# Convection-reaction-diffusion systems and interface dynamics

Matthieu Alfaro

N° D'ORDRE : 8392

**UNIVERSITE PARIS XI**

UFR Scientifique d'Orsay

THESE

présentée pour obtenir le grade de

Docteur en sciences de l'Université Paris XI Orsay, spécialité mathématiques

par

**Matthieu ALFARO**

---

Sujet :

# Systèmes de convection-réaction-diffusion et dynamique d'interface

---

Soutenue le 28 septembre 2006 devant la Commission d'examen :

| | | | |
|---|---|---|---|
| Mme | Aline | BONAMI | rapporteur |
| M. | François | HAMEL | rapporteur |
| M. | Bernard | HELFFER | président |
| Mme | Danielle | HILHORST | directrice de thèse |
| Mme | Elisabeth | LOGAK | examinateur |
| M. | Hiroshi | MATANO | examinateur |

## Remerciements

Je tiens d'abord à remercier ma directrice de thèse, Danielle Hilhorst. Elle m'a proposé de nombreux sujets de recherche passionnants dans un domaine mathématique très actif. Sa disponibilité, son énergie et son optimisme furent d'un grand secours dans les moments de doute. Enfin, je lui suis reconnaissant de m'avoir invité à de nombreux congrès, notamment au Japon, et permis d'entreprendre de stimulantes collaborations scientifiques avec des mathématiciens étrangers.

Je voudrais également remercier ici le professeur Hiroshi Matano de l'université de Tokyo. Ses idées, sa rigueur et sa recherche de la clarté m'ont beaucoup apporté. Nos échanges au jardin du Luxembourg, à Cassis ou encore à Yokohama ont été très fructueux. Son sens de l'hospitalité fut également très appréciable.

Aline Bonami et François Hamel ont accepté la lourde tâche d'être les rapporteurs de cette thèse. J'apprécie sincèrement l'intérêt qu'ils portent à mes travaux ainsi que leurs commentaires avisés. Je voudrais aussi exprimer ma gratitude à Bernard Helffer pour avoir accepté d'être le président du jury et à Elisabeth Logak pour avoir accepté d'en être membre.

Rémi Weidenfeld, dont j'ai en quelque sorte pris la relève, a été un soutien moral efficace ; d'autre part j'ai plus appris sur les mystères du TeX en deux heures avec Martin Vohralík qu'en plusieurs années de travail autodidacte ! Merci à eux et à toute l'équipe d'Analyse Numérique et Equations aux Dérivées Partielles de l'Université de Paris-Sud.

Enfin, je voudrais dire à ma famille et à mes proches combien j'ai apprécié la patience et l'affection dont ils ont fait preuve durant ces quatre années.

J'aimerais dédier cette thèse à Maïté, à mes côtés depuis le début, et à Malou, présente depuis un peu moins longtemps !

### Systèmes de convection-réaction-diffusion et dynamique d'interface


#### Résumé

Le sujet de cette thèse est la limite singulière d'équations et de systèmes d'équations paraboliques non-linéaires de type bistable, où intervient un petit paramètre $\varepsilon$, avec des conditions initiales générales. Nous obtenons une estimation nouvelle et optimale de l'épaisseur et de la localisation de la zone de transition.

Au Chapitre 1, nous étudions une équation d'Allen-Cahn ainsi qu'une famille de systèmes de réaction-diffusion, notamment le système de FitzHugh-Nagumo et certains systèmes prédateur-proie. Nous considérons d'abord l'équation d'Allen-Cahn. Nous montrons qu'à partir d'une condition initiale arbitraire, la solution devient rapidement proche d'une fonction en escalier, sauf dans un petit voisinage de l'interface initiale, créant ainsi une zone de transition abrupte (*génération de l'interface*). Notre estimation du temps nécessaire au développement de cette zone est optimale. Dans un deuxième temps, l'interface se déplace et la solution reste proche de la fonction en escalier (*déplacement de l'interface*). Le déplacement de l'interface, solution de la limite singulière du problème parabolique bistable, est induit par sa courbure moyenne et par un terme de pression. Pour obtenir ces résultats, nous construisons deux paires distinctes de sous- et sur-solutions : l'une pour démontrer la propriété de génération de l'interface, et l'autre pour analyser le déplacement de l'interface. En imbriquant ces paires de sous- et sur-solutions, nous estimons de façon optimale, d'une part l'épaisseur de la zone de transition, et d'autre part sa localisation. Ensuite, nous étendons nos résultats à une classe assez large de systèmes de réaction-diffusion : comme nos preuves ne s'appuyent pas sur le principe de comparaison, nous ne faisons pas d'hypothèse de monotonie sur les termes de réaction. L'idée est de considérer la première équation du système comme une perturbation de l'équation d'Allen-Cahn ; les preuves s'appuient sur une légère modification des résultats pour l'équation seule, sur une étude de la dépendance du déplacement de l'interface vis-à-vis de différents paramètres, et sur de fines estimations a priori.

Le Chapitre 2 est consacré à l'étude d'un système chimiotactique, dont la première équation est parabolique et non-linéaire, alors que la seconde équation est elliptique et linéaire. Il s'agit d'un modèle pour une agrégation d'amibes soumises à trois effets : la diffusion, la croissance et le chimiotactisme. Ce dernier phénomène est une propension de certaines espèces à se déplacer vers les plus forts gradients de substances chimiques, souvent produites par ces espèces elles-mêmes. En étudiant successivement la génération et le déplacement de l'interface, nous obtenons des estimations optimales de l'épaisseur de la zone de transition et de sa localisation.

Enfin, au Chapitre 3, nous considérons une équation quasi-linéaire anisotrope de type Allen-Cahn, qui intervient en science des matériaux et dont le terme de diffusion est inhomogène et singulier aux points où le gradient de la solution s'annule. Nous définissons une notion de solution faible et prouvons un principe de comparaison. Le déplacement de l'interface limite est induit par une version anisotrope de sa courbure moyenne. Nous effectuons l'analyse en utilisant la distance associée à une métrique de Finsler. Nous étudions la génération et le déplacement de l'interface, obtenant une estimation optimale de l'épaisseur de la zone de transition.



**Mots clés :** Systèmes de convection-réaction-diffusion – Equation d'Allen-Cahn – Système de FitzHugh-Nagumo – Chimiotactisme – Anisotropie – Génération d'interface – Propagation d'interface – Epaisseur d'interface.

**AMS subject classifications :** 35K57, 35K60, 35K50, 35K20, 35R35, 35B20.

**Convection-reaction-diffusion systems and interface dynamics**


**Abstract**

This thesis deals with the singular limit of systems of parabolic partial differential equations involving a small parameter $\varepsilon$, with bistable nonlinear reaction terms and general initial data. We obtain a new and optimal estimate of the thickness and the location of the transition layer that develops.

In Chapter 1, we study a perturbed Allen-Cahn equation and a class of reaction-diffusion systems, which includes the FitzHugh-Nagumo system and some prey-predator systems. We first consider the case of the single equation. We show that, leaving from arbitrary initial data, the solution quickly becomes close to a step function, except in a small neighborhood of the initial interface, creating a steep transition layer (*generation of interface*). Our estimation of the time needed to develop such a transition layer is optimal. In the second stage, the interface starts to move, and the solution remains close to the step function (*motion of interface*). The motion of the interface, solution of the singular limit of the original problem, is driven by its mean curvature and a pressure term. To prove these results, we construct two completely different pairs of sub- and super-solutions: one for the generation of interface, and the other for the motion of interface. Fitting these pairs of sub- and super-solutions into each other, we estimate, in an optimal way, the thickness of the transition layer, and its location. Then, we extend our results to a large class of reaction-diffusion systems: since our proofs do not rely on the comparison principle, we do not make any monotony assumptions on the reaction terms. The idea is to regard the first equation of the system as a perturbed Allen-Cahn equation; the proofs rely on a slight modification of the results for the single equation, a study of the dependence of the interface motion on various parameters together with some refined a priori estimates.

Chapter 2 is devoted to the study of a chemotaxis system, where the first equation is parabolic and nonlinear, whereas the second equation is elliptic and linear. This is a model for an aggregation of amoebae which are subjected to three effects: diffusion, growth and chemotaxis. This last phenomenon is a tendency of some species to move towards higher gradients of chemical substances which they often produce themselves. By successively studying the generation and the motion of interface, we obtain here as well optimal estimates of the thickness of the transition layer and of its location.

Finally, in Chapter 3, we consider a quasi-linear anisotropic Allen-Cahn equation, which arises for instance in material sciences, and whose diffusion term is spatially inhomogeneous and singular in the points where the gradient of the solution vanishes. We define a notion of weak solution and prove a comparison principle. The motion of the limit interface is driven by its anisotropic mean curvature. We perform the analysis using the distance function associated with a Finsler metric related to the anisotropic diffusion term. We study both the generation and the motion of interface and obtain an optimal estimate of the thickness of the transition layer.



**Key words:** Convection-reaction-diffusion systems – Allen-Cahn equation – FitzHugh-Nagumo system– Chemotaxis – Anisotropy – Generation of interface – Motion of interface – Thickness of interface.

**AMS subject classifications:** 35K57, 35K60, 35K50, 35K20, 35R35, 35B20.

# Table des matières

# Introduction

L'objet de cette thèse est l'étude d'équations aux dérivées partielles paraboliques non-linéaires qui interviennent, par exemple, en biologie et en sciences des matériaux. Lorsque le coefficient du terme de diffusion est petit ou lorsque celui du terme de réaction est grand, ces problèmes peuvent donner lieu à des zones de transition abruptes, aussi appelées interfaces, entre les différents états que peut atteindre la solution. Une équation modèle est donnée par l'équation d'Allen-Cahn :

$$u_t = \Delta u + \frac{1}{\varepsilon^2}(u - u^3),$$

où intervient le petit paramètre $\varepsilon > 0$. Dans un premier temps, dit de *génération de l'interface*, le terme de diffusion $\Delta u$ peut être négligé devant le terme de réaction $\varepsilon^{-2}(u-u^3)$. Dans l'échelle de temps $\tau = t/\varepsilon^2$, la solution de l'équation d'Allen-Cahn $u^\varepsilon$ se comporte comme la solution de l'équation différentielle ordinaire $u_\tau = f(u)$ ; ainsi, les valeurs de $u^\varepsilon$ deviennent rapidement proches de l'un des deux équilibres stables 1 ou $-1$ et une zone de transition se développe entre les deux régions $\{u^\varepsilon \approx 1\}$ et $\{u^\varepsilon \approx -1\}$. A son voisinage, le terme de diffusion ne peut plus être négligé et sa combinaison avec le terme de réaction induit, dans un deuxième temps, un *déplacement de l'interface*. On sait que l'épaisseur de la zone de transition est liée au paramètre $\varepsilon$.

De nombreux travaux ont porté sur le comportement asymptotique de l'équation d'Allen-Cahn. En 1979, les physiciens Allen et Cahn [2] obtiennent, par analyse formelle, l'équation du problème à frontière libre limite : l'interface se déplace selon sa courbure moyenne. Nous renvoyons également aux travaux de Kawasaki et Ohta [52], en 1982.

Sous l'hypothèse que l'interface initiale est une hypersurface, il y a existence et unicité, locales en temps, de la solution du problème à frontière libre associé. La convergence vers la solution classique sur son intervalle de temps d'existence est démontrée au début des années 90. Citons, par exemple, les résultats de Bronsard et Kohn [18], dans le cas de la symétrie sphérique, de de Mottoni et Schatzman [59, 60] et de X. Chen [20, 21].

En général, la solution du problème limite devient singulière en temps fini et il est nécessaire de considérer des solutions faibles. Dans les années 90, la notion de solution de viscosité est introduite, notamment par Y.G. Chen, Giga et Goto [25]. Puis, la convergence vers la solution de viscosité est démontrée, notamment par Barles, Soner et Souganidis [6], Evans, Soner et Souganidis [33], Ilmanen [48] et Barles et Souganidis [7].

Notons que l'intégrale de la fonction non-linéaire $f(u) = u - u^3$ entre les deux équilibres stables 1 et $-1$ est nulle. Si l'on perturbe cette fonction non-linéaire par un terme d'ordre $\varepsilon$, dépendant des variables d'espace, de temps et de la fonction $u$, alors un terme supplémentaire intervient dans l'équation du problème à frontière libre associé. Ce résultat a été obtenu de manière formelle par Rubinstein, Sternberg et Keller [66] dès 1989. En 1997, Ei, Iida et Yanagida [30], démontrent la convergence vers la solution du problème à frontière libre limite en supposant que la condition initiale a une zone de transition déjà bien développée dont le profil dépend de $\varepsilon$.

Cette thèse porte sur la limite singulière d'équations ou systèmes de convection-réaction-diffusion qui font intervenir des termes inhomogènes et anisotropes, et qui étendent l'équation d'Allen-Cahn. Nous démontrons des propriétés de génération et de déplacement de l'interface ; l'étude de la génération permet de considérer une condition initiale très générale. D'autre part, nous montrons que l'épaisseur de la zone de transition est d'ordre $\varepsilon$. Nous localisons également de manière optimale l'ensemble des points où la solution a pour valeur l'équilibre instable du terme non-linéaire ; plus précisément, nous démontrons que sa distance de Hausdorff à l'interface solution du problème à frontière libre limite est également d'ordre $\varepsilon$.

## Chapitre 1 : Limite singulière de l'équation d'Allen-Cahn et du système de FitzHugh-Nagumo

Ce Chapitre fait l'objet d'un article écrit en collaboration avec D. Hilhorst (Université de Paris-Sud) et H. Matano (Université de Tokyo), soumis prochainement pour publication dans *Journal of Differential Equations*.

Il comporte deux parties. D'abord nous étudions une équation d'Allen-Cahn dont le terme non-linéaire est perturbé. Ensuite, en exploitant notre étude de cette équation, ainsi que des estimations a priori supplémentaires, nous étendons nos résultats à des systèmes de réaction-diffusion, notamment au système de FitzHugh-Nagumo et à certains systèmes de type prédateur-proie.

### L'équation d'Allen-Cahn perturbée

Nous considérons le problème parabolique non-linéaire

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}(f(u) - \varepsilon g^\varepsilon(x,t,u)) & \text{dans } \Omega \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{sur } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{dans } \Omega, \end{cases} \tag{1}$$

où $\Omega$ est un ouvert borné et régulier de $\mathbb{R}^N$, $N \geq 2$. La fonction non-linéaire $f$ admet exactement trois zéros $\alpha_- < a < \alpha_+$, le caractère bistable étant assuré par des pentes strictement négatives aux équilibres stables $u = \alpha_-, \alpha_+$, et strictement positive à l'équilibre instable $u = a$ ; d'autre part, nous supposons que l'intégrale de $f$ entre $\alpha_-$ et $\alpha_+$ s'annule et que la fonction $g^\varepsilon$ est de la forme

$$g^\varepsilon(x,t,u) = g(x,t,u) + O(\varepsilon). \tag{2}$$

Le problème à frontière libre limite est donné par

$$\begin{cases} V_n = -(N-1)\kappa + c_0 \displaystyle\int_{\alpha_-}^{\alpha_+} g(x,t,r)dr & \text{sur } \Gamma_t, \\ \Gamma_t\big|_{t=0} = \Gamma_0, \end{cases} \tag{3}$$

où l'interface initiale $\Gamma_0$, définie par $\Gamma_0 := \{x \in \Omega,\ u_0(x) = a\}$, est une hypersurface sans bords régulière ; $V_n$ désigne la vitesse de déplacement de l'interface le long de la normale, $\kappa$ la courbure moyenne de l'interface et $c_0$ une constante liée à la fonction non-linéaire $f$. Ce problème possède une solution classique unique $\Gamma_t$ sur un intervalle de temps $[0,T]$.

Les résultats essentiels de cette partie sont les suivants. Lorsque $\varepsilon \to 0$, la solution classique $u^\varepsilon$ du problème (1) converge vers $\alpha_-$ ou $\alpha_+$, selon que l'on se trouve à l'intérieur ou à l'extérieur de l'interface, sur l'intervalle de temps $(0,T]$. De plus, l'épaisseur de la zone de transition est d'ordre $\varepsilon$, ainsi que la distance de Hausdorff entre l'ensemble des points $\Gamma_t^\varepsilon := \{x \in \Omega,\ u^\varepsilon(x,t) = a\}$, et l'interface $\Gamma_t$, solution du problème à frontière libre limite (3). Ces estimations sont optimales.

Concernant la condition initiale, nos hypothèses sont peu restrictives. Nous démontrons d'abord une propriété de génération d'interface. Dans ce but, nous construisons une paire de sous- et sur-solutions basées sur la solution d'une équation différentielle de la forme $Y_\tau = f(Y) + O(\varepsilon)$, obtenue en négligeant la diffusion et en travaillant dans l'échelle de temps $\tau = t/\varepsilon^2$. Par le principe de comparaison, nous démontrons alors que, après un temps $t^\varepsilon$ d'ordre $\varepsilon^2|\ln\varepsilon|$, la solution a déjà développé une zone de transition très abrupte, autour de l'interface initiale. Nous montrons également que ce temps de génération $t^\varepsilon$ est optimal, c'est-à-dire qu'une zone de transition escarpée ne peut s'être développée avant.

Dans un deuxième temps, pour démontrer une propriété de déplacement d'interface, nous construisons des sous- et sur-solutions basées, cette fois, sur les deux premiers termes du développement asymptotique formel de la solution, qui sont les solutions d'un problème stationnaire unidimensionnel associé et de sa version linéarisée.

Par imbrication des deux paires de sous- et sur-solutions construites, nous obtenons nos principaux résultats.

### Systèmes de réaction-diffusion

Nous étendons ensuite les résultats que nous avons obtenus pour l'équation seule à des systèmes de réaction-diffusion de la forme

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2} f^\varepsilon(u,v) & \text{dans } \Omega \times (0,+\infty), \\ v_t = D\Delta v + h(u,v) & \text{dans } \Omega \times (0,+\infty), \end{cases} \tag{4}$$

où le terme de réaction $f^\varepsilon$ est donné par

$$f^\varepsilon(u,v) = f(u) + \varepsilon f_1(u,v) + \varepsilon^2 f_2^\varepsilon(u,v),$$

avec des conditions aux limites de Neumann homogènes et des conditions initiales. Nos preuves ne s'appuyant pas sur un principe de comparaison, nous ne faisons aucune hypothèse de monotonie sur les termes de réaction non-linéaires $f^\varepsilon$ et $h$. A l'aide de la méthode des rectangles invariants, nous démontrons que la solution $(u^\varepsilon, v^\varepsilon)$ existe pour $t \geq 0$.

Ce système contient deux cas particuliers importants, à savoir le système de FitzHugh-Nagumo qui modélise la transmission nerveuse :

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}(f(u) - \varepsilon v), \\ v_t = D\Delta v + \alpha u - \beta v, \end{cases} \tag{5}$$

et certains systèmes prédateur-proie intervenant en écologie :

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}\big((1-u)(u-1/2) - \varepsilon v\big)u, \\ v_t = D\Delta v + (\alpha u - \beta v)v. \end{cases} \tag{6}$$

Le problème à frontière libre limite est constitué d'une équation de déplacement d'interface couplée à une équation parabolique :

$$V_n = -(N-1)\kappa - c_0 \int_{\alpha_-}^{\alpha_+} f_1(r, \tilde{v}(x,t))dr \qquad \text{sur} \ \ \Gamma_t, \tag{7a}$$

$$\tilde{v}_t = D\Delta\tilde{v} + h(\tilde{u}, \tilde{v}) \qquad \text{dans} \ \ \Omega \times (0,T], \tag{7b}$$

où la fonction en escalier $\tilde{u}$, valant $\alpha_-$ à l'intérieur de l'interface et $\alpha_+$ à l'extérieur, est complètement déterminée par l'interface $\Gamma_t$. Ce problème admet une solution classique unique $(\Gamma, \tilde{v})$ sur un intervalle de temps $[0,T]$.

Nous démontrons que nos estimations de l'épaisseur et de la localisation de la zone de transition de la solution $u^\varepsilon$ restent vraies pour les systèmes de réaction-diffusion considérés. Pour cela, nous considérons la première équation du système (4) comme une équation d'Allen-Cahn perturbée et cherchons à appliquer les résultats obtenus pour l'équation seule. Ceci nécessite de prouver l'analogue de (2), c'est-à-dire l'estimation a priori

$$v^\varepsilon(x,t) = \tilde{v}(x,t) + O(\varepsilon). \tag{8}$$

La difficulté tient au fait qu'ici, le terme de perturbation dépend de la fonction $v^\varepsilon$, dont le comportement n'est pas parfaitement connu. De plus, la solution $u^\varepsilon$ convergeant vers la fonction discontinue $\tilde{u}$, la fonction $\tilde{v}$ présente un déficit de régularité face à $v^\varepsilon$ ; plus précisément, par les estimations paraboliques, $v^\varepsilon$ est au moins de classe $C^{2,1}$ alors que $\tilde{v}$ est seulement de classe $C^{1+\vartheta, \frac{1+\vartheta}{2}}$. L'idée de la démonstration est la suivante : on réexamine d'abord l'équation pour le déplacement de l'interface (3) en perturbant le terme non-linéaire $g$ ainsi que l'interface initiale $\Gamma_0$. Ensuite, tout en accordant une certaine liberté aux conditions initiales, on construit une application $\Phi$ comme suit : à tout $v$, on fait correspondre un terme $f_1$ de perturbation du terme non-linéaire, et donc une interface solution d'une équation de la forme (7a) ; à cette interface, on associe une fonction en escalier, et donc une solution d'une équation de la forme (7b), notée $\Phi[v]$. Par construction, la solution $(\Gamma, \tilde{v})$ du système (7a)—(7b) est telle que $\tilde{v}$ est un point fixe de $\Phi$. En se basant sur des estimations de la solution fondamentale de l'équation de la chaleur $v_t = D\Delta v$, et sur le fait que $u^\varepsilon$ varie peu en dehors d'un voisinage d'ordre $\varepsilon$ de l'interface $\Gamma_t$, on démontre que $\Phi[v^\varepsilon] = v^\varepsilon + O(\varepsilon)$, ou encore que la fonction $v^\varepsilon$ est presque un point fixe de $\Phi$. Ceci, combiné au fait que $\Phi$ est contractante, nous permet alors de démontrer l'estimation essentielle (8), de laquelle découlent les résultats pour les systèmes.

## Chapitre 2 : Limite singulière d'un système de chimiotactisme-croissance avec condition initiale quelconque

Cette partie fait l'objet d'un article soumis pour publication dans *Advances in Differential Equations.*

Les Dictyostelides sont des organismes pouvant prendre alternativement une forme unicellulaire (amibe) ou une forme pluricellulaire. On les trouve dans les tapis de feuilles en décomposition. On a observé chez eux un cycle de vie assez complexe. Dans un premier temps, les amibes se dispersent et se nourrissent de bactéries. Lorsque ces dernières ont toutes été consommées, les amibes émettent un attracteur, dit chimiotactique, de façon à attirer les amibes voisines. Par agrégation, il se forme un organisme pluricellulaire de centaines de milliers de cellules, sorte de limace de quelques millimètres de longueur. Cet organisme est composé de trois parties, un disque basal, un pied et une masse de spores qui donnent naissance à de nouvelles amibes.

L'étude des méchanismes qui sous-tendent de tels phénomènes générant un organisme pluricellulaire est d'un grand intérêt en biologie. En 1970, Keller et Segel [53] ont proposé le système d'équations paraboliques

$$\begin{cases} u_t &= d_u \Delta u - \nabla \cdot (u \nabla \chi(v)), \\ \tau v_t &= d_v \Delta v + u - \gamma v, \end{cases} \tag{9}$$

pour la modélisation mathématique de ce processus d'agrégation ; la fonction $u$ représente la concentration d'amibes et $v$ celle de l'attracteur chimiotactique, dont le taux de dégradation est donné par la constante positive $\gamma$ ; $d_u$ et $d_v$ sont des coefficients de diffusion supposés constants ; $\tau$ est une constante positive ; la fonction strictement croissante $\chi$ exprime l'attraction des amibes par la substance chimiotactique. Les amibes sont ainsi soumises à deux phénomènes : la diffusion et le chimiotactisme, c'est-à-dire une propension à se diriger vers la substance attractrice qu'elles ont elles-mêmes sécrétée. Le problème est complété par des conditions initiales ainsi que des conditions aux limites de Neumann homogènes.

De nombreuses analyses mathématiques de ce modèle ont été faites. Il s'avère que l'agrégation, qui se traduit mathématiquement par un phénomène d'explosion en temps fini, n'est pas systématique. Par exemple, elle ne se produit jamais en dimension un d'espace alors qu'en dimension deux elle ne se produit que si le nombre initial d'amibes est suffisamment élevé.

Dans ce Chapitre, nous étudions un système d'équations proposé par Mimura et Tsujikawa [57], où interviennent un terme de diffusion, un terme de couplage lié au chimiotactisme ainsi qu'un terme de croissance. Plus précisément, on pose $\tau = 0$ et on étudie le problème de Neumann homogène

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)) + \dfrac{1}{\varepsilon^2} f_\varepsilon(u) & \text{dans } \Omega \times (0, +\infty), \\ 0 = \Delta v + u - \gamma v & \text{dans } \Omega \times (0, +\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{sur } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{dans } \Omega, \end{cases} \tag{10}$$

où $\varepsilon > 0$ est un petit paramètre ; le terme de réaction non-linéaire est donné par

$$f_\varepsilon(u) = u(1-u)(u - \frac{1}{2}) + \varepsilon \alpha u(1-u),$$

avec $\alpha$ constante positive.

Le processus d'agrégation sous-jacent est alors différent de celui du modèle sans terme de croissance. Sous l'hypothèse d'une condition initiale bien préparée, c'est-à-dire présentant déjà une interface, Bonami, Hilhorst, Logak et Mimura [17] ont montré que, lorsque $\varepsilon \to 0$, la solution $(u^\varepsilon, v^\varepsilon)$ converge vers $(u^0, v^0)$, où $u^0$ est une fonction en escalier prenant les valeurs 1 et 0. Le problème à frontière libre limite est donné par les équations couplées :

$$\begin{cases} V_n = -(N-1)\kappa + \dfrac{\partial \chi(v^0)}{\partial n} + \sqrt{2}\alpha & \text{sur } \Gamma_t, \\ \Gamma_t\big|_{t=0} = \Gamma_0 & \\ 0 = \Delta v^0 + u^0 - \gamma v^0 & \text{dans } \Omega \times (0, T], \\ \dfrac{\partial v^0}{\partial \nu} = 0 & \text{sur } \partial\Omega \times (0, T], \end{cases} \tag{11}$$

où l'interface initiale $\Gamma_0$, définie par $\Gamma_0 := \{x \in \Omega,\ u_0(x) = 1/2\}$, est une hypersurface sans bords régulière ; $n$ désigne le vecteur normal, unitaire, extérieur à $\Gamma_t$, $V_n$ la vitesse de déplacement de l'interface le long de la normale et $\kappa$ la courbure moyenne de l'interface. La première équation traduit le déplacement de la frontière libre séparant les régions $\{u^0 = 1\}$ et $\{u^0 = 0\}$. Ce problème admet une solution classique unique $(\Gamma, v^0)$ sur un intervalle $[0, T]$.

Dans ce Chapitre, nous étendons les résultats de [17] en supposant la condition initiale $u_0$ très générale. Après avoir rappelé le principe de comparaison utilisé dans [17], nous démontrons une propriété de génération d'interface. Nous nous appuyons pour cela sur une paire de sous- et sur-solutions construites à l'aide de la solution de l'équation différentielle ordinaire $u_t = \varepsilon^{-2} f(u)$, obtenue en négligeant la diffusion et le chimiotactisme. Nous étudions ensuite le déplacement de l'interface, en nous appuyant sur des sous- et sur-solutions construites à l'aide de la solution d'un problème stationnaire unidimensionnel associé ; la démonstration s'appuie sur des estimations de la fonction de Green associée au problème de Neumann homogène sur $\Omega$ pour l'opérateur $-\Delta + \gamma$. En imbriquant les deux paires de sous- et sur-solutions construites, on démontre que l'épaisseur de la zone de transition développée par la solution $u^\varepsilon$ est d'ordre $\varepsilon$, ainsi que la distance de Hausdorff entre l'ensemble des points $\Gamma_t^\varepsilon := \{x \in \Omega,\ u^\varepsilon(x,t) = 1/2\}$ et l'interface $\Gamma_t$, solution du problème à frontière libre limite (11). Ces résultats sont optimaux.

## Chapitre 3 : Limite singulière d'une equation d'Allen-Cahn inhomogène et anisotrope

Cette partie de la thèse correspond à des travaux réalisés en collaboration avec H. Garcke (Université de Regensburg), D. Hilhorst (Université de Paris-Sud), H. Matano (Université de Tokyo) et R. Schätzle (Université de Tübingen).

Le contexte de cette étude est la modélisation de mouvements d'interfaces en science des matériaux, où la vitesse normale de déplacement de l'interface dépend de l'angle du vecteur normal avec une direction fixe. On parle de mouvement anisotrope d'interface.

Nous étudions le problème de Neumann homogène pour une équation d'Allen-Cahn anisotrope et inhomogène :

$$\begin{cases} u_t = \nabla \cdot a_p(x, \nabla u) + \dfrac{1}{\varepsilon^2} f(u) & \text{dans } \Omega \times (0, +\infty), \\ a_p(x, \nabla u) \cdot \nu = 0 & \text{sur } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{dans } \Omega, \end{cases} \tag{12}$$

où $\varepsilon > 0$ est un petit paramètre. On suppose que le terme non-linéaire $f$ possède exactement trois zéros $0 < a < 1$, que sa pente est strictement négative aux équilibres $u = 0$ et $u = 1$, strictement positive à l'équilibre $u = a$, ce qui assure son caractère bistable ; nous supposons également que $f$ satisfait la condition intégrale

$$\int_0^1 f(u)du = 0.$$

L'anisotropie intervient dans le terme $\nabla \cdot a_p(x, \nabla u)$, où nous supposons la fonction $a(x, p)$ strictement positive, strictement convexe et 2 homogène (en la variable $p$) sur $\overline{\Omega} \times \mathbb{R}^N \setminus \{0\}$. Nous utilisons la notation $a_p(x, p)$ pour le vecteur gradient $(\dfrac{\partial a}{\partial p_1}, \cdots, \dfrac{\partial a}{\partial p_N})(x, p)$ et nous supposons que la fonction $a(x, p)$ est de classe $C^{3+\vartheta}_{loc}$ seulement sur $\overline{\Omega} \times \mathbb{R}^N \setminus \{0\}$, autrement dit que le terme de diffusion $\nabla \cdot a_p(x, \nabla u)$ est singulier aux points où le gradient de la solution s'annule. Il nous faut donc utiliser une notion de solution faible, pour laquelle nous démontrons un principe de comparaison.

L'équation parabolique dans le problème (12) contient, en particulier, l'équation inhomogène

$$u_t = \operatorname{div}(A(x)\nabla u) + \frac{1}{\varepsilon^2} f(u), \tag{13}$$

où $A(x)$ est une matrice symétrique définie positive, et l'équation anisotrope

$$u_t = \operatorname{div}\big(\mathcal{A}(\nabla u)\big) + \frac{1}{\varepsilon^2} f(u), \tag{14}$$

où les coefficients de la matrice $\nabla_p \otimes \mathcal{A} = \nabla_p\, {}^t\!\mathcal{A}$ peuvent être singuliers au point $p = 0$.

Nous commençons par construire une métrique adaptée au problème anisotrope, en nous inspirant de résultats de Bellettini, Paolini and Venturini sur une métrique de Finsler, [9] et [10].

Nous démontrons que, lorsque $\varepsilon \to 0$, la solution $u^\varepsilon$ converge presque partout vers $\tilde{u}$, où $\tilde{u}$ est une fonction en escalier prenant les valeurs 0 et 1, les régions $\{\tilde{u} = 0\}$ et $\{\tilde{u} = 1\}$ étant séparées par une interface limite qui se déplace. Le problème à frontière libre limite est donné par :

$$\begin{cases} V_{n,\phi} = -(N-1)\overline{\kappa}_\phi \quad \text{sur } \Gamma_t, \\ \Gamma_t\Big|_{t=0} = \Gamma_0, \end{cases} \tag{15}$$

où l'interface initiale $\Gamma_0$ est définie par $\Gamma_0 := \{x \in \Omega,\ u_0(x) = a\}$, où $V_{n,\phi}$ désigne la vitesse de déplacement anisotrope de l'interface le long de la normale anisotrope à $\Gamma_t$ et $\overline{\kappa}_\phi$ une version anisotrope de la courbure moyenne de l'interface. L'écriture du problème limite se complique sensiblement en géométrie euclidienne :

$$\begin{cases} \dfrac{1}{\sqrt{2a(x,n)}} V_n = -\nabla \cdot \Big[\dfrac{1}{\sqrt{2a(x,n)}}\, a_p(x,n)\Big] \quad \text{sur } \Gamma_t, \\ \Gamma_t\Big|_{t=0} = \Gamma_0. \end{cases} \tag{16}$$

Si $\Gamma_0$ est assez régulière, le problème limite admet une unique solution classique sur un intervalle de temps $[0, T]$.

Avec des hypothèses faibles sur le profil de la condition initiale, nous effectuons une analyse rigoureuse de la génération et du déplacement de l'interface. Nous nous appuyons pour cela sur deux paires distinctes de sous- et sur-solutions. Pour l'étude de la génération de l'interface, on perturbe la solution de l'équation différentielle ordinaire $u_t = \varepsilon^{-2} f(u)$, obtenue en négligeant la diffusion anisotrope. Pour l'étude du déplacement de l'interface, on utilise la solution d'un problème stationnaire unidimensionnel associé. En imbriquant ces deux paires de sous- et sur-solutions, on démontre que l'épaisseur de la zone de transition développée par la solution est d'ordre $\varepsilon$, améliorant ainsi des résultats connus [13], [70].

# Chapter 1

# The singular limit of the Allen-Cahn equation and the FitzHugh-Nagumo system

We consider an Allen-Cahn type equation of the form $u_t = \Delta u + \varepsilon^{-2} f^\varepsilon(x,t,u)$, where $\varepsilon > 0$ is a small parameter and $f^\varepsilon$ a bistable nonlinearity associated with a double-well potential whose well-depths are slightly unbalanced by order $\varepsilon$. Given a rather general initial datum $u_0$ that is independent of $\varepsilon$, we perform a rigorous analysis of both the generation and the motion of interface. More precisely we show that the solution develops a steep transition layer within the time scale of order $\varepsilon^2 |\ln \varepsilon|$, and that the layer obeys the law of motion that coincides with the formal asymptotic limit within an error margin of order $\varepsilon$. This is an optimal estimate that has not been known before for solutions with general initial datum, even in the case where $f^\varepsilon(x,t,u) = f(u)$.

Next we consider systems of reaction-diffusion equations of the form

$$\begin{cases} u_t = \Delta u + \varepsilon^{-2} f^\varepsilon(u,v), \\ v_t = D\Delta v + h(u,v), \end{cases}$$

which include the FitzHugh-Nagumo system as a special case. Given a rather general initial datum $(u_0, v_0)$, we show that the component $u$ develops a steep transition layer and that all the above-mentioned results remain true for the $u$-component of these systems.

## 1.1 Introduction

### 1.1.1 Perturbed Allen-Cahn equation

In some classes of nonlinear diffusion equations, solutions often develop sharp internal layers — or "interfaces" — that separate the spatial domain into different phase regions. This happens, in particular, when the diffusion coefficient is very small or the reaction term is very large. The motion of such interfaces is often driven by their curvature. A typical example is the Allen-Cahn equation $u_t = \Delta u + \varepsilon^{-2} f(u)$, where $\varepsilon > 0$ is a small parameter and $f(u)$ is a bistable nonlinearity, whose meaning is explained below. A usual strategy for studying such phenomena is to first derive the "sharp interface limit" as $\varepsilon \to 0$ by a formal analysis, then to check if this limit gives good approximation of the behavior of actual layers.

In this Chapter we consider a perturbed Allen-Cahn type equation of the form

$$(P^\varepsilon) \quad \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}(f(u) - \varepsilon g^\varepsilon(x,t,u)) & \text{in } \Omega \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

and study the behavior of layers near the sharp interface limit as $\varepsilon \to 0$. Here, $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ and $\nu$ is the Euclidian unit normal vector exterior to $\partial\Omega$. The nonlinearity is given by $f(u) := -W'(u)$, where $W(u)$ is a double-well potential with equal well-depth, taking its global minimum value at $u = \alpha_-, \alpha_+$. More precisely we assume that $f$ is of class $C^2$ on $\mathbb{R}$ and has exactly three zeros $\alpha_- < a < \alpha_+$ such that

$$f'(\alpha_\pm) < 0, \quad f'(a) > 0 \quad \text{(bistable nonlinearity)}, \tag{1.1}$$

and that

$$\int_{\alpha_-}^{\alpha_+} f(u)\, du = 0. \tag{1.2}$$

The condition (1.1) implies that the potential $W(u)$ attains its local minima at $u = \alpha_-, \alpha_+$, and (1.2) implies that $W(\alpha_-) = W(\alpha_+)$. In other words, the two stable zeros of $f$, namely $\alpha_-$ and $\alpha_+$, have "balanced" stability. A typical example is given by the cubic nonlinearity $f(u) = u(1-u^2)$.

The term $\varepsilon g^\varepsilon$ represents a small perturbation, where $g^\varepsilon(x,t,u)$ is a function defined on $\overline{\Omega} \times [0,+\infty) \times \mathbb{R}$. This has the role of breaking the balance of the two stable zeros slightly. In the special case where $g^\varepsilon \equiv 0$, Problem $(P^\varepsilon)$ reduces to the usual Allen-Cahn equation. As we will explain later, our main results are new even for this special case.

We assume that $g^\varepsilon$ is $C^2$ in $x$ and $C^1$ in $t$, $u$, and that, for any $T > 0$, there exist $\vartheta \in (0,1)$ and $C > 0$ such that, for all $(x,t,u) \in \overline{\Omega} \times [0,T] \times \mathbb{R}$,

$$|\Delta_x g^\varepsilon(x,t,u)| \leq C\varepsilon^{-1} \quad \text{and} \quad |g^\varepsilon_t(x,t,u)| \leq C\varepsilon^{-1}, \tag{1.3}$$

$$|g^\varepsilon_u(x,t,u)| \leq C, \tag{1.4}$$

$$\|g^\varepsilon(\cdot,\cdot,u)\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}(\overline{\Omega}\times[0,T])} \leq C. \tag{1.5}$$

Moreover, we assume that there exists a function $g(x,t,u)$ and a constant, which we denote again by $C$, such that

$$|g^\varepsilon(x,t,u) - g(x,t,u)| \leq C\varepsilon, \tag{1.6}$$

for all small $\varepsilon > 0$. Note that the estimate (1.5) and the pointwise convergence $g^\varepsilon \to g$ (as $\varepsilon \to 0$) imply that $g$ satisfies the same estimate as (1.5).

For technical reasons we also assume that

$$\frac{\partial g^\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times [0,T] \times \mathbb{R}, \tag{1.7}$$

which, in turn, implies the same Neumann boundary condition for $g$. Apart from these bounds and regularity requirements, we do not make any specific assumptions on the perturbation term $g^\varepsilon$.

**Remark 1.1.1.** *Since we will consider only bounded solutions in this Chapter, it is sufficient to assume* (1.3)—(1.5) *to hold in some bounded interval $-M \le u \le M$. Note that if $g^\varepsilon$ does not depend on $\varepsilon$, then assumptions* (1.3)—(1.5) *are automatically satisfied on any bounded interval $-M \le u \le M$.* □

**Remark 1.1.2.** *The reason why we do not assume more smoothness on $g$ is that we will later apply our results to systems of equations, including the FitzHugh-Nagumo system, in which $g^\varepsilon = g^\varepsilon(x,t)$ loses $C^{2,1}$-smoothness as $\varepsilon \to 0$.* □

**Remark 1.1.3.** *The equation in $(P^\varepsilon)$ can be expressed in the form*

$$u_t = \Delta u + \frac{1}{\varepsilon^2} f^\varepsilon(x,t,u),$$

*where $f^\varepsilon$ is $C^2$ in $x$, $\varepsilon$ and $C^1$ in $t$, $u$. Conversely, by setting*

$$g^\varepsilon(x,t,u) = -\frac{f^\varepsilon(x,t,u) - f(u)}{\varepsilon}, \qquad g(x,t,u) = -\left.\frac{\partial f^\varepsilon}{\partial \varepsilon}(x,t,u)\right|_{\varepsilon=0},$$

*the above equation is reduced to that in $(P^\varepsilon)$. The conditions* (1.3) *and* (1.6) *then follow automatically from the above regularity assumptions on $f^\varepsilon$. The condition* (1.5) *holds if we impose slightly stronger regularity on $f^\varepsilon$.* □

As for the initial datum $u_0(x)$, we assume $u_0 \in C^2(\overline{\Omega})$. Throughout the present Chapter the constant $C_0$ will stand for the following quantity:

$$C_0 := \|u_0\|_{C^0(\overline{\Omega})} + \|\nabla u_0\|_{C^0(\overline{\Omega})} + \|\Delta u_0\|_{C^0(\overline{\Omega})}. \tag{1.8}$$

Furthermore we define the "initial interface" $\Gamma_0$ by

$$\Gamma_0 := \{x \in \Omega,\ u_0(x) = a\}, \tag{1.9}$$

and suppose that $\Gamma_0$ is a $C^{3+\vartheta}$ hypersurface without boundary such that, $n$ being the outward unit normal vector to $\Gamma_0$,

$$\Gamma_0 \subset\subset \Omega \quad \text{and} \quad \nabla u_0(x) \cdot n(x) \neq 0 \quad \text{if } x \in \Gamma_0, \tag{1.10}$$

$$u_0 > a \quad \text{in } \Omega_0^+, \quad u_0 < a \quad \text{in } \Omega_0^-, \tag{1.11}$$

where $\Omega_0^-$ denotes the region enclosed by the hypersurface $\Gamma_0$ and $\Omega_0^+$ the region enclosed between the boundary of the domain $\partial\Omega$ and the hypersurface $\Gamma_0$.

It is standard that Problem $(P^\varepsilon)$ has a unique smooth solution, which we denote by $u^\varepsilon$. As $\varepsilon \to 0$, a formal asymptotic analysis shows the following: in the very early stage, the diffusion term $\Delta u$ is negligible compared with the reaction term $\varepsilon^{-2}(f(u) - \varepsilon g^\varepsilon(x,t,u))$; it follows that, in the rescaled time scale $\tau = t/\varepsilon^2$, the equation is well approximated by the ordinary differential equation $u_\tau = f(u) + O(\varepsilon)$. Hence, $f$ being a bistable nonlinearity, the value of $u^\varepsilon$ quickly becomes close to either $\alpha_+$ or $\alpha_-$ in most part of $\Omega$, creating a steep interface (transition layer) between the regions $\{u^\varepsilon \approx \alpha_-\}$ and $\{u^\varepsilon \approx \alpha_+\}$. Once such an interface develops, the diffusion term becomes large near the interface, and comes to balance with the reaction term. As a result, the interface ceases rapid development and starts to propagate in a much slower time scale.

To study such interfacial behavior, it is useful to consider a formal asymptotic limit of Problem $(P^\varepsilon)$ as $\varepsilon \to 0$. Then the limit solution $\tilde{u}(x,t)$ will be a step function taking the value $\alpha_+$ on one side of the interface, and $\alpha_-$ on the other side. This sharp interface, which we will denote by $\Gamma_t$, obeys a certain law of motion, which is expressed as follows (see Section 1.2 for details):

$$(P^0) \quad \begin{cases} V_n = -(N-1)\kappa + c_0 \displaystyle\int_{\alpha_-}^{\alpha_+} g(x,t,r)dr & \text{on } \Gamma_t, \\ \Gamma_t\big|_{t=0} = \Gamma_0, \end{cases}$$

where $V_n$ is the normal velocity of $\Gamma_t$ in the exterior direction, $\kappa$ the mean curvature at each point of $\Gamma_t$, $c_0$ the constant defined by

$$c_0 = \left[\sqrt{2}\int_{\alpha_-}^{\alpha_+} (W(s) - W(\alpha_-))^{1/2} ds\right]^{-1}, \tag{1.12}$$

with $W$ the double-well potential associated with $f$:

$$W(s) = -\int_a^s f(r)dr.$$

In the sequel, $\gamma$ will stand for:

$$\gamma(x,t) = c_0 \int_{\alpha_-}^{\alpha_+} g(x,t,r)dr. \tag{1.13}$$

It is well known that Problem $(P^0)$ possesses locally in time a unique smooth solution, say $\Gamma = \bigcup_{0\le t\le T}(\Gamma_t \times \{t\})$, for some $T > 0$. More precisely, so as $g$, the function $\gamma$ is in $C^{1+\vartheta, \frac{1+\vartheta}{2}}$, which implies, by the standard theory of parabolic equations, that $\Gamma$ is of class $C^{3+\vartheta, \frac{3+\vartheta}{2}}$. For more details, we refer to [23], Lemma 2.1.

Next we set

$$Q_T := \Omega \times [0,T],$$

and for each $t \in [0,T]$, we denote by $\Omega_t^-$ the region enclosed by $\Gamma_t$, and by $\Omega_t^+$ the region enclosed between $\partial\Omega$ and $\Gamma_t$. We define a step function $\tilde{u}(x,t)$ by

$$\tilde{u}(x,t) = \begin{cases} \alpha_+ & \text{in } \Omega_t^+ \\ \alpha_- & \text{in } \Omega_t^- \end{cases} \quad \text{for } t \in [0,T], \tag{1.14}$$

which represents the formal asymptotic limit of $u^\varepsilon$ (or the *sharp interface limit*) as $\varepsilon \to 0$.

The aim of the present Chapter is to make a rigorous and detailed study of the limiting behavior of the solution $u^\varepsilon$ of Problem $(P^\varepsilon)$ as $\varepsilon \to 0$. Our first main result, Theorem 1.1.4, describes the profile of the solution after a very short initial period. It asserts that: given a virtually arbitrary initial datum $u_0$, the solution $u^\varepsilon$ quickly becomes close to $\alpha_\pm$, except in a small neighborhood of the initial interface $\Gamma_0$, creating a steep transition layer around $\Gamma_0$ (*generation of interface*). The time needed to develop such a transition layer, which we will denote by $t^\varepsilon$, is of order $\varepsilon^2|\ln\varepsilon|$. The theorem then states that the solution $u^\varepsilon$ remains close to the step function $\tilde{u}$ on the time interval $[t^\varepsilon, T]$ (*motion of interface*); in other words, the motion of the transition layer is well approximated by the limit interface equation $(P^0)$.

**Theorem 1.1.4 (Generation and motion of interface).** *Let $\eta$ be an arbitrary constant satisfying $0 < \eta < \frac{1}{2}\min(a-\alpha_-, \alpha_+ - a)$ and set*

$$\mu = f'(a).$$

*Then there exist positive constants $\varepsilon_0$ and $C$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $t^\varepsilon \le t \le T$, where $t^\varepsilon := \mu^{-1}\varepsilon^2|\ln\varepsilon|$, we have*

$$u^\varepsilon(x,t) \in \begin{cases} [\alpha_- - \eta, \alpha_+ + \eta] & \text{if} \quad x \in \mathcal{N}_{C\varepsilon}(\Gamma_t) \\ [\alpha_- - \eta, \alpha_- + \eta] & \text{if} \quad x \in \Omega_t^- \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t) \\ [\alpha_+ - \eta, \alpha_+ + \eta] & \text{if} \quad x \in \Omega_t^+ \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t), \end{cases} \tag{1.15}$$

*where $\mathcal{N}_r(\Gamma_t) := \{x \in \Omega,\ dist(x, \Gamma_t) < r\}$ denotes the $r$-neighborhood of $\Gamma_t$.*

**Corollary 1.1.5 (Convergence).** *As $\varepsilon \to 0$, the solution $u^\varepsilon$ converges to $\tilde{u}$ everywhere in $\bigcup_{0<t\le T}(\Omega_t^\pm \times \{t\})$.*

The next theorem is concerned with the relation between the actual interface $\Gamma_t^\varepsilon := \{x \in \Omega,\ u^\varepsilon(x,t) = a\}$ and the formal asymptotic limit $\Gamma_t$, which is given as the solution of Problem $(P^0)$.

**Theorem 1.1.6 (Error estimate).** *There exists $C > 0$ such that*

$$\Gamma_t^\varepsilon \subset \mathcal{N}_{C\varepsilon}(\Gamma_t) \quad \text{for } 0 \le t \le T. \tag{1.16}$$

**Corollary 1.1.7 (Convergence of interface).** *There exists $C > 0$ such that*

$$d_{\mathcal{H}}(\Gamma_t^\varepsilon, \Gamma_t) \le C\varepsilon \quad \text{for } 0 \le t \le T, \tag{1.17}$$

*where*

$$d_{\mathcal{H}}(A,B) := \max\big\{\sup_{a\in A} d(a,B),\, \sup_{b\in B} d(b,A)\big\}$$

*denotes the Hausdorff distance between two compact sets $A$ and $B$. Consequently, $\Gamma_t^\varepsilon \to \Gamma_t$ as $\varepsilon \to 0$, uniformly in $0 \le t \le T$, in the sense of Hausdorff distance.*

Note that the estimates (1.16) and (1.17) follow from Theorem 1.1.4 in the range $t^\varepsilon \le t \le T$, but the range $0 \le t \le t^\varepsilon$ has to be treated by a separate argument. In fact, this is the time range in which a clear transition layer is formed rapidly from an arbitrarily given initial datum, therefore the behavior of the solution is quite different from that in the later time range $t^\varepsilon \le t \le T$, where things move more slowly.

The estimate (1.15) in our Theorem 1.1.4 implies that, once a transition layer is formed, its thickness remains within order $\varepsilon$ for the rest of time. Here, by "thickness of interface" we mean the smallest $r > 0$ satisfying

$$\{ x \in \Omega,\ u^\varepsilon(x,t) \notin [\alpha_- - \eta,\ \alpha_- + \eta] \cup [\alpha_+ - \eta,\ \alpha_+ + \eta] \} \subset \mathcal{N}_r(\Gamma_t^\varepsilon).$$

Naturally this quantity depends on $\eta$, but the estimates (1.15) and (1.17) assert that it is bounded by $2C\varepsilon$ (with the constant $C$ depending on $\eta$), hence it remains within $O(\varepsilon)$ regardless of the choice of $\eta > 0$.

**Remark 1.1.8 (Optimality of the thickness estimate).** *The above $O(\varepsilon)$ estimate is optimal, i.e., the interface cannot be thinner than this order. In fact, rescaling time and space as $\tau := t/\varepsilon^2$, $y := x/\varepsilon$, the equation reads as*

$$u_\tau = \Delta_y u + f(u) - \varepsilon\, g^\varepsilon.$$

*Thus, by the uniform boundedness of $u$ and by standard parabolic estimates, we have $|\nabla_y u| \leq M$ for some constant $M > 0$, which implies*

$$|\nabla_x u(x,t)| \leq \frac{M}{\varepsilon}.$$

*From this bound it is clear that the thickness of interface cannot be smaller than $M^{-1}(\alpha_+ - \alpha_-)\,\varepsilon$, hence, by (1.15), it has to be exactly of order $\varepsilon$. Intuitively, the order $\varepsilon$ estimate follows also from the formal asymptotic expansion (1.24), but the validity of such an expansion is far from obvious for solutions with arbitrary initial datum.* □

As far as we know, our $O(\varepsilon)$ estimate is new, even in the special case where $g^\varepsilon \equiv 0$, provided that $N \geq 2$. Previously, the best thickness estimate in the literature was of order $\varepsilon|\ln\varepsilon|$ (see [20]), except that Xinfu Chen has recently obtained an order $\varepsilon$ estimate for the case $N = 1$ by a different argument (private communication). We also refer to a forthcoming article [51] by Karali, Nakashima, Hilhorst and Matano, in which an order $\varepsilon$ estimate is established for a Lotka-Volterra competition-diffusion system, with large spatial inhomogeneity, whose nonlinearity is of the balanced bistable type.

**Remark 1.1.9 (Optimality of the generation time).** *The estimate* (1.15) *also implies that the generation of interface takes place within the time span of $t^\varepsilon$. This estimate is optimal. In other words, a well-developed interface cannot appear much earlier; see Proposition 1.3.10 for details.* □

The singular limit of Allen-Cahn equation was first studied in the pioneering work of Allen and Cahn [2] and, slightly later, in Kawasaki and Ohta [52] from the point of view of physicists. They derived the interface equation by formal asymptotic analysis, thereby revealing that the interface moves by the mean curvature. Triggered by these early observations, this problem has become a subject of extensive mathematical studies.

Let us mention for instance the results of Bronsard and Kohn [18] in the case of spherical symmetry, the articles of de Mottoni and Schatzman [59, 60] and those of Chen [20, 21]. These results prove convergence to the limit interface equation in a classical framework; that is, under the assumption that the limit interface $\Gamma_t$ is a smooth hypersurface. As for the case where

$\Gamma_t$ is a viscosity or a weak solution of the limit interface equation, we refer to the work of Barles, Soner and Souganidis [6], Evans, Soner and Souganidis [33], Ilmanen [48] and Barles and Souganidis [7].

As for Problem $(P^\varepsilon)$, whose nonlinearity is slightly unbalanced, the limit interface equation involves a pressure term as well as the curvature term as indicated in $(P^0)$. This fact has been long known on a formal level; see e.g. Rubinstein, Sternberg and Keller [66]. However, not much rigorous study has been made. Ei, Iida and Yanagida [30] proved rigorously that the motion of the layers of Problem $(P^\varepsilon)$ is well approximated by the limit interface equation $(P^0)$, on the condition that the initial datum has already a well developed transition layer whose profile depends on $\varepsilon$. In other words, they studied the motion of interface, but not the generation of interface.

### 1.1.2 Singular limit of reaction-diffusion systems

As a matter of fact, our results for the single equation can be extended to a important class of reaction-diffusion systems. More precisely, we consider systems of parabolic equations of the form:

$$(RD^\varepsilon)\quad \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2} f^\varepsilon(u,v) & \text{in } \Omega \times (0,+\infty), \\ v_t = D\Delta v + h(u,v) & \text{in } \Omega \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases}$$

where $D$ is a positive constant, and $f^\varepsilon$, $h$ are $C^2$ functions such that

**(F)** there exist $C^2$ functions $f_1(u,v)$, $f_2^\varepsilon(u,v)$ such that

$$f^\varepsilon(u,v) = f(u) + \varepsilon f_1(u,v) + \varepsilon^2 f_2^\varepsilon(u,v), \tag{1.18}$$

where $f(u)$ is a bistable nonlinearity satisfying (1.1), (1.2), and $f_2^\varepsilon$, along with its derivatives in $u, v$, remain bounded as $\varepsilon \to 0$;

**(H)** for any constant $L, M > 0$ there exists a constant $M_1 \geq M$ such that

$$h(u,-M_1) \geq 0 \geq h(u,M_1) \quad \text{for } |u| \leq L. \tag{1.19}$$

The conditions (F) and (H) imply that the system of ordinary differential equations

$$\dot{u} = \frac{1}{\varepsilon^2} f_\varepsilon(u,v), \qquad \dot{v} = h(u,v),$$

has a family of invariant rectangles of the form $\{|u| \leq L,\ |v| \leq M\}$, provided that $\varepsilon$ is sufficiently small. The maximum principle and standard parabolic estimates then guarantee that the solution $(u^\varepsilon, v^\varepsilon)$ of $(RD^\varepsilon)$ exists globally for $t \geq 0$ and remains bounded as $t \to \infty$ (see subsection 1.7.1 for details). Apart from (1.19), we do not make any specific assumptions on the function $h$.

Problem $(RD^\varepsilon)$ represents a large class of important reaction-diffusion systems including the FitzHugh-Nagumo system

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}(f(u) - \varepsilon v), \\ v_t = D\Delta v + \alpha u - \beta v, \end{cases} \tag{1.20}$$

which is a simplified model for nervous transmission, and the following type of prey-predator system that appears in mathematical ecology:

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}\big((1-u)(u-1/2) - \varepsilon v\big)u, \\ v_t = D\Delta v + (\alpha u - \beta v)v. \end{cases} \tag{1.21}$$

**Remark 1.1.10.** *In some equations such as the prey-predator system* (1.21)*, only nonnegative solutions are to be considered. In such a case, we replace the condition* (1.19) *by*

$$h(u,0) \geq 0 \geq h(u, M_1) \qquad \text{for } \; 0 \leq u \leq L,$$

*and assume* $f^\varepsilon(0,v) \geq 0$. *The rest of the argument remains the same.* □

Now the same formal analysis as is used to derive $(P^0)$ in Section 1.2 shows that the singular limit of $(RD^\varepsilon)$, as $\varepsilon \to 0$, is the following moving boundary problem:

$$(RD^0) \quad \begin{cases} V_n = -(N-1)\kappa - c_0\, F_1(\tilde{v}(x,t)) & \text{on } \; \Gamma_t, \\ \tilde{v}_t = D\Delta\tilde{v} + h(\tilde{u}, \tilde{v}) & \text{in } \; \Omega \times (0,T], \\ \dfrac{\partial \tilde{v}}{\partial \nu} = 0 & \text{on } \; \partial\Omega \times (0,T], \\ \Gamma_t\big|_{t=0} = \Gamma_0 & \\ \tilde{v}(x,0) = v_0(x) & \text{in } \; \Omega, \end{cases}$$

where $\tilde{u}$ is the step function defined in (1.14) and

$$F_1(v) = \int_{\alpha_-}^{\alpha_+} f_1(r,v)\, dr.$$

This is a system consisting of an equation of surface motion and a partial differential equation. Since $\tilde{u}$ is determined straightforwardly from $\Gamma_t$, in what follows, by a solution of $(RD^0)$ we mean the pair $(\Gamma, \tilde{v}) := (\Gamma_t, \tilde{v}(x,t))$. In the case of the FitzHugh-Nagumo system (1.20), the interface equation in $(RD^0)$ reduces to

$$\begin{cases} V_n = -(N-1)\kappa + c_0\,(\alpha_+ - \alpha_-)\tilde{v}(x,t), \\ \tilde{v}_t = D\Delta\tilde{v} + \alpha\tilde{u} - \beta\tilde{v}, \end{cases}$$

while in the prey-predator system (1.21), $(RD^0)$ reduces to

$$\begin{cases} V_n = -(N-1)\kappa + c_0\,\tilde{v}(x,t)/2, \\ \tilde{v}_t = D\Delta\tilde{v} + (\alpha\tilde{u} - \beta\tilde{v})\tilde{u}. \end{cases}$$

Note that the positive sign in front of the term $c_0\tilde{v}(x,t)$ in the interface equation implies an inhibitory effect on $\tilde{u}$, since the velocity $V_n$ is measured in the exterior normal direction, toward which $\tilde{u}$ decreases.

