Khokhlov-Zabolotskaya-Kuznetsov Equation. Mathematical Analyze, Validation of the Approximation and Contol Method

Anna Rozanova-Pierrat

To cite this version:

Anna Rozanova-Pierrat. Khokhlov-Zabolotskaya-Kuznetsov Equation. Mathematical Analyze, Validation of the Approximation and Contol Method. Mathematics [math]. Université Pierre et Marie Curie - Paris VI, 2006. English. NNT : . tel-00126487

HAL Id: tel-00126487
https://theses.hal.science/tel-00126487

Submitted on 25 Jan 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Laboratoire Jacques Louis Lions

THÈSE DE DOCTORAT DE L'UNIVERSITÉ PARIS 6

Spécialité

Mathématiques Appliquées

Présentée par

Mme Anna Rozanova

Pour obtenir le grade de

DOCTEUR de l'UNIVERSITÉ PARIS 6

Sujet de la thèse :

Equation de Khokhlov-Zabolotskaya-Kuznetsov

Analyse mathématique, validation de l'approximation et méthode de contrôle

Soutenue le 6 juillet 2006

devant le jury composé de :

MM.	Claude	BARDOS	Directeur de thèse
	Thierry	COLIN	Rapporteur
	François	GOLSE	
	Nikolay	TZVETKOV	Rapporteur
	Edriss	TITI	
	Mathias	FINK	
	Josselin	GARNIER	
	Alain	HARAUX	
	Mikhail Fedorovich	SUKHININ	Directeur de thèse

Université Pierre & Marie Curie - Paris VI
Bureau de la Scolarité Doctorat
Esc G, 2e étage
15 rue de l'École de Médecine
75270 PARIS Cedex 06

Tél Secrétariat : 01 42 34 68 35
Fax : 01 42 34 68 40
Tél pour les étudiants de A à ME : 01 42 34 68 41
Tél pour les étudiants de MG à Z : 01 42 34 68 51
E-mail : scolarite.doctorat@upmc.fr

À Romain mon amour.

Remerciements

Tout d'abord, je voudrais remercier mon directeur de thèse français Claude Bardos pour sa disponibilité, sa passion, ses conseils avisés et son optimisme inaltérable pendant l'attente des résultats. Merci Claude pour le support financier de ma thèse par le contrat d'ACI « Retournement Temporel » que vous codirigez avec Josselin Garnier. Je remercie également mon directeur de thèse russe Mikhail Fedorovich Sukhinin sous la direction duquel j'ai obtenu les résultats concernant la contrôlabilité (la seconde partie de ce manuscrit). Merci Mikhail Fedorovich pour votre compréhension, votre aide et vos conversations amicales.

Je remercie François Golse, Serge Alinhac, Nikolay Tzvetkov, Jeffrey Rauch, Benjamin Texier, Ilya Kostin et Grigory Panasenko pour leurs discussions et leurs suggestions si remarquables.

Je remercie encore Nikolai et Isabelle Tzvetkov pour leur si grande amitié et l'accueil chaleureux qui m'a été fait lors de ma venue à Lille.

Je remercie Ivan Straskraba et Antonin Novotny pour leur réponse par courrier électronique à mes questions.

Ce travail a été réalisé au Laboratoire Jaques-Louis Lions grâce à la bourse du président de la Fédération de Russie et en collaboration avec le Laboratoire Ondes et Acourstique de l'ESPCI de Mathias Fink et en particulier en collaboration avec Thierry Le Pollès qui a obtenu les résultats numériques sur KZK sous contrat d'ACI « Retournement Temporel », convention A020 de Ministère de la Recherche Français. Merci Thierry et Mathias pour votre collaboration et pour avoir eu les premiers l'idée de faire la recherche mathématique sur l'équation KZK.

Je remercie Josselin Garnier de m'avoir invitée au colloque à Luminy.

Je remercie Alain Haraux et Olivier Glasse pour leurs discussions intéressantes et leur intêret porté sur le résultat de la théorie de contrôle des moments.

Je fais part de mes remerciements également au directeur du laboratoire Yvon Maday pour son aide précieuse dans la résolution des problèmes administratifs.

Je remercie Madame Boulic et Madame Ruprecht pour leur chaleureux accueil au secrétariat et leur aide toujours attentive. Je remercie Henri Esser pour le traitement toujours avec le sourire de mes multiples dossiers de missions.

Je voudrais beaucoup remercier Madame Wasse et Madame Loriller pour le support moral et administratif dans les démarches d'inscription en école doctorale et la gestion du contrat de thèse en cotutelle.

Je remercie Filipa, Jean-Baptiste, Delphine, Antoine et Thérence pour leurs vraies amitiés et leurs disponibilités permanentes. Je remercie tous les thésards et membres du laboratoire pour leurs gentillesses et leurs joyeux sourires.

Je remercie mes parents Nina et Vladimir et ma grand-mère Nina pour leur amour et leurs soins pour moi.

Enfin je remercie infiniment Romain pour son immense aide pour la rédaction LaTeX de ma thèse et la vérification orthographique, ainsi que pour son si tendre amour pour moi.

Contents

List of Figures

Introduction française

Ce travail se compose de deux parties.

La **première partie** consiste en l'étude de l'équation quasi-linéaire d'acoustique non-linéaire appelée KZK (Khokhlov-Zabolotskaya-Kuznetsov). Dans le premier chapitre, l'accent est mis sur la dérivation de l'équation pour l'acoustique non-linéaire en raison de son application aux problèmes de retournement temporel en collaboration avec le LOA (Laboratoire Ondes et Acoustique) de l'ESPCI dirigé par M. Fink.

L'équation de KZK dans son interprétation initiale comme dans [11] est très étudiée par les physiciens mais jusqu'ici il n'y a aucune analyse mathématique de ce problème. Elle est utilisée dans des problèmes d'acoustique comme modèle mathématique décrivant la propagation non-linéaire d'une impulsion sonore d'amplitude finie dans un milieu thermo-visqueux (voir par exemple [1, 26, 9, 10, 33]). Plus tard elle a été employée dans plusieurs autres domaines et en particulier pour la description des grandes ondes dans les milieux ferromagnétiques [44].

L'équation KZK n'est pas intégrable contrairement à l'équation Kadomtsev-Petviashvili (KP) connue pour être intégrable. Les existences d'une onde de choc dans le cas de la propagation d'un faisceau dans un milieu non dispersif et d'une onde de quasi choc dans un milieu dispersif ont été obtenues numériquement dans [11]. Ce dernier phénomène correspond au fait que le front du faisceau de l'onde est proche de l'onde de choc mais la solution a tendance à être globale.

Nous avons obtenu la preuve de l'existence de l'onde de choc pour le problème sans viscosité. Nous avons établi l'existence globale en temps pour la propagation dans des milieux visqueux seulement pour des données initiales suffisamment petites.

Tout d'abord la dérivation de l'équation est empruntée à la littérature physique. On introduit un correcteur dans l'ansatz pour obtenir l'approximation de KZK ce qui permet de reconstruire la solution du système exact initial utilisé pour la dérivation de l'équation. Ensuite l'existence, l'unicité et la stabilité de l'équation sont analysées. De plus, un résultat sur le blow-up qui donne une limitation sur l'étendue des applications est donné comme adaptation d'un résultat de [3], [4] et [5]. En utilisant les résultats obtenus, on prouve une validité aux longs temps de l'approximation pour deux cas : pour des milieux thermoélastiques non visqueux et visqueux. Pour les besoins du résultat d'approximation, l'existence d'une solution régulière du système de Navier-Stokes isentropique dans le demi espace avec conditions aux limites périodiques et de valeur moyenne nulle en temps a été obtenue.

Plus précisément, nous étudions l'équation suivante :

$$(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0 \qquad \text{dans } \mathcal{R}_x \times \Omega, \tag{1}$$

appelée équation KZK, dans la classe des fonctions périodiques de variable x et de valeur moyenne nulle :

$$u(x+L, y, t) = u(x, y, t), \qquad \int_0^L u(x, y, t)dx = 0, \tag{2}$$

où β, γ sont des constantes positives. Ici les variables $(x, y) \in \mathcal{R}_x^+ \times (\Omega \subseteq \mathcal{R}^{n-1})$. Quand $\Omega \neq \mathcal{R}^{n-1}$, on suppose que la solution satisfait sur sa frontière la condition de Neumann.

L'equation (1) sous la forme

$$c\partial^2_{\tau z} I - \frac{(\gamma+1)}{4\rho_0}\partial^2_\tau I^2 - \frac{\nu}{2c^2\rho_0}\partial^3_\tau I - \frac{c^2}{2}\Delta_y I = 0, \tag{3}$$

avec $I(\tau, z, y) = I(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x')$, $x = (x_1, x')$, peut être obtenue à partir du système de Navier-Stokes isentropique pour les milieux visqueux

$$\partial_t \rho_\epsilon + \operatorname{div}(\rho_\epsilon u_\epsilon) = 0\,, \quad \rho_\epsilon[\partial_t u_\epsilon + (u_\epsilon \cdot \nabla)u_\epsilon] = -\nabla p(\rho_\epsilon) + \epsilon\nu\Delta u_\epsilon \tag{4}$$

avec l'équation d'état approchée

$$p = p(\rho_\epsilon) = c^2\epsilon\tilde{\rho}_\epsilon + \frac{(\gamma-1)c^2}{2\rho_0}\epsilon^2\tilde{\rho}_\epsilon^2 \tag{5}$$

et de l'équation d'Euler isentropique pour le cas non visqueux quand $\nu = \beta = 0$.

Pour les deux cas (qui sont différents parce que les domaines d'existence de la solution de KZK sont différents, dans le cas visqueux $\nu > 0$, le domaine est le demi espace $\{x_1 > 0, t > 0, x' \in \mathcal{R}^{n-1}\}$, et pour $\nu = 0$ c'est un cône), nous avons obtenu le résultat approximé pour la différence entre la solution exacte U et la fonction $\overline{U}_\epsilon = (\overline{\rho}_\epsilon, \overline{u}_\epsilon)$ définie par la solution de l'équation KZK (en utilisant l'ansatz approximé pour les ϵ suffisamment petits) périodique sur τ et de valeur moyenne nulle.

Précisément, il existe des constantes $C \geq 0$ et $T_0 = O(1)$ telles que pour tout temps fini $0 < t < T_0\frac{1}{\epsilon}\ln\frac{1}{\epsilon}$ et $\epsilon > 0$, il existe une solution régulière $U_\epsilon = (R_\epsilon, U_\epsilon)(x, t)$ du système de Navier-Stokes/Euler isentropique telle qu'on ait pour $s \geq 0$:

$$\|\overline{U}_\epsilon - U_\epsilon\|_{H^s} \leq \epsilon^{\frac{5}{2}}e^{\epsilon C t}.$$

Nous avons prouvé l'existence de la solution régulière du système de Navier-Stokes dans le demi espace avec conditions aux limites périodiques et de valeur moyenne nulle. Dans le cas visqueux, cette estimation perdure pour $s = 0$. Le temps $t < T_0\frac{1}{\epsilon}\ln\frac{1}{\epsilon}$ est le temps durant lequel $U_\epsilon - \overline{U}_\epsilon = O(\epsilon)$.

Pour prouver l'existence, l'unicité et la stabilité du résutat pour le problème de Cauchy pour l'équation KZK nous définissons l'inverse de la dérivée ∂_x^{-1} comme un opérateur agissant dans l'espace des fonctions périodiques de valeur moyenne nulle. Ceci donne la formule :

$$\partial_x^{-1}f = \int_0^x f(s)ds + \int_0^L \frac{s}{L}f(s)ds. \tag{6}$$

Cette formulation de l'opérateur ∂_x^{-1} conserve les deux propriétés de périodicité et de valeur moyenne nulle.

Dans cette situation, l'équation (1) est équivalente à

$$u_t - uu_x - \beta u_{xx} - \gamma \partial_x^{-1} \Delta_y u = 0 \quad \text{ in } \mathcal{R}_x \times \Omega. \tag{7}$$

Finalement, quand $\gamma = 0$, l'équation (1) se réduit à l'équation de Burgers-Hopf pour laquelle l'existence et l'unicité de la solution régulière est bien connue. Pour $\gamma = \beta = 0$, cela se réduit à l'équation de Burgers

$$\partial_t u - \partial_x \frac{u^2}{2} = 0,$$

qui laisse apparaître des singularités après un temps fini. Après le temps de choc, la solution peut être prolongée en une solution faible unique satisfaisant une condition élémentaire d'entropie (dans le cas actuel avec $\gamma \neq 0$, il semble que cette méthode ne puisse pas être adaptée à l'équation (1) avec $\beta = 0$ et $\gamma \neq 0$).

Notons également que la méthode de type J. Bourgain et l'introduction des espaces de Bourgain comme dans [46, 35, 36] et d'autres ne sont pas utiles pour le problème de KZK en raison de l'absence des termes avec derivée impaire comme par exemple u_{xxx} dans (7). La présence seulement de la seconde derivée rend impossible les principales estimations et égalités de cette méthode.

Il y a les travaux mathématiques [28], [29] pour l'équation de type KZK

$$\alpha u_{z\tau} = (f(u_\tau))_\tau + \beta u_{\tau\tau\tau} + \gamma u_\tau + \triangle_x u,$$

où $u_\tau = u_\tau(z, x, \tau)$ est la pression acoustique, $(z, x) \in \mathcal{R}^d \times \mathcal{R}$, $d = 1, 2$ sont des variables d'espace et τ est le temps retardé. L'équation est étudiée en supposant que la non-linéarité f ait des dérivées bornées ce qui permet de prouver l'existence globale pour le cas où les coefficients sont des fonctions fortement oscillantes de z. Ainsi ce problème n'est pas relié à notre problème « acoustique » pour l'équation KZK où il y a un choc illustrant l'existence d'une onde de choc comme nous le verrons plus tard.

Pour prouver l'existence du résultat pour l'équation KZK intégrée (7), nous déduisons en premier lieu une estimation à priori :

$$\text{Pour } s > [\frac{n}{2}] + 1 \quad \frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \beta\|\partial_x u\|_s^2 \leq C(s)\|u\|_s^3 \tag{8}$$

$$\text{et } \frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \beta C(L)\|u\|_s^2 \leq C(s)\|u\|_s^3. \tag{9}$$

L'estimation (8) est également vraie dans le cas non périodique et donne l'existence locale en temps. Mais la relation (9) perdure seulement dans le cas périodique et pas sur la droite des réels en entier (dans ce dernier cas, la norme H^s de $\partial_x u$ ne contrôle pas la norme H^s de u). Grâce à (9), nous prouvons l'existence globale en temps de la solution du problème de Cauchy $u \in C([0, T[, H^s) \cap C^1([0, T[, H^{s-2})$ $(s > [\frac{n}{2}] + 1)$ pour des données initiales suffisamment petites $u_0 \in H^s$.

La preuve est donnée par deux méthodes : par la méthode des pas fractionnaires dans le cas partiel $\mathcal{R}^3$ et $s = 3$ pour justifier les résultats numériques de Thierry Le Pollès (sous contrat ACI « Retournement Temporel », convention A020 du Ministère de la Recherche Français) et par la théorie des semigroupes et la meéthode de Kato dans le cas général.

Nous avons également établi en utilisant la méthode d'Alinhac le résultat du choc géométrique pour la première dérivée de la solution de l'équation KZK (1) avec $\beta = 0$.

D'une part nous notons que pour $\nu = 0$ (ou $\beta = 0$) et pour la fonction u indépendante de y, l'équation KZK

$$(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0 \quad \text{dans } \mathcal{R}_{t_+} \times \mathcal{R}_x \times \Omega \tag{10}$$

devient l'équation de Burgers qui est connue pour exhiber des singularités. D'autre part, la dérivation et les résultats d'approximation prouvent que n'importe quelle solution de l'équation KZK a dans son voisinage une solution de l'équation d'Euler isentropique. De nouveau on sait qu'une telle solution, même avec des données initiales régulières, peut laisser apparaître des singularités (cf. [15] ou [47]). Ces observations sont renforcées par le fait que pour $\beta = 0$ et $\gamma > 0$ l'équation (10) peut produire des singularités.

Nous prouvons le résultat du choc géométrique en utilisant la méthode d'Alinhac, qui est basée sur le fait que l'équation étudiée dégénère en l'équation de Burgers. En fait la méthode d'Alinhac est la méthode des caractéristiques généralisée pour l'équation de Burgers adaptée au cas multidimensionnel. Comme nous pouvons le voir, l'équation (10) possède toutes ces propriétés principales, et justifie son application.

Par conséquent, une équation du type (10) est introduite par Alinhac pour analyser le choc (dans $\mathcal{R}^{2+1}$) de l'équation d'ondes non-linéaire multidimensionnelle en suivant le cône d'onde

$$\partial_t^2 u - \triangle_x u + \sum_{0 \le i,j,k \le 2} g_{ij}^k \partial_k u \partial_{ij}^2 u = 0,$$

où

$$x_0 = t, \quad x = (x_1, x_2), \quad g_{ij}^k = g_{ji}^k,$$

avec des données initiales régulières et petites (voir [6]). En fait ceci correspond à la même échelle que l'équation KZK parce qu'à partir de cette équation d'ondes avec des changements de variables et en utilisant des approximations, Alinhac obtient (voir [4, 6, 7])

$$\partial_{xt}^2 u + (\partial_x u)(\partial_x^2 u) + \epsilon \partial_y^2 u = 0.$$

C'est la raison de l'analogie.

En particulier, nous avons le théorème suivant :

Theorem 1 *L'équation*

$$(u_t - uu_x)_x - \gamma \Delta_y u = 0 \quad \textit{dans } \mathcal{R}_{t_+} \times \mathcal{R}_x \times \Omega \tag{11}$$

avec la condition aux limites de Neumann sur $\partial\Omega$ n'a pas de solution régulière globale en temps si

$$\sup_{x,y} \partial_x u(x, y, 0)$$

est suffisamment grand par rapport à γ.

Comme nous pouvons le voir d'après [11], le résultat du théorème confirme parfaitement les résultats numériques. En pratique, à partir des figures I.7 et I.8, on observe que plus β devient petit (pour $\beta \to 0$), plus l'équation KZK a un quasi choc s'approchant de l'onde de choc, en lequel il dégénère pour $\beta = 0$.

La description des résultats obtenus sur l'équation KZK a été publiée dans [12, 13, 14].

L**a seconde partie** se compose des résultats sur la théorie de la contrôlabilité « des moments » pour l'équation non-linéaire abstraite d'évolution, pour l'équation quasi-linéaire de la chaleur et pour l'équation linéarisée KZK.

Le résultat pour l'équation non-linéaire abstraite (sans regarder la généralité) explique clairement la méthode de résolution de classe de problèmes de contrôlabilité pour des équations quasi-linéaires avec surdétermination intégrale. L'idée première pour obtenir ce résultat général consiste à éliminer les détails de la démonstration des cas particuliers et à essayer d'écrire un schéma de la méthode.

Cette méthode nous permet de prouver l'existence locale du problème inverse non-linéaire et de préciser la dimension du voisinage dans lequel il est possible de prendre une fonction à partir de la condition de surdétermination telle que le problème non-linéaire inverse initial ait une solution unique.

La méthode est basée sur les propriétés connues des solutions des problèmes linéaires (directs et inverses) et utilise deux fois le théorème des fonctions inverses dans l'espace fonctionnel correspondant. Dans un des deux cas, l'espace des antécédents est défini comme l'ensemble des solutions du problème correspondant. Après nous utilisons deux fois le raffinement du théorème des fonctions inverses (présenté sous sa forme la plus générale dans [48]) pour obtenir des conditions suffisantes pour que le problème inverse initial soit uniquement résolvable en terme de taille du voisinage dans lequel on peut choisir la fonction à partir de la condition de surdétermination.

Le résultat pour l'équation de la chaleur semilinéaire donne l'exemple de l'application de cette méthode, mais il ne peut pas être inclus sous la forme d'un théorème abstrait, en raison de la différence de la condition de surdétermination qui consiste dans ce cas-ci en l'intégrale spatiale connue à chaque instant et en raison de la dépendance uniquement en temps de la fonction de contrôle. Mais la théorie abstraite a été construite pour l'intégrale en temps comme condition de surdétermination et pour la fonction de contrôle dépendant de la variable d'espace.

Dans le dernier chapitre nous envisageons le problème de la contrôlabilité pour l'équation KZK linéarisée commençant par le problème linéaire et nous expliquons les difficultés pour obtenir le contrôle de l'équation KZK complète.

Plus précisément, notre but est de contrôler « un moment » sur la solution d'une équation d'évolution dans l'espace de Hilbert (ou l'espace de Banach réflectif) avec un opérateur A générateur infinitésimal du semigroupe C_0 de contractions.

Nous commençons à partir des problèmes de contrôle linéaires pour des équations de la forme :

$$u'_t(x,t) - (Au)(x,t) = f,$$

$$f \text{ est un contrôle et une fonction d'un seul argument } x \text{ ou } t,$$

ce qui est bien posé, c.-à-d. qu'il existe une unique fonction de contrôle telle que le problème direct soit uniquement résolvable et nous avons besoin également d'une estimation à priori pour la solution du problème direct linéaire (le problème de Cauchy ou le problème aux limites).

Les résultats des problèmes linéaires directs et inverses pour les cas des chapitres IV et V ont déjá été obtenus. Les résultats des problèmes inverses linéaires des chapitres IV et V ont été obtenus respectivement par Prilepko A.I., Tikhonov I.V. [40] et Kostin, A.B. [27].

Nous considérons deux problèmes modèles linéaires :

Problème 1.

$$u'(x,t) - [Au](x,t) = f(x) \qquad (0 \le t \le T), \tag{12}$$

$$u|_{t=0} = 0, \tag{13}$$

Nous nous proposons de trouver le contrôle f pour que

$$\int_0^T u(x,t)d\mu(t) = \psi(x) \qquad (\psi(x) \text{ et } \mu(t) \text{ sont donnés}), \tag{14}$$

et

Problème 2.

$$u_t - Au = h(x)f(t), \quad h(x) \text{ est donné}, \quad (x,t) \in \Omega \times [0,T], \tag{15}$$

$$u|_{t=0} = 0, \tag{16}$$

$$u|_{S_T} = 0, \quad S_T = \partial\Omega \times [0,T], \tag{17}$$

nous nous proposons de trouver le contrôle f pour que

$$\int_\Omega u(x,t)\omega(x)dx = \chi(t), \tag{18}$$

où ω et χ sont donnés.

Ici, pour la clarté, il y a des modèles simplifiés des problèmes que nous avons envisagés pour les membres de droite des équations dépendant du temps et de la variable d'espace de la forme :

$$h(x,t)f(x), \quad \text{or} \quad h(x,t)f(t).$$

Pour le problème 2, nous prenons le produit scalaire sur $\omega(x)$ dans $L_2(\Omega)$ et nous obtenons l'équation

$$\chi'(t) + (u(t), A^*\omega) = f(t)(h,\omega),$$

d'où nous tirons

$$f(t) - Kf(t) = \frac{\chi'(t)}{(h,\omega)}$$

avec

$$Kf(t) = \frac{1}{(h,\omega)} \int_0^t (e^{-(t-s)A}h, w) f(s) ds.$$

C'est pourquoi nous supposons que $(h,\omega)_{L_2(\Omega)}$ est strictement positif.

Pour le problème 1, nous appliquons l'opérateur de surdétermination $l : u(t) \mapsto \int_0^T u d\mu(t)$ à l'équation (15) et nous obtenons pour un opérateur A clos linéaire indépendant du temps

$$\int_0^T u'(t) d\mu(t) + A\left(\int_0^T u(s) d\mu(s)\right) = f(x) \int_0^T d\mu(t),$$

avec l'hypothèse $\int_0^T d\mu(t) > 0$.

Nous avons encore

$$f(x) - Kf(x) = \frac{A\psi(x)}{\int_0^T d\mu(t)}$$

avec

$$Kf(x) = \frac{\int_0^T e^{-tA} f(x) d\mu(t)}{\int_0^T d\mu(t)}.$$

Dans le second cas, le problème 2 peut être résolu (pour établir l'existence et l'unicité) en montrant qu'avec la norme suivante dans L_2

$$\|u\|^2_{L_2(0,T)} = \int_0^T \exp(-\beta t) \left|u(t)\right|^2 dt,$$

où le nombre $\beta > 0$ est à définir, l'opérateur K est une contraction stricte. Ce type de démonstration est décrit en détail dans la preuve du théorème 27 du chapitre VI pour la partie linéaire de l'équation KZK. Pour le problème 1, cela découle du fait qu'on démontre que le problème est bien posé (en utilisant l'alternative de Fredholm).

Nous utilisons le fait que les problèmes linéaires directs et inverses sont bien posés pour établir le résultat du contrôle de leur petite perturbation non-linéaire

$$u'(t) - (Au)(t) - G(u)(t) = f.$$

La non-linéarité G doit être strictement différentiable au sens de Fréchet et doit satisfaire les hypothèses principales $G(0) = 0$ et $G'(0) = 0$, qui sont suffisantes pour l'existence de la solution locale.

Nous prouvons le résultat de l'existence de la solution locale à l'aide de la double application du thèoreme des fonction inverses 16. La première fois nous l'appliquons pour prouver l'existence locale et l'unicité de la solution du problème direct non-linéaire et nous avons besoin d'introduire l'espace des solutions du problème linéaire initial ou aux limites H

$$H = \{v \in X | \exists F \in LH : v \quad \text{est une solution du problème linéaire initial ou aux limites}\}.$$

L'espace H est toujours relié à l'opérateur linéaire

$$L = d/dt - A, \quad Lu = F, \tag{19}$$

qui, grâce au fait connu que pour tout $F \in LH$, il existe une unique solution $u \in X$ et grâce à l'estimation à priori pour le problème direct linéaire initial ou aux limites

$$\|u\|_X \leq C\|F\|_{LH},$$

induit un isomorphisme isométrique de H sur LH avec la norme $\|u\|_H = \|Lu\|_{LH}$. Par conséquent, l'espace $(H, \|\cdot\|_H)$ est un espace de Banach ; de plus, grâce à l'estimation à priori, H est continûment inclus dans un espace initial X des fonctions u.

Cela nous permet d'appliquer le théorème des fonctions inverses 16 à l'application $\xi : u(t) \mapsto Lu(t) - Gu(t)$ qui est un difféomorphisme local de classe C^1 dans un voisinage de zéro de U' dans H sur un voisinage de zéro de V' dans LH.

En utilisant maintenant l'application inverse $\eta = \xi^{-1} : V' \to U'$ à ce difféomorphisme local, c.-à-d. $\eta : F \longmapsto u$ où u est une solution de l'équation non-linéaire et η est strictement différentiable sur V', et en appliquant l'opérateur de surdétermination à l'équation non-linéaire, nous démontrons que le problème inverse non-linéaire est équivalent à l'équation opératorielle

$$M(f) = \chi'.$$

Nous prouvons que M est strictement différentiable au sens de Fréchet dans un voisinage de zéro et, grâce à $G'(0) = 0$, $M'(0) = I - K$, où $I - K$ est l'opérateur du problème linéaire $(I - K)f = \chi'$. Comme le problème linéaire est bien posé, il existe $(M'(0))^{-1}$ et $\|(M'(0))^{-1}\| \leq 1/(1 - \left\|\hat{A}\right\|)$. Donc, en appliquant une nouvelle fois le théorème des fonctions inverses, nous concluons qu'il existe un voisinage ouvert de zéro U et V tels que M induise un difféomorphisme de classe C^1. Cela termine la démonstration de la contrôlabilité locale.

La deuxième question à considérer sur la contrôlabilité consiste à trouver les conditions suffisantes sur la taille du voisinage dans lequel la fonction de la condition de surdétermination peut être choisie de sorte que le problème inverse initial soit uniquement résolvable. Nous faisons ceci en trois étapes en utilisant le raffinement du theorème des fonctions inverses 18 de [49].

Nous établissons qu'avec une hypothèse additionnelle sur G

$$\|G'(u)\|_{\mathcal{L}(H,LH)} \leq \tilde{\vartheta}(r) \quad \text{pour } \|u\|_H \leq r,$$

où $\tilde{\vartheta} : [0, \infty[\to [0, \infty[$ est une fonction monotone non-décroissante,

1. pour le problème direct non-linéaire :
   $\forall r \in [0, r_*[\quad \forall\, F \in w(r)LU_H \quad \exists\,!\, u \in rU_H : Lu - Gu = F \quad \Leftrightarrow \quad \xi(u) = F$, où r_* est la racine de l'équation $1 - C_0\tilde{\vartheta}(r) = 0$ et C_0 est une constante d'une inclusion. La fonction $w(r)$ est calculée en terme de $\tilde{\vartheta}(r)$.

2. pour le problème en langage des opérateurs inverses :
   $\forall r' \in [0, r'_*[\ \forall v \in W(r')U_H\ \exists!\, F \in r'U_{LH} : L^{-1}F - Q(F) = v \quad \Leftrightarrow \quad \eta(F) = v$, où r'_* est la racine de $1 - 2C_0\vartheta(r) = 0$ et alors $r'_* < r_*$.
3. pour le problème inverse non-linéaire :
   $\forall\, \tilde{r} \in [0, \tilde{r}_*[\ \forall\, \chi' \in \hat{W}(\tilde{r})(I-K)U_E\ \exists!\, \varphi \in \tilde{r}U_E : M(f) = \chi'$ où $\tilde{r}$ est calculé en fonction de r'.

Les chapitres IV et V incluent ces deux types de résultats. Le but du résultat abstrait du chapitre IV est d'éliminer les détails techniques multiples des cas concrets pour expliquer le plus simplement possible cette méthode pour le problème modèle 1. Les résultats du chapitre IV peuvent être lus dans [42, 43]. Dans le chapitre V, il y a le résultat (publié dans [41]) pour cette technique dans le cas de l'équation de la chaleur correspondant au problème modèle 2.

Le chapitre VI présente l'application des idées du contrôle du moment a l'équation KZK étudiée dans la partie I. Nous avons obtenu le résultat sur la contrôlabilité pour l'équation KZK linéarisée. Ensuite, nous expliquons les difficultés et pourquoi la méthode n'est pas applicable à l'équation KZK non-linèaire. Cela se produit en raison de l'impossibilité de montrer que l'opérateur non-linéaire

$$\Phi(u) = uu_x, \quad \Phi : H^1((0,T); H^{s-2}(\Omega)) \to L_2((0,T); H^s(\Omega))$$

est strictement différentiable.

English introduction

This work consists of two parts.

The first part consists in studying the quasilinear equation of nonlinear acoustic named KZK (the Khokhlov-Zabolotskaya-Kuznetsov equation). In the first chapter the emphasis is put on the derivation of the equation for nonlinear acoustic in view of application to time reversal problems in collaboration with the LOA laboratory in ESPCI directed by M. Fink.

The KZK equation in its initial interpretation as in [11] is mostly studied by physicists but until now there is no mathematical analysis of this problem. It is using in acoustical problems as a mathematical model that describes the pulse finite amplitude sound beam nonlinear propagation in the thermo-viscous medium (see for example [1, 26, 9, 10, 33]). Later it has been used in several other fields and in particular in the description of long waves in ferromagnetic media [44].

The KZK equation is not an integrable one at variance Kadomtsev-Petviashvili (KP) equation known to be integrable. The existence of a shock wave in the case of propagation of the beam in nondissipative media and a quasi shock wave for the dissipative media have been obtained numerically in [11]. This last phenomenon corresponds to the fact that the beam's front is approaching the shock wave but the solution has the tentative to be global.

We have obtained the proof of the existence of the shock wave for the problem without viscosity. We have established the global existence in time of the propagation in viscous media only for sufficient small initial data.

First the derivation of the equation is borrowed from physical literature. We introduce a corrector in the ansatz to obtain the approximation of the KZK equation. This allows to reconstruct the solution of the initial exact system used for derivation. Then the existence, uniqueness and stability of the equation are analyzed. Moreover a blow-up result which gives a limitation to the range of application is given as an adaptation of a result of [3], [4] and [5]. Using obtained results one proves a large time validity of the approximation for two cases: for non viscous and viscous thermoellastic media. For the needs of the approximation result the existence of a regular solution of the isentropic Navier-Stokes system in the half space with periodic and mean zero in time boundary conditions was obtained.

More precisely, we study the following equation:

$$(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0 \quad \text{in } \mathcal{R}_x \times \Omega, \tag{20}$$

named KZK equation, in the class of periodic functions with respect to the variable x and which are of mean value zero:

$$u(x+L,y,t)=u(x,y,t), \quad \int_0^L u(x,y,t)dx=0, \tag{21}$$

where β, γ are some positive constants.

Here the variables $(x,y)\in\mathcal{R}_x^+\times(\Omega\subseteq\mathcal{R}^{n-1})$. When $\Omega\neq\mathcal{R}^{n-1}$ it is assumed that the solution satisfies on its boundary the Neumann boundary condition.

The equation (20) in the form

$$c\partial^2_{\tau z}I-\frac{(\gamma+1)}{4\rho_0}\partial^2_\tau I^2-\frac{\nu}{2c^2\rho_0}\partial^3_\tau I-\frac{c^2}{2}\Delta_y I=0, \tag{22}$$

with $I(\tau,z,y)=I(t-\frac{x_1}{c},\epsilon x_1,\sqrt{\epsilon}x')$, $x=(x_1,x')$, can be obtained from the isentropic Navier-Stokes system for the viscous media

$$\partial_t\rho_\epsilon+\mathrm{div}(\rho_\epsilon u_\epsilon)=0\,, \quad \rho_\epsilon[\partial_t u_\epsilon+(u_\epsilon\cdot\nabla)u_\epsilon]=-\nabla p(\rho_\epsilon)+\epsilon\nu\Delta u_\epsilon \tag{23}$$

with the approximate state equation

$$p=p(\rho_\epsilon)=c^2\epsilon\tilde{\rho}_\epsilon+\frac{(\gamma-1)c^2}{2\rho_0}\epsilon^2\tilde{\rho}^2_\epsilon \tag{24}$$

and from isentropic Euler equation for non viscous case when $\nu=\beta=0$.

For both cases (which are different because the domains of the existence of the KZK solution are different, in the viscous case $\nu>0$ the domain is the half space $\{x_1>0,t>0,x'\in\mathcal{R}^{n-1}\}$, and for $\nu=0$ it is a cone) we have obtained the approximation result for the difference between the exact solution U and the function $\overline{U}_\epsilon=(\overline{\rho}_\epsilon,\overline{u}_\epsilon)$ defined by the solution of KZK equation (using the approximation ansatz for rather small ϵ) periodic on τ and of mean value zero.

Precisely there exist constants $C\geq 0$ and $T_0=O(1)$, such that for any finite time $0<t<T_0\frac{1}{\epsilon}\ln\frac{1}{\epsilon}$ and $\epsilon>0$, there exists a smooth solution $U_\epsilon=(R_\epsilon,U_\epsilon)(x,t)$ of the isentropic Navier-Stokes/Euler system such that one has for some $s\geq 0$:

$$\|\overline{U}_\epsilon-U_\epsilon\|_{H^s}\leq\epsilon^{\frac{5}{2}}e^{\epsilon Ct}.$$

We have proved the existence of the smooth solution of Navier-Stokes system in the half space with periodic and mean value zero boundary conditions. In the viscous case the estimation holds with $s=0$. The time $t<T_0\frac{1}{\epsilon}\ln\frac{1}{\epsilon}$ is the time during which $U_\epsilon-\overline{U}_\epsilon=O(\epsilon)$.

To prove the existence, uniqueness and stability result for the Cauchy problem for KZK equation we define the inverse of the derivative ∂_x^{-1} as an operator acting in the space of periodic functions with mean value zero. This gives the formula:

$$\partial_x^{-1}f=\int_0^x f(s)ds+\int_0^L\frac{s}{L}f(s)ds. \tag{25}$$

This form of the operator ∂_x^{-1} preserves the both qualities: the periodicity and the mean value zero.

In this situation equation (20) is equivalent to the equation

$$u_t - uu_x - \beta u_{xx} - \gamma \partial_x^{-1} \Delta_y u = 0 \quad \text{ in } \mathcal{R}_x \times \Omega. \tag{26}$$

Finally when $\gamma = 0$ equation (20) reduces to the Burgers-Hopf equation for which existence smoothness and uniqueness of the solution are well known. For $\gamma = \beta = 0$ it reduces to the Burgers equation

$$\partial_t u - \partial_x \frac{u^2}{2} = 0,$$

which after a finite time exhibits singularities. After this "blow-up" time the solution can be uniquely continued into a weak solution satisfying an elementary entropy condition (in the present case with $\gamma \neq 0$, it seems that this construction cannot be adapted to equation (20) with $\beta = 0$ and $\gamma \neq 0$).

We also would like to notice that the J. Bourgain-type method and the introduction of the Bourgain spaces as in [46, 35, 36] and others are not useful for the KZK problem because of absence of the terms with an odd derivative as for example u_{xxx} in (26). The presence only of the second derivative makes impossible the main estimations and equalities of this method.

There are the mathematical works [28], [29] for KZK type equation

$$\alpha u_{z\tau} = (f(u_\tau))_\tau + \beta u_{\tau\tau\tau} + \gamma u_\tau + \triangle_x u,$$

where $u_\tau = u_\tau(z, x, \tau)$ is the acoustic pressure, $(z, x) \in \mathcal{R}^d \times \mathcal{R}$, $d = 1, 2$ are space variables and τ is the retarded time. The equation is studied with the hypothesis that the nonlinearity f has bounded derivative which allows to proof the global existence for the case when the coefficients are rapidly oscillating functions of z. So this problem is not related with our "acoustical" problem for the KZK equation where as we will see later there is a blow-up result illustrating the existence of a shock wave.

To prove the existence result for the integrated KZK equation (26) we firstly deduce the a priori estimates:

$$\text{For } s > [\frac{n}{2}] + 1 \quad \frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \beta\|\partial_x u\|_s^2 \leq C(s)\|u\|_s^3 \tag{27}$$

$$\text{and } \frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \beta C(L)\|u\|_s^2 \leq C(s)\|u\|_s^3. \tag{28}$$

The estimate (27) is also true in the nonperiodic case and gives the local existence in time. But the relation (28) holds only in the periodic case and not on the whole line (in this later case the H^s norm of $\partial_x u$ does not control the H^s norm of u). Thanks to (28) we prove the global existence in time of the solution of the Cauchy problem $u \in C([0, T[, H^s) \cap C^1([0, T[, H^{s-2})$ $(s > [\frac{n}{2}] + 1)$ for rather small initial data $u_0 \in H^s$.

The proof is given by two methods: by the fractional step method in the partial case $\mathcal{R}^3$ and $s = 3$ for justifying the numerical results of Thierry Le Pollès (during the grant of ACI "Retournement Temporel", convention A020 du Ministère de la Recherche Français) and by the theory of semigroups and the method of Kato in general case.

We have also established using the method of Alinhac the geometric blow-up result for the first derivative of the solution of KZK equation (20) with $\beta = 0$.

On the one hand we notice that for $\nu = 0$ (or $\beta = 0$) and for the function u independent of y the KZK equation

$$(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0 \quad \text{in } \mathcal{R}_{t_+} \times \mathcal{R}_x \times \Omega \tag{29}$$

becomes Burgers equation which is known to exhibit singularities. On the other hand the derivation and the approximation results show that any solution of the KZK equation has in its neighborhood a solution of the isentropic Euler equation. Once again it is known that such solution even with smooth initial data may exhibit singularities (cf. [15] or [47]). These observations are reflected by the fact that for $\beta = 0$ and $\gamma > 0$ the equation (29) may generate singularities.

We prove the geometric blow-up result using the method of Alinhac, which is based on the fact that the studied equation degenerates to the Burgers equation. In fact Alinhac's method is the generalized method of characteristics for the Burgers equation adapted to the multidimensional case. As we can see the equation (29) possess all this main properties, and gives us the reason to apply it.

Therefore, an equation of the type (29) is introduced by Alinhac to analyze the blow-up of multidimensional (in $\mathcal{R}^{2+1}$) nonlinear wave equation by following the wave cone

$$\partial_t^2 u - \triangle_x u + \sum_{0 \le i,j,k \le 2} g_{ij}^k \partial_k u \partial_{ij}^2 u = 0,$$

where

$$x_0 = t, \quad x = (x_1, x_2), \quad g_{ij}^k = g_{ji}^k,$$

with small smooth initial data (see [6]). In fact this corresponds to the same scaling as the KZK equation because from this wave equation with some changes of variable and approximate manipulations Alinhac obtains (see [4, 6, 7])

$$\partial_{xt}^2 u + (\partial_x u)(\partial_x^2 u) + \epsilon \partial_y^2 u = 0.$$

This is the reason for the analogy.

For instance one has the theorem:

Theorem 2 *The equation*

$$(u_t - uu_x)_x - \gamma \Delta_y u = 0 \quad \textit{in } \mathcal{R}_{t_+} \times \mathcal{R}_x \times \Omega \tag{30}$$

with Neumann boundary condition on $\partial\Omega$ has no global in time smooth solution if

$$\sup_{x,y} \partial_x u(x, y, 0)$$

is large enough with respect to γ.

As we can see from [11] the result of the theorem perfectly confirms the numerical results. In practically from figures I.7 and I.8 one observes that the more β becomes smaller (for

$\beta \to 0$), the more the KZK equation has a quasi shock approaching to the shock wave, into which it degenerates for $\beta = 0$.

The description of the results for KZK equation was published in [12, 13, 14].

The second part consists of results on controllability theory "of moments" for abstract nonlinear evolution equation, for quasilinear heat equation and for linearized KZK equation.

The result for the abstract nonlinear equation (without looking at the generality) clearly shows the method of resolution of a class controllability problems for quasilinear equations with integral overdetermination. The initial idea to obtain this general result is to eliminate the details of proof in a particular case and to try to write a schema of the method.

This method allows to prove the local existence of the inverse nonlinear problem and to specify the dimension of the neighborhood from which it is possible to take a function in condition of overdetermination such that the initial inverse nonlinear problem is uniquely resolvable.

The method is based on known properties of solutions of the linear (direct and inverse) problems and uses twice the inverse function theorem in the corresponding function spaces. In one of the two cases, the space of preimages is defined as the solution set of the corresponding problem. After it we use twice the refined inverse function theorem (represented in the most general form in [48]) to obtain sufficient conditions for the unique solvability of the original inverse problem in terms of the size of the neighborhood from which the function from the overdertermination condition can be taken so that the original inverse problem will be uniquely solvable.

The result for the semilinear heat equation gives the example of application of this method, but it cannot be included in the form of the abstract theorem, because of the difference of the condition of overdetermination which consists in this case of the space integral known in every moment of time and because the control function depends (only) on time. But the abstract theory has been constructed for the integral in time as a condition of overdetermination and for the control function depending on the space variable.

In the last chapter we envisage the controllability problem for linearized KZK equation starting with the linear problem and we explain the difficulties to obtain the control of full KZK equation.

More precisely, our goal is to control "a moment" on the solution of an evolution equation in Hilbert space (or reflective Banach space) with an operator A infinitesimal generator of a C_0-semigroup of contractions.

We start from the linear control problems for equations of the form:

$$u'_t(x,t) - (Au)(x,t) = f,$$

f is a control and is a function of only one argument x or t, which are well-posed, i.e. there exists a unique control function such that the direct problem is uniquely solvable and we also need an a priori estimate for the solution of the linear direct problem (the Cauchy problem or the boundary initial value problem).

The results of direct and inverse linear problems for the cases of chapter IV and chapter V have been already obtained. The results of linear inverse problem from chapter IV and chapter V have been obtained by Prilepko A.I., Tikhonov I.V. [40] and Kostin, A.B. [27] respectively.

We consider two model linear problems:

Problem 1.

$$u'(x,t) - [Au](x,t) = f(x) \qquad (0 \le t \le T), \tag{31}$$

$$u|_{t=0} = 0, \tag{32}$$

we offer to find the control f to accomplish

$$\int\limits_0^T u(x,t)d\mu(t) = \psi(x) \qquad (\psi(x) \text{ and } \mu(t) \text{ are given}), \tag{33}$$

and

Problem 2.

$$u_t - Au = h(x)f(t), \quad h(x) \text{ is given}, \quad (x,t) \in \Omega \times [0,T], \tag{34}$$

$$u|_{t=0} = 0, \tag{35}$$

$$u\big|_{S_T} = 0, \quad S_T = \partial\Omega \times [0,T], \tag{36}$$

we offer to find the control f to accomplish

$$\int\limits_\Omega u(x,t)\omega(x)dx = \chi(t), \tag{37}$$

with ω, χ given.

Here for the clarity, there are simplified models of the problems, which we have envisaged for the right-hand sides of equations depending on the time and the space variables of the form:

$$h(x,t)f(x), \quad \text{or} \quad h(x,t)f(t).$$

For the problem 2 we take the inner product on $\omega(x)$ in $L_2(\Omega)$ and we obtain the equation

$$\chi'(t) + (u(t), A^*\omega) = f(t)(h,\omega),$$

from where we have

$$f(t) - Kf(t) = \frac{\chi'(t)}{(h,\omega)}$$

with

$$Kf(t) = \frac{1}{(h,\omega)} \int_0^t (e^{-(t-s)A}h, w)f(s)ds.$$

So we suppose that $(h,\omega)_{L_2(\Omega)}$ is strictly positive.

For the problem 1 we apply the operator of overdetermination $l : u(t) \mapsto \int_0^T u d\mu(t)$ to the equation (34) and we obtain for a linear closed operator A not depending from time

$$\int_0^T u'(t)d\mu(t) + A\left(\int_0^T u(s)d\mu(s)\right) = f(x)\int_0^T d\mu(t),$$

with assumption that $\int_0^T d\mu(t) > 0$.

We have again

$$f(x) - Kf(x) = \frac{A\psi(x)}{\int_0^T d\mu(t)}$$

with

$$Kf(x) = \frac{\int_0^T e^{-tA} f(x)d\mu(t)}{\int_0^T d\mu(t)}.$$

In the second case the problem 2 can be resolved (to establish the existence and the uniqueness) showing that with the proper norm in L_2

$$\|u\|^2_{L_2(0,T)} = \int\limits_0^T \exp(-\beta t)\,|u(t)|^2\,dt,$$

where the number $\beta > 0$ is to define, the operator K is a strict contraction. This kind of proof is in details in the proof of the theorem 27 of chapter VI for linear part of KZK equation. For the problem 1 it follows from the proof of its well-posedness (using the alternative of Fredholm).

We use the properties of well-posedness for the direct and inverse linear problems to make the control result of their small nonlinear perturbation

$$u'(t) - (Au)(t) - G(u)(t) = f.$$

The nonlinearity G must be strictly differentiable in the sense of Fréchet and must satisfy the main hypotheses that $G(0) = 0$ and $G'(0) = 0$, which are sufficient for local solvability.

We prove the local solvability result with the help of two times application of the inverse function theorem 16. The first time we apply it to prove the local existence and the uniqueness of the solution of nonlinear direct problem which needs to introduce the space of solutions of linear initial/boundary problem H

$$H = \{v \in X | \exists F \in LH : v \quad \text{is a solution of linear initial/boundary problem}\}.$$

The space H is always related with the linear operator

$$L = d/dt - A, \quad Lu = F, \tag{38}$$

which, thanks to the known fact that for all $F \in LH$ there exists a unique solution $u \in X$ and thanks to a priory estimate for the direct linear initial/boundary problem

$$\|u\|_X \leq C\|F\|_{LH},$$

induces an isometric isomorphism of H on LH with norm $\|u\|_H = \|Lu\|_{LH}$. Therefore, the space $(H, \|\cdot\|_H)$ is Banach; moreover, by the a priori estimate H is continuously embedded in some initial space X of functions u.

This allows us to apply the inverse function theorem 16 to the map $\xi : u(t) \mapsto Lu(t) - Gu(t)$ which is a local diffeomorphism of class C^1 in a neighborhood of zero of U' in H onto a neighborhood of zero of V' in LH.

Using now the inverse mapping $\eta = \xi^{-1} : V' \to U'$ to this local diffeomorphism, i.e., $\eta : F \longmapsto u$, where u is a solution of the nonlinear equation and η is strictly differentiable on V', and applying the operator of overdetermination to the nonlinear equation we prove that the nonlinear inverse problem is equivalent to the operator equation

$$M(f) = \chi'.$$

We prove that M is strictly differentiable in the sense of Fréchet in a neighborhood of zero and, thanks to $G'(0) = 0$, $M'(0) = I - K$, where $I - K$ is the operator of the linear problem $(I - K)f = \chi'$. Since the linear problem is well defined, there exist $(M'(0))^{-1}$ and $\|(M'(0))^{-1}\| \leq 1/(1 - \left\|\hat{A}\right\|)$. So applying now again the inverse function theorem, we conclude that there exist open neighborhoods of zero U and V such that M induces a diffeomorphism of class C^1. This finishes the prove of local controllability.

The second considered question on the controllability is to find the sufficient conditions on the size of the neighborhood from which the function from the overdertermination condition can be taken so that the original inverse problem will be uniquely solvable. We do this in three steps using the refined inverse function theorem 18 from [49].

We establish that with additional hypothesis on G

$$\|G'(u)\|_{\mathcal{L}(H,LH)} \leq \tilde{\vartheta}(r) \quad \text{for } \|u\|_H \leq r,$$

where $\tilde{\vartheta} : [0, \infty[\to [0, \infty[$ is a monotone nondecreasing function,

1. for the direct nonlinear problem:

   $\forall r \in [0, r_*[\quad \forall F \in w(r)LU_H \quad \exists!\, u \in rU_H : Lu - Gu = F \quad \Leftrightarrow \quad \xi(u) = F$, where r_* is the root of the equation $1 - C_0\tilde{\vartheta}(r) = 0$ and C_0 is a constant of an embedding. The function $w(r)$ is calculated in terms of $\tilde{\vartheta}(r)$.

2. for the problem on inverse operators :

   $\forall r' \in [0, r'_*[\; \forall v \in W(r')U_H \; \exists!\, F \in r'U_{LH} : L^{-1}F - Q(F) = v \quad \Leftrightarrow \quad \eta(F) = v$, where r'_* is the root of $1 - 2C_0\vartheta(r) = 0$ and so $r'_* < r_*$.

3. for the inverse nonlinear problem:

   $\forall \tilde{r} \in [0, \tilde{r}_*[\; \forall \chi' \in \hat{W}(\tilde{r})(I - K)U_E \; \exists!\, \varphi \in \tilde{r}U_E : M(f) = \chi'$, where $\tilde{r}$ is calculated depending on r'.

The chapter IV and chapter V include this two kinds of results. The goal of the abstract result of chapter IV is eliminating multiple technical details of concrete cases to explain the most simply this method for the model problem 1. The results of chapter IV can be found in [42, 43]. In chapter V there is the result (published in [41]) for this technique in the case of heat equation corresponding to the model problem 2.

The chapter VI consists to apply the ideas of the control of moment to the KZK equation studied in the part I. We have obtained the controllability result for linearized KZK equation. After it we explain the difficulties why the method is not applicable to the nonlinear KZK equation. It happens because of impossibility to prove that the nonlinear operator

$$\Phi(u) = uu_x, \quad \Phi : H^1((0,T); H^{s-2}(\Omega)) \to L_2((0,T); H^s(\Omega))$$

is strictly differentiable.

Part I

Khokhlov-Zabolotskaya-Kuznetsov Equation

Chapter I

Introduction

The KZK equation, named after Khokhlov, Zabolotskaya and Kuznetsov, was originally derived as a tool for the description of nonlinear acoustic beams (cf for instance [11, 52]). It is using in acoustical problems as mathematical model that describes the pulse finite amplitude sound beam nonlinear propagation in the thermo-viscous medium, see for example [1, 26, 9, 10, 33]. Later it has been used in several other fields and in particular in the description of long waves in ferromagnetic media [44]. In the present chapter the emphasis is put on the derivation of the equation for nonlinear acoustic in view of application to time reversal problems in nonlinear media. The KZK equation in its initial interpretation as in [11] is mostly studied by physicists but until now there are is no mathematical analysis of this problem. The KZK equation is not an integrable equation at variance Kadomtsev-Petviashvili (KP) equation known to be integrable. Numerically in [11] has been obtained the existence of a shock wave in the case of propagation of the beam in nondissipative media and a quasi shock wave for the dissipative media. The last phenomenon corresponds to the approximation of the beam's front to the shock wave but the solution has the tentative to be global. We obtained the proof of the existence of the shock wave for the problem without viscosity. We have established the global existence in time of the propagation in viscous media only for rather small initial data. The announcement of the results can be found in [12, 13, 14].

This part is organized in the following way. First the derivation of the equation is borrowed from physical literature then the existence uniqueness stability of the equation is analyzed. Eventually a blow-up result which gives a limitation to the range of application is given as an adaptation of a result of [3], [4] and [5]. Using obtained results one proves a large time validity of the approximation for two cases: for non viscous thermoellastic media and viscous thermoellastic media.

Our main purpose is to prove existence and stability of solutions described by the KZK equation with the following properties

1. they are concentrated near the axis x_1;
2. they propagate along the x_1 direction;
3. they are generated either by initial condition or by a forcing on the boundary $x_1 = 0$.

This corresponds to the description of the quasi one d propagation of a signal in an homogenous but nonlinear isentropic media.

Therefore it is assumed that its variation in the direction

$$x' = (x_2, x_3, \dots, x_n)$$

perpendicular to the x_1 axis is much larger that its variation along the axis x_1.

For instance for the linear wave equation in $\mathcal{R}^n$ $(n > 1)$:

$$\frac{1}{c^2}\partial_t^2 u - \Delta u = 0\,, \tag{I.1}$$

the following ansatz

$$u_\epsilon = U(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x') \tag{I.2}$$

involving a “profile”

$$U(\tau, z, y)$$

(with ϵ) small leads to the formula:

$$\partial^2_{\tau,z} U - \frac{1}{2}\Delta_y U = O(\epsilon), \tag{I.3}$$

or for functions $U(\tau, z, y) = A(z, y)e^{i\omega\tau}$, to the equation

$$i\omega\partial_z A - \frac{1}{2}\Delta_y A = O(\epsilon). \tag{I.4}$$

Observe that with $\epsilon = 0$ (I.3) and (I.4) are two variants of the classical paraxial approximation and that equation (I.3) contains the linear non diffusive terms of the KZK equation which usually has the following form for some positive constants β and γ:

$$\partial^2_{\tau,z} U - \frac{1}{2}\partial^2_\tau U^2 - \beta\partial^3_\tau U - \gamma\Delta_y U = 0.$$

On the other hand the isentropic evolution of a thermo-elastic non viscous media is given by the following Euler Equation:

$$\partial_t\rho + \nabla(\rho v) = 0\,, \quad \rho(\partial_t v + v\cdot\nabla v) = -\nabla p(\rho)\,. \tag{I.5}$$

Any constant state (ρ_0, v_0) is a stationary solution of (I.5). Linearization near this state introduces the variables

$$\rho = \rho_0 + \epsilon\tilde{\rho}\,, \quad v = v_0 + \epsilon\tilde{v}$$

and the acoustic system:

$$\partial_t\tilde{\rho} + \rho_0\nabla\tilde{v} = 0\,, \quad \rho_0\partial_t\tilde{v} + p'(\rho_0)\nabla\tilde{\rho} = 0, \tag{I.6}$$

which is equivalent to the wave equation:

$$\frac{1}{c^2}\partial_t^2\tilde{\rho} - \Delta\tilde{\rho} = 0\,, \quad \partial_t\tilde{v} = -\frac{p'(\rho_0)}{\rho_0}\nabla\tilde{\rho}, \tag{I.7}$$

where $c = \sqrt{p'(\rho_0)}$ is the sound speed of the unperturbed media.

