$, where $\xi = \text{artanh}(u)$, because the singularities of $\tanh$ are $i(\frac{\pi}{2} + n\pi)$, n integer. As M is arbitrary, this defines the analytic continuation of $\mathcal{H}(d,.)$ for $-1 < u \leq 1$.

In the sequel we consider $u_0 \in]-1,0]$.

Lemma 1.6.1. *If we consider the above function $\mathcal{H}(d,u)$ and the operators defined in (1.2.22). Then, for $u_0 < u \leq 1$*

- *1.*
$$\mathcal{S}_2(\mathcal{H}) = (1-u^2)\mathcal{D}_1(d,u),$$
where the function $\mathcal{D}_1(d,u)$ is analytic, beginnings with d^8.

- *2.*
$$\mathcal{C}_2(\mathcal{H}) = -\mathcal{H}(d,u) + (1-u^2)\mathcal{D}_2(d,u),$$
where the function $\mathcal{D}_2(d,u)$ is analytic, beginnings with d^7.

- *3*
$$\mathcal{S}_2\Big(\frac{1}{\tau_2(u)}\mathcal{H}\Big) = -\frac{2u\varepsilon}{\tau_2(u)}\mathcal{H}(d,u) + \mathcal{D}_3(d,u)$$
where the function $\mathcal{D}_3(d,u)$ is analytic beginnings with d^8

Proof. (1)- If we replace u by T^+ and T^- in (1.6.5) we obtain

$$\mathcal{H}(d,T^+) - \mathcal{H}(d,T^-) = \int_0^{+\infty} e^{-\frac{t}{d}}\Big(h(t,T^+) - h(t,T^-)\Big)d\,t,$$

this implies

$$\mathcal{H}(d,T^+) - \mathcal{H}(d,T^-) = -\frac{1}{2\pi^2}\big(\mathcal{I}^+ - \mathcal{I}^-\big) + \sigma(d,T^+) - \sigma(d,T^-), \tag{1.6.7}$$

where

$$\begin{aligned}
\mathcal{I}^+ &= \int_0^{+\infty} e^{-\frac{t}{d}}\Big(g\big(\xi(T^+) + \frac{it}{\pi}\big) + g\big(\xi(T^+) - \frac{it}{\pi}\big)\Big)d\,t,\\
\mathcal{I}^- &= \int_0^{+\infty} e^{-\frac{t}{d}}\Big(g\big(\xi(T^-) + \frac{it}{\pi}\big) + g\big(\xi(T^-) - \frac{it}{\pi}\big)\Big)d\,t,\\
\sigma(d,u) &= \int_0^{+\infty} e^{-\frac{t}{d}}\mu(t,u)d\,t.
\end{aligned}$$

Using $\xi(T^\pm) = \xi \pm d$, where ξ and u are coupled by $\xi = \text{artanh}(u)$, we obtain

$$\mathcal{I}^+ = \int_0^{+\infty} e^{-\frac{t}{d}} g\big(\xi + d + \frac{it}{\pi}\big)d\,t + \int_0^{+\infty} e^{-\frac{t}{d}} g\big(\xi - d + \frac{it}{\pi}\big)d\,t.$$

If we split the integral and substitute $t+\pi i\, d$ in the first part, $t-\pi i\, d$ in the second part, we obtain

$$\mathcal{I}^{+} = -\int_{-\pi id}^{+\infty+\pi id} e^{-\frac{t}{d}} g\big(\xi+\frac{it}{\pi}\big)d\,t - \int_{\pi id}^{+\infty+\pi id} e^{-\frac{t}{d}} g\big(\xi-\frac{it}{\pi}\big)d\,t.$$

We apply Cauchy's theorem

$$\mathcal{I}^{+} = -\int_{0}^{+\infty} e^{-\frac{t}{d}}\Big(g\big(\xi+\frac{it}{\pi}\big)+g\big(\xi-\frac{it}{\pi}\big)\Big)d\,t \quad + \quad \int_{0}^{-\pi id} e^{-\frac{t}{d}} g\big(\xi+\frac{it}{\pi}\big)d\,t \\ + \quad \int_{0}^{\pi id} e^{-\frac{t}{d}} g\big(\xi-\frac{it}{\pi}\big)d\,t.$$

Substitute $t=-i\,s\,d$ in the second part, $t=i\,s\,d$ in the third part, we obtain

$$\mathcal{I}^{+} = -\int_{0}^{+\infty} e^{-\frac{t}{d}}\Big(g\big(\xi+\frac{it}{\pi}\big)+g\big(\xi-\frac{it}{\pi}\big)\Big)d\,t + 2d\int_{0}^{\pi}\sin(s) g\big(\xi+\frac{s\,d}{\pi}\big)d\,s.$$

This implies

$$\mathcal{I}^{+} \;=\; -\int_{0}^{+\infty} e^{-\frac{t}{d}}\Big(g\big(\xi+\frac{it}{\pi}\big)+g\big(\xi-\frac{it}{\pi}\big)\Big)d\,t - 4d(1-u^2)\mathcal{D}^{+}(d,u), \tag{1.6.8}$$

where

$$\mathcal{D}^{+}(d,u) = \int_{0}^{\pi}\sin(s)\frac{\big(u+\tanh(\frac{s\,d}{\pi})\big)\big(1-\tanh(\frac{s\,d}{\pi})^2\big)}{\big(1+u\tanh(\frac{ds}{\pi})\big)^3}d\,s.$$

We can also use the same method for $\mathcal{I}^{-}$ and obtain

$$\mathcal{I}^{-} \;=\; -\int_{0}^{+\infty} e^{-\frac{t}{d}}\Big(g\big(\xi+\frac{it}{\pi}\big)+g\big(\xi-\frac{it}{\pi}\big)\Big)d\,t - 4d(1-u^2)\mathcal{D}^{-}(d,u), \tag{1.6.9}$$

where

$$\mathcal{D}^{-}(d,u) = \int_{0}^{\pi}\sin(s)\frac{\big(u-\tanh(\frac{ds}{\pi})\big)\big(1-\tanh(\frac{s\,d}{\pi})^2\big)}{\big(1-u\tanh(\frac{s\,d}{\pi})\big)^3}d\,s.$$

Consequently

$$\mathcal{H}(d,T^{+})-\mathcal{H}(d,T^{-}) \;=\; \frac{2d(1-u^2)}{\pi^2}\Big(\mathcal{D}^{+}(d,u)-\mathcal{D}^{-}(d,u)\Big)+\sigma(d,T^{+})-\sigma(d,T^{-}).$$

Using (1.2.22), we find

$$\mathcal{S}_2(\mathcal{H}) = (1-u^2)\,\mathcal{D}_1(d,u), \tag{1.6.10}$$

where $\mathcal{D}_1(d,u)$ is analytic, beginnings with d^8.

(2). Using the same method, we obtain

$$\mathcal{H}(d,T^+) + \mathcal{H}(d,T^-) = -2\mathcal{H}(d,u) + 2(1-u^2)\mathcal{D}_2(d,u).$$

With (1.2.22) we find

$$\mathcal{C}_2(\mathcal{H}) = -\mathcal{H}(d,u) + (1-u^2)\mathcal{D}_2(d,u).$$

$\mathcal{D}_2(d,u)$ is analytic, beginnings with d^7.

(3). Using (1.5.7), (1.2.8), (1.2.22) and (1), (2) of this lemma, we obtain

$$\begin{aligned} \mathcal{S}_2\Big(\frac{1}{\tau_2(u)}\mathcal{H}\Big) &= \mathcal{S}_2\Big(\frac{1}{\tau_2(u)}\Big)\mathcal{C}_2(\mathcal{H}) + \mathcal{C}_2\Big(\frac{1}{\tau_2(u)}\Big)\mathcal{S}_2(\mathcal{H}) \\ &= -\frac{2u\varepsilon}{\tau_2(u)}\mathcal{H}(d,u) + \mathcal{D}_3(d,u) \end{aligned}$$

where

$$\mathcal{D}_3(d,u) := \big(1 + (1+u^2)\sinh(d)^2\big)(1-u^2)\mathcal{D}_1(d,u) + 2u\varepsilon(1-u^2)\mathcal{D}_2(d,u).$$

Proposition 1.6.2. *We have*

1. *For* $u_0 < u \leq 1$

$$\mathcal{H}(d,u) \sim \sum_{n=7}^{\infty}(n+1)\,!\Big(\frac{i}{\pi}\Big)^{n+1}\tau_{n+2}(u)d^n, \text{ as } d \searrow 0, \tag{1.6.11}$$

2. $\left|\frac{\partial\mathcal{H}}{\partial u}(d,u)\right| \leq Kd$ *for* $u_0 \leq u \leq 1 \;\; (d>0)$.

Proof.

1. To prove (1) we use Watson's lemma and (1.6.2). if $|t - k\pi^2| < \frac{\pi^2}{2}$, for $u_0 \leq u < 1$

2. • For $u_0 \le u \le \frac{1}{2}$

$$\Big|\mathcal{H}(d,u)\Big| = \frac{1}{d^6}\int_0^\infty e^{-t/d} h^{(-6)}(t,u)\,dt,$$

where $h^{(-6)}(t,u)$ satisfies

$$\Big(\frac{\partial}{\partial t}\Big)^6 h^{(-6)} = \frac{\partial h}{\partial u} \quad \text{and}\Big(\frac{\partial}{\partial t}\Big)^k h^{(-6)}(0,u) = 0, \quad \text{for } k = 0, ..., 5.$$

Because of

$$h^{(-6)}(t,u) \le K\frac{t^7}{7\,!} \qquad \text{for } t \ge 0,$$

we obtain

$$\Big|\frac{\partial \mathcal{H}}{\partial u}(d,u)\Big| \;\; \le \;\; \frac{K}{7\,!d^6}\int_0^\infty e^{-\frac{t}{d}} t^7 dt.$$

This implies

$$\Big|\frac{\partial \mathcal{H}}{\partial u}(d,u)\Big| \le Kd \qquad (d>0).$$

• for $\frac{1}{2} \le u \le 1$

$$\begin{aligned} \Big|\frac{\partial \mathcal{H}}{\partial u}(d,u)\Big| &= \Big|\int_0^\infty e^{-\frac{t}{d}} \frac{\partial h}{\partial u}(d,u)dt\Big|, \\ &\le K\int_0^\infty e^{-\frac{t}{d}} dt, \\ &\le Kd \qquad (d>0). \;\square \end{aligned}$$

Let us consider a sequence $R_n(u)$ of polynomials of degree at most n, such that

$$\|R_n\|_n = O\Big((n-5)\,!\pi^{-1}\log(n)\Big)$$

Lemma 1.6.3. *Let polynomial series difine $R \in \mathcal{Q}$ with the above estimate be given and define*

$$\begin{aligned} r(t,u) :&= \sum_{n=7}^{\infty} R_n(u)\frac{t^{n-1}}{(n-1)\,!}, \quad (t \in \mathbb{C}, |t| \le \frac{\pi^2}{2}, u_0 \le u \le 1), \\ r(t,u) :&= r\big(\frac{\pi^2}{2},u\big) + \big(t - \frac{\pi^2}{2}\big)\frac{\partial r}{\partial t}\big(\frac{\pi^2}{2},u\big), \quad (t > \frac{\pi^2}{2}, u_0 \le u \le 1), \\ \mathcal{R}(d,u) :&= \int_0^\infty e^{-\frac{t}{d}}\, r(t,u)dt. \end{aligned}$$

Then

1. *r is continuously differentiable function on the set B of all (t,u) such that u satisfies $u_0 \le u \le 1$ and t is a complex number and satisfies $|t| \le \frac{\pi^2}{2}$ or $t > \frac{\pi^2}{2}$. The restriction of r to $u_0 \le u \le 1, |t| \le \frac{\pi^2}{2}$ is twice continuously differentiable. For fixed $u_0 \le u \le 1$ the function $r(t,u)$ is analytic in $|t| < \frac{\pi^2}{2}$.*

2. *$\mathcal{R}(d,u)$ is continuous, partially differentiable with respect to u, has continuous partial derivative and*

$$\mathcal{R}(d,u) \sim \sum_{n=7}^{\infty} R_n(u) d^n, \text{ as } d \searrow 0. \tag{1.6.12}$$

3. $\big|\mathcal{R}(d,u)\big| \le K d^3, \ \big|\frac{\partial \mathcal{R}}{\partial u}(d,u)\big| \le K d^3, \quad$ *for* $\ u_0 \le u \le 1 \ (d > 0)$.

The importance of our definition of $\mathcal{R}$ lies in a certain compatibility with insertion of the functions T^+, T^- for u. First let

$$\begin{aligned} \sum_{n=7}^{\infty} R_n^+(u) d^n &:= \sum_{n=7}^{\infty} R_n\big(T^+(d,u)\big) d^n, \\ \sum_{n=7}^{\infty} R_n^-(u) d^n &:= \sum_{n=7}^{\infty} R_n\big(T^-(d,u)\big) d^n. \end{aligned}$$

We obtain a new sequences $R_n^+(u), R_n^-(u)$ of polynomials of degree at most n. Theorem 1.4.2, and (1.4.2) imply

$$\begin{aligned} \big\|R_n^+(u)\big\|_n &= O\Big((n-5)\,!\pi^{-n}\log(n)\Big) \\ \big\|R_n^-(u)\big\|_n &= O\Big((n-5)\,!\pi^{-n}\log(n)\Big) \end{aligned}$$

Therefore we can use lemma 1.6.3 for $R_n^+(u), \ R_n^-(u)$, and obtain functions $\mathcal{R}^+(d,u)$, $\mathcal{R}^-(d,u)$.

Theorem 1.6.4. *There is a positve constant K independent of d,u such that*

$$\begin{aligned} \big|\mathcal{R}^+(d,u) - \mathcal{R}(d,T^+)\big| &\le K d^3 e^{-\frac{\pi^2}{2d}} \quad \textit{for} \quad (0 < d < d_0, u_0 < u < 1) \\ \big|\mathcal{R}^-(d,u) - \mathcal{R}(d,T^-)\big| &\le K d^3 e^{-\frac{\pi^2}{2d}} \quad \textit{for} \quad (0 < d < d_0, u_0 < u < 1). \end{aligned}$$

Proof The proof is exactly the one of [10].

Definition 1.6.5. *Let $\mathcal{D}(d,u)$ be a function defined for $0<d<d_0$ and $u_0<u<1$. We say that $\mathcal{D}(d,u)$ has property G if*

$$\mathcal{D}(d,u)=\int_0^\infty e^{-\frac{t}{d}}\,q(t,u)dt \quad (0<d<d_0,\ u_0<u<1)$$

is the Laplace transform of some function $q(t,u)$ that has the following proprties:

1. *$q(t,u)$ is defined if $u_0<u<1$ and either t is complex and $|t|<\frac{\pi^2}{2}$ or t is real and $t\geq 0$,*
2. *$q(t,u)$ is analytic in $|t|<\frac{\pi^2}{2}$ for $u_0<u<1$,*
3. *$q(t,u)$ restricted to $0\leq t<\frac{\pi^2}{2}$ or $t\geq\frac{\pi^2}{2}$ is continuous and the $\lim_{t\to\frac{\pi^2}{2}}q(t,u)$ exists for every $u_0<u<1$,*
4. *there is a positve constant K such that*

$$|q(t,u)|\leq Ke^{Kt},\ \text{ for }\ t\geq 0,\ u_0<u<1,\ (0<d<d_0,\ u_0<u<1)$$

The set of all function with the property G will be denote by $\mathcal{G}$

Lemma 1.6.6. *For $u_0<u\leq 1$*

1. *If $\mathcal{H}(d,u)$ is the function defined in (1.6.6) then*

$$d^4\mathcal{H}(d,u)=(1-u^2)\tilde{\mathcal{H}}(d,u)+\mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{2d}}\Big)$$

   where $\tilde{\mathcal{H}}(d,u)$ has property G

2. *Let k be a psitive integer. If $\mathcal{D}_1,\mathcal{D}_2$ have property G and their first terms in the Taylor development at $d=0$, begin with d^k then*

$$\mathcal{D}_1(d,u)\mathcal{D}_2(d,u)=d^k\mathcal{D}(d,u)+\mathcal{O}\Big(d^ke^{-\frac{\pi^2}{2d}}\Big),$$

   where $\mathcal{D}(d,u)$ has property G

3. *If $\mathcal{H}(d,u)$ is the function defined in (1.6.6) then*

$$\mathcal{H}(d,u)^2 = (1-u^2)^2 d^3 \mathcal{E}(d,u) + \mathcal{O}\Big((1-u^2)^2 d^3 e^{-\frac{\pi^2}{2d}}\Big)$$

   where $\mathcal{E}(d,u)$ has property G

4. *Any function $\mathcal{D}(d,u)$ analytic in a neighborhood of $d=0$ has property G if $\mathcal{D}(0,u)=0$ for all u,*

5. *If $\mathcal{R}(d,u)$ is defined by lemma 1.6.3 then $\dfrac{1}{d^2}\mathcal{R}(d,u)$ has property G*

6. *If $\mathcal{D}_1, \mathcal{D}_2$ have property G then so do $\mathcal{D}_1+\mathcal{D}_2,\ \mathcal{D}_1-\mathcal{D}_2$ and $\mathcal{D}_1\cdot\mathcal{D}_2$*

7. *If $\mathcal{D}(d,u)$ has property G then*

$$\big|\mathcal{D}(d,u)\big| \leq Kd \quad \big(0<d<\frac{1}{K}) \tag{1.6.13}$$

   with some constant $K>0$ independent of u.

Proof.

1. For $u>0$, we have

$$d^4\mathcal{H} = (1-u^2)\int_0^\infty e^{-\frac{t}{d}} g_4(t,u)dt \tag{1.6.14}$$

   where

$$g_4(t,u) = \frac{1}{(1-u^2)}\int_0^t\int_0^\theta\int_0^\upsilon\int_0^\tau h(s,u)ds\,d\tau\,d\upsilon\,d\theta$$

   $g_4(t,u)$ has a logarithmic singularity at $t_k(s)=(2k+1)\dfrac{\pi^2}{2}\pm d^{\frac{\pi s}{\varepsilon}}i$ for $(k\geq 0, s>0)$. it is analytic in $|t|<\frac{\pi^2}{2}$ and $\lim_{t\to\frac{\pi^2}{2}} g_4(t,u)$ exists.

   If we put

$$\tilde{\mathcal{H}}(d,u) = \int_0^\infty e^{-\frac{t}{d}}\tilde{g}_4(t,u)d\,t$$

   where

$$\tilde{g}_4(t,u) = \begin{cases} g_4(t,u), & \text{if } t\leq\frac{\pi^2}{2} \\ g_4\big(\frac{\pi^2}{2},u\big), & \text{if } t\geq\frac{\pi^2}{2}\end{cases}$$

then $\tilde{\mathcal{H}}(d,u)$ has property G and

$$d^4\mathcal{H}(d,u) = (1-u^2)\tilde{\mathcal{H}}(d,u) + \mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{2d}}\Big).$$

Remark: By real analytic continuation the formula (1.6.14) is valid for $-1 < u_0 < u < 1$.

(ii)- For $-1 < u_0 < u < 1$, we have

$$\begin{aligned}
\mathcal{H}(d,u) &= \int_0^{\infty} e^{-\frac{t}{d}} h(t,u)d\,t \\
&= \int_0^{\infty e^{i\varphi}} e^{-\frac{t}{d}} h(t,u)d\,t + 2\pi i \sum_{k\geq 0} Res\Big(e^{-\frac{t}{d}} h(t,u), t_k(s)\Big) \\
&= \int_0^{\infty e^{i\varphi}} e^{-\frac{t}{d}} h(t,u)d\,t + \mathcal{O}\bigg(\frac{1}{d^2}(1-u^2)e^{-\frac{\pi^2}{2d}}\bigg)
\end{aligned}$$

where $\dfrac{\pi}{2} < \varphi < \dfrac{\pi}{4}$. This implis

$$d^4\mathcal{H}(d,u) = (1-u^2)\int_0^{\infty e^{i\varphi}} e^{-\frac{t}{d}} g_4(t,u)d\,t + \mathcal{O}\Big((1-u^2)d^2 e^{-\frac{\pi^2}{2d}}\Big)$$

we obtain

$$\begin{aligned}
d^4\mathcal{H}(d,u) &- (1-u^2)\tilde{\mathcal{H}}(d,u) = (1-u^2)\int_{\Gamma} e^{-\frac{t}{d}} g_2(t,u)d\,\Gamma \\
&+ \mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{2d}}\Big)
\end{aligned}$$

Since $g_4(.,u)$ is bounded on Γ, then

$$\Big|d^4\mathcal{H}(d,u) - (1-u^2)\tilde{\mathcal{H}}(d,u)\Big| \leq K(1-u^2)e^{-\frac{\pi^2}{2d}},$$

where K is positive constant. Finally

$$d^4\mathcal{H}(d,u) = (1-u^2)\tilde{\mathcal{H}}(d,u) + \mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{2d}}\Big) \quad \text{for} \quad (0 < u_0 < u \leq 1)$$

2. We assume that $\mathcal{D}_1(d,u), \mathcal{D}_2(d,u)$ have property G and their first terms in the Taylor development at $d=0$ begin with d^k. Then

$$\mathcal{D}_1 = \int_0^{\infty} e^{-\frac{t}{d}} f(t,u)d\,t$$
$$\mathcal{D}_2 = \int_0^{\infty} e^{-\frac{t}{d}} g(t,u)d\,t$$

where $f(t,u), g(t,u)$ are analytic in $|t| < \frac{\pi^2}{2}$ and $f(t,u) = O(t^{k-1}), g(t,u) = O(t^{k-1})$.

$$\mathcal{D}_1(d,u)\mathcal{D}_2(d,u) = \int_0^\infty e^{-\frac{t}{d}}(f*g)(t,u)d\,t \tag{1.6.15}$$

Since

$$\begin{aligned} h(t,u) &= (f*g)(t,u) = \int_0^s f(t,u)g(t-s,u)ds \\ &= \int_0^t f(t-s,u)g(s,u)ds \\ &= \int_{t/2}^t f(s,u)g(t-s,u)ds + \int_{t/2}^t f(t-s,u)g(s,u)ds. \end{aligned}$$

For $t < \frac{\pi^2}{2}$, the function $h(t,u)$ is k times differentiable with respect to t and

$$\begin{aligned} h'(t,u) &= f(t,u)g(0,u) + f(0,u)g(t,u) - f(\frac{t}{2},u)g(\frac{t}{2},u) \\ &+ \int_{t/2}^t f(s,u)g'(t-s,u)ds + \int_{t/2}^t f'(t-s,u)g(s,u)ds \\ &= \int_{t/2}^t f(s)g'(t-s)ds + \int_{t/2}^t f'(t-s,u)g(s,u)ds \\ &- f(\frac{t}{2},u)g(\frac{t}{2},u) \\ h^{(k)}(t,u) &= \int_{t/2}^t f(s)g^{(k-1)}(t-s)ds + \int_{t/2}^t f^{(k-1)}(t-s,u)g(s,u)ds \\ &- \sum_{n=0}^{k-1} f^{(n)}(\frac{t}{2},u)g^{(k-1-n)}(\frac{t}{2},u) \end{aligned}$$

Observe that $h^{(k)}(t,u)$ is contninuous on $[0,\pi^2[$, it is analytic for $|t| < \frac{\pi^2}{2}$. If we put

$$\tilde{h}(t,u) = \begin{cases} h(t,u), & \text{if } t < \frac{\pi^2}{2} \\ \left(t - \frac{\pi^2}{2}\right)^k. & \text{if } t \geq \frac{\pi^2}{2} \end{cases}$$

then

$$\int_0^\infty e^{-\frac{t}{d}}\tilde{h}^{(k)}(t,u)d\,t,$$

has property G and

$$
\begin{aligned}
(\mathcal{D}_1 \cdot \mathcal{D}_2)(d,u) &= \int_0^\infty e^{-\frac{t}{d}} h(t,u)d\,t = \int_0^\infty e^{-\frac{t}{d}} \tilde{h}(t,u)d\,t + \mathcal{O}\Big(d^k e^{-\frac{\pi^2}{2d}}\Big) \\
&= d^k \int_0^\infty e^{-\frac{t}{d}} \tilde{h}^{(k)}(t,u)d\,t + \mathcal{O}\Big(d^k e^{-\frac{\pi^2}{2d}}\Big) \\
&= d^k \mathcal{D}(d,u) + \mathcal{O}\Big(d^k e^{-\frac{\pi^2}{2d}}\Big)
\end{aligned}
$$

where $\mathcal{D}(d,u)$ has property G.

3. We have

$$
\mathcal{H}(d,u)^2 = \frac{1}{d^8}(d^4\mathcal{H}) \cdot (d^4\mathcal{H})
$$

Using (1) and (2) of this lemma, we obtain immediately the result.

For $0 \leq u \leq 1$, the proof of (4), (5), (6) and (7) is exactly the one of [10]. This proof is valid for $u_0 < u \leq 1$.

1.7 Approximate solution $\mathcal{A}(d,u)$ of the functional equation (1.2.4)

In this section we use the formal solution of (1.2.4). The estimates of section 5 for its coefficients and the preliminary results of section 6 allow to construct the function $\mathcal{A}(d,u)$ that almost satisfies (1.2.4), except an error which is exponentially small as $d \to 0$.