**Lemma 1.1.11 (Local existence).** *Assume that* $v_0 \in C^2(\overline{\Omega})$ *and that* $\Gamma_0$ *is a* $C^{2+\vartheta}$ *hypersurface which is the boundary of a domain* $D_0 \subset\subset \Omega$. *Then there exists* $T > 0$ *such that the limit free boundary Problem* $(RD^0)$ *has a unique solution* $(\Gamma, \tilde{v})$ *in the interval* $[0, T]$. *By the standard theory of parabolic equations,* $\Gamma$ *is of class* $C^{2+\vartheta, \frac{2+\vartheta}{2}}$ *and* $\tilde{v}$ *is of class* $C^{1+\vartheta, \frac{1+\vartheta}{2}}$.

The existence result was established in [24], Theorem 3.2 and following lemmas. The uniqueness can be obtained by using Theorem 2 in [21].

Our main results for the system $(RD^\varepsilon)$ are the following:

**Theorem 1.1.12 (Thickness of interface).** *Let* (1.18) *and* (1.19) *hold (or let the assumptions in Remark 1.1.10 hold). Assume also that* $u_0$ *satisfies* (1.10) *and* (1.11). *Then the same conclusion as in Theorem 1.1.4 holds for* $(RD^\varepsilon)$.

**Corollary 1.1.13 (Convergence).** *Under the assumptions of Theorem 1.1.12, the same conclusion as in Corollary 1.1.5 holds for* $(RD^\varepsilon)$.

**Theorem 1.1.14 (Error estimate).** *Let the assumptions of Theorem 1.1.12 hold. Then the same conclusion as in Theorem 1.1.6 holds for* $(RD^\varepsilon)$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|v^\varepsilon - \tilde{v}\|_{L^\infty(\Omega \times (0,T))} \leq C\varepsilon.$$

**Corollary 1.1.15 (Convergence of interface).** *Under the assumptions of Theorem 1.1.12, the same conclusion as in Corollary 1.1.7 holds for* $(RD^\varepsilon)$.

The organization of this Chapter is as follows. In Section 1.2, we derive the interface equation $(P^0)$ from $(P^\varepsilon)$ by formal asymptotic expansions which involve the so-called signed distance function. In Sections 1.3 and 1.4, we present basic estimates concerning the generation of interface for $(P^\varepsilon)$. For the clarity of underlying ideas, we first consider the special case where $g^\varepsilon \equiv 0$ in Section 1.3, and deal with the general case in Section 1.4. In Section 1.5 we prove a preliminary result on the motion of interface (Lemma 1.5.1), which implies that if the initial datum has already a well-developed transition layer, then the layer remains to exist for $0 \leq t \leq T$ and its motion is well approximated by the interface equation $(P^0)$. Our approach in Sections 1.3 to 1.5 is based on the sub- and super-solutions method, but we use two completely different sets of sub- and super-solutions. More precisely, the sub- and super-solutions for the motion of interface are constructed by using the first two terms of the formal asymptotic expansion (1.24), while those for the generation of interface are constructed by modifying the solution of the equation in the absence of diffusion: $u_t = \varepsilon^{-2} f(u)$. In Section 1.6, we prove our main results for Problem $(P^\varepsilon)$: Theorems 1.1.4, 1.1.6 and their respective corollaries.

In the final section, we study the reaction-diffusion system $(RD^\varepsilon)$ and prove Theorems 1.1.12, 1.1.14 and their corollaries. These results are obtained by applying a slightly modified version of the results for $(P^\varepsilon)$. The strategy is to regard $f_\varepsilon(u, v)$ as a perturbation of $f(u)$. Indeed, the equation for $u$ in $(RD^\varepsilon)$ is identical to that in $(P^\varepsilon)$ if we set $g^\varepsilon = -f_1 - \varepsilon f_2^\varepsilon$. However, what makes the analysis difficult is the fact that $g^\varepsilon$ is no longer a given function but a quantity that depends on the unknown function $v^\varepsilon$. In particular, the existence of the limit $g^\varepsilon \to g$ $(\varepsilon \to 0)$ is not a priori guaranteed, and the estimate (1.6) is far from obvious. As it turns out, the standard $L^p$ or Schauder estimates for $v^\varepsilon$ would not yield (1.6), because of the fact that $u^\varepsilon$ converges to a discontinuous function as $\varepsilon \to 0$. In order to overcome this difficulty, we derive a fine estimate of $v^\varepsilon$ that is based on estimates of the heat kernel and the fact that $u^\varepsilon$ remains uniformly smooth outside of an $O(\varepsilon)$ neighborhood of the smooth hypersurface $\Gamma_t$.

## 1.2 Formal derivation of the interface motion equation

In this section we derive the equation of interface motion corresponding to Problem $(P^\varepsilon)$ by using a formal asymptotic expansion. The resulting interface equation can be regarded as the singular limit of $(P^\varepsilon)$ as $\varepsilon \to 0$. Our argument is basically along the same lines with the formal derivation given by Nakamura, Matano, Hilhorst and Schätzle [63], who studied a similar but slightly different type of spatially inhomogeneous equations by formal analysis. Let us also mention some earlier papers [1], [36] and [66] involving the method of matched asymptotic expansions for problems that are related to ours.

As in [63], the first two terms of the asymptotic expansion determine the interface equation. Though our analysis in this section is for the most part formal, the observations we make here will help the rigorous analysis in later sections.

Let $u^\varepsilon$ be the solution of Problem $(P^\varepsilon)$. We recall that $\Gamma_t^\varepsilon := \{x \in \Omega, u^\varepsilon(x,t) = a\}$ is the interface at time $t$ and call $\Gamma^\varepsilon := \bigcup_{t\geq 0}(\Gamma_t^\varepsilon \times \{t\})$ the interface. Let $\Gamma = \bigcup_{0\leq t\leq T}(\Gamma_t \times \{t\})$ be the solution of the limit geometric motion problem and let $\widetilde{d}$ be the signed distance function to $\Gamma$ defined by:

$$\widetilde{d}(x,t) = \begin{cases} \operatorname{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^+ \\ -\operatorname{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^-, \end{cases} \tag{1.22}$$

where $\operatorname{dist}(x,\Gamma_t)$ is the distance from $x$ to the hypersurface $\Gamma_t$ in $\Omega$. We remark that $\widetilde{d} = 0$ on $\Gamma$ and that $|\nabla \widetilde{d}| = 1$ in a neighborhood of $\Gamma$. We then define

$$Q_T^+ = \bigcup_{0<t\leq T} (\Omega_t^+ \times \{t\}), \qquad Q_T^- = \bigcup_{0<t\leq T} (\Omega_t^- \times \{t\}).$$

We also assume that the solution $u^\varepsilon$ has the expansions

$$u^\varepsilon(x,t) = \alpha_\pm + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \cdots \tag{1.23}$$

away from the interface $\Gamma$ (the outer expansion) and

$$u^\varepsilon(x,t) = U_0(x,t,\xi) + \varepsilon U_1(x,t,\xi) + \varepsilon^2 U_2(x,t,\xi) + \cdots \tag{1.24}$$

near $\Gamma$ (the inner expansion). Here, the functions $U_k(x,t,z)$, $k = 0,1,2,\cdots$, are defined for $x \in \overline{\Omega}$, $t \geq 0$, $z \in \mathbb{R}$ and, by definition, $\xi := \widetilde{d}(x,t)/\varepsilon$. The stretched space variable $\xi$ gives exactly the right spatial scaling to describe the rapid transition between the regions $\{u^\varepsilon \approx \alpha_-\}$ and $\{u^\varepsilon \approx \alpha_+\}$. We normalize $U_k$ in such a way that

$$U_0(x,t,0) = a, \quad U_k(x,t,0) = 0,$$

for all $k \geq 1$ (normalization conditions). To make the inner and outer expansions consistent, we require that

$$\begin{aligned} U_0(x,t,+\infty) &= \alpha_+, \qquad U_k(x,t,+\infty) = 0, \\ U_0(x,t,-\infty) &= \alpha_-, \qquad U_k(x,t,-\infty) = 0, \end{aligned} \tag{1.25}$$

for all $k \geq 1$ (matching conditions).

In what follows we will substitute the inner expansion (1.24) into the parabolic equation of Problem $(P^\varepsilon)$ and collect the $\varepsilon^{-2}$ and $\varepsilon^{-1}$ terms. To that purpose we compute the needed terms and get

$$\begin{aligned}
u_t^\varepsilon &= U_{0t} + U_{0z}\frac{\widetilde{d}_t}{\varepsilon} + \varepsilon U_{1t} + U_{1z}\widetilde{d}_t + \cdots \\
\nabla u^\varepsilon &= \nabla U_0 + U_{0z}\frac{\nabla\widetilde{d}}{\varepsilon} + \varepsilon\nabla U_1 + U_{1z}\nabla\widetilde{d} + \cdots \\
\Delta u^\varepsilon &= \Delta U_0 + 2\frac{\nabla\widetilde{d}}{\varepsilon}\cdot\nabla U_{0z} + U_{0z}\frac{\Delta\widetilde{d}}{\varepsilon} + U_{0zz}\frac{|\nabla\widetilde{d}|^2}{\varepsilon^2} + \varepsilon\Delta U_1 \\
&\quad + 2\nabla\widetilde{d}\cdot\nabla U_{1z} + U_{1z}\Delta\widetilde{d} + U_{1zz}\frac{|\nabla\widetilde{d}|^2}{\varepsilon} + \cdots
\end{aligned}$$

where the functions $U_i$ $(i = 0, 1)$, as well as their derivatives, are taken at point $(x, t, \widetilde{d}(x, t)/\varepsilon)$. Here, $\nabla U_0$ denotes the derivative with respect to $x$ whenever we regard $U_0(x, t, z)$ as a function of three variables $x$, $t$ and $z$. The symbol $\Delta U_0$ is defined similarly and this convention applies to $U_{0z}$ and $U_{1zz}$ as well. We also use the expansions

$$\begin{aligned}
f(u^\varepsilon) &= f(U_0) + \varepsilon f'(U_0)U_1 + O(\varepsilon^2), \\
g^\varepsilon(x, t, u^\varepsilon) &= g(x, t, u^\varepsilon) + O(\varepsilon) \qquad \big(\leftarrow \text{ in view of (1.6)}\big) \\
&= g(x, t, U_0) + O(\varepsilon).
\end{aligned}$$

Next, we substitute the expressions above in the partial differential equation in Problem $(P^\varepsilon)$. Collecting the $\varepsilon^{-2}$ terms yields

$$U_{0zz} + f(U_0) = 0.$$

In view of the normalization and matching conditions, we can now assert that $U_0(x, t, z) = U_0(z)$, where $U_0(z)$ is the unique solution of the stationary problem

$$\begin{cases} U_0'' + f(U_0) = 0, \\ U_0(-\infty) = \alpha_-, \quad U_0(0) = a, \quad U_0(+\infty) = \alpha_+. \end{cases} \tag{1.26}$$

This solution represents the first approximation of the profile of a transition layer around the interface observed in the stretched coordinates. For example, in the special case where $f(u) = u(1 - u^2)$, we have $U_0(z) = \tanh(z/\sqrt{2})$. In the general case, the following standard estimates hold.

**Lemma 1.2.1.** *There exist positive constants $C$ and $\lambda$ such that the following estimates hold.*

$$\begin{aligned}
0 < \alpha_+ - U_0(z) &\le Ce^{-\lambda|z|} \quad \textit{for } z \ge 0, \\
0 < U_0(z) - \alpha_- &\le Ce^{-\lambda|z|} \quad \textit{for } z \le 0.
\end{aligned}$$

*In addition, $U_0$ is a strictly increasing function and, for $j = 1, 2$,*

$$|D^j U_0(z)| \le Ce^{-\lambda|z|} \quad \textit{for } z \in \mathbb{R}. \tag{1.27}$$

**Proof.** We only give an outline. Rewriting the equation in (1.26) as

$$\dot{u} = v, \qquad \dot{v} = -f(u),$$

we see that $(U_0(z), U_0'(z))$ is a heteroclinic orbit of the above system connecting the equilibria $(\alpha_-, 0)$ and $(\alpha_+, 0)$. These equilibria are saddle points, with the linearized eigenvalues $\{\lambda_-, -\lambda_-\}$ and $\{\lambda_+, -\lambda_+\}$, respectively, where

$$\lambda_- = \sqrt{-f'(\alpha_-)}, \qquad \lambda_+ = \sqrt{-f'(\alpha_+)}.$$

Consequently, we have

$$U_0(z) = \begin{cases} \alpha_- + C_1\, e^{\lambda_- z} + o(e^{\lambda_- z}) & \text{as } z \to -\infty, \\ \alpha_+ + C_2\, e^{-\lambda_+ z} + o(e^{-\lambda_+ z}) & \text{as } z \to +\infty \end{cases} \tag{1.28}$$

for some constants $C_1, C_2$. The desired estimates now follow by setting $\lambda = \min(\lambda_+, \lambda_-)$. □

Next we collect the $\varepsilon^{-1}$ terms. Since $U_0$ depends only on the variable $z$, we have $\nabla U_{0z} = 0$ which, combined with the fact that $|\nabla \widetilde{d}| = 1$ near $\Gamma_t$, yields

$$U_{1zz} + f'(U_0)U_1 = {U_0}'(\widetilde{d}_t - \Delta \widetilde{d}) + g(x,t,U_0). \tag{1.29}$$

This equation can be seen as a linearized problem for (1.26) with an inhomogeneous term. As is well known (see, for instance, [63]), the solvability condition for the above equation plays the key role in determining the equation of interface motion. The following lemma is rather standard, but we give an outline of the proof for the convenience of the reader.

**Lemma 1.2.2 (Solvability condition).** *Let $A(z)$ be a bounded function on $-\infty < z < \infty$. Then the problem*

$$\begin{cases} \psi_{zz} + f'(U_0(z))\psi = A(z) & z \in \mathbb{R}, \\ \psi(0) = 0, \quad \psi \in L^\infty(\mathbb{R}), \end{cases} \tag{1.30}$$

*has a solution if and only if*

$$\int_{\mathbf{R}} A(z){U_0}'(z)dz = 0. \tag{1.31}$$

*Moreover the solution, if it exists, is unique and satisfies, for some constant $C > 0$,*

$$|\psi(z)| \le C\|A\|_{L^\infty}, \tag{1.32}$$

*for all $z \in \mathbb{R}$.*

**Proof.** Multiplying the equation by ${U_0}'$ and integrating it by parts, we easily see that the condition (1.31) is necessary. Conversely, suppose that this condition is satisfied. Then, since ${U_0}'$ is a bounded positive solution to the homogeneous equation $\psi_{zz} + f'(U_0(z))\psi = 0$, one can use the method of variation of constants to find the above solution $\psi$ explicitly. More precisely,

$$\begin{aligned} \psi(z) &= \varphi(z) \int_0^z \Big(\varphi^{-2}(\zeta) \int_{-\infty}^{\zeta} A(\xi)\varphi(\xi)\, d\xi\Big) d\zeta \\ &= -\varphi(z) \int_0^z \Big(\varphi^{-2}(\zeta) \int_{\zeta}^{\infty} A(\xi)\varphi(\xi)\, d\xi\Big) d\zeta, \end{aligned} \tag{1.33}$$

where $\varphi := {U_0}'$. The estimate (1.32) now follows from the above expression and (1.28). □

From the above lemma, the solvability condition for (1.29) is given by

$$\int_{\mathbf{R}} \Big[{{U_0}'}^2(z)(\widetilde{d}_t - \Delta\widetilde{d})(x,t) + g(x,t,U_0(z)){U_0}'(z)\Big] dz = 0$$

for all $(x,t) \in Q_T$. Hence we get

$$\widetilde{d}_t - \Delta \widetilde{d} = -\frac{\displaystyle\int_{\mathbf{R}} g(x,t,U_0(z)) {U_0}'(z)\, dz}{\displaystyle\int_{\mathbf{R}} {U_0'}^2(z)\, dz},$$

which gives

$$\widetilde{d}_t = \Delta \widetilde{d} - \frac{\displaystyle\int_{\alpha_-}^{\alpha_+} g(x,t,r) dr}{\displaystyle\int_{\mathbf{R}} {U_0'}^2(z)\, dz}.$$

Moreover, multiplying equation (1.26) by ${U_0}'$ and integrating it from $-\infty$ to $z$, we obtain

$$\begin{aligned} 0 &= \int_{-\infty}^{z} ({U_0}''{U_0}' + f(U_0){U_0}')(s) ds \\ &= \frac{1}{2}{U_0'}^2(z) - W(U_0(z)) + W(\alpha_-), \end{aligned}$$

where we have also used the fact that $U_0(-\infty) = \alpha_-$ and ${U_0}'(-\infty) = 0$. This implies that

$${U_0}'(z) = \sqrt{2}(W(U_0(z)) - W(\alpha_-))^{1/2},$$

and therefore

$$\begin{aligned} \int_{\mathbf{R}} {U_0'}^2(z) dz &= \int_{\mathbf{R}} {U_0}'(z)\sqrt{2}\big(W(U_0(z)) - W(\alpha_-)\big)^{1/2} dz \\ &= \sqrt{2}\int_{\alpha_-}^{\alpha_+} (W(s) - W(\alpha_-))^{1/2} ds. \end{aligned} \tag{1.34}$$

It then follows, in view of the definition of $c_0$ in (1.12), that

$$\widetilde{d}_t = \Delta \widetilde{d} - c_0 \int_{\alpha_-}^{\alpha_+} g(x,t,r) dr. \tag{1.35}$$

We are now ready to derive the equation of interface motion. Since $\nabla \widetilde{d}$ $(= \nabla_x \widetilde{d}(x,t))$ coincides with the outward normal unit vector to the hypersurface $\Gamma_t$, we have $\widetilde{d}_t(x,t) = -V_n$, where $V_n$ is the normal velocity of the interface $\Gamma_t$. It is also known that the mean curvature $\kappa$ of the interface is equal to $\Delta \widetilde{d}/(N-1)$. Thus the equation of interface motion is given by:

$$V_n = -(N-1)\kappa + c_0 \int_{\alpha_-}^{\alpha_+} g(x,t,r) dr \quad \text{ on } \Gamma_t. \tag{1.36}$$

Summarizing, under the assumption that the solution $u^\varepsilon$ of Problem $(P^\varepsilon)$ satisfies

$$u^\varepsilon \to \begin{cases} \alpha_+ & \text{in } Q_T^+ \\ \alpha_- & \text{in } Q_T^- \end{cases} \quad \text{as } \varepsilon \to 0,$$

we have formally proved that the boundary $\Gamma_t$ between $\Omega_t^-$ and $\Omega_t^+$ moves according to the law (1.36).

To conclude this section, we give basic estimates for $U_1(x,t,z)$, which we will need in Section 1.5 to study the motion of interface. Substituting (1.35) into (1.29) gives

$$\begin{cases} U_{1zz} + f'(U_0(z))U_1 = g(x,t,U_0(z)) - \gamma(x,t){U_0}'(z), \\ U_1(x,t,0) = 0, \qquad\qquad U_1(x,t,\cdot) \in L^\infty(\mathbb{R}), \end{cases} \tag{1.37}$$

where $\gamma$ has been defined in (1.13). Thus $U_1(x,t,z)$ is a solution of (1.30) with

$$A = A_0(x,t,z) := g(x,t,U_0(z)) - \gamma(x,t){U_0}'(z), \tag{1.38}$$

where the variables $x,t$ are considered parameters. The problem (1.37) has a unique solution by virtue of Lemma 1.2.2. Moreover, since $A_0(x,t,z)$ remains bounded as $(x,t,z)$ varies in $\overline{\Omega} \times [0,T] \times \mathbb{R}$, the estimate (1.32) implies

$$|U_1(x,t,z)| \le M \qquad \text{for } \; x \in \overline{\Omega},\; t \in [0,T],\; z \in \mathbb{R}, \tag{1.39}$$

for some constant $M > 0$. Similarly, since $\nabla U_1$ is a solution of (1.30) with

$$A = \nabla_x A_0(x,t,z) \;\Big( = \nabla_x\big(g(x,t,U_0(z)) - \gamma(x,t){U_0}'(z)\big)\Big),$$

and since $g$ is assumed to be $C^1$ in $x$, we obtain

$$|\nabla_x U_1(x,t,z)| \le M \qquad \text{for } \; x \in \overline{\Omega},\; t \in [0,T],\; z \in \mathbb{R}, \tag{1.40}$$

for some constant $M > 0$.

To obtain estimates as $z \to \pm\infty$, we first observe that (1.28) implies

$$A_0(x,t,z) - g(x,t,\alpha_\pm) = O(e^{-\lambda|z|}) \qquad \text{as } \; z \to \pm\infty, \tag{1.41}$$

uniformly in $x \in \overline{\Omega}$, $t \in [0,T]$. We then apply the following general estimates.

**Lemma 1.2.3.** *Let the assumptions of Lemma 1.2.2 hold, and assume further that* $A(z) - A^\pm = O(e^{-\delta|z|})$ *as* $z \to \pm\infty$ *for some constants* $A^+$, $A^-$ *and* $\delta > 0$. *Then there exists a constant* $\lambda > 0$ *such that*

$$\psi(z) - \frac{A^\pm}{f'(\alpha_\pm)} = O(e^{-\lambda|z|}), \qquad |\psi'(z)| + |\psi''(z)| = O(e^{-\lambda|z|}), \tag{1.42}$$

*as* $z \to \pm\infty$.

**Proof.** We only state the outline. To derive the former estimate, we need a slightly more elaborate version of (1.28). Since $f(u)$ is $C^2$, we have $f(u) = (u-\alpha_\pm)f'(\alpha_\pm) + O\big((u-\alpha_\pm)^2\big)$. Consequently,

$$U_0(z) = \begin{cases} \alpha_- + C_1\, e^{\lambda_- z} + O(e^{2\lambda_- z}) & \text{as } z \to -\infty, \\ \alpha_+ + C_2\, e^{-\lambda_+ z} + O(e^{-2\lambda_+ z}) & \text{as } z \to +\infty. \end{cases} \tag{1.43}$$

Using the expression (1.33) along with the estimate $A(z) - A^\pm = O(e^{-\delta|z|})$ and (1.43), we see that

$$\psi(z) = -\frac{A^\pm}{(\lambda_\pm)^2} + O\big(|z|e^{-\lambda_\pm|z|}\big) + O\big(e^{-\min(\delta,\lambda_\pm)|z|}\big) \qquad \text{as } z \to \pm\infty.$$

This implies the former estimate in (1.42), where $\lambda$ can be any constant satisfying $0 < \lambda < \min(\lambda_-, \lambda_+, \delta)$. Substituting this into equation (1.30) gives the estimate for $\psi_{zz}$. Finally, the estimate for $\psi_z$ follows by integrating $\psi_{zz}$ from $\pm\infty$ to $z$. □

From the above lemma and (1.41) we obtain the estimate

$$|U_{1z}(x,t,z)| + |U_{1zz}(x,t,z)| \le Ce^{-\lambda|z|}, \tag{1.44}$$

for $x \in \overline{\Omega}$, $t \in [0,T]$, $z \in \mathbb{R}$. Similarly, since the definition of $A_0$ (1.38) and estimate (1.27) imply

$$(\nabla_x A_0)(x,t,z) - (\nabla_x g)(x,t,\alpha_\pm) = O(e^{-\lambda|z|}) \qquad \text{as} \quad z \to \pm\infty,$$

we can apply Lemma 1.2.3 to $\psi = \nabla_x U_1$, to obtain

$$|\nabla_x U_{1z}(x,t,z)| + |\nabla_x U_{1zz}(x,t,z)| \le Ce^{-\lambda|z|},$$

for $x \in \overline{\Omega}$, $t \in [0,T]$, $z \in \mathbb{R}$. As a consequence, there is a constant, which we denote again by $M$, such that

$$|\nabla_x U_{1z}(x,t,z)| \le M. \tag{1.45}$$

Finally we consider the boundary condition. Note that (1.7) implies

$$\frac{\partial}{\partial\nu} A_0 = \frac{\partial}{\partial\nu}\Big[g(x,t,U_0(z)) - \gamma(x,t){U_0}'(z)\Big] = 0 \quad \text{on} \ \ \partial\Omega \times [0,T] \times \mathbb{R}. \tag{1.46}$$

Consequently, from the expression (1.33), or equivalently the expression

$$U_1(x,t,z) = {U_0}'(z) \int_0^z \Big(\big({U_0}'(\zeta)\big)^{-2} \int_{-\infty}^{\zeta} A_0(x,t,\xi){U_0}'(\xi)\,d\xi\Big) d\zeta,$$

we see that

$$\frac{\partial U_1}{\partial \nu} = 0 \quad \text{on} \ \ \partial\Omega \times [0,T] \times \mathbb{R}. \tag{1.47}$$

## 1.3 Generation of interface: the case $g^\varepsilon \equiv 0$

This section deals with the generation of interface, namely the rapid formation of internal layers that takes place in a neighborhood of $\Gamma_0 = \{x \in \Omega,\ u_0(x) = a\}$ within the time span of order $\varepsilon^2 |\ln\varepsilon|$. For the time being we focus on the special case where $g^\varepsilon \equiv 0$. We will discuss the general case in Section 1.4. In the sequel, $\eta_0$ will stand for the following quantity:

$$\eta_0 := \frac{1}{2}\min(a - \alpha_-, \alpha_+ - a).$$

Our main result in this section is the following.

**Theorem 1.3.1.** *Let $\eta \in (0,\eta_0)$ be arbitrary and define $\mu$ as the derivative of $f(u)$ at the unstable equilibrium $u = a$, that is*

$$\mu = f'(a). \tag{1.48}$$

*Then there exist positive constants $\varepsilon_0$ and $M_0$ such that, for all $\varepsilon \in (0,\varepsilon_0)$,*

- *for all $x \in \Omega$,*

$$\alpha_- - \eta \le u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \le \alpha_+ + \eta, \tag{1.49}$$

- *for all* $x \in \Omega$ *such that* $|u_0(x) - a| \geq M_0\varepsilon$, *we have that*

$$\text{if} \quad u_0(x) \geq a + M_0\varepsilon \quad \text{then} \quad u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \geq \alpha_+ - \eta, \tag{1.50}$$

$$\text{if} \quad u_0(x) \leq a - M_0\varepsilon \quad \text{then} \quad u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \leq \alpha_- + \eta. \tag{1.51}$$

The above theorem will be proved by constructing a suitable pair of sub- and super-solutions.

### 1.3.1 The bistable ordinary differential equation

As mentioned in Section 1.1, the above sub- and super-solutions are constructed by modifying the solution of the problem without diffusion:

$$\bar{u}_t = \frac{1}{\varepsilon^2} f(\bar{u}), \qquad \bar{u}(x, 0) = u_0(x).$$

This solution is written in the form

$$\bar{u}(x, t) = Y\Big(\frac{t}{\varepsilon^2}, u_0(x)\Big),$$

where $Y(\tau, \xi)$ denotes the solution of the ordinary differential equation

$$\begin{cases} Y_\tau(\tau, \xi) &= f(Y(\tau, \xi)) \quad \text{for } \tau > 0, \\ Y(0, \xi) &= \xi. \end{cases} \tag{1.52}$$

Here $\xi$ ranges over the interval $(-2C_0, 2C_0)$, with $C_0$ being the constant defined in (1.8). We first study basic properties of $Y$.

**Lemma 1.3.2.** *We have* $Y_\xi > 0$, *for all* $\xi \in (-2C_0, 2C_0) \setminus \{\alpha_-, a, \alpha_+\}$ *and all* $\tau > 0$. *Furthermore,*

$$Y_\xi(\tau, \xi) = \frac{f(Y(\tau, \xi))}{f(\xi)}.$$

**Proof.** First, differentiating equation (1.52) with respect to $\xi$, we obtain

$$\begin{cases} Y_{\xi\tau} = Y_\xi f'(Y), \\ Y_\xi(0, \xi) = 1, \end{cases}$$

which can be integrated as follows:

$$Y_\xi(\tau, \xi) = \exp\Big[\int_0^\tau f'(Y(s, \xi))ds\Big] > 0. \tag{1.53}$$

We then differentiate equation (1.52) with respect to $\tau$ and obtain

$$\begin{cases} Y_{\tau\tau} = Y_\tau f'(Y), \\ Y_\tau(0, \xi) = f(\xi), \end{cases}$$

which in turn implies

$$\begin{aligned} Y_\tau(\tau, \xi) &= f(\xi) \exp\Big[\int_0^\tau f'(Y(s, \xi))ds\Big] \\ &= f(\xi) Y_\xi(\tau, \xi). \end{aligned}$$

This last equality, in view of (1.52), completes the proof of Lemma 1.3.2. □

We define a function $A(\tau,\xi)$ by

$$A(\tau,\xi) = \frac{f'(Y(\tau,\xi)) - f'(\xi)}{f(\xi)}. \tag{1.54}$$

**Lemma 1.3.3.** *We have, for all $\xi \in (-2C_0, 2C_0) \setminus \{\alpha_-, a, \alpha_+\}$ and all $\tau > 0$,*

$$A(\tau,\xi) = \int_0^\tau f''(Y(s,\xi))Y_\xi(s,\xi)ds.$$

**Proof.** Differentiating the equality of Lemma 1.3.2 with respect to $\xi$ leads to

$$Y_{\xi\xi} = A(\tau,\xi)Y_\xi, \tag{1.55}$$

whereas differentiating (1.53) with respect to $\xi$ yields

$$Y_{\xi\xi} = Y_\xi \int_0^\tau f''(Y(s,\xi))Y_\xi(s,\xi)ds.$$

These two last results complete the proof of Lemma 1.3.3. □

Next we need some estimates on the growth of $Y$, $A$ and theirs derivatives. We first consider the case where the initial value $\xi$ is far from the stable equilibria, more precisely when it lies between $\alpha_- + \eta$ and $\alpha_+ - \eta$.

**Lemma 1.3.4.** *Let $\eta \in (0,\eta_0)$ be arbitrary. Then there exist positive constants $\tilde{C}_1 = \tilde{C}_1(\eta)$, $\tilde{C}_2 = \tilde{C}_2(\eta)$ and $C_3 = C_3(\eta)$ such that, for all $\tau > 0$,*

- *if $\xi \in (a, \alpha_+ - \eta)$ then, for every $\tau > 0$ such that $Y(\tau,\xi)$ remains in the interval $(a, \alpha_+ - \eta)$, we have*
$$\tilde{C}_1 e^{\mu\tau} \le Y_\xi(\tau,\xi) \le \tilde{C}_2 e^{\mu\tau}, \tag{1.56}$$
*and*
$$|A(\tau,\xi)| \le C_3(e^{\mu\tau} - 1); \tag{1.57}$$
- *if $\xi \in (\alpha_- + \eta, a)$ then, for every $\tau > 0$ such that $Y(\tau,\xi)$ remains in the interval $(\alpha_- + \eta, a)$, (1.56) and (1.57) hold as well,*

*where $\mu$ is the constant defined in (1.48).*

**Proof.** We take $\xi \in (a, \alpha_+ - \eta)$ and suppose that for $s \in (0,\tau)$, $Y(s,\xi)$ remains in the interval $(a, \alpha_+ - \eta)$. Integrating the equality

$$\frac{Y_\tau(s,\xi)}{f(Y(s,\xi))} = 1$$

from 0 to $\tau$ yields

$$\int_0^\tau \frac{Y_\tau(s,\xi)}{f(Y(s,\xi))}ds \quad = \tau. \tag{1.58}$$

Hence by the change of variable $q = Y(s,\xi)$ we get

$$\int_\xi^{Y(\tau,\xi)} \frac{dq}{f(q)} \quad = \tau. \tag{1.59}$$

Moreover, the equality of Lemma 1.3.2 leads to

$$\begin{aligned}\ln Y_\xi(\tau,\xi) &= \int_\xi^{Y(\tau,\xi)} \frac{f'(q)}{f(q)}dq\\ &= \int_\xi^{Y(\tau,\xi)} [\frac{f'(a)}{f(q)} + \frac{f'(q)-f'(a)}{f(q)}]dq \\ &= \mu\tau + \int_\xi^{Y(\tau,\xi)} h(q)dq,\end{aligned} \tag{1.60}$$

where

$$h(q) = (f'(q)-\mu)/f(q).$$

Since

$$h(q) \to f''(a)/f'(a) \quad \text{as } q \to a,$$

$h$ is continuous on $[a, \alpha_+ - \eta]$. Hence we can define

$$H = H(\eta) := \|h\|_{L^\infty(a,\alpha_+-\eta)}.$$

Since $|Y(\tau,\xi)-\xi|$ takes its values in the interval $[0, \alpha_+ - a - \eta] \subset [0, \alpha_+ - a]$, it follows from (1.60) that

$$\mu\tau - H(\alpha_+ - a) \le \ln Y_\xi(\tau,\xi) \le \mu\tau + H(\alpha_+ - a),$$

which, in turn, proves (1.56). Next Lemma 1.3.3 and (1.56) yield

$$\begin{aligned}|A(\tau,\xi)| &\le \sup_{z\in[\alpha_-,\alpha_+]} |f''(z)| \int_0^\tau \tilde{C}_2 e^{\mu s} ds\\ &\le C_3(e^{\mu\tau}-1),\end{aligned}$$

which completes the proof of (1.57). The case where $\xi$ and $Y(\tau,\xi)$ are in $(\alpha_- + \eta, a)$ is similar and omitted. □

**Corollary 1.3.5.** *Let $\eta \in (0,\eta_0)$ be arbitrary. Then there exist positive constants $C_1 = C_1(\eta)$ and $C_2 = C_2(\eta)$ such that, for all $\tau > 0$,*

- *if $\xi \in (a, \alpha_+ - \eta)$ then, for every $\tau > 0$ such that $Y(\tau,\xi)$ remains in the interval $(a, \alpha_+ - \eta)$, we have*
$$C_1 e^{\mu\tau}(\xi - a) \le Y(\tau,\xi) - a \le C_2 e^{\mu\tau}(\xi - a); \tag{1.61}$$
- *if $\xi \in (\alpha_- + \eta, a)$ then, for every $\tau > 0$ such that $Y(\tau,\xi)$ remains in the interval $(\alpha_- + \eta, a)$, we have*
$$C_2 e^{\mu\tau}(\xi - a) \le Y(\tau,\xi) - a \le C_1 e^{\mu\tau}(\xi - a). \tag{1.62}$$

**Proof.** Since

$$f(q)/(q-a) \to f'(a) = \mu \quad \text{as } q \to a,$$

it is possible to find constants $B_1 = B_1(\eta) > 0$ and $B_2 = B_2(\eta) > 0$ such that, for all $q \in (a, \alpha_+ - \eta)$,

$$B_1(q-a) \le f(q) \le B_2(q-a). \tag{1.63}$$

We write this inequality for $a < Y(\tau,\xi) < \alpha_+ - \eta$ to obtain

$$B_1(Y(\tau,\xi)-a) \le f(Y(\tau,\xi)) \le B_2(Y(\tau,\xi)-a).$$

We also write this inequality for $a < \xi < \alpha_+ - \eta$ to obtain

$$B_1(\xi - a) \leq f(\xi) \leq B_2(\xi - a).$$

Next we use the equality $Y_\xi = f(Y)/f(\xi)$ of Lemma 1.3.2 to deduce that

$$\frac{B_1}{B_2}(Y(\tau,\xi) - a) \leq (\xi - a)Y_\xi(\tau,\xi) \leq \frac{B_2}{B_1}(Y(\tau,\xi) - a),$$

which, in view of (1.56), implies that

$$\frac{B_1}{B_2}\tilde{C}_1 e^{\mu\tau}(\xi - a) \leq Y(\tau,\xi) - a \leq \frac{B_2}{B_1}\tilde{C}_2 e^{\mu\tau}(\xi - a).$$

This proves (1.61). The proof of (1.62) is similar and omitted. □

We now present estimates in the case where the initial value $\xi$ is smaller than $\alpha_- + \eta$ or larger than $\alpha_+ - \eta$.

**Lemma 1.3.6.** *Let $\eta \in (0,\eta_0)$ and $M > 0$ be arbitrary. Then there exists a positive constant $C_4 = C_4(\eta, M)$ such that*

- *if $\xi \in [\alpha_+ - \eta, \alpha_+ + M]$, then, for all $\tau > 0$, $Y(\tau,\xi)$ remains in the interval $[\alpha_+ - \eta, \alpha_+ + M]$ and*
$$|A(\tau,\xi)| \leq C_4 \tau \quad \textit{for} \quad \tau > 0\,; \tag{1.64}$$
- *if $\xi \in [\alpha_- - M, \alpha_- + \eta]$, then, for all $\tau > 0$, $Y(\tau,\xi)$ remains in the interval $[\alpha_- - M, \alpha_- + \eta]$ and (1.64) holds as well.*

**Proof.** Since the two cases can be treated in the same way, we will only prove the former. The fact that $Y(\tau,\xi)$, the solution of the ordinary differential equation (1.52), remains in the interval $[\alpha_+ - \eta, \alpha_+ + M]$ directly follows from the bistable properties of $f$, or, more precisely, from the sign conditions $f(\alpha_+ - \eta) > 0$, $f(\alpha_+ + M) < 0$.

To prove (1.64), suppose first that $\xi \in [\alpha_+, \alpha_+ + M]$. In view of (1.1), $f'$ is strictly negative in an interval of the form $[\alpha_+, \alpha_+ + c]$ and $f$ is negative in $[\alpha_+, \infty)$. We denote by $-m < 0$ the maximum of $f$ on $[\alpha_+ + c, \alpha_+ + M]$. Then, as long as $Y(\tau,\xi)$ remains in the interval $[\alpha_+ + c, \alpha_+ + M]$, the ordinary differential equation (1.52) implies

$$Y_\tau \leq -m.$$

By integration, this means that, for any $\xi \in [\alpha_+, \alpha_+ + M]$, we have

$$Y(\tau,\xi) \in [\alpha_+, \alpha_+ + c] \qquad \text{for} \quad \tau \geq \overline{\tau} := \frac{M - c}{m}.$$

In view of this, and considering that $f'(Y) < 0$ for $Y \in [\alpha_+, \alpha_+ + c]$, we see from the expression (1.53) that

$$\begin{aligned}
Y_\xi(\tau,\xi) &= \exp\Big[\int_0^{\overline{\tau}} f'(Y(s,\xi))ds\Big] \exp\Big[\int_{\overline{\tau}}^{\tau} f'(Y(s,\xi))ds\Big] \\
&\leq \exp\Big[\int_0^{\overline{\tau}} f'(Y(s,\xi))ds\Big] \\
&\leq \exp\Big[\int_0^{\overline{\tau}} \sup_{z \in [\alpha_- - M, \alpha_+ + M]} |f'(z)| ds\Big] =: \tilde{C}_4 = \tilde{C}_4(M),
\end{aligned}$$

for all $\tau \geq \overline{\tau}$. It is clear from the same expression (1.53) that $Y_\xi \leq \tilde{C}_4$ holds also for $0 \leq \tau \leq \overline{\tau}$. We can then use Lemma 1.3.3 to deduce that

$$\begin{aligned} |A(\tau,\xi)| &\leq \tilde{C}_4 \int_0^\tau |f''(Y(s,\xi))| ds \\ &\leq \tilde{C}_4 \Big( \sup_{z\in[\alpha_- - M, \alpha_+ + M]} |f''(z)| \Big) \tau =: C_4 \tau. \end{aligned}$$

The case $\xi \in [\alpha_+ - \eta, \alpha_+]$ can be treated in the same way. This completes the proof of the lemma. □

Now we choose the constant $M$ in the above lemma sufficiently large so that $[-2C_0, 2C_0] \subset [\alpha_- - M, \alpha_+ + M]$, and fix $M$ hereafter. Then $C_4$ only depends on $\eta$. Using the fact that $\tau = O(e^{\mu\tau} - 1)$ for $\tau > 0$, one can easily deduce from (1.57) and (1.64) the following general estimate.

**Lemma 1.3.7.** *Let $\eta \in (0, \eta_0)$ be arbitrary and let $C_0$ be the constant defined in* (1.8)*. Then there exists a positive constant $C_5 = C_5(\eta)$ such that, for all $\xi \in (-2C_0, 2C_0)$ and all $\tau > 0$,*

$$|A(\tau,\xi)| \leq C_5(e^{\mu\tau} - 1).$$

### 1.3.2 Construction of sub- and super-solutions

We are now ready to construct the sub- and super-solutions for the study of generation of interface. For simplicity, we first consider the case where

$$\frac{\partial u_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.65}$$

In this case, our sub- and super-solutions are given by

$$w_\varepsilon^\pm(x,t) = Y\Big(\frac{t}{\varepsilon^2},\, u_0(x) \pm \varepsilon^2 C_6 (e^{\mu t/\varepsilon^2} - 1)\Big). \tag{1.66}$$

In the general case where (1.65) does not necessarily hold, we have to slightly modify $w_\varepsilon^\pm(x,t)$ near the boundary $\partial\Omega$. This will be discussed later.

**Lemma 1.3.8.** *Assume* (1.65)*. Then there exist positive constants $\varepsilon_0$ and $C_6$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, $(w_\varepsilon^-, w_\varepsilon^+)$ is a pair of sub- and super-solutions for Problem $(P^\varepsilon)$, in the domain*

$$\big\{(x,t) \in Q_T,\ x \in \Omega,\ 0 \leq t \leq \mu^{-1}\varepsilon^2 |\ln \varepsilon|\big\},$$

*satisfying $w_\varepsilon^-(x,0) = w_\varepsilon^+(x,0) = u_0(x)$. Consequently*

$$w_\varepsilon^-(x,t) \leq u^\varepsilon(x,t) \leq w_\varepsilon^+(x,t) \quad \text{for } x \in \overline{\Omega},\ 0 \leq t \leq \mu^{-1}\varepsilon^2 |\ln \varepsilon|. \tag{1.67}$$

**Proof.** The assumption (1.65) implies

$$\frac{\partial w_\varepsilon^\pm}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, +\infty).$$

Now we define the operator $\mathcal{L}_0$ by

$$\mathcal{L}_0 u := u_t - \Delta u - \frac{1}{\varepsilon^2} f(u),$$

and prove that $\mathcal{L}_0 w_\varepsilon^+ \geq 0$ . Straightforward computations yield

$$\mathcal{L}_0 w_\varepsilon^+ = \frac{1}{\varepsilon^2} Y_\tau + C_6\, \mu\, e^{\mu t/\varepsilon^2} Y_\xi - \Delta u_0 Y_\xi - |\nabla u_0|^2 Y_{\xi\xi} - \frac{1}{\varepsilon^2} f(Y),$$

therefore, in view of the ordinary differential equation (1.52),

$$\mathcal{L}_0 w_\varepsilon^+ = \Big[ C_6\, \mu\, e^{\mu t/\varepsilon^2} - \Delta u_0 - \frac{Y_{\xi\xi}}{Y_\xi} |\nabla u_0|^2 \Big] Y_\xi.$$

We note that, in the range $0 \leq t \leq \mu^{-1} \varepsilon^2 |\ln \varepsilon|$, we have, for $\varepsilon_0$ sufficiently small,

$$0 \leq \varepsilon^2 C_6 (e^{\mu t/\varepsilon^2} - 1) \leq \varepsilon^2 C_6 (\varepsilon^{-1} - 1) \leq C_0,$$

where $C_0$ is the constant defined in (1.8). Hence

$$\xi := u_0(x) \pm C_6 (e^{\mu t/\varepsilon^2} - 1) \in (-2C_0, 2C_0),$$

so that we can use the estimate of $A = Y_{\xi\xi}/Y_\xi$ in Lemma 1.3.7 and obtain

$$\begin{aligned} \mathcal{L}_0 w_\varepsilon^+ &\geq \Big[ C_6\, \mu e^{\mu t/\varepsilon^2} - |\Delta u_0| - C_5 (e^{\mu t/\varepsilon^2} - 1) |\nabla u_0|^2 \Big] Y_\xi \\ &\geq \Big[ (C_6\, \mu - C_5 |\nabla u_0|^2) e^{\mu t/\varepsilon^2} - |\Delta u_0| + C_5 |\nabla u_0|^2 \Big] Y_\xi. \end{aligned}$$

Since $Y_\xi > 0$, this inequality implies that, for $C_6$ large enough,

$$\mathcal{L}_0 w_\varepsilon^+ \geq \Big[ C_6 \mu - C_5 {C_0}^2 - C_0 \Big] Y_\xi \geq 0.$$

Hence $w_\varepsilon^+$ is a super-solution for Problem $(P^\varepsilon)$. Similarly $w_\varepsilon^-$ is a sub-solution. Obviously $w_\varepsilon^-(x,0) = w_\varepsilon^+(x,0) = Y\big(0, u_0(x)\big) = u_0(x)$. Lemma 1.3.8 is proved. □

In the more general case where (1.65) is not necessarily valid, one can proceed as follows: in view of (1.10) and (1.11) there exist positive constants $d_1$, $\rho$ such that $u_0(x) \geq a + \rho$ if $d(x, \partial\Omega) \leq d_1$. Let $\chi$ be a smooth cut-off function defined on $[0, +\infty)$ such that $0 \leq \chi \leq 1$, $\chi(0) = \chi'(0) = 0$ and $\chi(z) = 1$ for $z \geq d_1$. Then we define

$$\begin{aligned} u_0^+(x) &= \chi(d(x, \partial\Omega))\, u_0(x) + \big[1 - \chi(d(x, \partial\Omega))\big] \max_{x \in \overline{\Omega}} u_0(x), \\ u_0^-(x) &= \chi(d(x, \partial\Omega))\, u_0(x) + \big[1 - \chi(d(x, \partial\Omega))\big] (a + \rho). \end{aligned} \tag{1.68}$$

Clearly, $u_0^- \leq u_0 \leq u_0^+$, and both $u_0^+$ and $u_0^+$ satisfy (1.65). Now we set

$$\tilde{w}_\varepsilon^\pm(x,t) = Y\Big( \frac{t}{\varepsilon^2}, u_0^\pm(x) \pm \varepsilon^2 C_6 (e^{\mu t/\varepsilon^2} - 1) \Big).$$

Then the same argument as in Lemma 1.3.8 shows that $(\tilde{w}_\varepsilon^-, \tilde{w}_\varepsilon^+)$ is a pair of sub- and super-solutions for Problem $(P^\varepsilon)$. Furthermore, since $\tilde{w}_\varepsilon^-(x,0) = u_0^-(x) \leq u_0(x) \leq u_0^+(x) = \tilde{w}_\varepsilon^+(x,0)$, the comparison principle asserts that

$$\tilde{w}_\varepsilon^-(x,t) \leq u^\varepsilon(x,t) \leq \tilde{w}_\varepsilon^+(x,t) \quad \text{ for } x \in \overline{\Omega},\ 0 \leq t \leq \mu^{-1} \varepsilon^2 |\ln \varepsilon|. \tag{1.69}$$

### 1.3.3 Proof of Theorem 1.3.1

In order to prove Theorem 1.3.1 we first present a key estimate on the function $Y(\tau,\xi)$ after a time interval of order $\tau \sim |\ln\varepsilon|$.

**Lemma 1.3.9.** *Let $\eta \in (0,\eta_0)$ be arbitrary; there exist positive constants $\varepsilon_0$ and $C_7$ such that, for all $\varepsilon \in (0,\varepsilon_0)$,*

- *for all $\xi \in (-2C_0, 2C_0)$,*

$$\alpha_- - \eta \le Y(\mu^{-1}|\ln\varepsilon|,\xi) \le \alpha_+ + \eta, \tag{1.70}$$

- *for all $\xi \in (-2C_0, 2C_0)$ such that $|\xi - a| \ge C_7\varepsilon$, we have that*

$$\text{if} \quad \xi \ge a + C_7\varepsilon \quad \text{then} \quad Y(\mu^{-1}|\ln\varepsilon|,\xi) \ge \alpha_+ - \eta, \tag{1.71}$$
$$\text{if} \quad \xi \le a - C_7\varepsilon \quad \text{then} \quad Y(\mu^{-1}|\ln\varepsilon|,\xi) \le \alpha_- + \eta. \tag{1.72}$$

**Proof.** We first prove (1.71). For $\xi \ge a + C_7\varepsilon$, as long as $Y(\tau,\xi)$ has not reached $\alpha_+ - \eta$, we can use (1.61) to deduce that

$$\begin{aligned} Y(\tau,\xi) &\ge a + C_1 e^{\mu\tau}(\xi - a) \\ &\ge a + C_1C_7 e^{\mu\tau}\varepsilon \\ &\ge \alpha_+ - \eta \end{aligned}$$

provided that $\tau$ satisfies

$$\tau \ge \tau^\varepsilon =: \mu^{-1}\ln\frac{\alpha_+ - a - \eta}{C_1C_7\varepsilon}.$$

Choosing

$$C_7 = \frac{\max(a-\alpha_-, \alpha_+ - a) - \eta}{C_1},$$

we see that $\mu^{-1}|\ln\varepsilon| \ge \tau^\varepsilon$, which completes the proof of (1.71). Using (1.62), one easily proves (1.72).

Next we prove (1.70). First, by the bistable assumptions on $f$, if we leave from a initial value $\xi \in [\alpha_- - \eta, \alpha_+ + \eta]$ then $Y(\tau,\xi)$ will remain in $[\alpha_- - \eta, \alpha_+ + \eta]$. Now suppose that $\alpha_+ + \eta \le \xi \le 2C_0$. We check below that $Y(\mu^{-1}|\ln\varepsilon|,\xi) \le \alpha_+ + \eta$. First, in view of (1.1), we can find $p > 0$ such that

$$\begin{array}{llll} \text{if} & \alpha_+ \le u \le 2C_0 & \text{then} & f(u) \le p(\alpha_+ - u) \\ \text{if} & -2C_0 \le u \le \alpha_- & \text{then} & f(u) \ge -p(u - \alpha_-). \end{array} \tag{1.73}$$

We then use the ordinary differential equation to obtain, as long as $\alpha_+ + \eta \le Y \le 2C_0$, the inequality $Y_\tau \le p(\alpha_+ - Y)$. It follows that

$$\frac{Y_\tau}{Y - \alpha_+} \le -p.$$

Integrating this inequality from 0 to $\tau$ leads to

$$\begin{aligned} Y(\tau,\xi) &\le \alpha_+ + (\xi - \alpha_+)e^{-p\tau} \\ &\le \alpha_+ + (2C_0 - \alpha_+)e^{-p\tau}. \end{aligned}$$

Since $(2C_0 - \alpha_+)e^{-p\mu^{-1}|\ln\varepsilon|} \to 0$ as $\varepsilon \to 0$, the above inequality proves that, for $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 = \varepsilon_0(\eta)$ sufficiently small, $Y(\mu^{-1}|\ln\varepsilon|, \xi) \le \alpha_+ + \eta$, which completes the proof of (1.70). □

We are now ready to prove Theorem 1.3.1. By setting $t = \mu^{-1}\varepsilon^2|\ln\varepsilon|$ in (1.69), we obtain

$$Y\Big(\mu^{-1}|\ln\varepsilon|, u_0^-(x) - (C_6\varepsilon - C_6\varepsilon^2)\Big) \le u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \le Y\Big(\mu^{-1}|\ln\varepsilon|, u_0^+(x) + C_6\varepsilon - C_6\varepsilon^2\Big). \quad (1.74)$$

Furthermore, by the definition of $C_0$ in (1.8), we have, for $\varepsilon_0$ small enough,

$$-2C_0 \le u_0^\pm(x) \pm (C_6\varepsilon - C_6\varepsilon^2) \le 2C_0,$$

for $x \in \Omega$. Thus the assertion (1.49) of Theorem 1.3.1 is a direct consequence of (1.70) and (1.74).

Next we prove (1.50). We choose $M_0$ large enough so that $M_0\varepsilon - C_6\varepsilon + C_6\varepsilon^2 \ge C_7\varepsilon$. Then, for any $x \in \Omega$ such that $u_0^-(x) \ge a + M_0\varepsilon$, we have

$$u_0^-(x) - (C_6\varepsilon - C_6\varepsilon^2) \ge a + M_0\varepsilon - C_6\varepsilon + C_6\varepsilon^2 \ge a + C_7\varepsilon.$$

Combining this, (1.74) and (1.71), we see that

$$u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \ge \alpha_+ - \eta,$$

for any $x \in \Omega$ that satisfies $u_0^-(x) \ge a + M_0\varepsilon$. From the definition of $u_0^-$ in (1.68), it is clear that

$$u_0^-(x) \ge a + M_0\varepsilon \quad \text{if and only if} \quad u_0(x) \ge a + M_0\varepsilon,$$

provided that $\varepsilon$ is small enough. This proves (1.50). The inequality (1.51) can be shown the same way. This completes the proof of Theorem 1.3.1. □

### 1.3.4 Optimality of the generation time

To conclude this section we show that the generation time $t^\varepsilon := \mu^{-1}\varepsilon^2|\ln\varepsilon|$ that appears in Theorem 1.3.1 is optimal. In other words, the interface will not be fully developed much before $t^\varepsilon$.

**Proposition 1.3.10.** *Denote by $t^\varepsilon_{min}$ the smallest time such that* (1.15) *holds for all $t \in [t^\varepsilon_{min}, T]$. Then there exists a constant $b = b(C)$ such that*

$$t^\varepsilon_{min} \ge \mu^{-1}\varepsilon^2(|\ln\varepsilon| - b)$$

*for all $\varepsilon \in (0, \varepsilon_0)$.*

**Proof.** For simplicity, we deal with the case where (1.65) is valid. In that case, (1.67) holds for all small $\varepsilon > 0$. For each $b > 0$, we set

$$t^\varepsilon(b) := \mu^{-1}\varepsilon^2(|\ln\varepsilon| - b),$$

and evaluate $u^\varepsilon(x, t^\varepsilon(b))$ at a point $x \in \Omega_0^+$ where $\mathrm{dist}(x, \Gamma_0) = C\varepsilon$. Since $u_0 = a$ on $\Gamma_t$ and since $|\nabla u_0| \le C_0$ by (1.8), we have

$$u_0(x) \le a + C_0C\varepsilon. \quad (1.75)$$

It follows from this and (1.61) that

$$\begin{aligned} w_\varepsilon^+(x,t^\varepsilon(b)) &= Y\Big(\mu^{-1}(|\ln\varepsilon|-b), u_0(x)+\varepsilon C_6 e^{-b}-\varepsilon^2 C_6\Big) \\ &\leq a + C_2 e^{|\ln\varepsilon|-b}\big(u_0(x)+\varepsilon C_6 e^{-b}-\varepsilon^2 C_6 - a\big) \\ &\leq a + C_2\varepsilon^{-1}e^{-b}(C_0 C\varepsilon + \varepsilon C_6 e^{-b}) \\ &= a + C_2 e^{-b}(C_0 C + C_6 e^{-b}). \end{aligned}$$

Now we choose $b$ to be sufficiently large, so that

$$a + C_2 e^{-b}(C_0 C + C_6 e^{-b}) < \alpha_+ - \eta.$$

Then the above estimate and (1.67) yield

$$u^\varepsilon(x,t^\varepsilon(b)) \leq w_\varepsilon^+(x,t^\varepsilon(b)) < \alpha_+ - \eta.$$

This implies that (1.15) does not hold at $t = t^\varepsilon(b)$, hence $t^\varepsilon(b) < t^\varepsilon_{min}$. The lemma is proved. □

## 1.4 Generation of interface in the general case

In this section we extend Theorem 1.3.1 to the case where $g^\varepsilon \not\equiv 0$. The proof is more technical than the case $g^\varepsilon \equiv 0$, but the underlying ideas are the same. Hence we will basically follow the argument of Section 1.3, simply pointing out the main differences.

### 1.4.1 The perturbed bistable ordinary differential equation

We first consider a slightly perturbed nonlinearity:

$$f_\delta(u) = f(u) + \delta,$$

where $\delta$ is any constant. For $|\delta|$ small enough, this function is still bistable. More precisely, we claim that $f_\delta$ has the following properties.

**Lemma 1.4.1.** *Let $\delta_0$ be small enough. Then, for all $\delta \in (-\delta_0, \delta_0)$,*

- *$f_\delta$ has exactly three zeros, namely $\alpha_-(\delta) < a(\delta) < \alpha_+(\delta)$ and there exists a positive constant $C$ such that*

$$|\alpha_-(\delta) - \alpha_-| + |a(\delta) - a| + |\alpha_+(\delta) - \alpha_+| \leq C|\delta|. \tag{1.76}$$

- *We have that*

$$\begin{aligned} &f_\delta \quad \text{is strictly positive in} \quad (-\infty, \alpha_-(\delta)) \cup (a(\delta), \alpha_+(\delta)), \\ &f_\delta \quad \text{is strictly negative in} \quad (\alpha_-(\delta), a(\delta)) \cup (\alpha_+(\delta), +\infty). \end{aligned} \tag{1.77}$$

- *Set*

$$\mu(\delta) := f_\delta'(a(\delta)) = f'(a(\delta)),$$

*then there exists a positive constant, which we denote again by $C$, such that*

$$|\mu(\delta) - \mu| \leq C|\delta|. \tag{1.78}$$

Now, for each $\delta \in (-\delta_0, \delta_0)$, we define $Y(\tau, \xi; \delta)$ as the solution of the following ordinary differential equation:

$$\begin{cases} Y_\tau(\tau, \xi; \delta) &= f_\delta(Y(\tau, \xi; \delta)) \quad \text{for } \tau > 0, \\ Y(0, \xi; \delta) &= \xi, \end{cases} \tag{1.79}$$

where $\xi$ varies in $(-2C_0, 2C_0)$ with $C_0$ being the constant in (1.8).