And observe that the equation (I.3) which is the linearized and non viscous part of the KZK equation can be obtained in two steps. First consider small perturbations of a constant state for the isentropic Euler equation which are solution of the acoustic equation and then consider a paraxial approximation of such solutions.

The derivation of the full KZK equation follows almost the same line. It takes into account the viscosity and the size of the nonlinear terms. One starts from a Navier Stokes system:

$$\partial_t \rho + \nabla(\rho u) = 0\,, \quad \rho[\partial_t u + (u \cdot \nabla)\, u] = -\nabla p(\rho, S) + b\Delta u\,, \tag{I.8}$$

the pressure is given by the state law $p = p(\rho, S)$, where S is entropy.

First one assumes that the temperature T and the entropy S have the small increments $\tilde{T}$ and $\tilde{S}$. With the hypothesis of potential motion one introduces constant states

$$\rho = \rho_0, \quad u = u_0.$$

Next one assumes that the fluctuation of density (around the constant state ρ_0), of velocity (around u_0, which can be taken equal to zero with galilean), are of the same order ϵ:

$$\rho_\epsilon = \rho_0 + \epsilon\tilde{\rho}_\epsilon\,, \quad u_\epsilon = \epsilon\tilde{u}_\epsilon\,, \quad b = \epsilon\tilde{b},$$

here ϵ is a dimensionless parameter which characterizes the smallness of the perturbation. For instance in water with a initial power of the order of $0.3\,\mathrm{Vt/cm}^2$ $\epsilon = 10^{-5}$. Using the transport heat equation in the form

$$\rho_0 T_0 \frac{\partial \tilde{S}}{\partial t} = \kappa \triangle \tilde{T},$$

the approximate state equation

$$p = c^2 \epsilon \tilde{\rho}_\epsilon + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \rho^2}\right)_S \epsilon^2 \tilde{\rho}_\epsilon^2 + \left(\frac{\partial p}{\partial S}\right)_\rho \tilde{S}$$

(where the notation $(\cdot)_S$ means that the expression in brackets is constant on S), can be replaced [11], thanks to the relation

$$\tilde{S} = -\frac{\kappa}{T_0}\left(\frac{\partial T}{\partial p}\right)_S \operatorname{div} u_\epsilon,$$

by

$$p = c^2 \epsilon \tilde{\rho}_\epsilon + \frac{(\gamma - 1)c^2}{2\rho_0}\epsilon^2 \tilde{\rho}_\epsilon^2 - \kappa\left(\frac{1}{C_v} - \frac{1}{C_p}\right)\nabla . u_\epsilon. \tag{I.9}$$

The system (I.8) becomes an isentropic system

$$\partial_t \rho_\epsilon + \nabla(\rho_\epsilon u_\epsilon) = 0\,, \quad \rho[\partial_t u_\epsilon + (u_\epsilon \cdot \nabla)\, u_\epsilon] = -\nabla p(\rho_\epsilon) + \epsilon\nu\Delta u_\epsilon\,, \tag{I.10}$$

with the approximate state equation

$$p = p(\rho_\epsilon) = c^2 \epsilon \tilde{\rho}_\epsilon + \frac{(\gamma - 1)c^2}{2\rho_0}\epsilon^2 \tilde{\rho}_\epsilon^2 \tag{I.11}$$

and a rather small and positive viscosity coefficient:

$$\epsilon\nu = b + \kappa\left(\frac{1}{C_v} - \frac{1}{C_p}\right).$$

Next one reminds the direction of propagation of the beam say along the axis x_1, and therefore considers the following profiles:

$$\tilde{\rho}_\epsilon = I(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x'), \tag{I.12}$$

$$\tilde{u}_\epsilon = (u_{\epsilon,1}, u'_\epsilon) = (v(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x'), \sqrt{\epsilon}\vec{w}(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x')). \tag{I.13}$$

In (I.12) and (I.13) the argument of the functions will be denoted by $(\tau\,, z\,, y)$ and c is taken equal to the sound speed $c = \sqrt{p'(\rho_0)}$. Inserting the functions $\rho_\epsilon = \rho_0 + \epsilon I$, u_ϵ in the system (I.10) one obtains:

1 For the conservation of mass:

$$\begin{aligned}&\partial_t\rho_\epsilon + \nabla(\rho_\epsilon u_\epsilon) = \epsilon(\partial_\tau I - \frac{\rho_0}{c}\partial_\tau v) + \\ &+\epsilon^2\left(\rho_0(\partial_z v + \nabla_y \cdot \vec{w}) - \frac{1}{c}v\partial_\tau I - \frac{1}{c}I\partial_\tau v\right) + O(\epsilon^3) = 0\,.\end{aligned} \tag{I.14}$$

2 For the conservation of momentum in the x_1 direction:

$$\begin{aligned}&\rho_\epsilon\epsilon(\partial_t u_{\epsilon,1} + u_\epsilon\nabla u_{\epsilon,1}) + \partial_{x_1}p(\rho_\epsilon) - \epsilon^2\nu\Delta u_{\epsilon,1} = \epsilon(\rho_0\partial_\tau v - c\partial_\tau I) + \\ &+\epsilon^2\left(I\partial_\tau v - \tfrac{\rho_0}{c}v\partial_\tau v + c^2\partial_z I - \tfrac{(\gamma-1)}{2\rho_0}c\partial_\tau I^2 - \tfrac{\nu}{c^2}\partial_\tau^2 v\right) + O(\epsilon^3) = 0.\end{aligned} \tag{I.15}$$

And finally for the orthogonal (to the axis x_1) component of the momentum one has:

$$\begin{aligned}&\rho_\epsilon\epsilon(\partial_t u'_\epsilon + u_\epsilon\nabla u'_\epsilon) + \partial_{x'}p(\rho_\epsilon) - \epsilon^2\nu\Delta u'_\epsilon = \epsilon^{\frac{3}{2}}(\rho_0\partial_\tau\vec{w} + c^2\nabla_y I) + \\ &+\epsilon^{\frac{5}{2}}(-\tfrac{\rho_0 v}{c}\partial_\tau\vec{w} + I\partial_\tau\vec{w} + \tfrac{(\gamma-1)c^2}{2\rho_0}\nabla_y I^2 - \tfrac{\nu}{c^2}\Delta_y\vec{w}) + O(\epsilon^3) = 0.\end{aligned} \tag{I.16}$$

To eliminate the terms of the first order in ϵ we need to pose:

$$\partial_\tau I - \frac{\rho_0}{c}\partial_\tau v = 0, \tag{I.17}$$

which also implies

$$\rho_0\partial_\tau v - c\partial_\tau I = 0,$$

and therefore I and v should be related by the formula:

$$v = \frac{c}{\rho_0}I \tag{I.18}$$

and the second order terms of (I.14) and (I.15) by the formula:

$$\begin{aligned}&\rho_0(\partial_z v + \nabla_y \cdot \vec{w}) - \tfrac{1}{c}v\partial_\tau I - \tfrac{1}{c}I\partial_\tau v = \\ &= -\tfrac{1}{c}(I\partial_\tau v - \tfrac{\rho_0}{c}v\partial_\tau v + c^2\partial_z I - \tfrac{(\gamma-1)}{2\rho_0}c\partial_\tau I^2 - \tfrac{\nu}{c^2}\partial_\tau^2 v),\end{aligned} \tag{I.19}$$

which (with (I.18)) gives:

$$\rho_0\nabla_y \cdot \vec{w} + 2c\partial_z I - \frac{(\gamma+1)}{2\rho_0}\partial_\tau I^2 - \frac{\nu}{c^2\rho_0}\partial_\tau^2 I = 0. \tag{I.20}$$

Eventually one uses the equation for the orthogonal moment to eliminate the term $\rho_0 \nabla_y \cdot \vec{w}$. Assume in agreement with (I.16) that

$$\rho_0 \partial_\tau \vec{w} + c^2 \nabla_y I = 0, \tag{I.21}$$

take the divergence with respect to y of this equation. Differentiate (I.20) with respect to τ, and combine to obtain:

$$c\partial^2_{\tau z} I - \frac{(\gamma+1)}{4\rho_0}\partial^2_\tau I^2 - \frac{\nu}{2c^2\rho_0}\partial^3_\tau I - \frac{c^2}{2}\Delta_y I = 0. \tag{I.22}$$

The KZK equation (I.22) is written for the perturbation of density, but the same equation with only different constants can be also derived for the pressure and the velocity. The passage between these KZK equations is possible thanks to (I.11), (I.18) and (I.21). For example the equation for the pressure has the form

$$\partial^2_{\tau z} p - \frac{\beta}{2\rho_0 c^3}\partial^2_\tau p^2 - \frac{\delta}{2c^3}\partial^3_\tau p - \frac{c}{2}\triangle_y p = 0.$$

The above derivation is standard in physic articles however it does not imply that the function

$$\rho_\epsilon = \rho_0 + \epsilon I, u_\epsilon = \epsilon(v, \sqrt{\epsilon}\vec{w})$$

is a solution of the system (I.10) with an error term of the order of ϵ^3. In fact one can assume (I.17) and that (I.21) with (I.20) take place, but not the fact that this quantity which corresponds to the term of the order of ϵ^2 both in the conservation of mass and momentum along the axis x_1 is zero. To remedy to this fact and also to ensure an error of the order of $\epsilon^{\frac{5}{2}}$ in the moment orthogonal to the x_1 direction one introduces an Hilbert expansion type construction and writes

$$\rho_\epsilon = \rho_0 + \epsilon I\,, \quad u_\epsilon = \epsilon(v + \epsilon v_1, \sqrt{\epsilon}\vec{w}), \tag{I.23}$$

assuming that I is solution of the KZK equation (I.22), while v and w are given in term of I by (I.18) and (I.21), one obtains, modulo terms of order $\epsilon^{\frac{5}{2}}$, for the right hand side of the equations (I.14), (I.15) and (I.16):

$$\epsilon^2\left(-\frac{\rho_0}{c}\partial_\tau v_1 + \rho_0(\partial_z v + \nabla_y \cdot w) - \frac{1}{\rho_0}\partial_\tau I^2\right),$$
$$\epsilon^2\left(\rho_0\partial_\tau v_1 + c^2\partial_z I - \frac{\gamma-1}{2\rho_0}c\partial_\tau I^2 - \frac{\nu}{c\rho_0}\partial^2_\tau I\right).$$

Taking into account the KZK equation this implies for the “corrector” v_1 the relation:

$$\partial_\tau v_1 = \frac{\gamma-1}{2\rho_0^2}c\partial_\tau I^2 + \frac{\nu}{c\rho_0^2}\partial^2_\tau I - \frac{c^2}{\rho_0}\partial_z I. \tag{I.24}$$

At this point one can state a theorem with hypothesis to be specified later in chapter III (see theorems 10,11, 12).

Theorem 3 *Let I be a smooth solution of the KZK equation (I.22), define the functions v, w and v_1 by the known I. Define the function $\overline{U}_\epsilon = (\overline{\rho}_\epsilon, \overline{u}_\epsilon)$ by the formula:*

$$(\overline{\rho}_\epsilon, \overline{u}_\epsilon)(x_1, x', t) = (\rho_0 + \epsilon I, \epsilon(v + \epsilon v_1, \sqrt{\epsilon}\vec{w}))(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x').$$

Then there exist constants $C \geq 0$ and $T_0 = O(1)$, such that for any finite time $0 < t < T_0 \frac{1}{\epsilon} \ln \frac{1}{\epsilon}$ and $\epsilon > 0$, there exists a smooth solution $U_\epsilon = (R_\epsilon, U_\epsilon)(x, t)$ of the isentropic Navier-Stokes equation such that one has for some $s \geq 0$:

$$\|\overline{U}_\epsilon - U_\epsilon\|_{H^s} \leq \epsilon^{\frac{5}{2}} e^{\epsilon C t}.$$

It is interesting to notice that for the non viscous case, i.e., for the isentropic compressible Euler system, the KZK like equation with $\beta = 0$ have been obtained using the scaling of nonlinear diffractive geometric optic theory in [17, p. 1233] (in 2d) in the framework of nonlinear diffractive geometric optic with rectification. The initial goal of the article is to construct the nonlinear symmetric hyperbolic equation

$$L(u, \partial_x)u + F(u) = 0,$$

and the case of the isentropic compressible Euler system is given as an example. The basic ansatz in [17] has three scales

$$u_\epsilon(x) = \epsilon^2 a \left(\epsilon, \epsilon x, x, \frac{x \cdot \beta}{\epsilon}\right),$$

where

$$a(\epsilon, X, x, \theta) = a_0(X, x, \theta) + \epsilon a_1(X, x, \theta) + \epsilon^2 a_2(X, x, \theta).$$

Here $x = (t, y) \in \mathcal{R}^{1+d}$, $\beta = (\tau, \eta) \in \mathcal{R}^{1+d}$ and the profiles $a_j(X, x, \theta)$ are periodic in θ. The KZK like equation of the form

$$\partial_T a - \triangle_y \partial_\theta^{-1} a + \sigma a \partial_\theta a = 0,$$

with $\sigma \in \mathcal{R}$ determined from some identity and with $T = \frac{t}{\epsilon}$, holds for the profile a_0 with mean value zero on θ (for the proof see [17, pp. 1231, 1234]) which corresponds to vanishing non oscillatory part, if we have in our mind the notation of [17, p.1181]:

if (see [17, p.1181]) $\underline{a} := \frac{1}{2\pi} \int_0^{2\pi} a d\theta$, the oscillating part is denoted $a^* := a - \underline{a}$.

The analogue technique is used in [51] to study the short wave approximation for general symetric hyperbolic systems as

$$\begin{cases} L(\partial)u = \tilde{F}(u)\partial_x u, & \text{with } (x, y) \in \mathcal{R} \times \mathcal{R}, \\ u(0) = \epsilon u^0(x/\epsilon, y) \in \mathcal{R}^n. \end{cases} \tag{I.25}$$

with an hyperbolic operator $L(\partial) = \partial_t + A\partial_x + B\partial_y + E$. Short waves stands for short-wavelenght approximate solutions, or equivalently approximate solutions with initial data whose oscillatory frequencies are large compared to the paremeters of the system. For the variables

$$T = \frac{t}{\epsilon},\ X = \frac{x}{\epsilon},\ y, \tau = \epsilon t$$

in [51] one looks for the approximate solutions in the form

$$u^\epsilon(t,x,y) = \epsilon(u_0 + \epsilon u_1 + \epsilon^2 u_2)(T, X, y, \tau).$$

For the first profile u_0 one has the system of the form

$$\begin{cases} (\partial_T + c\partial_X)u_0 = 0, \\ (\partial_\tau \partial_X - \partial_Y^2)u_0 = \partial_X(u_0 \partial_X u_0). \end{cases} \tag{I.26}$$

corresponding to the KZK equation for the function $\tilde{u}_0(X - cT, \tau, y, X)$. The estimate of the approximate result in [51] between the exact solution v^ϵ of (I.25) and the solution u_0^ϵ of a system of the form (I.26) is following

$$\frac{1}{\epsilon}\|v^\epsilon - \epsilon u_0^\epsilon\|_{L_\infty([0,\tau_0/\epsilon]\times\mathcal{R}^2_{x,y})} = o(1).$$

To analyze common points between this work and KZK-approximation we can pass from the variables corresponding to our scaling

$$(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x')$$

to "$\widetilde{\text{variable}} = \sqrt{\epsilon}\,\text{variable}$" in following way

$$\left(\frac{1}{\sqrt{\epsilon}}(\tilde{t} - \frac{\tilde{x}_1}{c}), \sqrt{\epsilon}\tilde{x}_1, \tilde{x}'\right)$$

and supposing now that $\epsilon = \tilde{\epsilon}^2$, i.e., we obtain

$$\left(\frac{1}{\tilde{\epsilon}}(\tilde{t} - \frac{\tilde{x}_1}{c}), \tilde{\epsilon}\tilde{x}_1, \tilde{x}'\right),$$

and similar

$$(\rho, u) = (\rho_0 + \tilde{\epsilon}\tilde{I}, \tilde{\epsilon}(\tilde{v} + \tilde{\epsilon}^2\tilde{v}_1, \tilde{\epsilon}\tilde{w})).$$

This variables exactly correspond to Texier's case [51]

$$(T - \frac{X}{c}, \tau, y).$$

To the first profile ϵu_0 from [51] there corresponds to $(\rho_0 + \tilde{\epsilon}\tilde{I}, \tilde{\epsilon}\tilde{v})$ for which we have exactly the system (I.26) in the form of (I.17) and the KZK equation without viscous therm. The profile $\tilde{\epsilon}^2\tilde{w}$ is associated to $\epsilon^2 u_1$ and the profile $\tilde{\epsilon}^3\tilde{v}_1$ is associated to $\epsilon^3 u_2$. The result of [51] is obtained for nonperiodic case and without the vanishing mean condition important for physical reasons.

This small analysis of the abstract works shows that our approach is similar where the variables have been switched ϵ to balance the oscillate :

$$\frac{1}{\epsilon}\left(\tilde{t} - \frac{\tilde{x}_1}{c}\right) \mapsto \left(t - \frac{x_1}{c}\right).$$

In other words we can say that we have $O(1)$ oscillation.

The scaling of [44] for Landau-Lifshitz-Maxwell equation in $\mathcal{R}^3$ is very different.

Remark 1 *Several limits of the equation (I.31) leads to classical PDE.*

- *With $\rho_0 c \to \infty$ it becomes the paraxial approximation:*

$$\partial^2_{\tau z} I - \frac{c}{2}\Delta_y I = 0, \tag{I.27}$$

or in term of the pressure the equation

$$\frac{\partial p}{\partial z} = \frac{c}{2}\int_0^\tau \triangle_y p d\tau'.$$

The solutions of these equations have been numerically computed by Thierry Le Pollès (in Laboratoire Ondes et Acoustique, ESPCI, Paris) using a fractional step method. The proof of the validity of this method will be given in section II.1.2. The figures I.1 and I.2 have been simulated for the three dimensional problem for pressure p of a sound beam propagating in the water

$$\frac{\partial p}{\partial z} = \frac{c}{2}\int_0^\tau \triangle_y p d\tau', \quad y = (y_1, y_2),$$
$$p(\tau, 0, y) = g(\tau) \quad y \in \Omega, \quad \tau > 0,$$
$$\frac{\partial p}{\partial n} = 0 \text{ for } \partial\Omega, \quad \tau > 0.$$

Here $g(\tau)$ is the signal of the source situated in $z = 0$ and

$$g(\tau) = P_0 \exp[-(2\tau/T_d)^{2m}]\sin(w_0 t). \tag{I.28}$$

- *And when I does not depend on y it is the Burgers-Hopf equation*

$$c\partial_z I - \frac{(\gamma+1)}{4\rho_0}\partial_\tau I^2 - \frac{\nu}{2c^2\rho_0}\partial^2_\tau I = 0 \tag{I.29}$$

and eventually in this case with $\nu = 0$ the Burgers equation:

$$c\partial_z I - \frac{(\gamma+1)}{4\rho_0}\partial_\tau I^2 = 0.$$

In term of the pressure fluctuation, (I.29) is

$$\frac{\partial p}{\partial z} = \frac{\delta}{2c^3}\frac{\partial^2 p}{\partial \tau^2} + \frac{\beta}{2\rho_0 c^3}\frac{\partial p^2}{\partial \tau^2}. \tag{I.30}$$

The numerical simulation of the solution of (I.30) with the same initial and boundary data as in (I.27) is given in the figures I.3 and I.4.

- *The analogous 2d version of KZK equation is*

$$c\partial^2_{\tau z} I - \frac{(\gamma+1)}{4\rho_0}\partial^2_\tau I^2 - \frac{\nu}{2c^2\rho_0}\partial^3_\tau I - \frac{c^2}{2}\partial^2_y I = 0.$$

And for a "beam" (rotationally invariant around the x_1 axis) in 3 space variables it is:

$$c\partial^2_{\tau z}I - \frac{(\gamma+1)}{4\rho_0}\partial^2_\tau I^2 - \frac{\nu}{2c^2\rho_0}\partial^3_\tau I - \frac{c^2}{2}(\partial^2_r I + \frac{1}{r}\partial_r I) = 0. \quad \text{(I.31)}$$

The figures I.5 and I.6 represent the graph of the solution of the full KZK equation, composed of the both parts of (I.27) and (I.30) with the source (I.28)

$$\frac{\partial p}{\partial z} = \frac{c}{2}\int_0^\tau \triangle_y p d\tau' + \frac{\delta}{2c^3}\frac{\partial^2 p}{\partial \tau^2} + \frac{\beta}{2\rho_0 c^3}\frac{\partial p^2}{\partial \tau^2}. \quad \text{(I.32)}$$

All figures I.1-I.6 have been obtained by Thierry Le Pollès in Laboratoire Ondes et Acoustique, ESPCI, Paris, and are the illustrations of his numerical results calculated in C++.

Remark 2 *We would like also illustrate the case of "quasi-shock" using [11, pp.78-81]. This phenomenon appears for the KZK equation with small viscosity coefficient. According to [11] the wave is named a quasi shock wave if the breadth of the wave front $\triangle\tau \leq \pi/10$. The figures I.7 and I.8 have been obtained in [11] for the following problem for the density function of a beam rotationally invariant around the x_1 axis (cf. (I.31))*

$$\begin{aligned} &\frac{\partial^2\rho^2}{\partial\tau\partial z} - N\frac{\partial^2\rho^2}{\partial\tau^2} - \delta\frac{\partial^3\rho}{\partial\tau^3} - \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\rho = 0, \quad \text{(I.33)} \\ &\rho|_{z=0} = -e^{-r^2}\sin\tau. \end{aligned}$$

Remark 3 *There are the mathematical works [28], [29] for KZK type equation*

$$\alpha u_{z\tau} = (f(u_\tau))_\tau + \beta u_{\tau\tau\tau} + \gamma u_\tau + \triangle_x u,$$

where $u_\tau = u_\tau(z, x, \tau)$ is the acoustic pressure, $(z, x) \in \mathcal{R}^d \times \mathcal{R}$, $d = 1, 2$ are space variables and τ is the retarded time. The equation is studied with the hypothesis that the nonlinearity f has bounded derivative which allows to proof the global existence for the case when the coefficients are rapidly oscillating functions of z. So this problem is not related with our "acoustical" problem for the KZK equation where as we will see later there is a blow-up result illustrating the existence of a shock wave.

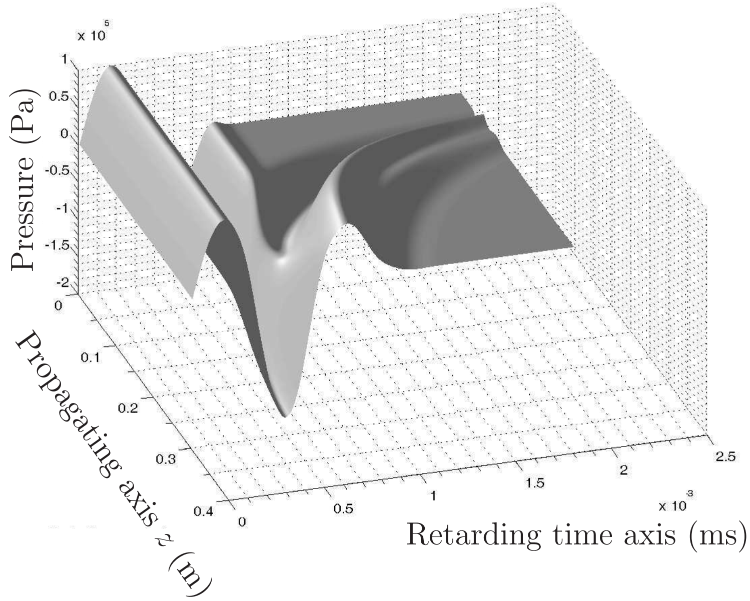

Figure I.1 – *The effects of diffraction (I.27) in 3d for the pressure with a square source* $3\,\mathrm{cm} \times 3\,\mathrm{cm}$. *The parameters of simulation for propagation in water:* $\triangle y_1 = \triangle y_2 = 3.75 \times 10^{-4}\,\mathrm{m}$, $\triangle z = 6 \times 10^{-4}\,\mathrm{m}$, $\triangle \tau = 6.6667 \times 10^{-9}\,\mathrm{s}$, $\rho_0 = 1000\,\mathrm{kg/m^3}$, $c = 1500\,\mathrm{m/s}$.

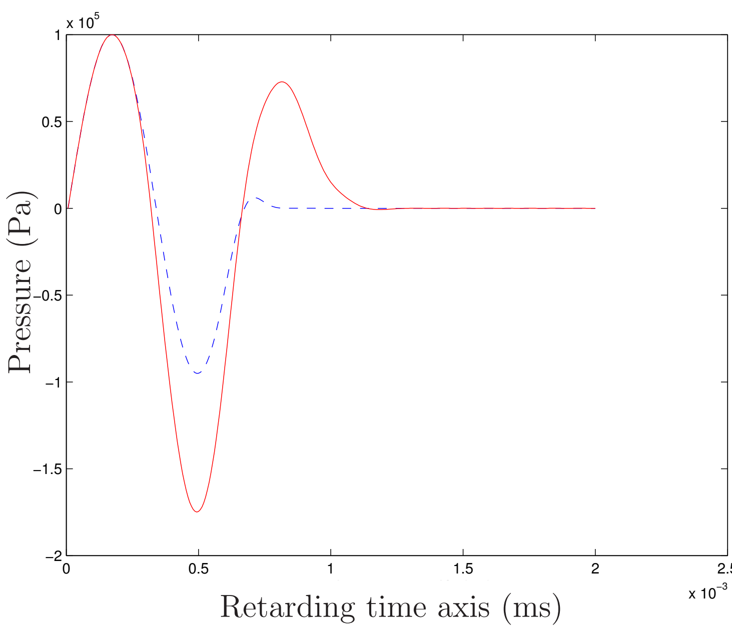

Figure I.2 – *The effects of diffraction (I.27) along the retarded axis* τ *in the two different places of the propagation axis* z *with a square source* $3\,\mathrm{cm} \times 3\,\mathrm{cm}$. *The dotted line corresponds to the signal source* $z = 0$ *and the solid line to the pressure at the distance* $z = 30\,\mathrm{cm}$. *The parameters of simulation are the same as in figure I.1.*

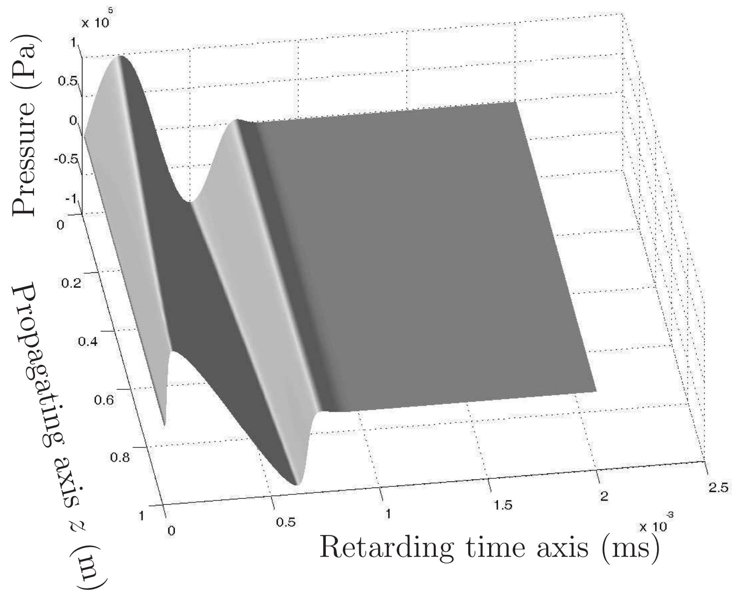

Figure I.3 – *The effects of absorption and nonlinear effects (I.30) in 3d for the pressure with a square source* $3\,\mathrm{cm} \times 3\,\mathrm{cm}$. *The parameters of simulation for propagation in water:* $\triangle y_1 = \triangle y_2 = 3.75 \times 10^{-4}\,\mathrm{m}$, $\triangle z = 5 \times 10^{-3}\,\mathrm{m}$, $\triangle \tau = 6.6667 \times 10^{-9}\,\mathrm{s}$, $\beta = 5$, $\delta = 4.1 \times 10^{-6}\,\mathrm{Np/m}$, $\rho_0 = 1000\,\mathrm{kg/m^3}$, $c = 1500\,\mathrm{m/s}$.

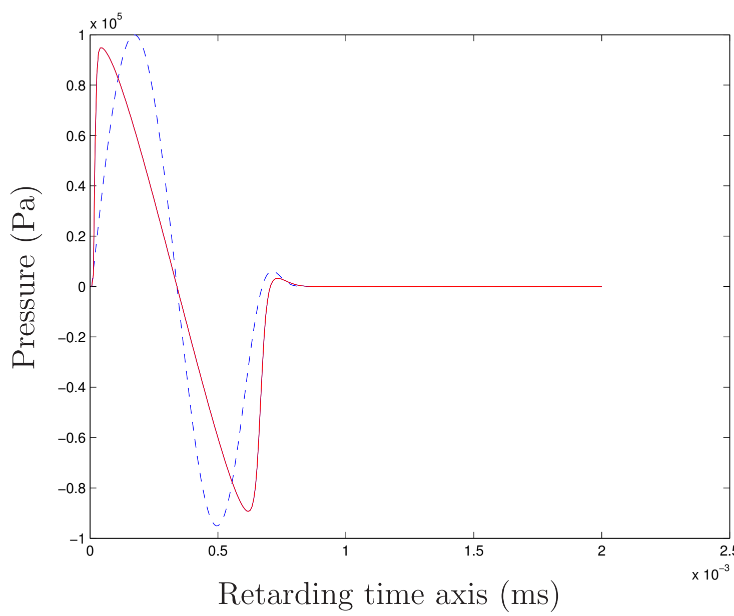

Figure I.4 – *The effects of absorption and nonlinear effects (I.30) along the retarded axis* τ *in the two different places of the propagation axis* z *with a square source* $3\,\mathrm{cm} \times 3\,\mathrm{cm}$. *The dotted line corresponds to the signal source* $z = 0$ *and the solid line to the pressure at the distance* $z = 1\,\mathrm{m}$. *The parameters of simulation are the same as in figure I.3.*

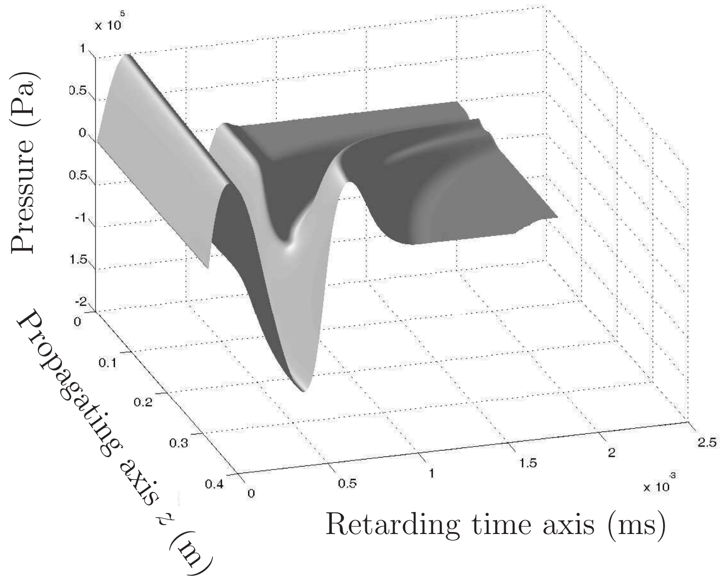

Figure I.5 – *Representation of the solution of KZK equation (I.32) for the pressure in 3d with a square source* $3\,\mathrm{cm} \times 3\,\mathrm{cm}$. *The parameters of simulation for propagation in water:* $\triangle y_1 = \triangle y_2 = 3.75 \times 10^{-4}\,\mathrm{m}$, $\triangle z = 1 \times 10^{-3}\,\mathrm{m}$, $\triangle \tau = 6.6667 \times 10^{-9}\,\mathrm{s}$, $\beta = 5$, $\delta = 4.1 \times 10^{-6}\,\mathrm{Np/m}$, $\rho_0 = 1000\,\mathrm{kg/m^3}$, $c = 1500\,\mathrm{m/s}$.

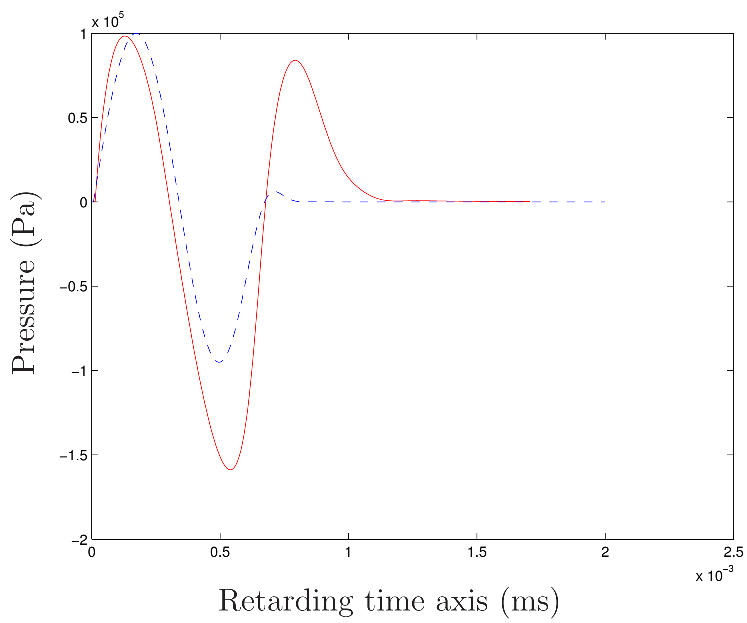

Figure I.6 – *Representation of the solution of KZK equation (I.32) along the retarded axis* τ *in the two different places of the propagation axis* z *with a square source* $3\,\mathrm{cm} \times 3\,\mathrm{cm}$. *The dotted line corresponds to the signal source* $z = 0$ *and the solid line to the pressure at the distance* $z = 31.2\,\mathrm{cm}$. *The parameters of simulation are the same as in figure I.5.*

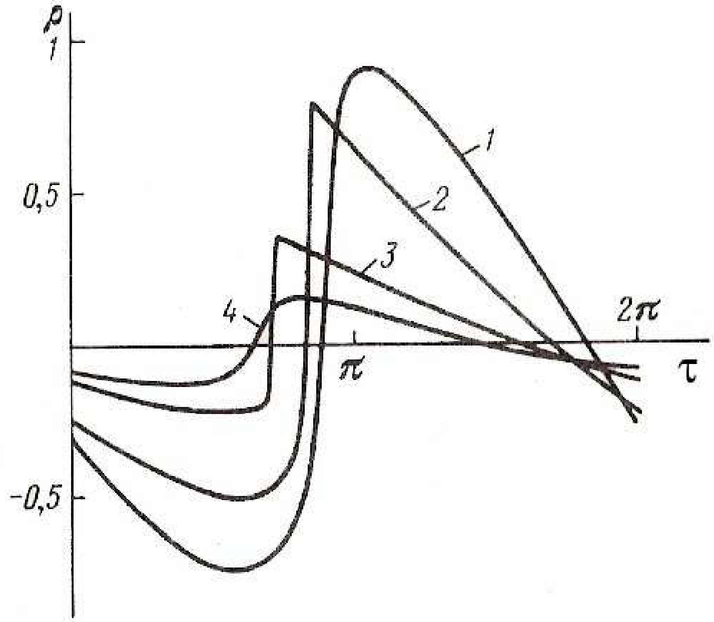

Figure I.7 – *Profiles of the solution of KZK equation (I.33) for the density along the axis τ with different values of z. The values of z on the curves 1-4 respectively are 0.15, 0.3, 0.7, 1.2;* $N = 3.25$, $\delta = 0.1$ *(see [11, p.80]).*

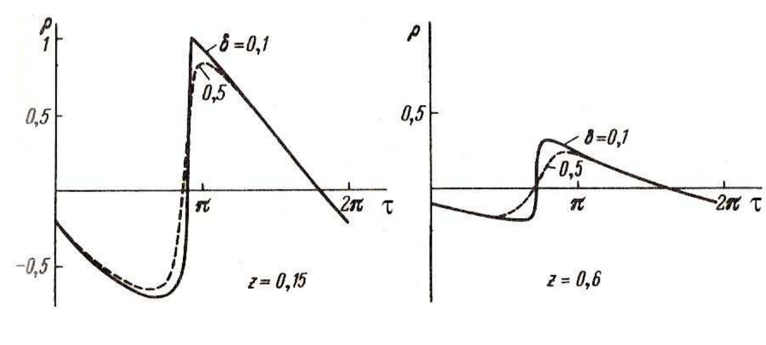

Figure I.8 – *Wave profiles of the solution of KZK equation (I.33) corresponding to different δ;* $N = 5$ *(see [11, p.81]).*

Chapter II

Mathematical Studies of the KZK Equation

Table of contents

II.1 Existence uniqueness and stability of solutions of the KZK equation

Following the mathematical tradition in this section and in the next one the unknown will be denoted by u, and the variables $(x, y) \in \mathcal{R}_x \times (\Omega \subseteq \mathcal{R}^{n-1})$. When $\Omega \neq \mathcal{R}^{n-1}$ it is assumed that the solution satisfies on its boundary the Neumann boundary condition. Multiplying u by a positive scalar one reduces the problem to an equation involving only two constants β and γ

$$(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0 \quad \text{ in } \mathcal{R}_x/(L\mathcal{Z}) \times \Omega. \tag{II.1}$$

For sake of clarity and because this also corresponds to practical situations [11, 52] we consider solutions which are periodic with respect to the variable x and which are of mean value zero:

$$u(x+L, y, t) = u(x, y, t), \quad \int_0^L u(x, y, t)dx = 0. \tag{II.2}$$

Observe that the conditions (II.2) are compatible with the flow and that the second one is "natural" because we consider fluctuations.

For these functions the norm of the space H^s ($s \in R$, $s \geq 0$) is denoted by

$$\|u\|_{H^s} = \left(\int\limits_{R^{n-1}} \sum_{k=-\infty}^{+\infty} (1 + k^2 + \eta^2)^s |\hat{u}(k, \eta)|^2 d\eta \right)^{\frac{1}{2}}.$$

If we introduce the operator $\Lambda = (1 - \Delta)^{\frac{1}{2}}$ as $\widehat{(\Lambda u)}(\zeta) = (1 + |\zeta|^2)^{\frac{1}{2}} \hat{u}(\zeta)$, then

$$\Lambda^s = (1 - \Delta)^{\frac{s}{2}}, \quad \|u\|_{H^s} = \|\Lambda^s u\|_{L_2}. \tag{II.3}$$

We define the inverse of the derivative ∂_x^{-1} as an operator acting in the space of periodic functions with mean value zero this gives the formula:

$$\partial_x^{-1} f = \int_0^x f(s)ds + \int_0^L \frac{s}{L} f(s)ds. \tag{II.4}$$

This form of the operator ∂_x^{-1} preserves the both qualities: the periodicity and having the mean value zero.

In this situation equation (II.1) is equivalent to the equation

$$u_t - uu_x - \beta u_{xx} - \gamma\partial_x^{-1}\Delta_y u = 0 \quad \text{in } \mathcal{R}_x/(L\mathcal{Z}) \times \Omega. \tag{II.5}$$

Finally when $\gamma = 0$ equation (II.1) reduces to the Burgers-Hopf equation for which existence smoothness and uniqueness of solution are well known. For $\gamma = \beta = 0$ it reduces to the Burgers equation

$$\partial_t u - \partial_x \frac{u^2}{2} = 0,$$

which after a finite time exhibits singularities. After this "blow-up" time the solution can be uniquely continued into a weak solution satisfying an elementary entropy condition (in the present case with $\gamma \neq 0$ it seems that this construction cannot be adapted to equation (II.1) with $\beta = 0$ and $\gamma \neq 0$).

We would like also to notice that the J. Bourgain-type method and introduction the Bourgain spaces as in [46, 35, 36] and others are not useful for the KZK problem because of absence of the terms with an odd derivative as for example u_{xxx} in (II.5). The presence only of the second derivative make impossible the main estimations and equalities of this method.

II.1.1 A priori estimates for smooth solutions

According to the standard approach we first establish a priori estimates for smooth solutions which are in particular a consequence of the relation:

$$\begin{aligned}\int_0^L \int_{\mathcal{R}_y^{n-1}} \partial_x^{-1}(\Delta_y u) u dx dy &= -\int_0^L \int_{\mathcal{R}_y^{n-1}} \partial_x^{-1}(\nabla_y u)\nabla_y u dx dy \\ &= \int_0^L \int_{\mathcal{R}_y^{n-1}} \partial_x^{-1}(\nabla_y u)\partial_x(\partial_x^{-1}(\nabla_y u)) dx dy = 0.\end{aligned} \tag{II.6}$$

The L_2 norm and the H^s in $(\mathcal{R}_x^+/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$ are denoted by $|u|$ and by $\|u\|_s$.

Proposition 1 *The following estimates are valid for solutions of the integrated KZK equation (II.5):*

$$\frac{1}{2}\frac{d}{dt}|u(\cdot,\cdot,t)|^2 + \beta|\partial_x u(\cdot,\cdot,t)|^2 = 0\,, \tag{II.7}$$

$$\textit{For } s > [\frac{n}{2}] + 1 \quad \frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \beta\|\partial_x u\|_s^2 \le C(s)\|u\|_s^3 \tag{II.8}$$

$$\textit{and } \ \frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \beta C(L)\|u\|_s^2 \le C(s)\|u\|_s^3. \tag{II.9}$$

The estimates (II.8), (II.9) are valid for $s > [\frac{n}{2}] + 1$ *which is the necessary condition because of application of the Sobolev theorem.*

Proof. To obtain the relation (II.7) multiply (II.5) by u, and integrate by part. It shows that for $\beta = 0$ we have the conservation low for the norm of u in $L_2(\mathcal{R}_x^+/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$. If $\beta > 0$ we also have according to the physical phenomena [11] the dissipation of energy.

For the clarity the proof of (II.8) is done firstly in 3 space variables, with $\Omega = \mathcal{R}^2$ and s an integer (i.e. in the present case $s = 3$) and after it we cite the proof in general case. In $2d$ in particular when $\Omega = S^1$ the proof is even simpler and in the periodic case. The proof in the whole is similar except for the relation (II.9) which holds only in the periodic case and not on the whole line. (In this later case the H^s norm of $\partial_x u$ does not control the H^s norm of u).

For the proof of general case $s \in \mathcal{R}$ one has used the representation of the norm in H^s with the help of the operator Λ by (II.3) and the technique demonstrated in [25] and [45] for periodic and nonperiodic cases, which allows to deduce

$$\frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \beta\|\partial_x u\|_s^2 \le C\|\nabla_{x,y}u\|_{L_\infty}\|u\|_s^2,$$

and this implies the necessity of our restriction for s:

$$\text{if } s > [\frac{n}{2}] + 1 \text{ then } \; H^{s-1} \subset L_\infty.$$

The elementary proof

The introduction of the H^3 norm for $n = 3$ comes from the control of the nonlinearity with the Sobolev theorem and it starts with the following relations and estimates

$$\begin{aligned}
\int_0^L\int_{\mathcal{R}_y^2} \partial_x^3(u\partial_x u)\partial_x^3 u dxdy &= \int_0^L\int_{\mathcal{R}_y^2} u\partial_x(\partial_x^3 u)\partial_x^3 u dxdy \\
&+ 3\int_0^L\int_{\mathcal{R}_y^2} (\partial_x^2 u)^2\partial_x^3 u dxdy \\
&+ 3\int_0^L\int_{\mathcal{R}_y^2} \partial_x u\partial_x^3 u\partial_x^3 u dxdy \\
&+ \int_0^L\int_{\mathcal{R}_y^2} (\partial_x^3 u)^2\partial_x u dxdy.
\end{aligned} \tag{II.10}$$

The first term of (II.10) is integrated by part (use the x periodicity):

$$\int_0^L\int_{\mathcal{R}_y^2} u\partial_x(\partial_x^3 u)\partial_x^3 u = -\frac{1}{2}\int_0^\infty\int_{\mathcal{R}_y^2} \partial_x u(\partial_x^3 u)^2 dxdy.$$

The second is a perfect derivative and therefore it disappears and finally one has:

$$|\int_0^L\int_{\mathcal{R}_y^2} \partial_x^3(u\partial_x u)\partial_x^3 u dxdy| \le C|\partial_x u|_{L^\infty(]0,L[,\times\mathcal{R}_y^2)}\|u\|^2_{H^3(\mathcal{R}_x\times\mathcal{R}_y^2)}.$$

In the same way with ∂_y denoting the derivative with respect to any orthogonal component one obtains:

$$\begin{aligned}\int_0^L\int_{\mathcal{R}_y^2}\partial_y^3(u\partial_x u)\partial_y^3 udxdy &= \int_0^L\int_{\mathcal{R}_y^2} u\partial_x(\partial_y^3 u)\partial_y^3 udxdy\\ &+ 3\int_0^L\int_{\mathcal{R}_y^2}(\partial_y u)(\partial_y^2\partial_x u)\partial_y^3 udxdy\\ &+ 3\int_0^L\int_{\mathcal{R}_y^2}\partial_y^2 u(\partial_y\partial_x u)\partial_y^3 udxdy\\ &+ \int_0^L\int_{\mathcal{R}_y^2}(\partial_y^3 u)^2\partial_x udxdy \qquad \text{(II.11)}\end{aligned}$$

and as above one has

$$\int_0^L\int_{\mathcal{R}_y^2} u\partial_x(\partial_y^3 u)\partial_y^3 u = -\frac{1}{2}\int_0^L\int_{\mathcal{R}_y^2}\partial_x u(\partial_y^3 u)^2 dxdy.$$

Therefore the sum of the first, second and last term of (II.11) are bounded by

$$C|\partial_y u|_{L^\infty(]0,L[\times\mathcal{R}_y^2)}\|u\|^2_{H^3(]0,L\times\mathcal{R}_y^2)}$$

and for the third term one can write:

$$\begin{aligned}\int_0^L\int_{\mathcal{R}_y^2}\partial_y^2 u(\partial_y\partial_x u)\partial_y^3 udxdy &= \frac{1}{2}\int_0^L\int_{\mathcal{R}_y^2}\partial_y(\partial_y^2 u)^2(\partial_y\partial_x u)dxdy\\ &= -\frac{1}{2}\int_0^L\int_{\mathcal{R}_y^2}(\partial_y^2 u)^2\partial_x(\partial_y^2 u)dxdy = 0. \qquad \text{(II.12)}\end{aligned}$$

Finally one has obtained the following estimate:

$$\begin{aligned}&\int_0^L\int_{\mathcal{R}_y^2}(\partial_x^3(u\partial_x u)(\partial_x^3 u) + \sum_{1\le i\le 2}\partial_{y_i}^3(u\partial_x u)(\partial_{y_i}^3 u))dxdy \le\\ &\le (\sup_{x,y}|\partial_x u(x,y,t)| + |\nabla_y u(x,y,t)|)\|u\|_3^2. \qquad \text{(II.13)}\end{aligned}$$

The choice of the index of derivation 3 comes from the Sobolev theorem which gives:

$$|\partial_x u| + |\partial_y u| \le C\|u\|_{H^3(]0,L[\times\mathcal{R}_y^2)}.$$

Eventually to obtain (II.8) write:

$$\begin{aligned}0 &= \int_0^L\int_{\mathcal{R}_y^2}[\partial_x^3(u_t - uu_x - \beta u_{xx} - \gamma\partial_x^{-1}\Delta_y u(s,y)ds).\partial_x^3 u\\ &+ \sum_{1\le i\le 2}\partial_{y_i}^3(u_t - uu_x - \beta u_{xx} - \gamma\partial_x^{-1}\Delta_y u(s,y)ds).\partial_{y_i}^3 u]dxdy\end{aligned}$$

and use the estimate (II.13). Finally to prove (II.9) one uses the fact that u is of x mean value 0 and therefore it is (cf: (II.4)) related to $\partial_x u$ by the formula

$$u = \partial_x^{-1}\partial_x u = \int_0^x \partial_x u(s,y)ds + \int_0^L \frac{s}{L}\partial_x u(s,y)ds, \qquad \text{(II.14)}$$

which implies the relation

$$\|u\|_{H^3(]0,L[\times\mathcal{R}_y^2)} \leq C\|\partial_x u\|_{H^3(]0,L[\times\mathcal{R}_y^2)}\,.$$

The general proof

We apply the operator Λ^s to equation (II.5) and multiply by $\Lambda^s u$ in L_2

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 - \beta(\Lambda^s u_{xx}, \Lambda^s u) - (\Lambda^s(uu_x), \Lambda^s u) = 0,$$

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 + \beta\|u_x\|_{H^s}^2 - (\Lambda^s(uu_x), \Lambda^s u) = 0.$$

Suppose that $[\Lambda^s, u]v = \Lambda^s(uv) - u\Lambda^s v$. Then

$$(\Lambda^s(uu_x), \Lambda^s u) = ([\Lambda^s, u]u_x, \Lambda^s u) + (u\partial_x\Lambda^s u, \Lambda^s u) = ([\Lambda^s, u]u_x, \Lambda^s u) - \frac{1}{2}(u_x\Lambda^s u, \Lambda^s u).$$

As soon as $2(s-1) > n$, the last term is estimated by

$$|(u_x\Lambda^s u, \Lambda^s u)| \leq \|u_x\|_{L_\infty}\|u\|_{H^s}^2 \leq C\|u_x\|_{H^{s-1}}\|u\|_{H^s}^2 \leq C\|u\|_{H^s}^3.$$

For the first term we have:

$$([\Lambda^s, u]u_x, \Lambda^s u) \leq \|[\Lambda^s, u]u_x\|_{L_2}\|u\|_{H^s} \leq C\|u\|_{H^s}\|u_x\|_{H^{s-1}}\|u\|_{H^s} \leq C\|u\|_{H^s}^3.$$

We need now the following proposition.

Proposition 2 *With the above notations we have the estimate*

$$\|[\Lambda^s, u]u_x\|_{L_2} \leq C\|u\|_{H^s}\|u_x\|_{H^{s-1}}.$$

Proof.

Non periodic case. Using the result from [25]

$$\|\Lambda^s(fg) - f(\Lambda^s g)\|_{L_p} \leq c(\|\nabla f\|_{L_\infty}\|\Lambda^{s-1}g\|_{L_p} + \|\Lambda^s f\|_{L_p}\|g\|_{L_\infty}),$$

where $1 < p < \infty$, $s \geq 0$, $\nabla = (\partial_1, ..., \partial_n)$, $\partial_j = \partial/\partial x_j$, we find for $p = 2$, $f = u$, $g = u_x$

$$\|[\Lambda^s, u]u_x\|_{L_2} \leq C(\|\nabla_{x,y}u\|_{L_\infty}\|\Lambda^{s-1}u_x\|_{L_2} + \|\Lambda^s u\|_{L_2}\|u_x\|_{L_\infty}).$$

Since

$$\|u_x\|_{L_\infty} \leq C\|u_x\|_{H^{s-1}},$$

(the embedding theorem for $2(s-1) > n$),

$$\|\nabla u\|_{L_\infty} \leq C\|u\|_{H^s},$$

$$\|\Lambda^s u\|_{L_2} = \|u\|_{H^s},$$

we have

$$\|[\Lambda^s, u]u_x\|_{L_2} \leq C\|u\|_{H^s}\|u_x\|_{H^{s-1}}.$$

Periodic case. Using the result of J.C. Saut and R. Temam from [45] which consists in the following:

if u, v are in $H^s(\mathcal{R}^n)$ or in $H^s(\mathcal{R}^n/\mathcal{Z}^n)$ and $s \in \mathcal{R}$, $s > 1$, $\gamma \in \mathcal{R}$, $\gamma > n/2$, then

$$\|D^s(uv) - uD^s v\|_{L_2} \leq c(\gamma, s)\{\|u\|_s\|v\|_\gamma + |u\|_{\gamma+1}\|v\|_{s-1}\},$$

what is easy to generalize for $H^s(\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}$.

The estimate remains true if we change D^s on Λ^s.

In our case $\gamma = s - 1$, from where the result follows. $\square$

To finish the proof for (II.9) we notice that

$$\|u\|_{H^s} \leq C\|\frac{\partial u}{\partial x}\|_{H^s},$$

because of (II.14). $\square$

II.1.2 Existence and uniqueness for smooth solutions

The following theorem is an easy consequence of the a priori estimates.

Theorem 4 *For the following Cauchy problem*

$$u_t - uu_x - \beta u_{xx} - \gamma\partial_x^{-1}(\Delta_y u) = 0\,, u(x, y, 0) = u_0 \tag{II.15}$$

considered in $(\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}$, i.e. in the class of x periodic functions with mean value 0 with the operator ∂_x^{-1} defined by the formula (II.4) and finally with $\beta \geq 0$ one has the following results.

1 For $s > [\frac{n}{2}] + 1$ ($s = 3$ for instance in dimension 3) there exists a constant $C(s, L)$ such that for any initial data $u_0 \in H^s$ the problem (II.15) has on an interval $[0, T[$ with

$$T \geq \frac{1}{C(s, L)\|u_0\|_{H^s}} \tag{II.16}$$

a solution in $C([0, T[, H^s) \cap C^1([0, T[, H^{s-2})$.

2 Let T^* be the biggest time on which such solution is defined then one has

$$\int_0^{T^*} \sup_{x,y}(|\partial_x u(x, y, t)| + |\nabla_y u(x, y, t)|)dt = \infty. \tag{II.17}$$

3 If $\beta > 0$ there exists a constant C_1 such that

$$\|u_0\|_s \leq C_1 \Rightarrow T^* = \infty. \tag{II.18}$$

4 For two solutions u and v of KZK equation, assume that $u \in L_\infty([0,T[;H^s)$, $v \in L^2([0,T[;L_2)$. Then one has the following stability uniqueness result:

$$|u(,t) - v(.,t)|_{L^2} \leq e^{\int_0^t sup_{x,y}|\partial_x u(x,y,s)|ds}|u(.,0) - v(.,0)|_{L^2}. \tag{II.19}$$

Remark 4 *The estimate (II.19) is of strong-weak form, as in [15] only the L_∞ norm of u_x is needed.*

Remark 5 *When there is no viscosity all the corresponding statements of the theorem 4 remain valid for $0 > t > -C$ with a convenient C.*

Remark 6 *As (II.1) is envisaged for $u(t,x,y)$ with $x \in \mathcal{R}/(L\mathcal{Z})$, the KZK equation can be also written for $u(t,x,y) = v(t,-x,y)$ in the equivalent form*

$$(v_t + vv_x - \beta v_{xx})_x + \gamma \triangle_y v = 0.$$

So it is important to keep invariant the sign $-\beta v_{xxx}$, $\beta \geq 0$, but all other signs can be changed.

Proof. To construct a solution one can proceed by regularization, by a fractional step method, or by any other type of approximation. In particular it was done for the general case with the help of Kato theory from [21, 22, 23, 24]. Since we intend to analyze the numerical methods, the fractional step is favored and once again the only case $n = 3$ and $s = 3$ with periodic solutions is analyzed. The idea of this kind of proof can be found in ([50]) and firstly have been introduced by Marchuk and Yanenko. Furthermore as for a priori estimates result we cite two proofs: one with the analysis of the fractional step method for the case $n = 3$ and $s = 3$ and an other proof for general case.

