



Etude de quelques EDP non linéaires sans compacité

Habib Yazidi

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Habib Yazidi. Etude de quelques EDP non linéaires sans compacité. Mathématiques [math]. Université Paris XII Val de Marne, 2006. Français. tel-00138957

HAL Id: tel-00138957

<https://tel.archives-ouvertes.fr/tel-00138957>

Submitted on 28 Mar 2007

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THÈSE

présentée par

Habib YAZIDI

pour obtenir le titre de

DOCTEUR DE L'UNIVERSITÉ PARIS XII

Spécialité: Mathématiques

Titre: *Etude de quelques EDP non linéaires sans compacité*

Soutenue le 27 Janvier 2006 devant le jury composé de

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Remerciements

Je tiens tout d'abord à exprimer ma gratitude envers mon Directeur de thèse Rejeb HADIJI, qui a su me guider pendant mon mémoire de DEA, puis pendant ma thèse. Ses encouragements, sa disponibilité, ses intuitions et sa vision claire des mathématiques, ont été pour moi très stimulants.

Je remercie infiniment Roger LEWANDOWSKI et Donato PASSASEO d'avoir rapporté sur ma thèse. Leurs remarques et leurs suggestions m'ont été très précieuses. Je suis très heureux et très honoré qu'ils aient tous les deux accepté de faire partie de mon jury.

Je tiens à exprimer ma sincère reconnaissance à Frank PACARD pour ses précieuses remarques. Je suis très honoré qu'il a accepté de faire partie de mon jury.

Sami BARAKET et Jean Michel CORON me font un immense honneur et un très grand plaisir d'être membre de mon jury, je les en remercie chaleureusement.

Je remercie le Laboratoire d'Analyse et de Mathématiques Appliqués de Paris XII, son Directeur Frank Pacard et tous les membres de m'avoir accueilli et de m'avoir donné des conditions de travail très favorables. Je remercie particulièrement, Colette Guillopé d'avoir faciliter mon service d'enseignement. Je remercie également Benoit Daniel et Jacques Printems pour les informations enrichissantes qu'ils m'ont fournies sur quelques questions mathématiques. Aussi je tiens à remercier vivement Eva Locherbach d'avoir eu la gentillesse de corriger l'anglais d'une grande partie de ce manuscrit de thèse, je suis néanmoins le seul responsable des erreurs que l'on rencontrera par la suite.

Je tiens à exprimer mon amitié envers tous les thésards en mathématiques de Paris XII, anciens et nouveaux.

J'adresse tout mon respect à tous mes professeurs, particulièrement, mes professeurs de la faculté des Sciences de Tunis et de DEA d'Analyse Numérique 2001-2002 de Paris VI.

Je salue tous mes amis d'hier, d'aujourd'hui et de demain. Leur présence et leur soutien constant m'ont permis, et j'espère me permettront toujours, de réaliser mes rêves.

Je dédie cette thèse à l'âme de mon père, à ma mère, à mes soeurs, à mes frères et à tous ceux qui comptent pour moi.

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Chapter 1

Introduction

Cette thèse a pour objet l'étude de quelques équations aux dérivées partielles non linéaires de type Dirichlet ou Neumann, à structure variationnelle, présentant un défaut de compacité. Ce sujet trouve son origine dans des problèmes de géométrie différentielle (problème de Yamabe avec ou sans bord) ou des problèmes de physique (les équations de Yang-Mills).

Etant donnée la structure variationnelle de ces problèmes, les solutions cherchées correspondent aux points critiques d'une fonctionnelle F , elles sont solutions de l'équation $F'(u) = 0$. La difficulté principale commune à tous ces problèmes est le manque de compacité. Pour remédier à cette difficulté, on utilise une forme déguisée de compacité : la condition de Palais-Smale (PS).

Définition 1.0.1.

Soit F une fonction de classe C^1 sur un Banach E . On dit que F vérifie la condition (PS) si pour toute suite (u_j) telle que $F(u_j)$ reste bornée et $\|F'(u_j)\| \rightarrow 0$, alors (u_j) est relativement compacte.

Cette condition empêche les points critiques de F de "fuir à l'infini". Cependant, la condition (PS) est trop forte, c'est à dire il existe des suites non relativement compactes pour lesquelles F reste bornée et sa dérivée tend vers zéro. Aussi, une fonction F peut atteindre son minimum sans vérifier la condition (PS). On introduit, alors, une condition plus faible $(PS)_c$ (voir Brézis-Coron-Nirenberg [BCN]).

Définition 1.0.2.

Soit $c \in \mathbb{R}$ fixé. On dit que F vérifie $(PS)_c$ si pour toute suite (u_j) telle que $F(u_j) \rightarrow c$ et $\|F'(u_j)\| \rightarrow 0$, alors (u_j) est relativement compacte.

Cette amélioration qui peut paraître négligeable est en fait extrêmement utile dans les applications. En pratique, on est conduit au programme suivant:

1) Identifier les valeurs de c pour lesquelles $(PS)_c$ tombe en défaut.

2) Montrer que $\inf F$ n'est pas une telle valeur.

Cette approche conduit à l'existence de solutions pour des équations à structures variationnelles. Le point d'orgue dans cette direction, est l'accomplissement de la preuve de la conjecture de Yamabe par Aubin [A] et Schoen. La résolution du problème de Yamabe marque une étape dans le développement de l'analyse non linéaire.

Depuis, plusieurs travaux fondateurs ont été mis au point, citons par exemples [BN1], [BaC], [C], [CHL], [Le1], [Pa], pour des problèmes avec condition de Dirichlet, et [AM], [AY], [Ch], [E], pour des problèmes avec condition de Neumann.

Cette thèse se répartit en quatre chapitres. Le chapitre 1 est une introduction générale. Dans le chapitre 2, on étudie un problème avec la condition de Dirichlet homogène et avec l'exposant critique de Sobolev.

Dans les chapitres 3 et 4, on s'intéresse à des problèmes de Neumann non linéaires avec une non-linéarité critique au bord.

1.1 Problèmes de Dirichlet non linéaires faisant intervenir l'exposant critique du Sobolev

On étudie une équation aux dérivées partielles elliptique non linéaire avec poids faisant intervenir l'exposant critique de Sobolev de type

$$(P1)_\Omega \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = u^{q-1} + \lambda u & \text{dans } \Omega, \\ u > 0 & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

où Ω est un ouvert borné régulier de \mathbb{R}^N , $N \geq 3$, $p : \bar{\Omega} \rightarrow \mathbb{R}$ est une fonction strictement positive avec $p \in H^1(\Omega) \cap C(\bar{\Omega})$, $\lambda \in \mathbb{R}$ et $q = \frac{2N}{N-2}$.

On rappelle que pour $t \in [1, q[$ l'injection de Sobolev $H_0^1(\Omega) \hookrightarrow L^t(\Omega)$ est compacte, et ne l'est plus lorsque $t = q$. L'exposant $q = \frac{2N}{N-2}$ est appelé exposant critique de Sobolev. On introduit alors la meilleure constante de Sobolev

$$S = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^q dx\right)^{\frac{2}{q}}}.$$

dont les propriétés essentielles sont (voir [BN1]): S ne dépend pas du domaine Ω , S n'est jamais atteinte si $\Omega \neq \mathbb{R}^N$, S est atteinte sur \mathbb{R}^N et les fonctions extrémales sont de la forme

$$U_{x_0, \varepsilon}(x) = \alpha_N \frac{\varepsilon^{\frac{N-2}{4}}}{(\varepsilon + |x - x_0|^2)^{\frac{N-2}{2}}}, \quad x \in \mathbb{R}^N$$

où $\varepsilon > 0$ est un paramètre de concentration, $x_0 \in \mathbb{R}^N$ est un paramètre de translation et α_N est une constante choisie de telle sorte que chacune de ces fonctions vérifie:

$$-\Delta U_{x_0, \varepsilon} = U_{x_0, \varepsilon}^{q-1} \quad \text{sur } \mathbb{R}^N.$$

Cette partie repose sur deux résultats fondamentaux d'existence concernant le problème $(P1)_\Omega$ dans le cas $p \equiv 1$. Le premier, dû à Brezis-Nirenberg [BN1], tient compte de la perturbation linéaire λu , le second, dû à Coron [C], tient compte de la géométrie de l'ouvert Ω lorsque $\lambda = 0$.

Les méthodes utilisées dans ce travail, font appel à la recherche de minima de la fonctionnelle associée à $(P1)_\Omega$ et au principe de Concentration-Compacité de Lions [L]. On utilise également une variante du Théorème d'Ambrosetti-Rabinowitz [AR].

1.1.1 Influence de la perturbation linéaire

On généralise les résultats de [BN1] concernant $(P1)_\Omega$ au cas où le poids $p \neq 1$. On remarque que l'existence de solutions dans ce cas dépend du paramètre λ et du comportement de la fonction p au voisinage de ses minima.

On considère alors $p_0 = \min\{p(x), x \in \bar{\Omega}\}$ et on suppose que $p^{-1}(\{p_0\}) \cap \Omega \neq \emptyset$.

Soit $a \in p^{-1}(\{p_0\}) \cap \Omega$, on suppose que, dans un voisinage de a

$$(1.1.1) \quad p(x) = p_0 + \beta_k |x - a|^k + |x - a|^k \theta(x),$$

avec $k > 0$, $\beta_k > 0$ et $\theta(x)$ tend vers 0 quand x tend vers a .

Dans le cas où $0 < k \leq 2$ la situation est plus délicate, alors on se restreint au cas où p vérifie la condition supplémentaire

$$(1.1.2) \quad k\beta_k \leq \frac{\nabla p(x) \cdot (x - a)}{|x - a|^k} \quad \text{p.p } x \in \Omega.$$

On a le théorème suivant

Théorème 1.

On suppose que $p \in H^1(\Omega) \cap C(\bar{\Omega})$ vérifie (1.1.1). Soit λ_1^{div} la première valeur propre de l'opérateur $\text{div}(p(x)\nabla \cdot)$ sur Ω avec condition de Dirichlet homogène. On a

1) Si $n \geq 4$ et $k > 2$, alors pour tout $\lambda \in]0, \lambda_1^{\text{div}}[$ il existe une solution de $(P1)_\Omega$.

2) Si $n \geq 4$ et $k = 2$, alors il existe une constante $\tilde{\gamma}(n) = \frac{(n-2)n(n+2)}{4(n-1)}\beta_2$ telle que pour tout $\lambda \in]\tilde{\gamma}(n), \lambda_1^{\text{div}}[$ il existe une solution de $(P1)_\Omega$.

3) Si $n = 3$ et $k \geq 2$, alors il existe une constante $\gamma(k) > 0$ telle que pour tout $\lambda \in]\gamma(k), \lambda_1^{\text{div}}[$ le problème $(P1)_\Omega$ admet une solution.

4) Si $n \geq 3$, $0 < k < 2$ et p vérifie la condition (1.1.2) alors il existe $\lambda^ \in [\tilde{\beta}_k \frac{n^2}{4}, \lambda_1^{\text{div}}[$, où $\tilde{\beta}_k = \beta_k \min[(\text{diam } \Omega)^{k-2}, 1]$, tel que pour tout $\lambda \in]\lambda^*, \lambda_1^{\text{div}}[$, le problème $(P1)_\Omega$*

admet une solution.

5) Si $n \geq 3$ et $k > 0$, alors pour tout $\lambda \leq 0$ le problème $(P1)_\Omega$ n'a pas de solution minimisante.

6) Si $n \geq 3$ et $k > 0$, alors le problème $(P1)_\Omega$ n'admet pas de solution pour tout $\lambda \geq \lambda_1^{div}$.

Dans la démonstration de ce théorème, l'outil de base est une technique de minimisation. Dans le cas $k > 2$, une estimation fine de l'énergie permet de conclure. Cependant, dans le cas $0 < k \leq 2$, on utilise en plus un lemme de type Hardy (voir [HLP]).

On obtient également grâce à l'identité de Pohozaev, des résultats de non existence pour le problème $(P1)_\Omega$ dans le cas d'un domaine étoilé.

1.1.2 Influence de la géométrie du domaine

Dans le cas où $p \equiv 1$ et $\lambda = 0$, l'identité de Pohozaev montre que le problème $(P1)_\Omega$ n'admet pas de solution pour un domaine étoilé. Cependant, Coron [C] a montré l'existence de solutions concernant le problème $(P1)_\Omega$ pour un domaine avec un petit trou.

Dans le cas où $p \neq 1$, on remarque, en utilisant l'identité de Pohozaev, que pour $\lambda = 0$ et pour $p \in H^1(\Omega) \cap C(\bar{\Omega})$ vérifiant $\nabla p(x) \cdot (x - a) \geq 0$ p.p sur Ω , le problème $(P1)_\Omega$ n'admet pas de solution pour un domaine étoilé par rapport à a . On modifie alors la géométrie du domaine afin d'obtenir une solution dans ce dernier cas.

Soit $\Omega \subset \mathbb{R}^N$, $N \geq 3$ un domaine borné régulier et étoilé par rapport à a , et soit $\varepsilon > 0$. On étudie le problème $(P1)_{\Omega_\varepsilon}$ pour $\lambda = 0$ avec $\Omega_\varepsilon = \Omega \setminus B(a, \varepsilon)$.

On montre le résultat suivant

Théorème 2.

On suppose que $p \in H^1(\Omega) \cap C(\bar{\Omega})$ vérifie $\nabla p(x) \cdot (x - a) \geq 0$ p.p Ω , alors il existe $\varepsilon_0 = \varepsilon_0(\Omega, p) > 0$ tel que pour tout $\varepsilon < \varepsilon_0$, le problème $(P1)_{\Omega_\varepsilon}$ admet une solution.

On s'inspire des méthodes utilisées dans [C] et dans [H]. L'idée de la démonstration consiste à étudier la fonctionnelle

$$F(u) = \frac{1}{2} \int_{\Omega_\varepsilon} p(x) |\nabla u|^2 dx - \frac{1}{q} \int_{\Omega_\varepsilon} |u|^q dx, \quad u \in H_0^1(\Omega).$$

Les solutions de $(P1)_{\Omega_\varepsilon}$ correspondent aux points critiques non nuls de F . En général F ne vérifie pas la condition de Palais Smale (PS), mais on montre que F vérifie une condition plus faible $(PS)_c$ pour $c \in]\frac{1}{N}(p_0 S)^{\frac{N}{2}}, \frac{2}{N}(p_0 S)^{\frac{N}{2}}[$. On applique alors une variante du Théorème d'Ambrosetti-Rabinowitz [AR] qui permet d'obtenir une solution.

1.2 Problèmes de Neumann non linéaires avec une non-linéarité critique au bord

On s'intéresse à des équations de Neumann non-linéaires avec poids et avec une non-linéarité critique au bord du type

$$(P2) \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = \lambda u + f(x, u) & \text{dans } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{sur } \partial\Omega, \end{cases}$$

où $\Omega \subset \mathbb{R}^N$, $N \geq 3$, est un domaine borné avec un bord régulier $\partial\Omega$, ν est la normale extérieure à $\partial\Omega$, Q est une fonction continue et positive dans $\partial\Omega$, $p \in H^1(\Omega)$ est continue et positive dans $\bar{\Omega}$, $q = \frac{2(N-1)}{N-2}$ et $f(\cdot, u)$ est un terme d'ordre inférieur à q . On sait que pour $t \in [1, q[$ l'injection de $H^1(\Omega) \hookrightarrow L^t(\partial\Omega)$ est compacte, et ne l'est plus lorsque $t = q$. L'exposant $q = \frac{2(N-1)}{N-2}$ est appelé exposant critique de Sobolev pour l'inclusion de trace de $H^1(\Omega)$ dans $L^q(\partial\Omega)$.

On introduit la meilleure constante de Sobolev de l'injection de trace $H^1(\mathbb{R}_+^N) \hookrightarrow L^q(\partial\mathbb{R}_+^N)$

$$S_1 = \inf_{u \in H^1(\mathbb{R}_+^N) \setminus \{0\}} \frac{\int_{\mathbb{R}_+^N} |\nabla u|^2 dx}{\left(\int_{\partial\mathbb{R}_+^N} |u|^q dx \right)^{\frac{2}{q}}}$$

où $\mathbb{R}_+^N = \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\}$.

On sait d'après Lions [L] que S_1 est atteinte, et d'après Escobar [E], que les fonctions minimisantes sont de la forme

$$W(x', x_N) = \frac{c_N}{[(1 + x_N)^2 + |x'|^2]^{\frac{N-2}{2}}}$$

où $c_N > 0$ est une constante qui ne dépend que de la dimension N .

De plus

$$S_1 = \frac{N-2}{2} \omega_N^{\frac{1}{N-1}}$$

où $\omega_N =$ volume de sphère unité S^{N-1} .

On considère la condition géométrique, suivante :

On dit que (g.c) est satisfaite au point $y \in \partial\Omega$ s'il existe un voisinage $U(y)$ tel que $\Omega \cap U(y)$ est contenu dans le demi-espace délimité par le plan tangent à $\partial\Omega$ au point y . Il s'agit donc d'un point où toutes les courbures principales sont positives.

On note qu'on a toujours l'existence d'un point $y \in \partial\Omega$ pour lequel (g.c) est satisfaite, voir [AY].

On sait d'après [AM] que la courbure moyenne de $\partial\Omega$ et la condition géométrique (g.c) en un point $y \in \partial\Omega$ jouent des rôles importants dans la résolution des problèmes de type (P2).

Dans le cas où $p \equiv 1 \equiv Q$ et $f(x, u) \equiv 0$, (P2) a été traité par plusieurs auteurs, on cite par exemple [AY], [Ch], [CY].

On se place dans le cas où p et Q sont des fonctions positives continues, et $f(x, u)$ est, soit de la forme αu^r , soit une fonction qui ne dépend pas de u .

Soit $x_0 \in \partial\Omega$ vérifiant

$$\frac{p(x_0)}{(Q(x_0))^{N-2}} = \min_{x \in \partial\Omega} \frac{p(x)}{(Q(x))^{N-2}}.$$

On suppose que

$$(1.2.1) \quad |p(x) - p(x_0)| = o(|x - x_0|)$$

et

$$(1.2.2) \quad |Q(x) - Q(x_0)| = o(|x - x_0|)$$

pour x proche de x_0 .

On définit

$$\tilde{S} = \frac{1}{2(N-1)} \frac{p(x_0)}{Q(x_0)^{N-2}} S_1^{N-1}.$$

1.2.1 Le cas où $f(x, u) = \alpha u^r$

On s'intéresse au problème suivant

$$(1.2.3) \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = \lambda u + \alpha|u|^{r-2}u & \text{dans } \Omega, \\ u \neq 0 & \text{dans } \Omega \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{sur } \partial\Omega, \end{cases}$$

avec $\alpha \in \mathbb{R}$ et $2 < r < q$.

On considère la fonctionnelle associée au problème (1.2.3)

$$I_\alpha = \frac{1}{2} \int_\Omega p(x)|\nabla u|^2 dx - \frac{\lambda}{2} \int_\Omega |u|^2 dx - \frac{\alpha}{r} \int_\Omega |u|^r dx - \frac{1}{q} \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x.$$

On commence par étudier l'existence de solutions au problème (1.2.3) dans le cas où le paramètre λ est strictement négatif et α un réel quelconque. Plus précisément on a le théorème suivant

Théorème 3.

Soit $\Omega \in \mathbb{R}^N$, $N \geq 3$ un domaine borné avec un bord $\partial\Omega$ régulier vérifiant (g.c) en x_0 et $H(x_0) > 0$. On suppose que p vérifie (1.2.1) et Q vérifie (1.2.2). Alors pour tout $\lambda < 0$ et pour tout $\alpha \in \mathbb{R}$ le problème (1.2.3) possède une solution.

Idée de la preuve: On vérifie que l'énergie I_α satisfait les hypothèses du Lemme du Col, d'où l'existence d'une suite (u_n) vérifiant $I_\alpha(u_n) \rightarrow c$ et $I'_\alpha(u_n) \rightarrow 0$ dans $H^{-1}(\Omega)$ avec c donné par le Lemme du Col.

Dans un premier temps, on vérifie que $c < \tilde{S}_1$. Ensuite, on montre que la suite (u_n) converge fortement vers u dans $H^1(\Omega)$. Pour se faire, on est conduit à distinguer les cas $\alpha \leq 0$ et $\alpha \geq 0$. Dans le cas où $\alpha \leq 0$, on conclut en utilisant les méthodes variationnelles classiques, par contre dans le cas où $\alpha \geq 0$ on fait appel au principe de Concentration-Compacité de Lions [L]. La limite u de la suite (u_n) sera une solution du problème (1.2.3).

Dans le cas où λ est strictement positif. On considère l'opérateur $-\operatorname{div}(p(x)\nabla \cdot)$ avec la condition de Neumann homogène au bord. On note la suite des valeurs propres correspondantes par $\{\lambda_k, k \in \mathbf{N} \setminus \{0\}\}$. Elles vérifient $0 = \lambda_1 < \lambda_2 \leq \dots$. De plus les fonctions propres qui correspondent à λ_1 sont les fonctions constantes. Soit $k \in \mathbf{N} \setminus \{0\}$ tel que $\lambda_{k-1} \neq \lambda_k$, on montre le résultat suivant

Théorème 4.

Soit $\Omega \in \mathbb{R}^N$, $N \geq 3$ un domaine borné avec un bord $\partial\Omega$ régulier vérifiant (g.c) en x_0 et $H(x_0) > 0$. On suppose que p vérifie (1.2.1) et Q vérifie (1.2.2). Alors pour tout $\lambda \in]\lambda_{k-1}, \lambda_k[$ et pour tout $\alpha \geq 0$ le problème (1.2.3) possède une solution.

Concernant la preuve de Théorème 4, on raisonne comme dans le Théorème 3 et on conclut à l'aide de principe de min-max (voir [W]).

1.2.2 Le cas où $f(x, u) = f(x)$

On considère l'équation suivante

$$(1.2.4) \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = \lambda u + f(x) & \text{dans } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{sur } \partial\Omega, \end{cases}$$

On s'intéresse à l'existence de solutions du problème (1.2.4) qui ne sont pas identiquement nulles sur le bord $\partial\Omega$. On suppose alors que f est telle que

$$(1.2.5) \quad \text{le problème } \begin{cases} -\operatorname{div}(p(x)\nabla u) = \lambda u + f(x) & \text{dans } \Omega, \\ \frac{\partial u}{\partial \nu} = u = 0 & \text{sur } \partial\Omega, \end{cases} \quad \text{n'admet pas de solution.}$$

Notons que l'hypothèse (1.2.5) est satisfaite, par exemple si f ne change pas de signe. En effet, si (1.2.5) n'est pas satisfaite pour une fonction f de signe constant, alors on obtient une contradiction grâce au Lemme de Hopf.

On distingue deux parties principales.

Dans la première partie, on s'intéresse au cas où λ est strictement négatif. Pour étudier l'existence de solutions du problème (1.2.4), certaines conditions sur f sont nécessaires pour cette approche. On les résume ici

$$(H1) \left\{ \begin{array}{l} f \in H^{-1}(\Omega) \cap C(\bar{\Omega}) \setminus \{0\} \text{ satisfait la condition (1.2.5)} \\ \text{et} \\ (*) \int_{\Omega} f(x) u \, dx < \sigma_N \left(\int_{\Omega} p(x) |\nabla u|^2 \, dx - \lambda \int_{\Omega} |u|^2 \, dx \right)^{\frac{N}{2}}, \\ \forall u \in H^1(\Omega), \int_{\partial\Omega} p(x) Q(x) |u|^q \, ds_x = 1. \end{array} \right.$$

avec $\sigma_N = (q - 2) \left[\frac{1}{q-1} \right]^{\frac{q-1}{q-2}}$.

On note que (*) est satisfaite par exemple si $\|f\|_{H^{-1}} < M$, où M est une constante connue qui ne dépend que de λ , de p , de Q et de l'injection $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$.

Le premier résultat de cette partie est le suivant

Théorème 5.

Soit $\Omega \in \mathbb{R}^N$, $N \geq 3$ un domaine borné avec un bord $\partial\Omega$ régulier vérifiant (g.c) en x_0 et tel que $H(x_0) > 0$. On suppose que f vérifie (H1), p vérifie (1.2.1) et Q vérifie (1.2.2). Alors pour tout $\lambda < 0$ on a

$$I_0 := \inf_{\left(\int_{\partial\Omega} p(x) Q(x) |u|^q \, ds_x\right)^{\frac{1}{q}} = 1} \left[\sigma_N \left(\int_{\Omega} p(x) |\nabla u|^2 \, dx - \lambda \int_{\Omega} |u|^2 \, dx \right)^{\frac{N}{2}} - \int_{\Omega} f(x) u \, dx \right]$$

est atteint et $I_0 > 0$.

Ce Théorème joue un rôle important dans la preuve de l'existence de solutions du problème (1.2.4).

La démonstration de ce Théorème repose sur des idées de Brezis-Nirenberg [BN2] concernant un problème de minimisation de type Dirichlet.

Le deuxième résultat de cette partie est le Théorème suivant

Théorème 6.

Soit $\Omega \in \mathbb{R}^N$, $N \geq 3$ un domaine borné avec un bord $\partial\Omega$ régulier vérifiant (g.c) en x_0 et $H(x_0) > 0$. On suppose que f vérifie (H1), p vérifie (1.2.1) et Q vérifie (1.2.2). Lorsque $N = 3$, on suppose de plus que f , p et Q sont assez régulières. On a alors pour tout $\lambda < 0$ le problème (1.2.4) admet au moins deux solutions.

L'idée de la preuve suit la stratégie développée par Tarantello dans [T] pour traiter un problème de type Dirichlet.

La fonctionnelle correspondante au problème (1.2.4) est donnée par

$$I(u) = \frac{1}{2} \int_{\Omega} p(x) |\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \int_{\Omega} f(x) u dx - \frac{1}{q} \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x, \quad u \in H^1(\Omega).$$

On note

$$I_1(u) = \int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx - (q-1) \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x.$$

On utilise les ensembles suivants:

$$\Lambda = \{u \in H^1(\Omega) : \langle I'(u), u \rangle = 0\}, \quad \Lambda^+ = \{u \in \Lambda : I_1(u) > 0\}, \quad \Lambda_0 = \{u \in \Lambda : I_1(u) = 0\}$$

et $\Lambda^- = \{u \in \Lambda : I_1(u) < 0\}$.

Pour montrer l'existence de deux solutions, on considère les problèmes de minimisation suivants

$$(1.2.6) \quad c_0 = \inf_{\Lambda} I(u)$$

et

$$(1.2.7) \quad c_1 = \inf_{\Lambda^-} I(u).$$

Pour montrer que l'infimum (1.2.6) est atteint pour une certaine fonction u_0 on fait appel au principe variationnel d'Ekeland [AE]. Par la suite, on vérifie que $u_0 \in \Lambda^+$ et qu'elle correspond à un minimum local. Ainsi on a une première solution.

Dans le but de montrer que (1.2.7) est atteint, on applique le principe variationnel d'Ekeland, on obtient une suite $(u_n) \subset \Lambda^-$ qui vérifie $I(u_n) \rightarrow c_1$ et $I'(u_n) \rightarrow 0$ dans $H^{-1}(\Omega)$. On montre en utilisant le principe de Concentration-Compacité [L] que si $c_1 < c_0 + \tilde{S}_1$, alors la suite (u_n) converge fortement vers une fonction u_1 dans $H^1(\Omega)$.

La comparaison entre c_1 et $c_0 + \tilde{S}_1$ présente une grande difficulté. Pour la surmonter, on compare séparément c_1 , $c_0 + \tilde{S}_1$ avec la valeur c donnée par le Lemme du Col. On obtient $c_1 \leq c < c_0 + \tilde{S}_1$.

Finalement, on vérifie que $u_1 \in \Lambda^-$. Ainsi on a une deuxième solution du problème (1.2.4).

Dans la deuxième partie, on s'intéresse au cas où le paramètre λ est strictement positif. On considère le spectre $\{\lambda_1, \lambda_2, \dots\}$ de l'opérateur $-\operatorname{div}(p(x)\nabla \cdot)$ avec la condition de Neumann homogène au bord.

Soit $k \in \mathbf{N} \setminus \{0\}$ tel que $\lambda_{k-1} \neq \lambda_k$. On pose $E_k^- = \operatorname{span}\{e_1, \dots, e_l\}$, où e_1, \dots, e_l sont les fonctions propres qui correspondent aux valeurs propres $\lambda_1, \dots, \lambda_{k-1}$. On a

la décomposition orthogonale suivante $H^1(\Omega) = E_k^- \oplus E_k^+$.

Les hypothèses sur f sont données par

$$(H2) \quad f \in E_k^+ \cap C(\bar{\Omega}) \setminus \{0\} \text{ satisfait (1.2.5).}$$

Le théorème principal de cette partie est le suivant

Théorème 7.

Soit $\Omega \in \mathbb{R}^N$, $N \geq 3$ un domaine borné avec un bord $\partial\Omega$ régulier vérifiant (g.c) en x_0 et telle que $H(x_0) > 0$. On suppose que p vérifie (1.2.1), Q vérifie (1.2.2) et que f vérifie (H2). Dans les cas où $N = 3$ et $N = 4$, on suppose de plus que $f(x_0) \neq 0$. Alors pour tout $\lambda \in]\lambda_{k-1}, \lambda_k[$, il existe une constante $\alpha = \alpha(p, Q, \lambda, N) > 0$ telle que si $\|f\|_{L^2} \leq \alpha$ alors le problème (1.2.4) admet au moins une solution.

Dans ce résultat, le fait que f soit dans E_k^+ permet de vérifier les hypothèses du théorème de min-max (voir [W]) et ainsi d'avoir une solution du problème (1.2.4).

Part I

Problèmes de Dirichlet non linéaires faisant intervenir l'exposant critique du Sobolev

Chapter 2

Problem with critical Sobolev exponent and with weight

(This work in collaboration with R. Hadiji has been submitted to the Chinese Annals of Mathematics serie B)

2.1 Introduction

In this paper we study the following problem:

$$(2.1.1) \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = u^{q-1} + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^n , $n \geq 3$, $p : \bar{\Omega} \rightarrow \mathbb{R}$ is a given positive weight such that $p \in H^1(\Omega) \cap C(\bar{\Omega})$, λ is a real constant and $q = \frac{2n}{n-2}$ is the critical exponent for the Sobolev embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$.

In [BN1], Brezis and Nirenberg treated the case where p is constant. They proved, in particular, the existence of a solution of (2.1.1) for $0 < \lambda < \lambda_1$ if $n \geq 4$ and for $\lambda^* < \lambda < \lambda_1$ if $n = 3$, where λ_1 is the first eigenvalue of $-\Delta$ on Ω with zero Dirichlet boundary condition and λ^* is a positive constant.

In this paper, we extend this result to the general case of where p is not constant. The study of problem (2.1.1), shows that the existence of solutions depends, apart from parameter λ , on the behavior of p near its minima and on the geometry of the domain Ω .

Set $p_0 = \min\{p(x), x \in \bar{\Omega}\}$, we suppose that $p^{-1}(\{p_0\}) \cap \Omega \neq \emptyset$ and let $a \in p^{-1}(\{p_0\}) \cap \Omega$.

In the first part of this work, we study the effect of the behavior of p near its minima on the existence of solution for our problem. The method that is mostly relied upon,

apart from the identities of Pohozeav, is the adaptations to the new context of the arguments developed in [BN].

We assume that, in a neighborhood of a , p behaves like

$$(2.1.2) \quad p(x) = p_0 + \beta_k |x - a|^k + |x - a|^k \theta(x),$$

with $k > 0$, $\beta_k > 0$ and $\theta(x)$ tends to 0 when x tends to a .

Note that the parameter k will play an essential role in the study of our problem. Indeed, 2 appears as a critical value for k . More precisely the case $k > 2$ is treated by a classical procedure, however the case $0 < k \leq 2$ is less easily accessible. Therefore, in this case, we restrict ourself to the case where p satisfies the additional condition

$$(2.1.3) \quad k\beta_k \leq \frac{\nabla p(x) \cdot (x - a)}{|x - a|^k} \quad \text{a.e } x \in \Omega.$$

Let us notice that if p is sufficiently smooth, then condition (2.1.2) follows directly from Taylor's expansion of p near a .

The fact that 2 is a critical value for k appears clearly in dimension $n = 4$, therefore, in this dimension and with the aim of obtaining more explicit results, we assume moreover that θ satisfies $\int_{B(a,1)} \frac{\theta(x)}{|x-a|^4} dx < \infty$. Let us emphasize that this last condition is not necessary to prove the existence of solutions.

Moreover, in dimension $n = 3$, the problem is more delicate, then we treat it in a particular case; more precisely for $p(x) = p_0 + \beta_k |x - a|^k$, $k > 0$.

The first result of this work is the following

Theorem 2.1.1.

Assume that $p \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfies (2.1.2). Let λ_1^{div} be the first eigenvalue of $\text{div}(p(x)\nabla \cdot)$ on Ω with zero Dirichlet boundary condition, we have

- 1) *If $n \geq 4$ and $k > 2$, then for every $\lambda \in]0, \lambda_1^{div}[$ there exists a solution of (2.1.1).*
- 2) *If $n \geq 4$ and $k = 2$, then there exists a constant $\tilde{\gamma}(n) = \frac{(n-2)n(n+2)}{4(n-1)}\beta_2$ such that for every $\lambda \in]\tilde{\gamma}(n), \lambda_1^{div}[$ there exists a solution of (2.1.1).*
- 3) *If $n = 3$ and $k \geq 2$, then there exists a constant $\gamma(k) > 0$ such that for every $\lambda \in]\gamma(k), \lambda_1^{div}[$ there exists a solution of (2.1.1).*
- 4) *If $n \geq 3$, $0 < k < 2$ and p satisfies the condition (2.1.3) then there exists $\lambda^* \in [\tilde{\beta}_k \frac{n^2}{4}, \lambda_1^{div}[$, where $\tilde{\beta}_k = \beta_k \min[(\text{diam } \Omega)^{k-2}, 1]$, such that for any $\lambda \in]\lambda^*, \lambda_1^{div}[$ problem (2.1.1) admits a solution.*
- 5) *If $n \geq 3$ and $k > 0$, then for every $\lambda \leq 0$ there is no minimizing solution of equation (2.1.1).*
- 6) *If $n \geq 3$ and $k > 0$, then there is no solution of problem (2.1.1) for every $\lambda \geq \lambda_1^{div}$.*

Remark 2.1.1.

In general, the intervals $] \tilde{\gamma}(n), \lambda_1^{div}[$ in 2) and $[\tilde{\beta}_k \frac{n^2}{4}, \lambda_1^{div}[$ in 4), may be empty. But

there are some sufficient conditions for which the above intervals are nonempty:

1) If $p_0 > \frac{n(n-4)}{(n-1)(n-2)^2} \beta_2 (\text{diam } \Omega)^2$, then $\tilde{\gamma}(n) < \lambda_1^{\text{div}}$.

Notice that this condition is always true if n is rather large.

2) If $p_0 > \frac{\tilde{\beta}_k n^2}{(n-2)^2 (\text{diam } \Omega)^2}$, then $\tilde{\beta}_k \frac{n^2}{4} < \lambda_1^{\text{div}}$.

We postpone the proof of 1) and 2) in this remark until the end of section 2.3.

The second part of this work is dedicated to the study of the effect of the geometry of the domain on the existence of solutions of our problem. More precisely, since for $\lambda = 0$ and $p \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfying $\nabla p(x) \cdot (x - a) > 0$ a.e in Ω , the problem (2.1.1) does not have a solution for a starshaped domain about a , we will modify the geometry of Ω in order to find a solution. Therefore, let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a starshaped domain about a and let $\varepsilon > 0$, we will study the existence of solution of the problem

$$(I_\varepsilon) \quad \begin{cases} -\text{div}(p(x)\nabla u) = u^{q-1} & \text{in } \Omega_\varepsilon, \\ u > 0 & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \Omega \setminus B(a, \varepsilon)$.

For $p \equiv 1$ and $\lambda = 0$, the problem (2.1.1) has been first investigated in [C] and an interesting result of existence has been proved for domains with holes. In [BaC], this last result is extended to all domains having "nontrivial" topology (in a suitable sense). This nontrivially condition (which covers a large class of domains) is only sufficient for the solvability but not necessary as shown by some examples of contractible domains Ω for which (2.1.1) has solutions (see [D], [Di], [Pa]).

In other direction, [Le1] shows that the solution of [C], on a domain with a hole of diameter ε and center x_0 , concentrates at the point x_0 . In [H], the author generalized the result of [C] for the case where u^q is replaced by $u^q + \mu u^\alpha$, where $\mu \in \mathbb{R}$ and $1 < \alpha < q$.

In this work, we consider the case where $p \in H^1(\Omega) \cap C(\bar{\Omega})$ and satisfying $\nabla p(x) \cdot (x - a) > 0$ a.e on $\Omega \setminus \{a\}$. The method we use in this part is an adaptation of those used in [C] and [H]. More particularly, we use the min-max techniques and a variant of the Ambrosetti-Rabinowitz theorem [AR].

The second result of this work is the following

Theorem 2.1.2.

There exists $\varepsilon_0 = \varepsilon_0(\Omega, p) > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the problem (I_ε) has at least one solution in $H_0^1(\Omega_\varepsilon)$.

The rest of this paper is divided into three sections. In Section 1 some preliminary results will be established. Section 2 and Section 3 are devoted respectively to the proof of Theorem 2.1.1 and the proof of Theorem 2.1.2.

2.2 Some preliminary results

We start by recalling some notations which will be frequently used throughout the rest of this paper. First, we define

$$S = \inf_{u \in H_0^1, \|u\|_q=1} \|\nabla u\|_2^2$$

that corresponds to the best constant for the Sobolev embedding $H_0^1(\Omega) \subset L^q(\Omega)$. Let us denote by $U_{a,\varepsilon}$ denote an extremal function for the Sobolev inequality

$$U_{a,\varepsilon}(x) = \frac{1}{(\varepsilon + |x - a|^2)^{\frac{n-2}{2}}}, \quad x \in \mathbb{R}^n.$$

We set

$$(2.2.1) \quad u_{a,\varepsilon}(x) = \zeta(x)U_{a,\varepsilon}(x), \quad x \in \mathbb{R}^n,$$

where $\zeta \in C_0^\infty(\bar{\Omega})$ is a fixed function such that $0 \leq \zeta \leq 1$, and $\zeta \equiv 1$ in some neighborhood of a included in Ω .

We know from [BN1] that

$$(2.2.2) \quad \|\nabla u_{a,\varepsilon}\|_2^2 = \frac{K_1}{\varepsilon^{\frac{n-2}{2}}} + O(1),$$

$$(2.2.3) \quad \|u_{a,\varepsilon}\|_q^2 = \frac{K_2}{\varepsilon^{\frac{n-2}{2}}} + O(\varepsilon)$$

and

$$(2.2.4) \quad \|u_{a,\varepsilon}\|_2^2 = \begin{cases} \frac{K_3}{\varepsilon^{\frac{n-4}{2}}} + O(1) & \text{if } n \geq 5 \\ \frac{\omega_4}{2} |\log \varepsilon| + O(1) & \text{if } n = 4 \end{cases}$$

where K_1 and K_2 are positive constants with $\frac{K_1}{K_2} = S$, ω_4 is the area of S^3 and $K_3 = \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^{n-2}} dx$.

We shall state some auxiliary results.

For $p \in C^1(\bar{\Omega})$ or $p \in H^1(\Omega) \cap C(\bar{\Omega})$ and $\nabla p(x) \cdot (x - a) \geq 0$ a.e $x \in \Omega$, we consider

$$\alpha(p) = \frac{1}{2} \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_{\Omega} \nabla p(x) \cdot (x - a) |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx}.$$

We easily see that $\alpha(p) \in [-\infty, +\infty[$, and we have the following result

Proposition 2.2.1.

- 1) If $p \in C^1(\Omega)$ and there exists $b \in \Omega$ such that $\nabla p(b)(b-a) < 0$, then $\alpha(p) = -\infty$.
 2) If $p \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfying (2.1.2) and $\nabla p(x) \cdot (x-a) \geq 0$ a.e $x \in \Omega$, we have
 2.a) If $k > 2$ and $p \in C^1(\Omega)$, then $\alpha(p) = 0$ for all $n \geq 3$.
 2.b) If $0 < k \leq 2$ and p satisfies condition (2.1.3) then for all $n \geq 3$ we have

$$\frac{k}{2} \beta_k \left(\frac{n+k-2}{2} \right)^2 (\text{diam } \Omega)^{k-2} \leq \alpha(p).$$

Proof.

We start by proving 1). Set $q(x) = \nabla p(x) \cdot (x-a) \forall x \in \Omega$ and let $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \varphi \leq 1$ on \mathbb{R}^n , $\varphi \equiv 1$ on the ball $\{x, |x| < r\}$, and $\varphi \equiv 0$ outside the ball $\{x, |x| < 2r\}$, where $r < 1$ is a positive constant .

Set $\varphi_j(x) = \varphi(j(x-b))$ for $j \in \mathbf{N}^*$. We have

$$\begin{aligned} \alpha(p) &\leq \frac{1}{2} \frac{\int_{\Omega} q(x) |\nabla \varphi_j(x)|^2 dx}{\int_{\Omega} |\varphi_j|^2 dx} \\ &\leq \frac{1}{2} \frac{\int_{B(b, \frac{2r}{j})} q(x) |\nabla \varphi_j(x)|^2 dx}{\int_{B(b, \frac{2r}{j})} |\varphi_j|^2 dx}. \end{aligned}$$

Using the change of variable $y = j(x-b)$, we obtain

$$\alpha(p) \leq \frac{j^2 \int_{B(0, 2r)} q(\frac{y}{j} + b) |\nabla \varphi(x)|^2 dx}{2 \int_{B(0, 2r)} |\varphi|^2 dx}.$$

Applying the Dominated Convergence Theorem, we obtain

$$\alpha(p) \leq \frac{j^2}{2} \left[q(b) \frac{\int_{B(0, 2r)} |\nabla \varphi(x)|^2 dx}{\int_{B(0, 2r)} |\varphi|^2 dx} + o(1) \right].$$

Letting $j \rightarrow \infty$, we deduce the desired result.

Now we will prove 2.a).

Using (2.1.2) and since $p \in C^1(\Omega)$ in a neighborhood V of a , we write

$$(2.2.5) \quad p(x) = p_0 + \beta_k |x-a|^k + \theta_1(x),$$

where $\theta_1 \in C^1(V)$ is such that

$$(2.2.6) \quad \lim_{x \rightarrow a} \frac{\theta_1(x)}{|x-a|^k} = 0.$$

Looking at (2.2.6), we deduce that there exists $0 < r < 1$, such that

$$(2.2.7) \quad \theta_1(x) \leq |x-a|^k \quad \forall x \in B(a, 2r).$$

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a function such that $0 \leq \varphi \leq 1$ on \mathbb{R}^n , $\varphi \equiv 1$ on the ball $\{x, |x| < r\}$, and $\varphi \equiv 0$ outside the ball $\{x, |x| < 2r\}$. Set $\varphi_j(x) = \varphi(j(x - a))$ for $j \in \mathbf{N}^*$, we have

$$0 \leq \alpha(p) \leq \frac{1}{2} \frac{\int_{\Omega} \nabla p(x) \cdot (x - a) |\nabla \varphi_j(x)|^2 dx}{\int_{\Omega} |\varphi_j|^2 dx}.$$

Using (2.2.5), we obtain

$$0 \leq \alpha(p) \leq \frac{k\beta_k \int_{B(a, \frac{2r}{j})} |x - a|^k |\nabla \varphi_j(x)|^2 dx}{2 \int_{B(a, \frac{2r}{j})} |\varphi_j|^2 dx} + \frac{1}{2} \frac{\int_{B(a, \frac{2r}{j})} \nabla \theta_1(x) \cdot (x - a) |\nabla \varphi_j(x)|^2 dx}{\int_{B(a, \frac{2r}{j})} |\varphi_j|^2 dx}.$$

Performing the change of variable $y = j(x - a)$, and integrating by parts the second term of the right hand side, we obtain

$$0 \leq \alpha(p) \leq \frac{k\beta_k}{2j^{k-2}} \frac{\int_{B(0, 2r)} |y|^k |\nabla \varphi(y)|^2 dx}{\int_{B(0, 2r)} |\varphi|^2 dx} + \frac{j}{2} \frac{\int_{B(0, 2r)} \theta_1(\frac{y}{j} + a) \nabla(y |\nabla \varphi(y)|^2) dx}{\int_{B(0, 2r)} |\varphi|^2 dx}.$$

Using (2.2.7), we write

$$0 \leq \alpha(p) \leq \frac{k\beta_k}{2j^{k-2}} \frac{\int_{B(0, 2r)} |y|^k |\nabla \varphi(y)|^2 dx}{\int_{B(0, 2r)} |\varphi|^2 dx} + \frac{1}{2j^{k-1}} \frac{\int_{B(0, 2r)} |y|^k \nabla(|\nabla \varphi(y)|^2 y) dx}{\int_{B(0, 2r)} |\varphi|^2 dx}.$$

Therefore, for $k > 2$ we deduce that $\alpha(p) = 0$, and this finishes the proof of this case. Now, in order to prove 2.b), we need to recall the following Hardy's inequality, see for example [CKN] or Theorem 330 in [HLP].

