Calcul stochastique via régularisation et applications financières

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Introduction

Le calcul d'Itô a été amplement employé en mathématiques financières pour modéliser l'évolution des actifs financiers et celle d'un portefeuille d'investisseur. Cette théorie s'appuie principalement sur l'hypothèse de propriété de semimartingale pour les processus décrivant le cours des prix d'un actif risqué.

Néanmoins, pour plusieurs raisons que nous décrirons plus loin, il semble pertinent de réfléchir à une théorie s'affranchissant de cette propriété. A l'origine, le but de cette thèse était l'application du *calcul stochastique via régularisation* au développement d'une telle théorie. En effet, ce calcul représente un outil essentiel puisqu'il étend le calcul d'Itô au-delà des semimartingales. Par la suite, nos recherches ont soulevé plusieurs problèmes techniques dont la résolution constitue une partie importante de la thèse. C'est pourquoi celle-ci s'inscrit dans la continuité du calcul via régularisation en contribuant à l'étude et à la généralisation des processus de Dirichlet.

Le calcul via régularisation a été introduit en 1991 dans [52] et a été développé dans les quinze dernières années par plusieurs auteurs ; un panorama de ce calcul se trouve dans [57]. Ses avantages principaux sont sa généralité et sa souplesse. Il intègre le cadre du calcul stochastique mais la structure probabiliste sur laquelle il se fonde est minimale. Il constitue, d'ailleurs, un pont entre le calcul stochastique classique et le calcul trajectoriel.

Dans ce contexte on introduit l'intégrale *forward*, l'intégrale *symétrique* et la *covariation*. Ces objets sont définis à travers une procédure de régularisation et convergence. On pose, pour $\varepsilon > 0$,

$$I(\varepsilon, Y, \xi, t) = \frac{1}{\varepsilon}\int_0^t Y_s(\xi_{s+\varepsilon} - \xi_s)ds, \quad I^\circ(\varepsilon, Y, \xi, t) = \frac{1}{2\varepsilon}\int_0^t Y_s(\xi_{s+\varepsilon} - \xi_{s-\varepsilon})ds$$

et

$$C(\varepsilon, Y, \xi, t) = \frac{1}{\varepsilon}\int_0^t (Y_{s+\varepsilon} - Y_s)(\xi_{s+\varepsilon} - \xi_s)ds.$$

Si les suites de processus $(I(\varepsilon, Y, \xi, t))_{\varepsilon>0}$, $(I^\circ(\varepsilon, Y, \xi, t))_{\varepsilon>0}$ et $(C(\varepsilon, Y, \xi, t))_{\varepsilon>0}$ convergent en probabilité pour tout t et le processus limite admet une modification continue, nous dirons respectivement que Y est ξ-forward intégrable, que l'intégrale symétrique de Y par rapport à ξ existe et que le vecteur (ξ, Y) admet sa covariation. L'intégrale forward, l'intégrale symétrique et la covariation serons symbolisées respectivement par $\int_0^{\cdot} Y_t d^- \xi_t$, $\int_0^t Y_t d^\circ \xi_t$, et $[\xi, Y]$. Si $Y = \xi$, ξ est dit processus à variation quadratique finie et $[\xi, \xi]$ est noté $[\xi]$. Si la suite de processus $\left(\frac{1}{\varepsilon}\int_0^t (\xi_{s+\varepsilon} - \xi_s)^3 ds\right)_{\varepsilon>0}$ converge en probabilité uniformément sur $[0,1]$, et pour tout suite $(\varepsilon_k)_{k\geq 0}$ il existe une sous-suite $(\bar{\varepsilon}_k)_{k\geq 0}$ telle que

$$\sup_{k\geq 0} \frac{1}{\bar{\varepsilon}_k} \int_0^1 |\xi_{s+\bar{\varepsilon}_k} - \xi_s|^3 \, ds < +\infty, \quad p.s.,$$

ξ est appelé processus à variation cubique finie (*strong cubic variation*, en anglais). Le processus limite est noté $[\xi,\xi,\xi]$.

L'intégrale *forward* est employée lorsque ξ est un processus à variation quadratique finie, alors que l'intégrale symétrique intervient si ξ a une variation cubique finie.

L'intégrale *forward* généralise l'intégrale d'Itô et peut être liée à l'intégrale de Skorohod lorsque l'intégrateur est le mouvement brownien. L'intégrale symétrique, lorsque les processus intervenants sont des semimartingales ou des processus gaussiens, coïncide avec l'intégrale de Fisk-Stratonovich. La covariation de deux semimartingales coïncide avec le crochet droit classique.

Un outil classique fondamental du calcul d'Itô est la dénommée *formule d'Itô* qui permet de développer $f(t,\xi_t)$ pour une certaine fonction $f:\mathbb{R}\to\mathbb{R}$ suffisamment régulière et une semimartingale ξ. A ce jour, il existe une quantité impressionnante de travaux généralisant la formule d'Itô et il serait impossible de tous les lister. Un premier niveau de généralisation survient lorsque ξ est à variation quadratique finie. L'état de l'art de cette formule appliquée à un processus à variation quadratique finie se trouve par exemple dans [25], [56], voir aussi [57]. Les généralisations ultérieures vont dans deux directions :

1. lorsque ξ n'est pas nécessairement à variation quadratique finie. Par exemple si ξ est un mouvement brownien fractionnaire d'indice de Hurst $H > \frac{1}{6}$, et f de classe C^6, voir e.g. [29, 11] ;
2. lorsque ξ est une semimartingale (*réversible*), donc essentiellement un intégrateur classique, mais f est moins régulière, par exemple de classe C^1, voir par exemple [26, 55].

Si ξ est un processus à variation quadratique finie et $f \in C^{1,2}([0,1]\times\mathbb{R})$, $f(t,\xi_t)$ se développe de la façon suivante :

$$f(t,\xi_t) = f(0,\xi_0) + \int_0^t \partial_s f(s,\xi_s)ds + \int_0^t \partial_x f(s,\xi_s)d^-\xi_s + \frac{1}{2}\int_0^t \partial_x^{(2)} f(s,\xi_s)d\,[\xi,\xi]_s\,.$$

Si ξ est un processus à variation cubique finie et $f \in C^{1,3}([0,1]\times\mathbb{R})$ alors ([21])

$$f(t,\xi_t) = f(0,\xi_0) + \int_0^t \partial_s f(s,\xi_s)ds + \int_0^t \partial_x f(s,\xi_s)d^\circ\xi_s - \frac{1}{12}\int_0^t \partial_x^{(3)} f(s,\xi_s)d\,[\xi,\xi,\xi]_s\,.$$

La formule d'Itô pour les processus à variation quadratique finie admet des généralisations de nature Itô-Wentzell, comme dans [24], où la dépendance en temps est de type semimartingale. Plus précisément, il est possible de développer $X_t(\xi_t)$ lorsque $X_t(x)$ est une famille de semimartingales, relatives à une filtration $\mathbb{F} = (\mathcal{F}_t)$, dépendant d'un paramètre x dans $\mathbb{R}$:

$$X(t,x) = f(x) + \sum_{i=1}^n \int_0^t a^i(s,x)dN_s^i, \tag{1}$$

où pour chaque x, f est $\mathcal{F}_0$-mesurable, a^i est $\mathbb{F}$-adapté et $(N^1, ..., N^n)$ est un vecteur de $\mathbb{F}$-semimartingales tel que ξ est $\mathbb{F}$-adapté et le vecteur $(\xi, N^1, ..., N^n)$ a tous ses crochets mutuels, i.e. $[N^i, N^j]$, $[\xi, N^j]$ et $[\xi, \xi]$ existent pour tout $i, j = 1, \ldots, n$. Plus précisément,

$$\begin{aligned} X(\cdot, \xi) &= X(0, \xi_0) + \sum_{i=1}^{n} \int_0^{\cdot} a^i(s, \xi_s) dN_s^i + \int_0^{\cdot} \partial_x X(s, \xi_s) d^- \xi_s \\ &+ \sum_{i=1}^{n} \int_0^{\cdot} \partial_x a^i(s, \xi_s) d\left[N^i, \xi\right]_s + \frac{1}{2} \int_0^{\cdot} \partial_x^{(2)} X(s, \xi_s) d\left[\xi, \xi\right]_s . \end{aligned} \tag{2}$$

L'approche du calcul via régularisation permet d'établir des formules de substitutions. Soient

$$(M(t,x), t \in [0,1], x \in \mathbb{R}) \quad \text{et} \quad (N(t,x), t \in [0,1], x \in \mathbb{R})$$

deux familles de $\mathbb{F}$-semimartingales dépendant d'un paramètre x et $(h(t,x), t \in [0,1], x \in \mathbb{R})$ une famille de processus $\mathbb{F}$-adaptés. Sous des conditions minimales il est possible de démontrer que, pour toute variable aléatoire $\mathcal{F}_1$-mesurable L on a

$$\int_0^{\cdot} h(t, L) d^- M(t, L) = \left(\int_0^{\cdot} h(t, x) dM(t, x) \right)_{x=L}, \tag{3}$$

et également

$$[M(\cdot, L), N(\cdot, L)] = ([M(\cdot, x), M(\cdot, x)])_{x=L} . \tag{4}$$

Nous rappelons que $[M(\cdot,x), N(\cdot,x)]$ coïncide avec le crochet droit classique entre deux semimartingales ; voir à ce sujet [54] et [56]. Ces formules deviennent utiles dans les contextes d'anticipation lorsque ni le calcul de Skorohod, ni les techniques de grossissement de filtration peuvent être appliquées.

Nous signalons que parmi les champs d'investigations du calcul via régularisation on trouve aussi les processus fractionnaires, les équations différentielles stochastiques à drift très irrégulier, voire distributionnel.

Concernant l'application financière, nous nous sommes placés dans le cadre où le prix d'un actif risqué est un processus à variation quadratique finie et nous avons remplacé l'intégrale d'Itô par l'intégrale *forward*. L'hypothèse d'existence d'une variation quadratique des prix apparaît comme la généralisation la plus naturelle de la propriété de semimartingale alors que l'intégrale forward est l'objet adéquat pour décrire la propriété d'autofinancement des stratégies. En effet, toute semimartingale est un processus à variation quadratique finie et, de plus, cette hypothèse se révèle indispensable si le processus des prix lui-même est compris parmi toutes les stratégies admissibles *autofinancées* (voir à ce propos la remarque 1.5.1).

Au-delà de ces considérations, la propriété de variation quadratique finie peut elle-aussi être une barrière. En effet, l'observation de certaines séries financières, voir [47], laisse apparaître que des processus n'étant pas à variation quadratique finie peuvent intervenir dans la modélisation. Dans ce contexte, il est naturel de considérer des équations différentielles stochastiques pour décrire l'évolution des prix ainsi que la richesse d'un investisseur. Pour cette raison la deuxième partie de cette thèse est consacrée à une classe d'équations faisant intervenir des processus à variation cubique finie mélangés à des martingales.

Ce mémoire est donc réparti en deux chapitres. Le premier établit les fondements de cette nouvelle approche à la modélisation financière et le deuxième se concentre sur des équations différentielles stochastiques générales où interviennent les intégrales via régularisation.

En 1900, Bachelier, dans sa thèse ([3]), montre la nécessité d'utiliser les probabilités pour construire les bases d'une "théorie de la spéculation". Il propose de modéliser le cours d'un actif financier risqué à l'aide d'un mouvement brownien avec tendance. En 1965, Samuelson ([58]) suggère de retenir cette modélisation et de l'appliquer aux rendements des actifs plutôt qu'aux cours, afin d'éviter des prix négatifs. En 1973 Black, Scholes ([8]) et Merton ([42]) renforcent cette théorie financière innovatrice, en introduisant le concept de prix d'un produit dérivé comme " le prix de sa couverture ". Ils proposent l'intégrale d' Itô comme objet mathématique apte à décrire la dynamique de la richesse d'un agent.

En particulier, dans tous les modèles proposés, le processus des prix d'un actif risqué S est une semimartingale et, de plus, il existe une probabilité équivalente à la probabilité de référence, qui rend S une martingale. L'existence d'une telle mesure de probabilité entraîne l'absence d'opportunités d'*arbitrage*. Ces modèles reposent donc sur l'**hypothèse fondamentale de la Finance** : dans un marché très liquide, ou il n'y a ni coûts de transactions, ni limitations sur l'achat et la vente des actifs, il n'est pas possible de gagner de l'argent sans risque, à partir d'un investissement nul.

Une quantité considérable de recherche a été produite dans le but de justifier la propriété de semimartingale mentionnée ci-dessus et de lui donner une interprétation économique. Elle a culminé en 1994 dans le *foundamental theorem of asset pricing* par Delbaen et Schachermayer [16]. Ce théorème établit le lien entre la propriété de semimartingale et une notion plus générale d'arbitrage, le *free lunch with vanishing risk (FLVR)* : en absence de FLVR sur l'ensemble des stratégies simples et prévisibles par rapport à la filtration $\mathbb{G}$ représentant le flux d'informations disponible pour l'agent, le prix de l'actif est une $\mathbb{G}$-semimartingale.

Par ailleurs, plusieurs considérations pratiques défient le remarquable théorème de Delbaen et Schachermayer.

Une des activités en finance est celle des *arbitrageurs*, qui sont des intervenants gagnant de l'argent en exploitant des opportunités d'arbitrage temporaires. Cette observation soulève des doutes sur la validité de l'hypothèse de non-arbitrage (NA) : les arbitrages existent, même marginalement. De plus, la présence de ces arbitrages temporaires est corroborée par les données économétriques : la structure microscopique de la source aléatoire des prix ne révèle pas la propriété de semimartingale, à cause des imperfections dûes aux effets *intra-day*. Un modèle qui prend en compte ces imperfections pourrait rajouter au mouvement brownien qui décrive le logarithme des prix, un mouvement brownien fractionnaire avec un indice de Hurst supérieur à $\frac{1}{2}$, voir [9] et [64].

A ce propos, des modèles de marché dans lequel le prix d'un actif financier est un mouvement brownien fractionnaire d'indice de Hurst $H \neq 1/2$, ont été développés en utilisant le produit de *Wick*, aussi bien dans l'équation qui décrit le cours, que dans la définition de stratégie admissible, voir par exemple [19] et [35]. La littérature relative au calcul stochastique via le produit de Wick est très riche lorsque l'intégrateur est un mouvement brownien, dans le cadre de la théorie du *white noise calculus*. Pour un calcul relatif au mouvement brownien fractionnaire nous signalons [4] et ses références.

L'introduction du produit de Wick dans la définition de stratégie admissible exclut les arbitrages. Malheureusement [7] montre que cette idée ne conduit pas à une interprétation économique raisonnable.

En dehors de ces considérations sur la structure des prix, les modèles sans semimartingale trouvent leur justification d'existence même quand la condition d'arbitrage est vraisemblable. En fait, la validité du théorème fondamentale dépend fortement du choix de l'ensemble des *stratégies admissibles*. Dans [16] les stratégies admissibles varient dans une classe suffisamment grande, d'un point de vue mathématique, dans le but de pouvoir établir le résultat recherché. Toutefois, la classe des stratégies admissibles est susceptible d'être restreinte à cause des régulations de marché ou pour des raisons purement pratiques. Par exemple, l'investisseur peut être obligé de limiter ses achats et de ne détenir qu'un nombre fixé d'actions. D'autre part, il pourrait être raisonnable d'imposer une distance temporelle minimale entre deux transactions, comme suggéré par Cheridito ([10]) (voir aussi [33]) : lorsque le logarithme des prix est un mouvement brownien fractionnaire géométrique, il n'y a pas de possibilité d'arbitrage satisfaisant cette hypothèse minimale. Nous rappelons que sans cette restriction, le marché admet des arbitrages, voir à ce sujet [50].

Par ailleurs, bien que la condition de (NA) pour un investisseur honnête soit réaliste, un *initié* pourrait réaliser des arbitrages par rapport à la filtration engendrée par $\mathbb{G}$ et l'information supplémentaire dont il dispose. La présence d'un initié fournit une situation typique où les modèles sans semimartingale peuvent être appliqués.

La littérature concernant l'initié, et plus généralement l'asymétrie d'information, a été considérablement enrichie par plusieurs auteurs lors des dernières années, parmi lesquels Pikowski et Karatzas ([46]), Grorud et Pontier ([31]), Amendinger, Imkeller, et Schweizer ([1]). Ces auteurs furent les premiers à adopter les techniques de *grossissement de filtration* pour modéliser la dynamique des prix des actifs par rapport à la filtration d'un initié.

Récemment, d'autres travaux ont abordé le problème de façon novatrice en utilisant l'intégrale *forward*, dans le cadre du calcul stochastique via régularisation. Leon, Navarro et Nualart, dans [39], ont résolu le problème de la maximisation logarithmique de la richesse terminale d'un initié, lorsqu'une information additionnelle est révélée à l'initié à l'instant initial de la période de négociation. Ils travaillent sous des conditions techniques qui, a priori, n'impliquent pas la condition classique (H') pour le grossissement de filtration considérée dans [36]. Néanmoins, a posteriori, ils découvrent que leurs conditions impliquent la propriété de semimartingale des prix.

Biagini et Øksendal ([5]) ont considéré une sorte d'implication inverse : s'il existe un portefeuille maximisant l'utilité terminale, alors le processus des prix est une semimartingale. Ankirchner et Imkeller ([2]), en se restreignant au cadre du grossissement de filtration, ont établi un lien entre le théorème fondamentale et l'utilité bornée de la richesse terminale d'un investisseur. En particulier, ils retrouvent un résultat comparable à celui contenu dans [5].

Dans le premier chapitre de la thèse, le marché considéré comporte un seul actif risqué, dont le prix est décrit par un processus strictement positif S, et un actif *moins risqué* de prix S^0, éventuellement sans risque, mais a priori seulement à variation finie.

Nous spécifions $\mathcal{A}$ comme un sous-espace vectoriel de **stratégies de portefeuille admissibles**. Si $\mathcal{A}$ n'est pas suffisamment riche pour contenir toutes les stratégies admissibles simples, alors S n'est pas nécessairement une semimartingale, même lorsque tout *free lunch with vanishing risk* est exclu parmi les stratégies de $\mathcal{A}$.

Ce chapitre essaie de poser les bases d'une théorie financière adéquate permettant de traiter des problèmes de valorisation de prix par couverture ou non-arbitrage, de maximisation d'utilité et de discuter les propriétés de viabilité et complétude du marché.

Par simplicité, dans la description suivante nous supposons que l'actif moins risqué S^0 est constant et égal à 1, et que $[\log(S)]_t = \sigma^2 t$, $\sigma > 0$.

Comme anticipé, un outil naturel pour décrire la condition d'autofinancement est l'intégrale *forward*. Soit $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le 1}$ une filtration sur un espace de probabilité $(\Omega, \mathcal{F}, P)$, avec $\mathcal{F} = \mathcal{G}_1$; $\mathbb{G}$ représente le flux d'information accessible à l'investisseur. Un **portefeuille autofinancé** est un couple (X_0, h) où X_0 est la valeur initiale du portefeuille et h est un processus $\mathbb{G}$-adapté et S-forward intégrable, qui spécifie le nombre de parts de l'actif S détenues dans le portefeuille. La valeur liquidative X d'un tel portefeuille est donnée par $X_0 + \int_0^{\cdot} h_s d^- S_s$, tandis que $h_t^0 = X_t - S_t h_t$ constitue la quantité d'actif moins risqué.

Cette formulation d'autofinancement est cohérente avec celle des modèles discrets. A cet effet, considérons une stratégie *buy-and-hold*, c'est-à-dire un couple (X_0, h) où $h = \eta I_{(t_0, t_1]}, 0 \le t_0 \le t_1 \le 1$, et η est une variable aléatoire $\mathcal{G}_{t_0}$-mesurable. En se servant de la définition d'intégrale *forward*, il n'est pas difficile d'obtenir que $X_{t_0} = X_0$, $X_{t_1} = X_0 + \eta(S_{t_1} - S_{t_0})$. Ceci implique que $h^0_{t_0+} = X_0 - \eta S_{t_0}$, $h^0_{t_1+} = X_0 + \eta(S_{t_1} - S_{t_0})$ et finalement

$$X_{t_0} = h_{t_0+} S_{t_0} + h^0_{t_0+}, \quad X_{t_1} = h_{t_1+} S_{t_1} + h^0_{t_1+}.$$

Aux instants de re-négociations t_0 et t_1, la valeur de l'ancien portefeuille doit être réinvestie pour constituer le nouveau portefeuille sans consommation ni apport de fonds. Ici h_{t+} symbolise $\lim_{s \downarrow t} h_s$.