The application of the fractional step method

To control the stability of the fractional step method one uses the following

Lemma 1 *Let X_0, C, T be three positive numbers with*

$$T < \frac{2}{C\sqrt{X_0}}.$$

Let N be a positive integer, $\Delta T = \frac{T}{N}$ and for $0 \leq k \leq N$ let X_k be a sequence of positive numbers which satisfy the estimate:

$$\textit{for } 0 \leq k \leq N-1\,, \quad X_{k+1} \leq \frac{X_k}{(1 - \frac{1}{2}C\Delta T\sqrt{X_k})^2},$$

then for any $0 \leq k \leq N$ one has

$$X_k \leq \frac{X_0}{(1 - \frac{1}{2}CT\sqrt{X_0})^2}.$$

Proof. The solution of the equation:

$$y' = Cy^{\frac{3}{2}}\,, y(0) = X_0$$

is given by the formula:

$$y(t) = \frac{X_0}{(1 - \frac{1}{2}Ct\sqrt{X_0})^2}$$

and is therefore positive and bounded on the interval $[0, T]$. Denote by y_k the value of this solution at the points $k\Delta T$ they satisfy the relation

$$y_{k+1} = \frac{y_k}{(1 - \frac{1}{2}C\Delta T\sqrt{y_k})^2}$$

and therefore for any $k \in [0, N-1]$ one has

$$0 \le X_k \le y_k$$

and the conclusion follows.

The operator $\partial_x^{-1}\Delta_y$ is the generator of a unitary group in the space of $L^2(\frac{\mathcal{R}}{\mathcal{Z}L} \times \Omega)$ with mean value zero and this unitary group

$$e^{-t\partial_x^{-1}\Delta_y}$$

preserves the H^s norm. In the mean time the solution of the Burgers equation:

$$\partial_t u - uu_x - \beta u_{xx} = 0$$

on the time interval $]k\Delta T, (k+1)\Delta T[$ with u given at the time $k\Delta T$ may increase (as in the proof of the a priori estimates) the H^3 according to the formula

$$\|u((k+1)\Delta T)\|_3 \le \frac{\|u(k\Delta T)\|_3}{(1 - \frac{1}{2}CT\sqrt{\|u(k\Delta T)\|_3})^2}.$$

According to the tradition one defines on the interval $[0, T]$ the functions u_N and $u_{N+\frac{1}{2}}$ by the following formula:

$$\begin{aligned}
&\text{for } t \in]k\Delta T, (k+1)\Delta T[\,, \quad u_{N+\frac{1}{2}}(0_-) = u_0,\\
&\partial_t u_N - u_N(u_N)_x - \beta(u_N)_{xx} = 0,\ u_N(k\Delta T) = u_{N+\frac{1}{2}}(k\Delta T_-),\\
&\partial_t u_{N+\frac{1}{2}} - \gamma\partial_x^{-1}\Delta_y u_{N+\frac{1}{2}} = 0,\ u_{N+\frac{1}{2}}(k\Delta T) = u_N((k+1)\Delta T_-).
\end{aligned}$$

The lemma 1 implies that the functions u_N and $u_{N+\frac{1}{2}}$ are uniformly bounded in

$$L^\infty(]0, T[, H^3)$$

and by a standard argument as it is done for instance in [50] they converge in

$$C(]0, T[, H^2)$$

to a function u which is solution of the KZK equation:

$$\partial_t u - uu_x - \beta u_{xx} - \gamma\partial_x^{-1}\Delta_y u = 0.$$

The fact that the solution $u \in C([0, T[, H^s) \cap C^1([0, T[, H^{s-2})$ can be easily shown as in [21].

This proof being invariant with respect to time translation shows also that whenever $u(t) \in H^s$ is finite the solution can be extended on a non zero time interval which is bounded below in term of $\|u(t)\|_s$. Now from the estimate (II.13) one deduces the relation:

$$\|u(t_2)\|_s^2 \leq 2\|u(t_1)\|_s^2 e^{\int_{t_1}^{t_2} \sup_{x,y}(|\partial_x u(x,y,s)|+|\nabla_y u(x,y,s)|)ds} \tag{II.20}$$

and this proves point 2.

To prove the next point one observes that periodic solutions with mean value 0 satisfies for t small enough the estimate:

$$\frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \|u\|_s^2(\beta C(L) - C(s)\|u\|_s) \leq 0. \tag{II.21}$$

Therefore if for $t = 0$ one has

$$\beta C(L) - C(s)\|u(0,\cdot)\|_s \geq 0 \quad \text{i.e.} \quad \|u(0,\cdot)\|_s \leq \frac{\beta C(L)}{C(s)}$$

the quantity $\|u(t,\cdot)\|_s^2$ will decay for $t > 0$ and therefore satisfies the same estimate on all the interval $[0, T^*[$, which and therefore can be extended after any finite value T^* and this proves point 3. Finally let u and v be two solutions. For the difference one has the relation:

$$\partial_t(u-v) - (u-v)\partial_x u + v\partial_x(v-u) - \beta\partial_x^2(u-v) - \gamma\partial_x^{-1}\Delta_y(u-v) = 0. \tag{II.22}$$

Multiplying this equation by $(u - v)$ integrating in x and y and performing standard integration by parts gives:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u-v|^2 - \int_0^L \int_{\mathcal{R}_y^{n-1}} \partial_x u(u-v)^2 dxdy + \\
&+ \int_0^L \int_{\mathcal{R}_y^{n-1}} v(u-v)\partial_x(u-v)dxdy + \\
&+\beta \int_0^L \int_{\mathcal{R}_y^{n-1}} (\partial_x(u-v))^2 dxdy = 0,
\end{aligned} \tag{II.23}$$

which, with the relation:

$$\begin{aligned}
&\int_0^L \int_{\mathcal{R}_y^{n-1}} v(u-v)\partial_x(u-v)dxdy = \int_0^L \int_{\mathcal{R}_y^{n-1}} [(v-u)+u]\frac{\partial_x(u-v)^2}{2}dxdy = \\
&= -\frac{1}{2}\int_0^L \int_{\mathcal{R}_y^{n-1}} \partial_x u\,(u-v)^2 dxdy
\end{aligned} \tag{II.24}$$

leads to the estimate

$$\frac{1}{2}\frac{d}{dt}|u-v|_{L_2}^2 \leq \sup_{x,y}|\partial_x u(x,y,t)|\,|u-v|_{L_2}^2, \tag{II.25}$$

and the Gronwall lemma gives (II.19).

II.1.3 General proof of the existence theorem

All functions in this part are supposed to have mean value zero:

$$\int_0^L u dx = 0.$$

The first we recall the theory of quasilinear evolution equations necessary to our proof (see [21, 22, 23, 24]).

Abstract frame and definitions

Let us consider the quasi linear evolution equation

$$\frac{du}{dt} + A(u)u = 0, t \geq 0; \quad u(0) = u_0, \tag{II.26}$$

and suppose that the following conditions are satisfied:

1. X and Y are reflexive Banach spaces such that $Y \subset X$ which continuous and dense immersion, and there exists a surjective isometry $S : Y \to X$.
2. If $W \subset Y$ is an open ball centered at the origin, then there exists $\beta \in \mathcal{R}$ such that for all $v \in W$, $-A(v)$ is the infinitesimal generator of a C_0- semigroup in X which satisfies
$$\|T_v(t)\| \leq e^{\beta t} \quad \text{for all } t \geq 0.$$
3. For all $v \in Y$, $SA(v)S^{-1} \supset A(v) + B(v)$, where $B(v) \in \mathcal{L}(X,X)$, the space of bounded linear operators on X, and $\|B(v)\|$ is uniformly bounded for $\|v\|$ bounded. Moreover, there is a constant μ_B such that
$$\|B(v_1) - B(v_2)\|_{\mathcal{L}(X,X)} \leq \mu_B \|v_1 - v_2\|_Y \quad \forall v_1, v_2 \in Y$$
4. For all $v \in Y$, $A(v) \in \mathcal{L}(Y,X)$. More precisely, $D(A(v)) \supset Y$ and the restriction to Y of $A(v)$ belongs to $\mathcal{L}(Y,X)$. Besides, there is a constant μ_A such that
$$\|A(v_1) - A(v_2)\|_{\mathcal{L}(Y,X)} \leq \mu_A \|v_1 - v_2\|_X \quad \forall v_1, v_2 \in Y$$

   The constants μ_A and μ_B depend only on $\max\{\|v_1\|_Y, \|v_2\|_Y\}$.

Theorem 5 *[23, 24] Under the hypothesis 1, 2, 3 and 4, given $u_0 \in Y$, there is $T > 0$, which depends only on $\|u_0\|_Y$, such that (II.26) has a unique solution $u \in C([0,T];Y) \cap C^1([0,T];X)$. The mapping $u_0 \mapsto u$ is continuous from Y to $C([0,T];Y)$.*

Besides, if $u \in C([0,T_{\max});Y) \cap C^1([0,T_{\max});X)$ is the unique noncontinuable solution of (II.26) and $0 < T < T_{\max}$, then there exists a neighborhood V of u_0 in Y such that the map $\tilde{u}_0 \mapsto \tilde{u}$is continuous from V to $C([0,T];Y)$.

An abstract problem associated to the KZK equation

For $s > [\frac{n}{2}] + 1$ we study the following problem in the Sobolev space $H^{s-2}((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$ of zero mean valued functions

$$\frac{du}{dt} + A(u)u = 0, \tag{II.27}$$

$$u(0) = u_0,$$

where $A(u)v = -\beta D_x^2 v - \gamma K v - u D_x v$ and K is defined by

$$\mathcal{F}(Ku)(m,\eta) = \begin{cases} \frac{-L\eta^2}{i2\pi m}\hat{u}(m,\eta), & \text{if } m \neq 0 \\ 0, & \text{if } m = 0. \end{cases} \tag{II.28}$$

The definition (II.4) of the operator ∂_x^{-1} which preserves the periodicity and having the mean value zero of considered functions will be reformulated now in terms of Fourier transform:

$$\partial_x^{-1} f = \sum_{k \neq 0} \frac{\widehat{f(k)}}{2\pi i \frac{k}{L}} e^{2i\pi \frac{kx}{L}}.$$

Besides, the function $f(x,y)$ is periodic on x and of mean value zero if and only if $\widehat{f}(0,\xi) = 0$ for all $\xi \in \mathcal{R}$.

The local existence

By using the method of proof from [22] (see also [21]), we obtain the existence of the KZK solution.

Suppose that $S = (1-\Delta)^{s/2} = \Lambda^s$.

Proposition 3 *Let the open ball $B_R \subset H^s([0,L] \times \mathcal{R}_y^{n-1}))$ $(s > [\frac{n}{2}]+1)$, then there exists $\beta(R) \in \mathbb{R}$ such that for all $v \in B_R$ the operator $\mp A$ is the infinitesimal generator of a C_0-semigroup in L_2 $T_v(t)$ which satisfies*

$$\|T_v(t)\| \leq e^{\beta(R)t} \qquad \forall t \geq 0.$$

Proof. Indeed, the linear operator K is the infinitesimal generator of a C_0-group of isometries in L_2.

1. $\overline{D(K)} = L_2$, which is obvious since $C^\infty \subset D(K)$.
2. K is conservative.

   If $u \in D(K)$, then

$$Re(u, Ku)_{L_2} = Re \int \sum_m \hat{u}(m,\eta) \left(-i\frac{\eta^2}{\frac{2\pi m}{L}}\right) \overline{\hat{u}}(m,\eta) d\eta = 0.$$

3. $I \pm K$ is surjective. We prove that $I+K$ is surjective, the other proof being similar.

   Let $f \in L_2$, then the function $u \in L_2$ having Fourier coefficients

$$\hat{u}(m,\eta) = \begin{cases} \hat{f}(m,\eta)\left[1 - i\frac{L\eta^2}{2\pi m}\right]^{-1}, & \text{if } m \neq 0 \\ 0, & \text{if } m = 0 \end{cases}$$

   belongs to $D(K)$ and satisfies $\hat{u}(n,\eta) + \widehat{Ku}(m,\eta) = \hat{f}(m,\eta)$. Therefore, $u + Ku = f$.

From what it follows that the operator K is the infinitesimal generator of a C_0-group of isometries in L_2.

By using the fact that $-(D_x^2 + \lambda I)$ is the infinitesimal generator of a C_0-semigroup of contractions in L_2 for $\lambda \geq 1$ [39, p.210] and by using the linearity of the operators, we see that the operator $\tilde{A} = -(K + D_x^2 + \lambda I)$ is the infinitesimal generator of a C_0-semigroup of contractions in L_2 (of functions with mean value zero).

Let $s > [\frac{n}{2}] + 1$ and let $v \in H^s([0,L] \times \mathcal{R}_y^{n-1}))$. We define the operator

$$\breve{A}(v) : D(\breve{A}(v)) \subset L_2 \to L_2$$

as follows:

$$D(\breve{A}(v)) = H_x^1, \quad \text{and}$$

$$\breve{A}(v)u = vD_xu + \frac{1}{L}\int_0^L vD_xuds, \quad u \in D(\breve{A}(v)).$$

Since $v \in C(\overline{]0,L[\times \mathcal{R}_y^{n-1}})$, we have $\breve{A}(v)u \in L_2$ and $D(\tilde{A}) \subset D(\breve{A}(v))$.

Let $v \in H^s([0,L] \times \mathcal{R}_y^{n-1})$ and $u \in D(\breve{A}(v))$ (x-periodic functions with mean value zero) then

$$(\breve{A}(v)u,u)_{L_2} = \int_0^L \int_{\mathcal{R}_y^{n-1}} v(D_xu)udxdy + \int_0^L \int_{\mathcal{R}_y^{n-1}} \left(\int_0^L vD_xuds\right) udxdy =$$
$$= \int_0^L \int_{\mathcal{R}_y^{n-1}} v(D_xu)udxdy + 0.$$

For $\phi \in C^\infty([0,L] \times \mathcal{R}_y^{n-1})$, an integration by parts gives for $\Omega = [0,L] \times \mathcal{R}_y^{n-1}$

$$\int\limits_\Omega v(D_x\phi)\phi dxdy = -\frac{1}{2}\int\limits_\Omega (D_xv)\phi^2 dxdy.$$

The density of $C^\infty(\Omega)$ in $H_x^1(\Omega)$ and the fact that v, $D_xv \in L_\infty(\Omega)$ show that the same equality holds with ϕ replaced by u (in the class of x-periodic functions with mean value zero). Thus

$$(\breve{A}(v)u,u)_{L_2} = -\frac{1}{2}\int\limits_\Omega (D_xv)u^2dxdy \geq -\frac{1}{2}\|D_xv\|_{L_\infty}\|u\|_{L_2}^2 \geq -C\|v\|_{H^s}\|u\|_{L_2}^2.$$

If $\tilde{\beta} \geq \tilde{\beta}_0(v) = C\|v\|_{H^s}$, then

$$((\breve{A}(v) + \tilde{\beta}I)u, u)_{L_2} \geq (-C\|v\|_{H^s} + \tilde{\beta})\|u\|_{L_2}^2 \geq 0$$

and therefore $-(\breve{A}(v) + \tilde{\beta}I)$ is dissipative.

Besides, for all $u \in D(\tilde{A})$,

$$\begin{aligned}
&\| - (\breve{A}(v) + \tilde{\beta}I)u\|_{L_2} \leq \|vD_x u\|_{L_2} + \frac{1}{L}\left\|\int_0^L vD_x u ds\right\|_{L_2} + \tilde{\beta}\|u\|_{L_2} \leq \\
&\leq \|v\|_{L_\infty}\|D_x u\|_{L_2} + C\|v\|_{L_\infty}\|D_x u\|_{L_2} + \tilde{\beta}\|u\|_{L_2} \leq \\
&\leq C\|v\|_{L_\infty}\|D_x u\|_{L_2} + \tilde{\beta}\|u\|_{L_2}.
\end{aligned}$$

By using the integration by parts one obtains

$$\|D_x u\|_{L_2} \leq C\|D_x^2 u\|_{L_2},$$

from which it follows that for all $\delta > 0$ there exists $C(\delta)$ such that

$$\begin{aligned}
\|D_x u\|_{L_2} &\leq \delta\|D_x^2 u\|_{L_2} + C(\delta) \leq \delta\|D_x^2 u\|_{L_2} + C(\delta)\|u\|_{L_2} \leq \\
&\leq \delta\|D_x^2 u\|_{L_2} + \delta\|Ku\|_{L_2} + \delta\|\lambda u\|_{L_2} + C(\delta)\|u\|_{L_2} = \delta\|\tilde{A}u\|_{L_2} + C(\delta)\|u\|_{L_2},
\end{aligned}$$

so since $\delta > 0$ we have

$$\|D_x u\|_{L_2} \leq \delta\|\tilde{A}u\|_{L_2} + C(\delta)\|u\|_{L_2}, \quad \forall u \in D(\tilde{A}).$$

If we set $\delta = (2C\|v\|_{L_\infty})^{-1}$, then we see that

$$\begin{aligned}
\| - (\breve{A}(v) + \tilde{\beta}I)u\|_{L_2} &\leq C\|v\|_{L_\infty}[\delta\|\tilde{A}u\|_{L_2} + C(\delta)\|u\|_{L_2}] + \tilde{\beta}\|u\|_{L_2} \leq \\
&\leq \frac{1}{2}\|\tilde{A}u\|_{L_2} + (CC(\delta)\|v\|_{L_\infty} + \tilde{\beta})\|u\|_{L_2}.
\end{aligned}$$

Therefore, by a well known perturbation theorem [39, p.82], we have that, for all $\tilde{\beta} \geq \tilde{\beta}_0(v)$, $\lambda \geq 1$ the operator $-(\tilde{A} + \breve{A}(v) + \tilde{\beta}I) = -(A + (\tilde{\beta} + \lambda)I)$ the infinitesimal generator of a C_0-semigroup of contractions in $L_2((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$ (in the class of x-periodic functions with mean value zero). Thus, if $T_v(t)$ is the semigroup generated by $-A$, then for $\beta = \tilde{\beta} + \lambda \geq \tilde{\beta}_0(v) + 1$

$$\|T_v(t)\| \leq e^{\beta t}.$$

Proposition 4 *Let $S = (1 - \Delta)^{s/2} = \Lambda^s : H^s \to L_2$ be a surjective isometry. Then for all $v \in H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$ (periodic in x with mean value zero) the operator $SA(v)S^{-1} - A(v)$ has a unique extension $B(v) \in \mathcal{L}(L_2(\Omega), L_2(\Omega))$:*

$$A(v) + B(v) \subset SA(v)S^{-1}.$$

Besides $\|B(v)\|$ is uniformed bounded for bounded $\|v\|_{H^s}$

$$\|B(v)\|_{\mathcal{L}(L_2(\Omega), L_2(\Omega))} \leq C\|v\|_{H^s},$$

and there exists a constant μ_B such that for all v_1, $v_2 \in H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$

$$\|B(v_1) - B(v_2)\|_{\mathcal{L}(L_2, L_2)} \leq \mu_B\|v_1 - v_2\|_{H^s}.$$

Proof. The proof repeats the proof of Lemma 2.3 in [22, p.389]. Using the estimate (II.9) we obtain that

$$\|(SA(v)S^{-1} - A(v))u\|_{L_2} = \|[\Lambda^s, v]S^{-1}D_x u\|_{L_2} \leq C\|v\|_s\|\Lambda^{-s}D_x u\|_{s-1} \leq$$

$$\leq C\|v\|_s\|\Lambda^{-1}D_x u\|_{L_2} \leq C\|v\|_s\|u\|_{L_2},$$

from where the result follows, if $\|v\|_s < CR$, then $\|B(v)\|_{\mathcal{L}(L_2,L_2)} \leq CR$ and

$$\|B(v_1) - B(v_2)\|_{\mathcal{L}(L_2,L_2)} \leq C\|v_1 - v_2\|_{H^s}.$$

Proposition 5 *Let $s > [\frac{n}{2}]+1$ and $v \in H^s((\mathcal{R}_x/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1})$ (periodic in x with mean value zero). Then $H^s((\mathcal{R}_x/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1}) \subset D(A(v))$, (the domain of definition of the operator A) and $A|_{H^s} \in \mathcal{L}(H^s, L_2)$. Besides, for all v, v_1, $v_2 \in H^s((\mathcal{R}_x/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1})$ there exist constants $\lambda(R)$, $\mu(R)$ such that*

$$\|A(v)\|_{\mathcal{L}(H^s,L_2)} \leq \lambda(R),$$

$$\|A(v_1) - A(v_2)\|_{\mathcal{L}(H^s,L_2)} \leq \mu(R)\|v_1 - v_2\|_{L_2}.$$

Proof. It is clear that $H^s \subset D(\tilde{A}) = D(A)$. Let $u \in H^s((\mathcal{R}_x/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1})$, then

$$\|A(v)u\|_{L_2} = \|-\beta D_x^2 u - \gamma K u - vD_x u\|_{L_2} \leq$$

$$\leq \beta\|D_x^2 u\| + \gamma\|Ku\| + \|vD_x u\| \leq C(\|u\|_{H^s} + \|v\|_{L_\infty}\|D_x u\|_{L_2}),$$

because

$$|\partial_x^{-1}u| = |\int\limits_0^x u dx + \frac{1}{L}\int_0^L s u ds| \leq C|\int_0^L u^2 dx|^{\frac{1}{2}} \leq C\|u\|_{L_2(\Omega)},$$

$$\|\partial_x^{-1}\triangle_y u\|_{L_2} \leq \|u\|_{H^2},$$

$$\|Ku\|_{L_2} = \|\widehat{Ku}\| = \|\frac{-L\eta^2}{2i\pi m}\hat{u}(m,\eta)\| \leq C\|\frac{L\eta^2}{2\pi m}\hat{u}(m,\eta)\|_{L_2} \leq C\|u\|_{H^s}, \quad (s > [\frac{n}{2}]+1).$$

Then

$$\|A(v)u\|_{L_2} \leq C(\|u\|_{H^s}+\|v\|_{L_\infty}\|D_x u\|_{L_2}) \leq C(\|u\|_{H^s}+\|v\|_{H^s}\|u\|_{H^1}) \leq C\|u\|_{H^s}(1+\|v\|_{H^s}),$$

thus $A(v) \in \mathcal{L}(H^s, L_2)$.

Now, for v, v_1, $v_2 \in H^s$, $s > [\frac{n}{2}]+1$ we see that

$$\|(A(v_1)-A(v_2))u\|_{L_2} \leq \|D_x u\|_{L_\infty}\|v_1-v_2\|_{L_2} \leq C\|D_x u\|_{H^{s-1}}\|v_1-v_2\|_{L_2} \leq C\|u\|_{H^s}\|v_1-v_2\|_{L_2}.$$

□

If we use the terminology of [22], with the help of the propositions 3, 4 and 5, the system $(L_2, H^s, \pm A, S)$ is admissible (i.e., it satisfies the hypothesis 1-4) and we can use the quasi linear theory of Kato from [23], [24] (see the paragraph II.1.3 and the theorem 5) which implies the following theorem:

Theorem 6 *Let $s > [\frac{n}{2}] + 1$ and let $u_0 \in H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$ (periodic in x with mean value zero). Then there exists $T > 0$, which depends only on $\|u_0\|_{H^s}$, such that the problem*

$$u_t - uD_xu - \beta D_x^2 u - \gamma \int_0^x \triangle_y u ds + \int_0^L \frac{s}{L}\triangle_y u ds = 0, \qquad \text{(II.29)}$$
$$u|_{t=0} = u_0$$

has a unique non-continuable solution

$$u \in C([0,T), H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})) \cap C^1([0,T), L_2((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}))$$

(periodic in x with mean value zero). Besides, if $0 < \bar{T} < T$, then the solution $u \in C([0,\bar{T}], H^s((\mathcal{R}_x/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1}))\cap C^1([0,\bar{T}], L_2((\mathcal{R}_x/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1}))$ depends continuously on the initial value u_0, i.e., the mapping $\tilde{u}_0 \mapsto \tilde{u}$ is continuous from $H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$ to $C([0,\bar{T}]; H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}))$.

Corollary 1 *Giving $u_0 \in H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$, $s > [\frac{n}{2}] + 1$, there exists $T > 0$ and a unique function $u \in C([0,T), H^s((\mathcal{R}_x/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1}))\cap C^1([0,T), L_2((\mathcal{R}_x/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1}))$ such that*

$$(u_t - uD_xu - \beta D_x^2 u)_x - \gamma\Delta_y u = 0, \qquad \text{(II.30)}$$
$$u|_{t=0} = u_0.$$

The solution depends continuously on the initial value u_0.

Proof. It is easy to verify that the solution of (II.29) is a solution of (II.30).

On the other hand, if $u \in C([0,T), H^s) \cap C^1([0,T), L_2)$ is a solution of (II.30), then

$$(u_t - uD_xu - \beta D_x^2 u)_x = \gamma\Delta_y u \in C([0,T); H^{s-2}) \hookrightarrow C([0,T); L_2).$$

Hence $u_t - uD_xu - \beta D_x^2 u \in H_x^1$, and

$$i\frac{2\pi}{L}m\mathcal{F}(u_t - \beta D_x^2 u - uD_xu)(m,\xi) = \mathcal{F}[(u_t - \beta D_x^2 u - uD_xu)](m,\xi) =$$
$$= \gamma\mathcal{F}(\triangle_y u)(m,\xi) = -\gamma\xi^2\widehat{u}(m,\xi).$$

Thus, for $m \neq 0$,

$$\widehat{(u_t)}(m,\xi) + i\frac{2\pi}{L}m\left(\beta\frac{4\pi^2}{L^2}m^2 + \gamma\frac{L\xi^2}{i2\pi m}\right)\widehat{u}(m,\xi) - \widehat{(uD_xu)}(m,\xi) = 0,$$

which means that, as from definition of the operator $\check{A}$ for $\check{A}(u)u \int_0^L uD_xudx = 0$,

$$(\widehat{u_t})(m,\xi) + \widehat{(\tilde{A}(u))}(m,\xi) + \widehat{(\check{A}(u), u)}(m,\xi) = 0,$$

equality which is also valid, in a trivial way, for $m = 0$. Therefore $u_t + A(u)u = 0$, and thus u is the solution of (II.29), which implies that the solution of (II.30) is unique.

Remark 7 *It may be seen from the equation that the solution u (periodic in x with mean value zero) belongs to $C^1([0,T), H^{s-2}((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}))$.*

The following theorem which can be proved in exactly the same way as theorem 2.3 from [21, p. 573], shows that $\bar{T}$ does not depend on s.

Theorem 7 (Regularity) *If $u \in C([0,\bar{T}], H^s(\Omega)) \cap C^1([0,\bar{T}], L_2(\Omega))$ is a solution of problem (II.30) and $u_0 \in H^{s'}$ with $s' > s > [\frac{n}{2}] + 1$ (for periodic on x mean value functions), then $u \in C([0,\bar{T}], H^{s'}) \cap C^1([0,\bar{T}], H^{s'-2})$ with the same $\bar{T}$.*

Remark 8 *For nonperiodic case the local existence can be easily proved using the estimate (II.8) in the form*

$$\|u\|_s^2 \leq C(s)\|u\|_s^3$$

and using the technique of [21] with regularization of system (II.27):

$$\frac{du}{dt} + A_\varepsilon(u)u = 0,$$

$$u(0) = u_0.$$

Here $A_\varepsilon(u)v = -D_x^2 v - K_\varepsilon v - uD_x v$ and K_ε is defined by

$$\mathcal{F}(K_\varepsilon u)(m,\xi) = \frac{-2\pi m\xi^2}{iL(\varepsilon + \frac{4\pi^2}{L^2}m^2)}\hat{u}(m,\xi).$$

By using the method of the proof from [21], we obtain the solution of KZK equation passing to the limit $\varepsilon \to 0$.

This regularization have been also done in [51].

Global existence in time of the solution for rather small initial data

Now let us prove that the maximal time of existence $T = \infty$ for rather small initial data.

Lemma 2 *For all $t \in [0,T)$,*

$$\|u\|_{H^s}^2 + \beta C_1(L)\int_0^t \|u\|_{H^s}^2 d\tau \leq C_2(s)\int_0^t \|u\|_{H^s}^3 d\tau + \|u_0\|_{H^s}^2, \tag{II.31}$$

where $\beta C_1(L)$ and $C_2(s)$ are positive constants. In particular, for all initial data u_0 satisfying

$$\|u_0\|_{H^s} \leq \frac{\beta C_1(L)}{C_2(s)},$$

the time of existence of the solution is $T = +\infty$ and

$$\|u\|_{C([0,+\infty),H^s)} \leq \frac{\beta C_1(L)}{C_2(s)}. \tag{II.32}$$

Proof. By using the regularity theorem for $u_0 \in H^{s+2}$, we have $u \in C([0,T), H^{s+2})$, and, thus, we apply the operator Λ^s to the equation

$$u'(t) + A(u(t))u(t) = 0, \quad t \in [0,T),$$

and take the inner product in L_2 with $\Lambda^s u$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2-(\Lambda^s(u_{xx}),\Lambda^s u)-(\Lambda^s(uu_x),\Lambda^s u)+\int\sum_m(1+\frac{4\pi^2}{L^2}m^2+\xi^2)^s\frac{L\xi^2}{i2\pi m}|\hat{u}(m,\xi)|^2d\xi=0.$$

Taking the real part of the former expression, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2+\|u_x\|_{H^s}^2-(\Lambda^s(uu_x),\Lambda^s u)=0.$$

Using now the proof of proposition 1 we obtain (II.31).

We define now $y(t)=\|u\|_{H^s}$, such that $y(0)=\|u_0\|_{H^s}$, thus we obtain the equation

$$\frac{d}{dt}(y^2)=C_2(s)y^3-\beta C_1(L)y^2.$$

Solving it we find that

$$y(t)=\left(\frac{C_2(s)}{\beta C_1(L)}-\left(\frac{C_2(s)}{\beta C_1(L)}-\frac{1}{\|u_0\|}\right)e^{\frac{\beta C_1(L)}{2}t}\right)^{-1},$$

from where, imposing $\|u_0\|_{H^s}\leq\frac{\beta C_1(L)}{C_2(s)}$, we obtain that $T=+\infty$ and it follows that

$$\|u(t)\|_{H^s}\leq y(t)\leq\frac{\beta C_1(L)}{C_2(s)}\quad\forall t\in[0,+\infty).$$

Lemma 3 *Let $s>[\frac{n}{2}]+1$ and suppose $u_0\in H^s((\mathcal{R}/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1})$ is such that $\partial_x^{-1}\triangle_y u_0=\phi_0\in H^{s-2}$ and $\|u_0\|_{H^s}\leq\frac{\beta C_1(L)}{C_2(s)}$. Then there exists a constant C such that*

$$\|u'(t)\|_{C([0,+\infty),H^{s-2})}\leq C. \tag{II.33}$$

Proof. For $t\in[0,+\infty)$ and $h>0$ sufficiently small, let $z(t)=h^{-1}[u(t+h)-u(t)]$. Then, having subtracted the KZK equation for $u(t)$ from the KZK equation for $u(t+h)$ and having divided by h, we obtain

$$z'(t)-D_x^2z(t)-Kz(t)-u(t)D_xz(t)=D_xu(t+h)z(t).$$

Let $l=s-2\geq 0$. Applying the operator Λ^l to the above equation, taking the inner product in L_2 with $\Lambda^l z(t)$, integrating by parts, and considering only real parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|_{H^l}^2+\|D_xz(t)\|_{H^l}^2-(\Lambda^l(u(t)D_xz(t)),\Lambda^l z(t))=(\Lambda^l f(t),\Lambda^l z(t)),$$

where $f(t)=D_xu(t+h)z(t)$. Thanks to [21, p.576, 577] one has the estimates

$$|(\Lambda^l(uD_xz),\Lambda^l z)|\leq C\|u\|_{H^s}\|z\|_{H^l}^2\quad\forall l>0,$$

$$|(\Lambda^l f,\Lambda^l z)|\leq C\|u(t+h)\|_{H^s}\|z\|_{H^l}^2\quad\forall l>0,$$

which with $-\|D_x z(t)\|_{H^l} \leq -\beta C_1(L)\|z(t)\|_{H^l}$ give

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2_{H^l} \leq C(\|u(t+h)\|_{H^s} + \|u(t)\|_{H^s})\|z(t)\|^2_{H^l} - \beta C_1(L)\|z(t)\|^2_{H^l} \leq$$

$$\leq (2C\frac{\beta C_1(L)}{C_2(s)} - \beta C_1(L))\|z(t)\|^2_{H^l} = \beta C_1(L)\left(\frac{2C}{C_2(s)} - 1\right)\|z(t)\|^2_{H^l},$$

from where, as the coefficient $\frac{2C}{C_2(s)} - 1$ can be, thanks to the choice of constants, negative, as well as positive or zero, we obtain that

$$\frac{d}{dt}\|z(t)\|^2_{H^l} \leq 0,$$

which implies using the Gronwall lemma that

$$\|z(t)\|_{H^l} \leq \|z(0)\|_{H^l}.$$

Passing to the limit for $h \to 0^+$ we find that

$$\|u'(t)\|_{H^l} \leq \|u'(0)\|_{H^l}.$$

But

$$\|u'(0)\|_{H^l} \leq \|D_x^2 u_0 + u_0 D_x u_0\|_{H^l} + \|Ku_0\|_{H^l},$$

and

$$\|Ku_0\|^2_{H^l} = \int\limits_{\mathbb{R}^{n-1}} \sum_m (1 + \frac{4\pi^2}{L^2}m^2 + \xi^2)^l \left|\frac{L\xi^2}{2\pi m}\hat{u}_0\right|^2 d\xi =$$

$$= \int\limits_{\mathbb{R}^{n-1}} \sum_m (1 + \frac{4\pi^2}{L^2}m^2 + \xi^2)^l \left|\frac{L}{2\pi m}\widehat{D_x\phi_0}\right|^2 d\xi =$$

$$= \int\limits_{\mathbb{R}^{n-1}} \sum_m (1 + \frac{4\pi^2}{L^2}m^2 + \xi)^l \left|\frac{i2\pi Lm}{2\pi Lm}\right|^2 |\hat{\phi}_0|^2 d\xi \leq \|\phi_0\|^2_{H^l}.$$

From where (II.33) follows. □

This concludes the proof in general case of theorem 4 and we can reformulate our result by the following theorem.

Theorem 8 *Let $u_0 \in H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}))$, $s > [\frac{n}{2}] + 1$, periodic in x with mean value zero, such that $D_x^{-1}\triangle_y u_0 \in H^{s-2}((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}))$, i.e., there exists $\varphi_0 \in H^{s-2}((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}))$ with $D_x\varphi_0 = \triangle_y u_0$, and the norm $\|u_0\|_{H^s} \leq \frac{\beta C_1(L)}{C_2(s)}$ is rather small. Then there exists a unique global solution of the problem*

$$(u_t - u_{xx} - uu_x)_x - \triangle_y u = 0,$$
$$u(0) = u_0$$

$u \in C([0, +\infty), H^s((\mathcal{R}_x/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1})))$ with $u' \overset{note}{=} du/dt \in L_\infty([0, +\infty), H^{s-2}((\mathcal{R}_x/(L\mathcal{Z}))\times \mathcal{R}_y^{n-1})))$.

II.2 Blow-up and singularities

The first remark is that for $\nu = 0$ (or $\beta = 0$) and for function independent of y the KZK equation

$$(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0 \qquad \text{in } \mathcal{R}_{t_+} \times \mathcal{R}_x \times \Omega \tag{II.34}$$

becomes Burgers equation which is known to exhibit singularities. On the other hand the derivation and the approximation results of the following section show that any solution of the KZK equation has in its neighborhood a solution of the isentropic Euler equation. Once again it is known that such solution even with smooth initial data may exhibit singularities (cf. [15] or [47]). These observations are reflected by the fact that for $\beta = 0$ and $\gamma > 0$ the equation (II.34) may generate singularities.

We prove the geometric blow-up result using the method of S. Alinhac, which is based on the fact that the studied equation degenerates to the Burgers equation. In fact Alinhac's method is the generalized method of characteristics for the Burgers equation adapted to the multidimensional case. As we can see the equation (II.34) possess all this main properties, and gives us the reason to apply it.

For instance one has the theorem:

Theorem 9 *The equation*

$$(u_t - uu_x)_x - \gamma \Delta_y u = 0 \qquad \textit{in } \mathcal{R}_{t_+} \times \mathcal{R}_x \times \Omega \tag{II.35}$$

with Neumann boundary condition on $\partial\Omega$ has no global in time smooth solution if

$$\sup_{x,y} \partial_x u(x, y, 0)$$

is large enough with respect to γ.

Remark 9 *As we can see from [11] the result of the theorem perfectly confirms the numerical results. In practically from figures I.7 and I.8 one observes that the KZK equation as soon as β becomes smaller (for $\beta \to 0$) has a quasi shock more approaching to the shock wave, into which it degenerates for $\beta = 0$.*

Proof. The proof follows the ideas of S. Alinhac ([3], [4] and [5]). First the blow-up is observed for $\gamma = 0$ and related to a singularity in the projection of an unfolded "blow-up system". Second the properties of this unfolded blow-up system are shown to be stable under small perturbations. One uses a Nash- Moser theorem with tamed estimates and this is the reason why will exists a T^* such that:

$$\lim_{t \to T^*} (T^* - t) \sup_{x,y} \partial_x u(x, y, t) > 0.$$

Remark 10 *The Nash-Moser theory and the definition of the tamed estimates can be found in [8].*

Remark 11 *An equation of the type (II.35) is introduced by Alinhac to analyze the blow-up of multidimensional (in $\mathcal{R}^{2+1}$) nonlinear wave equation by following the wave cone*

$$\partial_t^2 u - \triangle_x u + \sum_{0 \le i,j,k \le 2} g_{ij}^k \partial_k u \partial_{ij}^2 u = 0,$$

where

$$x_0 = t, \quad x = (x_1, x_2), \quad g_{ij}^k = g_{ji}^k,$$

with small smooth initial data (see [6]). In fact this corresponds to the same scaling as the KZK equation because from this wave equation with some changes of variable and approximate manipulations Alinhac obtains (see [4, 6, 7])

$$\partial_{xt}^2 u + (\partial_x u)(\partial_x^2 u) + \epsilon \partial_y^2 u = 0.$$

This is the reason for the analogy.

More precisely for a "beam" (rotationally invariant around the x axis) in 3 space variables the KZK equation has the form

$$\tilde{u}_{xt} - (\tilde{u}\tilde{u}_x)_x - \gamma \frac{1}{y} \tilde{u}_y - \gamma \tilde{u}_{yy} = 0, \tag{II.36}$$

$$\tilde{u}|_{t=0} = u_0(x, y), \quad \tilde{u}|_{x=K} = 0.$$

If we consider the KZK equation in $\mathcal{R}^3$ with $y = (y_1, y_2)$ for general case, the term $\frac{1}{y}\tilde{u}_y$ must be omitted and ∂_y be replaced by ∇_y.

Let $A_0 > 0$ be a fixed constant and $u_0 \in C^\infty$ be a function of variables x, y defined in the domain

$$\{(x, y) | x \in [-A_0, K], \quad y \in [r_0, r_1], \quad r_0 > 0\}.$$

For the reason of technical simplification, we assume

$$u_0(K, y) = \partial_x u_0(K, y) = 0.$$

Let the function $\partial_x u_0$ have on $[-A_0, K] \times [r_0, r_1]$ $(r_0 > 0)$ a unique positive maximum in the point

$$\begin{gathered} m_0 = (x_0, y_0), \quad -A_0 < x_0 < K, \quad \textit{such} \quad \textit{that} \quad \partial_x u_0(m_0) > 0, \\ \nabla_{x,y}(\partial_x u_0)(m_0) = 0, \quad \nabla_{x,y}^2(\partial_x u_0)(m_0) \ll 0. \end{gathered} \tag{II.37}$$

The condition (II.37) is the necessary condition for geometric blow-up.

For a $\bar{T} > 0$, which is the blow-up time and is unknown, and a function $A_0(y, t) > 0$ to be specified (with $A_0(y, 0) = A_0$), we have in the domain

$$D = \{(x, y, t) | x \in [-A_0(y, t), K], \quad y \in [r_0, r_1], \quad r_0 > 0, \quad t \in [0, \bar{T}[\} \tag{II.38}$$

a free boundary problem.

Let in (II.36) $\tilde{u} = u_x$, then

$$u_{xxt} - (u_x u_{xx})_x - \gamma \frac{1}{y} u_{yx} - \gamma u_{yyx} = 0,$$

and so we have

$$L(u) \equiv u_{xt} - u_x u_{xx} - \gamma \frac{1}{y} u_y - \gamma u_{yy} = 0, \tag{II.39}$$

$$u|_{t=0} = \partial_x^{-1} u_0, \quad u|_{x=K} = 0, \quad u_x|_{x=K} = 0.$$

The change of variables $\Phi : (s, Y, T) \to (x = \varphi(s, Y, T), y = Y, t = T)$, where $\varphi(s, y, t)$ is some unknown function such that $\partial_s \varphi > 0$, $\varphi|_{t=0} = s$, $\varphi|_{s=K} = 0$, allows to construct a blow-up system if we set

$$w(s, y, t) = u(\varphi(s, y, t), y, t), \quad v(s, y, t) = u_x(\varphi(s, y, t), y, t). \tag{II.40}$$

The method of introducing $x = \varphi(s, y, t)$ is based on the method of characteristics which naturally appears for Burgers' equation

$$(\partial_t - u\partial_x)u = 0, \quad u|_{t=0} = u_0(x). \tag{II.41}$$

Indeed, $x = \varphi(s, t) = s - tu_0(s)$, and the Cauchy problem

$$x_{tt} = 0, \quad x_t|_{t=0} = -u_0(s), \quad x|_{t=0} = s$$

with notation $x_t = \varphi_t = -u(\varphi(s, t), t) = -v$ becomes the blow-up system

$$v_t = 0, \quad v = -\varphi_t, \quad \varphi|_{t=0} = s, \quad v|_{t=0} = u_0. \tag{II.42}$$

Since $v = u(\varphi(s, t), t)$ and $v_s = u_x \varphi_s$, in the case where the solutions of (II.42) satisfy the conditions $v_s \neq 0$, $\varphi_s = 0$ in some point, $u_x = v_s/\varphi_s$ becomes infinite. According to the terminology of Alinhac, the solution u displays geometric blow-up, because the blow-up of u does not come from the blow-up of v, but from the singularity of the change of variables Φ.

From (II.40) we have

$$w_s = u_\varphi \varphi_s|_{\varphi=x} = u_x \varphi_s = v\varphi_s,$$
$$A \equiv w_s - v\varphi_s = 0. \tag{II.43}$$

Lets compute the derivatives in new variables which are present in (II.39)

$$u_{xx} = v_s(\varphi_s)^{-1}, \quad u_{xt} = v_t - v_s(\varphi_s)^{-1}\varphi_t,$$

$$u_y = w_y - v\varphi_y, \quad u_{yy} = w_{yy} - 2v_y\varphi_y + v_s(\varphi_s)^{-1}\varphi_y^2 - v\varphi_{yy}.$$

So we obtain

$$v_t + \frac{v_s}{\varphi_s}\left(-\varphi_t - v - \gamma\varphi_y^2\right) - \gamma(w_{yy} - 2v_y\varphi_y - v\varphi_{yy}) - \frac{\gamma}{y}(w_y - v\varphi_y) = 0,$$

$$L(u)(\varphi(s, y, t), y, t) = \mathcal{E}\frac{v_s}{\varphi_s} + \mathcal{R},$$

where

$$\mathcal{E} = -\varphi_t - v - \gamma\varphi_y^2, \tag{II.44}$$

$$\mathcal{R} = v_t - \gamma(w_{yy} - 2v_y\varphi_y - v\varphi_{yy}) - \frac{\gamma}{y}(w_y - v\varphi_y). \tag{II.45}$$

In this case the blow-up system is

$$\mathcal{A} = 0, \quad \mathcal{E} = 0, \quad \mathcal{R} = 0, \tag{II.46}$$

$$v|_{t=0} = u_0, \quad w|_{t=0} = \partial_s^{-1} u_0, \quad \varphi|_{t=0} = s, \quad w|_{s=K} = v|_{s=K} = \varphi|_{s=K} = 0.$$

From (II.40) it is easy to see that if we find the smooth solution of the blow-up system (II.46) such that in some point $\varphi_s = 0$, then the function u_{xx} has blow-up in this point which corresponds to the blow-up of $\tilde{u}_x$ of KZK equation's solution.

According to the change of variables we obtain that if the function $A_0(y,t)$ in definition of the domain D (II.38) is

$$A_0(y,t) = -\varphi(-A_0, y, t),$$

then

$$D_b = \{(s,y,t) | s \in [-A_0, K], \quad y \in [r_0, r_1], \quad r_0 > 0, \quad t \in [0, \bar{T}[\}.$$

In the domain D_b the blow-up system (II.46) has the form

$$\begin{cases} w_s - v\varphi_s = 0, \\ -\varphi_\tau - (v + \gamma\varphi_y^2) = 0, \\ v_\tau - \gamma(w_{yy} - 2v_y\varphi_y - v\varphi_{yy}) - \frac{\gamma}{y}(w_y - v\varphi_y) = 0, \\ v|_{\tau=0} = u_0, \quad w|_{\tau=0} = \partial_s^{-1}u_0, \quad \varphi|_{\tau=0} = s, \\ w|_{s=K} = v|_{s=K} = \varphi|_{s=K} = 0. \end{cases} \tag{II.47}$$

Note that for $\gamma = 0$ the problem (II.39) becomes the Burgers equation for $u_1 = u_x$ with the initial condition $u_1|_{t=0} = \partial_x \partial_x^{-1} u_0 = u_0$. This problem has a unique solution of C^∞ in D with the blow-up time

$$\bar{T} = \bar{T}_0 = \left(\sup_{x,y} \partial_x u_0 \right)^{-1}. \tag{II.48}$$

In this point we have

$$\partial_x u = \frac{\partial_x u_0(x_0, y_0)}{1 - \bar{T}_0 \partial_x u_0(x_0, y_0)} = \infty.$$

The blow-up system (II.46) for $\gamma = 0$

$$v_t = 0, \quad v = -\varphi_t, \quad w_s - v\varphi_s = 0$$

has an explicit solution

$$\varphi = s - tu_0(s,y), \quad v = u_0(s,y), \quad w = \partial_s^{-1}u_0 - \frac{u_0^2}{2}t,$$

from which it follows that $\partial_s \varphi$ has value zero "for the first time" at the time $\bar{T}_0$ (II.48) in the unique point $M_0 = (x_0, y_0, \bar{T}_0)$. This means with notation

$$\tilde{x}_0 = x_0 - \bar{T}_0 u_0(x_0, y_0), \quad \tilde{M}_0 = (\tilde{x}_0, y_0, \bar{T}_0),$$

that u_x has a blow-up in the point $\tilde{M}_0$.

We denote the blow-up system (II.47) by

$$\mathcal{L}(\varphi, v, w) = 0, \text{where } \mathcal{L} = (\mathcal{E}, \mathcal{R}, \mathcal{A}) \tag{II.49}$$

which we have to solve in the domain D_b. Set the field $Z = -\partial_t - 2\gamma\varphi_y\partial_y$, and the notation

$$Q = -\gamma\partial_y^2, \quad \dot{z} = \dot{w} - v\dot{\varphi}.$$

The choice of $\dot{z}$ is natural and can be explained by linearization of the relation $u(\Phi) = w$ which gives $\dot{u}(\Phi) + u'(\Phi)\dot{\Phi} = \dot{w}$ from what

$$\dot{z} \equiv \dot{u}(\Phi) = \dot{w} - v\dot{\varphi}.$$

Here the "physical" objects are u and $\dot{u}(\Phi)$ but not w and $\dot{w}$ which depend on the change of variables Φ. We can say that the introduction of $\dot{z}$ cancels the arbitrariness in the choices of w and Φ. Then

$$\mathcal{A} = w_s - v\varphi_s = 0, \quad \mathcal{E} = Z\varphi - Q\varphi - v = 0,$$

$$\mathcal{R} = -Zv - vQ\varphi + Qw - \frac{\gamma}{y}(w_y - v\varphi_y) = 0,$$

and the linearized blow-up system has the form (we note $\mathcal{L}'_{\varphi,v,w}(\dot{\varphi},\dot{v},\dot{w}) = \mathcal{L}'(\dot{\varphi},\dot{v},\dot{w})$)

$$\mathcal{E}'(\dot{\varphi},\dot{v},\dot{w}) = \dot{f}, \quad \mathcal{E}'(\dot{\varphi},\dot{v},\dot{w}) = -\dot{\varphi}_t - \dot{v} - 2\gamma\varphi_y\dot{\varphi}_y = Z\dot{\varphi} - \dot{v},$$

$$\mathcal{R}'(\dot{\varphi},\dot{v},\dot{w}) = \dot{g}, \ \mathcal{R}'(\dot{\varphi},\dot{v},\dot{w}) = -Z\dot{v} + Q\dot{z} + (Qv)\dot{\varphi} - (Q\varphi)\dot{v} - \frac{\gamma}{y}(\dot{z}_y - \dot{v}\varphi_y + \dot{\varphi}v_y),$$

$$\mathcal{A}'(\dot{\varphi},\dot{v},\dot{w}) = \dot{h}, \quad \mathcal{A}'(\dot{\varphi},\dot{v},\dot{w}) = \dot{z}_s + v_s\dot{\varphi} - \varphi_s\dot{v},$$

or simply

$$\mathcal{L}'(\dot{\varphi},\dot{v},\dot{w}) = \dot{l}. \tag{II.50}$$

Following the structure of [3, p.23] we find

$$\begin{aligned} &Z\partial_s\dot{z} - \varphi_s Q\dot{z} + ((Q\varphi)\partial_s + \frac{\gamma}{y}\varphi_s\partial_y)\dot{z} + \alpha_1 Z\dot{\varphi} + \alpha_2\dot{\varphi} = \\ &= -\varphi_s\mathcal{R}' + (Z + Q\varphi)\mathcal{A}' - (Z\varphi_s + \frac{\gamma}{y}\varphi_s v_y)\mathcal{E}', \end{aligned} \tag{II.51}$$

$$Z^2\dot{\varphi} - (Q_1\varphi)Z\dot{\varphi} + (Q_1 v)\dot{\varphi} + Q_1\dot{z} = (Z - Q_1\varphi)\mathcal{E}' - \mathcal{R}'. \tag{II.52}$$

Here

$$\alpha_1 = -\partial_s\mathcal{E} - \frac{\gamma}{y}\varphi_s v_y, \quad \alpha_2 = -Q_1\mathcal{A} - \partial_s\mathcal{R}, \quad Q_1 = -Q + \frac{\gamma}{y}\partial_y.$$

The coefficients α_1 and α_2 are small if $\mathcal{E}$, $\mathcal{R}$, $\mathcal{A}$ and their derivatives are small. So the system (II.51), (II.52) in $\dot{\varphi}$, $\dot{z}$ is almost decoupled. In a Nash-Moser scheme aimed at solving $\mathcal{E} = 0$, $\mathcal{R} = 0$, $\mathcal{A} = 0$ we could view these terms with the coefficients α_i as "quadratic errors". But we cannot just neglect them, because this would correspond to solving the linearized system up to quadratic errors divided by φ_s, which is not acceptable in the framework of smooth functions.

We need to introduce the identities (II.51), (II.52) with the help of which we will solve our linearized blow-up system.

The idea (see for example [3]) is the following. Suppose that we can solve exactly in D_b the system (II.51), (II.52) in $(\dot{z},\dot{\varphi})$ with $\mathcal{E}'$, $\mathcal{R}'$ and $\mathcal{A}'$ replaced by given quantities $\dot{f}$, $\dot{g}$

and $\dot{h}$. Determine now $\dot{v}$ from $\mathcal{E}' = \dot{f}$ using the relation $\mathcal{E}' = Z\dot{\varphi} - \dot{v}$. For the functions $(\dot{\varphi}, \dot{v}, \dot{w})$ thus obtained, we have then

$$\mathcal{E}' = \dot{f},\ \mathcal{R}' = \dot{g},\ (Z + \alpha_3)(\mathcal{A}' - \dot{h}) = 0.$$

Taking into account the boundary conditions on (φ, v, w) (hence on $(\dot{\varphi}, \dot{v}, \dot{w})$), suppose we can ensure that $\mathcal{A}' - \dot{h}$ vanishes on some part Γ of the boundary of D_b; suppose also that D_b is under the influence of Γ for Z; then we obtain $\mathcal{A}' = \dot{h}$, and the linearized blow-up system is exactly solved.

But before to do it, let us to pass from the free boundary domain to a fix one where we will solve our linearized blow-up system.

Consider the surface Σ through $\{t = 0, s = K\}$ which is characteristic for the operator $Z\partial_s - \varphi_s Q$

$$t = -\psi(s, y),$$

where ψ is solution of the Cauchy problem

$$(1 + \gamma(2\varphi_y)\psi_y)\psi_s + \gamma\varphi_s\psi_y^2 = 0, \quad \psi(K, y) = 0. \tag{II.53}$$

Equation (II.53) has, for small γ, a smooth solution in the appropriate domain. This solution is $O(\gamma)$ and decreases in s.

We now perform the change of variables

$$\tilde{x} = s, \quad \tilde{y} = y, \quad \tilde{t} = (\chi(\frac{t}{\eta}) - 1)t - (t + \psi)\chi(\frac{t}{\eta}), \tag{II.54}$$

where $\chi \in C^\infty$ is 0 near 1 and 1 near 0, and $\eta > 0$ is small enough. The still unknown domain

$$D_\psi = \{-A_0 \leq s \leq K, y \in [r_0, r_1], -\psi \leq t \leq \bar{t}_\gamma\}$$

is taken by this change into

$$\tilde{D} = \{-A_0 \leq \tilde{x} \leq K, \tilde{y} \in [r_0, r_1], -\bar{T} = -\bar{t}_\gamma \leq \tilde{t} \leq 0\}.$$

The change of variables (II.54) gives

$$\partial_s = \partial_{\tilde{x}} + \tilde{t}_s\partial_{\tilde{t}}, \quad \partial_y = \partial_{\tilde{y}} + \tilde{t}_y\partial_{\tilde{t}}, \quad \partial_t = \tilde{t}_t\partial_{\tilde{t}},$$

where

$$\tilde{t}_s = O(\gamma), \quad \tilde{t}_y = O(\gamma), \quad \tilde{t}_t = -1 + O(\gamma)$$

are known functions.

In the domain $\tilde{D}$ the blow-up time $\bar{T}$ is unknown and is the part of the problem, so we still have a free boundary problem. To handle this problem, we introduce a parameter λ close to zero and perform the change of variables

$$\tilde{x} = X, \quad \tilde{y} = Y, \quad \tilde{t} = \tilde{t}(T, \lambda) \equiv T + \lambda T(1 - \chi_1(T)), \tag{II.55}$$

where χ_1 is one near zero and zero near $-\bar{T}_0$ (defined in (II.48)).

This implies that

$$D_{bf} = \{(X,Y,T)|X \in [-A_0, K], \quad Y \in [r_0, r_1], \quad r_0 > 0, \quad T \in]-\bar{T}_0, 0]\}. \tag{II.56}$$

The two successive changes of variables (II.54), (II.55)

$$(s,y,t) \to (\tilde{x},\tilde{y},\tilde{t}) \to (X,Y,T)$$

imply that our blow-up system (II.49) are transformed into

$$\widetilde{\mathcal{L}}(\lambda,\varphi,v,w) = 0. \tag{II.57}$$

The system (II.51), (II.52) is also changed to some system $\widetilde{\text{(II.51)}}$, $\widetilde{\text{(II.52)}}$, the exact view of which we will find later.