To prove Theorem 1.3.1, we will construct a pair of sub- and super-solutions for $(P^\varepsilon)$ by simply replacing the function $Y(\tau, \xi)$ in (1.66) by $Y(\tau, \xi; \delta)$, with an appropriate choice of $\delta$. For this strategy to work, we have to check that the basic properties of $Y(\tau, \xi)$ in subsection 1.3.1 carry over to $Y(\tau, \xi; \delta)$.

First, it is clear that all the differential and integral identities in subsection 1.3.1 that follow directly from (1.52) are still valid for (1.79). In particular, Lemmas 1.3.2 and 1.3.3 remain to hold if we replace $Y(\tau, \xi)$ by $Y(\tau, \xi; \delta)$, $f$ by $f_\delta$ and $A(\tau, \xi)$ by $A(\tau, \xi; \delta)$, where

$$A(\tau, \xi, \delta) = \frac{f'_\delta(Y(\tau, \xi; \delta)) - f'_\delta(\xi)}{f_\delta(\xi)}.$$

Next let us show that the basic estimates which we have established in subsection 1.3.1 are also valid for the function $Y(\tau, \xi; \delta)$. The following lemma, which is an analogue of Lemma 1.3.4, is fundamental.

**Lemma 1.4.2.** *Let $\eta \in (0, \eta_0)$ be arbitrary. Then there exist positive constants $\delta_0 = \delta_0(\eta)$, $\widetilde{C}_1 = \widetilde{C}_1(\eta)$, $\tilde{C}_2 = \tilde{C}_2(\eta)$ and $C_3 = C_3(\eta)$ such that, for all $\delta \in (-\delta_0, \delta_0)$, for all $\tau > 0$,*

- *if $\xi \in (a(\delta), \alpha_+ - \eta)$ then, for every $\tau > 0$ such that $Y(\tau, \xi; \delta)$ remains in the interval $(a(\delta), \alpha_+ - \eta)$, we have*

$$\widetilde{C}_1 e^{\mu(\delta)\tau} \le Y_\xi(\tau, \xi; \delta) \le \tilde{C}_2 e^{\mu(\delta)\tau}, \tag{1.80}$$

*and*

$$|A(\tau, \xi; \delta)| \le C_3(e^{\mu(\delta)\tau} - 1); \tag{1.81}$$

- *if $\xi \in (\alpha_- + \eta, a(\delta))$ then, for every $\tau > 0$ such that $Y(\tau, \xi; \delta)$ remains in the interval $(\alpha_- + \eta, a(\delta))$, (1.80) and (1.81) hold as well.*

**Proof.** In view of (1.76), we can choose a small constant $\delta_0 = \delta_0(\eta) > 0$ such that $(a(\delta), \alpha_+ - \eta) \subset (a(\delta), \alpha_+(\delta))$ for every $\delta \in [-\delta_0, \delta_0]$. Therefore $f_\delta(q)$ does not change sign in the interval $(a(\delta), \alpha_+ - \eta)$. Thus, in order to prove the lemma, we just have to write again the proof of Lemma 1.3.4, simply replacing $Y(\tau, \xi)$ by $Y(\tau, \xi; \delta)$. We do not repeat the entire proof here. Instead, let us explain why $\widetilde{C}_1$, $\tilde{C}_2$ and $C_3$ are independent of $\delta$; in view of the proof of Lemma 1.3.4, it is sufficient to estimate, for $q \in [a(\delta), \alpha_+ - \eta]$, the modulus of the quantity

$$h_\delta(q) = \frac{f'_\delta(q) - f'_\delta(a(\delta))}{f_\delta(q)}$$

by a constant depending on $\eta$ but not on $\delta \in [-\delta_0, \delta_0]$. Since

$$h_\delta(q) \to \frac{f''_\delta(a(\delta))}{f'_\delta(a(\delta))} = \frac{f''(a(\delta))}{f'(a(\delta))} \quad \text{as } q \to a(\delta),$$

we see that the function $(q, \delta) \mapsto h_\delta(q)$ is continuous in the compact region $\{\, |\delta| \le \delta_0,\ a(\delta) \le q \le \alpha_+ - \eta \,\}$. It follows that $|h_\delta(q)|$ is bounded as $(q, \delta)$ varies in this region. This completes the proof of Lemma 1.4.2. □

**Corollary 1.4.3.** *Let $\eta \in (0, \eta_0)$ be arbitrary. Then there exist positive constants $\delta_0 = \delta_0(\eta)$, $C_1 = C_1(\eta)$ and $C_2 = C_2(\eta)$ such that, for all $\delta \in (-\delta_0, \delta_0)$, for all $\tau > 0$,*

- *if $\xi \in (a(\delta), \alpha_+ - \eta)$ then, for every $\tau > 0$ such that $Y(\tau, \xi; \delta)$ remains in the interval $(a(\delta), \alpha_+ - \eta)$, we have*

$$C_1 e^{\mu(\delta)\tau}(\xi - a(\delta)) \le Y(\tau, \xi; \delta) - a(\delta) \le C_2 e^{\mu(\delta)\tau}(\xi - a(\delta)), \tag{1.82}$$

- *if $\xi \in (\alpha_- + \eta, a(\delta))$ then, for every $\tau > 0$ such that $Y(\tau, \xi; \delta)$ remains in the interval $(\alpha_- + \eta, a(\delta))$, we have*

$$C_2 e^{\mu(\delta)\tau}(\xi - a(\delta)) \le Y(\tau, \xi; \delta) - a(\delta) \le C_1 e^{\mu(\delta)\tau}(\xi - a(\delta)). \tag{1.83}$$

**Proof.** We can simply follow the proof of Corollary 1.3.5. In order to prove that $C_1$ and $C_2$ are independent of $\delta$, all we have to do is to find constants $B_1 = B_1(\eta) > 0$ and $B_2 = B_2(\eta) > 0$ such that, for all $\delta \in [-\delta_0, \delta_0]$ and all $q \in (a(\delta), \alpha_+ - \eta)$,

$$B_1(q - a(\delta)) \le f_\delta(q) \le B_2(q - a(\delta)). \tag{1.84}$$

In view of (1.78), we can choose $\delta_0 > 0$ small enough so that, for all $\delta \in [-\delta_0, \delta_0]$, we have $\mu(\delta) \ge \mu/2 > 0$. Since

$$\frac{f_\delta(q)}{q - a(\delta)} \to \mu(\delta) \quad \text{as} \quad q \to a(\delta),$$

it follows that $(q, \delta) \mapsto f_\delta(q)/(q - a(\delta))$ is a strictly positive and continuous function on the compact region $\{\, |\delta| \le \delta_0,\ a(\delta) \le q \le \alpha_+ - \eta \,\}$, which insures the existence of the constants $B_1$ and $B_2$. This completes the proof of the corollary. □

Now, it is no trouble to establish an analogue of Lemmas 1.3.6 and 1.3.7 with constants independent of $\delta$. We claim, without proof, that:

**Lemma 1.4.4.** *Let $\eta \in (0, \eta_0)$ and $M > 0$ be arbitrary. Then there exist positive constants $\delta_0 = \delta_0(\eta, M)$ and $C_4 = C_4(\eta, M)$ such that, for all $\delta \in (-\delta_0, \delta_0)$,*

- *if $\xi \in [\alpha_+ - \eta, \alpha_+ + M]$, then, for all $\tau > 0$, $Y(\tau, \xi; \delta)$ remains in the interval $[\alpha_+ - \eta, \alpha_+ + M]$ and*

$$|A(\tau, \xi; \delta)| \le C_4 \tau \quad \textit{for} \quad \tau > 0\,; \tag{1.85}$$

- *if $\xi \in [\alpha_- - M, \alpha_- + \eta]$, then, for all $\tau > 0$, $Y(\tau, \xi; \delta)$ remains in the interval $[\alpha_- - M, \alpha_- + \eta]$ and (1.85) holds as well.*

**Lemma 1.4.5.** *Let $\eta \in (0, \eta_0)$ be arbitrary and let $C_0$ be the constant defined in (1.8). Then there exist positive constants $\delta_0 = \delta_0(\eta)$, $C_5 = C_5(\eta)$ such that, for all $\delta \in (-\delta_0, \delta_0)$, for all $\tau > 0$ and all $\xi \in (-2C_0, 2C_0)$,*

$$|A(\tau, \xi; \delta)| \le C_5(e^{\mu(\delta)\tau} - 1).$$

### 1.4.2 Construction of sub- and super-solutions

We now use $Y(\tau,\xi;\delta)$, the solution of the ordinary differential equation (1.79), to construct a pair of sub- and super-solutions. The same cut-off argument as in subsection 1.3.2 enables us to assume (1.65) for simplicity. We set

$$w_\varepsilon^{\pm}(x,t) = Y\Big(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 r(\pm\varepsilon\mathcal{G}, \frac{t}{\varepsilon^2}); \pm\varepsilon\mathcal{G}\Big), \tag{1.86}$$

where the function $r(\delta,\tau)$ is given by

$$r(\delta,\tau) = C_6(e^{\mu(\delta)\tau} - 1),$$

and the constant $\mathcal{G}$ is chosen such that, for all small $\varepsilon > 0$,

$$|g^\varepsilon(x,t,u)| \le \mathcal{G} \quad \text{for all} \quad (x,t,u) \in \overline{\Omega} \times [0,T] \times \mathbb{R},$$

which, in view of (1.5), is clearly possible.

**Lemma 1.4.6.** *There exist positive constants* $\varepsilon_0$ *and* $C_6$ *such that for all* $\varepsilon \in (0,\varepsilon_0)$, $(w_\varepsilon^-, w_\varepsilon^+)$ *is a pair of sub- and super-solutions for Problem* $(P^\varepsilon)$, *in the domain*

$$\big\{(x,t) \in Q_T,\ x \in \Omega,\ 0 \le t \le \mu^{-1}\varepsilon^2 |\ln\varepsilon|\big\},$$

*satisfying* $w_\varepsilon^-(x,0) = w_\varepsilon^+(x,0) = u_0(x)$. *Consequently*

$$w_\varepsilon^-(x,t) \le u^\varepsilon(x,t) \le w_\varepsilon^+(x,t) \quad \textit{for } x \in \overline{\Omega},\ 0 \le t \le \mu^{-1}\varepsilon^2|\ln\varepsilon|. \tag{1.87}$$

**Proof.** First, in view of (1.65), $w_\varepsilon^\pm$ satisfy the homogeneous Neumann boundary condition. We define the operator $\mathcal{L}$ by

$$\mathcal{L}u := u_t - \Delta u - \varepsilon^{-2}(f(u) - \varepsilon g^\varepsilon(x,t,u)),$$

and prove below that $\mathcal{L}w_\varepsilon^+ \ge 0$ by slightly modifying the argument which we have used to prove $\mathcal{L}_0 w_\varepsilon^+ \ge 0$ in subsection 1.3.2. A straightforward calculation yields

$$\mathcal{L}w_\varepsilon^+ = \frac{1}{\varepsilon^2}Y_\tau + Y_\xi\Big[C_6\mu(\varepsilon\mathcal{G})e^{\mu(\varepsilon\mathcal{G})t/\varepsilon^2} - \Delta u_0 - \frac{Y_{\xi\xi}}{Y_\xi}|\nabla u_0|^2\Big] - \frac{1}{\varepsilon^2}f(Y) + \frac{1}{\varepsilon}g^\varepsilon(x,t,Y).$$

If $\varepsilon_0$ is sufficiently small, we note that $\pm\varepsilon\mathcal{G} \in (-\delta_0,\delta_0)$ and that, in the range $0 \le t \le \mu^{-1}\varepsilon^2|\ln\varepsilon|$,

$$|\varepsilon^2 C_6(e^{\mu(\pm\varepsilon\mathcal{G})t/\varepsilon^2} - 1)| \le \varepsilon^2 C_6(\varepsilon^{-\mu(\pm\varepsilon\mathcal{G})/\mu} - 1) \le C_0,$$

which implies that

$$u_0(x) \pm \varepsilon^2 r(\pm\varepsilon\mathcal{G}, \frac{t}{\varepsilon^2}) \in (-2C_0, 2C_0).$$

These observations allow us to use the results of the previous subsection with $\tau = t/\varepsilon^2$, $\xi = u_0(x) + \varepsilon^2 r(\varepsilon\mathcal{G}, t/\varepsilon^2)$ and $\delta = \varepsilon\mathcal{G}$. In particular, the ordinary differential equation (1.79) yields $Y_\tau = f(Y) + \varepsilon\mathcal{G}$, which implies that

$$\mathcal{L}w_\varepsilon^+ = \frac{1}{\varepsilon}\Big[\mathcal{G} + g^\varepsilon(x,t,Y)\Big] + Y_\xi\Big[C_6\mu(\varepsilon\mathcal{G})e^{\mu(\varepsilon\mathcal{G})t/\varepsilon^2} - \Delta u_0 - \frac{Y_{\xi\xi}}{Y_\xi}|\nabla u_0|^2\Big].$$

By the choice of $\mathcal{G}$ the first term is positive. Using the estimate of $A = Y_{\xi\xi}/Y_\xi$ in Lemma 1.4.5, we obtain, for a constant $C_5$ that is independent of $\varepsilon$,

$$\begin{aligned}\mathcal{L}w_\varepsilon^+ &\geq Y_\xi\Big[C_6\mu(\varepsilon\mathcal{G})e^{\mu(\varepsilon\mathcal{G})t/\varepsilon^2} - |\Delta u_0| - C_5(e^{\mu(\varepsilon\mathcal{G})t/\varepsilon^2} - 1)|\nabla u_0|^2\Big]\\ &\geq Y_\xi\Big[(C_6\mu(\varepsilon\mathcal{G}) - C_5|\nabla u_0|^2)e^{\mu(\varepsilon\mathcal{G})t/\varepsilon^2} - |\Delta u_0| + C_5|\nabla u_0|^2\Big].\end{aligned}$$

In view of (1.78), this inequality implies that, for $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0$ small enough, and for $C_6$ large enough,

$$\mathcal{L}w_\varepsilon^+ \geq Y_\xi\Big[C_6\frac{1}{2}\mu - C_5{C_0}^2 - C_0\Big] \geq 0.$$

Hence $w_\varepsilon^+$ is a super-solution for Problem $(P^\varepsilon)$. Similarly $w_\varepsilon^-$ is a sub-solution. Obviously $w_\varepsilon^+(x,0) = w_\varepsilon^-(x,0) = Y\big(0, u_0(x); \pm\varepsilon\mathcal{G}\big) = u_0(x)$. Lemma 1.3.8 is proved. □

### 1.4.3 Proof of Theorem 1.3.1 for the general case

As in subsection 1.3.3, we first present a key estimate on the function $Y(\tau, \xi; \delta)$ after a time interval of order $\tau \sim |\ln\varepsilon|$. Roughly speaking, a perturbation $\delta$ of order $\varepsilon$ does not affect the result of Lemma 1.3.9.

**Lemma 1.4.7.** *Let $\eta \in (0, \eta_0)$ be arbitrary; there exist positive constants $\varepsilon_0$ and $C_7$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

- *for all $\xi \in (-2C_0, 2C_0)$,*

$$\alpha_- - \eta \leq Y(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon\mathcal{G}) \leq \alpha_+ + \eta, \tag{1.88}$$

- *for all $\xi \in (-2C_0, 2C_0)$ such that $|\xi - a| \geq C_7\varepsilon$, we have that*

$$\text{if} \quad \xi \geq a + C_7\varepsilon \quad \text{then} \quad Y(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon\mathcal{G}) \geq \alpha_+ - \eta, \tag{1.89}$$
$$\text{if} \quad \xi \leq a - C_7\varepsilon \quad \text{then} \quad Y(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon\mathcal{G}) \leq \alpha_- + \eta. \tag{1.90}$$

**Proof.** In view of (1.76), we have, for $C_7$ large enough, $a + C_7\varepsilon \geq a(\pm\varepsilon\mathcal{G}) + \frac{1}{2}C_7\varepsilon$, for all $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0$ sufficiently small. Hence for $\xi \geq a + C_7\varepsilon$, as long as $Y(\tau, \xi; \pm\varepsilon\mathcal{G})$ has not reached $\alpha_+ - \eta$, we can use (1.82) to deduce, as done in the proof of Lemma 1.3.9, that

$$\begin{aligned}Y(\tau, \xi; \pm\varepsilon\mathcal{G}) &\geq a(\pm\varepsilon\mathcal{G}) + C_1e^{\mu(\pm\varepsilon\mathcal{G})\tau}(\xi - a(\pm\varepsilon\mathcal{G}))\\ &\geq a - \varepsilon C\mathcal{G} + \tfrac{1}{2}C_1C_7e^{\mu(\pm\varepsilon\mathcal{G})\tau}\varepsilon\\ &\geq \alpha_+ - \eta\end{aligned}$$

provided that $\tau$ satisfies

$$\tau \geq \frac{1}{\mu(\pm\varepsilon\mathcal{G})}\ln\frac{m_0 - \eta + C\mathcal{G}\varepsilon}{\frac{1}{2}C_1C_7\varepsilon} =: \mu^{-1}(\varepsilon)|\ln\varepsilon|,$$

where $m_0 = \max(a - \alpha_-, \alpha_+ - a)$. To complete the proof of (1.89) we must choose $C_7$ so that $\mu^{-1}|\ln\varepsilon| - \mu^{-1}(\varepsilon)|\ln\varepsilon| \geq 0$. A simple computation shows that

$$\mu^{-1}|\ln\varepsilon| - \mu^{-1}(\varepsilon)|\ln\varepsilon| = \frac{\mu(\pm\varepsilon\mathcal{G}) - \mu}{\mu(\pm\varepsilon\mathcal{G})\mu}|\ln\varepsilon| - \frac{1}{\mu(\pm\varepsilon\mathcal{G})}\ln\frac{m_0 - \eta + C\mathcal{G}\varepsilon}{\frac{1}{2}C_1} + \frac{1}{\mu(\pm\varepsilon\mathcal{G})}\ln C_7.$$

Thanks to (1.78), as $\varepsilon \to 0$, the first term above is of order $\varepsilon|\ln\varepsilon|$ and the second one of order 1. Hence, for $C_7$ large enough, the upper quantity is positive for all $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0$ sufficiently small. The proof of (1.90) is similar and omitted.

Next we prove (1.88). First, by taking $\varepsilon_0$ sufficiently small, we can assume that the stable equilibria of $f_{\pm\varepsilon\mathcal{G}}$, namely $\alpha_-(\pm\varepsilon\mathcal{G})$ and $\alpha_+(\pm\varepsilon\mathcal{G})$, are in $[\alpha_- - \eta, \alpha_+ + \eta]$. Hence, $f_{\pm\varepsilon\mathcal{G}}$ being a bistable function, if we leave from a $\xi \in [\alpha_- - \eta, \alpha_+ + \eta]$ then $Y(\tau, \xi; \pm\varepsilon\mathcal{G})$ will remain in the interval $[\alpha_- - \eta, \alpha_+ + \eta]$. Now suppose that $\alpha_+ + \eta \le \xi \le 2C_0$. We check below that $Y(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon\mathcal{G}) \le \alpha_+ + \eta$. As done in the proof of Lemma 1.3.9, as long as $\alpha_+ + \eta \le Y \le 2C_0$, (1.73) leads to the inequality $Y_\tau \le p(\alpha_+ - Y) + \varepsilon\mathcal{G}$. It follows that

$$\frac{Y_\tau}{Y - \alpha_+} \le -p + \varepsilon\frac{\mathcal{G}}{\eta},$$

which implies, by integration from 0 to $\tau$, that

$$Y(\tau, \xi; \pm\varepsilon\mathcal{G}) \le \alpha_+ + (2C_0 - \alpha_+)e^{(-p+\varepsilon\frac{\mathcal{G}}{\eta})\tau}.$$

Since $(2C_0 - \alpha_+)e^{(-p+\varepsilon\mathcal{G}\eta^{-1})\mu^{-1}|\ln\varepsilon|} \to 0$ as $\varepsilon \to 0$, the above inequality proves that, for $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 = \varepsilon_0(\eta)$ sufficiently small, $Y(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon\mathcal{G}) \le \alpha_+ + \eta$, which completes the proof of (1.88). $\square$

We are now ready to prove Theorem 1.3.1 in the general case. By setting $t = \mu^{-1}\varepsilon^2|\ln\varepsilon|$ in (1.87), we get

$$\begin{aligned} &Y\Big(\mu^{-1}|\ln\varepsilon|, u_0(x) - \varepsilon^2 r(-\varepsilon\mathcal{G}, \mu^{-1}|\ln\varepsilon|); -\varepsilon\mathcal{G}\Big) \\ &\qquad \le u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \le Y\Big(\mu^{-1}|\ln\varepsilon|, u_0(x) + \varepsilon^2 r(\varepsilon\mathcal{G}, \mu^{-1}|\ln\varepsilon|); +\varepsilon\mathcal{G}\Big). \end{aligned} \tag{1.91}$$

The point is that, in view of (1.78),

$$\lim_{\varepsilon\to 0} \frac{\mu - \mu(\pm\varepsilon\mathcal{G})}{\mu} \ln\varepsilon = 0. \tag{1.92}$$

Hence we have, for $\varepsilon_0$ small enough,

$$\varepsilon^2 r(\pm\varepsilon\mathcal{G}, \mu^{-1}|\ln\varepsilon|) = C_6\varepsilon(\varepsilon^{(\mu-\mu(\pm\varepsilon\mathcal{G}))/\mu} - \varepsilon) \in (\frac{1}{2}C_6\varepsilon, \frac{3}{2}C_6\varepsilon).$$

Hence, as in subsection 1.3.3, the result (1.49) of Theorem 1.3.1 is a direct consequence of (1.88) and (1.91).

Next we prove (1.50). We take $x \in \Omega$ such that $u_0(x) \ge a + M_0\varepsilon$ so that

$$\begin{aligned} u_0(x) - \varepsilon^2 r(-\varepsilon\mathcal{G}, \mu^{-1}|\ln\varepsilon|) &\ge a + M_0\varepsilon - \tfrac{3}{2}C_6\varepsilon \\ &\ge a + C_7\varepsilon, \end{aligned}$$

if we choose $M_0$ large enough. Using (1.91) and (1.89) we obtain (1.50), which completes the proof of Theorem 1.3.1. $\square$

## 1.5 Motion of interface

In Sections 1.3 and 1.4, we proved that the solution $u^\varepsilon$ develops a clear transition layer within a very short time. The aim of the present section is to show that, once such a clear transition layer is formed, it persists for the rest of time and that its law of motion is well approximated by the interface equation $(P^0)$.

Let us formulate the above assertion more clearly. By taking the first two terms of the formal asymptotic expansion (1.24), we get a formal approximation of the solution $u^\varepsilon$ up to order $\varepsilon$:

$$u^\varepsilon(x,t) \approx \tilde{u}^\varepsilon(x,t) := U_0\Big(\frac{\widetilde{d}(x,t)}{\varepsilon}\Big) + \varepsilon U_1\Big(x,t,\frac{\widetilde{d}(x,t)}{\varepsilon}\Big). \tag{1.93}$$

Here $U_0$, $U_1$ are as defined in (1.26) and (1.37). The right-hand side has a clear transition layer which lies exactly on $\Gamma_t$. Our goal is to show that this function is a good approximation of the real solution; more precisely:

> *If $u^\varepsilon$ becomes rather close to $\tilde{u}^\varepsilon$ at some time moment $t = t_0$, then it stays close to $\tilde{u}^\varepsilon$ for the rest of time. Consequently, $\Gamma_t^\varepsilon$ evolves roughly like $\Gamma_t$.*

In order to prove such a result, we will construct a pair of sub- and super-solutions $u_\varepsilon^-$ and $u_\varepsilon^+$ for Problem $(P^\varepsilon)$ by slightly modifying the above function $\tilde{u}^\varepsilon$. It then follows that, if the solution $u^\varepsilon$ satisfies

$$u_\varepsilon^-(x,t_0) \le u^\varepsilon(x,t_0) \le u_\varepsilon^+(x,t_0),$$

for some $t_0 \ge 0$, then

$$u_\varepsilon^-(x,t) \le u^\varepsilon(x,t) \le u_\varepsilon^+(x,t),$$

for $t_0 \le t \le T$. As a result, since both $u_\varepsilon^+, u_\varepsilon^-$ stay close to $\tilde{u}^\varepsilon$, the solution $u^\varepsilon$ also stays close to $\tilde{u}^\varepsilon$ for $t_0 \le t \le T$.

The rest of this section is devoted to the construction of these sub- and super-solutions. We begin with some preparations.

### 1.5.1 A modified signed distance function

Rather than working with the usual signed distance function $\widetilde{d}$ that was introduced in (1.22), we define a "cut-off signed distance function" $d$ as follows. First, choose $d_0 > 0$ small enough so that $\widetilde{d}(\cdot,\cdot)$ is smooth in the tubular neighborhood of $\Gamma$

$$\{(x,t) \in \overline{Q_T},\ |\widetilde{d}(x,t)| < 3d_0\},$$

and such that

$$\operatorname{dist}(\Gamma_t, \partial\Omega) \ge 3d_0 \quad \text{for all } t \in [0,T]. \tag{1.94}$$

Next let $\zeta(s)$ be a smooth increasing function on $\mathbb{R}$ such that

$$\zeta(s) = \begin{cases} s & \text{if } |s| \le d_0 \\ -2d_0 & \text{if } s \le -2d_0 \\ 2d_0 & \text{if } s \ge 2d_0. \end{cases}$$

We then define the cut-off signed distance function $d$ by

$$d(x,t) = \zeta\big(\tilde{d}(x,t)\big). \tag{1.95}$$

Note that $|\nabla d| = 1$ in the region $\{(x,t) \in \overline{Q_T},\ |\widetilde{d}(x,t)| < d_0\}$ and that, in view of (1.94) and the above definition, $\nabla d = 0$ in a neighborhood of $\partial\Omega$. Note also that the equation of motion $(P^0)$, which is equivalent to (1.35), is now written as

$$d_t = \Delta d - \gamma(x,t) \quad \text{on } \ \Gamma_t, \tag{1.96}$$

where we recall that

$$\gamma(x,t) = c_0 \int_{\alpha_-}^{\alpha_+} g(x,t,r)dr. \tag{1.97}$$

### 1.5.2 Construction of sub- and super-solutions

As we stated earlier, we now construct sub- and super-solutions by modifying the function $\tilde{u}^\varepsilon$ in (1.93). Concerning the second term $U_1$, which is defined in (1.37), the terms $\Delta U_1$ and $U_{1t}$ do not make sense as we only assume that $g(\cdot,\cdot,u) \in C^{1+\vartheta,\frac{1+\vartheta}{2}}$. In order to cope with this lack of smoothness, as $g^\varepsilon(\cdot,\cdot,u) \in C^{2,1}$, we replace $U_1$ by a more smooth function $U_1^\varepsilon$, which is defined by

$$\begin{cases} U_{1zz}^\varepsilon + f'(U_0(z))U_1^\varepsilon = g^\varepsilon(x,t,U_0(z)) - \gamma^\varepsilon(x,t){U_0}'(z), \\ U_1^\varepsilon(x,t,0) = 0, \qquad\qquad\qquad U_1^\varepsilon(x,t,\cdot) \in L^\infty(\mathbb{R}), \end{cases} \tag{1.98}$$

where

$$\gamma^\varepsilon(x,t) = c_0 \int_{\alpha_-}^{\alpha_+} g^\varepsilon(x,t,r)dr. \tag{1.99}$$

Thus $U_1^\varepsilon(x,t,z)$ is a solution of (1.30) with

$$A = A_0^\varepsilon(x,t,z) := g^\varepsilon(x,t,U_0(z)) - \gamma^\varepsilon(x,t){U_0}'(z), \tag{1.100}$$

where the variables $x, t, \varepsilon$ are considered parameters. Using (1.5) and the same arguments as in the end of Section 1.2, we obtain estimates analogous to (1.39) and (1.40), with a constant $M$ independent of $\varepsilon$:

$$|U_1^\varepsilon(x,t,z)| \le M, \quad |\nabla_x U_1^\varepsilon(x,t,z)| \le M. \tag{1.101}$$

Moreover, $g^\varepsilon$ being $C^2$ in $x$ and $C^1$ in $t$, $\Delta_x U_1^\varepsilon$ and $U_{1t}^\varepsilon$ are solutions of (1.30) with $A = \Delta_x A_0^\varepsilon$ and $A = A_{0t}^\varepsilon$, respectively. Thus, in view of (1.3), we obtain

$$|\Delta_x U_1^\varepsilon(x,t,z)| \le C/\varepsilon, \quad |U_{1t}^\varepsilon(x,t,z)| \le C/\varepsilon, \tag{1.102}$$

for a constant $C$ independent of $\varepsilon$. Similarly, (1.4), (1.5) and Lemma 1.2.3 yield estimates analogous to (1.44) and (1.45) for $U_1^\varepsilon$, for constants $C$ and $M$ independent of $\varepsilon$:

$$|U_{1z}^\varepsilon(x,t,z)| + |U_{1zz}^\varepsilon(x,t,z)| \le Ce^{-\lambda|z|}, \tag{1.103}$$

$$|\nabla_x U_{1z}^\varepsilon(x,t,z)| \le M. \tag{1.104}$$

In the rest of this section, $C$ and $M$ will stand for the constants that appear in inequalities (1.101)—(1.104). Note also that, by the same arguments used to obtain (1.47), we see that (1.7) implies the homogeneous Neumann boundary condition for $U_1^\varepsilon$:

$$\frac{\partial U_1^\varepsilon}{\partial \nu} = 0 \quad \text{on } \ \partial\Omega \times [0,T] \times \mathbb{R}. \tag{1.105}$$

We look for a pair of sub- and super-solutions $u_\varepsilon^\pm$ for $(P^\varepsilon)$ of the form

$$u_\varepsilon^\pm(x,t) = U_0\Big(\frac{d(x,t) \pm \varepsilon p(t)}{\varepsilon}\Big) + \varepsilon U_1^\varepsilon\Big(x,t,\frac{d(x,t) \pm \varepsilon p(t)}{\varepsilon}\Big) \pm q(t), \tag{1.106}$$

where

$$\begin{aligned} p(t) &= -e^{-\beta t/\varepsilon^2} + e^{Lt} + K, \\ q(t) &= \sigma\big(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}\big). \end{aligned} \tag{1.107}$$

Note that $q = \sigma\varepsilon^2 p_t$. It is clear from the definition of $u_\varepsilon^\pm$ that

$$\lim_{\varepsilon\to 0} u_\varepsilon^\pm(x,t) = \begin{cases} \alpha_+ & \text{for all } (x,t) \in Q_T^+ \\ \alpha_- & \text{for all } (x,t) \in Q_T^-. \end{cases} \tag{1.108}$$

The main result of this section is the following.

**Lemma 1.5.1.** *Choose $\beta$, $\sigma > 0$ appropriately. For any $K > 1$, we can find positive constants $\varepsilon_0$ and $L$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, the functions $(u_\varepsilon^-, u_\varepsilon^+)$ are a pair of sub- and super-solutions for Problem $(P^\varepsilon)$ in the range $x \in \overline{\Omega}$, $0 \le t \le T$. In other words, $u_\varepsilon^-$ and $u_\varepsilon^+$ satisfy the homogeneous Neumann boundary condition and*

$$\mathcal{L}u_\varepsilon^- \le 0 \le \mathcal{L}u_\varepsilon^+,$$

*in the range $x \in \overline{\Omega}$, $0 \le t \le T$, where we recall that the operator $\mathcal{L}$ is defined by*

$$\mathcal{L}u := u_t - \Delta u - \varepsilon^{-2}(f(u) - \varepsilon g^\varepsilon(x,t,u)).$$

### 1.5.3 Proof of Lemma 1.5.1

By virtue of (1.105) and the fact that $\nabla d = 0$ near $\partial\Omega$, we have

$$\frac{\partial u_\varepsilon^\pm}{\partial \nu} = 0 \qquad \text{on } \ \partial\Omega \times [0,T].$$

In the following we prove inequality $\mathcal{L}u_\varepsilon^+ \ge 0$, the inequality $\mathcal{L}u_\varepsilon^- \le 0$ following by the same argument.

#### Computation of $\mathcal{L}u_\varepsilon^+$

Straightforward computations yield

$$\begin{aligned} (u_\varepsilon^+)_t &= U_0'(\frac{d_t}{\varepsilon} + p_t) + \varepsilon U_{1t}^\varepsilon + U_{1z}^\varepsilon(d_t + \varepsilon p_t) + q_t, \\ \nabla u_\varepsilon^+ &= U_0'\frac{\nabla d}{\varepsilon} + \varepsilon \nabla U_1^\varepsilon + U_{1z}^\varepsilon \nabla d, \\ \Delta u_\varepsilon^+ &= U_0''\frac{|\nabla d|^2}{\varepsilon^2} + U_0'\frac{\Delta d}{\varepsilon} + \varepsilon \Delta U_1^\varepsilon + 2\nabla U_{1z}^\varepsilon \cdot \nabla d + U_{1zz}^\varepsilon \frac{|\nabla d|^2}{\varepsilon} + U_{1z}^\varepsilon \Delta d, \end{aligned}$$

where the function $U_0$, as well as its derivatives, is evaluated at $\big(d(x,t) + \varepsilon p(t)\big)/\varepsilon$, whereas the function $U_1^\varepsilon$, as well as its derivatives, is evaluated at $\Big(x,t,\big(d(x,t)+\varepsilon p(t)\big)/\varepsilon\Big)$. Here, $\nabla U_1^\varepsilon$ denotes the derivative with respect to $x$ whenever we regard $U_1^\varepsilon(x,t,z)$ as a function of three variables $x$, $t$ and $z$. The symbol $\Delta U_1^\varepsilon$ is defined similarly. We also expand the reaction terms as follows.

$$f(u_\varepsilon^+) = f(U_0) + (\varepsilon U_1^\varepsilon + q) f'(U_0) + \frac{1}{2}(\varepsilon U_1^\varepsilon + q)^2 f''(\theta),$$
$$g(x,t,u_\varepsilon^+) = g(x,t,U_0) + (\varepsilon U_1^\varepsilon + q) g_u(x,t,\omega),$$

where $\theta(x,t)$ and $\omega(x,t)$ are some functions satisfying $U_0 < \theta < u_\varepsilon^+$, $U_0 < \omega < u_\varepsilon^+$. Writing $g^\varepsilon = g + g^\varepsilon - g$ and combining the above expressions with equations (1.26) and (1.98), we obtain

$$\mathcal{L} u_\varepsilon^+ = E_1 + \cdots + E_7,$$

where:

$$E_1 = -\frac{1}{\varepsilon^2} q \left( f'(U_0) + \frac{1}{2} q f''(\theta) \right) + {U_0}' p_t + q_t,$$

$$E_2 = \Big( \frac{{U_0}''}{\varepsilon^2} + \frac{U_{1zz}^\varepsilon}{\varepsilon} \Big)(1 - |\nabla d|^2),$$

$$E_3 = \Big( \frac{{U_0}'}{\varepsilon} + U_{1z}^\varepsilon \Big)(d_t - \Delta d + \gamma),$$

$$E_4 = \varepsilon U_{1z}^\varepsilon\, p_t + \frac{1}{\varepsilon} q \left( g_u(x,t,\omega) - U_1^\varepsilon f''(\theta) \right),$$

$$E_5 = -\gamma\, U_{1z}^\varepsilon - \frac{1}{2} (U_1^\varepsilon)^2 f''(\theta) + U_1^\varepsilon g_u(x,t,\omega) - 2\, \nabla U_{1z}^\varepsilon \cdot \nabla d,$$

$$E_6 = \varepsilon U_{1t}^\varepsilon - \varepsilon \Delta U_1^\varepsilon,$$

$$E_7 = \frac{1}{\varepsilon}(g^\varepsilon - g)(x,t,u_\varepsilon^+) - \frac{1}{\varepsilon}(g^\varepsilon - g)(x,t,U_0) + \frac{1}{\varepsilon}(\gamma^\varepsilon - \gamma)(x,t){U_0}'.$$

Before starting to estimate each of the above terms, let us present some useful inequalities. First, by the bistability assumption (1.1), there exist positive constants $b$, $m$ such that

$$f'(U_0(z)) \le -m \qquad \text{if} \quad U_0(z) \in [\alpha_-, \alpha_- + b] \cup [\alpha_+ - b, \alpha_+]. \tag{1.109}$$

On the other hand, since the region $\{z \in \mathbb{R} \,|\, U_0(z) \in [\alpha_- + b, \alpha_+ - b]\,\}$ is compact and since ${U_0}' > 0$ on $\mathbb{R}$, there exists a constant $a_1 > 0$ such that

$${U_0}'(z) \ge a_1 \qquad \text{if} \quad U_0(z) \in [\alpha_- + b, \alpha_+ - b]. \tag{1.110}$$

We set

$$\beta = \frac{m}{4}, \tag{1.111}$$

and choose $\sigma$ that satisfies

$$0 < \sigma \le \min(\sigma_0, \sigma_1, \sigma_2), \tag{1.112}$$

where

$$\sigma_0 := \frac{a_1}{m + F_1}, \quad \sigma_1 := \frac{1}{\beta + 1}, \quad \sigma_2 := \frac{4\beta}{F_2(\beta+1)},$$

with the constant $F_1$ and $F_2$ defined by

$$F_1 := \max_{\alpha_- \le u \le \alpha_+} |f'(u)|, \qquad F_2 := \max_{\alpha_- - 2 \le u \le \alpha_+ + 2} |f''(u)|.$$

Combining (1.109) and (1.110), and considering that $\sigma \leq \sigma_0$, we obtain

$$U_0{}'(z) - \sigma f'(U_0(z)) \geq \sigma m \qquad \text{for} \quad -\infty < z < \infty. \tag{1.113}$$

Now let $K > 1$ be arbitrary. In what follows we will show that $\mathcal{L}u_\varepsilon^+ \geq 0$ provided that the constants $\varepsilon_0$ and $L$ are appropriately chosen. We recall that $\alpha_- < U_0 < \alpha_+$. We go on under the following assumptions

$$\varepsilon_0 M \leq 1, \qquad \varepsilon_0^2 L e^{LT} \leq 1\,. \tag{1.114}$$

It follows from (1.101) that, for all $\varepsilon \in (0, \varepsilon_0)$, we have $\varepsilon|U_1^\varepsilon(x,t,z)| \leq 1$. Moreover, $\sigma \leq \sigma_1$ implies that $0 \leq q(t) \leq 1$, so that, in view of (1.106),

$$\alpha_- - 2 \leq u_\varepsilon^\pm(x,t) \leq \alpha_+ + 2\,. \tag{1.115}$$

**The term $E_1$**

A straightforward computation gives

$$E_1 = \frac{\beta}{\varepsilon^2}\, e^{-\beta t/\varepsilon^2}(I - \sigma\beta) + Le^{Lt}(I + \varepsilon^2\sigma L),$$

where

$$I = U_0{}' - \sigma f'(U_0) - \frac{\sigma^2}{2} f''(\theta)(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}).$$

In virtue of (1.113) and (1.115), we have

$$I \geq \sigma m - \frac{\sigma^2}{2} F_2(\beta + \varepsilon^2 L e^{LT}).$$

Combining this, (1.114) and the inequality $\sigma \leq \sigma_2$, we obtain

$$I \geq 2\sigma\beta.$$

Consequently, we have

$$E_1 \geq \frac{\sigma\beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + 2\sigma\beta L e^{Lt}.$$

**The term $E_2$**

First, in the region where $|d| < d_0$, we have $|\nabla d| = 1$, hence $E_2 = 0$. Next we consider the region where $|d| \geq d_0$. We deduce from Lemma 1.2.1 and from (1.103) that:

$$\begin{aligned} |E_2| \;&\leq C\Big(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\Big)(1 + \|\nabla d\|_\infty^2) e^{-\lambda|d+\varepsilon p|/\varepsilon} \\ &\leq \frac{2C}{\varepsilon^2}(1 + \|\nabla d\|_\infty^2) e^{-\lambda(d_0/\varepsilon - |p|)}. \end{aligned}$$

By the definition of $p$ in (1.107), we have that $0 < K - 1 \leq p \leq e^{LT} + K$. Consequently, if we assume

$$e^{LT} + K \leq \frac{d_0}{2\varepsilon_0}, \tag{1.116}$$

then $\dfrac{d_0}{\varepsilon} - |p| \geq \dfrac{d_0}{2\varepsilon}$, so that

$$\begin{aligned} |E_2| \;&\leq \frac{2C}{\varepsilon^2}(1 + \|\nabla d\|_\infty^2) e^{-\lambda d_0/(2\varepsilon)} \\ &\leq C_2 := \frac{32C}{(e\lambda d_0)^2}(1 + \|\nabla d\|_\infty^2). \end{aligned}$$

**The term $E_3$**

We recall that

$$(d_t - \Delta d + \gamma)(x,t) = 0 \qquad \text{on} \quad \Gamma_t = \{x \in \Omega,\ d(x,t) = 0\}.$$

By equality (1.97) and assumption (1.5), we see that $\gamma$ is in $C^{1+\vartheta,\frac{1+\vartheta}{2}}$ so that the interface $\Gamma$ is of class $C^{3+\vartheta,\frac{3+\vartheta}{2}}$. Therefore both $\Delta d$ and $d_t$ are Lipschitz continuous near $\Gamma_t$. It follows, from the mean value theorem applied separately on both sides of $\Gamma_t$, that there exists a constant $N_0 > 0$ such that:

$$|(d_t - \Delta d + \gamma)(x,t)| \le N_0 |d(x,t)| \quad \text{ for all } \ (x,t) \in Q_T.$$

Applying Lemma 1.2.1 and estimate (1.103) we deduce that

$$\begin{aligned} |E_3| &\le 2N_0 C \frac{|d(x,t)|}{\varepsilon} e^{-\lambda |d(x,t)/\varepsilon + p(t)|} \\ &\le 2N_0 C \max_{y\in\mathbb{R}} |y| e^{-\lambda|y+p(t)|} \\ &\le 2N_0 C \max\big(|p(t)|, \frac{1}{\lambda}\big) \\ &\le 2N_0 C \big(|p(t)| + \frac{1}{\lambda}\big). \end{aligned}$$

Thus, recalling that $|p(t)| \le e^{Lt} + K$, we obtain

$$|E_3| \le C_3(e^{Lt} + K) + C_3{}',$$

where $C_3 := 2N_0C$ and $C_3{}' := 2N_0 C/\lambda$.

**The term $E_4$**

In view of (1.4) and (1.103), both $g_u$ and $|U_{1z}^\varepsilon|$ are bounded by some constant $C$. Hence, substituting the expression for $p_t$ and $q$, we obtain

$$|E_4| \le C_4\big(\frac{1}{\varepsilon}\beta e^{-\beta t/\varepsilon^2} + \varepsilon L e^{Lt}\big),$$

where $C_4 := C + \sigma(C + MF_2)$.

**The term $E_5$**

In view of (1.97), the term $|\gamma|$ is bounded by $c_0(\alpha_+ - \alpha_-)C$ on $\overline{\Omega} \times [0,T]$. Using successively (1.103), (1.101), (1.4) and (1.104), we obtain

$$|E_5| \le c_0(\alpha_+ - \alpha_-)CM + \frac{1}{2}M^2 F_2 + MC + 2M\|\nabla d\|_\infty^2 =: C_5.$$

**The term $E_6$**

We use (1.102) to deduce that

$$|E_6| \le 2C =: C_6.$$

**Finally the term $E_7$**

We recall that $|g^\varepsilon - g| \le C\varepsilon$ so that $|\gamma^\varepsilon - \gamma| \le c_0(\alpha_+ - \alpha_-)C\varepsilon$. It then follows, in view of (1.27), that

$$|E_7| \le 2C + c_0(\alpha_+ - \alpha_-)C^2 =: C_7.$$

**Completion of the proof**

Collecting the above estimates of $E_1$—$E_7$ gives

$$\mathcal{L}u_\varepsilon^+ \ge (\frac{\sigma\beta^2}{\varepsilon^2} - \frac{C_4\beta}{\varepsilon})e^{-\beta t/\varepsilon^2} + (2\sigma\beta L - C_3 - \varepsilon C_4 L)e^{Lt} - C_8, \tag{1.117}$$

where $C_8 := C_2 + KC_3 + C_3' + C_5 + C_6 + C_7$. Now we set

$$L := \frac{1}{T}\ln\frac{d_0}{4\varepsilon_0},$$

which, for $\varepsilon_0$ small enough, validates assumptions (1.114) and (1.116). If $\varepsilon_0$ is chosen sufficiently small (i.e. $L$ large enough), we have, for all $0 < \varepsilon < \varepsilon_0$, that the first term of the right-hand side of (1.117) is positive, and that

$$\begin{aligned}\mathcal{L}u_\varepsilon^+ &\ge \left[\sigma\beta L - C_3\right]e^{Lt} - C_8\\ &\ge \tfrac{1}{2}\sigma\beta L - C_8\\ &\ge 0.\end{aligned}$$

The proof of Lemma 1.5.1 is now complete, with the choice of the constants $\beta, \sigma$ as in (1.111), (1.112). □

## 1.6 Proof of the main results

In this section, we prove our main results by fitting the two pairs of sub- and super-solutions, constructed for the study of the generation and the motion of interface, into each other.

### 1.6.1 Proof of Theorem 1.1.4

Let $\eta \in (0, \eta_0)$ be arbitrary. Choose $\beta$ and $\sigma$ that satisfy (1.111), (1.112) and

$$\sigma\beta \le \frac{\eta}{3}. \tag{1.118}$$

By the generation of interface Theorem 1.3.1, there exist positive constants $\varepsilon_0$ and $M_0$ such that (1.49), (1.50) and (1.51) hold with the constant $\eta$ replaced by $\sigma\beta/2$. Since $\nabla u_0 \cdot n \ne 0$ everywhere on the initial interface $\Gamma_0 = \{x \in \Omega,\ u_0(x) = a\}$ and since $\Gamma_0$ is a compact hypersurface, we can find a positive constant $M_1$ such that

$$\begin{aligned}&\text{if} \quad d_0(x) \ge M_1\varepsilon \quad \text{then} \quad u_0(x) \ge a + M_0\varepsilon,\\ &\text{if} \quad d_0(x) \le -M_1\varepsilon \quad \text{then} \quad u_0(x) \le a - M_0\varepsilon.\end{aligned} \tag{1.119}$$

Here $d_0(x) := d(x,0)$ denotes the cut-off signed distance function associated with the hypersurface $\Gamma_0$. Now we define functions $H^+(x), H^-(x)$ by

$$H^+(x) = \begin{cases} \alpha_+ + \sigma\beta/2 & \text{if } d_0(x) > -M_1\varepsilon \\ \alpha_- + \sigma\beta/2 & \text{if } d_0(x) \le -M_1\varepsilon, \end{cases}$$

$$H^-(x) = \begin{cases} \alpha_+ - \sigma\beta/2 & \text{if } d_0(x) \ge M_1\varepsilon \\ \alpha_- - \sigma\beta/2 & \text{if } d_0(x) < M_1\varepsilon. \end{cases}$$

Then from the above observation we see that

$$H^-(x) \le u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \le H^+(x) \qquad \text{for } x \in \Omega. \tag{1.120}$$

Next we fix a sufficiently large constant $K > 1$ such that

$$U_0(-M_1 + K) \ge \alpha_+ - \frac{\sigma\beta}{3} \quad \text{and} \quad U_0(M_1 - K) \le \alpha_- + \frac{\sigma\beta}{3}. \tag{1.121}$$

For this $K$, we choose $\varepsilon_0$ and $L$ as in Lemma 1.5.1. We claim that

$$u_\varepsilon^-(x,0) \le H^-(x), \quad H^+(x) \le u_\varepsilon^+(x,0) \qquad \text{for } x \in \Omega. \tag{1.122}$$

We only prove the former inequality, as the proof of the latter is virtually the same. Then it amounts to showing that

$$u_\varepsilon^-(x,0) = U_0\big(\frac{d_0(x)}{\varepsilon} - K\big) + \varepsilon U_1^\varepsilon\big(x, 0, \frac{d_0(x)}{\varepsilon} - K\big) - \sigma(\beta + \varepsilon^2 L) \le H^-(x). \tag{1.123}$$

By (1.101) we have $|U_1^\varepsilon| \le M$. Therefore, by choosing $\varepsilon_0$ small enough so that $\varepsilon_0 M \le \sigma\beta/6$, we see that

$$\begin{aligned} u_\varepsilon^-(x,0) &\le U_0\big(\frac{d_0(x)}{\varepsilon} - K\big) + \varepsilon M - \sigma(\beta + \varepsilon^2 L) \\ &\le U_0\big(\frac{d_0(x)}{\varepsilon} - K\big) - \frac{5}{6}\sigma\beta. \end{aligned}$$

In the range where $d_0(x) < M_1\varepsilon$, the second inequality in (1.121) and the fact that $U_0$ is an increasing function imply

$$U_0\big(\frac{d_0(x)}{\varepsilon} - K\big) - \frac{5}{6}\sigma\beta \le \alpha_- - \frac{\sigma\beta}{2} = H^-(x).$$

On the other hand, in the range where $d_0(x) \ge M_1\varepsilon$, we have

$$U_0\big(\frac{d_0(x)}{\varepsilon} - K\big) - \frac{5}{6}\sigma\beta \le \alpha_+ - \frac{5}{6}\sigma\beta \le H^-(x).$$

This proves (1.123), hence (1.122) is established.

Combining (1.120) and (1.122), we obtain

$$u_\varepsilon^-(x,0) \le u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \le u_\varepsilon^+(x,0).$$

Since $u_\varepsilon^-$ and $u_\varepsilon^+$ are sub- and super-solutions for Problem $(P^\varepsilon)$ thanks to Lemma 1.5.1, the comparison principle yields

$$u_\varepsilon^-(x,t) \le u^\varepsilon(x, t + t^\varepsilon) \le u_\varepsilon^+(x,t) \quad \text{for} \quad 0 \le t \le T - t^\varepsilon, \tag{1.124}$$

where $t^\varepsilon = \mu^{-1}\varepsilon^2|\ln\varepsilon|$. Note that, in view of (1.108), this is enough to prove Corollary 1.1.5. Now let $C$ be a positive constant such that

$$U_0(C - e^{LT} - K) \geq \alpha_+ - \frac{\eta}{2} \quad \text{and} \quad U_0(-C + e^{LT} + K) \leq \alpha_- + \frac{\eta}{2}. \tag{1.125}$$

One then easily checks, using successively (1.124), (1.106), (1.125) and (1.118), that, for $\varepsilon_0$ small enough, for $0 \leq t \leq T - t^\varepsilon$, we have

$$\begin{array}{llll} \text{if} & d(x,t) \geq \ \ C\varepsilon & \text{then} & u^\varepsilon(x, t + t^\varepsilon) \geq \alpha_+ - \eta, \\ \text{if} & d(x,t) \leq -C\varepsilon & \text{then} & u^\varepsilon(x, t + t^\varepsilon) \leq \alpha_- + \eta, \end{array} \tag{1.126}$$

and

$$u^\varepsilon(x, t + t^\varepsilon) \in [\alpha_- - \eta, \alpha_+ + \eta],$$

which completes the proof of Theorem 1.1.4. □

### 1.6.2 Proof of Theorem 1.1.6

In the case where $\mu^{-1}\varepsilon^2|\ln\varepsilon| \leq t \leq T$, the assertion of the theorem is a direct consequence of Theorem 1.1.4. All we have to consider is the case where $0 \leq t \leq \mu^{-1}\varepsilon^2|\ln\varepsilon|$. We shall use the sub- and super-solutions constructed for the study of the generation of interface in Section 1.4. To that purpose, we first prove the following lemma concerning $Y(\tau, \xi; \delta)$, the solution of the ordinary differential equation (1.79), in the initial time interval.

**Lemma 1.6.1.** *There exists a constant $C_8 > 0$ such that*

$$\begin{array}{llllll} \textit{if} & \xi \geq a + C_8\varepsilon & \textit{then} & Y(\tau, \xi; \pm\varepsilon\mathcal{G}) > a & \textit{for} & 0 \leq \tau \leq \mu^{-1}|\ln\varepsilon|, \\ \textit{if} & \xi \leq a - C_8\varepsilon & \textit{then} & Y(\tau, \xi; \pm\varepsilon\mathcal{G}) < a & \textit{for} & 0 \leq \tau \leq \mu^{-1}|\ln\varepsilon|. \end{array} \tag{1.127}$$

**Proof.** We only prove the first inequality. Assume $\xi \geq a + C_8\varepsilon$. By (1.76), for $C_8 \geq C\mathcal{G}$, we have that $\xi \geq a + C_8\varepsilon \geq a(\pm\varepsilon\mathcal{G})$. It then follows from (1.82) that

$$\begin{aligned} Y(\tau, \xi; \pm\varepsilon\mathcal{G}) &\geq a(\pm\varepsilon\mathcal{G}) + C_1 e^{\mu(\pm\varepsilon\mathcal{G})\tau}(a + C_8\varepsilon - a(\pm\varepsilon\mathcal{G})) \\ &\geq a - C\mathcal{G}\varepsilon + C_1(-C\mathcal{G}\varepsilon + C_8\varepsilon) \\ &\geq a + \varepsilon(C_1C_8 - C\mathcal{G}(C_1 + 1)) \\ &> a, \end{aligned}$$

provided that $C_8$ is sufficiently large. □

Now we turn to the proof of Theorem 1.1.6. We first claim that there exists a positive constant $M_2$ such that for all $t \in [0, \mu^{-1}\varepsilon^2|\ln\varepsilon|]$,

$$\Gamma_t^\varepsilon \subset \mathcal{N}_{M_2\varepsilon}(\Gamma_0). \tag{1.128}$$

To see this, we choose ${M_0}'$ large enough, so that ${M_0}' \geq C_8 + 2C_6$, where $C_6$ is as in Lemma 1.4.6. As is done for (1.119), there is a positive constant $M_2$ such that

$$\begin{array}{llll} \text{if} & d_0(x) \geq \ \ M_2\varepsilon & \text{then} & u_0(x) \geq a + {M_0}'\varepsilon, \\ \text{if} & d_0(x) \leq -M_2\varepsilon & \text{then} & u_0(x) \leq a - {M_0}'\varepsilon. \end{array} \tag{1.129}$$

In view of this last condition, we see that, if $\varepsilon_0$ is small enough and if $d_0(x) \geq M_2\varepsilon$, then for $0 \leq t \leq \mu^{-1}\varepsilon^2|\ln\varepsilon|$,

$$\begin{aligned} u_0(x) - \varepsilon^2 r(-\varepsilon\mathcal{G}, \frac{t}{\varepsilon^2}) &\geq a + M_0'\varepsilon - \varepsilon^2 C_6\big[e^{\mu(-\varepsilon\mathcal{G})|\ln\varepsilon|/\mu} - 1\big] \\ &\geq a + \varepsilon\big[M_0' - C_6\varepsilon^{(\mu-\mu(\pm\varepsilon\mathcal{G}))/\mu} + \varepsilon C_6\big] \\ &\geq a + \varepsilon(M_0' - 2C_6) \qquad \big(\leftarrow \text{ thanks to (1.92)}\big) \\ &\geq a + C_8\varepsilon. \end{aligned}$$

This inequality and Lemma 1.6.1 imply $w_\varepsilon^-(x,t) > a$, where $w_\varepsilon^-$ is the sub-solution defined in (1.86). Consequently, by (1.87),

$$u^\varepsilon(x,t) > a \qquad \text{if } \ d_0(x) \geq M_2\varepsilon.$$

In the case where $d_0(x) \leq -M_2\varepsilon$, similar arguments lead to $u^\varepsilon(x,t) < a$. This completes the proof of (1.128). Note that we have proved that, for all $0 \leq t \leq \mu^{-1}\varepsilon^2|\ln\varepsilon|$,

$$\begin{aligned} u^\varepsilon(x,t) > a \quad &\text{if} \quad x \in \Omega_0^+ \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0), \\ u^\varepsilon(x,t) < a \quad &\text{if} \quad x \in \Omega_0^- \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0). \end{aligned} \tag{1.130}$$

Next, since $\Gamma_t$ depends on $t$ smoothly, there is a constant $\tilde{C} > 0$ such that, for all $t \in [0, \mu^{-1}\varepsilon^2|\ln\varepsilon|]$,

$$\Gamma_0 \subset \mathcal{N}_{\tilde{C}\varepsilon^2|\ln\varepsilon|}(\Gamma_t), \tag{1.131}$$

and

$$\begin{aligned} \Omega_t^+ \setminus \mathcal{N}_{\tilde{C}\varepsilon}(\Gamma_t) \subset \Omega_0^+ \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0), \\ \Omega_t^- \setminus \mathcal{N}_{\tilde{C}\varepsilon}(\Gamma_t) \subset \Omega_0^- \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0). \end{aligned} \tag{1.132}$$

As a consequence of (1.128) and (1.131) we get

$$\Gamma_t^\varepsilon \subset \mathcal{N}_{M_2\varepsilon+\tilde{C}\varepsilon^2|\ln\varepsilon|}(\Gamma_t) \subset \mathcal{N}_{C\varepsilon}(\Gamma_t),$$

which completes the proof of Theorem 1.1.6. □

**Proof of Corollary 1.1.7.** In view of Theorem 1.1.6 and the definition of the Hausdorff distance, to prove this corollary we only need to show the reverse inclusion, that is

$$\Gamma_t \subset \mathcal{N}_{C'\varepsilon}(\Gamma_t^\varepsilon) \quad \text{ for } \quad 0 \leq t \leq T, \tag{1.133}$$

for some constant $C' > 0$. To that purpose let $C'$ be a constant satisfying $C' > \max(\tilde{C}, C)$, where $C$ is as in Theorem 1.1.4 and $\tilde{C}$ as in (1.132). Choose $t \in [0,T]$, $x_0 \in \Gamma_t$ arbitrarily and, $n$ being the Euclidian normal vector exterior to $\Gamma_t$ at point $x_0$, define a pair of points:

$$x^+ := x_0 + C'\varepsilon n \quad \text{and} \quad x^- := x_0 - C'\varepsilon n.$$

Since $C' > C$ and since the curvature of $\Gamma_t$ is uniformly bounded as $t$ varies over $[0,T]$, we see that

$$x^+ \in \Omega_t^+ \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t) \quad \text{and} \quad x^- \in \Omega_t^- \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t),$$

if $\varepsilon$ is sufficiently small. Therefore, if $t \in [\mu^{-1}\varepsilon^2|\ln\varepsilon|, T]$, then, by Theorem 1.1.4, we have

$$u^\varepsilon(x^-,t) < a < u^\varepsilon(x^+,t). \tag{1.134}$$

On the other hand, if $t \in [0, \mu^{-1}\varepsilon^2|\ln\varepsilon|]$, then from (1.130), (1.132) and the fact that $C' > \tilde{C}$, we again obtain (1.134). Thus (1.134) holds for all $t \in [0,T]$. Now, by the mean value theorem, we see that for each $t \in [0,T]$ there exists a point $x_1$ such that

$$x_1 \in [x^-, x^+] \quad \text{and} \quad u^\varepsilon(x_1, t) = a.$$

This implies $x_1 \in \Gamma_t^\varepsilon$. Furthermore we have $|x_0 - x_1| \leq C'\varepsilon$, since $x_1$ lies on the line segment $[x^-, x^+]$. This proves (1.133). □

## 1.7 Application to reaction-diffusion systems

In this section we discuss the singular limit of the reaction-diffusion system $(RD^\varepsilon)$ and prove Theorems 1.1.12, 1.1.14 and their corollaries. Our strategy is to regard the first equation of $(RD^\varepsilon)$ as a perturbed Allen-Cahn equation and apply what we have already proved for this equation. In order to make this strategy work, we will modify our former arguments slightly.