Par la suite, $\mathcal{A}$ dépendra du type de problème auquel nous serons confrontés, à savoir : couverture, maximisation d'utilité, modélisation d'un initié. Comme annoncé auparavant, si l'on demande que S appartienne à $\mathcal{A}$, S doit être un processus à variation quadratique finie. En fait, $\int_0^{\cdot} S d^- S$ existe si et seulement si la variation quadratique $[S]$ existe, voir [57] ; en particulier dans ce cas on trouve :

$$\int_0^{\cdot} S_s d^- S_s = S^2 - S_0^2 - \frac{1}{2}[S].$$

$\mathcal{L}$ est un sous-espace vectoriel de $L^0(\Omega)$ représentant un ensemble d'**actifs conditionnels** intéressants pour un investisseur. Un actif conditionnel $\mathcal{A}$**-réplicable** est une variable aléatoire C pour laquelle il existe un portefeuille autofinancé (X_0, h) avec h dans $\mathcal{A}$ et

$$C = X_0 + \int_0^1 h_s d^- S_s.$$

On appellera X_0 **le prix de réplication** de C. Le marché est $(\mathcal{A}, \mathcal{L})$-**complet** si chaque élément de $\mathcal{L}$ est $\mathcal{A}$-**réplicable**.

Dans cette introduction nous choisissons comme $\mathcal{L}$ l'ensemble de tous les actifs conditionnels européens $C = \psi(S_1)$ où ψ est continue à croissance polynomiale, et comme $\mathcal{A} = \mathcal{A}_S$ l'ensemble suivant :

$$\begin{aligned} \mathcal{A}_S &= \{(u(t, S_t)), 0 \leq t < 1 \mid u : [0,1] \times \mathbb{R} \to \mathbb{R}, \text{ borélienne} \\ &\qquad \text{à croissance polynomiale et bornée}\}. \end{aligned}$$

Un tel marché est $(\mathcal{A}, \mathcal{L})$-complet : en fait, une variable aléatoire $C = \psi(S_1)$ est un actif conditionnel $\mathcal{A}$-**réplicable**. Pour construire une stratégie de réplication, l'investisseur peut choisir v comme étant la solution du problème

$$\begin{cases} \partial_t v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^{(2)} v(t,x) &= 0 \\ v(1,x) &= \psi(x) \end{cases}$$

et $X_0 = v(0, S_0)$. Ce résultat est une conséquence directe de la formule d'Itô contenue dans la proposition 1.2.11 ; voir proposition 1.5.29 et remarque 1.5.30.

Cette méthode peut être adaptée pour couvrir également des options *asiatiques*. Ce résultat est le contenu de la proposition 1.5.31.

Une notion cruciale dans ce travail est celle de $\mathcal{A}$-**martingale**. Ce processus intervient naturellement dans la maximisation d'utilité, dans l'étude de la viabilité de marché, et dans la caractérisation de l'unicité des prix de réplication.

Un processus M est une $\mathcal{A}$-martingale si pour $Y \in \mathcal{A}$

$$E\left[\int_0^t Y_s d^- M_s\right] = 0, \quad \forall t \in [0,1].$$

Si $\mathcal{A}$ contient la classe des processus $\mathbb{F}$-previsibles bornés pour une filtration $\mathbb{F}$ par rapport à laquelle M est adapté, alors M est une $\mathbb{F}$-martingale.

Un exemple de $\mathcal{A}$-martingale est le **mouvement brownien faible d'ordre** $k = 1$ (en anglais **weak Brownian motion of order** $k = 1$) de variation quadratique égale à t. Cette notion a été introduite par [27] : un mouvement brownien faible d'ordre 1 est un processus X tel que la loi de X_t est $N(0, t)$ pour chaque $t \geq 0$.

Un portefeuille (X_0, h) est appelé $\mathcal{A}$-arbitrage si $h \in \mathcal{A}$, $X_1 \geq X_0$ presque sûrement et $P\{X_1 - X_0 > 0\} > 0$. Le marché est $\mathcal{A}$-**viable** s'il n'existe pas de $\mathcal{A}$-arbitrages. Nous notons par $\mathcal{M}$ l'ensemble des mesures de probabilité équivalentes à la probabilité P de référence par rapport auxquelles S est une $\mathcal{A}$-martingale. Si $\mathcal{M}$ n'est pas vide alors le marché est $\mathcal{A}$-viable. En effet, si un couple (X_0, h) constitue un $\mathcal{A}$-arbitrage et $Q \in \mathcal{M}$, alors $E^Q[X_1 - X_0] = E^Q[\int_0^1 h d^- S] = 0$. Dans un tel cas, le prix de réplication X_0 d'un actif conditionnel $\mathcal{A}$-**réplicable** C est unique, pourvu que pour chaque variable aléatoire bornée $\eta \in \mathcal{G}_0$ et h dans $\mathcal{A}$, le processus $h\eta$ appartienne encore à $\mathcal{A}$. De plus $X_0 = E^Q[C|\mathcal{G}_0]$. En réalité, le prix de réplication reste toujours unique sous l'hypothèse plus faible d'absence d'opportunité d'arbitrage, voir proposition 1.5.27.

Par ailleurs, si $\mathcal{M}$ n'est pas vide et $\mathcal{A} = \mathcal{A}_S$, comme c'est le cas dans cette introduction, la loi de S_t doit être équivalente à la mesure de Lebesgue pour chaque $0 < t \leq 1$, voir la proposition 1.5.21.

Si le marché est $(\mathcal{A}, \mathcal{L})$-complet, alors toute mesure de probabilité dans $\mathcal{M}$ coïncide avec $\sigma(\mathcal{L})$, voir proposition 1.5.28. Si $\sigma(\mathcal{L}) = \mathcal{F}$, alors $\mathcal{M}$ est un singleton et ce résultat recouvre le cas classique.

Soit une fonction d'utilité satisfaisant aux hypothèses habituelles ; la proposition 1.5.44 montre que son maximum π est atteint dans une classe $\mathcal{A}$ de proportions de portefeuilles si et seulement s'il existe une mesure de probabilité pour laquelle $\log(S) - \int_0^{\cdot} \left(\sigma^2 \pi_t - \frac{1}{2}\sigma^2\right) dt$ est une $\mathcal{A}$-martingale. Pour cette raison, si $\mathcal{A}$ est suffisamment grande pour vérifier les conditions de l'hypothèse $\mathcal{E}$ dans la définition 1.4.6, alors S est une semimartingale classique.

En conclusion, nous pouvons affirmer que la plupart des résultats fondamentaux de la théorie financière classique, admettent une généralisation naturelle à des modèles sans semimartingale.

Le premier chapitre est organisé de la façon suivante.

Après les préliminaires sur le calcul stochastique via régularisation pour les intégrales *forward*, nous présentons des exemples d'intégrateurs et intégrands pour lesquels l'intégrale *forward* existe et nous démontrons des propriétés importantes en prévision des applications financières. Ces exemples proviennent du calcul de Malliavin, des formules de substitution et des semimartingales dépendant d'un paramètre. Ces exemples serons largement utilisés tout au long du premier chapitre.

Concernant les formules de substitutions relatives à un processus à variation quadratique finie ξ nous faisons appel aux résultats contenus dans [24]. Nous définissons une classe de processus telle que si h et k appartiennent à celle-ci alors h et k sont *forward* intégrables par rapport à ξ, $\int_0^{\cdot} h_t d^- \int_0^t k_s d^- \xi_s$ existe, coincide avec $\int_0^{\cdot} h_t k_t d^- \xi_t$ et $\left[\int_0^{\cdot} h_t d^- \xi_t, \int_0^{\cdot} k_t d^- \xi_t\right] = \int_0^{\cdot} h_t k_t d\,[\xi, \xi]_t$.

Le deuxième exemple porte sur le lien entre l'intégrale *forward* et l'intégrale de Skorohod lorsque l'integrateur est un mouvement brownien. Nous introduisons le processus $D^- u$ comme la limite en L^p, $p \geq 2$, de la suite $(D_t u_{t-\varepsilon})_{\varepsilon>0}$, lorsque ε tend vers zéro, Du étant la dérivée dans le sens de Malliavin d'un processus u. Nous montrons que, sous des conditions opportunes, il est possible de décomposer l'intégrale *forward* d'un processus u comme la somme de l'intégrale de Skorohod et de l'intégrale de $D^- u$ par rapport au temps. En utilisant ce résultat préliminaire nous établissons des règles d'associativité pour l'intégrale *forward* dans ce contexte spécifique.

Le troisième exemple s'inscrit dans le cadre des formules de substitutions. Nous affinons des résultats obtenus dans [56] et [55] relatifs aux formules (3) et (4), en fournissant une version *localisée* des conditions de type Kolmogorov intervenant dans leurs hypothèses, voir propositions 1.3.33 et 1.3.30. Des conditions pour assurer l'associativité des intégrales définies à travers la substitution sont également établies.

Dans les applications à la finance, la classe des stratégies définie via le calcul de Malliavin est utile lorsque $\log(S)$ est un mouvement brownien géométrique par rapport à une filtration $\mathbb{F}$ contenue dans $\mathbb{G}$; les formules de substitutions sont employées lorsque l'investisseur est un initié disposant d'une information supplémentaire à la date $t = 0$; les semimartingales dépendant d'un paramètre sont utiles à chaque fois que S est un processus à variation quadratique finie.

La section 1.4 est dédiée à l'étude des $\mathcal{A}$-martingales : après avoir défini cette notion et établi ses propriétés fondamentales, nous avons exploré sa relation avec le mouvement brownien faible. Celle-ci est décrite dans un corollaire de la proposition 1.4.17 dont une version simplifiée est la suivante : un processus X à variation quadratique égale à t, est une $\mathcal{A}_X^1$-martingale si et seulement si sa loi est $\mathcal{N}(0,t)$ où

$$\begin{aligned}\mathcal{A}_X^1 \quad = \quad & \{(\psi(t,x), 0 \leq t \leq 1, \text{ à croissance polynomiale t.q. } \psi = \partial_x \Psi \\ & \Psi \in C^{1,2}([0,1] \times \mathbb{R}) \text{ avec } |\partial_t \Psi| + \left|\partial_{xx}^{(2)} \Psi\right| \text{ bornée } \}.\end{aligned}$$

Ceci implique que pour $k = 1$, X est une $\mathcal{A}_X^1$-martingale si et seulement si c'est un mouvement brownien faible d'ordre k. Si X est une $\mathcal{A}_X$-martingale alors X est un mouvement brownien faible d'ordre k ; de plus, si $k \geq 8$ alors la réciproque a lieu (proposition 1.4.21). Nous avons ensuite utilisé ce résultat pour explorer la transformation d'une $\mathcal{A}$-martingale à travers un changement de probabilité équivalente à la probabilité de référence P, voir proposition 1.4.19.

Successivement nous analysons le lien avec l'existence d'un maximum dans une classe $\mathcal{A}$ de processus pour un certain problème d'optimisation et la propriété de $\mathcal{A}$-martingale. Pour cela, nous avons utilisé la notion de dérivée de *Gâteaux* dont nous rappelons la définition et quelques propriétés.

A la section 1.5 nous traitons finalement les applications à la finance. Nous définissons la notion de stratégie autofinancée et nous fournissons des exemples. Par la suite, nous discutons l'absence de $\mathcal{A}$-arbitrages, la $(\mathcal{A}, \mathcal{L})$-complétude de marché et la couverture.

La fin de la section 1.5 est dédiée au problème de maximisation de l'utilité de la richesse terminale. Nous résolvons des problèmes techniques, liés à l'emploi de l'intégrale forward, dans le but de décrire l'évolution de la valeur liquidative d'un portefeuille à partir de la proportion de la richesse investie dans l'actif risqué. Ces difficultés proviennent du manque d'associativité de l'intégrale forward, voir la remarque 1.5.36. Nous introduisons une classe $\mathcal{A}^+$ de processus jouant le rôle de proportion de la richesse investie dans S. Nous démontrons que, s'il existe un processus π dans $\mathcal{A}^+$ maximisant l'utilité espérée de la richesse terminale, alors le processus $A - V - \int_0^\cdot \pi_t d\,[A]_t$, où

$$A = \log(S) - \log(S_0) + \frac{1}{2}\int_0^\cdot \frac{1}{S_t^2} d\,[S]_t\,, \quad V = \log(S^0)$$

est une $\mathcal{A}^+$-martingale par rapport à une mesure de probabilité Q^π dépendant de π et équivalente à la probabilité de référence. Sous l'hypothèse 1.5.32 cette proposition admet une réciproque. Si $\mathcal{A}^+$ est suffisamment grande alors l'existence d'un maximum entraîne la propriété de semimartingale pour S. Nous concluons la section en fournissant quelques exemples généralisant les résultats de [39] et [5].

Nous passons maintenant à la présentation de la deuxième partie qui discute de façon générale des équations différentielles stochastiques dirigées par un processus ξ qui n'est pas nécessairement à variation quadratique finie. Nous établissons une famille de résultats d'existence et d'unicité lorsque le processus ξ est muni d'un facteur multiplicatif σ. Le principe est que plus les trajectoires de ξ sont irrégulières, plus la régularité sur σ doit être importante. En particulier, si les réalisations de ξ sont hölder continues de paramètre $\gamma > \frac{1}{2}$, alors seul le caractère hölderien de σ est requis.

Nous généralisons la formule d'Itô-Wentzell (2) pour les processus à variation quadratique finie, en établissant une formule similaire dans le cas où ξ est un processus à variation cubique finie et $(\xi, N^1, \ldots, N^n)$ vérifie une condition technique introduite dans l' hypothèse $(\mathcal{D})$ de la définition 2.3.6. Nous supposons l'existence d'une filtration $\mathbb{H} \supseteq \mathbb{F}$, par rapport à laquelle le vecteur $(N^1, ..., N^n)$ est toujours un vecteur de semimartingales, tel que ξ se décompose en la somme de deux processus $\mathbb{H}$-adaptés Q et R, où $(Q, N^1, ..., N^n)$ a tous ses crochets mutuels et R est *fortement prévisible* (*strongly predictable*) par rapport à $\mathbb{H}$, voir définition 2.3.5. En particulier R est un processus $\mathbb{H}$-*weak Dirichlet* dans le sens de [21]. Nous rappelons qu'un processus $\mathbb{H}$-*weak Dirichlet* est la somme d'une $\mathbb{H}$-martingale locale continue et d'un processus $\mathbb{H}$-adapté Q tel que $[Q, N] = 0$ pour chaque $\mathbb{H}$-semimartingale N. De nouveaux développements sur le sujet sont parus dans [28] et [12]. L' hypothèse sur R mentionnée ci-dessus est vérifiée dans les cas suivants :

- lorsque R est $\mathcal{F}_0$-mesurable ;
- R est indépendant de $(N^1, ..., N^n)$ et la filtration engendrée par $(N^1, ..., N^n)$ et tout le processus R, contient $\mathbb{F}$.

Entre autre, le calcul mis au point autour de la formule d'Itô-Wentzell nous permet de clarifier la structure des processus $\mathbb{F}$-*weak Dirichlet* lorsque $\mathbb{F}$ est la filtration naturelle associée à un mouvement brownien W. Si Q est un processus $\mathbb{F}$-adapté et $[Q, W]$ a tous ses crochets mutuels, la covariation $[Q, L]$ peut être évaluée explicitement pour chaque $\mathbb{F}$-semimartingale continue L, voir proposition 2.3.9. Ceci nous permet de montrer qu'un processus A est $\mathbb{F}$-*weak Dirichlet* si et seulement si il est la somme d'une $\mathbb{F}$-martingale locale et d'un processus $\mathbb{F}$-adapté Q, avec $[Q, W] = 0$.

L'équation différentielle stochastique étudiée dans ce chapitre est de la forme suivante

$$d^\circ X_t = \sigma(t, X_t) \left[d^\circ \xi_t + \beta(t, X_t) d^\circ M_t + \alpha(t, X_t) dV_t \right], \tag{5}$$

où M est une martingale locale, V est un processus à variation finie et ξ un processus à variation cubique finie avec (ξ, M) vérifiant l'hypothèse $(\mathcal{D})$, par rapport à une filtration $\mathbb{H}$. Nous montrons, dans plusieurs situations, comment appliquer la formule d'Itô réduisant le coefficient de *diffusion* σ à 1. Ceci nous permet de discuter un théorème d'existence et d'unicité pour (5) en étudiant des équations où le processus ξ apparaît comme terme additif. La terminologie impropre de coefficient de diffusion sera utilisée dans tout le second chapitre. Un cas particulier de cette équation a été considéré par [21] où $\beta = 0$. Dans ce papier, σ était de classe C^3, et la notion de solution pour le processus X n'était pas naturelle car il était demandé que le couple (X, ξ) soit un *symmetric vector Itô process.* Lorsque σ est bornée inférieurement par une constante positive, cette équation peut être étudiée avec nos techniques. Ceci nous permet de relaxer les hypothèses sur les coefficients, d'agrandir la classe d'unicité et d'améliorer le sens de solution en évitant la notion de *symmetric vector Itô process.*

Dans la littérature, des équations différentielles stochastiques (EDS) de type forward

$$d^- X_t = \sigma(X_t) d^- \xi_t + \beta(t, X_t) dL_t,$$

ont été considérées et résolues en opérant des transformations $Y = h(X_t)$, pour une fonction h bien choisie. Une première tentative a été effectuée dans [56], lorsque L est à variation finie. Des résultats similaires ont été établis de façon indépendante dans [63]. Dans [24], l'existence et l'unicité ont été prouvées dans une classe de processus $(X(t, \xi_t))$, où $X(t, x)$ est une famille de semimartingales dépendant d'un paramètre et L est une semimartingale. Dans cet article, la régularité de σ était de classe C^4 avec σ', σ'' bornées. Dans un tel contexte, nos résultats permettent à nouveau d'agrandir la classe d'unicité et demandent moins de régularité sur les coefficients.

Les équations de type (5) ont été considérées par T. Lyons et ses collaborateurs, dans le cadre de la théorie des *rough paths*. A ce sujet dans le cas multidimensionnel on peut consulter [41], lorsque σ est Lipschitzienne, $\alpha = 0$, pour un processus X à p-variation déterministe strictement inférieure à 3. Dans [32], [23] on trouve d'autres formulations intéressantes de cette théorie et quelques applications aux EDS. L'analyse des *rough paths* est purement déterministe contrairement à la notre qui combine les techniques trajectorielles du calcul stochastique via régularisation et les concepts probabilistes, voir hypothèse $(\mathcal{D})$.

Nous considérons également l'équation

$$d^\circ X_t = \sigma(t, X_t) \left[d^\circ \xi_t + \alpha(t, X_t)dt\right], \tag{6}$$

où σ est localement hölderienne, α est localement lipschitzienne à croissance linéaire, et ξ est un processus continu à trajectoires γ-hölderiennes, avec $\gamma > \frac{1}{2}$.

Nous appliquons notre méthode à cette équation en utilisant une formule d'Itô établie dans [64], pour les intégrales de type fractionnaire, qui coïncide avec celle de Young [62] dans le contexte des intégrales et intégrands hölderiens respectivement de paramétre δ et γ avec $\gamma + \delta > 1$. Nous vérifions en fait que dans ce cas l'intégrale symétrique coïncide bien avec celle de Young, voir la proposition 2.4.26. Nous rappelons que les trajectoires du mouvement brownien fractionnaire d'indice de Hurst H sont γ-hölder continues pour chaque γ strictement inférieur à H. L'utilisation de la dite formule d'Itô nous permet d'étudier l'équation (6) lorsque ξ est un mouvement brownien fractionnaire d'indice de Hurst H supérieur à $\frac{1}{2}$. Nous transformons (6) en une équation du même type avec $\sigma = 1$, c'est à dire où le mouvement brownien fractionnaire intervient de façon additive, qui peut être traité grâce aux techniques de [44]. Nous pouvons ainsi améliorer notre résultat sur l'existence et l'unicité de l'équation (5) lorsque $\xi = B^H$ et B^H est un mouvement brownien fractionnaire avec $H > \frac{1}{2}$, i.e. :

$$d^\circ X_t = \sigma(t, X_t) \left[d^\circ B_t^H + \alpha(t, X_t)dt\right]. \tag{7}$$

Si $H = \frac{1}{2}$ le mouvement brownien fractionnaire B^H se réduit au mouvement brownien ; ainsi une formule d'Itô pour fonctions C^1 de semimartingales *réversibles*, voir la proposition 2.4.35, peut être utilisée. Cet outil nous fournit un théorème d'existence (2.4.36) pour l'équation (7), quand σ est juste continue et α est mesurable borné, voir le théorème 2.4.36. Si H est inférieur à $\frac{1}{2}$, la formule d'Itô pour les intégrales de type Young n'est plus applicable. Néanmoins, en partant de notre analyse, des conditions pour assurer l'existence et l'unicité de l'équation (7) peuvent être déduites. En effet, le mouvement brownien fractionnaire d'indice de Hurst $H \geq \frac{1}{3}$ a une variation cubique finie. Dans ce cas, le coefficient σ doit admettre, entre autre, une dérivée seconde continue par rapport à la variable d'espace.