We say that φ satisfies condition **(H)** in D_{bf} if, for some boundary point $M = (X,Y,-\bar{T}_0) \in \overline{D}_{bf}$

$$\begin{cases} \varphi_X \geq 0, \quad \varphi_X(X,Y,T) = 0 \Leftrightarrow (X,Y,T) = M, \\ \varphi_{XT}(M) < 0, \quad \nabla_{X,Y}(\varphi_X)(M) = 0, \nabla^2_{X,Y}(\varphi_X)(M) >> 0. \end{cases} \tag{II.58}$$

The problem we want to solve in D_{bf} is

1. $\widetilde{\mathcal{L}}(\lambda,\varphi,v,w) = 0,$
2. φ satisfies **(H)** in D_{bf}.

We are following the plan which is explained in details later:

1. We are assuming that we can solve the linearized system $\widetilde{\text{(II.51)}}$, $\widetilde{\text{(II.52)}}$ and so $\partial_{\varphi,v,w}\widetilde{\mathcal{L}}\dot{\Psi} = \dot{f}$ in flat functions with a tame estimate.
2. Resolution of $\widetilde{\mathcal{L}} = 0$ using the above fact by Nash-Moser iteration process reproducing in each step $\varphi^{(n)}$ satisfying the condition **(H)** $(\forall n)$ with the help of some techniques based on the structure of **(H)** and the implicit function theorem (fundamental lemma of Alinhac).
3. We prove the point 1 for the system $\widetilde{\text{(II.51)}}$, $\widetilde{\text{(II.52)}}$.

For the start point of the Nash-Moser iteration process we choose

$$\lambda^{(0)} = 0, \; \varphi^{(0)} = \bar{\varphi}^{(0)}, \; v^{(0)} = \bar{v}^{(0)}, \; w^{(0)} = \bar{w}^{(0)}.$$

Let us now determinate the functions denoted by $\bar{\varphi}^{(0)}$, $\bar{v}^{(0)}$, $\bar{w}^{(0)}$.

For $\gamma = 0$ and $\lambda = 0$ the exact solution of blow-up system with initial conditions

$$\varphi(X,Y,0) = X, \quad \partial_T\varphi(X,Y,0) = u_0(X,Y)$$

is

$$\bar{\varphi}_0 = X + Tu_0(X,Y), \quad \bar{v}_0 = u_0(X,Y), \quad \bar{w}_0 = \partial_s^{-1}u_0 + \frac{1}{2}Tu_0^2,$$

we can also notice collecting all change of variables that

$$\bar{\varphi}_0(s,y,T) = \bar{\varphi}_0(s,y,\tilde{t}(T,0)) = \bar{\varphi}_0(X,Y,T).$$

So for $\lambda = 0$ ($T = \tilde{t}$) the approximate solution of the first step of Nash-Moser process existing for the time $T \in]-\bar{T}_0, 0]$ can be obtained by gluing together the local true solution $(\bar{\varphi}, \bar{v}, \bar{w})$ of (II.57), which exists in a small strip $\{-\eta_1 \leq T \leq 0\}$ of D_{bf}, to $(\bar{\varphi}_0, \bar{v}_0, \bar{w}_0)$ in the following form:

$$\bar{\varphi}^{(0)}(X, Y, T) = \chi\left(\frac{-T}{\eta_1}\right)\bar{\varphi}(X, Y, T) + \left(1 - \chi\left(\frac{-T}{\eta_1}\right)\right)\bar{\varphi}_0(X, Y, T).$$

We have then [6, 7] that for this approximate solution

$$\widetilde{\mathcal{L}}(0, \bar{\varphi}^{(0)}, \bar{v}^{(0)}, \bar{w}^{(0)}) = \bar{l}^{(0)},$$

where $\bar{l}^{(0)}$ is smooth, flat on $\{X = M\}$, zero near $\{T = 0\}$, and zero for $\gamma = 0$.

Since the solution we start from has already all the good traces on $\{X = M\}$ and $\{T = 0\}$, we need only to solve the linearized system in flat functions.

The approximate solution $\bar{\varphi}^{(0)}$ satisfies, thanks to (II.37) and $\partial_X \bar{\varphi} > 0$ close to $\{T = 0\}$, the condition **(H)** (II.58) in point $M = (m_0, -\bar{T}_0)$, where $\bar{T}_0$ is from (II.48).

Solving $\widetilde{\mathcal{L}} = 0$ by a Nash-Moser iteration process, which we start from the point $(0, \bar{\varphi}^{(0)}, \bar{v}^{(0)}, \bar{w}^{(0)})$, further modifications of $\bar{\varphi}^{(0)}$ will yield functions not satisfying **(H)** anymore. So we have to make sure that we can reproduce at each step the new function φ satisfying the condition **(H)**. This is realized thanks to the "fundamental lemma" from [7].

For the multidimensional case when $n = 3$ and $y = (y^1, y^2)$ we use [6, p.16] for determining the final form of the fixed domain D_{bf} where we want to solve our blow-up system (II.57), and so the domain D_{bf} is the domain bounded by the planes $X = -A_0$, $X = M$, $T = -\bar{T}_0$, $T = 0$, the plan containing $(\delta_1, I_1 I_3)$, and the plane containing $(\delta_2, I_2 I_4)$. These planes have normal $n_\pm = (-\eta_1, \pm\nu, 1)$ and are described by

$$\delta_1 = \left\{T = 0, Y - (y_0 - y_1) = -\frac{\eta_1}{\nu}(X - M)\right\},\ I_1 = (y_0 - y_1 - \bar{T}_0/\nu, -\bar{T}_0),$$

$$\delta_2 = \left\{T = 0, Y - (y_0 - y_1) = \frac{\eta_1}{\nu}(X - M)\right\},\ I_2 = (y_0 + y_1 + \bar{T}_0/\nu, -\bar{T}_0),$$

$$I_3 = (Y = y_0 - y_1, T = 0), \quad I_4 = (y_0 + y_1, T = 0),$$

where y_0 is from (II.37), y_1 and ν are fixed such that $y_0 - y_1 \leq y \leq y_0 + y_1$, $0 < \bar{T}_0 < \frac{1}{2}\nu y_1$ (for the explication of the details see [6, p.16]). It is understood that A_0 and the small η_1 are chosen such that $\bar{\varphi}^{(0)}$ satisfies **(H)** for a point M interior to the lower boundary of D_{bf}.

The linearized operator of $\tilde{\mathcal{L}}$ at the point (λ, φ, v, w) is denoted by

$$\tilde{\mathcal{L}}'_{\lambda,\varphi,v,w}(\dot{\lambda}, \dot{\varphi}, \dot{v}, \dot{w}) = \partial_\lambda \tilde{\mathcal{L}}(\lambda, \varphi, v, w)\dot{\lambda} + \partial_\varphi \tilde{\mathcal{L}}(\lambda, \varphi, v, w)\dot{\varphi} + \partial_v \tilde{\mathcal{L}}(\lambda, \varphi, v, w)\dot{v} + \partial_w \tilde{\mathcal{L}}(\lambda, \varphi, v, w)\dot{w}.$$

We can see that with $q = \partial_\lambda \tilde{t} / \partial_T \tilde{t}$

$$\partial_\lambda \tilde{\mathcal{L}} + \partial_\varphi \tilde{\mathcal{L}}(\varphi_T q) + \partial_v \tilde{\mathcal{L}}(v_T q) + \partial_w \tilde{\mathcal{L}}(w_T q) = q \tilde{\mathcal{L}}_T.$$

Thus the linearized system

$$\tilde{\mathcal{L}}'_{(\lambda,\varphi,v,w)}(\dot{\lambda},\dot{\varphi},\dot{v},\dot{w}) = \dot{l},$$

is equivalent to

$$\widetilde{\mathcal{L}'}_{\varphi,v,w}(\dot{\Phi},\dot{V},\dot{W}) = \dot{l} - q\dot{\lambda}\tilde{\mathcal{L}}_T \tag{II.59}$$

with

$$\dot{\Phi} = \dot{\varphi} - \dot{\lambda}q\varphi_T, \quad \dot{V} = \dot{v} - \dot{\lambda}qv_T, \quad \dot{W} = \dot{w} - \dot{\lambda}qw_T, \quad \dot{Z} = \dot{W} - v\dot{\Phi},$$

here $\widetilde{\mathcal{L}'} = (\widetilde{\mathcal{E}'},\widetilde{\mathcal{R}'},\widetilde{\mathcal{A}'})$ denotes the linear system obtained from the linearized blow-up system (II.50) in the original variables s, y, t by the two successive changes of variables (II.54), (II.55).

Assume now that, at some stage of the Nash-Moser iteration process aimed at solving $\tilde{L} = 0$, the function φ satisfies **(H)**. We solve first (II.59), neglecting $q\dot{\lambda}\tilde{\mathcal{L}}_T$ in the right-hand side, since it is a quadratic error. We choose then, once $\dot{\phi}$ is known, $\dot{\lambda}$ such that

$$\varphi + \dot{\varphi} = \varphi + \dot{\Phi} + \dot{\lambda}q\varphi_T$$

satisfies again condition **(H)** for some point on the lower boundary of the fixed domain D_{bf}. It is possible using Alinhac's Fundamental lemma which can be found with the iteration scheme of following resolution of the problem in [7, p.110-112].

Hence, to finish our proof it is enough to solve the transformed linear system

$$\widetilde{\mathcal{E}'} = \dot{f},\ \widetilde{\mathcal{R}'} = \dot{g},\ \widetilde{\mathcal{A}'} = \dot{h}$$

in D_{fb}.

For this we use the system (II.51), (II.52) transformed by the two changes of variables into $\widetilde{(II.51)}$, $\widetilde{(II.52)}$ which we want to write now explicitly.

First let us introduce the following notations

$$\hat{Z} = \partial_T + \gamma z_0\partial_Y, \quad S = \partial_X + \gamma s_0\partial_T.$$

The composition of the two changes of variables operates the following transformation of operators (to avoid introducing unnecessary notation, we denote by $*$ known functions):

$$\partial_s = \partial_X + \gamma s_0(X,Y,T,\lambda)\partial_T \equiv S, \quad \partial_y = \partial_Y + \gamma * (X,Y,T,\lambda)\partial_T,$$

$$\partial_t = (-1 + \gamma * (X,Y,T,\lambda))(\partial_T\tilde{t})^{-1}\partial_T, \quad Z = (1 + \gamma * (X,Y,T,\lambda,\varphi,\varphi_Y,\varphi_T))\hat{Z}$$

and the transformed linearized system (II.51), (II.52) has the form

$$\hat{Z}S\dot{Z} + \gamma(S\varphi)N\dot{Z} + \gamma l_1(\dot{Z}) + \alpha_1\hat{Z}\dot{\Phi} + \alpha_2\dot{\Phi} = \dot{f}_1, \tag{II.60}$$

$$\hat{Z}^2\dot{\Phi} + \gamma\beta_1\hat{Z}\dot{\Phi} + \gamma\beta_2\dot{\Phi} + \gamma\hat{Z}H\dot{Z} + \gamma l'_1(\dot{Z}) = \dot{f}_2. \tag{II.61}$$

Here

1. $N = N_1\hat{Z}^2 + 2\gamma N_2\hat{Z}\partial_Y + N_3\partial_Y^2$, with $N_1 = -O(\gamma)$, $N_3 = \partial_T\tilde{t} + O(\gamma)$,
2. $l_1(\dot{Z})$, $l'_1(\dot{Z})$ are linear combinations of $\nabla\dot{Z}$ and $\dot{Z}$, for example $l_1(\dot{Z}) = (N\varphi)S\dot{Z} + \frac{1}{Y}(S\varphi)\partial_Y\dot{Z}$,

3. H is a linear combination of $\hat{Z}$ and ∂_Y.

We can also denote $\gamma = \varepsilon$ because of its smallness.

Also introducing

$$\tilde{P} = \hat{Z}S\hat{Z} + \epsilon(S\varphi)N\hat{Z},$$

we set $\dot{Z} = \hat{Z}\dot{k}$ in the linearized system (II.60), (II.61).

So we have now to solve the system

$$\begin{aligned}
&\tilde{P}\dot{k} + \epsilon l_1(\hat{Z}\dot{k}) + \alpha_1\hat{Z}\dot{\Phi} + \alpha_2\dot{\Phi} = \dot{f}_1,\\
&\hat{Z}^2\dot{\Phi} + \beta_1\hat{Z}\dot{\Phi} + \beta_2\dot{\Phi} + \epsilon\hat{Z}H\hat{Z}\dot{k} + \epsilon\beta_3\partial_Y^2\hat{Z}\dot{k} + \epsilon l_1'(\hat{Z}\dot{k}) = \dot{f}_2.
\end{aligned}$$

With the notations

$$A = S\varphi, \quad \delta = T - \bar{T}_0, \quad g = \exp\{h(x-t)\}, p^2 = \delta^\mu g, \quad |\cdot|_0 = |\cdot|_{L_2(D_{fb})},$$

we have the energy inequality (3.2.2) of [6, p.18] and the rest of the proof is absolutely identical to [6].

Having obtained the solution λ, φ, v, w of the blow-up system in the domain D_{bf} we construct as it is shown in [6, p.22-23] the solution u of (II.39) from which we go back to $\tilde{u}$ the KZK solution of (II.36) which will be periodic in x (thanks to the theorem 4 of the existing of the unique solution) if we take the initial data u_0 periodic in x, and $\tilde{u}_x$ has a blow-up at the point $(x_\gamma, y_\gamma, \bar{T}_\gamma)$.

Chapter III

Validity of the KZK approximation

Table of contents

The purpose of this chapter is to specify the theorem 3 and to show in which sense the KZK equation provides asymptotic solutions of the equation

$$\partial_t \rho + \nabla.(\rho u) = 0\,, \quad \rho(\partial_t u + (u.\nabla)u) = -\nabla p(\rho) + \epsilon\nu\Delta u\,.$$

The viscosity ν introduces some difference in the construction. With no viscosity both the nonlinear system of elasticity and the KZK equation are well posed for positive and negative but finite time.

With viscosity both problems are well posed only for positive time but under a smallness hypothesis of initial data up to infinity. This is the reason why both cases are treated separately.

III.1 Validity of the KZK approximation for non viscous thermoellastic media

On the one hand one considers the Euler system for $\tilde\rho_\epsilon(x_1, x', t)$, $\tilde u_\epsilon(x_1, x', t)$:

$$\partial_t \tilde\rho_\epsilon + \operatorname{div}(\tilde\rho_\epsilon \tilde u_\epsilon) = 0\,, \quad \tilde\rho_\epsilon[\partial_t \tilde u_\epsilon + (\tilde u_\epsilon.\nabla)\tilde u_\epsilon] = -\nabla p(\tilde\rho_\epsilon), \tag{III.1}$$

and on the other hand a non trivial solution I of the problem

$$c\partial^2_{\tau z} I - \frac{(\gamma+1)}{4\rho_0}\partial^2_\tau I^2 - \frac{c^2}{2}\Delta_y I = 0, \tag{III.2}$$

for some initial condition

$$I(\tau, 0, y) = I_0(\tau, y).$$

The solution I as a function of (τ, z, y) is periodic in τ of period L. One constructs in this case for the KZK approximation a solution for (x_1, t) positive and negative using initial data compact in x for $t = 0$.

The theorem 4 ensures for initial data $I(\tau, 0, y) \in H^{s'}$ with $s' > [\frac{n}{2}] + 1$ the existence of a solution $I(\tau, z, y) \in C(|z| < R; H^{s'}(\tau \times y))$ (for zero viscosity $\nu = 0$). The existence of the smooth solution $\tilde U_\epsilon = (\tilde\rho_\epsilon, \tilde u_\epsilon)(t, x_1, x')$ $(0 \le t \le T)$ of Euler equation (III.1) is due to the theorem 5.1.1 from [15, p. 62].

With the notation

$$\tilde\rho_\epsilon = \rho_0 + \epsilon\rho_\epsilon \quad \tilde u_\epsilon = \epsilon u_\epsilon, \tag{III.3}$$

we take

$$p = p(\tilde\rho_\epsilon) = c^2\epsilon\rho_\epsilon + \frac{(\gamma-1)c^2}{2\rho_0}\epsilon^2{\rho_\epsilon}^2. \tag{III.4}$$

Then one constructs according to the formulas the functions

$$v(\tau,z,y)=\frac{c}{\rho_0}I(\tau,z,y), \tag{III.5}$$

$$w(\tau,z,y)=-\frac{c^2}{\rho_0}\left(\int_0^\tau \nabla_y I(s,z,y)ds+\int_0^L \frac{s}{L}\nabla_y I(s,z,y)ds\right), \tag{III.6}$$

$$v_1(\tau,z,y)=-\frac{c^2}{\rho_0}\left(\int_0^\tau \partial_z I(s,z,y)ds+\int_0^L \frac{s}{L}\partial_z I(s,z,y)ds\right)+$$
$$+\frac{(\gamma-1)}{2\rho_0^2}cI^2(\tau,z,y)-\frac{c(\gamma-1)}{2L\rho_0^2}\int_0^L I^2(\tau,z,y)d\tau. \tag{III.7}$$

In the above formulas the terms containing $\int_0^L$ correspond to the definition of the operator ∂_τ^{-1}, which implies that all these functions are L-periodic in τ and of mean value 0.

Next introduce the densities and velocities

$$\overline{\rho}_\epsilon=\rho_0+\epsilon I(t-\frac{x_1}{c},\epsilon x_1,\sqrt{\epsilon}x'), \tag{III.8}$$

$$\overline{u}_{\epsilon,1}=\epsilon(v+\epsilon v_1)(t-\frac{x_1}{c},\epsilon x_1,\sqrt{\epsilon}x'), \tag{III.9}$$

$$\overline{u}'_\epsilon=\epsilon^{\frac{3}{2}}\vec{w}(t-\frac{x_1}{c},\epsilon x_1,\sqrt{\epsilon}x') \tag{III.10}$$

and eventually the expression:

$$\overline{U}_\epsilon=(\overline{\rho}_\epsilon,\overline{u}_\epsilon)=(\rho_0+\epsilon I,\epsilon(v+\epsilon v_1,\sqrt{\epsilon}\vec{w}))(t-\frac{x_1}{c},\epsilon x_1,\sqrt{\epsilon}x'). \tag{III.11}$$

We envisage the problem of approximation between the two systems: the exact system (III.1) and the approximate system obtained from a smooth solution of KZK equation (III.2):

$$\left\{\begin{array}{l}\partial_t\overline{\rho}_\epsilon+\nabla_x.(\overline{\rho}_\epsilon\overline{u}_\epsilon)=\\ \epsilon^3\left(\rho_0\partial_z v_1+\partial_z(Iv)-\frac{1}{c}\partial_\tau(Iv_1)+\nabla_y(Iw)\right)+\epsilon^4\partial_z(Iv_1)\end{array}\right. \tag{III.12}$$

$$\begin{cases} \bar{\rho}_\epsilon(\partial_t \bar{u}_{\epsilon,1} + \bar{u}_\epsilon \cdot \nabla . \bar{u}_{\epsilon,1}) + \partial_{x_1} p(\bar{\rho}_\epsilon) = \\ \epsilon^3 \left(I\partial_\tau v_1 - \frac{1}{2c} I \partial_\tau v^2 + \frac{\rho_0}{2}\partial_z v^2 - \frac{\rho_0}{c}\partial_\tau(vv_1) + \rho_0 w \nabla_y v + \frac{(\gamma-1)}{2\rho_0} c^2 \partial_z I^2 \right) + \\ \epsilon^4 \left(\frac{I}{2}\partial_z v^2 - \frac{1}{c} I \partial_\tau (vv_1) + I w \nabla_y v + \rho_0 \partial_z(vv_1) - \frac{\rho_0}{2c}\partial_\tau v_1^2 + \rho_0 w \nabla_y v_1 \right) + \\ \epsilon^5 \left(I\partial_z(vv_1) - \frac{1}{2c} I \partial_\tau v_1^2 + I w \nabla_y v_1 + \frac{\rho_0}{2}\partial_z v_1^2 \right) \end{cases} \quad \text{(III.13)}$$

$$\begin{cases} \bar{\rho}_\epsilon(\partial_t \bar{u}'_\epsilon + \bar{u}_\epsilon \cdot \nabla . \bar{u}'_\epsilon) + \partial_{x'} p(\bar{\rho}_\epsilon) = \epsilon^{\frac{5}{2}} \left(\frac{\gamma-1}{2\rho_0} c^2 \nabla_y I^2 \right) + \\ \epsilon^{\frac{7}{2}} \left(\rho_0 v \partial_z w - \frac{\rho_0}{c} v_1 \partial_\tau w + \frac{\rho_0}{2}\nabla_y w^2 - \frac{I}{c} v \partial_\tau w \right) + \\ \epsilon^{\frac{9}{2}} \left(\rho_0 v_1 \partial_z w + I v \partial_z w - \frac{I}{c} v_1 \partial_\tau w + \frac{I}{2}\nabla_y w^2 \right) + \epsilon^{\frac{11}{2}} I v_1 \partial_z w \end{cases} \quad \text{(III.14)}$$

The system (III.12)-(III.14) could be written in the form

$$\partial_t \overline{\rho}_\epsilon + \nabla.(\overline{\rho u}_\epsilon) = R_1\,,$$
$$\overline{\rho}_\epsilon(\partial_t \overline{u}_\epsilon + \overline{u}_\epsilon \cdot \nabla . \overline{u}_\epsilon) + \nabla p(\overline{\rho}_\epsilon) = \vec{R}_2,$$

with notation R_1 for the rest of (III.12), and $\vec{R}_2$ for the rest of (III.13) and of (III.14).

To ensure that R_1, $\vec{R}_2 \in L_\infty((-R,R);L_2)$ we need that $\partial_z^2 I \in L_\infty((-R,R);L_2)$ and choose in theorem 4 $s > \max\{4, [\frac{n}{2}]+1\}$.

The existence of the smooth "true" solution of Euler equation (III.1) $\widetilde{U}_\epsilon = (\tilde{\rho}_\epsilon, \tilde{u}_\epsilon)(t, x_1, x')$ $(0 \le t < T_0)$ with $\nabla.\widetilde{U}_\epsilon(\cdot,t) \in C^0([0,T_0);H^{s-1})$ for $s-1 > \frac{n}{2}$, with the same initial data $\widetilde{U}_\epsilon|_{t=0} = \overline{U}_\epsilon|_{t=0}$ is still due to the theorem 5.1.1 from [15, p. 62].

Remark 12 *As soon as the time of the existence of Euler system solution T_0 is finite, the interval $[0,T_0)$ is maximal (see [15, p. 62]) in the sense that*

$$\limsup_{t\uparrow T_0} \|\nabla.\widetilde{U}(\cdot,t)\|_{L_\infty} = \infty.$$

To precise the order of the blow-up time T_0 we can observe it in the simplified model of Euler equation, particulary the Burgers equation with, as in our case, the initial conditions of order ϵ:

$$\partial_t u + u\frac{\partial u}{\partial x} = 0, \quad u|_{t=0} = \epsilon u_0.$$

If we derive it once on x we obtain

$$\partial_t\left(\frac{\partial u}{\partial x}\right) + u\frac{\partial^2 u}{\partial x^2} = -\left(\frac{\partial u}{\partial x}\right)^2.$$

And if we envisage the solution $Y = \frac{\partial u}{\partial x}(x(t),t)$ of the equation

$$\partial_t Y + u\partial_x Y = -Y^2, \quad Y|_{t=0} = \epsilon \partial_x u_0$$

along the characteristic curve

$$\dot{x} = u(x(t), t),$$

we obtain the Riccaty equation

$$\frac{d}{dt}Y = -Y^2,$$

and so

$$\nabla_x u = \frac{\epsilon \nabla_x u_0}{1 - \epsilon \nabla_x u_0 t},$$

from where

$$t < \frac{1}{\epsilon |\nabla_x u_0|}.$$

Resulting, we will consider the solution of Euler equation for the time $t \in [0, \frac{T}{\epsilon})$, where T is a constant and ϵ is small.

So the solution given by the KZK approximation and a true solution of Euler system with the same data at time $t = 0$ can be compared according to the following theorem

Theorem 10 *Suppose that there exists the solution I of the KZK equation (for some initial data from H^s) such that*

- *$I(\tau, z, y)$ is L-periodic with respect to τ and defined for $|z| \leq R$ and $y \in \mathcal{R}^{n-1}$,*
- *assume that*

  $$z \mapsto I(\tau, z, y) \in C(]-R, R[; H^{s'}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1})) \cap C^1(]-R, R[; H^{s'-2}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1}))$$

  for $s > \max\{4, [\frac{n}{2}] + 1\}$.

(the existence of such solution is proved in theorem 4).

Let $\overline{U}_\epsilon$ be the approximate solution of the isentropic Euler equation deduced from a solution of the KZK equation with the help of (III.8)-(III.10), (III.5)-(III.7). Then the function $\overline{U}_\epsilon(x_1, x', t) = \overline{U}_\epsilon(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} x')$ given by the formula (III.11) is defined in

$$\mathcal{R}_t \times (\Omega_\epsilon = \{|x_1| < \frac{R}{\epsilon} - ct\} \times \mathcal{R}_{x'}^{n-1})$$

and is smooth enough according to the above procedure.

Consider now the solution of the Euler system (III.1) in a cone (see the figure III.1)

$$C(t) = \{0 < s < t\} \times Q_\epsilon(s) = \{x = (x_1, x') : |x_1| \leq \frac{R}{\epsilon} - Ms, \; M \geq c, \; x' \in \mathcal{R}^{n-1}\}$$

with the initial data

$$(\bar{\rho}_\epsilon - \rho_\epsilon)|_{t=0} = 0, \quad (\bar{u}_\epsilon - u_\epsilon)|_{t=0} = 0.$$

Then (see [15, p. 62]) there exists T_0 such that for the time interval $0 \leq t \leq \frac{T_0}{\epsilon}$ there exists the classical solution $U_\epsilon = (\rho_\epsilon, u_\epsilon)$ of the Euler system (III.1) in a cone

$$C(T) = \{0 < t < T | T < \frac{T_0}{\epsilon}\} \times Q_\epsilon(t) \tag{III.15}$$

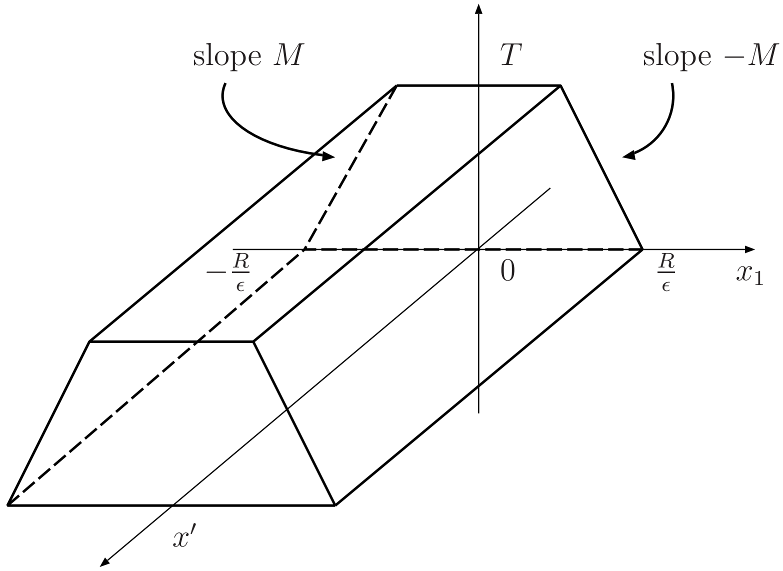

Figure III.1 – *The cone $C(T)$.*

with

$$\|\nabla.U_\epsilon\|_{L_\infty([0,\frac{T_0}{\epsilon}[;H^{s-1})} < \epsilon C \quad for \quad s > [\frac{n}{2}] + 1.$$

Then there exists a constant C such that for any ϵ small enough the solutions $\widetilde{U}_\epsilon \overset{note}{=} (\rho_\epsilon, \rho_\epsilon u_\epsilon)$ and $\bar{U}_\epsilon \overset{note}{=} (\overline{\rho}_\epsilon, \overline{\rho}_\epsilon \overline{u}_\epsilon)$, which have been determinate as above in the cone (III.15) with the same initial data (III.34) satisfy the estimate

$$\|\bar{U}_\epsilon - \widetilde{U}_\epsilon\|^2_{L_2(Q_\epsilon(t))} \leq \epsilon^5 e^{C\|\nabla.\bar{U}_\epsilon\|_{L_\infty(C(T))}t} \leq \epsilon^5 e^{C\epsilon t}. \tag{III.16}$$

Remark 13 *As soon as the booth solutions $\overline{U}_\epsilon$ and $\widetilde{U}_\epsilon$ are in $C([0, \frac{T_0}{\epsilon}[; H^s)$ in the cone for any $s > \max\{4, [\frac{n}{2}] + 1\}$, we can apply the operator $\Lambda^{s'}$ with $s' = s - 4$ and obtain the same estimate (III.16) but for the norm $\|\cdot\|_{H^{s'}(Q_\epsilon(t))}$.*

Proof. We need to use here a technique due to Dafermos [15].

As it is known the isentropic Euler equation admits a convex entropy $\eta(\widetilde{U}_\epsilon)$, which is the function:

$$\eta(\widetilde{U}_\epsilon) = \rho_\epsilon h(\rho_\epsilon) + \rho_\epsilon \frac{|u_\epsilon|^2}{2} \text{ with } h'(\rho_\epsilon) = \frac{p(\rho_\epsilon)}{\rho_\epsilon^2}. \tag{III.17}$$

Having assumed $\widetilde{U}_\epsilon = (\rho_\epsilon, \rho_\epsilon u_\epsilon)^T$, we can rewrite the Euler system

$$\partial_t \widetilde{U}_\epsilon + \nabla.F(\widetilde{U}_\epsilon) = 0, \quad \text{where } F(\widetilde{U}_\epsilon) = (\rho_\epsilon u_\epsilon, \rho_\epsilon u_\epsilon^2 + p(\rho_\epsilon))^T$$

in terms of entropy (III.17):

$$\partial_t \eta(\widetilde{U}_\epsilon) + \nabla.q(\widetilde{U}_\epsilon) = 0, \quad \text{where } q(\widetilde{U}_\epsilon) = u_\epsilon(\eta(\widetilde{U}_\epsilon) + p(\rho_\epsilon)).$$

So we have two systems

$$\begin{cases} \partial_t \eta(\widetilde{U}_\epsilon) + \nabla . q(\widetilde{U}_\epsilon) = 0, \\ \partial_t \widetilde{U}_\epsilon + \nabla . F(\widetilde{U}_\epsilon) = 0, \end{cases}$$

$$\begin{cases} \partial_t \eta(\overline{U}_\epsilon) + \nabla . q(\overline{U}_\epsilon) = \frac{\eta(\overline{U}_\epsilon)+p(\bar{\rho}_\epsilon)}{\bar{\rho}_\epsilon} R_1 + \bar{u}_\epsilon \vec{R}_2, \\ \partial_t \overline{U}_\epsilon + \nabla . F(\overline{U}_\epsilon) = \vec{R}, \end{cases}$$

where $R_1 = O(\epsilon^3)$ is the rest of the first equation of Euler system, $\vec{R}_2 = (O(\epsilon^3), O(\epsilon^{\frac{5}{2}})) = O(\epsilon^{\frac{5}{2}})$ is the rest of the second equation of Euler system in two directions x_1 and x', and $\vec{R} = (R_1, \vec{R}_2) = O(\epsilon^{\frac{5}{2}})$. So $\frac{\eta(\overline{U}_\epsilon)+p(\bar{\rho}_\epsilon)}{\bar{\rho}_\epsilon} R_1 + \bar{u}_\epsilon \vec{R}_2 = O(\epsilon^3)$.

Let us compute

$$\begin{aligned} &\frac{\partial}{\partial t}(\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon)) = -\nabla.(q(\widetilde{U}_\epsilon) - q(\overline{U}_\epsilon)) - \\ &-\vec{R}^T \eta''(\overline{U})(\widetilde{U} - \overline{U}) - \frac{\eta(\overline{U}_\epsilon) + p(\bar{\rho}_\epsilon)}{\bar{\rho}_\epsilon} R_1 - \bar{u}_\epsilon \vec{R}_2 + \\ &+\eta'(\overline{U}_\epsilon)\nabla.(F(\widetilde{U}_\epsilon) - F(\overline{U}_\epsilon)) - \eta'(\overline{U}_\epsilon)\vec{R} - (\partial_t \overline{U}_\epsilon)^T \eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon), \end{aligned} \tag{III.18}$$

and using the property for convex entropy $\eta''(U)F'(U) = (F'(U))^T \eta''(U)$ the last term is

$$-(\partial_t \overline{U}_\epsilon)^T \eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) = \nabla.\overline{U}_\epsilon^T (F'(\overline{U}_\epsilon))^T \eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) = \nabla.\overline{U}_\epsilon^T \eta''(\overline{U}_\epsilon) F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon).$$

Integrate (III.18) over the cone $C(t)$ (cf. (III.15)). The use of the Green formula gives:

$$\begin{aligned} &\int_{|x_1| < \frac{R}{\epsilon} - Mt} \int_{\mathcal{R}^{n-1}} (\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon)(x,t) dx \\ &- \int_{|x_1| < \frac{R}{\epsilon}} \int_{\mathcal{R}^{n-1}} (\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon)(x,0) dx \\ &= - \int_{\partial C(t)} (n_t(\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) d\sigma \\ &- \int_{\partial C(t)} n_{x_1}(q(\widetilde{U}_\epsilon) - q(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(F(\widetilde{U}_\epsilon) - F(\overline{U}_\epsilon))) d\sigma \\ &- \int_{C(t)} \nabla.\overline{U}_\epsilon^T \eta''(\overline{U}_\epsilon)(F(\widetilde{U}_\epsilon) - F(\overline{U}_\epsilon) - F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon)) dx ds - \\ &- \int_{C(t)} (\frac{\eta(\overline{U}_\epsilon) + p(\bar{\rho}_\epsilon)}{\bar{\rho}_\epsilon} R_1 + \bar{u}_\epsilon \vec{R}_2 + \eta'(\overline{U}_\epsilon)\vec{R} - \vec{R}^T \eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon)) dx ds . \end{aligned} \tag{III.19}$$

With the help of the facts that the entropy η is convex

$$\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) \geq \alpha |\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2,$$

and associated entropy flux q is related with η by relation [15, p. 52]

$$q'(U) = \eta'(U) F'(U),$$

we obtain that

$$q(\widetilde{U}_\epsilon) - q(\overline{U}_\epsilon) = q'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) + O(|\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2) = \eta'(\overline{U}_\epsilon)F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) + O(|\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2),$$

so, using the Taylor expansion

$$F(\widetilde{U}_\epsilon) - F(\overline{U}_\epsilon) = F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) + O(|\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2),$$

$$F(\widetilde{U}_\epsilon) - F(\overline{U}_\epsilon) - F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) \le C|\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2,$$

$$q(\widetilde{U}_\epsilon) - q(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(F(\widetilde{U}_\epsilon) - F(\overline{U}_\epsilon)) \le C|\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2.$$

At last one can always choose our cone with the help of a constant $M \ge c$ such that $\alpha n_t + C n_{x_1} > 0$ and

$$-\int_{\partial C(t)} (\alpha n_t + C n_x)\|\widetilde{U}_\epsilon - \overline{U}_\epsilon\|^2_{L_2(Q_\epsilon(s))} d\sigma < 0.$$

Taking the same initial data

$$\overline{U}_\epsilon|_{t=0} = \widetilde{U}_\epsilon|_{t=0},$$

we obtain

$$\|\widetilde{U}_\epsilon - \overline{U}_\epsilon\|^2_{L_2(Q_\epsilon(t))} \le C\|\nabla.\overline{U}_\epsilon\|_{L_\infty(C(T))} \int_0^t \|\widetilde{U}_\epsilon - \overline{U}_\epsilon\|^2_{L_2(Q_\epsilon(s))} ds + Kt\epsilon^5.$$

Here the constants K and C do not depend on t.

Therefore applying the Gronwall lemma one has

$$\|\widetilde{U}_\epsilon - \overline{U}_\epsilon\|^2_{L_2(Q_\epsilon(t))} \le K\epsilon^5 \int_0^t e^{C(t-s)\|\nabla.\overline{U}_\epsilon\|_{L_\infty(C(T))}} ds. \tag{III.20}$$

As soon as the difference of the solutions has the order $\widetilde{U}_\epsilon - \overline{U}_\epsilon = O(\epsilon)$, also $\nabla.\overline{U}_\epsilon = O(\epsilon)$, and the left side of (III.20) has the order $O(\epsilon^2)$, so we have that in the cone $C(T)$ our estimate always remains as ϵ^2

$$\epsilon^5 \int_0^t e^{C\epsilon(t-s)} ds < \epsilon^2.$$

III.2 Validity of the KZK approximation for viscous thermoellastic media

One has seen in theorem 4 that the solution $u(x, y, t)$ of the KZK equation with the term of viscosity $\beta > 0$ or $\nu > 0$ defines globally in time $t > 0$ for rather small initial data. Considering the KZK equation as the asymptotic form of Navier-Stokes system we note that its solution $I(\tau, z, y) = I(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} x')$ is defined for $x_1 > 0$ (as soon as z becomes the time variable according to the KZK derivation from chapter I). For this reason the approximate domain of validity of the KZK approximation for viscous thermoellastic media is the half space $x_1 > 0$, $t > 0$, $x' \in R^{n-1}$.

III.2.1 Linearized KZK equation

From the Navier-Stokes system

$$\partial_t\widetilde{\rho} + \mathrm{div}(\widetilde{\rho}\widetilde{u}) = 0\,, \quad \widetilde{\rho}(\partial_t\widetilde{u} + (\widetilde{u}\cdot\nabla)\widetilde{u}) = -\nabla p(\widetilde{\rho}) + \epsilon\nu\Delta\widetilde{u} \tag{III.21}$$

which is well posed for rather small initial data in half space $\{x_1 > 0, x' \in \mathcal{R}^{n-1}, t > 0\}$, the isentropic linear Navier-Stokes system can be obtained with the help of the choice

$$\widetilde{\rho} = \rho_0 + \epsilon^2\rho, \quad \widetilde{u} = \epsilon^2 u, \quad p = p(\widetilde{\rho}) = c^2\epsilon^2\rho$$

not taking into account the terms of order $O(\epsilon^3)$:

$$\partial_t\rho + \rho_0\nabla.u = 0, \tag{III.22}$$
$$\rho_0\partial_t u + \nabla p(\rho) = \epsilon\nu\Delta u, \tag{III.23}$$

which have a unique global solution.

Combining ∂_t and ∇ applied to (III.22) and (III.23) we obtain two decoupled linear equations

$$\partial_t^2\rho - c^2\triangle\rho = \epsilon\frac{\nu}{\rho_0}\triangle\partial_t\rho, \tag{III.24}$$
$$\partial_t^2 u - c^2\nabla\,\mathrm{div}\,u = \epsilon\frac{\nu}{\rho_0}\triangle\partial_t u. \tag{III.25}$$

The existence of the smooth solution u of (III.25) follows from the lemma.

Lemma 4 *The equation (III.25) has unique regular global on time solution in the half space $\{x_1 > 0, t > 0\}$, with regular initial $u|_{t=0} = u_0$, $u_t|_{t=0} = u_0$ and boundary $u|_{x_1=0} = u_b$ conditions (u_b has the same properties as the initial condition for the KZK equation $I(\tau, 0, y)$, and u_0 as $I(-\frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x')$).*

Proof. If $u_b = 0$, it follows from the relation

$$\frac{d}{dt}\left(\int_{x_1>0}|\partial_t u|^2dx + c^2\int_{x_1>0}|\nabla.u|^2dx\right) + \frac{\epsilon\nu}{\rho_0}\int_{x_1>0}|\nabla.\partial_t u|^2ds -$$
$$-c^2\int_{x_1=0}\nabla.u\partial_t u dx' - \frac{\epsilon\nu}{\rho_0}\int_{x_1=0}\nabla.\partial_t u\partial_t u dx' = 0,$$

which gives in this case

$$\frac{d}{dt}\int_{x_1>0}\left(|\partial_t u|^2 + c^2|\nabla.u|^2\right)dx \le 0.$$

If we take $u_b \neq 0$, then we can write that $u = v + \varphi$, such that $v|_{t=0}$, $v_t|_{t=0}$ are the same as for u and zero in the boundary, and φ is a function of a compact support with zero initial conditions and the same boundary condition as u. We can construct such φ using the trace theorem, thanks to the regularity of u_b.

Let us put $u = v + \varphi$ in (III.25)

$$\partial_t^2(v+\varphi) - c^2\nabla\,\mathrm{div}(v+\varphi) = \epsilon\frac{\nu}{\rho_0}\triangle\partial_t(v+\varphi).$$

From where we have

$$\partial_t^2 v - c^2 \nabla \operatorname{div} v - \epsilon \frac{\nu}{\rho_0} \triangle \partial_t v = g(\varphi).$$

We multiply now the equation on $\partial_t v$ and integrate on x:

$$\frac{d}{dt}\left(\int_{x_1>0} |\partial_t v|^2 dx + c^2 \int_{x_1>0} |\nabla . v|^2 dx \right) + \frac{\epsilon\nu}{\rho_0} \int_{x_1>0} |\nabla . \partial_t v|^2 ds \leq$$
$$\leq \frac{1}{2} \int_{x_1>0} |g(\varphi)|^2 dx + \frac{1}{2} \int_{x_1>0} |\partial_t v|^2 dx,$$

applying now the Gronwall lemma for the equality type $y' \leq a + y$ we obtain the global existence in time of the solution. □

Having the solution u of (III.25) with boundary condition $u|_{x_1=0} = u^1$ we can find ρ from (III.22) according to the formula

$$\rho(x,t) = \rho(x,0) - \rho_0 \int_0^t \nabla . u(x,s) ds. \tag{III.26}$$

This ρ satisfies (III.24) with boundary condition

$$\partial_t \rho|_{x_1=0} = -\rho_0 \nabla . u|_{x_1=0}.$$

And so we can envisage instead of the system (III.22), (III.23) the following system

$$\partial_t \rho + \rho_0 \nabla . u = 0, \tag{III.27}$$
$$\partial_t^2 u - c^2 \nabla \operatorname{div} u = \epsilon \frac{\nu}{\rho_0} \triangle \partial_t u. \tag{III.28}$$

If we pass to the variables (τ, z, y) in (III.22), we obtain the linear part of KZK for $\rho(t, x_1, x') = I(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} x')$

$$\partial_t^2 \rho - c^2 \triangle \rho - \epsilon \frac{\nu}{\rho_0} \triangle \partial_t \rho = \epsilon (2c \partial_{\tau z}^2 I - c^2 \triangle_y I - \frac{\nu}{\rho_0 c^2} \partial_\tau^3 I) +$$

$$+ \epsilon^2 (-c^2 \partial_z^2 I + 2 \frac{\nu}{\rho_0 c} \partial_\tau^2 \partial_z I - \frac{\nu}{\rho_0} \triangle_y \partial_\tau I) - \epsilon^3 \frac{\nu}{\rho_0} \partial_z^2 \partial_\tau I.$$

Suppose that I is the smooth periodic solution of linear KZK equation

$$2c \partial_{\tau z}^2 I - c^2 \triangle_y I - \frac{\nu}{\rho_0 c^2} \partial_\tau^3 I = 0, \tag{III.29}$$

then

$$\partial_t^2 \bar\rho - c^2 \triangle \bar\rho - \epsilon \triangle \partial_t \bar\rho = \epsilon^2 (-c^2 \partial_z^2 I + 2 \frac{\nu}{\rho_0 c} \partial_\tau^2 \partial_z I - \frac{\nu}{\rho_0} \triangle_y \partial_\tau I) - \epsilon^3 \frac{\nu}{\rho_0} \partial_z^2 \partial_\tau I. \tag{III.30}$$

And the equation of the same form holds for $\bar{u} = (v, \sqrt{\epsilon}\vec{w})$

$$\partial_t^2 \bar{u} - c^2 \nabla \operatorname{div} \bar{u} - \epsilon \triangle \partial_t \bar{u} = \begin{cases} \epsilon^2(-c^2\partial_z^2 v + 2\frac{\nu}{c}\partial_\tau^2\partial_z v - \nu\triangle_y\partial_\tau v) - \epsilon^3\nu\partial_z^2\partial_\tau v, \\ \\ \epsilon^{\frac{5}{2}}(2\frac{\nu}{c}\partial_\tau^2\partial_z\vec{w} - \nu\triangle_y\partial_\tau\vec{w}) - \epsilon^{\frac{7}{2}}\nu\partial_z^2\partial_\tau\vec{w}, \end{cases} \tag{III.31}$$

where the functions v and $\vec{w}$ are constructed according to the formulas (III.5), (III.6). So for the exact system (III.27), (III.28) the approximate system has the form

$$\partial_t\bar{\rho} + \rho_0\nabla.\bar{u} = \epsilon\rho_0(\partial_z v + \nabla_y \vec{w}), \tag{III.32}$$

$$\partial_t^2 \bar{u} - c^2 \nabla \operatorname{div} \bar{u} - \epsilon \frac{\nu}{\rho_0} \triangle \partial_t \bar{u} = \begin{cases} \epsilon^2(-c^2\partial_z^2 v + 2\frac{\nu}{c}\partial_\tau^2\partial_z v - \nu\triangle_y\partial_\tau v) - \epsilon^3\nu\partial_z^2\partial_\tau v, \\ \\ \epsilon^{\frac{5}{2}}(2\frac{\nu}{c}\partial_\tau^2\partial_z\vec{w} - \nu\triangle_y\partial_\tau\vec{w}) - \epsilon^{\frac{7}{2}}\nu\partial_z^2\partial_\tau\vec{w}, \end{cases} \tag{III.33}$$

In this case we easily obtain what follows

Theorem 11 *Let $I(\tau, z, y)$ be a solution of the linear KZK equation (III.29) L-periodic and mean value zero with respect to τ and defined in the half space $z > 0, \tau \in \mathcal{R}/L\mathcal{Z}, y \in \mathcal{R}^{n-1}$, decays for $z \to \infty$. Assume that*

$$z \mapsto I(\tau, z, y) \in C([0, \infty[; H^{s'}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1})) \cap C^1([0, \infty[; H^{s'-2}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1}))$$

for $s > \max\{6, [\frac{n}{2}] + 1\}$.

Let $\overline{U}_\epsilon = (\bar{\rho}, \bar{u})$ be the smooth solution of the approximate system (III.32), (III.33) deduced from a solution of the KZK equation with the help of (III.5)-(III.6). Then the function $\overline{U}_\epsilon(x_1, x', t) = \overline{U}_\epsilon(x_1 - ct, \epsilon x_1, \sqrt{\epsilon}x')$ given by the formula

$$\overline{U}_\epsilon(x_1, x', t) = (I, (v, \sqrt{\epsilon}\vec{w}))(x_1 - ct, \epsilon x_1, \sqrt{\epsilon}x')$$

is defined in the half space

$$\{x_1 > 0,\ x' \in \mathcal{R}^{n-1},\ t > 0\}$$

and is smooth enough according to the above procedure.

If $U = (\rho, u)$ is the solution of (III.27), (III.28) in $\{x_1 > 0,\ x' \in \mathcal{R}^{n-1},\ t > 0\}$ with the same rather small initial data and boundary condition

$$\begin{aligned} &(\bar{u} - u)|_{t=0} = \partial_t(\bar{u} - u)|_{t=0} = 0, \\ &(\bar{u} - u)|_{x_1=0} = 0, \quad \partial_t\rho|_{x_1=0} = -\rho_0\nabla.u|_{x_1=0}, \end{aligned} \tag{III.34}$$

then there exists a constant $C > 0$ such that the following estimates hold

$$\int_{x_1>0} |\partial_t(\bar{u}_\epsilon - u)|^2 + |\nabla.(\bar{u}_\epsilon - u)|^2 dx_1 dx' \le C\epsilon^4 t^2, \tag{III.35}$$

which remains smaller than the order ϵ^2 for the time $0 < t < \frac{T}{\epsilon^2}$, and

$$\int_{x_1>0} |\bar{\rho}_\epsilon - \rho|^2 dx_1 dx' \le \epsilon^2 e^{Ct}, \tag{III.36}$$

which remains smaller than the order ϵ^2 for the time $0 < t < T \ln \frac{1}{\epsilon}$.

The approximation result is true for the solution $\widetilde{U}$ of the linearized system (III.22), (III.23) since $\widetilde{U} \equiv U$ (with $\partial_t\rho|_{x_1=0} = -\rho_0\nabla.u|_{x_1=0}$).

Proof.

For the difference $\bar{u}-u$ we have

$$\partial_t^2(\bar{u}-u)-c^2\triangle(\bar{u}-u)=\epsilon\triangle\partial_t(\bar{u}-u)+\epsilon^2\vec{R},$$

where

$$\vec{R}=\begin{cases}\epsilon^2(-c^2\partial_z^2 v+2\frac{\nu}{c}\partial_\tau^2\partial_z v-\nu\triangle_y\partial_\tau v)-\epsilon^3\nu\partial_z^2\partial_\tau v,\\ \\ \epsilon^{\frac{5}{2}}(2\frac{\nu}{c}\partial_\tau^2\partial_z\vec{w}-\nu\triangle_y\partial_\tau\vec{w})-\epsilon^{\frac{7}{2}}\nu\partial_z^2\partial_\tau\vec{w},\end{cases}$$

the rest of (III.30) bounded, under the smoothness hypotheses of the KZK solution I, at least in $L_\infty([0,\infty[,L_2)$.

Multiplying the last equation by $\partial_t(\bar{u}-u)$ one obtains

$$\frac{d}{dt}\int_{x_1>0}(|\partial_t(\bar{u}-u)|^2+c^2|\nabla.(\bar{u}-u)|^2)dx_1dx'-c^2\int_{x_1=0}\partial_t(\bar{u}-u)\cdot\nabla.(\bar{u}-u)dx'=$$

$$=\epsilon\int_{x_1=0}\partial_t(\bar{u}-u)\nabla.\partial_t(\bar{u}-u)dx'-\epsilon\int_{x_1>0}|\nabla.\partial_t(\bar{u}-u)|^2dx_1dx'+\epsilon^2\int_{x_1>0}R\partial_t(\bar{u}-u)dx_1dx'.$$

In the same time

$$\int_{x_1=0}\partial_t(\bar{u}-u)\nabla.\partial_t(\bar{u}-u)dx'+c^2\int_{x_1=0}\partial_t(\bar{u}-u)\cdot\nabla.(\bar{u}-u)dx'=0,$$

because one can choose $\bar{u}|_{x_1=0}=u|_{x_1=0}$(cf. (III.34)). From where

$$\frac{d}{dt}\int_{x_1>0}(|\partial_t(\bar{u}-u)|^2+c^2|\nabla.(\bar{u}-u)|^2)dx_1dx'\leq\epsilon^2C\left(\int_{x_1>0}|\partial_t(\bar{u}-u)|^2+c^2|\nabla.(\bar{u}-u)|^2dx_1dx'\right)^{\frac{1}{2}}.$$

The above estimate has the form

$$\frac{d}{dt}(\sqrt{E})^2\leq C\epsilon^2\sqrt{E}\quad\Rightarrow\quad\frac{d}{dt}\sqrt{E}\leq C\epsilon^2$$

and then

$$\frac{d}{dt}\left(\int_{x_1>0}|\partial_t(\bar{u}-u)|^2+c^2|\nabla.(\bar{u}-u)|^2dx_1dx'\right)^{\frac{1}{2}}\leq C\epsilon^2,$$

which gives the estimate (III.35).

In our construction the expression $\sqrt{E}=O(1)$ because $\bar{u}=O(1)$, $u=O(1)$, $(\bar{u}-u)=O(1)$, which ensures that the estimate remains smaller the order ϵ^2 for the time $0<t<\frac{T}{\epsilon^2}$.

Passing now to $(\bar{\rho}-\rho)$ we have the relation

$$\partial_t(\bar{\rho}-\rho)+\rho_0\nabla.(\bar{u}-u)=\epsilon\rho_0(\partial_z v+\nabla_y\vec{w}).$$

We multiply the equation by $(\bar{\rho}-\rho)$ in $L_2(\{x_1>0\})$

$$\frac{d}{dt}\int_{x_1>0}|\bar{\rho}-\rho|^2dx+\rho_0\int_{x_1>0}(\bar{\rho}-\rho)\nabla.(\bar{u}-u)dx=\epsilon\rho_0\int_{x_1>0}\vec{R}(\bar{\rho}-\rho)dx,$$

and using the estimate (III.35) for the time $t<\frac{T}{\epsilon^2}$, we obtain the estimate

$$\|\bar{\rho}-\rho\|_{L_2(\{x_1>0\})}\leq\epsilon Ce^{Ct},$$

which remains smaller the order ϵ^2 for the time $t<T\ln\frac{1}{\epsilon}$. $\square$

III.2.2 The general nonlinear case

On the one hand one considers the system:

$$\partial_t \rho_\epsilon + \operatorname{div}(\rho_\epsilon u_\epsilon) = 0\,, \quad \rho_\epsilon[\partial_t u_\epsilon + (u_\epsilon \cdot \nabla) u_\epsilon] = -\nabla p(\rho_\epsilon) + \epsilon\nu\Delta u_\epsilon \tag{III.37}$$

and on the other hand a non trivial solution I of the problem

$$c\partial^2_{\tau z} I - \frac{(\gamma+1)}{4\rho_0}\partial^2_\tau I^2 - \frac{\nu}{2c^2\rho_0}\partial^3_\tau I - \frac{c^2}{2}\Delta_y I = 0, \tag{III.38}$$

for some initial data

$$I(\tau, 0, y) = I_0(\tau, y).$$

The solution I as a function of (τ, z, y) is periodic in τ of period L. The theorem 4 implies for any initial data I_0 $\mathcal{R}^{\text{per}}_\tau \times \Omega_y$ with small enough H^s $(s > [\frac{n}{2}] + 1)$ norm (with respect to ν) there exists a unique solution which decays for $z \to \infty$.

We still envisage our problem in the half space $x_1 > 0$, $t > 0$ with the assumption that $u \to 0, \rho \to \rho_0$ for $|x| \to \infty$.

Let us construct the approximate system to the Navier-Stokes system (III.37).

We take as the state equation

$$p = p(\rho_\epsilon) = c^2 \epsilon \tilde{\rho}_\epsilon + \frac{(\gamma-1)c^2}{2\rho_0}\epsilon^2 \tilde{\rho}^2_\epsilon.$$

Then one constructs according to the formulas the functions: v from (III.5), w from (III.6) and

$$v_1(\tau, z, y) = -\frac{c^2}{\rho_0}\left(\int_0^\tau \partial_z I(s, z, y)ds + \int_0^L \frac{s}{L}\partial_z I(s, z, y)ds\right) + \\ + \frac{(\gamma-1)}{2\rho_0^2}cI^2 - \frac{c(\gamma-1)}{2L\rho_0^2}\int_0^L I^2(\tau, z, y)d\tau + \frac{\nu}{c\rho_0^2}\partial_\tau I. \tag{III.39}$$

In the above formula the terms containing $\int_0^L$ correspond to the definition of the operator ∂_τ^{-1}, which implies that v_1 is L-periodic in τ and of mean value 0. To exclude the derivative on z from (III.39) we find from the KZK equation that

$$-\frac{c^2}{\rho_0}\partial_\tau^{-1}\partial_z I = -\frac{(\gamma+1)c}{4\rho_0^2}I^2 + \frac{c(\gamma+1)}{4\rho_0^2 L}\int_0^L I^2 ds + \frac{\nu}{2c\rho^2}\partial_\tau I + \frac{c^3}{2\rho_0}\partial_\tau^{-2}\triangle_y I,$$

and so

$$v_1(\tau, z, y) = \frac{c^3}{2\rho_0}\partial_\tau^{-2}\triangle_y I(\tau, z, y) + \frac{(\gamma-1)}{4\rho_0^2}cI^2(\tau, z, y) - \\ -\frac{c(\gamma-1)}{4L\rho_0^2}\int_0^L I^2(\tau, z, y)d\tau + \frac{3\nu}{2c\rho_0^2}\partial_\tau I(\tau, z, y). \tag{III.40}$$

Next we introduce the densities and velocities (III.8)-(III.10) and construct the function $\overline{U}$ (III.11).