Lemma 2.2.1.

Let $t \in \mathbb{R}$ such that $t + n > 0$, we have $\forall u \in H_0^1(\Omega)$

$$\int_{\Omega} |x|^t |u|^2 dx \leq \left(\frac{2}{n+t}\right)^2 \int_{\Omega} |x \cdot \nabla u|^2 |x|^t dx.$$

Moreover the constant $\left(\frac{2}{n+t}\right)^2$ is optimal and is not achieved.

Now we prove 2.b). Since p satisfies (2.1.3), we have for all $u \in H_0^1(\Omega) \setminus \{0\}$,

$$\frac{\int_{\Omega} \nabla p(x) \cdot (x - a) |\nabla u(x)|^2 dx}{\int_{\Omega} |u(x)|^2 dx} \geq k\beta_k \frac{\int_{\Omega} |x - a|^k |\nabla u(x)|^2 dx}{\int_{\Omega} |u(x)|^2 dx}.$$

By applying the last Lemma for $0 < k = 2 + t \leq 2$, we find

$$\frac{\int_{\Omega} \nabla p(x) \cdot (x - a) |\nabla u(x)|^2 dx}{\int_{\Omega} |u|^2 dx} \geq k\beta_k \left(\frac{n+k-2}{2}\right)^2 (\text{diam } \Omega)^{k-2}.$$

This implies that $\alpha(p) \geq \frac{k}{2}\beta_k \left(\frac{n+k-2}{2}\right)^2 (\text{diam } \Omega)^{k-2}$. □

Let us give the following non-existence result

Proposition 2.2.2.

We assume that $\alpha(p) > -\infty$. There is no solution for (2.1.1) when $\lambda \leq \alpha(p)$ and Ω is a starshaped domain about a .

Proof. This follows from Pohozev's identity. Suppose that u is a solution of (2.1.1). We first multiply (2.1.1) by $\nabla u(x) \cdot (x - a)$, next we integrate over Ω and we obtain

$$(2.2.8) \quad \int_{\Omega} u^{q-1} \nabla u(x) \cdot (x - a) dx = -\frac{n-2}{2} \int_{\Omega} |u(x)|^q dx,$$

$$(2.2.9) \quad \lambda \int_{\Omega} u \nabla u(x) \cdot (x - a) dx = -\frac{n}{2} \lambda \int_{\Omega} |u(x)|^2 dx$$

and

$$(2.2.10) \quad \begin{aligned} \int_{\Omega} -\operatorname{div}(p(x) \nabla u) \nabla u(x) \cdot (x - a) dx &= -\frac{n-2}{2} \int_{\Omega} p(x) |\nabla u(x)|^2 dx \\ &- \frac{1}{2} \int_{\Omega} \nabla p(x) \cdot (x - a) |\nabla u(x)|^2 dx \\ &- \frac{1}{2} \int_{\partial\Omega} p(x) (x - a) \cdot \nu \left| \frac{\partial u}{\partial \nu} \right|^2 dx, \end{aligned}$$

where ν denotes the outward normal to $\partial\Omega$.

Before continuing the proof, we give some details about (2.2.10). Indeed

$$\int_{\Omega} -\operatorname{div}(p(x) \nabla u) \nabla u(x) \cdot (x - a) dx = -\sum_{i=1}^n \sum_{j=1}^n \int_{\Omega} \partial_i(p(x) \partial_i u) \partial_j u \cdot (x_j - a_j) dx$$

where $\partial_i = \frac{\partial}{\partial x_i}$ and x_i is such that $x = (x_1, \dots, x_i, \dots, x_n)$

Integrating by parts the right hand side of the above equality, we obtain

$$(2.2.11) \quad \begin{aligned} - \int_{\Omega} \partial_i(p(x) \partial_i u) \partial_j u \cdot (x_j - a_j) dx &= \int_{\Omega} p(x) \partial_i u \partial_i \partial_j u (x_j - a_j) dx \\ &+ \int_{\Omega} p(x) \partial_i u \partial_j u \partial_i (x_j - a_j) dx \\ &- \int_{\partial\Omega} p(x) |\nabla u|^2 (x - a) \cdot \nu dx. \end{aligned}$$

On other hand, integrating by parts we see that

$$\begin{aligned} \int_{\Omega} p(x) \partial_i u \partial_i \partial_j u (x_j - a_j) dx &= -\frac{1}{2} \int_{\Omega} \partial_j p(x) \partial_i u \partial_i u \cdot (x_j - a_j) dx \\ &- \frac{1}{2} \int_{\Omega} p(x) \partial_i u \partial_i u \partial_i (x_j - a_j) dx \\ &+ \frac{1}{2} \int_{\partial\Omega} p(x) |\nabla u|^2 (x - a) \cdot \nu dx. \end{aligned}$$

Inserting the above equality in (2.2.11), we get

$$\begin{aligned}
-\int_{\Omega} \partial_i(p(x)\partial_i u)\partial_j u.(x_j - a_j)dx &= -\frac{1}{2}\int_{\Omega} \partial_j p(x)\partial_i u\partial_i u.(x_j - a_j)dx \\
&+ \frac{1}{2}\int_{\Omega} p(x)\partial_i u\partial_j u\partial_i(x_j - a_j)dx \\
&- \frac{1}{2}\int_{\partial\Omega} p(x)|\nabla u|^2(x - a).\nu dx.
\end{aligned}$$

By making the sum over i and j , knowing that when $u = 0$ on the boundary then $\nabla u = \frac{\partial u}{\partial \nu}.\nu$, and using the fact that $\partial_i(x_j - a_j) = 1$ if $i = j$ and 0 if not, we obtain (2.2.10).

Now we return to the proof of Proposition 2.2.2.

Combining (2.2.8), (2.2.9) and (2.2.10), we write

$$\begin{aligned}
(2.2.12) \quad &-\frac{n-2}{2}\int_{\Omega} p(x)|\nabla u(x)|^2 dx - \frac{1}{2}\int_{\Omega} \nabla p(x).(x - a)|\nabla u(x)|^2 dx = \\
&-\frac{n-2}{2}\int_{\Omega} |u(x)|^q dx - \frac{n}{2}\lambda\int_{\Omega} |u(x)|^2 dx.
\end{aligned}$$

On the other hand, we multiply (2.1.1) by $\frac{n-2}{2}u$ and we integrate by parts, we get

$$(2.2.13) \quad \frac{n-2}{2}\int_{\Omega} p(x)|\nabla u(x)|^2 dx = \frac{n-2}{2}\int_{\Omega} |u(x)|^q dx + \frac{n-2}{2}\lambda\int_{\Omega} |u(x)|^2 dx.$$

Combining (2.2.12) and (2.2.13), we obtain

$$\lambda\int_{\Omega} |u(x)|^2 dx - \frac{1}{2}\int_{\Omega} \nabla p(x).(x - a)|\nabla u(x)|^2 dx - \frac{1}{2}\int_{\partial\Omega} p(x)\left|\frac{\partial u}{\partial \nu}\right|^2(x - a).\nu dx = 0.$$

If Ω is starshaped about a , then $(x - a).\nu > 0$ on $\partial\Omega$, and

$$\lambda\int_{\Omega} |u(x)|^2 dx - \frac{1}{2}\int_{\Omega} \nabla p(x).(x - a)|\nabla u(x)|^2 dx > 0.$$

It follows that

$$\lambda > \frac{1}{2}\frac{\int_{\Omega} \nabla p(x).(x - a)|\nabla u(x)|^2 dx}{\int_{\Omega} |u|^2 dx}.$$

Then

$$\lambda \geq \frac{1}{2}\inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_{\Omega} \nabla p(x).(x - a)|\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx}$$

and

$$\lambda \geq \alpha(p).$$

Conclusion: If u is a solution of (2.1.1), then $\lambda \geq \alpha(p)$. \square

2.3 Existence of solutions

Let $\Omega \in \mathbb{R}^n$, $n \geq 3$ be a bounded domain. In this section, we show that (2.1.1) possesses a solution of lower energy less than $p_0 S$. We will use a minimization technique.

Set

$$(2.3.1) \quad Q_\lambda(u) = \frac{\int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx}{\|u\|_q^2}$$

the functional associated to (2.1.1).

We define

$$(2.3.2) \quad S_\lambda(p) = \inf_{u \in H_0^1(\Omega), u \neq 0} Q_\lambda(u).$$

Let us remark that

$$S_\lambda(p) = \inf_{u \in H_0^1(\Omega), \|u\|_q=1} \int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx.$$

The method used for the proof of Theorem 2.1.1 is the following : First we show that $S_\lambda(p) < p_0 S$, we then prove that the infimum $S_\lambda(p)$ is achieved.

We have the following result

Lemma 2.3.1.

If $S_\lambda(p) < p_0 S$ for $\lambda > 0$, then the infimum in (2.3.2) is achieved.

Proof.

Let $\{u_j\} \subset H_0^1(\Omega)$ be a minimizing sequence for (2.3.2) that is,

$$(2.3.3) \quad \|u_j\|_q = 1,$$

$$(2.3.4) \quad \int_\Omega p(x)|\nabla u_j(x)|^2 dx - \lambda \int_\Omega |u_j(x)|^2 dx = S_\lambda(p) + o(1) \quad \text{as } j \rightarrow \infty.$$

The sequence u_j is bounded in $H_0^1(\Omega)$. Indeed, from (2.3.4), we have

$$\int_\Omega p(x)|\nabla u_j(x)|^2 dx = S_\lambda(p) + \lambda \int_\Omega |u_j(x)|^2 dx + o(1).$$

Using the embedding of $L^q(\Omega)$ into $L^2(\Omega)$, there exists a positive constant C_1 such that

$$\int_\Omega p(x)|\nabla u_j(x)|^2 dx \leq S_\lambda(p) + \lambda C_1 \|u_j\|_q^2 + o(1).$$

Using the fact that (see (2.3.3))

$$\|u_j\|_q = 1,$$

we obtain

$$\int_{\Omega} p(x) |\nabla u_j(x)|^2 dx \leq S_{\lambda}(p) + \lambda C_1 + o(1).$$

Since $0 < p_0 \leq p(x)$ for every $x \in \Omega$, we deduce

$$\int_{\Omega} |\nabla u_j(x)|^2 dx \leq \frac{S_{\lambda}(p) + \lambda C_1}{p_0} + o(1).$$

This gives the desired result.

Since $\{u_j\}$ is bounded in $H_0^1(\Omega)$ we may extract a subsequence still denoted by u_j , such that

$$\begin{aligned} u_j &\rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \\ u_j &\rightarrow u \quad \text{strongly in } L^2(\Omega), \\ u_j &\rightarrow u \quad \text{a.e. on } \Omega, \end{aligned}$$

with $\|u\|_q \leq 1$. Set $v_j = u_j - u$, so that

$$\begin{aligned} v_j &\rightharpoonup 0 \quad \text{weakly in } H_0^1(\Omega) \\ v_j &\rightarrow 0 \quad \text{strongly in } L^2(\Omega), \\ v_j &\rightarrow 0 \quad \text{a.e. on } \Omega. \end{aligned}$$

Using (2.3.3), the definition of S and the fact that $\min_{\bar{\Omega}} p(x) = p_0 > 0$, we have

$$\int_{\Omega} p(x) |\nabla u_j(x)|^2 dx \geq p_0 S.$$

From (2.3.4) it follows that $\lambda \|u\|_2^2 \geq p_0 S - S_{\lambda}(p) > 0$ and therefore $u \neq 0$. Using again (2.3.4) we obtain

$$(2.3.5) \quad \int_{\Omega} p(x) |\nabla u(x)|^2 dx + \int_{\Omega} p(x) |\nabla v_j(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx = S_{\lambda}(p) + o(1),$$

since $v_j \rightharpoonup 0$ weakly in $H_0^1(\Omega)$. On the other hand, it follows from a result of Brezis and Lieb [BL] that

$$\|u + v_j\|_q^q = \|u\|_q^q + \|v_j\|_q^q + o(1),$$

(which holds since v_j is bounded in L^q and $v_j \rightarrow 0$ a.e.). Thus, by (2.3.3), we have

$$1 = \|u\|_q^q + \|v_j\|_q^q + o(1)$$

and therefore

$$1 \leq \|u\|_q^2 + \|v_j\|_q^2 + o(1),$$

which leads to

$$(2.3.6) \quad 1 \leq \|u\|_q^q + \frac{1}{p_0 S} \int_{\Omega} p(x) |\nabla v_j(x)|^2 dx + o(1).$$

We distinguish two cases:

(a) $S_{\lambda}(p) > 0$, which corresponds to $0 < \lambda < \lambda_1^{div}$,

(b) $S_{\lambda}(p) \leq 0$, which corresponds to $\lambda \geq \lambda_1^{div}$.

In case (a) we deduce from (2.3.6) that

$$(2.3.7) \quad S_{\lambda}(p) \leq S_{\lambda}(p) \|u\|_q^2 + \left(\frac{S_{\lambda}(p)}{p_0 S}\right) \int_{\Omega} p(x) |\nabla v_j(x)|^2 dx + o(1).$$

Combining (2.3.5) and (2.3.7) we obtain

$$\begin{aligned} \int_{\Omega} p(x) |\nabla u(x)|^2 - \lambda |u(x)|^2 dx + \int_{\Omega} p(x) |\nabla v_j(x)|^2 dx &\leq S_{\lambda}(p) \|u\|_q^2 \\ &+ \left(\frac{S_{\lambda}(p)}{p_0 S}\right) \int_{\Omega} p(x) |\nabla v_j(x)|^2 dx + o(1). \end{aligned}$$

Thus

$$\begin{aligned} \int_{\Omega} p(x) |\nabla u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx &\leq S_{\lambda}(p) \|u\|_q^2 \\ &+ \left[\frac{S_{\lambda}(p)}{p_0 S} - 1 \right] \int_{\Omega} p(x) |\nabla v_j(x)|^2 dx + o(1). \end{aligned}$$

Since $S_{\lambda}(p) < p_0 S$, we deduce

$$(2.3.8) \quad \int_{\Omega} p(x) |\nabla u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx \leq S_{\lambda}(p) \|u\|_q^2,$$

this means that u is a minimum of $S_{\lambda}(p)$.

In case (b), since $\|u\|_q^2 \leq 1$, we have $S_{\lambda}(p) \leq S_{\lambda}(p) \|u\|_q^2$. Again, we deduce (2.3.8) from (2.3.5). This concludes the proof of Lemma 2.3.1. \square

To prove assertion 1) and 2) of Theorem 2.1.1 (case $k \geq 2$), we need the following

Lemma 2.3.2.

a) For $n \geq 4$, we have

$$S_{\lambda}(p) < p_0 S \text{ for all } \lambda > 0 \text{ and for } k > 2.$$

b) For $n = 4$ and $k = 2$, we have

$$S_{\lambda}(p) < p_0 S \text{ for all } \lambda > 4\beta_2.$$

c) For $n \geq 5$ and $k = 2$, we have

$$S_{\lambda}(p) < p_0 S \text{ for all } \lambda > \frac{(n-2)n(n+2)}{4(n-1)} \beta_2.$$

d) For $n = 3$ and $k \geq 2$, we have

$$S_\lambda(p) < p_0 S \quad \text{for all } \lambda > \gamma(k) \text{ where } \gamma(k) \text{ is a positive constant.}$$

Proof.

We shall estimate the ratio $Q_\lambda(u)$ defined in (2.3.1), with $u(\cdot) = u_{a,\varepsilon}(\cdot)$.

We claim that, as $\varepsilon \rightarrow 0$, we have

$$(2.3.9) \quad \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx \leq \begin{cases} p_0 K_1 + O(\varepsilon^{\frac{n-2}{2}}) & \text{if } \begin{cases} n \geq 4 \text{ and} \\ n - 2 < k, \end{cases} \\ p_0 K_1 + A_k \varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}) & \text{if } \begin{cases} n \geq 4 \text{ and} \\ n - 2 > k, \end{cases} \\ p_0 K_1 + \frac{(n-2)^2 (\beta_{n-2} + M) \omega_n \varepsilon^{\frac{n-2}{2}} |\log \varepsilon|}{2} + o(\varepsilon^{\frac{n-2}{2}} |\log \varepsilon|) & \text{if } \begin{cases} n > 4 \text{ and} \\ k = n - 2, \end{cases} \\ p_0 K_1 + 2\beta_2 \omega_4 \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) & \text{if } \begin{cases} n = 4 \text{ and} \\ k = 2, \end{cases} \end{cases}$$

with $K_1 = (n-2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n} dy$, $s = \min(\frac{k}{2}, \frac{n-2}{2})$, $A_k = (n-2)^2 \beta_k \int_{\mathbb{R}^n} \frac{|x|^{k+2}}{(1+|x|^2)^n} dx$ and M is a positive constant.

Verification of (2.3.9)

1. Case $n \geq 4$ and $k > 0$, with $k \neq 2$ if $n = 4$.

We have

$$\begin{aligned} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx &= \int_{\Omega} \frac{p(x) |\nabla \zeta(x)|^2}{(\varepsilon + |x-a|^2)^{n-2}} dx + (n-2)^2 \int_{\Omega} \frac{p(x) |\zeta(x)|^2 |x-a|^2}{(\varepsilon + |x-a|^2)^n} dx \\ &\quad - 2(n-2) \int_{\Omega} \frac{p(x) \zeta(x) \nabla \zeta(x) (x-a)}{(\varepsilon + |x-a|^2)^{n-1}} dx. \end{aligned}$$

Since $\zeta \equiv 1$ on a neighborhood of a , we assume that $\varphi \equiv 1$ on $B(a, l)$ with l is a small positive constant. Therefore we get $|\nabla \varphi|^2 \equiv 0$ on $B(a, l)$ and $\nabla \varphi(x) \cdot (x-a) = 0$ on $B(a, l)$.

Thus, we obtain

$$(2.3.10) \quad \begin{aligned} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx &= \int_{\Omega \setminus B(a,l)} \frac{p(x) |\nabla \zeta(x)|^2}{(\varepsilon + |x-a|^2)^{n-2}} dx + (n-2)^2 \int_{\Omega} \frac{p(x) |\zeta(x)|^2 |x-a|^2}{(\varepsilon + |x-a|^2)^n} dx \\ &\quad - 2(n-2) \int_{\Omega \setminus B(a,l)} \frac{p(x) \zeta(x) \nabla \zeta(x) (x-a)}{(\varepsilon + |x-a|^2)^{n-1}} dx. \end{aligned}$$

On the other hand, we have

$$\frac{p(x)|\nabla\varphi|^2}{(\varepsilon + |x - a|^2)^{n-2}} \leq \frac{p(x)|\nabla\varphi|^2}{|x - a|^{2(n-2)}} \in L^1(\Omega \setminus B(a, l))$$

and

$$\left| \frac{p(x)\zeta(x)\nabla\zeta(x)(x - a)}{(\varepsilon + |x - a|^2)^{n-1}} \right| \leq \left| \frac{p(x)\zeta(x)\nabla\zeta(x)(x - a)}{|x - a|^{2(n-1)}} \right| \in L^1(\Omega \setminus B(a, l)).$$

Therefore, applying the Dominated Convergence Theorem, (2.3.10) becomes

$$\int_{\Omega} p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx = (n-2)^2 \int_{\Omega} \frac{p(x)|\zeta(x)|^2|x - a|^2}{(\varepsilon + |x - a|^2)^n} dx + O(1).$$

Using (2.1.2), a direct computation gives

$$\begin{aligned} \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx &= (n-2)^2 p_0 \varepsilon^{\frac{n-2}{2}} \int_{\Omega} \frac{|x - a|^2}{(\varepsilon + |x - a|^2)^n} dx \\ &+ (n-2)^2 \varepsilon^{\frac{n-2}{2}} \beta_k \int_{\Omega} \frac{|x - a|^{k+2}}{(\varepsilon + |x - a|^2)^n} dx \\ &+ (n-2)^2 \varepsilon^{\frac{n-2}{2}} \int_{\Omega} \frac{|x - a|^{k+2}\theta(x)}{(\varepsilon + |x - a|^2)^n} dx \\ &+ (n-2)^2 \varepsilon^{\frac{n-2}{2}} \int_{\Omega} \frac{|x - a|^{k+2}(\beta_k + \theta(x))(|\zeta(x)|^2 - 1)}{(\varepsilon + |x - a|^2)^n} dx \\ &+ O(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

Using again the definition of ζ , and applying the Dominated Convergence Theorem, we obtain

$$\begin{aligned} \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx &= (n-2)^2 p_0 \varepsilon^{\frac{n-2}{2}} \int_{\Omega} \frac{|x - a|^2}{(\varepsilon + |x - a|^2)^n} dx \\ &+ (n-2)^2 \varepsilon^{\frac{n-2}{2}} \beta_k \int_{\Omega} \frac{|x - a|^{k+2}}{(\varepsilon + |x - a|^2)^n} dx \\ &+ (n-2)^2 \varepsilon^{\frac{n-2}{2}} \int_{\Omega} \frac{|x - a|^{k+2}\theta(x)}{(\varepsilon + |x - a|^2)^n} dx + O(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

Here we will consider the following three subcases:

1.1. If $n - 2 > k$,

$$\begin{aligned} \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx &= p_0 (n-2)^2 \varepsilon^{\frac{n-2}{2}} \left[\int_{\mathbb{R}^n} \frac{|x - a|^2}{(\varepsilon + |x - a|^2)^n} dx - \int_{\mathbb{R}^n \setminus \Omega} \frac{|x - a|^2}{(\varepsilon + |x - a|^2)^n} dx \right] \\ &= (n-2)^2 \varepsilon^{\frac{n-2}{2}} \left[\int_{\mathbb{R}^n} \frac{|x - a|^{k+2}(\beta_k + \theta(x))}{(\varepsilon + |x - a|^2)^n} dx - \int_{\mathbb{R}^n \setminus \Omega} \frac{|x - a|^{k+2}(\beta_k + \theta(x))}{(\varepsilon + |x - a|^2)^n} dx \right] \\ &= O(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

Using a simple change of variable and applying the Dominated Convergence Theorem, we find

$$\begin{aligned} \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx &= p_0 \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n} + (n-2)^2 \varepsilon^{\frac{k}{2}} \int_{\mathbb{R}^n} \frac{|y|^{k+2} (\beta_k + \theta(a + \varepsilon^{\frac{1}{2}} y))}{(1+|y|^2)^n} dy \\ &\quad + o(\varepsilon^{\frac{k}{2}}). \end{aligned}$$

Using the fact that $\theta(x)$ tends to 0 when x tends to a , we obtain

$$\varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx = p_0 K_1 + A_k \varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}),$$

with $K_1 = (n-2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n} dy$ and $A_k = \beta_k \int_{\mathbb{R}^n} \frac{|y|^{k+2}}{(1+|y|^2)^n} dy$.

1.2. If $n-2 < k$,

$$\begin{aligned} \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx &= p_0 K_1 + (n-2)^2 \beta_k \varepsilon^{\frac{n-2}{2}} \int_{\Omega} \frac{|x-a|^{k+2}}{(\varepsilon + |x-a|^2)^n} dx \\ &\quad + (n-2)^2 \varepsilon^{\frac{n-2}{2}} \int_{\Omega} \frac{|x-a|^{k+2} \theta(x)}{(\varepsilon + |x-a|^2)^n} dx + O(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

Since Ω is a bounded domain, there exists some positive constant R such that $\Omega \subset B(a, R)$ and thus

$$\begin{aligned} \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx &= p_0 K_1 + O(\varepsilon^{\frac{n-2}{2}}) \\ &\quad + (n-2)^2 \varepsilon^{\frac{n-2}{2}} \left[\int_{B(a,R)} \frac{|x-a|^{k+2} (\beta_k + \theta(x))}{(\varepsilon + |x-a|^2)^n} dx - \int_{B(a,R) \setminus \Omega} \frac{|x-a|^{k+2} (\beta_k + \theta(x))}{(\varepsilon + |x-a|^2)^n} dx \right]. \end{aligned}$$

By a simple change of variable, we get

$$\begin{aligned} \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx &= p_0 K_1 + (n-2)^2 \varepsilon^{\frac{n-2}{2}} \int_{B(0,R)} \frac{|y|^{k+2} (\beta_k + \theta(a+y))}{(\varepsilon + |y|^2)^n} dy \\ &\quad + O(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

Using the definition of θ given by (2.1.2), there exists a positive constant M such that

$$\begin{aligned} \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx &\leq p_0 K_1 + (n-2)^2 \varepsilon^{\frac{n-2}{2}} (\beta_k + M) \int_{B(0,R)} \frac{|y|^{k+2}}{(\varepsilon + |y|^2)^n} dy \\ &\quad + O(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

Using the Dominated Convergence Theorem we deduce that

$$\varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx \leq p_0 K_1 + O(\varepsilon^{\frac{n-2}{2}})$$

and this completes the proof of (2.3.9) in this case.

1.2. If $k = n - 2$,

$$\begin{aligned} \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx &= p_0 K_1 + (n-2)^2 \beta_{n-2} \varepsilon^{\frac{n-2}{2}} \int_{\Omega} \frac{|x-a|^n}{(\varepsilon + |x-a|^2)^n} dx \\ &+ (n-2)^2 \varepsilon^{\frac{n-2}{2}} \int_{\Omega} \frac{|x-a|^n \theta(x)}{(\varepsilon + |x-a|^2)^n} dx + O(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

Since Ω is a bounded domain, there exists some positive constant R such that $\Omega \subset B(a, R)$ and thus

$$\begin{aligned} \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx &= p_0 K_1 + O(\varepsilon^{\frac{n-2}{2}}) \\ &+ (n-2)^2 \varepsilon^{\frac{n-2}{2}} \left[\int_{B(a,R)} \frac{|x-a|^n (\beta_{n-2} + \theta(x))}{(\varepsilon + |x-a|^2)^n} dx - \int_{B(a,R) \setminus \Omega} \frac{|x-a|^n (\beta_{n-2} + \theta(x))}{(\varepsilon + |x-a|^2)^n} dx \right]. \end{aligned}$$

Hence

$$\begin{aligned} \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx &= p_0 K_1 + (n-2)^2 \varepsilon^{\frac{n-2}{2}} \int_{B(a,R)} \frac{|x-a|^n (\beta_{n-2} + \theta(x))}{(\varepsilon + |x-a|^2)^n} dx \\ &+ O(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

Using the definition of θ given by (2.1.2), there exists a positive constant M such that

$$(2.3.11) \quad \begin{aligned} \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx &\leq p_0 K_1 + (n-2)^2 (\beta_{n-2} + M) \varepsilon^{\frac{n-2}{2}} \int_{B(a,R)} \frac{|x-a|^n}{(\varepsilon + |x-a|^2)^n} dx \\ &+ O(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

On the other hand, a easy computation gives

$$(2.3.12) \quad \varepsilon^{\frac{n-2}{2}} \int_{B(a,R)} \frac{|x-a|^n}{(\varepsilon + |x-a|^2)^n} dx = \omega_n \varepsilon^{\frac{n-2}{2}} \int_0^R \frac{r^{2n-1}}{(\varepsilon + r^2)^n} dr.$$

We see that, for $r > 0$ and for $\varepsilon > 0$ small enough

$$\begin{aligned} ((\varepsilon + r^2)^n)' &= 2n r (\varepsilon + r^2)^{n-1} = 2n r [r^{2n-2} (1 + \frac{\varepsilon}{r^2})^{n-1}] \\ &= 2n r^{2n-1} (1 + (n-1) \frac{\varepsilon}{r^2} + o(\frac{\varepsilon}{r^2})) \\ &= 2n r^{2n-1} + 2n(n-1) \varepsilon r^{2n-3} + r^{2n-3} o(\varepsilon). \end{aligned}$$

Thus

$$r^{2n-1} = \frac{1}{2n} ((\varepsilon + r^2)^n)' - (n-1) \varepsilon r^{2n-3} + r^{2n-3} o(\varepsilon).$$

Therefore, inserting the last equality into (2.3.12), we obtain

$$(2.3.13) \quad \begin{aligned} \varepsilon^{\frac{n-2}{2}} \int_{B(a,R)} \frac{|x-a|^n}{(\varepsilon + |x-a|^2)^n} dx &= \frac{\omega_n \varepsilon^{\frac{n-2}{2}}}{2n} \int_0^R \frac{((\varepsilon + r^2)^n)'}{(\varepsilon + r^2)^n} dr \\ &- \omega_n (n-1) \varepsilon^{\frac{n}{2}} \int_0^R \frac{r^{2n-3}}{(\varepsilon + r^2)^n} dr + o\left(\varepsilon^{\frac{n}{2}} \int_0^R \frac{r^{2n-3}}{(\varepsilon + r^2)^n} dr\right). \end{aligned}$$

Using the change of variable $t = \frac{r}{\varepsilon^{\frac{1}{2}}}$, we obtain

$$\varepsilon^{\frac{n}{2}} \int_0^R \frac{r^{2n-3}}{(\varepsilon + r^2)^n} dr = \varepsilon^{\frac{n-2}{2}} \int_0^{\frac{R}{\varepsilon^{\frac{1}{2}}}} \frac{t^{2n-3}}{(1+t^2)^n} dt$$

Applying the Dominated Convergence Theorem, we get

$$\begin{aligned} \varepsilon^{\frac{n}{2}} \int_0^R \frac{r^{2n-3}}{(\varepsilon+r^2)^n} dr &= \varepsilon^{\frac{n-2}{2}} \int_0^{\frac{R}{\varepsilon^{\frac{1}{2}}}} \frac{t^{2n-3}}{(1+t^2)^n} dt = \varepsilon^{\frac{n-2}{2}} \int_0^{+\infty} \frac{t^{2n-3}}{(1+t^2)^n} dt + o(\varepsilon^{\frac{n-2}{2}}) \\ &= O(\varepsilon^{\frac{n-2}{2}}). \end{aligned}$$

Thus, combining the above estimate with (2.3.13), we deduce

$$\varepsilon^{\frac{n-2}{2}} \int_{B(a,R)} \frac{|x-a|^n}{(\varepsilon + |x-a|^2)^n} dx = \frac{\omega_n}{2n} \varepsilon^{\frac{n-2}{2}} \int_0^R \frac{((\varepsilon + r^2)^n)'}{(\varepsilon + r^2)^n} dr + O(\varepsilon^{\frac{n-2}{2}})$$

and

$$(2.3.14) \quad \varepsilon^{\frac{n-2}{2}} \int_{B(a,R)} \frac{|x-a|^n}{(\varepsilon + |x-a|^2)^n} dx = \frac{\omega_n}{2} \varepsilon^{\frac{n-2}{2}} |\log \varepsilon| + o(\varepsilon^{\frac{n-2}{2}} |\log \varepsilon|).$$

Inserting (2.3.14) into (2.3.11) we obtain

$$\varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx \leq p_0 K_1 + \frac{(n-2)^2 (\beta_{n-2} + M) \omega_n}{2} \varepsilon^{\frac{n-2}{2}} |\log \varepsilon| + o(\varepsilon^{\frac{n-2}{2}} |\log \varepsilon|).$$

2) Case $n = 4$ and $k = 2$.

As we have announced in the introduction, we assume in this case the following additional condition on θ : $\int_{B(a,1)} \frac{\theta(x)}{|x-a|^4} dx < \infty$. We have

$$\begin{aligned} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}|^2 dx &= \int_{\Omega} \frac{p(x) |\nabla \zeta(x)|^2}{(\varepsilon + |x-a|^2)^2} dx + 4 \int_{\Omega} \frac{p(x) |\zeta(x)|^2 |x-a|^2}{(\varepsilon + |x-a|^2)^4} dx \\ &\quad - 4 \int_{\Omega} \frac{p(x) \zeta(x) \nabla \zeta(x) (x-a)}{(\varepsilon + |x-a|^2)^3} dx. \end{aligned}$$

Using (2.1.2) and the fact that $\zeta \equiv 1$ near a , it follows that

$$\begin{aligned} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}|^2 dx &= 4p_0 \int_{\Omega} \frac{|\zeta(x)|^2 |x-a|^2}{(\varepsilon + |x-a|^2)^4} dx + 4\beta_2 \int_{\Omega} \frac{|\zeta(x)|^2 |x-a|^4}{(\varepsilon + |x-a|^2)^4} dx \\ &\quad + 4 \int_{\Omega} \frac{|x-a|^4 \theta(x)}{(\varepsilon + |x-a|^2)^4} dx + O(1), \\ &= \frac{4p_0}{\varepsilon} \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^4} dy + 4 \int_{\Omega} \frac{|x-a|^4 (\beta_k + \theta(x))}{(\varepsilon + |x-a|^2)^4} dx + O(1). \end{aligned}$$

Using the fact that $\int_{B(a,1)} \frac{\theta(x)}{|x-a|^4} dx < \infty$ and applying the Dominated Convergence Theorem , we obtain

$$\begin{aligned} \int_{\Omega} \frac{|x-a|^4 \theta(x)}{(\varepsilon + |x-a|^2)^4} dx &= \int_{\Omega} \frac{\theta(x)}{|x-a|^4} dx + o(1) \\ &= O(1). \end{aligned}$$

Consequently

$$\int_{\Omega} p(x) |\nabla u_{a,\varepsilon}|^2 dx = \frac{4p_0}{\varepsilon} \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^4} dy + 4\beta_k \int_{\Omega} \frac{|x-a|^4}{(\varepsilon + |x-a|^2)^4} dx + O(1).$$

Let $R_i > 0$, $i = 1, 2$ such that

$$\int_{|x-a| \leq R_1} \frac{|x-a|^4}{(\varepsilon + |x-a|^2)^4} dx \leq \int_{\Omega} \frac{|x-a|^4}{(\varepsilon + |x-a|^2)^4} dx \leq \int_{|x-a| \leq R_2} \frac{|x-a|^4}{(\varepsilon + |x-a|^2)^4} dx.$$

We see that

$$\begin{aligned} \int_{|x-a| \leq R} \frac{|x-a|^4}{(\varepsilon + |x-a|^2)^4} dx &= \omega_4 \int_0^R \frac{r^7}{(\varepsilon + r^2)^4} dr, \\ &= \frac{1}{8} \omega_4 \int_0^R \frac{((\varepsilon + r^2)^4)'}{(\varepsilon + r^2)^4} dr - \omega_4 \int_0^R \frac{r\varepsilon^3 + 3r^3\varepsilon^2 + 3\varepsilon r^4}{(\varepsilon + r^2)^4} dr, \\ &= \frac{1}{2} \omega_4 |\log \varepsilon| - \omega_4 \int_0^{\frac{R}{\varepsilon^{\frac{1}{2}}}} \frac{t + 3t^3 + 3t^5}{(1+t^2)^4} dt + O(1), \\ &= \frac{1}{2} \omega_4 |\log \varepsilon| + O(1). \end{aligned}$$

Hence, we have

$$\int_{\Omega} p(x) |\nabla u_{a,\varepsilon}|^2 dx = \frac{p_0 K_1}{\varepsilon} + 2\beta_2 \omega_4 |\log \varepsilon| + O(1),$$

where $K_1 = \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^4} dy$. This completes the proof of (2.3.9).

Let us come back to the proof of Lemma 2.3.2.

It is convenient to rewrite (2.3.9) as

$$(2.3.15) \quad \varepsilon^{\frac{n-2}{2}} \int_{\Omega} p(x) |\nabla u_{a,\varepsilon}|^2 dx \leq \begin{cases} p_0 K_1 + o(\varepsilon) & \text{if } n \geq 5, \text{ and } k > 2, \\ p_0 K_1 + A_2 \varepsilon + o(\varepsilon) & \text{if } n \geq 5, \text{ and } k = 2, \\ p_0 K_1 + A_k \varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}) & \text{if } n \geq 4, \text{ and } k < 2, \\ p_0 K_1 + o(\varepsilon) & \text{if } n = 4, \text{ and } k > 2, \\ p_0 K_1 + 2\omega_4 \beta_2 \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) & \text{if } n = 4, \text{ and } k = 2. \end{cases}$$

Combining (2.3.15), (2.2.3) and (2.2.4), we obtain

$$(2.3.16) \quad S_\lambda(p) \leq Q_\lambda(u_{a,\varepsilon}) \leq \begin{cases} p_0 S - \lambda \frac{K_3}{K_2} \varepsilon + o(\varepsilon) & \text{if } n \geq 5, \text{ and } k > 2, \\ p_0 S - (\lambda - C) \frac{K_3}{K_2} \varepsilon + o(\varepsilon) & \text{if } n \geq 5, \text{ and } k = 2, \\ p_0 S + A_k \varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}) & \text{if } n \geq 4, \text{ and } k < 2, \\ p_0 S - \lambda \frac{\omega_4}{2K_2} \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) & \text{if } n = 4, \text{ and } k > 2, \\ p_0 S - \frac{\omega_4}{2K_2} [\lambda - 4\beta_2] \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) & \text{if } n = 4, \text{ and } k = 2, \end{cases}$$

$$\text{with } C = \frac{A_2}{K_3} = \frac{\beta_2(n-2)n(n+2)}{4(n-1)}.$$

Assertions a), b) and c) of Lemma 2.3.2 follow directly for ε small enough.

Now we prove d) in Lemma 2.3.2 (case $n = 3$ and $k \geq 2$). We will estimate the ratio

$$Q_\lambda(u) = \frac{\int_\Omega p(x) |\nabla u|^2 dx - \lambda \|u\|_2^2}{\|u\|_q^2}$$

with

$$u(x) = u_{\varepsilon,a}(r) = \frac{\zeta(r)}{(\varepsilon + r^2)^{\frac{1}{2}}}, r = |x|, \varepsilon > 0,$$

where ζ is a fixed smooth function satisfying $0 \leq \zeta \leq 1$, $\zeta = 1$ in $\{x, |x - a| < \frac{R}{2}\}$ and $\zeta = 0$ in $\{x, |x - a| \geq R\}$, where R is a positive constant such that $B(a, R) \subset \Omega$.

We claim that, as $\varepsilon \rightarrow 0$,

$$(2.3.17) \quad \int p(x) |\nabla u_{a,\varepsilon}(x)|^2 dx = \frac{p_0 K_1}{\varepsilon^{\frac{1}{2}}} + \omega_3 \int_0^R (p_0 + \beta_k r^k) |\zeta'(r)|^2 dr + \omega_3 k \int_0^R |\zeta|^2 r^{k-2} dr + o(1).$$

And from [BN1], we already have

$$(2.3.18) \quad \|\nabla u_{a,\varepsilon}\|_2^2 = \frac{K_1}{\varepsilon^{\frac{1}{2}}} + \omega_3 \int_0^R |\zeta'(r)|^2 dr + O(\varepsilon^{\frac{1}{2}}),$$

$$(2.3.19) \quad \|u_{a,\varepsilon}\|_6^2 = \frac{K_2}{\varepsilon^{\frac{1}{2}}} + O(\varepsilon^{\frac{1}{2}}),$$

$$(2.3.20) \quad \|u_{a,\varepsilon}\|_2^2 = \omega_3 \int_0^R \zeta^2(r) dr + O(\varepsilon^{\frac{1}{2}}),$$

where K_1 and K_2 are positive constants such that $\frac{K_1}{K_2} = S$ and ω_3 is the area of S^2 .

Verification of (2.3.17).

Using (2.1.2), (2.3.18) and the fact that $\zeta = 0$ in $\{x, |x - a| \geq R\}$, we write

$$\begin{aligned} \int p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx &= \frac{p_0 K_1}{\varepsilon^{\frac{1}{2}}} + \omega_3 p_0 \int_0^R |\zeta'(r)|^2 dr \\ &+ \omega_3 \beta_k \int_0^R \left[\frac{|\zeta'(r)|^2}{\varepsilon + r^2} - \frac{2r\zeta(r)\zeta'(r)}{(\varepsilon + r^2)^2} + \frac{r^2\zeta^2(r)}{(\varepsilon + r^2)^3} \right] r^{k+2} dr \\ &+ O(\varepsilon^{\frac{1}{2}}). \end{aligned}$$

Using the fact that $\zeta = 1$ in $\{x, |x - a| < \frac{R}{2}\}$, $\zeta'(0) = 0$ and $\zeta(R) = 0$, we obtain

$$-2 \int_0^R \frac{\zeta(r)\zeta'(r)r^{k+3}}{(\varepsilon + r^2)^2} dr = (k+3) \int_0^R \frac{|\zeta(r)|^2 r^{k+2}}{(\varepsilon + r^2)^2} dr - 4 \int_0^R \frac{|\zeta(r)|^2 r^{k+4}}{(\varepsilon + r^2)^3} dr.$$

Consequently

$$\begin{aligned} \int_{\Omega} p(x)|\nabla u_{a,\varepsilon}(x)|^2 dx &= \frac{p_0 K_1}{\varepsilon^{\frac{1}{2}}} + \omega_3 p_0 \int_0^R |\zeta'(r)|^2 dr + \omega_3 \beta_k \int_0^R \frac{|\zeta'(r)|^2 r^{k+2}}{\varepsilon + r^2} dr \\ &- 3\omega_3 \beta_k \int_0^R \frac{|\zeta(r)|^2 r^{k+4}}{(\varepsilon + r^2)^3} dr + (k+3)\omega_3 \beta_k \int_0^R \frac{|\zeta(r)|^2 r^{k+2}}{(\varepsilon + r^2)^2} dr \\ &+ O(\varepsilon^{\frac{1}{2}}). \end{aligned}$$

By using the Dominated Convergence Theorem, we get the desired result.

Combining (2.3.17), (2.3.19) and (2.3.20), we obtain

$$\begin{aligned} Q_{\lambda}(u_{a,\varepsilon}) &= p_0 S + \omega_3 \left[\int_0^R (p_0 + \beta_k r^k) |\zeta'(r)|^2 dr + k \beta_k \int_0^R |\zeta(r)|^2 r^{k-2} dr - \lambda \int_0^R \zeta^2(r) dr \right] \frac{\varepsilon^{\frac{1}{2}}}{K_2} \\ &+ O(\varepsilon), \end{aligned}$$

thus,

$$(2.3.21) \quad Q_{\lambda}(u_{a,\varepsilon}) = p_0 S + \frac{\omega_3 \int_0^R \zeta^2(r) dr}{K_2} \left[\frac{\int_0^R (p_0 + \beta_k r^k) |\zeta'(r)|^2 dr + k \int_0^R |\zeta(r)|^2 r^{k-2} dr}{\int_0^R |\zeta(r)|^2 dr} - \lambda \right] \varepsilon^{\frac{1}{2}} + O(\varepsilon).$$

$$\text{Set } D(k, \zeta) = \frac{\int_0^R (p_0 + \beta_k r^k) |\zeta'(r)|^2 dr + k \int_0^R |\zeta(r)|^2 r^{k-2} dr}{\int_0^R |\zeta(r)|^2 dr} \text{ and } \gamma(k) = \inf_H D(k, \zeta)$$

where H is defined by

$$H = \{ \zeta \in C_0^\infty(\bar{\Omega}), 0 \leq \zeta \leq 1, \zeta = 1 \text{ in } \{x, |x - a| < \frac{R}{2}\} \text{ and } \zeta = 0 \text{ in } \{x, |x - a| \geq R\} \}.$$

This finishes the proof of Lemma 2.3.2. \square

Now, we go back to proof of assertion 3) in Theorem 2.1.1 (case $0 < k < 2$). First of all, let us emphasize that if the domain Ω is starshaped about a , the assertion 3) is more interesting. Indeed, it gives a better estimate of the least value of the parameter λ over which there is a solution to problem (2.1.1).

In the case of a non-starshaped domain, combining the fact that $S_0(p) = p_0 S$ with the properties of $S_\lambda(p)$ (see the proof of lemma 2.3.4), we have that there exists $\lambda^* \in [0, \lambda_1^{div}[$ such that for all $\lambda \in]\lambda^*, \lambda_1^{div}[$, the problem (2.1.1) has a solution. Note that we have no other information on λ^* .

Therefore, throughout the rest of this proof, we assume that the domain Ω is starshaped about a .

We need two Lemmas. Let us start by the following

Lemma 2.3.3.

Assume $0 < k \leq 2$. Then there exists a constant $\tilde{\beta}_k = \beta_k \min[(\text{diam } \Omega)^{k-2}, 1]$ such that

$$(2.3.22) \quad S_\lambda(p) = p_0 S \text{ for every } \lambda \in]-\infty, \tilde{\beta}_k \frac{n^2}{4}]$$

and the infimum of $S_\lambda(p)$ is not achieved.

Proof. We know from (2.3.16) that

$$S_\lambda(p) \leq Q_\lambda(u_{a,\varepsilon}) \leq p_0 S + A_k \varepsilon^{\frac{k}{2}} + o(\varepsilon^{\frac{k}{2}}) \quad \text{with } A_k \text{ is a positive constant,}$$

thus

$$S_\lambda(p) \leq p_0 S.$$

On the other hand, we know from Lemma 2.2.2 and Proposition 2.2.1, that for $0 < k \leq 2$, for every $\lambda \leq \frac{k}{2} \beta_k (\frac{n+k-2}{2})^2 (\text{diam } \Omega)^{k-2}$, problem (2.1.1) has no solution. So we exclude the case $S_\lambda(p) < p_0 S$, otherwise, Lemma 2.3.1 will yield in a contradiction.