In theorem 1.2.1 we found that (1.2.4) has a uniquely determined formal power series solution

$$
A(d,u) = \frac{1}{\varepsilon}B(d,u) = \frac{1}{\varepsilon}\Bigg(U + \sum_{\substack{n=7 \\ n \text{ odd}}}^{\infty} F_n(u)d^n\Bigg), \tag{1.7.1}
$$

where

$$
\begin{aligned}
U(d,u) &= \varepsilon u + (u-u^3)d^3 + (\frac{10}{3}u^5 - \frac{16}{3}u^3 + 2u)d^5, \\
F_n &= \alpha n\,!\Big(\frac{i}{\pi}\Big)^{n+1}\tau'_{n+1}(u) + \beta(n-2)\,!\Big(\frac{i}{\pi}\Big)^{n-1}\tau'_{n-1}(u) \\
&+ \gamma(n-4)\,!\Big(\frac{i}{\pi}\Big)^{n-4}\tau'_{n-3}(u) + R_n(u), \\
\big\|R_n\big\|_n &= O\Big((n-5)\,!\pi^{-1}\log(n)\Big), \quad (n \geq 7).
\end{aligned}
$$

Consequently

$$
B(d,u) = \big(\alpha + \beta d^2 + \gamma d^4\big)\sum_{\substack{n=7 \\ n\text{ odd}}}^{\infty}(n+1)\,!\Big(\frac{i}{\pi}\Big)^{n+1}\frac{\tau_{n+2}(u)}{\tau_2(u)}d^n + \sum_{\substack{n=7 \\ n\text{ odd}}}^{\infty} R_n(u)d^n + U(d,u),
$$

Now we set

$$
\mathcal{B}(d,u) = \big(\alpha + \beta d^2 + \gamma d^4\big)\frac{1}{\tau_2(u)}\mathcal{H}(d,u) + \mathcal{R}(d,u) + U(d,u),\ (d>0, u_0<u<1), \tag{1.7.2}
$$

where $\mathcal{H}(d,u)$ is defined in (1.6.6) and $\tilde{\mathcal{R}}(d,u)$ is the function corresponding to $R_n,\ n = 7, 9, ..$ By lemma 1.6.3 and proposition 1.6.2, we have

$$
\mathcal{B}(d,u) \sim B(d,u), \quad \text{as } d \searrow 0 \text{ for every } \quad u_0 < u < 1. \tag{1.7.3}
$$

Theorem 1.7.1. *We have*

$$
\Big|\mathcal{A}(d,T^+) - \mathcal{A}(d,T^-) - f\big(\varepsilon, \mathcal{A}(d,u)\big)\Big| \leq \frac{K}{d}e^{-\frac{\pi^2}{2d}}, \quad \textit{for } (0<d<d_0, u_0 \leq u < 1),
$$

where K is a constant independent of d, u and

$$
\mathcal{A}(d,u) = \frac{1}{\varepsilon}\mathcal{B}(d,u) \tag{1.7.4}
$$

Proof. We set

$$
\mathcal{T}(d,u) = \mathcal{S}_2(\mathcal{B}) - \varepsilon^2 + \mathcal{B}(d,u)^2. \tag{1.7.5}
$$

then

$$\begin{aligned}\mathcal{T}(d,u) &= \big(\alpha+\beta d^2+\gamma d^4\big)\mathcal{S}_2\Big(\frac{1}{\tau_2(u)}\mathcal{H}\Big)-\varepsilon^2+\mathcal{B}(d,u)^2\\ &+ \frac{1}{2}\mathcal{R}(d,T^+)-\frac{1}{2}\mathcal{R}(d,T^-)+\mathcal{S}_2(U)\end{aligned}$$

With (1.7.2) and lemma 1.6.1, this implies

$$\mathcal{T}(d,u)=\mathcal{T}_0(d,u)\cdot\mathcal{H}(d,u)+\mathcal{T}_1(d,u)+\mathcal{T}_2(d,u),$$

where

$$\begin{aligned}\mathcal{T}_0(d,u) &= \big(\alpha+\beta d^2+\gamma d^4\big)\Big[-2\varepsilon u+2U(d,u)\Big]\frac{1}{\tau_2(u)},\\ \mathcal{T}_1(d,u) &= \big(\alpha+\beta d^2+\gamma d^4\big)^2\frac{1}{\tau_2(u)^2}\mathcal{H}(d,u)^2+\mathcal{R}(d,u)^2\\ &+ 2\big(\alpha+\beta d^2+\gamma d^4\big)\frac{1}{\tau_2(u)}\mathcal{H}(d,u)\mathcal{R}(d,u)\\ \mathcal{T}_2(d,u) &= 2U(d,u)\mathcal{R}(d,u)+\frac{1}{2}\mathcal{R}(d,T^+)-\frac{1}{2}\mathcal{R}(d,T^-),\\ \mathcal{T}_3(d,u) &= U(d,u)^2+\mathcal{S}_2(U)-\varepsilon^2+\frac{d}{2}\mathcal{D}(d,u)\end{aligned}$$

Observing that $\mathcal{T}_0(d,u)$ is analytic function beginnings with d^3. Using (1) of lemma 1.6.6, we obtain that $d\mathcal{T}_0(d,u)\mathcal{H}(d,u)$ has property G.

Applying lemma 1.6.6, we obtain

$$\mathcal{T}_1(d,u)=d^3\mathcal{D}_3(d,u)+\mathcal{O}\Big(d^3e^{-\frac{\pi^2}{2d}}\Big)$$

where $\mathcal{D}_3(d,u)$ has property G.

Using (5) of lemma 1.6.6 and theorem 1.6.4, we find

$$\mathcal{T}_2(d,u)=d^3\mathcal{D}_4(d,u)+\mathcal{O}\Big(d^3e^{-\frac{\pi^2}{2d}}\Big)$$

where $\mathcal{D}_4(d,u)$ has property G.

On the other hand, $\frac{1}{d}\mathcal{T}_3(d,u)$ has property G. This implies

$$\mathcal{T}(d,u)=\frac{1}{d}\mathcal{D}_5(d,u)+\mathcal{K}(d,u),$$

where $\mathcal{D}_5(d,u)$ has property G and

$$\big|\mathcal{K}(d,u)\big| \le K d^3 e^{\frac{-\pi^2}{2d}} \quad \text{for} \quad (0<d<d_0,\, u_0<u<1). \tag{1.7.6}$$

Altogether we have proved that

$$\mathcal{T}(d,u) = \frac{1}{d}\int_0^\infty e^{-\frac{t}{d}}\delta(t,u)dt + \mathcal{K}(d,u),$$

where $\delta(.,u)$ is analytic in $|t| < \frac{\pi^2}{2}$, it is continuous on $[0,\frac{\pi^2}{2}[$ and $]\frac{\pi^2}{2},\infty[$ and has a limit as $t \longrightarrow \frac{\pi^2}{2}$ for every $u_0 < u < 1$ and satisfaies

$$|\delta(t,u)| \le K\, e^{Kt}, \tag{1.7.7}$$

with a constant K independent of u. Now if $\delta(t,u) = \sum_{n=0}^\infty \delta_n(t)t^n$ is the power series of $\delta(t,u)$ near $t=0$, Watson's lemma implies with (1.7.6) that

$$d\mathcal{T}(d,u) \sim \sum_{n=0}^\infty n\,!\delta_n d^{n+1}, \quad \text{as } d \searrow 0 \text{ for every } u_0 < u < 1.$$

On the other hand, because of its definition

$$\begin{aligned} \mathcal{T}(d,u) &\sim \frac{1}{2}B(d,T^+) - \frac{1}{2}B(d,T^-) - \varepsilon^2 + B(d,u)^2 \\ &\sim 0 + 0d + \cdots, \end{aligned}$$

because the formal series B satisfy (1.5.1). But this means that all δ_n must vanish and $\delta(t,u) \equiv 0$ if
$|t| < \frac{\pi^2}{2},\ u_0 < u < 1$. With (1.7.7), thus we obtain for $u_0 < u < 1$

$$\int_0^\infty e^{-\frac{t}{d}}|\delta(t,u)|dt \le \int_{\frac{\pi^2}{2}}^\infty e^{-\frac{t}{d}}Ke^{Kt}dt \le Kde^{-\frac{\pi^2}{2d}}, \quad (0<d<d_0).$$

We obtain

$$\big|\mathcal{T}(d,u)\big| \le Ke^{-\frac{\pi^2}{2d}} \quad (0<d<d_0).$$

Finally, we have shown that there is a constant K such that

$$\Big|\mathcal{A}(d,T^+) - \mathcal{A}(d,T^-) - f\big(\varepsilon,\mathcal{A}(d,u)\big)\Big| = \frac{1}{\varepsilon}\big|\mathcal{T}(d,u)\big| \le \frac{K}{d}e^{-\frac{\pi^2}{2d}},$$

for $0<d<d_0,\ 0 \le u < 1$.

1.8 Distance between points of manifolds

Let $y(t), t \in]-1, \infty[$ an exact solution to the difference equation (1.2.1), so that the stable manifold W_s^+ is parametrized by $t \mapsto \big(y(t), y(t+\varepsilon)\big)$. We know that $y^-(t) = -y^-(-t)$ is also an exact solution of (1.2.1) with $y^-(t) \to 1$ as $t \to -\infty$. Morever the instable manifold W_i^- is parametrized by $t \mapsto \big(y^-(t), y^-(t+\varepsilon)\big)$. We introduced the quasi-solution $\mathcal{A}(d,u)$ in previous section, then $\mathcal{A}^-(d,u) = -\mathcal{A}(d,-u)$ is a quasi-solution close to the exact solution $y^-(t)$. We denote by $\tilde{W}_s$, $\tilde{W}_i$ the manifolds close to W_s^+ respectively W_i^- parametrized by $t \mapsto \big(\xi_+(t), \xi_+(t+\varepsilon)\big)$ respectively $t \mapsto \big(\xi_-(t), \xi_-(t+\varepsilon)\big)$, where $\xi_+(t) = \mathcal{A}\big(d, u(t)\big)$ and $\xi_-(t) = \mathcal{A}^-\big(d, u(t)\big)$.

In this section we will show that the vertical distance between a point of the stable manifold and the manifold $\tilde{W}_s$ is exponentially small as well as that between $\tilde{W}_i$ and W_i^-; both are smaller, as we will show than the distance between $\tilde{W}_s$ and $\tilde{W}_i$. This eventually proves that W_s^+ and W_i^- do not coincide.

In order to estimate the distance between some point (x_0, y_0) on the stable manifold W_s^+, we consider the sequence $P_n = (x_n, y_n)$, defined by $P_{n+1} = \phi(P_n)$, where

$$\phi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x + 2\varepsilon(1-y^2) \end{pmatrix}. \tag{1.8.1}$$

$P_n = (x_n, y_n)$ is a point of the stable manifold. We obtain

$$P_{n+1} = \phi\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} y_n \\ x_n + 2\varepsilon(1-y_n^2) \end{pmatrix} \tag{1.8.2}$$

Since $\Phi = \phi \circ \phi$, where Φ is defined in (1.1.3), and the stable manifold W_s^+ is invariant by the application Φ, then the stable manifold W_s^+ remains invariant by ϕ. Then theorem 1.7.1 implies

$$\xi_+(t+\varepsilon) = \xi_+(t-\varepsilon) + 2\varepsilon\big(1 - \xi_+(t)^2\big) + \frac{1}{\varepsilon}\eta(t)e^{-\frac{\pi^2}{2\varepsilon}} \tag{1.8.3}$$

where $\eta(t)$ is bounded. There is a sequence $t_n \in]-1, +\infty[$ such that

$$\xi_+(t_n - \varepsilon) = x_n = \mathcal{A}\Big(d, T^-\big(d, u(t_n)\big)\Big) = \mathcal{A}\Big(d, T^-\big(d, U_\varepsilon(n)\big)\Big) \tag{1.8.4}$$

$$\begin{aligned} \xi_+(t_n) &= \mathcal{A}\big(d, U_\varepsilon(n)\big) \\ \xi_+(t_n + \varepsilon) &= \mathcal{A}\Big(d, T^+\big(d, U_\varepsilon(n)\big)\Big) \end{aligned} \tag{1.8.5}$$

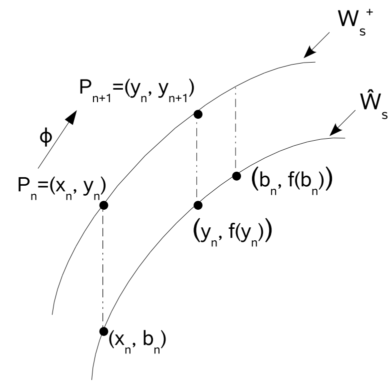

Figure 1.4: The vertical distance between a point of the stable manifold and the approximation defi ned using the quasi-solution $\mathcal{A}$ for the logistic equation .

where $U_\varepsilon(n) = u(t_n)$. The relation (1.8.4) implies that the point (x_n, b_n), where $b_n = \xi_+(t_n)$, the vertical projection of the point (x_n, y_n) on the curve $t \mapsto \big(\xi_+(t), \xi_+(t+\varepsilon)\big)$. But, according to (1.8.2), (1.8.4), we have

$$\xi_+(t_{n+1} - \varepsilon) = x_{n+1} = y_n. \tag{1.8.6}$$

Now consider the error quantities

$$d_\varepsilon(n) = y_n - \mathcal{A}\big(d, U_\varepsilon(n)\big) = y_n - b_n, \tag{1.8.7}$$

$d_\varepsilon(n)$ denotes the vertical distance between a point P_n and the manifold $\hat{W}_s$. The manifold $\tilde{W}_s^+$ is parameterized by the curve
$t \mapsto \big(\xi_+(t), \xi_+(t+\varepsilon)\big)$. We obtain

$$d_\varepsilon(n+1) = y_{n+1} - f(x_{n+1}) = y_{n+1} - f(y_n) = y_{n+1} - f\big(d_\varepsilon(n) + b_n\big), \tag{1.8.8}$$

where $f(x) = \xi_+\Big(\xi_+^{-1}(x) + \varepsilon\Big)$. The point $\big(y_n, f\big(y_n\big)\big)$ is the vertical projection of the point (x_{n+1}, y_{n+1}) on the curve $t \mapsto \big(\xi_+(t), \xi_+(t+\varepsilon)\big)$ (See FIG.4).

If we use Taylor expainson, (1.8.8) implies

$$d_\varepsilon(n+1) = y_{n+1} - f(b_n) - f'(\theta_n) d_\varepsilon(n), \qquad b_n < \theta_n < b_n + d_\varepsilon(n). \tag{1.8.9}$$

Then according to the definitions of f and b_n, we have $f(b_n) = \xi_+(t_n + \varepsilon)$, with (1.8.3) and (1.8.4), we obtain

$$f(b_n) = x_n + 2\varepsilon(1 - b_n^2) + \frac{1}{\varepsilon}\eta(t_n) e^{-\frac{\pi^2}{2\varepsilon}}. \tag{1.8.10}$$

Together with (1.8.9) and (1.8.2) this implies

$$d_\varepsilon(n+1) = -f'(\theta_n)d_\varepsilon(n) + 2\varepsilon(b_n^2 - y_n^2) - \frac{1}{\varepsilon}\eta(t_n)e^{-\frac{\pi^2}{2\varepsilon}}, \tag{1.8.11}$$

Using (1.8.7) we obtain

$$d_\varepsilon(n+1) = -\Big(f'(\theta_n) + 2\varepsilon(y_n + b_n)\Big)d_\varepsilon(n) - \frac{1}{\varepsilon}\eta(t_n)e^{-\frac{\pi^2}{2\varepsilon}}. \tag{1.8.12}$$

Now

$$f'(\theta_n) = \frac{\xi'_+(z_n+\varepsilon)}{\xi'_+(z_n)}, \quad \text{where} \quad z_n = \xi_+^{-1}(\theta_n),$$

then

$$f'(\theta_n) = 1 - 2u(z_n)d + \big(3u(z_n)^2 - 1\big)d^2 + O(d^3).$$

On the other hand $b_n < \theta_n < y_n$ implies

$$0 < b_n < \xi(z_n) < y_n$$

and thus

$$\begin{aligned} f'(\theta_n) + 2\varepsilon(y_n + b_n) &> 1 + 2\varepsilon\Big(\xi_+(z_n) + b_n\Big) - 2u(z_n)d \\ &+ \Big(3u(z_n)^2 - 1\Big)d^2 + O(d^3). \end{aligned} \tag{1.8.13}$$

We know that

$$\xi_+(z_n) = u(z_n) + \Big(u(z_n) - u(z_n)^3\Big)d^2 + O(d^4),$$

this with (1.2.8) implies

$$f'(\theta_n) + 2\varepsilon(y_n + b_n) > 1 + 2\varepsilon b_n + O(d^2).$$

As $b_n > 0$, then for sufficiently small d,

$$f'(\theta_n) + 2\varepsilon(y_n + b_n) > 1.$$

This implies

$$f'(\theta_n) + 2\varepsilon(y_n + b_n) = 1 + g_\varepsilon\big(u(z_n)\big), \quad \text{with some} \quad g_\varepsilon\big(u(z_n)\big) \geq \varepsilon c,$$

$g_\varepsilon\big(u(z_n)\big) = O(\varepsilon)$ and c is a positive constant. With (1.8.12), this implies

$$d_\varepsilon(n+1) = -\Big(1 + g_\varepsilon\big(u(z_n)\big)\Big) d_\varepsilon(n) - \frac{1}{\varepsilon}\eta(t_n) e^{-\frac{\pi^2}{2\varepsilon}}. \tag{1.8.14}$$

As $d_\varepsilon(n) \to 0$ as $n \to +\infty$, this implies that

$$|d_\varepsilon(n)| \leq \frac{1}{\varepsilon} e^{-\frac{\pi^2}{2\varepsilon}} \sum_{k=n}^{\infty} \left| \frac{\eta(s_k)}{\prod_i^{i=k} \Big(1 + g_\varepsilon\big(u(z_k)\big)\Big)} \right|, \tag{1.8.15}$$

provided this series converges. This follows from

$$\sum_{k=n}^{\infty} \left| \frac{\eta(s_k)}{\prod_i^{i=k} \Big(1 + g_\varepsilon\big(u(z_k)\big)\Big)} \right| \leq K_1 \sum_{k=0}^{+\infty} \big(1 + \varepsilon c\big)^{-k},$$

where we used that η is bounded. This imply that there is a constant $K > 0$ such that

$$\sum_{k=n}^{\infty} \left| \frac{\eta(s_k)}{\prod_i^{i=k} \Big(1 + g_\varepsilon\big(u(z_k)\big)\Big)} \right| \leq \frac{K}{\varepsilon}.$$

With (1.8.5) implies

$$d_\varepsilon(n) = O\bigg(\frac{1}{\varepsilon^2} \exp\Big(-\frac{\pi^2}{2\varepsilon}\Big)\bigg),$$

and in particular that the vertical distance $d_\varepsilon(0)$ of (x_0, y_0) to $\tilde{W}_s$ is

$$d_\varepsilon(0) = O\bigg(\frac{1}{\varepsilon^2} \exp\Big(-\frac{\pi^2}{2\varepsilon}\Big)\bigg). \tag{1.8.16}$$

Now we will calculate the vertical distance between the twomanifolds $\tilde{W}_s$ and $\tilde{W}_i$. To do this, we will need to extend both quasi-solutions in an adequate way. We define

$$\begin{aligned} \tilde{\xi}_+(t) &= \mathcal{A}\big(d, u(t)\big), \quad \text{for} \quad -\frac{1}{2} \leq u \leq 1, \\ \tilde{\xi}_-(t) &= \mathcal{A}^-\big(d, u(t)\big), \ \text{for} \quad -1 \leq u \leq \frac{1}{2}. \end{aligned} \tag{1.8.17}$$

First, we evaluate the quantities

$$d_\varepsilon\Big(\tilde{\xi}_+(t), \tilde{\xi}_-(t)\Big) = \mathcal{A}(d,u) - \mathcal{A}^-(d,u), \ \text{for} \ -\frac{1}{2} \leq u \leq \frac{1}{2} \tag{1.8.18}$$

using (1.7.2) and (1.7.4)

$$d_\varepsilon\Big(\tilde{\xi}_+(t),\tilde{\xi}_-(t)\Big) = \tilde{\xi}_+(t)-\tilde{\xi}_-(t), \tag{1.8.19}$$
$$= \frac{1}{\varepsilon}(\alpha+\beta d^2+\gamma d^4)\big(\tilde{\mathcal{H}}(d,u)-\tilde{\mathcal{H}}^-(d,u)\big),$$

where

$$\tilde{\mathcal{H}}(d,u) = \int_0^\infty e^{-\frac{s}{d}}h(s,u)\,ds, \quad \text{for} \quad -\frac{1}{2}\le u\le\frac{1}{2},$$
$$\tilde{\mathcal{H}}^-(d,u) = -\int_0^\infty e^{-\frac{s}{d}}h(s,-u)\,ds, \quad \text{for} \quad -\frac{1}{2}\le u\le\frac{1}{2}. \tag{1.8.20}$$

Modifying the path of integral, we obtain

$$\tilde{\mathcal{H}}(d,u) = \int_\Gamma e^{-\frac{s}{d}}h(s,u)\,ds + \sum_{Im(s_k(u))<0} 2\pi i Res\Big(e^{-\frac{s}{d}}h(s,u),s_k(t)\Big)$$
$$- \sum_{Im(s_k(u))>0} 2\pi i Res\Big(e^{-\frac{s}{d}}h(s,u),s_k(t)\Big),$$

where $s_k(t)=\frac{\pi^2}{2}\pm\frac{d\pi\, t}{\varepsilon}\,i+k\pi^2$ for $k\ge 0$, Γ is the sum of two paths Γ_1 and Γ_2. Here Γ_1 consist of two segments of the higher half-plane, one of these two segments is parallel to the axis $y=0$ and begins at the point $(\frac{\pi^2}{4},1)$, the other ends in the point (0,0), and the path Γ_2 is the symmetry of Γ_1 relatively to the axis $y=0$.

This implies

$$\tilde{\mathcal{H}}(d,u) = \int_\Gamma e^{-\frac{s}{d}}h(s,u)\,ds + \frac{\pi e^{\frac{\pi\, t}{\varepsilon}i}}{2d^2(1-u^2)}\sum_{k=0}^{\infty}\exp\left(-\frac{(k+1)\pi^2}{2d}\right)$$
$$+ \frac{\pi e^{-\frac{\pi\, t}{\varepsilon}i}}{2d^2(1-u^2)}\sum_{k=0}^{\infty}\exp\left(-\frac{(k+1)\pi^2}{2d}\right)$$

Therefore

$$\tilde{\mathcal{H}}(d,u) = \int_\Gamma e^{-\frac{s}{d}}h(s,u)\,ds + \frac{\pi\cos(\frac{\pi\, t}{\varepsilon})}{d^2(1-u^2)}\sum_{k=0}^{\infty}\exp\left(-\frac{(k+1)\pi^2}{2d}\right). \tag{1.8.21}$$

Similary for $\tilde{\mathcal{H}}^-(d,u)$

$$\tilde{\mathcal{H}}^-(d,u) = \int_\Gamma e^{-\frac{s}{d}}h(s,u)\,ds - \frac{\pi\cos(\frac{\pi\, t}{\varepsilon})}{d^2(1-u^2)}\sum_{k=0}^{\infty}\exp\left(-\frac{(k+1)\pi^2}{2d}\right).$$

Therefore

$$d_\varepsilon\Big(\tilde{\xi}_+(t), \tilde{\xi}_-(t)\Big) = \frac{2\pi(\alpha+\beta d^2+\gamma d^4)\cos(\frac{\pi t}{\varepsilon})}{\varepsilon^3\,(1-u^2)} \exp\left(-\frac{\pi^2}{2d}\right) + \mathcal{O}\left(\frac{\cos(\frac{\pi t}{\varepsilon})}{\varepsilon^3} e^{-\frac{3\pi^2}{2d}}\right),$$

for $-\text{arctanh}(\frac{1}{2}) < t < \text{arctanh}(\frac{1}{2})$. Consequently

$$d_\varepsilon\Big(\tilde{\xi}_+(t), \tilde{\xi}_-(t)\Big) = \frac{2\pi\alpha\cos(\frac{\pi t}{\varepsilon})}{\varepsilon^3\big(1-\tanh(\frac{d}{\varepsilon}t)^2\big)} e^{-\frac{\pi^2}{2\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon}\cos(\frac{\pi\, t}{\varepsilon}) e^{-\frac{\pi^2}{2d}}\right) \tag{1.8.22}$$

Now we want to estimate the vertical distance between a point on the quasimanifold $\tilde{W}_s$ and its vertical projection on the quasimanifold $\tilde{W}_i$. For this purpose let t such that $-\frac{1}{2} \le u(t) \le \frac{1}{2}$. Then the point $\Big(\tilde{\xi}_+(t), \tilde{\xi}_+(t+\varepsilon)\Big)$ is on the quasimanifold $\tilde{W}_s$. We suppose that $\Big(\tilde{\xi}_+(t), \tilde{\xi}^-(t_1+\varepsilon)\Big)$, where $-\frac{1}{2} \le u(t_1) \le \frac{1}{2}$, is its vertical projection on the quasimanifold $\tilde{W}_i$, then the vertical distance between these two points is

$$dist_v(t) = \tilde{\xi}_+(t+\varepsilon) - \tilde{\xi}_-(t_1+\varepsilon) = \tilde{\xi}_+(t+\varepsilon) - \tilde{f}\big(\tilde{\xi}_+(t)\big), \tag{1.8.23}$$

where $\tilde{f}(x) = \tilde{\xi}_-\Big(\tilde{\xi}_-^{-1}(x)+\varepsilon\Big)$, $\tilde{\xi}_-^{-1}$ the inverse function of $\tilde{\xi}_-$. Using (1.8.22) and (1.8.20) we obtain

$$dist_v(t) = \tilde{\xi}_+(t+\varepsilon) - \tilde{f}\big(\tilde{\xi}_-(t) + e_\varepsilon(t)\big),$$

where $e_\varepsilon(t) = d_\varepsilon\Big(\tilde{\xi}_+(t), \tilde{\xi}_-(t)\Big)$. We use Taylor expansion and obtain

$$\begin{aligned} dist_v(t) &= \tilde{\xi}_+(t+\varepsilon) - \tilde{f}\big(\tilde{\xi}_-(t)\big) - e_\varepsilon(t)\cdot \tilde{f}'(\theta), \\ &= \tilde{\xi}_+(t+\varepsilon) - \tilde{\xi}_-(t+\varepsilon) - e_\varepsilon(t)\cdot \tilde{f}'(\theta), \end{aligned} \tag{1.8.24}$$

where $\tilde{\xi}_-(t) < \theta < \tilde{\xi}_-(t) + e_\varepsilon(t)$. With (1.8.20) this implies

$$dist_v(t) = e_\varepsilon(t+\varepsilon) - e_\varepsilon(t)\cdot \tilde{f}'(\theta). \tag{1.8.25}$$

Since $e_\varepsilon(t+\varepsilon) = -e_\varepsilon(t) + \mathcal{O}\Big(\varepsilon\cdot e_\varepsilon(t)\Big)$ and $\tilde{f}'(\theta) = \dfrac{\tilde{\xi}_-'\Big(\tilde{\xi}_-^{-1}(\theta)+\varepsilon\Big)}{\tilde{\xi}_-'\Big(\tilde{\xi}_-^{-1}(\theta)\Big)} = 1 + \mathcal{O}(\varepsilon)$, we obtain

$$dist_v(t) = \big(-2 + \mathcal{O}(\varepsilon)\big)e_\varepsilon(t). \tag{1.8.26}$$

Consequently

$$dist_v(t) \sim -\frac{4\pi\alpha\cos(\frac{\pi\, t}{\varepsilon})}{\varepsilon^3\big(1-\tanh(\frac{d}{\varepsilon}t)^2\big)}e^{-\frac{\pi^2}{2\varepsilon}}. \tag{1.8.27}$$

Combining (1.8.16) and (1.8.27), we conclude

$$Dist_\varepsilon\big(w_s^+(t), W_i^-\big) = \frac{4\pi\alpha\cos(\frac{\pi}{\varepsilon}t+\pi)}{\varepsilon^3\big(1-\tanh(\frac{d}{\varepsilon}t)\big)}e^{\frac{-\pi^2}{2\varepsilon}} + O\left(\frac{1}{\varepsilon^2}e^{\frac{-\pi^2}{2\varepsilon}}\right), \quad \text{as } \varepsilon \searrow 0,$$

1.9 An estimate for the main asymptotic coefficients

In this section we show that the number α in the main theorem of the introduction is not zero and we want to find bounds for α_n and will then obtain estimats for α using the first α_n which can be computed explicitly. We recall that

$$\alpha = \frac{1}{\pi}\sum_{n=7}^{\infty}\alpha_n \tag{1.9.1}$$

where $\alpha_n(n-1)\,!\big(\frac{i}{\pi}\big)^{n-1}u^n$ is the leading term of the coefficient $\{\mathcal{C}(G)\}_n(u)$ of d^n in $\mathcal{C}(G)(d,u)$, where $G = Q(d,u).F(d,u)$, $F(d,u) = B(d,u) - U(d,u)$ (cf.(1.5.2) and (1.5.5)).