D'autre part, la nature hölderienne du mouvement brownien fractionnaire peut être exploitée pour étudier des équations de type (7) lorsque $H < \frac{1}{2}$, dans le cadre des techniques trajectorielles. Le prolongement naturel de l'intégrale et du calcul de Young est effectivement l'analyse des *rough paths*.

Cette théorie a été perfectionnée dans plusieurs travaux, voir [14], [32], [13]. Ces auteurs adaptent les résultats de la théorie des *rough paths* aux EDS dirigées par des processus γ-hölderiens avec $\gamma > \frac{1}{3}$, ou par le mouvement brownien fractionnaire avec un indice de Hurst $H > \frac{1}{4}$.

Dans [32] l'auteur aborde l'existence et l'unicité des EDS de type (6) avec $\alpha = 0$, dirigées par des trajectoires irrégulières avec un exposant d'hölder $\gamma > \frac{1}{3}$. Le coefficient multiplicatif σ doit être dérivable d'ordre 2 avec la dérivée seconde δ-hölderienne et $\delta > \frac{1}{\gamma} - 2$.

A notre connaissance, la première tentative d'appliquer la théorie des *rough paths* à l'étude des EDS de type (7), dirigées par le mouvement brownien fractionnaire B^H avec $H < \frac{1}{2}$, se trouve dans [14]. Ces auteurs considèrent le cas $\frac{1}{4} < H < \frac{1}{2}$, et $\alpha = 0$. Ils proposent une approche trajectorielle basée sur le *universal limit theorem* établi dans [40] et supposent que le coefficient multiplicatif σ est dérivable avec dérivées bornées jusqu'à l'ordre $\left[\frac{1}{H}\right] + 1$.

Dans les ouvrages pré-cités, il n'est pas évident de déduire la nature des intégrales intervenant dans les EDS étudiées. Par ailleurs, elle dépend fortement du contexte dans lequel le problème est traité.

Un premier résultat liant l'approche déterministe et stochastique est contenu dans [13]. Ici, l'équation (7) est considérée avec α et σ champs vectoriels dépendant du temps. On démontre que la solution provenant de la théorie des *rough paths* est bien une solution dans le sens de Stratonovich, en supposant σ dérivable et borné jusqu'à l'ordre $[\delta]$ avec ses $[\delta]$-dérivées $(\delta - [\delta])$-hölderiennes pour $\delta > \frac{1}{H}$.

Notre analyse sur l'unicité est inspirée par l'équation différentielle ordinaire de type

$$\frac{dX(t)}{dt} = \sigma(X(t)), \tag{8}$$

où σ est juste continue à croissance linéaire. Le théorème de Peano assure alors l'existence mais pas l'unicité. Dans le cas où $\{x_0\} = \{x \in \mathbb{R}, \text{ t.q. } \sigma(x) = 0\}$, il est possible de démontrer que l'équation (8) admet une solution unique, pour chaque condition initiale, si pour $\varepsilon > 0$

$$\int_{x_0}^{x_0+\varepsilon} \frac{1}{|\sigma|}(y)dy = \int_{x_0-\varepsilon}^{x_0} \frac{1}{|\sigma|}(y)dy = +\infty. \tag{9}$$

Par contre, il existe au moins deux solutions avec condition initiale x_0 si la condition précédente n'est pas vérifiée. Supposons, par exemple, que la deuxième intégrale est infinie et posons $H(x) = \int_{x_0}^{x} \frac{1}{\sigma(y)} dy$, $x > x_0$. Les fonctions

$$X(t) = H^{-1}(t) \qquad \text{et} \qquad X(t) \equiv x_0$$

sont deux solutions de l'équation (8) de condition initiale x_0.

Ce phénomène est décrit dans le cas stochastique, même pour σ inhomogène, voir, par exemple, la proposition 2.4.30 et la remarque 2.4.31.

Nous rappelons qu'une condition de type (9) apparaît lorsqu'on étudie l'équation stochastique uni-dimensionnelle de la forme $dX(t) = \sigma(X(t))dW(t)$, où W est un mouvement brownien classique. L'unicité est assurée, pour chaque condition initiale, si et seulement si

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{\sigma^2}(t)dt = +\infty,$$

pour $x_0 \in \mathbb{R}$, voir à ce propos [20].

Notre contribution à l'étude de l'équation (5) par rapport à la littérature existante peut être ainsi résumée.

- Nous supposons que ξ est un processus à variation cubique finie et que σ est inhomogène.
- La notion de solution est simplifiée. En effet, nous évitons d'introduire le concept de *symmetric vector Itô process.*
- Nous établissons une formule de type Itô-Wentzell pour des processus à variation cubique finie.
- Nous clarifions la structure des processus *weak Dirichlet* lorsque la filtration sous-jacente est brownienne.
- Lorsque les trajectoires de ξ sont γ-hölderiennes et $\gamma > \frac{1}{2}$, la régularité des coefficients est affaiblie.
- Nous formulons un théorème d'existence nouveau pour une équation de type (7) dirigée par un mouvement brownien classique.
- Les conditions classiques d'existence et d'unicité sont considérablement affaiblies. De même, nous proposons des hypothèses moins fortes que celles provenant de la théorie des *rough paths.*

L'organisation du deuxième chapitre est la suivante.

La section 2.2 rappelle des définitions et des résultats concernant le calcul stochastique par rapport à un processus à variation cubique finie. Elle inclut, entre autres, la formule d'Itô et un résultat de stabilité de la variation cubique sous des transformations de classe C^1. Nous démontrons également des propriétés techniques de l'intégrale symétrique. Plus précisément, nous clarifions le comportement de $\int_0^{\cdot} Y_t d^\circ X_t$ localisée à un sous-ensemble de probabilité de référence, et nous montrons que

$$\left(\int_0^{\cdot} Y_t d^\circ X_t\right)^\tau = \left(\int_0^{\tau\wedge\cdot} Y_t d^\circ X_t\right)^\tau, \quad \int_\tau^{\tau+\cdot} Y_t d^\circ X_t = \int_0^{\cdot} Y_{t+\tau} d^\circ X_{t+\tau},$$

pour un temps aléatoire $0 \leq \tau \leq 1$.

Dans la section 2.3 nous introduisons la classe $\mathcal{C}_\xi^k(\mathbb{H})$ de la forme :

$$Z_t = X_t(\xi_t), \quad 0 \leq t \leq 1,$$

où $(X_t(x), 0 \leq t \leq 1, x \in \mathbb{R})$ est une fonction définie par la formule (1), dérivable jusqu'à l'ordre k en x, ξ est un processus à variation cubique finie et $(\xi, N^1, \ldots, N^n)$ vérifie l'hypothèse $(\mathcal{D})$ par rapport à la filtration $\mathbb{H}$. Nous démontrons que si ξ a une variation cubique finie alors tout processus dans $\mathcal{C}_\xi^1(\mathbb{H})$ est un processus à variation cubique finie.

Nous proposons une formule d'Itô-Wentzell qui permet de développer les processus appartenant à $\mathcal{C}^3_\xi$. Dans cette section nous nous intéressons aussi aux processus *weak Dirichlet* : nous introduisons un critère de caractérisation des processus $\mathbb{F}$-*weak Dirichlet* lorsque $\mathbb{F}$ est une filtration brownienne. La section est complétée par la preuve de l'existence de $\int_0^\cdot Y_t d^\circ X_t$, X et Y étant deux éléments dans $\mathcal{C}^2_\xi(\mathbb{H})$, et par une formule d'associativité pour l'intégrale symétrique.

La section 2.4 est consacrée à l'unicité et à l'existence de l'équation (5). Elle est divisée en neuf sous-sections.

La première et la seconde spécifient la notion de solution et décrivent le cadre du travail : nous nous plaçons dans le cas où le support S de σ est indépendant du temps et où une condition de non-intégrabilité autour de ses zéros de type (9) est vérifiée. Sous cette condition il est possible de définir sur S une primitive $H(t,\cdot)$ de $\sigma(t,\cdot)^{-1}$, pour tout t.

La troisième se focalise sur les trajectoires d'une solution. Si X est une solution de l'équation (5) et $X_0 \in S$ alors ses trajectoires restent dans S et $H(t, X_t) = \xi + N$, où

$$\begin{aligned} N &= H(0,\eta) + \int_0^\cdot \beta(s,X_s)dM_s + \int_0^\cdot \alpha(s,X_s)dV_s + \int_0^\cdot \partial_s H(s,X_s)ds \\ &+ \frac{1}{2}\left[\beta(\cdot,X), M\right] + \frac{1}{12}\int_0^\cdot \left(\sigma\partial_x^{(2)}\sigma + (\partial_x\sigma)^2\right)(s,X_s)d\left[\xi,\xi,\xi\right]_s. \end{aligned}$$

De même, si les coefficients dirigeant l'équation ne dépendent pas du temps, une solution ayant la condition initiale η dans $D = \mathbb{R}/S$ est constante et égale à η. En combinant ces résultats, dans la quatrième sous-section, nous établissons une équivalence entre l'équation (5) et une équation du même type ayant un coefficient de diffusion égal à 1. Ce résultat fait l'objet de la proposition 2.4.8. Nous donnons ensuite des conditions d'existence et d'unicité pour cette équation. Dans la cinquième sous-section nous proposons un ensemble de processus parmi lesquels la solution intégrale de l'équation est unique : la régularité de σ et β est augmentée pour montrer l'unicité de la solution intégrale dans la classe $\mathcal{C}^2_\xi(\mathbb{H})$, en sachant que (ξ, M) vérifie l'hypothèse $(\mathcal{D})$ par rapport à la filtration $\mathbb{H}$. La sixième sous-section est dédiée à une révision des résultats précédents, lorsque ξ est supposé avoir une variation quadratique. L'hypothèse de variation quadratique finie nous permet de remplacer l'intégrale symétrique par l'intégrale forward et d'utiliser une formule d'Itô pour des fonctions de classe C^2. Les conditions de régularité sur σ sont ainsi affaiblies. Dans la septième sous-section nous appliquons notre méthode aux processus ξ à trajectoires hölderiennes. La huitième sous-section décrit comment notre méthode peut être combinée aux résultats de [44] pour traiter le cas spécifique d'une équation dirigée par un mouvement brownien fractionnaire. La fin du deuxième chapitre contient un résultat original sur l'existence d'une solution d'une équation de type Stratonovich dirigée par un mouvement brownien, à coefficient de diffusion continu et avec un *drift* continu mesurable borné.

Chapitre 1

Marchés financiers sans semimartingale

Dans ce chapitre nous ne supposons pas, a priori, que les prix d'un actif S soient des semimartingales. Puisque les stratégies admissibles d'un investisseur sont autofinancées, S est contraint d'être un processus à variation quadratique finie. En dépit du fait que S ne soit pas une semimartingale, la propriété de non-arbitrage est assurée si la classe $\mathcal{A}$ des stratégies admissibles est restreinte. Nous remplaçons la notion de semimartingale classique par celle de $\mathcal{A}$-semimartingale et nous développons un calcul relatif à celle-ci : des exemples sont proposés. Nous concluons avec quelques applications à la viabilité du marché, la couverture, et la maximisation de l'utilité d'un investisseur, potentiellement initié.[1]

1.1 Introduction

According to the fundamental theorem of asset pricing of Delbaen and Schachermayer in [16], in absence of *free lunches with vanishing risk* (NFLVR), when investing possibilities run only trough simple predictable strategies with respect to some filtration $\mathbb{G}$, the price process of the risky asset S is forced to be a semimartingale. However (NFLVR) condition could not be reasonable in several situations. In that case S may not be a semimartingale. We illustrate here some of those circumstances.

Generally, admissible strategies are let vary in a quite large class of predictable processes with respect to some filtration $\mathbb{G}$, representing the information flow available to the investor. As a matter of fact, the class of admissible strategies could be reduced because of different market regulations or for practical reasons. For instance, the investor could not be allowed to hold more than a certain number of stock shares. On the other hand it could be realistic to impose a minimal delay between two possible transactions as suggested by Cheridito ([10]) : when the logarithmic price $\log(S)$ is a geometric fractional Brownian motion (fbm), it is impossible to realize arbitrage possibilities satisfying that minimal requirement. We remind that without that restriction, the market admits arbitrages, see for

[1] Ce chapitre fait l'objet d'une prépublication en collaboration avec Francesco RUSSO

instance [50]. When the logarithmic price of S is a fbm or some particular strong Markov process, arbitrages can be excluded taking into account proportional transactions costs : Guasoni ([32]) has shown that, in that case, the class of admissible strategies has to be restricted to bounded variation processes and this rules out arbitrages.

Besides the restriction of the class of admissible strategies, the adoption of non-semimartingale models finds its justification when the no-arbitrage condition itself is not likely.

Empirical observations reveal, indeed, that S could fail to be a semimartingale because of market imperfections due to micro-structure noise, as intra-day effects. A model which considers those imperfections would add to W, the Brownian motion describing log-prices, a zero quadratic variation process, as a fractional Brownian motion of Hurst index greater than $\frac{1}{2}$, see for instance [61]. Theoretically arbitrages in very small time interval could be possible, which would be compatible with the lack of semimartingale property.

At the same way if (FLVR) are not possible for an *honest investor*, an *inside trader* could realize a free lunch with respect to the enlarged filtration $\mathbb{G}$ including the one generated by prices and the extra-information. Again in that case S may not be a semimartingale. The literature concerning inside trading and asymmetry of information has been extensively enriched by several papers in the last ten years ; among them we quote Pikowski and Karatzas ([46]), Grorud and Pontier ([31]), Amendinger, Imkeller and Schweizer ([1]). They adopt enlargement of filtration techniques to describe the evolution of stock prices in the insider filtration.

Recently, some authors approached the problem in a new way using in particular forward integrals, in the framework of stochastic calculus via regularizations. For a comprehensive survey of that calculus see [57]. Indeed, forward integrals could exist also for non-semimartingale integrators. Leon, Navarro and Nualart in [39], for instance, solve the problem of maximization of expected logarithmic utility of an agent who holds an initial information depending on the future of prices. They operate under technical conditions which, a priori, do not imply the classical assumption (H') for enlargement considered in [37]. Using forward integrals, they determine the utility maximum. However, a posteriori, they found out that their conditions let S be a semimartingale.

Biagini and Øksendal ([5]) considered somehow the converse implication. Supposing that the maximum utility is attained, they proved that S is a semimartingale. Ankkirchner and Imkeller ([2]) continue to develop the enlargement of filtrations techniques and show, among the others, a similar result as [5] using the fundamental theorem of asset pricing of Delbaen-Schachermayer. In particular they establish a link between that fundamental theorem and finite utility.

In our paper we treat a market where there are one risky asset, whose price is a strictly positive process S, and a *less risky* asset with price S^0, possibly riskless but a priori only with bounded variation. A class $\mathcal{A}$ of admissible trading strategies is specified. If $\mathcal{A}$ is not large enough to generate all predictable simple strategies, then S has no need to be a semimartingale, even requiring the absence of free lunches among those strategies. We try to build the basis of a corresponding financial theory which allows to deal with several problems as hedging and non-arbitrage pricing, viability and completeness as well as with utility maximization.

For the sake of simplicity in this introduction we suppose that the less risky asset S^0 is constant and equal to 1.

As anticipated, a natural tool to describe the self-financing condition is the forward integral of an integrand process Y with respect to an integrator X, denoted by $\int_0^t Y d^- X$; see section 1.2 for definitions. Let $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le 1}$ be a filtration on an underlying probability space $(\Omega, \mathcal{F}, P)$, with $\mathcal{F} = \mathcal{G}_1$; $\mathbb{G}$ represents the flow of information available to the investor. A **self-financing portfolio** is a pair (X_0, h) where X_0 is the initial value of the portfolio and h is a $\mathbb{G}$-adapted and S-forward integrable process specifying the number of shares of S held in the portfolio. The market value process X of such a portfolio, is given by $X_0 + \int_0^\cdot h_s d^- S_s$, while $h_t^0 = X_t - S_t h_t$ constitutes the number of shares of the less risky asset held.

This formulation of self-financing condition is coherent with the discrete-time case. Indeed, let we consider a *buy-and-hold strategy*, i.e. a pair (X_0, h) with $h = \eta I_{(t_0, t_1]}, 0 \le t_0 \le t_1 \le 1$, and η being a $\mathcal{G}_{t_0}$-measurable random variable. Using the definition of forward integral it is not difficult to see that : $X_{t_0} = X_0$, $X_{t_1} = X_0 + \eta(S_{t_1} - S_{t_0})$. This implies $h^0_{t_0^+} = X_0 - \eta S_{t_0}$, $h^0_{t_1^+} = X_0 + \eta(S_{t_1} - S_{t_0})$ and

$$X_{t_0} = h_{t_0^+} S_{t_0} + h^0_{t_0^+}, \quad X_{t_1} = h_{t_1^+} S_{t_1} + h^0_{t_1^+} :$$

at the *re-balancing* dates t_0 and t_1, the value of the old portfolio must be reinvested to build the new portfolio without exogenous withdrawal of money.

In this paper $\mathcal{A}$ will be a real linear subspace of all self-financing portfolios and it will constitute, by definition, the class of all **admissible** portfolios. $\mathcal{A}$ will depend on the kind of problems one has to face : hedging, utility maximization, modeling inside trading. If we require that S belongs to $\mathcal{A}$, then the process S is forced to be a finite quadratic variation process. In fact, $\int_0^\cdot S d^- S$ exists if and only if the quadratic variation $[S]$ exists, see [57] ; in particular one would have

$$\int_0^\cdot S_s d^- S_s = S^2 - S_0^2 - \frac{1}{2}[S].$$

$\mathcal{L}$ will be the sub-linear space of $L^0(\Omega)$ representing a set of **contingent claims of interest** for one investor. An $\mathcal{A}$**-attainable contingent claim** will be a random variable C for which there is a self-financing portfolio (X_0, h) with $h \in \mathcal{A}$ and

$$C = X_0 + \int_0^1 h_s d^- S_s.$$

X_0 will be called **replication price** for C. The market will be said $(\mathcal{A}, \mathcal{L})$**-complete** if every element of $\mathcal{L}$ is $\mathcal{A}$-attainable.

In these introductory lines we will focus only on one particular elementary situation.

For simplicity we illustrate the case where $[\log(S)]_t = \sigma^2 t$. We choose as $\mathcal{L}$ the set of all *European* contingent claims $C = \psi(S_1)$ where ψ is continuous with polynomial growth. We consider the case $\mathcal{A} = \mathcal{A}_S$, where

$$\begin{aligned} \mathcal{A}_S \quad = \quad & \{(u(t, S_t)), 0 \le t < 1 \mid u : [0,1] \times \mathbb{R} \to \mathbb{R}, \text{ Borel-measurable} \\ & \text{with polynomial growth and lower bounded}\} . \end{aligned}$$

Such a market is $(\mathcal{A}, \mathcal{L})$-complete : in fact, a random variable $C = \psi(S_1)$ is an $\mathcal{A}$-attainable contingent claim. To build a replicating strategy the investor has to choose v as solution of the following problem

$$\begin{cases} \partial_t v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial^{(2)}_{xx} v(t,x) & = & 0 \\ v(1,x) & = & \psi(x) \end{cases}$$

and $X_0 = v(0, S_0)$. This follows easily after application of Itô formula contained in proposition 1.2.11, see proposition 1.5.29.

We highlight that this method can be adjusted to hedge also *Asian* contingent claims.

A crucial concept is the one of $\mathcal{A}$-martingale processes. Those processes naturally intervene in utility maximization, arbitrage and uniqueness of hedging prices.