We conclude that for $0 < k \leq 2$, we have

$$(2.3.23) \quad S_\lambda(p) = p_0 S \quad \text{for every } \lambda \leq \frac{k}{2} \beta_k (\frac{n+k-2}{2})^2 (\text{diam } \Omega)^{k-2}.$$

Now, we consider \tilde{p} defined by

$$(2.3.24) \quad \begin{cases} \tilde{p}(x) = p(x) & \forall x \in \Omega \setminus B(a, r), \\ \tilde{p}(x) = p_0 + \beta_k |x - a|^2 & \forall x \in B(a, \frac{r}{2}), \\ p(x) \geq \tilde{p}(x) \geq p_0 + \beta_k |x - a|^2 & \forall x \in B(a, r) \setminus B(a, \frac{r}{2}), \end{cases}$$

where $r < 1$ is a positive constant.

Since $0 < k \leq 2$, we have $|x - a|^k \geq |x - a|^2$ for every $x \in B(a, r)$ and $p(x) \geq \tilde{p}(x)$

in Ω .

Let $u \in H_0^1(\Omega)$ with $\|u\|_q = 1$, then

$$\int_{\Omega} p(x)|\nabla u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx \geq \int_{\Omega} \tilde{p}(x)|\nabla u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx,$$

thus,

$$(2.3.25) \quad \int_{\Omega} p(x)|\nabla u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx \geq \int_{\Omega} (p_0 + \frac{1}{2}(\tilde{p}(x) - p_0))|\nabla u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx + \frac{1}{2} \int_{\Omega} (\tilde{p}(x) - p_0)|\nabla u(x)|^2 dx.$$

Set $\tilde{p}(x) = p_0 + \frac{1}{2}(\tilde{p}(x) - p_0)$.

From (2.1.3) we deduce that

$$(2.3.26) \quad p(x) - p_0 \geq \beta_k |x - a|^k \quad \text{a.e in } \Omega.$$

Indeed,

Set $f(t) = p(a + t(x - a))$, we have

$$p(x) - p(a) = f(t) - f(0) = \int_0^1 f'(t) dt = \int_0^1 \frac{1}{t} \nabla p(a + t(x - a)) \cdot t(x - a) dt.$$

From (2.1.3), we write

$$p(x) - p(a) = \int_0^1 \frac{1}{t} \nabla p(a + t(x - a)) \cdot t(x - a) dt \geq k \beta_k \int_0^1 t^{k-1} |x - a|^k dt = \beta_k |x - a|^k.$$

Using (2.3.24) and (2.3.26), a simple computation gives $\tilde{p}(x) - p_0 \geq \tilde{\beta}_k |x - a|^2$ a.e in Ω , with $\tilde{\beta}_k = \beta_k \min[(\text{diam } \Omega)^{k-2}, 1]$.

Applying Lemma 2.2.1, we find

$$\int_{\Omega} (\tilde{p}(x) - p_0)|\nabla u(x)|^2 dx \geq \tilde{\beta}_k \frac{n^2}{4} \int_{\Omega} |u(x)|^2 dx.$$

Inequality (2.3.25) becomes for every $u \in H_0^1(\Omega)$,

$$\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \geq \int_{\Omega} \tilde{p}(x)|\nabla u|^2 dx - \left(\lambda - \tilde{\beta}_k \frac{n^2}{8} \right) \int_{\Omega} |u|^2 dx.$$

Thus, we find

$$S_{\lambda}(p) \geq \inf_{\|u\|_q^2=1} \left[\int_{\Omega} \tilde{p}(x)|\nabla u|^2 dx - \left(\lambda - \tilde{\beta}_k \frac{n^2}{8} \right) \int_{\Omega} |u|^2 dx \right].$$

On the other hand $\lambda - \tilde{\beta}_k \frac{n^2}{8} \leq \frac{1}{2} \tilde{\beta}_k \frac{n^2}{4}$ since $\lambda \leq \tilde{\beta}_k \frac{n^2}{4}$, so by (2.3.23), we conclude that

$$\inf_{\|u\|_q=1} \left[\int_{\Omega} \tilde{p}(x) |\nabla u|^2 dx - (\lambda - \tilde{\beta}_k \frac{n^2}{8}) \int_{\Omega} |u|^2 dx \right] = p_0 S,$$

hence, (2.3.22) follows.

Now, we are able to prove that the infimum in (2.3.22) is not achieved. Suppose by contradiction that it is achieved by some u_0 . Let δ such that $\tilde{\beta}_k \frac{n^2}{4} \geq \delta > \lambda$. Using u_0 as a test function for S_δ , we obtain

$$S_\delta(p) \leq \frac{\int_{\Omega} p(x) |\nabla u_0|^2 dx - \delta \int_{\Omega} |u_0|^2 dx}{\|u_0\|_q^2} < \frac{\int_{\Omega} p(x) |\nabla u_0|^2 dx - \lambda \int_{\Omega} |u_0|^2 dx}{\|u_0\|_q^2}$$

and thus $S_\delta(p) < S_\lambda(p) = p_0 S$. This is a contradiction since $S_\delta(p) = p_0 S$ for $\delta \leq \tilde{\beta}_k \frac{n^2}{4}$. \square

The second Lemma on which the proof of assertion 3) in Theorem 2.1.1 is based is the following

Lemma 2.3.4.

There exists $\lambda^ \in [\tilde{\beta}_k \frac{n^2}{4}, \lambda_1^{div}[$, such that for all $\lambda \in]\lambda^*, \lambda_1^{div}[$ we have*

$$S_\lambda(p) < p_0 S.$$

Proof. The proof is based on a study of some properties of the function $\lambda \mapsto S_\lambda(p)$. We have $S_{\lambda_1^{div}}(p) = 0$. Indeed let φ_1 be the eigenfunction of $\text{div}(p\nabla \cdot)$ corresponding to λ_1^{div} , we have

$$S_{\lambda_1^{div}} \leq \frac{\int p(x) |\nabla \varphi_1|^2 dx - \lambda_1^{div} \int |\varphi_1|^2 dx}{(\int |\varphi_1|^q dx)^{\frac{2}{q}}} = 0.$$

Moreover, $\lambda \mapsto S_\lambda(p)$ is continuous and $S_{\tilde{\beta}_k \frac{n^2}{4}}(p) = p_0 S$. Then according to the Mean Value Theorem, there exists $\beta \in]\tilde{\beta}_k \frac{n^2}{4}, \lambda_1^{div}[$ such that $0 < S_\beta(p) < p_0 S$. But the function $\lambda \mapsto S_\lambda(p)$ is decreasing hence $\forall \lambda \in [\beta, \lambda_1^{div}[$ we have $S_\lambda(p) < p_0 S$, and the Lemma follows at once. \square

Now we have all the necessary ingredients for the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1 concluded: Concerning the proof of 1), 2), 3) and 4), let $u \in H_0^1(\Omega)$ be given by Lemma 2.3.1, that is,

$$\|u\|_q = 1 \quad \text{and} \quad \int_{\Omega} p(x) |\nabla u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx = S_\lambda(p).$$

We may as well assume that $u \geq 0$. Since u is a minimizer for (2.3.2) there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$-div(p\nabla u) - \lambda u = \mu u^{q-1} \text{ on } \Omega.$$

In fact, $\mu = S_\lambda(p)$, and $S_\lambda(p) > 0$ since $\lambda < \lambda_1^{div}$. It follows that γu satisfies (2.1.1) for some appropriate constant $\gamma > 0$ ($\gamma = (S_\lambda(p))^{\frac{1}{q-2}}$), note that $u > 0$ on Ω by the strong maximum principle.

Now we prove the assertion 5) of Theorem 2.1.1. From (2.3.16) and since $\lambda \leq 0$ we have

$$p_0 S \leq S_\lambda(p) \leq Q_\lambda(u_{a,\varepsilon}) \leq p_0 S + o(1).$$

Hence $S_\lambda(p) = p_0 S$ and the infimum is not achieved, indeed we suppose that $S_\lambda(p)$ is achieved by some function $u \in H_0^1(\Omega)$, in that case

$$S_\lambda(p) = \int_\Omega p(x)|\nabla u(x)|^2 dx - \lambda \int_\Omega |u(x)|^2 dx, \text{ with } \|u\|_q = 1.$$

Using the fact that S is not attained and since $\lambda \leq 0$, we deduce

$$p_0 S < p_0 \int_\Omega |\nabla u(x)|^2 dx \leq S_\lambda(p) = p_0 S,$$

then we obtain a contradiction.

Finally we prove assertion 6) in Theorem 2.1.1. Let φ_1 be the eigenfunction corresponding to λ_1^{div} with $\varphi_1 > 0$ on Ω . Suppose that u is a solution of (2.1.1). We have

$$\begin{aligned} - \int_\Omega div(p(x)\nabla u(x))\varphi_1(x) dx &= \lambda_1^{div} \int_\Omega u(x)\varphi_1(x) dx \\ &= \int_\Omega u^{q-1}(x)\varphi_1(x) dx + \lambda \int_\Omega u(x)\varphi_1(x) dx, \end{aligned}$$

thus

$$\lambda_1^{div} \int_\Omega u(x)\varphi_1(x) dx > \lambda \int_\Omega u(x)\varphi_1(x) dx$$

and

$$\lambda_1^{div} > \lambda.$$

This completes the proof of Theorem 2.1.1.

Proof of Remark 2.1.1

Proof of 1)

In this case, we have $k = 2$ and $n \geq 4$.

We have (see proof of (2.3.22) page 33)

$$p(x) - p_0 \geq \beta_2 |x - a|^2 \quad \text{a.e } \in \Omega.$$

Therefore

$$\frac{\int_{\Omega} p(x)|\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx} \geq p_0 \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx} + \beta_2 \frac{\int_{\Omega} |x-a|^2 |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx}.$$

On one hand, we apply, again, Lemma 2.2.1 (with $t = 0$), we obtain

$$\frac{\int_{\Omega} |x-a|^2 |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx} \geq \frac{n^2}{4}.$$

On other hand, we have

$$\frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx} \geq \frac{1}{(\text{diam } \Omega)^2} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |x|^{-2} |u|^2 dx}$$

We apply Lemma 2.2.1 (with $t = -2$), we deduce

$$\begin{aligned} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx} &\geq \frac{1}{(\text{diam } \Omega)^2} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |x|^{-2} |u|^2 dx} \\ &\geq \frac{1}{(\text{diam } \Omega)^2} \frac{(n-2)^2}{4}. \end{aligned}$$

Combining the above inequality, we get

$$\frac{\int_{\Omega} p(x)|\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx} \geq p_0 \frac{(n-2)^2}{4(\text{diam } \Omega)^2} + \beta_2 \frac{n^2}{4}$$

and

$$\lambda_1^{div} \geq p_0 \frac{(n-2)^2}{4(\text{diam } \Omega)^2} + \beta_2 \frac{n^2}{4}.$$

Therefore, if we have $p_0 > \frac{n(n-4)}{(n-1)(n-2)^2} \beta_2 (\text{diam } \Omega)^2$, we obtain the desired result.

Proof of 2)

Here we have $n \geq 3$ and $0 < k < 2$.

Using the definition of p_0 , we have

$$\frac{\int_{\Omega} p(x)|\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx} \geq p_0 \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx}.$$

We apply the Lemma 2.2.1 (with $t = -2$), we obtain

$$\begin{aligned} \frac{\int_{\Omega} p(x)|\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx} &\geq p_0 \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx} \\ &\geq p_0 \frac{(n-2)^2}{4(\text{diam } \Omega)^2} \end{aligned}$$

and

$$\lambda_1^{div} \geq p_0 \frac{(n-2)^2}{4(\text{diam } \Omega)^2}.$$

Therefore, if $p_0 > \frac{\tilde{\beta}_k n^2}{(n-2)^2 (\text{diam } \Omega)^2}$, we deduce that $\tilde{\beta}_k \frac{n^2}{4} < \lambda_1^{div}$.

2.4 The effect of the geometry of the domain

Let $\Omega \subset \mathbb{R}^n, n \geq 3$, be a bounded domain. We study the equation

$$(2.4.1) \quad \begin{cases} -\text{div}(p(x)\nabla u) = u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $q = \frac{2n}{n-2}$ and $p : \bar{\Omega} \rightarrow \mathbb{R}$ is a positive weight belonging to $C(\bar{\Omega}) \cap H_0^1(\Omega)$.

We assume in this section that p is such that $\nabla p(x) \cdot (x - a) \geq 0$ a.e $x \in \Omega$ and we set $p_0 = p(a)$.

Let us start by the following non-existence result

Lemma 2.4.1.

There is no solution of (2.4.1) if Ω is a starshaped domain about a .

Proof. This follows from Pohozaev's identity.

Suppose that u is a solution of (2.4.1), we have (see Lemma 2.2.2 Section 1 for $\lambda = 0$),

$$(2.4.2) \quad \int_{\Omega} \nabla p(x) \cdot (x - a) |\nabla u(x)|^2 dx + \int_{\partial\Omega} p(x) [(x - a) \cdot \nu] \left| \frac{\partial u}{\partial \nu} \right|^2 dx = 0.$$

Note that $(x - a) \cdot \nu > 0$ a.e on $\partial\Omega$ since Ω is starshaped about a .

Since $\nabla p(x) \cdot (x - a) \geq 0$ a.e $x \in \Omega$, we deduce from (2.4.2) that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, and then by (2.4.1) we have

$$\int_{\Omega} u^{q-1}(x) dx = - \int_{\Omega} \text{div}(p(x)\nabla u(x)) dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dx = 0,$$

thus

$$u \equiv 0.$$

□

Suppose that Ω is starshaped about a . In view of Lemma 2.4.1, we will modify the geometry of Ω in order to find a solution of problem 2.4.1. For a $\varepsilon > 0$ small enough, we set $\Omega_\varepsilon = \Omega \setminus \bar{B}(a, \varepsilon)$.

We investigate the problem (2.4.1) in the new domain Ω_ε , and, throughout the rest of this paper, we shall denote this new problem by (I_ε) .

Since p is a continuous function, then, $\forall \theta > 0, \exists r_0 > 0$ such that $\forall \sigma \in \Sigma$, where Σ designates the unit sphere of \mathbb{R}^n , we have $|p(a + r_0\sigma) - p_0| < \frac{\theta}{2S^{\frac{n}{2}}}$.

Throughout the rest of this Section, $\theta > 0$ is fixed, small enough, and $r_0 > 0$ is given as the previous definition.

We recall the main result of this section which we have already stated by theorem 2.1.2 in the introduction

Theorem 2.4.1.

There exists $\varepsilon_0 = \varepsilon_0(\Omega, p) \leq r_0$ such that for every $0 < \varepsilon < \varepsilon_0$, the problem (I_ε) has at least one solution in $H_0^1(\Omega_\varepsilon)$.

In order to prove the Theorem 2.4.1, we need to apply the following result, see [AR],

Theorem A 1.

Let E be a C^1 function defined on a Banach space X , and let K a compact metric space. We denote by K^ a nonempty subset of K , closed, different from K and we fix $f^* \in C(K^*, X)$.*

We define $\mathcal{P} = \{f \in C(K, X) / f = f^ \text{ on } K^*\}$ and $c = \inf_{f \in \mathcal{P}} \sup_{t \in K} E(f(t))$
Suppose that for every f of \mathcal{P} , we have*

$$\max_{t \in K} E(f(t)) > \max_{t \in K^*} E(f(t)),$$

then there exists a sequence $(u_j) \subset X$ such that $E(u_j) \rightarrow c$ and $E'(u_j) \rightarrow 0$ in X^ .*

We consider the functional

$$E(u) = \frac{1}{2} \int_{\Omega_\varepsilon} p(x) |\nabla u(x)|^2 dx - \frac{1}{q} \int_{\Omega_\varepsilon} |u(x)|^q dx.$$

In addition to Theorem A 1, the proof of Theorem 2.4.1 requires the following result (see [B] and Proposition 2.1 in [S])

Theorem A 2.

Suppose that for some sequence $(u_j) \subset H_0^1(\Omega_\varepsilon)$ we have $E(u_j) \rightarrow c \in]\frac{1}{n}(p_0S)^{\frac{n}{2}}, \frac{2}{n}(p_0S)^{\frac{n}{2}}[$ and $dE(u_j) \rightarrow 0$ in $H^{-1}(\Omega_\varepsilon)$. Then (u_j) contains a strongly convergent subsequence.

See Appendix for more details.

Now, we return to the proof of Theorem 2.4.1.

We shall need the following functions:

$$\begin{aligned} \Gamma & : H_0^1(\Omega_\varepsilon) \longrightarrow \mathbb{R}, & \Gamma(u) &= \int_{\Omega_\varepsilon} p(x)|\nabla u(x)|^2 dx - \int_{\Omega_\varepsilon} |u(x)|^q dx. \\ F & : H_0^1(\Omega_\varepsilon) \longrightarrow \mathbb{R}^n, & F(u) &= (p_0S)^{-\frac{n}{2}} \int_{\Omega_\varepsilon} xp(x)|\nabla u(x)|^2 dx. \end{aligned}$$

We have the following result

Lemma 2.4.2.

For every neighborhood V of $\bar{\Omega}_\varepsilon$ there exists $\eta > 0$ such that if $u \neq 0$, $\Gamma(u) = 0$ and $E(u) \leq \frac{1}{n}(p_0S)^{\frac{n}{2}} + 2\eta$, then $F(u) \in V$.

Proof.

We proceed by contradiction. We assume that there exists V a compact neighborhood of $\bar{\Omega}_\varepsilon$ not containing a , such that $\forall j \in \mathbf{N}^*$, we have

$$\begin{aligned} u_j & \neq 0, \\ \Gamma(u_j) & = 0, \\ E(u_j) & \leq \frac{1}{n}(p_0S)^{\frac{n}{2}} + \frac{1}{j}, \\ F(u_j) & \notin V. \end{aligned}$$

Using the fact that $\Gamma(u_j) = 0$, we obtain

$$\int_{\Omega_\varepsilon} p(x)|\nabla u_j|^2 dx = \int_{\Omega_\varepsilon} |u_j|^q dx$$

and

$$\int_{\Omega_\varepsilon} p(x)|\nabla u_j|^2 dx = \left(\frac{\int_{\Omega_\varepsilon} p(x)|\nabla u_j|^2 dx}{\left(\int_{\Omega_\varepsilon} |u_j|^q dx \right)^{\frac{2}{q}}} \right)^{\frac{n}{2}}.$$

Consequently

$$E(u_j) = \frac{1}{n} \int_{\Omega_\varepsilon} p(x)|\nabla u_j(x)|^2 dx.$$

Using the definition of u_j , the fact that $p_0 = \min_{\bar{\Omega}} p(x)$ and the definition of S , we write

$$\frac{1}{n}(p_0 S)^{\frac{n}{2}} \leq \frac{1}{n} \left(\frac{p_0 \int_{\Omega_\varepsilon} |\nabla u_j|^2 dx}{\left(\int_{\Omega_\varepsilon} |u_j|^q dx \right)^{\frac{2}{q}}} \right)^{\frac{n}{2}} \leq E(u_j) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \frac{1}{j}$$

and we deduce

$$\int_{\Omega_\varepsilon} p(x) |\nabla u_j(x)|^2 dx = (p_0 S)^{\frac{n}{2}} + o(1).$$

Using Theorem 2 in [C], (see also [L, Lemma I.4 and Lemma I.1]), for a subsequence of $(u_j)_j$ still denoted by $(u_j)_j$, there exists $x_0 \in \bar{\Omega}_\varepsilon$ such that

$$p(x) |\nabla u_j|^2 \longrightarrow (p_0 S)^{\frac{n}{2}} \delta_{x_0} \quad (j \rightarrow \infty),$$

where the above convergence is understood for the weak topology of bounded measures on $\bar{\Omega}_\varepsilon$ and where δ_{x_0} is the Dirac measure at x_0 .

As a consequence, $F(u_j) \in \bar{\Omega}_\varepsilon \subset V$, and this contradicts the hypothesis. \square

Let $R_0 > 0$ such that $B(a, 2R_0) \subset \Omega$.

For $k \in \mathbf{N}^*$, let $\varphi_k \in C^\infty(\mathbb{R}^n, [0, 1])$ such that

$$\begin{cases} \varphi_k(x) = 0 & \text{if } |x - a| \leq \frac{1}{4k^2} \text{ and if } |x - a| \geq 2R_0, \\ \varphi_k(x) = 1 & \text{if } \frac{1}{2k^2} \leq |x - a| \leq R_0. \end{cases}$$

We consider the family of functions

$$u_t^\sigma(x) = \left[\frac{1-t}{(1-t)^2 + |x-a-t\sigma|^2} \right]^{\frac{n-2}{2}},$$

where $t \in [0, 1[$, $\sigma \in \Sigma$ and where Σ denotes the unit sphere of \mathbb{R}^n .

We see easily that $\int_{\mathbb{R}^n} |\nabla u_t^\sigma|^2 dx$ and $\int_{\mathbb{R}^n} |u_t^\sigma|^q dx$ are independent of $t \in [0, 1[$ and of $\sigma \in \Sigma$. We also have

$$\int_{\mathbb{R}^n} |\nabla u_t^\sigma(x)|^2 dx = S \left(\int_{\mathbb{R}^n} |u_t^\sigma(x)|^q dx \right)^{\frac{2}{q}}.$$

Write

$$v_{t,k}^\sigma(x) = \frac{(1-t)^{\frac{n-2}{2}} k^{\frac{n-2}{2}} \varphi_k(x)}{\left((1-t)^2 + |k(x-a-tr_0\sigma)|^2 \right)^{\frac{n-2}{2}}},$$

we remark that $v_{t,k}^\sigma \in H_0^1(\Omega_\varepsilon)$. For $r > 0$, let $g(r) = E(rv_{t,k}^\sigma)$, then

$rg'(r) = \Gamma(rv_{t,k}^\sigma)$, $g(r) \rightarrow -\infty$, when $r \rightarrow +\infty$, $g(0) = 0$ and $g(r) > 0$ for $r > 0$ small enough.

We conclude, from the above, that g reaches its maximum at

$$r = \left[\frac{\int_{\Omega_\varepsilon} p(x) |\nabla v_{t,k}^\sigma|^2 dx}{\int_{\Omega_\varepsilon} |v_{t,k}^\sigma|^q dx} \right]^{\frac{1}{q-2}} > 0.$$

We set $w_{t,k}^\sigma = rv_{t,k}^\sigma$. We have

Lemma 2.4.3.

The following two statements are true:

- a) $\forall \delta > 0, \exists k_0 \geq 1$ such that $(\forall k \geq k_0)$ then
 $(\forall \sigma \in \Sigma \text{ and } \forall t \in [0, 1[, E(w_{t,k}^\sigma) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \delta)$
- b) $\forall \alpha > 0, \exists \mu > 0$ such that $(\mu < t < 1)$ then
 $(\forall \sigma \in \Sigma \text{ and } \forall k \geq 1, E(w_{t,k}^\sigma) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \alpha)$
and $|F(w_{t,k}^\sigma) - (a + r_0 \sigma)| \leq \alpha.$

Proof.

Before proving this Lemma, let us remark that the function $v_{t,k}^\sigma$ corresponds to the function $u_{a,\varepsilon}$ defined in the beginning of this paper, so for more details of calculus we refer to section 2.

We start by proving the assertion a). Let $t \in [0, 1[$, we have

$$\begin{aligned} E(w_{t,k}^\sigma) &= \frac{1}{2} \int_{\Omega_\varepsilon} p(x) |\nabla w_{t,k}^\sigma|^2 dx - \frac{1}{q} \int_{\Omega_\varepsilon} |w_{t,k}^\sigma|^q dx, \\ &= \frac{r^2}{2} \int_{\Omega_\varepsilon} p(x) |\nabla v_{t,k}^\sigma|^2 dx - \frac{r^q}{q} \int_{\Omega_\varepsilon} |v_{t,k}^\sigma|^q dx. \end{aligned}$$

Using the definition of r , the definition of φ_k and applying the Dominated Convergence Theorem, we obtain, as $k \rightarrow \infty$,

$$\begin{aligned} E(w_{t,k}^\sigma) &= \frac{1}{n} \left[\frac{k^n (n-2)^2 (1-t)^{n-2} \int_{\{\frac{1}{2k^2} \leq |x-a| \leq R_0\}} p(x) \frac{|k(x-a-tr_0\sigma)|^2}{((1-t)^2 + |k(x-a-tr_0\sigma)|^2)^n} dx}{\left[k^n (n-2)^2 (1-t)^n \int_{\{\frac{1}{2k^2} \leq |x-a| \leq R_0\}} \frac{1}{((1-t)^2 + |k(x-a-tr_0\sigma)|^2)^n} dx \right]^{\frac{2}{q}}} \right]^{\frac{n}{2}} \\ &+ o(1). \end{aligned}$$

By the following change of variable $y = \frac{k(x-a-tr_0\sigma)}{1-t}$, we see that

$$\begin{aligned} E(w_{t,k}^\sigma) &= \frac{1}{n} \left[\frac{(n-2)^2 \int_{\{\frac{1}{2k(1-t)} - \frac{tr_0}{1-t} \leq |y| \leq \frac{kR_0}{1-t} + \frac{tr_0}{1-t}\}} p\left(\frac{y(1-t)}{k} + a + tr_0\sigma\right) \frac{|y|^2}{(1+|y|^2)^n} dy}{\left[\int_{\{\frac{1}{2k(1-t)} - \frac{tr_0}{1-t} \leq |y| \leq \frac{kR_0}{1-t} + \frac{tr_0}{1-t}\}} \frac{1}{(1+|y|^2)^n} dy \right]^{\frac{2}{q}}} \right]^{\frac{n}{2}} \\ &+ o(1). \end{aligned}$$

Applying again the Dominated Convergence Theorem, we deduce, as $k \rightarrow \infty$, that

$$\begin{aligned} E(w_{t,k}^\sigma) &= \frac{1}{n} \left[\frac{(n-2)^2 p(a + tr_0\sigma) \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n} dy}{\left[\int_{\mathbb{R}^n} \frac{1}{(1-t)^2 + |y|^2} dy \right]^{\frac{2}{q}}} \right]^{\frac{n}{2}} + o(1), \\ &= \frac{1}{n} (p(a + tr_0\sigma))^{\frac{n}{2}} S^{\frac{n}{2}} + o(1). \end{aligned}$$

Now, using the definition of r_0 , a simple computation shows that $\forall \delta > 0, \exists k_0 \geq 1$ such that $\forall k \geq k_0$, we have

$$E(w_{t,k}^\sigma) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \delta,$$

which finishes the proof of a).

Now we return to the proof of b), let $k \in \mathbf{N}^*$, we have

$$\begin{aligned} E(w_{t,k}^\sigma) &= \frac{1}{2} \int_{\Omega_\varepsilon} p(x) |\nabla w_{t,k}^\sigma|^2 dx - \frac{1}{q} \int_{\Omega_\varepsilon} |w_{t,k}^\sigma|^q dx \\ &= \frac{r^2}{2} \int_{\Omega_\varepsilon} p(x) |\nabla v_{t,k}^\sigma|^2 dx - \frac{r^q}{q} \int_{\Omega_\varepsilon} |v_{t,k}^\sigma|^q dx. \end{aligned}$$

Using the definition of φ_k and r , we easily see, as $t \rightarrow 1$, that

$$E(w_{t,k}^\sigma) = \frac{1}{n} \left[\frac{k^n (n-2)^2 (1-t)^{n-2} \int_{\mathbb{R}^n} p(x) \frac{|k(x-a-tr_0\sigma)|^2}{((1-t)^2 + |k(x-a-tr_0\sigma)|^2)^n} dx}{\left[k^n (1-t)^n \int_{\mathbb{R}^n} \frac{1}{((1-t)^2 + |k(x-a-tr_0\sigma)|^2)^n} dx \right]^{\frac{2}{q}}} \right]^{\frac{n}{2}} + O((1-t)^{n-2}).$$

By the change of variable $y = \frac{k(x-a-tr_0\sigma)}{1-t}$, we get

$$E(w_{t,k}^\sigma) = \frac{1}{n} \left[\frac{(n-2)^2 \int_{\mathbb{R}^n} p\left(\frac{(1-t)y}{k} + a + tr_0\sigma\right) \frac{|y|^2}{(1+|y|^2)^n} dy}{\left[\int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^n} dy \right]^{\frac{2}{q}}} \right]^{\frac{n}{2}} + O((1-t)^{n-2}).$$

Applying the Dominated Convergence Theorem, we obtain

$$\begin{aligned} E(w_{t,k}^\sigma) &= \frac{1}{n} \left[\frac{(n-2)^2 p(a+r_0\sigma) \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n} dy}{\left[\int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^n} dy \right]^{\frac{2}{q}}} \right]^{\frac{n}{2}} + O((1-t)^{n-2}), \\ &= \frac{1}{n} (p(a+r_0\sigma))^{\frac{n}{2}} S^{\frac{n}{2}} + O((1-t)^{n-2}). \end{aligned}$$

Using the definition of r_0 , a simple computation shows that $\forall \alpha > 0, \exists \mu > 0$ such that $\forall \mu < t < 1$, we have

$$E(w_{t,k}^\sigma) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \alpha.$$

On the other hand

$$\begin{aligned} F(w_{t,k}^\sigma) &= (p_0 S)^{-\frac{n}{2}} \int_{\mathbb{R}^n} xp(x) |\nabla w_{t,k}^\sigma(x)|^2 dx, \\ &= (p_0 S)^{-\frac{n}{2}} r^2 \int_{\mathbb{R}^n} xp(x) |\nabla v_{t,k}^\sigma(x)|^2 dx. \end{aligned}$$

Using the definition of r , we get

$$F(w_{t,k}^\sigma) = (p_0 S)^{-\frac{n}{2}} \left[\frac{\int_{\Omega_\varepsilon} p(x) |\nabla v_{t,k}^\sigma|^2 dx}{\int_{\Omega_\varepsilon} |v_{t,k}^\sigma|^q dx} \right]^{\frac{2}{q-2}} \int_{\mathbb{R}^n} x p(x) |\nabla v_{t,k}^\sigma(x)|^2 dx.$$

Using the definition of $v_{t,k}^\sigma$, we write

$$F(w_{t,k}^\sigma) = (p_0 S)^{-\frac{n}{2}} \left[\frac{(1-t)^{n-2} (n-2)^2 \int_{\mathbb{R}^n} p(x) \frac{|k(x-a-tr_0\sigma)|^2}{((1-t)^2 + |k(x-a-tr_0\sigma)|^2)^n} dx}{(1-t)^n \int_{\mathbb{R}^n} \frac{1}{((1-t)^2 + |k(x-a-tr_0\sigma)|^2)^n} dx} \right]^{\frac{2}{q-2}} \times \\ (1-t)^{n-2} k^n (n-2)^2 \int_{\mathbb{R}^n} x p(x) \frac{|k(x-a-tr_0\sigma)|^2}{((1-t)^2 + |k(x-a-tr_0\sigma)|^2)^n} dx + o(1-t).$$

Using the change of variable $y = \frac{k(x-a-tr_0\sigma)}{1-t}$, we obtain

$$F(w_{t,k}^\sigma) = (p_0 S)^{-\frac{n}{2}} \left[\frac{(n-2)^2 \int_{\mathbb{R}^n} p \left(\frac{(1-t)y}{k} + a + tr_0\sigma \right) \frac{|y|^2}{(1+|y|^2)^n} dx}{\int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^n} dx} \right]^{\frac{2}{q-2}} \times \\ (n-2)^2 \int_{\mathbb{R}^n} \frac{\left(\frac{(1-t)y}{k} + a + tr_0\sigma \right) p \left(\frac{(1-t)y}{k} + a + tr_0\sigma \right) |y|^2}{(1+|y|^2)^n} dx + o(1-t).$$

Applying the Dominated Convergence Theorem, we deduce that

$$F(w_{t,k}^\sigma) = (p_0 S)^{-\frac{n}{2}} (p(a+r_0\sigma))^{\frac{n}{2}} \left[\frac{(n-2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n} dy}{\left[\int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^n} dy \right]^{\frac{2}{q}}} \right]^{\frac{q}{q-2}} (a+r_0\sigma) + o(1-t), \\ = (p_0 S)^{-\frac{n}{2}} (p(a+r_0\sigma))^{\frac{n}{2}} S^{\frac{n}{2}} (a+r_0\sigma) + o(1-t).$$

Using the definition of r_0 we get the desired result. \square

Consequences

Let V be a compact neighborhood of $\bar{\Omega}_\varepsilon$ not containing a . Let $0 < \eta < r_0$ small enough, which corresponds to V as in Lemma 2.4.2, verifying $r_0\sigma + \xi \neq a$ for $|\sigma| = 1$ and $|a - \xi| \leq \eta$.

By Lemma 2.4.3, there exists $k_0 \geq 1$ such that :

$$(2.4.3) \quad E(w_{t,k_0}^\sigma) \leq \frac{2}{n} (p_0 S)^{\frac{n}{2}} - \eta, \quad \forall \sigma \in \Sigma, \quad \forall t \in [0, 1].$$

Remark 2.4.1.

We choose $\varepsilon_0 = \varepsilon_0(\Omega, p) \leq \frac{1}{4k_0^2}$ small enough and such that $\forall \varepsilon < \varepsilon_0$ we have $\{x \mid |x - a| \leq \varepsilon\} \not\subset V$.

We fix $\lambda > 1$, large enough such that $E(\lambda w_{t,k_0}^\sigma) < 0, \forall \sigma \in \Sigma, \forall t \in [0, 1]$. In order to apply Theorem A 1, we define the sets K, K^* and the function f^* as

$$K = [0, 1] \times \bar{B}(a, r_0),$$

$$K^* = \partial K = [0, 1] \times \partial \bar{B}(a, r_0) \cup \{0, 1\} \times \bar{B}(a, r_0) \text{ and}$$

$$f^* : K \rightarrow H_0^1(\Omega_\varepsilon),$$

$$f^*(s, tr_0\sigma) = \lambda s w_{t,k_0}^\sigma.$$

The conclusion of Theorem 2.4.1 follows from the next

Lemma 2.4.4.

We have

$$\sup_K E(f) \geq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + 2\eta, \forall f \in \mathcal{P}.$$

We postpone the proof of Lemma 2.4.4 and complete the proof of Theorem 2.4.1.

From (2.4.3) we have

$$\max_{r \geq 0} E(rv_{t,k_0}^\sigma) = E(w_{t,k_0}^\sigma) \leq \frac{2}{n}(p_0 S)^{\frac{n}{2}} - \eta \quad \forall \sigma \in \Sigma, \quad \forall t \in [0, 1].$$

From assertion b) of Lemma 2.4.3 there exists μ , we fix $t_0 \in]\mu, 1[$ such that

$$\max_{r \geq 0} E(rv_{t_0,k_0}^\sigma) = E(w_{t_0,k_0}^\sigma) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \eta, \quad \forall \sigma \in \Sigma.$$

then

$$\max_{\partial K} E(f^*) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \eta \quad \text{and} \quad \sup_K E(f^*) < \frac{2}{n}(p_0 S)^{\frac{n}{2}}.$$

So, by Lemma 2.4.4,

$$\sup_K E(f) \geq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + 2\eta > \frac{1}{n}(p_0 S)^{\frac{n}{2}} + \eta \geq \sup_{\partial K} E(f^*)$$

and

$$c = \inf_{f \in \mathcal{P}} \sup_{t \in K} E(f) \in]\frac{1}{n}(p_0 S)^{\frac{n}{2}}, \frac{2}{n}(p_0 S)^{\frac{n}{2}}[.$$

Applying Theorem A 1 and Theorem A 2, we obtain the conclusion of Theorem 2.4.1.

Proof of Lemma 2.4.4. We argue by contradiction. Suppose that there exists $f \in C(K, H_0^1(\Omega_\varepsilon))$ with $f = f^*$ on ∂K , and $E(f(s, \xi)) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + 2\eta, \forall (s, \xi) \in K$. We consider the function $G : K \rightarrow \mathbb{R}^{n+1}$, defined by

$$G(s, \xi) = (s, F(f(s, \xi))).$$

We will prove that

$$(2.4.4) \quad \deg(G, K, (\lambda^{-1}, a)) = 1.$$

The map $H : [0, 1] \times K \longrightarrow \mathbb{R}^{n+1}$, defined by

$$H(t, s, \xi) = tG(s, \xi) + (1-t)(s, \xi) = (s, tF(f(s, \xi)) + (1-t)\xi)$$

is a homotopy between G and Id_K , where Id_K is the Identity application of K .

To get (2.4.4), we start by checking that $(\lambda^{-1}, a) \notin H(t, \partial K)$.

If not, there exists $(s, \xi) \in \partial K$ such that $H(t, s, \xi) = (\lambda^{-1}, a)$, as a consequence $s = \lambda^{-1}$ and $a = tF(f(\lambda^{-1}, \xi)) + (1-t)\xi = t(F(w_{t_0, k_0}^\sigma) - \xi) + \xi$.

Since $s = \lambda^{-1} \in]0, 1[$, we have $\xi \in \partial \bar{B}(a, r_0)$. But, since $|F(w_{t_0, k_0}^\sigma) - (a + r_0\sigma)| < \eta$ $\forall \sigma \in \Sigma$ (Lemma 2.4.3, b)), the fact that $t(F(w_{t_0, k_0}^\sigma) - \xi) + \xi = a$, $\xi \in \partial \bar{B}(a, r_0)$ leads to a contradiction. Then, we deduce that $(\lambda^{-1}, a) \notin H(t, \partial K)$ and consequently $\forall t \in [0, 1]$, $\deg(H(t, \cdot), K, (\lambda^{-1}, a))$ is well defined.

We consider the following sets:

$$K^+ = \{(s, \xi) \in K | \Gamma(f(s, \xi)) > 0\} \cup (0, \xi), \quad K^- = \{(s, \xi) \in K | \Gamma(f(s, \xi)) < 0\} \text{ and } \\ K^0 = \{(s, \xi) \in K | \Gamma(f(s, \xi)) = 0\}.$$

If $(s, \xi) \in \partial K$ then we have $f(s, \xi) = f^*(s, \xi) = \lambda s w_{t_0, k_0}^\sigma$ and

$$\Gamma(f(s, \xi)) = (s\lambda)^2 \int_{\Omega_\varepsilon} p(x) |\nabla w_{t_0, k_0}^\sigma(x)|^2 dx - (s\lambda)^q \int_{\Omega_\varepsilon} |w_{t_0, k_0}^\sigma(x)|^q dx$$

$$\Gamma(f(s, \xi)) = [(s\lambda)^2 - (s\lambda)^q] \int_{\Omega_\varepsilon} p(x) |\nabla w_{t_0, k_0}^\sigma(x)|^2 dx.$$

Since $\int_{\Omega_\varepsilon} p(x) |\nabla w_{t_0, k_0}^\sigma(x)|^2 dx > 0$, we see that

$$(2.4.5) \quad \text{If } (s, \xi) \in \partial K \text{ and if } 0 \leq s < \lambda^{-1}, \text{ then } (s, \xi) \in K^+$$

$$(2.4.6) \quad \text{If } (s, \xi) \in \partial K \text{ and if } \lambda^{-1} < s \leq 1, \text{ then } (s, \xi) \in K^-$$

$$(2.4.7) \quad (\lambda^{-1}, \xi) \in K^0, \quad \forall \xi \in \partial \bar{B}(a, r_0).$$

Let $(s, \xi) \in K^0$, we have $\Gamma(f(s, \xi)) = 0$. Moreover, since $E(f(s, \xi)) \leq \frac{1}{n}(p_0 S)^{\frac{n}{2}} + 2\eta$, looking at Lemma 2.4.2, we deduce that

$$F(f(s, \xi)) \in V.$$

Consequently $\forall (s, \xi) \in K^0$, $F(f(s, \xi)) \neq a$ since $a \notin V$.

Hence $(\lambda^{-1}, a) \notin G(K^0) = G(K \setminus (K^+ \cup K^-))$, then

$$(2.4.8) \quad \deg(G, K^+, (\lambda^{-1}, a)) + \deg(G, K^-, (\lambda^{-1}, a)) = \deg(G, K, (\lambda^{-1}, a)).$$

On the other hand, since $(\lambda^{-1}, a) \notin H(t, \partial K) \forall t \in [0, 1]$ we have

$$\deg(H(1, \cdot), K, (\lambda^{-1}, a)) = \deg(H(0, \cdot), K, (\lambda^{-1}, a)).$$

Using the fact that $H(0, \cdot) = G$, $H(1, \cdot) = Id_K$ and $\deg(Id_K, K, (\lambda^{-1}, a)) = 1$, we deduce (2.4.4).

Now, to finish we will prove that

$$(2.4.9) \quad \deg(G, K^+, (\lambda^{-1}, a)) = 0$$

$$(2.4.10) \quad \deg(G, K^-, (\lambda^{-1}, a)) = 0.$$

Fix $R > \lambda^{-1}$ let $y \in \mathbb{R}^{n+1}$ such that $|y| \geq R$ then $y \notin G(K)$.

We define the path $r(t) = (tR + (1-t)\lambda^{-1}, a)$, for $t \in [0, 1]$.

We claim that $r(t) \notin G(\partial K^+) \forall t \in [0, 1]$.

If not, there exists $(s, \xi) \in \partial K^+$ with $(Rt + (1-t)\lambda^{-1}, a) = (s, F(f(s, \xi)))$. Hence $s = tR + (1-t)\lambda^{-1} \geq \lambda^{-1}$ and $a = F(f(s, \xi))$. But $\forall (s, \xi) \in K^0$, we have $F(f(s, \xi)) \neq a$, then $(s, \xi) \notin K^0$. Hence $(s, \xi) \in \partial K \cap K^+$, (2.4.5) implies that $s < \lambda^{-1}$ and this contradicts the fact that $s \geq \lambda^{-1}$. Thus $r(t) \notin G(\partial K^+) \forall t \in [0, 1]$. Hence $\deg(G, K^+, r(t))$ is well defined and is independent of t .

Since $(R, a) \notin G(K)$ we obtain

$$\deg(G, K^+, (R, a)) = 0.$$

Using the fact that

$$\deg(G, K^+, r(t)) = \deg(G, K^+, (R, a)) \quad \forall t \in [0, 1],$$

we deduce (2.4.9).

Similarly, we prove (2.4.10) by using the path $q(t) = (-tR + (1-t)\lambda^{-1}, a)$, $t \in [0, 1]$.

We have that $\deg(G, K^-, q(t))$ is independent of t . Using the fact that $(-R, a) \notin G(K)$, we conclude that

$$\deg(G, K^-, (\lambda^{-1}, a)) = \deg(G, K^-, (-R, a)) = 0.$$

From (2.4.4), (2.4.8), (2.4.9) and (2.4.10) we obtain a contradiction, and Lemma 2.4.4 is proved.

2.5 Appendix

In this appendix, Ω_ε will be noted Ω .

2.5.1 The limiting problem of (2.4.1)

Let $u \in H_0^1(\Omega)$, a solution of

$$\begin{cases} -\operatorname{div}(p(x)\nabla u) = |u|^{q-2}u(x), & x \in \Omega, \\ u = 0 & \text{sur } \partial\Omega \end{cases}$$

We have

$$(2.5.1) \quad -p(x)\Delta u(x) - \nabla p(x) \cdot \nabla u(x) = |u|^{q-2}u(x), \quad x \in \Omega.$$

Let $r > 0$, we set

$$u(x) = r^{-\frac{n-2}{r}} v_r\left(\frac{x-a}{r}\right),$$

(u is extended by 0 outside Ω).

One can see that v_r satisfies

$$-r^{-\frac{n+2}{2}} p(x)\Delta v_r\left(\frac{x-a}{r}\right) - r^{-\frac{n-2}{2}-1} \nabla p(x) \cdot (\nabla v_r)\left(\frac{x-a}{r}\right) = r^{-\frac{n+2}{2}} |v_r|^{q-2} v_r\left(\frac{x-a}{r}\right), \quad x \in \Omega.$$

We set $y = \frac{x-a}{r}$ and we write the above equality according to y , we obtain

$$-r^{-\frac{n+2}{2}} p(a+ry)\Delta v_r(y) - r^{-\frac{n-2}{2}} (\nabla p)(a+ry) \cdot (\nabla v_r)(y) = r^{-\frac{n+2}{2}} |v_r|^{q-2} v_r(y), \quad y \in \Omega_r,$$

with $\Omega_r = \frac{\Omega-a}{r}$.