For a polynomial series $X(d,u) = \sum_{n=0}^{\infty} X_n(u)d^n$ where the degrees of $X_n(u)$ do not exceed n, we can write $X_n(u) = \sum_{k=0}^{n} x_{nk}u^n$. Then we denote by $\widehat{X}(z) = \sum_{n=0}^{\infty} x_{nn}z^n$.

The mapping $X(d,u) \mapsto \widehat{X}(z)$ extracts the leading terms of the series $X(d,u)$. It is compatible with addition and multiplication. If we define the following operator

$\widehat{D} = -z^2 \frac{d}{dz}$, we also find

$$
\begin{aligned}
\widehat{D}\widehat{X}(z) &= \widehat{DX}(z) = -z^2 \widehat{X}'(z) \\
\widehat{\mathcal{C}(X)}(z) &= \cosh\left(\frac{\widehat{D}}{2}\right)\widehat{X}(z) := \frac{1}{2}\left(\widehat{X}\Big(\frac{2z}{2+z}\Big) + \widehat{X}\Big(\frac{2z}{2-z}\Big)\right) \\
\widehat{\mathcal{S}(X)}(z) &= \sinh\left(\frac{\widehat{D}}{2}\right)\widehat{X}(z) := \frac{1}{2}\left(\widehat{X}\Big(\frac{2z}{2+z}\Big) - \widehat{X}\Big(\frac{2z}{2-z}\Big)\right) \\
\widehat{\mathcal{C}_2(X)}(z) &= \cosh(\widehat{D})\widehat{X}(z) := \frac{1}{2}\left(\widehat{X}\Big(\frac{z}{1+z}\Big) + \widehat{X}\Big(\frac{z}{1-z}\Big)\right) \\
\widehat{\mathcal{S}_2(X)}(z) &= \sinh(\widehat{D})\widehat{X}(z) := \frac{1}{2}\left(\widehat{X}\Big(\frac{z}{1+z}\Big) - \widehat{X}\Big(\frac{z}{1-z}\Big)\right)
\end{aligned}
$$

where the operators $\mathcal{S}, \mathcal{C}, \mathcal{S}_2, \mathcal{C}_2$ are defined in (1.2.18). If we apply the mapping $X(d,u) \mapsto \widehat{X}(z)$ to the equation (1.5.13) , we obtain

$$
\widehat{\mathcal{L}}\left(\frac{\widehat{E}}{\widehat{Q}_1}\right)(z) = \widehat{\mathcal{R}}\big(\widehat{E}\big)(z), \tag{1.9.2}
$$

where

$$
\begin{aligned}
\widehat{\mathcal{L}}(x) &:= -\frac{\widehat{V}_1}{z^2}\left(\widehat{\mathcal{S}}(x) - \frac{\widehat{\mathcal{S}}(z^2)}{\widehat{\mathcal{C}}(z^2)}\widehat{\mathcal{C}}(x)\right) \\
\widehat{\mathcal{R}}(x) &:= -\frac{\widehat{\mu}}{z^2} + \frac{\widehat{W}_5}{z^2}\widehat{\mathcal{C}}\left(\frac{x}{\widehat{Q}_1}\right) - \frac{\widehat{W}_1}{2z^2\widehat{Q}}\mathcal{Y} + \frac{\widehat{W}_4}{2z^2\widehat{Q}^2}\widehat{\mathcal{C}}\left(\frac{x}{\widehat{Q}_1}\right)\mathcal{Y}^2 \\
&- \frac{\widehat{W}_2}{2z^2}\widehat{\mathcal{C}}_2\left(\frac{\mathcal{Y}}{\widehat{Q}}\right), \\
\widehat{E} &= \widehat{\mathcal{C}(G)} = \sum_{n=0\,\text{odd}}^{\infty} \alpha_n (n-1)\,!\left(\frac{i}{\pi}\right)^{n-1} z^n, \\
\mathcal{Y} &= \widehat{\mathcal{C}}^{-1}(x), \\
\widehat{Q} &= -z^2 + z^4 - \frac{13}{6}z^6 + \frac{47}{18}z^8, \\
\widehat{Q}_1 &= -z^2 - \frac{3}{2}z^4, \\
\widehat{V}_1 &= 4z^2\Big[-192 - 240z^2 + 508z^4 + 1995z^6 + 3418z^8 \\
&+ 4389z^{10} + 217z^{12}\Big]\Big/\Big[3(z^2-4)^4\Big], \\
\widehat{W}_2 &= z^{11}\Big[591 - 15457z^2 + 21610z^4 + 59194z^6 + 36583z^8 \\
&+ 2759z^{10}\Big]\Big/\Big[27(z^2-1)^8\Big], \\
\widehat{W}_1 &= z^{11}\Big[2385 - 32832z^2 - 11790z^4 + 231917z^6 - 116267z^8 + 720682z^{10} \\
&+ 1018936z^{12} + 290089z^{14} + 2480z^{16}\Big]\Big/\Big[81(z^2-1)^8\Big], \\
\widehat{W}_4 &= \widehat{V}\widehat{\mathcal{C}}_2(\widehat{Q}) + \widehat{W}\widehat{\mathcal{S}}_2(\widehat{Q}) \\
&= -z^2\Big[54 - 432z^2 + 1449z^4 - 2724z^6 + 4449z^8 - 14719z^{10} - 12857z^{12} \\
&+ 95146z^{14} + 110905z^{16} + 29041z^{18} + 248z^{20}\Big]\Big/\Big[54(z^2-1)^8\Big], \\
\widehat{W}_5 &= \widehat{W}_3 + \widehat{P}\widehat{V}_3, \\
&= -8z^5\Big[-1152 + 1168z^2 + 9744z^4 + 18901z^6 + 14664z^8 \\
&+ 1240z^{10}\Big]\Big/\Big[3(z^2-4)^4(4+z^2)\Big],
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\mu} &= z^{10}\Big[-527000z^{30}-1988300z^{28}+14763570z^{26}-25488145z^{24}\\
&+ 11808105z^{22}+11381027z^{20}-17065562z^{18}-15479497z^{16}-15777878z^{14}\\
&- 21499321z^{12}+14844222z^{10}+4847178z^{8}-1419003z^{6}-541026z^{4}\\
&+ 320490z^{2}-50220\Big]\Big/\Big[972(z^2-1)^{12}\Big], \qquad (1.9.3)
\end{aligned}
$$

with $x = x(z)$ is a power series.

Now for integers $m > p \geq -1,\ m \geq 0$, we introduce the set

$$
\mathcal{M}(m,p) = \Big\{ x_n z^n \backslash\ x_n \big/ \big[(n-p)\,!\pi^{-n}\big], n = m, m+2... \text{ is bounded} \Big\}
$$

$\mathcal{M}(m,p)$ is a Banach space with the norm

$$
|x|_{m,p} = \sup\big\{\,|x_n|\big/\big[(n-p)\,!\pi^{-n}\big], n \geqq m, n \equiv m \mod 2\big\}. \qquad (1.9.4)
$$

Here the symbol for the summation means that only summands with $n \equiv m \mod 2$ are considered.

Proposition 1.9.1. *Let m,p,q be integers and* $m \geq p \geq -1, m \geq q \geq -1, m \geq 0$. *The norms (1.9.4) have the following property:*

1. *If* $q < p \leq m$ *then* $\mathcal{M}(m,p) \subset \mathcal{M}(m,q)$ *and*

$$
|x|_{m,q} \leq \frac{(m-p)\,!}{(m-q)\,!}|x|_{m,p} \quad \textit{for } x \in \mathcal{M}(m,p) \qquad (1.9.5)
$$

2. *If* $x \in \mathcal{M}(m,p)$ *then* $zx \in \mathcal{M}(m+1,p+1)$ *and*

$$
|zx|_{m+1,p+1} = \pi|x|_{m,p} \qquad (1.9.6)
$$

3. $\widehat{D} : \mathcal{M}(m,p) \mapsto \mathcal{M}(m+1,p)$ *and*

$$
\pi|x|_{m,p} \leq |\widehat{D}(x)|_{m+1,p} \leq \pi\frac{m}{m-p+1}|x|_{m,p} \qquad (1.9.7)
$$

Now we consider an even power series $f(z) = \sum_{j=0}^{\infty} * f_j z^j$ having a radius of convergence greater than π. Then $f(\widehat{D})$ maps $\mathcal{M}(m,p)$ into itself and

$$|f(\widehat{D})x|_{m,p} \leq \|f\|_{m,p}|x|_{m,p}, \quad \text{for } x \in \mathcal{M}(m,p) \tag{1.9.8}$$

where

$$\begin{aligned} \|f\|_{m,p} &= \sup\left\{|f_j|\pi^j \prod_{k=1}^{p-1} \frac{n-k}{n-j-k} \backslash n \geq m, n \equiv m \mod 2\right\} \text{ if } p \geq 1, \\ \|f\|_{m,p} &= \sup\left\{|f_j|\pi^j \backslash n \geq m, n \equiv m \mod 2\right\} \text{ if } p \leq 0. \end{aligned} \tag{1.9.9}$$

We have also

$$|f.x|_{m,p} \leq M_{m,p}(f)|x|_{m,p}, \quad \text{for } x \in \mathcal{M}(m,p), \tag{1.9.10}$$

where

$$M_{m,p}(f) = \sup\left\{|f_j|\pi^j \frac{(n-p-j)\,!}{(n-p)\,!} \backslash n \geq m, n \equiv m \mod 2\right\}. \tag{1.9.11}$$

We conclude that the inverse of $\widehat{\mathcal{C}}$ maps $\mathcal{M}(m,p)$ with $p \geq 3$ into $\mathcal{M}(m,1)$ and

$$|\widehat{\mathcal{C}}^{-1}(x)|_{m,1} \leq c_{m,p}|x|_{m,p}, \text{ with } c_{m,p} = \frac{4}{\pi}\frac{(n-p)\,!}{(n-1)\,!}. \tag{1.9.12}$$

Let $x \in \mathcal{M}(m,p)$ and $y \in \mathcal{M}(m,q)$ and assume that $m+q \leq n+p$. Then we obtain $x.y \in \mathcal{M}(n+m, q+m)$ and

$$|x.y|_{n+m,q+m} \leq \alpha_{m-p,n-q}|x|_{m,p}|y|_{n,q}, \tag{1.9.13}$$

with

$$\alpha_{r,s} = \sup\left\{\frac{j\,!(N-j)\,!}{(N-r)\,!} \backslash N \geq r+s, N-r-s \text{ even}\right\}.$$

Finally we need estimates for the inverse of the linear operator $\widehat{\mathcal{L}}$ in (1.9.2). $\widehat{\mathcal{L}}$ maps $\mathcal{M}(m-1,q)$ onto $\mathcal{M}(m,q)$ and is one to one.Now let $x \in \mathcal{M}(m-1,p)$ and $w = \widehat{\mathcal{L}}(x)$. If we proceed similar to the proof of theorem 1.4.6, we find that

$$\begin{aligned} \widehat{D}y &= f(\widehat{D})\left[-\frac{(4+z^2)}{4\widehat{V}_1}w\right], \\ x &= z^2 y. \end{aligned} \tag{1.9.14}$$

where $f(z) = z/\sinh(z/2)$.

We can apply (1.9.6)..(1.9.13) and obtain for $m \geq \max(p+1, 4)$

$$|\widehat{\mathcal{L}}^{-1}(w)|_{m-1,p} \leq L_{m,p}|w|_{m,p} \quad \text{for } w \in \mathcal{M}(m,p), \tag{1.9.15}$$

where

$$L_{m,p} = \frac{1}{\pi}\left\|\frac{z}{\sinh(z/2)}\right\|_{m-2,p-2} M_{m,p}\left(\frac{(4+z^2)z^2}{4\widehat{V}_1}\right). \tag{1.9.16}$$

Now we are in a position to use (1.9.2) to estimate $\widehat{E}$. We use that $\widehat{E}$ can be computed recursively from (1.9.2). If x is an odd power series and $x \equiv \widehat{E} \mod z^n$ with $n \geq 9$ then via $\widehat{\mathcal{C}}^{-1}(x) \equiv \widehat{G} \mod z^n$ we first obtain $\widehat{\mathcal{R}}(x) \equiv \widehat{\mathcal{R}}(\widehat{E}) \mod z^{n+1}$ and then $\widehat{\mathcal{L}}^{-1}\Big(\widehat{\mathcal{R}}(x)\Big) \equiv \frac{\widehat{E}}{\widehat{Q}_1} \mod z^{n+1}$.Thus if $x_0 \equiv 0 \mod z^9$ is arbitrary and we let

$$x_{n+1} := \widehat{Q}_1\widehat{\mathcal{L}}^{-1}\Big(\widehat{\mathcal{R}}(x_n)\Big) \quad \text{for } n = 0... \tag{1.9.17}$$

then $x_n \equiv \widehat{E} \mod z^{9+2n}$ for all n. We begin with the polynomial x_0 of degree ≤ 79 with $x_0 \equiv \widehat{E} \mod z^{81}$. We computed it explicitly and found that

$$|x_0|_{9,7} \leq K_0 := 308027.359777894414. \tag{1.9.18}$$

Then we consider the sequence x_n of polynomial series defined in (1.9.17), which now satisfies $x_n \equiv \widehat{E} \mod z^{81+2n}$. We will show

$$|x_n|_{9,7} \leq K_0 \quad \text{for all } n \geq 0 \tag{1.9.19}$$

and this implies $|\widehat{E}|_{9,7} \leq K_0$ because each coefficient of $\widehat{E}$ is also that of some x_n if n is sufficiently large.

We prove (1.9.19) by induction. For $n = 0$ it is true. Assume that $|x_n|_{9,7} \leq K_0$, for this Let us take again the equation (1.9.18) with (1.9.2). We obtain

$$\begin{aligned} x_{n+1} \quad &+ \quad \widehat{Q}_1\widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{\mu}}{z^2}\right) = \widehat{Q}_1\widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{W_5}}{z^2}\widehat{\mathcal{C}}\left(\frac{x_n}{\widehat{Q}_1}\right)\right) - \widehat{Q}_1\widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{W_1}}{2z^2\widehat{Q}}\mathcal{Y}_n\right) \\ &+ \quad \widehat{Q}_1\widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{W_4}}{2z^2\widehat{Q}^2}\widehat{\mathcal{C}}\left(\frac{x_n}{\widehat{Q}_1}\right)\mathcal{Y}_n^2\right) - \widehat{Q}_1\widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{W_2}}{2z^2}\widehat{\mathcal{C}}_2\left(\frac{\mathcal{Y}_n}{\widehat{Q}}\right)\right) \end{aligned} \tag{1.9.20}$$

where $\mathcal{Y}_n = \widehat{\mathcal{C}}^{-1}(x_n)$. Then using (1.9.2)..(1.9.17), we obtain

$$\begin{aligned}
\left| \widehat{Q}_1 \widehat{\mathcal{L}}^{-1} \left(\frac{\widehat{W}_5}{z^2} \widehat{\mathcal{C}} \left(\frac{x_n}{\widehat{Q}_1} \right) \right) \right|_{11,10} &\leq K_1 |x_0|_{9,7}, \\
\left| \widehat{Q}_1 \widehat{\mathcal{L}}^{-1} \left(\frac{\widehat{W}_1}{2z^2 \widehat{Q}} \mathcal{Y}_n \right) \right|_{17,10} &\leq K_2 |x_0|_{9,7}, \\
\left| \widehat{Q}_1 \widehat{\mathcal{L}}^{-1} \left(\frac{\widehat{W}_2}{2z^2} \widehat{\mathcal{C}}_2 \left(\frac{\mathcal{Y}_n}{\widehat{Q}} \right) \right) \right|_{17,10} &\leq K_3 |x_0|_{9,7}, \\
\left| \widehat{Q}_1 \widehat{\mathcal{L}}^{-1} \left(\frac{\widehat{W}_4}{2z^2 \widehat{Q}^2} \widehat{\mathcal{C}} \left(\frac{x_n}{\widehat{Q}_1} \right) \mathcal{Y}_n^2 \right) \right|_{15,10} &\leq K_4 |x_0|_{9,7}^2
\end{aligned}$$

where

$$\begin{aligned}
K_1 : &= \pi^2 M_{9,8} \left(1 + \frac{3}{2} z^2 \right) L_{10,8} M_{7,5} \left(\frac{\widehat{W}_5}{z^5} \right) M_{9,7} \left(\frac{z^2}{\widehat{Q}_1} \right) \left\| \cosh \left(\frac{z}{2} \right) \right\|_{7,5}, \\
K_2 : &= \pi^9 M_{15,8} \left(1 + \frac{3}{2} z^2 \right) L_{16,8} M_{9,1} \left(\frac{\widehat{W}_1}{2z^9 \widehat{Q}} \right) \; c_{9,7}, \\
K_3 : &= \pi^9 M_{15,8} \left(1 + \frac{3}{2} z^2 \right) L_{16,8} M_{7,-1} \left(\frac{\widehat{W}_2}{2z^{11}} \right) \| \cosh(z) \|_{7,-1} M_{9,1} \left(\frac{z^2}{\widehat{Q}} \right) \; c_{9,7}, \\
K_4 : &= \frac{1}{56\pi^2} M_{13,8} \left(1 + \frac{3}{2} z^2 \right) L_{14,8} M_{14,8} \left(\frac{\widehat{W}_4}{2z^2} \right) M_{18,12} \left(\frac{z^4}{\widehat{Q}^2} \right) \alpha_{8,8} \;\; c_{9,7}^2.
\end{aligned}$$

This imply

$$\left| x_{n+1} + \widehat{Q}_1 \widehat{\mathcal{L}}^{-1} \left(\frac{\widehat{\mu}}{z^2} \right) \right|_{11,10} \leq \big(K_1 + K_2 + K_3 + K_4 K_0 \big) K_0$$

Evaluation of these quantities yields approximately

K_1	K_2	K_3	K_4
375142.8501573	633.1622891589	10214.44411459	0.000011150281

and altogether

$$\left| x_{n+1} + \widehat{Q}_1 \widehat{\mathcal{L}}^{-1} \left(\frac{\widehat{\mu}}{z^2} \right) \right|_{11,10} \leq 118896679183. \qquad (1.9.21)$$

Now we consider $\mathrm{g}(z) := z^2\widehat{\mathcal{L}}^{-1}\left(\frac{\widehat{\mu}}{z^2}\right)$, letting $z^2y = z^{-2}\mathrm{g}(z)$, with (1.9.15) we obtain

$$\widehat{D}y = \widehat{D}\big(z^{-4}\mathrm{g}\big) = f\big(\widehat{D}\big)\left[\frac{\widehat{\mu}(z)(4+z^2)}{4z^4\widehat{V}_1}\right], \tag{1.9.22}$$

where

$$f(z) := \frac{z}{\sinh(z/2)}.$$

The coefficient of z^k in $f(z)$ is smaller than $4(2\pi)^k$ we obtain by comparison of the coefficients of z^j in (1.9.22)

$$(j-1)|\mathrm{g}_{j+3}| \leq 4\sum_{k=4}^{j} * \,(2\pi)^{k-j}\, h_k \frac{(j-1)\,!}{(k-1)\,!}$$

and hence

$$\frac{|\mathrm{g}_j|\pi^j 2^j}{(j-2)\,!} \leq 4\sum_{k=4}^{\infty} \frac{h_k(2\pi)^k}{(k-1)\,!} \leq 3668333.$$

We obtain for add $j \geq 9$

$$\frac{|\mathrm{g}_j|}{(j-10)\,!\pi^{-j}} \leq \pi^3 \cdot 3668333 \cdot (j-9)(j-8)(j-7)(j-6)(j-5)2^{-j+3}.$$

and

$$\frac{|\mathrm{g}_{j-2}|}{(j-10)\,!\pi^{-j}} \leq \pi^5 \cdot 3668333 \cdot (j-9)(j-8)(j-7)2^{-j+5}.$$

but, $\left\{\frac{\widehat{Q}_1}{z^2}\mathrm{g}(z)\right\}_j = \mathrm{g}_j + \frac{3}{2}\mathrm{g}_{j-2}$, with (1.9.20) this yields eventually

$$\frac{\big|\{x_{n+1}\}_j\big|}{(j-10)\,!\pi^{-j}} \leq 118896679184 \quad \text{for } \ j \geq 81.$$

This implies

$$\frac{\big|\{x_{n+1}\}_j\big|}{(j-7)\,!\pi^{-j}} \leq \frac{118896679184}{72 \cdot 73 \cdot 74} \leq 305691 \quad \text{for } \ j \geq 81$$

and with $x_{n+1} \equiv x_n \mod z^{81}$ we finally proved that $|x_{n+1}|_{9,7} \leq K_0 = 308027.35$. Thus the proof of (1.9.18) is complete.

Theorem 1.9.2.

$$|\alpha_n| \le \frac{98048.15}{(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)} \quad \textit{for} \;\; n \ge 9.$$

and even better

$$|\alpha_n| \le 37845988419 \frac{(n-10)\,!}{(n-1)\,!} \quad \textit{for odd} \;\; n \ge 81.$$

As a consequence we obtain estimates for α

$$\begin{aligned}
\left|\alpha - \frac{1}{\pi}\alpha_n\right| &\le 12046752267 \sum_{n=N}^{\infty} \frac{(n-10)\,!}{(n-1)\,!} \quad \text{for} \;\; n \ge 9, \\
&\le 12046752267 \cdot \frac{(N-10)\,!}{16(N-2)\,!}\left(1 + \frac{8}{N-1}\right), \\
&\le \frac{762333542}{N(N-1)(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}
\end{aligned}$$

We evaluated α_n for $n = 9, 11, ..., 81$ and obtain

$$|\alpha - 1.264150331| \le 6 \cdot 10^{-7}. \tag{1.9.23}$$

Chapter 2

On the distance between separatrices for the discretized pendulum equation

2.1 Introduction

We consider the following difference equation

$$q(t+\varepsilon) + q(t-\varepsilon) - 2q(t) = \varepsilon^2 \sin\big(q(t)\big). \tag{2.1.1}$$

This second order equation is a discretization of the pendulum equation $q'' = \sin(q)$. It is equivalent to the following system of first order difference equations

$$\begin{cases} q(t+\varepsilon) = q(t) + \varepsilon p(t+\varepsilon), \\ p(t+\varepsilon) = p(t) + \varepsilon \sin\big(q(t)\big). \end{cases} \tag{2.1.2}$$

which can be considered as a discretization of the system

$$\begin{cases} q' = p, \\ p' = \sin(q). \end{cases} \tag{2.1.3}$$

The latter system has two saddle points at $A = (0,0)$, $B = (2\pi, 0)$ and there exist stable and unstable manifolds. For the discretized equation (2.1.2) and sufficiently small $\varepsilon > 0$, these manifolds still exist.

The system (2.1.3) has $\big(q_0(t), q_0^{'}(t)\big)$, where $q_0(t) = 4\arctan\big(e^{-t}\big)$, as a heteroclinic orbit connecting the stationary points B and A; it is a parametrisation of the curve $p = -2\sin(q/2)$ and contains the stable manifold of (2.1.3) at the point A as well as its unstable manifold at B. This curve, together with $p = 2\sin(q/2)$, separates regions with periodic orbits from regions with non-periodic orbits and is therefore often called a *separatrix*. Our purpose is study the behavior of this separatrix under discretization of the equation – it turns out that there is no longer a heteroclinic orbit for system (2.1.2) and its the stable manifold at A and the unstable manifold at B no longer coincide. More precisely, we want to estimate the distance between the stable manifold $W^-_{s,\varepsilon}$ of (2.1.2) at A and the unstable manifold $W^+_{u,\varepsilon}$ of (2.1.2) at B as a function of the parameter ε.

Lazutkin et. al. [9], Gelfreich [4], (see also Lazutkin [7][8]) had given an asymptotic estimate of the splitting angle between the manifolds. Starting from a heteroclinic solution of the differential equation, they study the behavior of analytic solutions of the difference equation in the neighbourhood of its singularities $t = \pm\frac{\pi}{2}i$.

We show that the distance between these two manifolds is exponentially small but not zero and we give an asymptotic estimate of this distance. This result is similar to that of Lazutkin et. al. [9]; our method of proof, however, is quite different.

We use a method adapted from the article of Schäfke-Volkmer [10] using a formal power series solution and accurate estimates of the coefficients. This method was adapted for the logistic equation in Sellama[11]. It turns out that the adaptation of this method for the pendulum equation is more difficult than in the case of the logistic equation.

We will show

Theorem 2.1.1. *Given any positive t_0, it is known that for sufficiently smal $\varepsilon_0 > 0$ and all $t \in]-\infty, t_0]$ there is exactly one one point $w^+_{u,\varepsilon}(t) = (q_0(t), \tilde{p}^+_{u,\varepsilon}(t))$ on the stable unstable manifold having first coordinate $q_0(t)$. There exist constants $\alpha \neq 0$, such that for any positive t_0*

$$\mathrm{Dist}_v\big(w^+_{s,\varepsilon}(t), W^-_{s,\varepsilon}\big) = \frac{4\pi\alpha}{\varepsilon^2}\cosh(t)\sin\left(\frac{2\pi t}{\varepsilon}\right)e^{-\frac{\pi^2}{\varepsilon}} + O\left(\frac{1}{\varepsilon}e^{-\frac{\pi^2}{\varepsilon}}\right), \quad \textit{as } \varepsilon \searrow 0,$$

uniformly for $-t_0 < t < t_0$, where$\mathrm{Dist}_v\big(P, W^-_{u,\varepsilon}\big)$ denotes the vertical distance of a point P from the unstable manifolds $W^-_{u,\varepsilon}$.

This result corresponds to the result of Lazutkin et. al. [9] as the angle between the manifolds at an intersection point is asymptotically equivalent to $\frac{1}{q_0'(t)}\frac{d}{dt}\mathrm{Dist}_v\big(w^+_{s,\varepsilon}(t), W^-_{u,\varepsilon}\big)$, but we do not want to give any detail here.