In particular for $t = 0$ one has functions defined for $x_1 > 0$ because I was well defined for any $z > 0$

$$\overline{\rho}_\epsilon(0, x_1, x') = \rho_0 + \epsilon I(-\frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x'),$$
$$\overline{u}_\epsilon \mid_{t=0} = (\overline{u}_{\epsilon,1}, \overline{u}'_\epsilon)(-\frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x')$$

and for $x_1 = 0$ one has L-periodic functions of mean value zero

$$\overline{\rho}_\epsilon(t, 0, x') = \rho_0 + \epsilon I(t, 0, \sqrt{\epsilon}x'), \quad \text{(III.41)}$$
$$\overline{u}_\epsilon(t, 0, x') = (\overline{u}_{\epsilon,1}, \overline{u}'_\epsilon)(t, 0, \sqrt{\epsilon}x'). \quad \text{(III.42)}$$

Since for (III.37) in our case on the boundary $u|_{x_1=0} = \epsilon\tilde{u}_\epsilon|_{x_1=0}$ is small and so $|u|_{x_1=0}| < c$, we have only two cases in our boundary: a subsonic inflow boundary when the first velocity component is positive $u_1|_{x_1=0} > 0$, and a subsonic outflow boundary when the first component of velocity is negative $u_1|_{x_1=0} < 0$.

We also notice that, thanks to (III.40),

$$\bar{u}_1|_{x_1=0} = \left(\epsilon\frac{c}{\rho_0}I + \epsilon^2 G(I)\right)(t, 0, \sqrt{\epsilon}x') = \left(\epsilon\frac{c}{\rho_0}I + \epsilon^2 G(I)\right)\Bigg|_{z=0} =$$
$$= \epsilon\frac{c}{\rho_0}I_0(t, y) + \epsilon^2 G(I_0)(t, y),$$

with

$$G(I) = \frac{c^3}{2\rho_0}\partial_\tau^{-2}\triangle_y I + \frac{(\gamma-1)}{4\rho_0^2}cI^2 - \frac{c(\gamma-1)}{4L\rho_0^2}\int_0^L I^2 d\tau + \frac{3\nu}{2c\rho_0^2}\partial_\tau I, \quad \text{(III.43)}$$

so the boundary conditions for $\bar{u}_1$ are defined by the initial conditions for KZK equation and are L-periodic on t and of mean value zero. Therefore, the sign of $\bar{u}_1|_{x_1=0}$ is the same as the sign of I_0 (because the term $G(I_0)$ has higher order of smallness on ϵ).

The function $\overline{U}_\epsilon \overset{note}{=} (\bar{\rho}_\epsilon, \bar{u}_\epsilon)$, defined in (III.11) by densities and velocities from (III.8)-(III.10), is solution of the problem

$$\begin{cases} \partial_t\bar{\rho}_\epsilon + \nabla.(\bar{\rho}_\epsilon\bar{u}_\epsilon) = \\ \epsilon^3\left(\rho_0\partial_z v_1 + \partial_z(Iv) - \frac{1}{c}\partial_\tau(Iv_1) + \nabla_y(Iw)\right) + \epsilon^4\partial_z(Iv_1), \end{cases} \quad \text{(III.44)}$$

$$\begin{cases} \bar\rho_\epsilon(\partial_t \bar u_{\epsilon,1} + (\bar u_\epsilon\cdot\nabla)\bar u_{\epsilon,1}) + \partial_{x_1} p(\bar\rho_\epsilon) - \epsilon\nu\Delta \bar u_{\epsilon,1} = \\ \epsilon^3\Bigg(I\partial_\tau v_1 - \frac{1}{2c} I\partial_\tau v^2 + \frac{\rho_0}{2}\partial_z v^2 - \frac{\rho_0}{c}\partial_\tau(vv_1) + \rho_0 w\nabla_y v + \\ +\frac{(\gamma-1)}{2\rho_0}c^2\partial_z I^2 + \frac{2\nu}{c}\partial^2_{z\tau}v - \nu\Delta_y v - \frac{\nu}{c^2}\partial^2_\tau v_1\Bigg) + \\ \epsilon^4\Bigg(\frac{I}{2}\partial_z v^2 - \frac{1}{c}I\partial_\tau(vv_1) + Iw\nabla_y v + \rho_0\partial_z(vv_1) - \frac{\rho_0}{2c}\partial_\tau v_1^2 + \\ +\rho_0 w\nabla_y v_1 - \nu\partial^2_z v + \frac{2\nu}{c}\partial^2_{z\tau}v_1 - \nu\Delta_y v_1\Bigg) + \\ \epsilon^5\Bigg(I\partial_z(vv_1) - \frac{1}{2c}I\partial_\tau v_1^2 + Iw\nabla_y v_1 + \frac{\rho_0}{2}\partial_z v_1^2 - \nu\partial_z^2 v_1\Bigg) \end{cases} \tag{III.45}$$

$$\begin{cases} \bar\rho_\epsilon(\partial_t \bar u'_\epsilon + (u_\epsilon\cdot\nabla)\bar u'_\epsilon) + \partial_{x'} p(\bar\rho_\epsilon) - \epsilon\nu\Delta \bar u'_\epsilon = \\ \epsilon^{\frac52}\Bigg(\frac{\gamma-1}{2\rho_0}c^2\nabla_y I^2 + \frac{\nu}{\rho_0}\Delta_y I\Bigg) + \\ \epsilon^{\frac72}\Bigg(\rho_0 v\partial_z w - \frac{\rho_0}{c}v_1\partial_\tau w + \frac{\rho_0}{2}\nabla_y w^2 - \frac{I}{c}v\partial_\tau w - \nu\Delta w + \frac{2\nu}{c}\partial^2_{z\tau}w\Bigg) + \\ \epsilon^{\frac92}\Bigg(\rho_0 v_1\partial_z w + Iv\partial_z w - \frac{I}{c}v_1\partial_\tau w + \frac{I}{2}\nabla_y w^2 - \nu\partial_z^2 w\Bigg) + \\ \epsilon^{\frac{11}{2}}\Bigg(Iv_1\partial_z w\Bigg) \end{cases} \tag{III.46}$$

Here to control the terms in right sides we need that $\partial_z^3 I \in L_\infty([0,\infty[, L_2(\tau\times y))$, so in theorem 4 we take $s > \max\{6, [\frac{n}{2}]+1\}$. Then the rest of (III.44)-(III.46) is bounded in L_2.

So we have two systems: the system (III.37) and

$$\partial_t\bar\rho_\epsilon + \nabla.(\bar\rho_\epsilon\bar u_\epsilon) = \epsilon^{\frac52}R_1\,, \quad \bar\rho_\epsilon[\partial_t\bar u_\epsilon + (\bar u_\epsilon\cdot\nabla)\bar u_\epsilon] + \nabla p(\bar\rho_\epsilon) - \epsilon\nu\Delta\bar u_\epsilon = \epsilon^{\frac52}\vec R_2, \tag{III.47}$$

where $\epsilon^{\frac52}R_1 = O(\epsilon^3)$ is the rest of the first equation of Navier-Stokes system (III.44), $\epsilon^{\frac52}\vec R_2 = (O(\epsilon^3), O(\epsilon^{\frac52})) = O(\epsilon^{\frac52})$ is the rest of the second equation of Navier-Stokes system (III.45), (III.46) in two directions x_1 and x'.

Remark 14 *As we have the term of viscosity $\epsilon\nu\triangle u$, where ϵ is fixed rather small parameter, ν is a constant, then in our case $\epsilon\nu$ does not converge to zero and so the phenomenon of boundary layer is exclude.*

Theorem 12 *Suppose that a function $I_0(t,y) = I_0(t,\sqrt{\epsilon}x')$ is such that*

1. *it is periodic on t with the period L and of mean value zero,*
2. *for fixed t it has the same sign for all $y\in\mathcal{R}^{n-1}$, and for $t\in]0,L[$ change the sign, i.e., $I_0 = 0$, only finite number times,*
3. *$I_0(t,y)\in H^{s'}(\{t>0\}\times\mathcal{R}^{n-1})$ for $s' > \max\{6,[\frac{n}{2}]+1\}$,*

4. I_0 is sufficiently small in the sense of the theorem 4 and $I_0 = \epsilon^p \tilde{I}_0$, $p \geq 0$.

For all function $I_0(t, y)$ satisfying the properties 1-4

- *there exists the unique solution $I(\tau, z, y)$ of the KZK equation (III.38) with the initial condition $I|_{z=0} = I_0$ such that*
  - *$I(\tau, z, y)$ is L-periodic with respect to τ and defined in the half space $\{z > 0, \tau \in \mathcal{R}/L\mathcal{Z}, y \in \mathcal{R}^{n-1}\}$, decays for $z \to \infty$;*
  - *$z \mapsto I(\tau, z, y) \in C([0,\infty[; H^{s'}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1})) \cap C^1([0,\infty[; H^{s'-2}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1}))$ for $s' > \max\{6, [\frac{n}{2}] + 1\}$*

  (the existence of such solution is proved in theorem 4).
- *there exists a unique global in time solution $\overline{U}_\epsilon = (\bar{\rho}_\epsilon, \bar{u}_\epsilon)$ of the approximate system (III.44)-(III.46) deduced from a solution of the KZK equation with the help of (III.5), (III.6), (III.40). The function $\overline{U}_\epsilon(x_1, x', t) = \overline{U}_\epsilon(x_1 - ct, \epsilon x_1, \sqrt{\epsilon} x')$, given by the formula (III.11), is defined in the half space*

  $$\{x_1 > 0,\ x' \in \mathcal{R}^{n-1},\ t > 0\}.$$

  Moreover, according to its definition,

  $$\bar{\rho}_\epsilon \in C(]0,\infty[, H^{s'}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1})) \cap C^1([0,\infty[; H^{s'-2}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1})),$$

  $$\bar{u}_\epsilon \in C([0,\infty[; H^{s'-2}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1})) \cap C^1([0,\infty[; H^{s'-4}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1})).$$

Consider now the Navier-Stokes system (III.37) in the half space with the initial data

$$(\bar{\rho}_\epsilon - \rho_\epsilon)|_{t=0} = 0, \quad (\bar{u}_\epsilon - u_\epsilon)|_{t=0} = 0,$$

and following boundary conditions (III.42)

$$(\bar{u}_\epsilon - u_\epsilon)|_{x_1=0} = 0,$$

and when the first component of the velocity is positive $u_{\epsilon,1}|_{x_1=0} > 0$ (i.e. at points where the fluid enters the domain) the additional boundary condition (III.41)

$$(\bar{\rho}_\epsilon - \rho_\epsilon)|_{x_1=0} = 0.$$

When $u_{\epsilon,1}|_{x_1=0} \leq 0$ there is not any boundary condition for ρ_ϵ.

Suppose also that $u_\epsilon \to 0$, $\rho_\epsilon \to \rho_0$ as $|x| \to \infty$.

Then

- *there exists a constant $T_0 > 0$ such that for all $t < \frac{T_0}{\epsilon^{2+p}}$ there exists a unique solution $U_\epsilon = (\rho_\epsilon, u_\epsilon)$ of Navier-Stokes system (III.37) with the same smoothness as $\overline{U}_\epsilon$,*
- *there exists a constant C such that for all rather small ϵ the solutions $(\rho_\epsilon, u_\epsilon)$ of (III.37) and $(\overline{\rho}_\epsilon, \overline{u}_\epsilon)$ of (III.44)-(III.46) satisfy the following stability estimate*

  $$\|\rho_\epsilon - \overline{\rho}_\epsilon\|_{L_2} + \|\rho_\epsilon u_\epsilon - \overline{\rho}_\epsilon \overline{u}_\epsilon\|_{L_2} \leq \epsilon^{\frac{5}{2}} e^{C\|\nabla.\overline{U}_\epsilon\|_{L_\infty} t} \leq \epsilon^{\frac{5}{2}} e^{C\epsilon t}. \tag{III.48}$$

which remains any finite time

$$0 < t < \frac{T}{\epsilon} \ln \frac{1}{\epsilon}$$

smaller than the order ϵ (here T is a positive constant and $T = O(1)$).

Remark 15 *Since the boundary conditions for the Navier-Stokes system are periodic and of mean value zero on t, the first component of the velocity $u_1|_{x_1=0}$ changes the sign and the inflow part of the boundary goes after the inflow one and so on. On the variables x' we have the constant sign of $u_1|_{x_1=0}$. This hypothesis follows from the physical reason of works of Zabolotskaya (see [11]). In [11] one takes as the initial conditions for the KZK equation (which correspond to the boundary condition for u_1) the expression*

$$I(\tau, 0, y) = -F(y) \sin \tau.$$

The amplitude distribution $F(y)$ is taken two types:

- *for a Gaussian beam*
$$F(y) = e^{-y^2},$$
- *for a beam with a polynomial amplitude*
$$F(y) = \begin{cases} (1-y^2)^2, & y \leq 1, \\ 0, & y > 1. \end{cases}$$

Proof. Using the fact of the convex entropy for the isentropic Euler equation $\eta(\widetilde{U}_\epsilon)$, which is the function (see (III.17)):

$$\eta(\widetilde{U}_\epsilon) = \rho_\epsilon h(\rho_\epsilon) + \rho_\epsilon \frac{|u_\epsilon|^2}{2} = H(\rho_\epsilon) + \frac{1}{\rho_\epsilon}\frac{m^2}{2} \text{ with } h'(\rho_\epsilon) = \frac{p(\rho_\epsilon)}{\rho_\epsilon^2},\ u_\epsilon = \frac{m}{\rho_\epsilon},$$

and their first and second derivatives

$$\eta'(\widetilde{U}_\epsilon) = \begin{bmatrix} H'(\rho_\epsilon) - \frac{1}{\rho_\epsilon^2}\frac{m^2}{2} \\ \frac{m}{\rho_\epsilon} \end{bmatrix}^T = \begin{bmatrix} H'(\rho_\epsilon) - \frac{u_\epsilon^2}{2} \\ u_\epsilon \end{bmatrix}^T, \tag{III.49}$$

$$\eta''(\widetilde{U}_\epsilon) = \begin{bmatrix} H''(\rho_\epsilon) + \frac{m^2}{\rho_\epsilon^3} & -\frac{m}{\rho_\epsilon^2} \\ -\frac{m}{\rho_\epsilon^2} & \frac{1}{\rho_\epsilon} \end{bmatrix} = \begin{bmatrix} H''(\rho_\epsilon) + \frac{u_\epsilon^2}{\rho_\epsilon} & -\frac{u_\epsilon}{\rho_\epsilon} \\ -\frac{u_\epsilon}{\rho_\epsilon} & \frac{1}{\rho_\epsilon} \end{bmatrix}. \tag{III.50}$$

Have assumed $\widetilde{U}_\epsilon = (\rho_\epsilon, \rho_\epsilon u_\epsilon)^T = (\rho_\epsilon, m)^T$, we can rewrite the Navier-Stokes system

$$\partial_t \widetilde{U}_\epsilon + \nabla . F(\widetilde{U}_\epsilon) - \epsilon\nu \begin{bmatrix} 0 \\ \triangle u_\epsilon \end{bmatrix} = 0, \quad \text{where } F(\widetilde{U}_\epsilon) = \begin{bmatrix} \rho_\epsilon u_\epsilon \\ \rho_\epsilon u_\epsilon^2 + p(\rho_\epsilon) \end{bmatrix}$$

in terms of entropy (III.17):

$$\partial_t \eta(\widetilde{U}_\epsilon) + \nabla . q(\widetilde{U}_\epsilon) - \epsilon\nu \begin{bmatrix} 0 \\ u_\epsilon \triangle u_\epsilon \end{bmatrix} = 0, \quad \text{where } q(\widetilde{U}) = u_\epsilon(\eta(\widetilde{U}_\epsilon) + p(\rho_\epsilon)).$$

Entropy estimate for isentropic Navier-Stokes equation on the half space and existence result

Let us multiply the Navier-Stokes system, from the left, by $2\widetilde{U}_\epsilon^T\eta''(U_\epsilon)$

$$2\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\partial_t\widetilde{U}_\epsilon+2\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)\nabla.\widetilde{U}_\epsilon-\epsilon\nu 2\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\begin{bmatrix}0\\ \triangle u_\epsilon\end{bmatrix}=0.$$

Note that

$$\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\begin{bmatrix}0\\ \triangle u_\epsilon\end{bmatrix}=0,$$

and that

$$2\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\partial_t\widetilde{U}_\epsilon=\partial_t[\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon]-2\widetilde{U}_\epsilon^T\partial_t\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon.$$

Moreover, by virtue of $\eta''(U)F'(U)=(F'(U))^T\eta''(U)$,

$$2\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)\nabla.\widetilde{U}_\epsilon=\nabla.[\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon]-2\widetilde{U}_\epsilon^T\nabla.[\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)]\widetilde{U}_\epsilon.$$

Integrating by $[0,t]\times\{x_1>0\}$ $(x'\in\mathcal{R}^{n-1})$ we have

$$\int_0^t\int_{x_1>0}\partial_t[\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon]dxds+\int_0^t\int_{x_1>0}\nabla.[\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon]dxds-$$

$$-2\int_0^t\int_{x_1>0}\widetilde{U}_\epsilon^T\partial_t\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dxds-2\int_0^t\int_{x_1>0}\widetilde{U}_\epsilon^T\nabla.[\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)]\widetilde{U}_\epsilon dxds=0.$$

One integrates now by parts

$$-2\int_0^t\int_{x_1>0}\widetilde{U}_\epsilon^T\partial_t\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dxds=-2\int_{x_1>0}\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dx+2\int_{x_1>0}\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon|_{t=0}dx+$$

$$+4\int_0^t\int_{x_1>0}\partial_t\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dxds,$$

$$-2\int_0^t\int_{x_1>0}\widetilde{U}_\epsilon^T\nabla.[\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)]\widetilde{U}_\epsilon dxds=-2\int_0^t\int_{x_1=0}\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dx'ds+$$

$$+4\int_0^t\int_{x_1>0}\nabla.\widetilde{U}_\epsilon^T[\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)]\widetilde{U}_\epsilon dxds,$$

noticing that

$$4\int_0^t\int_{x_1>0}\partial_t\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dxds+\int_0^t\int_{x_1>0}\nabla.\widetilde{U}_\epsilon^T[\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)]\widetilde{U}_\epsilon dxds=0,$$

we result in

$$\int_{x_1>0}\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dx-\int_{x_1>0}\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon|_{t=0}dx-\int_0^t\int_{\mathcal{R}^{n-1}}\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon|_{x_1=0}dx'ds=0.$$

Recall that $\eta''(\widetilde{U}_\epsilon)$ is positive definite, so that

$$\widetilde{U}_\epsilon^T \eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon \geq \delta|\widetilde{U}_\epsilon|^2,$$

for some $\delta > 0$.

Therefore, we obtain for the initial data

$$\widetilde{U}_0 = \left[\begin{array}{c} \rho_0 + \epsilon I \\ \epsilon\left(\rho_0 + \epsilon I\right)\left(\frac{c}{\rho_0} I + \epsilon v_1, \sqrt{\epsilon}\vec{w}\right) \end{array} \right] \left(-\frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} x'\right)$$

and for the first component of velocity the relation

$$\int_{x_1>0} \widetilde{U}_\epsilon^2 dx - \int_{x_1>0} \widetilde{U}_0^2 dx - \int_0^t \int_{\mathcal{R}^{n-1}} \widetilde{U}_{1,\epsilon}^T \eta''(\widetilde{U}_{1,\epsilon}) F'(\widetilde{U}_{1,\epsilon}) \widetilde{U}_{1,\epsilon}|_{x_1=0} dx' ds \leq 0.$$

Let us now consider the integral on the boundary. With notation $u_1 = u_{1,\epsilon}$ for the first component of velocity, we see that

$$\widetilde{U}_{1,\epsilon}^T \eta''(\widetilde{U}_{1,\epsilon}) F'(\widetilde{U}_{1,\epsilon}) \widetilde{U}_{1,\epsilon} = (\rho_\epsilon, \rho_\epsilon u_1) \begin{pmatrix} H''(\rho_\epsilon) + \frac{u_\epsilon^2}{2} & -\frac{u_1}{\rho_\epsilon} \\ -\frac{u_1}{\rho_\epsilon} & \frac{1}{\rho_\epsilon} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -u_\epsilon^2 + p'(\rho_\epsilon) & 2u_1 \end{pmatrix} \begin{pmatrix} \rho_\epsilon \\ \rho_\epsilon u_1 \end{pmatrix} =$$

$$= \rho_\epsilon^2 u_1 \left(H''(\rho_\epsilon) + \frac{u_\epsilon^2}{2}\right) - u_1 \left(\left(-u_\epsilon^2 + p'(\rho_\epsilon)\right)\rho_\epsilon + 2\rho_\epsilon u_1^2\right) + \rho_\epsilon u_1 \left(-u_1^2 + \frac{1}{\rho_\epsilon}\left(\left(-u_\epsilon^2 + p'(\rho_\epsilon)\right)\rho_\epsilon + 2\rho_\epsilon u_1^2\right)\right) =$$

$$= u_1 \rho_\epsilon \left(p'(\rho_\epsilon) + \rho_\epsilon \frac{u_\epsilon^2}{2}\right) - u_1^3 \rho_\epsilon,$$

as soon as $H''(\rho) = \frac{p'(\rho)}{\rho}$.

Let us consider the initial condition $I_0(t, y)$ for the KZK equation of the type of the remark 15 and we suppose without loss of generality that $I_0 = 0$ for $t \in]0, L[$ only once in the point $\frac{L}{2}$, precisely we suppose that the sign of u_1 is changing in the following way:

- $u_1 \leq 0$ for $t \in [0 + (k-1)L, \frac{L}{2} + (k-1)L]$ $(k = 1, 2, 3, ...)$
- and $u_1 > 0$ for $t \in (\frac{L}{2} + (k-1)L, kL)$ $(k = 1, 2, 3, ...)$.

If $t \in [0, \frac{L}{2}]$ (for $k = 1$) the first component of velocity is negative

$$u_1|_{x_1=0} = \epsilon \frac{c}{\rho_0} I_0(t, y) + \epsilon^2 G(I_0)(t, y) < 0,$$

where $G(I_0)$ is L-periodic and of mean value zero from (III.43) ($u_1|_{x_1=0} = 0$ for $t = 0, \frac{L}{2}$), then

$$u_1 \rho_\epsilon \left(p'(\rho_\epsilon) + \rho_\epsilon \frac{u_\epsilon^2}{2}\right) - u_1^3 \rho_\epsilon < 0$$

since we have the negative term of order ϵ^2 and the positive term $-u_1^3 \rho_\epsilon$ of order ϵ^3. Therefore, for $t \in [0, \frac{L}{2}]$

$$\int_{x_1>0} \widetilde{U}_\epsilon^2 dx \leq \int_{x_1>0} \widetilde{U}_0^2 dx.$$

If $t \in (\frac{L}{2}, L)$ the first component of velocity is positive $u_1|_{x_1=0} > 0$, then we also impose $\rho_\epsilon|_{x_1=0} = \rho_0 + \epsilon I_0(t,y)$ where $I_0(t,y)$ is the initial condition for the KZK equation.Then we have that $u_1^3\rho_\epsilon$ has the good sign, for the term

$$u_1\rho_\epsilon\left(p'(\rho_\epsilon) + \rho_\epsilon\frac{u_\epsilon^2}{2}\right) > 0$$

we see that on the boundary it has the form

$$\epsilon\left(\frac{c}{\rho_0}I_0 + \epsilon v_1|_{z=0}\right)(\rho_0 + \epsilon I_0)\left(c^2\epsilon + \frac{(\gamma-1)c^2}{\rho_0}\epsilon^2 I_0 + (\rho_0 + \epsilon I_0)\left(\left(\epsilon\frac{c}{\rho_0}I_0 + \epsilon^2 v_1|_{z=0}\right)^2 + \right.\right.$$
$$\left.\left. + \epsilon^3\vec{w}^2|_{z=0}\right)\right) \le \epsilon^2 C_0 I_0,$$

for some constant $C_0 = c^3 + \delta$ and so

$$\int_0^t\left(u_1\rho_\epsilon\left(p'(\rho_\epsilon) + \rho_\epsilon\frac{u_\epsilon^2}{2}\right) - u_1^3\rho_\epsilon\right)ds =$$

$$= \int_0^{\frac{L}{2}}\left(u_1\rho_\epsilon\left(p'(\rho_\epsilon) + \rho_\epsilon\frac{u_\epsilon^2}{2}\right) - u_1^3\rho_\epsilon\right)ds + \int_{\frac{L}{2}}^t\left(u_1\rho_\epsilon\left(p'(\rho_\epsilon) + \rho_\epsilon\frac{u_\epsilon^2}{2}\right) - u_1^3\rho_\epsilon\right)ds \le$$

$$\le -\left|\int_0^{\frac{L}{2}}\left(u_1\rho_\epsilon\left(p'(\rho_\epsilon) + \rho_\epsilon\frac{u_\epsilon^2}{2}\right) - u_1^3\rho_\epsilon\right)ds\right| + \int_{\frac{L}{2}}^L u_1\rho_\epsilon\left(p'(\rho_\epsilon) + \rho_\epsilon\frac{u_\epsilon^2}{2}\right)ds - \int_{\frac{L}{2}}^L u_1^3\rho_\epsilon dt \le$$

$$\le \epsilon^2 C_0\int_{\frac{L}{2}}^L I_0 dt = \epsilon^{2+p}C_0\int_{\frac{L}{2}}^L \tilde{I}_0 dt = \epsilon^{2+p}\widetilde{K},$$

where $\widetilde{K} = O(1)$ is a positive constant non depending on time and $p \ge 0$ is the order of "the sufficient small" initial data I_0.

Since $\widetilde{U}_0 = O(1)$ we obtain for all $t \le L + \frac{L}{2}$ (for $t \in [L, L+\frac{L}{2}]$ $u_1|_{x_1=0} < 0$ and $\int_L^{L+\frac{L}{2}}\left(u_1\rho_\epsilon\left(p'(\rho_\epsilon) + \rho_\epsilon\frac{u_\epsilon^2}{2}\right) - u_1^3\rho_\epsilon\right)ds < 0$) the estimate

$$\int_{x_1>0}\widetilde{U}_\epsilon^2 dx \le \int_{x_1>0}\widetilde{U}_0^2 dx + \epsilon^{2+p}\widetilde{K}.$$

But for $t < 2L + \frac{L}{2}$ we have, thanks to the periodicity of I_0,

$$\int_{x_1>0}\widetilde{U}_\epsilon^2 dx \le \int_{x_1>0}\widetilde{U}_0^2 dx + 2\epsilon^{2+p}\widetilde{K}.$$

Then we conclude that for $t < \tau L + \frac{L}{2}$

$$\int_{x_1>0}\widetilde{U}_\epsilon^2 dx \le \int_{x_1>0}\widetilde{U}_0^2 dx + \tau\epsilon^{2+p}\widetilde{K}.$$

To keep the sense of the bounded a priori estimate we need to impose that

$$\tau\epsilon^{2+p}\widetilde{K} = O(1), \quad \text{or} \quad \tau < \frac{1}{\widetilde{K}\epsilon^{2+p}}.$$

So for $t < L\left(\frac{1}{\widetilde{K}\epsilon^{2+p}} + \frac{1}{2}\right) = T$, or shortly $t < \frac{T_0}{\epsilon^{2+p}}$ with a constant $T_0 = O(1)$, we obtain that $\widetilde{U}_\epsilon \in L_\infty(]0,T[, L_2(\{x_1 > 0\} \times \mathcal{R}^{n-1}))$.

If $I_0 = 0$ for $t \in]0, L[$ finite number times, m times, beginning for example by negative sign, then we have the a priori estimate for $t < L\tau = T$, where $\tau < \frac{T_0}{(\widetilde{K}_1 + \ldots + \widetilde{K}_r)\epsilon^{2+p}}$, $r = [\frac{m+1}{2}]$.

To prove now that $\widetilde{U}_\epsilon \in L_\infty(]0,T[, H^{s'-2}(\{x_1 > 0\} \times \mathcal{R}^{n-1}))$ with s' from the condition of the theorem and where $s' - 1$ corresponds to the regularity of the initial condition $\widetilde{U}_0$, we use the result of [20, p. 352] for incompletely parabolic problems. We also obtain that $\partial_t \widetilde{U}_\epsilon \in L_\infty(]0,T[, H^{s'-4}(\{x_1 > 0\} \times \mathcal{R}^{n-1}))$.

Using the standard Faedo-Galerkin method with the theorem about a sequential compactness of the unit ball in the Hilbert space we obtain the existence of the unique solution of Navier-Stokes system. More precisely $\rho_\epsilon \in C([0,T[, H^{s'}(\{x_1 > 0\} \times \mathcal{R}^{n-1})) \cap C^1([0,T[, H^{s'-2}(\{x_1 > 0\} \times \mathcal{R}^{n-1}))$ and $u_\epsilon \in C([0,T[, H^{s'-2}(\{x_1 > 0\} \times \mathcal{R}^{n-1})) \cap C^1([0,T[, H^{s'-4}(\{x_1 > 0\} \times \mathcal{R}^{n-1}))$.

The approximation result

So we have two systems

$$\begin{cases} \partial_t \eta(\widetilde{U}_\epsilon) + \nabla.q(\widetilde{U}_\epsilon) - \epsilon\nu \begin{bmatrix} 0 \\ u_\epsilon \triangle u_\epsilon \end{bmatrix} = 0, \\ \partial_t \widetilde{U}_\epsilon + \nabla.F(\widetilde{U}_\epsilon) - \epsilon\nu \begin{bmatrix} 0 \\ \triangle u_\epsilon \end{bmatrix} = 0, \end{cases}$$

$$\begin{cases} \partial_t \eta(\overline{U}_\epsilon) + \nabla.q(\overline{U}_\epsilon) - \epsilon\nu \begin{bmatrix} 0 \\ \bar{u}_\epsilon \triangle \bar{u}_\epsilon \end{bmatrix} = \epsilon^{\frac{5}{2}} \left(\frac{\eta(\overline{U}_\epsilon) + p(\bar{\rho}_\epsilon)}{\bar{\rho}_\epsilon} R_1 + \bar{u}_\epsilon \vec{R}_2 \right), \\ \partial_t \overline{U}_\epsilon + \nabla.F(\overline{U}_\epsilon) - \epsilon\nu \begin{bmatrix} 0 \\ \triangle \bar{u}_\epsilon \end{bmatrix} = \epsilon^{\frac{5}{2}} \vec{R}, \end{cases}$$

where $\vec{R} = (R_1, \vec{R}_2)$ is the rest from (III.47). Since we suppose that $\overline{U}_\epsilon = (\bar{\rho}_\epsilon, \bar{\rho}_\epsilon \bar{u}_\epsilon)^T$ is bounded we can denote again

$$\bar{R} = \frac{\eta(\overline{U}_\epsilon) + p(\bar{\rho}_\epsilon)}{\bar{\rho}_\epsilon} R_1 + \bar{u}_\epsilon \vec{R}_2.$$

Let us compute

$$\frac{\partial}{\partial t}(\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon)).$$

The first we find from initial systems that

$$\frac{\partial}{\partial t}(\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon)) = -\nabla.((q(\widetilde{U}_\epsilon) - q(\overline{U}_\epsilon)) + \epsilon\nu \begin{bmatrix} 0 \\ u_\epsilon \triangle u_\epsilon - \bar{u}_\epsilon \triangle \bar{u}_\epsilon \end{bmatrix} - \epsilon^{\frac{5}{2}} \bar{R}. \qquad \text{(III.51)}$$

Then we notice that

$$-\frac{\partial}{\partial t}\left(\eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon)\right) = -\left(\frac{\partial \overline{U}_\epsilon}{\partial t}\right)^T \eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)\left(\frac{\partial \widetilde{U}_\epsilon}{\partial t} - \frac{\partial \overline{U}_\epsilon}{\partial t}\right),$$

where

$$-\left(\frac{\partial \overline{U}_\epsilon}{\partial t}\right)^T \eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) = -\left(-\nabla.F(\overline{U}_\epsilon) + \begin{bmatrix} 0 \\ \epsilon\nu\triangle\bar{u}_\epsilon \end{bmatrix} + \epsilon^{\frac{5}{2}}\vec{R}\right)^T \eta''(\overline{U}_\epsilon)\left(\widetilde{U}_\epsilon - \overline{U}_\epsilon\right),$$

and

$$\begin{aligned} -\eta'(\overline{U}_\epsilon)\left(\frac{\partial \widetilde{U}_\epsilon}{\partial t} - \frac{\partial \overline{U}_\epsilon}{\partial t}\right) &= -\eta'(\overline{U}_\epsilon)\left(-\nabla.F(\widetilde{U}_\epsilon) + \nabla.F(\overline{U}_\epsilon)\right) - \\ &\quad -\eta'(\overline{U}_\epsilon)\epsilon\nu \begin{bmatrix} 0 \\ \triangle u_\epsilon - \triangle \bar{u}_\epsilon \end{bmatrix} + \epsilon^{\frac{5}{2}}\eta'(\overline{U}_\epsilon)\vec{R}. \end{aligned}$$

Using the property for convex entropy $\eta''(U)F'(U) = (F'(U))^T\eta''(U)$ we find that

$$\left(\nabla.F(\overline{U}_\epsilon)\right)^T \eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) = \nabla.\overline{U}_\epsilon^T(F'(\overline{U}_\epsilon))^T\eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) = \nabla.\overline{U}_\epsilon^T\eta''(\overline{U}_\epsilon)F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon).$$

So we obtain that

$$\begin{aligned} &\frac{\partial}{\partial t}(\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon)) = -\nabla.(q(\widetilde{U}_\epsilon) - q(\overline{U}_\epsilon)) + \epsilon\nu \begin{bmatrix} 0 \\ u_\epsilon\triangle u_\epsilon - \bar{u}_\epsilon\triangle\bar{u}_\epsilon \end{bmatrix} - \epsilon^{\frac{5}{2}}\bar{R} + \\ &+\nabla.\overline{U}_\epsilon^T\eta''(\overline{U}_\epsilon)F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) - \begin{bmatrix} 0 \\ \epsilon\nu\triangle\bar{u}_\epsilon \end{bmatrix}^T \eta''(\overline{U}_\epsilon)\left(\widetilde{U}_\epsilon - \overline{U}_\epsilon\right) - \epsilon^{\frac{5}{2}}\vec{R}^T\eta''(\overline{U}_\epsilon)\left(\widetilde{U}_\epsilon - \overline{U}_\epsilon\right) + \\ &-\eta'(\overline{U}_\epsilon)\left(-\nabla.F(\widetilde{U}_\epsilon) + \nabla.F(\overline{U}_\epsilon)\right) - \eta'(\overline{U}_\epsilon)\epsilon\nu \begin{bmatrix} 0 \\ \triangle u_\epsilon - \triangle\bar{u}_\epsilon \end{bmatrix} + \epsilon^{\frac{5}{2}}\eta'(\overline{U}_\epsilon)\vec{R}, \end{aligned} \quad \text{(III.52)}$$

where, thanks to (III.49), (III.50) (and $\frac{\rho}{\bar{\rho}_\epsilon} = s+1$),

$$-\begin{bmatrix} 0 \\ \epsilon\nu\triangle\bar{u}_\epsilon \end{bmatrix}^T \eta''(\overline{U}_\epsilon)\left(\widetilde{U}_\epsilon - \overline{U}_\epsilon\right) = -\epsilon\nu \begin{bmatrix} -\frac{\bar{u}_\epsilon}{\bar{\rho}_\epsilon}\triangle\bar{u}_\epsilon \\ \frac{1}{\bar{\rho}_\epsilon}\triangle\bar{u}_\epsilon \end{bmatrix}^T \begin{bmatrix} \rho_\epsilon - \bar{\rho}_\epsilon \\ \rho u - \bar{\rho}_\epsilon\bar{u}_\epsilon \end{bmatrix} =$$

$$= -\epsilon\nu\triangle\bar{u}_\epsilon\,(-\bar{u}_\epsilon, 1)\begin{bmatrix} s \\ u(s+1) - \bar{u}_\epsilon \end{bmatrix} = -\epsilon\nu\triangle\bar{u}_\epsilon(u_\epsilon - \bar{u}_\epsilon) - \epsilon\nu\triangle\bar{u}_\epsilon\frac{\rho_\epsilon - \bar{\rho}_\epsilon}{\bar{\rho}_\epsilon}(u_\epsilon - \bar{u}_\epsilon),$$

$$-\eta'(\overline{U}_\epsilon)\epsilon\nu\begin{bmatrix} 0 \\ \triangle u_\epsilon - \triangle\bar{u}_\epsilon \end{bmatrix} = -\epsilon\nu\bar{u}_\epsilon(\triangle u_\epsilon - \triangle\bar{u}_\epsilon)$$

Integrate (III.18) over the half space. The use of the integration by parts gives with notation q_1, F_1 for first components of vectors q and F:

$$\frac{d}{dt}\int_{x_1>0}(\eta(\widetilde{U}_\epsilon)-\eta(\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon)dx=$$
$$-\int_{x_1=0}(q_1(\widetilde{U}_\epsilon)-q_1(\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)^T(F_1(\widetilde{U}_\epsilon)-F_1(\overline{U}_\epsilon)))dx'-$$
$$-\int_{x_1>0}\nabla.\overline{U}_\epsilon^T\eta''(\overline{U}_\epsilon)(F(\widetilde{U}_\epsilon)-F(\overline{U}_\epsilon)-F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon))dx+$$
$$+\epsilon\nu\int_{x_1=0}\left(u_\epsilon\frac{\partial u_\epsilon}{\partial x_1}-\bar{u}_\epsilon\frac{\partial\bar{u}_\epsilon}{\partial x_1}\right)dx'-\epsilon\nu\int_{x_1>0}\left(|\nabla.u_\epsilon|^2-|\nabla.\bar{u}_\epsilon|^2\right)dx+$$
$$-\epsilon^{\frac{5}{2}}\int_{x_1>0}\left(\bar{R}-\eta'(\overline{U}_\epsilon)\vec{R}\right)dx-\epsilon^{\frac{5}{2}}\int_{x_1>0}\vec{R}^T\eta''(\overline{U}_\epsilon)\left(\widetilde{U}_\epsilon-\overline{U}_\epsilon\right)dx+$$
$$+\epsilon\nu\int_{x_1>0}\triangle\bar{u}_\epsilon\frac{\rho_\epsilon-\bar{\rho}_\epsilon}{\bar{\rho}_\epsilon}(u_\epsilon-\bar{u}_\epsilon)dx+\epsilon\nu\int_{x_1=0}\left(-\bar{u}_\epsilon\frac{\partial(u_\epsilon-\bar{u}_\epsilon)}{\partial x_1}-\frac{\partial\bar{u}_\epsilon}{\partial x_1}(u_\epsilon-\bar{u}_\epsilon)\right)dx'+$$
$$+\epsilon\nu\int_{x_1>0}\left(\nabla.\bar{u}_\epsilon\cdot\nabla.(u_\epsilon-\bar{u}_\epsilon)+\nabla.\bar{u}_\epsilon\cdot\nabla.(u_\epsilon-\bar{u}_\epsilon)\right)dx. \qquad \text{(III.53)}$$

It is easy to see that

$$\epsilon\nu\int_{x_1=0}\left(u_\epsilon\frac{\partial u_\epsilon}{\partial x_1}-\bar{u}_\epsilon\frac{\partial\bar{u}_\epsilon}{\partial x_1}\right)dx'+\epsilon\nu\int_{x_1=0}\left(-\bar{u}_\epsilon\frac{\partial(u_\epsilon-\bar{u}_\epsilon)}{\partial x_1}-\frac{\partial\bar{u}_\epsilon}{\partial x_1}(u_\epsilon-\bar{u}_\epsilon)\right)dx'=$$
$$=\epsilon\nu\int_{x_1=0}(u_\epsilon-\bar{u}_\epsilon)\frac{\partial(u_\epsilon-\bar{u}_\epsilon)}{\partial x_1}dx' \qquad \text{(III.54)}$$

and

$$-\epsilon\nu\int_{x_1>0}\left(|\nabla.u_\epsilon|^2-|\nabla.\bar{u}_\epsilon|^2\right)dx+\epsilon\nu\int_{x_1>0}\left(\nabla.\bar{u}_\epsilon\cdot\nabla.(u_\epsilon-\bar{u}_\epsilon)+\nabla.\bar{u}_\epsilon\cdot\nabla.(u_\epsilon-\bar{u}_\epsilon)\right)dx=$$
$$=-\epsilon\nu\int_{x_1>0}|\nabla.(u_\epsilon-\bar{u}_\epsilon)|^2dx.$$

Let us envisage now the boundary condition

$$-\int_{x_1=0}(q_1(\widetilde{U}_\epsilon)-q_1(\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)^T(F_1(\widetilde{U}_\epsilon)-F_1(\overline{U}_\epsilon)))dx'=$$
$$=(q_1(\widetilde{U}_\epsilon)-q_1(\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)^T(F_1(\widetilde{U}_\epsilon)-F_1(\overline{U}_\epsilon)))|_{x_1=0}. \qquad \text{(III.55)}$$

Explicitly (III.55) has the form:

$$(q_1(\widetilde{U}_\epsilon)-q_1(\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)^T(F_1(\widetilde{U}_\epsilon)-F_1(\overline{U}_\epsilon)))|_{x_1=0}=u_1(\eta(\widetilde{U}_\epsilon)+p(\rho_\epsilon))-\bar{u}_1(\eta(\overline{U}_\epsilon)+p(\bar{\rho}_\epsilon))-$$
$$-\begin{bmatrix}H'(\bar{\rho}_\epsilon)-\frac{\bar{u}_\epsilon^2}{2}\\ \bar{u}_1\end{bmatrix}^T\begin{bmatrix}\rho_\epsilon u_1-\bar{\rho}_\epsilon\bar{u}_1\\ \rho_\epsilon u_\epsilon^2+p(\rho_\epsilon)-\bar{\rho}_\epsilon\bar{u}_\epsilon^2-p(\bar{\rho}_\epsilon)\end{bmatrix}\Bigg|_{x_1=0}.$$

Choosing always $u_1|_{x_1=0} = \bar{u}_1|_{x_1=0}$ we obtain with the help of the facts that the entropy η is convex

$$\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) \geq \alpha|\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2,$$

$$(q_1(\widetilde{U}_\epsilon) - q_1(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)^T(F_1(\widetilde{U}_\epsilon) - F_1(\overline{U}_\epsilon)))|_{x_1=0} = u_1(\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) + p(\rho_\epsilon) - p(\bar{\rho}_\epsilon)) -$$
$$-u_1(\rho_\epsilon - \bar{\rho}_\epsilon)\left(H'(\bar{\rho}_\epsilon) - \frac{\bar{u}_\epsilon^2}{2}\right) - u_1\left[\rho_\epsilon u_\epsilon^2 - \bar{\rho}_\epsilon \bar{u}_\epsilon^2 + p(\rho_\epsilon) - p(\bar{\rho}_\epsilon)\right]|_{x_1=0} =$$
$$= u_1\left(\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)^T(\widetilde{U}_\epsilon - \overline{U}_\epsilon)\right)\Big|_{x_1=0} \tag{III.56}$$

and so this boundary condition take the same sign as the first component of velocity $u_1|_{x_1=0}$ since

$$\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)^T(\widetilde{U}_\epsilon - \overline{U}_\epsilon) \geq (\widetilde{U}_\epsilon - \overline{U}_\epsilon)^2 \geq 0$$

is always positive.

So for the boundary conditions we take $u_\epsilon|_{x_1=0} = \bar{u}_\epsilon|_{x_1=0}$ and if $u_1 > 0$ we also suppose $\rho_\epsilon|_{x_1=0} = \bar{\rho}_\epsilon|_{x_1=0}$. Then the first boundary condition (III.54) is always zero:

$$\epsilon\nu \int_{x_1=0} (u_\epsilon - \bar{u}_\epsilon)\frac{\partial(u_\epsilon - \bar{u}_\epsilon)}{\partial x_1}dx' = -\epsilon\nu(u_\epsilon - \bar{u}_\epsilon)\frac{\partial(u_\epsilon - \bar{u}_\epsilon)}{\partial x_1}|_{x_1=0} = 0.$$

For the second boundary condition (III.56), if $u_\epsilon|_{x_1=0} < 0$ then we have in the left-hand side of the estimate a positive quantity on the boundary and it can be omitted.

If $u_\epsilon|_{x_1=0} > 0$ then we have for $\rho_\epsilon|_{x_1=0} = \bar{\rho}_\epsilon|_{x_1=0}$

$$u_1\left(H(\rho_\epsilon) + \rho_\epsilon\frac{u_\epsilon^2}{2} - H(\bar{\rho}_\epsilon) - \bar{\rho}_\epsilon\frac{\bar{u}_\epsilon^2}{2} - (\rho_\epsilon - \bar{\rho}_\epsilon)\left(H'(\bar{\rho}_\epsilon) - \frac{\bar{u}_\epsilon^2}{2}\right) - \rho_\epsilon u_\epsilon^2 + \bar{\rho}_\epsilon\bar{u}_\epsilon^2\right)\Bigg|_{x_1=0} =$$
$$= u_1\left(H(\rho_\epsilon) - H(\bar{\rho}_\epsilon) - (\rho_\epsilon - \bar{\rho}_\epsilon)H'(\bar{\rho}_\epsilon) + \rho_\epsilon\left(\frac{\bar{u}_\epsilon^2}{2} - \frac{u_\epsilon^2}{2}\right)\right)\Bigg|_{x_1=0} =$$
$$= u_1\left(H(\rho_\epsilon) - H(\bar{\rho}_\epsilon) - (\rho_\epsilon - \bar{\rho}_\epsilon)H'(\bar{\rho}_\epsilon)\right)|_{x_1=0} = 0.$$

We use the Taylor expansion

$$F(\widetilde{U}_\epsilon) - F(\overline{U}_\epsilon) = F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) + O(|\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2),$$

and we have

$$F(\widetilde{U}_\epsilon) - F(\overline{U}_\epsilon) - F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) \leq C|\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2.$$

Taking the same initial data

$$\overline{U}_\epsilon|_{t=0} = \widetilde{U}_\epsilon|_{t=0},$$

we obtain

$$\frac{d}{dt}\int_{x_1>0}|\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2 dx \leq C\|\nabla.\overline{U}_\epsilon\|_{L_\infty}\int_{x_1>0}|\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2 dx + K\epsilon^5.$$

Here the constants K and C do not depend on t.

Therefore applying the Gronwall lemma since $\nabla.\overline{U}_\epsilon = O(\epsilon)$ one has

$$\|\widetilde{U}_\epsilon - \overline{U}_\epsilon\|^2_{L_2(\{x_1>0\})} \le K\epsilon^5 e^{\epsilon Ct}. \tag{III.57}$$

As soon as the difference of the solutions has the order $\widetilde{U}_\epsilon - \overline{U}_\epsilon = O(\epsilon)$ and the left side of (III.57) has the order $O(\epsilon^2)$, so we would like to have the inequality

$$\epsilon^5 \int_0^t e^{C\epsilon(t-s)} ds < \epsilon^2.$$

From $\epsilon^3 e^{C\epsilon t} < 1$ we obtain that the estimate (III.48) is smaller than the order ϵ^2 for time $t < T\frac{1}{\epsilon}\ln\frac{1}{\epsilon}$. $\square$

It is possible to extend the previous results to the case where $\widetilde{U}_\epsilon$ is only an admissible solution satisfying the boundary conditions.

Definition 1 *The pair of functions (ρ, u) is called an admissible weak solution of Navier-Stokes system (III.37) satisfying the boundary conditions in the half space if it satisfies the following properties:*

1. *it is a weak solution of (III.37),*
2. *it satisfies in the sense of distributions (see [15, p.52])*

$$\partial_t \eta(U_\epsilon) + \nabla.q(U_\epsilon) - \epsilon\nu \left[\begin{array}{c} 0 \\ u_\epsilon \triangle u_\epsilon \end{array}\right] \le 0,$$

or equivalently for all nonnegative twice differentiable test function ψ with compact support in the half space

$$\int_0^T \int_{x_1>0} \left(\partial_t\psi\eta(U_\epsilon) + \nabla.\psi q(U_\epsilon) - \epsilon\nu \left[\begin{array}{c} 0 \\ |\nabla.u_\epsilon|^2 \end{array}\right]\psi + \epsilon\nu \left[\begin{array}{c} 0 \\ \nabla\frac{u_\epsilon^2}{2} \end{array}\right]\nabla.\psi\right) dxdt +$$
$$+ \int_{x_1>0} \psi(x,0)\eta(U_0(x))dx + \int_0^T \int_{\mathcal{R}^{n-1}} \psi\left(q(U_\epsilon) - \epsilon\nu\left[\begin{array}{c} 0 \\ u_\epsilon \cdot \nabla.u_\epsilon \end{array}\right]\right)\Big|_{x_1=0} dx'dt \ge 0.$$

3. *it satisfies the equality*

$$-\int_{x_1>0} \frac{U_\epsilon^2}{2} dx + \int_0^t \int_{x_1>0} \left(\nabla.U_\epsilon F(U_\epsilon) + \epsilon\nu\left[\begin{array}{c} 0 \\ |\nabla.u_\epsilon|^2 \end{array}\right]\right) dxds + \int_{x_1>0} U_0^2(x)dx +$$
$$+ \int_0^t \int_{\mathcal{R}^{n-1}} U_\epsilon \left(F(U_\epsilon) - \epsilon\nu\left[\begin{array}{c} 0 \\ \nabla u_\epsilon \end{array}\right]\right)\Big|_{x_1=0} dx'dt = 0.$$

Theorem 13 *To have the estimate (III.48) it is sufficient to have an admissible weak solution of the Navier-Stokes system (III.37) satisfying the boundary conditions in the half space*

$$\partial_t U_\epsilon + \nabla.F(U_\epsilon) - \epsilon\nu\left[\begin{array}{c} 0 \\ \triangle u_\epsilon \end{array}\right] = 0,$$

such that [34, 38] $\rho_\epsilon \in L_\infty((0,T),L_2)\cap C([0,T],L_p)$ *for* $1\leq p\leq 2$, $\rho_\epsilon \geq 0$ *a.e.,* $\nabla.u_\epsilon \in L_2((0,T);L_2)$, $\rho_\epsilon|u_\epsilon|^2 \in L_\infty((0,T),L_1)$, $u_\epsilon \in L_2((0,T),H_0^1)$, $\rho_\epsilon u_\epsilon \in C([0,T],L_{\frac{4}{3}}-\omega)$, *here by* $C([0,T],L_{\frac{4}{3}}-\omega)$ *is denoted the space of continuous functions with values in a closed ball of* $L_{\frac{4}{3}}$ *endowed with the weak topology.*

III.3 Conclusion

The approximation result for nonlinear KZK equation

$$\|\bar{U}_\epsilon - U_\epsilon\|^2_{L_2} \leq \epsilon^5 e^{C\epsilon t}$$

is valid in the viscous and non viscous cases. The obtained estimate guarantees that the difference $\bar{U}_\epsilon - U_\epsilon$ stays of order $O(\epsilon)$ during the time of the order $\frac{1}{\epsilon}\ln\frac{1}{\epsilon}$.

Let us for the conclusion make a comparative table for the approximation results (see table III.1).

One use the notation: $A(u-v)\equiv|\partial_t(u-v)|^2+|\nabla.(u-v)|^2$.

Table III.1 – *The approximation's results*

<table>
<tr><th></th><th>Linearization of Navier-Stokes system</th><th>Navier-Stokes system</th><th>Euler system</th></tr>
<tr><td>Domain</td><td colspan="2">the half space
$\{x_1 > 0, x' \in \mathcal{R}^{n-1}\}$</td><td>the cone
$Q_\epsilon(t) = \{|x_1| < \frac{R}{\epsilon} - ct\} \times \mathcal{R}^{n-1}_{x'}$</td></tr>
<tr><td>Exact system</td><td>$\partial_t \rho + \nabla . u = 0,$
$\partial_t u + \nabla p(\rho) = \epsilon \nu \Delta u$</td><td>$\partial_t \rho_\epsilon + \mathrm{div}(\rho_\epsilon u_\epsilon) = 0,$
$\rho_\epsilon[\partial_t u_\epsilon + (u_\epsilon \cdot \nabla) u_\epsilon] = -\nabla p(\rho_\epsilon) + \epsilon \nu \Delta u_\epsilon$</td><td>$\partial_t \rho_\epsilon + \mathrm{div}(\rho_\epsilon u_\epsilon) = 0,$
$\rho_\epsilon[\partial_t u_\epsilon + (u_\epsilon \cdot \nabla) u_\epsilon] + \nabla p(\rho_\epsilon) = 0$</td></tr>
<tr><td>State equation</td><td>$p(\rho) = c^2 \rho$</td><td colspan="2">$p = p(\rho_\epsilon) = c^2 \epsilon \rho_\epsilon + \frac{(\gamma-1)c^2}{2\rho_0} \epsilon^2 {\rho_\epsilon}^2$</td></tr>
<tr><td>Exact solution</td><td>$U_\epsilon = (\rho, u)$</td><td colspan="2">$U_\epsilon = (\rho_\epsilon, \rho_\epsilon u_\epsilon)$</td></tr>
<tr><td>Approximate solution</td><td>$\bar{U}_\epsilon = (I, v + \sqrt{\epsilon} \vec{w})$</td><td colspan="2">$\bar{U}_\epsilon = (\rho_0 + \epsilon I, \epsilon(\rho_0 + \epsilon I)(v + \epsilon v_1, \sqrt{\epsilon} \vec{w}))$</td></tr>
<tr><td>$v_1(\tau, z, y) =$</td><td>0</td><td>$-\frac{c^2}{\rho_0}\left(\int_0^\tau \partial_z I ds + \int_0^L \frac{s}{L} \partial_z I ds \right) +$
$+ \frac{(\gamma-1)}{2\rho_0^2} c I^2 - \frac{c(\gamma-1)}{2L\rho_0^2} \int_0^L I^2 d\tau + \frac{\nu}{c\rho_0^2} \partial_\tau I$</td><td>$-\frac{c^2}{\rho_0}\left(\int_0^\tau \partial_z I ds + \int_0^L \frac{s}{L} \partial_z I ds \right) +$
$+ \frac{(\gamma-1)}{2\rho_0^2} c I^2 - \frac{c(\gamma-1)}{2L\rho_0^2} \int_0^L I^2 d\tau$</td></tr>
<tr><td>I is solution of</td><td>$2c\partial^2_{\tau z} I - c^2 \triangle_y I -$
$-\frac{\nu}{\rho_0 c^2} \partial^3_\tau I = 0$</td><td>$c\partial^2_{\tau z} I - \frac{(\gamma+1)}{4\rho_0} \partial^2_\tau I^2$-
$-\frac{\nu}{2c^2\rho_0} \partial^3_\tau I - \frac{c^2}{2} \Delta_y I = 0$</td><td>$c\partial^2_{\tau z} I - \frac{(\gamma+1)}{4\rho_0} \partial^2_\tau I^2 - \frac{c^2}{2} \Delta_y I = 0$</td></tr>
<tr><td>Smoothness of KZK solution</td><td colspan="2">$s > \max\{6, [\frac{n}{2}] + 1\}$</td><td>$s > \max\{4, [\frac{n}{2}] + 1\}$</td></tr>
<tr><td>Time of validation for $U_\epsilon - \overline{U}_\epsilon = O(\epsilon)$</td><td>$T \frac{1}{\epsilon^2}$ for $\bar{u}_\epsilon - u_\epsilon$,
$T \ln \frac{1}{\epsilon}$ for $\bar{\rho}_\epsilon - \rho_\epsilon$</td><td colspan="2">$\frac{T}{\epsilon} \ln \frac{1}{\epsilon}$</td></tr>
<tr><td>Estimation</td><td>$\int_{x_1>0} A(\bar{u}_\epsilon - u_\epsilon) dx \le \epsilon^4 t^2$
$\|\bar{\rho}_\epsilon - \rho_\epsilon\|_{L_2} \le \epsilon^2 e^{Ct}$</td><td>$\|\bar{U}_\epsilon - U_\epsilon\|^2_{L_2} \le \epsilon^5 e^{C\epsilon t}$</td><td>$s' = s - 4 \quad \|\bar{U}_\epsilon - U_\epsilon\|^2_{H^{s'}} \le \epsilon^5 e^{C\epsilon t}$</td></tr>
</table>

Part II

Controllability of Moments of Evolution Equations

Chapter IV

Controllability of the Moments $\int_0^T u(t,x)d\mu(t)$ on Solutions of Nonlinear Abstract Evolution Equation

Table of contents

This chapter consists of three parts. As soon as our goal was to make a schema of the using method by the most clear way, we envisage the abstract problem. Therefore, there is a rather general result of this kind for an linear inverse problem for an abstract equation by Prilepko A.I. and Tikhonov I.V., which we cite and use here. Since the proof of the main result for the nonlinear problem under consideration is totally based on the theory for the linear analog of the problem given in [40], in the first part of this chapter we introduce the main notations from [40] with some brief explanations and give statements of the theorems needed in what follows.