We multiply the above equality by $r^{\frac{n+2}{2}}$, we get

$$(2.5.2) \quad -p(a+ry)\Delta v_r(y) - r^2 (\nabla p)(a+ry) \cdot (\nabla v_r)(y) = |v_r|^{q-2} v_r(y), \quad y \in \Omega_r,$$

We see that $\{v_r\}_r$ is bounded in $H^1(\mathbb{R}^n)$, indeed

$$\int_{\Omega_r} |\nabla v_r(y)|^2 dy = \int_{\Omega} |\nabla u(x)|^2 dx.$$

Therefore $v_r \rightharpoonup w$ weakly in $H_{loc}^1(\mathbb{R}^n)$ and $v_r \rightharpoonup w$ weakly in $L_{loc}^q(\mathbb{R}^n)$, as $r \rightarrow 0$.

We claim that the limiting problem is

$$(2.5.3) \quad \begin{aligned} -p(a)\Delta w(y) &= |w|^{q-2}w(y), \quad y \in \mathbb{R}^n \\ w(x) &\rightarrow 0 \quad |y| \rightarrow +\infty. \end{aligned}$$

Indeed, let $\varphi \in C_c^\infty(\Omega_r)$. There exists $r_0 > 0$ such that $K = \text{support } \varphi \subset \Omega_{r_0} \subset \Omega_r$ for every $r \leq r_0$.

We multiply (2.5.2) by φ and we integrate by part, we obtain

$$\begin{aligned} \int_K p(a+ry)\nabla v_r(y)\nabla\varphi(y)dy + r \int_K \nabla p(a+ry) \cdot \nabla v_r(y)\varphi(y)dy \\ - r^2 \int_K \nabla p(a+ry) \cdot \nabla v_r(y)\varphi(y)dy = \int_K |v_r|^{q-2}v_r(y)\varphi(y)dy. \end{aligned}$$

Since $v_r \rightharpoonup w$ weakly in $H_{loc}^1(\mathbb{R}^n)$, we have $\int_K \nabla v_r \cdot \nabla \varphi(y) dy \rightarrow \int_K \nabla w \cdot \nabla \varphi(y) dy$.

Therefore we apply [Bre, Théorème IV.9, page 58], and we obtain for a subsequence,

$$(2.5.4) \quad \left\{ \begin{array}{l} \nabla v_r(y) \cdot \nabla \varphi(y) \rightarrow \nabla w(y) \cdot \nabla \varphi(y) \text{ a.e. on } K \\ \text{and} \\ \text{there exists } h \in L^1(K), \text{ with } h \geq 0, \text{ such that } |\nabla v_r(y) \cdot \nabla \varphi(y)| \leq h(y) \text{ a.e. on } K. \end{array} \right.$$

On the one hand, since $p \in C(\bar{\Omega})$, there exists $\beta > 0$ such that $p(x) \leq \beta \forall x \in \bar{\Omega}$, moreover using (2.5.4), we obtain

$$p(a + r y) \nabla v_r(y) \nabla \varphi(y) \rightarrow p(a) \nabla w(y) \nabla \varphi(y) \quad \text{a.e. on } K,$$

and

$$|p(a + r y) \nabla v_r(y) \nabla \varphi(y)| \leq \beta h(y) \in L^1(K).$$

We apply the Dominated Convergence Theorem to get

$$(2.5.5) \quad \int_K p(a + r y) \nabla v_r(y) \nabla \varphi(y) dy \rightarrow \int_K p(a) \nabla w(y) \nabla \varphi(y) dy.$$

On the other hand, Using Hölder's inequality, the fact that $p \in H^1(\Omega)$ and the fact that v_r is bounded in $H_{loc}^1(\mathbb{R}^n)$, we obtain

$$(2.5.6) \quad r \left| \int_K \nabla p(a + r y) \cdot \nabla v_r(y) \varphi(y) dy \right| \leq r \|\nabla p\|_{L^2(\Omega)} \|\varphi \nabla v_r\|_{L^2(K)} \rightarrow 0$$

and

$$(2.5.7) \quad r^2 \left| \int_K \nabla p(a + r y) \cdot \nabla v_r(y) \varphi(y) dy \right| \rightarrow 0.$$

Moreover, since $v_r \rightarrow w$ in $L_{loc}^q(\mathbb{R}^n)$, we see that

$$(2.5.8) \quad \int_K |v_r|^{q-2} v_r(y) \varphi(y) dy \rightarrow \int_K |w|^{q-2} w(y) \varphi(y) dy.$$

Finally, combining (2.5.5), (2.5.6), (2.5.7) and (2.5.8) we deduce, as $r \rightarrow 0$,

$$\int_K p(a) \nabla w(y) \nabla \varphi(y) dy = \int_K |w|^{q-2} w(y) \varphi(y) dy,$$

and we deduce (2.5.3).

2.5.2 Proof of Theorem A 2

We recall that $p_0 = p(a) = \min_{\bar{\Omega}} p(x)$.

We have also, (see page 35 line 5), that

$$\inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} p(x) |\nabla u|^2 dx}{\left(\int_{\Omega} |u|^q dx \right)^{\frac{2}{q}}} = p_0 S.$$

Theorem A 2

Suppose that $\{u_j\} \subset H_0^1(\Omega)$ satisfies:

$$E(u_j) \rightarrow c, \text{ with } \frac{1}{n}(p_0 S)^{\frac{n}{2}} < c < \frac{2}{n}(p_0 S)^{\frac{n}{2}}$$

and

$dE(u_j) \rightarrow 0$ strongly in $H^{-1}(\Omega)$ ($j \rightarrow \infty$).

Then $\{u_j\}$ contains a strongly convergent subsequence.

We start by proving the following

Lemma 2.5.1.

Suppose $\{u_j\} \subset H_0^1(\Omega)$ satisfies:

$E(u_j) \rightarrow l$, with $l < \frac{1}{n}(p_0 S)^{\frac{n}{2}}$

and

$dE(u_j) \rightarrow 0$ strongly in $H^{-1}(\Omega)$ ($j \rightarrow \infty$).

Then $\{u_j\} \rightarrow 0$ strongly in $H_0^1(\Omega)$, as $j \rightarrow \infty$, and $l = 0$.

Proof.

Let $\{u_j\}$ be a sequence such that $E(u_j) \rightarrow l < \frac{1}{n}(p_0 S)^{\frac{n}{2}}$ and $dE(u_j) \rightarrow 0$.

Then we have

$$(2.5.9) \quad \frac{1}{2} \int_{\Omega} p(x) |\nabla u_j|^2 dx - \frac{1}{q} \int_{\Omega} |u_j|^q dx = l + o(1)$$

and

$$(2.5.10) \quad -\operatorname{div}(p(x) \nabla u_j) = |u_j|^{q-2} u_j + \xi_j, \quad \text{with } \xi_j \rightarrow 0 \text{ in } H^{-1}.$$

We start by showing that $\{u_j\}$ is bounded in $H_0^1(\Omega)$.

Testing (2.5.10) with u_j , we obtain

$$(2.5.11) \quad \int_{\Omega} p(x) |\nabla u_j|^2 dx - \int_{\Omega} |u_j|^q dx = \langle \xi_j, u_j \rangle.$$

Computing (2.5.9) $-\frac{1}{q}$ (2.5.11), we get

$$\frac{1}{n} \int_{\Omega} p(x) |\nabla u_j|^2 dx = l - \frac{1}{q} \langle \xi_j, u_j \rangle + o(1)$$

and

$$\frac{1}{q} p_0 \|u_j\|^2 \leq l + \|\xi_j\|_{H^{-1}} \|u_j\| + o(1).$$

From the above inequality we deduce that $\{u_j\}$ is bounded in $H_0^1(\Omega)$.

We may extract a subsequence still denoted by u_j , such that

$$u_j \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega),$$

$$u_j \rightarrow u \quad \text{weakly in } L^q(\Omega).$$

We take, respectively, (2.5.9) $-\frac{1}{q}$ (2.5.11), and (2.5.9) $-\frac{1}{2}$ (2.5.11), we obtain

$$\int_{\Omega} p(x) |\nabla u_j|^2 dx = n l + o(1) \quad \text{and} \quad \int_{\Omega} |u_j|^q dx = n l + o(1).$$

Then

$$\lim_{j \rightarrow \infty} \int_{\Omega} p(x) |\nabla u_j|^2 dx = n l \quad \text{and} \quad \lim_{j \rightarrow \infty} \int_{\Omega} |u_j|^q dx = n l.$$

On the other hand, using Sobolev's inequality and the fact that $p(x) \geq p_0$ for all $x \in \bar{\Omega}$, we write

$$\begin{aligned} \int_{\Omega} p(x) |\nabla u_j|^2 dx &\geq p_0 \int_{\Omega} |\nabla u_j|^2 dx \\ &\geq p_0 S \left(\int_{\Omega} |u_j|^q dx \right)^{\frac{2}{q}}. \end{aligned}$$

Letting j tend to $+\infty$, we obtain

$$n l \geq p_0 S (n l)^{\frac{2}{q}}.$$

From the above inequality, we have either $l = 0$ or $l \geq \frac{1}{n} (p_0 S)^{\frac{n}{2}}$.

By assumption, $l < \frac{1}{n} (p_0 S)^{\frac{n}{2}}$, we deduce that $l = 0$ and $u_j \rightarrow 0$ strongly in $H_0^1(\Omega)$. \square

Proof of Theorem A 2

We use the procedure of the proof of Lemma 2 in ([B], page S33), (see also [S]).

Let $\{u_j\}$ be given as in the hypothesis of Theorem A 2. Therefore we have

$$(2.5.12) \quad \frac{1}{2} \int_{\Omega} p(x) |\nabla u_j|^2 dx - \frac{1}{q} \int_{\Omega} |u_j|^q dx = c + o(1)$$

and

$$(2.5.13) \quad -\operatorname{div}(p(x) \nabla u_j) = |u_j|^{q-2} u_j + \xi_j, \quad \text{with } \xi_j \rightarrow 0 \text{ in } H^{-1}.$$

As in proof of Lemma 2.5.1, we see that $\{u_j\}$ is bounded in $H_0^1(\Omega)$ and therefore $u_j \rightharpoonup u$ weakly in $H_0^1(\Omega)$.

We will obtain Theorem A 2, by contradiction. We suppose that

$$u_j \text{ does not strongly converge to } u \text{ in } H_0^1(\Omega).$$

Set $v_j = u_j - u$, we have $\int_{\Omega} |\nabla v_j|^2 dx \not\rightarrow 0$. Therefore, as in [L], (see also [B], [S] and [Le1]), there exists a sequence $\{x_j\}$ in Ω and a sequence $r_j \rightarrow 0$ such that

$$\int_{B(x_j, r_j)} p(x) |\nabla v_j|^2 dx = \mathbf{v} \quad \text{with } \mathbf{v} > 0 \text{ small enough.}$$

Since $\{x_j\} \subset \Omega$ and Ω is bounded, then for a subsequence, still denoted by x_j , we can assume that $x_j \rightarrow x_0$ in $\bar{\Omega}$. Following [Le1, Corollary 6], we have that $x_0 = a$ and then

$$(2.5.14) \quad x_j \rightarrow a \quad \text{in } \bar{\Omega}.$$

Set

$$\tilde{v}_j(x) = r^{\frac{n-2}{2}} v_j(r_j x + x_j), \quad x \in \mathbb{R}^n,$$

(v_j is extended by 0 outside Ω), since $\{v_j\}$ is bounded in $H_0^1(\Omega)$, then $\{\tilde{v}_j\}$ is bounded in $H^1(\mathbb{R}^n)$.

We have, (see the limiting problem), that \tilde{v}_j satisfies

$$(2.5.15) \quad -p(x_j + r_j y) \Delta \tilde{v}_j(y) - r_j^2 (\nabla p)(x_j + r_j y) \cdot (\nabla \tilde{v}_j)(y) = |\tilde{v}_j|^{q-2} \tilde{v}_j(y) + \tilde{\xi}_j \quad \text{on } \frac{\Omega - x_j}{r_j},$$

with $\tilde{\xi}_j \rightarrow 0$.

Passing to the limit, on a convenient subsequence of \tilde{v}_j , in (2.5.15), (see the limiting problem for more details), we obtain $\tilde{v}_j \rightharpoonup w$ weakly in $L_{loc}^q(\mathbb{R}^n)$, $\tilde{v}_j \rightharpoonup w$ weakly in $H_{loc}^1(\mathbb{R}^n)$, and w satisfies

$$(2.5.16) \quad -p_0 \Delta w(y) = |w|^{q-2} w(y) \quad \text{on } \mathbb{R}^n,$$

and as in [Le1, Proposition 3], (see also [Le2]), we have $w \neq 0$.

It is easy to see that $\tilde{w} = (p_0)^{-\frac{1}{q-2}} w$ satisfies

$$(2.5.17) \quad -\Delta \tilde{w}(y) = |\tilde{w}|^{q-2} \tilde{w}(y) \quad \text{on } \mathbb{R}^n,$$

Therefore by the uniqueness of solution of problem (2.5.17), (see Aubin [A]), we know that \tilde{w} is of the form $\frac{\gamma_n}{(1 + |x|^2)^{\frac{n-2}{2}}}$, where γ_n is a positive constant depending only on n .

Now, we set

$$w_j(x) = v_j(x) - \frac{1}{r_j^{\frac{n-2}{2}}} w \left(\frac{x - x_j}{r_j} \right), \quad x \in \Omega.$$

We have

$$(2.5.18) \quad E(w_j) = E(u_j) - E(u) - E \left(\frac{1}{r_j^{\frac{n-2}{2}}} w \left(\frac{x - x_j}{r_j} \right) \right) + o(1)$$

and

$$(2.5.19) \quad \begin{aligned} dE(w_j) &= dE(u_j) - dE(u) - dE \left(\frac{1}{r_j^{\frac{n-2}{2}}} w \left(\frac{x - x_j}{r_j} \right) \right) + o(1) \\ dE(w_j) &= o(1). \end{aligned}$$

On other hand, we see that

$$E \left(\frac{1}{r_j^{\frac{n-2}{2}}} w \left(\frac{x - x_j}{r_j} \right) \right) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{p(x)}{r_j^n} |\nabla w|^2 \left(\frac{x - x_j}{r_j} \right) dx - \frac{1}{q} \int_{\mathbb{R}^n} \frac{|w(\frac{x-x_j}{r_j})|^q}{r_j^n} dx.$$

Using the change of variable $y = \frac{x-x_j}{r_j}$, we write

$$E \left(\frac{1}{r_j^{\frac{n-2}{2}}} w \left(\frac{x-x_j}{r_j} \right) \right) = \frac{1}{2} \int_{\mathbb{R}^n} p(x_j + r_j y) |\nabla w(y)|^2 dy - \frac{1}{q} \int_{\mathbb{R}^n} |w(y)|^q dy.$$

Since p is a continuous function and $x_j \rightarrow a$ in $\bar{\Omega}$, applying the Dominated Convergence Theorem, we obtain

$$E \left(\frac{1}{r_j^{\frac{n-2}{2}}} w \left(\frac{x-x_j}{r_j} \right) \right) = \frac{1}{2} p_0 \int_{\mathbb{R}^n} |\nabla w(y)|^2 dy - \frac{1}{q} \int_{\mathbb{R}^n} |w(y)|^q dy + o(1).$$

Now, we use the fact that $w = (p_0)^{\frac{1}{q-2}} \tilde{w}$, we obtain

$$E \left(\frac{1}{r_j^{\frac{n-2}{2}}} w \left(\frac{x-x_j}{r_j} \right) \right) = (p_0)^{\frac{q}{q-2}} \left[\frac{1}{2} \int_{\mathbb{R}^n} |\nabla \tilde{w}(y)|^2 dy - \frac{1}{q} \int_{\mathbb{R}^n} |\tilde{w}(y)|^q dy \right] + o(1).$$

By definition of \tilde{w} , we know that $\left[\frac{1}{2} \int_{\mathbb{R}^n} |\nabla \tilde{w}(y)|^2 dy - \frac{1}{q} \int_{\mathbb{R}^n} |\tilde{w}(y)|^q dy \right] = \frac{1}{n} S^{\frac{n}{2}}$.

Therefore, we deduce that

$$(2.5.20) \quad E \left(\frac{1}{r_j^{\frac{n-2}{2}}} w \left(\frac{x-x_j}{r_j} \right) \right) = \frac{1}{n} (p_0 S)^{\frac{n}{2}}.$$

Inserting (2.5.20) into (2.5.18), we get

$$\begin{aligned} E(w_j) &= E(u_j) - E(u) - E \left(\frac{1}{r_j^{\frac{n-2}{2}}} w \left(\frac{x-x_j}{r_j} \right) \right) + o(1) \\ &= c - E(u) - \frac{1}{n} (p_0 S)^{\frac{n}{2}} + o(1). \end{aligned}$$

From the assumption in Theorem A 2, namely, $c \in]\frac{1}{n}(p_0 S)^{\frac{n}{2}}, \frac{2}{n}(p_0 S)^{\frac{n}{2}}[$, and the fact that $E(u) \geq 0$ we have

$$(2.5.21) \quad c - E(u) - \frac{1}{n} (p_0 S)^{\frac{n}{2}} < \frac{1}{n} (p_0 S)^{\frac{n}{2}}.$$

Therefore, we have

$$E(w_j) \rightarrow l = c - E(u) - \frac{1}{n} (p_0 S)^{\frac{n}{2}} < \frac{1}{n} (p_0 S)^{\frac{n}{2}}$$

and

$$dE(w_j) \rightarrow 0 \quad \text{in } H^{-1}.$$

Applying Lemma 2.5.1, we deduce that $l = 0$ and

$$(2.5.22) \quad c = E(u) + \frac{1}{n} (p_0 S)^{\frac{n}{2}}.$$

We distinguish two cases:

Case 1: If $u = 0$, then $E(u) = 0$ and (2.5.22) becomes $c = \frac{1}{n}(p_0 S)^{\frac{n}{2}}$ which is a contradiction since $c \in]\frac{1}{n}(p_0 S)^{\frac{n}{2}}, \frac{2}{n}(p_0 S)^{\frac{n}{2}}[$.

Case 2: If $u \neq 0$, then, since $c < \frac{2}{n}(p_0 S)^{\frac{n}{2}}$, from (2.5.22), we see that $E(u) < \frac{1}{n}(p_0 S)^{\frac{n}{2}}$ which is a contradiction.

Indeed, since $u_j \rightharpoonup u$, by the weak continuity of dE , we have $dE(u) = 0$.

Multiplying this last equality by u , we obtain

$$\int_{\Omega} p(x)|\nabla u|^2 dx = \int_{\Omega} |u|^q dx.$$

Therefore, since $u \neq 0$, we have

$$\begin{aligned} E(u) &= \frac{1}{2} \int_{\Omega} p(x)|\nabla u|^2 dx - \frac{1}{q} \int_{\Omega} |u|^q dx \\ &= \frac{1}{n} \int_{\Omega} p(x)|\nabla u|^2 dx \\ &= \frac{1}{n} \left(\frac{\int_{\Omega} p(x)|\nabla u|^2 dx}{\left(\int_{\Omega} |u|^q dx\right)^{\frac{2}{q}}} \right)^{\frac{n}{2}} \\ &> \frac{1}{n}(p_0 S)^{\frac{n}{2}}. \end{aligned}$$

Conclusion: $\{u_j\}$, for a subsequence, converge strongly to u .

This finishes the proof of Theorem A 2. □

Part II

Problèmes de Neumann non linéaires avec une non-linéarité critique au bord

Chapter 3

On some nonlinear Neumann problem with weight and with critical Sobolev trace maps

(This work is a paper in preparation)

3.1 Introduction

In this paper, we study the existence of solutions to the problem

$$(3.1.1) \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = \lambda u + \alpha|u|^{r-2}u & \text{in } \Omega, \\ u \neq 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with the smooth boundary $\partial\Omega$, ν is the outer normal on $\partial\Omega$, the coefficient Q is continuous and positive on $\partial\Omega$ and the coefficient $p \in H^1(\Omega)$ is continuous and positive in $\bar{\Omega}$. $q = \frac{2(N-1)}{N-2}$, $2 < r < q$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $\alpha \in \mathbb{R}$. The exponent q is the critical Sobolev exponent for the trace embedding of the space $H^1(\Omega)$ into $L^q(\partial\Omega)$. The embedding of $H^1(\Omega)$ into $L^q(\partial\Omega)$ is continuous, but not compact.

The case $p \equiv 1 \equiv Q$ and $\alpha = 0$ has an extensive literature and we refer to [AM], [AY], [CK] and [CY]. It was shown that the mean curvature of $\partial\Omega$ and the geometrical condition (g.c) at some point on $\partial\Omega$ play an important role in solving similar problems of (3.1.1).

The condition (g.c) is defined as follows (see [AM] and [AY]),

(g.c): $\partial\Omega$ is said to satisfy the geometrical condition at $y \in \partial\Omega$ if there exists a neighborhood $U(y)$ of y such that $\Omega \cap U(y)$ lies on one side of the tangent plane at

y .

Let us remark that there always exists at least one point $y \in \partial\Omega$ for which (g.c) is satisfied. Let $x_0 \in \partial\Omega$ satisfy

$$\frac{p(x_0)}{(Q(x_0))^{N-2}} = \min_{x \in \partial\Omega} \frac{p(x)}{(Q(x))^{N-2}}.$$

We assume that

$$(3.1.2) \quad |p(x) - p(x_0)| = o(|x - x_0|)$$

and

$$(3.1.3) \quad |Q(x) - Q(x_0)| = o(|x - x_0|)$$

for x near x_0 .

Throughout this paper, we suppose that $\partial\Omega$ satisfies the geometrical condition (g.c) at the point x_0 . This condition guarantees that all the principal curvatures at x_0 with respect to unit outward normal are non-negative.

By $\{\lambda_k(p)\}$ we denote the sequence of eigenvalues for $-\operatorname{div}(p(x)\nabla \cdot)$ with the Neumann boundary conditions

$$\begin{cases} -\operatorname{div}(p(x)\nabla u) = \mu u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

It is known that $0 = \lambda_1(p) < \lambda_2(p) \leq \dots$

In this paper, we treat the case where $\lambda < 0$ and the case where $\lambda_{k-1}(p) < \lambda < \lambda_k(p)$ for some $k \in \mathbf{N} \setminus \{0\}$.

Our main result is the following

Theorem 3.1.1.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ such that $\partial\Omega$ satisfies the geometrical condition (g.c) at x_0 and $H(x_0) > 0$. We assume that p and Q respectively satisfy (3.1.2) and (3.1.3). Then

- 1) for every $\lambda < 0$ and for every $\alpha \in \mathbb{R}$ there exists a solution of (3.1.1),*
- 2) for every $\lambda \in]\lambda_{k-1}(p), \lambda_k(p)[$ and for every $\alpha \geq 0$ the problem (3.1.1) possesses a solution.*

The rest of this paper is divided into three sections. In Section 1 we give some notations and some estimates which will be used throughout this paper. Section 2, Section 3 and section 4 are devoted respectively to the study of the case $\lambda \in]\lambda_{k-1}(p), \lambda_k(p)[$ and $\alpha \geq 0$, the case $\lambda < 0$ and $\alpha \geq 0$ and the case $\lambda < 0$ and $\alpha \leq 0$.

3.2 Preliminary and notations

Let

$$S_1 = \inf \left\{ \int_{\mathbb{R}_+^N} |\nabla u|^2 dx; u \in H^1(\mathbb{R}_+^N), \int_{\partial\mathbb{R}_+^N} |u|^q dx = 1 \right\}$$

be the best constant in the trace embedding for the space $H^1(\mathbb{R}_+^N)$ into $L^q(\partial\mathbb{R}_+^N)$, where $\mathbb{R}_+^N = \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\}$. Due to a result of [L], S_1 is achieved and from a result of [E], S_1 is given by

$$S_1 = \frac{N-2}{2} (\omega_N)^{\frac{1}{N-1}}, \quad \text{where } \omega_N \text{ is the area of } S^{N-1}.$$

The minimizing functions are of the form

$$(3.2.1) \quad W(x) = \frac{\gamma_N}{[|x'|^2 + (1+x_N)^2]^{\frac{N-2}{2}}},$$

where $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$ and γ_N is a positive constant depending on N .

We set

$$W_{\varepsilon, x_0}(x) = \varepsilon^{-\frac{N-2}{2}} \phi(x) W\left(\frac{x-x_0}{\varepsilon}\right),$$

where $x_0 \in \partial\Omega$ and ϕ is a radial C^∞ -function such that

$$\phi(x) = \begin{cases} 1 & \text{if } |x-x_0| \leq \frac{R}{4} \\ 0 & \text{if } |x-x_0| > \frac{R}{2} \end{cases}$$

with $R > 0$ is a small constant.

Using (3.1.2) and (3.1.3), from [AY], we have the following estimates

$$(3.2.2) \quad \int_{\Omega} p(x) |\nabla W_{\varepsilon, x_0}|^2 dx = p(x_0) A_1 - p(x_0) H(x_0) \begin{cases} A'_2 \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) & \text{if } N = 3 \\ A_2 \varepsilon + o(\varepsilon) & \text{if } N \geq 4, \end{cases}$$

$$(3.2.3) \quad \int_{\partial\Omega} p(x) Q(x) |W_{\varepsilon, x_0}|^q ds_x = p(x_0) Q(x_0) (B_1 - H(x_0) B_2 \varepsilon) + o(\varepsilon)$$

and

$$(3.2.4) \quad \int_{\Omega} |W_{\varepsilon, x_0}|^2 dx = \begin{cases} O(\varepsilon) & \text{if } N = 3 \\ O(\varepsilon^2 |\log \varepsilon|) & \text{if } N = 4 \\ D\varepsilon^2 + o(\varepsilon^2) & \text{if } N \geq 5. \end{cases}$$

where $A_1 = \frac{(\Gamma(\frac{N-1}{2}))^2}{2\Gamma(N-2)} \omega_{N-1}$, $A'_2 = \frac{\omega_2}{2}$, $A_2 = \frac{(N-1)(N-2)\Gamma(\frac{N-1}{2})\Gamma(\frac{N-3}{2})}{8\Gamma(N-2)} \omega_{N-1}$,

$$B_1 = \frac{(\Gamma(\frac{N-1}{2}))^2}{2\Gamma(N-1)}\omega_{N-2}, B_2 = \frac{(N-1)(\Gamma(\frac{N-1}{2}))^2}{4\Gamma(N-1)}\omega_{N-2} \text{ and } D = \frac{\Gamma(\frac{N-1}{2})\Gamma(\frac{N-3}{2})}{2(N-4)\Gamma(N-2)}\omega_{N-1}.$$

The function Γ is defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-t}t^{z-1}dt \quad \text{for every } z \in]0, +\infty[.$$

Let us notice that

$$S_1 = \frac{A_1}{B_1^q}.$$

We claim that, as $\varepsilon \rightarrow 0$, we have

$$(3.2.5) \quad \int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^{q-1}ds_x = p(x_0)Q(x_0)K_1\varepsilon^{\frac{N-2}{2}} + o(\varepsilon^{\frac{N-2}{2}}).$$

and

$$(3.2.6) \quad \int_{\Omega} |W_{\varepsilon,x_0}|^r dx = \begin{cases} o(\varepsilon|\log \varepsilon|) & \text{if } N = 3 \\ o(\varepsilon) & \text{if } N \geq 4. \end{cases}$$

The proof of these two claims are given in the appendix.

The norm in the Lebesgue space $L^t(\Omega)$ is denoted by $\|\cdot\|_t$. We endow the Sobolev space $H^1(\Omega)$ with the norm

$$\|u\| = \left[\int_{\Omega} (|\nabla u|^2 + |u|^2) dx \right]^{\frac{1}{2}}.$$

3.3 Case $\lambda_{k-1}(p) < \lambda < \lambda_k(p)$ and $\alpha \geq 0$

The main result of this section is the following

Theorem 3.3.1.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ such that $\partial\Omega$ satisfies the geometrical condition (g.c) at x_0 and such that $H(x_0) > 0$. We assume that p and Q respectively satisfy (3.1.2) and (3.1.3). Then for every $\lambda \in]\lambda_{k-1}(p), \lambda_k(p)[$ and for every $\alpha \geq 0$, the problem (3.1.1) has a solution.

The proof of Theorem 3.3.1 relies on the following min-max principle based on a topological linking, see [W].

Theorem 3.3.2.

Let $X = E^- \oplus E^+$ be a Banach space with $\dim E^- < \infty$. Let $\rho > \theta > 0$ and let $w \in E^+$ be such that $\|w\| = \theta$. Define

$$\begin{aligned} M &:= \{u = v + sw : \|u\| \leq \rho, s \geq 0, v \in E^-\}, \\ M_0 &:= \{u = v + sw : v \in E^-, \|u\| = \rho \text{ and } s \geq 0 \text{ or } \|u\| \leq \rho \text{ and } s = 0\}, \\ N &:= \{u \in E^+ : \|u\| = \theta\}. \end{aligned}$$

Let $\varphi \in C^1(X, \mathbb{R})$ be such that

$$b := \inf_N \varphi > a := \max_{M_0} \varphi.$$

If φ satisfies the Palais-Smale condition with

$$c := \inf_{\gamma \in \Gamma} \max_{u \in M} \varphi(\gamma(u)), \quad \text{where } \Gamma := \{\gamma \in C(M, X) : \gamma|_{M_0} = \text{id}\}.$$

Then c is a critical value of φ .

Let I_α be a variational functional for (3.1.1) given by

$$I_\alpha(u) = \frac{1}{2} \int_\Omega p(x) |\nabla u|^2 dx - \frac{\lambda}{2} \int_\Omega |u|^2 dx - \frac{\alpha}{r} \int_\Omega |u|^r dx - \frac{1}{q} \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x.$$

We have the following lemma

Lemma 3.3.1.

Let $\{u_n\} \subset H^1(\Omega)$ be a sequence satisfying

$$(3.3.1) \quad I_\alpha(u_n) \rightarrow c < \frac{p(x_0)}{2(N-1)(Q(x_0))^{N-2}} S_1^{N-1}$$

and

$$(3.3.2) \quad I'_\alpha(u_n) \rightarrow 0 \text{ in } H^{-1}(\Omega).$$

Then $\{u_n\}$ is relatively compact in $H^1(\Omega)$.

Proof. We start by showing that $\{u_n\}$ is bounded in $H^1(\Omega)$. Arguing by contradiction, we assume that $\|u_n\| \rightarrow \infty$. We set $v_n = \frac{u_n}{\|u_n\|}$. Since $\{v_n\}$ is bounded in $H^1(\Omega)$, we may assume that $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$.

Using (3.3.2), for every $\phi \in H^1(\Omega)$, we obtain

$$(3.3.3) \quad \int_\Omega (p(x) \nabla v_n \nabla \phi - \lambda v_n \phi) dx = \|u_n\|^{-1} \left[\int_{\partial\Omega} Q(x) |u_n|^{q-2} u_n \phi ds_x + \alpha \int_\Omega |u_n|^{r-2} u_n \phi dx \right] + o(1).$$

On the other hand, we have

$$(3.3.4) \quad \int_{\partial\Omega} p(x)Q(x)|u_n|^{q-2}u_n\phi ds_x \leq C_1 \left(\int_{\partial\Omega} |u_n|^q ds_x \right)^{\frac{q-1}{q}} \left(\int_{\partial\Omega} |\phi|^q ds_x \right)^{\frac{1}{q}}$$

and

$$(3.3.5) \quad \int_{\Omega} |u_n|^{r-2}u_n\phi dx \leq \left(\int_{\Omega} |u_n|^r dx \right)^{\frac{r-1}{r}} \left(\int_{\Omega} |\phi|^r dx \right)^{\frac{1}{r}}.$$

Combining the fact that $H^1(\Omega) \subset L^r(\Omega)$ with the relations (3.3.1) and (3.3.2), an easy computation yields

$$\frac{1}{2(N-1)} \int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x \leq c + \alpha \left(\frac{1}{r} - \frac{1}{2} \right) \int_{\Omega} |u_n|^r dx + o(\|u_n\|).$$

Since $\alpha \geq 0$ and $2 < r < q$, there exists a positive constant C_2 such that

$$(3.3.6) \quad \int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x \leq C_2 + o(\|u_n\|).$$

Inserting (3.3.4), (3.3.5) and (3.3.6) into (3.3.3), we deduce that, for a positive constant C_3 we have

$$\int_{\Omega} (p(x)\nabla v_n \nabla \phi - \lambda v_n \phi) dx \leq C_3 \left[\|u_n\|^{\frac{(q-1)}{q}-1} + \|u_n\|^{-\frac{1}{r}} \right] + o(1).$$

We obtain after letting $n \rightarrow \infty$

$$\int_{\Omega} (p(x)\nabla v \nabla \phi - \lambda v \phi) dx = 0$$

for every $\phi \in H^1(\Omega)$. Since $\lambda \in]\lambda_{k-1}(p), \lambda_k(p)[$, we see that $v \equiv 0$ on Ω . Consequently, we may assume that $v_n \rightarrow 0$ strongly in $L^2(\Omega)$ and in $L^r(\Omega)$ since $r < q < \frac{2N}{N-2}$. Hence, from (3.3.1) and (3.3.2), we can write

$$\frac{1}{2} \int_{\Omega} p(x)|\nabla v_n|^2 dx = \frac{\|u_n\|^{q-2}}{q} \int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x + o(1)$$

and

$$\int_{\Omega} p(x)|\nabla v_n|^2 dx = \|v_n\|^{q-2} \int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x + o(1).$$

These two relations imply that $\int_{\Omega} p(x)|\nabla v_n|^2 dx = o(1)$ and thus $v_n \rightarrow 0$ strongly in $H^1(\Omega)$, which is impossible since $\|v_n\| = 1$. Consequently $\{u_n\}$ is bounded in $H^1(\Omega)$ and we may assume that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$.

By the concentration-compactness principle [L], there exists some, at most countable, set J and a family $(x_j)_{j \in J}$ of distinct points in $\partial\Omega$, $(\mu_j)_{j \in J}$ and $(\nu_j)_{j \in J}$ in $]0, +\infty[$ such that

$$|\nabla u_n|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \sum_{j \in J} \mu_j \delta_{x_j}$$

and

$$|u_n|^q \rightharpoonup |u|^q + \sum_{j \in J} \nu_j \delta_{x_j}$$

weakly in the sense of measures. Moreover, for every $j \in J$ we have $S_1(\nu_j)^{\frac{2}{q}} \leq \mu_j$.

Let x_{j_0} be fixed. Testing (3.3.2) by a family of C^1 -functions concentrating at x_{j_0} we get

$$p(x_{j_0})\mu_{j_0} \leq p(x_{j_0})Q(x_{j_0})\nu_{j_0}.$$

If $\nu_{j_0} > 0$, then

$$(3.3.7) \quad \frac{S_1^{N-1}}{Q(x_{j_0})^{N-1}} \leq \nu_{j_0}.$$

On the other hand we have

$$I(u_n) - \frac{1}{2} \langle I'(u_n), u_n \rangle = \frac{1}{2(N-1)} \int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x + \alpha \left(\frac{1}{2} - \frac{1}{r} \right) \int_{\Omega} |u_n|^r dx + o(1).$$

Letting $n \rightarrow \infty$ we obtain

$$c = \frac{1}{2(N-1)} \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + \alpha \left(\frac{1}{2} - \frac{1}{r} \right) \int_{\Omega} |u|^r dx + \frac{1}{2(N-1)} \sum_{j \in J} p(x_j)Q(x_j)\nu_j,$$

Using (3.3.7), the fact that $\alpha \geq 0$, $2 < r < q$ and also $\min_{x \in \partial\Omega} \frac{p(x)}{(Q(x))^{N-2}} = \frac{p(x_0)}{Q(x_0)^{N-2}}$, we get

$$c \geq \frac{p(x_{j_0})S_1^{N-1}}{2(N-1)(Q(x_{j_0}))^{N-2}} \geq \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}$$

and we have a contradiction with the hypothesis $c < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}$. Hence $\nu_j = 0$ for every $j \in J$. This gives that $u_n \rightarrow u$ strongly in $L^q(\partial\Omega)$. Since $u_n \rightarrow u$ strongly in $L^2(\Omega)$ and in $L^r(\Omega)$, by testing (3.3.2) successively by u_n and u , an easy computation yields

$$\lim_{n \rightarrow \infty} \int_{\Omega} p(x)|\nabla u_n|^2 dx = \int_{\Omega} p(x)|\nabla u|^2 dx.$$

Since $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$, we deduce that $u_n \rightarrow u$ strongly in $H^1(\Omega)$ and $\{u_n\}$ is relatively compact in $H^1(\Omega)$. \square

We now apply the Theorem 3.3.2 to our case, with $X = H^1(\Omega)$, $\varphi = I_\alpha$ and $E^- = \text{span}\{e_1(p), \dots, e_l(p)\}$, where $e_1(p), \dots, e_l(p)$ are eigenfunctions corresponding to the eigenvalues $\lambda_1(p), \dots, \lambda_{k-1}(p)$. We have the orthogonal decomposition $X = H^1(\Omega) = E^- \oplus E^+$.

We have the following

Lemma 3.3.2.

There exist constants $\beta > 0$, $\rho > 0$ and $\rho > \theta$ such that

$$I_\alpha(u) \geq \beta \text{ for all } u \in N$$

and

$$I_\alpha(u) \leq 0 \text{ for all } u \in M_0$$

where $M_0 = \{u = v + sw : v \in E^-, \|u\| = \rho \text{ and } s \geq 0 \text{ or } \|u\| \leq \rho \text{ and } s = 0\}$ and $N = \{u \in E^+ : \|u\| = \theta\}$.

Proof.

We start by a useful remark: For every $u \in E^+$, we have

$$\int_{\Omega} p(x)|\nabla u|^2 dx \geq \lambda_k(p) \int_{\Omega} |u|^2 dx.$$

Therefore, there exists a positive constants C_1 and C_2 , such that for every $u \in E^+$ we have

$$(3.3.8) \quad \int_{\Omega} p(x)|\nabla u|^2 dx \geq C_1 \|u\|^2 \geq C_2 \int_{\Omega} p(x)|\nabla u|^2 dx.$$

Using (3.3.8) and Hölder's inequality, we write

$$\begin{aligned} I_\alpha(u) &= \frac{1}{2} \int_{\Omega} p(x)|\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \frac{1}{q} \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x - \frac{1}{\alpha} \int_{\Omega} |u|^r dx \\ &\geq \frac{(1 - \frac{\lambda}{\lambda_k(p)})}{2} C_1 \|u\|^2 - C_3 \|u\|^q - C_4 \|u\|^r, \end{aligned}$$

where C_3 and C_4 are some positive constants.

Therefore

$$I_\alpha(u) \geq \|u\|^2 \left(\frac{(1 - \frac{\lambda}{\lambda_k(p)})}{2} C_1 - C_3 \|u\|^{q-2} - C_4 \|u\|^{r-2} \right)$$

and it is easy to see that there exists $\beta > 0$ and $\theta > 0$ such that

$$I_\alpha(u) \geq \beta \quad \forall u \in N.$$

On the other hand, we have that

$$I_\alpha(u) \rightarrow -\infty, \text{ as } \|u\| \rightarrow +\infty,$$

thus, there exists $\rho > \theta$ such that $\max_{M_0} I_\alpha(u) \leq 0$. □

We now define

$$Z_\varepsilon = E^- \oplus \mathbb{R}W_{\varepsilon, x_0} = E^- \oplus \mathbb{R}W_{\varepsilon, x_0}^+,$$

where W_{ε, x_0}^+ denotes the projection of W_{ε, x_0} onto E^+ . From now on, we use W_{ε, x_0}^+ in place of w .

Proof of Theorem 3.3.1

We start by checking that $c < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}$.

Since $\alpha \geq 0$, we have

$$I(u) \leq \frac{1}{2} \left[\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right] - \frac{1}{q} \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x,$$

then

$$\max_{0 \leq t < \infty} I(tu) \leq \frac{[\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx]^{N-1}}{2(N-1) (\int_{\partial\Omega} p(x) Q(x) |u|^q ds_x)^{N-2}}$$

for $u \in H^1(\Omega)$ with $u \neq 0$ on $\partial\Omega$.

Set

$$m_\varepsilon = \sup_{u \in Z_\varepsilon} \int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx.$$

$$\int_{\partial\Omega} p(x) Q(x) |u|^q ds_x = 1$$

Therefore if

$$(3.3.9) \quad m_\varepsilon < \frac{(p(x_0))^{\frac{1}{N-1}} S_1}{2(N-1)(Q(x_0))^{\frac{N-2}{N-1}}},$$

then

$$\sup_{u \in M} I(u) < \frac{p(x_0) S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}.$$

Hence it is sufficient to show that (3.3.9) holds.

Suppose that $\int_{\partial\Omega} p(x) Q(x) |u|^q ds_x = 1$. We write $u = u^- + sW_{\varepsilon, x_0} = (u^- + sW_{\varepsilon, x_0}^-) + sW_{\varepsilon, x_0}^+$.

Since $\lambda \in]\lambda_{k-1}(p), \lambda_k(p)[$, we have

$$\int_{\Omega} p(x) |\nabla W_{\varepsilon, x_0}^-|^2 - \lambda |W_{\varepsilon, x_0}^-|^2 dx \leq 0,$$

and

$$\int_{\Omega} p(x) |\nabla W_{\varepsilon, x_0}^-|^2 dx \leq \lambda \int_{\Omega} |W_{\varepsilon, x_0}^-|^2 dx \leq \int_{\Omega} |W_{\varepsilon, x_0}|^2 dx \rightarrow 0$$

as $\varepsilon \rightarrow 0$. Moreover

$$\int_{\partial\Omega} p(x) Q(x) |W_{\varepsilon, x_0}^-|^q ds_x \leq C \left(\int_{\Omega} (|\nabla W_{\varepsilon, x_0}^-|^2 + |W_{\varepsilon, x_0}^-|^2) dx \right)^{\frac{q}{2}} \rightarrow 0$$

as $\varepsilon \rightarrow 0$.

On the other hand, it is easy to see that $\int_{\partial\Omega} |u^-|^q ds_x \leq C_1$ and $0 < s < C_2$ with C_1

and C_2 are a positive constants.

Using the convexity of $u \rightarrow \int_{\partial\Omega} p(x)Q(x)|u|^q dx$, we can write

$$\begin{aligned} 1 &= \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \\ &\geq \int_{\partial\Omega} p(x)Q(x)|sW_{\varepsilon,x_0}|^q ds_x + q \int_{\partial\Omega} p(x)Q(x)u^-(sW_{\varepsilon,x_0})^{q-1} ds_x \\ &\geq \int_{\partial\Omega} p(x)Q(x)|sW_{\varepsilon,x_0}|^q ds_x - C_3 \left(\int_{\partial\Omega} |W_{\varepsilon,x_0}|^{q-1} ds_x \right)^{\frac{1}{q}} \left(\int_{\partial\Omega} |u^-|^q ds_x \right)^{\frac{1}{q}} \end{aligned}$$

where C_3 is a positive constant.

Since $\int_{\partial\Omega} |W_{\varepsilon,x_0}|^{q-1} ds_x = O(\varepsilon^{\frac{N-2}{2}})$ (see (3.2.5)), we deduce from the above inequality that

$$(3.3.10) \quad \int_{\partial\Omega} p(x)Q(x)|sW_{\varepsilon,x_0}|^q dx \leq 1 + C_4 \varepsilon^{\frac{N-2}{2}}$$

for some constant $C_4 > 0$.

Since all the norms on E_k^- are equivalent, there exists a positive constant C_5 such that

$$(3.3.11) \quad \begin{aligned} \int_{\Omega} (p(x)Q(x)\nabla W_{\varepsilon,x_0}\nabla u^- - \lambda W_{\varepsilon,x_0}u^-) dx &\leq C_5(\|\nabla W_{\varepsilon,x_0}\|_1 + \lambda\|W_{\varepsilon,x_0}\|_1)\|u^-\| \\ &= O(\varepsilon^{\frac{N-2}{2}})\|u^-\|. \end{aligned}$$

Finally, using the above estimate, we can write

$$\begin{aligned} \int_{\Omega} p(x)|\nabla u|^2 - \lambda \int_{\Omega} |u|^2 dx &\leq \int_{\Omega} (p(x)|\nabla u^-|^2 - \lambda|u^-|^2) dx + O(\varepsilon^{\frac{N-2}{2}})\|u^-\| \\ &\quad + s^2 \int_{\Omega} (p(x)|\nabla W_{\varepsilon,x_0}|^2 - \lambda|W_{\varepsilon,x_0}|^2) dx \\ &\leq (\lambda_{k-1}(p) - \lambda) \int_{\Omega} |u^-|^2 dx + O(\varepsilon^{\frac{N-2}{2}})\|u^-\| \\ &\quad + \frac{\int_{\Omega} (p(x)|\nabla W_{\varepsilon,x_0}|^2 - \lambda|W_{\varepsilon,x_0}|^2) dx}{\left(\int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x \right)^{\frac{2}{q}}} \left(s^q \int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x \right)^{\frac{2}{q}}. \end{aligned}$$

Combining (3.2.2), (3.2.3), (3.2.4) and (3.3.10), an easy computation yields $m_\varepsilon <$

$\frac{(p(x_0))^{\frac{1}{N-1}} S_1}{2(N-1)(Q(x_0))^{\frac{N-2}{N-1}}}$ for $\varepsilon > 0$ sufficiently small.

