Quelques méthodes de résolution d'équations aux dérivées partielles elliptiques avec contrainte sur les espaces $W^{1,p}$ et BV.

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Quelques méthodes de résolution d’équations aux dérivées partielles elliptiques avec contrainte sur les espaces $W^{1,p}$ et BV

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par
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Contents

Chapter 1

Introduction

Cette thèse a pour sujet l'étude de quelques équations aux dérivées partielles singulières ou dégénérées, sous contraintes. Sont aussi traitées des équations dites pénalisées qui remplacent la contrainte par un terme qui assymptotiquement tend vers la contrainte, ceci permettant une approximation numériquement plus souple de l'edp avec contrainte. Les méthodes employées sont celles du calcul des variations, la convexité, la théorie de la dualité...

La première partie concerne l'approximation des premières fonctions propres et valeurs propres pour le 1-Laplacien. Lorsque Ω est un ouvert borné régulier de $\mathbb{R}^N$, $N > 1$, on définit la première valeur propre du 1-Laplacien, comme le réel positif

$$\lambda_1 := \inf_{\substack{u \in W_0^{1,1}(\Omega) \\ \|u\|_1 = 1}} \left\{ \int_\Omega |\nabla u| \right\}. \tag{1.1}$$

La recherche de l'existence d'une solution demande l'introduction de l'espace des fonctions à variations bornées défini par:

$$BV(\Omega) = \left\{ u \in L^1(\Omega), \nabla u \in \mathcal{M}^1(\Omega) \right\},$$

où $\mathcal{M}^1(\Omega)$ est l'ensemble des mesures bornées dans Ω. L'espace $BV(\Omega)$ est un espace de Banach, muni de la norme:

$$\|u\|_{BV} = \|\nabla u\|_1 + \|u\|_1,$$

où $\|\nabla u\|_1$ est la variation totale de ∇u.

On introduit ainsi un problème dit relaxé, donné par:

$$\inf_{\substack{u\in BV(\Omega)\\ \|u\|_1=1}} \left\{ \int_\Omega |\nabla u| + \int_{\partial\Omega} |u| \right\}, \tag{1.2}$$

qui prend en compte la propriété de non continuité de l'application trace. On montre alors qu'une solution $u \geq 0$ qui existe satisfait

$$\begin{cases} -\text{div } \sigma = \lambda_1, \\ \sigma \cdot \nabla u = |\nabla u| \text{ dans } \Omega, \\ \sigma \cdot \vec{n} u = -u, \text{ sur } \partial\Omega, \end{cases}$$

où σ est une fonction de $L^\infty(\Omega, \mathbb{R}^n)$ satisfaisant $\|\sigma\|_{L^\infty} \leq 1$ dans Ω, le produit $\sigma \cdot \nabla u$ ayant un sens à préciser. Des ouvrages sur la question font état de l'existence de fonctions propres qui sont des fonctions caractéristiques d'ensemble, mais le fait que toutes les fonctions propres sont proportionnelles est encore un problème ouvert, sauf dans le cas $N = 2$, où ([2], [3]) font d'ailleurs explicitement la construction d'un ensemble propre.

Le premier chapitre, qui a fait l'objet d'un article accepté pour publication aux Annales de la Faculté des sciences de Toulouse, concerne l'approximation de la première valeur propre et des premières fonctions propres par une méthode de pénalisation. Elle consiste à remplacer la contrainte $\|u\|_1 = 1$ par le terme $n\left(\int_\Omega |u| - 1\right)^2$ dans la fonctionnelle (1.1). Plus précisemment, on considère $\lambda_{1,n}$ définissant

$$\inf_{u\in W_0^{1,1}(\Omega)} \left\{ \int_\Omega |\nabla u| + n\left(\int_\Omega |u| - 1\right)^2 \right\}. \tag{1.3}$$

Puis la forme relaxée de $\lambda_{1,n}$ qui consiste à l'étendre à l'espace $BV(\Omega)$ et à ajouter à la fonctionnelle à minimiser un terme $\int_{\partial\Omega} |u|$. Ce procédé est classique en théorie des surfaces minimales et en plasticité. Il permet de pallier au manque de continuité de l'application trace pour la topologie faible. On montre donc dans un premier temps l'existence d'une solution u_n pour ce problème approché, défini par (1.3), solution que l'on peut prendre positive, et qui satisfait donc l'équation pénalisée suivante:

$$\begin{cases} -\text{div } \sigma_n + 2n \left(\displaystyle\int_\Omega |u_n| - 1 \right) \text{ sign}^+ u_n = 0, \\ \sigma_n \cdot \nabla u_n = |\nabla u_n| \text{ dans } \Omega, \\ \|\sigma_n\|_{L^\infty(\Omega)} \leq 1, \\ \sigma_n \cdot \vec{n} u = -u_n. \end{cases}$$

Par passage à la limite lorsque n tends vers $+\infty$, on obtient la convergence du terme $2n \left(\int_\Omega |u_n| - 1 \right) \text{sign}^+ u_n$ vers $-\lambda_1$, et la convergence d'une suite de solutions u_n, pour une topologie plus forte, intermédiaire entre la topologie faible et la topologie de la norme, vers une première fonction propre pour le 1-Laplacien.

Dans un deuxième chapitre on considère un problème d'obstacle sur $W^{1,p}(\Omega)$ puis le problème de contrôle optimal correspondant. Les résultats généralisent certains résultats obtenus dans le cas $p = 2$, dans les travaux bien connus de Kindherlerer et d'Adams Lennhart ([5], [6], [34]).

Plus précisemment, soit $1 < p < N$, Ω un ouvert borné régulier de $\mathbb{R}^N$, ψ une fonction de $W_0^{1,p}(\Omega)$ et f une fonction de $L^{p'}(\Omega)$ (p' étant le conjugué de p, tel que $\frac{1}{p} + \frac{1}{p'} = \frac{1}{N}$). On cherche u qui réalise:

$$\begin{cases} u \geq \psi, \quad u \in W_0^{1,p}(\Omega). \\ -\text{div}(|\nabla u|^{p-2} \nabla u) \geq f, \\ \displaystyle\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla(u - \psi) = \int_\Omega f(u - \psi). \end{cases}$$

Pour résoudre ce problème, on introduit classiquement le problème de minimisation suivant:

$$\inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \geq \psi}} \left\{ \int_\Omega |\nabla u|^p - \int_\Omega f u \right\}. \tag{1.4}$$

On montre l'existence d'un minimiseur par les méthodes variationnelles classiques. Puis on caractérise la solution comme la plus petite fonction f-surharmonique, plus grande que ψ, ce qui prouve l'unicité de la solution alors notée $T_f(\psi)$. On remarque la croissance de l'application qui à ψ associe

$T_f(\psi)$, ce qui permet de montrer une propriété de semicontinuité inférieure de T_f, pour la topologie faible de $W^{1,p}(\Omega)$.

Dans un deuxième temps on s'intéresse à un problème de contrôle optimal à savoir le problème suivant: une fonction z étant donnée dans $L^p(\Omega)$, appelée fonction de coût , on cherche à minimiser la fonctionnelle suivante:

$$J_f(\psi) = \frac{1}{p}\left\{\int |T_f(\psi) - z|^p + \int_\Omega |\nabla \psi|^p\right\}. \tag{1.5}$$

Lorsqu'un minimiseur ψ existe, $(\psi, T_f(\psi))$ est appelée une paire optimale. La difficulté de ce type de problème est dûe à l'absence de semicontinuité inférieure de J_f pour la topologie faible de $W^{1,p}(\Omega)$. Dans cette partie, on montre, prolongeant ainsi des résultats obtenus dans le cas $p = 2$ dans ([5], [6]), que:

1. Si $f \leq 0$, il existe une paire optimale de la forme $(u^\star, u^\star) = (T_f(\psi), T_f(\psi))$.
2. Si $f \geq 0$, et si on définit G_f comme l'unique fonction dans $W_0^{1,p}(\Omega)$ qui verifie
$$\begin{cases} -\Delta_p(G_f) = f, \text{ p.p. dans } \Omega, \\ G_f = 0, \text{ sur } \partial\Omega. \end{cases}$$
Si $z \leq G_f$, alors il existe une unique paire optimale donnée par $(0, G_f)$.

Le dernier chapitre est consacré d'une part à un problème d'obstacle sur $BV(\Omega)$, analogue au problème d'obstacle sur $W_0^{1,p}(\Omega)$ traité dans le chapitre prècedent et à quelques résultats de contrôle sur $BV(\Omega)$. On considère toujours Ω un ouvert borné régulier de $\mathbb{R}^N$, $N > 1$, une fonction $f \in L^\infty(\Omega)$, ψ dans $BV(\Omega)$, on cherche un minimum pour le problème de minimisation suivant:

$$\mathcal{P} = \inf_{\substack{u \in W_0^{1,1}(\Omega) \\ u \geq \psi}} \left\{\int_\Omega |\nabla u| - \int_\Omega f u\right\},$$

et à donner un sens à l'équation

$$-\text{div } \sigma = \tau + f,$$

où τ est une mesure positive, équation découlant naturellement de l'inéquation

$$-\text{div } \sigma \geq f,$$

où $\sigma = \frac{``(\nabla u)"}{|\nabla u|}$, est “satisfaite” par u lorsque u est solution de $\mathcal{P}$.

On définit tout d'abord la forme relaxée du problème défini par $\mathcal{P}$, notée $\mathcal{P}_{BV}$ tel que:

$$\mathcal{P}_{BV} = \inf_{\substack{u \in BV(\Omega) \\ u \geq \psi}} \left\{ \int_\Omega |\nabla u| + \int_{\partial\Omega} |u| - \int_\Omega fu \right\}.$$

On montre dans une première étape l'existence d'un minimiseur au problème défini par $\mathcal{P}_{BV}$, en utilisant une méthode de pénalisation, qui consiste à remplacer la contrainte “$u \geq \psi$” par le terme $\frac{1}{\delta}\int_\Omega (u-\psi)^-$ pour un $\delta > 0$.

Ainsi on définit le problème de minimisation suivant:

$$\mathcal{P}_\delta = \inf_{u \in BV(\Omega)} \left\{ \int_\Omega |\nabla u| + \int_{\partial\Omega} |u| + \frac{1}{\delta}\int_\Omega (u-\psi)^- - \int_\Omega fu \right\},$$

pour lequel on montre l'existence d'une solution u_δ et on montre par la suite que :

$$u_\delta \longrightarrow u \quad \text{quand } \delta \to 0,$$
$$\mathcal{P}_\delta \longrightarrow \mathcal{P} \quad \text{quand } \delta \to 0,$$

où u est un minimiseur du problème défini par $\mathcal{P}_{BV}$.

Ensuite, par une méthode de calcul du dual, on montre l'existence d'un couple de solution (σ, τ) au problème dual de $\mathcal{P}$, noté $\mathcal{P}^*$ et qui réalise:

$$\begin{cases} \text{div}\sigma + \tau + f = 0, \\ (\sigma, \tau) \in L^\infty(\Omega, \mathbb{R}^N) \times L^\infty(\Omega), \\ \tau \geq 0, \|\sigma\|_\infty \leq 1. \end{cases}$$

Dans la suite, on s'intéresse au cas unidimensionnel en présentant deux exemples explicites de résolution.

Dans la dernière partie, on s'intéresse à l'étude d'un problème de contrôle optimal sur $BV(\Omega)$ pour $f \leq 0$ et pour cela on introduit le problème de minimisation suivant:

$$\mathcal{P}_\lambda = \inf_{\substack{u \in W_0^{1,1}(\Omega) \\ u \geq \psi}} \left\{ \int_\Omega |\nabla u| - \lambda \int_\Omega fu \right\}. \tag{1.6}$$

On remarque que si $\lambda \in L^\infty(\Omega)$ est assez petit, $\mathcal{P}_\lambda$ est coercif sur $BV(\Omega)$. On montre l'existence d'une solution au problème d'obstacle, mais en l'absence de résultat d'unicité, $T_{\lambda f}$ –analogue de T_f dans le cas $p > 1$– est mal défini. Pour cette raison, on dira que $u \in E_{\lambda f}(\psi)$ quand u appartient à $BV(\Omega)$ et réalise l'infimum du problème d'obstacle définie par $\mathcal{P}_\lambda$.

On peut néanmoins montrer les propriétés suivantes de E_f:

$$\text{Si } \psi_k \rightharpoonup \psi \quad \text{faiblement} \quad \text{dans} \quad BV(\Omega),$$

alors

$$\mathcal{P}(\psi) \leq \liminf_{k\to\infty} \mathcal{P}(\psi_k),$$

et si

$$\psi_k \leq \psi, \quad \text{il y a egalité .}$$

D'autre part, on a la propriété de "croissance" des ensembles $E_f(\psi)$, analogue à la propriété de croissance de $T_f(\psi)$ pour le $p-$obstacle:

$$\psi_1 \geq \psi_2, \exists\ u_1 \in E_f(\psi_1), u_2 \in E_f(\psi_2), \text{ tels que } u_1 \geq u_2.$$

Enfin dans une dernière section, on définit le problème de contrôle optimal: soit z dans $BV(\Omega)$ appelée fonction de coût et $\psi \in W_0^{1,1}(\Omega)$ appelée variable de contrôle, on cherche le couple $(\psi, u \in E_f(\psi))$, solution du problème suivant:

$$\inf_{\substack{(u,\psi)\in(W_0^{1,1}(\Omega))^2 \\ u\in E_f(\psi)}} \left\{ \int_\Omega |u - z| + \int_\Omega |\nabla\psi|\ dx \right\}. \tag{1.7}$$

La solution (ψ^*, u^*) de ce problème est appelé couple optimal. On définit

$$\lambda_f = \inf_{\substack{(u,\psi)\in(W_0^{1,1}(\Omega))^2 \\ \int_\Omega fu=1}} \int_\Omega |\nabla u|\ dx. \tag{1.8}$$

Dans le cas $\lambda < \lambda_f$ et $f \leq 0$, on montre l'existence d'un couple optimal de la forme (u, u).

Pour le cas $\lambda = \lambda_f$ et $f \leq 0$, on donne quelques propriétés et caractérisation de la solution.

Chapter 2

Approximation of eigenvalues and eigenfunctions for the 1-Laplacian

2.1 Introduction: the first eigenvalue for the 1-Laplacian

In recent works, several authors were interested on the study of the "first eigenvalue" for the 1-Laplacian operator, that we shall denote as the not everywhere defined $u \mapsto -\mathrm{div}(\frac{\nabla u}{|\nabla u|})$.

Due to the singularity of this operator, the definition of the first eigenvalue can be correctly defined with the aid of a variational form: let λ_1 be defined as

$$\lambda_1 = \inf_{\substack{u \in W_0^{1,1}(\Omega) \\ \|u\|_1 = 1}} \int_\Omega |\nabla u|. \tag{2.1}$$

Notice that λ_1 is well defined and is positive, due to Poincaré's inequality.

In order to justify the term "eigenvalue" for λ_1, one must prove the existence of an associated "eigenfunction". As in the p-Laplacian case, an eigenfunction will be a solution of (2.1). Unfortunately, since $W^{1,1}(\Omega)$ is not a reflexive space, one cannot hope to obtain a solution for (2.1) by classical arguments.

This difficulty can be overcome by introducing the space $BV(\Omega)$, which is the weak closure of $W^{1,1}(\Omega)$, and by extending the infimum to that space, using the features of $BV(\Omega)$: Density of regular maps in $BV(\Omega)$, existence of the trace map on the boundary... However, these properties are not sufficient to obtain solutions by classical methods, since the trace map –which is well defined on $BV(\Omega)$– is not continuous for the weak topology.

This new difficulty can be "solved" by introducing – as it is the case in the theory of minimal surfaces and in plasticity and also for related problems – a "relaxed" formulation for (2.1). This relaxed formulation consists in replacing the condition $\{u = 0\}$ on the boundary by the addition of a term $\int_{\partial\Omega} |u|$ in the functional to minimize. The new formulation is then

$$\inf_{\substack{u \in BV(\Omega) \\ \|u\|_1 = 1}} \int_\Omega |\nabla u| + \int_{\partial\Omega} |u|. \tag{2.2}$$

This problem has an infimum equal to λ_1. It can be seen by approximating function in $BV(\Omega)$ by functions in $W^{1,1}(\Omega)$ for a topology related to the narrow topology of measures. This topology is specified in section 2.2.

Then we can prove the existence of a minimizer of (2.2) in $BV(\Omega)$ using classical arguments, which will be specified later in this chapter.

To obtain the partial differential equation satisfied by a minimizer of (2.2), equation which can be seen as an eigenvalue's equation, the author used in [20] an approximation of (2.1) by the following problem on $W_0^{1,1+\varepsilon}(\Omega)$:

$$\lambda_{1+\varepsilon} = \inf_{\substack{u \in W_0^{1,1+\varepsilon}(\Omega) \\ \|u\|_1 = 1}} \int_\Omega |\nabla u|^{1+\varepsilon}, \tag{2.3}$$

and proves that $\lambda_{1+\varepsilon}$ converges to λ_1. Moreover, if u_ε is a positive solution of the minimizing problem defined in (2.3), u_ε converges weakly in $BV(\Omega)$ to some u which satisfies

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \lambda_1,$$

in a sense which needs to be specified, and is detailed in the present chapter.

Let us note that it is also proved in [18] that there are characteristic functions of sets which are solutions. These sets are therefore called eigensets.

Another approach is used in [2], [3], where the authors use the concept of Cheeger sets [14]. In these papers, the authors present a remarkable construction of eigensets for $2-$dimensional convex sets Ω. Among their results, there is the uniqueness of eigensets in the case $N = 2$.

Our aim in the present chapter is to propose an approach of the first eigenvalue and the first eigenfunction of the 1-Laplacian operator, using a penalization method. This method has an obvious numerical advantage: the constraint $\int_\Omega |u| = 1$ has a higher cost than the introduction of a penalization term as $n\left(\int_\Omega |u| - 1\right)^2$. This provides in the same time, a new proof of the existence and uniqueness of a positive eigenfunction.

2.2 Survey on known results about the space $BV(\Omega)$

Let us recall the definition of the space of functions with bounded variation. Let Ω be an open regular domain in $\mathbb{R}^N$, $N > 1$, and let $\mathcal{M}^1(\Omega)$ be the space of bounded measures in Ω. We define

$$BV(\Omega) = \left\{u \in L^1(\Omega), \nabla u \in \mathcal{M}^1(\Omega)\right\}.$$

Endowed with the norm $\int_\Omega |\nabla u| + \int_\Omega |u|$, the space $BV(\Omega)$ is a Banach space.

Another topology is crucial when one wants to use variational technics. We define *the weak topology* with the aid of sequences as follows: we say that a sequence $u_n \rightharpoonup u$ weakly in $BV(\Omega)$ if the following two conditions are fulfilled:

- $\int_\Omega |u_n - u| \longrightarrow 0 \quad \text{in } L^1(\Omega) \quad \text{when } n \longrightarrow \infty,$
- $\int_\Omega \partial_i u_n \, \phi \longrightarrow \int_\Omega \partial_i u \, \phi, \quad \forall i = 1, 2, ...N \quad \forall \phi \in \mathcal{C}_c(\Omega) \quad \text{when } n \longrightarrow \infty.$

Let us note that the second convergence is also called *the vague convergence* of ∇u_n towards ∇u.

We shall also use the concept of *tight convergence* in $BV(\Omega)$: we say that a sequence u_n converges tightly to u in $BV(\Omega)$ if the following two conditions are fulfilled:

- $u_n \rightharpoonup u, \text{ weakly} \quad \text{in } BV(\Omega) \quad \text{when } n \longrightarrow \infty,$

- $\displaystyle\int_\Omega |\nabla u_n| \longrightarrow \int_\Omega |\nabla u| \quad \text{when } n \longrightarrow \infty.$

Let us note that the last assertion is equivalent to say that, for all $\phi \in \mathcal{C}(\bar{\Omega}, \mathbb{R}^N)$,

$$\int_\Omega \nabla u_n \cdot \phi \longrightarrow \int_\Omega \nabla u \cdot \phi, \quad \text{when } n \longrightarrow \infty.$$

We now recall some facts about embedding and compact embedding from BV to other L^q spaces:

- If Ω is an open $\mathcal{C}^1$ set, then $BV(\Omega)$ is continuously embedded in $L^p(\Omega)$ for all $p \leq \frac{N}{N-1}$.
- If Ω is also bounded and smooth, the embedding is compact in $L^p(\Omega)$ for every $p < \frac{N}{N-1}$.

Finally we recall the existence of a map, called *trace map* and defined on $BV(\Omega)$, which coincides with the restriction on $\partial\Omega$ of u when u belongs to $\mathcal{C}(\bar{\Omega}) \cap BV(\Omega)$ or less classically when $u \in W^{1,1}(\Omega)$. . This map is continuous under the strong topology. It is not continuous under the weak topology However the following property holds: if $u_n \to u$ tightly in $BV(\Omega)$, then

$$\int_{\partial\Omega} |u_n - u| \longrightarrow 0 \quad \text{for } n \to \infty.$$

We now state a generalization of the Green's formula : this allows us to give sense to the product $\sigma \cdot \nabla u$ when σ is in $L^\infty(\Omega, \mathbb{R}^N)$, div $\sigma \in L^N(\Omega)$ and $u \in BV(\Omega)$. This will be useful in the formulation of the partial differential equation associated to the eigenvalue.

Let us recall that $\mathcal{D}(\Omega)$ is the space of $\mathcal{C}^\infty$-functions, with support on Ω.

PROPOSITION 2.2.1. *Let $\sigma \in L^\infty(\Omega, \mathbb{R}^N)$, $\text{div}\sigma \in L^N(\Omega)$ and $u \in BV(\Omega)$. Define the distribution $\sigma \cdot \nabla u$ by the following formula : for any $\varphi \in \mathcal{D}(\Omega)$,*

$$\langle \sigma \cdot \nabla u, \varphi \rangle = -\int_\Omega (\text{div}\sigma) u\varphi - \int_\Omega (\sigma \cdot \nabla\varphi)\ u. \tag{2.4}$$

Then

$$|\langle \sigma \cdot \nabla u, \varphi \rangle| \leq \|\sigma\|_\infty \langle |\nabla u|, |\varphi| \rangle.$$

In particular, $\sigma \cdot \nabla u$ is a bounded measure which satisfies:

$$|\sigma \cdot \nabla u| \leq \|\sigma\|_\infty |\nabla u|.$$

In addition, if $\varphi \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)$*, the following Green's Formula holds:*

$$\langle \sigma \cdot \nabla u, \varphi \rangle = -\int_\Omega (\mathrm{div}\sigma) u\varphi - \int_\Omega (\sigma \cdot \nabla\varphi)\ u + \int_{\partial\Omega} \sigma \cdot \overrightarrow{n} u\ \varphi, \tag{2.5}$$

where $\overrightarrow{n}$ *is the unit outer normal to* $\partial\Omega$*.*

Suppose that $U \in BV(\mathbb{R}^N \setminus \overline{\Omega})$*, that* $u \in BV(\Omega)$ *and define the function* $\widetilde{u}$ *as:*

$$\widetilde{u} = \begin{cases} u \text{ in } \Omega, \\ U \text{ in } \mathbb{R}^N \setminus \overline{\Omega}. \end{cases}$$

Then $\widetilde{u} \in BV(\mathbb{R}^N)$ *and*

$$\nabla\widetilde{u} = \nabla u\ \chi_\Omega + \nabla U\ \chi_{(\mathbb{R}^N \setminus \overline{\Omega})} + (U - u)\ \delta_{\partial\Omega},$$

where in the last term, U *and* u *denote the trace of* U *and* u *on* $\partial\Omega$ *and* $\delta_{\partial\Omega}$ *denotes the uniform Dirac measure on* $\partial\Omega$*. Finally, we introduce the measure* $\sigma \cdot \nabla\widetilde{u}$ *on* $\overline{\Omega}$ *by the formula*

$$(\sigma \cdot \nabla\widetilde{u}) = (\sigma \cdot \nabla u)\chi_\Omega + \sigma \cdot \overrightarrow{n}(U - u)\ \delta_{\partial\Omega},$$

where $(\sigma \cdot \nabla u)\chi_\Omega$ *has been defined in* (2.4)*. Then* $\sigma \cdot \nabla\widetilde{u}$ *is absolutely continuous with respect to* $|\nabla\widetilde{u}|$*, with the inequality*

$$|\sigma \cdot \nabla\widetilde{u}| \leq \|\sigma\|_\infty |\nabla\widetilde{u}|.$$

For a proof the reader can consult [17], [35], [43].