In the second part, we prove the local solvability of the inverse problem for a nonlinear abstract evolution equation with integral overdetermination so as to be able to treat this problem as one of controllability of moments. A similar result for the linear problem was obtained earlier in [40]. The proof is based on well-known properties of solutions of linear (direct and inverse) problems as well as on the two-fold application of the inverse function theorem in the corresponding function spaces (in one of two cases, the space of preimages is defined as the solution set of the corresponding problem).

In the third part, twice applying the refined inverse function theorem in its most general version given in [48], we obtain sufficient conditions on the size of the neighborhood from which the function from the overdertermination condition can be taken so that the original inverse problem will be uniquely solvable.

The results of this chapter can be found in [42, 43].

Earlier, this approach to inverse problems was applied, for example, to a quasilinear heat equation with integral overdetermination condition [2].

IV.1 Preliminaries. Notation

In the present chapter, following [40], we use the following notation:

- E is a Banach space with norm $\|\cdot\|$;
- $\mathcal{L}(E)$ is the Banach algebra of linear bounded operators mapping E into E, endowed with ordinary operator norm;
- I is the unit operator in $\mathcal{L}(E)$;
- $A : D(A) \to E$ is a linear closed operator with a dense domain $D(A) \subset E$;
- $|||\cdot|||$ is the norm of a graph on $D(A)$, i.e., $|||\psi||| = \|\psi\| + \|A\psi\|$ for $\psi \in D(A)$;
- $\mathcal{D}(A)$ is the Banach space $D(A)$ with the norm of a graph;

- $\sigma(A)$ is the spectrum of the operator A;
- $X = C^1([0,T];E) \cap C([0,T];D(A))$, where $T > 0$.

By a *scalar function* we mean a function taking real or complex values according as the space E is real or complex. By a *vector function* we mean a function with values in a Banach space. By an *operator function* we mean a function with values in a Banach algebra $\mathcal{L}(E)$. Integrals of operator functions *a priori* in the *strong operator topology* $\mathcal{L}(E)$.

IV.2 Linear problem. Fundamentals

IV.2.1 Setting the linear problem

In [40], the problem of determining an element f in Banach space E from the relations

$$u'(t) = [Au](t) + \Phi(t)f \qquad (0 \le t \le T), \tag{IV.1}$$

$$u(0) = \psi_0 \qquad (\psi_0 \in D(A)), \tag{IV.2}$$

$$\int\limits_0^T u(t)d\mu(t) = \psi_1 \qquad (\psi_1 \in D(A)), \tag{IV.3}$$

was studied; here $T > 0$ is a number,

A generates a semigroup $S(t)$ of class C_0,

the function $\Phi(t) \in C^1([0,T];\mathcal{L}(E))$ is an operator function,

$\mu(t)$ is a scalar function of bounded variation on $[0,T]$,

and $f \in E$ is an unknown control;

the integral in condition (IV.3) is regarded as a vector Riemann-Stieltjes integral.

Definition 2 *By a solution of the problem (IV.1)-(IV.3) for fixed ψ_0, ψ_1 from $D(A)$ we mean an element $f \in E$ such that the solution of the Cauchy problem (IV.1), (IV.2) with given f satisfies condition (IV.3).*

By a *solution of the Cauchy problem* we mean a function $u(t) \in X$ satisfying equation (IV.1) for $0 \le t \le T$ and taking the value ψ_0 at $t = 0$. The requirements imposed on the operators A and $\Phi(t)$, guarantee the existence and uniqueness o the solution of the Cauchy problem (IV.1), (IV.2) for any f from E (see [30, 39]) and the estimate of well-posedness [39, p.107], [40, p.169]

$$\|u\|_{C^1([0,T],E)} \le K\left(\||\psi_0\|| + \|\Phi(t)f\|_{C^1([0,T],E)}\right)$$

with some constant K.

Remark 16 *The above estimate follows from the Cauchy formula*

$$u(t) = S(t)\psi_0 + \int_0^t S(t-s)\Phi(s)f ds$$

and with the help of [40, p.169]

$$A\left(\int_0^t S(t-s)\Phi(s)f ds\right) = S(t)\Phi(0)f - \Phi(t)f + \int_0^t S(t-s)\Phi'(s)f ds$$

from the relation

$$u'(t) = AS(t)\psi_0 + S(t)\Phi(0)f + \int_0^t S(t-s)\Phi'(s)f ds.$$

Indeed, we estimate $u(t)$ and $u'(t)$ in the $C([0,T],E)$ norm using that $\Phi(t) \in C^1([0,T];\mathcal{L}(E))$ and the semigroup $S(t)$ is bounded as it is of class C_0.

Therefore, problem (IV.1)- (IV.3) is to choose, by varying the element f, the function $u(t) = u(t;f)$ satisfying condition (IV.3), among all the solutions of the Cauchy problem (IV.1), (IV.2). By the accepted terminology, problem (IV.1)- (IV.3) belongs to the *controllability problems* or to the *inverse problems*, i.e., to those problems in which it is required to find the control (in other words, to reconstruct the exact form of the differential equation) by using special *overdetermination* (in our case, condition (IV.3)).

Remark 17 *In [40], it was noted that the overdetermination condition (IV.3) generalizes the well-known terminal and integral overdetermination conditions, which can be obtained from (IV.3) as special cases and thus allows one to consider problems with different overdetermination conditions within the framework of a single theory.*

So the terminal overdetermination condition $u(T) = \psi_1$ can be obtained as a special case in which the function $\mu(t)$ in (IV.3) is a function of jumps:

$$\alpha_1 u(t_1) + \alpha_2 u(t_2) + ... = \psi_1, \qquad \text{where} \qquad t_i \in [0,T],$$

and the numbers α_i satisfy $\Sigma|\alpha_i| < \infty$.

On the other hand, if the function $\mu(t)$ is absolutely continuous, then, instead of condition (IV.3), we obtain the following integral overdetermination condition:

$$(B)\int_0^T \omega(t)u(t)dt = \psi_1,$$

where $\omega(t) = \mu'(t)$ is an integrable function(here the Bochner integral for $\mu \in C^1[0,T]$ can be replaced by a Riemann integral).

IV.2.2 Well-posedness of the linear problem

Definition 3 *Problem (IV.1)- (IV.3) is said to be* well-posed *if for all ψ_0, ψ_1 from $D(A)$ it has a unique solution $f \in E$.*

Moreover, as was proved in [40], in that case the solution f corresponding to the elements ψ_0, ψ_1 satisfies the estimate

$$\|f\| \leq C(|||\psi_0||| + |||\psi_1|||)$$

with constant $C > 0$ independent of ψ_0, ψ_1.

Conditions for well-posedness were obtained in [40] for the following two cases: when the function $\Phi(t)$ in (IV.1) is a scalar function and when it is an operator one. For the clarity of subsequent arguments, we formulate these cases.

Case of a scalar function $\Phi(t)$

Suppose that $\Phi(t)$ is the multiplication operator by a scalar function $\phi(t)$, i.e., $\Phi(t)f \equiv \phi(t)f$, where $0 \le t \le T$, $f \in E$. Suppose that $\phi \in C^1[0,T]$.

For this case, the following theorem was proved in [40].

Theorem 14 *Suppose that $\mu(t)$ is a nondecreasing function on $[0,T]$ and is continuous on the right for $t = 0$. Suppose that $\phi \ge 0$ for $0 \le t \le T$ and*

$$\int_0^T \phi(t) d\mu(t) \neq 0,$$

and suppose that the semigroup $S(t)$ generated by the operator A satisfies the estimate

$$\|S(t)\| \le Mexp(-\beta t) \tag{IV.4}$$

with constant $M \ge 1$, $\beta > 0$. Also, suppose that any of the following conditions is satisfied:

(a) $\phi'(t) \ge 0$ for $0 \le t \le T$;

(b) the function $\mu(t)$ is convex up on $[0,T]$.

Then problem (IV.1)- (IV.3) is well posed.

Remark 18 *The condition (b) signifies that*

$$\mu\left(\frac{t_1 + t_2}{2}\right) \ge \frac{1}{2}(\mu(t_1) + \mu(t_2)) \quad for \quad 0 \le t_1,\ t_2 \le T.$$

In this case the function $\mu(t)$ is continuous on the interval $(0,T)$ (see [37, ch.X, §5]), and since $\mu(t)$ is continuous for $t = 0$ and nondecreasing, then $\mu(t)$ is continuous on all interval $[0,T]$. By [37, appendix III] there exists a such summable, nonnegative and non-increasing on $[0,T]$ the function $\omega(t)$, that

$$\mu(t) = \mu(0) + \int_0^t \omega(s) ds$$

(it is not excluded that $\omega(0) = +\infty$). So, if the property (b) holds then the operator

$$K = \int_0^T \left[S(t)\Phi(0) + \int_0^t S(t-s)\Phi'(s) ds \right] d\mu(t)$$

can be rewritten in the following way:

$$K = \omega(T) \int_0^T S(s)\Phi(T-s)ds - (B) \int_0^T S(s)ds \int_s^T \Phi(t-s)d\omega(t).$$

The last presentation of the operator K allows to estimate it in E.

Briefly, it is sufficient to suppose that $\mu \in C^1[0,T]$ and then we have

$$\mu(t) = \mu(0) + \int_0^t \mu'(s)ds.$$

Case of an operator function $\Phi(t)$

We assume that the space E is a Banach structure (a lattice) with cone E_+. The condition for the positivity of an operator $L \in \mathcal{L}(E)$, i.e., an operator such that $L(E_+) \subset E_+$, is denoted by the symbol $L \geq 0$. A semigroup $S(t)$ is said to be *positive* if $S(t) \geq 0$ for all $t \geq 0$.

Theorem 15 *Suppose that $\mu(t)$ is a nondecreasing function on $[0,T]$ and is continuous on the right for $t = 0$. Suppose that $\Phi \geq 0$ for $0 \leq t \leq T$; moreover, let the operator*

$$J = \int_0^T \Phi(t)d\mu(t)$$

satisfy the conditions $J^{-1} \in \mathcal{L}(E)$, $J^{-1} \geq 0$. Suppose that the semigroup $S(t)$, generated by the operator A is positive and compact for $t > 0$. Further, suppose that the spectrum of the operator A lies in the half-plane $\{\lambda \in C : Re\lambda < 0\}$. Finally, suppose that any of the following relations is satisfied:

(a) $\Phi'(t) \geq 0$ for $0 \leq t \leq T$;

(b) the function $\mu(t)$ is convex up on $[0,T]$.

Then problem (IV.1)- (IV.3) is well posed.

Remark 19 *[40]. Since the semigroup $S(t)$ is positive, the requirement, imposed on the spectrum $\sigma(A)$ in theorem 15, implies $s(A) < 0$, where $s(A) \equiv sup\{Re\lambda : \lambda \in \sigma(A)\}$ is the spectral boundary of the operator A.*

Remark 20 *[40]. The requirements for the semigroup $S(t)$ and $s(A) < 0$ in theorem 15 are equivalent to the estimate (IV.4) in theorem 14.*

IV.2.3 Inverse function theorem

Let us recall the inverse function theorem.

Theorem 16 *Suppose that X and Y are Banach spaces, U is an open set in X, the mapping $f : U \to Y$ is strictly differentiable on U, and $f'(x_0) : X \to Y$ is an isomorphism from X to Y for some point $x_0 \in U$. Then there exists a neighborhood U' of the point x_0, such that f induces a homeomorphism of U' onto the open set $f(U')$, $f'(x)$ is an* *isomorphism from* X *onto* Y *for* $x \in U'$, $f^{-1} : f(U') \to X$ *is strictly differentiable on* $f(U')$ *and* $(f^{-1})'\big(f(x)\big) = [f'(x)]^{-1}$ *for* $x \in U'$ *(i.e.,* f *is a diffeomorphism* U' *on* $f(U')$ *of class* C^1*).*

IV.3 Nonlinear problem. Local solvability

In what follows, we use the notation introduced above.

In (IV.1), we additionally denote

$$\varphi = Jf = \left[\int_0^T \Phi(t)d\mu(t)\right] f \in E, \tag{IV.5}$$

$$F(t) = \Phi(t)f \in C^1([0,T],E), \tag{IV.6}$$

whence

$$F(t) = [\Phi \circ J^{-1}](\varphi). \tag{IV.7}$$

Also, consider the linear operator

$$\hat{A} : E \to E, \quad \hat{A}\varphi = \int_0^T u'(t)d\mu(t), \tag{IV.8}$$

where $u(t)$ is the solution of the linear problem (IV.1)- (IV.3) for $\psi_0 = 0$, which can be obtained for $f = J^{-1}\varphi$ (see (IV.5)).

Remark 21 *It follows from the condition* $s(A) < 0$ *of theorem 15 that* $\lambda = 0$ *is a regular point of the operator* A*, i.e.,* A *is invertible on* $\mathcal{D}(A)$*, and then the homogeneous equation*

$$A\psi = 0 \tag{IV.9}$$

has only the zero solution or, equivalently, for any $\chi \in E$ *there exists a unique solution* $\psi \in \mathcal{D}(A)$ *of the equation* $A\psi = \chi$*, which will be denoted by* $\psi = A^{-1}\chi$*. By Remark 20, we can assume that equation (IV.9) always h as the zero solution.*

Let

$$H = \big\{v \in X | \exists F \in C^1([0,T],E) : v \text{ is the solution of the linear} \\ \text{Cauchy problem (IV.1)-(IV.2) for } \psi_0 = 0\big\}\,,$$

i.e., H is the solution space of the linear Cauchy problem.

Assuming $\psi_0 = 0$ in (IV.2), we note a well-known fact, appearing [39, p.107]: for any $F \in C^1([0,T],E)$, there exists a unique solution of the Cauchy problem (IV.1)- (IV.2) with estimate of well-posedness

$$\|u\|_{C^1([0,T],E)} \le K\|F\|_{C^1([0,T],E)},$$

where $K > 0$ is a constant independent of F.

Therefore, the operator

$$L = d/dt - A \tag{IV.10}$$

induces an isometric isomorphism of H on $LH = C^1([0,T], E)$ [39, p.107] with norm $\|u\|_H = \|Lu\|_{C^1([0,T],E)}$ and, therefore, the space $(H, \|\cdot\|_H)$ is Banach; moreover, by the *a priori* estimate H is continuously embedded in $C^1([0,T], E)$, and hence in $X = C^1([0,T]; E) \cap C([0,T]; D(A))$.

Theorem 17 *Suppose that the linear problem (IV.1)- (IV.3) is well posed(see theorems 14 and 15), $G : X \to C^1([0,T], E)$ is a nonlinear, strictly Fréchet-differentiable operator satisfying the conditions $G'(0) = 0$, $G(0) = 0$.*

Then the nonlinear problem

$$u'(t) = [Au](t) + Gu(t) + F(t) \qquad (0 \le t \le T), \tag{IV.11}$$

$$u(0) = 0 \qquad (\psi_0 = 0),$$

$$\int_0^T u(t)d\mu(t) = \psi_1 \qquad (\psi_1 \in D(A)),$$

where $F(t)$ is the same as in (VI.9), has a unique solution f in a neighborhood of zero in E for sufficiently small (in norm) ψ_1 from $D(A)$.

Remark 22 *In what follows, we assume everywhere that $\psi_0 = 0$ in (IV.2).*

Remark 23 *In the theorem the nonlinearity $G(u)$ can be rather general. Precisely, if (Ω, Σ, μ) is a space with measure with t, $s \in \Omega$ then the following types of operators satisfies the conditions of theorem on the operator G:*

1. *Nemytski's operator: $u(t) \mapsto g(t, u(t))$,*
2. *Urysohn's operator: $u(t) \mapsto \int_\Omega K(t, s, u(s))d\mu(s)$,*
3. *Hammerstein's operator: $u(t) \mapsto \int_\Omega K(t, s)g(s, u(s))d\mu(s)$.*

Proof. Both sides of equations (IV.11) and (IV.1) belong to $C([0,T], E)$; therefore to these equations we can apply the linear overdetermination operator $l \in \mathcal{L}(C([0,T], E), E)$, assigning the integral $\int_0^T v(t)d\mu(t)$ of the function $v(t)$:

$$\int_0^T u'(t)d\mu(t) = \int_0^T [Au](t)d\mu(t) + \int_0^T Gu(t)d\mu(t) + \int_0^T \Phi(t)fd\mu(t) \qquad \text{(for (IV.11))},$$

or

$$l(u'(t)) - l([Au](t)) = \varphi \qquad \text{(for (IV.1))}. \tag{IV.12}$$

Noting the fact that the operator A is closed, in view of (IV.3), we obtain

$$\int_0^T [Au](t)d\mu(t) = A\left(\int_0^T u(t)d\mu(t)\right) = A\psi_1$$

and

$$l(Au) = A(lu). \tag{IV.13}$$

Denote

$$A\psi_1 = -\chi. \tag{IV.14}$$

Then, taking into account the notation introduced in (IV.8), (IV.5) and (IV.14), we find that the linear problem (IV.1) - (IV.3) can be reduced to the operator equation

$$\chi = \varphi - \hat{A}\varphi = (I - \hat{A})\varphi. \tag{IV.15}$$

Let us show that problem (IV.1) - (IV.3) and equation (IV.15) are equivalent.

To do this, it suffices to prove that if φ is a solution of equation (IV.15) with some $\chi \in E$, then the solution $u(t)$ of equations (IV.1) - (IV.2) corresponding a given $f = J^{-1}\varphi$, satisfies equation (IV.3) with $\psi_1 = A^{-1}\chi$, i.e., we must establish the equality $lu = A^{-1}\chi$.

Indeed, noting that $\hat{A}\varphi = l(u'(t))$ by definition, we add (IV.12) and (IV.15) and, using (IV.13), we obtain $-A(lu) = \chi$; hence, by Remark 21, we see that problem (IV.1) - (IV.3) and equation (IV.15) are equivalent.

Let us return to the nonlinear problem (IV.11), (IV.2), (IV.3).

Since H is subset of X, it follows that the mapping $G : H \to C^1([0,T],E)$ is also strictly differentiable in the sense of Fréchet. Taking also into account the equality $G'(0) = 0$, the fact that $L : H \to C^1([0,T],E)$ is an isomorphism (see (IV.10)), and using the inverse function theorem (see Sec. IV.2.3), we see that the mapping $\xi : u(t) \mapsto Lu(t) - Gu(t)$ is a local diffeomorphism of class C^1 in a neighborhood of zero of U' in H onto a neighborhood of zero of V' in $C^1([0,T],E)$.

Suppose that $\eta = \xi^{-1} : V' \to U'$ is the mapping inverse to this local diffeomorphism, i.e., $\eta : F \longmapsto u$, where u is a solution of equation (IV.11) and η is strictly differentiable on V'.

We have $F(t) = (\Phi \circ J^{-1})\varphi$ (see (IV.7)).

Consider

$$P(\varphi) = \eta\left[(\Phi \circ J^{-1})\varphi\right], \quad P : E \to H. \tag{IV.16}$$

Since the mapping

$$\Lambda : \varphi \longmapsto (\Phi \circ J^{-1})\varphi, \quad \Lambda : E \to C^1([0,T],E) \tag{IV.17}$$

is linear and continuous, it follows that P is strictly differentiable in the sense of Fréchet in a neighborhood of zero in the space E as a mapping into H.

Further, we look for a solution of the nonlinear problem (IV.11), (IV.2), (IV.3) as

$$u = P(\varphi),$$

where u is a solution of (IV.11), (IV.2), (IV.3) with $F(t) = (\Phi \circ J^{-1})\varphi$ on the right-hand side of (IV.11).

Let us introduce the mapping

$$M : E \to E, \quad M : \varphi \longmapsto \left\{ \varphi - \int_0^T [P(\varphi)]_t d\mu(t) + \int_0^T G[P(\varphi)] d\mu(t) \right\}. \tag{IV.18}$$

Then the system (IV.11), (IV.2), (IV.3) can be written as

$$M\varphi = \chi. \tag{IV.19}$$

Let us now prove that problem (IV.11), (IV.2), (IV.3) is equivalent to the operator equation (IV.19), i.e., we must show that if φ is a solution of equation (IV.19) with some χ, then $u = P(\varphi)$ (obtained from φ), where u is a solution of the Cauchy problem (IV.11), (IV.2) with given $F(t) = (\Phi \circ J^{-1})\varphi$, satisfies condition (IV.3) with $\psi_1 = -A^{-1}\chi \in D(A)$.

Assume the converse: suppose that u satisfies (IV.3) with $\tilde{\psi}_1 \neq \psi_1$.

Deriving the operator equation for these φ and $\tilde{\psi}_1$, we find that φ also satisfies the equation

$$M\varphi = \tilde{\chi} = -A\tilde{\psi}_1. \tag{IV.20}$$

Subtracting (IV.20) from (IV.19), we obtain $\chi - \tilde{\chi} = -A\psi_1 + A\tilde{\psi}_1 = 0$, i.e., $A(\tilde{\psi}_1 - \psi_1) = 0$. By Remark 21, this equation has only a zero solution; therefore, $\tilde{\psi}_1 = \psi_1$, which contradicts the original assumption.

Let us show that M is strictly differentiable in the sense of Fréchet in a neighborhood of zero in E and $M'(0) = I - \hat{A}$ (see (IV.8)).

Note that the mapping $\varphi \longmapsto \int_0^T G[P(\varphi)] d\mu(t)$ is strictly differentiable in the sense of Fréchet in a neighborhood of zero by the theorem on the differentiability of a composite function and its derivative at zero is zero, since $G'(0) = 0$. Further, the mapping $\varphi \longmapsto \int_0^T [P(\varphi)]_t d\mu(t)$ strictly differentiable in a neighborhood of zero, since $v(\cdot) \longmapsto \int_0^T v'(t) d\mu(t)$ is a linear continuous operator from X to E and, especially, from H to E.

Moreover,

$$\int_0^T [P'(0)\varphi]_t d\mu(t) = \hat{A}\varphi. \tag{IV.21}$$

Therefore, M is strictly differentiable in a neighborhood of zero in E and $M'(0) = I - \hat{A}$ is the operator of the linear problem (IV.1)- (IV.3) (see (IV.15)). Since the linear problem is well defined, there exist $(M'(0))^{-1}$ and $\|(M'(0))^{-1}\| \leq 1/(1 - \left\|\hat{A}\right\|)$. By the inverse function theorem, there exist open neighborhoods of zero of U and V in E such that M induces a diffeomorphism of class C^1 of U onto V.

Let $\tilde{V} = \{\psi_1 \in D(A) | \chi = -A\psi_1 \in V\}$. Then $\tilde{V}$ is an open neighborhood of zero in $\mathcal{D}(A) \subset E$ and for all $\psi_1 \in \tilde{V}$ there exists a unique $\varphi \in U$ such that $M\varphi = -A\psi_1$.

Thus, we have proved the local unique solvability of the operator equation equivalent to problem (IV.11), (IV.2), (IV.3), which concludes the proof of theorem 17. $\square$

IV.4 Nonlinear Problem. Refinement of the Neighborhood of Local Solvability

IV.4.1 Preliminaries. Notation

In what follows, we shall use the following notation:

- U_H is the open unit ball in H;
- H is from Sec. IV.3 such that the following continuous embedding holds:

$$H \subset C^1([0,T], E); \tag{IV.22}$$

- $LH = C^1([0,T], E)$ (see (IV.10));
- $C_{X \to Y}$ is the embedding constant X in Y;
- $\Lambda : \varphi \longmapsto (\Phi \circ J^{-1})\varphi$, $\Lambda : E \to LH$ (see (IV.17));
- $\|\Lambda\| = \|\Lambda\|_{\mathcal{L}(E,LH)}$;
- $\mathcal{L}(X,Y)$is the set of linear continuous operators from X to Y;
- l is an operator of overdetermination

$$l \in \mathcal{L}(C^1([0,T],E),E), \quad lu = \int\limits_0^T u(t)d\mu(t); \tag{IV.23}$$

- B is an operator such that

$$B \in \mathcal{L}(H,E), \quad Bu = \int\limits_0^T u'(t)d\mu(t). \tag{IV.24}$$

We consider the inverse problem (IV.11), (IV.2), (IV.3), assuming that all the assumptions of theorem 17 (in Sec. IV.3) are valid and, additionally, the following estimate holds:

$$\|G'(u)\|_{\mathcal{L}(H,LH)} \le \tilde{\vartheta}(r) \quad \text{for} \quad \|u\|_H \le r, \tag{IV.25}$$

where $\tilde{\vartheta} : [0,\infty[\to [0,\infty[$ is a monotone nondecreasing function.

Let us recall a result from [49] (see also [48, p.33]):

Theorem 18 *Let X be a Banach space, let Y be a separable topological vector space, let $A : X \to Y$ be a linear continuous operator, let U be the open unit ball in X, let $\mathrm{P}_{AU} : AX \to [0,\infty[$ be the Minkowski functional of the set AU, and let $\Psi : X \to AX$ be a mapping satisfying the condition*

$$\mathrm{P}_{AU}\big(\Psi(x) - \Psi(\bar{x})\big) \leq \Theta(r)\,\|x - \bar{x}\| \quad \textit{for} \quad \|x - x_0\| \leqslant r, \quad \|\bar{x} - x_0\| \leq r$$

for some $x_0 \in X$, where $\Theta : [0,\infty[\to [0,\infty[$ is a monotone nondecreasing function. Set $b(r) = \max\big(1 - \Theta(r), 0\big)$ for $r \geq 0$.

Suppose that

$$w = \int\limits_0^\infty b(r)\,dr \in]0,\infty], \quad r_* = \sup\{r \geq 0 |\, b(r) > 0\},$$

$$w(r) = \int\limits_0^r b(t)dt \quad (r \geq 0) \quad \textit{and} \quad f(x) = Ax + \Psi(x) \quad \textit{for} \quad x \in X.$$

Then for any $r \in [0, r_*[$ and $y \in f(x_0) + w(r)AU$, there exists an $x \in x_0 + rU$ such that $f(x) = y$.

Remark 24 *If either A is injective or $Ker A$ has a topological complement E in X such that $A(E \cap U) = AU$, then the assertion of theorem 18 follows from the contraction mapping principle [49]. In particular, if A is injective, then the solution is unique.*

In what follows, we also need Hadamard's theorem.

Theorem 19 *Let X and Y be Banach spaces, let $A : X \to Y$ be an isomorphism of X onto Y ($AX = Y$, $\|A\| < \infty$, $\exists A^{-1}$, $\|A^{-1}\| < \infty$), and let $f : X \to Y$ satisfy the Lipschitz condition with constant $q/\left\|A^{-1}\right\|$, where $q < 1$. Set $F(x) = Ax + f(x)$. Then $F(X) = Y$, F is bijective, and*

$$\left\|F^{-1}(y) - F^{-1}(\bar{y})\right\| \leq \frac{\left\|A^{-1}\right\|}{1 - q}\,\|y - \bar{y}\|\,.$$

IV.4.2 Estimates of the nonlinear summands in the differential operators for the direct problem

The central point here is to derive of an estimate of the nonlinear part of the operator in problem (IV.11), (IV.2), (IV.3) in the class $C^1([0,T],E)$, which allows us to apply theorem 18 in what follows.

Suppose that $u(t) \in H$, $z(t) \in H$ (see (IV.22)) satisfy

$$\|u\|_H \leq r, \|u + z\|_H \leq r. \tag{IV.26}$$

Then

$$\|G(u+z) - G(u)\|_{C^1([0,T],E)} = \left\|\int\limits_0^1 G'(u+sz)zds\right\|_{C^1([0,T],E)} \leq$$

$$\leq \int_0^1 \|G'(u+sz)\|_{\mathcal{L}(H,LH)}\, ds \|z\|_{C^1([0,T],E)} \leq$$

$$\leq \tilde{\vartheta}(r) C_{H\to C^1([0,T],E)} \|z\|_H$$

(see (IV.22),(IV.25)). Thus,

$$\|G(u+z) - G(u)\|_{C^1([0,T],E)} \leq \vartheta(r)\, \|z\|_H\, ,$$

where

$$\vartheta(r) = C_{H\to C^1([0,T],E)} \tilde{\vartheta}(r). \tag{IV.27}$$

Note that $LU_H = U_{C^1([0,T],E)}$, since L is an isometric isomorphism of H on $C^1([0,T],E)$.

Further,

$$P_{LU_H}(G(u+z) - G(u)) = P_{U_{C^1([0,T],E)}}(G(u+z) - G(u)) \leq \vartheta(r)\, \|z\|_H\, .$$

Then

$$P_{LU_H}(G(u+z) - G(u)) \leq \vartheta(r)\, \|z\|_H\, . \tag{IV.28}$$

In this case, in theorem 18 we can set $A = L$, $X = H$, $Y = LH$ with unit ball LU_H, $b(r) = \bar{b}(r)$ and $\Psi = G$, where

$$\bar{b}(r) = \max(1 - \vartheta(r), 0).$$

If $r \in [0, r_*[$, then we take $\bar{b}(r) = 1 - \vartheta(r)$, where r_* is the root of the equation

$$1 - \vartheta(r) = 0, \tag{IV.29}$$

and, by theorem 18, we have

$$w(r) = \int_0^r b(t)dt = \int_0^r (1 - \vartheta(t))dt = r - \int_0^r \vartheta(t)dt. \tag{IV.30}$$

But if $r \in [r_*, \infty]$, then we take $\bar{b}(r) = 0$ and $w(r) = w(r_*)$.

Using theorem 18, we obtain the following theorem.

Theorem 20 *Suppose that G satisfies (IV.25) and the assumptions of theorem 17 hold. Further, suppose that $\vartheta(r)$ is from (IV.27), w is from (IV.30), and r_* is a root of equation (IV.29).*

Then

$$\forall r \in [0, r_*[\quad \forall F \in w(r) LU_H \quad \exists! u \in rU_H : \quad L[u(t)] - G[u(t)] = F(t).$$

IV.4.3 Estimates of the nonlinear summands in operators inverse to differential operators for the direct problem

It follows from Sec. IV.3 that

$$\xi(v) = L(v) - G(v) = F \Rightarrow \eta(F) = L^{-1}F - Q(F);$$

this yields

$$F = \xi(\eta(F)) = \xi(L^{-1}F - Q(F)) = F - L(Q(F)) - G(L^{-1}F - Q(F)).$$

Then $L(Q(F)) - G(L^{-1}F - Q(F)) = 0$, i.e.,

$$L(Q(F)) = G(L^{-1}F - Q(F)), \quad Q(F) = L^{-1}G(L^{-1}F - Q(F)) = L^{-1}G(\eta(F)).$$

Therefore,

$$Q(F + \Delta F) - Q(F) = L^{-1}[G(\eta(F + \Delta F)) - G(\eta(F))].$$

Using theorem 20, we find that

$$\|F\|_{LH} < w(r) \Rightarrow \|\eta(F)\|_H < r, \tag{IV.31}$$

$$\|F + \Delta F\|_{LH} < w(r) \Rightarrow \|\eta(F + \Delta F)\|_H < r,$$

where $w(r)$ is from (IV.30). Then, from (IV.28) we obtain

$$P_{LU_H}(G(\eta(F + \Delta F)) - G(\eta(F))) \leq \vartheta(r) \, \|z\|_H \,, \tag{IV.32}$$

where

$$z = \eta(F + \Delta F) - \eta(F).$$

Then

$$\|z\|_H = \|\eta(F + \Delta F) - \eta(F)\|_H = \|\xi^{-1}(F + \Delta F) - \xi^{-1}(F)\|_H \overset{\text{Th.19}}{\leq} \frac{1}{1 - \vartheta(r)} \|\Delta F\|_{LH}$$

(in the last inequality, we used Hadamard's theorem).

Hence it follows that inequality (IV.32) takes the form

$$P_{LU_H}(G(\eta(F + \Delta F)) - G(\eta(F))) \leq \vartheta(r) \cdot \frac{1}{1 - \vartheta(r)} \|\Delta F\|_{LH} \,.$$

Suppose that $r' = w(r)$. Then $r = w^{-1}(r')$,

$$P_{U_H}(Q(F + \Delta F) - Q(F)) \leq \vartheta(w^{-1}(r')) \cdot \frac{1}{1 - \vartheta(w^{-1}(r'))} \|\Delta F\|_{LH} \,,$$

i.e.,

$$P_{U_H}(Q(F + \Delta F) - Q(F)) \leq \Theta(r') \, \|\Delta F\|_{LH} \,,$$

where

$$\Theta(r') = \frac{\vartheta(w^{-1}(r'))}{1 - \vartheta(w^{-1}(r'))}. \tag{IV.33}$$

Under the assumptions of theorem 18, we can set $A = L^{-1}$, $X = LH$, $Y = H$ with the unit ball LU_H, $\Psi = Q$, $\Theta(r) = \Theta(r')$ and $b(r) = B(r')$, where $\Theta(r')$ from (IV.33) and $B(r') = \max(1 - \Theta(r'), 0)$.

If $r' \in [0, r'_*[$, then we take

$$B(r') = \frac{1 - 2\vartheta(w^{-1}(r'))}{1 - \vartheta(w^{-1}(r'))},$$

where r'_* is the root of the equation

$$1 - 2\vartheta(w^{-1}(r')) = 0, \tag{IV.34}$$

and, by theorem 18, we have

$$W(r') = \int_0^{r'} B(t)dt = r' - \int_0^{r'} \Theta(t)dt,$$

i.e.,

$$W(r') = r' - \int_0^{r'} \frac{\vartheta(w^{-1}(t))}{1 - \vartheta(w^{-1}(t))} dt. \tag{IV.35}$$

But if $r' \in [r'_*, \infty]$, then we take $B(r') = 0$ and $W(r') = W(r'_*)$.

Using theorem 18, we obtain the following theorem.

Theorem 21 *Suppose that G satisfies (IV.25) and the assumptions of theorem 17 hold. Further, suppose that $\vartheta(r)$ is from (IV.27), w is from (IV.30), $W(r')$ is from (IV.35) and r'_* is the root of equation (IV.34).*

Then

$$\forall r' \in [0, r'_*[\quad \forall v \in W(r')L^{-1}(LU_H) = W(r')U_H \quad \exists! F \in r'U_{LH}: \quad L^{-1}F - Q(F) = v.$$

IV.4.4 Estimates of the nonlinear summands in the operators for the inverse problem

From (IV.18), we obtain

$$M\varphi = \chi = \varphi - \int_0^T [P(\varphi)]_t d\mu(t) + \int_0^T G[P(\varphi)]d\mu(t) =$$

$$= I\varphi - \int_0^T [P'(0)\varphi]_t d\mu(t) + \int_0^T G[P(\varphi)] d\mu(t) - \int_0^T [P(\varphi) - P'(0)\varphi]_t d\mu(t),$$

or, if take into account the expression for the operator $\hat{A}$ (see (IV.21)),

$$M\varphi = \chi = (I - \hat{A})\varphi + (l \circ G)[P(\varphi)] + \hat{A}\varphi - B[P(\varphi)] = \hat{\hat{A}}\varphi + \hat{\Psi}(\varphi),$$

where

$\hat{A}$ is the linear operator from (IV.8), assigning to $\varphi \in E$ the solution $u(t)$ of the linear problem (IV.1)- (IV.3), P is the operator defined in (IV.16), χ was defined in (IV.14), l is the operator defined in (IV.23), B is the operator defined in (IV.24), and

$$\hat{\hat{A}}\varphi = (I - \hat{A})\varphi, \tag{IV.36}$$

$$\hat{\Psi}(\varphi) = \hat{A}\varphi + (l \circ G)[P(\varphi)] - B[P(\varphi)]. \tag{IV.37}$$

Note that

$$\|\varphi\|_E \le \tilde{r} \Rightarrow \|F\|_{LH} = \|\Lambda\varphi\|_{LH} \le \|\Lambda\|_{\mathcal{L}(E,LH)} \tilde{r} \Rightarrow$$

$$\|P(\varphi)\|_H = \|\eta(\Lambda\varphi)\|_H \le w^{-1}(\|\Lambda\| \tilde{r}),$$

(in the last inequality, we have used (IV.31)), i.e.,

$$\|\varphi\|_E \le \tilde{r} \Rightarrow \|P(\varphi)\|_H \le w^{-1}(\|\Lambda\| \tilde{r}),$$

$$\|\varphi + sz\|_E \le \tilde{r} \Rightarrow \|P(\varphi + sz)\|_H \le w^{-1}(\|\Lambda\| \tilde{r}). \tag{IV.38}$$

To find an estimate of $\|P'(\varphi + sz)\|_{\mathcal{L}(E,H)}$, which we shall need later, note that

$$P'(\varphi + sz) = \eta'(\Lambda(\varphi + sz)) \circ \Lambda = [\xi'(P(\varphi + sz))]^{-1} \circ \Lambda$$

by the inverse function theorem.

Further, we have

$$\xi'(P(\varphi + sz)) = (L' - G')[P(\varphi + sz)] = (L - G')[P(\varphi + sz)] = L(I - L^{-1}G')[P(\varphi + sz)].$$

Therefore,

$$[\xi'(P(\varphi + sz))]^{-1} = [(I - L^{-1}G')[P(\varphi + sz)]]^{-1} \circ L^{-1},$$

and hence

$$P'(\varphi + sz) = [(I - L^{-1}G')[P(\varphi + sz)]]^{-1} \circ L^{-1} \circ \Lambda. \tag{IV.39}$$

In this case

$$\|P'(\varphi + sz)\|_{\mathcal{L}(E,H)} \le \left\|L^{-1}\right\| \|\Lambda\| \left\| \left[\left(I - L^{-1}G'\right) [P(\varphi + sz)] \right]^{-1} \right\| \le$$

$$\le \left\|L^{-1}\right\|_{\mathcal{L}(LH,H)} \|\Lambda\| \cdot \frac{1}{1 - \|L^{-1}\|_{\mathcal{L}(LH,H)} \|G'(P(\varphi + sz))\|_{\mathcal{L}(H,LH)}}$$

(by Hadamard's theorem), and from the equality $\|L^{-1}\|_{\mathcal{L}(LH,H)} = 1$ we find that

$$\|P'(\varphi + sz)\|_{\mathcal{L}(E,H)} \le \frac{\|\Lambda\|}{1 - \|G'(P(\varphi + sz))\|_{\mathcal{L}(H,LH)}}. \tag{IV.40}$$

Now, to estimate $\|G'(P(\varphi + sz))\|_{\mathcal{L}(H,LH)}$, we transform (IV.28) into

$$P_{LH} \left(\frac{G[P(\varphi + sz) + te] - G[P(\varphi + sz)]}{t} \right) \le \vartheta(w^{-1}(\|\Lambda\| \, \tilde{r})) \, \|e\|_H \,,$$

where e is an arbitrary vector from H, and $t \ne 0$. As $t \to 0$, we have

$$P_{LH}(G'[P(\varphi + sz)](e)) \le \vartheta(w^{-1}(\|\Lambda\| \, \tilde{r})) \, \|e\|_H \,,$$

whence, since e is arbitrary, we obtain

$$\|G'[P(\varphi + sz)]\|_{\mathcal{L}(H,LH)} \le \vartheta(w^{-1}(\|\Lambda\| \, \tilde{r})), \tag{IV.41}$$

and, substituting (IV.41) into (IV.40), we can write

$$\|P'(\varphi + sz)\|_{\mathcal{L}(E,H)} \le \frac{\|\Lambda\|}{1 - \vartheta(w^{-1}(\|\Lambda\| \, \tilde{r}))}. \tag{IV.42}$$

Let $R(\varphi) = \hat{A}\varphi + B[P(\varphi)]$. Then

$$R(\varphi + z) - R(\varphi) = -\int_0^T [P(\varphi + z) - P'(0)(\varphi + z) - P(\varphi) + P'(0)\varphi]_t \, d\mu(t) =$$

$$= -\int_0^T [P(\varphi + z) - P'(0)z - P(\varphi)]_t \, d\mu(t) =$$

$$= -B \left[P(\varphi + z) - P'(0)z - P(\varphi) \right].$$

Note that $P'(0) = \eta'(0) \circ \Lambda = L^{-1} \circ \Lambda$ and

$P'(0)\varphi - P(\varphi) = (L^{-1} \circ \Lambda)\varphi - P(\varphi) = L^{-1}G(P(\varphi))$, i.e., $P(\varphi) = (L^{-1} \circ \Lambda)\varphi - L^{-1}G(P(\varphi))$, and, in that case, we have

$$P(\varphi + z) - P'(0)z - P(\varphi) = P(\varphi + z) - P(\varphi) - (L^{-1} \circ \Lambda)z =$$
$$= L^{-1}\left[G(P(\varphi + z)) - G(P(\varphi))\right] = L^{-1}\left[\int_0^1 G'(P(\varphi + sz))P'(\varphi + sz)zds\right] \tag{IV.43}$$

In this case,

$$\|R(\varphi + z) - R(\varphi)\|_E = \| - B\left[P(\varphi + z) - P'(0)(z) - P(\varphi)\right]\|_E \leq$$

$$\leq \|B\|_{\mathcal{L}(H,E)}\|P(\varphi + z) - P(\varphi) - L^{-1} \circ \Lambda z\|_H \leq$$

$$\leq \|B\|_{\mathcal{L}(H,E)}\left\|L^{-1}\left[\int_0^1 G'(P(\varphi + sz))P'(\varphi + sz)zds\right]\right\|_H \leq$$

$$\leq \|B\|_{\mathcal{L}(H,E)}\|L^{-1}\|_{\mathcal{L}(LH,H)}\left\|\left[\int_0^1 G'(P(\varphi + sz))P'(\varphi + sz)ds\right]z\right\|_{C^1([0,T],E)} \leq$$

$$\leq \|B\|_{\mathcal{L}(H,E)}\|L^{-1}\|_{\mathcal{L}(LH,H)}\int_0^1 \|G'(P(\varphi + sz))\|_{\mathcal{L}(H,LH)}\|P'(\varphi + sz)\|_{\mathcal{L}(E,H)}ds\|z\|_E \leq$$

$$\leq \|B\|_{\mathcal{L}(H,E)}\|L^{-1}\|_{\mathcal{L}(LH,H)}\frac{\|\Lambda\|\,\vartheta(w^{-1}(\|\Lambda\|\,\tilde{r}))}{1 - \vartheta(w^{-1}(\|\Lambda\|\,\tilde{r}))}\|z\|_E =$$

$$= \|B\|_{\mathcal{L}(H,E)}\frac{\|\Lambda\|\,\vartheta(w^{-1}(\|\Lambda\|\,\tilde{r}))}{1 - \vartheta(w^{-1}(\|\Lambda\|\,\tilde{r}))}\|z\|_E$$

(in passing to the next-to-last formula, we used the estimates (IV.41) and (IV.42), while for the last one we use the equality $\|L^{-1}\|_{\mathcal{L}(LH,H)} = 1$, which is valid, because L is an isometric isomorphism).

We denote

$$\zeta = \frac{\|\Lambda\|\,\vartheta(w^{-1}(\|\Lambda\|\,\tilde{r}))}{1 - \vartheta(w^{-1}(\|\Lambda\|\,\tilde{r}))}. \tag{IV.44}$$

Thus, we have obtained the estimate

$$\|R(\varphi + z) - R(\varphi)\|_E \leq \|B\|_{\mathcal{L}(H,E)}\zeta\|z\|_E. \tag{IV.45}$$

Further,

$$\left\| \int_0^T (G[P(\varphi+z)] - G[P(\varphi)])\, d\mu(t) \right\|_E = \|l\,(G[P(\varphi+z)] - G[P(\varphi)])\|_E \leq$$

$$\leq \|l\|_{\mathcal{L}(C^1([0,T],E),E)} \|G[P(\varphi+z)] - G[P(\varphi)]\|_{C^1([0,T],E)} \leq$$

$$\leq \|l\|_{\mathcal{L}(C^1([0,T],E),E)} \left\| \left(\int_0^1 (G'[P(\varphi+z)]P'(\varphi+z)ds \right) z \right\|_{C^1([0,T],E)} \leq$$

$$\leq \|l\|_{\mathcal{L}(C^1([0,T],E),E)} \zeta \|z\|_E,$$

i.e.,

$$\|l\,(G[P(\varphi+z)] - G[P(\varphi)])\|_E \leq \|l\|_{\mathcal{L}(C^1([0,T],E),E)} \zeta \|z\|_E. \tag{IV.46}$$

Then it follows from (IV.45) and (IV.46) that

$$\left\| \hat{\Psi}(\varphi+z) - \hat{\Psi}(\varphi) \right\|_E \leq \|(l \circ G)[P(\varphi+z)] - (l \circ G)[P(\varphi)]\|_E +$$

$$+ \|R(\varphi+z) - R(\varphi)\|_E \leq (\|B\|_{\mathcal{L}(H,E)} + \|l\|_{\mathcal{L}(C^1([0,T],E),E)})\zeta \|z\|_E.$$

Therefore, $P_{\hat{\hat{A}}U_E}(\hat{\Psi}(\varphi+z) - \hat{\Psi}(\varphi)) \leq \hat{\Theta}(\tilde{r})\, \|z\|_E$, where

$$\hat{\Theta}(\tilde{r}) = \left\| \hat{\hat{A}}^{-1} \right\| (\|B\|_{\mathcal{L}(H,E)} + \|l\|_{\mathcal{L}(C^1([0,T],E),E)})\zeta.$$

In this case, in the statement of theorem 18, we can set $A = \hat{\hat{A}}$, $X = E$, $Y = E$ with unit ball $\hat{\hat{A}}U_E$, $b(r) = \hat{B}(\tilde{r})$, $\Psi = \hat{\Psi}$, $\Theta(r) = \hat{\Theta}(\tilde{r})$, where $\hat{\hat{A}}$ is from (IV.36), $\hat{\Psi}$ is from (IV.37), and $\hat{B}(\tilde{r}) = \max\left[1 - \hat{\Theta}(\tilde{r}), 0\right]$.

If $\tilde{r} \in [0, \tilde{r}_*[$, then we take $\hat{B}(\tilde{r}) = 1 - \hat{\Theta}(\tilde{r})$, where $\tilde{r}_*$ is the root of the equation

$$1 - \hat{\Theta}(\tilde{r}) \ = \ 0.$$

Moreover, let $w(r) = \hat{W}(\tilde{r})$, where

$$\hat{W}(\tilde{r}) = \int_0^{\tilde{r}} b(t)dt = \tilde{r} - \left\| \hat{\hat{A}}^{-1} \right\| (\|B\|_{\mathcal{L}(H,E)} + \|l\|_{\mathcal{L}(C^1([0,T],E),E)}) \int_0^{\tilde{r}} \zeta(t)dt.$$

For the case in which $\tilde{r} \in [\tilde{r}_*, \infty]$, we take $\hat{B}(\tilde{r}) = 0$ and $\hat{W}(\tilde{r}) = \hat{W}(\tilde{r}_*)$.

Theorem 22 *Suppose that G satisfies (IV.25), i.e.,*

$$\|G'(u)\|_{\mathcal{L}(H,LH)} \le \tilde{\vartheta}(r) \quad for \quad \|u\|_H \le r,$$

where $\tilde{\vartheta} : [0,\infty[\to [0,\infty[$ — is a monotone nondecreasing function and the assumptions of theorem 17 are satisfied.

Further, suppose that (for convenience, we present all the formulas obtained above)

$$\vartheta(r) = C_{H \to C^1([0,T],E)} \tilde{\vartheta}(r),$$

$$w(r) = r - \int\limits_0^r \vartheta(t)dt,$$

$$\zeta = \frac{\|\Lambda\| \, \vartheta(w^{-1}(\|\Lambda\| \, \tilde{r}))}{1 - \vartheta(w^{-1}(\|\Lambda\| \, \tilde{r}))},$$

$$\hat{W}(\tilde{r}) = \int\limits_0^{\tilde{r}} b(t)dt = \tilde{r} - \left\| \hat{\hat{A}}^{-1} \right\| (\|B\|_{\mathcal{L}(H,E)} + \|l\|_{\mathcal{L}(C^1([0,T],E),E)}) \int\limits_0^{\tilde{r}} \zeta(t)dt$$

and $\tilde{r}_*$ is the root of the equation

$$1 - \left\| \hat{\hat{A}}^{-1} \right\| (\|B\|_{\mathcal{L}(H,E)} + \|l\|_{\mathcal{L}(C^1([0,T],E),E)})\zeta = 0.$$

Then

$$\forall \tilde{r} \in [0, \tilde{r}_*[\quad \forall \chi \in \hat{W}(\tilde{r})\hat{\hat{A}}U_E \quad \exists! \varphi \in \tilde{r}U_E : \quad M(\varphi) = \chi.$$

Thus, the operator equation (IV.19) is uniquely solvable and, the controllability problem (IV.11), (IV.2), (IV.3) equivalent to it also has unique solution in the neighborhood specified above.

Chapter V

Controllability of the Moments $\int_\Omega u(t,x)\omega(x)dx$ on Solutions of Nonlinear Parabolic Problem

Table of contents

The controllability of moment problem for the linear heat equation with an integral redetermination

$$\int_\Omega u(x,t)\omega(x)dx = \chi(t)$$

was studied in [27]. However, actual heat diffusion processes cannot be described by a linear equation, and hence we introduce a very simple nonlinear term for which the problem remains uniquely solvable. Needless to say, actual diffusion processes can be described by more complicated nonlinearities, and so the present work [41] can be treated as one of the first steps in this direction.

The chapter consists of two sections:

- In the first section, as in the chapter IV we prove the local solvability of the inverse problem, which can be treated as a controllability problem of moment, for the quasilinear heat equation with integral overdetermination. A similar result was obtained in [27] for the linear problem. The method of the proof follows the ideas of the chapter IV.

- In the second section, we use twice the refined inverse function theorem (represented in the most general form in [48]) to obtain sufficient conditions for the unique solvability of the original inverse problem in terms of the size of the neighborhood in which the function occurring in the overdetermination condition can be chosen. This method was earlier applied in [2] to the inverse problem for the quasilinear heat equation with final overdetermination.

V.1 The local existence of the controllability problem for the quasilinear heat equation with integral overdetermination

V.1.1 Preliminaries. Notation

Throughout the following, $\Omega \subset R^n$ is a bounded connected domain with boundary $\partial\Omega \in C^2$ $(n \geq 2)$, $Q_T = \Omega \times (0,T)$, where $T > 0$, and $S_T = \partial\Omega \times [0,T]$. Let us recall the results of [27]. Consider the inverse problem

$$u_t - \Delta u = F(x,t), \tag{V.1}$$

$$u|_{t=0} = 0, \tag{V.2}$$

$$u|_{S_T} = 0, \tag{V.3}$$

$$\int_\Omega u(x,t)\omega(x)dx = \chi(t), \tag{V.4}$$

where $F(x,t) = h(x,t)f(t)$, h, ω, and χ are given functions, and u and f are the unknown functions. The problem is to find a control $f(t)$ such that the integral of the product of the solution by a given function depending only on the spatial variables is a prescribed function of time. This may correspond to a continuous measurement of some averaged characteristic of the solution (say, mean temperature over some part of Ω).

Definition 4 *A generalized solution of the inverse problem (V.1)-(V.4) is a pair* (u,f), $u \in W_2^{2,1}(Q_T)$, $f \in L_2(0,T)$, *of functions such that equation (V.1) holds almost everywhere in* Q_T, *the traces of* $u(x,t)$ *satisfy (V.2), (V.3) and relation (V.4) is valid almost everywhere on* $[0,T]$.

Thus the controllability problem (V.1)-(V.4) is essentially the problem of varying the control $f \in L_2(0,T)$ so as to pick up a solution $u(x,t) = u(x,t;f)$ of the direct problem (V.1)-(V.3) satisfying condition (V.4), commonly referred to as an integral overdetermination condition in the theory of inverse problems. It was shown in [27, p.46] that problem (V.1)-(V.4) is well posed.

In the space $L_2(0,T)$, we introduce the equivalent norm

$$\|f\|^2_{L_2(0,T)} = \int_0^T \exp(-\beta t)\,|f(t)|^2\,dt, \tag{V.5}$$

where the number $\beta > 0$ is defined in [27]. (Formula (V.5) is used only in the proof of the unique solvability theorem [27, p.46] for problem (V.1)-(V.4) in Subsections V.1.4 and V.2.4; in all other cases, $L_2(0,T)$ is equipped with the standard norm.)

Remark 25 *Throughout the following, we use the notation* $\|u\|_{2,\Omega} = \|u\|_{L_2(\Omega)}$.

Wherever our calculations use the Hölder inequality, the Cauchy-Schwarz-Bunyakovskii inequality, the Minkowski inequality, the Fubini theorem, or the embedding theorem, related places are marked by the symbols “HI”, “CSBI”, “MI”, “FT”, and “ET”. We set $\varphi(t) = (h,\omega)_{2,\Omega} f$ and introduce an operator $A : L_2(0,T) \to L_2(0,T)$, by the formula

$$(A\varphi)(t) = -\int_\Omega u(x,t)\Delta\omega(x)dx, \tag{V.6}$$

where $\varphi(t) \in L_2(0,T)$ and $u(x,t)$ is the solution of the linear problem (V.1)-(V.4). The operator A map φ in the following way:

$$\varphi \to f \to u \to A\varphi,$$

where the solution of inverse problem (V.1)-(V.4) is found by the given f, and f is found by the formula

$$f(t) = \frac{\varphi(t)}{(h,\omega)_{L_2(\Omega)}}.$$

It was shown in [27, p.47] that $\|A\| < 1$, if $L_2(0,T)$ is equipped with the norm (V.5).