Combining Lemma 3.3.2, Theorem 3.3.2 and Lemma 3.3.1, we deduce the conclusion of Theorem 3.3.1.

3.4 Case $\lambda < 0$ and $\alpha \geq 0$

The main result of this section is the following.

Theorem 3.4.1.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ such that $\partial\Omega$ satisfies the geometrical condition (g.c) at x_0 and $H(x_0) > 0$. We assume that p and Q respectively satisfy (3.1.2) and (3.1.3). Then for every $\lambda < 0$ and for every $\alpha \geq 0$ there exists a solution of (3.1.1).

The proof of Theorem 3.4.1 relies on the following variant of the mountain pass Theorem of Ambrosetti and Rabinowitz without the (PS) condition:

Theorem 3.4.2.

Let Φ be a C^1 function on a Banach space E . Suppose that :

(3.4.1)

There exists a neighborhood U of 0 in E and a constant ρ such that $\Phi(u) \geq \rho$ for every u in the boundary of U ,

(3.4.2) $\Phi(0) < \rho$ and $\Phi(v) < \rho$ for some $v \notin U$.

Set

$$(3.4.3) \quad c = \inf_{\mathcal{P} \in \mathcal{A}} \max_{w \in \mathcal{P}} \Phi(w) \geq \rho,$$

where \mathcal{A} denotes the class of continuous paths joining 0 to v .

Then there is a sequence $\{u_j\}$ in E such that

$$\Phi(u_j) \rightarrow c \quad \text{and} \quad \Phi'(u_j) \rightarrow 0 \quad \text{in} \quad E^*.$$

In addition to Theorem 3.4.2, the proof of Theorem 3.4.1 requires the following two Lemmas

Lemma 3.4.1.

Let $\{u_n\} \subset H^1(\Omega)$ be a sequence satisfying

$$(3.4.4) \quad I_\alpha(u_n) \rightarrow c < \frac{p(x_0)}{2(N-1)(Q(x_0))^{N-2}} S_1^{N-1}$$

and

$$(3.4.5) \quad I'_\alpha(u_n) \rightarrow 0 \quad \text{in} \quad H^{-1}(\Omega).$$

Then $\{u_n\}$ is relatively compact in $H^1(\Omega)$.

The proof of Lemma 3.4.1 is identical to that of Lemma 3.3.1.

Lemma 3.4.2.

There exists $\varepsilon_0 > 0$ such that

$$\sup_{t \geq 0} I_\alpha(tW_{\varepsilon, x_0}) < \frac{p(x_0)}{2(N-1)(Q(x_0))^{N-2}} S_1^{N-1}$$

for every $0 < \varepsilon < \varepsilon_0$.

Proof.

We have

$$\begin{aligned} I_\alpha(tW_{\varepsilon, x_0}) &= \frac{t^2}{2} \int_{\Omega} p(x) |\nabla W_{\varepsilon, x_0}|^2 dx - \frac{\lambda t^2}{2} \int_{\Omega} |W_{\varepsilon, x_0}|^2 dx - \frac{\alpha t^r}{r} \int_{\Omega} |W_{\varepsilon, x_0}|^r dx \\ &\quad - \frac{t^q}{q} \int_{\partial\Omega} p(x) Q(x) |W_{\varepsilon, x_0}|^q ds_x. \end{aligned}$$

Set

$$I_0(tW_{\varepsilon, x_0}) = \frac{t^2}{2} \int_{\Omega} p(x) |\nabla W_{\varepsilon, x_0}|^2 dx - \frac{\lambda t^2}{2} \int_{\Omega} |W_{\varepsilon, x_0}|^2 dx - \frac{t^q}{q} \int_{\partial\Omega} p(x) Q(x) |W_{\varepsilon, x_0}|^q ds_x.$$

Since $\alpha \geq 0$, it is easy to see that

$$(3.4.6) \quad \sup_{t \geq 0} I_\alpha(tW_{\varepsilon, x_0}) \leq \sup_{t \geq 0} I_0(tW_{\varepsilon, x_0}).$$

Using (3.2.2), (3.2.3) and (3.2.4), an easy computation gives

$$I_0(tW_{\varepsilon, x_0}) = \begin{cases} \frac{t^2}{2} p(x_0) (A_1 - A'_2 \varepsilon |\log \varepsilon|) - \frac{t^q}{q} p(x_0) Q(x_0) B_1 + o(\varepsilon |\log \varepsilon|) & \text{if } N = 3 \\ \frac{t^2}{2} p(x_0) (A_1 - A_2 \varepsilon) - \frac{t^q}{q} p(x_0) Q(x_0) (B_1 - B_2 \varepsilon) + o(\varepsilon). & \text{if } N \geq 4 \end{cases}$$

Set

$$g(t) = \begin{cases} \frac{t^2}{2} p(x_0) (A_1 - A'_2 \varepsilon |\log \varepsilon|) - \frac{t^q}{q} p(x_0) Q(x_0) B_1 & \text{if } N = 3 \\ \frac{t^2}{2} p(x_0) (A_1 - A_2 \varepsilon) - \frac{t^q}{q} p(x_0) Q(x_0) (B_1 - B_2 \varepsilon). & \text{if } N \geq 4 \end{cases}$$

It is easy to see that g achieves its maximum at

$$t_0 = \begin{cases} \left(\frac{A_1 - A'_2 \varepsilon |\log \varepsilon|}{Q(x_0) B_1} \right)^{\frac{1}{q-2}} & \text{if } N = 3 \\ \left(\frac{A_1 - A_2 \varepsilon}{Q(x_0) (B_1 - B_2 \varepsilon)} \right)^{\frac{1}{q-2}} & \text{if } N \geq 4. \end{cases}$$

By a standard computation we obtain

$$(3.4.7) \quad t_0 = \begin{cases} \left(\frac{A_1}{Q(x_0) B_1} \right)^{\frac{1}{q-2}} \left[1 - \frac{1}{q-2} \frac{A'_2}{A_1} \varepsilon |\log \varepsilon| \right] + o(\varepsilon |\log \varepsilon|) & \text{if } N = 3 \\ \left(\frac{A_1}{Q(x_0) B_1} \right)^{\frac{1}{q-2}} \left[1 - \frac{1}{(q-2)A_1} \left(A_2 - \frac{A_1 B_2}{B_1} \right) \varepsilon \right] + o(\varepsilon) & \text{if } N \geq 4. \end{cases}$$

Hence, for ε small enough we have

$$I_0(t_0 W_{\varepsilon, x_0}) = \begin{cases} \frac{1}{2(N-1)} \frac{p(x_0)}{(Q(x_0))^{N-2}} \left(\frac{A_1}{B_1} \right)^{\frac{q}{q-2}} - \left(\frac{A_1}{Q(x_0)B_1} \right)^{\frac{2}{q-2}} p(x_0) A_2' \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) & \text{if } N = 3, \\ \frac{1}{2(N-1)} \frac{p(x_0)}{(Q(x_0))^{N-2}} \left(\frac{A_1}{B_1} \right)^{\frac{q}{q-2}} - \left(\frac{A_1}{Q(x_0)B_1} \right)^{\frac{2}{q-2}} \left[\frac{p(x_0)}{(q-2)} \left(A_2 - \frac{A_1 B_2}{B_1} \right) + \frac{p(x_0) A_2}{2} \right] \varepsilon \\ + \left(\frac{A_1}{Q(x_0)B_1} \right)^{\frac{q}{q-2}} \left[\frac{p(x_0) Q(x_0) B_1}{(q-2) A_1} \left(A_2 - \frac{A_1 B_2}{B_1} \right) + \frac{p(x_0) Q(x_0) B_2}{q} \right] \varepsilon + o(\varepsilon) & \text{if } N \geq 4. \end{cases}$$

Therefore

$$(3.4.8) \quad I_0(t_0 W_{\varepsilon, x_0}) = \begin{cases} \frac{1}{4} \frac{p(x_0)}{(Q(x_0))} S_1^2 - \left(\frac{A_1}{Q(x_0)B_1} \right)^{\frac{2}{q-2}} A_2' \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) & \text{if } N = 3, \\ \frac{1}{2(N-1)} \frac{p(x_0)}{(Q(x_0))^{N-2}} S_1^{N-1} - \frac{p(x_0)}{2} \left(A_2 - \frac{2 A_1 B_2}{q B_1} \right) \varepsilon \\ + o(\varepsilon) & \text{if } N \geq 4. \end{cases}$$

Using the fact that $\Gamma(z+1) = z\Gamma(z) \forall z \in]0, +\infty[$ together with the definitions of the constants A_1, A_2, B_1 and B_2 , we get

$$(3.4.9) \quad A_2 - \frac{A_1 B_2}{B_1} > 0.$$

Combining (3.4.9), the fact that $\frac{2}{q} < 1$ and (3.4.8) we deduce that there exists $\varepsilon_0 > 0$ such that

$$(3.4.10) \quad I_0(t_0 W_{\varepsilon, x_0}) < \frac{1}{2(N-1)} \frac{p(x_0)}{(Q(x_0))^{N-2}} S_1^{N-1}$$

for $0 < \varepsilon < \varepsilon_0$.

Finally, this together with (3.4.6) gives the desired result. \square

Proof of Theorem 3.4.1

We start by verifying the assumptions of the Theorem 3.4.2 with $E = H^1(\Omega)$ and $\Phi = I_\alpha$.

Verification of (3.4.1):

Let $u \in H^1(\Omega)$, we have

$$I_\alpha(u) = \frac{1}{2} \int_{\Omega} p(x) |\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \frac{\alpha}{r} \int_{\Omega} |u|^r dx - \frac{1}{q} \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x.$$

Using the embedding of $H^1(\Omega)$ into $L^q(\partial\Omega)$ and $L^r(\Omega)$ and the fact that the norm $\left(\int_{\Omega} p(x)|\nabla \cdot|^2 dx - \lambda \int_{\Omega} |\cdot|^2 dx\right)$ is equivalent to $\|\cdot\|$ we obtain

$$I_{\alpha}(u) \geq M_1 \|u\|^2 - \alpha M_2 \|u\|^r - M_3 \|u\|^q$$

where M_1, M_2 and M_3 are some positive constants.

This implies (3.4.1) with some $\rho > 0$ and U a small ball in $H^1(\Omega)$.

Verification of (3.4.2):

For any $u \in H^1(\Omega)$, $u \not\equiv 0$ on $\bar{\Omega}$, we have $\lim_{t \rightarrow +\infty} I_{\alpha}(tu) = -\infty$. Thus there are many v 's satisfying (3.4.2). However, it will be important for later purpose to use Theorem 3.4.2 with a special v , namely $v = t_1 W_{\varepsilon, x_0}$, where t_1 is chosen large enough so that $v \notin U$ and $I_{\alpha}(v) \leq 0$.

It follows from Lemma 3.4.2 that for $0 < \varepsilon < \varepsilon_0$ we have

$$\sup_{t \geq 0} I_{\alpha}(tv) < \frac{1}{2(N-1)} \frac{p(x_0)}{(Q(x_0))^{N-2}} S_1^{N-1}$$

and therefore we deduce

$$(3.4.11) \quad c < \frac{1}{2(N-1)} \frac{p(x_0)}{(Q(x_0))^{N-2}} S_1^{N-1}.$$

Applying Theorem 3.4.2 we obtain a sequence $\{u_n\}$ in $H^1(\Omega)$ such that $I_{\alpha}(u_n) \rightarrow c$ and $I'_{\alpha}(u_n) \rightarrow 0$ in $H^{-1}(\Omega)$.

From Lemma 3.4.1, for a subsequence of $\{u_n\}$ (which we still call $\{u_n\}$), we have $u_n \rightarrow u$ strongly in $H^1(\Omega)$ and $I_{\alpha}(u) = c$. Thus, u is a solution of (3.1.1). This finishes the proof of Theorem 3.4.1.

3.5 Case $\lambda < 0$ and $\alpha \leq 0$

The main result of this section is the following

Theorem 3.5.1.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ such that $\partial\Omega$ satisfies the geometric condition (g.c) at x_0 and $H(x_0) > 0$. We assume that p and Q respectively satisfy (3.1.2) and (3.1.3). Then for every $\lambda < 0$ and for every $\alpha \leq 0$ there exists a solution of (3.1.1).

As in section 3, the proof of Theorem 3.5.1 is a consequence of the following two Lemmas and Theorem 3.4.2.

Lemma 3.5.1.

Let $\{u_n\} \subset H^1(\Omega)$ be a sequence satisfying

$$(3.5.1) \quad I_{\alpha}(u_n) \rightarrow c < \frac{p(x_0)}{2(N-1)(Q(x_0))^{N-2}} S_1^{N-1}$$

and

$$(3.5.2) \quad I'_\alpha(u_n) \rightarrow 0 \quad \text{in } H^{-1}(\Omega).$$

Then $\{u_n\}$ is relatively compact in $H^1(\Omega)$.

Proof.

We rewrite (3.5.1) as

$$(3.5.3) \quad \frac{1}{2} \int_{\Omega} p(x) |\nabla u_n|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u_n|^2 dx - \frac{\alpha}{r} \int_{\Omega} |u_n|^r dx - \frac{1}{q} \int_{\partial\Omega} p(x) Q(x) |u_n|^q ds_x = c + o(1)$$

and multiplying (3.5.2) by u_n , we find

$$(3.5.4) \quad \int_{\Omega} p(x) |\nabla u_n|^2 dx = \lambda \int_{\Omega} |u_n|^2 dx + \alpha \int_{\Omega} |u_n|^r dx + \int_{\partial\Omega} p(x) Q(x) |u_n|^q ds_x + \langle \varepsilon_n, u_n \rangle$$

with $\varepsilon_n \rightarrow 0$ in $H^{-1}(\Omega)$.

We start by showing that (u_n) is bounded in $H^1(\Omega)$.

Computing (3.5.3) $-\frac{1}{q}$ (3.5.4), we obtain

$$\frac{1}{2(N-1)} \left(\int_{\Omega} p(x) |\nabla u_n|^2 dx - \lambda \int_{\Omega} |u_n|^2 dx \right) = c + \alpha \left(\frac{1}{r} - \frac{1}{q} \right) \int_{\Omega} |u_n|^r dx + \langle \varepsilon_n, u_n \rangle + o(1).$$

Since $\alpha \leq 0$ and $2 < r < q$, we have, for large n ,

$$\begin{aligned} \frac{1}{2(N-1)} \left(\int_{\Omega} p(x) |\nabla u_n|^2 dx - \lambda \int_{\Omega} |u_n|^2 dx \right) &\leq c + \langle \varepsilon_n, u_n \rangle + o(1) \\ &\leq c + \|u_n\| + o(1). \end{aligned}$$

Since $\lambda < 0$, the norm $\left(\int_{\Omega} p(x) |\nabla \cdot|^2 dx - \lambda \int_{\Omega} |\cdot|^2 dx \right)^{\frac{1}{2}}$ is equivalent to $\|\cdot\|$, therefore there exists a positive constant C such that

$$\|u_n\| \leq C$$

and $\{u_n\}$ is bounded in $H^1(\Omega)$. Hence we may extract a subsequence, still denoted by $\{u_n\}$, such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } H^1(\Omega), \\ u_n &\rightarrow u \quad \text{strongly in } L^2(\Omega), \\ u_n &\rightarrow u \quad \text{pointwise a.e in } \Omega, \\ u_n &\rightharpoonup u \quad \text{weakly in } L^q(\partial\Omega), \\ u_n &\rightarrow u \quad \text{pointwise a.e in } \partial\Omega, \end{aligned}$$

From (3.5.2) we have

$$\langle I'_\alpha(u), v \rangle = 0 \quad \text{for every } w \in H^1(\Omega)$$

and then u satisfies

$$\begin{cases} -\operatorname{div}(p(x)\nabla u) = \lambda u + \alpha|u|^{r-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

Hence we also have

$$(3.5.5) \quad \int_{\Omega} p(x)|\nabla u|^2 dx = \lambda \int_{\Omega} |u|^2 dx + \alpha \int_{\Omega} |u|^r dx + \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x.$$

We write

$$u_n = u + v_j.$$

From a result of [BL] we have

$$(3.5.6) \quad \int_{\partial\Omega} p(x)Q(x)|u_j|^q ds_x = \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + \int_{\partial\Omega} p(x)Q(x)|v_j|^q ds_x + o(1).$$

Combining (3.5.3) with (3.5.6) and (3.5.4) with (3.5.6) we obtain

$$(3.5.7) \quad I_\alpha(u) + \frac{1}{2} \int_{\Omega} p(x)|\nabla v_j|^2 dx - \frac{1}{q} \int_{\partial\Omega} p(x)Q(x)|v_j|^q ds_x = c + o(1)$$

and

$$(3.5.8) \quad \begin{aligned} \int_{\Omega} p(x)|\nabla u|^2 dx + \int_{\Omega} p(x)|\nabla v_j|^2 dx &= \lambda \int_{\Omega} |u|^2 dx + \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \\ &+ \int_{\partial\Omega} p(x)Q(x)|v_j|^q ds_x + \alpha \int_{\Omega} |u|^r dx + o(1). \end{aligned}$$

Then, using (3.5.5), we deduce that

$$(3.5.9) \quad \int_{\Omega} p(x)|\nabla v_j|^2 dx = \int_{\partial\Omega} p(x)Q(x)|v_j|^q ds_x + o(1).$$

We may therefore assume that, as $n \rightarrow +\infty$

$$\int_{\Omega} p(x)|\nabla v_j|^2 dx \rightarrow l \quad \text{and} \quad \int_{\partial\Omega} p(x)Q(x)|v_j|^q ds_x \rightarrow l.$$

From the result of [Z, Theorem 02] we know that there exists a constant $C(\Omega) > 0$, such that for every $w \in H^1(\Omega)$

$$\int_{\Omega} |\nabla w|^2 dx + C(\Omega) \int_{\Omega} |w|^k dx \geq S_1 \left(\int_{\partial\Omega} |w|^q dx \right)^{\frac{2}{q}}$$

with $k = \frac{2N}{N-1}$ if $N \geq 4$ and $k > 3 = \frac{2N}{N-1}$ if $N = 3$.

We apply this result for $w_n = (p(x))^{\frac{1}{2}}v_n$, and in particular for $N = 3$ we take k such that $6 = \frac{2N}{N-2} > k > 3$, we obtain, for n large enough

$$\begin{aligned} \int_{\Omega} p(x)|\nabla v_n|^2 dx + C(\Omega) \int_{\Omega} |(p(x))^{\frac{1}{2}}v_n|^k dx &= \int_{\Omega} |\nabla[(p(x))^{\frac{1}{2}}v_n]|^2 dx \\ &+ C(\Omega) \int_{\Omega} |(p(x))^{\frac{1}{2}}v_n|^k dx + o(1) \\ &\geq S_1 \left(\int_{\partial\Omega} |(p(x))^{\frac{1}{2}}v_n|^q ds_x \right)^{\frac{2}{q}} + o(1). \end{aligned}$$

Since $k < \frac{2N}{N-2}$ for every $N \geq 3$, thanks to the compact embedding $H^1(\Omega) \hookrightarrow L^k(\Omega)$ we have $v_n \rightarrow 0$ strongly in $L^k(\Omega)$, and we deduce

$$\int_{\Omega} p(x)|\nabla v_n|^2 dx + o(1) \geq S_1 \left(\int_{\partial\Omega} |(p(x))^{\frac{1}{2}}v_n|^q ds_x \right)^{\frac{2}{q}} + o(1).$$

Using the fact that

$$\frac{p(x)}{(Q(x))^{N-2}} \geq \frac{p(x_0)}{(Q(x_0))^{N-2}} \quad \forall x \in \partial\Omega,$$

an easy computation yields

$$(3.5.10) \quad \int_{\Omega} p(x)|\nabla v_n|^2 dx \geq \left(\frac{p(x_0)}{(Q(x_0))^{N-2}} \right)^{\frac{1}{N-1}} S_1 \left(\int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x \right)^{\frac{2}{q}} + o(1)$$

and at the limit we have $l \geq S_1 l^{\frac{2}{q}}$. It follows that either $l = 0$ or $l \geq S_1^{N-1}$.

We shall now see that $l \geq S_1^{N-1}$ is excluded, which implies that $l = 0$, i.e., $u_n \rightarrow u$ strongly in $H^1(\Omega)$ and the proof of Lemma 3.5.1 will be complete.

Suppose that $l \geq S_1^{N-1}$. Passing to the limit in (3.5.7) we obtain

$$I_{\alpha}(u) + \frac{1}{2(N-1)}l = c$$

and with the assumption on c in (3.5.1) we find

$$(3.5.11) \quad I_{\alpha}(u) < 0.$$

On the other hand we have, by definition of I_{α} and in view of (3.5.5),

$$\begin{aligned} I_{\alpha}(u) &= \frac{1}{2} \int_{\Omega} p(x)|\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \frac{\alpha}{r} \int_{\Omega} |u|^r dx - \frac{1}{q} \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \\ &= \frac{1}{2(N-1)} \left(\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) + \alpha \left(\frac{1}{q} - \frac{1}{r} \right) \int_{\Omega} |u|^r dx. \end{aligned}$$

Since $\alpha \leq 0$ and $r < q$ we obtain

$$I_\alpha(u) \geq 0$$

which contradicts (3.5.11). This achieves the proof of Lemma 3.5.1. \square

Lemma 3.5.2.

There exists $\varepsilon_0 > 0$ such that

$$\sup_{t \geq 0} I_\alpha(tW_{\varepsilon, x_0}) < \frac{p(x_0)}{2(N-1)(Q(x_0))^{N-2}} S_1^{N-1}$$

for every $0 < \varepsilon < \varepsilon_0$.

Proof.

We have

$$\begin{aligned} I_\alpha(tW_{\varepsilon, x_0}) &= \frac{t^2}{2} \int_{\Omega} p(x) |\nabla W_{\varepsilon, x_0}|^2 dx - \frac{\lambda t^2}{2} \int_{\Omega} |W_{\varepsilon, x_0}|^2 dx - \frac{\alpha t^r}{r} \int_{\Omega} |W_{\varepsilon, x_0}|^r dx \\ &\quad - \frac{t^q}{q} \int_{\partial\Omega} p(x) Q(x) |W_{\varepsilon, x_0}|^q ds_x. \end{aligned}$$

Using (3.2.2), (3.2.3), (3.2.4) and (3.2.6), an easy computation gives

$$I_\alpha(tW_{\varepsilon, x_0}) = \begin{cases} \frac{t^2}{2} p(x_0) (A_1 - A'_2 \varepsilon |\log \varepsilon|) - \frac{t^q}{q} p(x_0) Q(x_0) B_1 + o(\varepsilon |\log \varepsilon|) & \text{if } N = 3 \\ \frac{t^2}{2} p(x_0) (A_1 - A_2 \varepsilon) - \frac{t^q}{q} p(x_0) Q(x_0) (B_1 - B_2 \varepsilon) + o(\varepsilon). & \text{if } N \geq 4. \end{cases}$$

For the rest of this proof, we proceed as in the proof of Lemma 3.4.2 and we obtain the desired result. \square

Now, as in proof of Theorem 3.3.2, we apply Theorem 3.4.2 together with Lemma 3.5.1 and Lemma 3.5.2 we obtain the conclusion of Theorem 3.5.1.

Finally, combining Theorem 3.3.1, Theorem 3.3.2 and Theorem 3.4.1 we obtain the conclusion of Theorem 3.1.1.

3.6 Appendix

In this appendix we give the proof of (3.2.5) and (3.2.6).

Let $R > 0$ (see the definition of R in Section 1), using the ideas developed in [AY],

we introduce the following notations

$$\begin{aligned}
x &= (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \quad \text{and} \quad x_0 = (x'_0, x_{0N}), \\
B(x_0, R) &= \{x \in \mathbb{R}^N, |x - x_0| < R\}, \\
B(x_0, R) \cap \Omega &= \{(x', x_N) \in B(x_0, R), x_N - x_{0N} > \rho(x' - x'_0)\}, \\
B(x_0, R) \cap \partial\Omega &= \{(x', x_N) \in B(x_0, R), x_N - x_{0N} = \rho(x' - x'_0)\}, \\
B(x_0, R)^+ &= B(x_0, R) \cap \{x_N - x_{0N} > 0\} \\
\Sigma &= \{(x', x_N) \in B(x_0, R), 0 < x_N - x_{0N} < \rho(x' - x'_0)\},
\end{aligned}$$

where

$$\rho(x' - x'_0) = \frac{1}{2} \sum_{i=1}^{N-1} \alpha_i (x_i - x_{0i})^2 + o(|x' - x'_0|^2),$$

where $\{\alpha_i\}_{1 \leq i \leq N-1}$ are the principal curvatures at x_0 and $H(x_0) = \frac{1}{n-1} \sum_{i=1}^{N-1} \alpha_i$.

Since $\partial\Omega$ satisfies (e.g), then $\rho(\cdot) \geq 0$.

We recall the following relation

$$(3.6.1) \quad \int_0^{+\infty} \frac{r^\theta}{(1+r^2)^\beta} dr = \frac{\Gamma(\frac{\theta+1}{2})\Gamma(\frac{2\beta-\theta-1}{2})}{2\Gamma(\beta)} \quad \text{for } 2\beta - \theta > 1.$$

Proof of (3.2.5):

Set $v(x) = p(x)Q(x)$, $x \in \bar{\Omega}$ and $v = 0$ outside $\partial\Omega$. Then we have

$$\begin{aligned}
\int_{\partial\Omega} |W_{\varepsilon, x_0}|^{q-1} v \, ds_z &= \varepsilon^{\frac{N}{2}} \int_{|z'-x'_0| \leq a} \frac{(1 + |\nabla\rho|^2)^{\frac{1}{2}} v(z', x_{0N} + \rho(z' - x'_0))}{[(\varepsilon + \rho(z' - x'_0))^2 + |z' - x'_0|^2]^{\frac{N}{2}}} dx' + O(\varepsilon^{\frac{N}{2}}) \\
&= \varepsilon^{\frac{N}{2}} \int_{|x'| \leq a} \frac{(1 + |\nabla\rho|^2)^{\frac{1}{2}} v(x'_0 + x', x_{0N} + \rho(x'))}{[(\varepsilon + \rho(x'))^2 + |x'|^2]^{\frac{N}{2}}} dx' + O(\varepsilon^{\frac{N}{2}}).
\end{aligned}$$

Using Taylor's expansion of ρ at x' , we obtain

$$\begin{aligned}
\frac{(1 + |\nabla\rho|^2)^{\frac{1}{2}}}{[(\varepsilon + \rho(x'))^2 + |x'|^2]^{\frac{N}{2}}} &= \\
&= \frac{1 + \frac{1}{2}|\nabla\rho|^2 + o(|\nabla\rho|^2)}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} \left[1 - \frac{\frac{N}{2}\rho(x')(2\varepsilon + \rho(x'))}{(\varepsilon^2 + |x'|^2)} + o\left(\frac{\frac{N}{2}\rho(x')(2\varepsilon + \rho(x'))}{(\varepsilon^2 + |x'|^2)}\right) \right].
\end{aligned}$$

Consequently

$$\begin{aligned}
\varepsilon^{\frac{N}{2}} \int_{\partial\Omega} |W_{\varepsilon,y}|^{q-1} v \, ds_z &= \varepsilon^{\frac{N}{2}} \int_{|x'| \leq a} \frac{v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' \\
&+ \frac{\varepsilon^{\frac{N}{2}}}{2} \int_{|x'| \leq a} \frac{|\nabla \rho|^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' \\
&+ \varepsilon^{\frac{N+2}{2}} \int_{|x'| \leq a} \frac{N\rho(x') v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N+2}{2}}} dx' \\
&+ o\left(\varepsilon^{\frac{N+2}{2}} \int_{|x'| \leq a} \frac{N\rho(x') v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N+2}{2}}} dx' \right) \\
&= T_1 + T_2 + T_3 + o(T_3) + O(\varepsilon^{\frac{N}{2}}).
\end{aligned}$$

We have

$$\begin{aligned}
T_1 &= \varepsilon^{\frac{N}{2}} \int_{|x'| \leq a} \frac{v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' \\
&= \varepsilon^{\frac{N}{2}} \int_{\mathbb{R}^{N-1}} \frac{v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' + O(\varepsilon^{\frac{N}{2}}) \\
&= \varepsilon^{\frac{N-2}{2}} v(x_0) \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + |z'|^2)^{\frac{N}{2}}} dz' + o(\varepsilon^{\frac{N-2}{2}})
\end{aligned}$$

By changing to polar coordinates and using (3.6.1), we get

$$T_1 = \omega_{N-2} \varepsilon^{\frac{N-2}{2}} v(x_0) \int_0^{+\infty} \frac{r^{N-2}}{(1+r^2)^{\frac{N}{2}}} dr + o(\varepsilon^{\frac{N-2}{2}}),$$

$$(3.6.2) \quad T_1 = \frac{\omega_{N-2} \Gamma(\frac{N-1}{2}) \Gamma(\frac{1}{2}) v(x_0) \varepsilon^{\frac{N-2}{2}}}{2\Gamma(\frac{N}{2})} + o(\varepsilon^{\frac{N-2}{2}}).$$

Since $|\nabla \rho|^2 = \sum_{i=1}^{N-1} \alpha_i^2 x_i^2 + o(|x'|^2)$, then

$$\begin{aligned}
T_2 &= \varepsilon^{\frac{N}{2}} \sum_{i=1}^{N-1} \alpha_i^2 \int_{|x'|^2 \leq a} \frac{x_i^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' \\
&+ o\left(\varepsilon^{\frac{N}{2}} \int_{|x'|^2 \leq a} \frac{x_i^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' \right).
\end{aligned}$$

We have

$$\begin{aligned} \varepsilon^{\frac{N}{2}} \sum_{i=1}^{N-1} \alpha_i^2 \int_{|x'| \leq a} \frac{x_i^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' \\ \leq \frac{\sum_{i=1}^{N-1} \alpha_i \varepsilon^{\frac{N}{2}}}{(N-1)} \int_{|x'| \leq a} \frac{|x'|^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx'. \end{aligned}$$

Applying the Dominated Convergence Theorem we find that

$$\int_{|x'| \leq a} \frac{|x'|^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' = O(1)$$

and we deduce that

$$(3.6.3) \quad T_2 = O(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^{\frac{N-2}{2}}).$$

By definition of ρ , we have

$$\begin{aligned} T_3 &= \varepsilon^{\frac{N+2}{2}} \frac{N}{2} \sum_{i=1}^{N-1} \alpha_i \int_{|x'| \leq a} \frac{x_i^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N+2}{2}}} dx' \\ &\quad + o\left(\varepsilon^{\frac{N+2}{2}} \int_{|x'| \leq a} \frac{x_i^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N+2}{2}}} dx'\right). \end{aligned}$$

We have

$$\begin{aligned} \varepsilon^{\frac{N+2}{2}} \frac{N}{2} \sum_{i=1}^{N-1} \alpha_i \int_{|x'| \leq a} \frac{x_i^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N+2}{2}}} dx' \\ \leq \frac{N \sum_{i=1}^{N-1} \alpha_i}{2(N-1)} \varepsilon^{\frac{N}{2}} v(x_0) \int_0^{\frac{a}{\varepsilon}} \frac{r^N}{(1+r^2)^{\frac{N+2}{2}}} dr \\ = \frac{N v(x_0) \sum_{i=1}^{N-1} \alpha_i}{2(N-1)} \varepsilon^{\frac{N}{2}} \int_0^{+\infty} \frac{r^N}{(1+r^2)^{\frac{N+2}{2}}} dr + o(\varepsilon^{\frac{N}{2}}) \\ = \frac{N v(x_0) \Gamma(\frac{N+1}{2}) \Gamma(\frac{1}{2}) \sum_{i=1}^{N-1} \alpha_i}{4(N-1) \Gamma(\frac{N+2}{2})} \varepsilon^{\frac{N}{2}} + o(\varepsilon^{\frac{N}{2}}). \end{aligned}$$

Thus,

$$(3.6.4) \quad T_3 \leq \frac{N v(x_0) \Gamma(\frac{N+1}{2}) \Gamma(\frac{1}{2}) \sum_{i=1}^{N-1} \alpha_i}{4(N-1) \Gamma(\frac{N+2}{2})} \varepsilon^{\frac{N}{2}} + o(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^{\frac{N-2}{2}}).$$

Finally, combining (3.6.2), (3.6.3) and (3.6.4) we obtain

$$\int_{\partial\Omega} |W_{\varepsilon, x_0}|^{q-1} v ds_z = \frac{\omega_{N-2} \Gamma(\frac{N-1}{2}) \Gamma(\frac{1}{2}) u(x_0) \varepsilon^{\frac{N-2}{2}}}{2 \Gamma(\frac{N}{2})} + o(\varepsilon^{\frac{N-2}{2}}).$$

Proof of (3.2.6):

We have

$$\begin{aligned}
\int_{\Omega} |W_{\varepsilon, x_0}|^r dx &= \varepsilon^{\frac{r(N-2)}{2}} \int_{B(x_0, R)^+} \frac{1}{[(\varepsilon + x_N - x_{0N})^2 + |x' - x'_0|^2]^{\frac{r(N-2)}{2}}} dx \\
&\quad - \varepsilon^{\frac{r(N-2)}{2}} \int_{\Sigma} \frac{1}{[(\varepsilon + x_N - x_{0N})^2 + |x' - x'_0|^2]^{\frac{r(N-2)}{2}}} dx + O(\varepsilon^{\frac{r(N-2)}{2}}) \\
&= L_1 - L_2 + O(\varepsilon^{\frac{r(N-2)}{2}}).
\end{aligned}$$

Now, in order to estimate L_1 we distinguish two cases:

Case $N \geq 4$:

By a change of variables and using polar coordinates, we obtain

$$\begin{aligned}
L_1 &= \varepsilon^{\frac{2N-r(N-2)}{2}} \int_{\mathbb{R}^N} \frac{1}{[(1 + x_N)^2 + |x'|^2]^{\frac{r(N-2)}{2}}} dx \\
&\quad + o(\varepsilon^{\frac{2N-r(N-2)}{2}}) \\
&= \varepsilon^{\frac{2N-r(N-2)}{2}} \left(\int_0^{+\infty} \frac{1}{(1 + x_N)^{r(N-2)-(N-1)}} dx_N \right) \left(\int_0^{+\infty} \frac{\omega_{N-2} t^{N-2}}{(1 + t^2)^{\frac{r(N-2)}{2}}} dt \right) \\
&\quad + o(\varepsilon^{\frac{2N-r(N-2)}{2}}).
\end{aligned}$$

Since $2 < r < q$, we get

$$\begin{aligned}
(3.6.5) \quad L_1 &= \varepsilon^{\frac{2N-r(N-2)}{2}} \frac{\omega_{N-2}}{N-2} \int_0^{+\infty} \frac{\omega_{N-2} t^{N-2}}{(1 + t^2)^{\frac{r(N-2)}{2}}} dt + o(\varepsilon^{\frac{2N-r(N-2)}{2}}) \\
&= o(\varepsilon).
\end{aligned}$$

Case $N = 3$:

Here $q = 4$ and $2 < r < 4$. The cases $r > 3$, $r = 3$ and $r < 3$ are different, therefore we distinguish three cases.

If $r > 3$:

As in the above estimates we have

$$\begin{aligned}
L_1 &= \varepsilon^{\frac{6-r}{2}} \int_{\mathbb{R}^3} \frac{1}{[(1 + x_3)^2 + |x'|^2]^{\frac{r}{2}}} dx + o(\varepsilon^{\frac{6-r}{2}}) \\
&= \varepsilon^{\frac{6-r}{2}} \left(\int_0^{+\infty} \frac{1}{(1 + x_3)^{r-2}} dx_3 \right) \left(\int_0^{+\infty} \frac{\omega_1 t}{(1 + t^2)^{\frac{r}{2}}} dt \right) + o(\varepsilon^{\frac{6-r}{2}}) = o(\varepsilon)
\end{aligned}$$

and

$$(3.6.6) \quad L_1 = o(\varepsilon) = o(\varepsilon |\log \varepsilon|).$$

If $r = 3$:

We have

$$\begin{aligned}
L_1 &= \varepsilon^{\frac{3}{2}} \int_{B^+(x_0, R)} \frac{1}{[(\varepsilon + x_N - x_{0N})^2 + |x' - x'_0|^2]^{\frac{3}{2}}} dx \\
&= \varepsilon^{\frac{3}{2}} \int_0^a \int_{\{|y'| \leq a\}} \frac{1}{[(\varepsilon + y_N)^2 + |y'|^2]^{\frac{3}{2}}} dy' dy_N \\
&= \varepsilon^{\frac{3}{2}} \int_0^a \frac{1}{(\varepsilon + y_N)} \int_0^{\frac{a}{\varepsilon + y_N}} \frac{t}{(1 + t^2)^{\frac{3}{2}}} dt dy_N \\
&= \varepsilon^{\frac{3}{2}} \int_0^a \frac{1}{\varepsilon + y_N} dy_N - \varepsilon^{\frac{3}{2}} \int_0^a \frac{1}{(a^2 + (\varepsilon + y_N)^2)^{\frac{1}{2}}} dy_N
\end{aligned}$$

and

$$(3.6.7) \quad L_1 = o(\varepsilon |\log \varepsilon|).$$

If $r < 3$:

Using the Dominated Convergence Theorem, we obtain

$$\begin{aligned}
L_1 &= \varepsilon^{\frac{r}{2}} \int_{B^+(x_0, R)} \frac{1}{[(\varepsilon + x_N - x_{0N})^2 + |x' - x'_0|^2]^{\frac{r}{2}}} dx \\
&= \varepsilon^{\frac{r}{2}} \int_{B^+(x_0, R)} \frac{1}{|x - x_0|^r} dx + o(\varepsilon^{\frac{r}{2}})
\end{aligned}$$

and

$$(3.6.8) \quad L_1 = o(\varepsilon |\log \varepsilon|).$$

Now, we go back to estimate L_2 .

We have

$$\begin{aligned}
L_2 &= \varepsilon^{\frac{r(N-2)}{2}} \int_{\Sigma} \frac{1}{[(\varepsilon + x_N - x_{0N})^2 + |x' - x'_0|^2]^{\frac{r(N-2)}{2}}} dx \\
&= \varepsilon^{\frac{r(N-2)}{2}} \int_{\{|y'| \leq a\}} \int_0^{\rho(y')} \frac{1}{[(\varepsilon + y_N)^2 + |y'|^2]^{\frac{r(N-2)}{2}}} dy_N dy'.
\end{aligned}$$

Applying Taylor's formula to $\frac{1}{[(\varepsilon + (y_N)^2 + |y'|^2)^{\frac{r(N-2)}{2}}]}$ with respect to y_N , we obtain

$$\begin{aligned}
L_2 &= \varepsilon^{\frac{r(N-2)}{2}} \int_{\{|y'| \leq a\}} \frac{\rho(y')}{(\varepsilon^2 + |y'|^2)^{\frac{r(N-2)}{2}}} dy' \\
&\quad - r(N-2) \varepsilon^{\frac{r(N-2)}{2}+1} \int_{\{|y'| \leq a\}} \frac{\rho(y')}{(\varepsilon^2 + |y'|^2)^{\frac{r(N-2)}{2}+1}} dy' \\
&\quad + o\left(\varepsilon^{\frac{r(N-2)}{2}+1} \int_{\{|y'| \leq a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^{\frac{r(N-2)}{2}+1}} dy'\right) \\
&= L_{21} - L_{22} + o\left(\varepsilon^{\frac{r(N-2)}{2}+1} \int_{\{|y'| \leq a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^{\frac{r(N-2)}{2}+1}} dy'\right).
\end{aligned}$$

On one hand, we have

$$\begin{aligned}
L_{21} &= \frac{\sum_{i=1}^{N-1} \alpha_i}{2(N-1)} \varepsilon^{\frac{r(N-2)}{2}} \int_{\{|y'| \leq a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^{\frac{r(N-2)}{2}}} dy' \\
&\quad + o\left(\varepsilon^{\frac{r(N-2)}{2}} \int_{\{|y'| \leq a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^{\frac{r(N-2)}{2}}} dy'\right).
\end{aligned}$$

If $N \geq 5$:

By changing to polar coordinates, we obtain

$$\begin{aligned}
\varepsilon^{\frac{r(N-2)}{2}} \int_{\{|y'| \leq a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^{\frac{r(N-2)}{2}}} dy' &= \varepsilon^{\frac{2(N+1)-r(N-2)}{2}} \int_0^{+\infty} \frac{t^N}{(1+t^2)^{\frac{r(N-2)}{2}}} dt \\
&= o(\varepsilon)
\end{aligned}$$

and

$$(3.6.9) \quad L_{21} = o(\varepsilon).$$

If $N = 4$:

We distinguish three subcases:

If $r > \frac{5}{2}$:

As in the above case, we have

$$\begin{aligned}
\varepsilon^r \int_{\{|y'| \leq a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^r} dy' &= \varepsilon^{5-r} \int_0^{+\infty} \frac{t^4}{(1+t^2)^r} dt \\
&= o(\varepsilon)
\end{aligned}$$

and

$$(3.6.10) \quad L_{21} = o(\varepsilon).$$

If $r = \frac{5}{2}$:

Using polar coordinates, we obtain

$$\begin{aligned} L_{21} &= \varepsilon^{\frac{5}{2}} \int_{\{|y'| \leq a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^{\frac{5}{2}}} dy' \\ &= \omega_2 \varepsilon^{\frac{5}{2}} \int_0^a \frac{t^4}{(\varepsilon^2 + t^2)^{\frac{5}{2}}} dt \end{aligned}$$

and

$$(3.6.11) \quad L_{21} = O(\varepsilon^{\frac{5}{2}} |\log \varepsilon|) = o(\varepsilon).$$

If $r < \frac{5}{2}$:

Applying the Dominated Convergence Theorem, we get

$$\begin{aligned} L_{21} &= \varepsilon^r \int_{\{|y'| \leq a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^r} dy' \\ &= \varepsilon^r \int_{\{|y'| \leq a\}} \frac{|y'|^2}{|y'|^{2r-2}} dy' + o(\varepsilon^r) \end{aligned}$$

and

$$(3.6.12) \quad L_{21} = o(\varepsilon).$$

If $N = 3$

Using the Dominated Convergence Theorem, we obtain

$$\begin{aligned} L_{21} &= \frac{\sum_{i=1}^2 \alpha_i}{4} \varepsilon^{\frac{r}{2}} \int_{\{|y'| \leq a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^{\frac{r}{2}}} dy' \\ &= \frac{\sum_{i=1}^{N-1} \alpha_i}{2(N-1)} \varepsilon^{\frac{r}{2}} \int_{\{|y'| \leq a\}} \frac{1}{|y'|^{r-2}} dy' \end{aligned}$$

and

$$(3.6.13) \quad L_{21} = o(\varepsilon |\log \varepsilon|).$$

On other hand, using the definition of ρ , we write

$$\begin{aligned} L_{22} &= \frac{r(N-2)\varepsilon^{\frac{r(N-2)}{2}+1} \sum_{i=1}^{N-1} \alpha_i^2}{2(N-1)} \int_{\{|y'| \leq a\}} \frac{|y'|^2}{(\varepsilon^2 + |y'|^2)^{\frac{r(N-2)}{2}+1}} dy' \\ &\quad + o\left(\int_{\{|y'| \leq a\}} \frac{\varepsilon^{\frac{r(N-2)}{2}+1} |y'|^2}{(\varepsilon^2 + |y'|^2)^{\frac{r(N-2)}{2}+1}} dy'\right). \end{aligned}$$

By a change of variable and using polar coordinates, we obtain

$$L_{22} = \frac{r(N-2)\varepsilon^{\frac{2N-r(N-2)}{2}} \sum_{i=1}^{N-1} \alpha_i^2}{2(N-1)} \int_{\mathbb{R}^{N-1}} \frac{|x'|^2}{(1+|x'|^2)^{\frac{r(N-2)}{2}+1}} dx' \\ + o(\varepsilon^{\frac{2N-r(N-2)}{2}})$$

and

$$(3.6.14) \quad L_{22} = o(\varepsilon).$$

Finally, combining all the above estimates, we deduce

$$\int_{\Omega} |W_{\varepsilon, x_0}|^r dx = \begin{cases} o(\varepsilon |\log \varepsilon|) & \text{if } N = 3 \\ o(\varepsilon) & \text{if } N \geq 4 \end{cases}$$

which is the desired result.

Chapter 4

On nonhomogeneous Neumann problem with weight and with critical nonlinearity in the boundary

(This work is submitted to *Nonlinear Analysis: Theory, Methods and Applications*, series A.)

4.1 Introduction

In this paper, we are concerned with the problem of the form

$$(4.1.1) \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = \lambda u + f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with the smooth boundary $\partial\Omega$, ν is the outer normal on $\partial\Omega$, the coefficient Q is continuous and positive on $\partial\Omega$ and the coefficient $p \in H^1(\Omega)$ is continuous and positive in $\bar{\Omega}$. $q = \frac{2(N-1)}{N-2}$, λ is a constant and $f \not\equiv 0$. The assumptions on f will be formulated later. The exponent q is the critical Sobolev exponent for the trace embedding of the space $H^1(\Omega)$ into $L^q(\partial\Omega)$. The embedding of $H^1(\Omega)$ into $L^q(\partial\Omega)$ is continuous, but not compact.