2.3 Presentation of the main result

We now describe the approximation result here enclosed. Let $n \in \mathbb{N}^*$, let us consider the following minimization problem:

$$\lambda_{1,n} = \inf_{u \in W_0^{1,1}(\Omega)} \left\{ \int_\Omega |\nabla u| + n \left(\int_\Omega |u| - 1 \right)^2 \right\}. \tag{2.6}$$

Let us introduce the relaxed formulation

$$\widetilde{\lambda}_{1,n} = \inf_{u \in BV(\Omega)} \left\{ \int_\Omega |\nabla u| + \int_{\partial\Omega} |u| + n \left(\int_\Omega |u| - 1 \right)^2 \right\}. \tag{2.7}$$

We prove here the following result :

THEOREM 2.3.1. *Let Ω be a piecewise $\mathcal{C}^1$ bounded domain in $\mathbb{R}^N$, $N > 1$. For every $n \in \mathbb{N}^*$, the problem* (2.7) *possesses a solution u_n in $BV(\Omega)$ which can be chosen nonnegative. Moreover, u_n satisfies the following partial differential equation:*

$$\begin{cases} -\text{div } \sigma_n + 2n \left(\displaystyle\int_\Omega u_n - 1 \right) \text{sign}^+(u_n) = 0 \quad \text{in } \Omega, \\ \sigma_n \in L^\infty(\Omega, \mathbb{R}^N), \ \|\sigma_n\|_\infty \leq 1, \\ \sigma_n \cdot \nabla u_n = |\nabla u_n| \quad \text{in } \Omega, \\ u_n \text{ is not identically zero}, -\sigma_n \cdot \overrightarrow{n}(u_n) = u_n \quad \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where $\overrightarrow{n}$ denotes the unit outer normal to $\partial\Omega$ and $\sigma_n \cdot \nabla u_n$ is the measure defined in Proposition 2.2.1 *and* $\text{sign}^+(u_n)$ *is some function in $L^\infty(\Omega)$ such that* $\text{sign}^+(u_n) u_n = u_n$ *in Ω.*

Moreover $\lambda_{1,n}$ converges towards λ_1 and u_n converges towards the first eigenfunction u.

REMARK 2.3.1. Clearly, u_n is not identically zero for n large enough as soon as $n > \lambda_1$.

REMARK 2.3.2. From Proposition 2.2.1 (with $U = 0$), the conditions

$$\sigma_n \cdot \nabla u_n = |\nabla u_n| \quad \text{in } \Omega \ , \qquad -\sigma_n \cdot \overrightarrow{n}(u_n) = u_n \quad \text{on } \partial\Omega,$$

are equivalent to

$$\sigma_n \cdot \nabla \widetilde{u}_n = |\nabla \widetilde{u}_n| \quad \text{on } \Omega \cup \partial\Omega.$$

REMARK 2.3.3. The identity $\sigma_n \cdot \nabla u_n = |\nabla u_n|$ makes sense since

$$-\text{div } \sigma_n = -2n \left(\int_\Omega u_n - 1 \right) \text{sign}^+(u_n),$$

which implies that $\text{div}\sigma_n \in L^\infty(\Omega)$, therefore $\sigma_n \cdot \nabla u_n$ is well-defined by Proposition 2.2.1.

We subdivide the proof of Theorem 2.3.1 into several steps :

- First step: We use some kind of regularization of the minimization problem by introducing for some $\varepsilon > 0$ and small

$$\inf_{u \in W_0^{1,1+\varepsilon}(\Omega)} \left\{ \int_\Omega |\nabla u|^{1+\varepsilon} + n \left(\int_\Omega |u|^{1+\varepsilon} - 1 \right)^2 \right\}.$$

We prove that for n large enough, this problem possesses a solution which can be chosen nonnegative and denoted by $u_{n,\varepsilon}$, which satisfies

$$\left\{ -\operatorname{div}(|\nabla u_{n,\varepsilon}|^{\varepsilon-1}\nabla u_{n,\varepsilon}) + 2n\left(\int_\Omega u_{n,\varepsilon}^{1+\varepsilon} - 1\right) u_{n,\varepsilon}^{\varepsilon} = 0, \quad \text{in } \Omega, \right.$$

- Second step: We extend $u_{n,\varepsilon}$ by zero outside of Ω and observe that the sequence still denoted $(u_{n,\varepsilon})$ is uniformly bounded in $BV(\mathbb{R}^N)$, more precisely
$$\int_{\mathbb{R}^N} |\nabla u_{n,\varepsilon}|^{1+\varepsilon} \leq C.$$
Then we can extract from $u_{n,\varepsilon}$ a subsequence, such that $u_{n,\varepsilon} \rightharpoonup u_n$ weakly in $BV(\mathbb{R}^N)$. The limit function belongs to $BV(\mathbb{R}^N)$ and is zero outside of $\bar{\Omega}$.

- Third step: we prove that $\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1}\nabla u_{n,\varepsilon}$ is uniformly bounded in $L^q(\Omega)$ $\forall\, q < \infty$. Then we can extract from $\sigma_{n,\varepsilon}$ a subsequence, such that $\sigma_{n,\varepsilon} \rightharpoonup \sigma_n$ weakly in $L^q(\Omega)$ $\forall\, q < \infty$, such that $\|\sigma_n\|_\infty \leq 1$ and $\sigma_n \cdot \nabla u_n = |\nabla u_n|$ in $\Omega \cup \partial\Omega$.

- Fourth step: we prove that u_n is a solution of the minimizing problems (2.7) and (2.8). We also prove that σ_n satisfies the problem (2.8).

- Fifth step: we establish that $\lambda_{1,n}$ converges strongly to λ_1 when n goes to ∞ and that u_n converges strongly to the first eigenfunction associated to λ_1.

2.4 Proof of the main result

We provide here the proof of Theorem 2.3.1, outlined as above.

Step 1: We prove here the existence and uniqueness of a positive solution for the following approximation problem

$$\lambda_{1+\varepsilon,n} = \inf_{u \in W_0^{1,1+\varepsilon}(\Omega)} I_{1+\varepsilon,n}(u), \tag{2.9}$$

where $I_{1+\varepsilon,n}$ is the following functional

$$I_{1+\varepsilon,n}(u) = \int_\Omega |\nabla u|^{1+\varepsilon} + n\left(\int_\Omega |u|^{1+\varepsilon} - 1\right)^2, \tag{2.10}$$

for some positive ε given.

We first prove that $\lambda_{1+\varepsilon,n}$ is achieved, using standard variational technics: Let $(u_i)_i$ be a minimizing sequence for $\lambda_{1+\varepsilon,n}$. Without loss of generality, up to replace u_i by $|u_i|$, one may assume that u_i is nonnegative. Since $I_{1+\varepsilon,n}$ is coercive, (u_i) is bounded in $W_0^{1,1+\varepsilon}(\Omega)$.

As a consequence, we may extract from it a subsequence, still denoted $(u_i)_i$, which converges weakly in $W_0^{1,1+\varepsilon}(\Omega)$ to some function $u_{n,\varepsilon} \in W_0^{1,1+\varepsilon}(\Omega)$. Furthermore, by the Rellich-Kondrakov Theorem [10], [9], [1], $(u_i)_i$ converges to $u_{n,\varepsilon}$ in $L^{1+\varepsilon}(\Omega)$.

Using the weak lower semicontinuity of the semi-norm $\displaystyle\int_\Omega |\nabla u|^{1+\varepsilon}$ for the weak topology of $W_0^{1,1+\varepsilon}(\Omega)$, one has:

$$\begin{aligned}\lambda_{1+\varepsilon,n} &\leq \int_\Omega |\nabla u_{n,\varepsilon}|^{1+\varepsilon} + n\left(\int_\Omega |u_{n,\varepsilon}|^{1+\varepsilon} - 1\right)^2 \\ &\leq \liminf_{i\to+\infty}\left[\int_\Omega |\nabla u_i|^{1+\varepsilon} + n\left(\int_\Omega |u_i|^{1+\varepsilon} - 1\right)^2\right] = \lambda_{1+\varepsilon,n}.\end{aligned}$$

Hence, $u_{n,\varepsilon}$ is a solution of the minimization problem (2.9).

We now prove that this weak solution solves the following partial differential equation:

$$\begin{cases} -\mathrm{div}\sigma_{n,\varepsilon} + 2n\left(\displaystyle\int_\Omega u_{n,\varepsilon}^{1+\varepsilon} - 1\right) u_{n,\varepsilon}^{\varepsilon} = 0 \quad \text{in } \Omega, \\ \sigma_{n,\varepsilon}\cdot\nabla u_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{1+\varepsilon} \quad \text{in } \Omega, \\ u_{n,\varepsilon} > 0 \quad \text{in } \Omega,\ u_{n,\varepsilon} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

Indeed, for every $h \in \mathcal{D}(\Omega)$, we have:

$$\begin{aligned} &DI_{1+\varepsilon,n}(u_{n,\varepsilon})\cdot h \\ &= (1+\varepsilon)\left[\int_\Omega |\nabla u_{n,\varepsilon}|^{\varepsilon-1}\nabla u_{n,\varepsilon}\cdot\nabla h + 2n\left(\int_\Omega u_{n,\varepsilon}^{1+\varepsilon} - 1\right)\int_\Omega u_{n,\varepsilon}^{\varepsilon} h\right] \\ &= (1+\varepsilon)\int_\Omega\left[-\mathrm{div}\left(|\nabla u_{n,\varepsilon}|^{\varepsilon-1}\nabla u_{n,\varepsilon}\right) + 2n\left(\int_\Omega u_{n,\varepsilon}^{1+\varepsilon} - 1\right)u_{n,\varepsilon}^{\varepsilon}\right] h \\ &= 0. \end{aligned}$$

Thus, we get:

$$-\mathrm{div}\left(|\nabla u_{n,\varepsilon}|^{\varepsilon-1}\nabla u_{n,\varepsilon}\right) + 2n\left(\int_\Omega u_{n,\varepsilon}^{1+\varepsilon} - 1\right) u_{n,\varepsilon}^{\varepsilon} = 0, \tag{2.12}$$

in a distribution sense.

Since $u_{n,\varepsilon}$ is a weak solution of equation (2.12), by regularity results (as developed by Guedda-Veron [33], see also Tolksdorf [47]), one gets that $u_{n,\varepsilon} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})\ \forall\ \alpha \in (0,1)$. Moreover, since $u_{n,\varepsilon}$ is a nonnegative weak solution of the equation (2.12), by the strict maximum principle of Vazquez (see [48]), $u_{n,\varepsilon}$ is positive everywhere. Hence, setting $\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1}\nabla u_{n,\varepsilon}$, we have shown that $u_{n,\varepsilon} \in \mathcal{C}^{1,\alpha}(\bar{\Omega}) \cap W_0^{1,1+\varepsilon}(\Omega)$ is a positive solution of (2.11).

LEMMA 2.4.1. *The problem* (2.11) *has a unique positive solution*

Proof of Lemma 2.4.1. Let u and v be two positive solutions of (2.11). Then we have:

$$-\text{div}\,[\sigma_{n,\varepsilon}(u) - \sigma_{n,\varepsilon}(v)] + 2n\,[\alpha(u) - \alpha(v)]\,u^{\varepsilon} + 2n\,\alpha(v)\,(u^{\varepsilon} - v^{\varepsilon}) = 0, \quad (2.13)$$

where $\alpha(u) = \displaystyle\int_{\Omega} u^{1+\varepsilon} - 1.$

Case 1: $\|u\|_{1+\varepsilon} \geq \|v\|_{1+\varepsilon}$.
Let us multiply (2.13) by $(u-v)^+$ then integrate. It is clear that

$$2n\,[\alpha(u) - \alpha(v)] \int_{\Omega} u^{\varepsilon}(u-v)^+ \geq 0.$$

So we get that:

$$\int_{\Omega} [\sigma_{n,\varepsilon}(u) - \sigma_{n,\varepsilon}(v)] \cdot \nabla(u-v)^+ + 2n\,\alpha(v) \int_{\Omega} (u^{\varepsilon} - v^{\varepsilon})\,(u-v)^+ \leq 0. \quad (2.14)$$

We know that

$$\int_{\Omega} (\sigma_{n,\varepsilon}(u) - \sigma_{n,\varepsilon}(v)) \cdot \nabla(u-v) \geq 0. \quad (2.15)$$

On the other hand it is clear that

$$\int_{\Omega} (u^{\varepsilon} - v^{\varepsilon})\,(u-v) \geq 0. \quad (2.16)$$

By (2.15) and (2.16), we have that:

$$\int_{\Omega} [\sigma_{n,\varepsilon}(u) - \sigma_{n,\varepsilon}(v)] \cdot \nabla(u-v)^+ + 2n\,\alpha(v) \int_{\Omega} (u^{\varepsilon} - v^{\varepsilon})\,(u-v)^+ \geq 0. \quad (2.17)$$

So from (2.14) and (2.17), we obtain that

$$\int_\Omega [\sigma_{n,\varepsilon}(u) - \sigma_{n,\varepsilon}(v)] \cdot \nabla(u-v)^+ + 2n\ \alpha(v) \int_\Omega (u^\varepsilon - v^\varepsilon)\,(u-v)^+ = 0.$$

Then $\displaystyle\int_\Omega (u^\varepsilon - v^\varepsilon)\,(u-v)^+ = 0$, which implies $(u-v)^+ = 0$, i.e. $u \leq v$. Using $\|u\|_{1+\varepsilon} \geq \|v\|_{1+\varepsilon}$, one finally gets $u = v$ a.e.

Case 2: $\|u\|_{1+\varepsilon} \leq \|v\|_{1+\varepsilon}$.
We use the same arguments as in the Case 2, just replacing $(u-v)^+$ by $(v-u)^+$. □

Thus, we have proved the existence and uniqueness of a positive solution to the problem (2.9).

Step 2: We prove here that $\displaystyle\lim_{\varepsilon\to 0}\lambda_{1+\varepsilon,n} = \lambda_{1,n}$.

PROPOSITION 2.4.1. *For every $n \in \mathbb{N}^*$, we have:*

$$\limsup_{\varepsilon\to 0}\ \lambda_{1+\varepsilon,n} \leq \lambda_{1,n}$$

Proof of Proposition 2.4.1. Let $\delta > 0$ be given and $\varphi \in \mathcal{D}(\Omega)$ such that

$$I_{1,n}(\varphi) = \int_\Omega |\nabla\varphi| + n\left(\int_\Omega |\varphi| - 1\right)^2 \leq \lambda_{1,n} + \delta.$$

But $\displaystyle\lim_{\varepsilon\to 0} I_{1+\varepsilon,n}(\varphi) = I_{1,n}(\varphi)$, hence,

$$\limsup_{\varepsilon\to 0} \lambda_{1+\varepsilon,n} \leq \lambda_{1,n} + \delta.$$

δ being arbitrary, we get $\displaystyle\limsup_{\varepsilon\to 0}\lambda_{1+\varepsilon,n} \leq \lambda_{1,n}$. □

Let now $u_{n,\varepsilon}$ be the positive solution of the minimizing problem (2.9). Using Poincaré's and Hölder's inequalities, we get

$$\begin{aligned}\int_\Omega u_{n,\varepsilon}dx \leq C\int_\Omega |\nabla u_{n,\varepsilon}|dx &\leq C\left(\int_\Omega |\nabla u_{n,\varepsilon}|^{1+\varepsilon}dx\right)^{\frac{1}{1+\varepsilon}} |\Omega|^{\frac{\varepsilon}{1+\varepsilon}} \\ &\leq C'\lambda_{1+\varepsilon,n}.\end{aligned}$$

These inequalities show that $(u_{n,\varepsilon})_{\varepsilon>0}$ and $(\nabla u_{n,\varepsilon})_{\varepsilon>0}$ are both bounded in $L^1(\Omega)$. This means that $(u_{n,\varepsilon})_{\varepsilon>0}$ is bounded in $BV(\Omega)$.

Therefore, we may extract from it a subsequence, still denoted by $(u_{n,\varepsilon})$, which converges in BV for the weak topology, towards some limit denoted by u_n, such that

$$u_{n,\varepsilon} \longrightarrow u_n \quad \text{strongly in } L^k(\Omega), \forall\, k < 1^* = \frac{N}{N-1} \quad \text{when } \varepsilon \to 0,$$

$$\nabla u_{n,\varepsilon} \rightharpoonup \nabla u_n \quad \text{weakly in } \mathcal{M}^1(\Omega) \quad \text{when } \varepsilon \to 0,$$

where $\mathcal{M}^1(\Omega)$ denotes the space of bounded Radon measures on Ω.

In step (4) we shall precise this limit. In particular we shall obtain u_n as the restriction to Ω of some limit of extended functions $u_{n,\varepsilon}$ by zero outside of Ω.

Step 3: we obtain $\sigma_n = \frac{\text{“}\nabla u_n\text{”}}{|\nabla u_n|}$ as the weak limit of $\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1}\nabla u_{n,\varepsilon}$.

Let $\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1}\nabla u_{n,\varepsilon}$, one sees that $\sigma_{n,\varepsilon}$ is uniformly bounded in $L^{\frac{1+\varepsilon}{\varepsilon}}(\Omega)$. Let us prove that $\sigma_{n,\varepsilon}$ is uniformly bounded in every $L^q(\Omega)$, for all $q < \infty$. Indeed, let $q > 1$ be given and let ε be such that $q < \frac{1+\varepsilon}{\varepsilon}$. Then

$$\left(\int_\Omega |\sigma_{n,\varepsilon}|^q\right)^{\frac{1}{q}} \leq \left(\int_\Omega |\sigma_{n,\varepsilon}|^{\frac{1+\varepsilon}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}} |\Omega|^{\frac{1+\varepsilon(1-q)}{(1+\varepsilon)q}} \leq C.$$

Then we may extract from it a subsequence, still denoted by $\sigma_{n,\varepsilon}$, such that $\sigma_{n,\varepsilon}$ tends to some σ_n weakly in $L^q(\Omega)$, for all $q < \infty$ and $\sigma_{n,\varepsilon}$ tends to σ_n a.e., when ε tends to 0.

We need now to prove that $\|\sigma_n\|_\infty \leq 1$. For that aim, let η be in $\mathcal{D}(\Omega, \mathbb{R}^N)$. Then

$$\begin{aligned}
\left|\int_\Omega \sigma_n \cdot \eta\right| \leq \liminf_{\varepsilon\to 0}\left|\int_\Omega \sigma_{n,\varepsilon}\cdot\eta\right| &\leq \liminf_{\varepsilon\to 0}\int_\Omega |\nabla u_{n,\varepsilon}|^\varepsilon |\eta| \\
&\leq \liminf_{\varepsilon\to 0}\left(\int_\Omega |\nabla u_{n,\varepsilon}|^{1+\varepsilon}\right)^{\frac{\varepsilon}{1+\varepsilon}}\left(\int_\Omega |\eta|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \\
&\leq \liminf_{\varepsilon\to 0}\,(\lambda_{1+\varepsilon,n})^{\frac{\varepsilon}{1+\varepsilon}}\left(\int_\Omega |\eta|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \\
&\leq \int_\Omega |\eta|.
\end{aligned}$$

This implies that $\|\sigma_n\|_\infty \leq 1$.

Let us now observe that $u^\varepsilon_{n,\varepsilon}$ is uniformly bounded in every $L^q(\Omega), q < \infty$. Indeed, let q be given and let ε be small enough, such that $q < \frac{1+\varepsilon}{\varepsilon}$, then

$$\left(\int_\Omega |u^\varepsilon_{n,\varepsilon}|^q\right)^{\frac{1}{q}} \leq \left(\int_\Omega |u_{n,\varepsilon}|^{1+\varepsilon}\right)^{\frac{\varepsilon}{1+\varepsilon}} |\Omega|^{\frac{1+\varepsilon(1-q)}{q(1+\varepsilon)}} \leq C.$$

Then $w_{n,\varepsilon} = u^\varepsilon_{n,\varepsilon}$ converges weakly, in every $L^q(\Omega)$, $q < \infty$, up to a subsequence, to some w_n, when ε tends to 0.

Let us prove that $0 \leq w_n \leq 1$ and $(w_n - 1)u_n = 0$. For the first assertion, let $\eta \in \mathcal{D}(\Omega)$,

$$\begin{aligned}\left|\int_\Omega w_n \cdot \eta\right| &\leq \left(\int_\Omega |w_n|^{\frac{1+\varepsilon}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}} \left(\int_\Omega |\eta|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \\ &\leq \liminf_{\varepsilon\to 0} (\lambda_{1+\varepsilon,n})^{\frac{\varepsilon}{1+\varepsilon}} \left(\int_\Omega |\eta|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \\ &\leq \int_\Omega |\eta|.\end{aligned}$$

Hence $0 \leq w_n \leq 1, \quad \forall n \in \mathbb{N}^*$.

To prove that $(w_n - 1)u_n = 0$, let us observe that $u_{n,\varepsilon} \longrightarrow u_n$ in $L^k(\Omega)$ strongly for all $k < \frac{N}{N-1}$ and $w_{n,\varepsilon} \longrightarrow w_n$ in $L^{N+1}(\Omega)$ weakly, therefore

$$\int_\Omega w_{n,\varepsilon} u_{n,\varepsilon} \longrightarrow \int_\Omega w_n u_n \quad \text{when } \varepsilon \to 0$$

Finally,

$$\int_\Omega w_n u_n = \lim_{\varepsilon\to 0} \int_\Omega w_{n,\varepsilon} u_{n,\varepsilon} = \lim_{\varepsilon\to 0} \int_\Omega u^{1+\varepsilon}_{n,\varepsilon} = \int_\Omega u_n.$$

Using the fact that $0 \leq w_n \leq 1$, one gets the result.

Passing to the limit in (2.12), one gets:

$$-\text{div}\sigma_n + 2n\left(\int_\Omega u_n - 1\right) w_n = 0. \tag{2.18}$$

Step 4: Extension of $u_{n,\varepsilon}$ outside Ω and convergence towards a solution of (2.11).