A process M is said to be an $\mathcal{A}$-**martingale** if for any process $Y \in \mathcal{A}$,

$$E\left[\int_0^{\cdot} Y d^- M\right] = 0.$$

If for some filtration $\mathbb{F}$ with respect to which M is adapted, $\mathcal{A}$ contains the class of all bounded $\mathbb{F}$-predictable processes, then M is an $\mathbb{F}$-martingale.

An example of $\mathcal{A}$-martingale is the so called **weak Brownian motion of order** $k = 1$ and quadratic variation equal to t. That notion was introduced in [27] : a weak Brownian motion of order 1 is a process X such that the law of X_t is $N(0,t)$ for any $t \geq 0$.

A portfolio (X_0, h) is said to be an $\mathcal{A}$-arbitrage if $h \in \mathcal{A}$, $X_1 \geq X_0$ almost surely and $P\{X_1 - X_0 > 0\} > 0$. We denote by $\mathcal{M}$ the set of probability measures being equivalent to the initial probability P under which S is an $\mathcal{A}$-martingale. If $\mathcal{M}$ is non empty then the market is $\mathcal{A}$-arbitrage free. In fact if $Q \in \mathcal{M}$, given a pair (X_0, h) which is an $\mathcal{A}$-arbitrage, then $E^Q[X_1 - X_0] = E^Q[\int_0^1 h d^- S] = 0$. In that case the replication price X_0 of an $\mathcal{A}$-attainable contingent claim C is unique, provided that the process $h\eta$, for any bounded random variable η in $\mathcal{G}_0$ and h in $\mathcal{A}$, still belongs to $\mathcal{A}$. Moreover $X_0 = E^Q[C|\mathcal{G}_0]$. In reality, under the weaker assumption that the market is $\mathcal{A}$-arbitrage free, the replication price is still unique, see proposition 1.5.27. Furthermore if $\mathcal{M}$ is non empty and $\mathcal{A} = \mathcal{A}_S$, as assumed in this section, the law of S_t has to be equivalent to Lebesgue measure for every $0 \leq t \leq 1$, see proposition 1.5.21.

If the market is $(\mathcal{A}, \mathcal{L})$-complete then all the probabilities measures in $\mathcal{M}$ coincide on $\sigma(\mathcal{L})$, see proposition 1.5.28. If $\sigma(\mathcal{L}) = \mathcal{F}$ then $\mathcal{M}$ is a singleton : this result recovers the classical case.

Given an utility function satisfying usual assumptions, it is possible to show that the maximum π is attained on a class of portfolios fulfilling conditions related to assumption 1.5.37, if and only if there exists a probability measure under which $\log(S) - \int_0^{\cdot} \left(\sigma^2 \pi_t - \frac{1}{2}\sigma^2\right) dt$ is an $\mathcal{A}$-martingale, see proposition 1.5.44. Therefore if $\mathcal{A}$ is big enough to fulfill conditions related to assumption $\mathcal{D}$ in Definition 1.4.6, then S is a classical semimartingale.

Those considerations show that most of the classical results of basic financial theory admit a natural extension to non-semimartingale models.

The paper is organized as follows. After some preliminaries about stochastic calculus via regularizations for forward integrals, we provide in section 3 examples of integrators and integrands for which forward integrals exist and realize some important properties in view of financial applications : those examples appear in three essential situations coming from Malliavin calculus, substitution formulae and Itô-fields. Regarding finance applications, the class of strategies defined using Malliavin calculus are useful when $\log(S)$ is a geometric Brownian motion with respect to a filtration $\mathbb{F}$ contained in $\mathbb{G}$; the use of substitution formulae naturally appear when trading with an initial extra information, already available at time 0 ; Itô fields apply whenever S is a generic finite quadratic variation process.

Section 4 is devoted to the study of $\mathcal{A}$-martingales : after having defined and established basic properties, we explore the relation between $\mathcal{A}$-martingales and weak Brownian motion ; later we discuss the link between the existence of a maximum for a an optimization problem and the $\mathcal{A}$-martingale property.

In Section 5 we finally deal with applications to mathematical finance. We define self-financing portfolio strategies and we provide examples. Moreover we face technical problems related to the use of forward integral in order to describe the evolution of the wealth process. Those problems arise because of the lack of chain rule properties. Later, we discuss absence of $\mathcal{A}$-arbitrages, $(\mathcal{A}, \mathcal{L})$-completeness and hedging. We conclude the section analyzing the problem of maximizing expected utility from terminal wealth. We obtain results about the existence of an *optimal portfolio* generalizing those of [39] and [5].

1.2 Preliminaries

For the convenience of the reader we give some basic concepts and fundamental results about stochastic calculus with respect to finite quadratic variation processes which will be extensively used later. For more details we refer the reader to [57].

In the whole paper $(\Omega, \mathcal{F}, P)$ will be a fixed probability space. For a stochastic process $X = (X_t, 0 \le t \le 1)$ defined on $(\Omega, \mathcal{F}, P)$ we will adopt the convention $X_t = X_{(t \vee 0) \wedge 1}$, for t in $\mathbb{R}$. Let $0 \le T \le 1$. We will say that a sequence of processes $(X_t^n, 0 \le t \le T)_{n \in \mathbb{N}}$ **converges uniformly in probability (ucp) on** $[0,T]$ toward a process $(X_t, 0 \le t \le T)$, if $\sup_{t \in [0,T]} |X_t^n - X_t|$ converges to zero in probability.

Definition 1.2.1. *1. Let $X = (X_t, 0 \le t \le T)$ and $Y = (Y_t, 0 \le t \le T)$ be processes with paths respectively in $C^0([0,T])$ and $L^1([0,T])$. Set, for every $0 \le t \le T$,*

$$I(\varepsilon, Y, X, t) = \frac{1}{\varepsilon} \int_0^t Y_s \left(X_{s+\varepsilon} - X_s \right) ds,$$

and

$$C(\varepsilon, X, Y, t) = \frac{1}{\varepsilon} \int_0^t \left(Y_{s+\varepsilon} - Y_s \right) \left(X_{s+\varepsilon} - X_s \right) ds.$$

If $I(\varepsilon, Y, X, t)$ converges in probability for every t in $[0,T]$, and the limiting process admits a continuous version $I(Y, X, t)$ on $[0,T]$, Y is said to be ***X-forward integrable*** *on $[0,T]$. The process $(I(Y, X, t), 0 \le t \le T)$ is denoted by $\int_0^\cdot Y d^- X$. If $I(\varepsilon, Y, X, \cdot)$ converges ucp on $[0,T]$ we will say that the forward integral $\int_0^\cdot Y d^- X$ is the* ***limit ucp of its regularizations****.*

2. *If* $(C(\varepsilon, X, Y, t), 0 \leq t \leq T)$ *converges ucp on* $[0, T]$ *when* ε *tends to zero, the limit will be called the* ***covariation process*** *between* X *and* Y *and it will be denoted by* $[X, Y]$. *If* $X = Y$, $[X, X]$ *is called the* ***finite quadratic variation*** *of* X *: it will also be denoted by* $[X]$, *and* X *will be said to be a* ***finite quadratic variation process*** *on* $[0, T]$.

Definition 1.2.2. *We will say that a process* $X = (X_t, 0 \leq t \leq T)$, *is* ***localized by the sequence*** $\left(\Omega_k, X^k\right)_{k \in \mathbb{N}^*}$, *if* $P\left(\cup_{k=0}^{+\infty} \Omega_k\right) = 1$, $\Omega_h \subseteq \Omega_k$, *if* $h \leq k$, *and* $I_{\Omega_k} X^k = I_{\Omega_k} X$, *almost surely for every* k *in* $\mathbb{N}$.

Remark 1.2.3. *Let* $(X_t, 0 \leq t \leq T)$ *and* $(Y, 0 \leq t \leq T)$ *be two stochastic processes. The following statements are true.*

1. *Let* Y *and* X *be localized by the sequences* $\left(\Omega_k, X^k\right)_{k \in \mathbb{N}}$ *and* $\left(\Omega_k, Y^k\right)_{k \in \mathbb{N}}$, *respectively, such that* Y^k *is* X^k*-forward integrable on* $[0, T]$ *for every* k *in* $\mathbb{N}$. *Then* Y *is* X*-forward integrable on* $[0, T]$ *and*
$$\int_0^{\cdot} Y d^- X = \int_0^{\cdot} Y^k d^- X^k, \qquad \text{on } \Omega_k, \quad a.s..$$
2. *If* Y *is* X*-forward integrable on* $[0, T]$, *then* $Y I_{[0,t]}$ *is* X*-forward integrable for every* $0 \leq t \leq T$, *and*
$$\int_0^{\cdot} Y_s I_{[0,t]} d^- X_s = \int_0^{\cdot \wedge t} Y_s d^- X_s.$$
3. *If the covariation process* $[X, Y]$ *exists on* $[0, T]$, *then the covariation* $[X, Y I_{[0,t]}]$ *exists for every* $0 \leq s \leq t \leq T$, *and*
$$\left[X, Y I_{[0,t]}\right]_s = [X, Y]_{t \wedge s}.$$

Definition 1.2.4. *Let* $X = (X_t, 0 \leq t \leq T)$ *and* $Y = (Y_t, 0 \leq t < T)$ *be processes with paths respectively in* $C^0([0, T])$ *and* $L^1_{loc}([0, T))$, *i.e.* $\int_0^t |Y_s| \, ds < +\infty$ *for any* $t < T$.

1. *If* $Y I_{[0,t]}$ *is* X*-forward integrable for every* $0 \leq t < T$, Y *is said* ***locally*** X***-forward integrable on*** $[0, T)$. *In this case there exists a continuous process, which coincides, on every compact interval* $[0, t]$ *of* $[0, 1)$, *with the forward integral of* $Y I_{[0,t]}$ *with respect to* X. *That process will still be denoted with* $I(\cdot, Y, X) = \int_0^{\cdot} Y d^- X$.
2. *If* Y *is locally* X*-forward integrable and* $\lim_{t \to T} I(t, Y, X)$ *exists almost surely,* Y *is said* X***-improperly forward integrable*** *on* $[0, T]$.
3. *If the covariation process* $[X, Y I_{[0,t]}]$ *exists, for every* $0 \leq t < T$, *we say that the* ***covariation process*** $[X, Y]$ ***exists locally*** *on* $[0, T)$ *and it is still denoted by* $[X, Y]$. *In this case there exists a continuous process, which coincides, on every compact interval* $[0, t]$ *of* $[0, 1)$, *with the covariation process* $\left[X, Y I_{[0,t]}\right]$. *That process will still be denoted with* $[X, Y]$. *If* $X = Y$, *we will say that the* ***quadratic variation of*** X ***exists locally*** *on* $[0, T]$.
4. *If the covariation process* $[X, Y]$ *exists locally on* $[0, T)$ *and* $\lim_{t \to T} [X, Y]_t$ *exists, the limit will be called the* ***improper covariation process*** *between* X *and* Y *and it will still be denoted by* $[X, Y]$. *If* $X = Y$, *we will say that the* ***quadratic variation of*** X ***exists improperly*** *on* $[0, T]$.

Remark 1.2.5. *Let $X = (X_t, 0 \leq t \leq T)$ and $Y = (Y_t, 0 \leq t \leq T)$ be two stochastic processes being in $C^0([0,1])$ and $L^1([0,1])$, respectively. If Y is X-forward integrable on $[0,T]$ then its restriction to $[0,1)$ is X-improperly forward integrable and the improper integral coincides with the forward integral of Y with respect to X.*

Definition 1.2.6. *A vector $((X_t^1, ..., X_t^m), 0 \leq t \leq T)$ of continuous processes is said to have all its* ***mutual brackets*** *on $[0,T]$ if $[X^i, X^j]$ exists on $[0,T]$ for every $i,j = 1,...,m$.*

In the sequel if $T = 1$ we will omit to specify that objects defined above exist on the interval $[0,1]$ (or $[0,1)$, respectively).

Proposition 1.2.7. *Let $M = (M_t, 0 \leq t \leq T)$ be a continuous local martingale with respect to some filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ of $\mathcal{F}$. Then the following properties hold.*

1. *The process M is a finite quadratic variation process on $[0,T]$ and its quadratic variation coincides with the classical bracket appearing in the Doob decomposition of M^2.*
2. *Let $Y = (Y_t, 0 \leq t \leq T)$ be an $\mathbb{F}$-adapted process with left continuous and bounded paths. Then Y is M-forward integrable on $[0,T]$ and $\int_0^\cdot Y d^- M$ coincides with the classical Itô integral $\int_0^\cdot Y dM$.*

Proposition 1.2.8. *Let $V = (V_t, 0 \leq t \leq T)$ be a bounded variation and continuous process and $Y = (Y_t, 0 \leq t \leq T)$, be a process with paths being bounded and with at most countable discontinuities. Then the following properties hold.*

1. *The process Y is V-forward integrable on $[0,T]$ and $\int_0^\cdot Y d^- V$ coincides with the Lebesgue-Stieltjes integral denoted with $\int_0^\cdot Y dV$.*
2. *The covariation process $[Y,V]$ exists on $[0,T]$ and it is equal to zero. In particular a bounded variation process has zero quadratic variation.*

Corollary 1.2.9. *Let $X = (X_t, 0 \leq t \leq T)$ be a continuous process and $Y = (Y_t, 0 \leq t \leq T)$ a bounded variation and continuous process. Then*

$$XY - X_0 Y_0 = \int_0^\cdot X_s dY_s + \int_0^\cdot Y_s d^- X_s.$$

Proposition 1.2.10. *Let $X = (X_t, 0 \leq t \leq T)$ be a continuous finite quadratic variation process, and f a function in $C^1(\mathbb{R})$. Then $Y = f(X)$ has a finite quadratic variation on $[0,T]$ and $[Y] = \int_0^\cdot f'(X)^2 d[X]$.*

Proposition 1.2.11. *Let $X = (X_t, 0 \leq t \leq T)$ be a continuous finite quadratic variation process and $V = ((V_t^1, \ldots, V_t^m), 0 \leq t \leq T)$ be a vector of continuous bounded variation processes. Then for every u in $C^{1,2}(\mathbb{R}^m \times \mathbb{R})$, the process $(\partial_x u(V_t, X_t), 0 \leq t \leq T)$ is X-forward integrable on $[0,T]$ and*

$$\begin{aligned} u(V,X) &= u(V_0,X_0) + \sum_{i=1}^m \int_0^\cdot \partial_{v_i} u(V_t,X_t) dV_t^i + \int_0^\cdot \partial_x u(V_t,X_t) d^- X_t \\ &+ \frac{1}{2}\int_0^\cdot \partial_{xx}^{(2)} u(V_t,X_t) d[X]_t. \end{aligned}$$

Lemma 1.2.12. *Let $X = (X_t^1, \dots, X_t^m, 0 \le t \le T)$ be a vector of continuous processes having all its mutual brackets. Let $\psi : \mathbb{R}^m \to \mathbb{R}$ be of class $C^2(\mathbb{R}^m)$ and $Y = \psi(X)$. Then, if $Z\partial_{x^i}\psi(X)$ is X^i-forward integrable on $[0,T]$, for every $i = 1, ..., m$, Z is Y-forward integrable on $[0,T]$ and*

$$\int_0^\cdot Z d^- Y = \sum_{i=1}^m \int_0^\cdot Z\partial_{x^i}\psi(X) d^- X^i + \frac{1}{2}\sum_{i,j=0}^m \int_0^\cdot Z\partial^{(2)}_{x^i x^j}\psi(X) d\left[X^i, X^j\right].$$

Proof. The proof derives from proposition 4.3 of [56]. The result is a slight modification of that one. It should only be noted that there forward integral of a process Y with respect to a process X was defined as limit ucp of its regularizations.

□

1.3 Existence of forward integrals and properties

In this section we illustrate examples of processes for which forward integrals exist and we list some related properties which will be extensively used in further applications to finance.

1.3.1 Forward integrals of Itô fields

In this subsection ξ will be a $\mathbb{G}$-adapted process with finite quadratic variation, where $\mathbb{G}$ is some filtration of $\mathcal{F}$. The following definitions and results are extracted from [24].

Definition 1.3.1. *Let k be in $\mathbb{N}^*$. A random field $(H(t,x), 0 \le t \le 1, x \in \mathbb{R})$ is called a C^k $\mathbb{G}$-**Itô-semimartingale field** driven by the vector $N = (N^1, ..., N^n)$, if N is a vector of semimartingales with respect to $\mathbb{G}$, and*

$$H(t,x) = f(x) + \sum_{i=1}^n \int_0^t a^i(s,x) dN_s^i, \quad 0 \le t \le 1,$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ belongs to $C^k(\mathbb{R})$ almost surely and it is $\mathbb{G}^0$-measurable for every x, H and $a^i : [0,1] \times \mathbb{R} \times \Omega \to \mathbb{R}$, $i = 1, ..., n$ are $\mathbb{G}$-adapted for every x, almost surely continuous with their partial derivatives with respect to x in (t,x) up to order k, and for every index $h \le k$ it holds

$$\partial_x^{(h)} H(t,x) = \partial_x^{(h)} f(x) + \sum_{i=1}^n \int_0^t \partial_x^{(h)} a^i(s,x) dN_s^i, \quad 0 \le t \le 1.$$

Definition 1.3.2. *We denote with $\mathcal{C}_\xi^k(\mathbb{G})$ the set of processes of the form*

$$(H(t,\xi_t), 0 \le t \le 1),$$

being $(H(t,x), 0 \le t \le 1, x \in \mathbb{R})$ a C^k $\mathbb{G}$-Itô-semimartingale field driven by the vector $N = (N^1, ..., N^n)$, such that $(N^1, ..., N^n, \xi)$ has all its mutual brackets.

Remark 1.3.3. *1. The set $\mathcal{C}^1_\xi(\mathbb{G})$ is an algebra.*

2. Let ψ be in $C^\infty(\mathbb{R})$ and h in $\mathcal{C}^2_\xi(\mathbb{G})$. Itô formula implies that $\psi(h)$ belongs to $\mathcal{C}^2_\xi(\mathbb{G})$.

Proposition 1.3.4. *Let h and k be in $\mathcal{C}^1_\xi(\mathbb{G})$. Then the following statements are true.*

1. *The process h is ξ-forward integrable, the forward integral $\int_0^\cdot h_t d^-\xi_t$ is the limit ucp of its regularizations and it belongs to $\mathcal{C}^2_\xi(\mathbb{G})$.*
2. *The covariation process $\left(\left[\int_0^\cdot h_t d^-\xi_t, \int_0^\cdot k_t d^-\xi_t\right]\right)$ exists and it is equal to $\int_0^\cdot h_t k_t d\,[\xi]_t$.*
3. *The process $\int_0^\cdot h_t d^-\xi_t$ is forward integrable with respect to the process $\int_0^\cdot k_t d^-\xi_t$ and*

$$\int_0^\cdot h_t d^- \int_0^t k_s d^-\xi_s = \int_0^\cdot h_t k_t d^-\xi_t.$$

Using remark 1.2.3 it is not difficult to prove that proposition 1.3.4 extends to processes which are *simple combinations* of processes in $\mathcal{C}^1_\xi(\mathbb{G})$. We illustrate this result below.

Definition 1.3.5. *Let $\mathcal{S}(\mathcal{C}^k_\xi(\mathbb{G}))$ be the set of all processes h of type*

$$h = h^0 I_{\{0\}} + \sum_{i=1}^m h^i I_{(t_{i-1},t_i]}$$

where $0 = t_0 \leq t_1, \cdots, t_m = 1$, and h^i belongs to $\mathcal{C}^k_\xi(\mathbb{G})$, for $i = 1, ..., m$.

Remark 1.3.6. *Thanks to remark 1.3.3, if h belongs to $\mathcal{S}(\mathcal{C}^k_\xi(\mathbb{G}))$ and ψ is of class $C^\infty(\mathbb{R})$, then $\psi(h)$ is still in $\mathcal{S}(\mathcal{C}^k_\xi(\mathbb{G}))$.*

Proposition 1.3.7. *Let h and k be in $\mathcal{S}(\mathcal{C}^1_\xi(\mathbb{G}))$. Then the we can state the following.*

1. *The process h is ξ-forward integrable and it belongs to $\mathcal{S}(\mathcal{C}^2_\xi(\mathbb{G}))$.*
2. *The covariation process $\left(\left[\int_0^\cdot h_t d^-\xi_t, \int_0^\cdot k_t d^-\xi_t\right]\right)$ exists and it is equal to $\int_0^\cdot h_t k_t d\,[\xi]_t$.*
3. *The process $\left(\int_0^\cdot h_t d^-\xi_t, 0 \leq t \leq 1\right)$ is forward integrable with respect to the process $\left(\int_0^\cdot k_t d^-\xi_t, 0 \leq t \leq 1\right)$ and*

$$\int_0^\cdot h_t d^- \int_0^t k_s d^-\xi_s = \int_0^\cdot h_t k_t d^-\xi_t.$$

Proof. By linearity of forward integral and bilinearity of covariation it is sufficient to prove the statement for processes of type $hI_{[0,t]}$ and $kI_{[0,s]}$, with h and k in $\mathcal{C}^1_\xi(\mathbb{G})$ and $0 \leq s \leq t \leq 1$. The proof is a consequence of remark 1.2.3 and proposition 1.3.4.