The case $p \equiv 1 \equiv Q$ and $f \equiv 0$ is considered in [AM], [AY], [CK] and [CY]. As in Chapter 3, the geometrical condition (g.c) at some point on $\partial\Omega$ play an important role in solving problems similar to (4.1.1). It is convenient to recall the definition.

(g.c): $\partial\Omega$ is said to satisfy the geometrical condition at $y \in \partial\Omega$ if there exists a neighborhood $U(y)$ of y such that $\Omega \cap U(y)$ lies on one side of the tangent plane at

y .

Let $x_0 \in \partial\Omega$ satisfy

$$\frac{p(x_0)}{(Q(x_0))^{N-2}} = \min_{x \in \partial\Omega} \frac{p(x)}{(Q(x))^{N-2}}.$$

We assume that

$$(4.1.2) \quad |p(x) - p(x_0)| = o(|x - x_0|)$$

and

$$(4.1.3) \quad |Q(x) - Q(x_0)| = o(|x - x_0|)$$

for x near x_0 .

Throughout this paper, we suppose that $\partial\Omega$ satisfies the geometrical condition (g.c) at the point x_0 . This condition guarantees that the mean curvature at x_0 is non-negative.

In this work, we are interested in the existence of a solution of (4.1.1) which satisfies $u \not\equiv 0$ on $\partial\Omega$. Therefore we assume that f is chosen so that

$$(4.1.4) \quad \text{the equation} \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = \lambda u + f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{has no solution.}$$

Note that assumption (4.1.4) certainly holds if f has a constant sign on Ω , otherwise we obtain a contradiction thanks to Hopf's Lemma.

In the first part of this work we study the case where λ is negative. The assumptions on f are summarized in the following

$$(H1) \quad \begin{cases} f \in H^{-1}(\Omega) \cap C(\bar{\Omega}) \setminus \{0\} \text{ satisfies (4.1.4)} \\ \text{and} \\ (*) \int_{\Omega} f(x) u \, dx < \sigma_N \left(\int_{\Omega} p(x) |\nabla u|^2 \, dx - \lambda \int_{\Omega} |u|^2 \, dx \right)^{\frac{N}{2}}, \\ \forall u \in H^1(\Omega), \int_{\partial\Omega} p(x) Q(x) |u|^q \, ds_x = 1. \end{cases}$$

$$\text{with } \sigma_N = (q-2) \left[\frac{1}{q-1} \right]^{\frac{q-1}{q-2}}.$$

Let us remark that (*) certainly holds if

$$\|f\|_{H^{-1}} < \sigma_N \frac{\left[\min(-\lambda, \min_{\Omega} p(x)) \right]^{\frac{N}{2}}}{\left[\max_{\partial\Omega} p(x) Q(x) \right]^{\frac{N-1}{q}}} C^{N-1}$$

where

$$C = C(\Omega) = \inf_{H^1(\Omega) \setminus H_0^1(\Omega)} \frac{\left(\int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |u|^2 dx \right)^{\frac{1}{2}}}{\left(\int_{\partial\Omega} |u|^q ds_x \right)^{\frac{1}{q}}}$$

is the best constant in the trace embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$.

To obtain the existence of solutions of problem (4.1.1) we use the strategy developed by Tarantello, in [T], to study a nonhomogeneous Dirichlet problem with critical Sobolev exponent. The main result is the following

Theorem 4.1.1.

Let $\Omega \in \mathbb{R}^N$, $N \geq 3$ be a bounded domain with a smooth boundary $\partial\Omega$ satisfying (g.c) at x_0 and $H(x_0) > 0$. We assume that f satisfies (H1) and the functions p and Q respectively satisfy (4.1.2) and (4.1.3). When $N = 3$, we further assume that f , p and Q are sufficiently smooth. Then for every $\lambda < 0$ the problem (4.1.1) admits at least two weak solutions.

The second part of this work is dedicated to the case where λ interferes with the spectrum $\{\lambda_1, \lambda_2, \dots\}$ of the operator $\operatorname{div}(p(x)\nabla \cdot)$ with the Neumann boundary conditions.

It is known that $0 = \lambda_1 < \lambda_2 \leq \dots$ and that the eigenfunction corresponding to λ_1 is constant function.

Let $k \in \mathbb{N} \setminus \{0\}$ such that $\lambda_{k-1} \neq \lambda_k$. Set $E_k^- = \operatorname{span}\{e_1, \dots, e_l\}$, where e_1, \dots, e_l are eigenfunctions corresponding to eigenvalues $\lambda_1, \dots, \lambda_{k-1}$. We have the orthogonal decomposition $H^1(\Omega) = E_k^- \oplus E_k^+$.

The assumption on f is the following

$$(H2) \quad f \in E_k^+ \setminus \{0\} \cap C(\bar{\Omega}) \text{ satisfies (4.1.4).}$$

Applying the min-max principle argument based on the topological linking [W], we obtain the following result

Theorem 4.1.2.

Let $\Omega \in \mathbb{R}^N$, $N \geq 3$ be a bounded domain with a smooth boundary $\partial\Omega$ satisfying (g.c) at x_0 and $H(x_0) > 0$. We assume that f satisfies (H2) and the functions p and Q respectively satisfy (4.1.2) and (4.1.3). When $N = 3$ and $N = 4$ we further assume that $f(x_0) \neq 0$. For every $\lambda \in]\lambda_{k-1}, \lambda_k[$, there is a constant $\alpha = \alpha(p, Q, \lambda, N) > 0$ such that if $\|f\|_{L^2} \leq \alpha$ then the problem (4.1.1) admits at least one solution.

The rest of this paper is divided into three sections. In section 1 we give some notations which will be used throughout this paper. Section 2 and Section 3 are devoted respectively to the proof of Theorem 4.1.1 and the proof of Theorem 4.1.2.

4.2 Notation

We denote by $H(x_0)$ the mean curvature of $\partial\Omega$ at x_0 with respect to the unit outward normal. Let us recall that since $\partial\Omega$ satisfies (g.c) then $H(x_0) \geq 0$.

Let

$$S_1 = \inf \left\{ \int_{\mathbb{R}_+^N} |\nabla u|^2 dx; u \in H^1(\mathbb{R}_+^N), \int_{\mathbb{R}^{N-1}} |u|^q dx = 1 \right\}$$

be the best constant in the trace embedding for the space $H^1(\mathbb{R}_+^N)$ into $L^q(\partial\mathbb{R}_+^N)$, where $\mathbb{R}_+^N = \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\}$. By a result of [L], S_1 is achieved and by a result of [E], S_1 is given by

$$S_1 = \frac{N-2}{2} (\omega_N)^{\frac{1}{N-1}}, \quad \text{where } \omega_N \text{ is the area of } S^{N-1}.$$

The minimizing functions are of the form

$$(4.2.1) \quad W(x) = \frac{\gamma_N}{[|x'|^2 + (1+x_N)^2]^{\frac{N-2}{2}}},$$

where $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$ and γ_N is a positive constant depending only on N . We set

$$W_{\varepsilon, x_0}(x) = \varepsilon^{-\frac{N-2}{2}} \phi(x) W\left(\frac{x-x_0}{\varepsilon}\right),$$

where ϕ is a radial C^∞ -function such that

$$\phi(x) = \begin{cases} 1 & \text{if } |x-x_0| \leq \frac{R}{4} \\ 0 & \text{if } |x-x_0| > \frac{R}{2} \end{cases}$$

with $R > 0$ is a small constant.

Using (4.1.2) and (4.1.3), from [AY] we obtain the following estimates

$$(4.2.2) \quad \int_{\Omega} p(x) |\nabla W_{\varepsilon, x_0}|^2 dx = p(x_0) A_1 - p(x_0) H(x_0) \begin{cases} A'_2 \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) & \text{if } N = 3 \\ A_2 \varepsilon + o(\varepsilon) & \text{if } N \geq 4, \end{cases}$$

$$(4.2.3) \quad \int_{\partial\Omega} p(x) Q(x) |W_{\varepsilon, x_0}|^q ds_x = p(x_0) Q(x_0) (B_1 - H(x_0) B_2 \varepsilon) + o(\varepsilon)$$

and

$$(4.2.4) \quad \int_{\Omega} |W_{\varepsilon, x_0}|^2 dx = \begin{cases} O(\varepsilon) & \text{if } N = 3 \\ O(\varepsilon^2 |\log \varepsilon|) & \text{if } N = 4 \\ D\varepsilon^2 + o(\varepsilon^2) & \text{if } N \geq 5. \end{cases}$$

where $A_1 = \frac{(\Gamma(\frac{N-1}{2}))^2}{2\Gamma(N-2)}\omega_{N-1}$, $A'_2 = \frac{\omega_2}{2}$, $A_2 = \frac{(N-1)(N-2)\Gamma(\frac{N-1}{2})\Gamma(\frac{N-3}{2})}{8\Gamma(N-2)}\omega_{N-1}$,
 $B_1 = \frac{(\Gamma(\frac{N-1}{2}))^2}{2\Gamma(N-1)}\omega_{N-2}$, $B_2 = \frac{(N-1)(\Gamma(\frac{N-1}{2}))^2}{4\Gamma(N-1)}\omega_{N-2}$ and $D = \frac{\Gamma(\frac{N-1}{2})\Gamma(\frac{N-3}{2})}{2(N-4)\Gamma(N-2)}\omega_{N-1}$.
The function Γ is defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt \quad \text{for every } z \in]0, +\infty[.$$

Let us notice that

$$S_1 = \frac{A_1}{B_1^q}.$$

We endow the Sobolev space $H^1(\Omega)$ with the norm

$$\|u\| = \left[\int_{\Omega} (|\nabla u|^2 + |u|^2) dx \right]^{\frac{1}{2}}.$$

The norms in the Lebesgue space $L^t(\Omega)$ are denoted by $\|\cdot\|_t$.

4.3 Case $\lambda < 0$

In this section, we study the following problem

$$(4.3.1) \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = \lambda u + f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

Throughout this section we assume that f satisfies (H1), p satisfies (4.1.2), Q satisfies (4.1.3), $\lambda < 0$ and $\partial\Omega$ satisfies (g.c) at x_0 . We start by the following Theorem which plays an essential role in the study of existence of a solution of the problem (4.3.1)

Theorem 4.3.1.

Let $\Omega \in \mathbb{R}^N$, $N \geq 3$ be a bounded domain with a smooth boundary $\partial\Omega$ satisfying (g.c) at x_0 and $H(x_0) > 0$. We assume that f satisfies (H1) and the functions p and Q respectively satisfy (4.1.2) and (4.1.3). Then for every $\lambda < 0$ we have

$$(4.3.2) \quad I_0 := \inf_{\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{1}{q}}=1} \left[\sigma_N \left(\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right)^{\frac{N}{2}} - \int_{\Omega} f(x)u dx \right]$$

is achieved and $I_0 > 0$.

Proof.

We use a straightforward adaptation of the proof given in [BN2] for an analogous

minimization problem.

Let $\{u_n\}$ be a minimizing sequence for (4.3.2). It is easy to see that $\{u_n\}$ is bounded in $H^1(\Omega)$, thus we may extract a subsequence, still denoted by $\{u_n\}$, such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } H^1(\Omega), \\ u_n &\rightarrow u \quad \text{strongly in } L^2(\Omega), \\ u_n &\rightarrow u \quad \text{pointwise a.e in } \Omega, \\ u_n &\rightharpoonup u \quad \text{weakly in } L^q(\partial\Omega), \\ u_n &\rightarrow u \quad \text{pointwise a.e in } \partial\Omega, \end{aligned}$$

$$\text{with } \left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \right)^{\frac{1}{q}} \leq 1.$$

$$\text{It remains to prove that } \left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \right)^{\frac{1}{q}} = 1.$$

Suppose, by contradiction, that

$$\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \right)^{\frac{1}{q}} < 1.$$

Set $v_n = u_n - u$. Using the Brezis-Lieb Lemma [BL], we obtain

$$1 = \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + \int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x + o(1)$$

(which holds since v_n is bounded in $L^q(\partial\Omega)$ and $v_n \rightarrow 0$ a.e on $\partial\Omega$).

We deduce that

$$(4.3.3) \quad \left(\int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x \right)^{\frac{2}{q}} = \left[1 - \left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \right) \right]^{\frac{2}{q}} + o(1).$$

On the other hand, we have

$$(4.3.4) \quad \begin{aligned} I_0 + o(1) &= \sigma_N \left(\int_{\Omega} p(x)|\nabla u_n|^2 dx - \lambda \int_{\Omega} |u_n|^2 dx \right)^{\frac{N}{2}} - \int_{\Omega} f(x) u_n dx \\ &= \sigma_N \left(\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx + \int_{\Omega} p(x)|\nabla v_n|^2 dx \right)^{\frac{N}{2}} - \int_{\Omega} f(x) u dx. \end{aligned}$$

From the result of [Z, Theorem 02] we have the existence of a constant $C(\Omega) > 0$, such that for every $w \in H^1(\Omega)$ we have

$$\int_{\Omega} |\nabla w|^2 dx + C(\Omega) \int_{\Omega} |w|^k dx \geq S_1 \left(\int_{\partial\Omega} |w|^q dx \right)^{\frac{2}{q}}$$

with $k = \frac{2N}{N-1}$ if $N \geq 4$ and $k > 3 = \frac{2N}{N-1}$ if $N = 3$.

We apply this result for $w_n = (p(x))^{\frac{1}{2}}v_n$ and particularly for $N = 3$ we take k such that $6 = \frac{2N}{N-2} > k > 3$, we obtain, for n large enough

$$\begin{aligned} \int_{\Omega} p(x)|\nabla v_n|^2 dx + C(\Omega) \int_{\Omega} |(p(x))^{\frac{1}{2}}v_n|^k dx \\ = \int_{\Omega} |\nabla[(p(x))^{\frac{1}{2}}v_n]|^2 dx + C(\Omega) \int_{\Omega} |(p(x))^{\frac{1}{2}}v_n|^k dx + o(1) \\ \geq S_1 \left(\int_{\partial\Omega} |(p(x))^{\frac{1}{2}}v_n|^q ds_x \right)^{\frac{2}{q}} + o(1). \end{aligned}$$

Since $k < \frac{2N}{N-2}$ for every $N \geq 3$, thanks to the compact injection of $H^1(\Omega)$ into $L^k(\Omega)$ we have $v_n \rightarrow 0$ strongly in $L^k(\Omega)$, and we deduce

$$\int_{\Omega} p(x)|\nabla v_n|^2 dx + o(1) \geq S_1 \left(\int_{\partial\Omega} |(p(x))^{\frac{1}{2}}v_n|^q ds_x \right)^{\frac{2}{q}} + o(1).$$

Using the fact that

$$\frac{p(x)}{(Q(x))^{N-2}} \geq \frac{p(x_0)}{(Q(x_0))^{N-2}} \quad \forall x \in \partial\Omega,$$

an easy computation yields

$$(4.3.5) \quad \int_{\Omega} p(x)|\nabla v_n|^2 dx \geq \left(\frac{p(x_0)}{(Q(x_0))^{N-2}} \right)^{\frac{1}{N-1}} S_1 \left(\int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x \right)^{\frac{2}{q}} + o(1).$$

Inserting (4.3.3) and (4.3.5) into (4.3.4), we obtain after letting n tend to infinity

$$(4.3.6) \quad \begin{aligned} I_0 \geq - \int_{\Omega} f(x) u dx + \sigma_N \times \\ \left[\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx + \left(\frac{p(x_0)}{(Q(x_0))^{N-2}} \right)^{\frac{1}{N-1}} S_1 \left(1 - \left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \right) \right)^{\frac{2}{q}} \right]^{\frac{N}{2}}. \end{aligned}$$

We shall now prove the opposite inequality.

Since $\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \right)^{\frac{1}{q}} < 1$ there exists $t_\varepsilon > 0$ such that

$$\left(\int_{\partial\Omega} p(x)Q(x)|u + t_\varepsilon W_{\varepsilon, x_0}|^q ds_x \right)^{\frac{1}{q}} = 1.$$

For ε small enough we have

$$\begin{aligned} \int_{\Omega} p(x)|\nabla(u + t_\varepsilon W_{\varepsilon, x_0})|^2 dx - \lambda \int_{\Omega} |u + t_\varepsilon W_{\varepsilon, x_0}|^2 dx \\ = \int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx + t_\varepsilon^2 \left(\int_{\Omega} p(x)|\nabla W_{\varepsilon, x_0}|^2 dx - \lambda \int_{\Omega} |W_{\varepsilon, x_0}|^2 dx \right) + o(1) \end{aligned}$$

and

$$\begin{aligned}
1 &= \int_{\partial\Omega} p(x)Q(x)|u + t_\varepsilon W_{\varepsilon,x_0}|^q ds_x \\
&= \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q \int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x + o(1).
\end{aligned}$$

From (4.2.2), (4.2.3) and (4.2.4) we deduce that

$$\begin{aligned}
(4.3.7) \quad & \int_{\Omega} p(x)|\nabla(u + t_\varepsilon W_{\varepsilon,x_0})|^2 dx - \lambda \int_{\Omega} |u + t_\varepsilon W_{\varepsilon,x_0}|^2 dx \\
&= \int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx + t_\varepsilon^2 p(x_0)A_1 + o(1)
\end{aligned}$$

and

$$(4.3.8) \quad 1 = \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q p(x_0)Q(x_0)B_1 + o(1).$$

We use $u + t_\varepsilon W_{\varepsilon,x_0}$ as a test function in (4.3.2), we deduce

$$\begin{aligned}
I_0 &\leq c_N \left[\int_{\Omega} p(x)|\nabla(u + t_\varepsilon W_{\varepsilon,x_0})|^2 dx - \lambda \int_{\Omega} |u + t_\varepsilon W_{\varepsilon,x_0}|^2 dx \right]^{\frac{N}{2}} \\
&\quad - \int_{\Omega} f(x)(u + c_\varepsilon W_{\varepsilon,x_0}) dx \\
&\leq - \int_{\Omega} f(x)u dx + \sigma_N \left[\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right. \\
&\quad \left. - \left(\frac{p(x_0)}{(Q(x_0))^{N-2}} \right)^{\frac{1}{N-1}} S_1 \left(1 - \left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \right) \right)^{\frac{2}{q}} \right]^{\frac{N}{2}} + o(1).
\end{aligned}$$

Passing to the limit as $\varepsilon \rightarrow 0$, we get

$$\begin{aligned}
I_0 &\leq - \int_{\Omega} f(x)u dx + \sigma_N \left[\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right. \\
&\quad \left. - \left(\frac{p(x_0)}{(Q(x_0))^{N-2}} \right)^{\frac{1}{N-1}} S_1 \left(1 - \left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \right) \right)^{\frac{2}{q}} \right]^{\frac{N}{2}}.
\end{aligned}$$

Combining the last inequality with (4.3.6) we obtain

$$\begin{aligned}
(4.3.9) \quad I_0 &= - \int_{\Omega} f(x)u dx + \sigma_N \left[\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right. \\
&\quad \left. - \left(\frac{p(x_0)}{(Q(x_0))^{N-2}} \right)^{\frac{1}{N-1}} S_1 \left(1 - \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \right)^{\frac{2}{q}} \right]^{\frac{N}{2}}.
\end{aligned}$$

Set

$$K_1 = N\sigma_N \left[\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right. \\ \left. - \left(\frac{p(x_0)}{(Q(x_0))^{N-2}} \right)^{\frac{1}{N-1}} S_1 \left(1 - \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x \right)^{\frac{2}{q}} \right]^{\frac{N-2}{2}}$$

and

$$K_2 = \left(\frac{p(x_0)}{(Q(x_0))^{N-2}} \right)^{\frac{1}{N-1}} S_1 \left(1 - \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x \right)^{\frac{2-q}{q}}.$$

For every $w \in H^1(\Omega)$, $t \in \mathbb{R}$, we consider

$$J(u + tw) = - \int_{\Omega} f(x)(u + tw) dx + \sigma_N \left[\int_{\Omega} p(x) |\nabla(u + tw)|^2 dx - \lambda \int_{\Omega} |u + tw|^2 dx \right. \\ \left. - \left(\frac{p(x_0)}{(Q(x_0))^{N-2}} \right)^{\frac{1}{N-1}} S_1 \left(1 - \left(\int_{\partial\Omega} p(x) Q(x) |u + tw|^q ds_x \right) \right)^{\frac{2}{q}} \right]^{\frac{N}{2}}.$$

Differentiating with respect to t and passing to the limit as $t \rightarrow 0$, from (4.3.9) we get

$$K_1 \left[\int_{\Omega} p(x) \nabla u \cdot \nabla w dx - \lambda \int_{\Omega} u w dx - K_2 \int_{\partial\Omega} p(x) Q(x) |u|^{q-2} u w ds_x \right] - \int_{\Omega} f(x) w dx = 0$$

for every $w \in H^1(\Omega)$.

We obtain that u satisfies weakly

$$(4.3.10) \quad \begin{cases} -\operatorname{div}(p(x) \nabla u) = \lambda u + \frac{1}{K_1} f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = K_2 Q(x) |u|^{q-2} u & \text{on } \partial\Omega. \end{cases}$$

Since $f \not\equiv 0$ we easily see that $u \not\equiv 0$ on Ω . Moreover, since f satisfies (4.1.4), then $u \not\equiv 0$ on $\partial\Omega$.

Remark 4.3.1.

If $p \in C^\infty(\bar{\Omega})$, $Q \in C^\infty(\partial\Omega)$ and $f \in C^\infty(\bar{\Omega})$, then by [Ch], any solution u of (4.3.10) belongs to $C^\infty(\bar{\Omega})$.

Let us note that we do not know if $u(x_0) \neq 0$ or not, but if $u(x_0) \neq 0$ we can assume that $u(x_0) > 0$ (replace u with $-u$ and f with $-f$ if necessary). Therefore, throughout the rest of proof, we assume that $u(x_0) \geq 0$. Set

$$J(v) = \sigma_N \left(\int_{\Omega} p(x) |\nabla v|^2 dx - \lambda \int_{\Omega} |v|^2 dx \right)^{\frac{N}{2}} - \int_{\Omega} f(x) v dx,$$

we will obtain a contradiction by showing that

$$(4.3.11) \quad J(u + t_\varepsilon W_{\varepsilon, x_0}) < I_0$$

for $\varepsilon > 0$ small enough.

At this stage, we need to give a more explicit expression of t_ε .

From (4.3.8) and (4.2.3) it follows that $t_\varepsilon \rightarrow t_0$ as $\varepsilon \rightarrow 0$ where

$$(4.3.12) \quad t_0 = \left[\frac{1 - \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x}{p(x_0)Q(x_0)B_1} \right]^{\frac{1}{q}}.$$

Set

$$(4.3.13) \quad t_\varepsilon = t_0(1 + \delta_\varepsilon), \quad \text{with } \delta_\varepsilon \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

We shall give the precise rate at which $\delta_\varepsilon \rightarrow 0$.

We remark that if $q \leq 3$ then $N \geq 4$ and if $q > 3$ then $N < 4$, consequently we shall distinguish the cases $N \geq 4$ and $N = 3$.

Case $N \geq 4$

We have the following inequality :

For every $1 \leq q \leq 3$, there exists a constant $C = C(q)$ such that, for $x, y \in \mathbb{R}$, we have

$$(4.3.14) \quad ||x + y|^q - |x|^q - |y|^q| - qxy(|x|^{q-2} + |y|^{q-2}) \leq \begin{cases} C|x|^{q-1}|y| & \text{if } |x| \leq |y| \\ C|y|^{q-1}|x| & \text{if } |x| \geq |y|. \end{cases}$$

We apply (4.3.14) with $x = u$ and $y = W_{\varepsilon, x_0}$.

We need the following estimates which are proved in the appendix.

$$(4.3.15) \quad \begin{aligned} \int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon, x_0}|^{q-2}W_{\varepsilon, x_0}u ds_x &= \frac{\omega_{N-2}\Gamma(\frac{N-1}{2})\Gamma(\frac{1}{2})p(x_0)Q(x_0)u(x_0)\varepsilon^{\frac{N-2}{2}}}{2\Gamma(\frac{N}{2})} + o(\varepsilon^{\frac{N-2}{2}}) \\ &= u(x_0)D\varepsilon^{\frac{N-2}{2}} + o(\varepsilon^{\frac{N-2}{2}}), \end{aligned}$$

$$(4.3.16) \quad \int_{\partial\Omega} p(x)Q(x)|u|^{q-2}u W_{\varepsilon, x_0} ds_x = \varepsilon^{\frac{N-2}{2}} \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x - x_0|^{N-2}} ds_x + o(\varepsilon^{\frac{N-2}{2}}),$$

$$(4.3.17) \quad t_\varepsilon \int_{\partial\Omega \cap \{|u| \leq t_\varepsilon W_{\varepsilon, x_0}\}} p(x)Q(x)|u|^{q-1}W_{\varepsilon, x_0} ds_x = o(\varepsilon^{\frac{N-2}{2}})$$

and

$$(4.3.18) \quad t_\varepsilon^{q-1} \int_{\partial\Omega \cap \{|u| \geq t_\varepsilon W_{\varepsilon, x_0}\}} p(x)Q(x)|u||W_{\varepsilon, x_0}|^{q-1} ds_x = o(\varepsilon^{\frac{N-2}{2}}).$$

Looking at (4.2.3) and the above estimates, direct computation yields that the cases $N = 4$ and $N \geq 5$ are different and will be treated separately. Therefore we distinguish two cases:

If $N \geq 5$

Combining, (4.3.13), (4.3.14), (4.3.15), (4.3.16), (4.3.17) and (4.3.18) we deduce that

$$1 = \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q \int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x + O(\varepsilon^{\frac{N-2}{2}}).$$

Using (4.2.3), we obtain

$$1 = \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q [p(x_0)Q(x_0)(B_1 - H(x_0)B_2\varepsilon)] + o(\varepsilon).$$

Consequently

$$(4.3.19) \quad t_\varepsilon^2 = t_0^2 \left(1 + \frac{2}{q} H(x_0) \frac{B_2}{B_1} \varepsilon\right) + o(\varepsilon)$$

and

$$(4.3.20) \quad \delta_\varepsilon = \frac{1}{q} H(x_0) \frac{B_2}{B_1} \varepsilon + o(\varepsilon).$$

We have

$$\begin{aligned} J(u + t_\varepsilon W_{\varepsilon,x_0}) &= \sigma_N \left[\left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) \right. \\ &\quad + 2t_\varepsilon \left(\int_{\Omega} p(x) \nabla u \cdot \nabla W_{\varepsilon,x_0} dx - \lambda \int_{\Omega} u W_{\varepsilon,x_0} dx \right) \\ &\quad \left. + t_\varepsilon^2 \left(\int_{\Omega} p(x) |\nabla W_{\varepsilon,x_0}|^2 dx - \lambda \int_{\Omega} |W_{\varepsilon,x_0}|^2 dx \right) \right]^{\frac{N}{2}} \\ &\quad - \int_{\Omega} f(x) u dx - t_\varepsilon \int_{\Omega} f(x) W_{\varepsilon,x_0} dx. \end{aligned}$$

Using (4.2.2) and (4.2.4), we write

$$\begin{aligned} J(u + t_\varepsilon W_{\varepsilon,x_0}) &= \sigma_N \left[\left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) \right. \\ &\quad + 2t_\varepsilon \left(\int_{\Omega} p(x) \nabla u \cdot \nabla W_{\varepsilon,x_0} dx - \lambda \int_{\Omega} u W_{\varepsilon,x_0} dx \right) \\ (4.3.21) \quad &\quad \left. + t_0^2 (1 + 2\delta_\varepsilon) A_1 - H(x_0) t_0^2 A_2 \varepsilon + o(\varepsilon) + o(\delta_\varepsilon) \right]^{\frac{N}{2}} \\ &\quad - \int_{\Omega} f(x) u dx - \int_{\Omega} f(x) W_{\varepsilon,x_0} dx. \end{aligned}$$

From (4.3.20) we have

$$J(u + t_\varepsilon W_{\varepsilon, x_0}) = - \int_{\Omega} f(x) u \, dx + o(\varepsilon) \\ + \sigma_N \left[\left(\int_{\Omega} p(x) |\nabla u|^2 \, dx - \lambda \int_{\Omega} |u|^2 \, dx \right) + t_0^2 A_1 + \frac{2}{q} t_0^2 H(x_0) \frac{B_2}{B_1} A_1 \varepsilon - t_0^2 A_2 H(x_0) \varepsilon + o(\varepsilon) \right]^{\frac{N}{2}}.$$

Therefore

$$J(u + t_\varepsilon W_{\varepsilon, x_0}) \neq \sigma_N \left[\left(\int_{\Omega} p(x) |\nabla u|^2 \, dx - \lambda \int_{\Omega} |u|^2 \, dx \right) + t_0^2 A_1 \right]^{\frac{N}{2}} - \int_{\Omega} f(x) u \, dx \\ + \frac{N}{2} \sigma_N \left[\left(\int_{\Omega} p(x) |\nabla u|^2 \, dx - \lambda \int_{\Omega} |u|^2 \, dx \right) + t_0^2 A_1 \right]^{\frac{N-2}{2}} t_0^2 \left(\frac{2}{q} \frac{B_2}{B_1} A_1 - A_2 \right) H(x_0) \varepsilon + o(\varepsilon).$$

Combining the fact that $\Gamma(z+1) = z\Gamma(z) \forall z \in]0, +\infty[$ with the definition of the constants A_1, A_2, B_1 and B_2 , we deduce that

$$(4.3.22) \quad A_2 - \frac{A_1 B_2}{B_1} > 0.$$

Using (4.3.9), (4.3.22), the fact that $\frac{2}{q} < 1$, the expression of t_0 and the expression of K_1 we deduce that, for $\varepsilon > 0$ sufficiently small

$$J(u + t_\varepsilon W_{\varepsilon, x_0}) = I_0 + \frac{K_1}{2} t_0^2 \left(\frac{2}{q} \frac{B_2}{B_1} A_1 - A_2 \right) H(x_0) \varepsilon + o(\varepsilon) < I_0$$

which gives a contradiction in this case.

If $N = 4$

From (4.3.13), (4.3.14), (4.3.15), (4.3.16), (4.3.17) and (4.3.18) we have

$$1 = \int_{\partial\Omega} p(x) Q(x) |u|^q \, ds_x + t_\varepsilon^q \int_{\partial\Omega} p(x) Q(x) |W_{\varepsilon, x_0}|^q \, ds_x \\ - \varepsilon \left(qu(x_0) D t_0^{q-1} + q t_0 \int_{\partial\Omega} \frac{p(x) Q(x) |u(x)|^{q-2} u(x) \phi(x)}{|x - x_0|^2} \, ds_x \right) + o(\varepsilon).$$

Using (4.2.3) we obtain

$$1 = \int_{\partial\Omega} p(x) Q(x) |u|^q \, ds_x + t_\varepsilon^q [p(x_0) Q(x_0) (B_1 - H(x_0) B_2 \varepsilon)] \\ - \varepsilon \left(qu(x_0) D t_0^{q-1} + q \int_{\partial\Omega} \frac{p(x) Q(x) |u(x)|^{q-2} u(x) \phi(x)}{|x - x_0|^2} \, ds_x \right) + o(\varepsilon).$$

Consequently

$$(4.3.23) \quad t_\varepsilon^2 = t_0^2 \left[1 + \frac{2}{q} \left(H(x_0) \frac{B_2}{B_1} - \frac{qu(x_0) D t_0^{q-1} + q \int_{\partial\Omega} \frac{p(x) Q(x) |u(x)|^{q-2} u(x) \phi(x)}{|x - x_0|^2} \, ds_x}{B_1 t_0^q} \right) \varepsilon \right] \\ + o(\varepsilon)$$

and

(4.3.24)

$$\delta_\varepsilon = \frac{1}{q} \left(H(x_0) \frac{B_2}{B_1} - \frac{qu(x_0)Dt_0^{q-1} + q \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^2} ds_x}{B_1 t_0^q} \right) \varepsilon + o(\varepsilon).$$

Therefore

$$\begin{aligned} J(u+t_\varepsilon W_{\varepsilon,x_0}) &= \sigma_N \left[\left(\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) \right. \\ &\quad + 2t_\varepsilon \left(\int_{\Omega} p(x)\nabla u \cdot \nabla W_{\varepsilon,x_0} dx - \lambda \int_{\Omega} u W_{\varepsilon,x_0} dx \right) \\ &\quad \left. + t_0^2(1+2\delta_\varepsilon)A_1 - H(x_0)t_0^2 A_2 \varepsilon + o(\varepsilon) + o(\delta_\varepsilon) \right]^{\frac{N}{2}} \\ &\quad - \int_{\Omega} f(x)u dx - t_\varepsilon \int_{\Omega} f(x)W_{\varepsilon,x_0} dx \end{aligned}$$

and

$$\begin{aligned} J(u+t_\varepsilon W_{\varepsilon,x_0}) &= \\ &\sigma_N \left[\left(\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) + t_0^2 A_1 \right]^{\frac{N}{2}} \int_{\Omega} f(x)u dx - t_0 \int_{\Omega} f(x)W_{\varepsilon,x_0} dx \\ &+ \left[2t_0^2 \delta_\varepsilon A_1 - H(x_0)t_0^2 A_2 \varepsilon + o(\varepsilon) + 2t_0 \left(\int_{\Omega} p(x)\nabla u \cdot \nabla W_{\varepsilon,x_0} dx - \lambda \int_{\Omega} u W_{\varepsilon,x_0} dx \right) \right]^{\frac{N}{2}} \\ &\times \sigma_N \left[\left(\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) + t_0^2 A_1 \right]^{\frac{N-2}{2}}. \end{aligned}$$

Using (4.3.9) we get

$$\begin{aligned} J(u+t_\varepsilon W_{\varepsilon,x_0}) &= I_0 + t_0 \left[K_1 \left(\int_{\Omega} p(x)\nabla u \cdot \nabla W_{\varepsilon,x_0} dx - \lambda \int_{\Omega} u W_{\varepsilon,x_0} dx \right) \right. \\ &\quad \left. - \int_{\Omega} f(x)W_{\varepsilon,x_0} dx \right] + t_0^2 K_1 \delta_\varepsilon A_1 - H(x_0) \frac{K_1}{2} t_0^2 A_2 \varepsilon + o(\varepsilon). \end{aligned}$$

Thus from equation (4.3.10) we derive

$$\begin{aligned} J(u+t_\varepsilon W_{\varepsilon,x_0}) &= I_0 + t_0 K_1 K_2 \int_{\partial\Omega} p(x)Q(x)|u|^{q-2}u W_{\varepsilon,x_0} ds_x \\ &\quad + t_0^2 K_1 \delta_\varepsilon A_1 - H(x_0) \frac{K_1}{2} t_0^2 A_2 \varepsilon + o(\varepsilon). \end{aligned}$$

Inserting (4.3.16) in the above equality, we obtain

$$\begin{aligned} J(u+t_\varepsilon W_{\varepsilon,x_0}) &= I_0 + t_0 K_1 K_2 \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^2} ds_x \varepsilon \\ &\quad + t_0^2 K_1 \delta_\varepsilon A_1 - H(x_0) \frac{K_1}{2} t_0^2 A_2 \varepsilon + o(\varepsilon). \end{aligned}$$

Therefore

$$\begin{aligned} J(u+t_\varepsilon W_{\varepsilon,x_0}) &= I_0 \\ &\quad + K_1 \left[t_0 K_2 \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^2} ds_x \varepsilon + t_0^2 \delta_\varepsilon A_1 - \frac{H(x_0)t_0^2 A_2}{2} \varepsilon \right] + o(\varepsilon) \end{aligned}$$

A combination of the definition of t_0 together with the definition of S_1 , namely $S_1 = \frac{A_1}{B_1^q}$, and the definition of K_2 , in the last inequality, allows to write

$$\begin{aligned} J(u+t_\varepsilon W_{\varepsilon,x_0}) &= I_0 + \frac{K_1 A_1}{B_1 t_0^{q-2}} \left[\int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^2} ds_x \varepsilon + t_0^q \delta_\varepsilon B_1 - H(x_0) t_0^q B_1 \frac{A_2}{2A_1} \varepsilon \right] \\ &\quad + o(\varepsilon). \end{aligned}$$

It follows from (4.3.24) that

$$\begin{aligned} J(u+t_\varepsilon W_{\varepsilon,x_0}) &= I_0 + \frac{K_1 A_1}{B_1 t_0^{q-2}} \left[\frac{1}{q} H(x_0) t_0^q B_2 \varepsilon - u(x_0) t_0^{q-1} D \varepsilon - H(x_0) t_0^q B_1 \frac{A_2}{2A_1} \varepsilon \right] \\ &\quad + o(\varepsilon) \\ &= I_0 - \frac{H(x_0) t_0^2 K_1}{2} \left(A_2 - \frac{2 A_1 B_2}{q B_1} \right) \varepsilon - u(x_0) D \frac{t_0 K_1 A_1}{B_1} \varepsilon + o(\varepsilon). \end{aligned}$$

Since $H(x_0) > 0$ and $u(x_0) \geq 0$, from (4.3.22) and the fact that $\frac{2}{q} < 1$ we deduce the desired result in this case.

Case $N = 3$

In this case, we have $q = 4$ therefore we use the simple identity

$$(4.3.25) \quad |x+y|^4 = |x|^4 + |y|^4 + 4|x|^2xy + 4xy|y|^2 + 6|x|^2|y|^2.$$

By opposition with the other cases, in dimension $N = 3$, the case $u(x_0) = 0$ and $u(x_0) > 0$ will be treated separately. We distinguish two subcases.

Case 1: If $u(x_0) > 0$.

In order to apply (4.3.25) with $x = u$ and $y = W_{\varepsilon,x_0}$, we need the following new estimate which will be proved in the appendix.

$$(4.3.26) \quad \int_{\partial\Omega} p(x)Q(x)|u(x)|^2 |W_{\varepsilon,x_0}|^2 ds_x \leq M_1 \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) = o(\varepsilon^{\frac{1}{2}}),$$

$$(4.3.27) \quad \int_{\partial\Omega} p(x)Q(x)u(x)|W_{\varepsilon,x_0}|^3 ds_x = M_2 u(x_0)\varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}})$$

and

$$(4.3.28) \quad \int_{\partial\Omega} p(x)Q(x)|u(x)|^2 u(x) W_{\varepsilon,x_0} ds_x = \varepsilon^{\frac{1}{2}} \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^2 u(x)\phi(x)}{|x-x_0|} ds_x + o(\varepsilon^{\frac{1}{2}}).$$

where M_1 and M_2 are positive constants.

From (4.3.25), (4.3.26), (4.3.27) and (4.3.28) we have

$$\begin{aligned} 1 &= \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q \int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon,x_0}|^q ds_x \\ &\quad - q\varepsilon^{\frac{1}{2}} \left(u(x_0)Dt_0^{q-1} + t_0 \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2} u(x)\phi(x)}{|x-x_0|} ds_x \right) + o(\varepsilon^{\frac{1}{2}}). \end{aligned}$$

Using (4.2.3) we obtain

$$\begin{aligned} 1 &= \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x + t_\varepsilon^q p(x_0)Q(x_0)B_1 \\ &\quad - q\varepsilon^{\frac{1}{2}} \left(u(x_0)Dt_0^{q-1} + \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2} u(x)\phi(x)}{|x-x_0|} ds_x \right) + o(\varepsilon^{\frac{1}{2}}). \end{aligned}$$

Consequently

$$(4.3.29) \quad t_\varepsilon^2 = t_0^2 \left(1 - \frac{2}{q} \frac{u(x_0)Dt_0^{q-1} + \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2} u(x)\phi(x)}{|x-x_0|} ds_x}{B_1 t_0^q} \varepsilon^{\frac{1}{2}} \right) + o(\varepsilon^{\frac{1}{2}})$$

and

$$(4.3.30) \quad \delta_\varepsilon = - \frac{u(x_0)Dt_0^{q-1} + \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2} u(x)\phi(x)}{|x-x_0|} ds_x}{B_1 t_0^q} \varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}}).$$

Therefore

$$\begin{aligned}
& J(u + t_\varepsilon W_{\varepsilon, x_0}) = \\
& \sigma_3 \left[\left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) + t_0^2 (1 + 2\delta_\varepsilon) A_1 - H(x_0) t_0^2 A'_2 \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) \right. \\
& \left. + o(\delta_\varepsilon) + 2t_\varepsilon \left(\int_{\Omega} p(x) \nabla u \cdot \nabla W_{\varepsilon, x_0} dx - \lambda \int_{\Omega} u W_{\varepsilon, x_0} dx \right) \right]^{\frac{3}{2}} - \int_{\Omega} f(x) u dx - \int_{\Omega} f(x) W_{\varepsilon, x_0} dx \\
& = \sigma_3 \left[\left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) + t_0^2 A_1 \right]^{\frac{N}{2}} - \int_{\Omega} f(x) u dx - t_0 \int_{\Omega} f(x) W_{\varepsilon, x_0} dx \\
& + \left[2t_0^2 \delta_\varepsilon A_1 + o(\delta_\varepsilon) + 2t_0 \left(\int_{\Omega} p(x) \nabla u \cdot \nabla W_{\varepsilon, x_0} dx - \lambda \int_{\Omega} u W_{\varepsilon, x_0} dx \right) \right] \\
& \times \frac{N}{2} \sigma_3 \left[\left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) + t_0^2 A_1 \right]^{\frac{N-2}{2}}.
\end{aligned}$$

Using (4.3.9) we get

$$\begin{aligned}
J(u + t_\varepsilon W_{\varepsilon, x_0}) &= I_0 + t_0^2 K_1 \delta_\varepsilon A_1 + o(\delta_\varepsilon) \\
&+ t_0 \left[K_1 \left(\int_{\Omega} p(x) \nabla u \cdot \nabla W_{\varepsilon, x_0} dx - \lambda \int_{\Omega} u W_{\varepsilon, x_0} dx \right) - \int_{\Omega} f(x) W_{\varepsilon, x_0} dx \right].
\end{aligned}$$

Thus from equation (4.3.10) we derive

$$J(u + t_\varepsilon W_{\varepsilon, x_0}) = I_0 + t_0 K_1 K_2 \int_{\partial\Omega} p(x) Q(x) |u|^{q-2} u W_{\varepsilon, x_0} ds_x + t_0^2 K_1 \delta_\varepsilon A_1 + o(\delta_\varepsilon).$$

Inserting (4.3.16) in the above equality, we write

$$\begin{aligned}
J(u + t_\varepsilon W_{\varepsilon, x_0}) &= I_0 + t_0 K_1 K_2 \int_{\partial\Omega} \frac{p(x) Q(x) |u(x)|^{q-2} u(x) \phi(x)}{|x - x_0|} ds_x \varepsilon^{\frac{1}{2}} \\
&+ t_0^2 K_1 \delta_\varepsilon A_1 + o(\varepsilon^{\frac{1}{2}}).
\end{aligned}$$

Therefore

$$J(u + t_\varepsilon W_{\varepsilon, x_0}) = I_0 +$$

$$K_1 \left[t_0 K_2 \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|} ds_x \varepsilon^{\frac{1}{2}} + t_0^2 \delta_\varepsilon A_1 \right] + o(\varepsilon^{\frac{1}{2}})$$

$$= I_0 + o(\varepsilon^{\frac{1}{2}}) +$$

$$K_1 \left[\frac{\left(\frac{p(x_0)}{(Q(x_0))^{N-2}} \right)^{\frac{1}{N-1}} S_1 t_0 \varepsilon^{\frac{1}{2}}}{\left(1 - \left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \right) \right)^{\frac{q-2}{q}}} \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|} ds_x + t_0^2 \delta_\varepsilon A_1 \right].$$

Combining the definition of t_0 together with the definition of S_1 , namely $S_1 = \frac{A_1}{B_1^{\frac{q}{2}}}$, we write

$$J(u + t_\varepsilon W_{\varepsilon, x_0}) = I_0 + \frac{K_1 A_1}{B_1 t_0^{q-2}} \left[\int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|} ds_x \varepsilon^{\frac{1}{2}} + t_0^q \delta_\varepsilon B_1 \right] + o(\varepsilon^{\frac{1}{2}}).$$

It follows from (4.3.30) that

$$J(u + t_\varepsilon W_{\varepsilon, x_0}) = I_0 - u(x_0) D \frac{t_0 K_1 A_1}{B_1} \varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}}).$$

Since $u(x_0) > 0$ we deduce that for $\varepsilon > 0$ sufficiently small we have

$$J(u + t_\varepsilon W_{\varepsilon, x_0}) < I_0$$

and we obtain a contradiction in this case.