Let $\widetilde{u}_{n,\varepsilon}$ be the extension of $u_{n,\varepsilon}$ by 0 in $\mathbb{R}^N \setminus \overline{\Omega}$. Since $u_{n,\varepsilon} = 0$ on $\partial\Omega$, then $\widetilde{u}_{n,\varepsilon} \in W^{1,1+\varepsilon}(\mathbb{R}^N)$ and $(\widetilde{u}_{n,\varepsilon})$ is bounded in $BV(\mathbb{R}^N)$. Then one may extract from it a subsequence, still denoted $(\widetilde{u}_{n,\varepsilon})$, such that

$$\widetilde{u}_{n,\varepsilon} \longrightarrow v_n \quad \text{in } L^k(\mathbb{R}^N), \quad \forall\, k < \frac{N}{N-1} \quad \text{when } \varepsilon \longrightarrow 0,$$

with $v_n = 0$ outside of $\overline{\Omega}$ and

$$\nabla\widetilde{u}_{n,\varepsilon} \rightharpoonup \nabla v_n \quad \text{weakly in } \mathcal{M}^1(\mathbb{R}^N) \quad \text{when } \varepsilon \longrightarrow 0.$$

We denote by u_n the restriction of v_n to Ω. We use in the above some limit σ_n of $\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1}\nabla u_{n,\varepsilon}$ obtained in the third step. More precisely:

$$\sigma_{n,\varepsilon} = |\nabla u_{n,\varepsilon}|^{\varepsilon-1}\nabla u_{n,\varepsilon} \rightharpoonup \sigma_n \quad \text{weakly in } L^q(\Omega),\ \forall\, q < \infty \quad \text{when } \varepsilon \longrightarrow 0.$$

Multiplying the equation (2.12) by $\widetilde{u}_{n,\varepsilon}\varphi$, where $\varphi \in \mathcal{D}(\mathbb{R}^N)$, and integrating by parts, one obtains:

$$\int_{\Omega} \sigma_{n,\varepsilon} \cdot \nabla(\widetilde{u}_{n,\varepsilon}\varphi) + 2n\left(\int_{\Omega} \widetilde{u}_{n,\varepsilon}^{1+\varepsilon} - 1\right)\int_{\Omega} \widetilde{u}_{n,\varepsilon}^{1+\varepsilon}\varphi = 0\ ,$$

or equivalently

$$\int_{\mathbb{R}^N} |\nabla(\widetilde{u}_{n,\varepsilon})|^{1+\varepsilon}\varphi + \int_{\Omega} \sigma_{n,\varepsilon} u_{n,\varepsilon} \cdot \nabla\varphi + 2n\left(\int_{\mathbb{R}^N} \widetilde{u}_{n,\varepsilon}^{1+\varepsilon} - 1\right)\int_{\mathbb{R}^N} \widetilde{u}_{n,\varepsilon}^{1+\varepsilon}\varphi = 0\ . \tag{2.19}$$

Since $\sigma_{n,\varepsilon} \rightharpoonup \sigma_n$ in $L^q(\Omega)$ for all $q < \infty$, in particular for any $\alpha > 0$, $\sigma_{n,\varepsilon}$ tends weakly towards σ_n in $L^{N+\alpha}(\Omega)$. Since $\widetilde{u}_{n,\varepsilon}$ tends strongly towards v_n in $L^k(\Omega)$, $k < \frac{N}{N-1}$, one obtains that:

$$\int_{\Omega} \sigma_{n,\varepsilon} u_{n,\varepsilon} \cdot \nabla\varphi \longrightarrow \int_{\Omega} \sigma_n u_n \cdot \nabla\varphi, \qquad \text{when } \varepsilon \to 0.$$

By passing to the limit in the equation (2.19) and defining, up to extracting a subsequence, the measure μ on $\mathbb{R}^N$ by: $\lim_{\varepsilon\to 0} |\nabla(\widetilde{u}_{n,\varepsilon})|^{1+\varepsilon} = \mu$, one obtains:

$$\langle \mu, \varphi \rangle + \int_{\Omega} \sigma_n u_n \cdot \nabla\varphi + 2n\left(\int_{\mathbb{R}^N} v_n - 1\right)\int_{\mathbb{R}^N} v_n\varphi = 0. \tag{2.20}$$

On the other hand, multiplying equation (2.18) by $v_n\varphi$ where $\varphi \in \mathcal{D}(\mathbb{R}^N)$, one gets

$$\int_{\Omega\cup\partial\Omega} \sigma_n \cdot \nabla v_n\varphi + \int_{\Omega} \sigma_n u_n \cdot \nabla\varphi + 2n\left(\int_{\Omega} u_n - 1\right)\int_{\Omega} u_n\varphi = 0. \tag{2.21}$$

Subtracting (2.21) from (2.20), one gets

$$\mu = \sigma_n \cdot \nabla v_n \quad \text{in } \Omega \cup \partial\Omega. \tag{2.22}$$

This implies in particular, according to Proposition 2.2.1, that

$$|\mu| \leq |\nabla v_n| \quad \text{in } \Omega \cup \partial\Omega,$$

and

$$\int_{\mathbb{R}^N} |\nabla(\widetilde{u}_{n,\varepsilon})|^{1+\varepsilon} \longrightarrow \int_{\mathbb{R}^N} |\nabla v_n| \quad \text{when } \varepsilon \to 0.$$

Finally, according to proposition 2.2.1, one has $\nabla v_n \cdot \sigma_n \leq |\nabla v_n|$ on $\Omega \cup \partial\Omega$, one derives that

$$|\nabla v_n| = \sigma_n \cdot \nabla v_n \quad \text{in } \Omega \cup \partial\Omega.$$

Recall that from Proposition 2.2.1

$$\nabla v_n = \nabla u_n \chi_\Omega - u_n \, \delta_{\partial\Omega} \overrightarrow{n},$$

$$\sigma_n \cdot \nabla v_n = \sigma_n \cdot \nabla u_n \chi_\Omega - \sigma_n \cdot \overrightarrow{n} u_n \delta_{\partial\Omega},$$

we have obtained

$$\begin{cases} \sigma_n \cdot \nabla u_n = |\nabla u_n| & \text{in } \Omega, \\ \sigma_n \cdot \overrightarrow{n} u_n = -u_n & \text{on } \partial\Omega. \end{cases}$$

Then u_n is a nonnegative solution of (2.8). Moreover, the convergence of $|\nabla \widetilde{u}_{n,\varepsilon}|$ is tight on $\overline{\Omega}$, i.e.

$$\int_\Omega |\nabla u_{n,\varepsilon}| \longrightarrow \int_\Omega |\nabla u_n| + \int_{\partial\Omega} u_n, \qquad \text{when } \varepsilon \to 0.$$

Indeed, one has $\int_\Omega |\nabla u_{n,\varepsilon}|^{1+\varepsilon} \longrightarrow \int_\Omega |\nabla u_n| + \int_{\partial\Omega} u_n$ when $\varepsilon \longrightarrow 0$. Using the lower semicontinuity for the extension $u_{n,\varepsilon}$ and Hölder's inequality, we get

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_\Omega |\nabla u_{n,\varepsilon}|^{1+\varepsilon} = \int_\Omega |\nabla u_n| + \int_{\partial\Omega} u_n &\leq \liminf_{\varepsilon\to 0} \int_\Omega |\nabla u_{n,\varepsilon}| \\ &\leq \liminf_{\varepsilon\to 0} \left(\int_\Omega |\nabla u_{n,\varepsilon}|^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} |\Omega|^{\frac{\varepsilon}{1+\varepsilon}} \\ &= \lim_{\varepsilon\to 0} \int_\Omega |\nabla u_{n,\varepsilon}|^{1+\varepsilon} \end{aligned}$$

The result is proved.

Step 5: The convergence of $\lambda_{1+\varepsilon,n}$ towards λ_1

In this step we explicit the relation between the values $\lambda_{1+\varepsilon,n}$ when n is large and the first eigenvalue λ_1 defined in the first part.

THEOREM 2.4.1. *Let u_n be a nonnegative solution of (*2.18*), then, up to a subsequence, as $n \to \infty$, (u_n) converges to $u \in BV(\Omega)$, $u \geq 0$, $u \not\equiv 0$, which realizes the minimum defined in* (2.2)*. Moreover*

$$\lim_{n\to\infty} \lambda_{1,n} = \lambda_1.$$

Proof of the Theorem 2.4.1. For $\lambda_{1,n}$ and λ_1 defined as above, it is clear that we have:

$$\limsup_{n\to\infty} \lambda_{1,n} \leq \lambda_1. \tag{2.23}$$

Let $(u_n)_n$ be a sequence of positive solutions of the relaxed problem defined in (2.7). We begin to prove that $(u_n)_n$ is bounded in $BV(\Omega)$. For that aim let us note that by (2.23), one gets that $n\left(\int_\Omega u_n - 1\right)^2$ is bounded by λ_1, which implies that $\lim_{n\to\infty}\left(\int_\Omega u_n - 1\right)^2 = 0$. Then

$$\lim_{n\to\infty} \int_\Omega u_n = 1,$$

Hence, $(u_n)_n$ is bounded in $L^1(\Omega)$.
Using once more (2.23), one can conclude that $(u_n)_n$ is bounded in $BV(\Omega)$. Then, the extension of each u_n by zero outside of $\overline{\Omega}$ is bounded in $BV(\mathbb{R}^N)$. One can then extract from it a subsequence, still denoted u_n, such that

$$u_n \rightharpoonup u \quad \text{weakly in } BV(\mathbb{R}^N) \quad \text{when } n \to \infty,$$

obviously $u = 0$ outside of $\overline{\Omega}$ and $u > 0$ in Ω. By the compactness of the Sobolev embedding from $BV(\Omega)$ into $L^1(\Omega)$, one has $\|u\|_{L^1(\Omega)} = 1$. Using the lower semi continuity of the total variation $\int_{\mathbb{R}^N} |\nabla u|$ with respect to the weak topology, one has (since $u_n \to u$ in $L^1(\Omega)$)

$$\begin{aligned}\lambda_1 \le \int_{\mathbb{R}^N} |\nabla u| \le \int_{\mathbb{R}^N} |\nabla u| + n\left(\int_{\mathbb{R}^N} u - 1\right)^2 \\ \le \liminf_{n\to\infty}\left[\int_{\mathbb{R}^N} |\nabla u_n| + n\left(\int_{\mathbb{R}^N} u_n - 1\right)^2\right] \\ \le \limsup_{n\to\infty} \lambda_{1,n} \le \lambda_1.\end{aligned}$$

Then one gets that

$$\lim_{n\to\infty} \lambda_{1,n} = \lambda_1.$$

Since $u = 0$ outside of $\overline{\Omega}$, one has $\nabla u = \nabla u \chi_\Omega - u\overrightarrow{n}\ \delta_{\partial\Omega}$ and then

$$\int_{\mathbb{R}^N} |\nabla u| = \int_\Omega |\nabla u| + \int_{\partial\Omega} u.$$

Moreover, one obtains that:

$$\lim_{n\to\infty} n\left(\int_\Omega u_n - 1\right)^2 = 0,$$

and

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n| = \int_\Omega |\nabla u| + \int_{\partial\Omega} |u|.$$

Then, we get the tight convergence of u_n to u in $BV(\overline{\Omega})$.

Let us observe that $\text{sign}^+(u_n)$ converges to some w, $0 \le w \le 1$ in every $L^q(\Omega), \forall\ q < \infty$. Using the convergence of u_n to u in $L^q(\Omega)$, $\forall\ q < \frac{N}{N-1}$, one gets

$$\int_\Omega u_n = \int_\Omega u_n \text{sign}^+(u_n) \longrightarrow \int_\Omega u = 1 \quad \text{when} \quad n \to \infty.$$

As a consequence

$$-2n\left(\int_\Omega u_n - 1\right)\int_\Omega u_n \longrightarrow \lambda_1 \quad \text{when} \quad n \to \infty,$$

and then also

$$-2n\left(\int_\Omega u_n - 1\right) \longrightarrow \lambda_1 \quad \text{when} \quad n \to \infty.$$

This ends the proof of the main result. □

2.5 The duality method

We propose here another approach of the partial differential equation satisfied by u_n and σ_n, using convex analysis. This approach does not need the approximation in $W^{1,1+\varepsilon}(\Omega)$ developed in section 2.4

We begin by recalling some basic definitions and properties about theory of duality in convex analysis, which are necessary to study the variational problem defined above.

Let X, Y be some Banach spaces, X^* the topologic dual of X, and let $\varphi : X \to \bar{\mathbb{R}}$ be a function. We recall that the conjugate function $\varphi^* : X^* \to \bar{\mathbb{R}}$ of φ is defined by:

$$\varphi^*(y) = \sup_{x \in X} \{\langle y, x\rangle - \varphi(x)\}, \quad \forall\, y \in X^*.$$

Let us introduce the following minimizing problem:

$$\mathcal{P} : \inf_{u \in X} \{F(u) + G(\Lambda u)\},$$

where F is some convex function on X, G is convex on Y and $\Lambda \in \mathcal{L}(X, Y)$ is linear and continuous on X. We can introduce now the dual problem $\mathcal{P}^*$ of $\mathcal{P}$:

$$\mathcal{P}^* : \sup_{p^* \in X^*} \{-F^*(-\Lambda^* p^*) - G^*(p^*)\},$$

Proposition 2.5.1 (See [41], [28], [22]). *Let F be a function defined on X. If there exists $u_0 \in X$, such that $F(u_0) < \infty$ and if G is continuous on Λu_0, then:*

$$\inf_{u \in X} \{G(\Lambda u) + F(u)\} = \sup_{p^* \in Y^*} \{-G^*(p^*) - F^*(-\Lambda^* p^*)\}, \tag{2.24}$$

and $\mathcal{P}^*$ possesses a solution.

Let us apply these general technics to solve our primal problem defined in (2.6). Let us note $\lambda^*_{1,n}$ its dual problem. For that aim, let us introduce the following Banach spaces: $X = W_0^{1,1}(\Omega)$ and $Y = L^1(\Omega, \mathbb{R}^N) \times L^1(\Omega)$, and define the following functions:

$$F : X \longrightarrow \bar{\mathbb{R}},\ F(u) = 0, \tag{2.25}$$

the linear continuous operator Λ:

$$\begin{aligned} \Lambda : X &\longrightarrow Y \\ u &\longmapsto (\nabla u, u), \end{aligned}$$

and the function $G(\sigma,\tau) = G_1(\sigma) + G_2(\tau)$ where

$$G_1(\sigma) = \int_\Omega |\sigma|, \quad \forall\, \sigma \in L^1(\Omega, \mathbb{R}^N), \tag{2.26}$$

$$G_2(\tau) = n\left(\int_\Omega |\tau| - 1\right)^2, \quad \forall\, \tau \in L^1(\Omega). \tag{2.27}$$

Using these functions, we can write the minimization problem in (2.6) as:

$$\lambda_{1,n} : \inf_{u\in X} \{G(\Lambda u) + F(u)\}. \tag{2.28}$$

The dual problem is defined by

$$\begin{aligned} \lambda^*_{1,n} : &\sup_{(\sigma,\tau)\in Y^*} \{-G^*(\sigma,-\tau) - F^*(\Lambda^*(-\sigma,\tau))\} = \\ &= \sup_{(\sigma,\tau)\in Y^*} \{-G^*_1(\sigma) - G^*_2(-\tau) - F^*(\Lambda^*(-\sigma,\tau))\}, \end{aligned} \tag{2.29}$$

where $Y^* = L^\infty(\Omega,\mathbb{R}^N) \times L^\infty(\Omega)$. Let us compute first the conjugate F^*:

$$\begin{aligned} F^*(\Lambda^*(-\sigma,\tau)) &= \sup_{u\in X} \{\langle \Lambda^*(-\sigma,\tau), u\rangle - F(u)\} \\ &= \sup_{u\in X} \{\langle (-\sigma,\tau), \Lambda u\rangle - F(u)\} \\ &= \sup_{u\in X} \{\langle (-\sigma,\tau), (\nabla u, u)\rangle - F(u)\} \\ &= \sup_{u\in X} \left\{\int_\Omega -\sigma\cdot\nabla u + \int_\Omega u.\tau\right\} \\ &\geq \sup_{\varphi\in\mathcal{D}(\Omega)} \left\{\int_\Omega (\mathrm{div}\sigma + \tau)\varphi\right\}. \end{aligned}$$

This implies that if the left-hand side is finite, the right one is finite too, therefore $\mathrm{div}\sigma + \tau = 0$ in the distribution sens. By the definition of F^* we get

$$F^*(\Lambda^*(-\sigma,\tau)) = \begin{cases} 0 & \text{if } \mathrm{div}\sigma + \tau = 0, \\ +\infty & \text{elsewhere.} \end{cases} \tag{2.30}$$

On the other hand, we have

$$
\begin{aligned}
G_1^*(\sigma) &= \sup_{u\in L^1} \{\langle \sigma, u\rangle - G_1(u)\} \\
&= \sup_{u\in L^1} \left\{ \int_\Omega \sigma \cdot u - \int_\Omega |u| \right\} \\
&= \sup_{u\in L^1} \left\{ \int_\Omega |\sigma||u| - \int_\Omega |u| \right\} \\
&= \sup_{u\in L^1} \left\{ \int_\Omega |u|(\|\sigma\|_\infty - 1) \right\}, \quad \forall\, \sigma \in L^\infty(\Omega, \mathbb{R}^N).
\end{aligned}
$$

Then

$$
G_1^*(\sigma) = \begin{cases} 0 & \text{if } \|\sigma\|_\infty \le 1, \\ +\infty & \text{elsewhere.} \end{cases} \tag{2.31}
$$

And finally, we have

$$
\begin{aligned}
G_2^*(\tau) &= \sup_{u\in L^1} \{\langle \tau, u\rangle - G_2(u)\} \\
&= \sup_{u\in L^1} \left\{ \int_\Omega \tau u - n\left(\int_\Omega |u|\right)^2 + 2n\int_\Omega |u| - n \right\} \\
&= \sup_{u\in L^1} \left\{ \int_\Omega |\tau||u| - n\left(\int_\Omega |u|\right)^2 + 2n\int_\Omega |u| - n \right\} \\
&= \sup_{u\in L^1} \left\{ \int_\Omega (|\tau| + 2n)\,|u| - n\left(\int_\Omega |u|\right)^2 \right\} - n \\
&= \sup_{u\in L^1} \left\{ (\|\tau\|_\infty + 2n)\,\|u\|_1 - n\|u\|_1^2 - n \right\} \\
&= \sup_{\lambda\in\mathbb{R}^+} \left\{ (\|\tau\|_\infty + 2n)\lambda - n\lambda^2 - n \right\} \\
&= \sup_{\lambda\in\mathbb{R}^+} \left\{ -n\left(\lambda - \frac{\|\tau\|_\infty + 2n}{2n}\right)^2 - n + \frac{(\|\tau\|_\infty + 2n)^2}{4n} \right\} \\
&= \frac{(\|\tau\|_\infty + 2n)^2}{4n} - n \\
&= \frac{\|\tau\|_\infty^2}{4n} + \|\tau\|_\infty, \quad \forall\, \tau \in L^\infty(\Omega).
\end{aligned}
$$

Thus, we obtain the following dual problem

$$\lambda_{1,n}^{*}=\sup_{\substack{(\sigma,\tau)\in L^{\infty}(\Omega,\mathbb{R}^N)\times L^{\infty}(\Omega)\\ |\sigma|\le 1\\ \mathrm{div}\sigma+\tau=0}}\left\{-\|\tau\|_{\infty}-\frac{1}{4n}\|\tau\|_{\infty}^{2}\right\}. \tag{2.32}$$

The extremality relation

Let u_n be a positive solution of the relaxed problem (2.7) as obtained in the fourth section. Let (σ_n,τ_n) be a solution of the dual problem (2.32). Then one has the following extremality relation

$$\int_{\Omega}|\nabla u_n|+\int_{\partial\Omega}|u_n|+n\left(\int_{\Omega}|u_n|-1\right)^{2}=-\|\tau_n\|_{\infty}-\frac{1}{4n}\|\tau_n\|_{\infty}^{2}.$$

Using

$$\int_{\Omega\cup\partial\Omega}\nabla u_n\cdot\sigma_n=-\int_{\Omega}(\mathrm{div}\sigma_n)u_n=\int_{\Omega}\tau_n u_n,$$

one gets

$$\begin{aligned}
&\int_{\Omega\cup\partial\Omega}|\nabla u_n|-\int_{\Omega\cup\partial\Omega}\nabla u_n\cdot\sigma_n\\
&=-\|\tau_n\|_{\infty}-\frac{1}{4n}\|\tau_n\|_{\infty}^{2}-n\big(\|u_n\|_1-1\big)^{2}-\int_{\Omega}\tau_n u_n\\
&=-\int_{\Omega}\tau_n u_n-\|\tau_n\|_{\infty}\|u_n\|_1+\|\tau_n\|_{\infty}\left(\|u_n\|_1-1\right)-\frac{1}{4n}\|\tau_n\|_{\infty}^{2}-n\big(\|u_n\|_1-1\big)^{2}\\
&=\left(-\int_{\Omega}\tau_n u_n-\|\tau_n\|_{\infty}\|u_n\|_1\right)-\frac{1}{n}\left[\frac{1}{2}\|\tau_n\|_{\infty}-n\big(\|u_n\|_1-1\big)\right]^{2}.
\end{aligned}$$

Since the left hand side is nonnegative and the right hand side is the sum of two negative terms, we can conclude that the two sides are equals to zero. We obtain then

$$|\nabla u_n|=\nabla u_n\cdot\sigma_n \text{ in } \Omega\cup\partial\Omega,$$
$$-\tau_n\cdot u_n=\|\tau_n\|_{\infty}\|u_n\|_1$$

and

$$\|\tau_n\|_{\infty}=2n(\|u_n\|_1-1).$$

Using this in the fact that $-\mathrm{div}\sigma_n=\tau_n$, we get the following

$$-\mathrm{div}\sigma_n=-2n\ \mathrm{sign}^{+}\ (u)\ \left(\int_{\Omega}|u_n|-1\right).$$

2.6 Minima as Caccioppoli sets

Let us introduce λ_E as the value of the infimum

$$\lambda_E = \inf_{E \in C(\Omega)} \left\{ \int_\Omega |\nabla \chi_E| + n \left(\int_\Omega \chi_E - 1 \right)^2 \right\}, \tag{2.33}$$

where $C(\Omega) = \{E, E \text{ is Caccioppoli set } E \subset\subset \Omega\}$ (let us recall that a Caccioppoli set in Ω is merely a set whose characteristic function belongs to $BV(\Omega)$). We have the following:

THEOREM 2.6.1. *One has*

$$\begin{aligned} \lambda_E &= \inf_{E \in C(\Omega)} \left\{ \int_\Omega |\nabla \chi_E| + n \left(\int_\Omega \chi_E - 1 \right)^2 \right\} \\ &= \inf_{E \in C(\mathbb{R}^N)} \left\{ \int_\Omega |\nabla \chi_E| + \int_{\partial\Omega} \chi_E + n \left(\int_\Omega \chi_E - 1 \right)^2 \right\} \\ &= \inf_{E \in C(\mathbb{R}^N)} \left\{ P(E, \Omega) + |E \cap \partial\Omega| + n \left(|E \cap \Omega| - 1 \right)^2 \right\} \end{aligned}$$

and

$$\lambda_E \geq \lambda_{1,n},$$

where $\lambda_{1,n}$ is defined in (2.6).

REMARK 2.6.1. $P(E, \Omega)$ is the perimeter of E in Ω, see ([15], [32]).

PROPOSITION 2.6.1. *Let λ_E be defined as in Theorem* 2.6.1*, then λ_E is achieved.*

Proof of Proposition 2.6.1. Let (E_i) be a subsequence of Caccioppoli sets, $E_i \subset\subset \Omega$ such that

$$\int_\Omega |\nabla \chi_{E_i}| + n \left(\int_\Omega \chi_{E_i} - 1 \right)^2 \longrightarrow \lambda_E \; .$$

Then

$$\limsup_{i \to \infty} \left\{ \int_\Omega |\nabla \chi_{E_i}| + n \left(\int_\Omega \chi_{E_i} - 1 \right)^2 \right\} \leq \lambda_E \; .$$

It is clear that χ_{E_i} is bounded in $BV(\Omega)$ (same arguments as in Theorem 2.6.1). More precisely χ_{E_i} is bounded in $BV(\mathbb{R}^N)$.
Extracting from it a subsequence still denoted χ_{E_i}, one get that

$$\chi_{E_i} \rightharpoonup u \quad \text{weakly} \;\; \text{in } BV(\mathbb{R}^N).$$

By construction $u = 0$ outside of $\overline{\Omega}$. Moreover one can assume that χ_{E_i} tends to u a.e, and then u can only takes the values 0 and 1. As a consequence u is the characteristic function of some set E. By lower semicontinuity, one has that

$$\int_{\mathbb{R}^N} |\nabla \chi_E| + n \left(\int_{\mathbb{R}^N} \chi_E - 1 \right)^2 \leq \liminf_{i \to \infty} \left\{ \int_\Omega |\nabla \chi_{E_i}| + n \left(\int_\Omega \chi_{E_i} - 1 \right)^2 \right\}.$$

Then, one obtains that E is a solution for the relaxed problem (2.33). □

Chapter 3

The obstacle problem on $W^{1,p}$

An optimal control problem for an elliptic obstacle variational inequality with a source term was considered in the case $p = 2$ by Adams, Lenhart and Yong in ([5], [6]). The authors consider the following obstacle problem:

$$\begin{cases} u \in K(\psi), \\ \displaystyle\int_\Omega \sigma(u) \cdot \nabla(v-u)\ dx \geq \int_\Omega f(v-u)\ dx,\ \forall\ v \in K(\psi), \end{cases} \tag{3.1}$$

where

$$K(\psi) = \{v \in H_0^1(\Omega), v \geq \psi \text{ a.e. in } \Omega\}.$$

They produced existence, uniqueness and regularity as well as some characterizations of the solution $u =: T_f(\psi)$ to the obstacle problem (3.1). They consider then $\psi \in H_0^1(\Omega)$ as the control variable and $u =: T_f(\psi)$ as the corresponding state variable. The goal of their work is to find the optimal obstacle ψ from $H_0^1(\Omega)$ so that the corresponding state $u =: T_f(\psi)$ is close to some given desired profile, while ψ is not too large in $H_0^1(\Omega)$. For that aim, the authors introduce the following cost functional:

$$\inf_{\psi \in H_0^1(\Omega)} \left\{ \int_\Omega |T_f(\psi) - z|^2 + \int_\Omega |\nabla \psi|^2\ dx \right\}, \tag{3.2}$$

for some $z \in L^2(\Omega)$ is referred to as *the initial profile*, ψ as *the control variable* and $T_f(\psi)$ as *the state variable*. The pair $(\psi^*, T_f(\psi^*))$ where ψ^* is a solution to the problem (3.2) is called an optimal pair and ψ^* an optimal control.