### 1.7.1 Preliminaries: global existence

Before studying the singular limit of $(RD^\varepsilon)$, we first show that the solution of this system exists globally for $t \geq 0$, provided that $\varepsilon$ is sufficiently small. Recall that the system $(RD^\varepsilon)$ is written in the form

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}\big(f(u) + \varepsilon\, f_1(u,v) + O(\varepsilon^2)\big), \\ v_t = D\Delta v + h(u,v), \end{cases}$$

where $h(u,v)$ satisfies the hypothesis (**H**). The standard parabolic theory guarantees the existence of local solutions for $(RD^\varepsilon)$. In order to prove that the solution exists globally for $t \geq 0$, it suffices to show that the solution remains uniformly bounded. This will be done by using the method of invariant rectangles.

Given arbitrary $u_0$, $v_0 \in C(\overline{\Omega})$, we choose a constant $L > 0$ such that

$$f(-L) > 0 > f(L), \qquad -L \leq u_0(x) \leq L \quad \text{for} \;\; x \in \overline{\Omega}. \tag{1.135}$$

Such a constant $L$ exists since $f(u) > 0$ for $u < \alpha_-$, and $f(u) < 0$ for $u > \alpha_+$. By the hypothesis (**H**), we can choose a constant $M_1$ satisfying

$$M_1 \geq \|v_0\|_{L^\infty(\overline{\Omega})},$$

along with the condition (1.19), namely

$$h(u, -M_1) \geq 0 \geq h(u, M_1) \qquad \text{for} \;\; |u| \leq L. \tag{1.136}$$

Now we consider the rectangle

$$\mathcal{R} := \{\, (u,v) \in \mathbb{R}^2 \,\big|\, |u| \leq L,\, |v| \leq M_1 \,\}.$$

It follows from (1.135) that, for all sufficiently small $\varepsilon > 0$,

$$f^\varepsilon(-L, v) > 0 > f^\varepsilon(L, v) \qquad \text{for} \;\; |v| \leq M_1. \tag{1.137}$$

The inequalities (1.136) and (1.137) imply that the rectangle $\mathcal{R}$ is a positively invariant region for the system of ordinary differential equations:

$$\begin{cases} u_t = \dfrac{1}{\varepsilon^2} f^\varepsilon(u,v), \\ v_t = h(u,v), \end{cases}$$

since the vector field $(\varepsilon^{-2} f^\varepsilon(u,v), h(u,v))$ points inwards everywhere on the boundary of $\mathcal{R}$. The maximum principle then implies that $\mathcal{R}$ is also positively invariant for the system $(RD^\varepsilon)$. Consequently, since $(u_0(x), v_0(x)) \in \mathcal{R}$ for $x \in \overline{\Omega}$, we have

$$(u(x,t), v(x,t)) \in \mathcal{R} \qquad \text{for} \ \ x \in \overline{\Omega}, \ t \geq 0,$$

as long as the solution is defined. This uniform bound then implies that the solution exists globally for $t \geq 0$.

In the case of equations for which only nonnegative solutions are to be considered (see Remark 1.1.10), we can argue just similarly, by replacing $\mathcal{R}$ by the rectangle $\mathcal{R}_+ := \{(u,v) \,|\, 0 \leq u \leq L, \, 0 \leq v \leq M_1\}$. Summarizing, we have proved the following proposition:

**Proposition 1.7.1.** *Let $(u_0, v_0) \in C(\overline{\Omega}) \times C(\overline{\Omega})$. In the case where the conditions of Remark 1.1.10 apply, assume further that $u_0, v_0 \geq 0$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the solution $(u^\varepsilon, v^\varepsilon)$ of $(RD^\varepsilon)$ exists globally for $t \geq 0$ and is uniformly bounded.*

**Remark 1.7.2.** *For the details of the method of invariant rectangles, we refer the reader to the book [68], Chapter 14, Corollary 14.8. See also [28] and [27]. It should be noted that [50] makes a much earlier study of invariant rectangles for a finite-difference scheme for reaction-diffusion systems.* □

### 1.7.2 Re-examination of the Allen-Cahn equation

Now we turn to the singular limit of $(RD^\varepsilon)$. As we have mentioned earlier, our strategy is to regard the first equation of $(RD^\varepsilon)$ as a perturbed Allen-Cahn equation of the form $(P^\varepsilon)$. However, our results for Problem $(P^\varepsilon)$, Theorems 1.1.4 and 1.1.6, do not apply to the system $(RD^\varepsilon)$ directly, because of the assumption (1.6), which requires $|g^\varepsilon - g|$ to be of order $\varepsilon$. Naturally, such an assumption cannot be made *a priori* for the system $(RD^\varepsilon)$, since the perturbation term $g^\varepsilon := -f_1(u, v^\varepsilon(x,t)) + O(\varepsilon)$ depends on the unknown function $v^\varepsilon$.

In view of this, we will first re-examine our previous argument for the single equation $(P^\varepsilon)$ and see what we can say without the assumption (1.6).

We begin with some notation. Given any function $\bar{g}(x,t,u)$ satisfying the conditions (1.3) and (1.5), and any smooth hypersurface without boundary $\bar{\Gamma}_0$ such that $\bar{\Gamma}_0 \subset\subset \Omega$, we can define the classical solution of the interface equation $(P^0)$ on some time interval $0 \leq t < T(\bar{g}; \bar{\Gamma}_0)$. We denote this solution by $\Gamma_t[\,\bar{g}; \bar{\Gamma}_0]$ in order to clarify its dependence on $\bar{g}$ and $\bar{\Gamma}_0$. More specifically, $\Gamma_t[\bar{g}; \bar{\Gamma}_0]$ is the solution of the problem

$$(P^0_{\bar{g},\bar{\Gamma}_0}) \qquad \begin{cases} V_n = -(N-1)\kappa + c_0 \displaystyle\int_{\alpha_-}^{\alpha_+} \bar{g}(x,t,r)dr \qquad \text{on } \Gamma_t, \\ \Gamma_t\big|_{t=0} = \bar{\Gamma}_0. \end{cases}$$

Also, we denote by $u^\varepsilon[\bar g;\bar u_0](x,t)$ the solution of the problem

$$(P^\varepsilon_{\bar g,\bar u_0})\quad \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}(f(u)-\varepsilon \bar g(x,t,u)) & \text{in } \Omega\times(0,+\infty),\\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,+\infty),\\ u(x,0)=\bar u_0(x) & \text{in } \Omega,\end{cases}$$

and define

$$\Gamma^\varepsilon_t[\bar g;\bar u_0] := \{x\in\Omega,\ u^\varepsilon[\bar g;\bar u_0](x,t)=a\}.$$

Once the interface $\Gamma_t[\bar g;\bar\Gamma_0]$ is given, we denote by $\Omega^-_t[\bar g;\bar\Gamma_0]$, $\Omega^+_t[\bar g;\bar\Gamma_0]$ the region enclosed by $\Gamma_t[\bar g;\bar\Gamma_0]$ and the one enclosed between $\partial\Omega$ and $\Gamma_t[\bar g;\bar\Gamma_0]$, respectively. As in (1.14), we define the step function $\tilde u[\bar g;\bar\Gamma_0](x,t)$ by

$$\tilde u[\bar g;\bar\Gamma_0](x,t) = \begin{cases}\alpha_+ & \text{in } \Omega^+_t[\bar g;\bar\Gamma_0]\\ \alpha_- & \text{in } \Omega^-_t[\bar g;\bar\Gamma_0]\end{cases} \quad \text{for } t\in[0,T(\bar g;\bar\Gamma_0)). \tag{1.138}$$

With these notations, the solution $u^\varepsilon$, the interfaces $\Gamma^\varepsilon_t$, $\Gamma_t$ and the step function $\tilde u$ which we have defined in Section 1.1, can be expressed as follows:

$$u^\varepsilon = u^\varepsilon[g^\varepsilon;u_0],\quad \Gamma^\varepsilon_t=\Gamma^\varepsilon_t[g^\varepsilon;u_0],\quad \Gamma_t=\Gamma_t[g;\Gamma_0],\quad \tilde u=\tilde u[g;\Gamma_0].$$

Henceforth we will fix the initial datum $u_0$ (hence $\Gamma_0$) throughout this subsection.

Now let us consider what happens if we do not assume (1.6), i.e. $|g-g^\varepsilon| = O(\varepsilon)$. Sections 1.3 and 1.4, which deal with the generation of interface, remain unchanged since the assumption (1.6) is not used. The only places where this assumption has been used are the following.

- In the formal derivation of the interface equation in Section 1.2, the assumption (1.6) is used while collecting the $O(\varepsilon^{-1})$ terms, leading to the conclusion that the second term $U_1$ of the inner asymptotic expansion is given by a solution of equation (1.29);
- In subsection 1.5.3, the assumption (1.6) is used to show the boundedness of the term $E_7$.

Note that the term $E_7$ appears, so to say, as a result of discrepancy between the term $U^\varepsilon_1$ and the distance function $d$ that are used to define $u^\pm_\varepsilon$ in (1.106). More precisely, the distance function $d(x,t)$ is associated with the interface $\Gamma_t[g;\Gamma_0]$, whose law of motion is $(P^0_{g,\Gamma_0})$, while $U^\varepsilon_1$ is associated with $g^\varepsilon$ via (1.98). Therefore, when we calculate $\mathcal{L}u^\pm_\varepsilon$, both $g$ and $g^\varepsilon$ appear without cancelling each other.

On the other hand, if we replace $d$ by the signed distance function $d^\varepsilon$ associated with the interface $\Gamma_t[g^\varepsilon;\Gamma_0]$, and define $\hat u^\pm_\varepsilon$ by

$$\hat u^\pm_\varepsilon(x,t) = U_0\Big(\frac{d^\varepsilon(x,t)\pm\varepsilon p(t)}{\varepsilon}\Big) + \varepsilon U^\varepsilon_1\Big(x,t,\frac{d^\varepsilon(x,t)\pm\varepsilon p(t)}{\varepsilon}\Big) \pm q(t), \tag{1.139}$$

then the term $E_7$ does not appear in the calculation of $\mathcal{L}\hat u^\pm_\varepsilon$. Moreover the remaining terms $E_1$ to $E_6$ are virtually the same as those in subsection 1.5.3. Consequently, the new functions $\hat u^\pm_\varepsilon$ are sub- and super-solutions even without the assumption (1.6). Thus, arguing as in subsection 1.6.1, we obtain the following analogue of (1.124):

$$\hat u^-_\varepsilon(x,t) \le u^\varepsilon(x,t+t^\varepsilon) \le \hat u^+_\varepsilon(x,t) \quad \text{for} \quad 0\le t\le T'-t^\varepsilon, \tag{1.140}$$

where $t^\varepsilon = \mu^{-1}\varepsilon^2|\ln\varepsilon|$ and $T'$ is a constant such that $0<T'<T(g^\varepsilon;\Gamma_0)$.

Summarizing, the following proposition holds.

**Proposition 1.7.3.** *The conclusions of Theorems 1.1.4 and 1.1.6 hold without assuming* (1.6)*, provided that* $\Gamma_t[g;\Gamma_0]$ *is replaced by* $\Gamma_t[g^\varepsilon;\Gamma_0]$*. In particular,*

$$d_{\mathcal{H}}(\Gamma_t^\varepsilon[g^\varepsilon;u_0],\Gamma_t[g^\varepsilon;\Gamma_0]) \le C\varepsilon \quad for \quad 0\le t\le T', \tag{1.141}$$

*where* $d_{\mathcal{H}}$ *denotes the Hausdorff distance.*

### 1.7.3 Interface motion under various perturbations

Next we show that $\Gamma_t[\,\bar g;\bar\Gamma_0]$ depends on $\bar g$ and $\bar\Gamma_0$ continuously. To this end, we first fix constants $C_*>0$, $T'>0$, $\vartheta\in(0,1)$, and denote by $\mathcal{Y}$ the set of functions $\bar g(x,t,u)$ on $\overline{\Omega}\times[0,\,T']\times\mathbb{R}$ satisfying

$$\sup_{u\in[\alpha_-,\alpha_+]} \|\bar g(\cdot,\cdot,u)\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}(\overline{\Omega}\times[0,T'])} \le C_*. \tag{1.142}$$

We also fix $\mathcal{M}$ a $N-1$ dimensional manifold without boundaries. We denote by $\mathcal{Z}$ the set of $C^{3+\vartheta}$ hypersurfaces without boundary $\bar\Gamma_0$ satisfying $\bar\Gamma_0\subset\subset\Omega$ and such that

$$\|\bar\Lambda_0\|_{C^{3+\vartheta}(\mathcal{M})} \le C_*, \tag{1.143}$$

where the function $\bar\Lambda_0:\mathcal{M}\mapsto[-L,L]$ is a parametrization of $\bar\Gamma_0$. For more details we refer to [23].

**Proposition 1.7.4.** *Let* $\bar g\in\mathcal{Y}$ *and* $\bar\Gamma_0\in\mathcal{Z}$*. Let* $T\in(0,T(\bar g;\bar\Gamma_0))$*. Then there exist positive constants* $\delta$, $K$, $M$ *such that, for any* $\tilde g\in\mathcal{Y}$ *and any* $\tilde\Gamma_0\in\mathcal{Z}$ *satisfying*

$$\|\tilde g-\bar g\|_{L^\infty(\Omega\times[0,T]\times[\alpha_-,\alpha_+])}\le\delta \qquad and \qquad d_{\mathcal{H}}(\tilde\Gamma_0,\bar\Gamma_0)\le\delta,$$

*there holds that* $T(\tilde g;\tilde\Gamma_0)>T$*, where we recall that* $T(\tilde g;\tilde\Gamma_0)$ *is the maximum time of existence of a classical solution of Problem* $(P^0_{\tilde g,\tilde\Gamma_0})$*. Furthermore, for each* $t\in[0,T]$*,*

$$d_{\mathcal{H}}(\,\Gamma_t[\tilde g;\tilde\Gamma_0],\Gamma_t[\bar g;\bar\Gamma_0]\,) \le K(e^{Mt}-1)\,\|\tilde g-\bar g\|_{L^\infty}+e^{Mt}d_{\mathcal{H}}(\tilde\Gamma_0,\bar\Gamma_0)\,, \tag{1.144}$$

*where the* $L^\infty$ *norm on the right-hand side is taken in* $\Omega\times[0,t]\times[\alpha_-,\alpha_+]$*.*

**Proof.** By using the local coordinates, one can express $\Gamma_t[\bar g,\bar\Gamma_0]$, respectively $\Gamma_t[\tilde g,\tilde\Gamma_0]$, as a graph over $\mathcal{M}$ and transfer the motion equation $(P^0_{\bar g,\bar\Gamma_0})$, respectively $(P^0_{\tilde g,\tilde\Gamma_0})$, into a quasi-linear parabolic equation on the manifold $\mathcal{M}\times[0,T(\bar g,\bar\Gamma_0)]$, respectively $\mathcal{M}\times[0,T(\tilde g,\tilde\Gamma_0)]$. Since $\bar g$ and $\tilde g$ satisfy (1.142), and since the embedding

$$C^{1+\vartheta,\frac{1+\vartheta}{2}} \hookrightarrow C^{1+\vartheta',\frac{1+\vartheta'}{2}}$$

is compact if $0<\vartheta'<\vartheta$, the assumption $\|\tilde g-\bar g\|_{L^\infty}\le\delta$ implies

$$\|\tilde g(\cdot,\cdot,u)-\bar g(\cdot,\cdot,u)\|_{C^{1+\vartheta',\frac{1+\vartheta'}{2}}}\le C(\delta),$$

where $C(\delta)$ is a constant satisfying $C(\delta)\to0$, as $\delta\to0$. Consequently,

$$\|\int_{\alpha_-}^{\alpha_+}\tilde g(\cdot,\cdot,r)dr-\int_{\alpha_-}^{\alpha_+}\bar g(\cdot,\cdot,r)dr\,\|_{C^{1+\vartheta',\frac{1+\vartheta'}{2}}}\le(\alpha_+-\alpha_-)C(\delta).$$

By a similar argument, the assumption $d_{\mathcal{H}}(\tilde{\Gamma}_0, \bar{\Gamma}_0) \leq \delta$ implies

$$\|\bar{\Lambda}_0 - \tilde{\Lambda}_0\|_{C^{3+\vartheta'}(\mathcal{M})} \leq C'(\delta),$$

where $C'(\delta)$ is a constant satisfying $C'(\delta) \to 0$, as $\delta \to 0$. The assertion that $T(\tilde{g}, \tilde{\Gamma}_0) > T$ then follows from the standard local existence theory. For more details, we refer to [3] Theorem 7.1 and its corollary, which are concerned with smooth curvature flows with slightly perturbations of the velocities and the initial data.

Next we prove the estimate (1.144). This will be done by using the maximum principle. Let us introduce some notation. For each $\bar{g} \in \mathcal{Y}$ and each $\bar{\Gamma}_0 \in \mathcal{Z}$, we denote by $d(x, t; \bar{g}; \bar{\Gamma}_0)$ the signed distance function associated with the interface $\Gamma_t[\bar{g}; \bar{\Gamma}_0]$. By $\bar{\Gamma}_t \preceq \tilde{\Gamma}_t$ we mean that $\bar{\Gamma}_t$ lies inside of $\tilde{\Gamma}_t$. Clearly we have

$$\Gamma_t[\bar{g}; \bar{\Gamma}_0] \preceq \Gamma_t[\tilde{g}; \tilde{\Gamma}_0] \iff d(x, t; \bar{g}; \bar{\Gamma}_0) \geq d(x, t; \tilde{g}; \tilde{\Gamma}_0) \quad \text{for } x \in \overline{\Omega}. \tag{1.145}$$

Now we choose $t_0 \in [0, T]$ arbitrarily and put

$$\eta_0 := \|\tilde{g} - \bar{g}\|_{L^\infty(\Omega \times [0,t_0] \times [\alpha_-, \alpha_+])} \quad \text{and} \quad \eta_1 := d_{\mathcal{H}}(\tilde{\Gamma}_0, \bar{\Gamma}_0).$$

Then

$$\bar{g}(x, t, u) - \eta_0 \leq \tilde{g}(x, t, u) \leq \bar{g}(x, t, u) + \eta_0,$$

for $x \in \Omega$, $0 \leq t \leq t_0$, $\alpha_- \leq u \leq \alpha_+$, and

$$\bar{\Gamma}_0 - \eta_1 \preceq \tilde{\Gamma}_0 \preceq \bar{\Gamma}_0 + \eta_1,$$

where, by definition

$$\bar{\Gamma}_0 \pm \eta_1 := \{x \pm \eta_1 n; x \in \bar{\Gamma}_0\},$$

$n$ being the outward unit normal to $\bar{\Gamma}_0$. The comparison principle then yields

$$\Gamma_t[\bar{g} - \eta_0; \bar{\Gamma}_0 - \eta_1] \preceq \Gamma_t[\tilde{g}; \tilde{\Gamma}_0] \preceq \Gamma_t[\bar{g} + \eta_0; \bar{\Gamma}_0 + \eta_1] \quad \text{for} \quad 0 \leq t \leq t_0.$$

Thus, in order to prove (1.144), it suffices to show that there exists constants $K, M > 0$ such that, for all small $\eta_0 > 0$, $\eta_1 > 0$,

$$\begin{cases} d_{\mathcal{H}}(\Gamma_t[\bar{g} - \eta_0; \bar{\Gamma}_0 - \eta_1],\, \Gamma_t[\bar{g}; \bar{\Gamma}_0]) \leq K\eta_0\,(e^{Mt} - 1) + \eta_1\, e^{Mt}, \\ d_{\mathcal{H}}(\Gamma_t[\bar{g} + \eta_0; \bar{\Gamma}_0 + \eta_1],\, \Gamma_t[\bar{g}; \bar{\Gamma}_0]) \leq K\eta_0\,(e^{Mt} - 1) + \eta_1\, e^{Mt}, \end{cases} \tag{1.146}$$

for $0 \leq t \leq t_0$. We will only show the latter inequality for $\Gamma_t[\bar{g} + \eta_0; \bar{\Gamma}_0 + \eta_1]$ since the former can be shown in the same manner.

Recall that $d(x, t; \bar{g}; \bar{\Gamma}_0)$ satisfies the equation (1.96), namely

$$d_t = \Delta d - c_0 \int_{\alpha_-}^{\alpha_+} \bar{g}(x, t, r)dr \quad \text{on} \;\; \Gamma_t[\bar{g}; \bar{\Gamma}_0]. \tag{1.147}$$

Choose a constant $d_0 > 0$ such that $d(x, t; \bar{g}; \bar{\Gamma}_0)$ is smooth — say, $C^3$ in $x$ and $C^{3/2}$ in $t$ — in the neighborhood $\mathcal{N}_{d_0}(\Gamma_t[\bar{g}; \bar{\Gamma}_0])$, $0 \leq t \leq T$. By equality (1.147) and the mean value theorem applied separately on both sides of the interface, there exists a constant $N_0 > 0$ such that

$$\Big|d_t - \Delta d + c_0 \int_{\alpha_-}^{\alpha_+} \bar{g}(x, t, r)dr\Big| \leq N_0|d| \quad \text{in} \;\; \mathcal{N}_{d_0/2}(\Gamma_t[\bar{g}; \bar{\Gamma}_0]).$$

Now we put

$$d^{new}(x,t) := d(x,t;\bar g;\bar\Gamma_0) - K\eta_0\,(e^{2N_0t}-1) - \eta_1\,e^{2N_0t},$$

$$\tilde\Gamma_t := \{x\in\Omega \,|\, d^{new}(x,t)=0\},$$

where the constant $K$ is to be determined later. If

$$\eta_0 \le \eta_0^* := \frac{e^{-2N_0T}\,d_0}{4}K^{-1} \quad \text{and} \quad \eta_1 \le \eta_1^* := \frac{e^{-2N_0T}\,d_0}{4},$$

then $\tilde\Gamma_t$ lies within the neighborhood $\mathcal{N}_{d_0/2}(\Gamma_t[\bar g;\bar\Gamma_0])$. Observe that

$$\begin{aligned}(d^{new})_t - \Delta d^{new} &= d_t - 2N_0K\eta_0\,e^{2N_0t} - 2N_0\eta_1\,e^{2N_0t} - \Delta d\\ &\le -c_0\int_{\alpha_-}^{\alpha_+}\bar g(x,t,r)dr + N_0|d| - 2N_0K\eta_0\,e^{2N_0t} - 2N_0\eta_1\,e^{2N_0t}.\end{aligned}$$

Since $d = K\eta_0\,(e^{2N_0t}-1)+\eta_1\,e^{2N_0t}$ on $\tilde\Gamma_t$, we obtain

$$\begin{aligned}(d^{new})_t - \Delta d^{new} &\le -c_0\int_{\alpha_-}^{\alpha_+}\bar g(x,t,r)dr - N_0K\eta_0\,e^{2N_0t} - N_0\eta_1 e^{2N_0t} \quad \text{on } \tilde\Gamma_t\\ &\le -c_0\int_{\alpha_-}^{\alpha_+}\bar g(x,t,r)dr - N_0K\eta_0 \quad \text{on } \tilde\Gamma_t.\end{aligned}$$

Now we set

$$K = (\alpha_+-\alpha_-)c_0{N_0}^{-1}.$$

Then it follows from the above inequality that

$$(d^{new})_t \le \Delta d^{new} - c_0\int_{\alpha_-}^{\alpha_+}\bar g(x,t,r)dr - (\alpha_+-\alpha_-)c_0\eta_0 \quad \text{on } \tilde\Gamma_t.$$

This inequality and the fact that $d^{new}(x,0) = d(x,0\,;\bar g;\bar\Gamma_0) - \eta_1$ imply that $\tilde\Gamma_t$ satisfies

$$\begin{cases} V_n \ge -(N-1)\kappa + c_0\displaystyle\int_{\alpha_-}^{\alpha_+}\big(\bar g(x,t,r)+\eta_0\big)\,dr \quad \text{on } \tilde\Gamma_t,\\ \tilde\Gamma_t\big|_{t=0} = \bar\Gamma_0 + \eta_1.\end{cases}$$

On the other hand, $\Gamma_t[\bar g+\eta_0;\bar\Gamma_0+\eta_1]$ satisfies

$$\begin{cases} V_n = -(N-1)\kappa + c_0\displaystyle\int_{\alpha_-}^{\alpha_+}\big(\bar g(x,t,r)+\eta_0\big)\,dr \quad \text{on } \Gamma_t[\bar g+\eta_0;\bar\Gamma_0+\eta_1],\\ \Gamma_t[\bar g+\eta_0;\bar\Gamma_0+\eta_1]\big|_{t=0} = \bar\Gamma_0 + \eta_1.\end{cases}$$

By the comparison principle, we obtain

$$\Gamma_t[\bar g;\bar\Gamma_0] \preceq \Gamma_t[\bar g+\eta_0;\bar\Gamma_0+\eta_1] \preceq \tilde\Gamma_t \qquad \text{for } 0\le t\le t_0.$$

Consequently,

$$d_{\mathcal H}(\Gamma_t[\bar g+\eta_0;\bar\Gamma_0+\eta_1],\,\Gamma_t[\bar g;\bar\Gamma_0]) \le d_{\mathcal H}(\tilde\Gamma_t,\,\Gamma_t[\bar g;\bar\Gamma_0]) \le K\eta_0\,(e^{2N_0t}-1)+\eta_1\,e^{2N_0t},$$

for $0\le t\le t_0$. The proposition is proved. □

Before closing this subsection, we remark that our main results for the Allen-Cahn equation — Theorems 1.1.4 and 1.1.6 — can also be derived from Propositions 1.7.3 and 1.7.4.

### 1.7.4 Proof of the main results

Now we turn to the reaction-diffusion system $(RD^\varepsilon)$. In what follows, we fix the initial datum $(u_0,\, v_0)$ and denote the solution of this system by $(u^\varepsilon,\, v^\varepsilon)$. We need some more notation; given a function $v(x,t)$ on $\overline{\Omega}\times[0,\infty)$, we set

$$\begin{aligned} g^\varepsilon[v](x,t,u) &:= -f_1(u,v(x,t)) - \varepsilon\, f_2^\varepsilon(u,v(x,t)),\\ g[v](x,t,u) &:= -f_1(u,v(x,t)), \end{aligned} \tag{1.148}$$

where $f_1,\, f_2^\varepsilon$ are as in (1.18). The first equation of $(RD^\varepsilon)$ is then written in the form

$$u_t = \Delta u + \frac{1}{\varepsilon^2}\big(f(u) - \varepsilon\, g^\varepsilon[v^\varepsilon](x,t,u)\big). \tag{1.149}$$

The limit Problem $(RD^0)$ is decomposed into two parts:

$$\begin{cases} V_n = -(N-1)\kappa + c_0 \displaystyle\int_{\alpha_-}^{\alpha_+} g[\tilde v](x,t,r)\,dr & \text{on } \Gamma_t,\\ \Gamma_t\big|_{t=0} = \Gamma_0, \end{cases} \tag{1.150}$$

and

$$\begin{cases} \tilde v_t = D\Delta \tilde v + h(\tilde u,\tilde v) & \text{in } \ \Omega\times(0,T],\\ \dfrac{\partial \tilde v}{\partial \nu} = 0 & \text{on } \ \partial\Omega\times(0,T],\\ \tilde v(x,0) = v_0(x) & \text{in } \ \Omega, \end{cases} \tag{1.151}$$

where $\tilde u$ is the step function associated with the interface $\Gamma_t$. Using the notation given in subsection 1.7.2, the above interface $\Gamma_t$ in (1.150) can be written as $\Gamma_t[g[\tilde v];\Gamma_0]$.

In order for Theorems 1.1.4 and 1.1.6 to be applicable to Problem $(RD^\varepsilon)$, we have to verify the conditions (1.3) to (1.6). More precisely, we have to show that, for all small $\varepsilon>0$,

$$|\Delta_x g^\varepsilon[v^\varepsilon](x,t,u)| \le C\varepsilon^{-1} \quad \text{and} \quad |\partial_t\, g^\varepsilon[v^\varepsilon](x,t,u)| \le C\varepsilon^{-1},$$

$$|\partial_u\, g^\varepsilon[v^\varepsilon](x,t,u)| \le C,$$

$$\|g^\varepsilon[v^\varepsilon](\cdot,\cdot,u)\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}(\overline{\Omega}\times[0,T])} \le C,$$

$$\big|g^\varepsilon[v^\varepsilon](x,t,u) - g[\tilde v](x,t,u)\big| \le C\varepsilon.$$

Since $g[v]$, $g^\varepsilon[v]$ are defined by (1.148) and since $f_1$, $f_2^\varepsilon$ are smooth (see assumption **(F)** in subsection 1.1.2), it suffices to prove the following estimates for some $C>0$ and for all small $\varepsilon>0$:

$$|\Delta_x v^\varepsilon(x,t)| \le C\varepsilon^{-1} \quad \text{and} \quad |\partial_t v^\varepsilon(x,t)| \le C\varepsilon^{-1}, \tag{1.152}$$

$$\|v^\varepsilon\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}(\overline{\Omega}\times[0,T])} \le C, \tag{1.153}$$

$$|v^\varepsilon(x,t) - \tilde v(x,t)| \le C\varepsilon. \tag{1.154}$$

The first two estimates are elementary. In fact, since $v^\varepsilon$ satisfies

$$v_t^\varepsilon = D\Delta v^\varepsilon + h(u^\varepsilon, v^\varepsilon) \quad \text{in } \ \Omega\times(0,T] \tag{1.155}$$

along with the homogeneous Neumann boundary condition, it can be expressed as

$$v^\varepsilon(x,t) \ = \ I_1 + I_2, \tag{1.156}$$

where

$$I_1 := \int_\Omega G(x,y,t)v_0(y)dy,$$
$$I_2 := \int_0^t \int_\Omega G(x,y,t-s)\,h(u^\varepsilon(y,s),v^\varepsilon(y,s))\,dy\,ds,$$

with $G(x,y,t)$ being the fundamental solution for equation $v_t = D\Delta v$ under the homogeneous Neumann boundary condition. Since $h(u^\varepsilon, v^\varepsilon)$ is uniformly bounded, standard estimates of $G(x,y,t)$ imply (1.153) for any $\vartheta \in (0,1)$. In the meanwhile, the same rescaling argument as in Remark 1.1.8 yields

$$\|u^\varepsilon\|_{C^{\vartheta,\frac{\vartheta}{2}}(\overline{\Omega}\times[0,T])} \le C\varepsilon^{-\vartheta}. \tag{1.157}$$

Indeed, since $\nabla_y u^\varepsilon$, $u^\varepsilon_\tau$ are bounded, where $y = x/\varepsilon$, $\tau = t/\varepsilon^2$, we have $\nabla_x u^\varepsilon = O(1/\varepsilon)$, $u^\varepsilon_t = O(1/\varepsilon^2)$. Consequently we have

$$\begin{aligned}
\frac{|u^\varepsilon(x,t)-u^\varepsilon(x',t')|}{|x-x'|^\vartheta + |t-t'|^{\vartheta/2}} &\le \frac{|u^\varepsilon(x,t)-u^\varepsilon(x',t)|}{|x-x'|^\vartheta} + \frac{|u^\varepsilon(x',t)-u^\varepsilon(x',t')|}{|t-t'|^{\vartheta/2}} \\
&\le |u^\varepsilon(x,t)-u^\varepsilon(x',t)|^{1-\vartheta}\frac{|u^\varepsilon(x,t)-u^\varepsilon(x',t)|^\vartheta}{|x-x'|^\vartheta} \\
&\qquad + |u^\varepsilon(x',t)-u^\varepsilon(x',t')|^{1-\vartheta/2}\frac{|u^\varepsilon(x',t)-u^\varepsilon(x',t')|^{\vartheta/2}}{|t-t'|^{\vartheta/2}} \\
&\le (2\|u^\varepsilon\|_{L^\infty})^{1-\vartheta}\|\nabla_x u^\varepsilon\|^\vartheta_{L^\infty} + (2\|u^\varepsilon\|_{L^\infty})^{1-\vartheta/2}\|u^\varepsilon_t\|^{\vartheta/2}_{L^\infty} \\
&\le C\varepsilon^{-\vartheta}.
\end{aligned}$$

Combining (1.157) and (1.153), we see that $\|h(u^\varepsilon,v^\varepsilon)\|_{C^{\vartheta,\frac{\vartheta}{2}}(\overline{\Omega}\times[0,T])} \le C\varepsilon^{-\vartheta}$, hence, by the Schauder estimate,

$$\|I_2\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{\Omega}\times[0,T])} \le C\varepsilon^{-\vartheta}.$$

Here the constant $C$ may depend on the choice of $\vartheta \in (0,1)$. On the other hand, $I_1$ is bounded in $C^{2,1}(\overline{\Omega}\times[0,T])$ since $v_0 \in C^2(\overline{\Omega})$. Combining these, we obtain that $|\Delta_x v^\varepsilon(x,t)| = O(\varepsilon^{-\vartheta})$, hence $O(\varepsilon^{-1})$. By a similar argument, we obtain that $|\partial_t v^\varepsilon(x,t)| = O(\varepsilon^{-1})$.

It remains to prove (1.154). This requires more elaborate analysis. Let us introduce some more notation. Given functions $u(x,t)$ and $\bar{v}_0(x)$ on $\overline{\Omega}\times[0,\infty)$, we denote by $V[u;\bar{v}_0](x,t)$ the solution of the problem

$$\begin{cases} V_t = D\Delta V + h(u(x,t),V) & \text{in } \ \Omega\times(0,T], \\ \dfrac{\partial V}{\partial \nu} = 0 & \text{on } \ \partial\Omega\times(0,T], \\ V(x,0) = \bar{v}_0(x) & \text{in } \ \Omega. \end{cases} \tag{1.158}$$

The solution $v^\varepsilon$ of $(RD^\varepsilon)$ and $\tilde{v}$ of $(RD^0)$ can then be expressed as

$$v^\varepsilon = V[u^\varepsilon; v_0], \qquad \tilde{v} = V[\tilde{u}; v_0].$$

Recall also that, with the notation defined in subsection 1.7.2 and in (1.148), the solution $u^\varepsilon$ of $(RD^\varepsilon)$ and the step function $\tilde{u}$ in $(RD^0)$ are expressed as

$$u^\varepsilon = u^\varepsilon[g^\varepsilon[v^\varepsilon]; u_0], \qquad \tilde{u} = \tilde{u}[g[\tilde{v}]; \Gamma_0].$$

**First estimate.** Let us compare $\overline{u}^\varepsilon := u^\varepsilon[g^\varepsilon[v^\varepsilon]; \bar{u}_0]$ with the step function $\tilde{u}^\varepsilon := \tilde{u}[g^\varepsilon[v^\varepsilon]; \bar{\Gamma}_0]$. By (1.140), we have

$$\hat{u}_\varepsilon^-(x,t) \le \overline{u}^\varepsilon(x, t+t^\varepsilon) \le \hat{u}_\varepsilon^+(x,t) \quad \text{for} \quad 0 \le t \le T - t^\varepsilon,$$

where $\hat{u}_\varepsilon^\pm$ are as in (1.139), $d^\varepsilon$ being the signed distance function associated with the interface $\Gamma_t[g^\varepsilon[v^\varepsilon]; \bar{\Gamma}_0]$. Since the term $e^{-\beta t/\varepsilon^2}$ in $q(t)$, see (1.107), quickly becomes small,

$$|\overline{u}^\varepsilon(x,t) - \tilde{u}^\varepsilon(x,t)| \le \alpha_+ - U_0\Big(\frac{d^\varepsilon(x,t) - \varepsilon p(t)}{\varepsilon}\Big) + O(\varepsilon) \quad \text{in} \quad \Omega_t^+[g^\varepsilon[v^\varepsilon]; \bar{\Gamma}_0],$$
$$|\overline{u}^\varepsilon(x,t) - \tilde{u}^\varepsilon(x,t)| \le U_0\Big(\frac{d^\varepsilon(x,t) + \varepsilon p(t)}{\varepsilon}\Big) - \alpha_- + O(\varepsilon) \quad \text{in} \quad \Omega_t^-[g^\varepsilon[v^\varepsilon]; \bar{\Gamma}_0],$$

for $\mu_1\varepsilon^2|\ln\varepsilon| \le t \le T$, provided that we choose the constant $\mu_1$ large enough. Consequently, by Lemma 1.2.1, there exist constants $B$, $C > 0$ such that

$$|\overline{u}^\varepsilon(x,t) - \tilde{u}^\varepsilon(x,t)| \le B \exp\Big(-\lambda\frac{|d^\varepsilon(x,t)|}{\varepsilon}\Big) + C\varepsilon, \tag{1.159}$$

for $(x,t) \in \overline{\Omega} \times [\mu_1\varepsilon^2|\ln\varepsilon|,\ T]$.

**Second estimate.** Next we compare $\overline{v}^\varepsilon := V[\overline{u}^\varepsilon; \bar{v}_0]$ and $\tilde{v}^\varepsilon := V[\tilde{u}^\varepsilon; \tilde{v}_0]$. Set $w := \overline{v}^\varepsilon - \tilde{v}^\varepsilon$. Then

$$w_t = D\Delta w + \big(h(\overline{u}^\varepsilon, \overline{v}^\varepsilon) - h(\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)\big).$$

Since

$$|h(\overline{u}^\varepsilon, \overline{v}^\varepsilon) - h(\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)| \le C|w| + C|\overline{u}^\varepsilon(x,t) - \tilde{u}^\varepsilon(x,t)|,$$

for some constant $C > 0$, the function $\tilde{w} := e^{-Ct}w$ satisfies

$$\tilde{w}_t \le D\Delta\tilde{w} + Ce^{-Ct}|\overline{u}^\varepsilon(x,t) - \tilde{u}^\varepsilon(x,t)| + C(|\tilde{w}| - \tilde{w}),$$

hence

$$\tilde{w}_t \le D\Delta\tilde{w} + C|\overline{u}^\varepsilon(x,t) - \tilde{u}^\varepsilon(x,t)| + C(|\tilde{w}| - \tilde{w}). \tag{1.160}$$

Now let $W(x,t)$ be the solution of the equation

$$W_t = D\Delta W + C|\overline{u}^\varepsilon(x,t) - \tilde{u}^\varepsilon(x,t)| + C(|W| - W),$$

with initial datum $W(x,0) = |\bar{v}_0(x) - \tilde{v}_0(x)|$. Then since (1.160) implies that $\tilde{w}$ is a sub-solution of the above equation, and since

$$\tilde{w}(x,0) \le |\tilde{w}(x,0)| = |\bar{v}_0(x) - \tilde{v}_0(x)| = W(x,0),$$

we have

$$\tilde{w}(x,t) \le W(x,t) \qquad \text{for} \ \ x \in \overline{\Omega},\ t \ge 0. \tag{1.161}$$

Moreover, since $W \ge 0$, the above equation for $W$ can be reduced to

$$W_t = D\Delta W + C|\overline{u}^\varepsilon(x,t) - \tilde{u}^\varepsilon(x,t)|.$$

In view of this and $W(x,0) = |\bar{v}_0(x) - \tilde{v}_0(x)|$, we see that

$$W(x,t) = C\int_0^t\int_\Omega G(x,y,t-s)|\overline{u}^\varepsilon(y,s) - \tilde{u}^\varepsilon(y,s)|\,dy\,ds + \int_\Omega G(x,y,t)|\bar{v}_0(y) - \tilde{v}_0(y)|dy,$$

$G(x, y, t)$ being the fundamental solution that appears in (1.156). This and (1.161) yield

$$|w(x,t)| \le Ce^{Ct} \int_0^t \int_\Omega G(x,y,t-s)|\overline{u}^\varepsilon(y,s) - \tilde{u}^\varepsilon(y,s)|\, dy\, ds \\ + e^{Ct} \int_\Omega G(x,y,t)|\bar{v}_0(y) - \tilde{v}_0(y)| dy. \quad (1.162)$$

Combining this and (1.159), we obtain

$$|w(x,t)| \le BCe^{Ct} \int_0^t \int_\Omega G(x,y,t-s) \exp\Big(-\lambda \frac{|d^\varepsilon(y,s)|}{\varepsilon}\Big) dy\, ds \\ + O(\varepsilon) + C'\|\bar{v}_0 - \tilde{v}_0\|_{L^\infty(\Omega)}, \quad (1.163)$$

for some constant $C' > 0$. In order to estimate the above integral, we need the following lemma.

**Lemma 1.7.5.** *Let $\Gamma$ be a smooth closed hypersurface in $\Omega$ and denote by $d(x)$ the signed distance function associated with $\Gamma$. Then there exist constants $C, r_0 > 0$ such that for any function $\eta(r) \ge 0$ on $\mathbb{R}$, it holds that*

$$\int_{|d| \le r_0} G(x,y,t)\, \eta(d(y))\, dy \le \frac{C}{\sqrt{t}} \int_{-r_0}^{r_0} \eta(r)\, dr \quad \textit{for } \ 0 < t \le T. \quad (1.164)$$

The proof of this lemma will be given in the next subsection. As is easily seen from its proof, the above estimate remains to hold if $\Gamma$ depends on $t$ smoothly; in other words, the constant $C$ can be chosen uniformly as $\Gamma$ varies. Applying the above estimate to $\Gamma_t[g^\varepsilon[v^\varepsilon]; \bar{\Gamma}_0]$, $0 < t \le T$, we obtain

$$\begin{aligned} &\int_\Omega G(x,y,t-s) \exp\Big(-\lambda \frac{|d^\varepsilon(y,s)|}{\varepsilon}\Big) dy \\ &\quad = \int_{|d^\varepsilon| < r_0} + \int_{|d^\varepsilon| \ge r_0} G(x,y,t-s) \exp\Big(-\lambda \frac{|d^\varepsilon(y,s)|}{\varepsilon}\Big) dy \\ &\quad = O\big(\frac{\varepsilon}{\sqrt{t-s}}\big) + O\big(e^{-\lambda r_0/\varepsilon}\big) \\ &\quad = O\big(\frac{\varepsilon}{\sqrt{t-s}}\big). \end{aligned}$$

It follows from this and (1.163) that

$$|w(x,t)| = O\Big(\varepsilon \int_0^t \frac{1}{\sqrt{t-s}}\, ds\Big) + O(\varepsilon) + C'\|\bar{v}_0 - \tilde{v}_0\|_{L^\infty(\Omega)}, \quad (1.165)$$

hence

$$\overline{v}^\varepsilon(x,t) - \tilde{v}^\varepsilon(x,t) = O(\varepsilon) + O(\|\bar{v}_0 - \tilde{v}_0\|_{L^\infty(\Omega)}). \quad (1.166)$$

**Key estimate.** Finally we compare the two step functions $\tilde{u}^\varepsilon := \tilde{u}[g^\varepsilon[v^\varepsilon]; \bar{\Gamma}_0]$ and $\tilde{u}[g[v^\varepsilon]; \tilde{\Gamma}_0]$. By (1.148), we have $\|g^\varepsilon[v^\varepsilon] - g[v^\varepsilon]\|_{L^\infty} = O(\varepsilon)$. In the following we assume $d_{\mathcal{H}}(\tilde{\Gamma}_0, \bar{\Gamma}_0) \le \delta$, so that, by Proposition 1.7.4, we have

$$\sup_{0 \le t \le T} d_{\mathcal{H}}(\Gamma_t[g^\varepsilon[v^\varepsilon]; \bar{\Gamma}_0],\, \Gamma_t[g[v^\varepsilon]; \tilde{\Gamma}_0]) = O(\varepsilon) + O(d_{\mathcal{H}}(\tilde{\Gamma}_0, \bar{\Gamma}_0)).$$

This means that $\tilde{u}[g^\varepsilon[v^\varepsilon];\bar{\Gamma}_0]$ and $\tilde{u}[g[v^\varepsilon];\tilde{\Gamma}_0]$ differ only in a thin neighborhood of $\Gamma_t[g[v^\varepsilon];\tilde{\Gamma}_0]$ of thickness $O(\varepsilon)+O(d_{\mathcal{H}}(\tilde{\Gamma}_0,\bar{\Gamma}_0))$. Arguing as above and applying Lemma 1.7.5 again, we see that

$$\|V\big[\tilde{u}[g^\varepsilon[v^\varepsilon];\bar{\Gamma}_0];\tilde{v}_0\big]-V\big[\tilde{u}[g[v^\varepsilon];\tilde{\Gamma}_0];\tilde{v}_0\big]\|_{L^\infty(\Omega\times[0,T])}=O(\varepsilon)+O(d_{\mathcal{H}}(\tilde{\Gamma}_0,\bar{\Gamma}_0)).$$

Combining this and (1.166), we obtain

$$\|\,\overline{v}^\varepsilon-V\big[\tilde{u}[g[v^\varepsilon];\tilde{\Gamma}_0];\tilde{v}_0\big]\,\|_{L^\infty(\Omega\times[0,T])}=O(\varepsilon)+O(d_{\mathcal{H}}(\tilde{\Gamma}_0,\bar{\Gamma}_0))+O(\|\bar{v}_0-\tilde{v}_0\|_{L^\infty(\Omega)}).\quad(1.167)$$

**Conclusion.** In what follows we will show that (1.167) implies our desired estimate (1.154). Our proof is based on a contraction mapping argument, but this argument applies only to a certain time interval $[0,T_1]\subset[0,T]$. Once we obtain (1.154) for the interval $[0,T_1]$, we will repeat the same argument to derive (1.154) on an interval $[T_1,2T_1]$, and this "step by step" procedure eventually yields (1.154) on the whole interval $[0,T]$. To make the above strategy work, we first introduce some notation. Choose a constant $C^*>0$ sufficiently large so that the estimate (1.153) holds with $C=C^*$ for all small $\varepsilon>0$, and that $\|\tilde{v}\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}}\leq C^*$. We fix such $C^*>0$ and define

$$\begin{aligned}\mathcal{U}\ &:=\{\,v\in C^{1+\vartheta,\frac{1+\vartheta}{2}}(\overline{\Omega}\times[0,T]),\ \|v\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}}\leq C^*\,\},\\ \mathcal{U}_\delta\ &:=\{\,v\in\mathcal{U},\ \|v-\tilde{v}\|_{L^\infty}\leq\delta\,\}.\end{aligned}$$

Also we choose $C_*>0$ large enough so that $v\in\mathcal{U}$ implies that $g[v](x,t,u)$ satisfies (1.142). We remark that $\mathcal{U}_\delta$ is a closed subset of $L^\infty(\overline{\Omega}\times[0,T])$, since $\|v_n\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}}\leq C^*$ and $v_n\to v$ in $L^\infty$ implies $\|v\|_{C^{1+\vartheta,\frac{1+\vartheta}{2}}}\leq C^*$. Consequently $\mathcal{U}_\delta$ is a complete metric space with respect to the $L^\infty$ topology.

Fix $\tilde{v}_0$ and $\tilde{\Gamma}_0$. If $\delta>0$ is chosen small enough, then by Proposition 1.7.4 the classical solution of $(P^0_{g,\tilde{\Gamma}_0})$ with $g=g[v]$ exists on the entire interval $[0,T]$; we denote it by $\Gamma_t[g[v];\tilde{\Gamma}_0]$ as before. This determines the step function $\tilde{u}[g[v];\tilde{\Gamma}_0]$, which then determines $V\big[\tilde{u}[g[v];\tilde{\Gamma}_0];\tilde{v}_0\big]$. Combining these, we can define a mapping

$$\Phi: v\mapsto V\big[\tilde{u}[g[v];\tilde{\Gamma}_0];\tilde{v}_0\big]$$

from $\mathcal{U}_\delta$ into $L^\infty(\Omega\times[0,T])$. Roughly speaking, $\Phi$ is a contraction mapping on a time interval $[0,T_1]$. More precisely, the following result holds.

**Lemma 1.7.6.** *$\Phi$ is a Lipschitz continuous map from $\mathcal{U}_\delta$ into $L^\infty(\Omega\times[0,T])$. Moreover, there exist constants $T_1>0$ and $\theta\in(0,1)$ such that*

$$\|\Phi[v_1]-\Phi[v_2]\|_{L^\infty(Q_{T_1})}\leq\theta\,\|v_1-v_2\|_{L^\infty(Q_{T_1})},\quad(1.168)$$

*for any $v_1,\,v_2\in\mathcal{U}_\delta$.*

The proof of this lemma will be given in the next subsection. We are now ready to prove the key estimate (1.154).

- <u>1st step:</u> We first put $\bar{u}_0=u_0$, $\bar{v}_0=\tilde{v}_0=v_0$, $\bar{\Gamma}_0=\tilde{\Gamma}_0=\Gamma_0$. It follows that

$$\overline{u}^\varepsilon:=u^\varepsilon[g^\varepsilon[v^\varepsilon];\bar{u}_0]=u^\varepsilon[g^\varepsilon[v^\varepsilon];u_0]=u^\varepsilon,$$

$$\overline{v}^\varepsilon:=V[\overline{u}^\varepsilon;\bar{v}_0]=V[u^\varepsilon;v_0]=v^\varepsilon,$$

so that (1.167) yields

$$\|v^\varepsilon - \Phi[v^\varepsilon]\|_{L^\infty(\Omega\times[0,T_1])} = O(\varepsilon). \tag{1.169}$$

In other words, $v^\varepsilon$ is almost a fixed point of the mapping $\Phi$ by an error margin of $O(\varepsilon)$. Note that $\tilde v$ is a fixed point of the map $\Phi$:

$$\tilde v = \Phi[\,\tilde v\,].$$

These two observations are sufficient to conclude that $\tilde v$ and $v^\varepsilon$ are close to each other, by $O(\varepsilon)$, until time $T_1$. Indeed, by the above lemma, we have, working on $Q_{T_1}$,

$$\|\Phi[v^\varepsilon] - \tilde v\|_{L^\infty} = \|\Phi[v^\varepsilon] - \Phi[\tilde v]\|_{L^\infty} \le \theta\|v^\varepsilon - \tilde v\|_{L^\infty}.$$

On the other hand,

$$\|\Phi[v^\varepsilon] - \tilde v\|_{L^\infty} \ge \|v^\varepsilon - \tilde v\|_{L^\infty} - \|v^\varepsilon - \Phi[v^\varepsilon]\|_{L^\infty}.$$

Combining these, we obtain

$$\|v^\varepsilon - \tilde v\,\|_{L^\infty} \le \frac{1}{1-\theta}\|v^\varepsilon - \Phi[v^\varepsilon]\,\|_{L^\infty}. \tag{1.170}$$

In view of (1.169), this proves (1.154) on $\overline\Omega\times[0,T_1]$. Hence, conditions (1.152) to (1.154) are satisfied, at least until time $t = T_1$; we can then apply our results for the single equation to system $(RD^\varepsilon)$ and obtain, by Corollary 1.1.7,

$$d_{\mathcal H}(\Gamma_{T_1}, \Gamma^\varepsilon_{T_1}) = O(\varepsilon). \tag{1.171}$$

- 2nd step: We take $T_1$ as a new initial moment and put $\bar u_0 = u^\varepsilon(\cdot,T_1)$, $\bar v_0 = v^\varepsilon(\cdot,T_1)$, $\tilde v_0 = \tilde v(\cdot,T_1)$, $\bar\Gamma_0 = \Gamma^\varepsilon_{T_1} := \{x\in\Omega, u^\varepsilon(x,T_1) = a\}$, $\tilde\Gamma_0 = \Gamma_{T_1}$. It follows that

$$\overline{u}^\varepsilon := u^\varepsilon[g^\varepsilon[v^\varepsilon];\bar u_0] = u^\varepsilon[g^\varepsilon[v^\varepsilon];u^\varepsilon(\cdot,T_1)] = u^\varepsilon,$$

$$\overline{v}^\varepsilon := V[\overline{u}^\varepsilon;\bar v_0] = V[u^\varepsilon;v^\varepsilon(\cdot,T_1)] = v^\varepsilon.$$

By the result of the first step, we have

$$\|\bar v_0 - \tilde v_0\|_{L^\infty(\Omega)} = \|v^\varepsilon(\cdot,T_1) - \tilde v(\cdot,T_1)\|_{L^\infty(\Omega)} = O(\varepsilon).$$

Moreover, by (1.171),

$$d_{\mathcal H}(\tilde\Gamma_0,\bar\Gamma_0) = d_{\mathcal H}(\Gamma_{T_1},\Gamma^\varepsilon_{T_1}) = O(\varepsilon),$$

so that (1.167) leads to

$$\|v^\varepsilon - \Phi[v^\varepsilon]\|_{L^\infty(\Omega\times[T_1,2T_1])} = O(\varepsilon).$$

Then, using the same arguments as in the first step, we obtain estimate (1.154) on $\overline\Omega\times[T_1,2T_1]$, and also an analogue of estimate (1.171) at time $t = 2T_1$; repeating this procedure a finite number of times we obtain estimate (1.154) on $\overline\Omega\times[0,T]$.

Hence, all the conditions (1.152) to (1.154) are verified so that Theorems 1.1.12 and 1.1.14, along with their corollaries, follow directly from Theorems 1.1.4, 1.1.6 and their corollaries. This completes the proof of the main results for $(RD^\varepsilon)$. □

### 1.7.5 Proof of Lemmas 1.7.5 and 1.7.6

**Proof of Lemma 1.7.5.** We first show that

$$\int_\Gamma G(x,y,t)\,dS_y \le \frac{C}{\sqrt{t}} \qquad \text{for } \ x\in\Omega,\ 0<t\le T. \tag{1.172}$$

It suffices to prove this estimate on a small interval $[0,t_0]$, since the estimate for the remaining interval $[t_0,T]$ will follow by simply choosing a large constant $C$. Note also that, for $t$ sufficiently small, $G(x,y,t)$ is well approximated by the fundamental solution on the entire space $\mathbb{R}^N$:

$$G_0(x,y,t) := \frac{1}{(4\pi Dt)^{N/2}}\,\exp\Big(-\frac{|x-y|^2}{4Dt}\Big).$$

In particular, there exists a constant $C^*>0$ such that

$$0<G(x,y,t)\le C^*\,G_0(x,y,t) \qquad \text{for } \ x,y\in\overline{\Omega},\ 0<t\le t_0;$$

for this result, we refer to [35], Chapter I, Section IV.2. Thus it suffices to prove (1.172) for $G_0$ instead of $G$.

Given $x\in\Omega$, let $x_0$ be the point on $\Gamma$ that is closest to $x$, and let $\nu(x_0)$ be the outward normal to $\Gamma$ at $x_0$. Then $x-x_0=d(x)\nu(x_0)$. Define

$$\widetilde{Y} := \{\, y\in\mathbb{R}^N\ ,\ y\cdot\nu(x_0)=0\,\}, \qquad Y_0 := span\langle\nu(x_0)\rangle,$$

where $\cdot$ denotes the Euclidean inner product in $\mathbb{R}^N$ and $span\langle w\rangle$ the line spanned by the vector $w$. This gives an orthogonal decomposition $\mathbb{R}^N=\widetilde{Y}\oplus Y_0$, and $x_0+\widetilde{Y}$ is the tangent hyperplane of $\Gamma$ at $x_0$. Since $\Gamma$ is smooth, it is expressed locally as the graph of a map defined on a subset of $\widetilde{Y}$. More precisely, there exist a smooth map

$$h:\widetilde{Y}\to Y_0,$$

and a constant $\delta>0$ such that $h(0)=0$, $\nabla h(0)=0$, and that

$$\begin{aligned} &S := \{\, x_0+\tilde{y}+h(\tilde{y})\ ,\ \tilde{y}\in\widetilde{Y}, |\tilde{y}|<\delta\,\}\subset\Gamma,\\ &\mathrm{dist}(x_0,\Gamma\setminus S)\ge\delta. \end{aligned} \tag{1.173}$$

Now we decompose the integral (1.172) for $G_0$ as

$$\int_\Gamma G_0(x,y,t)\,dS_y = \frac{1}{(4\pi Dt)^{N/2}}\left(\int_S+\int_{\Gamma\setminus S}\exp\Big(-\frac{|x-y|^2}{4Dt}\Big)dS_y\right).$$

Since $|x-y|\ge|d(x)|$ for every $y\in\Gamma$ and since

$$|x-y|\ge\big|\,|x-x_0|-|y-x_0|\,\big| = \big|\,|d(x)|-|y-x_0|\,\big|,$$

we have

$$|x-y|\ge\frac{|d(x)|+\big|\,|d(x)|-|y-x_0|\,\big|}{2}\ge\frac{|y-x_0|}{2}.$$

This and (1.173) yield

$$|x-y|\ge\frac{\delta}{2} \qquad \text{for } \ y\in\Gamma\setminus S.$$

Consequently,

$$\int_{\Gamma\setminus S} \exp\Big(-\frac{|x-y|^2}{4Dt}\Big)dS_y \le e^{-\delta^2/16Dt}\,|\Gamma|, \tag{1.174}$$

where $|\Gamma|$ denotes the total area of $\Gamma$.