Case 2: If $u(x_0) = 0$

From (4.3.13) we have

$$t_\varepsilon = t_0(1 + \delta_\varepsilon) \quad \text{with } \delta_\varepsilon \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

We write

$$\begin{aligned}
J(u+t_\varepsilon W_{\varepsilon,x_0}) &= -\int_{\Omega} f(x) u \, dx - \int_{\Omega} f(x) W_{\varepsilon,x_0} \, dx + \sigma_3 \left[\left(\int_{\Omega} p(x) |\nabla u|^2 \, dx - \lambda \int_{\Omega} |u|^2 \, dx \right) \right. \\
&\quad \left. + 2t_\varepsilon \left(\int_{\Omega} p(x) \nabla u \cdot \nabla W_{\varepsilon,x_0} \, dx - \lambda \int_{\Omega} u W_{\varepsilon,x_0} \, dx \right) + t_0^2 (1 + 2\delta_\varepsilon + \delta_\varepsilon^2) p(x_0) A_1 \right. \\
&\quad \left. - H(x_0) t_0^2 p(x_0) A'_2 \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) \right]^{\frac{3}{2}} \\
&= \sigma_3 \left[\left(\int_{\Omega} p(x) |\nabla u|^2 \, dx - \lambda \int_{\Omega} |u|^2 \, dx \right) + t_0^2 A_1 \right]^{\frac{3}{2}} - \int_{\Omega} f(x) u \, dx - t_0 \int_{\Omega} f(x) W_{\varepsilon,x_0} \, dx \\
&\quad + \frac{3}{2} \sigma_3 \left[\left(\int_{\Omega} p(x) |\nabla u|^2 \, dx - \lambda \int_{\Omega} |u|^2 \, dx \right) + t_0^2 A_1 \right]^{\frac{1}{2}} \left[2t_0^2 \delta_\varepsilon p(x_0) A_1 + t_0^2 \delta_\varepsilon^2 p(x_0) A_1 \right. \\
&\quad \left. - H(x_0) t_0^2 p(x_0) A'_2 \varepsilon |\log \varepsilon| + 2t_0 \left(\int_{\Omega} p(x) \nabla u \cdot \nabla W_{\varepsilon,x_0} \, dx - \lambda \int_{\Omega} u W_{\varepsilon,x_0} \, dx \right) \right] + o(\varepsilon |\log \varepsilon|).
\end{aligned}$$

Using (4.3.9) we get

$$\begin{aligned}
J(u + t_\varepsilon W_{\varepsilon,x_0}) &= I_0 + t_0^2 K_1 \delta_\varepsilon p(x_0) A_1 + t_0^2 \frac{K_1}{2} \delta_\varepsilon^2 p(x_0) A_1 - H(x_0) t_0^2 p(x_0) A'_2 \varepsilon |\log \varepsilon| \\
&\quad + t_0 \left[K_1 \left(\int_{\Omega} p(x) \nabla u \cdot \nabla W_{\varepsilon,x_0} \, dx - \lambda \int_{\Omega} u W_{\varepsilon,x_0} \, dx \right) - \int_{\Omega} f(x) W_{\varepsilon,x_0} \, dx \right] \\
&\quad + o(\varepsilon |\log \varepsilon|).
\end{aligned}$$

Thus from equation (4.3.10) we derive

(4.3.31)

$$\begin{aligned}
J(u + t_\varepsilon W_{\varepsilon,x_0}) &= I_0 + t_0 K_1 K_2 \int_{\partial\Omega} p(x) Q(x) |u|^{q-2} u W_{\varepsilon,x_0} \, ds_x + t_0^2 K_1 \delta_\varepsilon p(x_0) A_1 \\
&\quad + t_0^2 \frac{K_1}{2} p(x_0) A_1 \delta_\varepsilon^2 - H(x_0) t_0^2 p(x_0) A'_2 \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|).
\end{aligned}$$

On the other hand, we apply (4.3.25) with $x = u$ and $y = W_{\varepsilon, x_0}$, we obtain

$$\begin{aligned}
& t_\varepsilon \int_{\partial\Omega} p(x)Q(x)|u(x)|^2 u(x) W_{\varepsilon, x_0} ds_x \\
& \leq \frac{1}{4} \left[1 - \int_{\partial\Omega} p(x)Q(x)|u(x)|^4 ds_x - t_\varepsilon^4 \int_{\partial\Omega} p(x)Q(x)|W_{\varepsilon, x_0}|^4 ds_x \right] \\
& \quad - t_\varepsilon^3 \int_{\partial\Omega} p(x)Q(x)u(x)|W_{\varepsilon, x_0}|^3 ds_x \\
& \leq \frac{1}{4} \left[1 - \int_{\partial\Omega} p(x)Q(x)|u(x)|^4 ds_x - t_0^4(1 + 4\delta_\varepsilon + 6\delta_\varepsilon^2)p(x_0)Q(x_0)(B_1 - B_2\varepsilon) \right] \\
& \quad - t_\varepsilon^3 \int_{\partial\Omega} p(x)Q(x)u(x)|W_{\varepsilon, x_0}|^3 ds_x + o(\varepsilon) + o(\delta_\varepsilon^2).
\end{aligned}$$

Using the expression of t_0 given by (4.3.12) we get

$$\begin{aligned}
t_\varepsilon \int_{\partial\Omega} p(x)Q(x)|u(x)|^2 u(x) W_{\varepsilon, x_0} ds_x & \leq -t_0^4 p(x_0)Q(x_0)B_1\delta_\varepsilon - \frac{3}{2}t_0^4 p(x_0)Q(x_0)B_1\delta_\varepsilon^2 + t_0^4 B_2\varepsilon \\
& \quad - t_\varepsilon^3 \int_{\partial\Omega} p(x)Q(x)u(x)|W_{\varepsilon, x_0}|^3 ds_x + o(\delta_\varepsilon^2) + o(\varepsilon).
\end{aligned}$$

Moreover we have (see appendix)

$$(4.3.32) \quad \int_{\partial\Omega} p(x)Q(x)u(x)|W_{\varepsilon, x_0}|^3 ds_x = o(\varepsilon|\log\varepsilon|).$$

Therefore

$$\begin{aligned}
(4.3.33) \quad t_\varepsilon \int_{\partial\Omega} p(x)Q(x)|u(x)|^2 u(x) W_{\varepsilon, x_0} ds_x & \leq -t_0^4 p(x_0)Q(x_0)B_1\delta_\varepsilon - \frac{3}{2}t_0^4 p(x_0)Q(x_0)B_1\delta_\varepsilon^2 \\
& \quad + o(\delta_\varepsilon^2) + o(\varepsilon|\log\varepsilon|).
\end{aligned}$$

Inserting (4.3.33) into (4.3.31), we obtain

$$\begin{aligned}
J(u + t_\varepsilon W_{\varepsilon, x_0}) & \leq I_0 - \left(\frac{3K_1}{2} K_2 t_0^4 p(x_0)Q(x_0)B_1 - t_0^2 \frac{K_1}{2} p(x_0)A_1 \right) \delta_\varepsilon^2 + o(\delta_\varepsilon^2) \\
& \quad - H(x_0)t_0^2 p(x_0)A_2' \varepsilon |\log\varepsilon| + o(\varepsilon|\log\varepsilon|).
\end{aligned}$$

Using the expression of K_1 and t_0 we deduce that

$$\begin{aligned}
J(u + t_\varepsilon W_{\varepsilon, x_0}) & \leq I_0 - K_1 t_0^2 p(x_0)A_1 \delta_\varepsilon^2 + o(\delta_\varepsilon^2) - H(x_0)t_0^2 p(x_0)A_2' \varepsilon |\log\varepsilon| \\
& \quad + o(\varepsilon|\log\varepsilon|).
\end{aligned}$$

Since $H(x_0) > 0$, we conclude that for $\varepsilon > 0$ small enough we have $J(u + t_\varepsilon W_{\varepsilon, x_0}) < I_0$, which gives a contradiction in this case.

This ends the proof of Theorem 4.3.1. \square

Now we return to the proof of Theorem 4.1.1. As announced in the introduction, we follow the steps of [T] in studying a Dirichlet problem with critical Sobolev exponent.

We consider the functional

$$I(u) = \frac{1}{2} \int_{\Omega} p(x) |\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \int_{\Omega} f(x) u dx - \frac{1}{q} \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x, \quad u \in H^1(\Omega).$$

We define the following manifolds

$$\begin{aligned} \Lambda &= \{u \in H^1(\Omega) : \langle I'(u), u \rangle = 0\}, \\ \Lambda^+ &= \left\{ u \in \Lambda : \int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx - (q-1) \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x > 0 \right\}, \\ \Lambda_0 &= \left\{ u \in \Lambda : \int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx - (q-1) \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x = 0 \right\}, \\ \Lambda^- &= \left\{ u \in \Lambda : \int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx - (q-1) \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x < 0 \right\}. \end{aligned}$$

We start by studying the existence of a first solution of (4.3.1) which will be a local minimum of I on Λ . We have the following

Theorem 4.3.2.

Let $\Omega \in \mathbb{R}^N$, $N \geq 3$ be a bounded domain with a smooth boundary $\partial\Omega$ satisfying (g.c) at x_0 and $H(x_0) > 0$. We assume that f satisfies (H1) and the functions p and Q respectively satisfy (4.1.2) and (4.1.3). Then for every $\lambda < 0$,

$$(4.3.34) \quad c_0 = \inf_{\Lambda} I$$

is achieved

In order to prove Theorem 4.3.2, we need some Lemmas

Lemma 4.3.1.

Let f satisfy (H1). For every $u \in H^1(\Omega) \setminus \{0\}$ such that $u \not\equiv 0$ in $\partial\Omega$, there exists a unique $t^+ = t^+(u) > 0$ such that $t^+u \in \Lambda^-$. In particular:

$$t^+ > \left[\frac{\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx}{(q-1) \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x} \right]^{\frac{1}{q-2}} := t_{\max}$$

and $I(t^+u) = \max_{t \geq t_{\max}} I(tu)$

Moreover, if $\int_{\Omega} f(x) u dx > 0$, then there exists a unique $t^- = t^-(u) > 0$ such that $t^-u \in \Lambda^+$.

In particular,

$$t^- < \left[\frac{\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx}{(q-1) \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x} \right]^{\frac{1}{q-2}}$$

and $I(t^-u) \leq I(tu)$, $\forall t \in [0, t^+]$.

Proof.

Set $\varphi(t) = t \left[\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right] - t^{q-1} \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x$.

It is easy to see that φ is concave and achieves its maximum at

$$t_{\max} = \left[\frac{\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx}{(q-1) \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x} \right]^{\frac{1}{q-2}}.$$

Moreover

$$\varphi(t_{\max}) = (q-2) \left[\frac{1}{q-1} \right]^{\frac{q-1}{q-2}} \left[\frac{(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx)^{q-1}}{\int_{\partial\Omega} p(x) Q(x) |u|^q ds_x} \right]^{\frac{1}{q-2}}$$

and

$$\varphi(t_{\max}) = \sigma_N \frac{(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx)^{\frac{N}{2}}}{(\int_{\partial\Omega} p(x) Q(x) |u|^q ds_x)^{N-1}}$$

with

$$(4.3.35) \quad \sigma_N = (q-2) \left[\frac{1}{q-1} \right]^{\frac{q-1}{q-2}}.$$

Therefore if $\int_{\Omega} f(x)u dx \leq 0$ then there exists a unique $t^+ > t_{\max}$ such that

$$\varphi(t^+) = \int_{\Omega} f(x)u dx \quad \text{and} \quad \varphi'(t^+) < 0.$$

This gives that $t^+u \in \Lambda^-$ and $I(t^+u) \geq I(tu) \forall t \geq t_{\max}$.

In case $\int_{\Omega} f(x)u dx > 0$, by assumption (*) of (H1) we have that necessarily $\int_{\Omega} f(x)u dx < \varphi(t_{\max})$ and there exist t^- and t^+ unique such that

$$0 < t^- < t_{\max} < t^+, \quad \varphi(t^+) = \int_{\Omega} f(x)u dx = \varphi(t^-) \quad \text{and} \quad \varphi'(t^-) > 0 > \varphi'(t^+).$$

This implies that $t^+u \in \Lambda^-$ and $t^-u \in \Lambda^+$. Moreover $I(t^+u) \geq I(tu), \forall t \geq t^-$ and $I(t^-u) \leq I(tu), \forall t \in [0, t^+]$. \square

For $u \not\equiv 0$ on $\partial\Omega$, let

$$(4.3.36) \quad \psi(u) = \sigma_N \frac{\left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right)^{\frac{N}{2}}}{\left[\left(\int_{\partial\Omega} p(x) Q(x) |u|^q ds_x \right)^{\frac{1}{q}} \right]^{N-1}} - \int_{\Omega} f(x)u dx.$$

Let $t > 0$ and $\int_{\partial\Omega} p(x) Q(x) |u|^q ds_x = 1$ we have:

$$\psi(tu) = t \left[\sigma_N \left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right)^{\frac{N}{2}} - \int_{\Omega} f(x)u dx \right],$$

given $\gamma > 0$, from Theorem 4.3.1 we deduce that

$$(4.3.37) \quad \inf_{\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x\right)^{\frac{1}{q}} \geq \gamma} \psi(u) \geq \gamma I_0 > 0.$$

Lemma 4.3.2.

For every $u \in \Lambda \setminus \{0\}$, $u \not\equiv 0$ on $\partial\Omega$ we have

$$\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx - (q-1) \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \neq 0$$

Proof.

Arguing by contradiction assume that for $u \in \Lambda \setminus \{0\}$, $u \not\equiv 0$ on $\partial\Omega$ we have

$$(4.3.38) \quad \int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx - (q-1) \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x = 0.$$

Combining the last equality with the fact that $u \in \Lambda$, we write

$$(4.3.39) \quad 0 = (q-2) \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x - \int_{\Omega} f(x) u dx$$

Condition (4.3.38) gives that

$$\begin{aligned} \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x &= \frac{1}{q-1} \left(\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) \\ &\geq \frac{M_1}{q-1} \left(\int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |u|^2 dx \right) = \frac{M_1}{q-1} \|u\|^2 \end{aligned}$$

where $M_1 = \min \left(\min_{x \in \Omega} p(x), \frac{-\lambda}{\min_{x \in \Omega} p(x)} \right) > 0$.

Using the embedding of $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ there exists a positive constant M_2 such that

$$\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \geq M_2 \left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \right)^{\frac{2}{q}}.$$

Thus

$$\left(\int_{\partial\Omega} p(x)Q(x)|u|^q ds_x \right)^{\frac{1}{q}} \geq M_2^{\frac{1}{q-2}} := \gamma.$$

Therefore, from (4.3.37) and (4.3.39) we obtain

$$0 < I_0\gamma \leq$$

$$\begin{aligned} \psi(u) &= \left[\frac{1}{q-1} \right]^{\frac{q-1}{q-2}} (q-2) \left[\frac{\left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right)^{\frac{q-1}{q-2}}}{\int_{\partial\Omega} p(x) Q(x) |u|^q ds_x} \right]^{\frac{1}{q-2}} - \int_{\Omega} f(x) u dx \\ &\leq (q-2) \left(\left[\frac{1}{q-1} \right]^{\frac{q-1}{q-2}} \left[\frac{\left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right)^{\frac{q-1}{q-2}}}{\int_{\partial\Omega} p(x) Q(x) |u|^q ds_x} \right]^{\frac{1}{q-2}} - \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x \right) \\ &\leq (q-2) \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x \left(\left[\frac{\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx}{(q-1) \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x} \right]^{\frac{q-1}{q-2}} - 1 \right) = 0 \end{aligned}$$

which yields a contradiction. \square

As a consequence of Lemma 4.3.2 we have the

Lemma 4.3.3.

Given $u \in \Lambda \setminus \{0\}$, $u \not\equiv 0$ on $\partial\Omega$ there exist $\varepsilon > 0$ and a differentiable function $t = t(w) > 0$, $w \in H^1(\Omega)$ and $\|w\| < \varepsilon$ satisfying

$$t(0) = 1, \quad t(w)(u - w) \in \Lambda, \quad \text{for } \|w\| < \varepsilon,$$

and

(4.3.40)

$$\begin{aligned} \langle t'(0), w \rangle &= \\ &= \frac{2 \left(\int_{\Omega} p(x) \nabla u \cdot \nabla w dx - \lambda \int_{\Omega} u w dx \right) - \int_{\Omega} f(x) w dx - q \int_{\partial\Omega} p(x) Q(x) |u|^{q-2} u w ds_x}{\left[\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx - (q-1) \int_{\partial\Omega} |u|^q ds_x \right]}. \end{aligned}$$

Proof.

Define $G : \mathbb{R} \times H^1(\Omega) \rightarrow \mathbb{R}$ as follows

$$\begin{aligned} G(t, w) &= t \left(\int_{\Omega} p(x) |\nabla(u-w)|^2 dx - \lambda \int_{\Omega} |u-w|^2 dx \right) - t^{q-1} \int_{\partial\Omega} p(x) Q(x) |u-w|^q ds_x \\ &\quad - \int_{\Omega} f(x) (u-w) dx. \end{aligned}$$

Since $G(1, 0) = 0$ and

$$\frac{\partial G}{\partial t}(1, 0) = \left(\int_{\Omega} p(x) |\nabla(u)|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) - (q-1) \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x \neq 0$$

by Lemma 4.3.2, we can apply the Implicit Function Theorem at the point $(1, 0)$ and get the desired result. \square

We are now in a position to give the

Proof of Theorem 4.3.2

We start by showing that I is bounded from below in Λ .

Let $u \in \Lambda$, we have

$$\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx - \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x - \int_{\Omega} f(x) u dx = 0.$$

Thus

$$\begin{aligned} I(u) &= \frac{1}{2(N-1)} \left(\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) - \frac{N}{2(N-1)} \int_{\Omega} f(x) u dx \\ &\geq \frac{1}{2(N-1)} \left(\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) - \frac{N}{2(N-1)} \|f\|_{H^{-1}} \|u\| \\ &\geq \frac{M}{2(N-1)} \|u\|^2 - \frac{N}{2(N-1)} \|f\|_{H^{-1}} \|u\| \end{aligned}$$

with M is a positive constant.

Consequently

$$(4.3.41) \quad c_0 \geq -\frac{N^2}{8(N-1)} M \|f\|_{H^{-1}}^2.$$

In order to obtain an upper bound for c_0 , let $v \in H^1(\Omega)$ be a solution for the following Neumann problem

$$\begin{cases} -\operatorname{div}(p(x)\nabla v) = \lambda v + f(x) & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us remark that, since f satisfies (H1), namely (4.1.4), then $v \not\equiv 0$ on $\partial\Omega$.

We have

$$\int_{\Omega} f(x) v dx = \int_{\Omega} p(x)|\nabla v|^2 dx - \lambda \int_{\Omega} |v|^2 dx.$$

Let $t_0 = t^-(v) > 0$ as defined by Lemma 4.3.1, we write

$$\begin{aligned} I(t_0 v) &= \frac{t_0^2}{2} \left(\int_{\Omega} p(x)|\nabla v|^2 dx - \lambda \int_{\Omega} |v|^2 dx \right) - \frac{t_0^q}{q} \int_{\partial\Omega} p(x)Q(x)|v|^q ds_x \\ &\quad - t_0 \left(\int_{\Omega} p(x)|\nabla v|^2 dx - \lambda \int_{\Omega} |v|^2 dx \right). \end{aligned}$$

Using the fact that $t_0 v \in \Lambda$ we obtain

$$I(t_0 v) = -\frac{t_0^2}{2} \left(\int_{\Omega} p(x)|\nabla v|^2 dx - \lambda \int_{\Omega} |v|^2 dx \right) + \frac{q-1}{q} t_0^q \int_{\partial\Omega} p(x)Q(x)|v|^q ds_x.$$

Since $t_0 v \in \Lambda^+$, from the above inequality we deduce

$$I(t_0 v) < -\frac{t_0^2}{2(N-1)} \left(\int_{\Omega} p(x)|\nabla v|^2 dx - \lambda \int_{\Omega} |v|^2 dx \right) = -\frac{t_0^2}{2(N-1)} \|f\|_{H^{-1}}^2$$

This implies that

$$(4.3.42) \quad c_0 < -\frac{t_0^2}{2(N-1)} \|f\|_{H^{-1}}^2.$$

Applying Ekeland's Variational principle (see [AE]) to the minimization problem (4.3.34), we obtain a minimizing sequence $\{u_n\} \subset \Lambda$ with the following properties

$$(4.3.43) \quad I(u_n) < c_0 + \frac{1}{n}$$

$$(4.3.44)$$

$$I(w) \geq I(u_n) - \frac{1}{n} \left(\int_{\Omega} p(x) |\nabla(w - u_n)|^2 dx - \lambda \int_{\Omega} |w - u_n|^2 dx \right)^{\frac{1}{2}}, \quad \forall w \in \Lambda.$$

Combining (4.3.42) and (4.3.43), we obtain for n large

$$(4.3.45) \quad \begin{aligned} I(u_n) &= \frac{1}{2(N-1)} \left(\int_{\Omega} p(x) |\nabla u_n|^2 dx - \lambda \int_{\Omega} |u_n|^2 dx \right) - \frac{N}{2(N-1)} \int_{\Omega} f(x) u_n dx \\ &< -\frac{t_0^2}{2(N-1)} \|f\|_{H^{-1}}^2 \end{aligned}$$

This yields

$$(4.3.46) \quad \int_{\Omega} f(x) u_n dx \geq \frac{t_0^2}{N} \|f\|_{H^{-1}}^2 > 0.$$

Consequently $u_n \not\equiv 0$ in Ω . Since $\lambda < 0$, putting together (4.3.45) and (4.3.46) we derive

$$(4.3.47) \quad \frac{t_0^2}{N} \|f\|_{H^{-1}} \leq \|u_n\| \leq \frac{M}{2} \|f\|_{H^{-1}}$$

with M is a positive constant.

We now prove that $\|I'(u_n)\| \rightarrow 0$ as $n \rightarrow +\infty$.

We argue by contradiction and assume that $\|I'(u_n)\| \geq \alpha > 0$, with α a constant.

Applying Lemma 4.3.3 with $u = u_n$ and $w = \delta \frac{I'(u_n)}{\|I'(u_n)\|}$, with $\delta > 0$ a small constant, there exists $t_n(\delta) := t \left(\delta \frac{I'(u_n)}{\|I'(u_n)\|} \right)$ such that

$$w_\delta = t_n(\delta) \left[u_n - \delta \frac{I'(u_n)}{\|I'(u_n)\|} \right] \in \Lambda.$$

From (4.3.44) we have

$$\frac{1}{n} \left(\int_{\Omega} p(x) |\nabla(w_\delta - u_n)|^2 dx - \lambda \int_{\Omega} |w_\delta - u_n|^2 dx \right)^{\frac{1}{2}} \geq I(u_n) - I(w_\delta).$$

Using Taylor's expansion we obtain

$$\begin{aligned} \frac{1}{n} \left(\int_{\Omega} p(x) |\nabla(w_\delta - u_n)|^2 dx - \lambda \int_{\Omega} |w_\delta - u_n|^2 dx \right)^{\frac{1}{2}} &\geq \langle I'(w_\delta), u_n - w_\delta \rangle + o(\delta) \\ &\geq (1 - t_n(\delta)) \langle I'(w_\delta), u_n \rangle + \delta t_n(\delta) \langle I'(w_\delta), \frac{I'(u_n)}{\|I'(u_n)\|} \rangle + o(\delta). \end{aligned}$$

Dividing by $\delta > 0$ and passing to the limit as $\delta \rightarrow 0$ we derive

$$\frac{1}{n} \left[1 + |t'_n(0)| \left(\int_{\Omega} p(x) |\nabla u_n|^2 dx - \lambda \int_{\Omega} |u_n|^2 dx \right)^{\frac{1}{2}} \right] \geq -t'_n(0) \langle I'(u_n), u_n \rangle + \|I'(u_n)\|$$

where we have set $t'_n = \langle t'(0), \frac{I'(u_n)}{\|I'(u_n)\|} \rangle$.

Since $u_n \in \Lambda$ we obtain

$$\frac{1}{n} \left[1 + |t'_n(0)| \left(\int_{\Omega} p(x) |\nabla(u_n)|^2 dx - \lambda \int_{\Omega} |u_n|^2 dx \right)^{\frac{1}{2}} \right] \geq \|I'(u_n)\|.$$

Using (4.3.47) we conclude

$$(4.3.48) \quad \|I'(u_n)\| \leq \frac{C}{n} (1 + |t'_n(0)|)$$

where C is a positive constant.

Now, to conclude it remains to show that $|t'_n(0)|$ is uniformly bounded in n .

From (4.3.40) and the estimate (4.3.47) we obtain

$$|t'_n(0)| \leq \frac{C_1}{\left| \int_{\Omega} p(x) |\nabla u_n|^2 dx - \lambda \int_{\Omega} |u_n|^2 dx - (q-1) \int_{\partial\Omega} p(x) Q(x) |u_n|^q ds_x \right|}$$

where C_1 is a positive constant.

Hence we need to show that

$$\left| \int_{\Omega} p(x) |\nabla u_n|^2 dx - \lambda \int_{\Omega} |u_n|^2 dx - (q-1) \int_{\partial\Omega} p(x) Q(x) |u_n|^q ds_x \right|$$

is bounded away from zero.

Arguing by contradiction, assume that for a subsequence (still denoted by u_n) we have

$$(4.3.49) \quad \int_{\Omega} p(x) |\nabla u_n|^2 dx - \lambda \int_{\Omega} |u_n|^2 dx - (q-1) \int_{\partial\Omega} p(x) Q(x) |u_n|^q ds_x = o(1).$$

Using (4.3.47) we obtain

$$\int_{\partial\Omega} p(x) Q(x) |u_n|^q ds_x \geq \gamma \quad (\gamma > 0 \text{ is a suitable constant})$$

and

$$\left[\frac{\int_{\Omega} p(x) |\nabla u_n|^2 dx - \lambda \int_{\Omega} |u_n|^2 dx}{q-1} \right]^{\frac{q-1}{q-2}} - \left[\int_{\partial\Omega} p(x) Q(x) |u_n|^q ds_x \right]^{\frac{q-1}{q-2}} = o(1).$$

On the other hand, combining (4.3.49) with the fact that $u_n \in \Lambda$ we see that

$$\int_{\Omega} f(x) u_n dx = (q-2) \int_{\partial\Omega} p(x) Q(x) |u_n|^q ds_x + o(1).$$

Finally, using the above together with (4.3.35), (4.3.36) and (4.3.37) we are led to

$$\begin{aligned}
0 &< I_0 \gamma^N \leq \left[\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x \right]^{\frac{1}{q-2}} \psi(u_n) \\
&\leq (q-2) \left[\left(\frac{\int_{\Omega} p(x)|\nabla u_n|^2 dx - \lambda \int_{\Omega} |u_n|^2 dx}{q-1} \right)^{\frac{q-1}{q-2}} - \left(\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x \right)^{\frac{q-1}{q-2}} \right] \\
&= o(1)
\end{aligned}$$

which is impossible and we conclude that

$$(4.3.50) \quad \|I'(u_n)\| \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Let $u_0 \in H^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H^1(\Omega)$.

From (4.3.50) we deduce that

$$\langle I'(u_0), w \rangle = 0, \quad \forall w \in H^1(\Omega)$$

and thus $u_0 \in \Lambda$ is a weak solution of (4.3.1).

Therefore

$$\begin{aligned}
c_0 \leq I(u_0) &= \frac{1}{2(N-1)} \left(\int_{\Omega} p(x)|\nabla u_0|^2 dx - \lambda \int_{\Omega} |u_0|^2 dx \right) - \int_{\Omega} f(x) u_0 dx \\
&\leq \lim_{n \rightarrow +\infty} I(u_n) = c_0.
\end{aligned}$$

Consequently $u_n \rightarrow u_0$ strongly in $H^1(\Omega)$ and also in $L^q(\partial\Omega)$.

Thus $\inf_{\Lambda} I = c_0 = I(u_0)$ and from Lemma 4.3.1 and (4.3.46) we deduce that $u_0 \in \Lambda^+$. Now we claim that u_0 is a local minimum for I . Indeed we know from Lemma 4.3.1 that for every $u \in H^1(\Omega)$ with $u \not\equiv 0$ on $\partial\Omega$ and $\int_{\Omega} f(x) u dx > 0$ there exists $t^- > 0$ such that

$$(4.3.51) \quad I(tu) \geq I(t^-u) \quad \text{for every } 0 < t < \left(\frac{\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx}{(q-1) \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x} \right)^{\frac{1}{q-2}}.$$

Since $u_0 \in \Lambda^+$,

$$(4.3.52) \quad t^- = 1 < \left(\frac{\int_{\Omega} p(x)|\nabla u|^2 dx - \lambda \int_{\Omega} |u_0|^2 dx}{(q-1) \int_{\partial\Omega} p(x)Q(x)|u_0|^q ds_x} \right)^{\frac{1}{q-2}}.$$

Let $\varepsilon > 0$ sufficiently small so that

$$1 < \frac{\int_{\Omega} p(x)|\nabla(u_0 - w)|^2 dx - \lambda \int_{\Omega} |u_0 - w|^2 dx}{(q-1) \int_{\partial\Omega} p(x)Q(x)|u_0 - w|^q ds_x} \quad \text{for } \|w\| < \varepsilon.$$

From Lemma 4.3.3, let $t(w) > 0$ such that $t(w)(u_0 - w) \in \Lambda$ for every $\|w\| < \varepsilon$.

Since $t(w) \rightarrow 1$ as $\|w\| \rightarrow 0$, we can always assume that

$$t(w) < \left(\frac{\int_{\Omega} p(x)|\nabla(u_0 - w)|^2 dx - \lambda \int_{\Omega} |u_0 - w|^2 dx}{(q-1) \int_{\partial\Omega} p(x)Q(x)|u_0 - w|^q ds_x} \right)^{\frac{1}{q-2}} \quad \text{for every } w : \|w\| < \varepsilon.$$

Moreover $t(w)(u_0 - w) \in \Lambda^+$ and we have

$$I(s(u_0 - w)) \geq I(t(w)(u_0 - w)) \geq I(u_0)$$

$$\text{for evry } 0 < s < \left(\frac{\int_{\Omega} p(x) |\nabla(u_0 - w)|^2 dx - \lambda \int_{\Omega} |u_0 - w|^2 dx}{(q-1) \int_{\partial\Omega} p(x) Q(x) |u_0 - w|^q ds_x} \right)^{\frac{1}{q-2}}.$$

From (4.3.52) we can take $s = 1$ and we obtain the desired conclusion.

This completes the proof. We now turn to the study of the existence of a second solution of problem (4.3.1). Then we are led to investigate a second minimization problem. Namely

$$(4.3.53) \quad c_1 = \inf_{\Lambda^-} I.$$

We have the following result

Theorem 4.3.3.

Let $\Omega \in \mathbb{R}^N$, $N \geq 3$ be a bounded domain with a smooth boundary $\partial\Omega$ satisfying (g.c) at x_0 and $H(x_0) > 0$. We assume that f satisfies (H1) and that the functions p and Q respectively satisfy (4.1.2) and (4.1.3). Then for every $\lambda < 0$ the infimum in (4.3.53) is achieved and we have $c_1 > c_0$.

In order to prove this last result we start by the following lemma

Lemma 4.3.4.

Let $\{u_n\} \subset H^1(\Omega)$ be a sequence satisfying

$$(4.3.54) \quad I(u_n) \rightarrow c < c_0 + \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}$$

and

$$(4.3.55) \quad I'(u_n) \rightarrow 0 \quad \text{in } H^{-1}(\Omega)$$

then $\{u_n\}$ is relatively compact in $H^1(\Omega)$.

Proof.

We start by showing that $\{u_n\}$ is bounded in $H^1(\Omega)$. Arguing by contradiction, we assume that $\|u_n\| \rightarrow \infty$. We set $v_n = \frac{u_n}{\|u_n\|}$. Since $\{v_n\}$ is bounded in $H^1(\Omega)$, we may assume that $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$.

Using (4.3.55), for every $\phi \in H^1(\Omega)$, we have

$$(4.3.56) \quad \int_{\Omega} (p(x) \nabla v_n \nabla \phi - \lambda v_n \phi) dx = \|u_n\|^{-1} \left[\int_{\partial\Omega} p(x) Q(x) |u_n|^{q-2} u_n \phi ds_x + \int_{\Omega} f(x) \phi dx \right] + o(1).$$

On the one hand, we have

$$(4.3.57) \quad \int_{\partial\Omega} p(x)Q(x)|u_n|^{q-2}u_n\phi \, ds_x \leq C_1 \left(\int_{\partial\Omega} |u_n|^q ds_x \right)^{\frac{q-1}{q}} \left(\int_{\partial\Omega} |\phi|^q ds_x \right)^{\frac{1}{q}}.$$

On the other hand, we compute (4.3.54) $- \frac{1}{2}$ (4.3.55), which gives

$$\frac{1}{2(N-1)} \int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x - \frac{1}{2} \int_{\Omega} f(x) u_n \, dx \leq c + o(\|u_n\|).$$

Therefore

$$\begin{aligned} \int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x &\leq M_1 [1 + \|f\|_{H^{-1}}\|u_n\|] + o(\|u_n\|) \\ \left(\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x \right)^{\frac{q-1}{q}} &\leq M_2^{\frac{q-1}{q}} \|f\|_{H^{-1}}^{\frac{q-1}{q}} \|u_n\|^{\frac{q-1}{q}} \left[1 + \frac{1}{\|u_n\|} + o(1) \right]^{\frac{q-1}{q}} \end{aligned}$$

where M_1, M_2 are some positive constants.

Since $\|u_n\| \rightarrow \infty$, we have

$$\begin{aligned} \left(\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x \right)^{\frac{q-1}{q}} &\leq M_2^{\frac{q-1}{q}} \|f\|_{H^{-1}}^{\frac{q-1}{q}} \|u_n\|^{\frac{q-1}{q}} + \frac{q-1}{q} M_2^{\frac{q-1}{q}} \|f\|_{H^{-1}}^{\frac{q-1}{q}} \|u_n\|^{\frac{-1}{q}} \\ &\quad + o(\|u_n\|^{\frac{q-1}{q}}) \end{aligned}$$

and

$$(4.3.58) \quad \|u_n\|^{-1} \left(\int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x \right)^{\frac{q-1}{q}} \leq M^{\frac{q-1}{q}} \|f\|_{H^{-1}}^{\frac{q-1}{q}} \|u_n\|^{\frac{-1}{q}} + M^{\frac{q-1}{q}} \|f\|_{H^{-1}}^{\frac{q-1}{q}} \|u_n\|^{\frac{-(q+1)}{q}} + o(\|u_n\|^{\frac{-1}{q}}).$$

Combining (4.3.57) with (4.3.58) we obtain

$$(4.3.59) \quad \|u_n\|^{-1} \left[\int_{\partial\Omega} Q(x)|u_n|^{q-2}u_n\phi \, ds_x + \int_{\Omega} f(x) \phi \, dx \right] \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Inserting (4.3.59) into (4.3.56), letting $n \rightarrow \infty$, we deduce that

$$\int_{\Omega} (p(x)\nabla v \nabla \phi - \lambda v \phi) \, dx = 0$$

for every $\phi \in H^1(\Omega)$. Since $\lambda < 0$ we see that $v \equiv 0$ on Ω . Consequently, we may assume that $v_n \rightarrow 0$ strongly in $L^2(\Omega)$. Hence, from (4.3.54) and (4.3.55), we can write

$$\frac{1}{2} \int_{\Omega} p(x)|\nabla v_n|^2 dx = \frac{\|u_n\|^{q-2}}{q} \int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x + o(1)$$

and

$$\int_{\Omega} p(x)|\nabla v_n|^2 dx = \|v_n\|^{q-2} \int_{\partial\Omega} p(x)Q(x)|v_n|^q ds_x + o(1).$$

These two relations imply that $\int_{\Omega} p(x)|\nabla v_n|^2 dx = o(1)$ and thus $v_n \rightarrow 0$ strongly in $H^1(\Omega)$, which is impossible since $\|v_n\| = 1$. Consequently $\{u_n\}$ is bounded in $H^1(\Omega)$ and we may assume that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$.

Hence, by going to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$.

By the concentration-compactness principle [L] we have

$$|\nabla u_n|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \sum_{j \in J} \mu_j \delta_{x_j}$$

and

$$|u_n|^q \rightharpoonup |u|^q + \sum_{j \in J} \nu_j \delta_{x_j}$$

weakly in the sense of measures, where $x_j \in \partial\Omega$, μ_j and ν_j are some positive constants satisfying

$$(4.3.60) \quad S_1 \nu_j^{\frac{2}{q}} \leq \mu_j$$

and J is a countable set.

Testing (4.3.55) by a family of C^1 -functions concentrating at x_j with some $j \in J$, we get

$$(4.3.61) \quad p(x_j)\mu_j \leq p(x_j)Q(x_j)\nu_j.$$

We claim that $\nu_j = 0$ for every $j \in J$, otherwise there exists $j_0 \in J$ such that $\nu_{j_0} > 0$ and from (4.3.60) and (4.3.61) we have

$$(4.3.62) \quad \frac{S_1^{N-1}}{Q(x_{j_0})^{N-1}} \leq \nu_{j_0}.$$

On the other hand we have

$$I(u_n) - \frac{1}{2} \langle I'(u_n), u_n \rangle = \frac{1}{2(N-1)} \int_{\partial\Omega} p(x)Q(x)|u_n|^q ds_x - \frac{1}{2} \int_{\Omega} f(x) u_n dx + o(1).$$

Letting $n \rightarrow +\infty$ we obtain

$$c = \frac{1}{2(N-1)} \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x - \frac{1}{2} \int_{\Omega} f(x) u dx + \frac{1}{2(N-1)} \sum_{j \in J} p(x_j)Q(x_j)\nu_j.$$

The fact that u is a weak solution of problem (4.3.1) gives

$$c \geq I(u) + \frac{1}{2(N-1)} p(x_{j_0})Q(x_{j_0})\nu_{j_0}.$$

Using (4.3.62) and the definition of c_0 we obtain

$$c \geq c_0 + \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}$$

which contradicts (4.3.54). Hence $\nu_j = 0$ for every $j \in J$. This yields that $u_n \rightarrow u$ strongly in $L^q(\partial\Omega)$. Easily we deduce that $u_n \rightarrow u$ strongly in $H^1(\Omega)$, which is the desired conclusion. \square

Analogously to the proof of Theorem 4.3.2, the Ekeland's Variational Principle (see [AE], Corollary 7.5.3) applied to minimizing problem (4.3.53) gives a sequence $\{u_n\} \subset \Lambda^-$ satisfying $I(u_n) \rightarrow c_1$ and $\|I'(u_n)\| \rightarrow 0$. Then the conclusion of Theorem follows from Lemma 4.3.4 if we have

$$(4.3.63) \quad c_1 < c_0 + \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}.$$

However it appears difficult to derive (4.3.63) directly. Then, as in [T], we shall obtain it by comparison with a mountain-pass value.

Let (\mathcal{F}) be the class of continuous paths joining u_0 to $u_0 + R_0U_{\varepsilon, x_0}$, with $R_0 > 0$ fixed. We have the following result

Lemma 4.3.5.

For a suitable choice of $R_0 > 0$ and $\varepsilon > 0$, the value

$$c = \inf_{h \in \mathcal{F}} \max_{t \in [0,1]} I(h(t))$$

defines a critical value for I , and $c \geq c_1$.

The proof of this last Lemma relies on the following variant of the Mountain Pass Theorem of Ambroseti and Rabinowitz [AR]

Theorem 4.3.4.

Let Φ be a C^1 function on a Banach space E . Suppose that :

$$(4.3.64)$$

There exists a neighborhood U of 0 in E and a constant ρ such that $\Phi(u) \geq \rho$ for every u in the boundary of U ,

$$(4.3.65) \quad \Phi(0) < \rho \quad \text{and} \quad \Phi(v) < \rho \quad \text{for some } v \notin U.$$

Set

$$(4.3.66) \quad c = \inf_{\mathcal{P} \in \mathcal{A}} \max_{w \in \mathcal{P}} \Phi(w) \geq \rho,$$

where \mathcal{A} denotes the class of continuous paths joining 0 to v .

Conclusion :

There exists a sequence $\{u_j\}$ in E such that

$$\Phi(u_j) \rightarrow c \quad \text{and} \quad \Phi'(u_j) \rightarrow 0 \quad \text{in } E^*.$$

Lemma 4.3.6.

For every $R > 0$, there exists $\varepsilon_0 = \varepsilon_0(R) > 0$ such that

$$I(u_0 + RW_{\varepsilon, x_0}) < c_0 + \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}$$

for every $0 < \varepsilon < \varepsilon_0$

Proof.

The proof is similar to the proof of (4.3.11) in Theorem 4.3.1. \square

Now, we are in position to give the

Proof of Lemma 4.3.5

We start by verifying the assumptions of the Theorem 4.3.4 with u_0 in the place of 0 , $\Psi = I - I(u_0)$, $E = H^1(\Omega)$ and $\mathcal{A} = \mathcal{F}$.

Verification of (4.3.64):

Let $u \in H^1(\Omega)$. For t small enough we have

$$\begin{aligned} I(u_0 + tu) - I(u_0) &= t\langle I'(u_0), u \rangle + t^2\langle I''(u_0)u, u \rangle + o(t^2) \\ &= t^2\langle I''(u_0)u, u \rangle + o(t^2). \end{aligned}$$

Since u_0 is a local minimum for I we deduce (4.3.64) with some $\rho > 0$ (and U a small ball centered at u_0 in H^1).

Verification of (4.3.65):

From the proof of Theorem 4.3.3, we have $I(u_0) = c_0 < 0$. On the other hand, it is easy to see that, for any $u \in H^1(\Omega) \setminus \{0\}$, $u \not\equiv 0$ on $\partial\Omega$, we have $\lim_{t \rightarrow +\infty} I(u_0 + tu) = -\infty$. Thus, there are many v 's satisfying (4.3.65). However, it will be important for later purpose to use Theorem 4.3.4 with a special v , namely $v = u_0 + R_0W_{\varepsilon, x_0}$, where $R_0 > 0$ is chosen large enough so that $v \notin U$ and $I(v) \leq 0$.

Applying Theorem 4.3.4 we obtain a sequence $\{u_j\}$ in $H^1(\Omega)$ such that $I(u_j) \rightarrow c$ and $I'(u_j) \rightarrow 0$ in $H^{-1}(\Omega)$.

Combining the results of Lemma 4.3.6 and Lemma 4.3.4 we deduce that c is a critical value for I .

Now, we will prove that $c \geq c_1 = \inf_{\Lambda^-} I(u)$. The proof of this last fact is based on two claims

Claim 1: Λ^- disconnects $H^1(\Omega)$ in exactly two connected components U_1 and U_2 .

Indeed, from Lemma 4.3.1, for every $u \in H^1(\Omega)$ there exists a unique $t^+(u) > 0$ such that $t^+(u)u \in \Lambda^-$ and $I(t^+(u)u) = \max_{t \geq t_{max}} I(tu)$.

The uniqueness of $t^+(u)$ and its extremal property give that $t^+(u)$ is a continuous function of u .

Define for $u \in H^1(\Omega)$

$$\|u\|_\Omega = \int_\Omega p(x)|\nabla u|^2 dx - \lambda \int_\Omega |u|^2 dx.$$

Let us remark that since $\lambda < 0$, $\|\cdot\|_\Omega$ is equivalent to the standard norm $\|\cdot\|$.

Set

$$U_1 = \left\{ u \equiv 0 \text{ or } u : \|u\|_\Omega < t^+ \left(\frac{u}{\|u\|_\Omega} \right) \right\}$$

and

$$U_2 = \left\{ u : \|u\|_\Omega > t^+ \left(\frac{u}{\|u\|_\Omega} \right) \right\}.$$

We have $H^1(\Omega) \setminus \Lambda^- = U_1 \cup U_2$ and $\Lambda^+ \subset U_1$.

Since $u_0 \in \Lambda^+$ we deduce that $u_0 \in U_1$.

Claim 2: For a suitable choice of R_0 , $u_0 + R_0 W_{\varepsilon, x_0} \in U_2$.

Indeed, let $u \not\equiv 0$ on $\partial\Omega$ with $\|u\|_\Omega = 1$, looking at the proof of Lemma 4.3.1, we have $t_{max} < t^+$ and

$$\varphi(t^+u) = t^+ - (t^+)^{q-1} \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x = \int_\Omega f(x)u dx.$$

Therefore, using the embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ and the fact that $t_{max} < t^+$, a standard computation yields that there exists a constant $M_1 > 0$ such that $0 < t^+ < M_1$.

We choose R_0 such that $R_0 > \left(\frac{1}{p(x_0)A_1} |M_1 - \|u_0\|_\Omega^2| \right)^{\frac{1}{2}}$. It follows from (4.2.2) and (4.2.4) that, for ε small enough we have

$$\|u_0 + R_0 W_{\varepsilon, x_0}\|_\Omega^2 = \|u_0\|_\Omega^2 + R_0^2 p(x_0)A_1 + o(1) > M_1 > \left[t^+ \left(\frac{u_0 + R_0 W_{\varepsilon, x_0}}{\|u_0 + R_0 W_{\varepsilon, x_0}\|_\Omega} \right) \right]^2,$$

which gives the result of Claim 2.