We consider here the analogous problem on $W_0^{1,p}(\Omega),\ p > 1$.

Let Ω be a bounded domain in $\mathbb{R}^N, N \geq 2$, whose boundary is $\mathcal{C}^1$ piece-wise. For $p > 1$ and for ψ given in $W_0^{1,p}(\Omega)$, define

$$K(\psi) = \{v \in W_0^{1,p}(\Omega), v \geq \psi \text{ a.e. in } \Omega\}.$$

It is clear that $K(\psi)$ is a convex and weakly closed set in $L^p(\Omega)$. Let p' be the conjugate of p, and $f \in L^{p'}(\Omega)$. We consider the following variational inequality called *the obstacle problem*:

$$\begin{cases} u \in K(\psi), \\ \displaystyle\int_\Omega \sigma(u) \cdot \nabla(v - u)\ dx \geq \int_\Omega f(v-u)\ dx,\ \forall v \in K(\psi), \end{cases} \tag{3.3}$$

where $\sigma(u) = |\nabla u|^{p-2}\nabla u$. We shall say that ψ is *the obstacle* and f is *the source term*.

We begin to prove existence and uniqueness of a solution u to (3.3), using variational formulation of the obstacle problem on the set $K(\psi)$. We shall then denote u by: $u = T_f(\psi)$. Secondly, we characterize $T_f(\psi)$ as the lowest $f-$superharmonic function greater than ψ.

3.1 Existence and uniqueness of the solution

PROPOSITION 3.1.1. *A function u is a solution to the problem* (3.3) *if and only if u satisfies the following:*

$$\begin{cases} u \in K(\psi), \\ -\Delta_p u \geq f,\ \ a.e. \ \ in\ \Omega, \\ \displaystyle\int_\Omega \sigma(u) \cdot \nabla(\psi - u)\ dx = \int_\Omega f(\psi - u)\ dx. \end{cases} \tag{3.4}$$

Proof of Proposition 3.1.1. Suppose that u satisfies (3.3). Then taking $v = u + \varphi \in K(\psi)$ for $\varphi \in \mathcal{D}(\Omega),\ \varphi \geq 0$, one gets that $-\Delta_p u \geq f$ in Ω. Moreover, For $v = \psi$ and $v = 2u - \psi$, one gets that

$$\int_\Omega \sigma(u) \cdot \nabla(\psi - u)\ dx = \int_\Omega f(\psi - u)\ dx,$$

hence u satisfies (3.4).

Conversely, let $u \in K(\psi)$ such that $-\Delta_p u \geq f$, let v be in $K(\psi)$ and $\varphi_n \in \mathcal{D}(\Omega), \varphi_n \geq 0$ such that $\varphi_n \to v - \psi$ in $W_0^{1,p}(\Omega)$. Then one gets

$$\begin{aligned}\int_\Omega \sigma(u) \cdot \nabla(v-\psi) &= \lim_{n\to\infty} \int_\Omega \sigma(u) \cdot \nabla \varphi_n \\ &= \lim_{n\to\infty} \int_\Omega -\Delta_p u \ \varphi_n \\ &\geq \lim_{n\to\infty} \int_\Omega f\varphi_n = \int_\Omega f(v-\psi), \ \forall \ v \in K(\psi).\end{aligned}$$

Using the last equality of (3.4), one gets that

$$\int_\Omega \sigma(u) \cdot \nabla(v-u) \geq \int_\Omega f(v-u), \ \forall \ v \in K(\psi),$$

hence u satisfies (3.3). □

Let us prove now the existence and uniqueness of a solution to the obstacle problem (3.3).

PROPOSITION 3.1.2. *There exists a solution to* (3.3)*, which can be obtained as the minimizer of the following minimization problem*

$$\inf_{v \in K(\psi)} I(v), \tag{3.5}$$

where I is the following energy functional

$$I(v) = \frac{1}{p} \int_\Omega |\nabla v|^p - \int_\Omega fv.$$

Proof of Proposition 3.1.2. Using classical arguments in the calculus of variations, since $K(\psi)$ is a weakly closed convex set in $W_0^{1,p}(\Omega)$, and the functional I is convex and coercive on $W_0^{1,p}(\Omega)$, then one obtains that there exists a solution u to (3.5). □

PROPOSITION 3.1.3. *The inequation* (3.3) *possesses a unique solution.*

Proof of Proposition 3.1.3. Suppose that $u_1, u_2 \in W_0^{1,p}(\Omega)$ are two solutions of the variational inequality (3.3)

$$u_i \in K(\psi) : \int_\Omega \sigma(u_i) \cdot \nabla(v-u_i) \ dx \geq \int_\Omega f(v-u_i) \ dx, \ \forall \ v \in K(\psi), \ i = 1, 2$$

Taking $v = u_1$ for $i = 2$ and $v = u_2$ for $i = 1$ and adding, we have

$$\int_\Omega [\sigma(u_1) - \sigma(u_2)] \cdot \nabla(u_1 - u_2) \leq 0.$$

Recall that we have

$$\int_\Omega [\sigma(u_1) - \sigma(u_2)] \cdot \nabla(u_1 - u_2) \geq 0,$$

which implies that

$$\int_\Omega [\sigma(u_1) - \sigma(u_2)] \cdot \nabla(u_1 - u_2) = 0,$$

and then, $u_1 = u_2$ a.e in Ω. □

Thus, we get the existence and uniqueness of a solution to (3.3).

Definition 3.1.1. We shall say that u is *$f-$superhamonic* in Ω, if $u \in W_0^{1,p}(\Omega)$ is a weak solution to $-\Delta_p u \geq f$, in the sense of distributions.

Proposition 3.1.4. *A function u is a solution of* (3.3)*, if and only if u is the lowest $f-$superharmonic function, greater than ψ.*

Proof of Proposition 3.1.4. Let u be a solution of (3.3) and v be an $f-$super harmonic function, greater than ψ. Let $\xi = \max(u, v),\ \xi \in K(\psi)$. Recalling that $v^- = \sup(0, -v)$, one has then $(\xi - u) = -(v - u)^-$. From (3.3), one gets

$$\int_\Omega \sigma(u) \cdot \nabla(\xi - u) \geq \int_\Omega f(\xi - u).$$

On the other hand, since $\xi - u \leq 0$ and $-\Delta_p v \geq f$, we have

$$\int_\Omega \sigma(v) \cdot \nabla(\xi - u) \leq \int_\Omega f(\xi - u).$$

We obtain, subtracting the above two inequalities:

$$\int_\Omega [\sigma(v) - \sigma(u)] \cdot \nabla(\xi - u) \leq 0,$$

which implies that

$$-\int_\Omega [\sigma(v) - \sigma(u)] \cdot \nabla(v - u)^- \leq 0,$$

and then $(v - u)^- = 0$, or equivalently $u \leq v$ in Ω. □

Recall that we define by $T_f(\psi)$ the lowest $f-$superharmonic function, greater than ψ.

LEMMA 3.1.1. *The mapping $\psi \mapsto T_f(\psi)$ is increasing.*

Proof of Lemma 3.1.1. Let $u_1 = T_f(\psi_1)$ and $u_2 = T_f(\psi_2)$, which are respectively solutions to the following variational inequalities

$$\begin{cases} -\Delta_p u_i \geq f \\ u_i \geq \psi_i, \ i = 1, 2 \end{cases}$$

and let $\psi_1 \leq \psi_2$. It is clear that $u_2 \geq \psi_1$. Hence u_2 is $f-$superharmonic and using Proposition 3.1.4, one obtains $u_1 \leq u_2$. □

PROPOSITION 3.1.5. *The mapping $\psi \mapsto T_f(\psi)$ is weak lower semicontinuous, in the sense that:*

- *If $\psi_k \rightharpoonup \psi$ weakly in $W_0^{1,p}(\Omega)$, then $T_f(\psi) \leq \liminf_{k\to\infty} T_f(\psi_k)$.*
- $\displaystyle\int_\Omega |\nabla(T_f(\psi))|^p \leq \liminf_{k\to\infty} \int_\Omega |\nabla(T_f(\psi_k))|^p.$

Proof of Proposition 3.1.5. Let (ψ_k) be a sequence in $W_0^{1,p}(\Omega)$ which converges weakly in $W_0^{1,p}(\Omega)$ to ψ, and let $\varphi_k = \min(\psi_k, \psi)$. Since T_f is increasing, one gets that $T_f(\varphi_k) \leq T_f(\psi_k)$. We now prove that $T_f(\varphi_k)$ converges strongly in $W_0^{1,p}(\Omega)$ towards $T_f(\psi)$. This will imply that

$$T_f(\psi) = \lim_{k\to\infty} T_f(\varphi_k) \leq \liminf_{k\to\infty} T_f(\psi_k).$$

We denote u_k as $T_f(\varphi_k)$. It is clear that u_k is bounded in $W_0^{1,p}(\Omega)$ since $\varphi_k \leq \psi$. Hence for a subsequence, still denoted u_k, there exists some u in $W_0^{1,p}(\Omega)$ such that

$$\nabla u_k \rightharpoonup \nabla u \text{ weakly in } L^p(\Omega), \ u_k \to u \text{ strongly in } L^p(\Omega). \tag{3.6}$$

On the other hand, using the fact that φ_k converges weakly to ψ in $W_0^{1,p}(\Omega)$ (see Lemma 3.1.2 below), one gets the following assertion:

$$u_k \geq \varphi_k \Longrightarrow u \geq \psi.$$

Let us prove now that u is a solution of the minimizing problem (3.5). For that aim, for $v \in K(\psi)$, since $v \geq \psi \geq \varphi_k$, we have

$$\begin{aligned}\frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega fu &\leq \liminf_{k\to\infty} \frac{1}{p}\int_\Omega |\nabla u_k|^p - \int_\Omega fu_k \\ &\leq \liminf_{k\to\infty} \inf_{w\geq\varphi_k} \left\{\frac{1}{p}\int_\Omega |\nabla w|^p - \int_\Omega fw\right\} \\ &\leq \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega fv.\end{aligned}$$

Then u realizes the infimum in (3.5). At the same time, since $u \in K(\psi)$, one has the following convergence

$$\frac{1}{p}\int_\Omega |\nabla u_k|^p - \int_\Omega fu_k \longrightarrow \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega fu, \text{ when } k \to \infty,$$

which implies that u_k converges strongly to u in $W_0^{1,p}(\Omega)$. We can conclude that $T_f(\varphi_k)$ converges strongly to $T_f(\psi)$. □

LEMMA 3.1.2. *Suppose that ψ_k converges weakly to some ψ in $W_0^{1,p}(\Omega)$. Then, $\varphi_k = \min(\psi_k, \psi)$ converges weakly to ψ in $W_0^{1,p}(\Omega)$.*

Proof of Lemma 3.1.2. We have

$$\psi_k \longrightarrow \psi \quad \text{in } L^p(\Omega).$$

Then

$$\varphi_k = \frac{\psi_k + \psi - |\psi_k - \psi|}{2} \longrightarrow \psi \quad \text{in } L^p(\Omega).$$

Let us prove now that $|\nabla\varphi_k|$ is bounded in $L^p(\Omega)$. For that aim, we write

$$\begin{aligned}\int_\Omega |\nabla\varphi_k|^p &= \int_\Omega \left|\nabla\left(\frac{\psi_k + \psi - |\psi_k - \psi|}{2}\right)\right|^p \\ &\leq C_p\left(\int_\Omega |\nabla\psi_k|^p + \int_\Omega |\nabla\psi|^p\right).\end{aligned}$$

Therefore the sequence φ_k is bounded in $W_0^{1,p}(\Omega)$, so it converges weakly, up to a subsequence, to ψ in $W_0^{1,p}(\Omega)$. □

PROPOSITION 3.1.6. *The mapping T_f is an involution, i.e. $T_f^2 = T_f$.*

Proof of Proposition 3.1.6. Up to replacing ψ by u in the variational inequalities (3.3), and using proposition 3.1.4, one gets that $u = T_f(u)$. Then, we conclude that $T_f^2(\psi) = T_f(\psi)$. □

3.2 A method of penalization

Let $\mathcal{M}^+(\Omega)$ be the set of all nonnegative Radon measures on Ω and $W^{-1,p'}(\Omega)$ be the dual space of $W^{1,p}(\Omega)$ on Ω where p' is the conjugate of p $(1 < p < \infty)$. Suppose that u solves (3.3). Using the fact that a nonnegative distribution on Ω is a nonnegative measure on Ω (cf. [24]), one gets the existence of $\mu \geq 0,\ \mu \in \mathcal{M}^+(\Omega)$, such that

$$\int_\Omega \sigma(u) \cdot \nabla\Phi \, dx - \int_\Omega f\Phi \, dx = \langle \mu, \Phi \rangle, \quad \forall \, \Phi \in \mathcal{D}(\Omega), \tag{3.7}$$

that we shall also write $-\Delta_p u = f + \mu, \quad \mu \geq 0$ in Ω.

Let us introduce

$$\beta(x) = \begin{cases} 0, & x > 0, \\ x, & x \leq 0. \end{cases} \tag{3.8}$$

Clearly, β is $\mathcal{C}^1$ piecewise, $\beta(x) \leq 0$ and is nondecreasing. Let us consider, for some $\delta > 0$, the following semilinear elliptic equation:

$$\begin{cases} -\Delta_p u + \frac{1}{\delta}\beta(u - \psi) = f, & \text{in } \Omega \\ u_{|\partial\Omega} = 0. \end{cases} \tag{3.9}$$

We have the following existence result:

Theorem 3.2.1. *For any given $\psi \in W_0^{1,p}(\Omega)$ and $\delta > 0$,* (3.9) *possesses a unique solution u^δ. Moreover,*

(1) $u^\delta \longrightarrow u$ strongly in $W_0^{1,p}(\Omega)$, *as $\delta \longrightarrow 0$, with $u := T_f(\psi)$.*

(2) *There exists a unique $\mu \in W^{-1,p'}(\Omega) \cap \mathcal{M}^+(\Omega)$ such that:*

  (i) $-\frac{1}{\delta}\beta(u^\delta - \psi) \rightharpoonup \mu$ in $W^{-1,p'}(\Omega) \cap \mathcal{M}^+(\Omega)$.

  (ii) $\langle \mu, T_f(\psi) - \psi \rangle = 0$.

Proof of Theorem 3.2.1. (1) Let B be defined as $B(r) = \displaystyle\int_0^r \beta(s)ds,\ \forall\, r \in \mathbb{R}$. We introduce the following variational problem

$$\inf_{v \in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p} \int_\Omega |\nabla v|^p + \frac{1}{\delta} \int_\Omega B(v - \psi) - \int_\Omega fv \right\}. \tag{3.10}$$

The functional in (3.10) is coercive, strictly convex and continuous. As a consequence it possesses a unique solution $u^\delta \in W_0^{1,p}(\Omega)$. Since $B(0) = 0$, one has

$$\frac{1}{p}\int_\Omega |\nabla u^\delta|^p + \frac{1}{\delta}\int_\Omega B(u^\delta - \psi) - \int_\Omega f u^\delta \leq \frac{1}{p}\int_\Omega |\nabla \psi|^p - \int_\Omega f\psi,$$

since $B \geq 0$, then u^δ is bounded in $W_0^{1,p}(\Omega)$. Extracting from u^δ a subsequence, there exists u in $W_0^{1,p}(\Omega)$, such that

$$\nabla u^\delta \rightharpoonup \nabla u \text{ weakly in } L^p(\Omega),\ u^\delta \to u \text{ strongly in } L^p(\Omega).$$

Using $\frac{1}{\delta}\displaystyle\int_\Omega B(u^\delta - \psi) \leq C$ and the continuity of B one has

$$0 \leq \int_\Omega B(u-\psi) \leq \liminf_{\delta\to 0}\int_\Omega B(u^\delta - \psi) = 0,$$

hence $u \in K(\psi)$.

We want to prove now that u solves (3.3). Let $v \in K(\psi)$, since $B(r) \geq 0,\ \forall\, r \in \mathbb{R}$ one gets:

$$\begin{aligned}
\frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega f u &\leq \liminf_{\delta\to 0}\left(\frac{1}{p}\int_\Omega |\nabla u^\delta|^p - \int_\Omega f u^\delta\right) \\
&\leq \liminf_{\delta\to 0}\left(\frac{1}{p}\int_\Omega |\nabla u^\delta|^p + \frac{1}{\delta}\int_\Omega B(u^\delta - \psi) - \int_\Omega f u^\delta\right) \\
&\leq \liminf_{\delta\to 0}\inf_{u\geq\psi}\left\{\frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{\delta}\int_\Omega B(u-\psi) - \int_\Omega f u\right\} \\
&\leq \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega f v.
\end{aligned}$$

Then, one concludes that $\nabla u^\delta \longrightarrow \nabla u$ strongly in $L^p(\Omega)$ and since $u \in K(\psi)$, then u solves (3.3).

(2) (i) let u^δ be the solution of (3.9), since ∇u^δ is uniformly bounded in $L^p(\Omega)$ by some constant C, we get that $-\Delta_p u^\delta - f$ is bounded in $W^{-1,p'}(\Omega)$, so it converges weakly, up to a subsequence, in $W^{-1,p'}(\Omega)$. Hence, $-\frac{1}{\delta}\beta(u^\delta - \psi)$ converges too, up to a subsequence, in $W^{-1,p'}(\Omega)$, and we have

$$-\frac{1}{\delta}\beta(u^\delta - \psi) \rightharpoonup \mu \text{ weakly in } W^{-1,p'}(\Omega),$$

where μ is a positive distribution, hence a positive measure. Then, by (1), we see that u and μ are linked by the relation (3.7).

We now prove (ii): let u be the solution of (3.3). Taking $\varphi = (\psi - u) \in W_0^{1,p}(\Omega)$ in the above inequalities, one gets

$$-\frac{1}{\delta}\int_\Omega \beta(u^\delta - \psi)\ (u-\psi)\ dx \le \|\nabla u^\delta\|_p^{p-1}\|\nabla(\psi - u)\|_p + \|f\|_{p'}\|\psi - u\|_p.$$

Since $u \in K(\psi)$, passing to the limit we obtain:

$$\langle \mu, \psi - u\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(\psi - u) - \int_\Omega f(\psi - u) = 0, \text{ by } (3.4)$$

Then (ii) follows. □

3.3 Differentiation properties of the solution of the penalization problem with respect to the obstacle

In this section, we assume that β is replaced by a differentiable function of the same type. For example $\beta(x) = -(x^-)^{\frac{Np}{Np-N+p}+\gamma+1}$, for some $\gamma > 0$, such that $\frac{Np}{Np-N+p} + \gamma > 0$. Then β is $\mathcal{C}^1(\Omega)$. We also assume that $f \equiv 0$.

Lemma 3.3.1. *Suppose that $1 < p \le 2$. There exists a constant C_p such that for all X and Y in $\mathbb{R}^N$*

$$\left||X|^{p-2}X - |Y|^{p-2}Y\right|^{\frac{p}{p-1}} \le C_p|X-Y|^p, \quad \forall\ X, Y \in \mathbb{R}^N. \tag{3.11}$$

Proof of Lemma 3.3.1. First, by homogeneity, one need only to prove the result for $|Y| \le |X| = 1$. We distinguish two cases:

(1) Assume first that $|X - Y| > \frac{1}{2}$. Then, suppose that the inequality in (3.11) is false. There would exist sequences X_n and Y_n in $\mathbb{R}^N$ such that $|X_n - Y_n| > \frac{1}{2}$ and $|Y_n| \le |X_n| = 1$ with

$$|X_n - |Y_n|^{p-2}Y_n|^{\frac{p}{p-1}} \ge n|X_n - Y_n|^p. \tag{3.12}$$

Passing to the limit, up to subsequences, one has $X_n \to X$ and $Y_n \to Y$. Then

$$|X-Y|^p \le \lim_{n\to\infty}\frac{1}{n}\left(1 + |Y_n|^{p-2}|Y_n|\right)^{\frac{p}{p-1}} \le \lim_{n\to\infty}\frac{2}{n} = 0. \tag{3.13}$$

This implies that $X = Y$ and contradicts the assumption $|X - Y| > \frac{1}{2}$.

(2) We now assume that $|X-Y| \leq \frac{1}{2}$. Then using the mean value theorem, we have for some $\Theta \in (0,1)$,

$$\begin{aligned} ||X|^{p-2}X - |Y|^{p-2}Y| &= (p-1)|X+\Theta(Y-X)|^{p-2}|X-Y| \\ &\leq (p-1)\left(\frac{1}{2}\right)^{p-2}|X-Y|, \end{aligned} \tag{3.14}$$

and then taking the power $\frac{p}{p-1}$:

$$\begin{aligned} ||X|^{p-2}X - |Y|^{p-2}Y|^{\frac{p}{p-1}} &\leq (p-1)^{\frac{p}{p-1}}\left(\frac{1}{2}\right)^{\frac{p(p-2)}{p-1}}|X-Y|^{\frac{p}{p-1}} \\ &\leq (p-1)^{\frac{p}{p-1}}\left(\frac{1}{2}\right)^{\frac{p(p-2)}{p-1}}\left(\frac{1}{2}\right)^{\frac{p(2-p)}{p-1}}|X-Y|^{p} \\ &= (p-1)^{\frac{p}{p-1}}|X-Y|^{p}. \end{aligned}$$

□

LEMMA 3.3.2. *For $1 < p \leq 2$, for all X and Y in $\mathbb{R}^N$, there exists a constant C_p such that*

$$\big(|X|^{p-2}X - |Y|^{p-2}Y\big)\cdot(X-Y) \geq C_p(|X|+|Y|)^{p-2}|X-Y|^2. \tag{3.15}$$

Proof of Lemma 3.3.2. First, by homogeneity, one need only to prove the result for $|X| = 1$. Then, we prove that for $X = 1$ and $Y > 0$ we have

$$(1-Y^{p-1})(1-Y) \geq k_p(1+Y)^{p-2}(1-Y)^2.$$

For that aim, we introduce the following function:

$$v(Y) = \begin{cases} \frac{(1-Y^{p-1})(1-Y)}{(1+Y)^{p-2}(1-Y)^2}, & \text{for } Y \neq 1, \\ (p-1)2^{2-p}, & \text{for } Y = 1. \end{cases}$$

v is continuous and positive on $\mathbb{R}$. In addition, one observes that $\lim_{0} v = \lim_{\infty} v = 1$. Let us denote by k_p the minimum of v. Then for $C_p = \inf(\frac{1}{2}; k_p)$

we get the desired result. Indeed, for $|X| = 1$ we have:

$$\begin{aligned}(X - |Y|^{p-2}Y)\cdot(X-Y)\\ &= 1 + |Y|^p - X\cdot Y(1+|Y|^{p-2})\\ &= (1-|Y|^{p-1})(1-|Y|) + (|Y| - (X\cdot Y))(1+|Y|^{p-2})\\ &\geq C_p(1+|Y|)^{p-2}(1-|Y|)^2 + 2C_p(|Y| - (X\cdot Y))(1+|Y|^{p-2})\\ &\geq C_p(1+|Y|)^{p-2}(1+|Y|^2 - 2X\cdot Y)\\ &= C_p(|X|+|Y|)^{p-2}|X-Y|^2.\end{aligned}$$

This ends the proof. □

Lemma 3.3.3. *Let $1 \leq p \leq 2$. The map $\psi \longrightarrow u = u(\psi)$ is weakly differentiable in the following sense: given $\psi \in W_0^{1,p}(\Omega)$ and $l \in \mathcal{D}(\Omega)$, there exists $\xi \in W_0^{1,p}(\Omega)$ such that*

$$\frac{u(\psi+\varepsilon l) - u(\psi)}{\varepsilon} \rightharpoonup \xi \text{ weakly in } W_0^{1,p}(\Omega) \text{ as } \varepsilon \to 0., \tag{3.16}$$

Furthermore, $\frac{\sigma_\varepsilon - \sigma}{\varepsilon} \rightharpoonup \tau$, weakly in $L^{\frac{p}{p-1}}(\Omega)$, where τ realizes the following

$$\begin{cases} -\text{div } \tau + \frac{1}{\delta}\beta'(u-\psi)(\xi - l) = 0 \text{ in } \Omega,\\ \xi_{|\partial\Omega} = 0.\end{cases} \tag{3.17}$$

Proof of Lemma 3.3.3. Note that

$$-\Delta_p u_\varepsilon + \frac{1}{\delta}\beta(u_\varepsilon - (\psi + \varepsilon l)) = 0, \tag{3.18}$$

where $u_\varepsilon = T_0(\psi + \varepsilon l) \in W_0^{1,p}(\Omega)$. Let

$$-\Delta_p u + \frac{1}{\delta}\beta(u-\psi) = 0, \tag{3.19}$$

where $u = T_0(\psi) \in W_0^{1,p}(\Omega)$.