On the other hand, for each $y \in S$, we can express $y - x_0$ as

$$y - x_0 = \tilde{y} + h(\tilde{y}) \qquad (\tilde{y} \in \widetilde{Y},\ h(\tilde{y}) \in Y_0),$$

and $\widetilde{Y}$ can be identified with $\mathbb{R}^{N-1}$. Thus

$$\begin{aligned}\int_S \exp\Big(-\frac{|x-y|^2}{4Dt}\Big)dS_y \\ = \int_{|\tilde{y}|<\delta} \exp\Big(-\frac{|x-x_0-\tilde{y}-h(\tilde{y})|^2}{4Dt}\Big)\sqrt{1+|\nabla h(\tilde{y})|^2}\,d\tilde{y}.\end{aligned}$$

Since $h(0) = 0$ and $\nabla h(0) = 0$, there exists a constant $C_1 > 0$ such that

$$|\nabla h(\tilde{y})| \le C_1|\tilde{y}| \qquad \text{for} \quad |\tilde{y}| < \delta. \tag{1.175}$$

Note also that the orthogonality $(x - x_0 - h(\tilde{y})) \perp \tilde{y}$ implies

$$|x-x_0-\tilde{y}-h(\tilde{y})|^2 = |x-x_0-h(\tilde{y})|^2 + |\tilde{y}|^2 \ge |\tilde{y}|^2.$$

Combining these, we obtain

$$\begin{aligned}\int_S \exp\Big(-\frac{|x-y|^2}{4Dt}\Big)dS_y &\le \int_{|\tilde{y}|<\delta} \exp\Big(-\frac{|\tilde{y}|^2}{4Dt}\Big)\sqrt{1+C_1^2|\tilde{y}|^2}\,d\tilde{y} \\ &= t^{(N-1)/2}\int_{|z|<\sqrt{t}^{-1}\delta} e^{-|z|^2/4D}\sqrt{1+t\,C_1^2|z|^2}\,dz,\end{aligned}$$

where $z := \tilde{y}/\sqrt{t}$. Observe that, as $t \to 0$,

$$\int_{|z|<\sqrt{t}^{-1}\delta} e^{-|z|^2/4D}\sqrt{1+t\,C_1^2|z|^2}\,dz \;\to\; \int_{\mathbb{R}^{N-1}} e^{-|z|^2/4D}\,dz = (4D\pi)^{(N-1)/2}.$$

Consequently,

$$\frac{1}{(4\pi Dt)^{N/2}}\int_S \exp\Big(-\frac{|x-y|^2}{4Dt}\Big)dS_y \le \frac{1}{\sqrt{4\pi Dt}} + o\Big(\frac{1}{\sqrt{t}}\Big).$$

Combining this and (1.174), we obtain

$$\int_\Gamma G_0(x,y,t)\,dS_y = O\Big(\frac{1}{\sqrt{t}}\Big) + O\Big(\frac{1}{(\sqrt{t}\,)^N}e^{-\delta^2/16Dt}\Big) = O\Big(\frac{1}{\sqrt{t}}\Big).$$

Since $\Gamma$ is a smooth compact hypersurface, its curvatures are bounded. Therefore, the constants $\delta$ and $C_1$ that appear in (1.174), (1.175) can be chosen independent of the choice of $x_0 \in \Gamma$. Hence the above $O(1/\sqrt{t})$ estimate is uniform with respect to the choice of $x \in \Omega$. This proves the estimate (1.172).

Now, choose a sufficiently small constant $r_0 > 0$ such that the signed distance function $d(x)$ is smooth in the region $\{\,d(x) < 2r_0\,\}$. For each $r \in [-r_0, r_0]$, we define a hypersurface $\Gamma(r)$ by

$$\Gamma(r) := \{\,x \in \Omega,\ d(x) = r\,\}.$$

Then the curvatures of $\Gamma(r)$ are uniformly bounded as $r$ varies, which implies that there exists some constant $C > 0$ such that

$$\int_{\Gamma(r)} G(x,y,t)\, dS_y \leq \frac{C}{\sqrt{t}} \qquad \text{for } \ 0 < t \leq T, \ \ r \in [-r_0, r_0].$$

The estimate (1.164) now follows by integrating in $r$. □

**Proof of Lemma 1.7.6.** For each $t \in [0,T]$, we put $Q_t := \Omega \times [0,t]$. Given $v_1, \ v_2 \in \mathcal{U}_\delta$, we have, in view of (1.148),

$$\|g[v_1] - g[v_2]\|_{L^\infty(Q_t \times [\alpha_-, \alpha_+])} \leq K_1 \|v_1 - v_2\|_{L^\infty(Q_t)},$$

where

$$K_1 = \max_{(u,v)\in\mathcal{R}} |\partial_v f_1(u,v)|,$$

with $\mathcal{R}$ being the rectangle defined in subsection 1.7.1. By Proposition 1.7.4,

$$d_{\mathcal{H}}(\, \Gamma_t[g[v_1]; \tilde{\Gamma}_0], \Gamma_t[g[v_2]; \tilde{\Gamma}_0]\,) \leq K(e^{Mt} - 1)\, \|g[v_1] - g[v_2]\|_{L^\infty(Q_t \times [\alpha_-, \alpha_+])}.$$

Combining these, we obtain

$$d_{\mathcal{H}}(\, \Gamma_t[g[v_1]; \tilde{\Gamma}_0], \Gamma_t[g[v_2]; \tilde{\Gamma}_0]\,) \leq KK_1(e^{Mt} - 1)\, \|v_1 - v_2\|_{L^\infty(Q_t)}\,. \tag{1.176}$$

Now we define the step functions $\tilde{u}_1 := \tilde{u}[g[v_1]; \tilde{\Gamma}_0]$ and $\tilde{u}_2 := \tilde{u}[g[v_2]; \tilde{\Gamma}_0]$. Since

$$|\tilde{u}_1 - \tilde{u}_2| \leq \alpha_+ - \alpha_-,$$

and since the two step functions differ only in the region enclosed between the two surfaces $\Gamma_t[\, g[v_1]; \ \tilde{\Gamma}_0]$ and $\Gamma_t[\, g[v_2]; \ \tilde{\Gamma}_0]$, the estimates (1.162) and (1.164) imply that there exists a constant $B_1 > 0$ such that

$$\|V[\tilde{u_1}; \tilde{v}_0] - V[\tilde{u_2}; \tilde{v}_0]\,\|_{L^\infty(Q_t)} \leq B_1 \int_0^t \frac{d_{\mathcal{H}}(\, \Gamma_s[g[v_1]; \tilde{\Gamma}_0], \Gamma_s[g[v_2]; \tilde{\Gamma}_0]\,)}{\sqrt{t-s}}\, ds.$$

Combining this and (1.176), we obtain

$$\|\, \Phi[v_1] - \Phi[v_2]\,\|_{L^\infty(Q_t)} \leq C_1 \int_0^t \frac{\|v_1 - v_2\|_{L^\infty(Q_s)}}{\sqrt{t-s}}\, ds, \tag{1.177}$$

where $C_1 = B_1 K K_1 (e^{MT} - 1)$. In particular,

$$\begin{aligned} \|\, \Phi[v_1] - \Phi[v_2]\,\|_{L^\infty(Q_{T_1})} &\leq C_1 \int_0^{T_1} \frac{1}{\sqrt{T_1 - s}}\, ds\, \|v_1 - v_2\|_{L^\infty(Q_{T_1})} \\ &= \theta\, \|v_1 - v_2\|_{L^\infty(Q_{T_1})}, \end{aligned}$$

where $\theta := 2C_1\sqrt{T_1} < 1$ for $T_1$ small enough. This proves Lemma 1.7.6. □

# Chapter 2

# The singular limit of a chemotaxis-growth system with general initial data

We consider a system of partial differential equations which is a model for an aggregation of amoebae subjected to three competitive effects: diffusion, growth and chemotaxis, i.e. the tendency of the specie to move towards higher gradients of a chemical substance. The system involves a small parameter $\varepsilon > 0$ and a cubic nonlinearity whose stable equilibria are 0 and 1. We consider rather general initial data $u_0$ that are independent of $\varepsilon$. We denote by $(u^\varepsilon, v^\varepsilon)$ the solution. First we prove a generation of interface result namely that, after a time of order $\varepsilon^2|\ln\varepsilon|$ , $u^\varepsilon$ develops a thin transition layer that separates the regions $\{u^\varepsilon \approx 1\}$ and $\{u^\varepsilon \approx 0\}$. Then, we make an analysis of the motion of interface: in a much slower time scale, the layer starts to propagate. As a consequence, as $\varepsilon \to 0$, the solution $u^\varepsilon$ converges to 0 in $\Omega_t^{(0)}$ and to 1 in $\Omega_t^{(1)}$, where $\Omega_t^{(0)}$ and $\Omega_t^{(1)}$ are sub-domains separated by an interface $\Gamma_t$, whose motion is driven by its mean curvature and a nonlocal drift term. We also show that the thickness of the transition layer is of order $\varepsilon$.

## 2.1 Introduction

Let us start by a short description of life-cycles of the cellular slime molds (*amoebae*). The cells feed and divide until exhaustion of food supply. Then, the amoebae aggregate to form a multicellular assembly called a slug. It migrates to a new location, then forms into a fruiting body, consisting of a stalk formed from dead amoebae and spores on the top (fruiting bodies that are visible to the naked eye are often referred to as mushrooms). Under suitable conditions of moisture, temperature, spores release new amoebae. The cycle then repeats itself.

It is known that the aggregation stage is mediated by *chemotaxis*, i.e. the tendency of biological individuals to direct their movements according to certain chemicals in their environment. The chemotactant (*acrasin*) is produced by the amoebae themselves and degraded by an extracellular enzyme (*acrasinase*). Moreover, acrasin and acrasinase react to form a complex whose concentration is assumed to be at a steady state. For more details on the biological background, we refer to [53], [62] or [38].

So the amoebae have a random motion analogous to diffusion coupled with an oriented chemotactic motion in the direction of a positive gradient of acrasin. In 1970, Keller and Segel [53] proposed the following system as a model to describe such movements leading to slime mold aggregation:

$$(KS) \quad \begin{cases} u_t &= \nabla \cdot (D_2 \nabla u) - \nabla \cdot (D_1 \nabla v), \\ v_t &= D_v \Delta v + f(v)u - k(v)v, \end{cases}$$

inside a closed region $\Omega$. Here, $u$, respectively $v$, denotes the concentration of amoebae, respectively of acrasin; $f(v)$ is the production rate of acrasin, and $k(v)$ the degradation rate of acrasin (due to acrasinase); $D_2 = D_2(u,v)$, respectively $D_1 = D_1(u,v)$, measures the vigor of the random motion of the amoebae, respectively the strength of the influence of the acrasin gradient on the flow of amoebae; $D_v$ is a positive and constant diffusion coefficient. The problem is completed by initial data $u_0$ and $v_0$ and, assuming that there is now flow of the amoebae or the acrasin across the boundary $\partial\Omega$, by homogeneous Neumann boundary conditions

$$\nabla u \cdot \nu = \nabla v \cdot \nu = 0 \quad \text{on } \partial\Omega \times (0, +\infty),$$

$\nu$ being the unit outward normal to $\partial\Omega$.

An often used simplified model is obtained as follows. By some receptor mechanism, cells do not measure the gradient of $v$ but of some $\chi(v)$, with a sensitive function $\chi$ satisfying $\chi' > 0$, so that $D_1(u,v) = u\chi'(v)$. By taking $D_2$, $f$ and $k$ as constant functions and using some rescaling arguments, the system reduces to

$$(KS') \quad \begin{cases} u_t &= d_u \Delta u - \nabla \cdot (u \nabla \chi(v)), \\ \tau v_t &= d_v \Delta v + u - \gamma v, \end{cases}$$

with $d_u$, $d_v$, $\tau$ and $\gamma$ some positive constants.

Many analysis of the Keller-Segel model for the aggregation process were proposed. Chemotaxis having some features of "negative diffusion", Nanjundiah [62] suggests that the whole population concentrates in a single point; we refer to this phenomenon as the *chemotactic collapse*. In mathematical terms, this means formation of a Dirac delta-type singularity in finite time. As a matter of fact, it turns out that the possibility of collapse depends upon the space dimension. In particular it never happens in the one-dimensional case whereas in two space dimensions, assuming radially symmetric situations, it only occurs if the total amoebae number is sufficiently large. The problem of global existence and blow up of solutions has been intensively studied; we refer in particular to [26], [67], [55], [49], [61], [43], [44].

In a different framework, Mimura and Tsujikawa [57], consider aggregating pattern-dynamics arising in the following chemotaxis model with growth:

$$(MT^\varepsilon) \quad \begin{cases} u_t &= \varepsilon^2 \Delta u - \varepsilon \nabla \cdot (u \nabla \chi(v)) + f(u), \\ \tau v_t &= \Delta v + u - \gamma v, \end{cases}$$

where $\varepsilon > 0$ is a small parameter. The function $f$ is cubic, 0 and 1 being its stable zeros, and satisfies $\int_0^1 f > 0$. In this model, the population is subjected to three competing effects: diffusion, growth and chemotaxis. The diffusion rate and the chemotactic rate are both very small compared with the growth rate. They observe that, in a first stage, internal layers — which describe the boundaries of aggregating regions — develop; in a second stage, the motion of the aggregating regions — which can be described by that of internal layers — takes place. The balance of the three effects (diffusion, growth and chemotaxis) makes the aggregation mechanism possible. Taking the limit $\varepsilon \to 0$, they formally derive the equation for the motion of the limit interface and study the stability of radially symmetric equilibrium solutions.

The purpose of this Chapter is to extend some of the results obtained by Bonami, Hilhorst, Logak and Mimura [17] about the singular limit of a variant of system $(MT^\varepsilon)$, where the second equation is elliptic ($\tau = 0$):

$$(P^\varepsilon) \quad \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)) + \dfrac{1}{\varepsilon^2} f_\varepsilon(u) & \text{in } \Omega \times (0, +\infty), \\ 0 = \Delta v + u - \gamma v & \text{in } \Omega \times (0, +\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, $\nu$ is the Euclidian unit normal vector exterior to $\partial\Omega$. We assume that $\gamma$ is a positive constant and that the nonlinearity $f_\varepsilon$ is given by

$$\begin{aligned} f_\varepsilon(u) &= u(1-u)(u - \frac{1}{2}) + \varepsilon \alpha u(1-u) \\ &=: f(u) + \varepsilon g(u), \end{aligned} \tag{2.1}$$

with $\alpha > 0$. The role of the function $g$ is to slightly break the balance of the two stable zeros. The sensitive function $\chi$ is smooth and satisfies $\chi'(v) > 0$ for $v > 0$.

We also assume that the initial datum satisfies $u_0 \in C^2(\overline{\Omega})$ and $u_0 \geq 0$. Throughout the present Chapter, we fix a constant $C_0 > 1$ that satisfies

$$\|u_0\|_{C^0(\overline{\Omega})} + \|\nabla u_0\|_{C^0(\overline{\Omega})} + \|\Delta u_0\|_{C^0(\overline{\Omega})} \leq C_0. \tag{2.2}$$

Furthermore we define the "initial interface" $\Gamma_0$ by

$$\Gamma_0 := \{x \in \Omega,\ u_0(x) = 1/2\}.$$

We suppose that $\Gamma_0$ is a $C^{2+\vartheta}$ hypersurface without boundary, for a $\vartheta \in (0,1)$, such that, $n$ being the Euclidian unit normal vector exterior to $\Gamma_0$,

$$\Gamma_0 \subset\subset \Omega \quad \text{and} \quad \nabla u_0(x) \cdot n(x) \neq 0 \quad \text{if } x \in \Gamma_0, \tag{2.3}$$

$$u_0 > 1/2 \quad \text{in } \Omega_0^{(1)}, \quad u_0 < 1/2 \quad \text{in } \Omega_0^{(0)}, \tag{2.4}$$

where $\Omega_0^{(1)}$ denotes the region enclosed by $\Gamma_0$ and $\Omega_0^{(0)}$ the region enclosed between $\partial\Omega$ and $\Gamma_0$.

The existence of a unique smooth solution to Problem $(P^\varepsilon)$ is proved in [17], Lemma 4.2:

**Lemma 2.1.1.** *There exists* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$*, there exists a unique solution* $(u^\varepsilon, v^\varepsilon)$ *to Problem* $(P^\varepsilon)$ *on* $\Omega \times [0, +\infty)$*, with* $0 \le u^\varepsilon \le C_0$ *on* $Q_T$*.*

To study the interfacial behavior associated with this model, it is useful to consider a formal asymptotic limit of Problem $(P^\varepsilon)$ as $\varepsilon \to 0$. Then the limit solution $u^0(x,t)$ will be a step function taking the value 1 on one side of the interface, and 0 on the other side. This sharp interface, which we will denote by $\Gamma_t$, obeys a law of motion, which can be obtained by formal analysis (see Section 2.2):

$$(P^0) \quad \begin{cases} V_n = -(N-1)\kappa + \dfrac{\partial \chi(v^0)}{\partial n} + \sqrt{2}\alpha & \text{on } \Gamma_t, \\ \Gamma_t\big|_{t=0} = \Gamma_0 & \\ -\Delta v^0 + \gamma v^0 = u^0 & \text{in } \Omega \times (0,T], \\ \dfrac{\partial v^0}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T], \end{cases}$$

where $V_n$ is the normal velocity of $\Gamma_t$ in the exterior direction, $\kappa$ the mean curvature at each point of $\Gamma_t$. We set $Q_T := \Omega \times [0,T]$ and for each $t \in [0,T]$, we define $\Omega_t^{(1)}$ as the region enclosed by the hypersurface $\Gamma_t$ and $\Omega_t^{(0)}$ as the region enclosed between $\partial\Omega$ and $\Gamma_t$. The step function $u^0$ is determined straightforwardly from $\Gamma_t$ by

$$u^0(x,t) = \begin{cases} 1 & \text{in } \Omega_t^{(1)} \\ 0 & \text{in } \Omega_t^{(0)} \end{cases} \quad \text{for } t \in [0,T]. \tag{2.5}$$

By a contraction fixed-point argument in suitable Hölder spaces, the well-posedness, locally in time, of the free boundary Problem $(P^0)$ is proved in [17], Theorem 2.1:

**Lemma 2.1.2.** *There exists a time* $T > 0$ *such that* $(P^0)$ *has a unique solution* $(v^0, \Gamma)$ *on* $[0,T]$*, with*

$$\Gamma = \bigcup_{0 \le t \le T} (\Gamma_t \times \{t\}) \in C^{2+\vartheta, \frac{2+\vartheta}{2}},$$

*and* $v^0|_\Gamma \in C^{2+\vartheta, \frac{2+\vartheta}{2}}$.

Bonami, Hilhorst, Logak and Mimura [17] have proved a motion of interface property; more precisely, for some prepared initial data, they show that $(u^\varepsilon, v^\varepsilon)$ converges to $(u^0, v^0)$ as $\varepsilon \to 0$, on the interval $(0,T)$. So the evolution of $\Gamma_t$ determines the aggregating patterns of the individuals. Here we consider the case of arbitrary initial data. Our first main result, Theorem 2.1.3, describes the profile of the solution after a very short initial period. It asserts that, given a virtually arbitrary initial datum $u_0$, the solution $u^\varepsilon$ quickly becomes close to 1 or 0, except in a small neighborhood of the initial interface $\Gamma_0$, creating a steep transition layer around $\Gamma_0$ (*generation of interface*). The time needed to develop such a transition layer, which we will denote by $t^\varepsilon$, is of order $\varepsilon^2 |\ln \varepsilon|$. The theorem then states that the solution $u^\varepsilon$ remains close to the step function $u^0$ on the time interval $[t^\varepsilon, T]$ (*motion of interface*). Moreover, as is clear from the estimates in the theorem, the "thickness" of the transition layer is of order $\varepsilon$.

**Theorem 2.1.3 (Generation and motion of interface).** *Let* $\eta \in (0, 1/4)$ *be arbitrary and set*

$$\mu = f'(1/2) = 1/4.$$

*Then there exist positive constants* $\varepsilon_0$ *and* $C$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$*, for all* $t^\varepsilon \le t \le T$*, where* $t^\varepsilon = \mu^{-1}\varepsilon^2|\ln\varepsilon|$*, we have*

$$u^\varepsilon(x,t) \in \begin{cases} [-\eta, 1+\eta] & \text{if } \; x \in \mathcal{N}_{C\varepsilon}(\Gamma_t) \\ [-\eta, \eta] & \text{if } \; x \in \Omega_t^{(0)} \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t) \\ [1-\eta, 1+\eta] & \text{if } \; x \in \Omega_t^{(1)} \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t), \end{cases} \tag{2.6}$$

*where* $\mathcal{N}_r(\Gamma_t) := \{x \in \Omega, dist(x, \Gamma_t) < r\}$ *denotes the* $r$*-neighborhood of* $\Gamma_t$*.*

**Corollary 2.1.4 (Convergence).** *As* $\varepsilon \to 0$*, the solution* $(u^\varepsilon, v^\varepsilon)$ *converges to* $(u^0, v^0)$ *everywhere in* $\bigcup_{0<t\le T}(\Omega_t^{(0 \; or \; 1)} \times \{t\})$*.*

The next theorem deals with the relation between the set $\Gamma_t^\varepsilon := \{x \in \Omega, u^\varepsilon(x,t) = 1/2\}$ and the solution $\Gamma_t$ of Problem $(P^0)$.

**Theorem 2.1.5 (Error estimate).** *There exists* $C > 0$ *such that*

$$\Gamma_t^\varepsilon \subset \mathcal{N}_{C\varepsilon}(\Gamma_t) \quad \text{for } \; 0 \le t \le T. \tag{2.7}$$

**Corollary 2.1.6 (Convergence of interface).** *There exists* $C > 0$ *such that*

$$d_{\mathcal{H}}(\Gamma_t^\varepsilon, \Gamma_t) \le C\varepsilon \quad \text{for } \; 0 \le t \le T, \tag{2.8}$$

*where*

$$d_{\mathcal{H}}(A, B) := \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$$

*denotes the Hausdorff distance between two compact sets* $A$ *and* $B$*. Consequently,* $\Gamma_t^\varepsilon \to \Gamma_t$ *as* $\varepsilon \to 0$*, uniformly in* $0 \le t \le T$*, in the sense of Hausdorff distance.*

As far as we know, the best thickness estimate in the literature was of order $\varepsilon|\ln\varepsilon|$ (see [20], [21]). We refer to a forthcoming article [51] in which an order $\varepsilon$ estimate is established for a Lotka-Volterra competition-diffusion system.

The organization of this Chapter is as follows. Section 2.2 is devoted to preliminaries: we recall the method of asymptotic expansions to derive the equation of the interface motion; we also recall a relaxed comparison principle used in [17]. In Section 2.3, we prove a generation of interface property. The corresponding sub- and super-solutions are constructed by modifying the solution of the ordinary differential equation $u_t = \varepsilon^{-2}f(u)$, obtained by neglecting diffusion and chemotaxis. In Section 2.4, in order to study the motion of interface, we construct a pair of sub- and super-solutions that rely on a related one-dimensional stationary problem. Finally, in Section 2.5, by fitting the two pairs of sub- and super-solutions into each other, we prove Theorem 2.1.3, Theorem 2.1.5 and theirs corollaries.

## 2.2 Some preliminaries

### 2.2.1 Formal derivation

A formal derivation of the equation of interface motion was given in [16]. Nevertheless we briefly present it in a slightly different way: we use arguments similar to those in [63] where the first two terms of the asymptotic expansion determine the interface equation in $(P^0)$, which can be regarded as the singular limit of $(P^\varepsilon)$ as $\varepsilon \to 0$. The observations we make here will help the rigorous analysis in later sections, in particular for the construction of sub- and super-solutions for the study of the motion of interface in Section 2.4.

Let $(u^\varepsilon, v^\varepsilon)$ be the solution of Problem $(P^\varepsilon)$. We recall that $\Gamma_t^\varepsilon := \{x \in \Omega, u^\varepsilon(x,t) = 1/2\}$ is the interface at time $t$ and call $\Gamma^\varepsilon := \bigcup_{t\geq 0}(\Gamma_t^\varepsilon \times \{t\})$ the interface. Let $\Gamma = \bigcup_{0\leq t\leq T}(\Gamma_t \times \{t\})$ be the solution of the limit geometric motion problem and let $\widetilde{d}$ be the signed distance function to $\Gamma$ defined by:

$$\widetilde{d}(x,t) = \begin{cases} \quad \operatorname{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^{(0)} \\ - \quad \operatorname{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^{(1)}, \end{cases} \tag{2.9}$$

where $\operatorname{dist}(x,\Gamma_t)$ is the distance from $x$ to the hypersurface $\Gamma_t$ in $\Omega$. We remark that $\widetilde{d} = 0$ on $\Gamma$ and that $|\nabla \widetilde{d}| = 1$ in a neighborhood of $\Gamma$. We then define

$$Q_T^{(1)} = \bigcup_{0<t\leq T} (\Omega_t^{(1)} \times \{t\}), \qquad Q_T^{(0)} = \bigcup_{0<t\leq T} (\Omega_t^{(0)} \times \{t\}).$$

We assume that the solution $u^\varepsilon$ has the expansions

$$u^\varepsilon(x,t) = \{0 \text{ or } 1\} + \varepsilon u_1(x,t) + \cdots \tag{2.10}$$

away from the interface $\Gamma$ (the outer expansion) and

$$u^\varepsilon(x,t) = U_0(x,t,\frac{\widetilde{d}(x,t)}{\varepsilon}) + \varepsilon U_1(x,t,\frac{\widetilde{d}(x,t)}{\varepsilon}) + \cdots \tag{2.11}$$

near $\Gamma$ (the inner expansion). Here, the functions $U_k(x,t,z)$, $k = 0,1,\cdots$, are defined for $x \in \overline{\Omega}$, $t \geq 0$, $z \in \mathbb{R}$. The stretched space variable $\xi := \widetilde{d}(x,t)/\varepsilon$ gives exactly the right spatial scaling to describe the rapid transition between the regions $\{u^\varepsilon \approx 1\}$ and $\{u^\varepsilon \approx 0\}$. We use the normalization conditions

$$U_0(x,t,0) = 1/2, \quad U_k(x,t,0) = 0,$$

for all $k \geq 1$. The matching conditions between the outer and the inner expansion are given by

$$\begin{aligned} U_0(x,t,+\infty) &= 0, & U_k(x,t,+\infty) &= 0, \\ U_0(x,t,-\infty) &= 1, & U_k(x,t,-\infty) &= 0, \end{aligned} \tag{2.12}$$

for all $k \geq 1$. We also assume that the solution $v^\varepsilon$ has the expansion

$$v^\varepsilon(x,t) = v_0(x,t) + \varepsilon v_1(x,t) + \cdots \tag{2.13}$$

in $\Omega \times (0,T)$.

We now substitute the inner expansion (2.11) and the expansion (2.13) into the parabolic equation of $(P^\varepsilon)$ and collect the $\varepsilon^{-2}$ terms. We omit the calculations and, using $|\nabla \tilde{d}| = 1$ near $\Gamma_t$, the normalization and matching conditions, we deduce that $U_0(x,t,z) = U_0(z)$ is the unique solution of the stationary problem

$$\begin{cases} {U_0}'' + f(U_0) = 0 \\ U_0(-\infty) = 1, \quad U_0(0) = 1/2, \quad U_0(+\infty) = 0. \end{cases} \tag{2.14}$$

This solution represents the first approximation of the profile of a transition layer around the interface observed in the stretched coordinates. Recalling that the nonlinearity is given by $f(u) = u(1-u)(u-1/2)$, we have

$$U_0(z) = \frac{1}{2}\big(1 - \tanh\frac{z}{2\sqrt{2}}\big) = \frac{e^{-z/\sqrt{2}}}{1+e^{-z/\sqrt{2}}}\ . \tag{2.15}$$

We claim that $U_0$ has the following properties.

**Lemma 2.2.1.** *There exist positive constants $C$ and $\lambda$ such that the following estimates hold.*

$$\begin{aligned} 0 < \quad U_0(z) \quad &\le Ce^{-\lambda|z|} \quad \textit{for } z \ge 0,\\ 0 < 1 - U_0(z) \quad &\le Ce^{-\lambda|z|} \quad \textit{for } z \le 0. \end{aligned}$$

*In addition, $U_0$ is a strictly decreasing function and*

$$|{U_0}'(z)| + |{U_0}''(z)| \le Ce^{-\lambda|z|} \quad \textit{for } z \in \mathbb{R}.$$

Next we collect the $\varepsilon^{-1}$ terms. Since $U_0$ depends only on the variable $z$, we have $\nabla U_{0z} = 0$ which, combined with the fact that $|\nabla \widetilde{d}| = 1$ near $\Gamma_t$, yields

$$U_{1zz} + f'(U_0)U_1 = {U_0}'(\widetilde{d}_t - \Delta\widetilde{d} + \nabla\widetilde{d}\cdot\nabla\chi(v^0)) - g(U_0), \tag{2.16}$$

a linearized problem for (2.14). The solvability condition for the above equation, which can be seen as a variant of the Fredholm alternative, plays the key role for deriving the equation of interface motion. It is is given by

$$\int_{\mathbf{R}} \Big[{U_0}'^2(z)(\widetilde{d}_t - \Delta\widetilde{d} + \nabla\widetilde{d}\cdot\nabla\chi(v^0))(x,t) - g(U_0(z)){U_0}'(z)\Big]dz = 0,$$

for all $(x,t) \in Q_T$. By the definition of $g$ in (2.1), we compute

$$\int_{\mathbb{R}} g(U_0(z)){U_0}'(z)dz = -\int_0^1 g(u)du = -\alpha/6,$$

whereas the equality (2.15) yields

$$\int_{\mathbb{R}} {U_0}'^2(z)dz = \frac{1}{\sqrt{2}}\int_0^{+\infty}\frac{u}{1+u^4}du = 1/6\sqrt{2}.$$

Combining the above expressions, we obtain

$$\Big(\widetilde{d}_t - \Delta\widetilde{d} + \nabla\widetilde{d}\cdot\nabla\chi(v^0)\Big)(x,t) = -\sqrt{2}\alpha. \tag{2.17}$$

Since $\nabla\widetilde{d}\ (= \nabla_x\,\widetilde{d}(x,t))$ coincides with the outward normal unit vector to the hypersurface $\Gamma_t$, we have $\widetilde{d}_t(x,t) = -V_n$, where $V_n$ is the normal velocity of the interface $\Gamma_t$. It is also known that the mean curvature $\kappa$ of the interface is equal to $\Delta\widetilde{d}/(N-1)$. Thus the above equation reads as

$$V_n = -(N-1)\kappa + \frac{\partial\chi(v^0)}{\partial n} + \sqrt{2}\alpha \quad \text{on } \Gamma_t, \tag{2.18}$$

that is the equation of interface motion in $(P^0)$. Summarizing, under the assumption that the solution $u^\varepsilon$ of Problem $(P^\varepsilon)$ satisfies

$$u^\varepsilon \to \begin{cases} 1 & \text{in } Q_T^{(1)} \\ 0 & \text{in } Q_T^{(0)} \end{cases} \quad \text{as } \varepsilon \to 0,$$

we have formally showed that the boundary $\Gamma_t$ between $\Omega_t^{(0)}$ and $\Omega_t^{(1)}$ moves according to the law (2.18).

One can note that, using the equality (2.15), we clearly have $\sqrt{2}\alpha U_0{}' + g(U_0) \equiv 0$ so that, substituting (2.17) into (2.16) yields $U_1 \equiv 0$.

### 2.2.2 A comparison principle

The definition of sub- and super-solutions is the one proposed in [17].

**Definition 2.2.2.** *Let $(u_\varepsilon^-, u_\varepsilon^+)$ be two smooth functions with $u_\varepsilon^- \leq u_\varepsilon^+$ on $Q_T$ and*

$$\frac{\partial u_\varepsilon^-}{\partial \nu} \leq 0 \leq \frac{\partial u_\varepsilon^+}{\partial \nu} \quad \text{on } \partial\Omega \times (0,T).$$

*By definition, $(u_\varepsilon^-, u_\varepsilon^+)$ is a pair of sub- and super-solutions if, for any $v^\varepsilon$ which satisfies*

$$\begin{cases} u_\varepsilon^- \leq -\Delta v^\varepsilon + \gamma v^\varepsilon \leq u_\varepsilon^+ & \text{on } Q_T, \\ \dfrac{\partial v^\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{2.19}$$

*we have*

$$L_{v^\varepsilon}[u_\varepsilon^-] \leq 0 \leq L_{v^\varepsilon}[u_\varepsilon^+],$$

*where the operator $L_{v^\varepsilon}$ is defined by*

$$L_{v^\varepsilon}[\phi] = \phi_t - \Delta\phi + \nabla \cdot (\phi \nabla \chi(v^\varepsilon)) - \frac{1}{\varepsilon^2} f_\varepsilon(\phi).$$

As proved in [17], the following comparison principle holds.

**Proposition 2.2.3.** *Let a pair of sub- and super-solutions be given. Assume that, for all $x \in \Omega$,*

$$u_\varepsilon^-(x,0) \leq u_0(x) \leq u_\varepsilon^+(x,0).$$

*Then, if we denote by $(u^\varepsilon, v^\varepsilon)$ the solution of Problem $(P^\varepsilon)$, the function $u^\varepsilon$ satisfies, for all $(x,t) \in Q_T$,*

$$u_\varepsilon^-(x,t) \leq u^\varepsilon(x,t) \leq u_\varepsilon^+(x,t).$$

## 2.3 Generation of interface

In this section we study the rapid formation of internal layers in a neighborhood of $\Gamma_0 = \{x \in \Omega, u_0(x) = 1/2\}$ within a very short time interval of order $\varepsilon^2 |\ln \varepsilon|$. In the sequel, we shall always assume that $0 < \eta < 1/4$. The main result of this section is the following.

**Theorem 2.3.1.** *Let $\eta$ be arbitrary and define $\mu$ as the derivative of $f(u)$ at the unstable equilibrium $u = 1/2$, that is*

$$\mu = f'(1/2) = 1/4. \tag{2.20}$$

*Then there exist positive constants $\varepsilon_0$ and $M_0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

- *for all $x \in \Omega$,*

$$-\eta \leq u^\varepsilon(x, \mu^{-1}\varepsilon^2 |\ln \varepsilon|) \leq 1 + \eta, \tag{2.21}$$

- *for all $x \in \Omega$ such that $|u_0(x) - \frac{1}{2}| \geq M_0\varepsilon$, we have that*

$$\text{if} \quad u_0(x) \geq 1/2 + M_0\varepsilon \quad \text{then} \quad u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln \varepsilon|) \geq 1 - \eta, \tag{2.22}$$

$$\text{if} \quad u_0(x) \leq 1/2 - M_0\varepsilon \quad \text{then} \quad u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln \varepsilon|) \leq \eta. \tag{2.23}$$

The above theorem will be proved by constructing a suitable pair of sub and super-solutions.

### 2.3.1 The perturbed bistable ordinary differential equation

We first consider a slightly perturbed nonlinearity:

$$f_\delta(u) = f(u) + \delta,$$

where $\delta$ is any constant. For $|\delta|$ small enough, this function is still cubic and bistable; more precisely, we claim that it has the following properties.

**Lemma 2.3.2.** *Let $\delta_0 > 0$ be small enough. Then, for all $\delta \in (-\delta_0, \delta_0)$,*

- *$f_\delta$ has exactly three zeros, namely $\alpha_-(\delta) < a(\delta) < \alpha_+(\delta)$. More precisely,*

$$f_\delta(u) = (u - \alpha_-(\delta))(\alpha_+(\delta) - u)(u - a(\delta)), \tag{2.24}$$

*and there exists a positive constant $C$ such that*

$$|\alpha_-(\delta)| + |a(\delta) - 1/2| + |\alpha_+(\delta) - 1| \leq C|\delta|. \tag{2.25}$$

- *We have that*

$$\begin{array}{lll} f_\delta & \text{is strictly positive in} & (-\infty, \alpha_-(\delta)) \cup (a(\delta), \alpha_+(\delta)), \\ f_\delta & \text{is strictly negative in} & (\alpha_-(\delta), a(\delta)) \cup (\alpha_+(\delta), +\infty). \end{array} \tag{2.26}$$

- *Set*

$$\mu(\delta) := f'_\delta(a(\delta)) = f'(a(\delta)),$$

*then there exists a positive constant, which we denote again by $C$, such that*

$$|\mu(\delta) - \mu| \leq C|\delta|. \tag{2.27}$$

In order to construct a pair of sub and super-solutions for Problem $(P^\varepsilon)$ we define $Y(\tau, \xi; \delta)$ as the solution of the ordinary differential equation

$$\begin{cases} Y_\tau(\tau, \xi; \delta) &= f_\delta(Y(\tau, \xi; \delta)) \quad \text{for} \quad \tau > 0, \\ Y(0, \xi; \delta) &= \xi, \end{cases} \tag{2.28}$$

for $\delta \in (-\delta_0, \delta_0)$ and $\xi \in (-2C_0, 2C_0)$, where $C_0$ has been chosen in (2.2). We present below basic properties of $Y$.

**Lemma 2.3.3.** *We have* $Y_\xi > 0$*, for all* $\xi \in (-2C_0, 2C_0) \setminus \{\alpha_-(\delta), a(\delta), \alpha_+(\delta)\}$*, all* $\delta \in (-\delta_0, \delta_0)$ *and all* $\tau > 0$*. Furthermore,*

$$Y_\xi(\tau, \xi; \delta) = \frac{f_\delta(Y(\tau, \xi; \delta))}{f_\delta(\xi)}.$$

**Proof.** We differentiate (2.28) with respect to $\xi$ to obtain

$$\begin{cases} Y_{\xi\tau} = Y_\xi f'(Y), \\ Y_\xi(0, \xi; \delta) = 1, \end{cases}$$

which is integrated as follows:

$$Y_\xi(\tau, \xi; \delta) = \exp\Big[\int_0^\tau f'(Y(s, \xi; \delta))ds\Big] > 0. \tag{2.29}$$

Then differentiating (2.28) with respect to $\tau$, we obtain

$$\begin{cases} Y_{\tau\tau} = Y_\tau f'(Y), \\ Y_\tau(0, \xi; \delta) = f_\delta(\xi), \end{cases}$$

which in turn implies

$$Y_\tau(\tau, \xi; \delta) = f_\delta(\xi) \exp\Big[\int_0^\tau f'(Y(s, \xi; \delta))ds\Big],$$

which enables to conclude. □

We define a function $A(\tau, \xi; \delta)$ by

$$A(\tau, \xi; \delta) = \frac{f'(Y(\tau, \xi; \delta)) - f'(\xi)}{f_\delta(\xi)}. \tag{2.30}$$

**Lemma 2.3.4.** *We have, for all* $\xi \in (-2C_0, 2C_0) \setminus \{\alpha_-(\delta), a(\delta), \alpha_+(\delta)\}$*, all* $\delta \in (-\delta_0, \delta_0)$ *and all* $\tau > 0$*,*

$$A(\tau, \xi; \delta) = \int_0^\tau f''(Y(s, \xi; \delta))Y_\xi(s, \xi; \delta)ds.$$

**Proof.** We differentiate the equality of Lemma 2.3.3 with respect to $\xi$ to obtain

$$Y_{\xi\xi}(\tau, \xi; \delta) = A(\tau, \xi; \delta)Y_\xi(\tau, \xi; \delta). \tag{2.31}$$

Then differentiating (2.29) with respect to $\xi$ yields

$$Y_{\xi\xi} = Y_\xi \int_0^\tau f''(Y(s, \xi; \delta))Y_\xi(s, \xi; \delta)ds.$$

These two last results complete the proof of Lemma 2.3.4. □

Next we prove estimates on the growth of $Y$, $A$ and theirs derivatives. We first consider the case where the initial value $\xi$ is far from the stable equilibria, more precisely when it lies between $\eta$ and $1 - \eta$.

**Lemma 2.3.5.** *Let* $\eta$ *be arbitrary. Then there exist positive constants* $\delta_0 = \delta_0(\eta)$*,* $\widetilde{C}_1 = \widetilde{C}_1(\eta)$*,* $\tilde{C}_2 = \tilde{C}_2(\eta)$ *and* $C_3 = C_3(\eta)$ *such that, for all* $\delta \in (-\delta_0, \delta_0)$*, for all* $\tau > 0$*,*

- *if* $\xi \in (a(\delta), 1-\eta)$ *then, for every* $\tau > 0$ *such that* $Y(\tau, \xi; \delta)$ *remains in the interval* $(a(\delta), 1-\eta)$, *we have*
$$\tilde{C}_1 e^{\mu(\delta)\tau} \leq Y_\xi(\tau, \xi; \delta) \leq \tilde{C}_2 e^{\mu(\delta)\tau}, \tag{2.32}$$
*and*
$$|A(\tau, \xi; \delta)| \leq C_3(e^{\mu(\delta)\tau} - 1); \tag{2.33}$$
- *if* $\xi \in (\eta, a(\delta))$ *then, for every* $\tau > 0$ *such that* $Y(\tau, \xi; \delta)$ *remains in the interval* $(\eta, a(\delta))$, (2.32) *and* (2.33) *hold as well.*

**Proof.** We take $\xi \in (a(\delta), 1-\eta)$ and suppose that for $s \in (0, \tau)$, $Y(s, \xi; \delta)$ remains in the interval $(a(\delta), 1-\eta)$. Integrating the equality
$$\frac{Y_\tau(s, \xi; \delta)}{f_\delta(Y(s, \xi; \delta))} = 1$$
from 0 to $\tau$ and using the change of variable $q = Y(s, \xi; \delta)$ leads to
$$\int_\xi^{Y(\tau,\xi;\delta)} \frac{dq}{f_\delta(q)} = \tau. \tag{2.34}$$
Moreover, the equality in Lemma 2.3.3 enables to write
$$\begin{aligned}
\ln Y_\xi(\tau, \xi; \delta) &= \int_\xi^{Y(\tau,\xi;\delta)} \frac{f'(q)}{f_\delta(q)} dq \\
&= \int_\xi^{Y(\tau,\xi;\delta)} [\frac{f'(a(\delta))}{f_\delta(q)} + \frac{f'(q) - f'(a(\delta))}{f_\delta(q)}] dq \\
&= \mu(\delta)\tau + \int_\xi^{Y(\tau,\xi;\delta)} h_\delta(q) dq,
\end{aligned} \tag{2.35}$$
where
$$h_\delta(q) = \frac{f'(q) - f'(a(\delta))}{f_\delta(q)}.$$
In view of (2.27), respectively (2.25), we can choose $\delta_0 = \delta_0(\eta) > 0$ small enough so that, for all $\delta \in [-\delta_0, \delta_0]$, we have $\mu(\delta) \geq \mu/2 > 0$, respectively $(a(\delta), 1-\eta] \subset (a(\delta), \alpha_+(\delta))$. Since
$$h_\delta(q) \to \frac{f''_\delta(a(\delta))}{f'_\delta(a(\delta))} = \frac{f''(a(\delta))}{f'(a(\delta))} \quad \text{as} \quad q \to a(\delta),$$
we see that the function $(q, \delta) \mapsto h_\delta(q)$ is continuous in the compact region $\{\, |\delta| \leq \delta_0,\ a(\delta) \leq q \leq 1-\eta \,\}$. It follows that $|h_\delta(q)|$ is bounded by a constant $H = H(\eta)$ as $(q, \delta)$ varies in this region. Since $|Y(\tau, \xi; \delta) - \xi|$ takes its values in the interval $[0, 1-\eta-a(\delta)] \subset [0, 1]$, it follows from (2.35) that
$$\mu(\delta)\tau - H \leq \ln Y_\xi(\tau, \xi; \delta) \leq \mu(\delta)\tau + H,$$
which, in turn, proves (2.32). Next Lemma 2.3.4 and (2.32) yield
$$\begin{aligned}
|A(\tau, \xi; \delta)| &\leq \|f''\|_{L^\infty(0,1)} \int_0^\tau \tilde{C}_2 e^{\mu(\delta)s} ds \\
&\leq \frac{\|f''\|_{L^\infty(0,1)} \tilde{C}_2}{\mu(\delta)} (e^{\mu(\delta)\tau} - 1) \\
&\leq \frac{2}{\mu} \|f''\|_{L^\infty(0,1)} \tilde{C}_2 (e^{\mu(\delta)\tau} - 1),
\end{aligned}$$

which completes the proof of (2.33). The case where $\xi$ and $Y(\tau,\xi;\delta)$ are in $(\eta, a(\delta))$ is similar and omitted. □

**Corollary 2.3.6.** *Let $\eta$ be arbitrary. Then there exist positive constants $\delta_0 = \delta_0(\eta)$, $C_1 = C_1(\eta)$ and $C_2 = C_2(\eta)$ such that, for all $\delta \in (-\delta_0, \delta_0)$, for all $\tau > 0$,*

- *if $\xi \in (a(\delta), 1-\eta)$ then, for every $\tau > 0$ such that $Y(\tau,\xi;\delta)$ remains in the interval $(a(\delta), 1-\eta)$, we have*

$$C_1 e^{\mu(\delta)\tau}(\xi - a(\delta)) \leq Y(\tau,\xi;\delta) - a(\delta) \leq C_2 e^{\mu(\delta)\tau}(\xi - a(\delta)), \tag{2.36}$$

- *if $\xi \in (\eta, a(\delta))$ then, for every $\tau > 0$ such that $Y(\tau,\xi;\delta)$ remains in the interval $(\eta, a(\delta))$, we have*

$$C_2 e^{\mu(\delta)\tau}(\xi - a(\delta)) \leq Y(\tau,\xi;\delta) - a(\delta) \leq C_1 e^{\mu(\delta)\tau}(\xi - a(\delta)). \tag{2.37}$$

**Proof.** In view of (2.27), respectively (2.25), we can choose $\delta_0 = \delta_0(\eta) > 0$ small enough so that, for all $\delta \in [-\delta_0, \delta_0]$, we have $\mu(\delta) \geq \mu/2 > 0$, respectively $(a(\delta), 1-\eta] \subset (a(\delta), \alpha_+(\delta))$. Since

$$\frac{f_\delta(q)}{q - a(\delta)} \to \mu(\delta) \quad \text{as} \quad q \to a(\delta),$$

it follows that $(q,\delta) \mapsto f_\delta(q)/(q - a(\delta))$ is a strictly positive and continuous function in the compact region $\{\, |\delta| \leq \delta_0,\ a(\delta) \leq q \leq 1-\eta \,\}$, which insures the existence of constants $B_1 = B_1(\eta) > 0$ and $B_2 = B_2(\eta) > 0$ such that, for all $q \in (a(\delta), 1-\eta)$, all $\delta \in (-\delta_0, \delta_0)$,

$$B_1(q - a(\delta)) \leq f_\delta(q) \leq B_2(q - a(\delta)). \tag{2.38}$$

We write the inequalities (2.38) for $q = Y(\tau,\xi;\delta) \in (a(\delta), 1-\eta)$ and then for $q = \xi \in (a(\delta), 1-\eta)$, which, together with Lemma 2.3.3, implies that

$$\frac{B_1}{B_2}(Y(\tau,\xi;\delta) - a(\delta)) \leq (\xi - a(\delta))Y_\xi(\tau,\xi;\delta) \leq \frac{B_2}{B_1}(Y(\tau,\xi;\delta) - a(\delta)).$$

In view of (2.32), this completes the proof of inequalities (2.36). The proof of (2.37) is similar and omitted. □

We now present estimates in the case that the initial value $\xi$ is smaller than $\eta$ or larger than $1-\eta$.

**Lemma 2.3.7.** *Let $\eta$ and $M > 0$ be arbitrary. Then there exist positive constants $\delta_0 = \delta_0(\eta, M)$ and $C_4 = C_4(M)$ such that, for all $\delta \in (-\delta_0, \delta_0)$,*

- *if $\xi \in [1-\eta, 1+M]$, then, for all $\tau > 0$, $Y(\tau,\xi;\delta)$ remains in the interval $[1-\eta, 1+M]$ and*

$$|A(\tau,\xi;\delta)| \leq C_4 \tau \quad \textit{for} \quad \tau > 0\,; \tag{2.39}$$

- *if $\xi \in [-M, \eta]$, then, for all $\tau > 0$, $Y(\tau,\xi;\delta)$ remains in the interval $[-M, \eta]$ and (2.39) holds as well.*

**Proof.** Since the two statements can be treated in the same way, we will only prove the former. The fact that $Y(\tau,\xi;\delta)$, the solution of the ordinary differential equation (2.28), remains in the interval $[1-\eta, 1+M]$ directly follows from the bistable properties of $f_\delta$, or, more precisely, from the sign conditions $f_\delta(1-\eta) > 0$, $f_\delta(1+M) < 0$ valid if $\delta_0 = \delta_0(\eta, M)$ is small enough.

To prove (2.39), suppose first that $\xi \in [\alpha_+(\delta), 1+M]$. By the above arguments, $Y(\tau,\xi;\delta)$ remains in this interval. Moreover $f'$ is negative in this interval. Hence, it follows from (2.29) that $Y_\xi(\tau,\xi;\delta) \leq 1$. We then use Lemma 2.3.4 to deduce that

$$|A(\tau,\xi;\delta)| \leq \|f''\|_{L^\infty(-M,1+M)}\tau =: C_4\tau.$$

The case $\xi \in [1-\eta, \alpha_+(\delta)]$ being similar, this completes the proof of the lemma. □

Now we choose the constant $M$ in the above lemma sufficiently large so that $[-2C_0, 2C_0] \subset [-M, 1+M]$, and fix $M$ hereafter. Therefore the constant $C_4$ is fixed as well. Using the fact that $\tau \mapsto \tau(e^{\mu(\delta)\tau}-1)^{-1}$ is uniformly bounded for $\delta \in (-\delta_0, \delta_0)$, with $\delta_0$ small enough (see (2.27)), and for $\tau > 0$, one can easily deduce from (2.33) and (2.39) the following general estimate.

**Lemma 2.3.8.** *Let $\eta$ be arbitrary and let $C_0$ be the constant defined in* (2.2)*. Then there exist positive constants $\delta_0 = \delta_0(\eta)$, $C_5 = C_5(\eta)$ such that, for all $\delta \in (-\delta_0, \delta_0)$, all $\xi \in (-2C_0, 2C_0)$ and all $\tau > 0$,*

$$|A(\tau,\xi;\delta)| \leq C_5(e^{\mu(\delta)\tau}-1).$$

### 2.3.2 Construction of sub and super-solutions

We now use $Y$ to construct a pair of sub- and super-solutions for the proof of the generation of interface theorem. We set

$$w_\varepsilon^\pm(x,t) = Y\Big(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 r(\pm\varepsilon G, \frac{t}{\varepsilon^2}); \pm\varepsilon G\Big), \tag{2.40}$$

where the constant $G$ is defined by

$$G = \sup_{u\in[-2C_0,2C_0]} |g(u)|,$$

and the function $r(\delta,\tau)$ is given by

$$r(\delta,\tau) = C_6(e^{\mu(\delta)\tau}-1).$$

For simplicity, we make the following additional assumption:

$$\frac{\partial u_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{2.41}$$

In the general case where (2.41) does not necessary hold, we have to slightly modify $w_\varepsilon^\pm$ near the boundary $\partial\Omega$. This will be discussed in the next remark.

**Lemma 2.3.9.** *There exist positive constants $\varepsilon_0$ and $C_6$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the functions $w_\varepsilon^-$ and $w_\varepsilon^+$ are respectively sub- and super-solutions for Problem $(P^\varepsilon)$, in the domain*

$$\big\{(x,t) \in Q_T,\ x \in \Omega,\ 0 \leq t \leq \mu^{-1}\varepsilon^2|\ln\varepsilon|\big\}.$$

**Proof.** First, (2.41) implies the homogeneous Neumann boundary condition

$$\frac{\partial w_\varepsilon^\pm}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,+\infty).$$

Let $v^\varepsilon$ be such that

$$\begin{cases} w_\varepsilon^- \leq -\Delta v^\varepsilon + \gamma v^\varepsilon \leq w_\varepsilon^+ \\ \dfrac{\partial v^\varepsilon}{\partial \nu} = 0. \end{cases} \tag{2.42}$$

According to Definition 2.2.2, what we have to show is

$$L_{v^\varepsilon}[w_\varepsilon^+] := (w_\varepsilon^+)_t - \Delta w_\varepsilon^+ + \nabla \cdot (w_\varepsilon^+ \nabla \chi(v^\varepsilon)) - \frac{1}{\varepsilon^2} f_\varepsilon(w_\varepsilon^+) \geq 0.$$

Let $C_6$ be a positive constant which does not depend on $\varepsilon$. If $\varepsilon_0$ is sufficiently small, we note that $\pm\varepsilon G \in (-\delta_0, \delta_0)$ and that, in the range $0 \leq t \leq \mu^{-1}\varepsilon^2|\ln \varepsilon|$,

$$|\varepsilon^2 C_6(e^{\mu(\pm\varepsilon G)t/\varepsilon^2} - 1)| \leq \varepsilon^2 C_6(\varepsilon^{-\mu(\pm\varepsilon G)/\mu} - 1) \leq C_0,$$

which implies that

$$u_0(x) \pm \varepsilon^2 r(\pm\varepsilon G, t/\varepsilon^2) \in (-2C_0, 2C_0).$$

These observations allow us to use the results of the previous subsection with $\tau = t/\varepsilon^2$, $\xi = u_0(x) + \varepsilon^2 r(\varepsilon G, t/\varepsilon^2)$ and $\delta = \varepsilon G$. In particular, setting $F_1 := \|f'\|_{L^\infty(-2C_0, 2C_0)}$, this implies, using (2.29), that

$$e^{-F_1 T} \leq Y_\xi \leq e^{F_1 T}.$$

Straightforward computations yield

$$\begin{aligned} L_{v^\varepsilon}[w_\varepsilon^+] &= \frac{1}{\varepsilon^2} Y_\tau + C_6 \mu(\varepsilon G) e^{\mu(\varepsilon G)t/\varepsilon^2} Y_\xi - \Delta u_0 Y_\xi - |\nabla u_0|^2 Y_{\xi\xi} \\ &\quad + Y_\xi \nabla u_0 \cdot \nabla \chi(v^\varepsilon) + Y \Delta \chi(v^\varepsilon) - \frac{1}{\varepsilon^2} f(Y) - \frac{1}{\varepsilon} g(Y), \end{aligned}$$

and then, in view of the ordinary differential equation (2.28), $\varepsilon G$ playing the role of $\delta$,

$$\begin{aligned} L_{v^\varepsilon}[w_\varepsilon^+] = \frac{1}{\varepsilon}\big[G - g(Y)\big] + Y_\xi \quad &\Big[C_6 \mu(\varepsilon G) e^{\mu(\varepsilon G)t/\varepsilon^2} - \Delta u_0 \\ &- \frac{Y_{\xi\xi}}{Y_\xi}|\nabla u_0|^2 + \nabla u_0 \cdot \nabla \chi(v^\varepsilon) + \frac{Y}{Y_\xi} \Delta \chi(v^\varepsilon)\Big]. \end{aligned}$$

By the definition of $G$ the first term is positive. Now, using the choice of $C_0$ in (2.2), the fact that $Y_{\xi\xi}/Y_\xi = A$ and Lemma 2.3.8, we obtain, for a $C_5$ independent of $\varepsilon$,

$$\begin{aligned} L_{v^\varepsilon}[w_\varepsilon^+] \geq \quad & Y_\xi \Big[C_6 \mu(\varepsilon G) e^{\mu(\varepsilon G)t/\varepsilon^2} - C_0 - C_5(e^{\mu(\varepsilon G)t/\varepsilon^2} - 1)C_0^2 \\ & - C_0|\nabla \chi(v^\varepsilon)| - 2C_0 e^{F_1 T}|\Delta \chi(v^\varepsilon)|\Big]. \end{aligned}$$

Moreover, the inequalities in (2.42) can be written as $-\Delta v^\varepsilon + \gamma v^\varepsilon = h^\varepsilon$, with $-2C_0 \leq h^\varepsilon \leq 2C_0$, so that the standard theory of elliptic equations gives a uniform bound $M$ for $|v^\varepsilon|$, $|\nabla v^\varepsilon|$ and $|\Delta v^\varepsilon|$. Hence, using the smoothness of $\chi$, we have a uniform bound $M'$ for $|\nabla \chi(v^\varepsilon)|$ and $|\Delta \chi(v^\varepsilon)|$. It follows that

$$L_{v^\varepsilon}[w_\varepsilon^+] \geq Y_\xi \Big[(C_6 \mu(\varepsilon G) - C_5 C_0^2) e^{\mu(\varepsilon G)t/\varepsilon^2} - C_0 + C_5 C_0^2 - C_0 M' - 2C_0 e^{F_1 T} M'\Big].$$

Hence, in view of (2.27), we have, for $\varepsilon_0$ small enough (recall that $Y_\xi > 0$),

$$L w_\varepsilon^+ \geq Y_\xi \Big[(C_6 \frac{1}{2}\mu - C_5 C_0^2) - C_0 - C_0 M' - 2C_0 e^{F_1 T} M'\Big] \geq 0,$$

for $C_6$ large enough, so that $w_\varepsilon^+$ is a super-solution for Problem $(P^\varepsilon)$. We omit the proof that $w_\varepsilon^-$ is a sub-solution. □

Now, since $w_\varepsilon^\pm(x,0) = Y(0, u_0(x); \pm\varepsilon G) = u_0(x)$, the comparison principle set in Proposition 2.2.3 asserts that, for all $x \in \Omega$, for all $0 \le t \le \mu^{-1}\varepsilon^2 |\ln \varepsilon|$,

$$w_\varepsilon^-(x,t) \le u^\varepsilon(x,t) \le w_\varepsilon^+(x,t). \tag{2.43}$$

**Remark 2.3.10.** *In the more general case where* (2.41) *is not necessarily valid, one can proceed in the following way: in view of* (2.3) *and* (2.4) *there exist positive constants* $d_1$ *and* $\rho$ *such that* $u_0(x) \le 1/2 - \rho$ *if* $d(x, \partial\Omega) \le d_1$. *Let* $\chi$ *be a smooth cut-off function defined on* $[0, +\infty)$ *such that* $0 \le \chi \le 1$, $\chi(0) = \chi'(0) = 0$ *and* $\chi(z) = 1$ *for* $z \ge d_1$. *Then define*

$$u_0^+(x) = \chi(d(x,\partial\Omega))u_0(x) + (1 - \chi(d(x,\partial\Omega))(1/2 - \rho),$$

$$u_0^-(x) = \chi(d(x,\partial\Omega))u_0(x) + (1 - \chi(d(x,\partial\Omega)) \min_{x\in\overline{\Omega}} u_0(x).$$

*Clearly,* $u_0^- \le u_0 \le u_0^+$, *and both* $u_0^\pm$ *satisfy* (2.41). *Now we set*

$$\tilde{w}_\varepsilon^\pm(x,t) = Y\Big(\frac{t}{\varepsilon^2}, u_0^\pm(x) \pm \varepsilon^2 r(\pm\varepsilon G, \frac{t}{\varepsilon^2}); \pm\varepsilon G\Big).$$

*Then the same argument as in Lemma 2.3.9 shows that* $(\tilde{w}_\varepsilon^-, \tilde{w}_\varepsilon^+)$ *is a pair of sub and super-solutions for Problem* $(P^\varepsilon)$. *Furthermore, since* $\tilde{w}_\varepsilon^-(x,0) = u_0^-(x) \le u_0(x) \le u_0^+(x) = \tilde{w}_\varepsilon^+(x,0)$, *Proposition 2.2.3 asserts that, for all* $x \in \Omega$, *for all* $0 \le t \le \mu^{-1}\varepsilon^2|\ln\varepsilon|$, *we have* $\tilde{w}_\varepsilon^-(x,t) \le u^\varepsilon(x,t) \le \tilde{w}_\varepsilon^+(x,t)$. □

### 2.3.3 Proof of Theorem 2.3.1

In order to prove Theorem 2.3.1 we first present a key estimate on the function $Y$ after a time of order $\tau \sim |\ln\varepsilon|$.