Our proof uses the following steps. First, we construct a formal solution for the difference equation (2.1.1) in the form of a power series in $d = 2\mathrm{arsinh}(\varepsilon/2)$, whose coefficients are polynomials in $u = \tanh(dt/\varepsilon)$. This is done in section 2; the introduction of d is necessary because polynomials are desired as coefficients. Then, we give asymptotic approximations of these coefficients using appropriate norms on spaces of polynomials. To that purpose we introduce operators on polynomials series. In section 6 we use the truncated Laplace transform to construct a function which satisfies (2.1.1) except for an exponentially small error. The next and last step is to give an asymptotic estimate for the distance of some point of the stable manifold from the unstable manifold. A calculation shows that $\alpha = 89.0334$ and therefore $4\pi\alpha = 1118.8267$ (See Remark 2.5.4); the corresponding constants of Lazutkin have already been calculated with high precision (See Lazutkin et. al. [9]). A *proof* that $\alpha \neq 0$ as in [10] or [11] would be possible. Y.B. Suris [12] had shown that $\alpha \neq 0$.

2.2 Formal solutions

The purpose of this section is to find a convenient formal solution for equation (2.1.1). First, we need some preparations. We put

$$\begin{aligned} u : &= \tanh\left(\frac{d}{\varepsilon}t\right), \\ q_{0d}(t) : &= 4\arctan\left(\exp\left(-\frac{d}{\varepsilon}t\right)\right), \\ q_d(t) &= \sqrt{1-u^2}A_d(u) + q_{0d}(t), \quad A_d(u) = \sum_{n=1}^{\infty} A_n(u)d^n \end{aligned}$$

for a formal solution of (2.1.1), where $d = \varepsilon + \sum_{n=3}^{\infty} d_n\varepsilon^n$ is a formal powers series in ε to be determined.

Remark. The linearization of equation (2.1.1) at the point A gives the following equation

$$Z(t+\varepsilon) + Z(t-\varepsilon) - 2Z(t) = \varepsilon^2 Z(t).$$

The parameter d is such that $Z(t) = e^{-dt}$ is a solution of this equation, therefore ε and d are coupled by the relation $d = 2\text{arcsinh}(\varepsilon/2)$.

By Taylor expansion, we obtain

$$q_{0d}(t+\varepsilon) + q_{0d}(t-\varepsilon) - 2q_{0d}(t) = 2\sum_{n=1}^{+\infty} \frac{1}{(2n)\,!} q_{0d}^{(2n)}(t)\,\varepsilon^{2n}, \tag{2.2.1}$$

where $\dfrac{2}{(2n)!}q_{0d}^{(2n)}(t)\varepsilon^{2n}/d^{2n}$ is an odd polynomial $I_{2n-1}(u)$ multiplied by $\sqrt{1-u^2}$; we find $I_{2n-1}(1) = 4/(2n)!$.

Using $\cos(q_{0d}) = 2u^2 - 1$, $\sin(q_{0d}) = 2u\sqrt{1-u^2}$, we can express our equation (2.1.1) in the form

$$A_d(T^+)\sqrt{\frac{1-(T^+)^2}{1-u^2}} + A_d(T^-)\sqrt{\frac{1-(T^-)^2}{1-u^2}} - 2A_d(u) = f\big(\varepsilon, u, A_d(u)\big) \quad (2.2.2)$$

or equivalently

$$\frac{A_d(T^+)}{\cosh(d) + u\sinh(d)} + \frac{A_d(T^-)}{\cosh(d) - u\sinh(d)} - 2A_d(u) = f\big(\varepsilon, u, A_d(u)\big) \quad (2.2.3)$$

where

$$\begin{aligned}
f\big(\varepsilon, u, A_d(u)\big) &= \varepsilon^2\Bigg(2u\cos\Big(A_d(u)\sqrt{1-u^2}\Big) + \frac{2u^2-1}{\sqrt{1-u^2}}\sin\Big(A_d(u)\sqrt{1-u^2}\Big)\Bigg) \\
&\quad - \sum_{n=1}^{+\infty} I_{2n-1}(u)\, d^{2n}, \\
T^+ &= T^+(d,u) = \frac{u + \tanh(d)}{1 + u\tanh(d)} = \tanh\Big(\frac{d}{\varepsilon}(t+\varepsilon)\Big), \\
T^- &= T^-(d,u) = \frac{u - \tanh(d)}{1 - u\tanh(d)} = \tanh\Big(\frac{d}{\varepsilon}(t-\varepsilon)\Big).
\end{aligned}$$

As $u \to 1$, the expressions T^+ and T^- reduce to 1, the denominators in (2.2.3) simplify to $e^{\pm d}$ and hence equation (2.2.3) reduces to

$$(e^{-d} + e^d - 2)A_d(1) = \varepsilon^2(2 + A_d(1)) - 4(\cosh(d) - 1).$$

This is equivalent to $(2\cosh(d) - 2 - \varepsilon^2)(2 + A_d(1)) = 0$ and hence we have necessarily $\varepsilon = 2\sinh(d/2)$ if we want a formal solution such that the coefficients have limits as $u \to 1$.

Theorem 2.2.1. ***(On the formal solution)*** *If $\varepsilon = 2\sinh(d/2)$, then equation (2.2.2) has a unique formal solution of the form*

$$A_d(u) = \sum_{n=1}^{+\infty} A_{2n-1}(u) d^{2n}, \quad (2.2.4)$$

where $A_{2n-1}(u)$ are odd polynomials of degree $\leq 2n-1$.

Remark: A similar formal solution was found using another method in [12]. **Proof.** We will use the Induction Principle to show that there exist unique odd polynomials $A_1, A_3, A_5 ... A_{2n-1}$ such that

$$Z_n(d,u) = \sum_{k=1}^{n} A_{2k-1}(u) d^{2k} \tag{2.2.5}$$

satisfy

$$R_n(d,u) = O(d^{2n+4}) \tag{2.2.6}$$

where

$$\begin{aligned} R_n(d,u) &= Z_{n,d}\big(T^+\big)\sqrt{\frac{1-(T^+)^2}{1-u^2}} + Z_{n,d}\big(T^-\big)\sqrt{\frac{1-(T^-)^2}{1-u^2}} \\ &- 2Z_{n,d}(u) - f\big(\varepsilon, u, Z_{n,d}(u)\big) \end{aligned} \tag{2.2.7}$$

For $n = 1$, a short calculation shows that we must have $A_1(u) = -\frac{1}{4}u$ and hence $Z_{1,d}(u) = -\frac{1}{4}ud^2$. We obtain

$$R_1(d,u) = (\frac{-91}{48}u^5 + \frac{137}{48}u^3 - \frac{23}{24})d^6 + O(d^8).$$

Suppose now that there exists $A_1, A_3, A_5 ... A_{2n-1}$ such that

$$Z_n(d,u) = \sum_{k=1}^{n} A_{2k-1}(u) d^{2k} \tag{2.2.8}$$

satisfies (2.2.6), (2.2.7). We show that there is a unique polynomial $A_{2n+1}(u)$ such that

$$Z_{n+1}(d,u) = Z_n(d,u) + A_{2n+1}(u) d^{2n+2} \tag{2.2.9}$$

satisfies (2.2.6). We put

$$R_n(d,u) = R_{2n+3}(u) d^{2n+4} + O\big(d^{2n+6}\big) \tag{2.2.10}$$

where $R_{2n+3}(u)$ is odd and $\deg(R_{2n+3}(u)) \leq 2n+3$.

We substitute $Z_{n+1}(d,u)$ in equation (2.2.7). Using Taylor expansion, (2.2.9), (2.2.10) and $\varepsilon = 2\sinh(d/2)$, we obtain

$$Z_{n+1,d}\big(T^+\big)\sqrt{\frac{1-(T^+)^2}{1-u^2}} - Z_{n+1,d}\big(T^-\big)\sqrt{\frac{1-(T^-)^2}{1-u^2}} - 2Z_{n+1,d}(u) -$$

$$f\big(\varepsilon, u, Z_{n+1,d}\big) = \Big[(u^4 - 2u^2 + 1)A''_{2n+1}(u) + (4u^3 - 4u)A'_{2n+1}(u) +$$

$$R_{2n+3}(u)\Big] d^{2n+4} + O\big(d^{2n+6}\big)$$

We notice that (2.2.10) is satisfied if only if

$$\left[(1-u^2)^2 A'_{2n+1}(u)\right]' + R_{2n+3}(u) = 0 \tag{2.2.11}$$

This differential equation has a unique solution vanishing at $u=0$ without singularity at $u=1$, namely

$$A_{2n+1}(u) = -\int_0^u \frac{\int_1^t R_{2n+3}(s)ds}{(1-t^2)^2}dt. \tag{2.2.12}$$

We now show that this solution is an odd polynomial of u. It is clear that $\int_1^t R_{2n+3}(s)ds$ vanishes for $t=1$ and as $R_{2n+3}(s)$ is odd, it also vanishes for $t=-1$. It suffices to show that $R_{2n+3}(s)$ also vanishes at $t=\pm 1$. Indeed, taking the limit of (2.2.7) as $u \to 1$ as we did for (2.2.3) and using

$$\lim_{u\to 1} f\big(\varepsilon, u, Z(d,u)\big) \;=\; \varepsilon^2 Z(d,1)$$

we obtain

$$R_{2n+3}(1)d^{2n+4} = \left(e^d + e^{-d} - 2 - \varepsilon^2\right) Z(d,1) + O(d^{2n+6}).$$

By our choice of $\varepsilon = 2\sinh(d/2)$, we obtain $R_{2n+3}(1)d^{2n+4} = O(d^{2n+6})$. Consequently $R_{2n+3}(1)=0$. As $R_{2n+3}(u)$ is odd, we also have $R_{2n+3}(-1) = -R_{2n+3}(1) = 0$. This proves that $A_{2n+1}(u)$ is an odd polynomial of degree$\big(A_{2n+1}(u)\big) \le 2n+1$ and $A_{2n+1}(0)=0$.

The first polynomials $A_{2n-1}(u)$ with $n>0$ can be calculated using Maple.

n	1	2	3
$A_{2n-1}(u)$	$-\frac{1}{4}u$	$\left(\frac{91}{864}u^3 - \frac{47}{576}u\right)$	$\left(-\frac{319}{2880}u^5 + \frac{185}{1152}u^3 - \frac{3703}{69120}u\right)$

Now, we ntroduce the operators $\mathcal{C}_2, \mathcal{C}, \mathcal{S}_2, \mathcal{S}$ defined by

$$\begin{aligned}
\mathcal{C}(Z)(d,u) &= \tfrac{1}{2}\big(Z(d,T^{+\frac{1}{2}}) + Z(d,T^{-\frac{1}{2}})\big) \\
\mathcal{S}(Z)(d,u) &= \tfrac{1}{2}\big(Z(d,T^{+\frac{1}{2}}) - Z(d,T^{-\frac{1}{2}})\big) \\
\mathcal{C}_1(Z)(d,u) &= \tfrac{1}{2}\big(Z(d,T^{+}) + Z(d,T^{-})\big) \\
\mathcal{S}_1(Z)(d,u) &= \tfrac{1}{2}\big(Z(d,T^{+}) - Z(d,T^{-})\big)
\end{aligned} \tag{2.2.13}$$

where $T^{+\frac{1}{2}} = T^{+}(\frac{d}{2}, u)$, $T^{-\frac{1}{2}} = T^{-}(\frac{d}{2}, u)$ and $Z(d, u)$ is a formal power series in d whose coefficients are polynomials. We can show that

$$\begin{aligned} \mathcal{C}_1 &= 2\mathcal{S}^2 + Id \\ \mathcal{S}_1 &= 2\mathcal{S}\mathcal{C} \end{aligned} \tag{2.2.14}$$

and

$$\begin{aligned} \mathcal{C}_1(Q \cdot G) &= \mathcal{C}_1(Q)\mathcal{C}_1(G) + \mathcal{S}_1(Q)\mathcal{S}_1(G) \\ \mathcal{S}_1(Q \cdot G) &= \mathcal{C}_1(Q)\mathcal{S}_1(G) + \mathcal{S}_1(Q)\mathcal{C}_1(G)) \\ \mathcal{C}(Q \cdot G) &= \mathcal{C}(Q)\mathcal{C}(G) + \mathcal{S}(Q)\mathcal{S}(G) \\ \mathcal{S}(Q \cdot G) &= \mathcal{C}(Q)\mathcal{S}(G) + \mathcal{S}(Q)\mathcal{C}(G) \end{aligned} \tag{2.2.15}$$

if Q, G are formal power series in d whose coefficients are polynomials of u.

2.3 Norms for polynomials and basis

In this section we recall some definitions and results of [10]. Using a certain suquence of polynomials. we define convenient norms on spaces of polynomials which satisfies some useful proprieties. We denote by

- $\mathcal{P}$ the set of all polynomial whose coefficents are complex
- $\mathcal{P}_n$ the spaces of all polynomials of degree less than or equal to n

Proposition 2.3.1. *[10]. We define the sequence of polynomials $\tau_n(u)$ by*

$$\tau_0(u) = 1,\ \tau_1(u) = u,\ \tau_{n+1}(u) = \frac{1}{n} D\tau_n(u) \text{ for } n \geq 1,$$

where the operator D is defined by

$$D := (1 - u^2)\frac{\partial}{\partial u}.$$

Then we have

1. $T^{+}(d, u) = \sum_{n=0}^{\infty} \tau_{n+1}(u) d^n$,

2. $\tau_n(u)$ *has exactly degree* n *and hence* $\tau_0(u), ..., \tau_n(u)$ *form a basis of* $\mathcal{P}_n$*,*

3. $\tau_n(\tanh(z)) = \frac{1}{(n-1)!}\big(\frac{d}{dz}\big)^{n-1}\big(\tanh(z)\big)$

Definition 2.3.2. *Let* $p \in \mathcal{P}_n$*. As* $\tau_0(u), ..., \tau_n(u)$ *form a basis of* $\mathcal{P}_n$*, we can write* $p \in \mathcal{P}_n$ *as*

$$p = \sum_{k=0}^{n} a_k \tau_k(u).$$

Then we define the norms

$$\|p\|_n = \sum_{i=0}^{n} |a_i| \left(\frac{\pi}{2}\right)^{n-i}. \tag{2.3.1}$$

Theorem 2.3.3. *[10]. Let n,m be positive integers and* $p \in \mathcal{P}_n$*,* $q \in \mathcal{P}_m$*. The norms (2.3.1) have the following properties:*

1. $\|Dp\|_{n+1} \leq n\|p\|_n$*.*

2. *If the constant term of* p *in the basis* $\{\tau_0, \tau_1 .., \tau_n\}$ *is zero, we have*

$$\|p\|_n \leq \|Dp\|_{n+1}.$$

3. *There exists a constant* M_2 *such that* $\|pq\|_{n+m} \leq M_2\|p\|_n\|q\|_m$*.*

4. *There is a constant* M_3 *such that that for all* $n > 1$*,* $|p(u)| \leq M_3 \left(\frac{2}{\pi}\right)^n \|p\|_n$ $(-1 \leq u \leq 1)$*.*

5. *There is a constant* M_4 *such that for all* $n > 1$ *with* $p(1) = p(-1) = 0$

$$\left\|\frac{p}{\tau_2}\right\|_{n-2} \leq M_4\|p\|_n$$

2.4 Operators

In this section we will use some definitions of Schäfke-Volkmer[10] and adapt their results on operators on polynomial series to our context. Let

$$\mathcal{Q} := \left\{ Q(d,u) = \sum_{n=0}^{\infty} Q_n(u) d^n, \text{ where } Q_n(u) \in \mathcal{P}_n, \text{ for all } n \in \mathbb{N} \right\}.$$

By abuse of notation, let $\|Q\|_n = \|Q_n\|_n$ for a polynomial series

$$Q(d,u) = \sum_{n=0}^{\infty} Q_n(u) d^n.$$

Definition 2.4.1. *Let f be formal power series of z whose coefficients are complex. We define a linear operator $f(dD)$ on $\mathcal{Q}$ by*

$$f(dD)Q(d,u) = \sum_{n=0}^{\infty} \Big(\sum_{i=0}^{n} f_i D^i Q_{n-i}(u) \Big) d^n \tag{2.4.1}$$

where $f(z) = \sum_{i=0}^{\infty} f_i z^i$ and $Q \in \mathcal{Q}$.

By the above Definition and (1) of Proposition 2.3.1 we can show that

$$Q(d, T^+(\theta d, u)) = \big(\exp(\theta dD)Q\big)(d,u) \text{ for } Q \in \mathcal{Q} \text{ and all } \theta \in \mathbb{C}$$

Thus with (2.2.14) and (1) of Proposition 2.3.1 we obtain

$$\begin{aligned} \mathcal{C}(Q) &= \cosh(\tfrac{d}{2}D)Q, \; \mathcal{S}(Q) = \sinh(\tfrac{d}{2}D)Q \\ \mathcal{C}_1(Q) &= \cosh(dD)Q, \; \mathcal{S}_1(Q) = \sinh(dD)Q \end{aligned} \tag{2.4.2}$$

for polynomial series Q in $\mathcal{Q}$.

Remark. According to the definition of norms in (2.3.1), we have

$$\text{If } \; Q \in \mathcal{Q}, \; \text{ then } \; dQ \in \mathcal{Q} \; \text{ and } \; \|dQ\|_n = \frac{\pi}{2}\|Q\|_{n-1} \; \text{ for all } \; n \geq 1. \tag{2.4.3}$$

Theorem 2.4.2. *[10] Let $f(z)$ be formal power series having a radius of convergence greater than 2π and let k be a positve integer. There is a constant K such that: If Q is a polynomial series having the following property*

$$\|Q\|_n \le \begin{cases} 0 & \textit{for } n < k \\ M(n-k)\,!(2\pi)^{-n} & \textit{for } n \ge k \end{cases}$$

where M is independent of n and $Q \in \mathcal{Q}$ then the polynomial series $f(dD)Q$ satisfies

$$\|f(dD)Q\|_n \le \begin{cases} 0 & \textit{for } n < k \\ MK(n-k)\,!(2\pi)^{-n} & \textit{for } n \ge k \end{cases}$$

Now we define on $\mathcal{Q}$ the following operator

$$\mathcal{J} = \frac{\mathcal{S}}{dD}. \tag{2.4.4}$$

where the notation $\frac{\mathcal{S}}{dD}$ means simply $F(dD)$ with $F(z) = \dfrac{1}{z}\sinh(\frac{z}{2})$.

Lemma 2.4.3. *For each integer k there exist a positive constant K such that: If Q is a polynomial series with odd Q_n of degree at most n, $\|Q\|_n = 0$ for $n < k$ in case of positive k and*

$$\|dDQ\|_n \le M(n-k)\,!(2\pi)^{-n} \quad \textit{for } n \ge \max(0,k),$$

where M is independent of n, then the polynomial series $\mathcal{J}^{-1}(Q)$ satisfies

$$\|\mathcal{J}^{-1}(Q)\|_n \le MK(2\pi)^{-n} \begin{cases} (n-k+1)\,! & \textit{for } k \le 1 \\ (n-1)\,!\log(n) & \textit{for } k = 2 \\ (n-1)\,! & \textit{for } k \ge 3 \end{cases}$$

Proof. We can see easily that $\mathcal{J}^{-1} = \pi\,\tilde{\mathcal{C}}^{-1} + g(dD)$, where $\tilde{\mathcal{C}} = \cosh(\frac{1}{4}dD)$ and $g(z)$ is analytic for $|z| < 4\pi$, and use the proof of [10].

We have $\mathcal{S} = dD\,\mathcal{J} = \mathcal{J}\,dD$, but using this relation for the inversion of $\mathcal{S}$ would give an insufficient result. Using of the formula

$$1 = \frac{2}{z}\sinh(\frac{z}{2}) + F(z)z, \text{ where } F(z) = z^{-2}(z - 2\sinh(\frac{z}{2}))$$

is an entire function, we obtain the relation

$$Q = 2\mathcal{J}Q + F(dD)\,dDQ \tag{2.4.5}$$

for polynomial series $Q \in \mathcal{Q}$. This will be essential in the proof of

Theorem 2.4.4. *For each integer k there exist a positive constant K such that: If Q is a polynomial series with odd Q_n of degree at most n, $\|Q\|_n = 0$ for $n < k$ in case of positive k, and*

$$\|\mathcal{S}(Q)\|_n \leq M(n-k)\,!(2\pi)^{-n} \quad \textit{for } n \geq \max(0, k),$$

where M is independent of n, then the polynomial series Q satisfies

$$\|Q\|_n \leq MK(2\pi)^{-n} \begin{cases} (n-k+1)\,! & \textit{for } k \leq 1 \\ (n-1)\,!\log(n) & \textit{for } k = 2 \\ (n-1)\,! & \textit{for } k \geq 3 \end{cases}$$

Proof. By the preceding theorem, we have the wanted inequalities for $dDQ = \mathcal{J}^{-1}\mathcal{S}Q$ in the place of Q. Here we used again $\|dDZ\|_n \leq (n-1)\|Z\|_{n-1}$ for any polynomial series $Z \in \mathcal{Q}$. Using theorem 2.4.2 implies the same for $F(dD)dDQ$ with the entire function F of (2.4.5) As $\|Z\|_n \leq \|dDZ\|_{n+1}$ by theorem 2.3.3, we find the wanted inequalities (and even something better in the cases $k \geq 2$) also for $\mathcal{J}Q$ because $dD\mathcal{J} = \mathcal{S}$. Thus formula (2.4.5) yields the result □

In order to obtain an asymptotic approximation for the coefficients of the formal solution, we will need to reverse some operators. This is not possible for the operators $\mathcal{S}$ and dD on the set $\mathcal{Q}$, but we can define a subset $\mathcal{Q}^*$ of $\mathcal{Q}$ on which these operators have a right inverses.

If we define

$$\mathcal{Q}^* := \left\{ Q(d,u) = \sum_{n=1}^{\infty} P_n(u)d^n, \text{ where } P_n(u) \in \mathcal{P}_n^*, \text{ for all } n \geq 1 \right\}.$$

where $\mathcal{P}_n^*$ is the subspace of $\mathcal{P}_n$ defined by

$$\mathcal{P}_n^* := \{ \sum_{i=0}^{n} \alpha_i \tau_i \in \mathcal{P}_n, \mid \alpha_0 = 0 \}$$

Then, the restrictions of the operators $dD, \mathcal{S}$ to $\mathcal{Q}^*$, denoted here by the same symbols

$$\begin{aligned} dD &: \mathcal{Q}^* \to (1-u^2)d^2\mathcal{Q} \\ \mathcal{S} &: \mathcal{Q}^* \to (1-u^2)d^2\mathcal{Q} \end{aligned}$$

are bijective. We denote by $\mathcal{T}$ the inverse of the restriction of $\mathcal{S}$ to $\mathcal{Q}^*$, , and we have

$$\mathcal{T}\mathcal{S} = \text{Id on } \mathcal{Q}^*$$

Theorem 2.4.5. *[10] We consider a polynomial series*

$$Q_\alpha(d,u) = \sum_{n=2}^{\infty} \alpha_n (n-1)!\Big(\frac{i}{2\pi}\Big)^{n-1} \tau_n(u) d^n$$

where $\alpha_n = \mathcal{O}(n^{-k})$ *as* $n \to \infty$ *with some integer* $k \geq 2$*. Let*

$$\alpha := \frac{4}{\pi} \sum_{n=1}^{\infty} \alpha_n,$$

then the coefficients $\{\mathcal{T}(Q_\alpha)\}_n$ *of* $\mathcal{T}(Q_\alpha)$ *satisfy*

$$\left\| \{\mathcal{T}(Q_\alpha)\}_n - \alpha(n-1)!\Big(\frac{i}{2\pi}\Big)^{n-1}\tau_n \right\|_n = \mathcal{O}\Big((n-k)!(2\pi)^{-n}\Big)$$

as $n \to \infty$ *for* $n.$

Proof. The proof of this theorem is completely analogous to that of [10].

Theorem 2.4.6. *[10] Let* k, l, p, q *be integer with* $p \geq k$ *and* $q \geq l$*. Define* m *as the minimum of* $k + q$ *and* $l + p$*. then there is a constant* K *with the following property:*

If P *and* Q *are polynomial series such that* $\|P\|_n = 0$ *for* $n < p$*,* $\|Q\|_n = 0$ *for* $n < q$ *and*

$$\begin{aligned} \|P\|_n &\leq M_1(n-k)!\,(2\pi)^{-n} \quad \textit{for } n \geq p \\ \|Q\|_n &\leq M_2(n-l)!\,(2\pi)^{-n} \quad \textit{for } n \geq q \end{aligned}$$

then

$$\|PQ\|_n \leq K M_1 M_2 (n-m)!\,(2\pi)^{-n} \quad \textit{for } n \geq p+q.$$

Remark 2.4.7. Observe that the results of this section can also be applied, if the constants M are replaced by any increasing sequence $(M_n)_{n\in N}$. In theorems 2.4.2 and 2.4.6 the first n terms of the resulting polynomial series only depend of the first n terms of the given series, so the "M" in the result simply has to be replaced by "M_n". In lemma 2.4.3 and theorem 2.4.4, the first n terms of the result depend of the first $n+1$ given terms, so "M" in the result has to be replaced by "M_{n+1}".

2.5 Asymptotic approximation of the coefficients of the formal solution

In this section we will estimate the coefficients of the formal solution obtained previously (section 2). The idea is to write equation (2.2.2) essentially in the form

$$V(d,u)\mathcal{S}(Q_1\mathcal{S}(Q_2\,A))(d,u) = g\big(d,u,A(d,u)\big), \tag{2.5.1}$$

where V, Q_1 and Q_2 are known polynomials of d and u and g is a certain function of d, u and A involving the operators $\mathcal{S}, \mathcal{C}$ and $\mathcal{J}$ multiplied by sufficiently high powers of d.

Thanks to this equation, we will estimate the coefficients of the formal solution using the results of the previous section. We show that the coefficients of this formal solution is Gevrey-1, more precisely $\|A\|_n = O\big(n\,!\,(2\pi)^{-n}\big)$.