Consequently, since $u_0 \in \Lambda^+ \subset U_1$ and $u_0 + R_0 W_{\varepsilon, x_0} \in U_2$, we see that the range of any $h \in \mathcal{F}$ intersects Λ^- . Therefore $c \geq c_1 = \inf_{\Lambda^-} I(u)$ and this completes the proof of Lemma 4.3.5. From Lemma 4.3.5 and Lemma 4.3.6 we obtain (4.3.63) and this gives the conclusion of Theorem 4.3.3.

Finally, combining the conclusions of Theorem 4.3.2 and Theorem 4.3.3 we obtain Theorem 4.1.1. This completes this section.

4.4 Case $\lambda > 0$

It is convenient to rewrite (4.1.1) as

$$(4.4.1) \quad \begin{cases} -\operatorname{div}(p(x)\nabla u) = \lambda u + f(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

Throughout this section we assume that f satisfies (H2), p satisfies (4.1.2), Q satisfies (4.1.3) and $\partial\Omega$ satisfies (g.c) at x_0 and $H(x_0) > 0$.

We recall some notations given in the introduction. By $\{\lambda_i\}$ we denote the sequence of eigenvalues for $\operatorname{div}(p(x)\nabla \cdot)$ with Neumann boundary conditions

$$\begin{cases} -\operatorname{div}(p(x)\nabla u) = \beta u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $k \in \mathbb{N} \setminus \{0\}$ such that $\lambda_{k-1} \neq \lambda_k$. Set $E_k^- = \operatorname{span}\{e_1, \dots, e_l\}$, where e_1, \dots, e_l are eigenfunctions corresponding to eigenvalues $\lambda_1, \dots, \lambda_{k-1}$. We have the orthogonal decomposition $H^1(\Omega) = E_k^- \oplus E_k^+$.

Remark 4.4.1.

Since f satisfies (H2), namely $f \in E_k^+ \setminus \{0\}$, it is easy to see that, if there exists a solution u of (4.4.1), then $u \in E_k^+$.

The proof of Theorem 4.1.2 relies on the following min-max principle based on a topological linking, see [W],

Theorem 4.4.1.

Let $X = X^- \oplus X^+$ be a Banach space with $\dim X^- < \infty$. Let $\rho > r > 0$ and let $w \in X^+$ be such that $\|w\| = \delta$. Define

$$\begin{aligned} M &:= \{u = v + sw : \|u\| \leq \rho, s \geq 0, v \in X^-\}, \\ M_0 &:= \{u = v + sw : v \in X^-, \|u\| = \rho \text{ and } s \geq 0 \text{ or } \|u\| \leq \rho \text{ and } s = 0\}, \\ N &:= \{u \in X^+ : \|u\| = r\}. \end{aligned}$$

Let $\varphi \in C^1(X, \mathbb{R})$ be such that

$$b := \inf_N \varphi > a := \max_{M_0} \varphi.$$

If φ satisfies the Palais-Smale condition with

$$c := \inf_{\gamma \in \Gamma} \max_{u \in M} \varphi(\gamma(u)), \quad \text{where } \Gamma := \{\gamma \in C(M, X) : \gamma|_{M_0} = \operatorname{id}\},$$

then c is a critical value of φ .

We consider the functional

$$I(u) = \frac{1}{2} \int_{\Omega} p(x) |\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \int_{\Omega} f(x) u dx - \frac{1}{q} \int_{\partial\Omega} p(x) Q(x) |u|^q ds_x, \quad u \in H^1(\Omega).$$

Set

$$K = \frac{N^2 \|f\|_2^2}{4(N-1)(\lambda_k - \lambda)}.$$

We need the following two Lemmas

Lemma 4.4.1.

Let $\{u_n\} \subset H^1(\Omega)$ be a sequence satisfying

$$(4.4.2) \quad I(u_n) \rightarrow c < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - K$$

and

$$(4.4.3) \quad I'(u_n) \rightarrow 0 \quad \text{in } H^{-1}(\Omega)$$

then $\{u_n\}$ is relatively compact in $H^1(\Omega)$.

Proof.

We argue as in the proof of Lemma 4.3.4 and we obtain that $\{u_n\}$ is bounded in $H^1(\Omega)$. Hence, by choosing a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$.

By the Concentration-Compactness principle [L] we have

$$|\nabla u_n|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \sum_{j \in J} \mu_j \delta_{x_j}$$

and

$$|u_n|^q \rightharpoonup |u|^q + \sum_{j \in J} \nu_j \delta_{x_j}$$

weakly in the sense of measures. Here $x_j \in \partial\Omega$, μ_j and ν_j are some positive constants satisfying

$$(4.4.4) \quad S_1 \nu_j^{\frac{2}{q}} \leq \mu_j,$$

and J is a countable set.

Testing (4.4.3) by a family of C^1 -functions concentrating at x_j with the same $j \in J$ we get

$$(4.4.5) \quad p(x_j)\mu_j \leq p(x_j)Q(x_j)\nu_j.$$

We claim that $\nu_j = 0$ for every $j \in J$, otherwise there exists $j_0 \in J$ such that $\nu_{j_0} > 0$ and from (4.4.4) and (4.4.5) we have

$$(4.4.6) \quad \frac{S_1^{N-1}}{Q(x_{j_0})^{N-1}} \leq \nu_{j_0}.$$

We write

$$\begin{aligned} I(u_n) - \frac{1}{q} \langle I'(u_n), u_n \rangle &= \frac{1}{2(N-1)} \left(\int_{\Omega} p(x) |\nabla u_n|^2 dx - \lambda \int_{\Omega} |u_n|^2 dx \right) \\ &\quad - \frac{N}{2(N-1)} \int_{\Omega} f(x) u_n dx + o(1). \end{aligned}$$

Letting $n \rightarrow +\infty$ we obtain

$$\begin{aligned}
c &\geq \frac{1}{2(N-1)} \left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) - \frac{N}{2(N-1)} \int_{\Omega} f(x) u dx \\
&\quad + \frac{1}{2(N-1)} \sum_{j \in J} p(x_j) \mu_j \\
&\geq \frac{1}{2(N-1)} \left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) - \frac{N}{2(N-1)} \int_{\Omega} f(x) u dx \\
&\quad + \frac{1}{2(N-1)} \sum_{j \in J} p(x_j) S_1 \nu_j^{\frac{2}{q}} \\
&\geq \frac{1}{2(N-1)} \left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) - \frac{N}{2(N-1)} \int_{\Omega} f(x) u dx \\
&\quad + \frac{1}{2(N-1)} p(x_{j_0}) S_1 \nu_{j_0}^{\frac{2}{q}}.
\end{aligned}$$

Inserting (4.4.6) into the above inequality and using the definition of x_0 we find

$$\begin{aligned}
c &\geq \frac{1}{2(N-1)} \left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) - \frac{N}{2(N-1)} \int_{\Omega} f(x) u dx \\
&\quad + \frac{1}{2(N-1)} \frac{p(x_{j_0})}{Q(x_{j_0})^{N-2}} S_1^{N-1}
\end{aligned}$$

and

$$\begin{aligned}
(4.4.7) \quad c &\geq \frac{1}{2(N-1)} \left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) - \frac{N}{2(N-1)} \int_{\Omega} f(x) u dx \\
&\quad + \frac{1}{2(N-1)} \frac{p(x_0)}{Q(x_0)^{N-2}} S_1^{N-1}.
\end{aligned}$$

On the other hand, combining the fact that $\langle I'(u), v \rangle = 0$ for every $v \in H^1(\Omega)$ and the Remark 4.4.1 we deduce that $u \in E_k^+$ and we have

$$\begin{aligned}
\frac{1}{2(N-1)} \left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) - \frac{N}{2(N-1)} \int_{\Omega} f(x) u dx \\
\geq \frac{1}{2(N-1)} (\lambda_k - \lambda) \int_{\Omega} |u|^2 dx - \frac{N}{2(N-1)} \int_{\Omega} f(x) u dx.
\end{aligned}$$

Using the Hölder inequality we write

$$\begin{aligned} & \frac{1}{2(N-1)} \left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) - \frac{N}{2(N-1)} \int_{\Omega} f(x) u dx \\ & \geq \frac{(\lambda_k - \lambda)}{2(N-2)} \|u\|_2^2 - \frac{N}{2(N-2)} \|f\|_2 \|u\|_2. \end{aligned}$$

Thus

$$(4.4.8) \quad \frac{1}{2(N-1)} \left(\int_{\Omega} p(x) |\nabla u|^2 dx - \lambda \int_{\Omega} |u|^2 dx \right) - \frac{N}{2(N-1)} \int_{\Omega} f(x) u dx \geq h(\|u\|),$$

where

$$h(t) = \frac{(\lambda_k - \lambda)}{2(N-1)} t^2 - \frac{N}{2(N-1)} \|f\|_2 t \quad \text{for } t \geq 0.$$

Since $\lambda_k - \lambda > 0$, it follows that h achieves its minimum at $t_0 = \frac{N}{2(\lambda_k - \lambda)} \|f\|_2$ and

$$(4.4.9) \quad h(t_0) = \min_{t \geq 0} h(t) = -K.$$

Inserting (4.4.9) into (4.4.7) we obtain

$$c \geq \frac{1}{2(N-1)} \frac{p(x_0)}{Q(x_0)^{N-2}} S_1^{N-1} - K.$$

This contradicts the hypothesis. Consequently, $\nu_j = 0$ for each j and we deduce the desired result as in the proof of Lemma 4.3.4. \square

We now apply the Theorem 4.4.1 with $X = H^1(\Omega)$, $X^- = E_k^-$, $X^+ = E_k^+$ and $\varphi = I$.

We need the following Lemma

Lemma 4.4.2.

There exist constants $\beta > 0$, $\rho > 0$ and $\rho > r$ such that

$$I(u) \geq \beta \quad \text{for all } u \in N$$

and

$$I(u) \leq 0 \quad \text{for all } u \in M_0.$$

Proof.

We start by this useful remark: For every $u \in E_k^+$, we have

$$\int_{\Omega} p(x)|\nabla u|^2 dx \geq \lambda_k \int_{\Omega} |u|^2 dx.$$

Therefore, there exists constants $C_1 > 0$ and $C_2 > 0$ such that for every $u \in E_k^+$ we have

$$(4.4.10) \quad \int_{\Omega} p(x)|\nabla u|^2 dx \geq C_1 \|u\|^2 \geq C_2 \int_{\Omega} p(x)|\nabla u|^2 dx.$$

Using (4.4.10) and the Hölder inequality, we write

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\Omega} p(x)|\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \frac{1}{q} \int_{\partial\Omega} p(x)Q(x)|u|^q ds_x - \int_{\Omega} f(x)u dx \\ &\geq \frac{(1 - \frac{\lambda}{\lambda_k})}{2} C_1 \|u\|^2 - \frac{C_3}{q} \|u\|^q - \|f\|_2 \|u\|, \end{aligned}$$

where C_3 is positive constant.

Thus, we have

$$(4.4.11) \quad I(u) \geq h(\|u\|),$$

where

$$h(t) = \frac{(1 - \frac{\lambda}{\lambda_k})}{2} C_1 t^2 - \frac{C_3}{q} t^q - \|f\|_2 t.$$

We see easily that

$$h'(t) = (1 - \frac{\lambda}{\lambda_k(p)}) C_1 t - C_3 t^{q-1} - \|f\|_2$$

and

$$h''(t) = (1 - \frac{\lambda}{\lambda_k(p)}) C_1 - (q-1) C_3 t^{q-2}.$$

We have

$$h''(t_0) = 0 \quad \text{with} \quad t_0 = \left[\frac{(1 - \frac{\lambda}{\lambda_k(p)}) C_1}{(q-2) C_3} \right]^{\frac{1}{q-2}}.$$

If we let

$$(4.4.12) \quad \|f\|_2 < \frac{(1 - \frac{\lambda}{\lambda_k}) C_1 (q-3)}{2(q-2)} t_0,$$

a easy computation gives that

$$h(t_0) > (q-2) C_1 t_0^2.$$

Consequently, since $q > 2$, using this last inequality and (4.4.11) we obtain the conclusion of Lemma 4.4.2 with $\beta = (q-2) C_1 t_0^2$ and $\delta = t_0$. \square

Proof of Theorem 4.1.2

We start by checking that

$$c < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - K.$$

We have

$$\begin{aligned} I(tW_{\varepsilon,x_0}) &= \frac{t^2}{2} \int_{\Omega} p(x) |\nabla W_{\varepsilon,x_0}|^2 dx - \frac{t^2 \lambda}{2} \int_{\Omega} |W_{\varepsilon,x_0}|^2 dx - t \int_{\Omega} f(x) W_{\varepsilon,x_0} dx \\ &\quad - \frac{t^q}{q} \int_{\partial\Omega} p(x) Q(x) |W_{\varepsilon,x_0}|^q ds_x. \end{aligned}$$

Using the Dominated Convergence Theorem, we easily see that

$$(4.4.13) \quad \int_{\Omega} f(x) W_{\varepsilon,x_0} dx = \varepsilon^{\frac{N-2}{2}} \int_{\Omega} \frac{f(x)\phi(x)}{|x-x_0|^{N-2}} dx + o(\varepsilon^{\frac{N-2}{2}})$$

Looking at (4.2.2), (4.2.3), (4.2.4) and (4.4.13), a direct computation shows that the cases $N = 3$, $N = 4$ and $N \geq 5$ are different. Therefore we distinguish three cases.

When $N \geq 5$:

We have

$$(4.4.14) \quad I(tW_{\varepsilon,x_0}) = \frac{t^2}{2} p(x_0) (A_1 - A_2 H(x_0) \varepsilon) - \frac{t^q}{q} p(x_0) Q(x_0) (B_1 - B_2 H(x_0) \varepsilon) + o(\varepsilon)$$

Define

$$g(t) = \frac{t^2}{2} p(x_0) (A_1 - A_2 H(x_0) \varepsilon) - \frac{t^q}{q} p(x_0) Q(x_0) (B_1 - B_2 H(x_0) \varepsilon).$$

It is easy to see that g achieves its maximum at

$$t_0 = \left[\frac{(A_1 - A_2 H(x_0) \varepsilon)}{Q(x_0) (B_1 - B_2 H(x_0) \varepsilon)} \right]^{\frac{1}{q-2}}.$$

An easy computation yields

$$(4.4.15) \quad t_0 = \left(\frac{A_1}{Q(x_0) B_1} \right)^{\frac{1}{q-2}} - \frac{H(x_0)}{(q-2) A_1} \left(\frac{A_1}{Q(x_0) B_1} \right)^{\frac{1}{q-2}} \left(A_2 - \frac{A_1 B_2}{B_1} \right) \varepsilon + o(\varepsilon).$$

From (4.3.22) we have $A_2 - \frac{A_1 B_2}{B_1} > 0$.

Consequently, for $\varepsilon > 0$ sufficiently small we have

$$\begin{aligned} \sup_{t \geq 0} I(tW_{\varepsilon,x_0}) &\leq I(t_0 W_{\varepsilon,x_0}) = \frac{t_0^2}{2} (A_1 - A_2 H(x_0) \varepsilon) - \frac{t_0^q}{q} (B_1 - H(x_0) B_2 \varepsilon) + o(\varepsilon) \\ &\leq \frac{p(x_0) S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - MH(x_0) \varepsilon + o(\varepsilon) \end{aligned}$$

where M is a positive constant.

Therefore, for $\varepsilon > 0$ sufficiently small, if we let

$$(4.4.16) \quad \|f\|_2^2 < \varepsilon \frac{2H(x_0)M(N-1)(\lambda_k - \lambda)}{N^2}$$

we obtain

$$\sup_{t \geq 0} I(tW_{\varepsilon, x_0}) < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - K.$$

Thus we deduce that

$$c < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - K$$

which is the desired result in this case.

When $N = 4$

Combining (4.2.2), (4.2.3), (4.2.4) and (4.4.13), we obtain

$$(4.4.17) \quad \begin{aligned} I(tW_{\varepsilon, x_0}) &= \frac{t^2}{2}p(x_0)(A_1 - A_2H(x_0)\varepsilon) - \frac{t^q}{q}p(x_0)Q(x_0)(B_1 - B_2H(x_0)\varepsilon) \\ &\quad - \varepsilon t \int_{\Omega} \frac{f(x)\phi(x)}{|x - x_0|^2} dx + o(\varepsilon). \end{aligned}$$

Since in the case where $N = 4$ we have supposed that $f(x_0) \neq 0$, and using the fact that $f \in C(\bar{\Omega})$, without loss of generality, we can suppose that $f > 0$ in some neighborhood of x_0 . Then (4.4.17) becomes

$$I(tW_{\varepsilon, x_0}) \leq \frac{t^2}{2}p(x_0)(A_1 - A_2H(x_0)\varepsilon) - \frac{t^q}{q}p(x_0)Q(x_0)(B_1 - B_2H(x_0)\varepsilon) + o(\varepsilon)$$

and we conclude as in the above case.

When $N = 3$:

From (4.2.2), (4.2.3), (4.2.4) and (4.4.13), we have

$$(4.4.18) \quad I(tW_{\varepsilon, x_0}) = \frac{t^2}{2}p(x_0)A_1 - \frac{t^q}{q}p(x_0)Q(x_0)B_1 - \varepsilon^{\frac{1}{2}}t \int_{\Omega} \frac{f(x)\phi(x)}{|x - x_0|} dx + o(\varepsilon^{\frac{1}{2}}).$$

For $\varepsilon > 0$ sufficiently small, we set

$$h(t) = \frac{t^2}{2}p(x_0)A_1 - \frac{t^q}{q}p(x_0)Q(x_0)B_1 - \varepsilon^{\frac{1}{2}}t \int_{\Omega} \frac{f(x)\phi(x)}{|x - x_0|} dx \quad \text{for } t \geq 0.$$

Since $h(t)$ goes to $-\infty$ as t goes to $+\infty$, $\sup_{t \geq 0} h(t)$ is achieved at some $t_{\varepsilon} \geq 0$.

We have

$$h'(t_{\varepsilon}) = t_{\varepsilon}p(x_0)(A_1 - t_{\varepsilon}^{q-2}Q(x_0)B_1) - \varepsilon^{\frac{1}{2}} \int_{\Omega} \frac{f(x)\phi(x)}{|x - x_0|} dx = 0 \quad \text{and} \quad h''(t_{\varepsilon}) \leq 0,$$

thus

$$(4.4.19) \quad \left[\frac{A_1}{(q-1)Q(x_0)B_1} \right]^{\frac{1}{q-2}} \leq t_{\varepsilon} \leq \left[\frac{A_1}{Q(x_0)B_1} \right]^{\frac{1}{q-2}}.$$

Let

$$(4.4.20) \quad t_0 = \frac{1}{2} \left[\frac{A_1}{(q-1)Q(x_0)B_1} \right]^{\frac{1}{q-2}} < t_\varepsilon.$$

On the other hand, since the function $t \rightarrow \frac{t^2}{2}p(x_0)A_1 - \frac{t^q}{q}p(x_0)Q(x_0)B_1$ is increasing on the interval $[0, \left(\frac{A_1}{Q(x_0)B_1}\right)^{\frac{1}{q-2}}]$, we get

$$h(t_\varepsilon) \leq \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - \varepsilon^{\frac{1}{2}}t_\varepsilon \int_\Omega \frac{f(x)\phi(x)}{|x-x_0|} dx.$$

Using (4.4.20) we deduce

$$(4.4.21) \quad h(t_\varepsilon) \leq \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - \varepsilon^{\frac{1}{2}}t_0 \int_\Omega \frac{f(x)\phi(x)}{|x-x_0|} dx.$$

Inserting (4.4.21) into (4.4.18) we obtain

$$I(tW_{\varepsilon, x_0}) \leq \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}} - \varepsilon^{\frac{1}{2}}t_0 \int_\Omega \frac{f(x)\phi(x)}{|x-x_0|} dx + o(\varepsilon^{\frac{1}{2}}).$$

Since in the case where $N = 3$ we have supposed that $f(x_0) \neq 0$ and using the fact that $f \in C(\bar{\Omega})$, without loss of generality, we can suppose that $f > 0$ in some neighborhood of x_0 . Then, for $\varepsilon > 0$ small enough, if we let

$$\frac{8(\lambda_k - \lambda)t_0}{9} \varepsilon^{\frac{1}{2}} \int_\Omega \frac{f(x)\phi(x)}{|x-x_0|} dx > \|f\|_2^2,$$

we deduce that

$$\sup_{t \geq 0} I(tW_{\varepsilon, x_0}) < \frac{p(x_0)S_1^{N-1}}{2(N-1)(Q(x_0))^{N-2}}.$$

This completes the proof in this case.

Finally, Theorem 4.1.2 follows from the combination of Lemma 4.4.1, Lemma 4.4.2 and Theorem 4.4.1. This completes the proof.

4.5 Appendix

Let $R > 0$ (see the definition of R in Section 1), using the ideas developed in Adimurthi-Yadava [AY], we introduce the following notations

$$x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \quad \text{and} \quad x_0 = (x'_0, x_{0N}),$$

$$B(x_0, R) = \{x \in \mathbb{R}^N, |x - x_0| < R\},$$

$$B(x_0, R) \cap \Omega = \{(x', x_N) \in B(x_0, R), x_N - x_{0N} > \rho(x' - x'_0)\},$$

$$B(x_0, R) \cap \partial\Omega = \{(x', x_N) \in B(x_0, R), x_N - x_{0N} = \rho(x' - x'_0)\},$$

$$B(x_0, R)^+ = B(x_0, R) \cap \{x_N - x_{0N} > 0\}$$

$$\Sigma = \{(x', x_N) \in B(x_0, R), 0 < x_N - x_{0N} < \rho(x' - x'_0)\},$$

where

$$\rho(x' - x'_0) = \frac{1}{2} \sum_{i=1}^{N-1} \alpha_i (x_i - x_{0i})^2 + o(|x' - x'_0|^2),$$

and $\{\alpha_i\}_{1 \leq i \leq N-1}$ are the principal curvatures at x_0 and $H(x_0) = \frac{1}{n-1} \sum_{i=1}^{N-1} \alpha_i$.

The following relation will be used frequently

$$(4.5.1) \quad \int_0^{+\infty} \frac{r^\theta}{(1+r^2)^\beta} dr = \frac{\Gamma(\frac{\theta+1}{2})\Gamma(\frac{2\beta-\theta-1}{2})}{2\Gamma(\beta)} \quad \text{for } 2\beta - \theta > 1.$$

Let $a = \frac{R}{k}$ for some $k \in [1, N^2]$ and set $v(x) = p(x)Q(x)u(x)$, $x \in \bar{\Omega}$ and $v = 0$ outside $\partial\Omega$.

4.5.1 Appendix 1

Here we give the proof of (4.3.15), (4.3.16), (4.3.17), (4.3.18).

Proof of (4.3.15):

We have

$$\begin{aligned} \int_{\partial\Omega} |W_{\varepsilon, x_0}|^{q-1} v \, ds_z &= \varepsilon^{\frac{N}{2}} \int_{|z'-x_0| \leq a} \frac{(1 + |\nabla\rho|^2)^{\frac{1}{2}} v(z', x_{0N} + \rho(z' - x'_0))}{[(\varepsilon + \rho(z' - x'_0))^2 + |z' - x'_0|^2]^{\frac{N}{2}}} dx' + O(\varepsilon^{\frac{N}{2}}) \\ &= \varepsilon^{\frac{N}{2}} \int_{|x'| \leq a} \frac{(1 + |\nabla\rho|^2)^{\frac{1}{2}} v(x'_0 + x', x_{0N} + \rho(x'))}{[(\varepsilon + \rho(x'))^2 + |x'|^2]^{\frac{N}{2}}} dx' + O(\varepsilon^{\frac{N}{2}}). \end{aligned}$$

Using Taylor's expansion of ρ , we obtain

$$\begin{aligned} \frac{(1 + |\nabla\rho|^2)^{\frac{1}{2}}}{[(\varepsilon + \rho(x'))^2 + |x'|^2]^{\frac{N}{2}}} &= \\ &= \frac{1 + \frac{1}{2}|\nabla\rho|^2 + o(|\nabla\rho|^2)}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} \left[1 - \frac{\frac{N}{2}\rho(x')(2\varepsilon + \rho(x'))}{(\varepsilon^2 + |x'|^2)} + o\left(\frac{\frac{N}{2}\rho(x')(2\varepsilon + \rho(x'))}{(\varepsilon^2 + |x'|^2)}\right) \right]. \end{aligned}$$

Consequently

$$\begin{aligned}
\varepsilon^{\frac{N}{2}} \int_{\partial\Omega} |W_{\varepsilon,y}|^{q-1} v \, ds_z &= \varepsilon^{\frac{N}{2}} \int_{|x'| \leq a} \frac{u(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' \\
&+ \frac{\varepsilon^{\frac{N}{2}}}{2} \int_{|x'| \leq a} \frac{|\nabla \rho|^2 u(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' \\
&+ \varepsilon^{\frac{N+2}{2}} \int_{|x'| \leq a} \frac{N \rho(x') v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N+2}{2}}} dx' \\
&+ o \left(\varepsilon^{\frac{N+2}{2}} \int_{|x'| \leq a} \frac{N \rho(x') v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N+2}{2}}} dx' \right) \\
&= T_1 + T_2 + T_3 + o(T_3).
\end{aligned}$$

We have

$$\begin{aligned}
T_1 &= \varepsilon^{\frac{N}{2}} \int_{|x'| \leq a} \frac{v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' \\
&= \varepsilon^{\frac{N}{2}} \int_{\mathbb{R}^{N-1}} \frac{v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' + O(\varepsilon^{\frac{N}{2}}) \\
&= \varepsilon^{\frac{N-2}{2}} v(x_0) \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + |z'|^2)^{\frac{N}{2}}} dz' + o(\varepsilon^{\frac{N-2}{2}})
\end{aligned}$$

By changing to polar coordinates and using (4.5.1), we get

$$\begin{aligned}
T_1 &= \omega_{N-2} \varepsilon^{\frac{N-2}{2}} v(x_0) \int_0^{+\infty} \frac{r^{N-2}}{(1 + r^2)^{\frac{N}{2}}} dr + o(\varepsilon^{\frac{N-2}{2}}) \\
(4.5.2) \quad &= \frac{\omega_{N-2} \Gamma(\frac{N-1}{2}) \Gamma(\frac{1}{2}) v(x_0) \varepsilon^{\frac{N-2}{2}}}{2 \Gamma(\frac{N}{2})} + o(\varepsilon^{\frac{N-2}{2}}).
\end{aligned}$$

Since $|\nabla \rho|^2 = \sum_{i=1}^{N-1} \alpha_i^2 x_i^2 + o(|x'|^2)$, then

$$\begin{aligned}
T_2 &= \varepsilon^{\frac{N}{2}} \sum_{i=1}^{N-1} \alpha_i^2 \int_{|x'|^2 \leq a} \frac{x_i^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' \\
&+ o \left(\varepsilon^{\frac{N}{2}} \int_{|x'|^2 \leq a} \frac{x_i^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' \right).
\end{aligned}$$

We have

$$\begin{aligned} \varepsilon^{\frac{N}{2}} \sum_{i=1}^{N-1} \alpha_i^2 \int_{|x'| \leq a} \frac{x_i^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' \\ \leq \frac{\sum_{i=1}^{N-1} \alpha_i^2 \varepsilon^{\frac{N}{2}}}{(N-1)} \int_{|x'| \leq a} \frac{|x'|^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx'. \end{aligned}$$

Applying the Dominated Convergence Theorem we find that

$$\int_{|x'| \leq a} \frac{|x'|^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N}{2}}} dx' = O(1)$$

and we deduce that

$$(4.5.3) \quad T_2 = O(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^{\frac{N-2}{2}}).$$

By the definition of ρ , we have

$$\begin{aligned} T_3 &= \varepsilon^{\frac{N+2}{2}} \frac{N}{2} \sum_{i=1}^{N-1} \alpha_i \int_{|x'| \leq a} \frac{x_i^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N+2}{2}}} dx' \\ &\quad + o\left(\varepsilon^{\frac{N+2}{2}} \int_{|x'| \leq a} \frac{|x'|^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N+2}{2}}} dx'\right). \end{aligned}$$

We have

$$\begin{aligned} \varepsilon^{\frac{N+2}{2}} \frac{N}{2} \sum_{i=1}^{N-1} \alpha_i \int_{|x'| \leq a} \frac{x_i^2 v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{N+2}{2}}} dx' \\ = \frac{N \sum_{i=1}^{N-1} \alpha_i}{2(N-1)} \varepsilon^{\frac{N}{2}} v(x_0) \int_0^{\frac{a}{\varepsilon}} \frac{r^N}{(1+r^2)^{\frac{N+2}{2}}} dr \\ \leq \frac{N v(x_0) \sum_{i=1}^{N-1} \alpha_i}{2(N-1)} \varepsilon^{\frac{N}{2}} \int_0^{+\infty} \frac{r^N}{(1+r^2)^{\frac{N+2}{2}}} dr + o(\varepsilon^{\frac{N}{2}}) \\ = \frac{N v(x_0) \Gamma(\frac{N+1}{2}) \Gamma(\frac{1}{2}) \sum_{i=1}^{N-1} \alpha_i}{4(N-1) \Gamma(\frac{N+2}{2})} \varepsilon^{\frac{N}{2}} + o(\varepsilon^{\frac{N}{2}}). \end{aligned}$$

Thus,

$$(4.5.4) \quad T_3 \leq \frac{N v(x_0) \Gamma(\frac{N+1}{2}) \Gamma(\frac{1}{2}) \sum_{i=1}^{N-1} \alpha_i}{4(N-1) \Gamma(\frac{N+2}{2})} \varepsilon^{\frac{N}{2}} + o(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^{\frac{N-2}{2}}).$$

Finally, combining (4.5.2), (4.5.3) and (4.5.4) we obtain

$$\int_{\partial\Omega} |W_{\varepsilon, x_0}|^{q-1} v ds_z = \frac{\omega_{N-2} \Gamma(\frac{N-1}{2}) \Gamma(\frac{1}{2}) u(x_0) \varepsilon^{\frac{N-2}{2}}}{2\Gamma(\frac{N}{2})} + o(\varepsilon^{\frac{N-2}{2}}).$$

Proof of (4.3.16):

Using the Dominated Convergence Theorem, it follows easily that

$$\int_{\partial\Omega} p(x)Q(x)|u(x)|^{q-2}u(x)W_{\varepsilon,x_0}ds_x = \varepsilon^{\frac{N-2}{2}} \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^{q-2}u(x)\phi(x)}{|x-x_0|^{N-2}} ds_x + o(\varepsilon^{\frac{N-2}{2}}).$$

Proof of (4.3.17) and (4.3.18):

Since

$$\{|u| \leq t_\varepsilon W_{\varepsilon,x_0}\} \subset \{|x| \leq k_1 \varepsilon^{\frac{1}{2}}\}$$

and

$$\{|u| \geq t_\varepsilon W_{\varepsilon,x_0}\} \subset \{k_2 \geq |x| \geq k_1 \varepsilon^{\frac{1}{2}}\}$$

following the arguments of the previous calculus, we write

$$\begin{aligned} \int_{\partial\Omega \cap \{|u| \geq t_\varepsilon W_{\varepsilon,x_0}\}} |u(x)|^{q-2}|v(x)|W_{\varepsilon,x_0}ds_x &\leq M_1 \varepsilon^{\frac{N}{2}} \int_0^{k_1 \varepsilon^{\frac{1}{2}}} \frac{r^{N-2}}{(\varepsilon^2 + r^2)^{\frac{N-2}{2}}} dr \\ &= M_2 \varepsilon^{\frac{N}{2}} + o(\varepsilon^{\frac{N}{2}}), \end{aligned}$$

$$\begin{aligned} \int_{\partial\Omega \cap \{|u| \leq t_\varepsilon W_{\varepsilon,x_0}\}} |v||W_{\varepsilon,x_0}|^{q-1}ds_x &\leq M_3 \varepsilon^{\frac{N}{2}} \int_{k_1 \varepsilon^{\frac{1}{2}}}^{k_2} \frac{r^{N-2}}{(\varepsilon^2 + r^2)^{\frac{N}{2}}} dr \\ &= M_3 \varepsilon^{\frac{N-2}{2}} \int_{k_1 \varepsilon^{-\frac{1}{2}}}^{k_2 \varepsilon^{-1}} \frac{r^{N-2}}{(1+r^2)^{\frac{N}{2}}} dr \\ &= o(\varepsilon^{\frac{N-2}{2}}), \end{aligned}$$

for some constants M_i , $i = 1, 2, 3, 4$. This gives the desired result.

4.5.2 Appendix 2

Here we suppose that $N = 3$ and $q = 4$.

We distinguish two cases:

Case $u(x_0) > 0$

Proof of (4.3.26), (4.3.27) and (4.3.28):

Using the arguments of the derivation of the estimates in Appendix 1, we can write

$$\begin{aligned} \int_{\partial\Omega} p(x)Q(x)|u(x)|^2|W_{\varepsilon,x_0}|^2ds_x &\leq M_0 \varepsilon \int_{\{|x'| \leq a\}} \frac{1}{\varepsilon^2 + |x'|} dx' \\ &\leq M_0 \omega_2 \varepsilon \int_0^a \frac{r}{\varepsilon^2 + r^2} dr \\ &\leq M_1 \varepsilon |\log \varepsilon| + o(\varepsilon |\log \varepsilon|) = o(\varepsilon^{\frac{1}{2}}), \end{aligned}$$

where M_0 and M_1 are some positive constants. This gives (4.3.26).

$$\begin{aligned}
\int_{\partial\Omega} v(x)|W_{\varepsilon,x_0}|^3 ds_x &= \varepsilon^{\frac{3}{2}} \int_{|x'|\leq a} \frac{v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{3}{2}}} dx' \\
&+ o\left(\varepsilon^{\frac{3}{2}} \int_{|x'|\leq a} \frac{v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{3}{2}}} dx'\right) \\
&= \varepsilon^{\frac{3}{2}} \int_{\mathbb{R}^{N-1}} \frac{v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{3}{2}}} dx' \\
&+ o\left(\varepsilon^{\frac{3}{2}} \int_{|x'|\leq a} \frac{v(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{3}{2}}} dx'\right) \\
&+ \varepsilon^{\frac{3}{2}} \left[\int_{\mathbb{R}^{N-1} \setminus \{|x'|\leq a\}} \frac{p(x)Q(x)u}{|x'|^3} dx' + \int_{\partial\Omega} \frac{p(x)Q(x)(\phi(x) - 1)u}{|x'|^3} dx' \right] \\
&+ o(\varepsilon^{\frac{3}{2}})
\end{aligned}$$

Using a simple change of variable and applying the Dominated Convergence Theorem we deduce that

$$\int_{\partial\Omega} v(x)|W_{\varepsilon,x_0}|^3 ds_x = M_2 v(x_0) \varepsilon^{\frac{1}{2}} + o(\varepsilon^{\frac{1}{2}})$$

where M_2 is a positive constant. This gives (4.3.27).

By the Dominated Convergence Theorem we have

$$\int_{\partial\Omega} p(x)Q(x)|u(x)|^2 u(x) W_{\varepsilon,x_0} ds_x = \varepsilon^{\frac{1}{2}} \int_{\partial\Omega} \frac{p(x)Q(x)|u(x)|^2 u(x) \phi(x)}{|x - x_0|} ds_x + o(\varepsilon^{\frac{1}{2}}).$$

and this gives (4.3.28).

Case $u(x_0) = 0$

Proof of (4.3.32)

Using the arguments of Appendix 1, we remark that in order to estimate

$$\int_{\partial\Omega} p(x)Q(x)u(x)|W_{\varepsilon,x_0}|^3 ds_x$$

it is enough to estimate

$$\varepsilon^{\frac{3}{2}} \int_{\mathbb{R}^{N-1}} \frac{p(x'_0 + x', x_{0N} + \rho(x'))Q(x'_0 + x', x_{0N} + \rho(x'))u(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{3}{2}}} dx'.$$

We have

$$\begin{aligned}
(4.5.5) \quad &\varepsilon^{\frac{3}{2}} \int_{\mathbb{R}^2} \frac{p(x'_0 + x', x_{0N} + \rho(x'))Q(x'_0 + x', x_{0N} + \rho(x'))u(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{3}{2}}} dx' = \\
&\varepsilon^{\frac{1}{2}} \int_{\mathbb{R}^2} \frac{p(x'_0 + \varepsilon y', x_{0N} + \rho(\varepsilon y'))Q(x'_0 + \varepsilon y', x_{0N} + \rho(\varepsilon y'))u(x'_0 + \varepsilon y', x_{0N} + \rho(\varepsilon y'))}{(1 + |y'|^2)^{\frac{3}{2}}} dy'.
\end{aligned}$$

Since p , Q and f are smooth enough then by the result of [Ch] u is smooth enough. Therefore, we have

$$(4.5.6) \quad |u(x) - u(x_0)| \leq C \|u\|_{W^{1,6}(\mathbb{R}^3)} |x - x_0|^{\frac{1}{2}} \quad \forall x \in \mathbb{R}^3$$

where C is positive constant and $W^{1,6}(\mathbb{R}^3)$ is the Sobolev space.

Inserting (4.5.6) into (4.5.5) and using the fact that $u(x_0)$, we obtain

$$\begin{aligned} \varepsilon^{\frac{3}{2}} \int_{\mathbb{R}^2} \frac{p(x'_0 + x', x_{0N} + \rho(x')) Q(x'_0 + x', x_{0N} + \rho(x')) u(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{3}{2}}} dx' \\ \leq \varepsilon p(x_0) Q(x_0) C \|u\|_{W^{1,6}} \int_{\mathbb{R}^2} \frac{|y'|^{\frac{1}{2}}}{(1 + |y'|^2)^{\frac{3}{2}}} dy' + o(\varepsilon) \end{aligned}$$

Therefore

$$\begin{aligned} \varepsilon^{\frac{3}{2}} \int_{\mathbb{R}^{N-1}} \frac{p(x'_0 + x', x_{0N} + \rho(x')) Q(x'_0 + x', x_{0N} + \rho(x')) u(x'_0 + x', x_{0N} + \rho(x'))}{(\varepsilon^2 + |x'|^2)^{\frac{3}{2}}} dx' \\ = o(\varepsilon |\log \varepsilon|) \end{aligned}$$

and

$$\int_{\partial\Omega} p(x) Q(x) u(x) |W_{\varepsilon, x_0}|^3 ds_x = o(\varepsilon |\log \varepsilon|)$$

which is the desired result.

Bibliography

- [A] Th. Aubin, Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, *J. Math. Pures et Appl.* 55, 1976, pp 269-296.
- [A.E] J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis, Pure and Applied Mathematics*, Wiley Interscience Publications, 1984.
- [AM] Adimurthi and G. Mancini, The Neumann problem for elliptic equation with critical non linearity, 65th birthday of Prof. Prodi, Scuola Normale superiore, Pisa, Ed. by Ambrosetti and Marino (1991).
- [AY] Adimurthi and S. L. Yadava, Positive solution for Neumann problem with critical non linearity on boundary, *Comm. In Partial Diff. Equations*, 16, (11), 1991, pp 1733-1760.
- [AR] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* 14, 1973, pp. 349-381.
- [Ba] A. Bahri, *Critical point at infinity in some variational problems*, Longman Scientific and Technic, 1989.
- [BaC] A. Bahri and J. M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain, *Comm. Pure Appl. Math.* 41, 1988, pp. 253-294.
- [B] H. Brezis, Elliptic equations with limiting Sobolev exponents. The impact of topology, *Comm. Pure Appl. Math.*, 39, 1986, pp. S17-S39.

- [BCN] H. Brezis, J. M. Coron and L. Nirenberg, Free vibrations for a nonlinear wave equation and a theorem of R. Rabinowitz, *Comm. Pure Appl. Math.* **33**, 1980, pp 667-689.
- [BL] H. Brezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88**, 1983, pp. 486-490.
- [BN1] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36**, 1983, pp. 437-477.
- [BN2] H. Brezis and L. Nirenberg, A minimization problem with critical exponent and non-zero data, In *Symmetry in nature*, Scuola Superiore Pisa, **1**, 1989, pp 129-140.
- [Bre] H. Brezis, *Analyse fonctionnelle, Théorie et applications*, (1983), Masson.
- [C] J. M. Coron, Topologie et cas limite des injections de Sobolev, *C.R. acad. Sc.*, Paris. **299**, 1985, pp. 209-211.
- [Ch] P. Cherrier, Problèmes de Neumann non linéaires sur les variétés riemanniennes, *J. Funct. Anal.* **57**, 1984, pp. 154-206.
- [CK] M. Comte and M. C. Knaap, Solutions of elliptic equations involving Critical Sobolev exponents with Neumann boundary conditions, *Manuscripta Math.*, **69**, 1990, pp. 43-70.
- [CHL] R. Crouau, R. Hadji and R. Lewandowski, Critical Sobolev exponent and the dimension three, *Houston Journal of Mathematics*. Vol. **17**, N.3, 1991, pp. 1-16.
- [CY] J. Chabrowski and J. Yang, Sharp Sobolev Inequality Involving a Critical Nonlinearity on a Boundary, *Top. Meth. in Nonlinear Analysis*, vol **25**, N **1**,

2005, pp 135-155.

- [CKN] L. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequalities with weights, *Compositio Math.* 53, 1984, pp. 259-275.
- [D] E. N. Dancer, A note on an equation with critical exponent, *Bull. London Math. Soc.* 20 (6), 1988, pp 600-602.
- [Di] W. Y. Ding, Positive solutions of $\Delta u + u^{\frac{n+2}{n-2}} = 0$ on contractible domains, *J. Partial Differential Equations*, 2 (4), 1989, pp 83-88.
- [E] J. F. Escobar, Sharp constant in a Sobolev trace inequality, *Indiana Univ. Math. Journal.* 37, 1988, pp 687-698.
- [H] R. Hadiji, Solutions positives de l'équation $-\Delta u = u^p + \mu u^q$ dans un domaine à trou, *Annales de la faculté des Sciences de Toulouse.* Vol XI, n°3, 1990, pp. 55-71.
- [HLP] G. H. Hardy, J. E. Littlewood and G. Polya: *Inequalities*. Cambridge University Press, 1952.
- [GT] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd Edition, Springer-Verlag, (1983).
- [K] O. Kavian, *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, Springer-Verlag, (1993).
- [L] P. L. Lions, The concentration-compactness principle in the calculus of variations. The limit case, part 1, *Rev. Mat. Iberoamericana*, Vol 1, N° 1 (1985), pp. 145-201.
- [Le1] R. Lewandowski, Little holes and convergence of solutions of $-\Delta u = u^{\frac{N+2}{N-2}}$, *Nonlinear Analysis, Theory, Methods and Applications*, Vol. 14, 1990, No. 10,

pp 873-888.

- [Le2] R. Lewandowski, *Phd thesis*, Université Paris VI, (1990).
- [P] R. S. J. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Doklady. 6, 1965, pp. 1408-1411.
- [Pa] D. Passaseo, Multiplicity of positive solutions of nonlinear elliptic equations with critical Sobolev exponent in some contractible domains, Manuscripta Math. 65 (2), 1989, pp 147-165.
- [S] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, Math. Z. 187, 1984, pp. 511-517.
- [T] G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponent, Ann. Inst. H. Poincaré, 3, 1992, pp 281-304.
- [Z] M. Zhu, Sharp Sobolev inequality with interior norm, Calc. Var 8, 1999, pp 27-43.
- [W] M. Willem, *Minimax Theorem*, Progress in Nonlinear Differential Equations and Their Applications, 1996.

Résumé

Cette thèse est consacrée à l'étude de quelques équations aux dérivées partielles non linéaires de type Dirichlet ou Neumann sur un domaine borné régulier, qui sont à structure variationnelle, et qui présentent un défaut de compacité. Dans la première partie, nous étudions une EDP homogène avec un opérateur non linéaire faisant intervenir un poids strictement positif, une non-linéarité critique au sens de Sobolev et un paramètre λ . Nous établissons des résultats d'existence et de non-existence de solutions qui dépendent du comportement du poids au voisinage de ses minima, du paramètre λ et de la géométrie du domaine. Dans la seconde partie, nous nous intéressons à des EDP non homogènes avec poids et avec une non-linéarité critique au bord au sens de l'inclusion de trace. Nous montrons des résultats d'existence qui dépendent des différents coefficients des EDP étudiées et de la courbure moyenne en un point minimum de poids.

Mots-clés

Exposant critique de Sobolev, non-linéarité critique au bord, principe de Concentration-Compacité, principe variationnel d'Ekeland.

Abstract

This thesis is devoted to the study of some nonlinear partial differential equations of Dirichlet or Neumann type, with a non compact variational structure. In the first part, we study homogeneous PDE with a positive weight, with the critical Sobolev exponent and a parameter λ . We establish some existence and non-existence results which depend on the behavior of the weight near its minima, the parameter λ and the geometry of the domain. In the second part, we are interested in some non-homogeneous PDE with weight and with a critical nonlinearity on the boundary. We show some existence results which depend on the various coefficients of the studied PDE, and of the mean curvature of the boundary of the domain.