□

1.3.2 Forward integrals via Malliavin calculus

We work in the Malliavin calculus framework. To this extent we recall some basic notations and definitions from [45] and [43].

We suppose that $(\Omega, \mathbb{F}, \mathcal{F}, P)$ is the canonical probability space, meaning that $\Omega = C([0,1], \mathbb{R})$, P is the Wiener measure, W is the Wiener process, $\mathbb{F}$ is the filtration generated by W and the P-null sets and $\mathcal{F}$ is the completion of the Borel σ-algebra with respect to P.

Let $\mathcal{S}$ be the space of all random variables on $(\Omega, \mathcal{F}, P)$, of the form

$$F = f(W(t_1), ..., W(t_n)), \quad 0 \leq t_0, \cdots, t_n \leq 1,$$

with f in $C^\infty(\mathbb{R}^n)$ being bounded with its derivatives of all orders. The iterated derivative of order k operator is denoted by D^k. Then $D^k : \mathbb{D}^{k,p} \to L^p\left(\Omega \times [0,1]^k\right)$, where $\mathbb{D}^{k,p}$, $p \geq 2$, $k \in \mathbb{N}^*$, is the closure of $\mathcal{S}$ with respect to the norm

$$|| \, F \, ||^p_{\mathbb{D}^{k,p}} = ||F||^p_{L^p(\Omega)} + \sum_{j=1}^k \left|\left| \, ||D^j F||_{L^2([0,1]^j)} \right|\right|^p_{L^p(\Omega)}.$$

For any $p \geq 2$, $L^{1,p}$ denotes the space of all functions u in $L^p(\Omega \times [0,1])$ such that u_t belongs to $\mathbb{D}^{1,p}$ for every $0 \leq t \leq 1$ and there exists a measurable version of $(D_s u_t, 0 \leq s, t \leq 1)$ with $\int_0^1 \mathbb{E}\left[||Du_t||^p_{L^2([0,1])}\right] dt < \infty$. For every u in $L^{1,p}$ we denote $||u||^p_{L^{1,p}} = \int_0^1 ||u_t||^p_{\mathbb{D}^{1,p}} \, dt$. Similarly, for $p \geq 2$, $L^{2,p}$ denotes the space of all functions u in $L^p(\Omega \times [0,1])$, such that u_t belongs to $\mathbb{D}^{2,p}$ for every $0 \leq t \leq 1$ and there exist measurable versions of $(D_s u_t, 0 \leq s, t \leq 1)$ and $(D_r D_s u_t, 0 \leq s, t, r \leq 1)$ with

$$\int_0^1 \mathbb{E}\left[||Du_t||^p_{L^2([0,1])}\right] + \mathbb{E}\left[||D^2 u_t||^p_{L^2([0,1]^2)}\right] dt < \infty.$$

For every u in $L^{2,p}$ we denote $||u||^p_{L^{2,p}} = \int_0^1 ||u_t||^p_{\mathbb{D}^{2,p}} \, dt$.

The Skorohod integral δ is the adjoint of the derivative operator D; its domain is denoted by $Dom\delta$. An element u belonging to $Dom\delta$ is said Skorohod integrable. We recall that $\mathbb{D}^{1,2}$ is dense in $L^2(\Omega)$, $L^{1,2} \subset Dom\delta$, and that if u belongs to $L^{1,2}$ then, for each $0 \leq t \leq 1$, $uI_{[0,t]}$ is still in $L^{1,2}$. In particular it is Skorohod integrable. We will use the notation $\delta\left(uI_{[0,t]}\right) = \int_0^t u_s \delta W_s$, for each u in $L^{1,2}$. The process $\left(\int_0^t u_s \delta W_s, 0 \leq t \leq 1\right)$ is mean square continuous and then it admits a continuous version, which will be still denoted by $\int_0^\cdot u_t \delta W_t$. We finally recall that for every u in $L^{1,p}$ there exists a positive constant c_p such that

$$\begin{aligned} ||\delta(u)||^p_{L^p(\Omega)} &\leq c_p \left[\left(\int_0^1 |\mathbb{E}[u_t]|^2 \, dt\right)^{\frac{p}{2}} + \left|\left| \, ||Du||_{L^2([0,1]^2)} \right|\right|^p_{L^p(\Omega)}\right] \\ &\leq c_p \, ||u||^p_{L^{1,p}}. \end{aligned} \tag{1.1}$$

It is useful to remind the following result contained in [43], exercise 1.2.13.

Lemma 1.3.8. *Let F and G be two random variables in $\mathbb{D}^{1,2}$. Suppose that both G and $||DG||_{L^2([0,1])}$ are bounded. Then FG is still in $\mathbb{D}^{1,2}$ and $D(FG) = FDG + GDF$.*

Remark 1.3.9. *1. Let u and v be processes in $L^{1,p}$, for some $p \geq 2$, and in $L^{1,2}$, respectively, such that the random variable*

$$\sup_{t\in[0,1]} \left(|v_t| + \int_0^1 (D_s v_t)^2 ds \right)$$

is bounded. By lemma 1.3.8 the process uv belongs to $L^{1,p}$ and $Duv = uDv + vDu$.

2. Let u be a process in $L^{2,p}$, for some $p \geq 2$ and v in $L^{2,2}$ such that the random variable

$$\sup_{t\in[0,1]} \left(|v_t| + \int_0^1 (D_s v_t)^2 ds + \int_0^1 \int_0^1 (D_r D_s v_t)^2 dr ds \right)$$

is bounded. Then the process uv belongs to $L^{2,p}$.

In order to state a chain rule formula we will need the *Fubini*-type lemma below.

Lemma 1.3.10. *Let u be in $L^2\left(\Omega \times [0,1]^2\right)$. Assume that for every $0 \leq t \leq 1$, the process $u(\cdot,t)$ belongs to $L^{1,2}$, that there exist measurable versions of the two processes $(\delta(u(\cdot,t), 0 \leq t \leq 1)$ and $(D_r u(s,t), 0 \leq r,s,t \leq 1)$ and that*

$$\mathbb{E}\left[\int_0^1 ||Du(\cdot,t)||^2_{L^2([0,1]^2)}\, dt\right] < +\infty. \tag{1.2}$$

Then the process $\left(\int_0^1 u(s,t)dt, 0 \leq s \leq 1\right)$ belongs to $L^{1,2}$ and

$$\delta\left(\int_0^1 u(\cdot,t)dt\right) = \int_0^1 \delta(u(\cdot,t))dt.$$

Proof. Consider the process $(g_s, 0 \leq s \leq 1)$ so defined : $g_s = \int_0^1 u(s,t)dt$. Let $0 \leq s \leq 1$ be fixed. Since $(u(s,t), 0 \leq t \leq 1)$ is in $L^{1,2}$, g_s is in $\mathbb{D}^{1,2}$ and $Dg_s = \int_0^1 Du(s,t)dt$. By Fubini theorem $\left(\int_0^1 D_r u(s,t)dt, 0 \leq r,s \leq 1\right)$ admits a measurable version. Thanks to inequality (1.2), $\int_0^1 \mathbb{E}\left[||Dg_s||_{L^2([0,1])}\right] ds < +\infty$. This implies that g is in $L^{1,2}$. The conclusion of the proof is achieved using exercise 3.2.8, page 174 of [43].

□ □

Definition 1.3.11. *For every $p \geq 2$, $L^{1,p}_-$ will be the space of all processes u belonging to $L^{1,p}$ such that $\lim_{\varepsilon\to 0} D_t u_{t-\varepsilon}$ exists in $L^p(\Omega \times [0,1])$. The limiting process will be denoted by $\left(D_t^- u_t, 0 \leq t \leq 1\right)$.*

Remark 1.3.12. *1. If u belongs to $L_-^{1,p}$ then*

$$\mathbb{E}\left[\int_0^1 \left(\frac{1}{\varepsilon}\int_s^{s+\varepsilon} |D_r u_s - D_r^- u_r|^p\, dr\right) ds\right] \tag{1.3}$$

converges to zero when ε tends to zero. Indeed term (1.3) equals $\frac{1}{\varepsilon}\int_0^\varepsilon f(z)dz$, with $f(z) = \mathbb{E}\left[\int_0^1 |D_r u_{r-z} - D_r^- u_r|^p\, dr\right]$, and $\lim_{z\to 0} f(z) = 0$.

2. Let u and v be two left continuous processes respectively in $L_-^{1,p}$ and $L_-^{1,2}$ with $p \geq 2$. Suppose, furthermore, that $\sup_{t\in[0,1]} |u_t|$ belongs to $L^p(\Omega)$ and that the random variable $\sup_{t\in[0,1]} \left(|v_t| + \sup_{s\in[0,1]} |D_s v_t|\right)$ is bounded. Then uv belongs to $L_-^{1,q}$, for every $2 \leq q < p$. Moreover $D^- uv = uD^- v + vD^- u$. In particular v belongs to $L_-^{1,q}$, for every $q \geq 2$.

The hypothesis on the left continuity of u and v on point 2. of previous remark allows us to show that

$$\lim_{\varepsilon\to 0}\left[\int_0^1 |z_{t-\varepsilon} - z_t|^p\, dt\right] = 0, \quad z = u, v, \quad a.s. \tag{1.4}$$

That condition could be relaxed. It would be enough to suppose that,

$$\lambda((0 \leq t \leq 1, s.t.\, |z_t - z_{t^-}| \neq 0)) = 0,$$

almost surely, for $z = u, v$, being λ the Lebesgue measure on $\mathcal{B}([0,1])$. Nevertheless, convergence in (1.4) does not hold for every bounded process. To see this it is sufficient to consider, for instance, $z = I_{\mathbb{Q}\cap[0,1]}$.

Lemma 1.3.13. *Let u and v be respectively in $L_-^{1,p}$, $p \geq 2$, and $L_-^{1,2}$. Suppose that the random variable $\sup_{t\in[0,1]} \left(|v_t| + \sup_{s\in[0,1]} |D_s v_t|\right)$ is bounded. Then the sequence of processes*

$$\left(\frac{1}{\varepsilon}\int_0^1 u_t v_t (W_{t+\varepsilon} - W_t) dt - \frac{1}{\varepsilon}\int_0^1 u_t \left(\int_t^{t+\varepsilon} v_s \delta W_s\right) dt\right)_{\varepsilon > 0}$$

converges in $L^q(\Omega)$ to $\int_0^1 u_t D_t^- v_t dt$, for every $2 \leq q < p$.

Proof. Set

$$A_\varepsilon = \frac{1}{\varepsilon}\int_0^1 u_t v_t \left(W_{t+\varepsilon} - W_t\right) dt, \quad B_\varepsilon = \frac{1}{\varepsilon}\int_0^1 u_t \left(\int_t^{t+\varepsilon} v_s \delta W_s\right) dt.$$

Proposition 1.3.4 in section 1.3 of [43] permits to rewrite A_ε in the following way :

$$A_\varepsilon = \frac{1}{\varepsilon}\int_0^1 u_t v_t \left(\int_t^{t+\varepsilon} I_{[0,1]}(s) \delta W_s\right) dt.$$

Moreover, by point 1. of remark 1.3.9, $Duv = vDu + uDv$. For every $0 \leq t \leq 1$, the random variables $\int_t^{t+\varepsilon} D_s(u_t v_t) ds$ and $\int_t^{t+\varepsilon} D_s u_t v_s ds$ are square integrable. Therefore property (4) in section 1.3 of [43] can be exploited to write

$$A_\varepsilon = \frac{1}{\varepsilon}\int_0^1 \left(\int_t^{t+\varepsilon} u_t v_t \delta W_s\right) dt + \frac{1}{\varepsilon}\int_0^1 \left(\int_t^{t+\varepsilon} D_s(u_t v_t) ds\right) dt,$$

and

$$B_\varepsilon = \frac{1}{\varepsilon}\int_0^1 \left(\int_t^{t+\varepsilon} u_t v_s \delta W_s\right) dt + \frac{1}{\varepsilon}\int_0^1 \left(\int_t^{t+\varepsilon} D_s u_t v_s ds\right) dt.$$

This implies

$$\begin{aligned} A_\varepsilon - B_\varepsilon &= \int_0^1 \left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon} u_t(v_t - v_s)\delta W_s\right) dt \\ &+ \int_0^1 \left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon} (v_t - v_s) D_s u_t ds\right) dt + \int_0^1 \left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon} u_t D_s v_t ds\right) dt \\ &= I_\varepsilon^1 + I_\varepsilon^2 + I_\varepsilon^3. \end{aligned}$$

We observe that the function $(\omega, s, t) \longmapsto I_{(t,t+\varepsilon]}(s) u_t(\omega)(v_t - v_s)(\omega)$, for every (ω, s, t) in $\Omega \times [0,1]^2$, satisfies the hypotheses of lemma 1.3.10. Therefore I_ε^1 can be rewritten as follows

$$I_\varepsilon^1 = \int_0^1 \left(\frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} u_t(v_t - v_s) dt\right) \delta W_s.$$

Using inequality (1.1) it is possible to prove that there exists a positive constant c such that

$$\mathbb{E}\left[\left|I_\varepsilon^1\right|^p\right] \le c\mathbb{E}\left[\int_0^1 \left(|u_t|^p + \left(\int_0^1 (D_r u_t)^2 dr\right)^{\frac{p}{2}}\right) h_t^\varepsilon dt\right],$$

with

$$h_t^\varepsilon = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \left(|v_t - v_s|^p + \left(\int_0^1 \left|D_r v_t - D_r v_s\right|^2 dr\right)^{\frac{p}{2}}\right) ds.$$

Since $\sup_{t\in[0,1]} \left(|v_t| + \sup_{s\in[0,1]} |D_s v_t|\right)$ is a bounded random variable, for almost all (ω, t), h_t^ε converges to zero when ε goes to zero. Consequently, Lebesgue dominated convergence theorem applies to conclude that $\mathbb{E}\left[\left|I_\varepsilon^1\right|^p\right]$ converges to zero.

Considering the term I_ε^2, hölder inequality and the boundedness of v lead to

$$\begin{aligned} \mathbb{E}\left[\left|I_\varepsilon^2\right|^p\right] &\le c\mathbb{E}\left[\int_0^1 \left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \left|D_s u_t - D_s^- u_s\right|^p ds\right) dt\right] \\ &+ c\mathbb{E}\left[\int_0^1 \left(\frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} |v_t - v_s|^p dt\right) \left|D_s^- u_s\right|^p ds\right], \end{aligned}$$

for some positive constant c. The first term of previous sum converges to zero by point 1. of remark 1.3.12 ; the second by Lebesgue dominated convergence theorem.

Finally I_ε^3 may be rewritten as follows :

$$\begin{aligned} I_\varepsilon^3 &= \int_0^1 \left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon} (D_s v_t - D_s^- v_s) ds\right) u_t dt \\ &+ \int_0^1 \frac{1}{\varepsilon}\int_t^{t+\varepsilon} D_s^- v_s (u_t - u_s) ds dt + \int_0^1 u_s D_s^- v_s ds. \end{aligned}$$

Hölder inequality and again remark 1.3.12 implies the convergence to zero in $L^q(\Omega)$ of the first term of the sum for every $2 \leq q < p$. The convergence to zero of the second term of the sum in $L^p(\Omega)$ is due to the boundedness of $|D^- v|$ and the following maximal inequality contained in [60], theorem 1. :

$$\int_0^1 \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon} \int_t^{t+\varepsilon} |z_s|^p \, ds \right) dt \leq \int_0^1 |z_t|^p \, dt, \quad z \in L^p(\Omega \times [0,1]).$$

This leads to the conclusion.

□

We omit the proof of the following lemma which is, indeed, a slight modification of the proof of previous one.

Lemma 1.3.14. *Let v be in $L_-^{1,p}$, $p \geq 2$. Then the sequence of processes*

$$\left(\frac{1}{\varepsilon} \int_0^1 v_t (W_{t+\varepsilon} - W_t) dt - \frac{1}{\varepsilon} \int_0^1 \left(\int_t^{t+\varepsilon} v_s \delta W_s \right) dt \right)_{\varepsilon>0}$$

converges in $L^p(\Omega)$ to $\int_0^1 D_t^- v_t dt$.

Lemma 1.3.15. *Let u be a process in $L^{1,p}$ with $p \geq 2$. Then the process*

$$\left(\int_0^t u_s ds, 0 \leq t \leq 1 \right)$$

belongs to $L_-^{1,p}$, and $D^- \left(\int_0^\cdot u_t dt \right) = \int_0^\cdot D u_t dt$.

Proof. We set $g_t = \int_0^t u_s ds$. Clearly g is in $L^p(\Omega \times [0,1])$. As already observed for the proof of lemma 1.3.10, since the process u belongs to $L^{1,2}$ for every $0 \leq t \leq 1$, g_t is in $\mathbb{D}^{1,2}$ and $Dg_t = \int_0^t D u_s ds$. Moreover Hölder inequality implies

$$\mathbb{E} \left[\int_0^1 ||Dg_t||^p_{L^2([0,1])} \, dt \right] \leq \mathbb{E} \left[\int_0^1 ||Du_s||^p_{L^2([0,1])} \, ds \right] < +\infty.$$

Then g belongs to $L^{1,p}$. To conclude it is sufficient to observe that

$$\mathbb{E} \left[\int_0^1 \int_{t-\varepsilon}^t |D_t u_s|^p \, ds dt \right] = \mathbb{E} \left[\int_0^1 \int_s^{s+\varepsilon} |D_t u_s|^p \, dt ds \right],$$

and that the right hand side of previous equality converges to zero when ε goes to zero by point 1. of remark 1.3.12.

□

Lemma 1.3.16. *Let u be a process in $L^{2,p}$, with $p \geq 2$. Suppose furthermore that*

$$\int_0^1 \left(\mathbb{E} \left[||Du_t||^p_{L^p([0,1])} \right] + \mathbb{E} \left[||D^2 u_t||^p_{L^p([0,1]^2)} \right] \right) dt < +\infty. \tag{1.5}$$

Then the process $\left(\int_0^t u_s \delta W_s, 0 \leq t \leq 1 \right)$ is in $L_-^{1,p}$, and

$$D^- \left(\int_0^\cdot u_t \delta W_t \right) = \int_0^\cdot D u_t \delta W_t.$$

Proof. We set $g = \int_0^\cdot u_t \delta W_t$. By proposition 5.5 of [45], for every t in $[0,1]$, g_t belongs to $\mathbb{D}^{1,2}$ and $D_r g_t = \delta\left(D_r u I_{[0,t]}\right) + u_r I_{[0,t]}(r)$, for every r, almost surely. Using inequality (1.1) it is possible to find a positive constant c such that

$$||g||^p_{L^p(\Omega\times[0,1])} \leq c\,||u||^p_{L^{1,p}} < +\infty.$$

To prove that g belongs to $L^{1,p}$ we still have to show that $\mathbb{E}\left[\int_0^1 ||Dg_t||^p_{L^2([0,1])}\,dt\right]$ is finite. Clearly, $\mathbb{E}\left[\int_0^1 \left|\left|uI_{[0,t]}\right|\right|^p_{L^2([0,1])}\,dt\right] \leq ||u||_{L^p(\Omega\times[0,1])}$, which is finite. It remains to prove that $\mathbb{E}\left[\int_0^1 \left(\int_0^1 \left|\delta\left(D_r u I_{[0,t]}\right)\right|^2 dr\right)^{\frac{p}{2}} dt\right] < +\infty$. Applying again inequality (1.1) we obtain, for some $c > 0$,

$$\begin{aligned}
\mathbb{E}\left[\int_0^1 \left(\int_0^1 \left|\delta\left(D_r u I_{[0,t]}\right)\right|^2 dr\right)^{\frac{p}{2}} dt\right] &\leq c\int_0^1 \mathbb{E}\left[\int_0^1 \left|\delta\left(D_r u I_{[0,t]}\right)\right|^p dt\right] dr \\
&\leq c\int_0^1\int_0^1 \left|\left|D_r u I_{[0,t]}\right|\right|^p_{L^{1,p}}\,dr dt \\
&\leq c\int_0^1 ||D_r u||^p_{L^{1,p}}\,dr.
\end{aligned}$$

Last term in the expression above is bounded by the integral appearing in inequality (1.5). This permits to get the result.