Set $\sigma_\varepsilon = |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon$ and $\sigma = |\nabla u|^{p-2}\nabla u$. Multiplying ((3.18) – (3.19)) by $(u_\varepsilon - u)$ and integrating, since $0 \le \beta' \le 1$, one obtains,

$$\begin{aligned}
&\int_\Omega (\sigma_\varepsilon - \sigma)\cdot(\nabla u_\varepsilon - \nabla u)\\
&= -\frac{1}{\delta}\int_\Omega\int_0^1 \beta'\left((1-s)(u-\psi)+s(u_\varepsilon-\psi-\varepsilon l)\right)\ ds\ (u_\varepsilon - u - \varepsilon l)(u_\varepsilon - u)\ dx\\
&\le \frac{1}{\delta}\int_\Omega\int_0^1 \beta'\left((1-s)(u-\psi)+s(u_\varepsilon-\psi-\varepsilon l)\right)\ ds\ \varepsilon l(u_\varepsilon - u)\ dx\\
&\le \frac{\varepsilon}{\delta}\|l\|_{p'}\|u_\varepsilon - u\|_p,
\end{aligned} \tag{3.20}$$

Since $1 \le p \le 2$, we deduce by Hölder's inequality and Lemma 3.3.2 that

$$\begin{aligned}
&\int_\Omega |\nabla u_\varepsilon - \nabla u|^p\\
&= \int_\Omega (|\nabla u_\varepsilon - \nabla u|^2)^{\frac{p}{2}}\\
&\le \int_\Omega [(\sigma_\varepsilon - \sigma)\cdot(\nabla u_\varepsilon - \nabla u)]^{\frac{p}{2}}\left(|\nabla u_\varepsilon| + |\nabla u|\right)^{\frac{(2-p)p}{2}}\ dx\\
&\le \left[\int_\Omega (\sigma_\varepsilon - \sigma)\cdot(\nabla u_\varepsilon - \nabla u)\ dx\right]^{\frac{p}{2}}\left[\int_\Omega (|\nabla u_\varepsilon| + |\nabla u|)^p\ dx\right]^{\frac{2-p}{2}}\\
&\le \left(C\varepsilon\|l\|_{p'}\|u_\varepsilon - u\|_p\right)^{\frac{p}{2}},
\end{aligned}$$

for some constant C. Using first Poincaré's inequality, we gets that

$$\begin{aligned}
\|\nabla u_\varepsilon - \nabla u\|_p^p &\le \left(C\varepsilon\|l\|_{p'}\|u_\varepsilon - u\|_p\right)^{\frac{p}{2}}\\
&\le \left(C'\varepsilon\|l\|_{p'}\|\nabla u_\varepsilon - \nabla u\|_p\right)^{\frac{p}{2}}.
\end{aligned}$$

Then, we obtains

$$\|\nabla u_\varepsilon - \nabla u\|_p \le C'\varepsilon\|l\|_{p'}. \tag{3.21}$$

This proves that $\frac{u_\varepsilon - u}{\varepsilon}$ is weakly convergent in $W_0^{1,p}(\Omega)$ towards some function that we denote ξ.

We now prove that $\frac{\sigma_\varepsilon - \sigma}{\varepsilon}$ is bounded in $L^{\frac{p}{p-1}}(\Omega)$. For that aim, we use inequality (3.11), valid for all X and Y in $\mathbb{R}^N$, and for $p \le 2$:

$$\left||X|^{p-2}X - |Y|^{p-2}Y\right|^{\frac{p}{p-1}} \le C_p|X - Y|^p. \tag{3.22}$$

With this inequality, we get

$$\|\sigma_\varepsilon - \sigma\|_{\frac{p}{p-1}} \leq C_p \|\nabla u_\varepsilon - \nabla u\|_p \leq C'\varepsilon \|l\|_{p'}. \tag{3.23}$$

Then we can extract from $\frac{\sigma_\varepsilon - \sigma}{\varepsilon}$ a subsequence converging in $L^{\frac{p}{p-1}}(\Omega)$ to some $\tau \in L^{\frac{p}{p-1}}(\Omega)$. Now let us prove that

$$\int_0^1 \beta'\big((1-s)(u-\psi)+s(u_\varepsilon-\psi-\varepsilon l)\big)\, ds\, \left(\frac{u_\varepsilon - u}{\varepsilon} - l\right) - \beta'(u-\psi)(\xi - l) \to 0.$$

Let us recall that β' is continuous and note that $u_\varepsilon \longrightarrow u$ a.e. when $\varepsilon \longrightarrow 0$. Let us observe that since (3.21) holds, $\frac{u_\varepsilon - u}{\varepsilon}$ converges to ξ strongly in every $L^q(\Omega)$, $q < \frac{Np}{N-p}$. In particular, denoting $q' = \frac{N-p}{Np-N+p} + \gamma$, $\gamma > 0$, $\frac{u_\varepsilon - u}{\varepsilon}$ converges to ξ in $L^q(\Omega)$ for $\frac{1}{q} + \frac{1}{q'} = 1$. Using the increasing behavior of β', one has

$$\left|\beta'\big((1-s)(u-\psi) + s(u_\varepsilon - \psi - \varepsilon l)\big)\right| \leq C\left(|u_\varepsilon - \psi| + \varepsilon |l| + |u-\psi|\right)^{q'},$$

which is hence bounded in $L^{\frac{Np}{(N-p)q'}}(\Omega)$.

Using on the other hand $\left(\frac{u_\varepsilon - u}{\varepsilon} - l\right) \longrightarrow \xi - l$ strongly in $L^q(\Omega)$, $\forall\, q < \frac{Np}{N-p}$ and Lebesgue's dominated convergence theorem, one finally gets

$$\int_0^1 \beta'\big((1-s)(u-\psi)+s(u_\varepsilon-\psi-\varepsilon l)\big)\, ds \left(\frac{u_\varepsilon - u}{\varepsilon} - l\right) \longrightarrow \int_0^1 \beta'(u-\psi)(\xi - l),$$

a.e. in Ω. Then, for any $\varphi \in W_0^{1,p}(\Omega)$, passing to the limit:

$$\begin{aligned} &\int_\Omega \left(\frac{\sigma_\varepsilon - \sigma}{\varepsilon}\right) \cdot \nabla\varphi \\ &= -\frac{1}{\delta} \int_\Omega \int_0^1 \beta'\big((1-s)(u-\psi) + s(u_\varepsilon - \psi - \varepsilon l)\big)\, ds\, \left(\frac{u_\varepsilon - u}{\varepsilon} - l\right) \varphi\, dx, \end{aligned} \tag{3.24}$$

finally one gets the desired equation

$$-\mathrm{div}\tau + \frac{1}{\delta}\beta'(u-\psi)(\xi - l) = 0 \text{ in } \Omega.$$

□

3.4 Optimal control

3.4.1 Optimal control for a non positive source term

PROPOSITION 3.4.1. *Let f, ψ and $T_f(\psi)$ be as in* (3.3)*. One has*

$$\frac{1}{p}\int_\Omega |\nabla T_f(\psi)|^p \leq \frac{1}{p}\int_\Omega |\nabla\psi|^p \, dx + \int_\Omega f[T_f(\psi) - \psi] \, dx.$$

Proof of Proposition 3.4.1. From (3.3) taking $v = \psi$ and using Hölder's inequality, we have

$$\int_\Omega |\nabla T_f(\psi)|^p \leq \frac{p-1}{p}\|\nabla T_f(\psi)\|_p^p + \frac{1}{p}\|\nabla\psi\|_p^p + \int_\Omega f[T_f(\psi) - \psi] \, dx.$$

□

Note that since $T_f(\psi) \geq \psi$, it follows that if $f \leq 0$, then

$$\int_\Omega |\nabla T_f(\psi)|^p \, dx \leq \int_\Omega |\nabla\psi|^p \, dx. \tag{3.25}$$

Let us now introduce the following problem, said "*optimal control problem*":

$$\inf_{\widetilde{\psi} \in W_0^{1,p}(\Omega)} J_f(\widetilde{\psi}), \tag{3.26}$$

where

$$J_f(\widetilde{\psi}) = \frac{1}{p}\int_\Omega \left\{ |T_f(\widetilde{\psi}) - z|^p + |\nabla\widetilde{\psi}|^p \right\} \, dx, \tag{3.27}$$

for some given $z \in L^p(\Omega)$. z is said to be *the initial profile*, ψ is *the control variable* and $T_f(\psi)$ is *the state variable.* The pair $(\psi^*, T_f(\psi^*))$ where ψ^* is a solution for (3.26) is called an optimal pair and ψ^* an optimal control.

In this section, we establish the existence and uniqueness of the optimal pair in the case where $f \leq 0$.

THEOREM 3.4.1. *If $f \in L^{p'}(\Omega)$, $f \leq 0$ on Ω, then there exists a unique optimal control $\psi^* \in W_0^{1,p}(\Omega)$ for* (3.26)*. Moreover, the corresponding state u^* coincides with ψ^*, i.e. $T_f(\psi^*) = \psi^*$.*

Proof of Theorem 3.4.1. In a first time we prove that there exists a pair of solutions of the form (u^*, u^*), hence $(u^* = T_f(u^*))$. Let $(\psi_k)_k$ be a minimizing sequence for (3.27), then $T_f(\psi_k)$ is bounded in $W^{1,p}(\Omega)$, therefore $T_f(\psi_k)$ converges for a subsequence towards some $u^* \in W_0^{1,p}(\Omega)$. Moreover, using the lower semicontinuity of T_f as in proposition 3.1.5, one gets

$$T_f(u^*) \leq \liminf_{k\to\infty} T_f(T_f(\psi_k)) \leq \lim_{k\to\infty} T_f(\psi_k) = u^*,$$

and by the definition of T_f, $T_f(u^*) \geq u^*$. Hence $u^* = T_f(u^*)$.

We prove that (u^*, u^*) is an optimal pair. Using proposition 3.1.5, by the lower semicontinuity in $W_0^{1,p}(\Omega)$ of T_f:

$$\begin{aligned} J_f(u^*) &= \frac{1}{p}\int_\Omega \{|u^* - z|^p + |\nabla u^*|^p\}\ dx \\ &\leq \liminf_{k\to\infty} \frac{1}{p}\int_\Omega \{|T_f(\psi_k) - z)|^p + |\nabla \psi_k|^p\}\ dx \\ &= \inf_{\psi\in W_0^{1,p}(\Omega)} J_f(\psi). \end{aligned}$$

Secondly, we prove that every optimal pair is of the form (u^*, u^*). Observe that if $(\psi^*, T_f(\psi^*))$ is a solution then $(T_f(\psi^*), T_f(\psi^*))$ is a solution. Indeed

$$\int_\Omega \{|T_f(\psi^*) - z)|^p + |\nabla T_f(\psi^*)|^p\}\ dx \leq \int_\Omega \{|T_f(\psi^*) - z)|^p + |\nabla \psi^*|^p\}\ dx.$$

So

$$\int_\Omega |\nabla T_f(\psi^*)|^p\ dx = \int_\Omega |\nabla \psi^*|^p\ dx, \tag{3.28}$$

by inequality (3.25), using the Hölder's inequality, one obtains then

$$\begin{aligned} 0 &\leq \int_\Omega f(\psi^* - T_f(\psi^*))\ dx \\ &\leq \int_\Omega \sigma(T_f(\psi^*)) \cdot \nabla(\psi^* - T_f(\psi^*))\ dx \\ &\leq \int_\Omega |\nabla T_f(\psi^*)|^{p-2}\nabla T_f(\psi^*) \cdot \nabla\psi^* - \int_\Omega |\nabla T_f(\psi^*)|^p \\ &\leq \left(\int_\Omega |\nabla T_f(\psi^*)|^p\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla T_f(\psi^*)|^p\right)^{\frac{1}{p}} - \int_\Omega |\nabla T_f(\psi^*)|^p = 0, \end{aligned}$$

which implies

$$\int_\Omega |\nabla T_f(\psi^*)|^{p-2}\nabla T_f(\psi^*)\cdot\nabla\psi^* - \int_\Omega |\nabla T_f(\psi^*)|^p = 0.$$

Let us recall that by convexity, one has the following inequality

$$\frac{1}{p}\int_\Omega |\nabla\psi^*|^p + \frac{p-1}{p}\int_\Omega |\nabla T_f(\psi^*)|^p - \int_\Omega |\nabla T_f(\psi^*)|^{p-2}\nabla T_f(\psi^*)\cdot\nabla\psi^* \geq 0.$$

Then the equality holds and by the strict convexity, one gets $\nabla(\psi^*) = \nabla(T_f(\psi^*))$ a.e., hence $\psi^* = T_f(\psi^*)$. Finally, we deduce from the two previous steps that the pair is unique. Suppose that (u_1, u_1) and (u_2, u_2) are two solutions, and consider $(\frac{u_1+u_2}{2}, T_f(\frac{u_1+u_2}{2}))$. We prove that it is also a solution. Indeed:

$$\begin{aligned}
\int_\Omega \left|\frac{u_1+u_2}{2} - z\right|^p + \left|\nabla T_f(\frac{u_1+u_2}{2})\right|^p \, dx \\
&\leq \int_\Omega \left|\frac{u_1+u_2}{2} - z\right|^p + \left|\nabla\left(\frac{u_1+u_2}{2}\right)\right|^p \, dx \\
&\leq \frac{1}{2}\left(J_f(u_1) + J_f(u_2)\right) = \inf_{\psi\in W_0^{1,p}(\Omega)} J_f(\psi),
\end{aligned}$$

which implies that $u_1 = u_2$. Thus, the uniqueness of the optimal pair for $f \leq 0$ holds. □

3.4.2 Optimal control for a nonnegative source term

We are interested here to the case $f \geq 0$ on Ω. In what follows we will denote by Gf the unique function in $W_0^{1,p}(\Omega)$ which verifies

$$\begin{cases} -\Delta_p(Gf) = f, & \text{in } \Omega \text{ a.e.} \\ Gf = 0, & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^{p'}(\Omega)$ and $Gf \in W_0^{1,p}(\Omega)$.

THEOREM 3.4.2. *Suppose that $f \in L^{p'}(\Omega)$ is a nonnegative function. Suppose that $z \in L^p(\Omega)$, satisfying $z \leq Gf$ a.e on Ω. Then the minimizing problem* (3.26) *has a unique optimal pair* $(0, Gf)$.

LEMMA 3.4.1. *Let $T_f(\psi)$ be a solution to* (3.3) *and Gf defined as above. Then $T_f(\psi)$ is greater than Gf.*

Proof of Lemma 3.4.1. We have that $-\Delta_p(Gf) = f$, and $T_f(\psi)$ realizes $-\Delta_p(T_f(\psi)) \geq f$. Then, by the Comparison Theorem for $-\Delta_p$ we get that $Gf \leq T_f(\psi)$. □

Proof of Theorem 3.4.2. In a first time we prove that $(0, Gf)$ is an optimal pair. Indeed, for all $\psi \in W_0^{1,p}(\Omega)$

$$\begin{aligned} J_f(\psi) &= \frac{1}{p}\int_\Omega \{|Gf - z + T_f(\psi) - Gf|^p + |\nabla\psi|^p\} \\ &\geq \frac{1}{p}\int_\Omega \big\{|Gf - z|^p + p|Gf - z|^{p-2}(Gf - z)(T_f(\psi) - Gf)\big\} \\ &\geq \frac{1}{p}\int_\Omega \{|Gf - z|^p\} \\ &= J_f(0). \end{aligned}$$

The equality with $(\psi^*, T_f(\psi^*))$ implies that we have equality in each step, so we get $\|\nabla\psi^*\|_p = 0$, then $\psi^* = 0$ a.e. in Ω. Thus, $(0, Gf)$ is the unique optimal control pair. □

Chapter 4

The obstacle problem on BV

4.1 Introduction

Let Ω be a bounded regular domain in $\mathbb{R}^N$, $N > 1$. The obstacle problem for the 1-Laplacian operator, as modeled in the last chapter on the obstacle problem on $W_0^{1,p}(\Omega)$, can be stated as follows: let $W_0^{1,1}(\Omega)$ be the Sobolev space

$$W_0^{1,1}(\Omega) = \{u \in L^1(\Omega), \nabla u \in L^1(\Omega), u = 0 \text{ on } \partial\Omega\},$$

let ψ be in $W_0^{1,1}(\Omega)$, $f \in L^\infty(\Omega)$ and $\mathcal{P}$ be defined as

$$\mathcal{P} = \inf_{\substack{u \in W_0^{1,1}(\Omega) \\ u \geq \psi}} J(u), \tag{4.1}$$

where J is the following functional

$$J(u) = \int_\Omega |\nabla u| - \int_\Omega fu.$$

In order to make the problem "solvable", we need to impose some smallness condition on f. For that aim, we must recall the definition of the first eigenvalue for the 1-Laplacian operator:

$$\inf_{\substack{u \in W_0^{1,1}(\Omega) \\ \|u\|_1 = 1}} \int_\Omega |\nabla u|.$$

This value, denoted as λ_1, is positive by Poincaré's inequality.

The main result of this chapter is the following:

THEOREM 4.1.1. *Let Ω be a bounded open $\mathcal{C}^1$ set in $\mathbb{R}^N$, $N \geq 1$. Suppose that $f \in L^\infty(\Omega)$ satisfies $\|f\|_\infty < \lambda_1$, and $\psi \in W_0^{1,1}(\Omega)$. Then there exists a solution in $BV(\Omega)$ to the relaxed formulation of* (4.1)*, given by*

$$\mathcal{P}_{BV} = \inf_{\substack{u \in BV(\Omega) \\ u \geq \psi}} J_{BV}(u), \tag{4.2}$$

where J_{BV} is the following functional defined on $BV(\Omega)$ by

$$J_{BV}(u) = \int_\Omega |\nabla u| + \int_{\partial\Omega} |u| - \int_\Omega fu.$$

Moreover, let $\delta > 0$ be given, and $\mathcal{P}_\delta$ be defined as

$$\mathcal{P}_\delta = \inf_{u \in BV(\Omega)} \left\{ \int_\Omega |\nabla u| + \int_{\partial\Omega} |u| + \frac{1}{\delta} \int_\Omega (u - \psi)^- - \int_\Omega fu \right\}. \tag{4.3}$$

Then $\mathcal{P}_\delta$ converges to $\mathcal{P}$, and from every sequence of solutions (u_δ) of $\mathcal{P}_\delta$, one can extract a subsequence, which converges tightly in $BV(\Omega)$ to a solution of the relaxed problem.

This approximation permits, using the dual problem of $\mathcal{P}_{BV}$ and a pair (σ, τ) of its solution, to give sense to the partial differential equation satisfied by u.

In the second part of this chapter, we present explicit solutions in the one dimensional case. When $f \geq 0$, the uniqueness of u is proved. Depending on the fact that ψ achieves its maximum on one point or several, uniqueness and non uniqueness is proved for the couple of solutions (σ, τ) of the dual problem.

In the last part, we are interested by an optimal control problem on $BV(\Omega)$, for $f \leq 0$.

4.2 Existence of solutions for a relaxed formulation of the obstacle problem

4.2.1 Some properties and definitions about BV

Let Ω be a bounded open set in $\mathbb{R}^N$ $(N \geq 1)$, which is piecewise $\mathcal{C}^1$. Let us recall the definition of the space of functions with bounded variations:

$$BV(\Omega) = \left\{ u \in L^1(\Omega), \nabla u \in \mathcal{M}^1(\Omega) \right\},$$

where $\mathcal{M}^1(\Omega)$ denotes the space of bounded Radon measures on Ω. The space $BV(\Omega)$ is a Banach space, endowed with the following norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^1(\Omega)},$$

where $\|\nabla u\|_{L^1(\Omega)}$ is the total variation of ∇u.

In the following, we shall use the absolutely continuous part $(\nabla u)^{ac}$ of ∇u, and $(\nabla u)^s$ its singular part (see [22], [27]).

We recall some properties of $BV(\Omega)$ inherited from $W^{1,1}(\Omega)$:

1. $BV(\Omega)$ is continuously embedded in $L^q(\Omega)$, for $q \leq \frac{N}{N-1}$, and these embeddings are compact for $q < \frac{N}{N-1}$ if $N \geq 2$; for $q = \infty$, if $N = 1$.

2. There exists a map: $BV(\Omega) \longrightarrow L^1(\partial\Omega)$, which extends the notion of trace when u belongs to $W^{1,1}(\Omega)$. This map is not continuous for the weak topology on BV. However, it is continuous for an intermediate topology, linked to the tight convergence of measures. More precisely, for $u \in BV(\Omega)$, there exists (u_n) in $W^{1,1}(\Omega)$, such that

$$\begin{aligned} &\int_\Omega |u_n - u| \longrightarrow 0, \\ &\int_\Omega |\nabla u_n| \longrightarrow \int_\Omega |\nabla u|, \end{aligned} \tag{4.4}$$

(which means that u_n converges tightly in BV towards u). Then,

$$\int_{\partial\Omega} |u_n - u| \longrightarrow 0.$$

3. We shall need also in the following some density results of $\mathcal{C}^\infty(\Omega) \cap W^{1,1}(\Omega)$ in $BV(\Omega)$:

   - For each $u \in BV(\Omega)$, there exists $(u_n) \in \mathcal{C}^\infty(\Omega) \cap W^{1,1}(\Omega)$, such that $u_n = u$ on $\partial\Omega$ and

$$\begin{aligned} &\int_\Omega |u_n - u| \longrightarrow 0, \\ &\int_\Omega |\nabla u_n| \longrightarrow \int_\Omega |\nabla u|. \end{aligned} \tag{4.5}$$

Moreover, one can impose, when u is non negative, that u_n be also non negative, and that

$$\int_\Omega |\nabla u_n - (\nabla u)^{ac}| \longrightarrow \int_\Omega |(\nabla u)^s|. \tag{4.6}$$

- For each $u \in BV(\Omega)$, there exists $(u_n) \in W_0^{1,1}(\Omega)$ (with null trace on the boundary), such that

$$\begin{aligned} &\int_\Omega |u_n - u| \longrightarrow 0, \\ &\int_\Omega |\nabla u_n| \longrightarrow \int_\Omega |\nabla u| + \int_{\partial\Omega} |u|, \end{aligned} \tag{4.7}$$

and

$$\int_\Omega |\nabla u_n - (\nabla u)^{ac}| \longrightarrow \int_\Omega |(\nabla u)^s| + \int_{\partial\Omega} |u|. \tag{4.8}$$

4.2.2 Introduction of the relaxed problem

Let us recall that $W^{1,1}(\Omega)$ is not a reflexive space. For that reason, it is not possible to prove the existence of solutions of (4.1) in $W_0^{1,1}(\Omega)$. The convenient space in which one can look for a minimizer, is the space $BV(\Omega)$, which is the weak closure of $W^{1,1}(\Omega)$. Moreover, since the trace map, which is well defined on $BV(\Omega)$, is not weakly continuous (cf. (4.4)), one is led to replace the problem by the relaxed form defined in (4.2). To prove that (4.2) has the same infimum as (4.1), we will need the following approximation result:

Proposition 4.2.1. *Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 1$, which is piecewise $\mathcal{C}^1$. Let ψ be in $W_0^{1,1}(\Omega)$ and u be in $BV(\Omega)$, such that $u \geq \psi$. Then, there exists a sequence $u_n \in W_0^{1,1}(\Omega)$, $u_n \geq \psi$ such that*

$$\int_\Omega |u_n - u| \longrightarrow 0,$$

$$\int_\Omega |\nabla u_n| \longrightarrow \int_\Omega |\nabla u| + \int_{\partial\Omega} |u|.$$

Proof of Proposition 4.2.1. Suppose that $u \geq \psi$, $u \in BV(\Omega)$. Using the approximation result in (4.8), there exists v_n in $W_0^{1,1}(\Omega)$ such that $v_n \geq 0$ converges to $u - \psi \geq 0$, with

$$\int_\Omega |\nabla v_n - [\nabla(u-\psi)]^{ac}| \to \int_\Omega |[\nabla(u-\psi)]^s| + \int_{\partial\Omega} |u| = \int_\Omega |(\nabla u)^s| + \int_{\partial\Omega} |u|.$$

Then the sequence $u_n = v_n + \psi$ satisfies $u_n \rightharpoonup u$ weakly in $BV(\Omega)$ with

$$\int_\Omega |\nabla u_n - (\nabla u)^{ac}| \longrightarrow \int_\Omega |(\nabla u)^s| + \int_{\partial\Omega} |u|.$$

This implies that

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n| \leq \int_\Omega |(\nabla u)^{ac}| + \int_\Omega |(\nabla u)^s| + \int_{\partial\Omega} |u| = \int_\Omega |\nabla u| + \int_{\partial\Omega} |u|.$$

On the other hand, we always have

$$\int_\Omega |\nabla u| + \int_{\partial\Omega} |u| \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|,$$

this ends the proof. □

We are now in position to prove that $\mathcal{P}$ and $\mathcal{P}_{BV}$ coincide. First, since the infimum defining $\mathcal{P}_{BV}$ is taken on $BV(\Omega)$, one has $\mathcal{P}_{BV} \leq \mathcal{P}$. Now, suppose that $\varepsilon > 0$ is given and let u be in $BV(\Omega)$, $u \geq \psi$, such that

$$\int_\Omega |\nabla u| + \int_{\partial\Omega} |u| - \int_\Omega fu \leq \mathcal{P}_{BV} + \varepsilon.$$

Using proposition 4.2.1, there exists $v \in W_0^{1,1}(\Omega)$, $v \geq \psi$, such that

$$\int_\Omega |\nabla v| - \int_\Omega |\nabla u| - \int_{\partial\Omega} |u| - \int_\Omega f(v-u) \leq \varepsilon.$$

Then

$$\mathcal{P} \leq \int_\Omega |\nabla v| - \int_\Omega fv \leq \mathcal{P}_{BV} + 2\varepsilon.$$

This proves that $\mathcal{P}_{BV} = \mathcal{P}$.