**Lemma 2.3.11.** *Let* $\eta$ *be arbitrary; there exist positive constants* $\varepsilon_0 = \varepsilon_0(\eta)$ *and* $C_7 = C_7(\eta)$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$,

- *for all* $\xi \in (-2C_0, 2C_0)$,

$$-\eta \le Y(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon G) \le 1 + \eta, \tag{2.44}$$

- *for all* $\xi \in (-2C_0, 2C_0)$ *such that* $|\xi - \frac{1}{2}| \ge C_7\varepsilon$, *we have that*

$$\text{if} \quad \xi \ge 1/2 + C_7\varepsilon \quad \text{then} \quad Y(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon G) \ge 1 - \eta, \tag{2.45}$$

$$\text{if} \quad \xi \le 1/2 - C_7\varepsilon \quad \text{then} \quad Y(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon G) \le \eta. \tag{2.46}$$

**Proof.** We first prove (2.45). In view of (2.25), we have, for $C_7$ large enough, $1/2 + C_7\varepsilon \ge a(\pm\varepsilon G) + \frac{1}{2}C_7\varepsilon$, for all $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0$ small enough. Hence for $\xi \ge 1/2 + C_7\varepsilon$, as long as $Y(\tau, \xi; \pm\varepsilon G)$ has not reached $1 - \eta$, we can use (2.36) to deduce that

$$\begin{aligned} Y(\tau,\xi;\pm\varepsilon G) &\ge a(\pm\varepsilon G) + C_1 e^{\mu(\pm\varepsilon G)\tau}(\xi - a(\pm\varepsilon G)) \\ &\ge a(\pm\varepsilon G) + \tfrac{1}{2}C_1C_7\varepsilon e^{\mu(\pm\varepsilon G)\tau} \\ &\ge \tfrac{1}{2} - \varepsilon CG + \tfrac{1}{2}C_1C_7\varepsilon e^{\mu(\pm\varepsilon G)\tau} \\ &\ge 1 - \eta \end{aligned}$$

provided that

$$\tau \geq \tau^\varepsilon := \frac{1}{\mu(\pm\varepsilon G)} \ln \frac{1/2-\eta+CG\varepsilon}{C_1C_7\varepsilon/2}.$$

To complete the proof of (2.45) we must choose $C_7$ so that $\mu^{-1}|\ln\varepsilon| - \tau^\varepsilon \geq 0$. A simple computation shows that

$$\begin{aligned}\mu^{-1}|\ln\varepsilon| - \tau^\varepsilon = \frac{\mu(\pm\varepsilon G)-\mu}{\mu(\pm\varepsilon G)\mu}|\ln\varepsilon| \quad &-\frac{1}{\mu(\pm\varepsilon G)}\ln\frac{1/2-\eta+CG\varepsilon}{C_1/2}\\ &+\frac{1}{\mu(\pm\varepsilon G)}\ln C_7.\end{aligned}$$

Thanks to (2.27), as $\varepsilon \to 0$, the first term above is of order $\varepsilon|\ln\varepsilon|$ and the second one of order 1. Hence, for $C_7$ large enough, the quantity $\mu^{-1}|\ln\varepsilon| - \tau^\varepsilon$ is positive, for all $\varepsilon \in (0,\varepsilon_0)$, with $\varepsilon_0$ small enough. The proof of (2.46) is similar and omitted.

Next we prove (2.44). First note that, by taking $\varepsilon_0$ small enough, the stable equilibria of $f_{\pm\varepsilon G}$, namely $\alpha_-(\pm\varepsilon G)$ and $\alpha_+(\pm\varepsilon G)$, are in $[-\eta, 1+\eta]$. Hence, $f_{\pm\varepsilon G}$ being a bistable function, if we leave from a $\xi \in [-\eta, 1+\eta]$ then $Y(\tau,\xi;\pm\varepsilon G)$ will remain in the interval $[-\eta,1+\eta]$. Now suppose that $1+\eta \leq \xi \leq 2C_0$ (note that this work is useless if $2C_0 < 1+\eta$). We check below that $Y(\mu^{-1}|\ln\varepsilon|,\xi;\pm\varepsilon G) \leq 1+\eta$. As long as $1+\eta \leq Y \leq 2C_0$, (2.28) leads to the inequality $Y_\tau \leq f(1+\eta)+\varepsilon G \leq \frac{1}{2}f(1+\eta) < 0$, for $\varepsilon_0 = \varepsilon_0(\eta)$ small enough. By integration from 0 to $\tau$, it follows that

$$\begin{aligned}Y(\tau,\xi;\pm\varepsilon G) \quad &\leq \xi + \tfrac{1}{2}f(1+\eta)\tau\\ &\leq 2C_0 + \tfrac{1}{2}f(1+\eta)\tau\\ &\leq 1+\eta,\end{aligned}$$

provided that

$$\tau \geq \frac{2C_0-1-\eta}{-f(1+\eta)/2},$$

and a fortiori for $\tau = \mu^{-1}|\ln\varepsilon|$, which completes the proof of (2.44). □

We are now ready to prove Theorem 2.3.1. By setting $t = \mu^{-1}\varepsilon^2|\ln\varepsilon|$ in (2.43), we obtain

$$\begin{aligned}Y\Big(\mu^{-1}|\ln\varepsilon|, u_0(x) - \varepsilon^2 r(-\varepsilon G, \mu^{-1}|\ln\varepsilon|); -\varepsilon G\Big)&\\ \leq u^\varepsilon(x,\mu^{-1}\varepsilon^2|\ln\varepsilon|) \leq Y\Big(\mu^{-1}|\ln\varepsilon|, u_0(x)+\varepsilon^2 r(\varepsilon G,\mu^{-1}|\ln\varepsilon|); +\varepsilon G\Big).&\qquad (2.47)\end{aligned}$$

In view of (2.27),

$$\lim_{\varepsilon\to 0}\frac{\mu-\mu(\pm\varepsilon G)}{\mu}\ln\varepsilon = 0, \qquad (2.48)$$

so that, for $\varepsilon_0$ small enough, we have

$$\varepsilon^2 r(\pm\varepsilon G, \mu^{-1}|\ln\varepsilon|) = C_6\varepsilon(\varepsilon^{(\mu-\mu(\pm\varepsilon G))/\mu} - \varepsilon) \in (\frac{1}{2}C_6\varepsilon, \frac{3}{2}C_6\varepsilon).$$

It follows that $u_0(x) \pm \varepsilon^2 r(\pm\varepsilon G, \mu^{-1}|\ln\varepsilon|) \in (-2C_0, 2C_0)$. Hence the result (2.21) of Theorem 2.3.1 is a direct consequence of (2.44) and (2.47).

Next we prove (2.22). We take $x \in \Omega$ such that $u_0(x) \geq 1/2 + M_0\varepsilon$ so that

$$\begin{aligned}u_0(x) - \varepsilon^2 r(-\varepsilon G, \mu^{-1}|\ln\varepsilon|) \quad &\geq 1/2 + M_0\varepsilon - \tfrac{3}{2}C_6\varepsilon\\ &\geq 1/2 + C_7\varepsilon,\end{aligned}$$

if we choose $M_0$ large enough. Using (2.47) and (2.45) we obtain (2.22), which completes the proof of Theorem 2.3.1. □

## 2.4 Motion of interface

We have seen in Section 2.3 that, after a very short time, the solution $u^\varepsilon$ develops a clear transition layer. In the present section, we show that it persists and that its law of motion is well approximated by the interface equation in $(P^0)$ obtained by formal asymptotic expansions in subsection 2.2.1.

More precisely, take the first term of the formal asymptotic expansion (2.11) as a formal expansion of the solution:

$$u^\varepsilon(x,t) \approx \tilde{u}^\varepsilon(x,t) := U_0\Big(\frac{\widetilde{d}(x,t)}{\varepsilon}\Big), \tag{2.49}$$

where $U_0$ is defined in (2.14). The right-hand side is a function having a well-developed transition layer, and its interface lies exactly on $\Gamma_t$. We show that this function is a very good approximation of the solution; more precisely:

*If $u^\varepsilon$ becomes very close to $\tilde{u}^\varepsilon$ at some time moment $t = t_0$, then it stays close to $\tilde{u}^\varepsilon$ for the rest of time. Consequently, $\Gamma_t^\varepsilon$ evolves roughly like $\Gamma_t$.*

To that purpose, we will construct a pair of sub- and super-solutions $u_\varepsilon^-$ and $u_\varepsilon^+$ for Problem $(P^\varepsilon)$ by slightly modifying $\tilde{u}^\varepsilon$. It then follows that, if the solution $u^\varepsilon$ satisfies

$$u_\varepsilon^-(x,t_0) \le u^\varepsilon(x,t_0) \le u_\varepsilon^+(x,t_0),$$

for some $t_0 \ge 0$, then

$$u_\varepsilon^-(x,t) \le u^\varepsilon(x,t) \le u_\varepsilon^+(x,t),$$

for $t_0 \le t \le T$. As a result, since both $u_\varepsilon^+, u_\varepsilon^-$ stay close to $\tilde{u}^\varepsilon$, the solution $u^\varepsilon$ also stays close to $\tilde{u}^\varepsilon$ for $t_0 \le t \le T$.

### 2.4.1 Construction of sub- and super-solutions

To begin with we present mathematical tools which are essential for the construction of sub and super-solutions.

**A modified signed distance function.** Rather than working with the usual signed distance function $\widetilde{d}$, defined in (2.9), we define a "cut-off signed distance function" $d$ as follows. Choose $d_0 > 0$ small enough so that $\widetilde{d}(\cdot,\cdot)$ is smooth in the tubular neighborhood of $\Gamma$

$$\{(x,t) \in \overline{Q_T},\ |\widetilde{d}(x,t)| < 3d_0\},$$

and such that

$$\mathrm{dist}(\Gamma_t, \partial\Omega) > 4d_0 \quad \text{for all } t \in [0,T]. \tag{2.50}$$

Next let $\zeta(s)$ be a smooth increasing function on $\mathbb{R}$ such that

$$\zeta(s) = \begin{cases} s & \text{if } |s| \le 2d_0 \\ -3d_0 & \text{if } s \le -3d_0 \\ 3d_0 & \text{if } s \ge 3d_0. \end{cases}$$

We define the cut-off signed distance function $d$ by

$$d(x,t) = \zeta\big(\tilde{d}(x,t)\big). \tag{2.51}$$

Note that $|\nabla d| = 1$ in the region $\{(x,t) \in \overline{Q_T},\ |d(x,t)| < 2d_0\}$ and that, in view of the above definition, $\nabla d = 0$ in a neighborhood of $\partial\Omega$. Note also that the equation of motion interface in $(P^0)$, which is equivalent to (2.17), is now written as

$$d_t = \Delta d - \nabla d \cdot \nabla \chi(v^0) - \sqrt{2}\alpha \quad \text{on } \Gamma_t. \tag{2.52}$$

**Construction.** We look for a pair of sub- and super-solutions $u_\varepsilon^\pm$ for Problem $(P^\varepsilon)$ of the form

$$u_\varepsilon^\pm(x,t) = U_0\Big(\frac{d(x,t) \mp \varepsilon p(t)}{\varepsilon}\Big) \pm q(t), \tag{2.53}$$

where $U_0$ is the solution of (2.14), and where

$$\begin{aligned} p(t) &= -e^{-\beta t/\varepsilon^2} + e^{Lt} + K, \\ q(t) &= \sigma(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}). \end{aligned} \tag{2.54}$$

Note that $q = \sigma\varepsilon^2 p_t$. Let us remark that the construction (2.53) is more precise than the several procedure of only taking a zeroth order term of the form $U_0$, since we have shown in the formal derivation that the first order term $U_1$ vanishes in (2.11). It is clear from the definition of $u_\varepsilon^\pm$ that

$$\lim_{\varepsilon\to 0} u_\varepsilon^\pm(x,t) = \begin{cases} 1 & \text{for all } (x,t) \in Q_T^{(1)} \\ 0 & \text{for all } (x,t) \in Q_T^{(0)}. \end{cases} \tag{2.55}$$

The main result of this section is the following.

**Lemma 2.4.1.** *There exist positive constants $\beta$, $\sigma$ with the following properties. For any $K > 1$, we can find positive constants $\varepsilon_0$ and $L$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, the functions $u_\varepsilon^-$ and $u_\varepsilon^+$ are respectively sub- and super-solutions for Problem $(P^\varepsilon)$ in the range $x \in \overline{\Omega}$, $0 \le t \le T$.*

### 2.4.2 Proof of Lemma 2.4.1

First, since $\nabla d = 0$ in a neighborhood of $\partial\Omega$, we have the homogeneous Neumann boundary condition

$$\frac{\partial u_\varepsilon^\pm}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times [0,T].$$

Let $v^\varepsilon$ be such that (2.19) holds. We have to show that

$$L_{v^\varepsilon}[u_\varepsilon^+] := (u_\varepsilon^+)_t - \Delta u_\varepsilon^+ + \nabla u_\varepsilon^+ \cdot \nabla\chi(v^\varepsilon) + u_\varepsilon^+ \Delta\chi(v^\varepsilon) - \frac{1}{\varepsilon^2} f_\varepsilon(u_\varepsilon^+) \ge 0,$$

the proof of inequality $L_{v^\varepsilon}[u_\varepsilon^-] \le 0$ following by the same arguments.

**Computation of $L_{v^\varepsilon}[u_\varepsilon^+]$**

By straightforward computations we obtain the following terms:

$$
\begin{aligned}
(u_\varepsilon^+)_t &= U_0{}'(\frac{d_t}{\varepsilon} - p_t) + q_t, \\
\nabla u_\varepsilon^+ &= U_0{}'\frac{\nabla d}{\varepsilon}, \\
\Delta u_\varepsilon^+ &= U_0{}''\frac{|\nabla d|^2}{\varepsilon^2} + U_0{}'\frac{\Delta d}{\varepsilon},
\end{aligned}
$$

where the function $U_0$, as well as its derivatives, is taken at the point $\big(d(x,t) - \varepsilon p(t)\big)/\varepsilon$. We also use expansions of the reaction terms:

$$
\begin{aligned}
f(u_\varepsilon^+) &= f(U_0) + qf'(U_0) + \frac{1}{2}q^2 f''(\theta), \\
g(u_\varepsilon^+) &= g(U_0) + qg'(\omega),
\end{aligned}
$$

where $\theta(x,t)$ and $\omega(x,t)$ are some functions satisfying $U_0 < \theta < u_\varepsilon^+$, $U_0 < \omega < u_\varepsilon^+$. Combining the above expressions with equation (2.14) and the fact that $\sqrt{2}\alpha U_0{}' + g(U_0) \equiv 0$, we obtain

$$
L_{v^\varepsilon}[u_\varepsilon^+] = E_1 + \cdots + E_5,
$$

where:

$$
\begin{aligned}
E_1 &= -\frac{1}{\varepsilon^2}q[f'(U_0) + \frac{1}{2}qf''(\theta)] - U_0{}'p_t + q_t, \\
E_2 &= \frac{U_0{}''}{\varepsilon^2}(1 - |\nabla d|^2), \\
E_3 &= \frac{U_0{}'}{\varepsilon}(d_t - \Delta d + \nabla d \cdot \nabla\chi(v^0) + \sqrt{2}\alpha), \\
E_4 &= -\frac{1}{\varepsilon}qg'(\omega), \\
E_5 &= \frac{U_0{}'}{\varepsilon}\nabla d \cdot \nabla(\chi(v^\varepsilon) - \chi(v^0)) + u_\varepsilon^+\Delta\chi(v^\varepsilon).
\end{aligned}
$$

In order to estimate the terms above, we first present some useful inequalities. As $f'(0) = f'(1) = -1/2$, we can find strictly positive constants $b$ and $m$ such that

$$
\text{if} \quad U_0(z) \in [0,b] \cup [1-b,1] \qquad \text{then} \quad f'(U_0(z)) \le -m. \tag{2.56}
$$

On the other hand, since the region $\{z \in \mathbb{R} \,|\, U_0(z) \in [b, 1-b]\,\}$ is compact and since $U_0{}' < 0$ on $\mathbb{R}$, there exists a constant $a_1 > 0$ such that

$$
\text{if} \quad U_0(z) \in [b, 1-b] \quad \text{then} \quad U_0{}'(z) \le -a_1. \tag{2.57}
$$

We then define

$$
F = \sup_{-1 \le z \le 2} |f(z)| + |f'(z)| + |f''(z)|, \tag{2.58}
$$

$$
\beta = \frac{m}{4}, \tag{2.59}
$$

and choose $\sigma$ that satisfies

$$0 < \sigma \leq \min(\sigma_0, \sigma_1, \sigma_2), \tag{2.60}$$

where

$$\sigma_0 := \frac{a_1}{m+F}, \quad \sigma_1 := \frac{1}{\beta+1}, \quad \sigma_2 := \frac{4\beta}{F(\beta+1)}.$$

Hence, combining (2.56) and (2.57), we obtain, using that $\sigma \leq \sigma_0$,

$$-{U_0}'(z) - \sigma f'(U_0(z)) \geq 4\sigma\beta \qquad \text{for} \quad -\infty < z < \infty. \tag{2.61}$$

Now let $K > 1$ be arbitrary. In what follows we will show that $L_{v^\varepsilon}[u_\varepsilon^+] \geq 0$ provided that the constants $\varepsilon_0$ and $L$ are appropriately chosen. From now on, we suppose that the following inequality is satisfied:

$$\varepsilon_0^2 L e^{LT} \leq 1\,. \tag{2.62}$$

Then, given any $\varepsilon \in (0, \varepsilon_0)$, since $\sigma \leq \sigma_1$, we have $0 \leq q(t) \leq 1$, hence, recalling that $0 < U_0 < 1$,

$$-1 \leq u_\varepsilon^\pm(x,t) \leq 2\,. \tag{2.63}$$

**We first estimate the term $E_1$**

A direct computation gives

$$E_1 = \frac{\beta}{\varepsilon^2}\, e^{-\beta t/\varepsilon^2}(I - \sigma\beta) + Le^{Lt}(I + \varepsilon^2\sigma L),$$

where

$$I = -{U_0}' - \sigma f'(U_0) - \frac{\sigma^2}{2} f''(\theta)(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}).$$

In virtue of (2.61) and (2.63), we obtain

$$I \geq 4\sigma\beta - \frac{\sigma^2}{2} F(\beta + \varepsilon^2 L e^{LT}).$$

Then, in view of (2.62), using that $\sigma \leq \sigma_2$, we have

$$I \geq 2\sigma\beta.$$

Consequently, the following inequality holds.

$$E_1 \geq \frac{\sigma\beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + 2\sigma\beta L e^{Lt} =: \frac{C_1}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + {C_1}' L e^{Lt}.$$

**As for the term $E_2$**

First, in the points where where $|d(x,t)| < d_0$, we have that $|\nabla d| = 1$ so that $E_2 = 0$. Next we consider the points where $|d(x,t)| \geq d_0$. We deduce from Lemma 2.2.1 that:

$$\begin{aligned} |E_2| &\leq \frac{C}{\varepsilon^2}(1 + \|\nabla d\|_\infty^2) e^{-\lambda|d+\varepsilon p|/\varepsilon} \\ &\leq \frac{C}{\varepsilon^2}(1 + \|\nabla d\|_\infty^2) e^{-\lambda(d_0/\varepsilon - |p|)}. \end{aligned}$$

In view of the definition of $p$ in (2.54), we have that $0 < K - 1 \le p \le e^{LT} + K$, and suppose from now that the following assumption holds:

$$e^{LT} + K \le \frac{d_0}{2\varepsilon_0}. \tag{2.64}$$

Then $\dfrac{d_0}{\varepsilon} - |p| \ge \dfrac{d_0}{2\varepsilon}$, so that

$$\begin{aligned} |E_2| &\le \frac{C}{\varepsilon^2}(1 + \|\nabla d\|_\infty^2) e^{-\lambda d_0/(2\varepsilon)} \\ &\le C_2 := \frac{16C}{(e\lambda d_0)^2}(1 + \|\nabla d\|_\infty^2). \end{aligned}$$

**Next we consider the term $E_3$**

We recall that

$$d_t - \Delta d + \nabla d \cdot \nabla \chi(v^0) + \sqrt{2}\alpha = 0 \quad \text{ on } \Gamma_t = \{x \in \Omega, d(x,t) = 0\}.$$

Since $v^0$ is of class $C^{1+\vartheta', \frac{1+\vartheta'}{2}}$, for any $\vartheta' \in (0,1)$, and since the interface $\Gamma_t$ is of class $C^{2+\vartheta, \frac{2+\vartheta}{2}}$, the functions $\nabla d$, $\Delta d$, $d_t$ and $\nabla \chi(v^0)$ are Lipschitz continuous near $\Gamma_t$. It then follows, from the mean value theorem applied separately on both sides of $\Gamma_t$, that there exists $N_0 > 0$ such that:

$$|(d_t - \Delta d + \nabla d \cdot \nabla \chi(v^0) + \sqrt{2}\alpha)(x,t)| \le N_0 |d(x,t)| \quad \text{ for all } \ (x,t) \in Q_T.$$

Applying Lemma 2.2.1 we deduce that

$$\begin{aligned} |E_3| &\le N_0 C \frac{|d(x,t)|}{\varepsilon} e^{-\lambda |d(x,t)/\varepsilon + p(t)|} \\ &\le N_0 C \max_{y \in \mathbb{R}} |y| e^{-\lambda |y + p(t)|} \\ &\le N_0 C \max\big(|p(t)|, \frac{1}{\lambda}\big) \\ &\le N_0 C \big(|p(t)| + \frac{1}{\lambda}\big). \end{aligned}$$

Taking the expression of $p$ into account, we see that $|p(t)| \le e^{Lt} + K$, which implies

$$|E_3| \le C_3 (e^{Lt} + K) + {C_3}',$$

where $C_3 := N_0 C$ and ${C_3}' := N_0 C/\lambda$.

**The term $E_4$**

Substituting the expression for $q$ and setting $G_1 := \sup\{|g'(z)|; -1 \le z \le 2\}$ leads to

$$\begin{aligned} |E_4| &\le \sigma G_1 \Big(\frac{\beta}{\varepsilon} e^{-\beta t/\varepsilon^2} + \varepsilon L e^{Lt}\Big) \\ &\le \frac{C_4}{\varepsilon} e^{-\beta t/\varepsilon^2} + {C_4}' \varepsilon L e^{Lt}. \end{aligned}$$

**We continue with the term $E_5$**

This term requires a more delicate analysis. We need a precise estimate of $v^\varepsilon - v^0$. We recall that $v^0$ satisfies $-\Delta v^0 + \gamma v^0 = u^0$, with $u^0$ a step function discontinuous when crossing the interface.

**Lemma 2.4.2.** *There exists a positive constant $C_G$ such that, for all $(x,t) \in Q_T$,*

$$\Big(|v^\varepsilon| + |\nabla v^\varepsilon| + |\Delta v^\varepsilon|\Big)(x,t) \le C_G, \tag{2.65}$$

$$\Big(|v^\varepsilon - v^0| + |\nabla d \cdot \nabla (v^\varepsilon - v^0)|\Big)(x,t) \le C_G(\varepsilon p(t) + q(t)). \tag{2.66}$$

We postpone the proof of this lemma and pursue the proof of Lemma 2.4.1. Using the smoothness of $\chi$ and (2.65), we obtain a uniform bound $C_G{}'$ for $\Delta\chi(v^\varepsilon)$. Moreover, we can write

$$\nabla d \cdot \nabla\big(\chi(v^\varepsilon) - \chi(v^0)\big) = \chi'(v^\varepsilon)\nabla d \cdot \nabla(v^\varepsilon - v^0) + \big(\chi'(v^\varepsilon) - \chi'(v^0)\big)\nabla d \cdot \nabla v^0. \tag{2.67}$$

Since $v^0$ is of class $C^{1+\vartheta', \frac{1+\vartheta'}{2}}$, for any $\vartheta' \in (0,1)$, there exists a constant, which we denote again by $C_G$, such that

$$\|v^0\|_{L^\infty(Q_T)} + \|\nabla v^0\|_{L^\infty(Q_T)} \le C_G,$$

which, combined with (2.67), yields

$$|\nabla d \cdot \nabla\big(\chi(v^\varepsilon) - \chi(v^0)\big)| \le \|\chi'\|_\infty |\nabla d \cdot \nabla(v^\varepsilon - v^0)| + |v^\varepsilon - v^0| \; \|\chi''\|_\infty \|\nabla d\|_\infty C_G,$$

where the $L^\infty$-norms of $\chi'$ and $\chi''$ are considered on the interval $(-C_G, C_G)$. It follows from the above inequality and (2.66) that there exists a constant $C_G{}''$ such that, for all $(x,t) \in Q_T$,

$$|\nabla d \cdot \nabla\big(\chi(v^\varepsilon) - \chi(v^0)\big)|(x,t) \le C_G{}''(\varepsilon p(t) + q(t)).$$

Hence, using (2.63) and the above estimates, we obtain,

$$|E_5| \le \frac{C}{\varepsilon} C_G{}''(\varepsilon p(t) + q(t)) + 2C_G{}'.$$

Then, substituting the expressions for $p$ and $q$, we easily obtain positive constants $C_5$, $C_5{}'$ and $C_5{}''$ such that

$$|E_5| \le C_5 + \frac{C_5{}'}{\varepsilon} e^{-\beta t/\varepsilon^2} + C_5{}''(1 + \varepsilon L)e^{Lt}.$$

**Completion of the proof**

Collecting the above estimates of $E_1$—$E_5$ yields

$$L_{v^\varepsilon}[u_\varepsilon^+] \ge \Big(\frac{C_1}{\varepsilon^2} - \frac{C_4 + C_5{}'}{\varepsilon}\Big)e^{-\beta t/\varepsilon^2} + \Big(L(C_1{}' - \varepsilon C_4{}' - \varepsilon C_5{}'') - C_3 - C_5{}''\Big)e^{Lt} - C_7,$$

where $C_7 := C_2 + KC_3 + C_3{}' + C_5$. Now, we set

$$L := \frac{1}{T} \ln \frac{d_0}{4\varepsilon_0},$$

which, for $\varepsilon_0$ small enough, validates assumptions (2.62) and (2.64). If $\varepsilon_0$ is chosen sufficiently small (i.e. $L$ large enough), $C_1/\varepsilon^2 - (C_4 + C_5')/\varepsilon$ is positive, $C_1' - \varepsilon C_4' - \varepsilon C_5'' \geq \frac{1}{2}C_1'$, and

$$\begin{aligned} L_{v^\varepsilon}[u_\varepsilon^+] &\geq \left[\tfrac{1}{2}LC_1' - C_3 - C_5''\right]e^{Lt} - C_7 \\ &\geq \tfrac{1}{4}LC_1' - C_7 \\ &\geq 0. \end{aligned}$$

The proof of Lemma 2.4.1 is now completed, with the choice of the constants $\beta, \sigma$ as in (2.59), (2.60). □

### 2.4.3 Proof of Lemma 2.4.2

Lemma 2.4.2 is very inspired by Lemma 4.9 in [17]. We present the proof for the convenience of the reader. Since our pair of sub- and super-solutions is different from the one in [17], we need to perform some minor changes. First we give a useful estimate on "shifted $U_0$".

**Lemma 2.4.3.** *For all $a \in \mathbb{R}$, all $z \in \mathbb{R}$, we have*

$$|U_0(z + a) - \chi_{]-\infty,0]}(z)| \leq Ce^{-\lambda|z+a|} + \chi_{[-a,a]}(z)$$

**Proof.** Let us give the proof for $a > 0$. We distinguish three cases and use the estimates of Lemma 2.2.1. For $z \leq -a$, we have $|U_0(z + a) - 1| \leq Ce^{-\lambda|z+a|}$. For $-a < z \leq 0$, we have $|U_0(z+a) - 1| \leq |U_0(z+a)| + 1 \leq Ce^{-\lambda|z+a|} + 1$. For $z > 0$, we have $|U_0(z + a)| \leq Ce^{-\lambda|z+a|}$. We proceed in the same way for $a < 0$. □

We turn to the proof of Lemma 2.4.2. First, we recall that $v^\varepsilon$ is such that (2.19) holds; hence, in view of (2.63), the estimate (2.65) is a direct consequence of the standard theory of elliptic equations. Next we prove (2.66). The function $w = w^\varepsilon := v^\varepsilon - v^0$ is solution of

$$\begin{cases} -\Delta w + \gamma w = h & \text{on } Q_T, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{2.68}$$

with $u_\varepsilon^- - u^0 \leq h = h^\varepsilon \leq u_\varepsilon^+ - u^0$, where $u^0$ is the step function defined by $u^0(x, t) = \chi_{\{d(x,t) \leq 0\}}$. The key idea of the proof is the fact that $h$ is exponentially small with respect to $\varepsilon$, except possibly in a thin neighborhood of $\Gamma_t$ of width of order $\varepsilon p(t)$. More precisely, from the definitions of $u_\varepsilon^\pm$ in (2.53) and from the above lemma for $z = d(x, t)/\varepsilon$ and $a = \pm p(t)$, we deduce that

$$|h(x, t)| \leq C(e^{-\lambda|d(x,t)/\varepsilon + p(t)|} + e^{-\lambda|d(x,t)/\varepsilon - p(t)|}) + \chi_{\{|d(x,t)| \leq \varepsilon p(t)\}} + q(t). \tag{2.69}$$

By linearity, we successively consider equation (2.68) with the various terms appearing in the right-hand side of (2.69). By the standard elliptic estimates, the solution $w$ of (2.68) satisfies

$$|w(x, t)| + |\nabla w(x, t)| \leq C' \sup_{y \in \Omega} |h(y, t)|, \tag{2.70}$$

which gives the term $C_G q(t)$ that appears in the right-hand side of inequality (2.66) for $h(y, t) = q(t)$. We now suppose that the function $h$ satisfies one of the three following assumptions:

$$(H_1) \qquad |h(y, t)| \leq \chi_{\{|d(y,t)| \leq \varepsilon p(t)\}}$$

$$(H_2^\pm) \qquad |h(y, t)| \leq \exp\Big( -\lambda \Big|\frac{d(y, t)}{\varepsilon} \pm p(t)\Big|\Big),$$

and write

$$h(y,t) = h(y,t)\chi_{\{|d(y,t)|\le d_0\}} + h(y,t)\chi_{\{|d(y,t)|>d_0\}}.$$

We first consider the term $h(y,t)\chi_{\{|d(y,t)|>d_0\}}$. In virtue of (2.64), we have

$$0 < K - 1 \le p(t) \le d_0/2\varepsilon_0. \tag{2.71}$$

Under assumption $(H_1)$, it follows that $h$ is supported in $\{|d(y,t)| \le d_0/2\}$, which implies $h(y,t)\chi_{\{|d(y,t)|>d_0\}} = 0$. Moreover, under assumption $(H_2^{\pm})$, using again (2.71),

$$\begin{aligned} |h(y,t)|\chi_{\{|d(y,t)|>d_0\}} &\le \exp\big[-\lambda(d_0/\varepsilon - p(t))\big] \\ &\le \exp(-\lambda d_0/2\varepsilon) \\ &\le \frac{2}{\lambda d_0 e}\,\varepsilon \\ &\le \frac{2}{\lambda d_0 e}\frac{1}{K-1}\varepsilon p(t). \end{aligned}$$

Thus, under either of the assumptions $(H_1)$ or $(H_2^{\pm})$, the estimate (2.66) — for the term $h(y,t)\chi_{\{|d(y,t)|>d_0\}}$ — directly follows from inequality (2.70).

From now on, we assume that $h$ is supported in $\{|d(y,t)| \le d_0\}$. We have that

$$w(x,t) = \int_{|d(y,t)|\le d_0} G(x,y)h(y,t)dy,$$

and

$$\nabla d(x,t)\cdot\nabla w(x,t) = \int_{|d(y,t)|\le d_0} (\nabla_x G(x,y)\cdot\nabla d(x,t))h(y,t)dy,$$

where $G$ is the Green's function associated to the homogeneous Neumann boundary value problem on $\Omega$ for the operator $-\Delta+\gamma$. More precisely, $G(x,y) = g_\gamma(|x-y|) + H_\gamma(x,y)$, where $g_\gamma(|x-y|)$ is the Green's function associated to the operator $-\Delta+\gamma$ on $\mathbb{R}^N$ and where $H_\gamma(x,y)$ is smooth for $x$ and $y$ far away from $\partial\Omega$. It is known that $g_\gamma$ is the Bessel function defined by

$$g_\gamma(r) = c_N \int_0^{+\infty} e^{-\frac{r^2}{2s}} e^{-\gamma\frac{s}{2}} s^{\frac{-N+2}{2}}\frac{ds}{s},$$

with $c_N > 0$ a normalization constant. We use the following estimates (see [17]):

$$|G(x,y)| \le \begin{cases} \dfrac{C}{|y-x|^{N-2}} & \text{for } N\ge 3 \\ C|\ln|y-x|| & \text{for } N=2, \end{cases} \tag{2.72}$$

$$|\nabla_x G(x,y)\cdot\nabla d(x,t)| \le \frac{C|d(y,t)-d(x,t)|}{|y-x|^N} + \frac{C}{|y-x|^{N-2}} \quad \text{for } N\ge 2. \tag{2.73}$$

This last inequality follows from

$$|\nabla_x G(x,y)\cdot\nabla d(x,t)| \le \frac{C|\nabla d(x,t)\cdot(y-x)|}{|y-x|^N},$$

and from $d(y,t)-d(x,t)=\nabla d(x,t)\cdot(y-x)+O(|y-x|^2)$. Now, under respectively assumptions $(H_1)$, $(H_2^{\pm})$, we define a function $\tilde h=\tilde h^\varepsilon$ on $\mathbb{R}\times[0,T]$, respectively by

$$\tilde h(r,t):=\begin{cases}\chi\{|r|\le\varepsilon p(t)\}\\ \exp\Big(-\lambda\big|\dfrac{r}{\varepsilon}\pm p(t)\big|\Big).\end{cases}\tag{2.74}$$

Note that $|h(y,t)|\le\tilde h\big(d(y,t),t\big)$. Moreover, recalling (2.71), straightforward computations show that, under either of the assumptions $(H_1)$ or $(H_2^{\pm})$, there exists $\tilde C>0$ such that

$$0\le\int_{-d_0}^{d_0}\tilde h(r,t)dr\le\tilde C\varepsilon p(t).\tag{2.75}$$

Finally we have

$$|w(x,t)|+|\nabla d(x,t)\cdot\nabla w(x,t)|\le C[A(x,t)+B(x,t)],\tag{2.76}$$

with

$$A(x,t)=\begin{cases}\displaystyle\int_{|d(y,t)|\le d_0}\frac{2\tilde h\big(d(y,t),t\big)}{|y-x|^{N-2}}dy & \text{if } N\ge3\\[2ex] \displaystyle\int_{|d(y,t)|\le d_0}\tilde h\big(d(y,t),t\big)\Big[1+|\ln|y-x||\Big]dy & \text{if } N=2,\end{cases}$$

and

$$B(x,t)=\int_{|d(y,t)|\le d_0}\frac{|d(y,t)-d(x,t)|\tilde h\big(d(y,t),t\big)}{|y-x|^N}dy.$$

We now distinguish two cases according to the distance from $x$ to the interface.

- If $\mathrm{dist}(x,\Gamma_t)\ge2d_0$, then $|y-x|\ge d_0$ for all $y$ with $|d(y,t)|\le d_0$ so that

  $$A(x,t)+B(x,t)\le C\int_{|d(y,t)|\le d_0}\tilde h\big(d(y,t),t\big)dy,$$

  for a constant $C=C(d_0)$. Taking $d$ as one of the coordinates in $\{|d(y,t)|\le d_0\}$, we get

  $$A(x,t)+B(x,t)\le C\int_{-d_0}^{d_0}\tilde h(r,t)dr,$$

  and the estimate (2.66) follows from (2.76) and (2.75).

- If $\mathrm{dist}(x,\Gamma_t)\le2d_0$, then we can make, for any fixed $t\in[0,T]$, the change of variables

  $$\{y\in\Omega,|d(y,t)|\le2d_0\}\to\Gamma_t\times[-2d_0,2d_0]$$
  $$y\mapsto(s,r),$$

  where $s=s(y,t)$ is the projection of $y$ on $\Gamma_t$ along the normal of $\Gamma_t$, and $r=d(y,t)$ is the signed distance function from $y$ to $\Gamma_t$. We write this change of variables as $y=X(s,r)$. In the case $N\ge3$, we have

  $$A(x,t)\le C\int_{-d_0}^{d_0}dr\tilde h(r,t)\int_{\Gamma_t}\frac{ds}{|X(s,r)-X(s_0,r_0)|^{N-2}},$$

with $x = X(s_0, r_0)$. Since $X$ is a smooth diffeomorphism, there exists $C > 0$ such that for all $(s, r) \in \Gamma_t \times [-2d_0, 2d_0]$, we have

$$|X(s,r) - X(s_0,r_0)| \geq C(|s-s_0| + |r-r_0|) \geq C|s-s_0|.$$

Hence

$$A(x,t) \leq C \int_{-d_0}^{d_0} dr\tilde{h}(r,t) \int_{\Gamma_t} \frac{ds}{|s-s_0|^{N-2}}.$$

Since the singularity in $1/|s|^{N-2}$ is integrable on the $(N-1)$-dimensional hypersurface $\Gamma_t$, we have

$$\int_{\Gamma_t} \frac{ds}{|s-s_0|^{N-2}} \leq C,$$

for some $C > 0$ which does not depend on $t \in [0,T]$ and $s_0 \in \Gamma_t$. Therefore,

$$A(x,t) \leq C \int_{-d_0}^{d_0} \tilde{h}(r,t)dr. \tag{2.77}$$

The same estimate is obtained in the case where $N = 2$, using the fact that $\ln|s|$ is integrable on a finite line.

As for $B$, we have

$$B(x,t) \leq C \int_{-d_0}^{d_0} dr\tilde{h}(r,t) \int_{\Gamma_t} \frac{|r-r_0|}{\big[|s-s_0| + |r-r_0|\big]^N} ds.$$

Making the change of variables $s - s_0 = |r-r_0|s'$, we get

$$\int_{\Gamma_t} \frac{|r-r_0|}{[|s-s_0| + |r-r_0|]^N} ds \leq \int_{\Gamma_t'} \frac{ds'}{[|s'|+1]^N} \leq C,$$

where $\Gamma_t'$ is the image of $\Gamma_t$ by the mapping $s \mapsto s'$ and where $C > 0$ does not depend on $|r| \leq d_0$, $s_0 \in \Gamma_t$, $|r_0| \leq 2d_0$ and $t \in [0,T]$. Thus

$$B(x,t) \leq C \int_{-d_0}^{d_0} \tilde{h}(r,t)dr. \tag{2.78}$$

Then the estimate (2.66) follows from inequalities (2.76), (2.77), (2.78) and (2.75).

The proof of Lemma 2.4.2 is now complete. □

## 2.5 Proof of the main results

In this section, we prove our main results by fitting the two pairs of sub- and super-solutions, constructed for the study of the generation and the motion of interface, into each other.

### 2.5.1 Proof of Theorem 2.1.3

Let $\eta \in (0,1/4)$ be arbitrary. Choose $\beta$ and $\sigma$ that satisfy (2.59), (2.60) and

$$\sigma\beta \leq \frac{\eta}{3}. \tag{2.79}$$

By the generation of interface Theorem 2.3.1, there exist positive constants $\varepsilon_0$ and $M_0$ such that (2.21), (2.22) and (2.23) hold with the constant $\eta$ replaced by $\sigma\beta/2$. Since $\nabla u_0 \cdot n \neq 0$ everywhere on the initial interface $\Gamma_0 = \{x \in \Omega,\ u_0(x) = 1/2\}$ and since $\Gamma_0$ is a compact hypersurface, we can find a positive constant $M_1$ such that

$$\begin{array}{llll} \text{if} & d_0(x) \geq \ \ M_1\varepsilon & \text{then} & u_0(x) \leq 1/2 - M_0\varepsilon, \\ \text{if} & d_0(x) \leq -M_1\varepsilon & \text{then} & u_0(x) \geq 1/2 + M_0\varepsilon. \end{array} \tag{2.80}$$

Here $d_0(x) := d(x,0)$ denotes the cut-off signed distance function associated with the hypersurface $\Gamma_0$. Now we define functions $H^+(x), H^-(x)$ by

$$H^+(x) = \begin{cases} 1 + \frac{1}{2}\sigma\beta & \text{if } d_0(x) < M_1\varepsilon \\ \frac{1}{2}\sigma\beta & \text{if } d_0(x) \geq M_1\varepsilon, \end{cases}$$

$$H^-(x) = \begin{cases} 1 - \frac{1}{2}\sigma\beta & \text{if } d_0(x) \leq -M_1\varepsilon \\ -\frac{1}{2}\sigma\beta & \text{if } d_0(x) > -M_1\varepsilon. \end{cases}$$

Then from the above observation we see that

$$H^-(x) \ \leq\ u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \ \leq\ H^+(x) \qquad \text{for } x \in \Omega. \tag{2.81}$$

Next we fix a sufficiently large constant $K > 1$ such that

$$U_0(M_1 - K) \geq 1 - \frac{\sigma\beta}{3} \quad \text{and} \quad U_0(-M_1 + K) \leq \frac{\sigma\beta}{3}. \tag{2.82}$$

For this $K$, we choose $\varepsilon_0$ and $L$ as in Lemma 2.4.1. We claim that

$$u_\varepsilon^-(x,0) \leq H^-(x), \qquad H^+(x) \leq u_\varepsilon^+(x,0) \qquad \text{for } x \in \Omega. \tag{2.83}$$

We only prove the former inequality, as the proof of the latter is virtually the same. Then it amounts to showing that

$$u_\varepsilon^-(x,0) = U_0\big(\frac{d_0(x)}{\varepsilon} + K\big) - \sigma(\beta + \varepsilon^2 L) \ \leq\ H^-(x). \tag{2.84}$$

In the range where $d_0(x) > -M_1\varepsilon$, the second inequality in (2.82) and the fact that $U_0$ is a decreasing function imply

$$U_0\big(\frac{d_0(x)}{\varepsilon} + K\big) - \sigma(\beta + \varepsilon^2 L) \leq \frac{1}{3}\sigma\beta - \sigma\beta \ \leq\ H^-(x).$$

On the other hand, in the range where $d_0(x) \leq -M_1\varepsilon$, we have

$$U_0\big(\frac{d_0(x)}{\varepsilon} + K\big) - \sigma(\beta + \varepsilon^2 L) \leq\ 1 - \sigma\beta \ \leq\ H^-(x).$$

This proves (2.84), hence (2.83) is established.

Combining (2.81) and (2.83), we obtain

$$u_\varepsilon^-(x,0) \le u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \le u_\varepsilon^+(x,0).$$

Since $u_\varepsilon^-$ and $u_\varepsilon^+$ are sub- and super-solutions for Problem $(P^\varepsilon)$ thanks to Lemma 2.4.1, the comparison principle yields

$$u_\varepsilon^-(x,t) \le u^\varepsilon(x, t+t^\varepsilon) \le u_\varepsilon^+(x,t) \quad \text{for} \quad 0 \le t \le T - t^\varepsilon, \tag{2.85}$$

where $t^\varepsilon = \mu^{-1}\varepsilon^2|\ln\varepsilon|$. Note that, in view of (2.55), this is enough to prove Corollary 2.1.4. Now let $C$ be a positive constant such that

$$U_0(-C + e^{LT} + K) \ge 1 - \frac{\eta}{2} \quad \text{and} \quad U_0(C - e^{LT} - K) \le \frac{\eta}{2}. \tag{2.86}$$

One then easily checks, using successively (2.85), (2.53), (2.86) and (2.79), that, for $\varepsilon_0$ small enough, for $0 \le t \le T - t^\varepsilon$, we have

$$\begin{array}{llll} \text{if} & d(x,t) \ge \phantom{-}C\varepsilon & \text{then} & u^\varepsilon(x,t+t^\varepsilon) \le \eta, \\ \text{if} & d(x,t) \le -C\varepsilon & \text{then} & u^\varepsilon(x,t+t^\varepsilon) \ge 1-\eta, \end{array} \tag{2.87}$$

and

$$u^\varepsilon(x,t+t^\varepsilon) \in [-\eta, 1+\eta],$$

which completes the proof of Theorem 2.1.3. □

### 2.5.2 Proof of Theorem 2.1.5

In the case where $\mu^{-1}\varepsilon^2|\ln\varepsilon| \le t \le T$, the assertion of the theorem is a direct consequence of Theorem 2.1.3. All we have to consider is the case where $0 \le t \le \mu^{-1}\varepsilon^2|\ln\varepsilon|$. We shall use the sub- and super-solutions constructed for the study of the generation of interface in Section 2.3. To that purpose, we first prove the following lemma concerning $Y(\tau,\xi;\delta)$, the solution of the ordinary differential equation (2.28), in the initial time interval.

**Lemma 2.5.1.** *There exist constants $C_8 > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0)$,*

$$\begin{array}{llllll} \textit{if} & \xi \ge 1/2 + C_8\varepsilon & \textit{then} & Y(\tau,\xi;\pm\varepsilon G) > 1/2 & \textit{for} & 0 \le \tau \le \mu^{-1}|\ln\varepsilon|, \\ \textit{if} & \xi \le 1/2 - C_8\varepsilon & \textit{then} & Y(\tau,\xi;\pm\varepsilon G) < 1/2 & \textit{for} & 0 \le \tau \le \mu^{-1}|\ln\varepsilon|. \end{array} \tag{2.88}$$

**Proof.** We only prove the first inequality. Assume $\xi \ge 1/2 + C_8\varepsilon$. By (2.25), for $C_8 \ge CG$, we have that $\xi \ge 1/2 + C_8\varepsilon \ge a(\pm\varepsilon G)$. It then follows from (2.36) that

$$\begin{aligned} Y(\tau,\xi;\pm\varepsilon G) &\ge a(\pm\varepsilon G) + C_1 e^{\mu(\pm\varepsilon G)\tau}(1/2 + C_8\varepsilon - a(\pm\varepsilon G)) \\ &\ge 1/2 - CG\varepsilon + C_1(-CG\varepsilon + C_8\varepsilon) \\ &\ge 1/2 + \varepsilon(C_1C_8 - CG(C_1+1)) \\ &> 1/2, \end{aligned}$$

provided that $C_8$ is sufficiently large. □

Now we turn to the proof of Theorem 2.1.5. We first claim that there exists a positive constant $M_2$ such that for all $t \in [0, \mu^{-1}\varepsilon^2|\ln\varepsilon|]$,

$$\Gamma_t^\varepsilon \subset \mathcal{N}_{M_2\varepsilon}(\Gamma_0). \tag{2.89}$$

To see this, we choose ${M_0}'$ large enough, so that ${M_0}' \geq C_8 + 2C_6$, where $C_6$ is as in Lemma 2.3.9. As is done for (2.80), there is a positive constant $M_2$ such that

$$\begin{array}{llll} \text{if} & d_0(x) \geq M_2\varepsilon & \text{then} & u_0(x) \leq 1/2 - {M_0}'\varepsilon, \\ \text{if} & d_0(x) \leq -M_2\varepsilon & \text{then} & u_0(x) \geq 1/2 + {M_0}'\varepsilon. \end{array} \tag{2.90}$$

In view of this last condition, we see that, if $\varepsilon_0$ is small enough, if $d_0(x) \geq M_2\varepsilon$, then for all $0 \leq t \leq \mu^{-1}\varepsilon^2|\ln\varepsilon|$,

$$\begin{aligned} u_0(x) + \varepsilon^2 r(\varepsilon G, \frac{t}{\varepsilon^2}) &\leq 1/2 - {M_0}'\varepsilon + \varepsilon^2 C_6\big[e^{\mu(\varepsilon G)|\ln\varepsilon|/\mu} - 1\big] \\ &\leq 1/2 + \varepsilon\big[-{M_0}' + C_6\varepsilon^{(\mu-\mu(\pm\varepsilon G))/\mu} - \varepsilon C_6\big] \\ &\leq 1/2 + \varepsilon(-{M_0}' + 2C_6) \qquad \leftarrow \text{thanks to (2.48)} \\ &\leq 1/2 - C_8\varepsilon. \end{aligned}$$

This inequality and Lemma 2.5.1 imply $w_\varepsilon^+(x,t) < 1/2$, where $w_\varepsilon^+$ is the sub-solution defined in (2.40). Consequently, by (2.43),

$$u^\varepsilon(x,t) < 1/2 \qquad \text{if } \; d_0(x) \geq M_2\varepsilon.$$

In the case where $d_0(x) \leq -M_2\varepsilon$, similar arguments lead to $u^\varepsilon(x,t) > 1/2$. This completes the proof of (2.89). Note that we have proved that, for all $0 \leq t \leq \mu^{-1}\varepsilon^2|\ln\varepsilon|$,

$$\begin{array}{ll} u^\varepsilon(x,t) > 1/2 & \text{if} \quad x \in \Omega_0^{(1)} \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0), \\ u^\varepsilon(x,t) < 1/2 & \text{if} \quad x \in \Omega_0^{(0)} \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0). \end{array} \tag{2.91}$$

Next, since $\Gamma_t$ depends on $t$ smoothly, there is a constant $\tilde{C} > 0$ such that, for all $t \in [0, \mu^{-1}\varepsilon^2|\ln\varepsilon|]$,

$$\Gamma_0 \subset \mathcal{N}_{\tilde{C}\varepsilon^2|\ln\varepsilon|}(\Gamma_t), \tag{2.92}$$

and

$$\begin{array}{l} \Omega_t^{(1)} \setminus \mathcal{N}_{\tilde{C}\varepsilon}(\Gamma_t) \subset \Omega_0^{(1)} \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0), \\ \Omega_t^{(0)} \setminus \mathcal{N}_{\tilde{C}\varepsilon}(\Gamma_t) \subset \Omega_0^{(0)} \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0). \end{array} \tag{2.93}$$

As a consequence of (2.89) and (2.92) we get

$$\Gamma_t^\varepsilon \subset \mathcal{N}_{M_2\varepsilon + \tilde{C}\varepsilon^2|\ln\varepsilon|}(\Gamma_t) \subset \mathcal{N}_{C\varepsilon}(\Gamma_t),$$

which completes the proof of Theorem 2.1.5. □

**Proof of Corollary 2.1.6.** In view of Theorem 2.1.5 and the definition of the Hausdorff distance, to prove this corollary we only need to show the reverse inclusion, that is

$$\Gamma_t \subset \mathcal{N}_{C'\varepsilon}(\Gamma_t^\varepsilon) \quad \text{for} \quad 0 \leq t \leq T, \tag{2.94}$$

for some constant $C' > 0$. To that purpose let $C'$ be a constant satisfying $C' > \max(\tilde{C}, C)$, where $C$ is as in Theorem 2.1.3 and $\tilde{C}$ as in (2.93). Choose $t \in [0,T]$, $x_0 \in \Gamma_t$ arbitrarily and, $n$ being the Euclidian normal vector exterior to $\Gamma_t$ at point $x_0$, define a pair of points:

$$x^{(0)} := x_0 + C'\varepsilon n \quad \text{and} \quad x^{(1)} := x_0 - C'\varepsilon n.$$

Since $C' > C$ and since the curvature of $\Gamma_t$ is uniformly bounded as $t$ varies over $[0,T]$, we see that, if $\varepsilon_0$ is sufficiently small,

$$x^{(0)} \in \Omega_t^{(0)} \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t) \quad \text{and} \quad x^{(1)} \in \Omega_t^{(1)} \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t).$$

Therefore, if $t \in [\mu^{-1}\varepsilon^2|\ln\varepsilon|, T]$, then, by Theorem 2.1.3, we have

$$u^\varepsilon(x^{(0)},t) < 1/2 < u^\varepsilon(x^{(1)},t). \tag{2.95}$$

On the other hand, if $t \in [0, \mu^{-1}\varepsilon^2|\ln\varepsilon|]$, then from (2.91), (2.93) and the fact that $C' > \tilde{C}$, we again obtain (2.95). Thus (2.95) holds for all $t \in [0,T]$. Now, by the mean value theorem, we see that for each $t \in [0,T]$ there exists a point $\tilde{x}$ such that

$$\tilde{x} \in [x^{(0)}, x^{(1)}] \quad \text{and} \quad u^\varepsilon(\tilde{x},t) = 1/2.$$

This implies $\tilde{x} \in \Gamma_t^\varepsilon$. Furthermore we have $|x_0 - \tilde{x}| \le C'\varepsilon$, since $\tilde{x}$ lies on the line segment $[x^{(0)}, x^{(1)}]$. This proves (2.94). □

# Chapter 3

# The singular limit of a spatially inhomogeneous and anisotropic Allen-Cahn equation

We consider a spatially inhomogeneous and anisotropic reaction-diffusion equation, involving a small parameter $\varepsilon > 0$ and a bistable nonlinear term whose stable equilibria are 0 and 1, which arises for instance in material sciences. The diffusion term may be singular in the points where the gradient of the solution vanishes. We define a notion of weak solution and prove a comparison principle. We perform the analysis using the distance function associated with a Finsler metric. We consider rather general initial data $u_0$ that are independent of $\varepsilon$. We perform a rigorous analysis of both the generation and the motion of interface. More precisely, we show that, within the time scale of order $\varepsilon^2|\ln\varepsilon|$, the solution $u^\varepsilon$ develops a steep transition layer that separates the regions $\{u^\varepsilon \approx 0\}$ and $\{u^\varepsilon \approx 1\}$. Then, in a much slower time scale, the layer starts to propagate. As a consequence, as $\varepsilon \to 0$, the solution $u^\varepsilon$ converges almost everywhere to 0 in $\Omega_t^-$ and 1 in $\Omega_t^+$, where $\Omega_t^-$ and $\Omega_t^+$ are sub-domains of $\Omega$ separated by an interface $\Gamma_t$, whose motion is driven by its anisotropic mean curvature. We also prove that the thickness of the transition layer is of order $\varepsilon$.

## 3.1 Introduction

The background of our study is the modelling of anisotropic interface motion, where the normal velocity of displacement of the interface depends on the angle of the normal vector with a fixed direction [4], [65]. Related nonlinear reaction-diffusion equations give rise to sharp internal layers, or *interfaces*, when the coefficient of the diffusion term is very small or the one of the reaction term very large. In this Chapter we consider the following anisotropic parabolic problem involving a spatially inhomogeneous reaction-diffusion equation:

$$(P^\varepsilon) \qquad \begin{cases} u_t = \nabla \cdot a_p(x, \nabla u) + \dfrac{1}{\varepsilon^2} f(u) & \text{in } \Omega \times (0, +\infty), \\ a_p(x, \nabla u) \cdot \nu = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega$ is a smooth bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ and $\nu$ the Euclidian unit normal vector exterior to $\partial\Omega$.