□

Proposition 1.3.17. *Let v be a process in $L^{1,p}_-$, with $p > 4$. Then v is both forward and Skorohod integrable with respect to W and*

$$\int_0^\cdot v_t d^- W_t = \int_0^\cdot v_t \delta W_t + \int_0^\cdot D_t^- v_t dt.$$

Furthermore, if v is also left continuous with right limit, then $\int_0^\cdot v_t d^- W_t$ has finite quadratic variation equal to $\int_0^\cdot v_t^2 dt$.

Proof. First of all we observe that if a process v belongs to $L^{1,p}_-$ then $I_{[0,t]}v$ inherits the property for every t in $[0,1]$. Using lemma 1.3.14 and lemma 1.3.10 we find that $I(\varepsilon, v, W, t) - \int_0^t v_s \delta W_s$ converges in $L^p(\Omega)$ toward $\int_0^t D_s^- v_s ds$, for every $0 \leq t \leq 1$. If v belongs to $L^{1,p}_-$, with $p > 4$, by theorem 5.2 in [45], the Skorohod integral process $\int_0^\cdot v_t \delta W_t$ admits a continuous version. At the same time, thanks to theorem 1.1 of [54] we know that $\int_0^\cdot v_t \delta W_t$ has finite quadratic variation equal to $\int_0^\cdot v_t^2 dt$. The proof is complete.

□

Proposition 1.3.18. *Let u and v be left continuous processes, respectively in $L^{1,p}_-$ and $L^{1,2}_-$, with $p > 4$. Suppose that $\sup_{t\in[0,1]}|u_t|$ belongs to $L^p(\Omega)$, and that the random variable $\sup_{t\in[0,1]}\left(|v_t| + \sup_{s\in[0,1]}|D_s v_t|\right)$ is bounded. Then uv and v are forward integrable with* *respect to W. Furthermore u is forward integrable with respect to $\int_0^\cdot v_t d^- W_t$ and*

$$\begin{aligned} \int_0^\cdot u_t d^- \left(\int_0^t v_s d^- W_s \right) &= \int_0^\cdot u_t v_t d^- W_t \\ &= \int_0^\cdot u_t v_t \delta W_t + \int_0^\cdot (v_t D_t^- u_t + u_t D_t^- v_t) dt. \end{aligned}$$

Proof. By point 2. of remark 1.3.12 the process uv belongs to $L_-^{1,q}$, for every $4 < q < p$, and $D^- uv = vD^- u + uD^- v$. Proposition 1.3.17 immediately implies that

$$\int_0^\cdot u_t v_t d^- W_t = \int_0^\cdot u_t v_t \delta W_t + \int_0^\cdot (v_t D_t^- u_t + u_t D_t^- v_t) dt.$$

Lemma 1.3.13 permits to write, for every $0 \le t \le 1$,

$$\begin{aligned} I\left(\varepsilon, u, \int_0^\cdot v_t d^- W_t, t\right) &= \frac{1}{\varepsilon} \int_0^t u_s \int_s^{s+\varepsilon} v_r d^- W_r ds \\ &= \frac{1}{\varepsilon} \int_0^t u_s \left(\int_s^{s+\varepsilon} v_r \delta W_r \right) ds \\ &+ \frac{1}{\varepsilon} \int_0^t u_s \left(\int_s^{s+\varepsilon} D_r^- v_r dr \right) ds. \end{aligned}$$

Since $\sup_{t\in[0,1]} \left| D_t^- v_t \right|$ belongs to $L^p(\Omega)$ the second term of previous sum converges toward $\int_0^t u_s D_s^- v_s ds$ in $L^q(\Omega)$, for every $2 \le q < p$. As a consequence of this, by lemma 1.3.13, $I\left(\varepsilon, u, \int_0^\cdot v_t d^- W_t, t\right)$ converges toward $\int_0^t u_s v_s d^- W_s$ in $L^2(\Omega)$. The proof is then complete. □

Definition 1.3.19. *We say that a process u belongs to $L_{-,loc}^{1,p}$ if it is localized by a sequence $\left(\Omega_k, u^k\right)_{k\in\mathbb{N}}$, with u^k belonging to $L_-^{1,p}$ for every k in $\mathbb{N}$.*

Lemma 1.3.20. *Let $u = (u^1, \ldots, u^n)$, $n > 1$, be a vector of left continuous processes with bounded paths and in $L_-^{1,p}$, for some $p \ge 2$. Then, for every ψ in $C^1(\mathbb{R}^n)$ the process $\psi(u)$ belongs to $L_{-,loc}^{1,p}$. Moreover, the localizing sequence $\left(\Omega_k, \psi(u)^k\right)_{k\in\mathbb{N}}$ can be chosen such that $\psi(u)^k$ is left continuous, and $\sup_{t\in[0,1]} \left|\psi(u)_t^k\right|$ belongs to $L^p(\Omega)$ for every k in $\mathbb{N}$.*

Proof. For k in $\mathbb{N}^*$, set $\Omega_k = \left\{\sup_{0\le t\le 1} ||u_t||_{\mathbb{R}^n} \le k\right\}$ and $\psi(u)^k = \psi(u) f_k(u)$, being $f_k(u) = f(\frac{u}{k})$, and f a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$, with compact support and $f(x) = 1$, for every $||x|| \le 1$. Clearly $\psi(u)$ is localized by $\left(\Omega_k, \psi(u)^k\right)_{k\in\mathbb{N}}$. By [45], proposition 4.8, $\psi(u)^k$ belongs to $L^{1,2}$, for every k in $\mathbb{N}^*$. Since $\psi \circ f_k$ has bounded first partial derivatives, proposition 1.2.2 of [43] implies that

$$D\psi(u)_s^k = \sum_{i=1}^n \partial_i (\psi \circ f_k)(u_s) Du_s^i.$$

In particular $\psi(u)^k$ belongs to $L^{1,p}$. Using the continuity of all first partial derivatives of $\psi \circ f_k$ and the left continuity of u^i for every $i = 1, ..., n$, it is possible to prove that

$\psi(u)^k$ belongs indeed to $L^{1,p}_-$, and $D^-\psi(u)^k = \sum_{i=1}^n \partial_i(\psi \circ f_k(u))D^-u^i$. The proof is then complete.

□

We conclude this section giving a generalization of proposition 1.3.18.

Proposition 1.3.21. *Let $u = (u^1, \ldots, u^n)$, $n > 1$, be a vector of left continuous processes with bounded paths and in $L^{1,p}_-$, with $p > 4$. Let v be a process in $L^{1,2}_-$ with left continuous paths such that the random variable $|v_t| + \sup_{s\in[0,1]} |D_s v_t|$ is bounded. Then for every ψ in $C^1(\mathbb{R}^n)$ $\psi(u)v$ and v are forward integrable with respect to W. Furthermore $\psi(u)$ is forward integrable with respect to $\int_0^\cdot v_t d^-W_t$ and*

$$\int_0^\cdot \psi(u_t)d^-\left(\int_0^t v_s d^-W_s\right) \;=\; \int_0^\cdot \psi(u_t)v_t d^-W_t.$$

Proof. Let $\left(\Omega^k, \psi(u)^k\right)_{k\in\mathbb{N}}$ be a localizing sequence for $\psi(u)$ such that $\psi(u)^k$ is left continuous and $\sup_{t\in[0,1]} \left|\psi(u)^k_t\right|$ belongs to $L^p(\Omega)$ for every k in $\mathbb{N}$. Such a sequence exists thanks to lemma 1.3.20. Clearly $\left(\Omega^k, \psi(u)^k v\right)_{k\in\mathbb{N}}$ localizes $\psi(u)v$. For every k in $\mathbb{N}$, thanks to proposition 1.3.18, $\psi(u)^k$ and $\psi(u)^k v$ are forward integrable with respect to W and

$$\int_0^\cdot \psi(u)^k_t d^- \int_0^t v_s d^-W_s = \int_0^\cdot \psi(u)^k_t v_t d^-W_t.$$

The conclusion follows by remark 1.2.3.

□

1.3.3 Forward integrals of anticipating processes : substitution formulae

Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ be a filtration on $(\Omega, \mathcal{F}, P)$, with $\mathcal{F}_1 = \mathcal{F}$, and L an $\mathcal{F}$-measurable random variable with values in $\mathbb{R}^d$. We set $\mathcal{G}_t = (\mathcal{F}_t \vee \sigma(L))$, and we suppose that $\mathbb{G}$ is right continuous :

$$\mathcal{G}_t = \bigcap_{\varepsilon>0} (\mathcal{F}_{t+\varepsilon} \vee \sigma(L)).$$

In this section $\mathcal{P}^{\mathbb{F}}$ ($\mathcal{P}^{\mathbb{G}}$, resp.) will denote the σ-algebra of $\mathbb{F}$(of $\mathbb{G}$, resp.)-predictable processes. E will be the Banach space of all continuous functions on $[0,1]$ equipped with the uniform norm $||f||_E = \sup_{t\in[0,1]} |f(t)|$.

Preliminary results

We state in the sequel some results about forward integrals involving processes that are random field evaluated at L. To be more precise we will establish conditions to insure the existence of such integrals, their quadratic variation and a related associativity property.

Definition 1.3.22. *An increasing sequence of random times $(T_k)_{k\in\mathbb{N}}$ is said **suitable** if $P\left(\cup_{k=0}^{+\infty} \{T_k = 1\}\right) = 1$.*

Definition 1.3.23. *For every random time $0 \leq S \leq 1$, $p > 0$, and $\gamma > 0$, we define $\mathcal{C}_S^{p,\gamma}$ as the set of all families of continuous processes $((F(t,x), 0 \leq t \leq 1); x \in \mathbb{R}^d)$ such that for each compact set C of $\mathbb{R}^d$ there exists a constant $c > 0$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,S]} |F(t,x) - F(t,y)|^p\right] \leq c\,|x-y|^\gamma, \quad \forall x, y \in C.$$

If $S = 1$, $\mathcal{C}^{p,\gamma}$ will stand for $\mathcal{C}_S^{p,\gamma}$.

We begin recalling a result stated in [54], lemma 1.2, page 93.

Lemma 1.3.24. *Let $\left\{(F_n(t,x), 0 \leq t \leq 1), (F(t,x), 0 \leq t \leq 1); n \geq 1, x \in \mathbb{R}^d\right\}$ be a family of continuous processes such that F_n and F are $\mathcal{F} \otimes \mathcal{B}([0,1]) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. Suppose that for each x in $\mathbb{R}^d$, $F_n(\cdot, x)$ converges to $F(\cdot, x)$ ucp and that there exist $p > 1$, $\gamma > d$, with*

$$\mathbb{E}\left[\sup_{t\in[0,1]} |F_n(t,x) - F_n(t,y)|^p\right] \leq c\,|x-y|^\gamma, \quad \forall x, y \in C, \quad \forall n \in \mathbb{N},$$

and C compact set in $\mathbb{R}^d$. Then $x \mapsto F(\cdot, x)$ admits a continuous version $\bar{F}(\cdot, x)$ from $\mathbb{R}^d$ to E and $F_n(\cdot, L)$ converges toward $F(\cdot, L)$ ucp.

Definition 1.3.25. $bL(\mathcal{P}^{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}^d))$ *will denote the set of all functions*

$$((h(t,x), 0 \leq t \leq 1); x \in \mathbb{R}^d)$$

which are $\mathcal{P}^{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, such that for every x in $\mathbb{R}^d$, $h(\cdot, x)$ has left continuous and bounded paths.

Definition 1.3.26. *Let $p > 1, \gamma > 0$. We define $\mathcal{A}^{p,\gamma}$ as the set of all functions h in $bL(\mathcal{P}^{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}^d))$ satisfying the following assumption. There exists a suitable sequence of stopping times $(T_k)_{k\in\mathbb{N}}$ such that h belongs to $\bigcap_{k\in\mathbb{N}} \mathcal{C}_{T_k}^{p,\gamma}$.*

We state this lemma which will be useful later.

Lemma 1.3.27. *Let h and g be respectively in $\mathcal{A}^{p,\gamma_p}$ and $\mathcal{A}^{q,\gamma_q}$ for some $p, q > 1$, $\gamma_p, \gamma_q > 0$. Then the following statements hold.*

1. *If ψ belongs to $C^1(\mathbb{R})$ and it has bounded derivative, then $\psi(h)$ belongs to $\mathcal{A}^{p,\gamma_p}$.*
2. *The process hg belongs to $\mathcal{A}^{\alpha, \frac{\gamma_p \alpha}{p} \wedge \frac{\gamma_q \alpha}{q}}$ with $\alpha = \frac{pq}{p+q}$.*
3. *If N is a continuous $\mathbb{F}$-semimartingale, then the function*

$$\left(\int_0^t h(s,x) dN_s, 0 \leq t \leq 1, x \in \mathbb{R}\right)$$

belongs to $\mathcal{A}^{p,\gamma_p}$.

Proof. Let $(T_k)_{k\in\mathbb{N}}$ be a suitable sequence of stopping times such that h and k belong to $\bigcap_{k\in\mathbb{N}} \mathtt{C}^{p,\gamma_p}_{T_k}$ and $\bigcap_{k\in\mathbb{N}} \mathtt{C}^{q,\gamma_q}_{T_k}$, respectively.

The conclusion of the first point is straightforward.

Concerning the second point we set, for every k in $\mathbb{N}$,

$$S_k = \inf\{0 \le t \le 1, | \, |h(t,0)| + |g(t,0)| \ge k\} \wedge T_k \wedge 1.$$

If C is a compact set of $\mathbb{R}^d$, using Hölder inequality we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,S_k]} |hg(t,x) - hg(t,y)|^\alpha\right] &\le c\left(\mathbb{E}\left[\sup_{t\in[0,S_k]} |h(t,x)-h(t,y)|^p\right]\right)^{\frac{\alpha}{p}} \\ &+ c\left(\mathbb{E}\left[\sup_{t\in[0,S_k]} |g(t,x)-g(t,y)|^q\right]\right)^{\frac{\alpha}{q}} \\ &\le c\,|x-y|^{\frac{\gamma\alpha}{p}\wedge\frac{\gamma_q\alpha}{q}},\end{aligned}$$

where $c = \sup_{x,y\in C}\left(\mathbb{E}\left[\sup_{t\in[0,S_k]} |h(t,x)|^p\right]\right)^{\frac{\alpha}{p}} + \left(\mathbb{E}\left[\sup_{t\in[0,S_k]} |g(t,x)|^q\right]\right)^{\frac{\alpha}{q}}$ is bounded thanks to the choice of the sequence $(S_k)_{k\in\mathbb{N}}$ and the compactness of C.

To prove point 3. it is sufficient to define $S_k = \inf\{0 \le t \le 1, | \, |V|_t + [M]_t \ge k\} \wedge T_k$, for every k in $\mathbb{N}$, where $N = M + V$, M is an $\mathbb{F}$-local martingale, V is a bounded variation process, and $|V|$ denotes the total variation of V.

□

Existence

Let $M, V : \left(\Omega \times [0,1] \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}([0,1]) \otimes \mathcal{B}(\mathbb{R}^d)\right) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be measurable functions such that for each x in $\mathbb{R}^d$, $M(0,x) = V(0,x) = 0$, $(M(t,x), 0 \le t \le 1)$ is an $\mathbb{F}$-continuous local martingale, and $(V(t,x), 0 \le t \le 1)$ a continuous bounded variation process.

Remark 1.3.28. *If h is in $\mathtt{bL}(\mathcal{P}^{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}^d))$, the process $(h(t,L), 0 \le t \le 1)$ is left continuous with bounded paths and the process $(V(t,L), 0 \le t \le 1)$ is continuous with bounded variation. Then, by proposition 1.2.8, $\int_0^\cdot h(t,L)d^-V(t,L)$ exists and coincides with the Lebesgue-Stieltjes integral $\int_0^\cdot h(t,L)dV(t,L)$. Moreover*

$$\int_0^\cdot h(t,L)dV(s,L) = \left(\int_0^\cdot h(t,x)dV(t,x)\right)_{x=L}.$$

Lemma 1.3.29. *Let h be in $\mathtt{bL}(\mathcal{P}^{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}^d))$. Suppose that both $\sup_{t\in[0,1]} |h(t,0)|$ and $\sup_{t\in[0,1]} |M(t,0)|$ are bounded, and that there exist $p > 1$, $q > \frac{p}{p-1}$, $\gamma_p > \frac{d(q+p)}{q}$, $\gamma_q > \frac{d(q+p)}{p}$ such that M belongs to $\mathcal{C}^{p,\gamma_p}$ and h to $\mathcal{C}^{q,\gamma_q}$. Then the function $x \mapsto \int_0^\cdot h(s,x)dM(s,x)$ admits a continuous version, $\int_0^\cdot h(s,L)d^-M(s,L)$ exists as limit ucp of its regularizations and*

$$\int_0^\cdot h(t,L)d^-M(t,L) = \left(\int_0^\cdot h(t,x)dM(t,x)\right)_{x=L}.$$

Proof. For every $x \in \mathbb{R}^d$ and $0 \le t \le 1$ we set

$$F_\varepsilon(t,x) = \frac{1}{\varepsilon}\int_0^t h(s,x)(M(s+\varepsilon,x) - M(s,x))ds,$$

and

$$F(t,x) = \int_0^t h(s,x)dM(s,x).$$

To prove our statement we verify that lemma 1.3.24 applies to the families defined above.

Let x and y in $\mathbb{R}^d$ be fixed. Point 2. of proposition 1.2.7 implies that $F_\varepsilon(\cdot,x)$ converges *ucp* to $F(\cdot,x)$. Set $\alpha = \frac{pq}{p+q}$. Using theorem 45 in chapter IV of [48], we can write, for every $\varepsilon > 0$,

$$\begin{aligned} F_\varepsilon(\cdot,x) - F_\varepsilon(\cdot,y) &= \int_0^\cdot \left(\frac{1}{\varepsilon}\int_{(r-\varepsilon)^+}^r (h(s,x) - h(s,y))ds\right) dM(r,x) \\ &+ \int_0^\cdot \left(\frac{1}{\varepsilon}\int_{(r-\varepsilon)^+}^r h(s,y)ds\right) d\left(M(r,y) - M(r,x)\right). \end{aligned}$$

Thanks to theorem 2 in chapter V of [48], we find a positive constant a, depending only on p and q such that, for every $\varepsilon > 0$,

$$\mathbb{E}\left[\sup_{t\in[0,1]} |F(\varepsilon,t,x) - F(\varepsilon,t,y)|^\alpha\right] \le a(\delta_1 + \delta_2)$$

with

$$\delta_1 = \mathbb{E}\left[\sup_{t\in[0,1]} |h(t,x) - h(t,y)|^q\right]^{\frac{\alpha}{q}} \mathbb{E}\left[\sup_{t\in[0,1]} |M(t,x)|^p\right]^{\frac{\alpha}{p}}$$

and

$$\delta_2 = \mathbb{E}\left[\sup_{t\in[0,1]} |M(t,x) - M(t,y)|^p\right]^{\frac{\alpha}{p}} \mathbb{E}\left[\sup_{t\in[0,1]} |h(t,y)|^q\right]^{\frac{\alpha}{q}}.$$

Thanks to the hypotheses on M and h it is possible to find a constant b depending on C such that

$$\delta_1 \le b\,|x-y|^{\frac{\gamma_q\alpha}{q}}, \quad \delta_2 \le b\,|x-y|^{\frac{\gamma_p\alpha}{p}}, \quad \forall\varepsilon > 0.$$

Consequently, there will exist $c > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,1]} |F(\varepsilon,t,x) - F(\varepsilon,t,y)|^\alpha\right] \le c\,|x-y|^\gamma, \qquad \forall x,y \in C, \quad \forall\varepsilon > 0,$$

with $\gamma = \frac{\gamma_p\alpha}{p} \wedge \frac{\gamma_q\alpha}{q} > d$ and proof is complete.

□

The following proposition represents a generalization of previous lemma.