We have now the following existence result:

THEOREM 4.2.1. *There exists $u \in BV(\Omega)$, $u \geq \psi$ such that u realizes the infimum in* (4.2).

Proof of Theorem 4.2.1. Let u_n be a minimizing sequence for (4.2) in $W_0^{1,1}(\Omega)$, as it is allowed be in Proposition 4.2.1, and let $\tilde{u}_n$ be the extension of u_n by 0 outside Ω:

$$\widetilde{u}_n = \begin{cases} u_n, & \text{in } \Omega, \\ 0, & \text{in } \mathbb{R}^N \setminus \bar{\Omega}. \end{cases}$$

Using the fact that $\|f\|_\infty < \lambda_1$, one can observe that $\widetilde{u}_n$ is bounded in $W^{1,1}(\mathbb{R}^N)$. Then extracting from it a subsequence, still denoted $\widetilde{u}_n$, there exists v in $BV(\mathbb{R}^N)$, such that $\widetilde{u}_n \rightharpoonup v$ weakly in $BV(\mathbb{R}^N)$. Moreover, $v = 0$ outside $\bar{\Omega}$, and by lower semicontinuity of the total variation of measure with respect to the vague convergence, we get

$$\int_{\mathbb{R}^N} |\nabla v| = \int_\Omega |\nabla v| + \int_{\partial\Omega} |v| \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla \widetilde{u}_n|.$$

Using $\widetilde{u}_n \geq \widetilde{\psi}$ and the almost everywhere convergence of $\widetilde{u}_n$ to v, one gets that $v \geq \widetilde{\psi}$ and then $v \geq \psi$ in Ω. Finally, $u = v_{|\Omega}$ is admissible for $\mathcal{P}_{BV}$ and,

$$\mathcal{P}_{BV} \leq \int_{\Omega\cup\partial\Omega} |\nabla v| - \int_\Omega fv \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n| - \int_\Omega fu_n = \mathcal{P}_{BV}.$$

Hence, $u = v_{|\Omega}$ realizes the infimum of $\mathcal{P}_{BV}$. $\square$

In the following section, we approximate $\mathcal{P}$ by the analogous obstacle problem on $W_0^{1,1+\varepsilon}(\Omega)$, for ε small.

4.2.3 Approximation problem

We propose here to approximate $\mathcal{P}$ by an analogous problem defined on $W_0^{1,1+\varepsilon}(\Omega)$. Let $\varepsilon > 0$ be given, and let f be in $L^\infty(\Omega)$ such that $\|f\|_\infty < \lambda_1$. Let us define the following approximation problem:

$$\mathcal{P}_\varepsilon = \inf_{\substack{u\in W_0^{1,1+\varepsilon}(\Omega) \\ u\geq\psi}} J_\varepsilon(u), \tag{4.9}$$

where J_ε is the following functional

$$J_\varepsilon(u) = \frac{1}{1+\varepsilon} \int_\Omega |\nabla u|^{1+\varepsilon} - \int_\Omega fu. \tag{4.10}$$

We prove here that $\mathcal{P}_\varepsilon$ tends to $\mathcal{P}$, and that if u_ε is the unique solution of (4.9), u_ε converges, up to a subsequence, towards a solution u, which realizes $\mathcal{P}$, in the following sense:

$$\int_\Omega |u_\varepsilon - u| \longrightarrow 0,$$

and

$$\int_\Omega |\nabla u_\varepsilon|^{1+\varepsilon} \longrightarrow \int_\Omega |\nabla u| + \int_{\partial\Omega} |u|.$$

PROPOSITION 4.2.2. *Let u_ε be the solution of* (4.9). *Then u_ε converges weakly in $BV(\Omega)$ towards some u, which is a solution to the obstacle problem defined by $\mathcal{P}$. In the same time, one gets $\lim\limits_{\varepsilon\to 0}\mathcal{P}_\varepsilon = \mathcal{P}$.*

Proof of Proposition 4.2.2. Let $J_\varepsilon(\varphi)$ be as (4.10) and $\delta > 0$ be given and $\varphi \in \mathcal{D}(\Omega)$ such that:

$$\frac{1}{1+\varepsilon}\int_\Omega |\nabla\varphi|^{1+\varepsilon} - \int_\Omega f\varphi \le \mathcal{P} + \delta, \quad \forall\varepsilon > 0 \quad \text{small enough} \tag{4.11}$$

so $\lim_{\varepsilon\to 0} J_\varepsilon(\varphi) = J(\varphi)$ and $\mathcal{P}_\varepsilon \le J_\varepsilon(\varphi)$. Hence

$$\limsup_{\varepsilon\to 0}\mathcal{P}_\varepsilon \le \mathcal{P} + \delta, \tag{4.12}$$

δ being arbitrary, we get $\limsup\limits_{\varepsilon\to 0}\mathcal{P}_\varepsilon \le \mathcal{P}$.

To prove the reverse inequality, let u_ε be the infimum of $\mathcal{P}_\varepsilon$. We prove first that u_ε is bounded in $W_0^{1,1}(\Omega)$. Indeed, using Hölder's inequality, we obtain

$$\int_\Omega |\nabla u_\varepsilon|^{1+\varepsilon} \ge \left(\int_\Omega |\nabla u_\varepsilon|\right)^{1+\varepsilon} |\Omega|^{-\varepsilon} \ge \left(\lambda_1 \int_\Omega |u_\varepsilon|\right)^{1+\varepsilon} |\Omega|^{-\varepsilon}, \tag{4.13}$$

and then as $\varepsilon \longrightarrow 0$

$$\begin{aligned}\frac{1}{1+\varepsilon}\int_\Omega |\nabla u_\varepsilon|^{1+\varepsilon} - \int_\Omega f u_\varepsilon &\ge \frac{1}{1+\varepsilon}\left(\lambda_1 \int_\Omega |u_\varepsilon|\right)^{1+\varepsilon} |\Omega|^{-\varepsilon} - \|f\|_\infty \|u_\varepsilon\|_1 \\ &\ge [\lambda_1 - \|f\|_\infty + o(1)]\int_\Omega |u_\varepsilon|.\end{aligned}$$

Since the left hand side equals $\mathcal{P}_\varepsilon$, which is bounded, one sees that $\displaystyle\int_\Omega |u_\varepsilon|$ is bounded, and using once more (4.12), one sees that $\displaystyle\int_\Omega |\nabla u_\varepsilon|$ is also bounded uniformly and then u_ε is bounded in $BV(\Omega)$. Let $\widetilde{u}_\varepsilon$ be the extension of u_ε by 0 outside of Ω:

$$\widetilde{u}_\varepsilon = \begin{cases} u_\varepsilon, & \text{in } \Omega, \\ 0, & \text{in } \mathbb{R}^N \setminus \bar{\Omega}. \end{cases}$$

One can observe that $\widetilde{u}_\varepsilon$ is bounded in $BV(\mathbb{R}^N)$. Then extracting from it a subsequence, still denoted by $\widetilde{u}_\varepsilon$, there exists v in $BV(\mathbb{R}^N)$ such that $\widetilde{u}_\varepsilon \rightharpoonup v$ weakly in $BV(\mathbb{R}^N)$, with $v = 0$ outside of $\bar{\Omega}$. Using $\widetilde{u}_\varepsilon \geq \widetilde{\psi}$ and the almost everywhere convergence of $\widetilde{u}_\varepsilon$ to v, one gets that $v \geq \widetilde{\psi}$. Hence, we get for $u = v_{|\Omega}$,

$$\begin{aligned} \mathcal{P} = \mathcal{P}_{BV} &\leq \int_\Omega |\nabla u| + \int_{\partial\Omega} |u| - \int_\Omega fu \\ &\leq \liminf_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon| - \int_\Omega f u_\varepsilon \\ &\leq \liminf_{\varepsilon\to 0} \frac{1}{1+\varepsilon} \int_\Omega |\nabla u_\varepsilon|^{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon} |\Omega|^{\frac{\varepsilon}{1+\varepsilon}} - \int_\Omega f u_\varepsilon \\ &\leq \liminf_{\varepsilon\to 0} \mathcal{P}_\varepsilon. \end{aligned}$$

This ends the proof of Proposition 4.2.2. □

In order to characterize u_ε as solution of a P.D.E., as it is done in the case of the obstacle problem on $W_0^{1,p}(\Omega)$, $p > 1$, we can use the dual problem, computed in the next section.

4.3 Dual computation and first equality

We propose here to use the theory of convex analysis [41], [28], to compute the dual problem of (4.1). Let us denote by $\mathcal{P}^*$ the dual problem of $\mathcal{P}$ (cf. [28]). To compute $\mathcal{P}^*$, we introduce the following function,

$$F(u) = -\int_\Omega fu, \tag{4.14}$$

and the linear operator Λ:

$$\begin{aligned}\Lambda : W^{1,1}(\Omega) &\longrightarrow L^1(\Omega,\mathbb{R}^N)\times L^1(\Omega)\\ u &\longmapsto (\nabla u, u).\end{aligned}$$

We also define the function $G(\sigma,\tau) = G_1(\sigma) + G_2(\tau)$, where

$$G_1(\sigma) = \int_\Omega |\sigma| \quad \text{in } L^1(\Omega,\mathbb{R}^N), \tag{4.15}$$

$$G_2(\tau) = \begin{cases} 0, & \text{if } \tau \geq \psi \text{ in } L^1(\Omega),\\ +\infty, & \text{elsewhere.}\end{cases} \tag{4.16}$$

With these notations, we can write the minimization problem defined in (4.1), as

$$\mathcal{P} = \inf_{u\in W_0^{1,1}(\Omega)} \left\{G(\Lambda u) + F(u)\right\}. \tag{4.17}$$

The dual problem is then defined as

$$\begin{aligned}\mathcal{P}^* &= \sup_{(\sigma,\tau)\in(L^\infty(\Omega,\mathbb{R}^N),\ L^\infty(\Omega))} \left\{-G^*(\sigma,-\tau)) - F^*(\Lambda^*(-\sigma,\tau))\right\}\\ &= \sup_{(\sigma,\tau)\in(L^\infty(\Omega,\mathbb{R}^N),\ L^\infty(\Omega))} \left\{-G_1^*(\sigma) - G_2^*(-\tau) - F^*(\Lambda^*(-\sigma,\tau))\right\}.\end{aligned}$$

Let us compute $F^*(\Lambda^*(-\sigma,\tau))$. By the definition of the conjugate F^*, we have

$$\begin{aligned}F^*(\Lambda^*(-\sigma,\tau)) &= \sup_{u\in W_0^{1,1}(\Omega)} \left\{\langle \Lambda^*(-\sigma,\tau), u\rangle - F(u)\right\}\\ &= \sup_{u\in W_0^{1,1}(\Omega)} \left\{\langle (-\sigma,\tau), \Lambda u\rangle - F(u)\right\}\\ &= \sup_{u\in W_0^{1,1}(\Omega)} \left\{\langle (-\sigma,\tau), (\nabla u, u)\rangle - F(u)\right\}\\ &= \sup_{u\in W_0^{1,1}(\Omega)} \left\{\int_\Omega -\sigma\cdot\nabla u + \int_\Omega u\tau + \int_\Omega fu\right\}\\ &\geq \sup_{\varphi\in\mathcal{D}(\Omega)} \left\{\int_\Omega (\text{div}\sigma + \tau + f)\varphi\right\}.\end{aligned}$$

This implies that if the left-hand side is finite, the right one is finite too. In that case, the distribution $\text{div}\sigma + \tau + f$ is zero. Then Green's formula implies that for every $u \in W_0^{1,1}(\Omega)$,

$$\int_\Omega -\sigma \cdot \nabla u + \int_\Omega \tau u + \int_\Omega f u = 0.$$

We finally get

$$F^*(\Lambda^*(-\sigma, \tau)) = \begin{cases} 0, & \text{if } \operatorname{div}\sigma + \tau + f = 0 \text{ in } \mathcal{D}'(\Omega), \\ +\infty, & \text{elsewhere.} \end{cases} \tag{4.18}$$

On the other hand, we have

$$\begin{aligned} G_1^*(\sigma) &= \sup_{\tau \in L^1} \{\langle \sigma, \tau \rangle - G_1(\tau)\} \\ &= \sup_{\tau \in L^1} \left\{ \int_\Omega \sigma\tau - \int_\Omega |\tau| \right\} \\ &= \sup_{\tau \in L^1} \left\{ \int_\Omega |\sigma||\tau| - \int_\Omega |\tau| \right\} \\ &= \sup_{\tau \in L^1} \left\{ \int_\Omega |\tau|(|\sigma| - 1) \right\}. \end{aligned}$$

Then,

$$G_1^*(\sigma) = \begin{cases} 0, & \text{if } |\sigma| \le 1, \\ +\infty, & \text{elsewhere.} \end{cases} \tag{4.19}$$

Finally, we compute G_2^*:

$$\begin{aligned} G_2^*(-\tau) &= \sup_{u \in L^1,\ u \ge \psi} \{\langle u, -\tau \rangle - G_2(u)\} \\ &= \sup_{u \in L^1,\ u \ge \psi} \left\{ -\int_\Omega \tau(u - \psi) - \int_\Omega \tau\psi \right\} \\ &= \sup_{\varphi \in L^1,\ \varphi \ge 0} \left\{ -\int_\Omega \tau\varphi \right\} - \int_\Omega \tau\psi. \end{aligned}$$

Hence,

$$G_2^*(-\tau) = \begin{cases} -\displaystyle\int_\Omega \tau\psi, & \text{if } \tau \ge 0, \\ +\infty, & \text{elsewhere.} \end{cases} \tag{4.20}$$

Thus, we obtain that

$$\mathcal{P}^* = \sup_{\substack{(\sigma,\tau)\in L^\infty(\Omega,\mathbb{R}^N)\times L^\infty(\Omega)\\ \tau\geq 0, |\sigma|\leq 1\\ \mathrm{div}\sigma+\tau+f=0}} \left\{ \int_\Omega \tau\psi \right\}. \tag{4.21}$$

We are now interested in the equality between $\mathcal{P}$ and $\mathcal{P}^*$.

The case where $\psi \leq 0$ gives some trivial solution, as it is proved in the following proposition:

PROPOSITION 4.3.1. *Suppose that $\psi \leq 0$, then $\mathcal{P}^* = \mathcal{P} = 0$ and 0 is the unique solution of the problem $\mathcal{P}$.*

Proof of Proposition 4.3.1. The dual problem is

$$\mathcal{P}^* = \sup_{\substack{(\sigma,\tau)\in L^\infty(\Omega,\mathbb{R}^N)\times L^\infty(\Omega)\\ \tau\geq 0, |\sigma|\leq 1\\ \mathrm{div}\sigma+\tau+f=0}} \int_\Omega \tau\psi.$$

The primal–dual relations imply that

$$\mathcal{P} \geq \mathcal{P}^*. \tag{4.22}$$

The left hand side is non positive since $\psi \leq 0$ implies that $u = 0 \geq \psi$ is admissible for $\mathcal{P}$.

We now observe that $\mathcal{P}^* \geq 0$. Indeed, one has the following result:

LEMMA 4.3.1.

$$\inf_{\int_\Omega fu=1} \left\{ \int_\Omega |\nabla u| \right\} = \sup\left\{ \lambda : \exists\ \sigma \in L^\infty(\Omega,\mathbb{R}^N),\ \|\sigma\|_\infty \leq 1, -\mathrm{div}\sigma = \lambda f \right\}. \tag{4.23}$$

Proof of Lemma 4.3.1. Let us introduce the following functions,

$$F(u) = \begin{cases} 0, & \text{if } \displaystyle\int_\Omega fu = 1, \\ +\infty, & \text{if not.} \end{cases} \tag{4.24}$$

$$G_1(\sigma) = \int_\Omega |\sigma| \text{ in } L^1(\Omega,\mathbb{R}^N), \tag{4.25}$$

and the linear operator Λ,

$$\begin{aligned} \Lambda : W^{1,1}(\Omega) &\longrightarrow L^1(\Omega, \mathbb{R}^N) \\ u &\longmapsto \nabla u. \end{aligned}$$

Let us compute $F^*(\Lambda^*(-\sigma))$. Fix U is such that $\displaystyle\int_\Omega fU = 1$. Then

$$\begin{aligned} F^*(\Lambda^*(-\sigma)) &= \sup_{\substack{u \in W_0^{1,1}(\Omega) \\ \int_\Omega fu = 1}} \{\langle \Lambda^*(-\sigma), u\rangle - 0\} \\ &= \langle \Lambda^*(-\sigma), U\rangle + \sup_{\substack{u \in W_0^{1,1}(\Omega) \\ \int_\Omega fu = 0}} \langle \Lambda^*(-\sigma), u\rangle . \end{aligned}$$

The last supremum is the supremum of a linear functional on the kernel of the linear functional $u \longmapsto \int_\Omega fu$. Then, if the supremum is finite, the functional $u \longmapsto \langle \Lambda^*(-\sigma), u\rangle$ must be proportional to $u \mapsto \int_\Omega fu$. As a consequence, there exists $\lambda \in \mathbb{R}$ such that $\langle \Lambda^*(-\sigma), u\rangle = -\lambda \int_\Omega fu$ for all $u \in W_0^{1,1}$. In particular this implies that $-\text{div}\sigma = \lambda f$ in the distribution sense, and then using the Green's formula on $W_0^{1,1}$, one gets

$$F^*(\Lambda^*(-\sigma)) = \langle \Lambda^*(-\sigma), U\rangle + 0 = -\lambda \int_\Omega fU = -\lambda.$$

We conclude that

$$F^*(\Lambda^*(-\sigma)) = \begin{cases} -\lambda, & \text{if } -\text{div}\sigma = \lambda f, \\ +\infty, & \text{elsewhere.} \end{cases}$$

On the other hand, as we already did before, we have

$$G_1^*(\sigma) = \begin{cases} 0, & \text{if } |\sigma| \leq 1, \\ +\infty, & \text{elsewhere.} \end{cases}$$

Then, using the theorem of Ekeland-Temam [28], we get the inf-sup equality (4.23). □

As a consequence the supremum $\mathcal{P}^*$ is non negative. Indeed, let us note that $\tau = 0$ is admissible for the dual problem $\mathcal{P}^*$, since we have the equality in (4.23).

The proposition 4.3.1 is now proved. □

We come back to the general case where ψ is positive somewhere. We want to prove also that

$$\mathcal{P} = \mathcal{P}^*.$$

For that aim let us note first that one cannot apply directly the theorem of Ekeland and Temam since the functional G_2 defined above is not continuous in $L^1(\Omega)$.

To overcome this difficulty and prove the inf-sup equality , we shall use some perturbation problem, analogous to the one employed in section 3, which regularizes in some sense the constraint "$u \geq \psi$".

4.4 Some penalization problems, dual computations and consequences for the primal problem

Let δ be some positive number. The data ψ being as in the last section, we define

$$\mathcal{P}_\delta(\psi) = \inf_{v \in W_0^{1,1}(\Omega)} \left\{ \int_\Omega |\nabla v| + \frac{1}{\delta} \int_\Omega (v - \psi)^- - \int_\Omega f v \right\}, \tag{4.26}$$

where f is supposed to be in $L^\infty(\Omega)$, $\|f\|_\infty < \lambda_1(\Omega)$.

We shall prove in this section that $\mathcal{P}_\delta$ converges towards $\mathcal{P}$ in a sense to be specified. We shall compute in a first time the dual of $\mathcal{P}_\delta$, that we denote by $\mathcal{P}_\delta^*$.

In a second time we check the equality $\mathcal{P}_\delta = \mathcal{P}_\delta^*$. Finally, we let δ go to zero and prove the convergence of $\mathcal{P}_\delta$ and $\mathcal{P}_\delta^*$ towards, respectively, $\mathcal{P}$ and $\mathcal{P}^*$.

First Step: computation of the dual of $\mathcal{P}_\delta$.