2.5.1 Rewriting of equation (2.2.2)

Consider the decomposition

$$A(d,u) = U(d,u) + F(d,u) \tag{2.5.2}$$

where U is the initial part of A calculated before

$$U(d,u) = -\frac{1}{4}ud^2 + \left(\frac{91}{864}u^3 - \frac{47}{576}u\right)d^4 + \left(-\frac{319}{2880}u^5 + \frac{185}{1152}u^3 - \frac{3703}{69120}u\right)d^6.$$

We insert this into (2.2.3), with (2.2.14) and (2.2.15), and obtain

$$2\cosh(d)\cdot\mathcal{C}_1(F) - 2u\,\sinh(d)\cdot\mathcal{S}_1(F) = W_0\cdot F + f_1\Big(d,u,F(d,u)\Big) \tag{2.5.3}$$

where

$$
\begin{aligned}
W_0 &= \Big(\cosh^2(d) - u^2\sinh^2(d)\Big)\Big[2 + (2u^2-1)\varepsilon^2\cos\Big(U\cdot\sqrt{1-u^2}\Big) \\
&- 2u\,\varepsilon^2\sin\Big(U\cdot\sqrt{1-u^2}\Big)\cdot\sqrt{1-u^2}\Big] \\
f_1\Big(d,u,F(d,u)\Big) &= \varepsilon^2\Big(\cosh^2(d) - u^2\sinh^2(d)\Big)\Bigg[\Bigg(\frac{2u^2-1}{\sqrt{1-u^2}}\sin\Big(U\cdot\sqrt{1-u^2}\Big) \\
&+ 2u\cos\Big(U\cdot\sqrt{1-u^2}\Big)\Bigg)\cos\Big(F(d,u)\sqrt{1-u^2}\Big) + \\
&\Bigg(\frac{2u^2-1}{\sqrt{1-u^2}}\cos\Big(U\cdot\sqrt{1-u^2}\Big) - 2u\sin\Big(U\cdot\sqrt{1-u^2}\Big)\Bigg)\times \\
&\Bigg(\sin\Big(F(d,u)\sqrt{1-u^2}\Big) - F(d,u)\sqrt{1-u^2}\Bigg)\Bigg] \\
&- \big(\cosh^2(d) - u^2\sinh^2(d)\big)\sum_{n=1}^{+\infty} I_{2n-1}(u)d^{2n} - 2\cosh(d)\mathcal{C}_1(U) \\
&- 2u\,\sinh(d)\mathcal{S}_1(U) - 2\Big(\cosh^2(d) - u^2\sinh^2(d)\Big)\cdot U.
\end{aligned}
$$

Observ that f_1 has the form

$$
\begin{aligned}
f_1\big(d,u,F(d,u)\big) &= y_0(d,u) + y_1(d,u)\sum_{n=1}^{\infty}\frac{1}{(2n)!}(1-u^2)^n F(d,u)^{2n} + \\
&\quad y_2(d,u)\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}(1-u^2)^n F(d,u)^{2n+1},
\end{aligned}
\tag{2.5.4}
$$

where $y_n(d,u), n = 1,2,3$ are convergent polynomial series.

Now, we let

$$
\begin{aligned}
F(d,u) &= Q(d,u)\cdot G(d,u), \\
J(d,u) &= Q_1(d,u)\cdot\mathcal{S}(G),
\end{aligned}
\tag{2.5.5}
$$

where G is a formal power series whose the first term contains d^8 and

$$
\begin{aligned}
Q(d,u) &= 1+\tfrac{1}{4}(1-u^2)d^2+\Big(\tfrac{91}{432}u^4-\tfrac{13}{48}u^2+\tfrac{13}{216}\Big)d^4\\
&+ \Big(-\tfrac{319}{960}u^6+\tfrac{1079}{1728}u^4-\tfrac{937}{2880}u^2+\tfrac{287}{8640}\Big)d^6,\\
Q_1(d,u) &= (u^2-1)d^2+\tfrac{1}{4}(1-u^4)d^4-\frac{5}{48}(1-u^2)\Big(\tfrac{4}{9}u^4+u^2+1\Big)d^6\\
&+ (1-u^2)\Big(-\tfrac{367}{2160}u^6+\tfrac{185}{432}u^4-\frac{997}{4320}u^2\Big)d^8.
\end{aligned}
\tag{2.5.6}
$$

The choice of $Q_1(d,u)$ and $Q(d,u)$ depends in a precise way of the form of the equation (2.5.1) and has been determined using Maple.

Using (2.5.5), (2.5.6) and (2.2.15), we can rewrite equation (2.5.3) in the form

$$
\begin{aligned}
W_0Q+f_1\Big(d,u,F_d(u)\Big) &= \Big[2\cosh(d)\mathcal{C}_1(Q)-2u\,\sinh(d)\mathcal{S}_1(Q)\Big]\mathcal{C}_1(G)\\
&+ \Big[2\cosh(d)\mathcal{S}_1(Q)-2u\,\sinh(d)\mathcal{C}_1(Q)\Big]\mathcal{S}_1(G).
\end{aligned}
$$

Using (2.2.14), we obtain

$$
V\cdot\mathcal{S}^2(G)+W\cdot\mathcal{SC}(G)=W_1G+f_2\Big(d,u,F_d(u)\Big)
\tag{2.5.7}
$$

where $f_2\Big(d,u,F_d(u)\Big)=\frac{1}{4}f_1\Big(d,u,F_d(u)\Big)$

$$
\begin{aligned}
V(d,u) &= \cosh(d)\mathcal{C}_1(Q)-u\,\sinh(d)\mathcal{S}_1(Q)\\
W(d,u) &= \cosh(d)\mathcal{S}_1(Q)-u\,\sinh(d)\mathcal{C}_1(Q)\\
W_1(d,u) &= \tfrac{1}{4}\Big(-2\cosh(d)\mathcal{C}_1(Q)+2u\,\sinh(d)\mathcal{S}_1(Q)+W_0Q\Big)
\end{aligned}
\tag{2.5.8}
$$

The calculation of the first terms of the series W_1 by Maple shows that the convergent polynomial series $W_1(d,u)$ begins with a term containing d^{10}.

Using (2.5.5) and (2.2.15), we find

$$
\mathcal{S}(J)=\mathcal{S}\Big(Q_1\mathcal{S}(G)\Big)=\mathcal{C}(Q_1)\mathcal{S}^2(G)+\mathcal{S}(Q_1)\mathcal{SC}(G).
\tag{2.5.9}
$$

Using

$$
\begin{aligned}
V_1(d,u) &= 1+(1-u^2)d^2+\Big(-\frac{71}{432}u^4-\frac{1}{12}u^2+\frac{107}{432}\Big)d^4\\
&+ \Big(\frac{1351}{2160}u^6-\frac{193}{144}u^4+\frac{49}{60}u^2-\frac{11}{108}\Big)d^6,
\end{aligned}
$$

we obtain

$$\begin{aligned} V_1 \cdot \mathcal{C}(Q_1) &= Q_1 V + W_2 \\ V_1 \cdot \mathcal{S}(Q_1) &= Q_1 W + W_3 \end{aligned}$$

where $W_2(d,u)$ and $W_3(d,u)$ are convergent polynomials series beginning with d^{10}. With (2.5.7) and (2.5.9), this implies

$$V_1 \cdot \mathcal{S}\Big(Q_1\mathcal{S}(G)\Big) = W_2 \cdot \mathcal{S}^2(G) + W_3\mathcal{S}\mathcal{C}(G) + Q_1W_1G + Q_1f_2\Big(d,u,Q_1G(d,u)\Big). \tag{2.5.10}$$

This allows us to prove the following theorem

Theorem 2.5.1.

$$\begin{aligned} G(d,u) &= \Big(\frac{\alpha}{d^2} + (\beta + \frac{\alpha}{3})\Big)\frac{u\,H_0(d,u)}{\tau_2(u)} - \big(\beta d + \frac{\alpha}{d}\big)H_2(d,u) \\ &+ \delta d H_1(d,u) + S(d,u) \end{aligned} \tag{2.5.11}$$

where α, β, δ are constants and the polynomial series H_0, H_1, H_2, S are defined by

$$\begin{aligned} H_0(d,u) :&= \sum_{\substack{n=10 \\ n\ even}}^{\infty} (n-1)\,!\Big(\frac{i}{2\pi}\Big)^n \tau_n(u)\, d^n \\ H_1(d,u) :&= \sum_{\substack{n=9 \\ n\ odd}}^{\infty} (n-1)\,!\Big(\frac{i}{2\pi}\Big)^{n+1} \tau_n(u)\, d^n \\ H_2(d,u) :&= \sum_{\substack{n=9 \\ n\ odd}}^{\infty} n\,!\Big(\frac{i}{2\pi}\Big)^{n+1} \tau_n(u)\, d^n \end{aligned} \tag{2.5.12}$$

and $S(d,u)$ is a polynomial series satisfying

$$\|S\|_n = \mathcal{O}\Big((n-3)!(2\pi)^n\Big).$$

To prove this theorem we need to make some overvaluations on the coefficients of the polynomial series $\mathcal{S}\big(Q_1\mathcal{S}(G)\big)$. This will be the subject of the following paragraph.
Remark: Observe that the series F, G are odd in u, even in d and beginning with d^8. The series J is even in u, odd in d and beginning with d^{11} . In the series F, G, A, J, the degree of the polynomial that is the coefficient of d^n is at most $n-1$; thus the results of section 4 can still be applied and $d^{-1}F, d^{-1}G, d^{-1}A, d^{-1}J \in \mathcal{Q}$

2.5.2 Upper bounds for the coefficients of $\mathcal{S}\big(Q_1\mathcal{S}(G)\big)$

In this paragraph, we will use equation (2.5.10), together with the definitions of V_1 and Q_1, J and G, $W_i, i = 1, 2, 3$, to prove

Lemma 2.5.2.

$$\Big\|\frac{1}{d}\mathcal{S}\big(Q_1\mathcal{S}(G)\big)\Big\|_n = \mathcal{O}\Big((n-8)\,!(2\pi)^{-n}\Big) \quad \textit{as } n \to \infty.$$

Proof. We set

$$e_n := \frac{(2\pi)^n\big\|\mathcal{S}(J)\big\|_n}{(n-8)\,!} \ \text{ for } \ n \geq 12 \tag{2.5.13}$$

We must show that $e_n = \mathcal{O}(n^{-1})$. In the sequel, we will use the following convention: if $a_n, n = 0, 1, ...$ is any sequence of positve real numbers, then

$$a_n^+ := \max(a_0, a_1, ...a_n) \text{ for all } n \geq 0$$

We have

$$\big\|\mathcal{S}(J)\big\|_n \leq e_n(n-8)\,!(2\pi)^{-n} \quad \text{for } \ n \geq 12. \tag{2.5.14}$$

Using Theorem 2.4.4 and Remark 2.4.7, we obtain

$$\big\|J\big\|_n \leq K_1 e_{n+1}^+(n-1)\,!(2\pi)^{-n}, \quad \text{for } \ n \geq 11. \tag{2.5.15}$$

where K_1 denotes the constant associated with the operator $\mathcal{S}$ in Theorem 2.4.4, it is independent of the present context. In this proof $K_i, i = 1, ..9$ will always denote constants independent of n and the sequence e_n.

Using (2.5.5), we obtain

$$\big\|Q_1\mathcal{S}(G)\big\|_n \leq K_1 e_{n+1}^+(n-1)\,!(2\pi)^{-n} \quad \text{for } \ n \geq 11 \tag{2.5.16}$$

We use (5) of Theorem 2.3.3 and 2.4.6. Since $\dfrac{Q_1(d,u)}{\tau_2(u)d^2}$ is a convergent power series beginning with 1, there is a constant K_2 such that

$$\big\|\mathcal{S}(G)\big\|_n \leq K_2\, e_{n+3}^+(n+1)\,!(2\pi)^{-n}, \quad \text{for } \ n \geq 9. \tag{2.5.17}$$

Using again Theorem 2.4.4 (and remark 2.4.7) and the fact that $F = G/Q$ where Q is given in (2.5.6), we obtain

$$\begin{aligned} \left\|G\right\|_n &\le K_3 e^+_{n+4}(n+2)\,!(2\pi)^{-n} \quad \text{for } \ n \ge 8, \\ \left\|F\right\|_n &\le K_3 e^+_{n+4}(n+2)\,!(2\pi)^{-n} \quad \text{for } \ n \ge 8, \end{aligned} \tag{2.5.18}$$

where K_3 is a constant independent of n.

This together with theorem 2.4.6 implies that there are constants K_4, L such that for all $k \ge 2$

$$\left\|F^k\right\|_n \le K_4 L_1^k f_n^{(k)}(n-5)\,!(2\pi)^{-n} \quad \text{for } n \ge 8k \tag{2.5.19}$$

where

$$\begin{aligned} f_n^{(2)} &= \sum_{i=8}^{n-8} e^+_{i+4} e^+_{n-i+4} \frac{(i+2)\,!\,(n-i+2)\,!}{(n-5)\,!}, \quad \text{for } n \ge 16, \\ f_n^{(k+1)} &= \sum_{i=8}^{n-8k} e^+_{i+4} f^{(k)}_{n-i} \frac{(i+2)\,!\,(n-i-5)\,!}{(n-5)\,!}, \quad \text{for } n \ge 8(k+1), \end{aligned}$$

with $f_n^{(k)} := 0$ for $n < 8k$.

Using Theorems 2.4.2 and 2.4.6 and $W_i = O(d^{10}), i = 1, 2, 3$, we obtain

$$\left\|W_2 \cdot \mathcal{S}^2(G)\right\|_n \le K_5 e^+_{n-6}(n-9)\,!(2\pi)^{-n} \quad \text{for } n \ge 20 \tag{2.5.20}$$

$$\left\|W_3 \mathcal{SC}(G)\right\|_n \le K_6 e^+_{n-6}(n-9)\,!(2\pi)^{-n} \quad \text{for } n \ge 19, \tag{2.5.21}$$

$$\left\|Q_1 W_1 G\right\|_n \le K_7 e^+_{n-8}(n-10)\,!(2\pi)^{-n} \quad \text{for } n \ge 20, \tag{2.5.22}$$

$$\left\|Q_1 f_2\Big(d, u, F_d(u)\Big)\right\|_n \le K_8(1 + \sum_{k \ge 2} \frac{L_2^k}{k!} f^{(k)+}_{n-4})(n-9)\,!(2\pi)^{-n} \quad \text{for } n \ge 12, \tag{2.5.23}$$

Now, let us take the equation (2.5.10)

$$\left\|V_1 \cdot \mathcal{S}(J)\right\|_n \le \left\|W_2 \cdot \mathcal{S}^2(G)\right\|_n + \left\|W_3 \mathcal{SC}(G)\right\|_n + \left\|Q_1 W_1 G\right\|_n + \left\|Q_1 f_2\Big(d, u, F_d(u)\Big)\right\|_n$$

Using (2.5.20), (2.5.21), (2.5.22) and (2.5.23), we obtain

$$\left\|V_1 \cdot \mathcal{S}(J)\right\|_n \le K_9 \Big(1 + e^+_{n-6} + \sum_{k \ge 2} \frac{L_2^k}{k!} f^{(k)+}_{n-4}\Big)(n-9)!\,(2\pi)^{-n} \tag{2.5.24}$$

Since, V_1 is a convergent polynomial series begins with 1 , we also have

$$\left\|\mathcal{S}(J)\right\|_n \le K_{10}\Big(1+e^+_{n-6}+\sum_{k\ge 2}\frac{L_2^k}{k!}f^{(k)+}_{n-4}\Big)(n-9)!\,(2\pi)^{-n} \tag{2.5.25}$$

Using (2.5.13), we obtain

$$e^+_n \le \frac{K}{n}\Big(1+e^+_{n-6}+\sum_{k\ge 2}\frac{L_2^k}{k!}f^{(k)+}_{n-4}\Big) \quad \text{for } n\ge 12 \tag{2.5.26}$$

lem5.2

Lemma 2.5.3. *Under the condition (2.5.26), we have* $e_n=\mathcal{O}(n^{-1})$ *as* $n\to\infty$.

Proof. Let $K_1 \ge 10\,!e^+_{12}$ an arbitrary number. We assume that

$$e_n \le \frac{K_1(n+p-1)\,!}{(n-2)\,!(p+11)\,!} \qquad \text{for } 12\le n\le N-4 \tag{2.5.27}$$

with some $p\ge -1,\ N\ge 16$. This gives for $16\le n\le N$

$$(n-5)\,!f^{(2)}_n \le K_1^2\sum_{i=8}^{n-8}\frac{(i+p+3)\,!(n+p-i+3)\,!}{\big((p+11)\,!\big)^2}.$$

The first and last term of the above sum are the largest, so we can easily estimate

$$\sum_{i=8}^{n-8}(i+p+3)\,!(n+p-i+3)\,! \le (p+11)\,!(n+p-4)\,! \quad .$$

We obtain

$$f^{(2)}_n \le K_1^2\frac{(n+p-4)\,!}{(p+11)\,!(n-5)\,!} \qquad \text{for } 16\le n\le N.$$

In a similar way, we can prove by induction that

$$f^{(k)}_n \le K_1^k\frac{(n+p-4)\,!}{(p+11)\,!(n-5)\,!} \le K_1^k\frac{(n+p-1)\,!}{(p+11)\,!(n-2)\,!} \qquad \text{for } 8k\le n\le N.$$

Using the assumption of the lemma, we obtain

$$e_n \le \frac{K}{n}e^{(1+L_2)K_1}\frac{(n+p-1)\,!}{(p+11)\,!(n-2)\,!} \qquad \text{for } 16\le n\le N.$$

Now we choose $N_0 \geq 16$ so large that $\frac{K\exp((1+L_2)K_1)}{N_0} \leq K_1$ and then p so large that (2.5.27) holds for $N = N_0$. In a first step, our considerations imply by induction over N that (2.5.27) holds for *all* N and hence

$$e_n = \mathcal{O}\Big(\frac{(n+p-1)\,!}{(n-2)\,!}\Big) \text{ as } n \to \infty$$

for this possibly large value of p.

As K_1 is arbitrary in (2.5.27), we have also shown for any $p \geq -1$ that

$$e_n = \mathcal{O}\Big(\frac{(n+p-1)\,!}{(n-2)\,!}\Big) \text{ as } n \to \infty$$

implies that

$$e_n = \mathcal{O}\Big(\frac{(n+p-2)\,!}{(n-2)\,!}\Big) \text{ as } n \to \infty \quad .$$

Consequently the last assertion is proved for $p = -1$ and we have shown

$$e_n = \mathcal{O}\big(n^{-1}\big) \text{ as } n \to \infty \quad .$$

Finally we have proved that

$$\big\|\mathcal{S}(J)\big\|_n = \mathcal{O}\big((n-9)\,!(2\pi)^{-n}\big) \text{ as } n \to \infty$$

and hence that

$$\big\|\frac{1}{d}\mathcal{S}(J)\big\|_n = \mathcal{O}\big((n-8)\,!(2\pi)^{-n}\big) \text{ as } n \to \infty$$

which completes the proof of the lemma. □

2.5.3 Proof of theorem 2.5.1

Let $E := \dfrac{1}{d}\mathcal{S}(J) = \mathcal{S}(d^{-1}J)$. The polynomial series E is odd in d and its coefficients are odd in u. We partition it

$$\begin{aligned} E_n(u) \;\; = \;\; & \alpha_n(n-1)\,!\Big(\frac{i}{2\pi}\Big)^{n-1}\tau_n(u) + \beta_{n-2}(n-3)\,!\Big(\frac{i}{2\pi}\Big)^{n-3}\tau_{n-2}(u) + \\ & \gamma_{n-4}(n-5)\,!\Big(\frac{i}{2\pi}\Big)^{n-5}\tau_{n-4}(u) + \overline{E}_{n-6}(u) \end{aligned} \qquad (2.5.28)$$

for odd $n \geq 11$, where α_n , β_n and γ_n are real numbers and also $\overline{E}_n$ have at most degree n for all n. For the whole series E this is equivalent to

$$\mathcal{S}(d^{-1}J) = E = E_1 + d^2 E_2 + d^4 E_3 + d^6 \overline{E} \tag{2.5.29}$$

where

$$\begin{aligned} E_1 &= \sum_{n=11}^{+\infty} \alpha_n (n-1)!\Big(\frac{i}{2\pi}\Big)^{n-1} \tau_n(u) d^n \\ E_2 &= \sum_{n=9}^{+\infty} \beta_n (n-1)!\Big(\frac{i}{2\pi}\Big)^{n-1} \tau_n(u) d^n \\ E_3 &= \sum_{n=7}^{+\infty} \gamma_n (n-1)!\Big(\frac{i}{2\pi}\Big)^{n-1} \tau_n(u) d^n \\ \overline{E} &= \sum_{n=5}^{+\infty} \overline{E}_n(u) d^n \end{aligned}$$

Lemma 2.5.2 implies that

$$\alpha_n = \mathcal{O}(n^{-7}),\ \beta_n = \mathcal{O}(n^{-5}),\ \gamma_n = \mathcal{O}(n^{-3}) \text{ and } \|\overline{E}_n\|_n = \mathcal{O}\big((n-2)!(2\pi)^{-n}\big).$$

Applying $\mathcal{T}$ to (2.5.29) we obtain

$$\frac{1}{d} J = \mathcal{T}(E_1) + d^2 \mathcal{T}(E_2) + d^4 \mathcal{T}(E_3) + d^6 \mathcal{T}(\overline{E}) \tag{2.5.30}$$

To the first three summands we apply Theorem 2.4.5. Thus we obtain

$$\begin{aligned} \Big\|\{\mathcal{T}(E_1)\}_n - \alpha(n-1)!\Big(\frac{i}{2\pi}\Big)^n \tau_n(u)\Big\|_n &= \mathcal{O}\big((n-7)!(2\pi)^{-n}\big) \\ \Big\|\{\mathcal{T}(E_2)\}_n - \beta(n-1)!\Big(\frac{i}{2\pi}\Big)^n \tau_n(u)\Big\|_n &= \mathcal{O}\big((n-5)!(2\pi)^{-n}\big) \\ \Big\|\{\mathcal{T}(E_3)\}_n - \gamma(n-1)!\Big(\frac{i}{2\pi}\Big)^n \tau_n(u)\Big\|_n &= \mathcal{O}\big((n-3)!(2\pi)^{-n}\big) \end{aligned}$$

where

$$\alpha = \frac{4}{\pi} \sum_{\substack{n=11 \\ n\ odd}}^{\infty} \alpha_n,\ \beta = \frac{4}{\pi} \sum_{\substack{n=9 \\ n\ odd}}^{\infty} \beta_n, \gamma = \frac{4}{\pi} \sum_{\substack{n=7 \\ n\ odd}}^{\infty} \gamma_n.$$

To the last part of (2.5.30) we apply Theorem 2.4.4 and obtain

$$\Big\|\{\mathcal{T}(\overline{E})\}_n\Big\|_n \quad = \quad \mathcal{O}\big((n-1)\,!(2\pi)^{-n}\log(n)\big),$$

thus altogether

$$\Big\|\{(\frac{1}{d}J)\}_n \quad - \quad \alpha(n-1)\,!\Big(\frac{i}{2\pi}\Big)^n\tau_n - \beta(n-3)\,!\Big(\frac{i}{2\pi}\Big)^{n-2}\tau_{n-2} - \gamma(n-5)\,!\Big(\frac{i}{2\pi}\Big)^{n-4}\tau_{n-4}\Big\|_n = \mathcal{O}\Big((n-7)\,!(2\pi)^{-n}\log(n)\Big).$$

Using (2.4.3) we obtain

$$\Big\|\{J\}_n \quad - \quad \alpha(n-2)\,!\Big(\frac{i}{2\pi}\Big)^{n-1}\tau_{n-1} - \beta(n-4)\,!\Big(\frac{i}{2\pi}\Big)^{n-3}\tau_{n-3} - \gamma(n-6)\,!\Big(\frac{i}{2\pi}\Big)^{n-5}\tau_{n-5}\Big\|_n = \mathcal{O}\Big((n-8)\,!(2\pi)^{-n}\log(n)\Big) \ .$$

Remark 2.5.4. *The asymptotic of J_n gives a good approximation of α; its suffice to calculate , using a formal calculation software (for example: Pari), the first 40 terms of $A(d,u)$ by the recurrence of Section 2 and to evaluate the highest coefficients of J_n to get the approximation* $\alpha = 89.0334$.

Next we observe that $J = Q_1\mathcal{S}(G)$, where Q_1 is given in (2.5.6) Using part 5. of Theorem 2.3.3, we obtain

EA

$$\begin{aligned}\Big\|\big\{\mathcal{S}(G)\big\}_n \quad &+ \quad \alpha n\,!\Big(\frac{i}{2\pi}\Big)^{n+1}\frac{\tau_{n+1}}{\tau_2} + \big(\beta-\alpha p_2(u)\big)(n-2)\,!\Big(\frac{i}{2\pi}\Big)^{n-1}\frac{\tau_{n-1}}{\tau_2} \\ &+(\gamma-\beta p_2(u)-\alpha p_4(u)\big)(n-4)\,!\Big(\frac{i}{2\pi}\Big)^{n-3}\frac{\tau_{n-3}}{\tau_2}\Big\|_n \\ &= \quad \mathcal{O}\Big((n-6)\,!(2\pi)^{-n}\log(n)\Big) \end{aligned} \tag{2.5.31}$$

where

$$\begin{aligned} p_2(u) &= -\frac{1}{4}-\frac{u^2}{4} = -\frac{1}{2}+\frac{\tau_2}{4} \\ p_4(u) &= \frac{1}{24}-\frac{u^2}{48}-\frac{7u^4}{432} = \frac{1}{216}+\frac{55}{1296}\tau_2+\frac{7}{432}\tau_4. \end{aligned}$$

Remark: Observe that the approximation (2.5.31) of the coefficients $\{\mathcal{S}(G)\}_n$ is polynomial. Indeed; the polynomials $\tau_n(u), n \geq 2$ are divisible by $\tau_2(u)$.