We set $W_{2,0}^{2,1}(Q_T) = W_2^{2,1}(Q_T) \cap W_{2,0}^1(Q_T)$ with the norm of $W_2^{2,1}(Q_T)$, where $W_{2,0}^1(Q_T)$ is the subspace of $W_2^1(Q_T)$, in which smooth functions vanishing near S_T form a dense set [31].

Further, we set

$$L = \partial/\partial t - \Delta_x,$$

$$H = \left\{ v \in W_{2,0}^{2,1}(Q_T) | \exists F \in L_2(Q_T) : v \quad \text{is a solution of problem (V.1)-(V.3)} \right\}$$

with the norm $\|v\|_H = \|Lv\|_{L_2(Q_T)}$. Then

$$L : H \to L_2(Q_T) \quad \text{is an isometric isomorphism of } H \quad \text{on } L_2(Q_T) \tag{V.7}$$

by virtue of the a priori estimate in [31, p.189].

Note that $H = \left\{ v \in W_{2,0}^{2,1}(Q_T) | v|_{t=0} = 0, v|_{S_T} = 0 \right\}$, and $\|\cdot\|_H$ is equivalent to $\|\cdot\|_{W_{2,0}^{2,1}(Q_T)}$. (This implies that $(H, \|\cdot\|_H)$ is complete, the embedding $H \subset W_{2,0}^{2,1}(Q_T)$ is continuous, and hence $H \subset W_2^1(Q_T)$).

As in chapter IV we use in what follows the inverse function theorem (IFT) in the form of theorem 16.

We also use the estimate [18, p.177]

$$\forall u(x) \in \overset{\circ}{W}{}_p^l(\Omega) \quad \sum_{|\beta|=r} \left\| D^\beta u \right\|_{q,\Omega_m} \le \hat{C}_1 h^\theta \sum_{|\alpha|=l} \| D^\alpha u \|_{p,\Omega} + \hat{C}_2 h^{\theta - l} \| u \|_{p,\Omega}, \tag{V.8}$$

where Ω is an arbitrary domain in the space $R^{\tilde{n}}$, Ω_m is the intersection of Ω with an arbitrary m-dimensional plane $E^m \subset R^{\tilde{n}}$, h is an arbitrary positive number, $1 \le p \le q$, $0 \le r < l$, $0 \le \tilde{n} - p(l-r) < m$ (m, r, and l - are nonnegative integers) $\theta = l - r - \tilde{n}/p + m/q > 0$.

V.1.2 Statement of the Main Result

Let $n \ge 2$, and let $\Phi : R \to R$ be a continuously differentiable function on $\mathcal{R}$ such that

$$|\Phi(u)| \le C_1 |u|^{\alpha_1} + C_2 |u|^{\alpha_2}, \tag{V.9}$$

$$|\Phi'(u)| \le \alpha_1 C_1 |u|^{\alpha_1 - 1} + \alpha_2 C_2 |u|^{\alpha_2 - 1}, \tag{V.10}$$

where $1 < \alpha_1 \le \alpha_2 \le (n+1)/(n-1)$.

We set ${}_\circ W_2^1(0,T) = \{\chi \in W_2^1(0,T) | \chi(0) = 0\}$ with the norm $\|\cdot\|_{W_2^1(0,T)}$.

Theorem 23 *Let* $\omega \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$, $\chi \in {}_\circ W_2^1(0,T)$. *Further, let* $h(x,t)$ *be a function such that* $h \in L_2(Q_T)$, $\|h(\cdot,t)\|_{2,\Omega}$ *is bounded on* $[0,T]$, *and* $\left|\int\limits_\Omega h(x,t)\omega(x)dx\right| \geq \delta > 0$ *for almost all* $t \in [0,T]$. *Then the problem*

$$u_t - \Delta u - \Phi(u) = F(x,t), \tag{V.11}$$

$$u|_{t=0} = 0, \tag{V.12}$$

$$u|_{S_T} = 0, \tag{V.13}$$

with the overdetermination condition (V.4) and with $F(x,t) = h(x,t)f(t)$ *has a unique solution in a neighborhood of zero in* $H \times L_2(0,T)$ *for* $\chi \in {}_\circ W_2^1(0,T)$ *with sufficiently small norm.*

Remark 26 *To prove the theorem, it suffices to verify the assumptions of the inverse function theorem. The key points are the verification of the strict differentiability of the corresponding mappings and the choice of function spaces related to problem (V.11)-(V.13), (V.4). To this end, the conditions on* Φ *(namely,* $1 < \alpha_1 \leq \alpha_2$*) have been chosen so as to ensure that* $\Phi'(0) = 0$. *We carry out the proof of theorem 23 for the case in which* $C_2 = 0$ *in (V.9) and (V.10). (Accordingly, we write* α *instead of* α_1 *to simplify the notation.) If we admit inequalities (V.9) and (V.10) with* $C_2 \neq 0$, *then all estimates given below obviously remain valid; at the same time, the case* $C_2 \neq 0$ *covers a wider class of nonlinearities, since it weakens constraints imposed on the behavior of* Φ *at infinity.*

V.1.3 Strict Differentiability of the Operator $G : W_2^{2,1}(Q_T) \to L_2(0,T)$, $[G(u)](t) = \int\limits_\Omega \Phi(u(x,t))\omega(x)dx$

Proposition 6 *The operator* $G : W_2^{2,1}(Q_T) \to L_2(0,T)$,

$$[G(u)](t) = \int\limits_\Omega \Phi(u(x,t))\omega(x)dx \tag{V.14}$$

is strictly differentiable on $W_2^{2,1}(Q_T)$.

Proof. First, we show that G maps $\mathbf{W_2^{2,1}(Q_T)}$ into $\mathbf{L_2(0,T)}$.

Let $u(x,t) \in W_2^1(Q_T)$ and $\omega(x) \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$. It follows from the embedding theorem that $W_2^1(Q_T)$ is continuously embedded in $L_{\tilde q}(Q_T)$, where $\tilde q = 2(n+1)/(n-1)$, for $n \geq 2$, and that $W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ is continuously embedded in $L_s(\Omega)$, where $s = 2n/(n-4)$ for $n \geq 5$, $1 \leq s < \infty$ for $n = 4$ and $1 \leq s \leq \infty$ for $2 \leq n < 4$.

Since $\|[G(u)](t)\|^2_{L_2(0,T)} \leq \int\limits_0^T \left|\int\limits_\Omega C_1 |u(x,t)|^\alpha |\omega(x)|\, dx\right|^2 dt$, by (V.9) and (V.14), we should estimate the integral $\int\limits_0^T \left|\int\limits_\Omega |u(x,t)|^\alpha |\omega(x)|\, dx\right|^2 dt$.

First, consider **the case $\mathbf{n = 4}$.**

Note that $u(x,\cdot) \in L_{\tilde{p}}(\Omega)$, where $\tilde{p} = 2n/(n-1) = 8/3$, for each t fixed by the embedding theorem. Let $\alpha < 8/3$; then there exists an $\epsilon > 0$ such that $\alpha < 8/3 - \epsilon = q$.

We set $k = q/\alpha$ and $k' = q/(q-\alpha)$ and note that $k > 1$, $1 < k' < \infty$ and $k^{-1} + k^{'-1} = 1$.

We use the theorem in [18, p. 177]. In our case,$r = 0$, $l = 1$, $\tilde{n} = 5$, $m = 4$, $p = 2$, $q = 8/3 - \epsilon$, $\theta = 1 - 0 - 5/2 + 4/(8/3 - \epsilon) > 0$ and $h = 1$. Consequently,

$$\begin{aligned}
&\int_0^T \left| \int_\Omega |u(x,t)|^\alpha \, |\omega(x)| \, dx \right|^2 dt \overset{HI}{\leq} \\
&\leq \int_0^T [(\int_\Omega |u(x,t)|^{\alpha \cdot q/\alpha} \, dx)^{\alpha/q} (\int_\Omega |\omega(x)|^{q/(q-\alpha)} \, dx)^{(q-\alpha)/q}]^2 dt = \\
&= \|\omega\|^2_{q/(q-\alpha),\Omega} \int_0^T (\int_\Omega |u(x,t)|^q \, dx)^{2\alpha/q} dt \overset{(V.8)}{\leq} \\
&\leq \|\omega\|^2_{q/(q-\alpha),\Omega} \int_0^T C^{2\alpha} (\|u\|_{W_2^1(Q_T)})^{2\alpha} dt < \infty.
\end{aligned}$$

The choice of ϵ is affected by the conditions $\alpha < 8/3 - \epsilon \equiv q$ and $p \equiv 2 \leq 8/3 - \epsilon = q$, i.e., $\max(\alpha, 2) < 8/3 - \epsilon \equiv q$.

Now let us consider **the case** $\mathbf{n \geq 5}$.

Then $k = 2n/(n-4)$ and $k' = 2n/(n+4)$. We use the estimate (V.8). In our case,

$$r = 0, \quad l = 1, \quad \tilde{n} = n + 1, \quad m = n, \quad p = 2, \quad q = 2n\alpha/(n+4) \geq 2 = p,$$

$$\theta = 1 - 0 - (n+1)/2n/(2n\alpha/(n+4)) = -(n-1)/2 + (n+4)/2\alpha > 0 \quad \text{and} \quad h = 1.$$

From the last two inequalities, we obtain $(n+4)/n \leq \alpha < (n+4)/(n-1)$. Then

$$\begin{aligned}
&\int_0^T \left| \int_\Omega |u(x,t)|^\alpha \, |\omega(x)| \, dx \right|^2 dt \overset{HI}{\leq} \\
&\leq \int_0^T [(\int_\Omega |u(x,t)|^{2n\alpha/(n+4)} \, dx)^{(n+4)/2n} \cdot (\int_\Omega |\omega(x)|^{2n/(n-4)} \, dx)^{(n-4)/2n}]^2 dt = \\
&= \|\omega\|^2_{2n/(n-4),\Omega} \int_0^T (\int_\Omega |u(x,t)|^{2n\alpha/(n+4)} \, dx)^{(n+4)/n} dt \overset{(V.8)}{\leq} \\
&\leq \|\omega\|^2_{2n/(n-4),\Omega} \int_0^T C^{2\alpha} (\|u\|_{W_2^1(Q_T)})^{2\alpha} dt < \infty.
\end{aligned}$$

Is it possible to take $\alpha < (n+4)/n$? It turns out that the answer is "yes". It suffices to use the following well-known fact.

Assertion 1 *Let $\mu(A) < \infty$ and $1 \le \beta \le \gamma \le \infty$. Then $\|u\|_{\beta,A} \le \tilde{C}\,\|u\|_{\gamma,A}$.*

Suppose that $\alpha < (n+4)/n$, i.e., $2n\alpha/(n+4) < 2$. Then there exists an $\exists\tilde{\alpha} > \alpha$ such that

$$(n+4)/n \;\le\; \tilde{\alpha} \;<\; (n+4)/(n-1).$$

By analogy with the preceding, we have

$$\int_0^T \left| \int_\Omega |u(x,t)|^\alpha\, |\omega(x)|\, dx \right|^2 dt \overset{HI}{\le}$$

$$\le \int_0^T \left[\left(\int_\Omega |u(x,t)|^{2n\alpha/(n+4)}\, dx \right)^{\frac{(n+4)}{2n}} \cdot \left(\int_\Omega |\omega(x)|^{2n/(n-4)}\, dx \right)^{\frac{(n-4)}{2n}} \right]^2 dt =$$

$$= \|\omega\|^2_{2n/(n-4),\Omega} \int_0^T \left(\int_\Omega |u(x,t)|^{2n\alpha/(n+4)}\, dx \right)^{\frac{2\alpha(n+4)}{2\alpha n}} dt \overset{\alpha<\tilde{\alpha}}{\le}$$

$$\le \|\omega\|^2_{2n/(n-4),\Omega} \int_0^T \left(\int_\Omega |u(x,t)|^{2n\tilde{\alpha}/(n+4)}\, dx \right)^{\frac{2\alpha(n+4)}{2\tilde{\alpha} n}} \tilde{C}^{2\alpha} dt \overset{(V.8)}{\le}$$

$$\le \|\omega\|^2_{2n/(n-4),\Omega} \int_0^T \tilde{C}^{2\alpha} \hat{C}^{2\alpha} \left(\|u\|_{W_2^1(Q_T)} \right)^{2\alpha} dt < \infty.$$

Therefore, G maps $W_2^{2,1}(Q_T)$ into $L_2(0,T)$ for $n \ge 4$ and $1 < \alpha < (n+4)/(n-1)$.

The case $\mathbf{n = 3}$ can be treated in a similar way.

Here

$$r = 0, \quad l = 1, \quad \tilde{n} = 4, \quad m = 3, \quad p = 2, \quad q = \tilde{\alpha} \ge p = 2,$$
$$\theta = 1 - 0 - 4/2 + 3/\tilde{\alpha} = -1 + 3/\tilde{\alpha} > 0 \quad \text{and} \quad h = 1.$$

It follows from the last two inequalities that $2 \le \tilde{\alpha} < 3$, i.e., $1 < \alpha < 3$, and then

$$\int_0^T \left| \int_\Omega |u(x,t)|^\alpha\, |\omega(x)|\, dx \right|^2 dt \overset{ET}{\le}$$

$$\le \|\omega\|^2_{C(\bar{\Omega})} \int_0^T \left(\int_\Omega |u(x,t)|^\alpha\, dx \right)^2 dt \le \|\omega\|^2_{C(\bar{\Omega})} \int_0^T \tilde{C}^{2\alpha} \left(\int_\Omega |u(x,t)|^{\tilde{\alpha}}\, dx \right)^{\frac{2\alpha}{\tilde{\alpha}}} dt \overset{(V.8)}{\le}$$

$$\le \|\omega\|^2_{C(\bar{\Omega})} \int_0^T \tilde{C}^{2\alpha} \left(\|u\|_{W_2^1(Q_T)} \right)^{2\alpha} dt < \infty.$$

If $\mathbf{n=2}$, then in a similar way, we obtain $2 \le \tilde{\alpha} < 4$ and $1 < \alpha < 4$.

We have thereby shown that $G(u) \in L_2(0,T)$.

Let us proceed to **the proof of the strict differentiability of the operator $\mathbf{G(u)}$.**

Let u, u_k, $\hat{u}_k \in W_2^1(Q_T)$, $\hat{u}_k \neq 0$, $u_k \to u$ and $\hat{u}_k \to 0$ in $W_2^1(Q_T)$ as $k \to \infty$.

We set

$$[R(u_k,\hat{u}_k)]\,(t) = \int\limits_{\Omega} \left[\Phi(u_k(x,t)+\hat{u}_k(x,t)) - \Phi(u_k(x,t)) - \Phi'(u(x,t))\hat{u}_k(x,t)\right]\omega(x)dx.$$

Let us show that $R(u_k,\hat{u}_k)/\left\|\hat{u}_k\right\|_{W_2^1(Q_T)} \to 0$ in $L_2(0,T)$, which implies that

$$\left\|R(u_k,\hat{u}_k)/\left\|\hat{u}_k\right\|_{W_2^{2,1}(Q_T)}\right\|_{L_2(0,T)} \to 0,$$

since $\left\|\hat{u}_k\right\|_{W_2^1(Q_T)} \le \left\|\hat{u}_k\right\|_{W_2^{2,1}(Q_T)}$.

Passing to a subsequence, we can assume that $u_k \to u$, $\hat{u}_k \to 0$, $u_{kt} \to u_t$ and $\hat{u}_{kt} \to 0$ almost everywhere and there exists a $z \in L_{2(n+1)/(n-1)}(Q_T)$, such that [48, p. 162]

$$|u_k| + |\hat{u}_k| \le z \quad \text{almost everywhere on } Q_T. \tag{V.15}$$

Then $\omega \in W_2^2(\Omega) \subset L_q(\Omega)$ for $n > 4$ and $C_2 = 0$, where $q = 2n/(n-4)$, and

$$\int\limits_0^T \left|[R\,(u_k,u_k)]\,(t)\right|^2 dt =$$

$$= \int\limits_0^T \left|\int\limits_{\Omega}\left[\Phi\left(u_k(x,t)+\hat{u}_k(x,t)\right) - \Phi(u_k(x,t) - \Phi'(u(x,t))\hat{u}_k(x,t)\right]\omega(x)dx\right|^2 dt =$$

$$= \int\limits_0^T \left|\int\limits_{\Omega}\int\limits_0^1 \left[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))\right]\hat{u}_k(x,t)\omega(x)d\theta dx\right|^2 dt \overset{HI}{\le}$$

$$\le \int\limits_0^T \left(\int\limits_{\Omega}\left\{\int\limits_0^1 \left|\left[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))\right]\hat{u}_k(x,t)\right| d\theta\right\}^{\frac{2n}{(n+4)}} dx\right)^{\frac{2(n+4)}{2n}} \cdot$$

$$\cdot\left(\int\limits_{\Omega}|\omega(x)|^{2n/(n-4)}\,dx\right)^{\frac{2(n-4)}{2n}}dt=\|\omega\|^2_{2n/(n-4),\Omega}\cdot$$

$$\cdot\int\limits_0^T\left(\int\limits_{\Omega}\left\{\int\limits_0^1|[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t))-\Phi'(u(x,t))]\,\hat{u}_k(x,t)|\cdot 1d\theta\right\}^{\frac{2n}{(n+4)}}dx\right)^{\frac{(n+4)}{n}}dt\overset{HI}{\leq}$$

$$\leq\|\omega\|^2_{2n/(n-4),\Omega}\int\limits_0^T\left\{\int\limits_{\Omega}\left(\int\limits_0^1|[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t))-\Phi'(u(x,t))]\,\hat{u}_k(x,t)|^{\frac{2n}{(n+4)}}\,d\theta\cdot\right.\right.$$

$$\left.\left.\cdot\left(\int\limits_0^1 1^{2n/(n-4)}d\theta\right)^{\frac{(n-4)}{(n+4)}}\right)dx\right\}^{\frac{(n+4)}{n}}dt=\|\omega\|^2_{2n/(n-4),\Omega}\cdot$$

$$\cdot\int\limits_0^T\left\{\int\limits_{\Omega}\int\limits_0^1|[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t))-\Phi'(u(x,t))]\,\hat{u}_k(x,t)|^{2n/(n+4)}\,d\theta dx\right\}^{\frac{n+4}{n}}dt\overset{FT}{=}$$

$$=\|\omega\|^2_{2n/(n-4),\Omega}\cdot$$

$$\cdot\int\limits_0^T\left\{\int\limits_0^1\int\limits_{\Omega}|[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t))-\Phi'(u(x,t))]\,\hat{u}_k(x,t)|^{2n/(n+4)}\,dxd\theta\right\}^{\frac{n+4}{n}}dt\overset{HI}{\leq}$$

$$\leq\|\omega\|^2_{2n/(n-4),\Omega}\cdot$$

$$\cdot\int\limits_0^T\left\{\int\limits_0^1\int\limits_{\Omega}|[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t))-\Phi'(u(x,t))]\,\hat{u}_k(x,t)|^{2n(n+4)/n(n+4)}\,dxd\theta\right\}^{\frac{n(n+4)}{n(n+4)}}\cdot$$

$$\cdot\left(\int\limits_0^1\int\limits_{\Omega}1^{(n+4)/4}dxd\theta\right)^{4(n+4)/n(n+4)}dt\leq\|\omega\|^2_{2n/(n-4),\Omega}\cdot\mu(\Omega)^{4/n}\cdot$$

$$\cdot\int\limits_0^T\int\limits_0^1\int\limits_{\Omega}|[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t))-\Phi'(u(x,t))]|^2\,(\hat{u}_k(x,t))^2dxd\theta dt=\|\omega\|^2_{2n/(n-4),\Omega}\cdot$$

$$\cdot\mu(\Omega)^{4/n}\int\limits_0^1\int\limits_{Q_T}|[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t))-\Phi'(u(x,t))]|^2\,(\hat{u}_k(x,t))^2dxdtd\theta\overset{HI}{\leq}$$

$$\leq\|\omega\|^2_{2n/(n-4),\Omega}\cdot\mu(\Omega)^{4/n}\int\limits_0^1\left(\int\limits_{Q_T}|[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t))-\Phi'(u(x,t))]|^{n+1}\,dxdt\right)^{\frac{2}{n+1}}\cdot$$

$$\cdot\left(\int\limits_{Q_T}|\hat{u}_k(x,t)|^{2(n+1)/(n-1)}\,dxdt\right)^{(n-1)/(n+1)}d\theta=$$

$$= \|\omega\|^2_{2n/(n-4),\Omega} \cdot \mu(\Omega)^{4/n} \cdot \|\hat{u}_k(x,t)\|^2_{2(n+1)/(n-1),Q_T} \cdot$$
$$\cdot \int_0^1 \left(\int_{Q_T} |[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))]|^{n+1}\, dxdt \right)^{\frac{2}{n+1}} d\theta.$$

Note that $|[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))]|^{n+1} \to 0$ almost everywhere in Q_T and

$$|[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))]|^{n+1} \overset{(V.15)}{\le}$$
$$\le \left(2C_1\alpha z^{\alpha-1}\right)^{n+1} \le (2C_1\alpha)^{n+1} (z+1)^{2(n+1)/(n-1)} \in L_1(Q_T) \tag{V.16}$$

(since $z \in L_{2(n+1)/(n-1)}(Q_T)$) uniformly with respect to $\theta \in [0,1]$.

Remark 27 *The final estimate in (V.16) is intended for the subsequent application of the Lebesgue theorem and hence imposes the constraint $\alpha \le (n+1)/(n-1)$ on α.*

By applying the Lebesgue theorem twice, we obtain $\left\| R(u_k,\hat{u}_k)/\|\hat{u}_k\|_{W_2^1(Q_T)} \right\|_{L_2(0,T)} \to 0$ as $k \to \infty$.

For $n = 2$ and $n = 3$, we have

$$\int_0^T |[R(u_k,\hat{u}_k)](t)|^2\, dt =$$
$$= \int_0^T \left| \int_\Omega [\Phi(u_k(x,t)+\hat{u}_k(x,t)) - \Phi(u_k(x,t)) - \Phi'(u(x,t))\hat{u}_k(x,t)]\, \omega(x) dx \right|^2 dt \overset{ET}{\le}$$
$$\le \|\omega\|^2_{C(\bar\Omega)} \int_0^T \left| \int_\Omega \int_0^1 [\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))]\, \hat{u}_k(x,t) d\theta dx \right|^2 dt \overset{CSBI}{\le}$$
$$\le \|\omega\|^2_{C(\bar\Omega)} \cdot \mu(\Omega) \int_0^T \int_\Omega \left(\int_0^1 |[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))]\, \hat{u}_k(x,t)|\, d\theta \right)^2 dxdt \overset{CSBI}{\le}$$
$$\le \|\omega\|^2_{C(\bar\Omega)} \cdot \mu(\Omega) \int_0^T \int_\Omega \int_0^1 |\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))|^2\, (\hat{u}_k(x,t))^2 d\theta dxdt \overset{FT}{=}$$
$$= \|\omega\|^2_{C(\bar\Omega)} \cdot \mu(\Omega) \int_0^1 \int_{Q_T} |\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))|^2\, (\hat{u}_k(x,t))^2 dxdtd\theta.$$

Starting from this place, the preceding argument applies.

Finally, consider the case $n = 4$. We have

$$\int_0^T |[R(u_k,\hat{u}_k)](t)|^2\, dt =$$
$$= \int_0^T \left| \int_\Omega [\Phi(u_k(x,t)+\hat{u}_k(x,t)) - \Phi(u_k(x,t)) - \Phi'(u(x,t)\hat{u}_k(x,t))]\, \omega(x) dx \right|^2 dt =$$
$$= \int_0^T \left| \int_\Omega \int_0^1 [\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))]\, \hat{u}_k(x,t) d\theta \omega(x) dx \right|^2 dt \overset{FT}{=}$$

$$= \int\limits_0^T \left| \int\limits_0^1 \int\limits_\Omega \left[\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))\right] \hat{u}_k(x,t)\omega(x)dxd\theta \right|^2 dt \overset{HI}{\leq}$$

$$\leq \int\limits_0^T \left| \int\limits_0^1 \left(\int\limits_\Omega \left[|\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))| \cdot |\hat{u}_k(x,t)| \right]^{q/\alpha} dx \right)^{\frac{\alpha}{q}} \cdot \right.$$

$$\left. \cdot \left(\int\limits_\Omega |\omega(x)|^{q/(q-\alpha)} dx \right)^{\frac{q-\alpha}{q}} d\theta \right|^2 dt = \|\omega\|^2_{q/(q-\alpha),\Omega} \cdot$$

$$\cdot \int\limits_0^T \left| \int\limits_0^1 \left(\int\limits_\Omega \left[|\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))| \, |\hat{u}_k(x,t)| \right]^{q/\alpha} dx \right)^{\frac{\alpha}{q}} \cdot 1 d\theta \right|^2 dt \overset{HI}{\leq}$$

$$\leq \|\omega\|^2_{q/(q-\alpha),\Omega} \int\limits_0^T \left| \left\{ \int\limits_0^1 \int\limits_\Omega \left[|\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))| \cdot |\hat{u}_k(x,t)| \right]^{q/\alpha} dxd\theta \right\}^{2\alpha/q} \cdot \right.$$

$$\left. \cdot \left(\int\limits_0^1 1^{q/(q-\alpha)} d\theta \right) \right| dt \overset{HI}{\leq}$$

$$\leq \|\omega\|^2_{q/(q-\alpha),\Omega} \int\limits_0^T \left| \left\{ \int\limits_0^1 \int\limits_\Omega \left[|\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))| \cdot |\hat{u}_k(x,t)| \right]^2 dxd\theta \right\}^1 \right| \cdot$$

$$\cdot \left(\int\limits_0^1 \int\limits_\Omega 1^{2\alpha/(2\alpha-q)} dxd\theta \right)^{\frac{2\alpha-q}{2\alpha} \cdot \frac{2\alpha}{q}} dt \overset{FT}{=}$$

$$= \|\omega\|^2_{q/(q-\alpha),\Omega} \cdot \mu(\Omega)^{(2\alpha-q)/q} \int\limits_0^1 \int\limits_{Q_T} \left[|\Phi'(u_k(x,t)+\theta\hat{u}_k(x,t)) - \Phi'(u(x,t))| \cdot |\hat{u}_k(x,t)| \right]^2 dxdtd\theta.$$

The subsequent considerations are similar to those given above. Therefore, G is a strictly differentiable operator. The proof of the proposition is complete.

Proposition 7 *The mapping*

$$\Phi : W_2^1(Q_T) \to L_2(Q_T), [\Phi(u)](x,t) = \Phi(u(x,t)) \tag{V.17}$$

is strictly differentiable.

Proof. Recall that we carry out the proof for $C_2 = 0$.

We set $\left[R(u_k,\hat{u}_k)\right](x,t) = \Phi(u_k(x,t)+\hat{u}_k(x,t)) - \Phi(u_k(x,t)) - \Phi'(u(x,t))\hat{u}_k(x,t)$.

Then

$$\int\limits_0^T \int\limits_\Omega \left| \left[R(u_k,\hat{u}_k)\right](x,t) \right|^2 dxdt =$$

$$= \int\limits_0^T \int\limits_\Omega \left| \Phi(u_k(x,t)+\hat{u}_k(x,t)) - \Phi(u_k(x,t)) - \Phi'(u(x,t))\hat{u}_k(x,t) \right|^2 dxdt =$$

$$= \int\limits_0^T \int\limits_\Omega \left(\int\limits_0^1 \left[\Phi'(u_k(x,t)) + \theta\hat{u}_k(x,t)) - \Phi'(u(x,t))\right] \hat{u}_k(x,t) \cdot 1 d\theta \right)^2 dxdt \overset{CSBI}{\leq}$$

$$\leq \int\limits_{Q_T} \left(\int\limits_0^1 \left[\Phi'(u_k(x,t) + \theta\hat{u}_k(x,t)) - \Phi'(u(x,t))\right]^2 (\hat{u}_k(x,t))^2 d\theta \right) \left(\int\limits_0^1 1^2 d\theta \right) dxdt =$$

$$= \int\limits_{Q_T} \int\limits_0^1 \left[\Phi'(u_k(x,t) + \theta\hat{u}_k(x,t)) - \Phi'(u(x,t))\right]^2 (\hat{u}_k(x,t))^2 \, d\theta dxdt \overset{FT}{=}$$

$$= \int\limits_0^1 \int\limits_{Q_T} \left[\Phi'(u_k(x,t) + \theta\hat{u}_k(x,t)) - \Phi'(u(x,t))\right]^2 (\hat{u}_k(x,t))^2 \, dxdtd\theta \overset{HI}{\leq}$$

$$\leq \int\limits_0^1 \left(\int\limits_{Q_T} \left|\Phi'(u_k(x,t) + \theta\hat{u}_k(x,t)) - \Phi'(u(x,t))\right|^{n+1} dxdt \right)^{\frac{2}{n+1}} \cdot$$

$$\cdot \left(\int\limits_{Q_T} \left|\hat{u}_k(x,t)\right|^{2(n+1)/(n-1)} dxdt \right)^{\frac{n-1}{n+1}} d\theta =$$

$$= \left\|\hat{u}_k(x,t)\right\|^2_{2(n+1)/(n-1),Q_T} \int\limits_0^1 \left(\int\limits_{Q_T} \left|\Phi'(u_k(x,t) + \theta\hat{u}_k(x,t)) - \Phi'(u(x,t))\right|^{n+1} dxdt \right)^{\frac{2}{n+1}} d\theta.$$

By a similar argument, we arrive at the estimate (V.16). By using the Lebesgue theorem twice, we obtain $\left\| R(u_k, \hat{u}_k) / \left\|\hat{u}_k\right\|_{W_2^1(Q_T)} \right\|_{L_2(Q_T)} \to 0$ as $k \to \infty$.

Remark 28 *The exponent $2(n+1)/(n-1)$ determined by the embedding theorem was artificially introduced in the proof with the use of the Hölder inequality. Therefore, we obtain the following constraint for $\alpha : \alpha \leq (n+1)/(n-1)$ for $n \geq 2$. Note that, for the same α, the operator G defined in subsection V.1.3 is strictly differentiable.*

V.1.4 End of Proof of the Main Result

It follows from equation (V.11) that

$$(u_t - \Delta u - \Phi(u), \omega)_{2,\Omega} = (h, \omega)_{2,\Omega} \, f \tag{V.18}$$

for almost all $t \in [0,T]$.

We set

$$\varphi(t) = (h, \omega)_{2,\Omega} \, f.$$

By virtue of the assumptions of theorem 23, f can be uniquely determined on the basis of φ. Let us assume that the solution of the problem exists and derive an operator equation.

Since $(u, \omega)_{2,\Omega} \in W_2^1(0,T)$, we have

$$(u_t, \omega)_{2,\Omega} = d/dt(u, \omega)_{2,\Omega} = \chi'(t).$$

Since $u \in W_{2,0}^{2,1}(Q_T)$ and $\omega \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$, it follows that

$$(-\Delta u, \omega)_{2,\Omega} = -(u, \Delta\omega)_{2,\Omega},$$

for $t \in [0, T]$.

Then relation (V.18) implies that

$$\chi'(t) - \int_\Omega u(x,t)\Delta\omega(x)dx - \int_\Omega \Phi(u(x,t))\omega(x)dx = \varphi(t).$$

We set $\Psi(t) = \chi'(t)$, $\Psi \in L_2(0,T)$. Then

$$\Psi(t) = \int_\Omega [\Phi(u)](x,t)\omega(x)dx + \int_\Omega u(x,t)\Delta\omega(x)dx + \varphi(t). \tag{V.19}$$

Since $L : H \to L_2(Q_T)$ is an isomorphism (see (V.7)), $\Phi : W_2^1(Q_T) \to L_2(Q_T)$ is strictly Fréechet differentiable, and $\Phi'(0) = 0$, it follows from the inverse function theorem that the mapping

$$\xi : u \longmapsto u_t - \Delta u - \Phi(u)$$

is a local homeomorphism of a neighborhood U' of zero in H onto a neighborhood V' of zero in $L_2(Q_T)$.

Let $\eta = \xi^{-1}$ be the inverse of this local homeomorphism, i.e., $\eta : F \longmapsto u$, where u is a solution of the equation, and η is a strictly differentiable mapping (in H and hence in $W_2^{2,1}(Q_T)$) in a neighborhood of zero in $L_2(Q_T)$. We have

$$F(x,t) \;=\; h(x,t)f(t) = h(x,t)\varphi(t)/(h,\omega)_{2,\Omega}.$$

Let

$$P(\varphi) = \eta[(h(x,t)\varphi(t)/(h,\omega)_{2,\Omega})]. \tag{V.20}$$

Since the mapping

$$\Lambda : \varphi(t) \longmapsto h(x,t)\varphi(t)/(h,\omega)_{2,\Omega},\ \Lambda : L_2(0,T) \to L_2(Q_T) \tag{V.21}$$

is linear and continuous, it follows that P (treated as a mapping into $W_2^{2,1}(Q_T)$) is strictly Fréchet differentiable in a neighborhood of zero in $L_2(0,T)$. Further, we seek a solution of the nonlinear problem (V.11)-(V.13), (V.4) in the form $u = P(\varphi)$, where u is a solution of the nonlinear problem (V.11)-(V.13), (V.4) with right-hand side $F(x,t) = h(x,t)\varphi(t)/(h,\omega)_{2,\Omega}$. We introduce the mapping

$$M : L_2(0,T) \to L_2(0,T), \tag{V.22}$$

$$M : \varphi \longmapsto \left\{ \varphi(t) + \int_\Omega \Phi([P(\varphi)](x,t))\omega(x)dx + \int_\Omega [P(\varphi)](x,t)\Delta\omega(x)dx \right\}, \tag{V.23}$$

then an argument similar to that in [27] shows that problem (V.11)-(V.13), (V.4) is equivalent to the operator equation

$$M(\varphi) = \Psi.$$

Let us show that M is strictly Fréchet differentiable in a neighborhood of zero in $L_2(0,T)$ and $M'(0) = I - A$ (see (V.6)).

Note that the mapping

$$\varphi \longmapsto \int\limits_{\Omega} \Phi([P(\varphi)](x,t))\omega(x)dx$$

is strictly Fréchet differentiable in a neighborhood of zero by the composite function theorem, and its derivative vanishes at zero, since $\Phi'(0) = 0$. Further, the mapping

$$\varphi \longmapsto \int\limits_{\Omega} [P(\varphi)](x,t)\Delta\omega(x)dx$$

is strictly differentiable in a neighborhood of zero, since

$$v \longmapsto \int\limits_{\Omega} v(x,t)\Delta\omega(x)dx$$

is a linear continuous operator from $L_2(Q_T)$ to $L_2(0,T)$ and so much the more from $W_2^{2,1}(Q_T)$ to $L_2(0,T)$. Moreover (see (V.6) and [27, p.47]),

$$\int\limits_{\Omega} P'(0)\varphi\Delta\omega(x)dx = -A\varphi. \tag{V.24}$$

Consequently, M is strictly differentiable in a neighborhood of zero in $L_2(0,T)$, and $M'(0) = I - A$ is the operator of the linear problem.

Since $\|A\| < 1$ [27, p.47], it follows that $(M'(0))^{-1}$ exists and

$$\left\|(M'(0))^{-1}\right\| \leq 1/(1 - \|A\|).$$

By the inverse function theorem, there exist open neighborhoods U and V of zero in $L_2(0,T)$ such that M is a C^1-diffeomorphism of U onto V.

We set

$$\tilde{V} = \left\{\chi \in {}_{\circ}W_2^1(0,T) | \chi' \in V\right\}.$$

Then $\tilde{V}$ is an open neighborhood of zero in ${}_{\circ}W_2^1(0,T)$, and for each $\chi \in \tilde{V}$, there exists a unique $\varphi \in U$ such that $M(\varphi) = \chi'$.

We have thereby proved the local unique solvability of an operator equation equivalent to problem (V.11)-(V.13), (V.4), which completes the proof of the theorem.

V.2 The precision of the local solvability of the inverse problem for the nonlinear heat equation with integral overdetermination

V.2.1 Preliminaries. Notation

We shall use the following notation:

U_H is the open unit ball in H,

where H is the space defined in subsection V.1.1 such that

$$H \subset W_2^{2,1}(Q_T); \tag{V.25}$$

$C_{X \to Y}$ is the constant of the embedding of X in Y;

$\Lambda : L_2(0,T) \to LH,\ \Lambda : \varphi(t) \longmapsto \varphi(t)h(x,t)/(h,\omega)_{2,\Omega}$ (see (V.21));

$\|\Lambda\| = \|\Lambda\|_{\mathcal{L}(L_2(0,T),LH)}$;

$\mathcal{L}(X,Y)$ is the Banach algebra of linear continuous operators from X to Y.

We consider the inverse problem (V.11)-(V.13), (V.4) under the assumptions of subsection V.1.2. As in chapter IV (section IV.4.1) we use in what follows a result from [49] (see also [48, p.33]). We use mainly the theorems 18 and 19.

V.2.2 Estimates of Nonlinear Terms in the Differential Operators for the Direct Problem

Here the key point is the derivation of an estimate for the nonlinear part of the operator in problem (V.11)-(V.13), (V.4) in the class $L_2(Q_T)$, which enables us to use theorem 18. Let $u(x,t) \in H$ and $z(x,t) \in H$ (see (V.25)) satisfy the condition

$$\|u\|_H \le r, \|u+z\|_H \le r. \tag{V.26}$$

To estimate

$$\|\Phi(u+z) - \Phi(u)\|^2_{L_2(Q_T)} = \int\limits_{Q_T} |\Phi(u+z) - \Phi(u)|^2\, dxdt,$$

we note that

$$H \subset W_2^1(Q_T) \subset L_{2(n+1)/(n-1)}(Q_T)$$

by the embedding theorem. Then

$$\|\Phi(u+z)-\Phi(u)\|^2_{L_2(Q_T)} = \int\limits_{Q_T} |\Phi(u+z)-\Phi(u)|^2\,dxdt =$$

$$= \int\limits_{Q_T}[\int\limits_0^1 \Phi'(u+sz)zds]^2 dxdt \overset{CSBI}{\le}$$

$$\le \int\limits_{Q_T}\int\limits_0^1 [\Phi'(u+sz)]^2 z^2 dsdxdt \overset{FT}{=} \int\limits_0^1\int\limits_{Q_T}[\Phi'(u+sz)]^2 z^2 dxdtds \overset{(V.10)}{\le}$$

$$\le \int\limits_0^1\int\limits_{Q_T}[\alpha_1 C_1 |u+sz|^{\alpha_1-1} + \alpha_2 C_2 |u+sz|^{\alpha_2-1}]^2 |z|^2\,dxdtds \overset{MI}{\le}$$

$$\le \left\{ \left[\int\limits_0^1\int\limits_{Q_T} \left|\alpha_1 C_1 |u+sz|^{\alpha_1-1}|z|\right|^2 dxdtds\right]^{\frac12} + \right.$$

$$\left. + \left[\int\limits_0^1\int\limits_{Q_T} \left|\alpha_2 C_2 |u+sz|^{\alpha_2-1}|z|\right|^2 dxdtds\right]^{\frac12} \right\}^2 .$$

Let us estimate the first term:

$$\left[\int\limits_0^1\int\limits_{Q_T} \left|\alpha_1 C_1 |u+sz|^{\alpha_1-1}|z|\right|^2 dxdtds\right]^{\frac12} \overset{HI}{\le}$$

$$\le \left[\int\limits_0^1 \alpha_1^2 C_1^2 \left(\int\limits_{Q_T} |u+sz|^{(\alpha_1-1)\cdot 2\cdot(n+1)/2}\,dxdt\right)^{\frac{2}{n+1}} ds \left(\int\limits_{Q_T}|z|^{2\cdot(n+1)/(n-1)}\,dxdt\right)^{\frac{n-1}{n+1}}\right]^{\frac12} =$$

$$= \alpha_1 C_1 \left[\int\limits_0^1 \left(\int\limits_{Q_T}|u+sz|^{(n+1)(\alpha_1-1)}\,dxdt\right)^{2(\alpha_1-1)/(n+1)(\alpha_1-1)} ds\cdot\right.$$

$$\left.\cdot \left(\int\limits_{Q_T}|z|^{2(n+1)/(n-1)}\,dxdt\right)^{\frac{n-1}{2(n+1)}\cdot\frac{n-1}{n+1}\cdot\frac{2(n+1)}{n-1}}\right]^{\frac12} =$$

$$= \alpha_1 C_1 \left[\int\limits_0^1 \|u+sz\|^{2(\alpha_1-1)}_{(n+1)(\alpha_1-1),Q_T}\,ds\right]^{\frac12} \|z\|_{2(n+1)/(n-1),Q_T} \overset{ET}{\le}$$

$$\leq \alpha_1 C_1 C^{\alpha_1-1}_{H\to L_{(n+1)(\alpha_1-1)}(Q_T)} \left[\int\limits_0^1 \|u+sz\|_H^{2(\alpha_1-1)}\, ds\right]^{\frac{1}{2}} C_{H\to L_{2(n+1)/(n-1)}(Q_T)} \cdot \|z\|_H \overset{(V.26)}{\leq}$$

$$\leq \alpha_1 \cdot C_1 \cdot C^{\alpha_1-1}_{H\to L_{(n+1)(\alpha_1-1)}(Q_T)} \cdot C_{H\to L_{2(n+1)/(n-1)}(Q_T)} \cdot r^{\alpha_1-1} \cdot \|z\|_H .$$

The second term can be estimated in a similar way:

$$\left[\int\limits_0^1 \int\limits_{Q_T} \left|\alpha_2 C_2 \left|u+sz\right|^{\alpha_2-1}|z|\right|^2 dxdtds\right]^{\frac{1}{2}} \leq$$
$$\leq \alpha_2 \cdot C_2 \cdot C^{\alpha_2-1}_{H\to L_{(n+1)(\alpha_2-1)}(Q_T)} \cdot C_{H\to L_{2(n+1)/(n-1)}(Q_T)} \cdot r^{\alpha_2-1} \cdot \|z\|_H .$$

Therefore,

$$\|\Phi(u+z)-\Phi(u)\|_{L_2(Q_T)} \leq \vartheta_n(r)\,\|z\|_H ,$$

where

$$\vartheta_n(r) = \alpha_1 \cdot C_1 \cdot C^{\alpha_1-1}_{H\to L_{(n+1)(\alpha_1-1)}(Q_T)} \cdot C_{H\to L_{2(n+1)/(n-1)}(Q_T)} \cdot r^{\alpha_1-1} +$$
$$+\alpha_2 \cdot C_2 \cdot C^{\alpha_2-1}_{H\to L_{(n+1)(\alpha_2-1)}(Q_T)} \cdot C_{H\to L_{2(n+1)/(n-1)}(Q_T)} \cdot r^{\alpha_2-1}.$$

Here and in what follows, the subscript n indicates the dimension of Ω.

Note that $LU_H = U_{L_2(Q_T)}$, since L is an isometric isomorphism of H onto $L_2(Q_T)$.

Next,

$$P_{LU_H}(\Phi(u+z)-\Phi(u)) = P_{U_{L_2(Q_T)}}(\Phi(u+z)-\Phi(u)) \leq \vartheta_n(r)\,\|z\|_H .$$

Then

$$P_{LU_H}(\Phi(u+z)-\Phi(u)) \leq \vartheta_n(r)\,\|z\|_H , \tag{V.27}$$

where

$$\vartheta_n(r) = a_n r^{\alpha_1-1} + b_n r^{\alpha_2-1}, \tag{V.28}$$

$$a_n = \alpha_1 \cdot C_1 \cdot C^{\alpha_1-1}_{H\to L_{(n+1)(\alpha_1-1)}(Q_T)} \cdot C_{H\to L_{2(n+1)/(n-1)}(Q_T)}, \tag{V.29}$$

$$b_n = \alpha_2 \cdot C_2 \cdot C^{\alpha_2-1}_{H\to L_{(n+1)(\alpha_2-1)}(Q_T)} \cdot C_{H\to L_{2(n+1)/(n-1)}(Q_T)}. \tag{V.30}$$

In this case, in theorem 18, one can set $A = L$, $X = H$, $Y = LH$ with the unit ball LU_H, $b(r) = \bar{b}_n(r)$ and $\Psi = \Phi$, where

$$\bar{b}_n(r) = \max(1-\vartheta_n(r), 0) = \max(1-(a_n r^{\alpha_1-1} + b_n r^{\alpha_2-1}), 0).$$

If $r \in [0, r_*[$, then we set

$$\bar{b}_n(r) = 1-(a_n r^{\alpha_1-1} + b_n r^{\alpha_2-1}),$$

where r_* is the root of the equation

$$1-(a_n r^{\alpha_1-1} + b_n r^{\alpha_2-1}) = 0. \tag{V.31}$$

Then, by theorem 18, we have

$$w_n(r) = \int_0^r b(t)dt = \int_0^r (1 - (a_n r^{\alpha_1 - 1} + b_n r^{\alpha_2 - 1}))dt = r - \frac{a_n r^{\alpha_1}}{\alpha_1} - \frac{b_n r^{\alpha_2}}{\alpha_2}. \tag{V.32}$$

If $r \in [r_*, \infty]$, then we set $\bar{b}_n(r) = 0$ and $w_n(r) = w_n(r_*)$.

By using theorem 18, we obtain the following assertion.

Theorem 24 *Let $n \geq 2$, let*

$$1 < \alpha_1 \leq \alpha_2 \leq (n+1)/(n-1),$$

and let the assumptions of theorem 23 be valid.

Further, let a_n, b_n, and w_n be given by (V.29), (V.30), and (V.32), respectively, and let r_* be the root of equation (V.31).

Then for all $r \in [0, r_*[$ and $F \in w_n(r)LU_H$ there exists a unique solution $u \in rU_H$ of the equation

$$Lu - \Phi(u) = F(x,t),$$

where $F(x,t)$ is the right-hand side of equation (V.11).

V.2.3 Estimates of Nonlinear Terms in the Inverses of the Differential Operators of the Direct Problem

It follows from the results of subsection V.1.4 that

$$\xi(v) \ = \ L(v) \ - \ \Phi(v) = F \Rightarrow \eta(F) = L^{-1}F - Q(F),$$

which implies that

$$F = \xi(\eta(F)) = \xi(L^{-1}F - Q(F)) = F - L(Q(F)) - \Phi(L^{-1}F - Q(F)).$$

Then

$$L(Q(F)) - \Phi(L^{-1}F - Q(F)) = 0,$$

i.e.,

$$L(Q(F)) = \Phi(L^{-1}F - Q(F)),$$
$$Q(F) = L^{-1}\Phi(L^{-1}F - Q(F)) = L^{-1}\Phi(\eta(F)).$$

Consequently,

$$Q(F + \Delta F) - Q(F) = L^{-1}[\Phi(\eta(F + \Delta F)) - \Phi(\eta(F))].$$

By theorem 24, we obtain

$$\|F\|_{LH} < w_n(r) \Rightarrow \|\eta(F)\|_H < r, \tag{V.33}$$

$$\|F + \Delta F\|_{LH} < w_n(r) \Rightarrow \|\eta(F + \Delta F)\|_H < r,$$

where $w_n(r)$ is given by (V.32). Then from (V.27), we have

$$P_{LU_H}(\Phi(\eta(F + \Delta F)) - \Phi(\eta(F))) \le \vartheta_n(r)\, \|z\|_H\,, \tag{V.34}$$

where $z = \eta(F + \Delta F) - \eta(F)$.

Then

$$\|z\|_H = \|\eta(F+\Delta F)-\eta(F)\|_H = \|\xi^{-1}(F+\Delta F)-\xi^{-1}(F)\|_H \overset{Hadamard'sT}{\le} \frac{1}{1-\vartheta_n(r)}\, \|\Delta F\|_{LH}\,.$$

Therefore, relation (V.34) acquires the form

$$P_{LU_H}(\Phi(\eta(F + \Delta F)) - \Phi(\eta(F))) \le \vartheta_n(r) \cdot \frac{1}{1-\vartheta_n(r)}\, \|\Delta F\|_{LH}\,.$$

Let $r' = w_n(r)$. Then $r = w_n^{-1}(r')$,

$$P_{U_H}(Q(F + \Delta F) - Q(F)) \le \vartheta_n(w_n^{-1}(r')) \cdot \frac{1}{1-\vartheta_n(w_n^{-1}(r'))}\, \|\Delta F\|_{LH}\,,$$

i.e.,

$$P_{U_H}(Q(F + \Delta F) - Q(F)) \le \Theta_n(r')\, \|\Delta F\|_{LH}\,,$$

where

$$\Theta_n(r') = \frac{\vartheta_n(w_n^{-1}(r'))}{1-\vartheta_n(w_n^{-1}(r'))} = \frac{a_n(w_n^{-1}(r'))^{\alpha_1-1} + b_n(w_n^{-1}(r'))^{\alpha_2-1}}{1-a_n(w_n^{-1}(r'))^{\alpha_1-1} - b_n(w_n^{-1}(r'))^{\alpha_2-1}}, \tag{V.35}$$

since it follows from (V.28) that

$$\vartheta_n(w_n^{-1}(r')) = a_n(w_n^{-1}(r'))^{\alpha_1-1} + b_n(w_n^{-1}(r'))^{\alpha_2-1}.$$

In theorem 18, one can set $A = L^{-1}$, $X = LH$, $Y = H$ with the unit ball LU_H, $\Psi = Q$, $\Theta(r) = \Theta_n(r')$ and $b(r) = B_n(r')$,where $\Theta_n(r')$ from (V.35) and $B_n(r') = \max(1-\Theta_n(r'), 0)$.

If $r' \in [0, r'_*]$, then we set

$$B_n(r') = \frac{1-2[a_n(w_n^{-1}(r'))^{\alpha_1-1} + b_n(w_n^{-1}(r'))^{\alpha_2-1}]}{1-a_n(w_n^{-1}(r'))^{\alpha_1-1} - b_n(w_n^{-1}(r'))^{\alpha_2-1}},$$

where r'_* is the root of the equation

$$1-2[a_n(w_n^{-1}(r'))^{\alpha_1-1} + b_n(w_n^{-1}(r'))^{\alpha_2-1}] = 0. \tag{V.36}$$

Then by theorem 18, we have

$$W_n(r') = \int_0^{r'} B_n(t)dt = r' - \int_0^{r'} \Theta_n(t)dt,$$

i.e.,

$$W_n(r') = r' - \int_0^{r'} \frac{a_n(w_n^{-1}(t))^{\alpha_1-1} + b_n(w_n^{-1}(t))^{\alpha_2-1}}{1 - a_n(w_n^{-1}(t))^{\alpha_1-1} - b_n(w_n^{-1}(t))^{\alpha_2-1}} dt. \qquad \text{(V.37)}$$

If $r' \in [r'_*, \infty]$, then we set $B_n(r') = 0$ and $W_n(r') = W_n(r'_*)$.

By theorem 18, we obtain the following assertion.

Theorem 25 *Let $n \geq 2$, let*

$$1 < \alpha_1 \leq \alpha_2 \leq (n+1)/(n-1),$$

and let the assumptions of theorem 23 be valid.

Further, let a_n, b_n, w_n, and $W_n(r')$ be given by (V.29), (V.30), (V.32), and (V.37), respectively, and let r'_* be the root of equation (V.36).

Then for any $r' \in [0, r'_*[$ and $v \in W_n(r')L^{-1}(LU_H) = W_n(r')U_H$ there exists a unique $F \in r'U_{LH}$ such that $L^{-1}F - Q(F) = v$.

V.2.4 Estimates of Nonlinear Terms in the Operators of the Inverse Problem

It follows from (V.23) that

$$[M(\varphi)](t) = \Psi(t) = \varphi(t) + \int_\Omega \Phi([P(\varphi)](x,t))\omega(x)dx + \int_\Omega [P(\varphi)](x,t)\Delta\omega(x)dx =$$

$$= [I(\varphi)](t) + \int_\Omega [P'(0)\varphi](x,t)\Delta\omega(x)dx + \int_\Omega \Phi([P(\varphi)](x,t))\omega(x)dx +$$

$$+ \int_\Omega [P(\varphi) - P'(0)\varphi](x,t)\Delta\omega(x)dx$$

or, with regard to (V.24),

$$M(\varphi) = \Psi = (I - A)\varphi + G[P(\varphi)] + A\varphi + B[P(\varphi)] = \hat{A}\varphi + \hat{\Psi}(\varphi),$$

where $B : W_2^{2,1}(Q_T) \to L_2(0,T)$ and

$$B : v \longmapsto \int_\Omega v(x,t)\Delta\omega(x)dx,$$

here v is the solution of the nonlinear problem (V.11)-(V.13), (V.4), A is the linear operator occurring in (V.6), which takes each $\varphi(t) \in L_2(0,T)$ to the solution $u(x,t)$ of the linear problem (V.1)-(V.4) with $a = b = g = 0$, G is given by (V.14), P is given by (V.20), and

$$\hat{A}\varphi = (I - A)\varphi, \tag{V.38}$$

$$\hat{\Psi}(\varphi) = A\varphi + G[P(\varphi)] + B[P(\varphi)]. \tag{V.39}$$

Note that

$$\|\varphi\|_{L_2(0,T)} \le \tilde{r} \Rightarrow \|F\|_{LH} = \|\Lambda\varphi\|_{LH} \le \|\Lambda\|_{\mathcal{L}(L_2(0,T),LH)}\, \tilde{r} \Rightarrow$$

$$\Rightarrow \|P(\varphi)\|_H = \|\eta(\Lambda\varphi)\|_H \overset{(2.9)}{\le} w_n^{-1}(\|\Lambda\|\, \tilde{r}),$$

i.e.,

$$\|\varphi\|_{L_2(0,T)} \le \tilde{r} \Rightarrow \|P(\varphi)\|_H \le w_n^{-1}(\|\Lambda\|\, \tilde{r}),$$

$$\|\varphi + sz\|_{L_2(0,T)} \le \tilde{r} \Rightarrow \|P(\varphi + sz)\|_H \le w_n^{-1}(\|\Lambda\|\, \tilde{r}). \tag{V.40}$$

To derive an estimate for $\|P'(\varphi + sz)\|_{\mathcal{L}(L_2(0,T),H)}$, which will be used in what follows, we note that

$$P'(\varphi + sz) = \eta'(\Lambda(\varphi + sz)) \circ \Lambda \overset{FIT}{=} [\xi'(P(\varphi + sz))]^{-1} \circ \Lambda.$$

Further, we have

$$\xi'(P(\varphi + sz)) = (L' - \Phi')[P(\varphi + sz)] = (L - \Phi')[P(\varphi + sz)] = L(I - L^{-1}\Phi')[P(\varphi + sz)].$$

Consequently,

$$[\xi'(P(\varphi + sz))]^{-1} = [(I - L^{-1}\Phi')[P(\varphi + sz)]]^{-1} \circ L^{-1},$$

and therefore,

$$P'(\varphi + sz) = [(I - L^{-1}\Phi')[P(\varphi + sz)]]^{-1} \circ L^{-1} \circ \Lambda. \tag{V.41}$$

Here

$$\|P'(\varphi + sz)\|_{\mathcal{L}(L_2(0,T),H)} \le \left\|L^{-1}\right\| \|\Lambda\| \left\| \left[\left(I - L^{-1}\Phi'\right)[P(\varphi + sz)]\right]^{-1} \right\| \overset{T19}{\le}$$

$$\le \left\|L^{-1}\right\| \|\Lambda\| \cdot \frac{1}{1 - \|L^{-1}\| \, \|\Phi'(P(\varphi + sz))\|_{\mathcal{L}(H,LH)}},$$

and from the inequality $\|L^{-1}\| = 1$, we obtain

$$\|P'(\varphi + sz)\|_{\mathcal{L}(L_2(0,T),H)} \le \frac{\|\Lambda\|}{1 - \|\Phi'(P(\varphi + sz))\|_{\mathcal{L}(H,LH)}}. \tag{V.42}$$

To estimate $\|\Phi'(P(\varphi+sz))\|_{\mathcal{L}(H,LH)}$, we now use (V.27) in the form

$$P_{LH}\left(\frac{\Phi[P(\varphi+sz)+te]-\Phi[P(\varphi+sz)]}{t}\right) \leq \vartheta_n(w_n^{-1}(\|\Lambda\|\,\tilde{r}))\,\|e\|_H\,,$$

where e is an arbitrary vector in H and $t \neq 0$.