Proposition 1.3.30. *Suppose that M belongs to $\mathcal{A}^{p,\gamma_p}$ and that h belongs to $\mathcal{A}^{q,\gamma_q}$ for some $p>1$, $q>\frac{p}{p-1}$, $\gamma_p>\frac{d(q+p)}{q}$ and $\gamma_q>\frac{d(q+p)}{p}$. Then $x\mapsto\int_0^{\cdot} h(s,x)dM(s,x)$ admits a continuous version, $\int_0^{\cdot} h(s,L)d^-M(s,L)$ exists as limit ucp of its regularizations and*

$$\int_0^{\cdot} h(s,L)d^-M(s,L)=\left(\int_0^{\cdot} h(s,x)dM(s,x)\right)_{x=L}.$$

Proof. We observe that we do not loose generality assuming that there exists a suitable sequence of $\mathbb{F}$-stopping times $(T_k)_{k\in\mathbb{R}}$ such that M and h belong respectively to $\bigcap_{k\in\mathbb{N}}\mathtt{C}^{p,\gamma_p}_{T_k}$ and $\bigcap_{k\in\mathbb{N}}\mathtt{C}^{q,\gamma_q}_{T_k}$. Let $(S_k)_{k\in\mathbb{N}}$ be a suitable sequence of $\mathbb{F}$-stopping times such that, for every k in $\mathbb{M}$, S^k is the first instant, between 0 and 1, the process $|M(\cdot,0)|+|h(\cdot,0)|$ is greater than k. Set, for every k in $\mathbb{N}$,

$$R_k=S_k\wedge T_k,\quad M^k=M^{R_k},\quad h^k=h^{R_k},$$

and

$$\Omega_k=\left\{\sup_{t\in[0,1]}|M(t,0)|\le k\right\}\cap\left\{\sup_{t\in[0,1]}|h(t,0)|\le k\right\}\cap\{R_k=1\}.$$

Let k be fixed. It is clear that $\sup_{t\in[0,1]}\left|h^k(t,0)\right|$ and $\sup_{t\in[0,1]}\left|M^k(\cdot,0)\right|$ are bounded and that M^k and h^k belong, respectively, to $\mathtt{C}^{p,\gamma_p}$ and $\mathtt{C}^{q,\gamma_q}$. We can thus apply lemma 1.3.29 to state that the function $x\mapsto\int_0^{\cdot} h^k(t,x)dM^k(t,x)$ admits a continuous version $\bar F^k(\cdot,x)$ and

$$\int_0^{\cdot} h^k(t,L)dM^k(t,L)=\left(\int_0^{\cdot} h^k(t,x)dM^k(t,x)\right)_{x=L}.$$

By the *local character* of the classical stochastic integral, see [48], theorem 26, $F^k(\cdot,x)=F^h(\cdot,x)=\int_0^{\cdot} h(t,x)dM(t,x)$, for every x in $\mathbb{R}^d$, almost surely on Ω_h, for every $h\le k$. Therefore it is possible to define $\left(\bar F(t,x),0\le t\le 1,x\in\mathbb{R}^d\right)$ such that $x\mapsto\bar F(\cdot,x)$ is continuous, and for every k in $\mathbb{N}$

$$\bar F(\cdot,x)=\bar F^k(\cdot,x)=\int_0^{\cdot} h(t,x)dM(t,x),\quad\forall x\in\mathbb{R}^d,\quad\text{on }\Omega_k.$$

Furthermore, remark 1.2.3 implies that

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_0^{\cdot} h(s,L)(M(s+\varepsilon,L)-M(s,x))ds=\bar F(\cdot,L),$$

ucp, since the convergence holds on every on Ω_k, for every k in $\mathbb{N}$.

□

Previous proposition implies directly the following corollary.

Corollary 1.3.31. *Let N be a continuous $\mathbb{F}$-local martingale. Let h be in $\mathcal{A}^{q,\gamma_q}$, for $q>1$, $\gamma_q>d$. Then $x\mapsto\int_0^{\cdot} h(t,x)dN_t$ admits a continuous version, $\int_0^{\cdot} h(t,L)d^-N_t$ exists as limit ucp of its regularizations and*

$$\int_0^{\cdot} h(t,L)d^-N_t=\left(\int_0^{\cdot} h(t,x)dN_t\right)_{x=L}.$$

Quadratic variation

We examine the existence of the quadratic variation of forward integrals of the anticipating processes considered in the present subsection. We start giving a generalization of a substitution formula proved in [54], proposition 1.3. We furnish, in fact, a *localized* version of that result in view of further applications to finance. We omit the details of its proof, which are indeed similar to those used in the proof of proposition 1.3.30.

Proposition 1.3.32. *Suppose that M belongs to $\mathcal{A}^{p,\gamma}$ with $p > 2$ and $\gamma > 2d$. Then $x \mapsto [M(\cdot,x), M(\cdot,x)]$ admits a continuous version, the process $M(\cdot,L)$ has finite quadratic variation and*

$$[M(\cdot,L), M(\cdot,L)] = [M(\cdot,x), M(\cdot,x)]_{x=L}\,.$$

A consequence of previous proposition is the following.

Proposition 1.3.33. *Suppose that $M(t,x) = \int_0^\cdot h(t,x)dN(t,x)$, for every x in $\mathbb{R}^d$, where h and N verify the following assumption. The functions h and N are, respectively, in $\mathcal{A}^{q,\gamma_q}$ and $\mathcal{A}^{p,\gamma_p}$ with $p > 2$, $q > \frac{2p}{p-2}$, $\gamma_p > \frac{2d(q+p)}{q}$ and $\gamma_q > \frac{2d(q+p)}{p}$; for every x in $\mathbb{R}^d$, $N(0,x) = 0$, and $(N(t,x), 0 \le t \le 1)$ is a continuous $\mathbb{F}$-local martingale. Then $M(\cdot,L)$ has finite quadratic variation and $[M(\cdot,L)] = [M(\cdot,x)]_{x=L}$.*

Proof. We have to show that hypotheses of proposition 1.3.32 are satisfied. We denote with $(T_k)_{k\in\mathbb{N}}$ a suitable sequence of $\mathbb{F}$-stopping times such that N and h belong respectively to $\bigcap_{k\in\mathbb{N}} C^{p,\gamma_p}_{T_k}$ and $\bigcap_{k\in\mathbb{N}} C^{q,\gamma_q}_{T_k}$. Let $(S_k)_{k\in\mathbb{N}}$ be a suitable sequence of $\mathbb{F}$-stopping times such that, for every k in $\mathbb{N}$, S^k is the first instant, between 0 and 1, the process $|N(\cdot,0)|+|h(\cdot,0)|$ is greater than k. Set, for every k in $\mathbb{N}$, $R_k = S_k \wedge T_k$, $\quad N^k = N^{R_k}$, $M^k = M^{R_k}$ $h^k = h^{R_k}$, and $\Omega_k = \left\{\sup_{t\in[0,1]} |N(t,0)| \le k\right\} \cap \left\{\sup_{t\in[0,1]} |h(t,0)| \le k\right\} \cap \left\{R_k = 1\right\}$. Let C be a compact set of $\mathbb{R}^d$, x, y in C, and k in $\mathbb{N}$. Using arguments already employed in the proof of lemma 1.3.29, it is not difficult to show that if $\alpha = \frac{pq}{p+q} > 2$, there exists a constant $d_k > 0$, depending on C and k, such that,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,R_k]} |M(t,x) - M(t,y)|^\alpha\right] &= \mathbb{E}\left[\sup_{t\in[0,1]} \left|M^k(t,x) - M^k(t,y)\right|^\alpha\right] \\ &\le d_k\,|x-y|^\gamma\,,\end{aligned}$$

with $\gamma = \frac{\gamma_p\alpha}{p} \wedge \frac{\gamma_q\alpha}{q} > 2d$. This concludes the proof.

□

From proposition 1.3.32 and lemma 1.3.27 we can easily derive the following corollary.

Corollary 1.3.34. *Suppose that $M(\cdot,x) = \int_0^\cdot h(t,x)dN$ where N is a continuous $\mathbb{F}$-local martingale and h belongs to $\mathcal{A}^{q,\gamma_q}$, $q > 2$, $\gamma_q > 2d$. Then the function $x \mapsto [M(\cdot,x), M(\cdot,x)]$ admits a continuous version, the process $M(\cdot,L)$ has finite quadratic variation and*

$$[M(\cdot,L), M(\cdot,L)] = [M(\cdot,x), M(\cdot,x)]_{x=L}\,.$$

Chain rule formula

We conclude this section by proving the associative property of forward integrals for the processes studied in this part of the paper.

Proposition 1.3.35. *Let h and k be respectively in $\mathcal{A}^{p,\gamma_p}$ and $\mathcal{A}^{q,\gamma_q}$, with $p > 1$, $q > \frac{p}{p-1}$, $\gamma_p > \frac{d(q+p)}{q}$, $\gamma_q > \frac{d(q+p)}{p}$. Let N be a continuous $\mathbb{F}$-local martingale. Then $x \mapsto \int_0^\cdot k(t,x)d^- \int_0^t h(s,x)dN_s = \int_0^\cdot h(t,x)k(t,x)dN_s$ admits a continuous version, the process $\int_0^\cdot k(t,L)d^- \int_0^t h(s,L)d^-N_s$ exists as limit ucp of its regularizations and*

$$\int_0^\cdot k(t,L)d^- \int_0^t h(s,L)dN_s \quad = \quad \int_0^\cdot k(t,L)h(t,L)d^-N_t.$$

Proof. By point 3. of lemma 1.3.27 we know that $\int_0^\cdot h(t,x)dN_t$ belongs to $\mathcal{A}^{p,\gamma_p}$. Then, by proposition 1.3.30, $x \mapsto \int_0^\cdot k(t,x)d\int_0^t h(s,x)dN_s = \int_0^\cdot k(t,x)h(t,x)dN_t$ admits a continuous version and

$$\begin{aligned}\int_0^\cdot k(t,L)d^- \int_0^t h(s,L)dN_s &= \left(\int_0^\cdot k(t,x)d^- \int_0^t h(s,x)dN_s\right)_{x=L} \\ &= \left(\int_0^\cdot k(t,x)h(t,x)dN_t\right)_{x=L}.\end{aligned}$$

Point 2. of lemma 1.3.27 again shows that hk satisfies the hypotheses of corollary 1.3.31. As a consequence of of this

$$\int_0^\cdot h(t,L)k(t,L)d^-N_t = \left(\int_0^\cdot k(t,x)h(t,x)dN_t\right)_{x=L},$$

and we achieve the end of the proof.

□

Definition 1.3.36. *1. For $p > 1$ and $\gamma > d$, we define $\mathcal{A}^{p,\gamma}(L)$ as the set of all processes $(h(t,L), 0 \le t \le 1)$ with h belonging to $\mathcal{A}^{p,\gamma}$.*

2. $\mathcal{S}(\mathcal{A}^{p,\gamma}(L))$ will be the set of all processes $h = h^0 I_{\{0\}} + \sum_{i=1}^m h^i I_{(t_{i-1},t_i]}$ where $0 = t_0 \le t_1, \cdots, t_m = 1$, and h^i belongs to $\mathcal{A}^{p,\gamma}(L)$ for $i = 1, ..., m$.

Using similar arguments employed in the proof of proposition 1.3.7, it is possible to demonstrate the following one.

Proposition 1.3.37. *Let N be a continuous $\mathbb{F}$-local martingale and h be a process in $\mathcal{S}(\mathcal{A}^{p,\gamma}(L))$ for some $p > 1$, and $\gamma > d$. Then the following statements are true.*

1. *h is N-forward integrable and the forward integral $\int_0^\cdot h_t d^-N_t$ still belongs to $\mathcal{S}(\mathcal{A}^{p,\gamma}(L))$.*
2. *If $p > 2$ and $\gamma > 2d$, $\int_0^\cdot h_t d^-N_t$ has finite quadratic variation equal to $\int_0^\cdot h_t^2 d[N]_t$.*
3. *If k belongs to $\mathcal{S}(\mathcal{A}^{q,\gamma_q}(L))$ with $q > \frac{p}{p-1}$, $\gamma_p > \frac{d(p+q)}{q}$ and $\gamma_q > \frac{d(p+q)}{p}$, then k is forward integrable with respect to $\int_0^\cdot h_t d^-N_t$ and*

$$\int_0^\cdot k_t d^- \int_0^t h_s d^-N_s = \int_0^\cdot k_t h_t d^-N_t.$$

1.4 $\mathcal{A}$-martingales

Throughout this section $\mathcal{A}$ will be a real linear set of measurable processes indexed by $[0,1)$ with paths which are bounded on each compact interval of $[0,1)$. We also require that $\mathcal{A}$ must contain all real constants.

We will denote with $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ a filtration indexed by $[0,1]$ and with $\mathcal{P}(\mathbb{F})$ the σ-algebra generated by all left continuous and $\mathbb{F}$-adapted processes. In the remainder of the paper we will adopt the notations $\mathbb{F}$ and $\mathcal{P}(\mathbb{F})$ even when the filtration $\mathbb{F}$ is indexed by $[0,1)$. At the same way, if X is a process indexed by $[0,1]$, we shall continue to denote with X its restriction to $[0,1)$.

1.4.1 Definitions and properties

Definition 1.4.1. *A process $X = (X_t, 0 \le t \le 1)$ is said **$\mathcal{A}$-martingale** if every θ in $\mathcal{A}$ is X-improperly forward integrable and $\mathbb{E}\left[\int_0^t \theta_s d^- X_s\right] = 0$ for every $0 \le t \le 1$.*

Definition 1.4.2. *A process $X = (X_t, 0 \le t \le 1)$ is said **$\mathcal{A}$-semimartingale** if it can be written as the sum of an $\mathcal{A}$-martingale M and a bounded variation process V, with $V_0 = 0$.*

Remark 1.4.3.

1. *If X is a continuous $\mathcal{A}$-martingale with X belonging to $\mathcal{A}$, its quadratic variation exists improperly. In fact, if $\int_0^\cdot X_t d^- X_t$ exists improperly, it is possible to show that $[X,X]$ exists improperly and $[X,X] = X^2 - X_0^2 - 2\int_0^\cdot X_s d^- X_s$. We refer to proposition 4.1 of [56] for details.*
2. *Let X a continuous square integrable martingale with respect to some filtration $\mathbb{F}$. Suppose that every process in $\mathcal{A}$ is the restriction to $[0,1)$ of a process $(\theta_t, 0 \le t \le 1)$ which is $\mathbb{F}$-adapted, it has left continuous with right limit paths and $\mathbb{E}\left[\int_0^1 \theta_t^2 d[X]_t\right] < +\infty$. Then X is an $\mathcal{A}$-martingale.*
3. *In [27] the authors introduced the notion of **weak-martingale**. A semimartingale X is a weak-martingale if $\mathbb{E}\left[\int_0^t f(s,X_s)dX_s\right] = 0$, $0 \le t \le 1$, for every $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, bounded Borel-measurable. Clearly we can affirm the following. Suppose that $\mathcal{A}$ contains all processes of the form $f(\cdot, X)$, with f as above. Let X be a semimartingale X which is an $\mathcal{A}$-martingale. Then X is a weak-martingale.*

Proposition 1.4.4. *Let X be a continuous $\mathcal{A}$-martingale. The following statements hold true.*

1. *If X belongs to $\mathcal{A}$, $X_0 = 0$ and $[X,X] = 0$. Then $X \equiv 0$.*
2. *Suppose that $\mathcal{A}$ contains all bounded $\mathcal{P}(\mathbb{F})$-measurable processes. Then X is an $\mathbb{F}$-martingale.*

Proof. From point 1. of remark 1.4.3, $\mathbb{E}[X_t^2] = 0$, for all $0 \le t \le 1$.

Regarding point 2. it is sufficient to observe that processes of type $I_A I_{(s,t]}$, with $0 \le s \le t \le 1$, and A in $\mathcal{F}_s$ belong to $\mathcal{A}$. Moreover $\int_0^1 I_A I_{(s,t]}(r) d^- X_r = I_A(X_t - X_s)$. This imply $\mathbb{E}[X_t - X_s \mid \mathcal{F}_s] = 0$, $0 \le s \le t \le 1$.

□

Corollary 1.4.5. *The decomposition of an $\mathcal{A}$-semimartingale X in definition 1.4.2 is unique among the class of processes of type $M + V$, being M a continuous $\mathcal{A}$-martingale in $\mathcal{A}$ and V a bounded variation process.*

Proof. If $M + V$ and $N + W$ are two decompositions of that type, then $M - N$ is a continuous $\mathcal{A}$-martingale in $\mathcal{A}$ starting at zero with zero quadratic variation. Point 1. of proposition 1.4.4 permits to conclude.

□

The following proposition gives sufficient conditions for an $\mathcal{A}$-martingale to be a martingale with respect to some filtration $\mathbb{F}$, when $\mathcal{A}$ is made up of $\mathcal{P}(\mathbb{F})$-measurable processes. It constitutes a generalization of point 2. in proposition 1.4.4.

Definition 1.4.6. *We will say that $\mathcal{A}$ satisfies **assumption** $\mathcal{E}$ with respect to a filtration $\mathbb{F}$ if*

1. *Every θ in $\mathcal{A}$ is $\mathbb{F}$-adapted;*
2. *For every $0 \leq s < 1$ there exists a basis $\mathcal{B}_s$ for $\mathcal{F}_s$, with the following property. For every A in $\mathcal{B}_s$ there exists a sequence of $\mathcal{F}_s$-measurable random variables $(\Theta_n)_{n\in\mathbb{N}}$, such that for each n the process $\Theta_n I_{[s,1)}$ belongs to $\mathcal{A}$, $\sup_{n\in\mathbb{N}} |\Theta_n| \leq 1$, almost surely and*
$$\lim_{n\to+\infty} \Theta_n = I_A, \quad a.s.$$

Proposition 1.4.7. *Let $X = (X_t, 0 \leq t \leq 1)$ be a continuous $\mathcal{A}$-martingale adapted to some filtration $\mathbb{F}$, with X_t belonging to $L^1(\Omega)$ for every $0 \leq t \leq 1$. Suppose that $\mathcal{A}$ satisfies assumption $\mathcal{E}$ with respect to $\mathbb{F}$. Then X is an $\mathbb{F}$-martingale.*

Proof. We have to show that for all $0 \leq s \leq t \leq 1$, $\mathbb{E}\left[I_A (X_t - X_s)\right] = 0$, for all A in $\mathcal{F}_s$. We fix $0 \leq s < t \leq 1$ and A in $\mathcal{B}_s$. Let $(\Theta_n)_{n\in\mathbb{N}}$ be a sequence of random variables converging almost surely to I_A as in the hypothesis. Since X is an $\mathcal{A}$-martingale, $\mathbb{E}\left[\Theta_n (X_t - X_s)\right] = 0$, for all n in $\mathbb{N}$. We note that $X_t - X_s$ belongs to $L^1(\Omega)$, then, by Lebesgue dominated convergence theorem,
$$|\mathbb{E}\left[I_A (X_t - X_s)\right]| \leq \lim_{n\to+\infty} \mathbb{E}\left[|I_A - \Theta_n| \, |X_t - X_s|\right] = 0.$$
Previous result extends to the whole σ-algebra $\mathcal{F}_s$ and this permits to achieve the end of the proof.

□

Some interesting properties can be derived taking inspiration from [27].
For a process X, we will denote
$$\mathcal{A}_X = \{(\psi(t, X_t)), 0 \leq t < 1 \,|\, \psi : [0,1] \times \mathbb{R} \to \mathbb{R}, \text{ Borel-measurable with polynomial growth and lower bounded}\}. \tag{1.6}$$

Remark 1.4.8. *At this stage we could avoid to impose a lower bound on functions in $\mathcal{A}_X$. Nevertheless, we prefer to consider this qualitative restriction in view of further applications to finance. Indeed, $\mathcal{A}_X$ will play the role of a possible class of admissible portfolios* *and we are interested in excluding among them the so called doubling strategies. Generally speaking, a doubling strategy is an arbitrage which can be realized if unbounded accumulation of losses are allowed. For more details about this arguments the reader is referred to Harrison and Pliska (1979).*

Proposition 1.4.9. *Let X be a continuous $\mathcal{A}$-martingale with $\mathcal{A} = \mathcal{A}_X$.*
Then, for every ψ in $C^2(\mathbb{R})$ with bounded first and second derivatives, the process

$$\psi(X) - \frac{1}{2}\int_0^{\cdot} \psi''(X_s) d\,[X, X]_s$$

is an $\mathcal{A}$-martingale.