In order to find an asymptotic estimation for the coefficients of the formal solution, we need to apply the inverse of operator $\mathcal{S}$. To this purpose, we show the following lemma

Lemma 2.5.5. *If H_0, H_1, H_2 are the polynomial series defined in (2.5.12). Then*

- *1 If we define the operator $\mathcal{C}_{\frac{1}{4}} = \cosh\left(\dfrac{dD}{4}\right)$, then the polynomial series $\mathcal{C}_{\frac{1}{4}}(H_0), \mathcal{C}_{\frac{1}{4}}(H_1)$ are converging.*
- *2. the polynomial series $\mathcal{S}(H_0), \mathcal{S}(H_1)$ are converging.*
- *3. $\mathcal{C}(H_0) = -H_0 + \mu_1(d,u)$, where $\mu_1(d,u)$ is a convergent series.*
- *4. $\mathcal{S}(H_2) = \dfrac{1}{2}H_0 + \mu_2(d,u)$, where $H_2 = d\dfrac{\partial}{\partial d}H_1$ and $\mu_2(d,u)$ is a convergent series .*
- *5. $\mathcal{S}\left(\dfrac{H_0}{\tau_2(u)}\right) = -u\sinh(d)\dfrac{H_0}{\tau_2(u)} + \mu_3(d,u)$, where $\mu_3(d,u)$ is a convergent series .*
- *6. $\mathcal{S}\left(\dfrac{u\,H_0}{\tau_2(u)}\right) = -\frac{1}{2}(u^2+1)\sinh(d)\dfrac{H_0}{\tau_2(u)} + \mu_4(d,u)$, where $\mu_4(d,u)$ is a convergent series .*
- *7. $\mathcal{S}\left(\dfrac{u\,H_0}{\sinh(d)\tau_2(u)} - H_2\right) = -\dfrac{H_0}{\tau_2(u)} + \mu_5(d,u)$, where $\mu_5(d,u)$ is a convergent series .*

Proof. (1)- We have

$$\mathcal{C}_{\frac{1}{4}}(H_0) = \sum_{\substack{m=0 \\ m\ even}}^{\infty} \frac{1}{4^m m!} d^m D^m \Bigg(\sum_{\substack{n=10 \\ n\ even}}^{\infty} (n-1)!\left(\frac{i}{2\pi}\right)^n \tau_n(u)\, d^n \Bigg)$$

using the definition of the operator D in Proposition 2.3.1, we obtain

$$\begin{aligned} \mathcal{C}_{\frac{1}{4}}(H_0) &= \sum_{m,n} \frac{1}{4^m m!}(n+m-1)!\left(\frac{i}{2\pi}\right)^n \tau_{n+m}(u)\, d^{n+m} \\ &= \sum_{\substack{k=10 \\ k\ even}}^{\infty} \gamma_k (k-1)!\left(\frac{i}{2\pi}\right)^k \tau_k d^k \end{aligned} \qquad (2.5.32)$$

where

$$\gamma_k := \sum_{\substack{m=0 \\ m\ even}}^{k-10} \frac{1}{4^m m!}\Big(\frac{2\pi}{i}\Big)^m.$$

Hence

$$\gamma_k = \sum_{\substack{m=0 \\ m\ even}}^{\infty} \frac{1}{4^m m!}\Big(\frac{2\pi}{i}\Big)^m - \sum_{\substack{m=k-8 \\ m\ even}}^{\infty} \frac{1}{4^m m!}\Big(\frac{2\pi}{i}\Big)^m.$$

Using

$$\sum_{\substack{m=0 \\ m\ even}}^{\infty} \frac{1}{4^m m!}\Big(\frac{2\pi}{i}\Big)^m = \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!}\Big(\frac{\pi}{2}\Big)^{2l} = \cos\Big(\frac{\pi}{2}\Big) = 0,$$

we find

$$\gamma_k = -\sum_{\substack{m=k-8 \\ m\ even}}^{\infty} \frac{1}{m!}\Big(\frac{\pi}{2i}\Big)^m$$

which implies

$$\begin{aligned} |\gamma_k| &\leq \frac{1}{(k-8)!}\left(\frac{\pi}{2}\right)^{k-8} \sum_{\substack{m=0 \\ m\ even}}^{\infty} \frac{1}{m!}\Big(\frac{\pi}{2}\Big)^m \\ &\leq \frac{\cosh(\pi/2)}{(k-8)!}\left(\frac{\pi}{2}\right)^{k-8} \end{aligned}$$

This with (2.5.32) imply that $\mathcal{C}_{\frac{1}{4}}(H_0) = \mu(d,u)$ is convergent. For $\mathcal{C}_{\frac{1}{4}}(H_1)$, we can use the same method.

(2)- As $\mathcal{S} = 2\mathcal{S}_{\frac{1}{4}}\mathcal{C}_{\frac{1}{4}}$, where $\mathcal{S}_{\frac{1}{4}} = \sinh\left(\frac{dD}{4}\right)$, we obtain using (1),

$$\mathcal{S}(H_0) = 2\mathcal{S}_{\frac{1}{4}}\Big(\mathcal{C}_{\frac{1}{4}}(H_0)\Big) = 2\mathcal{S}_{\frac{1}{4}}(\mu)$$

This implies that $\mathcal{S}(H_0)$ is convergent. For $\mathcal{S}(H_1)$, we can use the same method.

(3)- We have $\mathcal{C}(H_0) = (2\mathcal{C}_{\frac{1}{4}}^2 - Id)H_0 = -H_0 + 2\mathcal{C}_{\frac{1}{4}}^2(H_0)$. This with (1) imply

$$\mathcal{C}(H_0) = -H_0 + \mu_1(d,u)$$

where $\mu_1(d,u)$ is a convergent series.

(4)- We differentiate the equation $\mathcal{S}(H_1) = C_1$ with respect to d. As

$$z\frac{d}{dz}\Big(\sinh(z/2)\Big) = \frac{z}{2}\cosh(z/2),$$

we obtain

$$\mathcal{S}(H_2) + \frac{1}{2}\mathcal{C}(dDH_1) = d\frac{\partial C_1}{\partial d}$$

Using (3) of this lemma, we obtain

$$\mathcal{S}(H_2) = \frac{1}{2}dDH_1 + \mu_2(d,u) = \frac{1}{2}H_0 + \mu_2(d,u),$$

where $\mu_2(d,u)$ is a convergent series.

(5)- Using (2.2.15) and (2), (3) of this lemma we obtain

$$\begin{aligned}
\mathcal{S}(H_0) = \mathcal{S}\left(\tau_2(u)\frac{H_0}{\tau_2(u)}\right) &= \mathcal{S}\big(\tau_2(u)\big)\mathcal{C}\left(\frac{H_0}{\tau_2}\right) + \mathcal{C}(\tau_2(u))\mathcal{S}\left(\frac{H_0}{\tau_2}\right) \\
&= \mu_3(d,u), \\
\mathcal{C}(H_0) = \mathcal{C}\left(\tau_2(u)\frac{H_0}{\tau_2(u)}\right) &= \mathcal{C}\big(\tau_2(u)\big)\mathcal{C}\left(\frac{H_0}{\tau_2}\right) + \mathcal{S}(\tau_2(u))\mathcal{S}\left(\frac{H_0}{\tau_2}\right) \\
&= -H_0.
\end{aligned}$$

This implis

$$\begin{aligned}
\mathcal{S}\left(\frac{H_0}{\tau_2}\right) &= \frac{\mathcal{S}\big(\tau_2(u)\big)}{\mathcal{C}\big(\tau_2(u)\big)^2 - \mathcal{S}\big(\tau_2(u)\big)^2}H_0 + \mu_4(d,u) \\
&= -u\sinh(d)\frac{H_0}{\tau_2(u)} + \mu_4(d,u),
\end{aligned}$$

where $\mu_3(d,u)$, $\mu_4(d,u)$ are converging series.

(6)- The proof of (6) is similar to that of (5).

(7)- Using (4) and (6), we obtain

$$\begin{aligned}
\mathcal{S}\left(\frac{u\,H_0}{\sinh(d)\tau_2(u)} - H_2\right) &= \left(-\frac{1}{2}(u^2+1) - \frac{1}{2}\tau_2(u)\right)\frac{H_0}{\tau_2(u)} + \mu_5(d,u) \\
&= -\frac{H_0}{\tau_2(u)} + \mu_5(d,u),
\end{aligned}$$

where $\mu_5(d,u)$ is a convergent series. This completes the proof of Lemma.

Using the definition of H_0 in (2.5.12) and (2.5.31), we can rewrite $\frac{1}{d}\mathcal{S}(G)$ in the form

$$\frac{1}{d}\mathcal{S}(G) = -\alpha\frac{H_0}{d^2\tau_2(u)} - (\beta + \frac{1}{2}\alpha)\frac{H_0}{\tau_2(u)} + \frac{1}{4}\alpha H_0 + \overline{X}, \qquad (2.5.33)$$

where

$$\|\overline{X}\|_n = \mathcal{O}\Big((n-3)\,!(2\pi)^{-n}\Big).$$

Using (2) and (5) of the previous Lemma and applying also the inverse of the operator $\mathcal{S}$ in (2.5.33), we obtain

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$$\frac{1}{d}G = \frac{1}{d^2}\Big(\alpha + (\beta + \frac{1}{2}\alpha)d^2\Big)\Big[\frac{1}{\sinh(d)}\frac{u\,H_0}{\tau_2(u)} - H_2\Big] + \frac{1}{2}\alpha H_2 + \delta H_1 + \overline{X}_1 \qquad (2.5.34)$$

where, δ is a constant and

$$H_2(d,u) = d\frac{\partial H_1}{\partial d}(d,u) = \sum_{\substack{n=9\\ n\ odd}}^{\infty} n\,!\Big(\frac{i}{2\pi}\Big)^{n+1}\tau_n(u)\,d^n \qquad (2.5.35)$$

$$\|\overline{X}_1\|_n = \mathcal{O}\Big((n-2)\,!(2\pi)^{-n}\Big) \qquad (2.5.36)$$

$$\delta = \frac{4}{\pi}\sum_{n=1}^{\infty}\delta_n, \qquad (2.5.37)$$

with $\delta_n = O(n^{-2})$. In the expression of $\frac{1}{d}G$, the term δH_1 comes from the fact that the series X can be written

$$X(d,u) = D_1(d,u) + d^2 D_2(d,u)$$

where

$$D_1(d,u) = \sum_{n=8}^{\infty}\delta_n(n-1)!\Big(\frac{i}{2\pi}\Big)^n\tau_n(u)d^n$$

$$\|D_2\|_n = \mathcal{O}\Big((n-1)\,!(2\pi)^{-n}\Big)$$

if we apply theorem 2.4.5 on the series $D_1(d,u)$ and Theorem 2.4.4 on $D_1(d,u)$, the term δH_1 appears in the expression of $\frac{1}{d}G$.

Since $\frac{1}{\sinh(d)} = d^{-1} - \frac{d}{6} + \mathcal{O}(d^3)$, we obtain

$$\begin{aligned} G(d,u) &= \Big(\frac{\alpha}{d^2} + (\beta + \frac{\alpha}{3})\Big)\frac{u\,H_0(d,u)}{\tau_2(u)} - \big(\beta d + \frac{\alpha}{d}\big)H_2(d,u) \\ &+ \delta d H_1(d,u) + S(d,u) \end{aligned} \tag{2.5.38}$$

where $\|S\|_n = \mathcal{O}\Big((n-3)!(2\pi)^n\Big)$ $\Box$

Observe that in $d^{-2}\frac{u\,H_0}{\tau_2(u)}$, $d^{-1}H_2$ the degree of the coefficients of d^n exceeds n. This is due to the fact that the expressions $u\frac{\tau_{n+2}}{\tau_2} - \tau_{n+1}$ etc., which are of degree $n-1$, were split.

It is not necessary (but would not be difficult) to write down asymptotic approximations for the coefficients of F, because equations (2.5.5) and (2.5.3) can be used. This completes the proof of the theorem 2.5.1 $\Box$

2.6 Functions and quasi-solutions

So far, we have shown that equation (2.2.2) has a formal solution and we have found an asymptotic approximation of the coefficients of the formal solution. We will use this to construct a quasi-solution, i.e. a function that satisfies equation (2.2.2) except for some exponentially small error. To that purpose, we define the functions

$$H_n(u) := (n-1)\,!\Big(\frac{i}{2\pi}\Big)^n \tau_n(u) \tag{2.6.1}$$

and

$$h_0(t,u) := \sum_{\substack{n=10 \\ n\ even}}^{\infty} H_n(u)\frac{t^{n-1}}{(n-1)\,!} \tag{2.6.2}$$

$$h_1(t,u) := \frac{i}{2\pi}\sum_{\substack{n=9 \\ n\ odd}}^{\infty} H_n(u)\frac{t^{n-1}}{(n-1)\,!} \tag{2.6.3}$$

$$h_2(t,u) := \frac{i}{2\pi}\sum_{\substack{n=9 \\ n\ odd}}^{\infty} n\,H_n(u)\frac{t^{n-1}}{(n-1)\,!} \quad . \tag{2.6.4}$$

This means that

$$h_0(t,u) = \frac{i}{2\pi}\sum_{\substack{n=1\\ n\ odd}}^{\infty}\Big(\frac{i}{2\pi}\Big)^n\tau_{n+1}(u)\,t^n - \Big(H_2\,t + H_4\frac{t^3}{3!} + H_6\,\frac{t^5}{5!} + H_8\,\frac{t^7}{7!}\Big)$$

$$h_1(t,u) = (\frac{i}{2\pi})^2\sum_{\substack{n=0\\ n\ even}}^{\infty}\Big(\frac{i}{2\pi}\Big)^n\tau_{n+1}(u)t^n - \frac{i}{2\pi}\Big(H_1 + H_3\frac{t^2}{2!} + H_5\,\frac{t^4}{4!}7H_7\,\frac{t^6}{6!}\Big) \tag{2.6.5}$$

Using part 4. of the proposition (2.3.1), we obtain

$$\frac{i}{2\pi}\sum_{\substack{n=1\\ n\ odd}}^{\infty}\Big(\frac{i}{2\pi}\Big)^n\tau_{n+1}(u)\,t^n = \frac{i}{2\pi}\sum_{\substack{n=1\\ n\ odd}}^{\infty}\frac{1}{n\,!}\Big(\frac{it}{2\pi}\Big)^n\frac{d^n}{d^n\xi}\Big(\tanh(\xi)\Big)$$

This is obviously the difference of two Taylor expansion and thus we can write

$$\begin{aligned} h_0(t,u) &= \frac{i}{4\pi}\Big[\tanh\big(\xi + \frac{it}{2\pi}\big) - \tanh\big(\xi - \frac{it}{2\pi}\big)\Big] \\ &- \Big(H_2\,t + H_4\frac{t^3}{3!} + H_6\,\frac{t^5}{5!} + H_8\,\frac{t^7}{7!}\Big). \end{aligned} \tag{2.6.6}$$

Similarly,

$$\begin{aligned} h_1(t,u) &= -\frac{1}{4\pi^2}\sum_{\substack{n=0\\ n\ even}}^{\infty}\frac{1}{n\,!}\Big(\frac{it}{2\pi}\Big)^n\frac{d^n}{d^n\xi}\Big(\tanh(\xi)\Big) \\ &- \frac{i}{2\pi}\Big(H_1 + H_3\frac{t^2}{2!} + H_5\,\frac{t^4}{4!}7H_7\,\frac{t^6}{6!}\Big) \end{aligned}$$

or equivalently

$$\begin{aligned} h_1(t,u) &= \frac{-1}{8\pi^2}\Big[\tanh\big(\xi + \frac{it}{2\pi}\big) + \tanh\big(\xi - \frac{it}{2\pi}\big)\Big] \\ &- \frac{i}{2\pi}\Big(H_1 + H_3\frac{t^2}{2!} + H_5\,\frac{t^4}{4!}7H_7\,\frac{t^6}{6!}\Big). \end{aligned} \tag{2.6.7}$$

The functional equations for the trigonometric and hyperbolic functions imply that

$$\begin{aligned}
h_0(t,u) &= -\frac{1}{2\pi}\frac{(1-u^2)\,\sin\left(\frac{t}{2\pi}\right)\cos\left(\frac{t}{2\pi}\right)}{\cos\left(\frac{t}{2\pi}\right)^2+u^2\sin\left(\frac{t}{2\pi}\right)^2} \\
&+ \frac{\tau_2}{4\pi^2}t-\frac{\tau_4}{16\pi^4}t^3+\frac{\tau_6}{64\pi^6}t^5-\frac{\tau_8}{256\pi^8}t^7 \\
h_1(t,u) &= -\frac{1}{4\pi^2}\frac{(u-u^3)\sin\left(\frac{t}{2\pi}\right)^2}{\cos\left(\frac{t}{2\pi}\right)^2+u^2\sin\left(\frac{t}{2\pi}\right)^2} \\
&+ \frac{\tau_1}{4\pi^2}-\frac{\tau_3}{16\pi^4}t^2+\frac{\tau_5}{64\pi^6}t^4-\frac{\tau_7}{256\pi^8}t^6
\end{aligned} \tag{2.6.8}$$

For fixed real u the functions $h_k(.,u), k=0,1,2$ are analytic in $|t|<\rho$, where $\rho=\pi^2$. In the subsequent definition, we consider real values of u, $0<u\leq 1$, here $h_k, k=0,1,2$ are also analytic with respect to t on the positive real axis.

We define the functions $\mathcal{H}_k(d,u), k=0..3$, by

$$\begin{aligned}
\mathcal{H}_0(d,u): &= \int_0^{+\infty} e^{-\frac{t}{d}}h_0(t,u)d\,t \text{ for } 0<u\leq 1 \\
\mathcal{H}_1(d,u): &= \int_0^{+\infty} e^{-\frac{t}{d}}h_1(t,u)d\,t \text{ for } 0<u\leq 1 \\
\mathcal{H}_2(d,u): &= \int_0^{+\infty} e^{-\frac{t}{d}}h_2(t,u)d\,t \text{ for } 0<u\leq 1.
\end{aligned} \tag{2.6.9}$$

We have $\mathcal{H}_2(d,u)=d\dfrac{\partial\mathcal{H}_1}{\partial d}(d,u)$. Indeed

$$\begin{aligned}
d\frac{\partial\mathcal{H}_1}{\partial d}(d,u) &= \int_0^{+\infty}\left(\frac{1}{d}e^{-\frac{t}{d}}\right)t\cdot h_1(t,u)d\,t \\
&= -\int_0^{+\infty}\frac{\partial}{\partial t}\left(e^{-\frac{t}{d}}\right)\cdot t\ \cdot h_1(t,u)d\,t \\
&= \int_0^{+\infty} e^{-\frac{t}{d}}\left(h_1(t,u)+t\cdot\frac{\partial}{\partial t}h_1(t,u)\right)d\,t \\
&= \int_0^{+\infty} e^{-\frac{t}{d}}h_2(t,u)d\,t
\end{aligned}$$

The functions $\mathcal{H}_k(d,.)$ are real analytic; they can be continued analytically to the interval $-1<u\leq 1$ in the following way. Choose some positive number M and let Γ_1 the path consisting of the segment from 0 to Mi and of the ray $t\mapsto t+Mi, t\geq 0$. Let

Γ_2 the symmetric path that could also be obtained using $-M$ instead of M. Recalling (2.6.6), we can also define

$$\begin{aligned}\mathcal{H}_0(d,u): &= \frac{i}{4\pi}\left[\int_{\Gamma_2} e^{-\frac{t}{d}}\tanh\left(\xi+\frac{it}{2\pi}\right)dt - \int_{\Gamma_1} e^{-\frac{t}{d}}\tanh\left(\xi-\frac{it}{2\pi}\right)dt\right]\\ &+ \mu_0(d,u),\end{aligned} \tag{2.6.10}$$

where

$$\mu_0(d,u) := \frac{1}{4\pi^2}\tau_2(u)d^2 - \frac{3}{8\pi^4}\tau_4(u)d^4 + \frac{15}{8\pi^6}\tau_6(u)d^6 - \frac{315}{16\pi^8}\tau_8(u)d^8,$$

for $-\tanh(\frac{2}{\pi}M) < u \leq 1$, where $\xi = \text{artanh}(u)$, because the singularities of $\tanh$ are $i(\frac{\pi}{2}+n\pi)$, n integer. As M is arbitrary, this defines the analytic continuation of $\mathcal{H}_0(d,.)$ for $-1 < u \leq 1$. Similarly, the real analytic continuations of $\mathcal{H}_k, k=1,2$ are defined.

In the sequel, we use the operator $\mathcal{C}, \mathcal{S}$ also for functions.

Lemma 2.6.1. *Consider the functions $\mathcal{H}_k(d,u), k = 0..2$, defined in (2.6.9). Then, for $-1 < u \leq 1$*

- *1 For $k = 0,1$,*

$$\mathcal{H}_k(d, T^{\pm\frac{1}{2}}) = -\mathcal{H}_k(d,u) + \mu_k^{\pm}(d,u), \tag{2.6.11}$$

  where $T^{+\frac{1}{2}}, T^{-\frac{1}{2}}$ are defined in (2.2.13) and the functions $\mu_k^{\pm}(d,u), k = 0,1,$ are analytic, beginning with d^{10}, resp d^9

- *2. For $k = 0,1$, $\mathcal{S}(\mathcal{H}_k) = \mu_k(d,u)$, where the functions $\mu_k(d,u), k = 0,1,$ are analytic, beginning with d^{11}, resp d^{10} .*

- *3. For $k = 0,1$, $\mathcal{C}(\mathcal{H}_k) = -\mathcal{H}_k(d,u) + \lambda_k(d,u)$, where the functions $\lambda_k(d,u), k = 0,1,$ are analytic, beginning with d^{10}, resp d^9.*

- *4. $\mathcal{S}(\mathcal{H}_2) = \frac{1}{2}\mathcal{H}_0(d,u) + \mu_2(d,u)$, where $\mu_2(d,u)$ is a analytic function, beginnings with d^{10} .*

- *5. $\mathcal{S}\left(\frac{u\,\mathcal{H}_0}{\tau_2(u)}\right) = -\frac{1}{2}(u^2+1)\sinh(d)\frac{\mathcal{H}_0}{\tau_2(u)} + \mu_4(d,u)$, where the function $\mu_4(d,u)$ is analytic, beginnings with d^{11} .*

Proof. (1)- For $k = 0$ we replace u by $T^{+\frac{1}{2}}$ in (2.6.6). Using (2.6.10) and $\xi(T^{+\frac{1}{2}}) = \xi(u) + \frac{1}{2}d$ we obtain for $0 < u \leq 1$

$$\mathcal{H}_0(d, T^{+\frac{1}{2}}) = \int_0^{+\infty} e^{-\frac{t}{d}} h_0(t, T^{+\frac{1}{2}}) d\,t = \frac{i}{4\pi}\mathcal{I}^+ + \mu(d, T^{+\frac{1}{2}}) \tag{2.6.12}$$

where

$$\mathcal{I}^+ = \int_0^{+\infty} e^{-\frac{t}{d}} \tanh\big(\xi + \frac{d}{2} + \frac{it}{2\pi}\big) d\,t - \int_0^{+\infty} e^{-\frac{t}{d}} \tanh\big(\xi - \frac{d}{2} + \frac{it}{2\pi}\big) d\,t$$

If we substitute $t + \pi i\, d$ in the first part, $t - \pi i\, d$ in the second part we obtain

$$\mathcal{I}^+ = -\int_{-\pi id}^{+\infty - \pi id} e^{-\frac{t}{d}} \tanh\big(\xi + \frac{it}{2\pi}\big) d\,t + \int_{\pi id}^{+\infty + \pi id} e^{-\frac{t}{d}} \tanh\big(\xi - \frac{it}{\pi}\big) d\,t \ \ .$$

Now, we apply Cauchy's theorem

$$\begin{aligned} \mathcal{I}^+ &= -\int_0^{+\infty} e^{-\frac{t}{d}} \Big(\tanh\big(\xi + \frac{it}{2\pi}\big) - \tanh\big(\xi - \frac{it}{2\pi}\big)\Big) d\,t \\ &+ \int_0^{-\pi id} e^{-\frac{t}{d}} \tanh\big(\xi + \frac{it}{2\pi}\big) d\,t - \int_0^{\pi id} e^{-\frac{t}{d}} \tanh\big(\xi - \frac{it}{2\pi}\big) d\,t \ \ . \end{aligned}$$

Substituting $t = -i\,ds$ in the second part, $t = i\,ds$ in the third part, we obtain

$$\begin{aligned} \mathcal{I}^+ &= \int_0^{+\infty} e^{-\frac{t}{d}} \Big(\tanh\big(\xi + \frac{it}{2\pi}\big) - \tanh\big(\xi - \frac{it}{2\pi}\big)\Big) d\,t \\ &- 2i\,d \int_0^{\pi} \cos(s) \tanh\big(\xi + d\frac{s}{2\pi}\big) ds. \end{aligned}$$

With (2.6.12) this implies for $0 < u \leq 1$

$$\mathcal{H}_0(d, T^+) = -\mathcal{H}_0(d, u) + \mu_0^+(d, u) \tag{2.6.13}$$

where

$$\mu_0^+(d, u) = \frac{d}{2\pi} \int_0^{\pi} \cos(s) \tanh\big(\xi + d\frac{s}{2\pi}\big) ds + \mu(d, T^{+\frac{1}{2}}).$$

By real analytic continuation, this formula is valid for $-1 < u \leq 1$. We use the same method for $\mathcal{H}_0(d, T^{-\frac{1}{2}}), \mathcal{H}_1(d, T^{\pm\frac{1}{2}})$ and obtain for $-1 < u \leq 1$

$$\begin{aligned} \mathcal{H}_0(d, T^{-\frac{1}{2}}) &= -\mathcal{H}_0(d, u) + \mu_0^-(d, u) \\ \mathcal{H}_1(d, T^{+\frac{1}{2}}) &= -\mathcal{H}_1(d, u) + \mu_1^+(d, u) \\ \mathcal{H}_1(d, T^{-\frac{1}{2}}) &= -\mathcal{H}_1(d, u) + \mu_1^-(d, u) \end{aligned} \tag{2.6.14}$$

where

$$\begin{aligned}
\mu_0^-(d,u) &= \frac{d}{2\pi}\int_0^\pi \cos(s)\tanh\big(\xi - \frac{d\,s}{2\pi}\big)d\,s + \mu_0(d,T^{-\frac{1}{2}})\\
\mu_1^+(d,u) &= -\frac{d}{4\pi^2}\int_0^\pi \sin(s)\tanh\big(\xi + \frac{d\,s}{2\pi}\big)d\,s + \mu_1(d,T^{+\frac{1}{2}})\\
\mu_1^-(d,u) &= -\frac{d}{4\pi^2}\int_0^\pi \sin(s)\tanh\big(\xi - \frac{d\,s}{2\pi}\big)d\,s + \mu_1(d,T^{-\frac{1}{2}})\\
\mu_1(d,u) &= \frac{1}{4\pi^2}\tau_1(u)d - \frac{1}{8\pi^4}\tau_3(u)d^3 + \frac{3}{8\pi^6}\tau_5(u)d^5 - \frac{45}{16\pi^8}\tau_7(u)d^7
\end{aligned}$$

(2)- Using the definition of the operator $\mathcal{S}$ in (2.2.13) and (1) of this Lemma, the result is immediate.

(3)- The proof of (3) is similar to that of (2).

(4)- For $k=1$, we differentiate (2.6.11) with respect to d. Because

$$\frac{\partial T^{\pm\frac{1}{2}}}{\partial d} = \pm\frac{1}{2}\Big(1-(T^{\pm\frac{1}{2}})^2\Big),$$

then

$$\mathcal{H}_2(d,u) \pm \frac{d}{2}\Big(1-(T^{\pm\frac{1}{2}})^2\Big)\frac{\partial \mathcal{H}_1}{\partial u}(d,T^{\pm\frac{1}{2}}) = -\mathcal{H}_2(d,u) + d\mu_1'^{\pm}(d,u)$$

implies

$$\begin{aligned}
\mathcal{S}\big(\mathcal{H}_2\big) &= -\frac{d}{2}\mathcal{C}\Big((1-u^2)\frac{\partial\mathcal{H}_1}{\partial u}\Big) + d\big(\mu_1'^{+}(d,u) - \mu_1'^{-}(d,u)\big)\\
&= \frac{d}{2}(1-u^2)\frac{\partial\mathcal{H}_1}{\partial u} + \mu_2(d,u)\\
&= \frac{1}{2}\mathcal{H}_0(d,u) + \mu_2(d,u)
\end{aligned}$$

where $\mu_2(d,u)$ is analytic function beginings with d^{10}.