We have

$$P_{LH}(\Phi'[P(\varphi+sz)](e)) \leq \vartheta_n(w_n^{-1}(\|\Lambda\|\,\tilde{r}))\,\|e\|_H\,,$$

as $t \to 0$, which implies that

$$\|\Phi'[P(\varphi+sz)]\|_{\mathcal{L}(H,LH)} \leq \vartheta_n(w_n^{-1}(\|\Lambda\|\,\tilde{r})), \tag{V.43}$$

since e is arbitrary. If we substitute (V.43) into (V.42), then we obtain

$$\|P'(\varphi+sz)\|_{\mathcal{L}(L_2(0,T),H)} \leq \frac{\|\Lambda\|}{1-\vartheta_n(w_n^{-1}(\|\Lambda\|\,\tilde{r}))}. \tag{V.44}$$

We set $R(\varphi) = A\varphi + B[P(\varphi)]$. Then

$$R(\varphi+z)-R(\varphi) = \int\limits_\Omega (P(\varphi+z)-P'(0)(\varphi+z)-P(\varphi)+P'(0)\varphi)\Delta\omega(x)dx =$$

$$= \int\limits_\Omega (P(\varphi+z)-P'(0)z-P(\varphi))\Delta\omega(x)dx.$$

Further, we have

$$\|R(\varphi+z)-R(\varphi)\|^2_{L_2(0,T)} = \int\limits_0^T \left[\int\limits_\Omega (P(\varphi+z)-P(\varphi)-P'(0)z)\Delta\omega(x)dx\right]^2 dt =$$

$$= \int\limits_0^T \left[\int\limits_\Omega (\int\limits_0^1 [P'(\varphi+sz)-P'(0)]zds)\Delta\omega(x)dx\right]^2 dt \overset{CSBI}{\leq}$$

$$\leq \int\limits_0^T \left[\int\limits_\Omega [\int\limits_0^1 (P'(\varphi+sz)-P'(0))zds]^2dx \int\limits_\Omega |\Delta\omega(x)|^2dx\right] dt \overset{CSBI}{\leq}$$

$$\leq \int\limits_0^T \left[\int\limits_\Omega (\int\limits_0^1 |P'(\varphi+sz)-P'(0)|^2ds \int\limits_0^1 |z|^2ds)dx \int\limits_\Omega |\Delta\omega(x)|^2dx\right] dt \overset{FT}{=}$$

$$= \int\limits_\Omega |\Delta\omega(x)|^2\,dx \cdot \int\limits_0^T |z|^2\,dt \cdot \int\limits_0^T \int\limits_\Omega \int\limits_0^1 |P'(\varphi+sz)-P'(0)|^2\,dsdxdt =$$

$$= \|\Delta\omega\|^2_{2,\Omega}\, \|z\|^2_{L_2(0,T)} \int_0^1 \|P'(\varphi + sz) - P'(0)\|^2_{L_2(Q_T)}\, ds.$$

By using (V.41) and the relation $P'(0) = \eta'(0) \circ \Lambda = L^{-1} \circ \Lambda$, we obtain

$$\begin{aligned} P'(\varphi + sz) - P'(0) &= \big[(I - L^{-1}\Phi')(P(\varphi + sz))\big]^{-1} \circ L^{-1} \circ \Lambda - L^{-1} \circ \Lambda = \\ &= \Big(\big[(I - L^{-1}\Phi')(P(\varphi + sz)\big]^{-1} - I\Big) \circ L^{-1} \circ \Lambda. \end{aligned} \tag{V.45}$$

Let us now use the Neumann series expansion: if $\|S\| < 1$, then

$$(I - S)^{-1} = I + S + S^2 + S^3 + \ldots.$$

It follows from (V.40) and the fact that $P(\varphi)$ and $\Phi(\varphi)$, are continuously differentiable and $\Phi'(0) = 0$ that

$$\big\|(L^{-1}\Phi')(P(\varphi + sz))\big\| \; < \; 1$$

and

$$(I - L^{-1}\Phi')^{-1} = I + (L^{-1}\Phi') + (L^{-1}\Phi')^2 + (L^{-1}\Phi')^3 + \ldots;$$

therefore,

$$||(I - L^{-1}\Phi')^{-1} - I|| = ||L^{-1}\Phi' + (L^{-1}\Phi')^2 + (L^{-1}\Phi')^3 + \ldots|| =$$

$$= ||L^{-1}\Phi'(I + L^{-1}\Phi' + (L^{-1}\Phi')^2 + \ldots)|| \le \frac{||L^{-1}\Phi'||}{||I - L^{-1}\Phi'||}.$$

Consequently,

$$\begin{aligned} \Big\|\big[\big(I - L^{-1}\Phi'\big)\,(P(\varphi + sz))\big]^{-1} - I\Big\|_{\mathcal{L}(H,H)} &\le \frac{\|L^{-1}\Phi'(P(\varphi + sz))\|_{\mathcal{L}(H,H)}}{1 - \|L^{-1}\Phi'(P(\varphi + sz))\|_{\mathcal{L}(H,H)}} \le \\ \le \frac{\|L^{-1}\|\, \|\Phi'(P(\varphi + sz))\|_{\mathcal{L}(H,LH)}}{1 - \|L^{-1}\|\, \|\Phi'(P(\varphi + sz))\|_{\mathcal{L}(H,LH)}}. \end{aligned} \tag{V.46}$$

It follows from (V.45) and (V.46) that

$$\begin{aligned} \|P'(\varphi + sz) - P'(0)\|_H &= \Big\|(\big[\big(I - L^{-1}\Phi'\big)\,(P(\varphi + sz))\big]^{-1} - I) \circ L^{-1} \circ \Lambda\Big\| \le \\ &\le \frac{\|\Lambda\|\, \|L^{-1}\|^2\, \|\Phi'\,(P(\varphi + sz))\|_{\mathcal{L}(H,LH)}}{1 - \|L^{-1}\|\, \|\Phi'\,(P(\varphi + sz))\|_{\mathcal{L}(H,LH)}} \overset{(V.33), ||L^{-1}||=1}{\le} \\ &\le \frac{\|\Lambda\|\, \vartheta_n(w_n^{-1}(\|\Lambda\|\, \tilde{r}))}{1 - \vartheta_n(w_n^{-1}(\|\Lambda\|\, \tilde{r}))}, \end{aligned} \tag{V.47}$$

where we have used the notation

$$\zeta = \frac{\|\Lambda\|\,\vartheta_n(w_n^{-1}(\|\Lambda\|\,\tilde r))}{1-\vartheta_n(w_n^{-1}(\|\Lambda\|\,\tilde r))}. \tag{V.48}$$

Then

$$\begin{aligned}
&\|R(\varphi+z)-R(\varphi)\|^2_{L_2(0,T)} \le \|z\|^2_{L_2(0,T)}\,\|\Delta\omega\|^2_{2,\Omega}\int_0^1 \|P'(\varphi+sz)-P'(0)\|^2_{L_2(Q_T)}\,ds \overset{H\subset L_2(Q_T)}{\le}\\
&\le \|z\|^2_{L_2(0,T)}\,\|\Delta\omega\|^2_{2,\Omega}\,C^2_{H\to L_2(Q_T)}\int_0^1 \|P'(\varphi+sz)-P'(0)\|^2_{H}\,ds \overset{(V.47)}{\le}\\
&\le \|z\|^2_{L_2(0,T)}\,\|\Delta\omega\|^2_{2,\Omega}\,C^2_{H\to L_2(Q_T)}\zeta^2.
\end{aligned} \tag{V.49}$$

Then

$$\|G[P(\varphi+z)]-G[P(\varphi)]\|^2_{L_2(0,T)} = \int_0^T\left[\int_\Omega(\Phi[P(\varphi+z)]-\Phi[P(\varphi)])\omega(x)dx\right]^2 dt \overset{CSBI}{\le}$$

$$\le \int_0^T\left[\int_\Omega(\Phi[P(\varphi+z)]-\Phi[P(\varphi)])^2dx\int_\Omega|\omega(x)|^2\,dx\right]dt =$$

$$= \int_0^T\int_\Omega\left[\int_0^1\Phi'[P(\varphi+sz)]P'(\varphi+sz)zds\right]^2 dxdt\cdot\int_\Omega|\omega(x)|^2\,dx \overset{CSBI}{\le}$$

$$\le \|\omega\|^2_{2,\Omega}\int_0^T\int_\Omega\left[\int_0^1|\Phi'[P(\varphi+sz)]|^2\,|P'(\varphi+sz)|^2\,|z|^2\,ds\right]dxdt =$$

$$= \|\omega\|^2_{2,\Omega}\int_0^T|z|^2\,dt\int_0^1\|\Phi'[P(\varphi+sz)]\|^2_{L_2(Q_T)}\,ds\cdot\int_0^1\|P'(\varphi+sz)\|^2_{L_2(Q_T)}\,ds \le$$

$$\le\ (L_2(Q_T)=LH\ \text{(because of (V.7)) and (V.43)})\ \le$$

$$\le \|\omega\|^2_{2,\Omega}\,\|z\|^2_{L_2(0,T)}\,\vartheta_n^2(w_n^{-1}(\|\Lambda\|\,\tilde r))\int_0^1\|P'(\varphi+sz)\|^2_{L_2(Q_T)}\,ds \overset{ET}{\le}$$

$$\leq \|\omega\|^2_{2,\Omega} \|z\|^2_{L_2(0,T)} \vartheta^2_n(w_n^{-1}(\|\Lambda\| \tilde{r}))C^2_{H\to L_2(Q_T)} \int\limits_0^1 \|P'(\varphi + sz)\|^2_H \, ds \overset{(V.44),\ (V.48)}{\leq}$$

$$\leq C^2_{H\to L_2(Q_T)}\zeta^2 \|\omega\|^2_{2,\Omega} \|z\|^2_{L_2(0,T)} . \tag{V.50}$$

Now relations (V.49) and (V.50) imply that

$$\left\|\hat{\Psi}(\varphi + z) - \hat{\Psi}(\varphi)\right\|_{L_2(0,T)} \leq \|G[P(\varphi + z)] - G[P(\varphi)]\|_{L_2(0,T)} + \|R(\varphi + z) - R(\varphi)\|_{L_2(0,T)} \leq$$

$$\leq C_{H\to L_2(Q_T)}\zeta \cdot (\|\omega\|_{2,\Omega} + \|\Delta\omega\|_{2,\Omega}) \|z\|_{L_2(0,T)} .$$

Therefore,

$$P_{\hat{A}U_{L_2(0,T)}}(\hat{\Psi}(\varphi + z) - \hat{\Psi}(\varphi)) \leq \hat{\Theta}_n(\tilde{r}) \|z\|_{L_2(0,T)} ,$$

where

$$\hat{\Theta}_n(\tilde{r}) = \left\|\hat{A}^{-1}\right\| C_{H\to L_2(Q_T)}(\|\omega\|_{2,\Omega} + \|\Delta\omega\|_{2,\Omega}) \frac{\|\Lambda\| \vartheta_n(w_n^{-1}(\|\Lambda\| \tilde{r}))}{1 - \vartheta_n(w_n^{-1}(\|\Lambda\| \tilde{r}))}.$$

Further, it follows from (V.28) and (V.48) that

$$\vartheta_n(w_n^{-1}(\|\Lambda\| \tilde{r})) = a_n \cdot [w_n^{-1}(\|\Lambda\| \tilde{r})]^{\alpha_1 - 1} + b_n \cdot [w_n^{-1}(\|\Lambda\| \tilde{r})]^{\alpha_2 - 1},$$

and then

$$\hat{\Theta}_n(\tilde{r}) = \left\|\hat{A}^{-1}\right\| C_{H\to L_2(Q_T)}(\|\omega\|_{2,\Omega} + \|\Delta\omega\|_{2,\Omega}) \frac{\|\Lambda\| \{a_n[w_n^{-1}(\|\Lambda\| \tilde{r})]^{\alpha_1 - 1} + b_n[w_n^{-1}(\|\Lambda\| \tilde{r})]^{\alpha_2 - 1}\}}{1 - (a_n[w_n^{-1}(\|\Lambda\| \tilde{r})]^{\alpha_1 - 1} + b_n[w_n^{-1}(\|\Lambda\| \tilde{r})]^{\alpha_2 - 1})}.$$

In this case, in theorem 18, one can set $A = \hat{A}$, $X = L_2(0,T)$, $Y = L_2(0,T)$ with the unit ball $\hat{A}U_{L_2(0,T)}$, $b(r) = \hat{B}_n(\tilde{r})$, $\Psi = \hat{\Psi}$ and $\Theta(r) = \hat{\Theta}_n(\tilde{r})$, where $\hat{A}$ is given by (V.38) and $\hat{\Psi}$ is given by (V.39), and

$$\hat{B}_n(\tilde{r}) = \max\left[1 - \hat{\Theta}_n(\tilde{r}), 0\right].$$

If $\tilde{r} \in [0, \tilde{r}_*[$, then we take

$$\hat{B}_n(\tilde{r}) = 1 - \hat{\Theta}_n(\tilde{r}),$$

where $\tilde{r}_*$ is the root of the equation $1 - \hat{\Theta}_n(\tilde{r}) \ = \ 0$.

Moreover, we set $w(r) = \hat{W}_n(\tilde{r})$,where

$$\hat{W}_n(\tilde{r}) = \int\limits_0^{\tilde{r}} b(t)dt = \tilde{r} - \left\|\hat{A}^{-1}\right\| C_{H\to L_2(Q_T)}(\|\omega\|_{2,\Omega} + \|\Delta\omega\|_{2,\Omega}) \|\Lambda\| \cdot$$
$$\cdot \int\limits_0^{\tilde{r}} \frac{a_n[w_n^{-1}(\|\Lambda\|\, t)]^{\alpha_1-1} + b_n[w_n^{-1}(\|\Lambda\|\, t)]^{\alpha_2-1}dt}{1 - a_n[w_n^{-1}(\|\Lambda\|\, t)]^{\alpha_1-1} - b_n[w_n^{-1}(\|\Lambda\|\, t)]^{\alpha_2-1}}. \tag{V.51}$$

If $\tilde{r} \in [\tilde{r}_*, \infty]$, then we take $\hat{B}_n(\tilde{r}) = 0$ and $\hat{W}_n(\tilde{r}) = \hat{W}_n(\tilde{r}_*)$.

Theorem 26 *Let $n \geq 2$,*

$$1 < \alpha_1 \leq \alpha_2 \leq (n+1)/(n-1)$$

and let the assumptions of theorem 23 be valid.

Further, let a_n, b_n, w_n, and $\hat{W}_n(\tilde{r})$ be given by (V.29), (V.30), (V.32), and (V.51), and let $\tilde{r}_*$ be the root of the equation

$$1 - \hat{\Theta}_n(\tilde{r}) \ = \ 0.$$

Then for any $\tilde{r} \in [0, \tilde{r}_*[$ and $\Psi \in \hat{W}_n(\tilde{r})\hat{A}U_{L_2(0,T)}$ there exists a unique $\varphi \in \tilde{r}U_{L_2(0,T)}$ such that

$$M(\varphi) = \Psi.$$

Consequently, the operator equation $M(\varphi) = \Psi$ is uniquely solvable, and hence the equivalent controllability problem (V.11)-(V.13), (V.4) also has a unique solution in the same neighborhood.

Chapter VI

Controllability for the KZK Equation

Table of contents

Having a goal to prove a controllability of moments result for the KZK equation first of all we need the results of well-posedness of the direct and inverse problems for the linear equation, which can be obtained from the KZK equation by omitting the nonlinear term uu_x.

VI.1 The direct problem for linearized KZK equation

As it can be easily seen, thanks to proved estimates for full KZK equation and to the proof of the theorem 4 of existence and uniqueness of the solution, the problem

$$u_t - \beta u_{xx} - \gamma \partial_x^{-1} \triangle_y u = F(x,y,t), \tag{VI.1}$$

$$u|_{t=0} = u_0, \quad u(x+L,y,t) = u(x,y,t), \quad \int_0^L u dx = 0 \tag{VI.2}$$

has a unique global solution in H^s for all $s \geq 0$. In particular for the homogeneous equation it follows from the estimate

$$\frac{d}{dt}\|u\|_{H^s}^2 \leq \beta C(L)\|u\|_{H^s}^2,$$

which takes place for all $s \geq 0$, and it follows also from the fact that the operator $\partial_x^{-1}\triangle_y$ is generator of a unitary C_0-group in L_2 with mean value zero [39, p.41] and this unitary group $e^{-t\partial_x^{-1}\triangle_y}$ preserves the H^s norm. For nonhomogeneous problem we can use the theorem from [39, p.107], supposing $F \in C^1([0,T], H^s)$ $(T \leq \infty)$. Then for the solution $u \in C([0,T], H^s) \cap C^1([0,T], H^{s-2})$ $(s-2 \geq 0)$ of the problem (VI.1) the Cauchy formula holds

$$u = S(t)u_0 + \int_0^t S(t-s)F ds,$$

which gives the estimate

$$\|u\|_{C^1([0,T],H^{s-2})} \leq C(\|u_0\|_{H^s} + \|F\|_{C^1([0,T],H^s)}) \tag{VI.3}$$

for some $s \geq 0$.

On the other hand it can be easily shown with the estimate

$$\frac{d}{dt}\|u\|_{L_2(\Omega)} + \frac{\beta}{C^2(\Omega)}\|u\|_{L_2(\Omega)} \leq \|F(\cdot,\cdot,t)\|_{L_2(\Omega)}$$

(see (VI.17)) and the Galerkin method as in [31] that for all $F(x,y,t) \in L_2((0,T), H^2(\Omega))$ and $u_0 \in H^2(\Omega)$ there exists a unique solution of (VI.1) $u \in W_{2,0}^{0,1}(Q_T)$ such that (see [31, pp.167,189])

$$\|u\|_{H^{0,1}(Q_T)} \leq C(\|u_0\|_{H^2(\Omega)} + \|F\|_{L_2((0,T),H^2(\Omega))}). \tag{VI.4}$$

To obtain the result of nonlinear controllability for the KZK equation it is natural to use the fact that for all $F(x, y, t) \in L_2((0, T); H^s(\Omega))$ and $u_0 \in H^s(\Omega)$ there exists unique solution of (VI.1) $u \in W_{2,0}^{s-2,1}(Q_T)$ such that

$$\|u\|_{H^{s-2,1}(Q_T)} \leq C(\|u_0\|_{H^s(\Omega)} + \|F\|_{L_2((0,T);H^s(\Omega))}). \tag{VI.5}$$

VI.2 The inverse problem for linearized KZK equation

We consider the controllability problem for the equation (VI.1) in a domain $Q_T = [0, T] \times \Omega_{x,y}$, where $\Omega_{x,y}$ can be bounded domain: $\Omega_{x,y} = [0, L] \times \Omega_y$ with some $\Omega_y \subset \mathcal{R}^{n-1}$; or can be unbounded domain $\Omega = \Omega_{x,y} = \mathcal{R}/(L\mathcal{Z}) \times \Omega_y$ with $\Omega_y \subseteq \mathcal{R}^{n-1}$. The boundary of Ω is denoted $\partial\Omega$. If $\Omega = \Omega_{x,y}$ is not bounded, than the constant of Poincaré-Friedrichs in the proof of the theorem 27 must be replaced by a constant of the periodicity on x $C(L)$. So we envisage the controllability problem (VI.1) in a domain $Q_T = [0, T] \times \Omega_{x,y}$ with an additional condition, called the condition of overdetermination,

$$\int_0^L \int_\Omega u(x, y, t)\omega(x, y)dxdy = \chi(t), \tag{VI.6}$$

with homogeneous boundary conditions and mean value zero on x in the case of bounded domain

$$u|_{t=0} = 0, \quad u|_{\partial\Omega} = 0, \quad \int_0^L udx = 0, \tag{VI.7}$$

and with

$$u|_{t=0} = 0, \quad u(x + L, y, t) = u(x, y, t), \quad \int_0^L udx = 0, \tag{VI.8}$$

for unbounded domain (if it is bounded on y we always suppose that $u|_{\partial\Omega_y} = 0$).

We suppose in what follows

$$F(x, y, t) = h(x, y, t)f(t). \tag{VI.9}$$

Here the functions h, ω, χ are imposed, and f - is an unknown function, which we call the control.

Remark 29 *The problem (VI.1), (VI.7), (VI.6) can be easily, thanks to its linearity, generalized on nonhomogeneous case of following form*

$$u_t - \beta u_{xx} - \gamma\partial_x^{-1}\triangle_y u = h(x, y, t)f(t) + g(x, y, t)$$
$$u|_{t=0} = u_0(x, y), \quad u|_{\partial\Omega} = u_1(x, y, t), \quad \int_0^L udx = 0,$$
$$\int_0^L \int_\Omega u(x, y, t)\omega(x, y)dxdy = \chi(t)$$

if the known functions g, u_0, u_1 are sufficiently smooth and the matching condition

$$\int_0^L \int_\Omega u_0(x, y)\omega(x, y)dxdy = \chi(0)$$

is satisfied.

Definition 5 *The function $f \in L_2(0,T)$ is called by the solution of the inverse problem (VI.1), (VI.7) (or (VI.8)), (VI.6) if the solution u of the problem (VI.1), (VI.7) (or (VI.8)) with this f satisfies the condition of overdetermination (VI.6) almost everywhere on $[0,T]$.*

We define in $L_2(0,T)$ an equivalent norm by the expression

$$\|f\|^2_{L_2(0,T)} = \int_0^T e^{-\alpha t}|f(t)|^2 dt, \tag{VI.10}$$

where $\alpha > 0$ is some number the choice of which will be done later.

Theorem 27 *Suppose that $\omega \in H^2(\Omega_{xy}) \cap \overset{\circ}{H}{}^1(\Omega_{xy})$, $\chi \in H^1(0,T)$, $\chi(0) = 0$. Further, let $h(x,t)$ be a function such that $h \in L_2(Q_T)$, $\|h(\cdot,\cdot,t)\|_{L_2(\Omega_{xy})}$ is bounded on $[0,T]$ and $\left|\int_{\Omega_{xy}} h(x,y,t)\omega(x,y)dxdy\right| \geq \delta > 0$ for almost all $t \in [0,T]$. Then there exists a unique solution of the problem (VI.1), (VI.7) (or (VI.8)), (VI.6) and the stability estimate holds:*

$$\delta \|f\|_{L_2(0,T)} \leq \|\chi'\|_{L_2(0,T)} /(1-m), \tag{VI.11}$$

where

$$m = C_1 \left(\beta\|\omega_{xx}\|_{L_2(\Omega_{xy})} + \gamma\|\partial_x^{-1}\Delta_y\omega\|_{L_2(\Omega_{xy})}\right) \frac{e^{\frac{T}{2}}}{(\alpha + 2/C^2(\Omega))^{\frac{1}{2}}},$$

$C_1 > 0$ such that

$$\frac{\|h(\cdot,\cdot,t)\|_{L_2(\Omega_{xy})}}{(h,\omega)_{L_2(\Omega_{xy})}} \leq C_1,$$

$C(\Omega)$ is a constant of Poincaré-Friedrichs (in the case of unbounded domain $C(\Omega)$ is replaced by $C(L)$) , $\alpha > 0$ is chosen from the condition $m < 1$.

Proof. It follows from equation (VI.1) that

$$\left(u_t - \beta u_{xx} - \gamma\partial_x^{-1}\Delta_y u, \omega\right)_{L_2(\Omega_{xy})} = (h,\omega)_{L_2(\Omega_{xy})} f \tag{VI.12}$$

for almost all $t \in [0,T]$.

We set

$$\varphi(t) = (h,\omega)_{L_2(\Omega_{xy})} f.$$

By virtue of the assumptions of theorem 27, f can be uniquely determined on the basis of φ. Let us assume that the solution of the problem exists and derive an operator equation. As $(u,\omega)_{L_2(\Omega_{xy})} \in H^1(0,T)$, then

$$(u_t,\omega)_{L_2(\Omega_{xy})} = d/dt(u,\omega)_{L_2(\Omega_{xy})} = \chi'(t).$$

Since $u \in W^{2,1}_{2,0}(Q_T)$, $\omega \in H^2(\Omega) \cap \overset{\circ}{H}{}^1(\Omega)$, it follows that

$$\left(-\gamma\partial_x^{-1}\Delta_y u, \omega\right)_{L_2(\Omega_{xy})} = \left(u, \gamma\partial_x^{-1}\Delta_y\omega\right)_{L_2(\Omega_{xy})},$$

$$(-\beta u_{xx},\omega)_{L_2(\Omega_{xy})} = -\left(u,\beta\omega_{xx}\right)_{L_2(\Omega_{xy})}$$

for $t \in [0,T]$.

Then relation (VI.12) implies that

$$\chi'(t) + \int\limits_0^L \int_{\Omega_y} u(x,y,t)(-\beta\omega_{xx} + \gamma\partial_x^{-1}\Delta_y\omega)(x,y)dxdy = \varphi(t).$$

We denote $\Psi(t) = \chi'(t)$, $\Psi \in L_2(0,T)$.

We introduce a linear operator

$$A : L_2(0,T) \to L_2(0,T),$$
$$(A\varphi)(t) = \int_0^L \int_{\Omega_y} u(x,y,t)(-\beta\omega_{xx} + \gamma\partial_x^{-1}\Delta_y\omega)(x,y)dxdy, \tag{VI.13}$$

which map φ according to the following way : $\varphi \to f \to u \to A\varphi$, where the solution of inverse problem (VI.1), (VI.7), (VI.6) is found by the given f, and f is found by the formula

$$f(t) = \frac{\varphi(t)}{(h,\omega)_{L_2(\Omega_{xy})}}.$$

Consequently, we obtain the operator equation

$$\varphi - A\varphi = \Psi. \tag{VI.14}$$

Let us prove that $A \in \mathcal{L}(L_2(0,T))$ and $\|A\| < 1$.

We estimate $\|A\|$ for all $\varphi \in L_2(0,T)$

$$\|A\varphi\|^2_{L_2(0,T)} = \int_0^T e^{-\alpha t}\left(\int_0^L \int_{\Omega_y} u(x,y,t)(-\beta\omega_{xx} + \gamma\partial_x^{-1}\Delta_y\omega)(x,y)dxdy\right)^2 dt. \tag{VI.15}$$

By Cauchy-Schwarz-Bunyakovkii inequality we have for $t \in [0,T]$

$$\left(\int_0^L \int_{\Omega_y} u(x,y,t)(-\beta\omega_{xx} + \gamma\partial_x^{-1}\Delta_y\omega)(x,y)dxdy\right)^2 \leq$$

$$\leq \left(\beta\|\omega_{xx}\|_{L_2(\Omega_{xy})} + \gamma\|\partial_x^{-1}\Delta_y\omega\|_{L_2(\Omega_{xy})}\right)^2 \|u(\cdot,\cdot,t)\|^2_{L_2(\Omega_{xy})}.$$

We substitute it into (VI.15), and obtain:

$$\|A\varphi\|^2_{L_2(0,T)} \leq \left(\beta\|\omega_{xx}\|_{L_2(\Omega_{xy})} + \gamma\|\partial_x^{-1}\Delta_y\omega\|_{L_2(\Omega_{xy})}\right)^2 \int_0^T e^{-\alpha t}\|u(\cdot,\cdot,t)\|^2_{L_2(\Omega_{xy})}dt. \tag{VI.16}$$

We find the estimate for $\|u(\cdot,\cdot,t)\|_{L_2(\Omega_{xy})}$ (see [31, p.167], [27, p.47]). Taking the inner product in $L_2(\Omega_{xy})$ with u the equation (VI.1), we have, noting in what follows $\Omega = \Omega_{xy}$,

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L_2(\Omega)} + \beta\|u_x\|^2_{L_2(\Omega)} \leq \|u\|_{L_2(\Omega)}\|f(t)h(\cdot,\cdot,t)\|_{L_2(\Omega)}.$$

Since the domain Ω is bounded in $\mathcal{R}^{n+1}$, we can apply the equality of Poincaré-Friedrichs, from which it is obviously following that

$$\frac{1}{C^2(\Omega)}\|u\|^2_{L_2(\Omega)} \leq \|u_x\|^2_{L_2(\Omega)}$$

and

$$\|u\|_{L_2(\Omega)}\frac{d}{dt}\|u\|_{L_2(\Omega)} + \frac{\beta}{C^2(\Omega)}\|u\|^2_{L_2(\Omega)} \leq \|u\|_{L_2(\Omega)}\|f(t)h(\cdot,\cdot,t)\|_{L_2(\Omega)}.$$

And then

$$\frac{d}{dt}\|u\|_{L_2(\Omega)} + \frac{\beta}{C^2(\Omega)}\|u\|_{L_2(\Omega)} \leq \|f(t)h(\cdot,\cdot,t)\|_{L_2(\Omega)}. \tag{VI.17}$$

The last inequality can be rewritten with the help of integrable factor in the way

$$\frac{d}{dt}\left(e^{\frac{\beta}{C^2(\Omega)}t}\|u\|_{L_2(\Omega)}\right) \leq e^{\frac{\beta}{C^2(\Omega)}t}|f(t)|\|h(\cdot,\cdot,t)\|_{L_2(\Omega)}.$$

Since the initial data has been chosen equivalent to zero, in the end we obtain

$$\|u\|_{L_2(\Omega)} \leq e^{-\frac{\beta}{C^2(\Omega)}t}\int_0^t e^{\frac{\beta}{C^2(\Omega)}\tau}|f(\tau)|\|h(\cdot,\cdot,\tau)\|_{L_2(\Omega)}d\tau. \tag{VI.18}$$

Let us transform the right-hand part of inequality in such way that it depends on the function $\varphi(t)$. For it we multiply and divide on $\int_\Omega |h(x,y,t)\omega(x,y)|dxdy$, using the assumption of the theorem about separability from zero of this integral, and using the fact of the existence of a positive constant C_1 such that $\|h(\cdot,\cdot,t)\|_{L_2(\Omega_{xy})}/(h,\omega)_{L_2(\Omega_{xy})} \leq C_1$:

$$\begin{aligned}
&\int_0^t e^{\frac{\beta}{C^2(\Omega)}\tau}|f(\tau)|\|h(\cdot,\cdot,\tau)\|_{L_2(\Omega)}d\tau = \\
&= \int_0^t e^{\frac{\beta}{C^2(\Omega)}\tau}\frac{\|h(\cdot,\cdot,\tau)\|_{L_2(\Omega)}}{\int_\Omega |h(x,y,\tau)\omega(x,y)|dxdy}\left(|f(\tau)|\int_\Omega |h(x,y,\tau)\omega(x,y)|dxdy\right)d\tau \\
&\leq C_1\int_0^t e^{\frac{\beta}{C^2(\Omega)}\tau}|\varphi(\tau)|d\tau.
\end{aligned}$$

I.e. (VI.18) takes the form

$$\|u\|_{L_2(\Omega)} \leq C_1 e^{-\frac{\beta}{C^2(\Omega)}t}\int_0^t e^{\frac{\beta}{C^2(\Omega)}\tau}|\varphi(\tau)|d\tau. \tag{VI.19}$$

Let us envisage the integral in the right-hand side of the inequality (VI.19)

$$\int_0^t e^{\frac{\beta}{C^2(\Omega)}\tau}|\varphi(\tau)|d\tau \leq \left(\int_0^t e^{(\frac{2\beta}{C^2(\Omega)}+\alpha)\tau}d\tau\right)^{\frac{1}{2}}\|\varphi\|_{L_2(0,T)}. \tag{VI.20}$$

Returning now to (VI.16), with the help of (VI.19), (VI.20), we obtain that

$$\|A\varphi\|^2_{L_2(0,T)} \leq N^2\frac{C_1^2}{\frac{2\beta}{C^2(\Omega)}+\alpha}\|\varphi\|^2_{L_2(0,T)}\int_0^T\left(e^{\frac{2\beta}{C^2(\Omega)}t} - e^{-\alpha t}\right)dt \leq$$

$$\leq N^2 C \frac{C_1^2}{\frac{2\beta}{C^2(\Omega)} + \alpha} e^T \|\varphi\|^2_{L_2(0,T)},$$

where $N = \left(\beta\|\omega_{xx}\|_{L_2(\Omega_{xy})} + \gamma\|\partial_x^{-1}\Delta_y\omega\|_{L_2(\Omega_{xy})}\right)^2$.

From where we conclude that $A \in \mathcal{L}(L_2(0,T))$ and there exists $\alpha > 0$ such that $\|A\| < 1$.

The condition $\|A\| < 1$ guarantees the one-valued solvability in $L_2(0,T)$ of the operator equation (VI.14).

Let us prove that (VI.14) is equivalent to the inverse problem (VI.1), (VI.7), (VI.6). Indeed, let φ is a solution of the equation (VI.14) with given in condition of the theorem function χ. We unequivocally define $f = \varphi/(h,\omega)_{L_2(\Omega)}$. By virtue of the assumptions of theorem 27 $f \in L_2(0,T)$. Let us show that the solution $u \in W^{2,1}_{2,0}(Q_T)$ founded by f of the direct problem (VI.1), (VI.7) satisfies the condition of overdetermination (VI.6).

Assume the converse:

$$\int_0^L \int_\Omega u(x,y,t)\omega(x,y)dxdy = \chi_1(t) \in H^1(0,T).$$

Since $u|_{t=0} = 0$, then $\chi_1(0) = 0$. Deriving the operator equation for these φ and χ_1, we find that φ also satisfies the equation

$$\varphi - A\varphi = \chi_1'. \tag{VI.21}$$

We subtract (VI.14) from (VI.21), and we obtain

$$\chi'(t) = \chi_1'(t), \quad \chi_1(0) = \chi(0) = 0.$$

Then $\chi(t) = \chi_1(t)$, $t \in [0,T]$, which contradicts the original assumption.

Let us prove that the solution of the problem (VI.1), (VI.7), (VI.6) is unique. Assume the converse. Then repeating the derivation of the operator equation (VI.14) for the difference $u - u_1$, we obtain that φ satisfies the homogeneous equation.

By virtue of the uniqueness of solution of the operator equation (VI.14) we obtain $f = 0$. By virtue of the uniqueness of solution of the direct problem $u - u_1 = 0$.

Let us show now the stability estimate (VI.11).

Indeed, if we envisage the relation between φ and f, then we can notice that

$$\|\varphi\|^2_{L_2(0,T)} = \int_0^T e^{-\alpha t}\left|\int_\Omega h(x,y,t)\omega(x,y)dxdy\right|^2 |f(t)|^2 dt \geq$$
$$\geq \delta^2\|f\|^2_{L_2(0,T)}.$$

Since $\|A\varphi\| \leq m\|\varphi\|$ for all $\varphi \in L_2(0,T)$, i.e.,

$$\|A\| = m,$$

and $\varphi - A\varphi = \chi'$, we obtain that

$$(1-m)\|\varphi\|_{L_2(0,T)} = \|\chi'\|_{L_2(0,T)},$$

from where it follows (VI.11). This completes the proof of the theorem 27.$\square$

We note that if we suppose in assumptions of the theorem 27 the additional regularity on (x, y) we obtain the following theorem.

Theorem 28 *Suppose that for $s > [\frac{n}{2}] + 1$*

$$\omega \in H^{2s-2}(\Omega_{xy}) \cap \overset{\circ}{H}{}^{s-1}(\Omega_{xy}),$$

$\chi \in H^1(0,T)$, $\chi(0) = 0$ and, let $h(x,t)$ be a function such that $h \in (L_2((0,T); H^{s-2}(\Omega_{xy}))$, $\|h(\cdot,\cdot,t)\|_{H^{s-2}(\Omega_{xy})}$ is bounded on $[0,T]$ and

$$(h(x,y,t), \omega(x,y))_{H^{s-2}(\Omega_{xy})} \geq \delta > 0$$

for almost all $t \in [0,T]$. Then there exists a unique solution of the problem (VI.1), (VI.7), (or (VI.8)) and

$$(u(x,y,t), \omega(x,y))_{H^{s-2}(\Omega_{xy})} = \chi(t), \tag{VI.22}$$

which is equivalent, thanks to the smoothness of ω, to

$$\int_{\Omega_{x,y}} u(x,y,t)\Lambda^{2(s-2)}\omega(x,y)dxdy = \chi(t),$$

and then the stability estimate holds:

$$\delta\,\|f\|_{L_2(0,T)} \leq \|\chi'\|_{L_2(0,T)}/(1-m),$$

where

$$m = C_1\left(\beta\|\omega_{xx}\|_{H^{s-2}(\Omega_{xy})} + \gamma\|\partial_x^{-1}\Delta_y\omega\|_{H^{s-2}(\Omega_{xy})}\right)\frac{e^{\frac{T}{2}}}{(\alpha + 2/C^2(\Omega))^{\frac{1}{2}}},$$

$C_1 > 0$ such that

$$\frac{\|h(\cdot,\cdot,t)\|_{H^{s-2}(\Omega_{xy})}}{(h,\omega)_{H^{s-2}(\Omega_{xy})}} \;\leq\; C_1,$$

$C(\Omega)$ is a constant of Poincaré-Friedrichs,(or it is the periodicity constant $C(L)$) $\alpha > 0$ is chosen from the condition $m < 1$.

Remark 30 *To prove the theorem it is sufficient to replace the norm in $L_2(\Omega)$ in theorem 27 by $\|\cdot\|_{H^{s-2}} = \|\Lambda^{s-2}\cdot\|_{L_2}$.*

VI.3 The difficulty on the way to the controllability for nonlinear KZK equation

We consider now the inverse problem in the domain $(0,T)\times\Omega$ with $S_T=\partial\Omega\times[0,T]$:

$$u_t - uu_x - \beta u_{xx} - \gamma\partial_x^{-1}\triangle_y u = F(x,y,t), \tag{VI.23}$$

$$u|_{t=0}=0, \quad u|_{S_T}=0, \quad \int_0^L u dx = 0 \text{ (if } S_T=\emptyset \text{ then}$$

$$u(x+L,y,t)=u(x,y,t)), \tag{VI.24}$$

$$(u, w(x,y))_{H^{s-2}} = \chi(t), \tag{VI.25}$$

with F from (VI.9).

Having the result of previous section for linear part of KZK in the form of the theorem 28, we would like to use the method of two times application of inverse function theorem demonstrated in the chapters IV and V.

Remark 31 *Unfortunately we cannot use the result of the theorem 27 because a simple reason: for construction of the space of solutions H for linear direct problem for (VI.1) we need an isomorphism $Lu=F\in L_2(Q_T)$, but we have it only for $u\in W_{2,0}^{0,1}$ which is insufficient to control the nonlinearity $\|uu_x\|_{L_2}$.*

From [8, p. 100] we have the estimate

$$\|\Phi(u)\|_{L_2(\Omega)} = \|uu_x\|_{L_2(\Omega)} \leq C\|u\|^2_{H^1(\Omega)} \text{ for } u\in H^{s'}, \quad s' > [\frac{n}{2}]+1,$$

which requests to have the solutions more regular on (x,y).

So the idea is to use the theorem 28.

We can introduce now the operator

$$L=\partial_t - \beta\partial_x^2 - \gamma\partial_x^{-1}\triangle_y, \tag{VI.26}$$

and the space of the solutions of linear direct problem

$$H=\{v\in H^1((0,T);H^{s-2}(\Omega)) | \exists F\in L_2((0,T);H^s(\Omega)) : v \text{ is a solution of problem (VI.1), (VI.7) with } u_0=0\} \tag{VI.27}$$

with the norm $\|v\|_H = \|Lv\|_{L_2((0,T);H^s(\Omega))}$. Then

$$L: H\to L_2((0,T);H^s(\Omega)) \text{ is an isometric isomorphism of } H \text{ on } L_2((0,T);H^s(\Omega)) \tag{VI.28}$$

by virtue of the a priori estimate (VI.5) and that for all $F(x,y,t)\in L_2((0,T);H^s(\Omega))$ $(u_0=0)$ there exists unique solution of (VI.1) $u\in W_{2,0}^{s-2,1}(Q_T)$.

Note that $H=\{v\in H^1((0,T);H^{s-2}(\Omega)) | v|_{t=0}=0,\ v|_{S_T}=0\}$, and $\|\cdot\|_H$ is equivalent to $\|\cdot\|_{W_{2,0}^{s-2,1}(Q_T)}$.

This implies that $(H,\|\cdot\|_H)$ is complete, the embedding $H\subset W_{2,0}^{s-2,1}(Q_T)$ is continuous.

Since from theorem 28 (and 27) the control f (and also φ) is from $L_2(0,T)$, then for applying the known technique we need that the operator defined as

$$\Phi(u) = uu_x, \quad \Phi : H^1((0,T); H^{s-2}(\Omega)) \to L_2((0,T); H^s(\Omega))$$

was strictly differentiable, which, it is seems, is impossible to prove.

Bibliography

[1] *S.I. Aanonsen et al.* Distortion and harmonic generation in the nearfield of finite amplitude sound beam. J. Acoust. Soc. Am., vol. 75, No. 3, 1984, p. 749-768.

[2] *Akpata, E.* Controllabillity in Nonlinear Parabolic Problems, Cand. Sci. (Phys..Math.) Dissertation, Moscow, 1999.

[3] *S.Alinhac* A minicourse on global existence and blowup of classical solutions to multidimensional quasilinear wave equations. Journées Équations aux dérivées partielles, Forges-les-Eaux, 3-7 juin 2002.

[4] *S. Alinhac* Explosion des solutions d'une équation d'ondes quasi-linéaire en deux dimensions d'espace. Comm. PDE 21 (5,6), 1996, p.923-969.

[5] *S. Alinhac* Blowup for nonlinear hyperbolic equations, Progress in Nonlinear Differential Equations and their Applications, Birkhäuser, Boston, 1995.

[6] *S. Alinhac* Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions, II. Acta Math., 182 (1999), 1-23.

[7] *S. Alinhac* Blowup of small data solutions for a quasilinear wave equation in two space dimensions, Annals of Math., 149 (1999), 97-127.

[8] *S. Alinhac, P. Gérard* Opérateurs pseudo-differentiels et théorème de Nash-Moser. InterEditions, Paris, 1991.

[9] *M.A. Averkiou, Y.-S. Lee, M.F. Hamilton* Self-demodulation of amplitude- and frequency-modulated pulses in a thermoviscous fluid. J. Acoust. Soc. Am. 94 (5), 1993, p. 2876-2883.

[10] *M.A. Averkiou, M.F. Hamilton* Nonlinear distortion of short pulses radiated by plane and focused circular pistons. J. Acoust. Soc. Am. 102 (5), Pt.1, 1997, p. 2539-2548.

[11] *N.S. Bakhvalov, Ya. M. Zhileikin and E.A. Zabolotskaya* Nonlinear Theory of Sound Beams, American Institute of Physics, New York, 1987 (Nelineinaya teoriya zvukovih puchkov, Moscow "Nauka", 1982).

[12] *Bardos C., Rozanova A.* KZK equation. International Conference Crimean Autumn Mathematical School-Symposium (KROMSH-2004) Ukraine, Crimea, Laspi-Batiliman, September 18-29, 2004. p.154-159.

[13] *Bardos C., Rozanova A.* KZK equation. International conference and workshop "Function Spaces, Approximation Theory, Nonlinear Analysis" dedicated to the cen-

tennial of Sergei Mikhailovich Nikolskii, Russian Academy of Sciences V. A. Steklov Mathematical Institute, Moscow, Russia, May 23-29, 2005, p.258.

[14] *Bardos C., Rozanova A.* Khokhlov-Zabolotskaya-Kuznetsov Equation. The Fourth International Conference on Differential and Functional Differential Equations, Moscow, Russia, August 14-21, 2005, p.19.

[15] *C. Dafermos* Hyperbolic Conservation Laws. Continuum Physics, Springer-Verlag, 2000, Vol. 325.

[16] *P. Donnat, J.-L. Joly, G. Metivier, J. Rauch* Diffractive nonlinear geometric optics, in Séminaire Équations aux Dérivées Partielles, École Polytechnique, 1995-1996, XVII 1-23, available at `http://www.math.lsa.umich.edu/~rauch`.

[17] *P. Donnat, J.-L. Joly, G. Metivier, J. Rauch* Diffractive nonlinear geometric optics with rectification, Indiana Univ. Math. J. 47, (1998), 1167-1241.

[18] *Faddeev, D.K., Vulikh, B.Z., Uraltseva, N.N., et al.* Izbrannye glavy analiza i vysshei algebry (Selected Chapters of Analysis and Higher Algebra), Leningrad, 1981.

[19] *K.O. Friedrichs and P.D. Lax* Systems of conservatio equations with a convex extension. Proc. Natl. Acad. Sci. USA 68, (1971), 1686-1688.

[20] *B. Gustafsson, A. Sundström* Incompletely parabolic problems in fluid dynamics. SIAM J. Appl. Math. Vol. 35, No. 2, (1978), 343-357.

[21] *P. Isaza, J. Mejia, V. Stallbohm* Local solution for the Kadomtsev-Petviashvili equation in R^2. J. of mathematical Analysis and Applications 196, 1995, p. 566-587.

[22] *P. Isaza, J. Mejia, V. Stallbohm* Local solution for the Kadomtsev-Petviashvili equation with periodic conditions. Manuscripta Math. 75, 1992, p. 383-393.

[23] *T. Kato* Quasilinear Equations of Evolution, with Applications to partial differential equations, in "Lecture Notes in Mathematics No. 448," pp. 27-50, Springer-Verlag, Berlin/ New York, 1975.

[24] *T. Kato* On the Korteweg-de Vries equation, Manuscripta Math. 28,1979,p. 89-99.

[25] *T. Kato, G. Ponce* Commutator Estimates and the Euler and Navier-Stokes Equations. Comm. Pure Appl. Math., 1988, p. 891-907.

[26] *A. Kitkauskaite, A. Kopustinskas* Nonlinear Acoustics Caused of Medium. available at `http://www.etf.rtu.lv/Latvieshu%20lapa/pasn_str/konf/p_7.pdf`

[27] *Kostin, A.B.* Obratnye zadachi dlya matematicheskikh modelei fizicheskikh protsessov (Inverse Problems for Mathematical Models of Physical Processes), Moscow, 1991.

[28] *I. Kostin, G. Panasenko* "Analysis and homogenization of the Khokhlov-Zabolotskaya-Kuznetsov type equation" oral communication in the International Conference on Advanced Problems in Mechanics, June 2005, St Petersburg.

[29] *I. Kostin, G. Panasenko* Khokhlov-Zabolotskaya-Kuznetsov type equation: nonlinear acoustics in heterogeneous media. CR Mécanique (submitted).

[30] *Krein S.G.* Linear Differential Equations in Banach Space [in Russian], Nauka, Moscow, 1967.

[31] *Ladyzhenskaya, O.A.* Kraevye zadachi matematicheskoi fiziki (Boundary Value Problems of Mathematical Physics, New York; Berlin ; London Springer-Verlag. 1985), Moscow, 1973.

[32] *Ladyzhenskaya, O.A., Solonnikov, V.A., and Uraltseva, N.N.* Lineinye i kvazilineinye uravneniya parabolicheskogo tipa (Linear and Quasilinear Equations of Parabolic Type), Moscow, 1967.

[33] *Y.-S. Lee, M.F. Hamilton* Time-domain modeling of pulsed finite-amplitude sound beams. J. Acoust. Soc. Am. 97 (2), 1995, p.906-917.

[34] *P.L. Lions* Mathematical Topics in Fluid Mechanics. Vol. 2. Compressible Models, Oxford Science Publications, 1998.

[35] *L. Molinet* The Cauchy Problem For The (Generalized) Kadomtsev-Petviashvili-Burgers Equation. Differential and Integral Equations. Vo 13(1-3), 2000, p. 189-216.

[36] *L. Molinet, F. Ribaud* On the Low Regularity of the Korterweg-de Vries-Burgers Equation. IMRN 2002, No.37, p. 1979-2005.

[37] *Natanson, I.P.* Theory of Functions of a Real Variable. Revised Edition, Volume I, Ungar, New-York, 1961.

[38] *S. Novo* Compressible Navier-Stokes Model with Inflow-Outflow Boundary Conditions. J. Math. Fluid Mech. 7 (2005), p. 485-514.

[39] *A. Pazy* Semigroups of linear operators and applications to Partial Differential Equations. Applied mathematical sciences, Vol. 44, Berlin-Heidelberg-New York: Springer, 1983.

[40] *Prilepko A.I., Tikhonov I.V.* Reconstruction of an inhomogeneous summand in an abstract evolution equation, Izv. Ross. Akad. Nauk Ser. Mat. [Russian Acad. Sci. Izv. Math.], vol. 58, 1994, no. 2, p. 167-188.

[41] *Rozanova A.V.* Controllability in a Nonlinear Parabolic Problem with Integral Overedetermination. Differential Equations, vol. 40, No. 6, 2004, pp. 853-872.

[42] *Rozanova A.V.* Controllability for a Nonlinear Abstract Evolution Equation. Mathematical Notes, vol. 76, No. 4, 2004, pp. 511-524.

[43] *Rozanova A.V.* Letter to the Editor. Mathematical Notes, Vol. 78, No 5-6, 2005, p.745.

[44] *D. Sanchez* Long waves in ferromagnetic media, Zabolotskaya-Khokhlov equation.J. Differential Equations 210, 2005, p. 263-289.

[45] *J.C. Saut, R. Temam* Remarks on the Korteweg-de Vries equation, Israel Journal of Mathematics, Vol. 24, No 1, 1976.

[46] *J.C. Saut, N. Tzvetkov* On Periodic KP-I Type Equations. Commun. Math. Phys. 221, 2001, p.451-476.

[47] *T. Sideris* Formation of Singularities in Three-dimensional compressible fluids. Comm. Math. Phys. 101 1985, 475-485.

[48] *Sukhinin, M.F.* Izbrannye glavy nelineinogo analiza (Selected Chapters of Nonlinear Analysis), Moscow, 1992.

[49] *Sukhinin, M.F.* On the solvability of nonlinear stationary transport equations, Teoret. Mat. Fiz., 1995, vol. 103, no. 1, pp. 23-31.

[50] *R. Temam* Sur la stabilité et la convergence de la méthode des pas fractionnaires. (French) Ann. Mat. Pura Appl. (4) 79, 1968, p.191-379.

[51] *B. Texier* The short-wave limit for nonlinear, symmetric, hyperbolic systems. Adv. Diff. equation 9 (2004), no. 1, 1-52. 2.

[52] *E. A. Zabolotskaya, R. V. Khokhlov* Quasi-planes waves in the nonlinear acoustic of confined beams, Sov. Phys. Acoust. 15(1) (1969) 35-40.

Résumé :

Ce travail se compose de deux parties. Dans la première, nous considérons l'équation de Khokhlov-Zabolotskaya-Kuznetsov (KZK) $(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0$ dans les espaces de Sobolev des fonctions périodiques sur x de valeur moyenne nulle. La dérivation de l'équation KZK à partir des équations de Navier-Stokes isentropiques non linéaires et de l'approximation de leurs solutions (pour les cas visqueux et non visqueux), les résultats de l'existence, de l'unicité, de la stabilité et du blow-up de la solution de KZK sont obtenus ainsi qu'un résultat sur l'existence d'une solution régulière du système de Navier-Stokes dans le demi espace avec conditions aux limites périodiques en temps et de valeur moyenne nulle. Dans la deuxième partie, nous prouvons la contrôlabilité locale des moments de deux systèmes décrits par une équation non-linéaire d'evolution dans un espace de Banach et par une équation non-linéaire de la chaleur quand le contrôle est un multiplicateur du membre de droite. Pour les deux systèmes avec une surdétermination intégrale nous obtenons des conditions suffisantes sur la taille du voisinage duquel nous pouvons prendre la fonction de la condition de surdétermination de sorte que le problème inverse ait une solution unique. Nous prouvons également le résultat de contrôlabilité pour l'équation KZK linéarisée.

Mots-clés : équation KZK, faisceaux sonors non-linéaires, périodicité, valeur moyenne nulle, approximation, demi espace, méthode des pas fractionnaires, systèmes de Navier-Stokes et d'Euler isentropiques, équation d'évolution non linéaire, équation de la chaleur, problème de contrôlabilité des moments, surdétermination intégrale, problème de Cauchy, dérivée de Fréchet

Abstract:

This work consists of two parts. In the first part we consider the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation $(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0$ in Sobolev spaces of functions periodic on x and with mean value zero. The derivation of KZK from the nonlinear isentropic Navier Stokes equations and approximation their solutions (for viscous and non viscous cases), the results of the existence, uniqueness, stability and blow-up of solution of KZK equation are obtained, also a result of existence of a smooth solution of Navier-Stokes system in the half space with periodic in time mean value zero boundary conditions. In the second part we prove the local controllability of moments for two systems described by a nonlinear evolution equation in Banach space and by a nonlinear heat equation when the control is a multiplier on the right-hand side. For this two systems with integral overdetermination we obtain sufficient conditions on the size of the neighborhood from which we can take the function from the overdetermination condition so that the inverse problem is uniquely solvable. We also prove the controllability result for linearized KZK equation.

Key words: KZK equation, nonlinear sound beams, periodic, mean value zero, approximation, half space, fractional step method, Navier-Stokes and Euler isentropic systems, nonlinear evolution equation, heat equation, controllability problem of moments, integral overdetermination, Cauchy problem, Fréchet derivative

Аннотация :

Работа состоит из двух частей. В первой части рассматривается уравнение Хохлова-Заболотской-Кузнецова (ХЗК) $(u_t - uu_x - \beta u_{xx})_x - \gamma\Delta_y u = 0$ в пространствах Соболева периодических по x функций и со средним значением ноль. Проведены вывод уравнения ХЗК из нелинейного изентропной системы Навье-Стокса и аппроксимация их решений (для вязкого и невязкого случаев), доказаны результаты существования, единственности, устойчивости и существования ударной волны для решения уравнения ХЗК. Также получен результат существования гладкого решения системы Навье-Стокса в полупространстве с периодическими по времени и нулевым средним по периоду значению граничными условиями. Во второй части получен результат локальной управляемости для двух систем описываемых нелинейным абстрактным эволюционным уравнением в банаховом пространстве и нелинейным уравнением теплопроводности когда управлением является множитель в правой части. Для этих двух систем с интегральным переопределением были получены достаточные условия на размер окрестности, из которой можно брать функцию из условия переопределения, с тем чтобы обратная задача была однозначно разрешима. Также доказана управляемость для линеаризованного уравнения ХЗК.

Ключевые слова : уравнение ХЗК, нелинейные акустические пучки, периодичность, нулевое среднее значение, аппроксимация, полупространство, метод дробных шагов, изентропные системы Эйлера и Навье-Стокса, нелинейное эволюционное уравнение, уравнение теплопроводности, задача управляемости, интегральное переопределение, задача Коши, дифференцируемость по Фреше