We introduce the following functions

$$F(u) = -\int f u, \tag{4.27}$$

$$G_1(\sigma) = \int_\Omega |\sigma| \quad \text{in } L^1(\Omega, \mathbb{R}^N), \tag{4.28}$$

$$G_2(\tau) = \frac{1}{\delta} \int_\Omega (\tau - \psi)^-. \tag{4.29}$$

Using the same definition as before, we obtain that

$$F^*(\Lambda^*(-\sigma,\tau)) = \begin{cases} 0, & \text{if } \operatorname{div}\sigma + \tau + f = 0, \\ +\infty, & \text{elsewhere.} \end{cases} \tag{4.30}$$

and

$$G_1^*(\sigma) = \begin{cases} 0, & \text{if } |\sigma| \le 1, \\ +\infty, & \text{elsewhere.} \end{cases} \tag{4.31}$$

And finally, we have

$$\begin{aligned} G_2^*(-\tau) &= \sup_{u\in L^1} \left\{ -\int_\Omega \tau u - G_2(u) \right\} \\ &= \sup_{u\in L^1} \left\{ -\int_\Omega \tau(u-\psi) - \int_\Omega \tau\psi - \frac{1}{\delta}\int_\Omega (u-\psi)^- \right\} \\ &= \sup_{\varphi\in L^1} \left\{ -\int_\Omega \tau\varphi - \frac{1}{\delta}\int_\Omega \varphi^- \right\} - \int_\Omega \tau\psi \\ &\ge \sup_{\varphi\in L^1,\ \varphi\ge 0} \left\{ -\int_\Omega \tau\varphi \right\} - \int_\Omega \tau\psi. \end{aligned}$$

We remark that $G_2^*(-\tau) = +\infty$, if $\tau^- \not\equiv 0$. For $\tau \ge 0$ we have

$$\begin{aligned} G_2^*(-\tau) &= \sup_{u\in L^1} \{\langle u, -\tau\rangle - G_2(u)\} \\ &= \sup_{\varphi\in L^1} \left\{ -\int_\Omega \tau\varphi - \frac{1}{\delta}\int_\Omega \varphi^- \right\} - \int_\Omega \tau\psi \\ &\ge \sup_{\varphi\in L^1,\ \varphi\le 0} \left\{ -\int_\Omega \tau\varphi - \frac{1}{\delta}\int_\Omega \varphi^- \right\} - \int_\Omega \tau\psi \\ &= \sup_{\varphi\in L^1,\ \varphi\le 0} \left\{ \int_\Omega \tau\varphi^- - \frac{1}{\delta}\int_\Omega \varphi^- \right\} - \int_\Omega \tau\psi \\ &= \sup_{\varphi\in L^1,\ \varphi\ge 0} \left\{ \int_\Omega (\tau - \frac{1}{\delta})\varphi \right\} - \int_\Omega \tau\psi. \end{aligned}$$

Then we obtain

$$G_2^*(-\tau) = \begin{cases} -\displaystyle\int_\Omega \tau\psi, & \text{if } \ 0 \le \tau \le \frac{1}{\delta}, \\ +\infty, & \text{elsewhere.} \end{cases} \tag{4.32}$$

Hence, the dual problem $\mathcal{P}_\delta^*$ of $\mathcal{P}_\delta$ can be written as:

$$\mathcal{P}_\delta^* = \sup_{\substack{(\sigma,\tau)\in L^\infty(\Omega,\mathbb{R}^N)\times L^\infty(\Omega)\\ 0\le\tau\le\frac{1}{\delta},|\sigma|\le 1\\ \mathrm{div}\sigma+\tau+f=0.}} \left\{ \int_\Omega \tau\psi \right\}. \tag{4.33}$$

Moreover, we have, using the theorem of Ekeland-Temam (cf. [28])

$$\mathcal{P}_\delta^* = \mathcal{P}_\delta.$$

Second Step: We now let δ go to zero.

PROPOSITION 4.4.1. *For $\delta > 0$, we have the following convergence for $\mathcal{P}_\delta$ and $\mathcal{P}$ defined in* (4.26) *and* (4.1)*:*

$$\lim_{\delta\to 0} \mathcal{P}_\delta = \mathcal{P}. \tag{4.34}$$

Proof of Proposition 4.4.1. Let us remark first that

$$\mathcal{P}_\delta^* = \mathcal{P}_\delta \le \mathcal{P}. \tag{4.35}$$

Hence

$$\limsup_{\delta\to 0} \mathcal{P}_\delta \le \mathcal{P}. \tag{4.36}$$

We now prove that $\mathcal{P}_\delta \longrightarrow \mathcal{P}$. Let u_δ be in $W_0^{1,1}(\Omega)$, such that

$$\int_\Omega |\nabla u_\delta| + \frac{1}{\delta}\int_\Omega (u_\delta - \psi)^- - \int_\Omega f u_\delta \le \mathcal{P}_\delta + \delta. \tag{4.37}$$

Using Poincaré's inequality, we get

$$\int_\Omega |\nabla u_\delta| - \int_\Omega f u_\delta \ge \left(\frac{\lambda_1 - \|f\|_\infty}{2}\right)\int_\Omega |u_\delta|. \tag{4.38}$$

This implies

$$\left(\frac{\lambda_1 - \|f\|_\infty}{2}\right)\int_\Omega |u_\delta| + \frac{1}{\delta}\int_\Omega (u_\delta - \psi)^- \le \mathcal{P}_\delta + \delta. \tag{4.39}$$

Hence, $(u_\delta)_\delta$ is bounded in $L^1(\Omega)$ as well as $(u_\delta - \psi)^-$. On the other hand, we have

$$\int_\Omega |\nabla u_\delta| \le \mathcal{P}_\delta + \delta + \|f\|_\infty \int_\Omega |u_\delta| + \frac{1}{\delta}\int_\Omega (u_\delta - \psi)^-,$$

which implies that $(u_\delta)_\delta$ is bounded in $W_0^{1,1}(\Omega)$. We define

$$\widetilde{u}_\delta = \begin{cases} u_\delta, & \text{in } \Omega, \\ 0, & \text{outside.} \end{cases}$$

then $\widetilde{u}_\delta$ is bounded in $W^{1,1}(\mathbb{R}^N)$. One can extract from it a subsequence, still denoted $\widetilde{u}_\delta$, such that

$$\widetilde{u}_\delta \rightharpoonup v \quad \text{weakly in } BV(\mathbb{R}^N). \tag{4.40}$$

Obviously $v = 0$ outside of $\overline{\Omega}$ and $v \ge \psi$. Since $\|(u_\delta - \psi)^-\|_{L^1} \le C\delta$, by (4.36) then by Fatou's lemma we have,

$$\int_\Omega (v-\psi)^- \le \liminf_{\delta\to 0} \int_\Omega (u_\delta - \psi)^- = 0. \tag{4.41}$$

Finally, by lower semicontinuity, we obtain:

$$\begin{aligned} \mathcal{P} \le \int_{\mathbb{R}^N} |\nabla v| - fv &= \int_\Omega |\nabla v| + \int_{\partial\Omega} |v| - \int_\Omega fv \\ &\le \liminf_{\delta\to 0} \int_\Omega |\nabla u_\delta| + \frac{1}{\delta}\int_\Omega (u_\delta - \psi)^- - \int_\Omega f u_\delta \\ &\le \liminf_{\delta\to 0} (\mathcal{P}_\delta + \delta) \\ &\le \limsup_{\delta\to 0} (\mathcal{P}_\delta + \delta) \le \mathcal{P}. \end{aligned}$$

Hence, $u = v_{|\Omega}$ realizes the infimum, and

$$\liminf_{\delta\to 0} \mathcal{P}_\delta = \mathcal{P}. \tag{4.42}$$

□

Let us note that $\mathcal{P}_\delta^* \le \mathcal{P}^*$. We have obtained

$$\mathcal{P}^* \le \mathcal{P} \le \liminf_{\delta\to 0} \mathcal{P}_\delta = \liminf_{\delta\to 0} \mathcal{P}_\delta^* \le \mathcal{P}^*.$$

Which implies in particular that

$$\mathcal{P} = \mathcal{P}^*.$$

REMARK 4.4.1. Let us note that if $(\sigma_\delta, \tau_\delta)$ is a pair of solutions for the dual problem $\mathcal{P}^*_\delta$. We have:

$$\int_\Omega \tau_\delta = \int_\Omega -\mathrm{div}\sigma_\delta - \int_\Omega f = \int_{\partial\Omega} \sigma_\delta \cdot \overrightarrow{n} - \int_\Omega f \leq |\partial\Omega| - \int_\Omega f.$$

Since $\tau_\delta \geq 0$, the sequence τ_δ is then bounded in $\mathcal{M}^1(\Omega)$. Thus τ_δ converges, up to a subsequence, to some $\tau \geq 0$, weakly in $\mathcal{M}^1(\Omega)$. On the other hand, since $|\sigma_\delta| \leq 1$, it converges, up to a subsequence, to some σ, such that $|\sigma| \leq 1$, and (σ, τ) satisfy

$$-\mathrm{div}\sigma = f + \tau \text{ in } \Omega. \tag{4.43}$$

On the other hand, since $\psi \in W^{1,1}_0(\Omega)$, we have

$$\begin{aligned} \int_\Omega \tau_\delta \psi &= -\int_\Omega (\mathrm{div}\sigma_\delta)\psi - \int_\Omega f\psi \\ &= \int_\Omega \sigma_\delta \cdot \nabla\psi - \int_\Omega f\psi \longrightarrow \int_\Omega \sigma \cdot \nabla\psi - \int_\Omega f\psi = \int_\Omega \tau\psi. \end{aligned}$$

As a consequence, (σ, τ) is a solution of $\mathcal{P}^*$.

REMARK 4.4.2. One could also prove that $\mathcal{P} = \mathcal{P}^*$ by choosing differently the functions F and G. Indeed, let us define

$$F(u) = \begin{cases} -\displaystyle\int_\Omega f u, & \text{if } u \geq \psi, \\ +\infty, & \text{if not.} \end{cases}$$

$$G_1(\sigma) = \int_\Omega |\sigma|.$$

Now, G_1 is continuous, so one can apply the theorem of Ekeland-Temam. Moreover, the previous method is interesting in itself, since it gives some kind of regularization of the problem.

4.5 Examples in the one dimensional case

We now present some explicit examples in the one dimensional case. We suppose here that $\Omega =]0, 1[$ and ψ is in $W^{1,1}(\Omega)$. We can prove however that the inf-sup equality holds, by exhibiting solutions u of the primal problem and (σ, τ) of the dual problem.

4.5.1 First example

PROPOSITION 4.5.1. *Suppose that ψ achieves its maximum only at one point γ, such that $\max \psi > 0$. Then*

$$\sup_{\substack{\tau \geq 0, \tau \in \mathcal{M}^1 \\ \int_0^1 \tau = 1}} \int_0^1 \tau \psi = \max \psi,$$

is achieved at $\bar{\tau}$ if and only if $\bar{\tau}$ is the Dirac mass centered at γ: i.e.

$$\bar{\tau} = \delta_\gamma.$$

Proof of Proposition 4.5.1. Let $\delta > 0$ be arbitrarily small and let $\bar{\tau}$ which realizes the supremum above. We prove that $\bar{\tau} = 0$ on $]0, \gamma - \delta[\cup]\gamma + \delta, 1[$. Indeed since the maximum is strict on γ for ψ, there exists ε_δ such that $\psi \leq \max \psi - \varepsilon_\delta$ therein. Then

$$\begin{aligned} \max \psi = \int_0^1 \bar{\tau} \psi &\leq \left(\int_0^{\gamma - \delta} \bar{\tau} + \int_{\gamma + \delta}^1 \bar{\tau} \right) (\max \psi - \varepsilon_\delta) + \int_{\gamma - \delta}^{\gamma + \delta} \bar{\tau} \psi \\ &\leq \int_0^1 \bar{\tau} \max \psi - \varepsilon_\delta \left(\int_0^{\gamma - \delta} \bar{\tau} + \int_{\gamma + \delta}^1 \bar{\tau} \right), \end{aligned}$$

and then this implies that $\left(\int_0^{\gamma - \delta} \bar{\tau} + \int_{\gamma + \delta}^1 \bar{\tau} \right) = 0$. We can let δ go to zero and obtain that the measure $\bar{\tau}$ has support on γ. Since $\int_0^1 \bar{\tau} = 1$ then $\bar{\tau}$ is a Dirac mass at γ and one gets the result. □

THEOREM 4.5.1. *Suppose that f is continuous and positive on $\Omega =]0, 1[$, such that $\|f\|_\infty < 2 = \lambda_1(\Omega)$, and let $\psi \in W^{1,1}(\Omega)$, which possesses a maximum strictly positive on only one point $\gamma \in]0, 1[$. Then there exists a unique solution for $\inf \mathcal{P}$ given by $u = \chi_{]0,1[} \max \psi$. Moreover $\inf \mathcal{P}$ has value $(2 - \int_0^1 f) \max \psi$ and $\sup \mathcal{P}^*$ has a unique pair (σ, τ) of solutions given by:*

$$\tau = \left(2 - \int_0^1 f \right) \delta_\gamma,$$

and

$$\sigma(x) = 1 - \int_0^x f - \int_0^x \tau.$$

Proof of Theorem 4.5.1. Using the computation performed in the last section one has

$$\inf_{\substack{u\in W_0^{1,1}(]0,1[)\\ u\geq\psi}} \left\{\int_{]0,1[} |u'| - \int_{]0,1[} fu\right\} \geq \sup_{\substack{(\sigma,\tau)\in L^\infty(]0,1[)\times L^\infty(]0,1[)\\ \tau\geq 0,\ |\sigma|\leq 1\\ -\sigma'=\tau+f}} \int_0^1 \tau\psi.$$

Let us note that since ψ is not necessarily zero on the boundary, one cannot use the result in section 3 to say that $\inf\mathcal{P} = \sup\mathcal{P}^\star$. The equality will be proved later, and will appear when the solution will be determined.

Let us remark first that the supremum on the right is greater than $(2 - \int_0^1 f)\max\psi$. Indeed, taking $\sigma(x) = 1 - \int_0^x f - \int_0^x \tau$ and $\tau = (2 - \int_0^1 f)\ \delta_\gamma$, which is an admissible pair for the dual problem, one has

$$\begin{aligned}
\inf_{\substack{u\in BV(]0,1[)\\ u\geq\psi}} \left\{\int_{]0,1[} |u'| - \int_{]0,1[} fu\right\} &\geq \sup_{\substack{(\sigma,\tau)\in L^\infty(]0,1[)\times L^\infty(]0,1[)\\ \tau\geq 0,\ |\sigma|\leq 1\\ -\sigma'=\tau+f}} \int_0^1 \tau\psi \\
&\geq \int_0^1 \tau\psi \\
&= \left(2 - \int_0^1 f\right)\int_0^1 \psi\delta_\gamma \\
&= \left(2 - \int_0^1 f\right)\ \psi(\gamma) \\
&= \left(2 - \int_0^1 f\right)\ \max\psi.
\end{aligned}$$

On the other hand, using

$$\sigma' + f = -\tau,$$

and integrating, we obtain

$$\int_0^1 \sigma' + \int_0^1 f = -\int_0^1 \tau.$$

This implies, using $|\sigma| \leq 1$,

$$\begin{aligned}
\int_0^1 \tau &= \sigma(0) - \sigma(1) - \int_0^1 f \\
&\leq 2 - \int_0^1 f.
\end{aligned}$$

Then, we deduce

$$\int_0^1 \tau\psi \le \left(\int_0^1 \tau\right) \max\psi \le \left(2 - \int_0^1 f\right) \max\psi.$$

Therefore the supremum $\mathcal{P}^*$ is smaller than $(2 - \int_0^1 f) \max\psi$, one gets

$$\mathcal{P}^* = \left(2 - \int_0^1 f\right) \max\psi.$$

Moreover, the supremum is achieved when τ is a Dirac mass, with $\tau = (2 - \int_0^1 f)\delta_\gamma$. In addition, we obtain that for a couple (σ, τ) which realizes the supremum, $\sigma(0) - \sigma(1) = 2$. Since σ is decreasing and $|\sigma| \le 1$, this can be obtained only if $\sigma(0) = 1$, and $\sigma(1) = -1$. Using

$$\sigma' + f = -\tau,$$

and integrating, one gets that

$$\bar{\sigma}(x) = \begin{cases} 1 - \displaystyle\int_0^x f, & \text{if } x < \gamma, \\ 1 - \displaystyle\int_0^x f - \max\psi, & \text{if } x > \gamma. \end{cases}$$

From this, one gets that the problem defining $\mathcal{P}^*$ has a unique pair of solutions $\tau = (2 - \int_0^1 f) \max\psi$, and $\sigma = \bar{\sigma}$.

We prove then $\mathcal{P} = \mathcal{P}^*$ and $u = \chi_{]0,1[} \max\psi$ is a solution for $\mathcal{P}$. Indeed, $u = \chi_{]0,1[} \max\psi$ is admissible for the problem defining $\mathcal{P}$ and then

$$\begin{aligned} \mathcal{P} \le \int_0^1 |u'| - \int_0^1 fu &\le 2\max\psi - \int_0^1 f \max\psi \\ &= \left(2 - \int_0^1 f\right) \max\psi \\ &= \mathcal{P}. \end{aligned}$$

This implies that $\mathcal{P} = \mathcal{P}^*$ and $u = \chi_{]0,1[} \max\psi$ is a solution.

We now show the uniqueness of the solution: for this purpose, we will need some more specified results about the convergence of $\mathcal{P}_\varepsilon$ towards $\mathcal{P}_{BV}$ in the one dimensional case. In particular, we precise the behavior of $\sigma_\varepsilon = |u'_\varepsilon|^{\varepsilon-1} u'_\varepsilon$.

Let us recall the approximation problem on the space $W_0^{1,1+\varepsilon}(]0,1[)$, already presented in (4.9), adapted to the one dimensional case:

$$\inf_{\substack{u\in W_0^{1,1+\varepsilon}(]0,1[)\\ u\geq\psi}} \frac{1}{1+\varepsilon}\int_0^1 |u'|^{1+\varepsilon} - \int_0^1 fu. \tag{4.44}$$

We use in the following the approximation of u by u_ε, to derive some properties of (σ,τ) defined before: the relations between u and (σ,τ) can be obtained by passing to the limit in the relations linking u_ε to $(\sigma_\varepsilon,\tau_\varepsilon)$, the solution of the dual of (4.44). This is the reason why we compute this dual.

We introduce the following function,

$$F(u) = -\int_0^1 fu, \tag{4.45}$$

the linear, continuous operator Λ:

$$\begin{aligned}\Lambda : W^{1,1+\varepsilon}(]0,1[) &\longrightarrow L^{1+\varepsilon}(]0,1[,\mathbb{R}) \times L^{1+\varepsilon}(]0,1[)\\ u &\longmapsto (u',u)\end{aligned}$$

and G defined as $G(\sigma,\tau) = G_1(\sigma) + G_2(\tau)$, where

$$G_1(\sigma) = \frac{1}{1+\varepsilon}\int_0^1 |\sigma|^{1+\varepsilon} \text{ in } L^{1+\varepsilon}(]0,1[,\mathbb{R}) \tag{4.46}$$

$$G_2(\tau) = \begin{cases} 0, & \text{if } \tau\geq\psi \text{ in } L^{1+\varepsilon}(]0,1[),\\ +\infty, & \text{elsewhere.}\end{cases} \tag{4.47}$$

One easily obtains that the dual problem of $\inf \mathcal{P}_\varepsilon$ is

$$\mathcal{P}_\varepsilon^* = \sup_{\substack{(\sigma,\tau)\in(L^{\frac{1+\varepsilon}{\varepsilon}}(]0,1[,\mathbb{R})\times L^{\frac{1+\varepsilon}{\varepsilon}}(]0,1[))\\ 0\leq\tau,\ |\sigma|\leq 1\\ \sigma'+\tau+f=0}} \left\{-\frac{\varepsilon}{1+\varepsilon}\int_0^1 |\sigma|^{\frac{1+\varepsilon}{\varepsilon}} + \int_0^1 \tau\psi\right\}. \tag{4.48}$$

Here, the theorem of Ekeland-Temam [28] can be applied and provide $\mathcal{P}_\varepsilon = \mathcal{P}_\varepsilon^*$. Moreover if $(\sigma_\varepsilon,\tau_\varepsilon)$ is a solution for the problem defining $\mathcal{P}_\varepsilon^*$, and u_ε is the solution of the problem defining $\mathcal{P}_\varepsilon$, one has

$$-\frac{\varepsilon}{1+\varepsilon}\int_0^1 |\sigma_\varepsilon|^{\frac{1+\varepsilon}{\varepsilon}} + \int_0^1 \tau_\varepsilon\psi = \frac{1}{1+\varepsilon}\int_0^1 |u_\varepsilon'|^{1+\varepsilon} - \int_0^1 fu_\varepsilon. \tag{4.49}$$

Using the equation $\sigma'_\varepsilon + \tau_\varepsilon + f = 0$ we obtain

$$\frac{\varepsilon}{1+\varepsilon}\int_0^1 |\sigma_\varepsilon|^{\frac{1+\varepsilon}{\varepsilon}} + \frac{1}{1+\varepsilon}\int_0^1 |u'_\varepsilon|^{1+\varepsilon} - \int_0^1 \sigma_\varepsilon u'_\varepsilon = \int_0^1 \tau_\varepsilon(\psi - u_\varepsilon). \tag{4.50}$$

Since the right hand side is non positive and the left hand side is nonnegative by the convexity of the map $x \longmapsto \frac{1}{p}|x|^p$ for $p > 1$, we obtain that the two sides are equals to zero. Then we get that $\int_0^1 \tau_\varepsilon(\psi - u_\varepsilon) = 0$, which implies that τ_ε is supported in the set $\{x, u_\varepsilon - \psi = 0\}$. Moreover, writing that the left hand side is zero, one gets that $\sigma_\varepsilon = |u'{}_\varepsilon|^{\varepsilon-1}u'{}_\varepsilon$.

PROPOSITION 4.5.2. *Let u_ε be the solution of* (4.44). *Then u_ε converges weakly in $BV(]0,1[)$ towards some u, which is a solution of the problem defining $\mathcal{P}$, and $\lim_{\varepsilon\to 0}\mathcal{P}_\varepsilon = \mathcal{P}$. Moreover, σ_ε converges in $BV(]0,1[)$ to σ, which is such that $\|\sigma\|_\infty \leq 1$, and there exists a non negative measure τ, such that*

$$-\sigma' = f + \tau.$$

REMARK 4.5.1. By Proposition 4.5.2, (σ, τ) is a solution for the dual problem.

Proof of Proposition 4.5.2. The first part has already been proved in the N-dimensional case. Observing that $\sigma_\varepsilon u'{}_\varepsilon = |u'{}_\varepsilon|^{\varepsilon+1}$, one sees that σ_ε is bounded in $L^{\frac{1+\varepsilon}{\varepsilon}}$. Let us prove that σ_ε is then bounded in every L^q, $q < \infty$. Indeed, q being given, let ε be such that $q < \frac{1+\varepsilon}{\varepsilon}$. Then

$$\left(\int_\Omega |\sigma_\varepsilon|^q\right)^{\frac{1}{q}} \leq \left(\int |\sigma_\varepsilon|^{\frac{1+\varepsilon}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}} |\Omega|^{\frac{1+\varepsilon(1-q)}{q(1+\varepsilon)}}.$$

Therefore, there exists σ, such that for every $q < \infty$, there exists a subsequence, still denoted by σ_ε, such that $\sigma_\varepsilon = |u'_\varepsilon|^{\varepsilon-1}u'_\varepsilon$ converges towards some σ in L^q. Let us observe that $\|\sigma\|_\infty \leq 1$. Indeed, let η be in $\mathcal{D}(\Omega, \mathbb{R})$. Then

$$\begin{aligned}\left|\int_\Omega \sigma\eta\right| \leq \liminf_{\varepsilon\to 0}\left|\int_\Omega \sigma_\varepsilon\eta\right| &\leq \liminf_{\varepsilon\to 0}\int_\Omega |u'_\varepsilon|^\varepsilon|\eta| \\ &\leq \liminf_{\varepsilon\to 0}\left(\int_\Omega |u'_\varepsilon|^{1+\varepsilon}\right)^{\frac{\varepsilon}{1+\varepsilon}}\left(\int_\Omega |\eta|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \\ &\leq \liminf_{\varepsilon\to 0} C^{\frac{\varepsilon}{1+\varepsilon}}\left(\int_\Omega |\eta|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} \\ &\leq \int_\Omega |\eta|.\end{aligned}$$

From this, one gets that $\|\sigma\|_\infty \leq 1$. We now prove that $-\sigma' = f + \tau$, for some nonnegative, bounded measure τ. For that aim, we use the fact that by passing to the limit in

$$-\sigma'_\varepsilon = f + \tau_\varepsilon,$$

$-\sigma' - f$ is a non negative distribution. Hence it is also a non negative measure. Since $\tau \geq 0$, to prove that this measure is bounded, it is enough to prove that for all a and b in $]0,1[$, $a < b$, we have

$$\int_{]a,b[} \tau \leq 2 + \int_0^1 f.$$

This is a mere consequence of the following lemma:

LEMMA 4.5.1. *Suppose that σ is monotone on $]0,1[$, $\sigma \in L^\infty(\Omega)$. Then $\sigma \in BV(]0,1[)$ with*

$$\int_{]0,1[} |\sigma'| \leq 2\|\sigma\|_\infty.$$

Proof of Lemma 4.5.1. One can assume without loss of generality that σ is non decreasing, and that $\|\sigma\|_\infty = 1$.