The parabolic equation in Problem $(P^\varepsilon)$ contains, on the one hand, the inhomogeneous partial differential equation

$$u_t = \operatorname{div}(A(x)\nabla u) + \frac{1}{\varepsilon^2} f(u), \tag{3.1}$$

where $A(x)$ is a positively definite symmetric matrix depending on the spatial location, and, on the other hand, the anisotropic equation

$$u_t = \operatorname{div}\big(\mathcal{A}(\nabla u)\big) + \frac{1}{\varepsilon^2} f(u), \tag{3.2}$$

where the coefficients of the matrix $\nabla_p \otimes \mathcal{A} = \nabla_p\, {}^t\mathcal{A}$ may be singular in the point $p = 0$.

We suppose that the nonlinear reaction function $f$ is such that $f(u) = -W'(u)$, where $W(u)$ is a double-well potential with equal well-depth, taking its global minimum value at $u = 0$ and $u = 1$. More precisely we assume that $f$ is smooth and has exactly three zeros $0 < a < 1$ such that

$$f'(0) < 0, \quad f'(a) > 0, \quad f'(1) < 0, \tag{3.3}$$

and that

$$\int_0^1 f(u)du = 0. \tag{3.4}$$

The assumptions concerning the anisotropic term are the following.

1. $a(x, p)$ is a real valued function, of class $C^{3+\vartheta}_{loc}$ (for some $0 < \vartheta < 1$) on $\overline{\Omega} \times \mathbb{R}^N \backslash \{0\}$;
2. $a(x, p) > 0$ for all $(x, p) \in \overline{\Omega} \times \mathbb{R}^N \backslash \{0\}$;
3. $a(x, \cdot)$ is strictly convex for all $x \in \overline{\Omega}$;
4. $a(x, p)$ is 2 homogeneous in the $p$ variable, i.e.

$$a(x, \alpha p) = \alpha^2 a(x, p) \qquad \text{for all } (x, p) \in \overline{\Omega} \times \mathbb{R}^N \backslash \{0\}, \text{ all } \alpha \neq 0. \tag{3.5}$$

If $p$ is given by $p = (p_1, \cdots, p_N)$, the vector valued function $a_p$ is defined by

$$a_p(x, p) = \Big(\frac{\partial a}{\partial p_1}(x, p), \cdots, \frac{\partial a}{\partial p_N}(x, p)\Big),$$

and the matrix valued function $a_{pp}$ by

$$a_{pp}(x,p) = \Big(\frac{\partial^2 a}{\partial p_j \partial p_i}(x,p)\Big)_{1\le i,j\le N},$$

for all $(x,p) \in \overline{\Omega} \times (\mathbb{R}^N \setminus \{0\})$. Moreover, for a vector $p = (p_1, \cdots, p_N)$ and a matrix $A = (a_{ij})_{1\le i,j\le N}$, we use the notations

$$|p| = \max_i |p_i| \qquad \text{and} \qquad |A| = \max_{i,j} |a_{ij}|.$$

The special case (3.1) is obtained by taking $a(x,p) = \frac{1}{2}A(x)p\cdot p$, where $A(x)$ is a positively definite symmetric matrix; here, $a(x,p)$ is of class $C^{3+\vartheta}_{loc}$ on the whole $\overline{\Omega}\times\mathbb{R}^N$, $a_p(x,p) = A(x)p$ and $a_{pp}(x,p) = A(x)$. Setting $A(x) = \mathcal{I}$ leads to the classical Allen-Cahn equation. The special case (3.2) is obtained by assuming that $a(x,p) = a(p)$ and defining $\mathcal{A}(p) = a_p(p)$.

**Remark 3.1.1.** *By differentiating* (3.5) *with respect to $p$ and to $\alpha$, we see that, for all $(x,p) \in \overline{\Omega} \times \mathbb{R}^N\backslash\{0\}$, all $\alpha \neq 0$,*

$$\begin{aligned} a_p(x,\alpha p) &= \alpha a_p(x,p),\\ a_p(x,\alpha p)\cdot p &= 2\alpha a(x,p),\\ a_{pp}(x,\alpha p) &= a_{pp}(x,p),\\ a_{pp}(x,\alpha p)p &= a_p(x,p),\\ a_{pp}(x,\alpha p)p\cdot p &= 2a(x,p). \end{aligned}$$

*We may define $a(x,0) = 0$ and $a_p(x,0) = 0$, which implies, in view of* (3.5) *— i.e. $a(x,\cdot)$ is 2 homogeneous — and the first equality above — i.e. $a_p(x,\cdot)$ is 1 homogeneous — that $a(x,p)$ is of class $C^1$ on the whole $\overline{\Omega} \times \mathbb{R}^N$.* □

We also assume that the initial datum $u_0 \in C^2(\overline{\Omega})$, and define $C_0$ as

$$C_0 := \|u_0\|_{C^0(\overline{\Omega})} + \|\nabla u_0\|_{C^0(\overline{\Omega})} + \|D^2 u_0\|_{C^0(\overline{\Omega})}, \tag{3.6}$$

where $D^2u_0(x) = \Big(\frac{\partial^2 u_0}{\partial x_j \partial x_i}(x)\Big)_{1\le i,j\le N}$. Furthermore we define the "initial interface" $\Gamma_0$ by

$$\Gamma_0 := \{x \in \Omega,\ u_0(x) = a\},$$

and suppose that $\Gamma_0$ is a $C^{4+\vartheta}$ hypersurface without boundary such that, $n$ being the Euclidian unit normal vector exterior to $\Gamma_0$,

$$\Gamma_0 \subset\subset \Omega \quad \text{and} \quad \nabla u_0(x)\cdot a_p(x,n(x)) \neq 0 \quad \text{if } x \in \Gamma_0, \tag{3.7}$$

$$u_0 > a \quad \text{in } \Omega_0^+, \quad u_0 < a \quad \text{in } \Omega_0^-, \tag{3.8}$$

where $\Omega_0^-$ denotes the region enclosed by $\Gamma_0$ and $\Omega_0^+$ the region enclosed between $\partial\Omega$ and $\Gamma_0$.

For $T > 0$, we set $Q_T = \Omega \times (0,T)$. We define below a notion of weak solution for Problem $(P^\varepsilon)$. For this definition, it is sufficient to only suppose that $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$.

**Definition 3.1.2.** *A function $u^\varepsilon \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T)$ is a weak solution of Problem $(P^\varepsilon)$, if*

- $u_t^\varepsilon \in L^2(Q_T)$,
- $a_p(x, \nabla u^\varepsilon(x,t)) \in L^\infty(0,T; L^2(\Omega))$,
- $u^\varepsilon(x,0) = u_0(x)$ *for almost all* $x \in \Omega$,
- $u^\varepsilon$ *satisfies the integral equality*

$$\int_0^t \int_\Omega \Big[ u_t^\varepsilon \varphi + a_p(x, \nabla u^\varepsilon) \cdot \nabla \varphi - \frac{1}{\varepsilon^2} f(u^\varepsilon)\varphi \Big] = 0, \tag{3.9}$$

*for all nonnegative function* $\varphi \in L^2(0,T; H^1(\Omega)) \cap L^\infty(Q_T)$ *and for all* $t \in [0,T]$.

One may prove, using monotonicity and compactness arguments as is done in [12] and [14], that Problem $(P^\varepsilon)$ possesses a unique weak solution which we denote by $u^\varepsilon$. As $\varepsilon \to 0$, the qualitative behavior of this solution is the following. In the very early stage, the anisotropic diffusion term is negligible compared with the reaction term $\varepsilon^{-2} f(u)$. Hence, rescaling time by $\tau = t/\varepsilon^2$, the equation is well approximated by the ordinary differential equation $u_\tau = f(u)$. Since $f$ is a bistable function, $u^\varepsilon$ quickly approaches the values 0 or 1, the stable equilibria of $f$, and an interface is formed between the regions $\{u^\varepsilon \approx 0\}$ and $\{u^\varepsilon \approx 1\}$. Once such an interface is developed, the anisotropic diffusion term becomes large near the interface, and comes to balance with the reaction term so that the interface starts to propagate, in a much slower time scale.

To understand such interfacial behavior, we have to study the singular limit of $(P^\varepsilon)$ as $\varepsilon \to 0$. Then the limit solution $\tilde{u}(x,t)$ is a step function taking the values 0 and 1 on the sides of the moving interface $\Gamma_t$. In the case of the usual Allen-Cahn equation, it is well known that $\Gamma_t$ evolves according to the mean curvature flow $V_n = -(N-1)\kappa$. Here we will prove that the interface evolves according to the law

$$(P^0) \quad \begin{cases} \dfrac{1}{\sqrt{2a(x,n)}} V_n = -\nabla \cdot \Big[ \dfrac{1}{\sqrt{2a(x,n)}}\, a_p(x,n) \Big] & \text{on } \Gamma_t, \\ \Gamma_t \Big|_{t=0} = \Gamma_0. \end{cases}$$

where $n$ is the Euclidian unit normal vector exterior to $\Gamma_t$ and $V_n$ the normal velocity of $\Gamma_t$. We will show below that this equation can be rewritten in the relative geometry associated with a Finsler metric; it then has the form

$$(P^0) \quad \begin{cases} V_{n,\phi} = -(N-1)\overline{\kappa}_\phi & \text{on } \Gamma_t, \\ \Gamma_t \Big|_{t=0} = \Gamma_0. \end{cases}$$

where $V_{n,\phi}$ is the anisotropic normal velocity of $\Gamma_t$ in the anisotropic exterior direction, and $\overline{\kappa}_\phi$ an analogue of the anisotropic mean curvature at each point of $\Gamma_t$.

Almgren, Taylor and Wang [3] have proved that a problem related to the limit Problem $(P^0)$ possesses locally in time a unique smooth solution. Here we will suppose that there exists $T > 0$ such that Problem $(P^0)$ has a unique solution $\Gamma = \bigcup_{0 \le t \le T} (\Gamma_t \times \{t\})$ which satisfies $\Gamma \in C^{3+\vartheta, (3+\vartheta)/2}$. For proofs of the local in time existence of solutions of related limit problems, we also refer to [41].

For each $t \in (0,T)$, we define $\Omega_t^-$ as the region enclosed by the hypersurface $\Gamma_t$ and $\Omega_t^+$ as the region enclosed between $\partial\Omega$ and $\Gamma_t$. Further we define a function $\tilde{u}(x,t)$ by

$$\tilde{u}(x,t) = \begin{cases} 1 & \text{in } \Omega_t^+ \\ 0 & \text{in } \Omega_t^- \end{cases} \quad \text{for } t \in (0,T). \tag{3.10}$$

The aim of this Chapter is to study the limiting behavior of the solution $u^\varepsilon$ of Problem $(P^\varepsilon)$ as $\varepsilon \to 0$. We extend previous studies [13], [70] about a related anisotropic parabolic equation. It is convenient to present our main result, Theorem 3.1.3, in the form of a convergence theorem, mixing generation and propagation of interface. It describes the profile of the solution after a very short initial period. It asserts that: given a virtually arbitrary initial datum $u_0$, the solution $u^\varepsilon$ quickly becomes close to 0 or 1, except in a small neighborhood of the initial interface $\Gamma_0$, creating a steep transition layer around $\Gamma_0$ (*generation of interface*). The time $t^\varepsilon$ for the generation of interface is of order $\varepsilon^2|\ln\varepsilon|$. The theorem then states that the solution $u^\varepsilon$ remains close to the step function $\tilde{u}$ on the time interval $(t^\varepsilon, T)$ (*motion of interface*). Moreover, as is clear from the estimates in the theorem, the thickness of the transition layer is of order $\varepsilon$.

**Theorem 3.1.3 (Generation and motion of interface).** *Let $\eta$ be an arbitrary constant satisfying $0 < \eta < \frac{1}{2}\min(a, 1-a)$ and set*

$$\mu = f'(a).$$

*Then there exist positive constants $\varepsilon_0$ and $C$ such that, for all $\varepsilon \in (0,\varepsilon_0)$, for almost all $(x,t)$ such that $t^\varepsilon \le t \le T$, where*

$$t^\varepsilon := \mu^{-1}\varepsilon^2|\ln\varepsilon|, \tag{3.11}$$

*we have,*

$$u^\varepsilon(x,t) \in \begin{cases} [-\eta, 1+\eta] & \text{if} \quad x \in \mathcal{N}_{C\varepsilon}(\Gamma_t) \\ [-\eta, \eta] & \text{if} \quad x \in \Omega_t^- \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t) \\ [1-\eta, 1+\eta] & \text{if} \quad x \in \Omega_t^+ \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t), \end{cases} \tag{3.12}$$

*where $\mathcal{N}_r(\Gamma_t) := \{x \in \Omega, dist_\phi(x,\Gamma_t) < r\}$ denotes the $r$-neighborhood of $\Gamma_t$; by $dist_\phi(x,\Gamma_t)$, we mean the $\delta_\phi$ distance to the set $\Gamma_t$, where $\delta_\phi$ is the distance associated to a Finsler metric, whose definition is given in Section 3.2.*

**Corollary 3.1.4 (Convergence).** *As $\varepsilon \to 0$, the solution $u^\varepsilon$ converges to $\tilde{u}$ almost everywhere in $\bigcup_{0<t\le T}(\Omega_t^\pm \times \{t\})$.*

The organization of this Chapter is as follows. In Section 3.2, we recall notations and results concerning the Finsler metrics that turn out to be very efficient in the anisotropic context. In Section 3.3, we perform asymptotic expansions in order to derive the equation of the interface motion. In Section 3.4 we prove a weak comparison principle for Problem $(P^\varepsilon)$. In Section 3.5, we prove a generation of interface property. The sub- and super-solutions for this time range are constructed by modifying the solution of the ordinary differential equation $u_t = \varepsilon^{-2}f(u)$. In Section 3.6, we construct a pair of sub- and super-solutions for the study of the motion of interface, by using a related one-dimensional stationary problem. In Section 3.7, by fitting these two pairs of sub- and super-solutions into each other, we prove our main results for $(P^\varepsilon)$: Theorem 3.1.3 and its corollary.

For the proof of a propagation of interface property in the case of a related problem we refer to Bellettini, Colli Franzone and Paolini [8], who also give a precise approximation of the moving interface. We also refer to articles about the convergence to classical and viscosity solutions, in the case of well prepared initial data, by Elliott and Schätzle [31], [32], for a homogeneous function $a = a(p)$.

## 3.2 Finsler metrics and the anisotropic context

For the convenience of the reader, we first recall properties stated by Bellettini, Paolini and Venturini, [9] and [10]. For more details and proofs, see these references where the idea is to endow $\mathbb{R}^N$ with the distance obtained by integrating the Finsler metric and to work in relative geometry.

### 3.2.1 Finsler metrics

Suppose that $\phi : \Omega \times \mathbb{R}^N \to [0, +\infty[$ is a continuous function satisfying the properties

$$\phi(x, \alpha\xi) = |\alpha|\phi(x, \xi) \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R}^N \text{ and all } \alpha \in \mathbb{R}, \tag{3.13}$$

$$\lambda_0|\xi| \leq \phi(x, \xi) \leq \Lambda_0|\xi| \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R}^N, \tag{3.14}$$

for two suitable constants $0 < \lambda_0 \leq \Lambda_0 < +\infty$. We say that $\phi$ is strictly convex if, for any $x \in \Omega$, the map $\xi \mapsto \phi^2(x, \xi)$ is strictly convex on $\mathbb{R}^N$. We shall indicate by

$$B_\phi(x) = \{\xi \in \mathbb{R}^N, \phi(x, \xi) \leq 1\}$$

the unit sphere of $\phi$ at $x \in \Omega$.

The dual function $\phi^0 : \Omega \times \mathbb{R}^N \to [0, +\infty[$ of $\phi$ is defined by

$$\phi^0(x, \xi^*) = \sup\left\{\xi^* \cdot \xi, \ \xi \in B_\phi(x)\right\} \tag{3.15}$$

for any $(x, \xi) \in \Omega \times \mathbb{R}^N$. One can prove that $\phi^0$ is continuous, convex, satisfies properties (3.13) and (3.14), and that $\phi^{00}$, the dual function of $\phi^0$, coincides with the convex envelope of $\phi$ with respect to $\xi$.

We say that $\phi$ is a (strictly convex smooth) Finsler metric, and we shall write $\phi \in \mathcal{M}(\Omega)$ if, in addition to properties (3.13) and (3.14), $\phi$ and $\phi^0$ are strictly convex and of class $C^2$ on $\Omega \times (\mathbb{R}^N \setminus \{0\})$. In particular $\phi^{00} = \phi$.

We denote by $\delta_\phi$ the integrated distance associated to $\phi \in \mathcal{M}(\Omega)$, that is, for any $(x, y) \in \Omega$, we set

$$\delta_\phi(x, y) = \inf\left\{\int_0^1 \phi(\gamma(t), \dot{\gamma}(t))dt \ , \ \gamma \in W^{1,1}\big([0,1]; \Omega\big) \ , \ \gamma(0) = x \ , \ \gamma(1) = y\right\}. \tag{3.16}$$

In the special case of the Euclidian metric, the function $\phi$ is given by $\phi(x, p) = \phi(p) = ({p_1}^2 + \cdots + {p_N}^2)^{1/2}$, so that $\delta_\phi$ reduces to the usual distance.

Given $\phi \in \mathcal{M}(\Omega)$ and $x \in \Omega$, let $T^0(x, \cdot) : \mathbb{R}^N \to \mathbb{R}^N$ be the map defined by

$$T^0(x, \xi^*) = \begin{cases} \phi^0(x, \xi^*)\phi^0_p(x, \xi^*) & \text{if } \xi^* \in \mathbb{R}^N \setminus \{0\} \\ 0 & \text{if } \xi^* = 0. \end{cases} \tag{3.17}$$

Here, $\phi^0_p$ denotes the gradient with respect to $p$ whenever we regard $\phi^0(x, p)$ as a function of two variables $x$ and $p$. In the following, for better readability, some $x$ and $t$ dependencies are omitted. If $u : \Omega \to \mathbb{R}$ is a smooth function with non vanishing gradient, we define the anisotropic gradient by

$$\nabla_\phi u = T^0(x, \nabla u) = \phi^0(x, \nabla u)\phi^0_p(x, \nabla u). \tag{3.18}$$

In a similar way as in the isotropic case, if $\Gamma_t$ is a smooth hypersurface of $\Omega$ at time $t$, and $n$ the outer normal vector to $\Gamma_t$ (in the Euclidian sense), we define $n_\phi$ the $\phi$-normal vector to $\Gamma_t$ and $\overline{\kappa}_\phi$, an analogue of the $\phi$-mean curvature of $\Gamma_t$ — which differs from $\kappa_\phi$ defined in [9], [10] — by

$$n_\phi = \phi^0_p(x, n), \quad \overline{\kappa}_\phi = \frac{1}{N-1} \operatorname{div} n_\phi. \tag{3.19}$$

Furthermore, we have the following formulas: if $\psi$ is a smooth function with non vanishing gradient such that $\Gamma_t = \{x \in \Omega,\ \psi(x,t) = 0\}$, and $\psi$ is positive outside $\Gamma_t$ and negative inside, then

$$n = \frac{\nabla\psi}{|\nabla\psi|}, \qquad n_\phi = \phi^0_p(x, \nabla\psi), \tag{3.20}$$

$$\kappa = \frac{1}{N-1} \operatorname{div} \frac{\nabla\psi}{|\nabla\psi|}, \qquad \overline{\kappa}_\phi = \frac{1}{N-1} \operatorname{div} \phi^0_p(x, \nabla\psi), \tag{3.21}$$

on $\Gamma_t$. We also define the normal velocity of $\Gamma_t$ and the $\phi$-normal velocity of $\Gamma_t$ by

$$V_n = -\frac{\psi_t}{|\nabla\psi|}, \qquad V_{n,\phi} = -\frac{\psi_t}{\phi^0(x, \nabla\psi)}. \tag{3.22}$$

To conclude these preliminaries, we quote a theorem proved in [10].

**Theorem 3.2.1.** *Let $\Omega$ be connected, and let $\phi \in \mathcal{M}(\Omega)$. Let $\delta_\phi$ be the integrated distance associated to $\phi$. Let $C \subseteq \Omega$ be a closed set, and let $dist_\phi(x, C)$ be the $\delta_\phi$ distance to the set $C$ defined by*

$$dist_\phi(x, C) = \inf \big\{\delta_\phi(x, y)\ ,\ y \in C\big\}. \tag{3.23}$$

*Then*

$$\phi^0(x, \nabla dist_\phi(x, C)) = 1, \tag{3.24}$$

*at each point $x \in \Omega \setminus C$ where $dist_\phi(\cdot, C)$ is differentiable.*

In the special case of the Euclidian metric, (3.24) reduces to the property that $|\nabla d| = 1$.

### 3.2.2 Application to the anisotropic Allen-Cahn equation

We set, for all $(x, p) \in \Omega \times \mathbb{R}^N$,

$$\phi^0(x, p) = \sqrt{2a(x, p)}. \tag{3.25}$$

First, since $a(x, \cdot)$ is 2 homogeneous, $\phi^0$ satisfies assumptions (3.13) and (3.14) with the constants

$$\lambda_0 = [2 \min_{x \in \overline{\Omega}, |p|=1} a(x, p)]^{1/2} > 0 \qquad \text{and} \qquad \Lambda_0 = [2 \max_{x \in \overline{\Omega}, |p|=1} a(x, p)]^{1/2} > 0, \tag{3.26}$$

also using that $a$ is continuous and strictly positive on the compact set $\overline{\Omega} \times S^{N-1}$, with $S^{N-1}$ the unit sphere of $\mathbb{R}^N$. By the hypotheses on $a(x, p)$, we see that $\phi^0$ is strictly convex and of class $C^2$ on $\Omega \times (\mathbb{R}^N \setminus \{0\})$; moreover, by Remark 3.1.1, $\phi^0$ is continuous on $\Omega \times \mathbb{R}^N$. It follows that $\phi$ is a Finsler metric and the theory of the above subsection applies. We have

$$T^0(x,p) = \begin{cases} a_p(x,p) & \text{if } p \in \mathbb{R}^N \setminus \{0\} \\ 0 & \text{if } p = 0. \end{cases} \tag{3.27}$$

Let $\Gamma = \bigcup_{0 \le t \le T} (\Gamma_t \times \{t\})$ be the unique solution of the limit geometric motion Problem $(P^0)$ and let $\widetilde{d}$ be the signed distance function to $\Gamma$ defined by

$$\widetilde{d}(x,t) = \begin{cases} \ \ \operatorname{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^+, \\ - \ \operatorname{dist}(x,\Gamma_t) & \text{for } x \in \Omega_t^-, \end{cases} \tag{3.28}$$

where $\operatorname{dist}(x,\Gamma_t)$ is the distance from $x$ to the hypersurface $\Gamma_t$ in $\Omega$. Let $\widetilde{d}_\phi$ be the anisotropic signed distance function to $\Gamma$ defined by

$$\widetilde{d}_\phi(x,t) = \begin{cases} \ \ \operatorname{dist}_\phi(x,\Gamma_t) & \text{for } x \in \Omega_t^+, \\ - \ \operatorname{dist}_\phi(x,\Gamma_t) & \text{for } x \in \Omega_t^-, \end{cases} \tag{3.29}$$

where $\operatorname{dist}_\phi(x,\Gamma_t)$ denotes the $\delta_\phi$ distance to the set $\Gamma_t$ defined in (3.23). By Theorem 3.2.1, the following equality holds

$$2a(x, \nabla \widetilde{d}_\phi(x,t)) = 1 \qquad \text{in a neighborhood of } \Gamma_t. \tag{3.30}$$

We then write the second equalities in (3.20), (3.21), (3.22), once with $\psi = \widetilde{d}$ and once with $\psi = \widetilde{d}_\phi$ to obtain two formulas to express the $\phi$-normal vector $n_\phi$, the analogue of the $\phi$-mean curvature $\overline{\kappa}_\phi$ and the $\phi$-normal velocity $V_{n,\phi}$:

$$n_\phi = \frac{1}{\sqrt{2a(x,\nabla \widetilde{d})}}\, a_p(x, \nabla \widetilde{d}) = \ a_p(x, \nabla \widetilde{d}_\phi), \tag{3.31}$$

$$\overline{\kappa}_\phi = \frac{1}{N-1} \operatorname{div} \left[ \frac{1}{\sqrt{2a(x,\nabla \widetilde{d})}}\, a_p(x,\nabla \widetilde{d}) \right] = \frac{1}{N-1} \operatorname{div} \left[ a_p(x, \nabla \widetilde{d}_\phi) \right], \tag{3.32}$$

$$V_{n,\phi} = - \frac{1}{\sqrt{2a(x,\nabla \widetilde{d})}}\, \widetilde{d}_t = -(\widetilde{d}_\phi)_t. \tag{3.33}$$

The end of this section is devoted to the operator

$$\operatorname{div} \nabla_\phi u = \operatorname{div} T^0(x, \nabla u) = \nabla \cdot a_p(x, \nabla u), \tag{3.34}$$

which differs from the anisotropic Laplacian $\Delta_\phi u$ defined in [9], [10]. In the case of the Finsler metric, it turns out that the term $\operatorname{div} \nabla_\phi u$ may be less regular than $\Delta u$. Nevertheless, we show below a boundedness property.

**Lemma 3.2.2.** *There exists a positive constant $C_L$ such that, for all $u \in C^{2,1}(\overline{\Omega} \times [0,T])$, the following inequality holds.*

$$\left| \nabla \cdot a_p(x, \nabla u(x,t)) \right| \le C_L(|\nabla u(x,t)| + |D^2 u(x,t)|) \qquad \textit{for all } (x,t) \in \overline{Q_T}. \tag{3.35}$$

**Proof.** We can, with no loss of generality, ignore the dependence in time. First, assume that $x$ is such that $\nabla u(x) \neq 0$. Regarding $a(x,p)$ as a function of two variables $x$ and $p = (p_1, \cdots, p_n)$, we obtain, by a straightforward calculation, that

$$\nabla \cdot a_p(x, \nabla u(x)) = \sum_j \frac{\partial^2 a}{\partial x_j \partial p_j}(x, \nabla u(x)) + \sum_{i,j} \frac{\partial^2 a}{\partial p_i \partial p_j}(x, \nabla u(x)) \frac{\partial^2 u}{\partial x_i \partial x_j}(x). \tag{3.36}$$

We recall that $a_p(x,\cdot)$ is 1 homogeneous, and therefore $\frac{\partial^2 a}{\partial x_j \partial p_j}(x,\cdot)$ is 1 homogeneous as well, and that $a_{pp}(x,\cdot)$ is 0 homogeneous. It follows that

$$\begin{aligned}
|\nabla \cdot a_p(x, \nabla u(x))| \leq |\nabla u(x)| \sum_j \Big| \frac{\partial^2 a}{\partial x_j \partial p_j}(x, \frac{\nabla u(x)}{|\nabla u(x)|}) \Big| \\
+ |D^2 u(x)| \sum_{i,j} \Big| \frac{\partial^2 a}{\partial p_i \partial p_j}(x, \frac{\nabla u(x)}{|\nabla u(x)|}) \Big| \\
\leq |\nabla u(x)| \sum_j \max_{y \in \overline{\Omega}, |p|=1} \Big| \frac{\partial^2 a}{\partial x_j \partial p_j}(y,p) \Big| \\
+ |D^2 u(x)| \sum_{i,j} \max_{y \in \overline{\Omega}, |p|=1} \Big| \frac{\partial^2 a}{\partial p_i \partial p_j}(y,p) \Big|,
\end{aligned}$$

where we have used that $a \in C^{3+\vartheta}\big(\overline{\Omega} \times \mathbb{R}^N \setminus \{0\}\big) \subset C^2\big(\overline{\Omega} \times S^{N-1}\big)$. This proves (3.35) under the assumption $\nabla u(x) \neq 0$.

Now assume that $x$ is such that $\nabla u(x) = 0$. We have to proceed in a slightly different way since $a_{pp}(x, 0)$ does not make sense. The operator $a_p(x,\cdot)$ is 1 homogeneous so that, for any direction $\zeta$,

$$t^{-1}(a_p(x, t\zeta) - a_p(x, 0)) = a_p(x, \zeta).$$

We denote by $(e_1, \cdots, e_N)$ the Euclidian basis of $\mathbb{R}^N$. It follows from the above equality that $a_p(x,\cdot)$ admits at the point 0 partial derivatives in any direction $e_i$ and

$$\frac{\partial a_p(x,\cdot)}{\partial p_i}(0) = a_p(x, e_i), \tag{3.37}$$

which, in turn, implies that

$$\frac{\partial}{\partial p_i} \frac{\partial a}{\partial p_j}(x, 0) = \frac{\partial a}{\partial p_j}(x, e_i). \tag{3.38}$$

Note that, since $a_p(x,\cdot)$ is 1 homogeneous, and therefore $\frac{\partial^2 a}{\partial x_j \partial p_j}(x,\cdot)$ is 1 homogeneous as well, the first term in (3.36) vanishes at point $(x, 0)$. It follows, from (3.36) and (3.38) that, in the case where $\nabla u(x) = 0$,

$$\begin{aligned}
|\nabla \cdot a_p(x, \nabla u(x))| &= \Big| \sum_{i,j} \frac{\partial a}{\partial p_j}(x, e_i) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \Big| \\
&\leq |D^2 u(x)| \sum_{i,j} \max_{y \in \overline{\Omega}, |p|=1} \Big| \frac{\partial a}{\partial p_j}(y,p) \Big|,
\end{aligned}$$

which gives (3.35) in this case as well. □

**Remark 3.2.3.** *By similar arguments, one can obtain a positive constant $C_T$ such that, for all $u \in C^{2,1}(\overline{\Omega} \times [0,T])$,*

$$\Big| \frac{\partial}{\partial t} \big[ a_p(x, \nabla u(x,t)) \big] \Big| \leq C_T \big| D_x^1 D_t^1 u(x,t) \big| \qquad \textit{for all } (x,t) \in \overline{Q_T}. \tag{3.39}$$

## 3.3 Formal derivation of the interface motion equation

In this section we derive the equation of interface motion corresponding to Problem $(P^\varepsilon)$ by using a formal asymptotic expansion. The resulting interface equation can be regarded as the singular limit of $(P^\varepsilon)$ as $\varepsilon \to 0$. Our argument goes basically along the same lines with the formal derivation given by Nakamura, Matano, Hilhorst and Schätzle [63]: the first two terms of the asymptotic expansion determine the interface equation. Though our analysis in this section is for the most part formal, the results we obtain will help the rigorous analysis in later sections.

Let $u^\varepsilon$ be the solution of $(P^\varepsilon)$. Let $\Gamma = \bigcup_{0\le t\le T}(\Gamma_t \times \{t\})$ be the solution of the limit geometric motion problem and let $\widetilde{d}_\phi$ be the anisotropic signed distance function to $\Gamma$ defined in (3.29). We then define

$$Q_T^+ = \bigcup_{0<t\le T} (\Omega_t^+ \times \{t\}), \qquad Q_T^- = \bigcup_{0<t\le T} (\Omega_t^- \times \{t\}).$$

We also assume that the solution $u^\varepsilon$ has the expansions

$$u^\varepsilon(x,t) = 0 \text{ or } 1 + \varepsilon u_1(x,t) + \cdots, \tag{3.40}$$

away from the interface $\Gamma$ (the outer expansion) and

$$u^\varepsilon(x,t) = U_0(x,t,\xi) + \varepsilon U_1(x,t,\xi) + \varepsilon^2 U_2(x,t,\xi) + \cdots, \tag{3.41}$$

near $\Gamma$ (the inner expansion), where $U_j(x,t,z)$, $j = 0,1,2,\cdots$, are defined for $x \in \overline{\Omega}$, $t \ge 0$, $z \in \mathbb{R}$ and $\xi := \widetilde{d}_\phi(x,t)/\varepsilon$. The stretched space variable $\xi$ gives exactly the right spatial scaling to describe the sharp transition between the regions $\{u^\varepsilon \approx 0\}$ and $\{u^\varepsilon \approx 1\}$. We normalize $U_k$ in such a way that

$$U_0(x,t,0) = a, \quad U_k(x,t,0) = 0,$$

for all $k \ge 1$ (normalization conditions). To make the inner and outer expansions consistent, we require that

$$\begin{array}{ll} U_0(x,t,+\infty) = 1, & U_k(x,t,+\infty) = 0, \\ U_0(x,t,-\infty) = 0, & U_k(x,t,-\infty) = 0, \end{array} \tag{3.42}$$

for all $k \ge 1$ (matching conditions).

In what follows we will substitute the inner expansion (3.41) into the parabolic equation of Problem $(P^\varepsilon)$ and collect the $\varepsilon^{-2}$ and $\varepsilon^{-1}$ terms. To that purpose, note that if $V = V(x,t,z)$ and $v(x,t) = V(x,t,\xi)$ are real valued functions then we have $\nabla v = \frac{1}{\varepsilon} V_z \nabla \widetilde{d}_\phi + \nabla_x V$ and $v_t = \frac{1}{\varepsilon}(\widetilde{d}_\phi)_t V_z + V_t$; if $v$ and $V$ are vector valued functions we obtain $\nabla\cdot v = \frac{1}{\varepsilon}\nabla\widetilde{d}_\phi \cdot V_z + \nabla_x \cdot V$. A straightforward computation yields

$$\begin{aligned}
u_t^\varepsilon &= \frac{1}{\varepsilon}(\widetilde{d}_\phi)_t U_{0z} + U_{0t} + (\widetilde{d}_\phi)_t U_{1z} + \varepsilon U_{1t} + \cdots \\
\nabla u^\varepsilon &= \frac{1}{\varepsilon} U_{0z} \nabla \widetilde{d}_\phi + \nabla_x U_0 + U_{1z} \nabla \widetilde{d}_\phi + \varepsilon \nabla_x U_1 + \cdots \\
a_p(x, \nabla u^\varepsilon) &= \frac{1}{\varepsilon} a_p(x, U_{0z}\nabla\widetilde{d}_\phi + \varepsilon\nabla_x U_0 + \varepsilon U_{1z}\nabla\widetilde{d}_\phi + \varepsilon^2 \nabla_x U_1 + \cdots) \\
&= \frac{1}{\varepsilon} a_p(x, U_{0z}\nabla\widetilde{d}_\phi) + a_{pp}(x, U_{0z}\nabla\widetilde{d}_\phi)(\nabla_x U_0 + U_{1z}\nabla\widetilde{d}_\phi) + \cdots \\
&= \frac{1}{\varepsilon} U_{0z} a_p(x, \nabla\widetilde{d}_\phi) + a_{pp}(x, \nabla\widetilde{d}_\phi)(\nabla_x U_0 + U_{1z}\nabla\widetilde{d}_\phi) + \cdots,
\end{aligned}$$

where we have used the various homogeneity properties of $a$ and its derivatives. It follows that

$$\begin{aligned}\nabla\cdot a_p(x,\nabla u^\varepsilon) =& \frac{1}{\varepsilon}\nabla\widetilde{d}_\phi\cdot\partial_z(a_p(x,\nabla u^\varepsilon))+\nabla_x\cdot(a_p(x,\nabla u^\varepsilon))\\ =& \frac{1}{\varepsilon}\nabla\widetilde{d}_\phi\cdot\Big[\frac{1}{\varepsilon}U_{0zz}a_p(x,\nabla\widetilde{d}_\phi)+a_{pp}(x,\nabla\widetilde{d}_\phi)(\nabla_xU_{0z}+U_{1zz}\nabla\widetilde{d}_\phi)\Big]\\ &+\frac{1}{\varepsilon}\big[\nabla_xU_{0z}\cdot a_p(x,\nabla\widetilde{d}_\phi)+U_{0z}\nabla\cdot a_p(x,\nabla\widetilde{d}_\phi)\big]+\cdots\\ =& \frac{1}{\varepsilon^2}U_{0zz}2a(x,\nabla\widetilde{d}_\phi)+\frac{1}{\varepsilon}\Big[2a(x,\nabla\widetilde{d}_\phi)U_{1zz}+2\nabla_xU_{0z}\cdot a_p(x,\nabla\widetilde{d}_\phi)\\ &+U_{0z}\nabla\cdot a_p(x,\nabla\widetilde{d}_\phi)\Big]+\cdots,\end{aligned}$$

where we have used Remark 3.1.1 and where the functions $U_i$ $(i=0,1)$, as well as their derivatives, are taken at the point $(x,t,\dfrac{\widetilde{d}_\phi(x,t)}{\varepsilon})$. Hence, in view of (3.30), we obtain, in a neighborhood of $\Gamma_t$,

$$\nabla\cdot a_p(x,\nabla u^\varepsilon)=\frac{1}{\varepsilon^2}U_{0zz}+\frac{1}{\varepsilon}\Big[U_{1zz}+2\nabla_xU_{0z}\cdot a_p(x,\nabla\widetilde{d}_\phi)+U_{0z}\nabla\cdot a_p(x,\nabla\widetilde{d}_\phi)\Big]+\cdots$$

We also use the expansion

$$f(u^\varepsilon)=f(U_0)+\varepsilon U_1f'(U_0)+\cdots.$$

Next, we substitute the expressions above in the partial differential equation in Problem $(P^\varepsilon)$. Collecting the $\varepsilon^{-2}$ terms yields

$$U_{0zz}+f(U_0)=0. \tag{3.43}$$

In view of the normalization and matching conditions, we can now assert that $U_0(x,t,z)=U_0(z)$, where $U_0$ is the unique solution of the one-dimensional stationary problem

$$\begin{cases}U_0''+f(U_0)=0,\\ U_0(-\infty)=0,\quad U_0(0)=a,\quad U_0(+\infty)=1.\end{cases} \tag{3.44}$$

This solution represents the first approximation of the profile of a transition layer around the interface observed in the stretched coordinates. We recall standard estimates on $U_0$, see Chapter 1 for more details.

**Lemma 3.3.1.** *There exist positive constants $C$ and $\lambda$ such that the following estimates hold.*

$$\begin{aligned}0<1-U_0(z)&\le Ce^{-\lambda|z|}\quad \text{for } z\ge 0,\\ 0<U_0(z)&\le Ce^{-\lambda|z|}\quad \text{for } z\le 0.\end{aligned}$$

*In addition to this $U_0'>0$ and, for all $j=1,2$,*

$$|D^jU_0(z)|\le Ce^{-\lambda|z|}\quad \text{for } z\in\mathbb{R}.$$

Since $U_0$ depends only on the variable $z$, we have $\nabla_xU_0'=0$. Then, by collecting the $\varepsilon^{-1}$ terms, we obtain

$$U_{1zz}+f'(U_0)U_1=(\widetilde{d}_\phi)_tU_0'-\nabla\cdot a_p(x,\nabla\widetilde{d}_\phi)U_0', \tag{3.45}$$

which can be seen as a linearized problem for (3.43). The solvability condition for the above equation plays the key role in deriving the equation of interface motion. By Lemma 1.2.2 in Chapter 1, which is a variant of the Fredholm alternative, it is given by

$$\int_{\mathbf{R}} \Big[(\widetilde{d}_\phi)_t(x,t){U_0}'(z) - \nabla\cdot a_p(x,\nabla\widetilde{d}_\phi(x,t)){U_0}'(z)\Big]{U_0}'(z)dz = 0,$$

for all $(x,t)\in Q_T$. Since $\int_{\mathbf{R}}({U_0}')^2>0$, it follows that

$$(\widetilde{d}_\phi)_t = \nabla\cdot a_p(x,\nabla\widetilde{d}_\phi). \tag{3.46}$$

In virtue of the expressions of $\overline{\kappa}_\phi$ and $V_{n,\phi}$ in (3.32) and (3.33), the above equation, written in relative geometry, reads as

$$V_{n,\phi} = -(N-1)\overline{\kappa}_\phi \quad \text{on } \Gamma_t, \tag{3.47}$$

that is the interface motion equation $(P^0)$. Using again the formulas (3.32) and (3.33), one can come back to the Euclidian geometry and obtain the equivalent interface motion equation

$$\frac{1}{\sqrt{2a(x,n)}}\,V_n = -\nabla\cdot\Big[\frac{1}{\sqrt{2a(x,n)}}\,a_p(x,n)\Big] \quad \text{on } \Gamma_t. \tag{3.48}$$

Summarizing, under the assumption that the solution $u^\varepsilon$ of Problem $(P^\varepsilon)$ satisfies

$$u^\varepsilon \to \begin{cases} 1 & \text{in } Q_T^+ \\ 0 & \text{in } Q_T^- \end{cases} \quad \text{as } \varepsilon\to 0, \ \text{ almost everywhere,}$$

we have formally proved that the boundary $\Gamma_t$ between $\Omega_t^-$ and $\Omega_t^+$ moves according to the law (3.47) or (3.48).

**Remark 3.3.2.** *To conclude this section, note that combining* (3.46) *with* (3.45) *yields* $U_1=0$. *In fact, the second term of the asymptotic expansion vanishes because the two stable zeros of the nonlinearity $f$ have "balanced" stability, or more precisely because of the assumption* $\int_0^1 f(u)du=0$. *If we perturb the non linearity by order $\varepsilon$, say $f(u)\longleftarrow f(u)-\varepsilon g(u)$, the equation of the free boundary problem contains an additional term and $U_1$ no longer vanishes.*

## 3.4 A comparison principle

In this section, we prove a comparison principle for Problem $(P^\varepsilon)$. To begin with, we define a notion of sub- and super-solution for Problem $(P^\varepsilon)$. To that purpose, we suppose that $u_0\in H^1(\Omega)\cap L^\infty(\Omega)$.

**Definition 3.4.1.** *A function $u_\varepsilon^+\in L^2(0,T;H^1(\Omega))\cap L^\infty(Q_T)$ is a weak super-solution for Problem $(P^\varepsilon)$, if*

- $(u_\varepsilon^+)_t\in L^2(Q_T)$,
- $\nabla_\phi u_\varepsilon^+(x,t) = a_p(x,\nabla u_\varepsilon^+(x,t))\in L^\infty(0,T;L^2(\Omega))$,
- $u_\varepsilon^+(x,0)\geq u_0(x)$ *for almost all* $x\in\Omega$,
- $u^\varepsilon$ *satisfies the integral inequality*

$$\int_0^t\int_\Omega \Big[(u_\varepsilon^+)_t\varphi + a_p(x,\nabla u_\varepsilon^+)\cdot\nabla\varphi - \frac{1}{\varepsilon^2}f(u_\varepsilon^+)\varphi\Big] \geq 0, \tag{3.49}$$

*for all nonnegative function* $\varphi\in L^2(0,T;H^1(\Omega))\cap L^\infty(Q_T)$ *and for all* $t\in[0,T]$.

*We define a weak sub-solution* $u_\varepsilon^-$ *in a similar way, by changing* $\geq$ *in (3.49) by* $\leq$*, and with the condition* $u_\varepsilon^-(x,0) \leq u_0(x)$*, for almost all* $x \in \Omega$*.*

The following remark will reveal efficient when constructing our smooth sub- and super-solutions in later sections.

**Remark 3.4.2.** *Note that, by Lemma 3.2.2, if* $u \in C^{2,1}(\overline{Q_T})$*, then the function*

$$\mathcal{L}_0 u := u_t - \nabla \cdot a_p(x, \nabla u) - \frac{1}{\varepsilon^2} f(u),$$

*is well-defined in* $Q_T$*. Also, using Lemma 3.2.2, we deduce that* $\nabla \cdot a_p(x, \nabla u) \in L^\infty(Q_T)$*.*

*Integrating by parts, we deduce that if* $u_\varepsilon^+ \in C^{2,1}(\overline{Q_T})$ *satisfies* $\mathcal{L}_0 u_\varepsilon^+ \geq 0$ *almost everywhere, the anisotropic Neumann boundary condition* $a_p(x, \nabla u_\varepsilon^+) \cdot \nu = 0$ *on* $\partial\Omega \times (0,T)$*, and* $u_\varepsilon^+(x,0) \geq u_0(x)$ *for almost all* $x \in \Omega$*, then* $u_\varepsilon^+$ *is a super-solution for Problem* $(P^\varepsilon)$*; an analogous remark stands for a sub-solution* $u_\varepsilon^- \in C^{2,1}(\overline{Q_T})$*.* □

We prove below an inequality which expresses the strong monotonicity of the function $T^0(x,p) = a_p(x,p)$.

**Lemma 3.4.3.** *There exists a constant* $\beta > 0$ *such that, for all* $x \in \overline{\Omega}$*, for all* $p_1, p_2 \in \mathbb{R}^N$*,*

$$(a_p(x,p_2) - a_p(x,p_1)) \cdot (p_2 - p_1) \geq \beta |p_2 - p_1|^2. \tag{3.50}$$

**Proof.** First we consider the case that $sp_1 + (1-s)p_2 \neq 0$ for all $s \in [0,1]$. Then, the function $s \mapsto a(x, sp_1 + (1-s)p_2)$ is of class $C^2$ on $[0,1]$ and there exist $s_1, s_2$ such that

$$a(x,p_2) - a(x,p_1) = a_p(x,p_1) \cdot (p_2 - p_1) + \frac{1}{2}(p_2 - p_1) \cdot a_{pp}(x, s_1 p_1 + (1-s_1)p_2)(p_2 - p_1),$$

and

$$a(x,p_1) - a(x,p_2) = a_p(x,p_2) \cdot (p_1 - p_2) + \frac{1}{2}(p_1 - p_2) \cdot a_{pp}(x, s_2 p_1 + (1-s_2)p_2)(p_1 - p_2).$$

We claim that there exist $0 < \lambda_2 \leq \Lambda_2$ such that, for all $x \in \overline{\Omega}$, all $p \in \mathbb{R}^N \setminus \{0\}$, all $\bar{p} \in \mathbb{R}^N$,

$$\lambda_2 |\bar{p}|^2 \leq a_{pp}(x,p)\bar{p} \cdot \bar{p} \leq \Lambda_2 |\bar{p}|^2. \tag{3.51}$$

Indeed, it follows from the strict convexity of $a(x,\cdot)$ that $a_{pp}(x,p)$ is a positively definite symmetric matrix. Hence the function $(x,p,\bar{p}) \mapsto a_{pp}(x,p)\bar{p} \cdot \bar{p}$ is strictly positive and continuous on the compact set $\overline{\Omega} \times S^{N-1} \times S^{N-1}$ which, combined with the fact that $a_{pp}(x,\cdot)$ is 0 homogeneous, proves (3.51). It then follows that

$$a(x,p_2) - a(x,p_1) \geq a_p(x,p_1) \cdot (p_2 - p_1) + \frac{\lambda_2}{2}|p_2 - p_1|^2, \tag{3.52}$$

$$a(x,p_1) - a(x,p_2) \geq a_p(x,p_2) \cdot (p_1 - p_2) + \frac{\lambda_2}{2}|p_2 - p_1|^2. \tag{3.53}$$

Adding up inequalities (3.52) and (3.53) yields the desired inequality, with the constant $\beta = \lambda_2$.

In the case that $sp_1 + (1-s)p_2 = 0$ for some $s \in [0,1]$, $p_1$ and $p_2$ are colinear and we may suppose that there exists $l \in \mathbb{R}$ such that $p_2 = lp_1$. We can assume $l \neq 0$, $l \neq 1$ and $p_1 \neq 0$. By using the different homogeneity properties in Remark 3.1.1, we obtain that

$$\begin{aligned}(a_p(x,p_2) - a_p(x,p_1)) \cdot (p_2 - p_1) &= (l-1)^2 a_p(x,p_1) \cdot p_1 \\ &= 2(l-1)^2 a(x,p_1) \\ &= 2a(x,(l-1)p_1) \\ &\geq {\lambda_0}^2 |(l-1)p_1|^2 = {\lambda_0}^2 |p_2 - p_1|^2,\end{aligned}$$

where $\lambda_0$ has been defined in (3.26). The proof is now completed. □

We are now ready to prove the following comparison principle.

**Lemma 3.4.4.** *Suppose that $u_\varepsilon^+$, respectively $u_\varepsilon^-$, is a super-solution, respectively a sub-solution, for Problem $(P^\varepsilon)$; we have that*

$$u_\varepsilon^- \leq u^\varepsilon \leq u_\varepsilon^+ \quad \textit{almost everywhere in } Q_T.$$

**Proof.** By subtracting equality (3.9) for the solution $u^\varepsilon$ and inequality (3.49) for the super-solution $u_\varepsilon^+$, we obtain that, for all $\varphi \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T)$ such that $\varphi \geq 0$, and for all $t \in [0,T]$,

$$\begin{aligned}\int_0^t \int_\Omega \Big[ (u^\varepsilon - u_\varepsilon^+)_t \varphi + (a_p(x,\nabla u^\varepsilon) - a_p(x,\nabla u_\varepsilon^+)) \cdot \nabla\varphi \Big] & \\ \leq \int_0^t \int_\Omega \frac{1}{\varepsilon^2} \Big( f(u^\varepsilon) - f(u_\varepsilon^+) \Big) \varphi & \\ \leq C_1 \int_0^t \int_\Omega |u^\varepsilon - u_\varepsilon^+| \varphi, & \end{aligned} \tag{3.54}$$

where $C_1$ is the positive constant defined by

$$C_1 = \varepsilon^{-2} \|f'\|_{L^\infty\left(-\max(\|u^\varepsilon\|_\infty, \|u_\varepsilon^+\|_\infty), \max(\|u^\varepsilon\|_\infty, \|u_\varepsilon^+\|_\infty)\right)}.$$

Next we set $\varphi = (u^\varepsilon - u_\varepsilon^+)^+$, which belongs to $L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T)$; it follows from (3.50) that

$$\begin{aligned}\int_0^t \int_\Omega (a_p(x,\nabla u^\varepsilon) - a_p(x,\nabla u_\varepsilon^+)) \cdot \nabla\varphi & \\ = \int_0^t \int_{\{u^\varepsilon - u_\varepsilon^+ \geq 0\}} (a_p(x,\nabla u^\varepsilon) - a_p(x,\nabla u_\varepsilon^+)) \cdot (\nabla u^\varepsilon - \nabla u_\varepsilon^+) & \\ \geq \beta \int_0^t \int_{\{u^\varepsilon - u_\varepsilon^+ \geq 0\}} |\nabla u^\varepsilon - \nabla u_\varepsilon^+|^2. & \end{aligned}$$

Then, substituting this inequality into (3.54) yields

$$\frac{1}{2} \int_0^t \frac{d}{dt} \int_\Omega \Big( (u^\varepsilon - u_\varepsilon^+)^+ \Big)^2 + \beta \int_0^t \int_{\{u^\varepsilon - u_\varepsilon^+ \geq 0\}} |\nabla u^\varepsilon - \nabla u_\varepsilon^+|^2 \leq C_1 \int_0^t \int_{\{u^\varepsilon - u_\varepsilon^+ \geq 0\}} (u^\varepsilon - u_\varepsilon^+)^2,$$

and therefore

$$\int_\Omega \Big( (u^\varepsilon - u_\varepsilon^+)^+ \Big)^2 (t) \leq 2C_1 \int_0^t \int_\Omega \Big( (u^\varepsilon - u_\varepsilon^+)^+ \Big)^2 + \int_\Omega \Big( (u^\varepsilon - u_\varepsilon^+)^+ \Big)^2 (0).$$

Using Gronwall's Lemma, we find that

$$\int_\Omega \Big((u^\varepsilon - u_\varepsilon^+)^+\Big)^2(t) \le e^{2C_1 t}\int_\Omega \Big((u^\varepsilon - u_\varepsilon^+)^+\Big)^2(0),$$

Since $u^\varepsilon(x,0) \le u_\varepsilon^+(x,0)$ for almost all $x \in \Omega$, it follows that

$$u^\varepsilon \le u_\varepsilon^+ \quad \text{a.e. in } Q_T.$$

Similarly one can show that $u_\varepsilon^- \le u^\varepsilon$ a.e. in $Q_T$. □

**Lemma 3.4.5.** *Let $u^\varepsilon$ be a solution of Problem $(P^\varepsilon)$. Then*

$$-\|u_0\|_{L^\infty(\Omega)} \le u^\varepsilon \le \max(1, \|u_0\|_{L^\infty(\Omega)}) \qquad a.e.\ in\ \ Q_T.$$

**Proof.** We remark that $-\|u_0\|_{L^\infty(\Omega)}$ and that $\max(1, \|u_0\|_{L^\infty(\Omega)})$ are sub- and super-solutions for Problem $(P^\varepsilon)$. □

## 3.5 Generation of interface

This section deals with the generation of interface, namely the rapid formation of internal layers that takes place in a neighborhood of $\Gamma_0 = \{x \in \Omega,\, u_0(x) = a\}$ within the time span of order $\varepsilon^2|\ln\varepsilon|$. In the sequel, $\eta_0$ will stand for the quantity

$$\eta_0 := \frac{1}{2}\min(a, 1-a).$$

Our main result in this section is the following.

**Theorem 3.5.1.** *Let $\eta \in (0, \eta_0)$ be arbitrary and define $\mu$ as the derivative of $f(u)$ at the unstable equilibrium $u = a$, that is*

$$\mu = f'(a). \tag{3.55}$$

*Then there exist positive constants $\varepsilon_0$ and $M_0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

- *for almost all $x \in \Omega$,*
$$-\eta \le u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \le 1 + \eta, \tag{3.56}$$

- *for almost all $x \in \Omega$ such that $|u_0(x) - a| \ge M_0\varepsilon$, we have that*

$$\text{if} \quad u_0(x) \ge a + M_0\varepsilon \quad \text{then} \quad u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \ge 1 - \eta, \tag{3.57}$$
$$\text{if} \quad u_0(x) \le a - M_0\varepsilon \quad \text{then} \quad u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \le \eta. \tag{3.58}$$

We will prove this result by constructing a suitable pair of sub and super-solutions.

### 3.5.1 The bistable ordinary differential equation

The sub- and super-solutions mentioned above will be constructed by modifying the solution of the problem without diffusion:

$$\bar u_t = \frac{1}{\varepsilon^2} f(\bar u), \qquad \bar u(x, 0) = u_0(x).$$

This solution is written in the form

$$\bar{u}(x,t) = Y\Big(\frac{t}{\varepsilon^2},\, u_0(x)\Big),$$

where $Y(\tau,\xi)$ denotes the solution of the ordinary differential equation

$$\begin{cases} Y_\tau(\tau,\xi) &= f(Y(\tau,\xi)) \quad \text{for } \tau > 0, \\ Y(0,\xi) &= \xi. \end{cases} \tag{3.59}$$

Here $\xi$ ranges over the interval $(-2C_0, 2C_0)$, with $C_0$ being the constant defined in (3.6). We first collect basic properties of $Y$.

**Lemma 3.5.2.** *We have $Y_\xi > 0$, for all $\xi \in (-2C_0, 2C_0)\backslash\{0, a, 1\}$ and all $\tau > 0$. Furthermore,*

$$Y_\xi(\tau,\xi) = \frac{f(Y(\tau,\xi))}{f(\xi)}.$$

**Proof.** First, differentiating equation (3.59) with respect to $\xi$, we obtain

$$\begin{cases} Y_{\xi\tau} = Y_\xi f'(Y), \\ Y_\xi(0,\xi) = 1, \end{cases} \tag{3.60}$$

which can be integrated as follows:

$$Y_\xi(\tau,\xi) = \exp\Big[\int_0^\tau f'(Y(s,\xi))ds\Big] > 0. \tag{3.61}$$

We then differentiate equation (3.59) with respect to $\tau$ and obtain

$$\begin{cases} Y_{\tau\tau} = Y_\tau f'(Y), \\ Y_\tau(0,\xi) = f(\xi), \end{cases} \tag{3.62}$$

which in turn implies

$$\begin{aligned} Y_\tau(\tau,\xi) &= f(\xi)\exp\Big[\int_0^\tau f'(Y(s,\xi))ds\Big] \\ &= f(\xi) Y_\xi(\tau,\xi). \end{aligned} \tag{3.63}$$

This last equality, in view of (3.59), completes the proof of Lemma 3.5.2. □

We define a function $A(\tau,\xi)$ by

$$A(\tau,\xi) = \frac{f'(Y(\tau,\xi)) - f'(\xi)}{f(\xi)}. \tag{3.64}$$

**Lemma 3.5.3.** *We have, for all $\xi \in (-2C_0, 2C_0) \setminus \{0, a, 1\}$ and all $\tau > 0$,*

$$A(\tau,\xi) = \int_0^\tau f''(Y(s,\xi))Y_\xi(s,\xi)ds.$$

**Proof.** Differentiating the equality of Lemma 3.5.2 with respect to $\xi$ leads to

$$Y_{\xi\xi} = A(\tau, \xi) Y_\xi, \tag{3.65}$$

whereas differentiating (3.61) with respect to $\xi$ yields

$$Y_{\xi\xi} = Y_\xi \int_0^\tau f''(Y(s,\xi)) Y_\xi(s,\xi) ds.$$

These two last results complete the proof of Lemma 3.5.3. □

Next we need some estimates on $Y$ and its derivatives. First, we perform some estimates when the initial value $\xi$ lies between $\eta$ and $1-\eta$.

**Lemma 3.5.4.** *Let $\eta \in (0, \eta_0)$ be arbitrary. Then there exist positive constants $\tilde{C}_1 = \tilde{C}_1(\eta)$, $\tilde{C}_2 = \tilde{C}_2(\eta)$ and $C_3 = C_3(\eta)$ such that, for all $\tau > 0$,*

- *if $\xi \in (a, 1-\eta)$ then, for every $\tau > 0$ such that $Y(\tau, \xi)$ remains in the interval $(a, 1-\eta)$, we have*
$$\tilde{C}_1 e^{\mu\tau} \le Y_\xi(\tau, \xi) \le \tilde{C}_2 e^{\mu\tau}, \tag{3.66}$$
*and*
$$|A(\tau, \xi)| \le C_3 (e^{\mu\tau} - 1); \tag{3.67}$$
- *if $\xi \in (\eta, a)$ then, for every $\tau > 0$ such that $Y(\tau, \xi)$ remains in the interval $(\eta, a)$,* (3.66) *and* (3.67) *hold as well,*

*where $\mu$ is the constant defined in* (3.55).