(5)- Using (2.2.15) and (2), (3) of previous lemma , we obtain

$$\begin{aligned}
\mathcal{S}\Big(\frac{u}{\tau_2}\mathcal{H}_0\Big) &= \mathcal{S}\Big(\frac{u}{\tau_2}\Big)\mathcal{C}\big(\mathcal{H}_0\big) + \mathcal{C}\Big(\frac{u}{\tau_2}\Big)\mathcal{S}\big(\mathcal{H}_0\big)\\
&= -\frac{1}{2}(u^2+1)\sinh(d)\frac{\mathcal{H}_0}{\tau_2(u)} + \mu_4(d,u),
\end{aligned}$$

where the function $\mu_4(d,u)$ is analytic, beginnings with d^{11}. This completes proof of the Lemma.

In the sequel we consider $u_0 \in]-1,0]$.

Proposition 2.6.2. *We have*

1. *Uniformly for* $u_0 \le u \le 1$*,*

$$\begin{aligned}
\mathcal{H}_0(d,u) &\sim \sum_{\substack{n=2\\ n\ even}}^{\infty} (n-1)\,!\Big(\frac{i}{2\pi}\Big)^n \tau_n(u) d^n \text{ as } d \searrow 0 \\
\mathcal{H}_1(d,u) &\sim \sum_{\substack{n=3\\ n\ odd}}^{\infty} (n-1)\,!\Big(\frac{i}{2\pi}\Big)^{n+1} \tau_n(u) d^n \text{ as } d \searrow 0 \\
\mathcal{H}_2(d,u) &\sim \sum_{\substack{n=3\\ n\ odd}}^{\infty} n\,!\Big(\frac{i}{2\pi}\Big)^{n+1} \tau_n(u) d^n \text{ as } d \searrow 0
\end{aligned} \tag{2.6.15}$$

2. *For* $i = 1..3$*,* $\left|\frac{\partial \mathcal{H}_i}{\partial u}(d,u)\right| \le Kd$ *for* $u_0 < u \le 1 \ (d > 0)$

Proof. The proof of this proposition is similar to that of [11].

With the aim of applying the results of [10], we consider $S_{n-2}(u) = R_n(u)$, where $S_n(u)$ is the remainder term in (2.5.38). Then $R_n(u), n$ is a sequence of polynomials of degree at most n and

$$\|R_n\|_n = O\Big((n-5)\,!(2\pi)^{-1}\log(n)\Big)$$

Lemma 2.6.3. *[10] If we define*

$$\begin{aligned}
r(t,u) :&= \sum_{n=10}^{\infty} R_n(u)\frac{t^{n-1}}{(n-1)\,!} \quad (t \in \mathbb{C}, |t| \le \pi^2, u_0 \le u \le 1) \\
r(t,u) :&= r\big(\pi^2, u\big) + \big(t - \pi^2\big)\frac{\partial r}{\partial t}\big(\pi^2, u\big) \quad (t > \pi^2, u_0 \le u \le 1) \\
\mathcal{R}(d,u) :&= \int_0^{\infty} e^{-\frac{t}{d}}\, r(t,u) dt
\end{aligned}$$

then

1. r *is continuously differentiable function on the set* B *of all* (t,u) *such that* u *satisfies* $u_0 \le u \le 1$ *and* t *is a complex number and satisfies* $|t| \le \pi^2$ *or* $t > \pi^2$*. The restriction of* r *to* $u_0 \le u \le 1, |t| \le \pi^2$ *is twice continuously differentiable. for fixed* $u_0 \le u \le 1$ *the function* $r(t,u)$ *is analytic in* $|t| < \pi^2$

2. *$\mathcal{R}(d,u)$ is continuous, partially differentiable with respect to u, has continuous partial derivative and*

$$\mathcal{R}(d,u) \sim \sum_{n=10}^{\infty} R_n(u) d^n \text{ as } d \searrow 0 \tag{2.6.16}$$

3. $\left|\mathcal{R}(d,u)\right| \leq K d^3,\ \left|\frac{\partial \mathcal{R}}{\partial u}(d,u)\right| \leq K d^3$ *for* $u_0 \leq u \leq 1\ \ (d > 0)$

The importance of our definition of $\mathcal{R}$ lies in a certain compatibility with insertion of the functions T^+, T^- for u. First let

$$\begin{aligned}
\sum_{n=10}^{\infty} R_n^+(u) d^n &= \sum_{n=11}^{\infty} R_n(T^+) d^n \\
\sum_{n=10}^{\infty} R_n^-(u) d^n &= \sum_{n=11}^{\infty} R_n(T^-) d^n
\end{aligned}$$

We obtain a new sequences $R_n^+(u), R_n^-(u)$ of polynomials of degree at most n. This follows from the relation

$$p\big(T^+(d,u)\big) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k p(u) d^k \ \ . \tag{2.6.17}$$

Theorem 2.4.2 implies

$$\begin{aligned}
\left\|R_n^+(u)\right\|_n &= O\Big((n-5)\,!(2\pi)^{-n}\log(n)\Big) \\
\left\|R_n^-(u)\right\|_n &= O\Big((n-5)\,!(2\pi)^{-n}\log(n)\Big)
\end{aligned}$$

Therefore we can use the previous lemma for $R_n^+(u),\ R_n^-(u)$ and obtain functions $\mathcal{R}^+(d,u),\ \mathcal{R}^-(d,u)$.

Theorem 2.6.4. *There is a positive constant K independent of d, u such that*

$$\begin{aligned}
\left|\mathcal{R}^+(d,u) - \mathcal{R}(d,T^+)\right| &\leq K d^3 e^{-\frac{\pi^2}{d}}, \quad \text{for } (d>0, u_0 < u \leq 1), \\
\left|\mathcal{R}^-(d,u) - \mathcal{R}(d,T^-)\right| &\leq K d^3 e^{-\frac{\pi^2}{d}} \quad \text{for } (d>0, u_0 < u \leq 1).
\end{aligned}$$

Proof. The proof is exactly the one of [10] .

Definition 2.6.5. *Let $\mathcal{D}(d,u)$ be a function defined for $0 < d < d_0$ and $u_0 < u < 1$. We say that $\mathcal{D}(d,u)$ has property G if*

$$\mathcal{D}(d,u) = \int_0^\infty e^{-\frac{t}{d}}\, q(t,u)dt \quad (0 < d < d_0,\ u_0 < u < 1)$$

is the Laplace transform of some function $q(t,u)$ that has the following properties

1. *$q(t,u)$ is defined if $u_0 < u < 1$ and either t is complex and $|t| < \pi^2$ or t is real and $t \geq 0$,*
2. *$q(t,u)$ is analytic in $|t| < \pi^2$ for $u_0 < u < 1$,*
3. *$q(t,u)$ restricted to $0 \leq t < \pi^2$ or $t \geq \pi^2$ is continuous and the $\lim_{t\to\pi^2} q(t,u)$ exists for every $u_0 < u < 1$,*
4. *there is a positve constant K such that*

$$|q(d,u)| \leq Ke^{Kt} \quad \textit{for} \quad t \geq 0, (0 < d < d_0,\ u_0 < u < 1)$$

Lemma 2.6.6. *For $u_0 < u \leq 1$, we have*

1. *If $\mathcal{H}_i(d,u), i = 0,1,2$ are the functions of (2.6.9) then*

$$d^2\mathcal{H}_k(d,u) = (1-u^2)\tilde{\mathcal{H}}_k(d,u) + \mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{d}}\Big),\ \ k = 0,1$$

   and

$$d^3\mathcal{H}_2(d,u) = (1-u^2)\tilde{\mathcal{H}}_2(d,u) + \mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{d}}\Big),$$

   where $\tilde{\mathcal{H}}_i(d,u), i = 1,2,3$ have property G.

2. *Let k be a psitive integer. If $\mathcal{D}_1, \mathcal{D}_2$ have property G and their first terms in the Taylor development at $d = 0$, begin with d^k then*

$$\mathcal{D}_1(d,u)\mathcal{D}_2(d,u) = d^k\mathcal{D}(d,u) + \mathcal{O}\Big(d^k e^{-\frac{\pi^2}{d}}\Big),$$

   where $\mathcal{D}(d,u)$ has property G

3. *Any function $\mathcal{D}(d,u)$ analytic in a neighborhood of $d = 0$ has property G if $\mathcal{D}(0,u) = 0$ for all u,*

4. *If $\mathcal{R}(d,u)$ is defined by lemma 2.6.3 then $\dfrac{1}{d^2}\mathcal{R}(d,u)$ has property G*

5. *If $\mathcal{D}_1, \mathcal{D}_2$ have property G then so do $\mathcal{D}_1 + \mathcal{D}_2,\ \mathcal{D}_1 - \mathcal{D}_2$ and $\mathcal{D}_1 \cdot \mathcal{D}_2$*

6. *If $\mathcal{D}(d,u)$ has property G then*

$$\big|\mathcal{D}(d,u)\big| \leq Kd \quad \big(0 < d < \frac{1}{K}) \tag{2.6.18}$$

with some constant $K > 0$ independent of u.

Proof

1. For $i = 0$,

(i)-If $u > 0$, we have

$$d^2\mathcal{H}_0 = (1-u^2)\int_0^{\infty} e^{-\frac{t}{d}} g_2(t,u)dt \tag{2.6.19}$$

where

$$g_2(t,u) = \frac{1}{(1-u^2)}\int_0^t\int_0^{\tau} h_0(s,u)ds\, d\tau$$

$g_2(t,u)$ has a logarithmic singularity at $t_k(s) = (2k+1)\pi^2 \pm d\dfrac{2\pi\, s}{\varepsilon}i$ for $(k \geq 0, s > 0)$. it is analytic in $|t| < \pi^2$ and $\lim_{t\to\pi^2} g_2(t,u)$ exists.

If we put

$$\tilde{\mathcal{H}}_0(d,u) = \int_0^{\infty} e^{-\frac{t}{d}}\tilde{g}_2(t,u)d\,t$$

where

$$\tilde{g}_2(t,u) = \begin{cases} g_2(t,u), & \text{if } t \leq \pi^2 \\ g_2(\pi^2,u), & \text{if } t \geq \pi^2 \end{cases}$$

then $\tilde{\mathcal{H}}_0(d,u)$ has property G and

$$d^2\mathcal{H}_0(d,u) = (1-u^2)\tilde{\mathcal{H}}_0(d,u) + \mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{d}}\Big).$$

Remark: By real analytic continuation the formula (2.6.19) is valid for $-1 < u_0 < u < 1$.

(ii)- For $-1 < u_0 < u < 1$, we have

$$\begin{aligned}
\mathcal{H}_0(d,u) &= \int_0^{\infty} e^{-\frac{t}{d}} h_0(t,u)d\,t \\
&= \int_0^{\infty e^{i\varphi}} e^{-\frac{t}{d}} h_0(t,u)d\,t + 2\pi i \sum_{k\geq 0} Res\Big(e^{-\frac{t}{d}} h_0(t,u), t_k(s)\Big) \\
&= \int_0^{\infty e^{i\varphi}} e^{-\frac{t}{d}} h_0(t,u)d\,t + \mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{d}}\Big)
\end{aligned}$$

where $\dfrac{\pi}{2} < \varphi < \dfrac{\pi}{4}$. This implis

$$d^2\mathcal{H}_0(d,u) = (1-u^2)\int_0^{\infty e^{i\varphi}} e^{-\frac{t}{d}} g_2(t,u)d\,t + \mathcal{O}\Big((1-u^2)d^2 e^{-\frac{\pi^2}{d}}\Big)$$

we obtain

$$\begin{aligned}
d^2\mathcal{H}_0(d,u) &- (1-u^2)\tilde{\mathcal{H}}_0(d,u) = (1-u^2)\int_\Gamma e^{-\frac{t}{d}} g_2(t,u)d\,\Gamma \\
&+ \mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{d}}\Big)
\end{aligned}$$

Since $g_2(.,u)$ is bounded on Γ, then

$$\Big| d^2\mathcal{H}_0(d,u) - (1-u^2)\tilde{\mathcal{H}}_0(d,u)\Big| \leq K(1-u^2)e^{-\frac{\pi^2}{d}},$$

where K is positive constant. Finally

$$d^2\mathcal{H}_0(d,u) = (1-u^2)\tilde{\mathcal{H}}_0(d,u) + \mathcal{O}\Big((1-u^2)e^{-\frac{\pi^2}{d}}\Big) \quad \text{for} \quad (0 < u_0 < u \leq 1)$$

For $i = 1$ we can use the same method.

For $i = 2$, we use the same method with

$$d^3\mathcal{H}_2 = (1-u^2)\int_0^{\infty} e^{-\frac{t}{d}} g_2(t,u)d\,t \tag{2.6.20}$$

where

$$g_2(t,u) = \frac{1}{(1-u^2)}\int_0^t \int_0^\sigma \int_0^\tau h_2(s,u)ds\,d\sigma\,d\tau.$$

2. We assume that $\mathcal{D}_1(d,u), \mathcal{D}_2(d,u)$ have property G and their first terms in the Taylor development at $d=0$ begin with d^k. Then

$$\mathcal{D}_1 = \int_0^\infty e^{-\frac{t}{d}} f(t,u)d\,t$$
$$\mathcal{D}_2 = \int_0^\infty e^{-\frac{t}{d}} g(t,u)d\,t$$

where $f(t,u), g(t,u)$ are analytic in $|t| < \pi^2$ and $f(t,u) = O(t^{k-1}), g(t,u) = O(t^{k-1})$.

$$\mathcal{D}_1(d,u)\mathcal{D}_2(d,u) = \int_0^\infty e^{-\frac{t}{d}} (f*g)(t,u)d\,t \qquad (2.6.21)$$

Since

$$\begin{aligned} h(t,u) &= (f*g)(t,u) = \int_0^s f(t,u)g(t-s,u)ds \\ &= \int_0^t f(t-s,u)g(s,u)ds \\ &= \int_{t/2}^t f(s,u)g(t-s,u)ds + \int_{t/2}^t f(t-s,u)g(s,u)ds. \end{aligned}$$

For $t < \pi^2$, the function $h(t,u)$ is k times differentiable with respect to t and

$$\begin{aligned} h'(t,u) &= f(t,u)g(0,u) + f(0,u)g(t,u) - f(\frac{t}{2},u)g(\frac{t}{2},u) \\ &+ \int_{t/2}^t f(s,u)g'(t-s,u)ds + \int_{t/2}^t f'(t-s,u)g(s,u)ds \\ &= \int_{t/2}^t f(s)g'(t-s)ds + \int_{t/2}^t f'(t-s,u)g(s,u)ds \\ &- f(\frac{t}{2},u)g(\frac{t}{2},u) \\ h^{(k)}(t,u) &= \int_{t/2}^t f(s)g^{(k-1)}(t-s)ds + \int_{t/2}^t f^{(k-1)}(t-s,u)g(s,u)ds \\ &- \sum_{n=0}^{k-1} f^{(n)}(\frac{t}{2},u)g^{(k-1-n)}(\frac{t}{2},u) \end{aligned}$$

Observe that $h^{(k)}(t,u)$ is contninuous on $[0, 2\pi^2[$, it is analytic for $|t| < \pi^2$. If we put

$$\tilde{h}(t,u) = \begin{cases} h(t,u), & \text{if } t < \pi^2 \\ (t-\pi^2)^k. & \text{if } t \geq \pi^2 \end{cases}$$

then

$$\int_0^\infty e^{-\frac{t}{d}} \tilde{h}^{(k)}(t,u) d\, t,$$

has property G and

$$\begin{aligned} (\mathcal{D}_1 \cdot \mathcal{D}_2)(d,u) &= \int_0^\infty e^{-\frac{t}{d}} h(t,u) d\, t = \int_0^\infty e^{-\frac{t}{d}} \tilde{h}(t,u) d\, t + \mathcal{O}\Big(d^k e^{-\frac{\pi^2}{d}}\Big) \\ &= d^k \int_0^\infty e^{-\frac{t}{d}} \tilde{h}^{(k)}(t,u) d\, t + \mathcal{O}\Big(d^k e^{-\frac{\pi^2}{d}}\Big) \\ &= d^k \mathcal{D}(d,u) + \mathcal{O}\Big(d^k e^{-\frac{\pi^2}{d}}\Big) \end{aligned}$$

where $\mathcal{D}(d,u)$ has property G.

For $0 \leq u \leq 1,$ the proof of (3), (4), (5) and (6) is exactly the one of [10]. This proof is valid for $u_0 < u \leq 1.$

Now we have a formal solution of the equation (2.2.2) and an asymptotic estimate for its coefficients. With the results on the functions in the beginning of this section we have enough information to be able to give a precise function which satisfies the equation (2.2.2) with an exponentially small error as $d \to 0.$

In Theorem 2.2.1 we found that (2.2.2) has a uniquely determined formal power series solution

$$A(d,u) = U(d,u) + Q(d,u) G(d,u) \tag{2.6.22}$$

where U, Q are defined in (2.5.2), (2.5.6) and G is given by (2.5.38). This suggest that we put

$$\begin{aligned} \mathcal{G}(d,u) &= \Big(\frac{\alpha}{d^2} + (\beta + \frac{\alpha}{3})\Big) \frac{u\, \mathcal{H}_0(d,u)}{\tau_2(u)} - \big(\beta d + \frac{\alpha}{d}\big) \mathcal{H}_2(d,u) \\ &+ \delta d \mathcal{H}_1(d,u) + d^{-2} \mathcal{R}(d,u) \\ \mathcal{A}(d,u) &= U(d,u) + Q(d,u) \mathcal{G}(d,u) \end{aligned} \tag{2.6.23}$$

for $d > 0, u_0 < u \leq 1$, where $\mathcal{H}_i(d,u), i = 0..2$ are defined in (2.6.9) and $\mathcal{R}(d,u)$ is the function corresponding to $R_n,\ n = 8.10..$ according to lemma 2.6.3. Using (1) of proposition 2.6.2 and (2) of lemma 2.6.3, we obtain

$$\mathcal{G}(d,u) \sim G(d,u) \quad \text{as } d \searrow 0 \text{ for every } u_0 < u \leq 1. \tag{2.6.24}$$

Consequently

$$\mathcal{A}(d,u) \quad \sim \quad A(d,u) \quad \text{as } d \searrow 0 \text{ for every } u_0 < u \leq 1. \tag{2.6.25}$$

Theorem 2.6.7. *The function $\mathcal{G}(d,u)$ satisfies (2.5.10) except for an exponentially small error. More precisely*

$$\left| V_1 \cdot \mathcal{S}\Big(Q_1 \mathcal{S}(\mathcal{G})\Big) - W_2 \cdot \mathcal{S}^2(\mathcal{G}) - W_3 \mathcal{S}\mathcal{C}(\mathcal{G}) - Q_1 W_1 \mathcal{G} - Q_1 f_2\Big(d,u,\mathcal{G}\Big)\right| \leq K d^3 (1-u^2) e^{-\frac{\pi^2}{d}},$$

uniformly for $(u_0 < u < 1, 0 < d < d_0,)$, where K is a constant independent of d and u.

In the proof of this Theorem the functions $\mathcal{D}_i(d,u), i = 1.2..$ have property G. **Proof.** We set

$$\mathcal{F}(d,u) = V_1 \cdot \mathcal{S}\Big(Q_1 \mathcal{S}(\mathcal{G})\Big) - W_2 \cdot \mathcal{S}^2(\mathcal{G}) - W_3 \mathcal{S}\mathcal{C}(\mathcal{G}) - Q_1 W_1 \mathcal{G} - Q_1 f_2\Big(d,u,\mathcal{G}\Big). \tag{2.6.26}$$

Using (2), (4), (5) of Lemma 2.6.1, and (2.6.23), we obtain

$$\mathcal{S}(\mathcal{G}) = -\Big(\alpha_1 \sinh(d)\frac{1}{\tau_2} - (\frac{\alpha_1}{2}\sinh(d) - \frac{\beta_1}{2})\Big)\mathcal{H}_0 + (1-u^2)d^8\mathcal{D}_1(d,u) + d^{-2}\mathcal{S}(\mathcal{R})$$

where $\mathcal{D}_1(d,u)$ has property G and

$$\begin{aligned} \alpha_1 : &= \frac{\alpha}{d^2} + (\beta + \frac{\alpha}{3}) \\ \beta_1 : &= \frac{\alpha}{d} + \beta d \end{aligned}$$

This with lemma 2.6.1 imply

$$\begin{aligned} V_1 \mathcal{S}\Big(Q_1 \mathcal{S}(\mathcal{G})\Big) &= d^5 \mathcal{D}_2(d,u)\mathcal{H}_0 + (1-u^2)d^{11}\mathcal{D}_3(d,u) + d^{-2}V_1\mathcal{S}(Q_1)\mathcal{C}\mathcal{S}(\mathcal{R}) \\ &+ d^{-2}V_1\mathcal{C}(Q_1)\mathcal{S}^2(\mathcal{R}), \end{aligned}$$

where $\mathcal{D}_2(d,u), \mathcal{D}_3(d,u)$ have property G. Because

$$\begin{aligned} \mathcal{C}(Q_1) &= (1-u^2)d^2\big(1+\mathcal{O}(d^2)\big), \\ \mathcal{S}(Q_1) &= (1-u^2)d^3\big(u+\mathcal{O}(d^2)\big), \\ V_1 &= 1+\mathcal{O}(d^2) \end{aligned}$$

it is sufficient to apply Theorem 2.6.4, (4) of Lemma 2.6.6 for $\mathcal{R}$ and (1) of Lemma 2.6.6 for $\mathcal{H}_0$ we obtain

$$V_1\mathcal{S}\Big(Q_1\mathcal{S}(\mathcal{G})\Big) = d^5\mathcal{D}_2(d,u)\mathcal{H}_0 + (1-u^2)d^2\mathcal{D}_4(d,u) + \mathcal{O}\Big((1-u^2)d^3e^{-\frac{\pi^2}{d}}\Big)$$

where $\mathcal{D}_4(d,u)$ has property G. With (1) of lemma 2.6.6 this implies

$$V_1\mathcal{S}\Big(Q_1\mathcal{S}(\mathcal{G})\Big) = (1-u^2)d^2\mathcal{D}_5(d,u) + \mathcal{O}\Big((1-u^2)d^3e^{-\frac{\pi^2}{d}}\Big) \tag{2.6.27}$$

Using the same method for the terms $W_2\cdot\mathcal{S}^2(\mathcal{G})$, $W_3\mathcal{S}\mathcal{C}(\mathcal{G})$ and $Q_1W_1\mathcal{G}$, we obtain

$$\begin{aligned} W_2\cdot\mathcal{S}^2(\mathcal{G}) &= (1-u^2)d^{10}\mathcal{D}_6(d,u) + \mathcal{O}\Big((1-u^2)d^{10}e^{-\frac{\pi^2}{d}}\Big), \\ W_3\mathcal{S}\mathcal{C}(\mathcal{G}) &= (1-u^2)d^9\mathcal{D}_7(d,u) + \mathcal{O}\Big((1-u^2)d^{10}e^{-\frac{\pi^2}{d}}\Big), \\ Q_1W_1\mathcal{G} &= (1-u^2)d^8\mathcal{D}_8(d,u) + \mathcal{O}\Big((1-u^2)d^{10}e^{-\frac{\pi^2}{d}}\Big). \end{aligned} \tag{2.6.28}$$

To study the term $Q_1f_2\Big(d,u,\mathcal{G}\Big)$, we first treat $\mathcal{G}^2$ and $\mathcal{G}^3$.

$$\begin{aligned} \mathcal{G}^2(d,u) &= \alpha_1^2\frac{u^2}{\tau_2^2}\mathcal{H}_0^2 + \beta_1^2\mathcal{H}_2^2 + \delta^2d^2\mathcal{H}_1^2 + d^{-4}\mathcal{R}^2 - 2\alpha_1\beta_1\frac{u}{\tau_2}\mathcal{H}_0\mathcal{H}_2 + 2\alpha_1\delta\frac{u}{\tau_2}\mathcal{H}_0\mathcal{H}_1 \\ &+ 2d^{-2}\alpha_1\frac{u}{\tau_2}\mathcal{H}_0\mathcal{R} - 2\beta_1\delta d\mathcal{H}_1\mathcal{H}_2 - 2d^{-2}\beta_1\mathcal{R}\mathcal{H}_2 + 2d^{-1}\delta\mathcal{R}\mathcal{H}_1 \end{aligned} \tag{2.6.29}$$

Using (1), (2) and (4) of the lemme 2.6.6 we obtain

$$\mathcal{G}(d,u)^2 = d^2\mathcal{D}_9(d,u) + \mathcal{O}\Big(d^2e^{-\frac{\pi^2}{d}}\Big) \tag{2.6.30}$$

With (2) of lemma 2.6.6 this implies

$$\mathcal{G}(d,u)^3 = \mathcal{G}(d,u)\mathcal{G}(d,u)^2 = d^8\mathcal{D}_{10}(d,u) + \mathcal{O}\Big(d^8e^{-\frac{\pi^2}{d}}\Big). \tag{2.6.31}$$

We can rewrite

$$f_2\big(d,u,\mathcal{G}\big) = y_0(d,u) + y_1(d,u)\mathcal{G}^2f_{1,1}\big(d,u,\mathcal{G}^2\big) + y_2(d,u)d^2\mathcal{G}^3f_{1,2}\big(d,u,\mathcal{G}^2\big) \tag{2.6.32}$$

where $f_{1,1}, f_{1,2}$ and $y_i, i=0,1,2$ are analytic .

Lemma 2.6.8. *For* $i = 1, 2$*,*

$$f_{1,i}(d, u, \mathcal{G}^2) = \int_0^\infty e^{-\frac{t}{d}} f_i(t, u) d\,t + \mathcal{O}\Big(d^2 e^{-\frac{\pi^2}{d}}\Big) \quad \textit{for} (d > 0, u_0 < u \leq 1) \tag{2.6.33}$$

where $f_i(., u)$ *is analytic in* $|t| < \pi^2$ *and continuous in* $[0, \pi^2[$ *and* $[\pi^2, \infty[$

Proof. Using (2.6.30), we can rewrite

$$\mathcal{G}^2(d, u) = \int_0^\infty e^{-\frac{t}{d}} g(t, u) d\,t + \mathcal{O}\Big(d^2 e^{-\frac{\pi^2}{d}}\Big)$$

where $g(., u)$ is analytic in $|t| < \pi^2$, it is continuous in $[0, \pi^2[$ and $[\pi^2, \infty[$, twice differentiable in $|t| < \pi^2$. We obtain

$$f_{1,i}(d, u, \mathcal{G}^2) = \sum_{n=1}^{\infty} f_{n,i}(d, u) \mathcal{G}^{2n}$$

where

$$f_{n,i}(d, u) = \int_0^\infty e^{-\frac{t}{d}} \varphi_{n,i}(t, u) d\,t$$

and $\varphi_{n,i}(t, u)$ are entire functions.