It is sufficient to prove that for all a, b in $]0,1[$, $a < b$

$$\int_{]a,b[} \sigma' \leq 2.$$

In order to show it, let us fix $\delta > 0$, and let ϕ_n be defined for $\frac{1}{n} \leq \frac{\inf(a,1-b)}{2}$ by

$$\phi_n = \begin{cases} n(x - a + \frac{1}{n}) \text{ on } [a - \frac{1}{n}, a], \\ 1 \text{ on } [a,b], \\ 1 - n(x-b) \text{ on } [b, b+\frac{1}{n}]. \end{cases}$$

which satisfies $\displaystyle\int_a^b |\phi'_n| = 2$. Let $\phi_\delta = \varrho_\delta * \phi_n$ be a regularization of ϕ_n, for some $\delta > 0$, $\delta < \frac{1}{n}$, such that $\displaystyle\int |\varrho_\delta * \phi'_n| \leq 2 + \delta$.

Since $\sigma' \geq 0$, we have,

$$\int_a^b \sigma' \leq \int_a^b \sigma'\phi_\delta = -\int_a^b \sigma\phi'_\delta =\leq |\sigma|_\infty \int_a^b |\phi'_\delta| \leq (2+\delta)|\sigma|_\infty.$$

This ends the proof of Lemma 4.5.1, since δ is arbitrary. □

Applying this Lemma with $-\sigma$, this ends the proof of Proposition 4.5.2.
□

We now prove the uniqueness of u, by observing that $u \in BV(\Omega)$, $u' \equiv 0$ a.e. $[0,1]$, and $\int_{[0,1]} |u'| = \left(2 - \int_0^1 f\right) \max \psi$. This can be obtained using lemma 4.5.2 below.

LEMMA 4.5.2. *Suppose that* $u_\varepsilon \in W^{1,1+\varepsilon}(]0,1[)$, $\forall\ \varepsilon > 0$ *and* $u_\varepsilon \rightharpoonup u$ *in* $BV(]0,1[)$ *weakly. Let* $\sigma_\varepsilon = |u_\varepsilon'|^{\varepsilon-1}u_\varepsilon'$ *with* $\int_0^1 |u_\varepsilon'|^{1+\varepsilon} \to \int_0^1 |u'|$, $-\sigma_\varepsilon' = f + \tau_\varepsilon$ *where* τ_ε *is, for each* $\varepsilon > 0$, *a bounded positive measure. Finally, assume that* τ_ε *converges weakly to some measure* $\tau \geq 0$. *We assume that* $|\sigma| \leq \alpha < 1$ *on some subset* $[a,b] \subset]0,1[$. *Then for* ε *small enough* $|\sigma_\varepsilon| < (\frac{\alpha+1}{2})$ *and* $|u'| = 0$ *on that subset.*

Proof of Lemma 4.5.2. One already has the strong convergence of σ_ε towards σ in $L^1(]0,1[)$, since it converges in $BV(]0,1[)$ weakly. In particular, up to a subsequence, σ_ε converges almost everywhere towards σ. Let us note that since σ is strictly decreasing and $\|\sigma\|_\infty \leq 1$ on $]0,1[$, $\|\sigma\|_\infty = \sup_{[a,b]}|\sigma| < 1$. Let x be a point in $[a,b]$ such that $\sigma_\varepsilon(x) \to \sigma(x)$, when $\varepsilon \to 0$. One has by the upper semicontinuity for the vague topology on compact sets, for all $y > x,\ y \in [a,b]$:

$$
\begin{aligned}
\sigma(x) - \sigma(y^-) &= \int_{[x,y]} -\sigma'(t)\ dt \\
&\geq \limsup_{\varepsilon \to 0} \int_{[x,y]} -\sigma_\varepsilon'(t)\ dt \\
&= \limsup_{\varepsilon \to 0}\ (\sigma_\varepsilon(x^+) - \sigma_\varepsilon(y^-)) \\
&= \sigma(x) - \limsup_{\varepsilon \to 0}\ \sigma_\varepsilon(y^-),
\end{aligned}
$$

which implies that, for ε small enough,

$$-\sigma_\varepsilon(y^-) \leq \alpha + o(1).$$

In the same manner, taking now $x > y$ such that $\sigma_\varepsilon(x) \to \sigma(x)$, one gets

$$\begin{aligned}
\sigma(y^+) - \sigma(x) &= \int_{[y,x]} -\sigma'(t)\ dt \\
&\geq \limsup_{\varepsilon\to 0} \int_{[y,x]} -\sigma'_\varepsilon(t)\ dt \\
&= \limsup_{\varepsilon\to 0} (\sigma_\varepsilon(y^+) - \sigma_\varepsilon(x)) \\
&= \limsup_{\varepsilon\to 0}\ \sigma_\varepsilon(y^+) - \sigma(x),
\end{aligned}$$

Moreover, since σ_ε is decreasing

$$\sigma_\varepsilon(y^+) \leq \sigma_\varepsilon(y^-) \leq \alpha + o(1).$$

The first announced result is done. We now prove that $u' = 0$ on the set $]a, b[$. Let $[a', b']$ which are no mass points for the measure u'. Then

$$\int_{]a',b'[} |u'_\varepsilon| \to \int_{]a',b'[} |u'|$$

and

$$\begin{aligned}
\int_{]a',b'[} |u'| &\leq \liminf_{\varepsilon\to 0} \int_{]a',b'[} |u'_\varepsilon|^{1+\varepsilon} \\
&\leq \liminf_{\varepsilon\to 0} \int_{]a',b'[} |u'_\varepsilon|^\varepsilon |u'_\varepsilon| \\
&\leq \left(\frac{\alpha+1}{2}\right) \liminf_{\varepsilon\to 0} \int_{]a',b'[} |u'_\varepsilon| \\
&\leq \left(\frac{\alpha+1}{2}\right) \int_{]a',b'[} |u'|.
\end{aligned}$$

This implies that $u' = 0$ on $]a', b'[$, hence $u = C$ on $]0, 1[$, $C \geq \max\psi$. Coming back to the fact that it must realizes the infimum of the problem, one gets that $u = \max\psi$ on $]a, b[$. This prove the uniqueness of u. □

This ends the proof of Theorem 4.5.1. □

4.5.2 Second example

We now present an example for the dual problem.

THEOREM 4.5.2. *Suppose that f is continuous and positive on* $]0,1[$. *Suppose that ψ achieves its maximum on $]\alpha,\beta[\subset]0,1[,\ \alpha<\beta$. Then there exists a solution for the problem defining $\mathcal{P}$ given by $u=\chi_{]0,1[}\max\psi$. Moreover $\mathcal{P}$ takes the $(2-\int_0^1 f)\max\psi$. Let σ be given by:*

$$\sigma(x)=1-\int_0^x f-\int_0^x \tau,$$

where $\sigma\cdot u'=|u'|$ on $]0,1[$.

To prove the non uniqueness result, let us note that if τ is a positive measure with support in $]\alpha,\beta[$, such that $\int_0^1\tau=2-\int_0^1 f$, and $-\sigma'=f+\tau$, then (σ,τ) is a solution of $\mathcal{P}^*$.

Proof of Theorem 4.5.2. One always has $\mathcal{P}^\star\geq(2-\int_0^1 f)\max\psi$, by taking as admissible couple (τ,σ), $\tau=\delta_\gamma$ where $\gamma\in]\alpha,\beta[$ is some point on which ψ achieves its maximum. σ is necessarily strictly decreasing since f is positive and $\tau\geq 0$. Using Lemma 4.5.2, one gets that $u'=0$ inside $]0,1[$. Then $u=\chi_{]0,1[}(2-\int_0^1 f)$. □

4.6 Some properties of the solution of the obstacle problem on BV

4.6.1 Technical results and analogy of the optimal control

We begin with some properties of the set of solutions for the obstacle problem. Let:

$$E_f(\psi)=\{u\in BV(\Omega), u \text{ realizes } \mathcal{P}(\psi)\}.$$

PROPOSITION 4.6.1. *Suppose that $\psi^1\geq\psi^2$ in $W_0^{1,1}(\Omega)$. Then there exists u^1 and u^2, which satisfy*

$$u^i\in E_f(\psi^i),\ i\in\{1,2\},\ \textit{and } u^1\geq u^2.$$

Proof of Proposition 4.6.1. Let $\psi^1 \geq \psi^2$, we show that there exist u^1, u^2 such that $u^i \in E_f(\psi^i)$, and $u^1 \geq u^2$. For that, let $T_\varepsilon(\psi)$ be the unique solution, of the corresponding obstacle problem for the $(1+\varepsilon)$-Laplacian, as it is proved in Proposition 3.1.3 and let $u^i_\varepsilon = T_\varepsilon(\psi^i)$, $i = 1, 2$. Acting as in the previous section, the extension of sequences (u^i_ε), $i = 1, 2$ by zero in $\mathbb{R}^N \setminus \Omega$, are bounded in $BV(\mathbb{R}^N)$. Then one can extract from it subsequences, denoted $\widetilde{u}^i_\varepsilon$, such that

$$\widetilde{u}^i_\varepsilon \rightharpoonup u^i \quad \text{weakly in } BV(\mathbb{R}^N), i = 1, 2$$

and $u^i \in E_f(\psi^i), i = 1, 2$. Since $u^1_\varepsilon \geq u^2_\varepsilon$ one gets that $u^1 \geq u^2$. □

Proposition 4.6.2. *Let $\psi_k \rightharpoonup \psi$ weakly in $BV(\Omega)$, then,*

$$\mathcal{P}(\psi) \leq \liminf_{k\to\infty} \mathcal{P}(\psi_k).$$

Moreover, if $\psi_k \leq \psi$, the previous convergence holds with

$$\mathcal{P}(\psi) = \lim_{k\to\infty} \mathcal{P}(\psi_k),$$

and there exist $u_k \in E_f(\psi_k)$ which is tightly convergent in $BV(\Omega)$ to $u \in E_f(\psi)$.

Proof of Proposition 4.6.2. Let (ψ_k) be a sequence which converges weakly in $BV(\Omega)$ to ψ, and $u_k \in E_f(\psi_k)$ for all k. By the coerciveness of the functional defined by $\mathcal{P}(\psi)$, and using the fact that $\mathcal{P}(\psi_k) \leq \int_\Omega |\nabla\psi_k| - \int_\Omega f\psi_k$, one gets that u_k is bounded in $BV(\Omega)$. Then, the extension of u_k by zero outside of $\bar{\Omega}$ is bounded in $BV(\mathbb{R}^N)$. One can extract from it a subsequence, denoted $\widetilde{u}_k$, such that

$$\widetilde{u}_k \rightharpoonup u \quad \text{weakly in } BV(\mathbb{R}^N).$$

Obviously $u = 0$ outside of $\bar{\Omega}$.

On the other hand, using the fact that ψ_k converges weakly to ψ, one gets the following assertion:

$$u_k \geq \psi_k \Longrightarrow u \geq \psi.$$

Using the lower semicontinuity, one has:

$$\int_{\partial\cup\Omega} |\nabla u| - \int_\Omega fu \leq \liminf_{k\to\infty} \int_{\partial\cup\Omega} |\nabla u_k| - \int_\Omega fu_k.$$

Since $u \geq \psi$, one gets

$$\mathcal{P}(\psi) \leq \liminf_{k\to\infty} \mathcal{P}(\psi_k). \tag{4.51}$$

We now assume that $\psi_k \leq \psi$ and prove that if $u_k \in E_f(\psi_k)$, for a subsequence, u_k converges to u which belongs to $E_f(\psi)$. In the same time, we get that $\lim_{k\to\infty} \mathcal{P}(\psi_k) = \mathcal{P}(\psi)$.

Let us note that if $\psi_k \leq \psi$, one has

$$\mathcal{P}(\psi_k) \leq \mathcal{P}(\psi). \tag{4.52}$$

Then, for $u_k \in E_f(\psi_k)$, one proves as before that $\widetilde{u}_k$, the extension of u_k by zero outside Ω, being bounded in $BV(\mathbb{R}^N)$, one can extract from it a subsequence converging to u which satisfies,

$$\begin{aligned}\int_{\Omega\cup\partial\Omega} |\nabla u| - \int_\Omega fu &\leq \liminf_{k\to\infty} \int_{\Omega\cup\partial\Omega} |\nabla u_k| - \int_\Omega fu_k \\ &\leq \liminf_{k\to\infty} \mathcal{P}(\psi_k) \\ &= \mathcal{P}(\psi) \leq \liminf_{k\to\infty} \mathcal{P}(\psi_k).\end{aligned}$$

Then $u \in E_f(\psi)$, and we obtain in the same time that

$$\int_{\Omega\cup\partial\Omega} |\nabla u_k| - \int_\Omega fu_k \longrightarrow \int_{\partial\cup\Omega} |\nabla u| - \int_\Omega fu, \quad \text{when } k \to \infty,$$

which implies that u_k converges tightly to u in $BV(\Omega)$. □

4.6.2 Optimal control on BV for non positive source term

Let us recall first that the optimal control problem for an elliptic obstacle variational inequality with a source term was considered in the case $p = 2$ by Adams, D. R., Lenhart, S. M., and Yong, J. in ([5], [6]). The authors produced an optimal control pair for the following problem

$$\inf_{\psi\in H_0^1(\Omega)} \left\{ \int_\Omega |T_f(\psi) - z|^2 + \int_\Omega |\nabla\psi|^2 \, dx \right\}, \tag{4.53}$$

for some $z \in L^2(\Omega)$ is referred to as *the initial profile*, ψ as *the control variable* and $T_f(\psi)$ as *the state variable.* The pair $(\psi^*, T_f(\psi^*))$ where ψ^* is a solution to the problem (4.53) is called an optimal pair and ψ^* an optimal control.

In the case $p > 1$, treated in the previous chapter, these results were generalized, and an optimal control pair is defined by the form (u, u) if $f \leq 0$ and by $(0, Gf)$ for $f \geq 0$.

In an analogous manner, and adapted to our context, let us consider the following minimization problem

$$\inf_{\substack{(u,\psi)\in(W_0^{1,1}(\Omega))^2 \\ u\in E_f(\psi)}} J_f(\psi), \tag{4.54}$$

where

$$J_f(\psi) = \left\{ \int_\Omega |u - z| + \int_\Omega |\nabla\psi| \, dx \right\},$$

for some given $z \in BV(\Omega)$ and $f \in L^\infty(\Omega)$.

We want to prove that under some conditions, there exists a solution to this problem. We shall assume in what follows that $f \leq 0$, $f \not\equiv 0$, and we establish existence of the optimal pair. For that aim, we introduce the following relaxed formulation of the problem (4.54)

$$\inf_{\substack{(u,\psi)\in(BV(\Omega))^2 \\ u\in E_f(\psi)}} \left\{ \int_\Omega |u - z| + \int_\Omega |\nabla\psi| + \int_{\partial\Omega} |\psi| \, dx \right\}, \tag{4.55}$$

We prove first an existence's result, when we are in the coercive case (this will be specified later). Secondly we make an analysis in the resonance case. To be more precise, let us observe that the following functional

$$\int_\Omega |\nabla u| - \lambda \int_\Omega fu, \quad u \in W_0^{1,1}(\Omega),$$

is coercive if $|\lambda| \leq \lambda_f$, where

$$\lambda_f = \inf_{\substack{u\in W_0^{1,1}(\Omega) \\ \int_\Omega fu=1}} \int_\Omega |\nabla u|. \tag{4.56}$$

We prove in a first time that the optimal control problem in $BV(\Omega)$ has a pair of solutions of the form (u^*, u^*), when $\lambda < \lambda_f$ (called the coercive case), and we consider next the case $\lambda = \lambda_f$ (called the resonance case).

We define first the set:

$$E_{\lambda f}(\psi) = \{u \in BV(\Omega), u \text{ realizes } \mathcal{P}_\lambda(\psi)\},$$

where
$$\mathcal{P}_\lambda(\psi) = \inf_{\substack{u\in BV(\Omega)\\ u\geq\psi}} \left\{ \int_\Omega |\nabla u| - \lambda \int_\Omega fu \right\}.$$

The case $\lambda < \lambda_f$

A key ingredient in the main result of this section is the following remark

Remark. If $f \leq 0, \forall\lambda > 0$, and $u \in E_{\lambda f}(\psi)$, we have
$$\int_{\partial\Omega\cup\Omega} |\nabla u|\; dx \leq \int_{\partial\Omega\cup\Omega} |\nabla\psi|\; dx. \tag{4.57}$$

Indeed, let us note that for f, ψ defined as before, and for all $u \in E_{\lambda f}(\psi)$, we get
$$\int_{\partial\Omega\cup\Omega} |\nabla u| - \lambda \int_\Omega fu \leq \int_{\partial\Omega\cup\Omega} |\nabla\psi|\; dx - \lambda \int_\Omega f\psi\; dx,$$
which is equivalent to
$$\int_{\partial\Omega\cup\Omega} |\nabla u| \leq \int_{\partial\Omega\cup\Omega} |\nabla\psi|\; dx + \lambda \int_\Omega f(u-\psi)\; dx \leq \int_{\partial\Omega\cup\Omega} |\nabla\psi|.$$

Theorem 4.6.1. *If $f \in L^\infty(\Omega)$, $f \leq 0$ on Ω, then there exists an optimal control $\psi^* \in BV(\Omega)$ for (4.55). Moreover, the corresponding state u^* coincides with ψ^*, i.e. $u^* = \psi^*$.*

Proof of Theorem 4.6.1. In a first time we prove that there exists a pair of solutions of the form (u^*, u^*) hence $(u^* \in E_f(\psi^*))$: Let $(\psi_k)_k$ be a minimizing sequence for (4.55), ψ_k is bounded in $BV(\Omega)$, then ψ_k converges, for a subsequence, towards some $\psi^* \in BV(\Omega)$. Let $\widetilde{\psi}_k$ be the extension of ψ_k by zero in $\mathbb{R}^N \setminus \bar{\Omega}$. Then $\widetilde{\psi}_k$ is bounded in $BV(\mathbb{R}^N)$, and we can extract from it a subsequence, still denoted $\widetilde{\psi}_k$, such that
$$\widetilde{\psi}_k \rightharpoonup \psi^* \quad \text{weakly} \;\; \text{in } BV(\mathbb{R}^N)$$
obviously $\psi^* = 0$ outside of $\bar{\Omega}$. In addition, we get that u_k is bounded in $BV(\Omega)$ Then, the extension $\widetilde{u}_k$ by zero outside of $\bar{\Omega}$ converges weakly towards u^* in $BV(\mathbb{R}^N)$, such that
$$\widetilde{u}_k \rightharpoonup u^* \quad \text{weakly} \;\; \text{in } BV(\mathbb{R}^N),$$

$u^* = 0$ outside of $\bar{\Omega}$ and $u^* \geq \psi^*$.

We prove now that (u^*, u^*) is an optimal pair. Using the lower semicontinuity as in proposition 4.6.2, and (4.57), one gets

$$\begin{aligned}\int_{\partial\Omega\cup\Omega} |\nabla u^*| + \int_\Omega |u^* - z| &\leq \liminf_{k\to\infty} \left\{ \int_{\partial\Omega\cup\Omega} |\nabla u_k| + \int_\Omega |u_k - z| \right\} \\ &\leq \liminf_{k\to\infty} \left\{ \int_{\partial\Omega\cup\Omega} |\nabla \psi_k| + \int_\Omega |u_k - z| \right\} \\ &= \inf_{\substack{(u,\psi)\in(BV(\Omega))^2 \\ u\in E_f(\psi)}} \left\{ \int_{\partial\Omega\cup\Omega} |\nabla \psi| + \int_\Omega |u - z| \right\}\end{aligned}$$

Hence, (u^*, u^*) realizes (4.55).

We prove now that every $v^* \in E_f(u^*)$ is then a solution to the minimization problem (4.55). Indeed one has

$$\begin{aligned}\int_{\partial\Omega\cup\Omega} |\nabla v^*| + \int_\Omega |u^* - z| &\leq \int_{\partial\Omega\cup\Omega} |\nabla u^*| + \int_\Omega |u^* - z| \\ &= \inf_{\substack{(u,\psi)\in(BV(\Omega))^2 \\ u\in E_f(\psi)}} \left\{ \int_{\partial\Omega\cup\Omega} |\nabla \psi| + \int_\Omega |u - z| \right\}.\end{aligned}$$

This prove that (u^*, v^*) is a pair of solution. □

The case $\lambda = \lambda_f$

We shall denote as Gf any "eigenfunction" relative to the problem defined in (4.56), with $\int_\Omega fGf = 1$. Let us note that Gf satisfies

$$\begin{cases} -\mathrm{div}(\sigma(Gf)) = \lambda_f f, \text{ in } \Omega, \\ \sigma \cdot \nabla G_f = |\nabla G_f|, \text{ in } \Omega \cup \partial\Omega, \\ |\sigma|_{L^\infty} \leq 1, \\ Gf = 0, \text{ on } \partial\Omega, \end{cases}$$

Let us see that $Gf \leq 0$. Indeed, multiplying the equation by Gf^+, one gets

$$0 \leq \int_\Omega \sigma(Gf) \cdot \nabla(Gf^+) = \lambda_f \int_\Omega fGf^+ \leq 0.$$

This implies that $\nabla G f^+ = 0$ and then $G f^+ = 0$.

Let us consider now the following minimization problem

$$\mathcal{P}_{\lambda_f} = \inf_{\substack{u \in W_0^{1,1}(\Omega) \\ u \geq \psi}} \left\{ \int_\Omega |\nabla u| - \lambda_f \int_\Omega f u \right\}, \tag{4.58}$$

THEOREM 4.6.2. *Suppose that ψ is positive somewhere, then the infimum in* (4.58) *above is finite and different from zero. Moreover, there exists a solution in $BV(\Omega)$ to the following relaxed formulation*

$$\inf_{\substack{u \in BV(\Omega) \\ u \geq \psi}} \left\{ \int_\Omega |\nabla u| + \int_{\partial\Omega} |u| - \lambda_f \int_\Omega f u \right\}, \tag{4.59}$$

Proof of Theorem 4.6.2. We assume that ψ is positive somewhere, and we prove that the infimum is zero. Let u_n be a minimizing sequence, then u_n is bounded in $BV(\Omega)$, since one has that for $f \leq 0$

$$\int_\Omega |\nabla u_n| \leq \int_\Omega |\nabla \psi| - \lambda_f \int_\Omega f(u_n - \psi) \leq \int_\Omega |\nabla \psi|.$$

Let us denote by $\widetilde{u}_n$ the extension of u_n by zero outside of $\bar{\Omega}$, $\widetilde{u}_n$ is bounded in $BV(\mathbb{R}^N)$. One can extract from it a subsequence, still denoted in the same manner, such that

$$\widetilde{u}_n \rightharpoonup u \text{ weakly in } BV(\mathbb{R}^N),$$

obviously $u = 0$ outside of $\bar{\Omega}$. Using classical arguments, we prove that u realizes the infimum (4.59). If the infimum (4.58) is zero, one would get

$$\int_\Omega |\nabla u| = \lambda_f \int_\Omega f u,$$

and then $\frac{u}{\int_\Omega f u}$ realizes the following

$$\lambda_f = \inf_{\substack{v \in W_0^{1,1}(\Omega) \\ \int_\Omega f v = 1}} \int_\Omega |\nabla v|.$$

But since $f \leq 0$, and $\int_\Omega f u \geq 0$, one gets that a solution u to this problem is nonnegative. If ψ is nonnegative, this is impossible. Since ψ is positive somewhere, then the infimum is strictly positive. □

THEOREM 4.6.3. *We denote by Gf a solution of λ_f, the infimum $\mathcal{P}_{\lambda_f} = 0$ if and only if there exists some $\mu > 0$, with $\psi \leq \mu Gf$.*

Proof of Theorem 4.6.3. The infimum λ_f is positive. If there exits $\mu > 0$ such that $\psi \leq \mu Gf$, then μGf satisfies

$$\int_\Omega |\nabla \mu Gf| - \lambda_f \int_\Omega f \mu Gf = 0, \tag{4.60}$$

and then

$$\mathcal{P}_{\lambda_f}(\psi) = 0.$$

Conversely, if the infimum $\mathcal{P}_{\lambda_f} = 0$, then using the process below to prove that the infimum is achieved, one gets that there exists $u \geq \psi$ such that

$$\int_{\Omega \cup \partial\Omega} |\nabla u| = \lambda_f \int_\Omega f u.$$

This implies that $\frac{u}{\int_\Omega fu}$ is a solution for the f-eigenvalue problem defined by λ_f. Then, since $\int_\Omega fu > 0$, by the previous equation (4.60), $\mu = \frac{1}{\int_\Omega fu}$ is convenient. $\square$

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