**Proof.** We take $\xi \in (a, 1-\eta)$ and suppose that for $s \in (0, \tau)$, $Y(s, \xi)$ remains in the interval $(a, 1-\eta)$. Integrating the equality

$$\frac{Y_\tau(s,\xi)}{f(Y(s,\xi))} = 1$$

from 0 to $\tau$ yields

$$\int_0^\tau \frac{Y_\tau(s,\xi)}{f(Y(s,\xi))} ds \quad = \tau. \tag{3.68}$$

Hence by the change of variable $q = Y(s, \xi)$ we get

$$\int_\xi^{Y(\tau,\xi)} \frac{dq}{f(q)} = \tau. \tag{3.69}$$

Moreover, the equality of Lemma 3.5.2 leads to

$$\begin{aligned} \ln Y_\xi(\tau, \xi) &= \int_\xi^{Y(\tau,\xi)} \frac{f'(q)}{f(q)} dq \\ &= \int_\xi^{Y(\tau,\xi)} [\frac{f'(a)}{f(q)} + \frac{f'(q) - f'(a)}{f(q)}] dq \\ &= \mu\tau + \int_\xi^{Y(\tau,\xi)} h(q) dq, \end{aligned} \tag{3.70}$$

where

$$h(q) = (f'(q) - \mu)/f(q).$$

Since

$$h(q) \to \frac{f''(a)}{f'(a)} \quad \text{as } q \to a,$$

the function $h$ is continuous on $[a, 1-\eta]$. Hence we can define

$$H = H(\eta) := \|h\|_{L^\infty(a,1-\eta)}.$$

Since $|Y(\tau,\xi) - \xi|$ takes its values in the interval $[0, 1-a-\eta] \subset [0, 1-a]$, it follows from (3.70) that

$$\mu\tau - H(1-a) \le \ln Y_\xi(\tau,\xi) \le \mu\tau + H(1-a),$$

which, in turn, proves (3.66). Lemma 3.5.3 and (3.66) yield

$$\begin{aligned} |A(\tau,\xi)| &\le \sup_{z\in[0,1]} |f''(z)| \int_0^\tau \tilde{C}_2 e^{\mu s} ds \\ &\le C_3(e^{\mu\tau} - 1), \end{aligned}$$

which completes the proof of (3.67). The case where $\xi$ and $Y(\tau,\xi)$ are in $(\eta, a)$ is similar and omitted. □

**Corollary 3.5.5.** *Let $\eta \in (0, \eta_0)$ be arbitrary. Then there exist positive constants $C_1 = C_1(\eta)$ and $C_2 = C_2(\eta)$ such that, for all $\tau > 0$,*

- *if $\xi \in (a, 1-\eta)$ then, for every $\tau > 0$ such that $Y(\tau,\xi)$ remains in the interval $(a, 1-\eta)$, we have*

$$C_1 e^{\mu\tau}(\xi - a) \le Y(\tau,\xi) - a \le C_2 e^{\mu\tau}(\xi - a); \tag{3.71}$$

- *if $\xi \in (\eta, a)$ then, for every $\tau > 0$ such that $Y(\tau,\xi)$ remains in the interval $(\eta, a)$, we have*

$$C_2 e^{\mu\tau}(\xi - a) \le Y(\tau,\xi) - a \le C_1 e^{\mu\tau}(\xi - a). \tag{3.72}$$

**Proof.** Since

$$f(q)/(q-a) \to f'(a) = \mu \quad \text{as } q \to a,$$

it is possible to find $B_1 = B_1(\eta) > 0$ and $B_2 = B_2(\eta) > 0$ such that, for all $q \in (a, 1-\eta)$,

$$B_1(q-a) \le f(q) \le B_2(q-a). \tag{3.73}$$

We write this inequality for $a < Y(\tau,\xi) < 1-\eta$ to obtain

$$B_1(Y(\tau,\xi) - a) \le f(Y(\tau,\xi)) \le B_2(Y(\tau,\xi) - a).$$

We also write this inequality for $a < \xi < 1-\eta$ to obtain

$$B_1(\xi - a) \le f(\xi) \le B_2(\xi - a).$$

Next we use the equality $Y_\xi = f(Y)/f(\xi)$ of Lemma 3.5.2 to deduce that

$$\frac{B_1}{B_2}(Y(\tau,\xi) - a) \le (\xi - a) Y_\xi(\tau,\xi) \le \frac{B_2}{B_1}(Y(\tau,\xi) - a),$$

which, in view of (3.66), implies that

$$\frac{B_1}{B_2}\tilde{C}_1 e^{\mu\tau}(\xi - a) \le Y(\tau,\xi) - a \le \frac{B_2}{B_1}\tilde{C}_2 e^{\mu\tau}(\xi - a).$$

This proves (3.71). The proof of (3.72) is similar and omitted. □

Next we present estimates in the case where the initial value $\xi$ is smaller than $\eta$ or larger than $1-\eta$.

**Lemma 3.5.6.** *Let $\eta \in (0, \eta_0)$ and $M > 0$ be arbitrary. Then there exists a positive constant $C_4 = C_4(\eta, M)$ such that*

- *if $\xi \in [1-\eta, 1+M]$, then, for all $\tau > 0$, $Y(\tau, \xi)$ remains in the interval $[1-\eta, 1+M]$ and*
$$|A(\tau,\xi)| \le C_4\tau \quad \text{for} \quad \tau > 0\,; \tag{3.74}$$
- *if $\xi \in [-M, \eta]$, then, for all $\tau > 0$, $Y(\tau, \xi)$ remains in the interval $[-M, \eta]$ and (3.74) holds as well.*

**Proof.** Since the two statements can be treated in the same way, we will only prove the former. The fact that $Y(\tau, \xi)$, the solution of the ordinary differential equation (3.59), remains in the interval $[1-\eta, 1+M]$ directly follows from the bistable properties of $f$, or, more precisely, from the sign conditions $f(1-\eta) > 0$, $f(1+M) < 0$.

To prove (3.74), suppose first that $\xi \in [1, 1+M]$. In view of (3.3), $f'$ is strictly negative in an interval of the form $[1, 1+c]$ and $f$ is negative in $[1, \infty)$. We denote by $-m < 0$ the maximum of $f$ on $[1+c, 1+M]$. Then, as long as $Y(\tau, \xi)$ remains in the interval $[1+c, 1+M]$, the ordinary differential equation (3.59) implies

$$Y_\tau \le -m.$$

By integration, this means that, for any $\xi \in [1, 1+M]$, we have

$$Y(\tau,\xi) \in [1, 1+c] \qquad \text{for} \quad \tau \ge \overline{\tau} := \frac{M-c}{m}.$$

In view of this, and considering that $f'(Y) < 0$ for $Y \in [1, 1+c]$, we see from the expression (3.61) that

$$\begin{aligned} Y_\xi(\tau,\xi) &= \exp\Big[\int_0^{\overline{\tau}} f'(Y(s,\xi))ds\Big] \exp\Big[\int_{\overline{\tau}}^{\tau} f'(Y(s,\xi))ds\Big] \\ &\le \exp\Big[\int_0^{\overline{\tau}} f'(Y(s,\xi))ds\Big] \\ &\le \exp\Big[\int_0^{\overline{\tau}} \sup_{z\in[-M,1+M]} |f'(z)|ds\Big] =: \tilde{C}_4 = \tilde{C}_4(M), \end{aligned}$$

for all $\tau \ge \overline{\tau}$. It is clear from the same expression (3.61) that $Y_\xi \le \tilde{C}_4$ holds also for $0 \le \tau \le \overline{\tau}$. We can then use Lemma 3.5.3 to deduce that

$$\begin{aligned} |A(\tau,\xi)| &\le \tilde{C}_4 \int_0^{\tau} |f''(Y(s,\xi))|ds \\ &\le \tilde{C}_4\Big(\sup\nolimits_{z\in[-M,1+M]} |f''(z)|\Big)\tau =: C_4\tau. \end{aligned}$$

The case $\xi \in [1-\eta, 1]$ can be treated in the same way. This completes the proof of the lemma. □

Now we choose the constant $M$ in the above lemma sufficiently large so that $[-2C_0, 2C_0] \subset [-M, 1+M]$, and fix $M$ hereafter. Then $C_4$ only depends on $\eta$. Using the fact that $\tau = O(e^{\mu\tau} - 1)$ for $\tau > 0$, one can easily deduce from (3.67) and (3.74) the following general estimate.

**Lemma 3.5.7.** *Let $\eta \in (0, \eta_0)$ be arbitrary and let $C_0$ be the constant defined in (3.6). Then there exists a positive constant $C_5 = C_5(\eta)$ such that, for all $\xi \in (-2C_0, 2C_0)$ and all $\tau > 0$,*

$$|A(\tau,\xi)| \le C_5(e^{\mu\tau} - 1).$$

### 3.5.2 Construction of sub- and super-solutions

We are now ready to construct the sub- and super-solutions for the study of generation of interface. To make the proof less technical, we make the additional assumption

$$a_p(x, \nabla u_0(x)) \cdot \nu = 0 \quad \text{on } \partial\Omega. \tag{3.75}$$

In this case, our sub- and super-solutions are given by

$$w_\varepsilon^\pm(x,t) = Y\Big(\frac{t}{\varepsilon^2},\, u_0(x) \pm \varepsilon^2 C_6(e^{\mu t/\varepsilon^2} - 1)\Big). \tag{3.76}$$

In the general case where (3.75) does not necessarily hold, we have to slightly modify $w_\varepsilon^\pm(x,t)$ near the boundary $\partial\Omega$. This can be done by using some cut-off initial data $u_0^\pm$ (see Chapter 1, Section 1.3).

**Lemma 3.5.8.** *There exist positive constants $\varepsilon_0$ and $C_6$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, $(w_\varepsilon^-, w_\varepsilon^+)$ is a pair of sub- and super-solutions for Problem $(P^\varepsilon)$, in the domain*

$$\big\{(x,t) \in Q_T,\ x \in \Omega,\ 0 \le t \le \mu^{-1}\varepsilon^2 |\ln \varepsilon|\big\}.$$

**Proof.** First, we remark that $w^\pm(x,0) = Y\Big(\frac{t}{\varepsilon^2},\, u_0(x)\Big) = u_0(x)$. Next we define the operator $\mathcal{L}_0$ by

$$\mathcal{L}_0 u := u_t - \nabla \cdot a_p(x, \nabla u) - \frac{1}{\varepsilon^2} f(u), \tag{3.77}$$

and prove that $\mathcal{L}_0 w_\varepsilon^+ \ge 0$. Straightforward calculations yield

$$\begin{aligned} ({w_\varepsilon}^+)_t &= \frac{1}{\varepsilon^2} Y_\tau + \mu C_6 e^{\mu t/\varepsilon^2} Y_\xi, \\ \nabla w_\varepsilon^+ &= \nabla u_0(x) Y_\xi. \end{aligned}$$

First, using (3.75) and the fact that $a_p(x,\cdot)$ is 1 homogeneous, we see that $w_\varepsilon^\pm$ satisfy the anisotropic Neumann boundary condition

$$a_p(x, \nabla w_\varepsilon^\pm) \cdot \nu = 0 \quad \text{on } \partial\Omega \times (0, +\infty).$$

In view of the ordinary differential equation (3.59), we obtain

$$\mathcal{L}_0 w_\varepsilon^+ = \mu C_6 e^{\mu t/\varepsilon^2} Y_\xi - \nabla \cdot a_p(x, \nabla w_\varepsilon^+).$$

By the estimate in Lemma 3.2.2, it follows that

$$\mathcal{L}_0 w_\varepsilon^+ \ge \mu C_6 e^{\mu t/\varepsilon^2} Y_\xi - C_L(|\nabla w_\varepsilon^+(x,t)| + |D^2 w_\varepsilon^+(x,t)|), \tag{3.78}$$

where we recall that

$$|D^2 w_\varepsilon^+(x,t)| = \max_{i,j} |\partial_i \partial_j w_\varepsilon^+(x,t)|.$$

A straightforward calculation yields

$$\partial_i \partial_j w_\varepsilon^+(x,t) = (\partial_i \partial_j u_0) Y_\xi + (\partial_i u_0 \partial_j u_0) Y_{\xi\xi}.$$

Recalling that $Y_\xi > 0$, we now combine the expression of $\nabla w_\varepsilon^+$, the above expression and inequality (3.78) to obtain

$$\mathcal{L}_0 w_\varepsilon^+ / Y_\xi \ge \mu C_6 e^{\mu t/\varepsilon^2} - C_L C_0 - C_0 - {C_0}^2 \frac{|Y_{\xi\xi}|}{Y_\xi}, \tag{3.79}$$

where $C_0$ is the constant defined in (3.6). We note that, in the range $0 \le t \le \mu^{-1}\varepsilon^2|\ln\varepsilon|$, we have

$$0 \le \varepsilon^2 C_6(e^{\mu t/\varepsilon^2} - 1) \le \varepsilon^2 C_6(\varepsilon^{-1} - 1) \le C_0,$$

if $\varepsilon_0$ is sufficiently small. Hence

$$\xi := u_0(x) + C_6(e^{\mu t/\varepsilon^2} - 1) \in (-2C_0, 2C_0),$$

so that, by the results of the previous subsection, $Y$ remains in $(-2C_0, 2C_0)$. In view of (3.65), $Y_{\xi\xi}/Y_\xi$ is equal to $A$ so that, combining the estimate of $A$ in Lemma 3.5.7 and (3.79) yield

$$\mathcal{L}_0 w_\varepsilon^+/Y_\xi \ge (\mu C_6 - {C_0}^2 C_5)e^{\mu t/\varepsilon^2} - C_L C_0 - C_0.$$

Now, choosing

$$C_6 \ge \frac{2}{\mu}\max\left({C_0}^2 C_5, C_0(C_L + 1)\right)$$

proves $\mathcal{L}_0 w_\varepsilon^+/Y_\xi \ge 0$. Since $Y_\xi > 0$, it follows that $\mathcal{L}_0 w_\varepsilon^+ \ge 0$. Hence, by Remark 3.4.2, $w_\varepsilon^+$ is a super-solution for Problem $(P^\varepsilon)$. Similarly $w_\varepsilon^-$ is a sub-solution. Lemma 3.5.8 is proved. □

Consequently, by the comparison principle proved in Lemma 3.4.4,

$$w_\varepsilon^-(x,t) \le u^\varepsilon(x,t) \le w_\varepsilon^+(x,t), \tag{3.80}$$

for almost all $(x,t) \in Q_T$ that satisfies $0 \le t \le \mu^{-1}\varepsilon^2|\ln\varepsilon|$.

### 3.5.3 Proof of Theorem 3.5.1

In order to prove Theorem 3.5.1 we first present a key estimate on the function $Y$ after a time interval of order $\tau \sim |\ln\varepsilon|$.

**Lemma 3.5.9.** *Let $\eta \in (0, \eta_0)$ be arbitrary; there exist positive constants $\varepsilon_0$ and $C_7$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

- *for all $\xi \in (-2C_0, 2C_0)$,*
$$-\eta \le Y(\mu^{-1}|\ln\varepsilon|, \xi) \le 1 + \eta, \tag{3.81}$$
- *for all $\xi \in (-2C_0, 2C_0)$ such that $|\xi - a| \ge C_7\varepsilon$, we have that*

$$\text{if} \quad \xi \ge a + C_7\varepsilon \quad \text{then} \quad Y(\mu^{-1}|\ln\varepsilon|, \xi) \ge 1 - \eta, \tag{3.82}$$
$$\text{if} \quad \xi \le a - C_7\varepsilon \quad \text{then} \quad Y(\mu^{-1}|\ln\varepsilon|, \xi) \le \eta. \tag{3.83}$$

**Proof.** We first prove (3.82). For $\xi \ge a + C_7\varepsilon$, as long as $Y(\tau, \xi)$ has not reached $1 - \eta$, we can use (3.71) to deduce that

$$\begin{aligned} Y(\tau,\xi) &\ge a + C_1 e^{\mu\tau}(\xi - a) \\ &\ge a + C_1 C_7 e^{\mu\tau}\varepsilon \\ &\ge 1 - \eta, \end{aligned}$$

provided that $\tau$ satisfies

$$\tau \ge \mu^{-1}\ln\frac{1 - a - \eta}{C_1 C_7 \varepsilon} =: \tau^\varepsilon.$$

Choosing

$$C_7 = \frac{\max(a, 1-a) - \eta}{C_1},$$

we see that $\mu^{-1}|\ln\varepsilon| \geq \tau^\varepsilon$, which completes the proof of (3.82). Using (3.72), one easily proves (3.83).

Next we prove (3.81). First, by the bistable assumptions on $f$, if we leave from a $\xi \in [-\eta, 1+\eta]$ then $Y(\tau, \xi)$ will remain in $[-\eta, 1+\eta]$. Now suppose that $1+\eta \leq \xi \leq 2C_0$. We check below that $Y(\mu^{-1}|\ln\varepsilon|, \xi) \leq 1+\eta$. First, in view of (3.3), we can find $p > 0$ such that

$$\begin{array}{llll} \text{if} & 1 \leq u \leq 2C_0 & \text{then} & f(u) \leq p(1-u) \\ \text{if} & -2C_0 \leq u \leq 0 & \text{then} & f(u) \geq -pu. \end{array} \tag{3.84}$$

We then use the ordinary differential equation (3.59) to obtain, as long as $1+\eta \leq Y \leq 2C_0$, the inequality $Y_\tau \leq p(1-Y)$. It follows that

$$\frac{Y_\tau}{Y-1} \leq -p.$$

Integrating this inequality from 0 to $\tau$ leads to

$$\begin{aligned} Y(\tau, \xi) &\leq 1 + (\xi - 1)e^{-p\tau} \\ &\leq 1 + (2C_0 - 1)e^{-p\tau}. \end{aligned}$$

Since $(2C_0 - 1)e^{-p\mu^{-1}|\ln\varepsilon|} \to 0$ as $\varepsilon \to 0$, the above inequality proves that, for $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 = \varepsilon_0(\eta)$ sufficiently small, $Y(\mu^{-1}|\ln\varepsilon|, \xi) \leq 1+\eta$, which completes the proof of (3.81). □

We are now ready to prove Theorem 3.5.1. By setting $t = \mu^{-1}\varepsilon^2|\ln\varepsilon|$ in (3.80), we obtain, for almost all $x \in \Omega$,

$$\begin{aligned} &Y\Big(\mu^{-1}|\ln\varepsilon|, u_0(x) - (C_6\varepsilon - C_6\varepsilon^2)\Big) \\ &\qquad \leq u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \leq Y\Big(\mu^{-1}|\ln\varepsilon|, u_0(x) + C_6\varepsilon - C_6\varepsilon^2\Big). \end{aligned} \tag{3.85}$$

Furthermore, by the definition of $C_0$ in (3.6), we have, for $\varepsilon_0$ small enough,

$$-2C_0 \leq u_0(x) \pm (C_6\varepsilon - C_6\varepsilon^2) \leq 2C_0,$$

for $x \in \Omega$. Thus the assertion (3.56) of Theorem 3.5.1 is a direct consequence of (3.81) and (3.85).

Next we prove (3.57). We choose $M_0$ large enough so that $M_0\varepsilon - C_6\varepsilon + C_6\varepsilon^2 \geq C_7\varepsilon$. Then, for any $x \in \Omega$ such that $u_0(x) \geq a + M_0\varepsilon$, we have

$$u_0(x) - (C_6\varepsilon - C_6\varepsilon^2) \geq a + M_0\varepsilon - C_6\varepsilon + C_6\varepsilon^2 \geq a + C_7\varepsilon.$$

Combining this, (3.85) and (3.82), we see that

$$u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \geq 1 - \eta,$$

for almost all $x \in \Omega$ that satisfies $u_0(x) \geq a + M_0\varepsilon$. This proves (3.57). The inequality (3.58) can be shown the same way. This completes the proof of Theorem 3.5.1. □

## 3.6 Motion of interface

We have seen in Section 3.5 that, after a very short time, the solution $u^\varepsilon$ develops a clear transition layer. In the present section, we show that it persists and that its law of motion is well approximated by the interface equation $(P^0)$.

More precisely, take the first term of the formal asymptotic expansion (3.41) as a formal expansion of the solution:

$$u^\varepsilon(x,t) \approx \tilde{u}^\varepsilon(x,t) := U_0\Big(\frac{\widetilde{d}_\phi(x,t)}{\varepsilon}\Big). \tag{3.86}$$

The right-hand side of (3.86) is a function having a well-developed transition layer, and its interface lies exactly on $\Gamma_t$. We show that this function is a very good approximation of the solution; therefore the following holds:

*If $u^\varepsilon$ becomes rather close to $\tilde{u}^\varepsilon$ at some time moment, then it stays close to $\tilde{u}^\varepsilon$ for the rest of time.*

To that purpose, we will construct a pair of sub- and super-solutions $u_\varepsilon^-$ and $u_\varepsilon^+$ for Problem $(P^\varepsilon)$ by slightly modifying $\tilde{u}^\varepsilon$. It then follows that, if the solution $u^\varepsilon$ satisfies

$$u_\varepsilon^-(x,t_0) \le u^\varepsilon(x,t_0) \le u_\varepsilon^+(x,t_0),$$

for some $t_0 \ge 0$ and for almost all $x \in \Omega$, then

$$u_\varepsilon^-(x,t) \le u^\varepsilon(x,t) \le u_\varepsilon^+(x,t),$$

for almost $(x,t) \in Q_T$ that satisfies $t_0 \le t \le T$. As a result, since both $u_\varepsilon^+, u_\varepsilon^-$ stay close to $\tilde{u}^\varepsilon$, the solution $u^\varepsilon$ also stays close to $\tilde{u}^\varepsilon$ for $t_0 \le t \le T$.

### 3.6.1 Construction of sub and super-solutions

To begin with we present mathematical tools which are essential for the construction of sub and super-solutions.

**A modified anisotropic signed distance function.** Rather than working with the anisotropic signed distance function $\widetilde{d}_\phi$, defined in (3.29), we define a "cut-off anisotropic signed distance function" $d_\phi$ as follows. Choose $d_0 > 0$ small enough so that $\widetilde{d}_\phi(\cdot,\cdot)$ is smooth in the tubular neighborhood of $\Gamma$

$$\{(x,t) \in \overline{Q_T},\ |\widetilde{d}_\phi(x,t)| < 3d_0\},$$

and that

$$\mathrm{dist}_\phi(\Gamma_t, \partial\Omega) > 3d_0 \quad \text{for all } t \in [0,T]. \tag{3.87}$$

Next let $\zeta(s)$ be a smooth increasing function on $\mathbb{R}$ such that

$$\zeta(s) = \begin{cases} s & \text{if } |s| \le d_0 \\ -2d_0 & \text{if } s \le -2d_0 \\ 2d_0 & \text{if } s \ge 2d_0. \end{cases}$$

We define the cut-off anisotropic signed distance function $d_\phi$ by

$$d_\phi(x,t) = \zeta\big(\widetilde{d}_\phi(x,t)\big). \tag{3.88}$$

Note that, in view of (3.30),

$$2a(x,\nabla d_\phi(x,t)) = 1 \qquad \text{in a neighborhood of } \Gamma_t, \tag{3.89}$$

more precisely in the region $\{(x,t) \in \overline{Q_T},\, |d_\phi(x,t)| < d_0\}$. Moreover, in view of (3.87), we have

$$2a(x,\nabla d_\phi(x,t)) = 0 \qquad \text{far away from } \Gamma_t, \tag{3.90}$$

i.e. in the region $\{(x,t) \in \overline{Q_T},\, |d_\phi(x,t)| \geq 2d_0\}$. Furthermore, we recall, see (3.46), that an equation for $\Gamma$ is given by

$$(d_\phi)_t = \nabla \cdot a_p(x,\nabla d_\phi) \quad \text{ on } \Gamma_t. \tag{3.91}$$

**Construction.** We look for a pair of sub- and super-solutions $u_\varepsilon^\pm$ for $(P^\varepsilon)$ of the form

$$u_\varepsilon^\pm(x,t) = U_0\Big(\frac{d_\phi(x,t) \pm \varepsilon p(t)}{\varepsilon}\Big) \pm q(t), \tag{3.92}$$

where $U_0$ is the solution of (3.43), and where

$$\begin{aligned} p(t) &= -e^{-\beta t/\varepsilon^2} + e^{Lt} + K,\\ q(t) &= \sigma(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}). \end{aligned} \tag{3.93}$$

Note that $q = \sigma\varepsilon^2 p_t$. It is clear from the definition of $u_\varepsilon^\pm$ that

$$\lim_{\varepsilon\to 0} u_\varepsilon^\pm(x,t) = \begin{cases} 1 & \text{for all } (x,t) \in Q_T^+ \\ 0 & \text{for all } (x,t) \in Q_T^-. \end{cases} \tag{3.94}$$

The main result of this section is the following.

**Lemma 3.6.1.** *There exist positive constants $\beta$, $\sigma$ with the following properties. For any $K > 1$, we can find positive constants $\varepsilon_0$ and $L$ such that, for any $\varepsilon \in (0,\varepsilon_0)$, the functions $u_\varepsilon^-$ and $u_\varepsilon^+$ satisfy the homogeneous anisotropic Neumann boundary condition and*

$$\mathcal{L}_0 u_\varepsilon^- \leq 0 \leq \mathcal{L}_0 u_\varepsilon^+ \quad \text{ in } \Omega \times [0,T],$$

*where the operator $\mathcal{L}_0$ has been defined in* (3.77).

### 3.6.2 Proof of Lemma 3.6.1

We show below that $\mathcal{L}_0 u_\varepsilon^+ := (u_\varepsilon^+)_t - \nabla \cdot a_p(x,\nabla u_\varepsilon^+) - \frac{1}{\varepsilon^2} f(u_\varepsilon^+) \geq 0$, the proof of inequality $\mathcal{L}_0 u_\varepsilon^- \leq 0$ following by the same arguments.

#### Computation of $\mathcal{L}_0 u_\varepsilon^+$

In the sequel, the function $U_0$ and its derivatives are taken at the point $(d_\phi(x,t) + \varepsilon p(t))/\varepsilon$. Straightforward computations yield

$$\begin{aligned} (u_\varepsilon^+)_t &= (\frac{1}{\varepsilon}(d_\phi)_t + p_t)U_0' + q_t,\\ \nabla u_\varepsilon^+ &= \frac{1}{\varepsilon} U_0' \nabla d_\phi,\\ \nabla \cdot a_p(x,\nabla u_\varepsilon^+) &= \frac{1}{\varepsilon^2} U_0'' \nabla d_\phi \cdot a_p(x,\nabla d_\phi) + \frac{1}{\varepsilon} U_0' \nabla \cdot a_p(x,\nabla d_\phi)\\ &= \frac{1}{\varepsilon^2} U_0'' 2a(x,\nabla d_\phi) + \frac{1}{\varepsilon} U_0' \nabla \cdot a_p(x,\nabla d_\phi), \end{aligned}$$

where we have used properties from Remark 3.1.1. Note that, $d_\phi$ being constant in a neighborhood of $\partial\Omega$, we have that $\nabla u_\varepsilon^+ = 0$ on $\partial\Omega \times (0,T)$ and $u_\varepsilon^+$ satisfies the homogeneous anisotropic Neumann boundary condition

$$a_p(x, \nabla u_\varepsilon^+) \cdot \nu = 0 \quad \text{on } \partial\Omega \times (0,T).$$

Further, we use the expansion

$$f(u_\varepsilon^+) = f(U_0) + qf'(U_0) + \frac{1}{2}q^2 f''(\theta),$$

for some function $\theta(x,t)$ satisfying $U_0 < \theta < u_\varepsilon^+$.

Combining the above expressions with equation (3.43), we obtain

$$\mathcal{L}_0 u_\varepsilon^+ = E_1 + E_2 + E_3,$$

where:

$$E_1 = -\frac{1}{\varepsilon^2} q\big(f'(U_0) + \frac{1}{2} q f''(\theta)\big) + {U_0}' p_t + q_t,$$

$$E_2 = \frac{{U_0}''}{\varepsilon^2}\Big(1 - 2a(x, \nabla d_\phi)\Big),$$

$$E_3 = \frac{{U_0}'}{\varepsilon}\Big((d_\phi)_t - \nabla \cdot a_p(x, \nabla d_\phi)\Big).$$

In order to estimate the above terms, we first present some useful inequalities. As $f'(0)$ and $f'(1)$ are strictly negative, we can find strictly positive constants $b$ and $m$ such that

$$\text{if} \quad U_0(z) \in [0,b] \cup [1-b, 1] \qquad \text{then} \quad f'(U_0(z)) \le -m. \tag{3.95}$$

On the other hand, since the region $\{(x,z) \in \overline{\Omega} \times \mathbb{R} \,|\, U_0(z) \in [b, 1-b]\,\}$ is compact and since ${U_0}' > 0$ on $\mathbb{R}$, there exists a constant $a_1 > 0$ such that

$$\text{if} \quad U_0(z) \in [b, 1-b] \quad \text{then} \quad {U_0}'(z) \ge a_1. \tag{3.96}$$

We define

$$F = \sup_{-1 \le z \le 2} |f(z)| + |f'(z)| + |f''(z)|, \tag{3.97}$$

$$\beta = \frac{m}{4}, \tag{3.98}$$

and choose $\sigma$ that satisfies

$$0 < \sigma \le \min(\sigma_0, \sigma_1, \sigma_2), \tag{3.99}$$

where

$$\sigma_0 := \frac{a_1}{4\beta + F}, \quad \sigma_1 := \frac{1}{\beta + 1}, \quad \sigma_2 := \frac{4\beta}{F(\beta+1)}.$$

Hence, combining (3.95) and (3.96), we obtain, using that $\sigma \le \sigma_0$,

$${U_0}'(z) - \sigma f'(U_0(z)) \ge 4\sigma\beta \qquad \text{for } z \in \mathbb{R}. \tag{3.100}$$

Now let $K > 1$ be arbitrary. In what follows we will show that $\mathcal{L}_0 u_\varepsilon^+ \ge 0$ provided that the constants $\varepsilon_0$ and $L$ are appropriately chosen. From now on, we suppose that the following inequality is satisfied:

$$\varepsilon_0^2 L e^{LT} \le 1\,. \tag{3.101}$$

Then, given any $\varepsilon \in (0, \varepsilon_0)$, since $\sigma \le \sigma_1$, we have $0 \le q(t) \le 1$, hence, recalling that $0 < U_0 < 1$,

$$-1 \le u_\varepsilon^\pm(x,t) \le 2\,. \tag{3.102}$$

**We first estimate the term $E_1$**

A direct computation gives

$$E_1 = \frac{\beta}{\varepsilon^2} e^{-\beta t/\varepsilon^2}(I - \sigma\beta) + Le^{Lt}(I + \varepsilon^2 \sigma L),$$

where

$$I = U_0{}' - \sigma f'(U_0) - \frac{\sigma^2}{2} f''(\theta)(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}).$$

In virtue of (3.100) and (3.102), we obtain

$$I \geq 4\sigma\beta - \frac{\sigma^2}{2} F(\beta + \varepsilon^2 L e^{LT}).$$

Then, in view of (3.101), using that $\sigma \leq \sigma_2$, we have

$$I \geq 2\sigma\beta.$$

Consequently, we have

$$E_1 \geq \frac{\sigma\beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + 2\sigma\beta L e^{Lt} =: \frac{C_1}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + C_1{}' L e^{Lt}.$$

**As for the term $E_2$**

First, in the points where $|d_\phi| < d_0$, by (3.89), we have $E_2 = 0$. Next we consider the points where $|d_\phi| \geq d_0$. We deduce from the definition of $\Lambda_0$ in (3.26) that

$$\begin{aligned} 0 \leq 2a(x, \nabla d_\phi(x,t)) &\leq (\Lambda^0)^2 |\nabla d_\phi(x,t)|^2 \\ &\leq (\Lambda^0)^2 \|\nabla d_\phi\|_\infty^2 := D < \infty. \end{aligned}$$

Applying Lemma 3.3.1 yields

$$\begin{aligned} |E_2| &\leq \frac{C}{\varepsilon^2}(1 + D) e^{-\lambda |d_\phi + \varepsilon p|/\varepsilon} \\ &\leq \frac{C}{\varepsilon^2}(1 + D) e^{-\lambda(d_0/\varepsilon - |p|)}. \end{aligned}$$

By the definition of $p$ in (3.93) we have that $0 < K - 1 \leq p \leq e^{LT} + K$; we suppose from now that the following assumption holds:

$$e^{LT} + K \leq \frac{d_0}{2\varepsilon_0}. \tag{3.103}$$

Then $\dfrac{d_0}{\varepsilon} - |p| \geq \dfrac{d_0}{2\varepsilon}$ so that, defining $C' := C(1 + D)$,

$$\begin{aligned} |E_2| &\leq \frac{C'}{\varepsilon^2} e^{-\lambda d_0/(2\varepsilon)} \\ &\leq C_2 := \frac{16 C'}{(e\lambda d_0)^2}. \end{aligned}$$

**Next we consider the term $E_3$**

We set
$$\mathcal{F}(x,t) = (d_\phi)_t(x,t) - \nabla \cdot a_p(x, \nabla d_\phi(x,t)).$$
We recall that $d_\phi \in C^{3+\vartheta,(3+\vartheta)/2}$ in a neighborhood $\mathcal{V}$ of $\Gamma$, say
$$\mathcal{V} = \{(x,t) \in Q_T,\ |d_\phi(x,t)| < d_0\}.$$
Combining the fact that
$$2a(x, \nabla d_\phi(x,t)) = 1 \quad \text{in } \mathcal{V},$$
with the definition of $\Lambda^0$ in (3.26), we see that
$$|\nabla d_\phi| \geq \frac{1}{\Lambda^0} \quad \text{in } \mathcal{V}. \tag{3.104}$$
We also recall that $(x,p) \mapsto a(x,p)$ is of class $C^{3+\vartheta}$ on $\overline{\Omega} \times \mathbb{R}^N \setminus \{0\}$. Since by (3.104) $|\nabla d_\phi|$ is bounded away from zero, it follows that $x \mapsto \nabla \cdot a_p(x, \nabla d_\phi(x,t))$ is in $C^{1+\vartheta}(\mathcal{V}_t)$, where
$$\mathcal{V}_t := \{x \in \Omega,\ (x,t) \in \mathcal{V}\}.$$
Moreover the function $x \mapsto (d_\phi)_t(x,t)$ is in $C^{1+\vartheta}(\mathcal{V}_t)$. Therefore the function $x \mapsto \mathcal{F}(x,t)$ is Lipschitz continuous on $\mathcal{V}_t$. By equation (3.91), we have that
$$\mathcal{F}(x,t) = 0 \quad \text{on } \Gamma_t = \{x \in \Omega, d_\phi(x,t) = 0\},$$
and it follows from the mean value theorem applied separately on both sides of $\Gamma_t$ that there exists a constant $N_1$ such that
$$|\mathcal{F}(x,t)| \leq N_1 |d_\phi(x,t)| \quad \text{for all } (x,t) \in \mathcal{V}. \tag{3.105}$$
Next, using Lemma 3.2.2, we remark that $\mathcal{F}$ is bounded on $\overline{\Omega} \times [0,T] \backslash \mathcal{V}$ so that there exists a constant $N_2$ such that
$$\sup_{\overline{\Omega} \times [0,T] \backslash \mathcal{V}} |\mathcal{F}(x,t)| \leq N_2. \tag{3.106}$$
By the inequalities (3.105) and (3.106), we deduce that
$$|\mathcal{F}(x,t)| = |(d_\phi)_t(x,t) - \nabla \cdot a_p(x, \nabla d_\phi(x,t))| \leq N_0 |d_\phi(x,t)| \quad \text{in } Q_T,$$
with $N_0 := \max(N_1, N_2/d_0)$. Applying Lemma 3.3.1 we deduce that
$$\begin{aligned} |E_3| &\leq N_0 C \frac{|d_\phi(x,t)|}{\varepsilon} e^{-\lambda |d_\phi(x,t)/\varepsilon + p(t)|} \\ &\leq N_0 C \max_{y \in \mathbb{R}} |y| e^{-\lambda |y + p(t)|} \\ &\leq N_0 C \max\big(|p(t)|, \frac{1}{\lambda}\big) \\ &\leq N_0 C \big(|p(t)| + \frac{1}{\lambda}\big). \end{aligned}$$
Thus, recalling that $|p(t)| \leq e^{Lt} + K$, we obtain
$$|E_3| \leq C_3 (e^{Lt} + K) + {C_3}',$$
where $C_3 := N_0 C$ and ${C_3}' := N_0 C / \lambda$.

**Completion of the proof**

Collecting the above estimates of $E_1$, $E_2$ and $E_3$ yields

$$\mathcal{L}_0 u_\varepsilon^+ \geq \frac{C_1}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + (L{C_1}' - C_3)e^{Lt} - C_4,$$

where $C_4 := C_2 + KC_3 + {C_3}'$. Now, we set

$$L := \frac{1}{T} \ln \frac{d_0}{4\varepsilon_0},$$

which, for $\varepsilon_0$ small enough, validates assumptions (3.101) and (3.103). If $\varepsilon_0$ is chosen sufficiently small (i.e. $L$ sufficiently large), we obtain, for all $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned} \mathcal{L}_0 u_\varepsilon^+ &\geq (L{C_1}' - C_3)e^{Lt} - C_4 \\ &\geq \tfrac{1}{2} L{C_1}' - C_4 \\ &\geq 0. \end{aligned}$$

The proof of Lemma 3.6.1 is now completed. □

## 3.7 Proof of the main results

In this section, we prove Theorem 3.1.3 and Corollary 3.1.4 by fitting the two pairs of sub- and super-solutions, constructed for the study of the generation and the motion of interface, into each other.

Let $\eta \in (0, \eta_0)$ be arbitrary. Choose $\beta$ and $\sigma$ that satisfy (3.98), (3.99) and

$$\sigma\beta \leq \frac{\eta}{3}. \tag{3.107}$$

By the generation of interface Theorem 3.5.1, there exist positive constants $\varepsilon_0$ and $M_0$ such that (3.56), (3.57) and (3.58) hold with the constant $\eta$ replaced by $\sigma\beta/2$. Since, by the hypothesis (3.7) and the equality (3.31), $\nabla u_0(x) \cdot n_\phi(x) \neq 0$ everywhere on the initial interface $\Gamma_0 = \{x \in \Omega,\ u_0(x) = a\}$ and since $\Gamma_0$ is a compact hypersurface, we can find a positive constant $M_1$ such that

$$\begin{array}{llll} \text{if} & d_\phi(x,0) \geq \ \ M_1\varepsilon & \text{then} & u_0(x) \geq a + M_0\varepsilon, \\ \text{if} & d_\phi(x,0) \leq -M_1\varepsilon & \text{then} & u_0(x) \leq a - M_0\varepsilon. \end{array} \tag{3.108}$$

Now we define functions $H^+(x), H^-(x)$ by

$$H^+(x) = \begin{cases} 1 + \sigma\beta/2 & \text{if } d_\phi(x,0) > -M_1\varepsilon \\ \sigma\beta/2 & \text{if } d_\phi(x,0) \leq -M_1\varepsilon, \end{cases}$$

$$H^-(x) = \begin{cases} 1 - \sigma\beta/2 & \text{if } d_\phi(x,0) \geq M_1\varepsilon \\ -\sigma\beta/2 & \text{if } d_\phi(x,0) < M_1\varepsilon. \end{cases}$$

Then from the above observation we see that

$$H^-(x) \ \leq \ u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \ \leq \ H^+(x), \tag{3.109}$$

for almost all $x \in \Omega$.

Next we fix a sufficiently large constant $K > 1$ such that

$$U_0(-M_1 + K) \geq 1 - \frac{\sigma\beta}{3} \quad \text{and} \quad U_0(M_1 - K) \leq \frac{\sigma\beta}{3}. \tag{3.110}$$

For this $K$, we choose $\varepsilon_0$ and $L$ as in Lemma 3.6.1. We claim that

$$u_\varepsilon^-(x,0) \leq H^-(x), \qquad H^+(x) \leq u_\varepsilon^+(x,0), \tag{3.111}$$

for all $x \in \Omega$. We only prove the former inequality, as the proof of the latter is virtually the same. Then it amounts to showing that

$$u_\varepsilon^-(x,0) = U_0\big(\frac{d_\phi(x,0)}{\varepsilon} - K\big) - \sigma(\beta + \varepsilon^2 L) \ \leq \ H^-(x). \tag{3.112}$$

In the range where $d_\phi(x,0) < M_1\varepsilon$, the second inequality in (3.110) and the fact that $U_0$ is an increasing function imply

$$\begin{aligned} U_0\big(\frac{d_\phi(x,0)}{\varepsilon} - K\big) - \sigma(\beta + \varepsilon^2 L) &\leq U_0(M_1 - K) - \sigma\beta - \sigma\varepsilon^2 L \\ &\leq \frac{\sigma\beta}{3} - \sigma\beta \\ &\leq H^-(x). \end{aligned}$$

On the other hand, in the range where $d_\phi(x,0) \geq M_1\varepsilon$, we have

$$\begin{aligned} U_0\big(\frac{d_\phi(x,0)}{\varepsilon} - K\big) - \sigma(\beta + \varepsilon^2 L) &\leq 1 - \sigma\beta \\ &\leq H^-(x). \end{aligned}$$

This proves (3.112), so that (3.111) is established.

Combining (3.109) and (3.111), we obtain

$$u_\varepsilon^-(x,0) \leq u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \leq u_\varepsilon^+(x,0),$$

for almost all $x \in \Omega$. Since, by Lemma 3.6.1, $u_\varepsilon^-$ and $u_\varepsilon^+$ are sub- and super-solutions for Problem $(P^\varepsilon)$, the comparison principle yields

$$u_\varepsilon^-(x,t) \leq u^\varepsilon(x, t + t^\varepsilon) \leq u_\varepsilon^+(x,t), \tag{3.113}$$

for almost all $(x,t) \in Q_T$ that satisfies $0 \leq t \leq T - t^\varepsilon$, where we recall that $t^\varepsilon = \mu^{-1}\varepsilon^2|\ln\varepsilon|$. Note that, in view of (3.94), this is sufficient to prove Corollary 3.1.4. Now let $C$ be a positive constant such that

$$U_0(C - e^{LT} - K) \geq 1 - \frac{\eta}{2} \quad \text{and} \quad U_0(-C + e^{LT} + K) \leq \frac{\eta}{2}. \tag{3.114}$$

One then easily checks, using (3.113), (3.92) and (3.107), that, for $\varepsilon_0$ small enough, for almost all $(x,t) \in Q_T$ with $0 \leq t \leq T - t^\varepsilon$, we have

$$\begin{array}{llll} \text{if} & d_\phi(x,t) \geq C\varepsilon & \text{then} & u^\varepsilon(x, t + t^\varepsilon) \geq 1 - \eta \\ \text{if} & d_\phi(x,t) \leq -C\varepsilon & \text{then} & u^\varepsilon(x, t + t^\varepsilon) \leq \eta, \end{array} \tag{3.115}$$

and

$$u^\varepsilon(x, t + t^\varepsilon) \in [-\eta, 1 + \eta],$$

which completes the proof of Theorem 3.1.3. □

# Bibliographie


[1] N. D. Alikakos, P. W. Bates and X. Chen, *Convergence of the Cahn-Hilliard equation to the Hele-Shaw model*, Arch. Rational Mech. Anal. **128** (1994), 165–205.

[2] S. Allen and J. Cahn, *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening*, Acta Metall. **27** (1979), 1084–1095.

[3] F. Almgren, J. E. Taylor and L. Wang, *Curvature-driven flows : a variational approach*, SIAM J. Control Optim. **31** (1993), 387–437.

[4] S. Angenent and M. E. Gurtin, *Multiphase thermomechanics with interfacial structure. II. Evolution of an isothermal interface*, Arch. Rational Mech. Anal. **108** (1989), 323–391.

[5] G. Barles and F. Da Lio, *A geometrical approach to front propagation problems in bounded domains with Neumann-type boundary conditions*, Interfaces Free Bound. **5** (2003), no. 3, 239–274.

[6] G. Barles, H. M. Soner and P. E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control Optim. **31** (1993), 439–469.

[7] G. Barles and P. E. Souganidis, *A new approach to front propagation problems : theory and applications*, Arch. Rat. Mech. Anal. **141** (1998), 237–296.

[8] G. Bellettini, P. Colli Franzone and M. Paolini, *Convergence of front propagation for anisotropic bistable reaction-diffusion equations*, Asympt. Anal. **15** (1997), 325–358.

[9] G. Bellettini and M. Paolini, *Anisotropic motion by mean curvature in the context of Finsler geometry*, Hokkaido Math. J. **25** (1996), 537–566.

[10] G. Bellettini, M. Paolini and S. Venturini, *Some results on surface measures in calculus of variations*, Ann. Mat. Pura Appl. (4). **170** (1996), 329–357.

[11] M. Beneš, *Mathematical and computational aspects of solidification of pure substances*, Acta Math. Univ. Comenian. **70** (2001), 123–151.

[12] M. Beneš, *Mathematical analysis of phase-field equations with gradient coupling term*, In Partial Differential Equations– Theory and Numerical Solution (W. Jäger, J. Nečas, O. John, K. Najzar and J. Stará, eds.), pages 25–33, New York, 2000.

[13] M. Beneš, D. Hilhorst and R. Weidenfeld, *Interface dynamics for an anisotropic Allen-Cahn equation*, Nonlocal elliptic and parabolic problems, Banach Center Publ. 66, Polish. Acad. Sci., Warsaw (2004), 39–45.

[14] M. Beneš and K. Mikula, *Simulation of anisotropic motion by mean curvature—comparison of phase field and sharp interface approaches*, Acta Math. Univ. Comenian. **67** (1998), 17–42.

[15] A. Bonami, D. Hilhorst and E. Logak, *Modified motion by mean curvature : local existence and uniqueness and qualitative properites*, Differential and Integral Equations, **13** (2000), 1371–1392.

[16] A. Bonami, D. Hilhorst, E. Logak and M. Mimura, *A free boundary problem arising in a chemotaxis model*, in "Free Boundary Problems, Theory and Applications", M. Niezgódka and P. Strzelecki Eds, Pitman Res. Notes in Math. Series **363** (1996).

[17] A. Bonami, D. Hilhorst, E. Logak and M. Mimura, *Singular limit of a chemotaxis-growth model*, Advances in Differential Equations, **6** (2001), 1173–1218.

[18] L. Bronsard and R. V. Kohn, *Motion by mean curvature as the singular limit of Ginzburg–Landau model*, J. Diff. Eqns. **90** (1991), 211–237.

[19] J. W. Cahn and D. W. Hoffman, *A vector thermodynamics for anisotropic surfaces*, Surface Science **31** (1972), 368–388.

[20] X. Chen, *Generation and propagation of interfaces for reaction-diffusion equations*, J. Differential equations **96** (1992), 116–141.

[21] X. Chen, *Generation and propagation of interfaces for reaction-diffusion systems*, Trans. Amer. Math. Soc. **334** (1992), 877–913.

[22] X. Chen, D. Hilhorst and E. Logak, *Asymptotic behavior of solutions of an Allen-Cahn equation with a nonlocal term*, Nonlinear Anal. **28** (1997), no. 7, 1283–1298.

[23] X. Chen and F. Reitich, *Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling*, J. Math. Anal. Appl. **164(2)** (1992), 350–362.

[24] X. Y. Chen, *Dynamics of interfaces in reaction diffusion systems*, Hiroshima Math. J. **21** (1991), 47–83.

[25] Y. G. Chen, Y. Giga and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Diff. Geom. **33** (1991), 749–786.

[26] S. Childress and J. K. Percus, *Nonlinear aspects of chemotaxis*, Math. Biosci. **56** (1981), 217–237.

[27] K. N. Chueh, *A compact positively invariant set of solutions of the Nagumo equation*, J. Differential Equations **28** (1978), no. 1, 35–42.

[28] K. N. Chueh, C. C. Conley, J. A. Smoller, *Positively invariant regions for systems of nonlinear diffusion equations,* Indiana Univ. Math. J. **26** (1977), 373–392.

[29] E. DiBenedetto, *Degenerate parabolic equations*, Springer-Verlag, New York, 1993.

[30] S. I. Ei, M. Iida and E. Yanagida, *Dynamics of interfaces in a scalar parabolic equation with variable diffusion coefficients*, Japanese J. Ind. Appl. Math. **14** (1997), 1–25.

[31] C. M. Elliott and R. Schätzle, *The limit of the anisotropic double-obstacle Allen-Cahn equation*, Proc. Roy. Soc. Edinburgh A **126** (1996), 1217–1234.

[32] C. M. Elliott and R. Schätzle, *The limit of the fully anisotropic double-obstacle Allen-Cahn equation in the nonsmooth case*, SIAM. J. Math. Anal. **28** (1997), 274–303.

[33] L. C. Evans, H. M. Soner and P. E. Souganidis, *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math. **45** (1992), 1097–1123.

[34] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature*, J. Diff. Geom. **33** (1991), 635–681.

[35] S. D. Eydel'man, *Parabolic systems*, North-Holland, Amsterdam-London, 1969.

[36] P. C. Fife, *Dynamics of internal layers and diffusive interfaces*, CBMS-NSF Regional Conf. Ser. in Appl. Math., SIAM, Philadelphia, 1988.

[37] P. C. Fife and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Rational Mech. Anal. **65** (1977), 335–361.

[38] R. M. Ford and D. A. Lauffenburger, *Analysis of chemotactic bacterial distributions in population migration assays using a mathematical model applicable to steep or shallow attractant gradients*, Bull. Math. Biol. **53** (1991), 721–749.

[39] M. Gage and R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Diff. Geom. **23** (1986), 69–96.

[40] H. Garcke, B. Nestler and B. Stoth, *On anisotropic order parameter models for multi-phase systems and their sharp interface limits*, Physica D **115** (1998), 87–108.

[41] Y. Giga and S. Goto, *Geometric evolution of phase-boundaries*, Mathematics and its applications 43, Springer-Verlag, Berlin, New York, 1992.

[42] M. Henry, D. Hilhorst and R. Schätzle, *Convergence to a viscosity solution for an advection-reaction-diffusion equation arising from a chemotaxis-growth model*, Hiroshima Math. J. **29** (1999), no. 3, 591–630.

[43] M. A. Herrero and J. Velazquez, *Chemotactic collapse for the Keller-Segel model*, J. Math. Biol. **35** (1996), no. 2, 177–194.

[44] M. A. Herrero and J. Velazquez, *A blow-up mechanism for a chemotaxis model*, Ann. Scuola. Norm. Sup. Pisa Cl. Sci. **24** (1997), no. 4, 633–683.

[45] D. Hilhorst and J. Hulshof, *An elliptic-parabolic problem in combustion theory : convergence to travelling waves*, Nonlinear Anal. **17** (1991), no. 6, 519–546.

[46] D. Hilhorst, E. Logak and R. Schätzle, *Global existence for a non-local mean curvature flow as a limit of a parabolic-elliptic phase transition model*, Interfaces Free Bound. **2** (2000), no. 3, 267–282.

[47] D. Hilhorst, H. Matano and R. Schätzle, *Singular limit of an inhomogeneous reaction-diffusion equation*, in preparation.

[48] T. Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Memoirs of the American Mathematical Society, **108** (1994).

[49] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc., **329** (1992), 819–824.

[50] Y. Kametaka, M. Mimura and M. Yamaguti, *On a certain difference scheme for some semilinear diffusion system*, Proc. Japan Academy **47** (1971), 385–387.

[51] G. Karali, K. Nakashima, D. Hilhorst and H. Matano, *Singular limit of a spatially inhomogeneous Lotka-Volterra competition-diffusion system*, to be published.

[52] K. Kawasaki and T. Ohta, *Kinetic drumhead model of interface I*, Progress of Theoretical Physics **67** (1982) 147–163.

[53] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol., **26** (1970), 399–415.

[54] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.

[55] C. S. Lin, W. M. Ni and I. Takagi, *Large amplitude stationary solutions to a chemotaxis system*, J. Differential Equations **72** (1988), 1–27.

[56] E. Logak, *Singular limit of reaction-diffusion systems and modified motion by mean curvature*, Proc. Roy. Soc. Edinburgh A **132** (2002), no. 4, 951–973.

[57] M. Mimura and T. Tsujikawa, *Aggregating pattern dynamics in a chemotaxis model including growth*, Physica A, **230** (1996), 499–543.

[58] M. Mimura, T. Tsujikawa, R. Kobayashi and D. Ueyama, *Dynamics of aggregation patterns in a chemotaxis-diffusion-growth model equation. Proceedings of the Workshop on Principles of Pattern Formation and Morphogenesis in Biological Systems (Kasugai, 1992/93)*, Forma, **8** (1993), 179–195.

[59] P. de Mottoni and M. Schatzman, *Development of interfaces in* $\mathbb{R}^n$, Proc. Royal Soc. Edin. **116A** (1990), 207–220.

[60] P. de Mottoni and M. Schatzman, *Geometrical evolution of developed interfaces*, Trans. Amer. Math. Soc. **347** (1995), 1533–1589.

[61] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis pattern*, Advances in Math. Sciences and Appl., Vol. 5, no. 2 (1995), 581-601.

[62] V. Nanjundiah, *Chemotaxis, signal relaying and aggregation morphology*, J. Theor. Biol., **42** (1973), 63–105.

[63] K.-I. Nakamura, H. Matano, D. Hilhorst and R. Schätzle, *Singular limit of a reaction-diffusion equation with a spatially inhomogeneous reaction term*, J. Stat. Phys. **95** (1999), 1165-1185.

[64] M. Paolini, *Fattening in two dimensions obtained with a nonsymmetric anisotropy : numerical simulations*, Acta Math. Univ. Comenian. (N.S.). **67** (1998), 43–55.

[65] P. Pelcé, *Théorie des formes de croissance*, CNRS Editions, EDP Sciences, 2000.

[66] J. Rubinstein, P. Sternberg and J. B. Keller, *Fast reaction, slow diffusion and curve shortening*, SIAM J. Appl. Math., **49** (1989), 116–133.

[67] R. Schaaf, *Stationary solutions of chemotaxis systems*, Trans. Amer. Math. Soc., **292** (1985), 531–556.

[68] J. Smoller, *Shock waves and reaction-diffusion equations*, Springer-Verlag, New York, 1994.

[69] F. Verhulst, *Nonlinear differential equations and dynamical systems*, Springer-Verlag, Berlin Heidelberg, 1990.

[70] R. Weidenfeld, *Limite singulière et comportement en temps long de systèmes de réaction-diffusion en dynamique des populations*, Thèse de doctorat de l'université Paris Sud (2002).

**Systèmes de convection-réaction-diffusion et dynamique d'interface**

Cette thèse porte sur la limite singulière d'équations et de systèmes d'équations paraboliques non-linéaires de type bistable, avec des conditions initiales générales. Nous prouvons des propriétés de *génération d'interface* et analysons le *déplacement d'interface*. Nous obtenons une estimation nouvelle et optimale de l'épaisseur et de la localisation de la zone de transition, améliorant ainsi des résultats connus pour différents problèmes modèles.

Au Chapitre 1, nous considérons d'abord une équation d'Allen-Cahn. Le déplacement de l'interface limite est induit par sa courbure moyenne et par un terme de pression. Nous étendons ensuite nos résultats à une classe assez large de systèmes de réaction-diffusion. Pour cela, nous considérons la première équation du système comme une perturbation de l'équation d'Allen-Cahn, étudions la dépendance du déplacement de l'interface vis-à-vis de différents paramètres, et prouvons de fines estimations a priori. Le Chapitre 2 est consacré à l'étude d'un système qui modélise une agrégation d'amibes soumises à la diffusion, à la croissance et au chimiotactisme. Ce dernier phénomène est une propension de certaines espèces à se déplacer vers les plus forts gradients de substances chimiques, souvent produites par ces espèces elles-mêmes. Enfin, au Chapitre 3, nous considérons une équation anisotrope, qui intervient en science des matériaux et dont le terme de diffusion est inhomogène et singulier aux points où le gradient de la solution s'annule. Nous définissons une notion de solution faible et prouvons un principe de comparaison. Le déplacement de l'interface limite est induit par une version anisotrope de sa courbure moyenne. Nous utilisons la distance associée à une métrique de Finsler.

**Mots clés :** Systèmes de convection-réaction-diffusion – Equation d'Allen-Cahn – Système de FitzHugh-Nagumo – Chimiotactisme – Anisotropie – Génération d'interface – Propagation d'interface – Epaisseur d'interface.

**Convection-reaction-diffusion systems and interface dynamics**

This thesis deals with the singular limit of systems of parabolic partial differential equations, with bistable nonlinear reaction terms and general initial data. We prove some *generation of interface* properties and study the *motion of interface*. We revisit a variety of model problems and obtain a new and optimal estimate of the thickness and the location of the transition layer that develops.

In Chapter 1, we first consider a perturbed Allen-Cahn equation. The motion of the limit interface is driven by its mean curvature and a pressure term. Then, we extend our results to a large class of reaction-diffusion systems. The idea is to regard the first equation of the system as a perturbed Allen-Cahn equation ; the proofs are based upon a study of the dependence of the interface motion on various parameters together with some refined a priori estimates. Chapter 2 is devoted to the study of a chemotaxis system. This is a model for the aggregation of amoebae in the presence of diffusion, growth and chemotaxis. This last phenomenon is a tendency of some species to move towards higher gradients of chemical substances which they often produce themselves. Finally, in Chapter 3, we consider an anisotropic equation, which arises for instance in material sciences, and whose diffusion term is spatially inhomogeneous and singular in the points where the gradient of the solution vanishes. We define a notion of weak solution and prove a comparison principle. The motion of the limit interface is driven by its anisotropic mean curvature. We use the distance function associated with a Finsler metric.

**Key words :** Convection-reaction-diffusion systems – Allen-Cahn equation – FitzHugh-Nagumo system– Chemotaxis – Anisotropy – Generation of interface – Motion of interface – Thickness of interface.

**AMS subject classifications :** 35K57, 35K60, 35K50, 35K20, 35R35, 35B20.