Using the proof of theorem 5.1 from [1], we find

$$f_{1,i}(d, u, \mathcal{G}^2) = \int_0^\infty e^{-\frac{t}{d}} f_i(t, u) d\,t + \mathcal{O}\Big(d^2 e^{-\frac{\pi^2}{d}}\Big) \tag{2.6.34}$$

where the series

$$\sum_{n=1}^{\infty} (\varphi_{n,i} * g^{*n})(t, u)\Big), \quad g^{*n} = g * ... * g, n \text{ times}$$

is uniformly convergent to a function $f_i(t, u)$ analytic in $|t| < \pi^2$ and continuous in $[0, \pi^2[$ and $[\pi^2, \infty[$, satisfies also

$$|f_i(t, u)| \leq K \exp(Kt) \quad \text{for} \;\; t \geq 0, u_0 < u \leq 1.$$

Then

$$f_{1,i}(d, u, \mathcal{G}^2) = \mathcal{Y}_i(d, u) + \mathcal{O}\Big(d^2 e^{-\frac{\pi^2}{d}}\Big) \quad \text{for} (d > 0, u_0 < u \leq 1),$$

where $\mathcal{Y}_i(d, u), i = 1, 2$ have property G.

This lemma with (2.6.32) and (2) of lemma 2.6.6 imply

$$Q_1 f_2\Big(d,u,\mathcal{G}\Big) = d^6(1-u^2)\mathcal{D}_{10}(d,u) + \mathcal{O}\Big(d^6(1-u^2)e^{-\frac{\pi^2}{d}}\Big) \qquad (2.6.35)$$

Combining (2.6.27), (2.6.28) and (2.6.35), we find

$$\mathcal{F}(d,u) \;=\; d^2(1-u^2)\mathcal{D}(d,u) + \mathcal{K}(d,u)$$

where $\mathcal{D}(d,u)$ has property G and

$$\Big|\mathcal{K}(d,u)\Big| \le K d^3(1-u^2)de^{\frac{-\pi^2}{d}} \quad \text{for} \quad (0<d<d_0,\, u_0<u<1). \qquad (2.6.36)$$

Hence

$$\mathcal{F}(d,u) = (1-u^2)d^2\int_0^\infty e^{-\frac{t}{d}}q(t,u)dt + \mathcal{K}(d,u)$$

where $q(t,u)$ is analytic in $|t|<\pi^2$, it is continuous on $[0,\pi^2[$ and $]\pi^2,\infty[$, has a limit as $t\to\pi^2$ for every $u_0<u<1$ and satisfies

$$|q(t,u)| \le K\, e^{Kt}, \qquad (2.6.37)$$

with a constant K independent of u. If $q(t,u)=\sum_{n=0}^\infty q_n(u)t^n$ is the power series of $q(t,u)$ near $t=0$, Watson's lemma with (2.6.36) imply

$$\mathcal{F}(d,u) \sim \sum_{n=0}^\infty n!(1-u^2)q_n(u)d^{n+3} \quad \text{as } d\searrow 0 \text{ for every } u_0<u<1 \;.$$

On the other hand because of its definition

$$\begin{aligned}\mathcal{F}(d,u) \;&\sim\; V_1\cdot\mathcal{S}\Big(Q_1\mathcal{S}(\mathcal{G})\Big) - W_2\cdot\mathcal{S}^2(\mathcal{G}) - W_3\mathcal{S}\mathcal{C}(\mathcal{G})\\ &-\; Q_1W_1\mathcal{G} - Q_1f_2\Big(d,u,\mathcal{G}\Big) = 0 + 0d + \cdots \qquad ,\end{aligned}$$

since the formal series G satisfies (2.5.10). This means that all $q_n\equiv 0$. Thus we obtain for $u_0<u<1$ with (2.6.37)

$$\begin{aligned}(1-u^2)d^2\int_0^\infty e^{-\frac{t}{d}}|q(t,u)|dt \;&\le\; (1-u^2)d^2\int_{\pi^2}^\infty e^{-\frac{t}{d}}Ke^{Kt}dt\\ &\le\; K(1-u^2)d^3e^{-\frac{\pi^2}{d}} \quad (0<d<d_0)\end{aligned}$$

and thus

$$\left|\mathcal{F}(d,u)\right| \le K(1-u^2)d^3 e^{-\frac{\pi^2}{d}} \quad (0 < d < d_0)\,.$$

We have proved that

$$\begin{aligned}\Big|V_1 \cdot \mathcal{S}\Big(Q_1\mathcal{S}(\mathcal{G})\Big) \;&-\; W_2 \cdot \mathcal{S}^2(\mathcal{G}) - W_3\mathcal{S}\mathcal{C}(\mathcal{G}) - Q_1W_1\mathcal{G} - Q_1 f_2\Big(d,u,\mathcal{G}\Big)\Big| \\ &\le\; K(1-u^2)d^3e^{-\frac{\pi^2}{d}},\end{aligned}$$

for $0 < d < d_0,\ u_0 < u < 1$, i.e. $\mathcal{Q}(d,u)$ is a quasi-solution of (2.5.10) on this interval. This implies that the function $\mathcal{A}(d,u)$ defined in (2.6.23) is a quasi-solution of (2.2.2), more precisely

$$\begin{aligned}\left|\sqrt{\frac{1-(T^+)^2}{1-u^2}}\mathcal{A}(d,T^+) \;\right. &+\; \left.\sqrt{\frac{1-(T^-)^2}{1-u^2}}\mathcal{A}(d,T^-) - 2\mathcal{A}(d,u) - f\Big(\varepsilon,\mathcal{A}(d,u)\Big)\right| \\ &\le\; Kde^{-\frac{\pi^2}{d}},\end{aligned} \tag{2.6.38}$$

2.7 Distance Between Points of Manifolds

Clearly, if $q_\varepsilon(t)$ is an exact solution of the difference equation (2.1.1), then $(q_\varepsilon(t), p_\varepsilon(t))$, where $p_\varepsilon(t) = \frac{1}{\varepsilon}\big(q_\varepsilon(t) - q_\varepsilon(t-\varepsilon)\big)$, is an exact solution of the the system (2.1.2). In the introduction, we have mentioned that the stable manifold W_s^- of this system at $A = (0,0)$ is parametrized by $t \to (q_\varepsilon^-(t), p_\varepsilon^-(t))$ and the unstable manifold W_u^+ of (2.1.2) at $B = (2\pi, 0)$ is parametrized by $t \to (q_\varepsilon^+(t), p_\varepsilon^+(t))$, where $(q_\varepsilon^-(t), p_\varepsilon^-(t))$ is an exact solution of (2.1.2) and $q_\varepsilon^+(t) = 2\pi - q_\varepsilon^-(-t), p_\varepsilon^+(t) = p_\varepsilon^-(-t+\varepsilon)$.

In the previous section, we have constructed a quasi-solution $\mathcal{A}(d,u)$ for equation (2.2.2), i.e. it satisfies this equation with an exponentially small error. We denote by $\tilde{W}_s$, $\tilde{W}_u$ the manifolds close to W_s^- respectively W_u^+ parametrized by $t \mapsto \big(\xi_-(t), \varphi_-(t)\big)$ respectively $t \mapsto \big(\xi_+(t), \varphi_+(t)\big)$, where $\xi_-(t) = \sqrt{1-u(t)^2}\mathcal{A}\big(d,u(t)\big) + q_{0d}(t)$ and $\xi_+(t) = 2\pi - \xi_-(-t)$, $\varphi_-(t) = \dfrac{1}{\varepsilon}\big(\xi_-(t) - \xi_-(t-\varepsilon)\big)$, $\varphi_+(t) = \dfrac{1}{\varepsilon}\big(\xi_+(t) - \xi_+(t-\varepsilon)\big)$. Here and in the sequel, we often omit to indicate the dependence with respect to ε for the sake of simplicity of notation.

We will first show that the vertical distance between some point (q_1, p_1) of the stable manifold W_s^- and the manifold $\tilde{W}_s$ is exponentially small. For this purpose, we consider the sequence $Z_n = (q_n, p_n)$ on the stable manifold W_s^-, defined by

$$Z_{n+1} = \begin{pmatrix} q_{n+1} \\ p_{n+1}\end{pmatrix} = \phi\begin{pmatrix} q_n \\ p_n\end{pmatrix} = \begin{pmatrix} q_n + \varepsilon p_{n+1} \\ p_n + \varepsilon\sin(q_n)\end{pmatrix}, \text{ for } n = 1,2,3,.... \tag{2.7.1}$$

There is a sequence t_n such that

$$\xi_-(t_n) = q_n = \sqrt{1-u(t_n)^2}\,\mathcal{A}\big(d, u(t_n)\big) + q_{0d}(t_n). \tag{2.7.2}$$

The vertical projection of the point (q_n, p_n) on the manifold $\tilde{W}_s$ is the point (q_n, b_n), where $b_n = \varphi_-(t_n) = g(q_n) := \varphi_-\big(\xi_-^{-1}(q_n)\big)$ for $n = 1, 2, 3, \ldots$. We denote by $\triangle_\varepsilon(n)$ the vertical distance between the point (q_n, p_n) and the manifold $\tilde{W}_s$. Then

$$\triangle_\varepsilon(n) = p_n - b_n. \tag{2.7.3}$$

For the ϕ-image of the point (q_n, b_n) on $\tilde{W}_s$, we find

$$\begin{aligned}
\begin{pmatrix}\tilde{q}_{n+1}\\ \tilde{b}_{n+1}\end{pmatrix} &= \phi\begin{pmatrix}q_n\\ b_n\end{pmatrix} = \begin{pmatrix}q_n + \varepsilon\tilde{b}_{n+1}\\ b_n + \varepsilon\sin(q_n)\end{pmatrix},\\
&= \begin{pmatrix}\xi_-(t_n) + \varepsilon\tilde{b}_{n+1}\\ \varepsilon^{-1}\big(\xi_-(t_n) - \xi_-(t_n-\varepsilon)\big) + \varepsilon\sin\big(\xi_-(t_n)\big)\end{pmatrix}\\
&= \begin{pmatrix}2\xi_-(t_n) - \xi_-(t_n-\varepsilon) + \varepsilon^2\sin\big(\xi_-(t_n)\big)\\ \varepsilon^{-1}\big(\xi_-(t_n) - \xi_-(t_n-\varepsilon)\big) + \varepsilon\sin\big(\xi_-(t_n)\big)\end{pmatrix}
\end{aligned} \tag{2.7.4}$$

As the function ξ_- satisfies equation (2.1.1) except for an exponentially small error because of (2.6.38) and (2.7.2), we have

$$\begin{aligned}
\tilde{q}_{n+1} &= \xi_-(t_n+\varepsilon) + e_{n+1}(\varepsilon)\\
\tilde{b}_{n+1} &= \varphi_-(t_n+\varepsilon) + \frac{1}{\varepsilon}e_{n+1}(\varepsilon),
\end{aligned} \tag{2.7.5}$$

where $|e_{n+1}(\varepsilon)| \leq K\varepsilon\exp(-\dfrac{\pi^2}{\varepsilon})$ with some K independent of ε and n. Since $g\Big(\xi_-(t_n+\varepsilon)\Big) = \varphi_-(t_n+\varepsilon)$, we have

$$\begin{aligned}
\tilde{b}_{n+1} &= g\Big(\xi_-(t_n+\varepsilon)\Big) + \frac{1}{\varepsilon}e_{n+1}(\varepsilon)\\
&= g\Big(\tilde{q}_{n+1} - e_{n+1}(\varepsilon)\Big) + \frac{1}{\varepsilon}e_{n+1}(\varepsilon)
\end{aligned} \tag{2.7.6}$$

On the other hand, using (2.7.3) and the definition of ϕ, we obtain

$$\begin{aligned}
\begin{pmatrix}\tilde{q}_{n+1}\\ \tilde{b}_{n+1}\end{pmatrix} &= \phi\begin{pmatrix}q_n\\ p_n - \triangle_\varepsilon(n)\end{pmatrix} = \begin{pmatrix}q_n + \varepsilon p_n + \varepsilon^2\sin(q_n) - \varepsilon\triangle_\varepsilon(n)\\ p_n + \varepsilon\sin(q_n) - \triangle_\varepsilon(n)\end{pmatrix},\\
&= \begin{pmatrix}q_{n+1} - \varepsilon\triangle_\varepsilon(n)\\ p_{n+1} - \triangle_\varepsilon(n)\end{pmatrix}.
\end{aligned} \tag{2.7.7}$$

With (2.7.3) and (2.7.6) this implies

$$\begin{aligned}\triangle_\varepsilon(n+1) &= p_{n+1} - b_{n+1} = p_{n+1} - g(q_{n+1}) \\ &= \triangle_\varepsilon(n) + \tilde{b}_{n+1} - g\Big(\tilde{q}_{n+1} + \varepsilon\triangle_\varepsilon(n)\Big) \\ &= \triangle_\varepsilon(n) + g\Big(\tilde{q}_{n+1} - e_{n+1}(\varepsilon)\Big) - g\Big(\tilde{q}_{n+1} + \varepsilon\triangle_\varepsilon(n)\Big) + \frac{1}{\varepsilon}e_{n+1}(\varepsilon).\end{aligned}$$

Using Taylor expansion we obtain

$$\triangle_\varepsilon(n+1) = \big(1 - \varepsilon g'(\theta_{n+1})\big)\triangle_\varepsilon(n) + \frac{1}{\varepsilon}\big(1 - \varepsilon g'(\theta_{n+1})\big)e_{n+1}(\varepsilon) \tag{2.7.8}$$

where $\tilde{q}_{n+1} - e_{n+1}(\varepsilon) < \theta_{n+1} < \tilde{q}_{n+1} + \varepsilon\triangle_\varepsilon(n)$.

Now g is ε-close to the curve $p = -2\sin(q/2)$, hence

$$g'(\theta_{n+1}) = -\cos\left(\frac{\theta_{n+1}}{2}\right) + O(d).$$

Thus given any positive $\mu < \pi$, there is a positive constant c such that for all $q_1 \le \mu$, all n and sufficiently small d,

$$1 - \varepsilon g'(\theta_{n+1}) \ge 1 + \varepsilon c \ .$$

It is now convenient to write (2.7.8) in the form

$$\triangle_\varepsilon(n) = \big(1 - \varepsilon g'(\theta_{n+1})\big)^{-1}\triangle_\varepsilon(n+1) - \frac{1}{\varepsilon}e_{n+1}(\varepsilon)$$

As $\triangle_\varepsilon(n) \to 0$ as $n \to \infty$, this implies that that there is a positive constant K such that

$$\big|\triangle_\varepsilon(n)\big| \le Ke^{-\frac{\pi^2}{\varepsilon}}\sum_{k=0}^{\infty}(1+\varepsilon c)^{-k}$$

Consequently

$$\triangle_\varepsilon(n) = O\bigg(\frac{1}{\varepsilon}\exp\Big(-\frac{\pi^2}{\varepsilon}\Big)\bigg). \tag{2.7.9}$$

In particular

$$\mathrm{Dist}_v((q_1,p_1),\tilde{W}_s) = O\bigg(\frac{1}{\varepsilon}\exp\Big(-\frac{\pi^2}{\varepsilon}\Big)\bigg), \tag{2.7.10}$$

where (q_1, p_1) is any point on the stable manifold W_s^-, provided $q_1 \le \mu < \pi$; here Dist_v denotes the vertical distance.

The estimate (2.7.10) can be extended to any $\mu < 2\pi$ and a starting point (q_1, p_1) with $q_1 \le \mu$ in the following way. The relation (2.7.8) remains valid, only now we just have the existence of some constant $c > 0$ such that $1 - \varepsilon g'(\theta_{n+1}) \ge 1 - \varepsilon c$ for all n. As system (2.1.2) can be regarded as a one-step numerical method for the system (2.1.3) of differential equations and the starting point is at a distance $\mathcal{O}(\varepsilon)$ of its solution $(q_0(t), q_0'(t))$, results on the convergence of one-step methods can be applied and yield that $q_k = q_0(t_1 + (k-1)\varepsilon) + \mathcal{O}(\varepsilon)$, where $q_0(t_1) = q_1$, uniformly for integer k, $1 \le k \le L/\varepsilon$, where L is any positive constant. We choose L such $q_0(t_1 + L) \le \mu/2 < \pi$. Repeated application of (2.7.8) now gives

$$|\triangle_\varepsilon(n) - \triangle_\varepsilon(n + [L/\varepsilon])| \le K e^{-\frac{\pi^2}{\varepsilon}} \sum_{k=0}^{[L/\varepsilon]} (1 - \varepsilon c)^{-k} \le \frac{K e^{Lc}}{c\varepsilon} e^{-\frac{\pi^2}{\varepsilon}} \quad .$$

To the quantity $\triangle_\varepsilon(n + [L/\varepsilon])$, inequality (2.7.9) can be applied, because $q_{n+[L/\varepsilon]} \le \mu/2 < \pi$. Thus (2.7.9) and hence also (2.7.10) remain valid also uniformly for $0 < q_1 \le \mu$, provided $\mu < 2\pi$.

The analogous reasoning applies to the vertical distance of a point $(\tilde{q}_1, \tilde{p}_1)$ on the unstable manifold W_u^+ from the manifold $\tilde{W}_u$ and yields

$$\mathrm{Dist}_v((\tilde{q}_1, \tilde{p}_1), \tilde{W}_u) = O\left(\frac{1}{\varepsilon} \exp\left(-\frac{\pi^2}{\varepsilon}\right)\right). \tag{2.7.11}$$

Another method to obtain (2.7.11) consists in using (2.7.10) and symmetry.

Now we will estimate the vertical distance between the two manifolds $\tilde{W}_s$ and $\tilde{W}_u$. As the quasi-solution $\mathcal{A}(d, u)$ is defined for $-1 < u =: \tanh(t) < 1$, we can define

$$\begin{aligned} \mathcal{A}^+(d, u) &= -\mathcal{A}(d, -u), \\ \xi_+(t) &= \sqrt{1 - u(t)^2}\mathcal{A}^+\big(d, u(t)\big) + 2\pi - q_{0d}(-t) = 2\pi - \xi_-(-t), \end{aligned} \tag{2.7.12}$$

$$D_\varepsilon(t) = \xi_+(t) - \xi_-(t) \quad \text{for } -\frac{4}{3} < t < \frac{4}{3} \quad . \tag{2.7.13}$$

Using (2.7.12) and the definition of $\xi_-(t)$ we find

$$\begin{aligned} D_\varepsilon(t) &= -\sqrt{1 - u(t)^2}\Big(\mathcal{A}\big(d, u(t)\big) + \mathcal{A}\big(d, -u(t)\big)\Big) - q_{0d}(t) - q_{0d}(-t) + 2\pi \\ &= -\sqrt{1 - u(t)^2}\Big(\mathcal{A}\big(d, u(t)\big) + \mathcal{A}\big(d, -u(t)\big)\Big) \end{aligned} \tag{2.7.14}$$

With (2.5.5) and (2.6.23) this implies

$$
\begin{aligned}
D_\varepsilon(t) &= -\sqrt{1-u(t)^2}\,Q(d,u)\Big[\alpha_1 \frac{u}{\tau_2}\Big(\mathcal{H}_0(d,u)-\mathcal{H}_0(d,-u)\Big) - \\
&\qquad \beta_1\Big(\mathcal{H}_2(d,u)+\mathcal{H}_2(d,-u)\Big)+\delta d\Big(\mathcal{H}_1(d,u)+\mathcal{H}_1(d,-u)\Big)\Big]
\end{aligned}
$$

where $Q(d,u)$ is defined in (2.5.6), α_1, β_1 are defind in (2.6.27) and $\mathcal{H}_i(d,u)$ are defined in (2.6.9) and can be continued analytically to $-1 < u \leq 1$ as in (2.6.10).

Using the fact that the functions $u\,h_0(s,u), h_1(s,u), h_2(s,u)$ in (2.6.9) are odd, we can apply the residue theorem and obtain for $-1 < u < 1$ that

$$
\begin{aligned}
\mathcal{H}_i(d,u)+\mathcal{H}_i(d,-u) &= \sum_{Im(s_k(t))<0} 2\pi i \,\mathrm{Res}\Big(e^{-\frac{s}{d}}h_i(s,u), s_k(t)\Big) \\
&\quad - \sum_{Im(s_k(t))>0} 2\pi i \,\mathrm{Res}\Big(e^{-\frac{s}{d}}h_i(s,u), s_k(t)\Big), \quad i=0,1,2,
\end{aligned}
$$

where $s_k(t) = \pi^2 \pm \frac{2d\pi\, t}{\varepsilon}\, i + 2k\pi^2$ for $k \geq 0$. We obtain

$$
\begin{aligned}
\mathrm{Res}\Big(e^{-\frac{s}{d}}h_0(s,u), s_k(t)\Big) &= \frac{1}{2} e^{-\dfrac{(k+1)\pi^2}{d}} e^{\mp\dfrac{2\pi t i}{\varepsilon}} \\
\mathrm{Res}\Big(e^{-\frac{s}{d}}h_1(s,u), s_k(t)\Big) &= \pm\frac{i}{4\pi} e^{-\dfrac{(k+1)\pi^2}{d}} e^{\mp\dfrac{2\pi t i}{\varepsilon}} \\
\mathrm{Res}\Big(e^{-\frac{s}{d}}h_2(s,u), s_k(t)\Big) &= \pm\frac{i}{4\varepsilon d}\big(\pi\varepsilon i \mp 2dt\big) e^{-\dfrac{(k+1)\pi^2}{d}} e^{\mp\dfrac{2\pi t i}{\varepsilon}}
\end{aligned}
$$

and hence

$$
D_\varepsilon(t) = \frac{1}{d^2}\phi_1(t,\varepsilon)\exp\left(-\frac{\pi^2}{d}\right) + O\Big(e^{-\frac{\pi^2}{d}}\Big). \tag{2.7.15}
$$

where

$$
\begin{aligned}
\phi_1(t,\varepsilon) &= 2\pi\alpha\left[\sinh\left(\frac{dt}{\varepsilon}\right) + t/\cosh\left(\frac{dt}{\varepsilon}\right)\right]\sin\left(\frac{2\pi t}{\varepsilon}\right) \\
&+ \left[\frac{\pi\alpha}{\cosh\left(\frac{dt}{\varepsilon}\right)}\right]\cos\left(\frac{2\pi t}{\varepsilon}\right)
\end{aligned} \tag{2.7.16}
$$

As a consequence of (2.7.15) and (2.7.12), we obtain immediately that

$$\xi^{\pm}(0) = \pi + \mathcal{O}(\varepsilon^{-2} e^{-\pi^2/\varepsilon}) \ . \tag{2.7.17}$$

Now, let us take a point $(\xi_+(t), \varphi_+(t))$ on the manifold $\tilde{W}_u$. We suppose that the point $(\xi_-(t_1), \varphi_-(t_1))$ is its vertical projection on the manifold $\tilde{W}_s$. We will evaluate the vertical distance between these two points

$$\mathrm{Dist}_v(t) = \varphi_+(t) - \varphi_-(t_1) = \varphi_+(t) - g\big(\xi_+(t)\big), \tag{2.7.18}$$

where $g(x) = \varphi_-\Big(\xi_-^{-1}(x)\Big)$. Thus by (2.7.13)

$$\mathrm{Dist}_v(t) = \varphi_+(t) - g\big(\xi_-(t) + D_\varepsilon(t)\big). \tag{2.7.19}$$

Using Taylor expansion, we find

$$\mathrm{Dist}_v(t) = \varphi_+(t) - g\big(\xi_-(t)\big) - D_\varepsilon(t) g'\big(\eta(t)\big), \tag{2.7.20}$$

where $\xi_-(t) < \eta(t) < \xi_-(t) + D_\varepsilon(t)$. Here

$$\eta(t) \ = \ q_{0d}(t) + O(d), \ \ \text{hence} \tag{2.7.21}$$

$$g'\big(\eta(t)\big) \ = \ -\cos\left(\frac{\eta(t)}{2}\right) + O(d) = -\tanh\left(\frac{dt}{\varepsilon}\right) + O(d). \tag{2.7.22}$$

As $g\big(\xi_-(t)\big) = \varphi_-(t)$, this yields

$$\begin{aligned} \mathrm{Dist}_v(t) \ &= \ \varphi_+(t) - \varphi_-(t) - D_\varepsilon(t) g'_-\big(\eta(t)\big), \\ &= \ \frac{1}{\varepsilon}\big(\xi_+(t) - \xi_+(t-\varepsilon)\big) - \frac{1}{\varepsilon}\big(\xi_-(t) - \xi_-(t-\varepsilon)\big) - D_\varepsilon(t) g'_-\big(\eta(t)\big), \\ &= \ \frac{1}{\varepsilon}\big(D_\varepsilon(t) - D_\varepsilon(t-\varepsilon)\big) - D_\varepsilon(t) g'\big(\eta(t)\big). \end{aligned} \tag{2.7.23}$$

Now formula (2.7.15) applies and we obtain

$$D_\varepsilon(t-\varepsilon) \ = \ D_\varepsilon(t) + \frac{1}{d}\phi_2(t,\varepsilon) \exp\left(-\frac{\pi^2}{d}\right) + O\left(e^{-\frac{\pi^2}{d}}\right) \tag{2.7.24}$$

where

$$\begin{aligned} \phi_2(t,\varepsilon) \ &= \ -2\pi\alpha\left[\frac{\cosh\left(\frac{dt}{\varepsilon}\right)^2 + 1}{\cosh\left(\frac{dt}{\varepsilon}\right)} - t\frac{\tanh\left(\frac{dt}{\varepsilon}\right)}{\cosh\left(\frac{dt}{\varepsilon}\right)}\right] \sin\left(\frac{2\pi t}{\varepsilon}\right) \\ &+ \ \pi\alpha\left[\frac{\tanh\left(\frac{dt}{\varepsilon}\right)}{\cosh\left(\frac{dt}{\varepsilon}\right)}\right] \cos\left(\frac{2\pi t}{\varepsilon}\right) \ . \end{aligned} \tag{2.7.25}$$

With (2.7.23) and (2.7.22) this implies

$$\mathrm{Dist}_v(t) = \frac{1}{d^2}\left[-\phi_2(t,\varepsilon) + \tanh\left(\frac{dt}{\varepsilon}\right)\phi_1(t,\varepsilon)\right] e^{-\frac{\pi^2}{d}} + O\left(\frac{1}{d} e^{-\frac{\pi^2}{d}}\right) \ . \quad (2.7.26)$$

Consequently, for $-\dfrac{4}{3} \leq t \leq \dfrac{4}{3}$

$$\mathrm{Dist}_v(t) = \frac{4\pi\alpha}{\varepsilon^2}\cosh(t)\sin\left(\frac{2\pi t}{\varepsilon}\right) e^{-\frac{\pi^2}{\varepsilon}} + O\left(\frac{1}{\varepsilon} e^{-\frac{\pi^2}{\varepsilon}}\right) \quad \text{as } \varepsilon \searrow 0. \quad (2.7.27)$$

Combining this result with (2.7.10) and (2.7.11), we finally obtain our main result theorem 1.1, because $\xi_\pm(t) = q_0(t) + \mathcal{O}(\varepsilon^2)$ uniformly with respect to t on any finite interval.

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