Proof. The process X belongs to $\mathcal{A}$. In particular, X admits improper quadratic variation. We set $Y = \psi(X) - \frac{1}{2}\int_0^{\cdot} \psi''(X_s) d\,[X, X]_s$. Let θ in $\mathcal{A}_X$. By lemma 1.2.12, for every $0 \leq t < 1$

$$\int_0^t \theta_s d^- Y_s = \int_0^t \theta_s \psi'(X_s) d^- X_s.$$

Since $\theta\psi'(X)$ still belongs to $\mathcal{A}$, θ is Y-improperly integrable and

$$\int_0^{\cdot} \theta_t d^- Y_t = \int_0^{\cdot} \theta_t \psi'(X_t) d^- X_t. \tag{1.7}$$

We conclude taking the expectation in equality (1.7).

□

Proposition 1.4.10. *Suppose that $\mathcal{A}$ is an algebra. Let X and Y be two continuous $\mathcal{A}$-martingales with X and Y in $\mathcal{A}$.*
Then the process $XY - [X, Y]$ is an $\mathcal{A}$-martingale .

Proof. Since $\mathcal{A}$ is a real linear space, $(X + Y)$ belongs to $\mathcal{A}$. In particular by point 1. of remark 1.4.3, $[X + Y, X + Y]$, $[X, X]$ and $[Y, Y]$ exist improperly. This implies that $[X, Y]$ exists improperly too and that it is a bounded variation process. Therefore the vector (X, Y) admits all its mutual brackets on each compact set of $[0, 1)$. Let θ be in $\mathcal{A}$. Since $\mathcal{A}$ is an algebra, θX and θY belong to $\mathcal{A}$ and so both $\int_0^{\cdot} \theta_s X_s d^- Y_s$ and $\int_0^{\cdot} \theta_s Y_s d^- X_s$ locally exist. By lemma 1.2.12 $\int_0^{\cdot} \theta_t d^- (X_t Y_t - [X, Y])$ exists improperly too and

$$\int_0^{\cdot} \theta_t d^- (X_t Y_t - [X, Y]_t) = \int_0^{\cdot} Y_t \theta_t d^- X_t + \int_0^{\cdot} X_t \theta_t d^- Y_t.$$

Taking the expectation in the last expression we then get the result.

□

We recall a notion and a related result of [15].
A process R is *strongly predictable* with respect to a filtration $\mathbb{F}$, if

$$\exists\ \delta > 0, \text{ such that } R_{\varepsilon+\cdot} \text{ is } \mathbb{F}\text{-adapted}, \quad \text{for every } \varepsilon \leq \delta.$$

Proposition 1.4.11. *Let R be an $\mathbb{F}$-strongly predictable continuous process. Then for every continuous $\mathbb{F}$-local martingale Y, $[R, Y] = 0$.*

Proposition 1.4.11 combined with proposition 1.4.10 implies proposition 1.4.12 and corollary 1.4.13.

Proposition 1.4.12. *Let $\mathcal{A}$, X and Y be as in proposition 1.4.10. Assume, moreover, that X is an $\mathbb{F}$-local martingale, and that Y is strongly predictable with respect to $\mathbb{F}$. Then XY is an $\mathcal{A}$-martingale.*

Corollary 1.4.13. *Let $\mathcal{A}$, X and Y be as in proposition 1.4.10. Assume that X is a local martingale with respect to some filtration $\mathbb{G}$ and that Y is either $\mathbb{F}$-independent, or $\mathcal{G}_0$-measurable. Then XY is an $\mathcal{A}$-martingale.*

Proof. If Y is $\mathbb{G}$-independent, it is sufficient to apply previous proposition with $\mathbb{F} = \left(\bigcap_{\varepsilon>0} \mathcal{G}_{t+\varepsilon} \vee \sigma(Y)\right)_{t\in[0,1]}$.

□

1.4.2 $\mathcal{A}$-martingales and Weak Brownian motion

We proceed defining and discussing processes which are *weak-Brownian motions* in order to exhibit explicit examples of $\mathcal{A}$-martingales.

Definition 1.4.14. *([27]) A stochastic process $(X_t, 0 \le t \le 1)$ is a* ***weak Brownian motion of order*** *k if for every k-tuple $(t_1, t_2, ..., t_k)$*

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_k}) \overset{law}{=} (W_{t_1}, W_{t_2}, \ldots, W_{t_k})$$

where $(W_t, 0 \le t \le 1)$ is a Brownian motion.

We set, for a process $(X_t, 0 \le t \le 1)$,

$$\begin{aligned} \mathcal{A}_X^1 \quad = \quad & \{(\psi(t, X_t), 0 \le t \le 1, \text{ with polynomial growth s.t } \psi = \partial_x \Psi \\ & \Psi \in C^{1,2}([0,1]\times\mathbb{R}) \text{ with } |\partial_t \Psi| + |\partial_{xx}\Psi| \text{ bounded } \}. \end{aligned}$$

Assumption 1.4.15. *We suppose that $\sigma : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a Borel-measurable and bounded function such that the following equation has a unique solution $(\nu_t)_{t\in[0,1]}$ in the sense of distribution*

$$\begin{cases} \partial_t \nu_t(dx) = \frac{1}{2}\left(\sigma(t,x)^2 \nu_t(dx)\right)'' \\ \nu_0(dx) = \delta_0. \end{cases} \tag{1.8}$$

Remark 1.4.16. *Assumption 1.4.15 is verified for $\sigma(t,x) \equiv \sigma$, being σ a positive real constant and, in that case, $\nu_t = N(0, \sigma^2 t)$, for every $0 \le t \le 1$.*

Proposition 1.4.17. *Let $(X_t, 0 \le t \le 1)$ be a continuous finite quadratic variation process with $X_0 = 0$, and $d[X]_t = (\sigma(t, X_t))^2 dt$, where σ fulfills assumption 1.4.15. Then the following statements are true.*

1. *Suppose that $\mathcal{A} = \mathcal{A}_X^1$. Then X is an $\mathcal{A}$-martingale if and only if, for every $0 \le t \le 1$, $X_t \overset{law}{=} Z_t$, for every (Z, B) solution of equation $dZ = \sigma(\cdot, Z)dB, Z_0 = 0$, in the sense of definition 1.2 in chapter IX of [49]. In particular, if $\sigma \equiv 1$, X is a weak Brownian motion of order 1, if and only if it is an $\mathcal{A}_X^1$-martingale.*

2. *Suppose that* $d[X]_t = f_t dt$, *with* f $\mathcal{B}([0,1])$*-measurable and bounded. If* X *is a weak Brownian motion of order* $k = 1$, *then* X *is an* $\mathcal{A}$*-semimartingale. Moreover the process*

$$X + \int_0^{\cdot} \frac{(1-f_s)X_s}{2s} ds.$$

is an $\mathcal{A}$*-martingale.*

Proof.

1. Using Itô inverse formula recalled in proposition 1.2.11 we can write, for every $0 \le t \le 1$ and $\psi = \partial_x \Psi$ in $\mathcal{A}_X^1$

$$\begin{aligned} \int_0^t \psi(s, X_s) d^- X_s &= \Psi(t, X_t) - \Psi(0, X_0) \\ &- \int_0^t \left(\partial_s \Psi + \frac{1}{2} \partial_{xx}^{(2)} \Psi \sigma^2 \right)(s, X_s) ds. \end{aligned} \tag{1.9}$$

For every $0 \le t \le 1$, we denote with $\mu_t(dx)$ the law of X_t. If X is an $\mathcal{A}_X^1$-martingale, from (1.9) we derive

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \Psi(t,x) d\mu_t(x) - \int_{\mathbb{R}} \Psi(0,x)\mu_0(dx) - \int_0^t \int_{\mathbb{R}} \partial_s \Psi(s,x)\mu_s(dx)ds \\ &- \frac{1}{2} \int_0^t \int_{\mathbb{R}} \partial_{xx}^{(2)} \Psi(s,x)\sigma(s,x)^2 \mu_s(dx)ds. \end{aligned} \tag{1.10}$$

In particular, the law of X solves equation (1.8).
On the other hand, let (Z, B) be a solution of equation $Z = \int_0^{\cdot} \sigma(s, Z_s) dB_s$. The law of Z fulfills equation (1.10) too. Indeed, Z is a finite quadratic variation process with $d[Z]_t = (\sigma(t, Z_t))^2 dt$ which is an $\mathcal{A}_Z^1$-martingale by point 2. of remark 1.4.3. By assumption (1.4.15) X_t must have the same law of Z_t. This establishes the converse implication of point 1.
Suppose, on the contrary, that X_t has the same law of Z_t, for every $0 \le t \le 1$. Using the fact that Z is an $\mathcal{A}_Z^1$-martingale which solves equation (1.9) we get

$$\mathbb{E}\left[\Psi(t, Z_t) - \Psi(0, Z_0) - \int_0^t \left(\partial_s \Psi + \frac{1}{2} \partial_{xx}^{(2)} \Psi \sigma^2 \right)(s, X_s) ds \right] = 0.$$

for every Ψ in $C^{1,2}([0,1] \times \mathbb{R})$ such that $\psi(\cdot, X)$ is in $\mathcal{A}_X^1$, with $\psi = \partial_x \Psi$. Since X_t has the same law of Z_t, for every $0 \le t \le 1$, equality (1.9) implies that

$$\mathbb{E}\left[\int_0^{\cdot} \psi(t, X_t) d^- X_t \right] = \mathbb{E}\left[\int_0^{\cdot} \psi(t, Z_t) d^- Z_t \right] = 0,$$

The proof of the first point is now achieved.

2. Suppose that $\sigma(t,x)^2 = f_t$, for every (t,x) in $[0,1]\times\mathbb{R}$. Let Ψ be in $C^{1,2}([0,1]\times\mathbb{R})$ such that $\psi = \partial_x\Psi$ belongs to $\mathcal{A}^1_X$. Proposition 1.2.11 yields

$$\int_0^t \psi(s,X_s)d^-X_s = Y_t^\Psi + \frac{1}{2}\int_0^t \partial^{(2)}_{xx}\Psi(s,X_s)(1-f_s)ds, \quad 0\le t\le 1,$$

with

$$Y_t^\Psi = \Psi(t,X_t) - \Psi(0,X_0) - \int_0^t \partial_s\Psi_s(s,X_s)ds - \frac{1}{2}\int_0^t \partial^{(2)}_{xx}\Psi(s,X_s)ds.$$

Moreover X is a weak Brownian motion of order 1. This implies $\mathbb{E}\left[Y_t^\Psi\right] = 0$, for every $0\le t\le 1$. We derive that

$$\mathbb{E}\left[\int_0^t \psi(s,X_s)d^-X_s + \frac{1}{2}\int_0^t \partial^{(2)}_{xx}\Psi(s,X_s)(f_s-1)ds\right] = \mathbb{E}\left[Y_t^\Psi\right] = 0.$$

Since the law of X_t is $N(0,t)$, by Fubini theorem and integration by parts on the real line we obtain

$$\mathbb{E}\left[\int_0^t \partial^{(2)}_{xx}\Psi(s,X_s)(f_s-1)ds\right] = \mathbb{E}\left[\int_0^t \psi(s,X_s)\frac{(1-f_s)X_s}{s}ds\right].$$

This concludes the proof of the second point.

□

From [27] we can extract an example of an $\mathcal{A}$-semimartingale which is not a semimartingale.

Example 1.4.18. *Suppose that $(B_t, 0\le t\le 1)$ is a Brownian motion on the probability space $(\Omega,\mathbb{G},P)$, being $\mathbb{G}$ some filtration on $(\Omega,\mathcal{F},P)$. Set*

$$X_t = \begin{cases} B_t, & 0\le t\le \frac{1}{2} \\ B_{\frac{1}{2}} + (\sqrt{2}-1)B_{t-\frac{1}{2}}, & \frac{1}{2} < t \le 1. \end{cases}$$

Then X is a continuous weak Brownian motion of order 1, which is not a $\mathbb{G}$-semimartingale. Moreover it is possible to show that $d[X]_t = f_t dt$, with $f = I_{[0,\frac{1}{2}]} + (\sqrt{2}-1)^2 I_{[\frac{1}{2},1]}$. In particular, thanks to point 2. of previous proposition, $X + \int_0^\cdot \frac{(1-f_s)X_s}{2s}ds$ is an $\mathcal{A}^1_X$-martingale.

A natural question is the following. Supposing that X is an $\mathcal{A}$-martingale with respect to a probability measure Q equivalent to P, what can we say about the nature of X under P?. The following proposition provides a partial answer to this problem when $\mathcal{A} = \mathcal{A}^1_X$.

Proposition 1.4.19. *Let X be as in proposition 1.4.17, and σ satisfy assumption 1.4.15. Suppose furthermore that $\nu_t << \lambda, 0 < t \le 1$, λ being the Lebesgue measure, and that X is an $\mathcal{A}^1_X$-martingale under a probability measure Q with $P << Q$, Then the law of X_t is absolutely continuous with respect to Lebesgue measure, for every $0 < t \le 1$.*

Proof. Since $P << Q$, for every $0 < t \leq 1$, $X_t(P) << X_t(Q)$. Then it is sufficient to observe that by proposition 1.4.17, for every $0 < t \leq 1$, the law of X_t under Q is absolutely continuous with respect to Lebesgue.

□

Corollary 1.4.20. *Let X be as in proposition 1.4.17, and σ satisfy assumption 1.4.15. Suppose furthermore that $\nu_t << \lambda, 0 < t \leq 1$, λ being the Lebesgue measure, and that X is an $\mathcal{A}_X$-martingale under a probability measure Q with $P << Q$. Then the law of X_t is absolutely continuous with respect to Lebesgue measure, for every $0 < t \leq 1$.*

Proof. Clearly $\mathcal{A}_X^1$ is contained in $\mathcal{A}_X$. The result is then a consequence of previous proposition.

□

Proposition 1.4.21. *Let $(X_t, 0 \leq t \leq 1)$ be a continuous weak Brownian motion of order 8. Then, for every $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}$, Borel measurable with polynomial growth, the forward integral $\int_0^\cdot \psi(t, X_t) d^- X_t$, exists and*

$$\mathbb{E}\left[\int_0^\cdot \psi(t, X_t) d^- X_t,\right] = 0.$$

In particular, X is an $\mathcal{A}_X$-martingale.

Proof. Let $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}$ be Borel measurable and t in $0 \leq t \leq 1$ be fixed. Set

$$I_\varepsilon^X(t) = I(\varepsilon, \psi(\cdot, X), X) \quad I_\varepsilon^B(t) = I(\varepsilon, \psi(\cdot, B), B),$$

being B a Brownian motion on a filtered probability space $(\Omega^B, \mathbb{F}^B, P^B)$.

Since X is a weak Brownian motion of order 8, it follows that

$$\mathbb{E}\left[\left|I_\varepsilon^X(t) - I_\delta^X(t)\right|^4\right] = \mathbb{E}^{P^B}\left[\left|I_\varepsilon^B(t) - I_\delta^B(t)\right|^4\right], \quad \forall\, \varepsilon, \delta > 0.$$

We show now that $I_\varepsilon^B(t)$ converges in $L^4(\Omega)$. This implies that $I_\varepsilon^X(t)$ is of Cauchy in $L^4(\Omega)$.

In [57], chapter 3.5, it is proved that $I_\varepsilon^B(t)$ converges in probability when ε goes to zero, and the limit equals the Itô integral $\int_0^t \psi(s, B_s) dB_s$. Applying Fubini theorem for Itô integrals, theorem 45 of [48], chapter IV and Burkholder-Davies-Gundy inequality, we can perform the following estimate, for every $p > 4$:

$$\begin{aligned}
\mathbb{E}^{P^B}\left[\left|I_\varepsilon^B(t)\right|^p\right] &= \mathbb{E}^{P^B}\left[\left|\int_0^t \left(\frac{1}{\varepsilon}\int_{r-\varepsilon}^r \psi(s, B_s) ds\right) dB_r\right|^p\right] \\
&\leq c\mathbb{E}^{P^B}\left[\int_0^1 \frac{1}{\varepsilon}\int_{r-\varepsilon}^r |\psi(s, B_s)|^p \, ds dr\right] \\
&\leq c \sup_{t \in [0,1]} \mathbb{E}^{P^B}\left[|\psi(t, B_t)|^p\right] < +\infty,
\end{aligned}$$

for some positive constant c. This implies the uniformly integrability of the family of random variables $\left((I_\varepsilon^B(t))^4\right)_{\varepsilon > 0}$ and therefore the convergence in $L^4(\Omega^B, P^B)$ of $\left(I_\varepsilon^B(t)\right)_{\varepsilon > 0}$.

Consequently, $\left(I_\varepsilon^X(t)\right)_{\varepsilon>0}$ converges in $L^4(\Omega)$ toward a random variable $I(t)$. It is clear that $\mathbb{E}\left[I(t)\right] = 0$, being $I(t)$ the limit in $L^2(\Omega)$ of random variables having zero expectation.

To conclude we show that Kolmogorov lemma applies to find a continuous version of $(I(t), 0 \leq t \leq 1)$. Let $0 \leq s \leq t \leq 1$. Applying the same arguments used above

$$\begin{aligned}
\mathbb{E}\left[|I(t) - I(s)|^4\right] &= \lim_{\varepsilon\to 0} \mathbb{E}^{P^B}\left[\left|\int_s^t \left(\frac{1}{\varepsilon}\int_u^{u+\varepsilon} \psi(u, B_u) dB_r\right) du\right|^4\right] \\
&\leq c\mathbb{E}^{P^B}\left[\left|\int_s^t \left(\frac{1}{\varepsilon}\int_{r-\varepsilon}^{r} \psi(u, B_u) du\right)^2 dr\right|^2\right] \\
&\leq |t-s|\,\mathbb{E}^{P^B}\left[\int_s^t \frac{1}{\varepsilon}\int_{r-\varepsilon}^{r} |\psi(u, B_u)|^4\, dudr\right] \\
&\leq \sup_{u\in[0,1]} \mathbb{E}^{P^B}\left[|\psi(u, B_u)|^4\right] |t-s|^2, \quad c > 0.
\end{aligned}$$

□

1.4.3 Optimization problems and $\mathcal{A}$-martingale property

Gâteaux-derivative : recalls

In this part of the paper we recall the notion of Gâteaux differentiability and we list some related properties.

Definition 1.4.22. *A function $f : \mathcal{A} \to \mathbb{R}$ is said* ***Gâteaux-differentiable*** *at $\pi \in \mathcal{A}$, if there exists $Df_\pi : \mathcal{A} \to \mathbb{R}$ such that*

$$\lim_{\varepsilon\to 0} \frac{f(\pi + \varepsilon\theta) - f(\pi)}{\varepsilon} = Df_\pi(\theta), \quad \forall\theta \in \mathcal{A}.$$

If f is Gâteaux-differentiable at every $\pi \in \mathcal{A}$, then f is said Gâteaux-differentiable on $\mathcal{A}$.

Definition 1.4.23. *Let $f : \mathcal{A} \to \mathbb{R}$. A process π is said* ***optimal*** *for f in $\mathcal{A}$ if*

$$f(\pi) \geq f(\theta), \quad \forall\theta \in \mathcal{A}.$$

We state this useful lemma omitting its straightforward proof.

Lemma 1.4.24. *Let $f : \mathcal{A} \to \mathbb{R}$. For every π and θ in $\mathcal{A}$ define $f_{\pi,\theta} : \mathbb{R} \longrightarrow \mathbb{R}$ in the following way :*

$$f_{\pi,\theta}(\lambda) = f(\pi + \lambda(\theta - \pi)).$$

Then it holds :

1. *f is Gâteaux-differentiable if and only if for every π and θ in $\mathcal{A}$, $f_{\pi,\theta}$ is differentiable on $\mathbb{R}$. Moreover $f'_{\pi,\theta}(\lambda) = Df_{\pi+\lambda(\theta-\pi)}(\theta - \pi)$.*
2. *f is concave if and only if $f_{\pi,\theta}$ is concave for every π and θ in $\mathcal{A}$.*

Proposition 1.4.25. *Let $f : \mathcal{A} \to \mathbb{R}$ be Gâteaux-differentiable. Then, if π is optimal for f in $\mathcal{A}$, $Df_\pi = 0$. If f is also concave*

$$\pi \text{ is optimal for } f \text{ in } \mathcal{A} \iff Df_\pi = 0.$$

Proof. It is immediate to prove that π is optimal for f if and only if $\lambda = 0$ is a maximum for $f_{\pi,\theta}$, for every θ in $\mathcal{A}$. By lemma 1.4.24 $f'_{\pi,\theta}(0) = Df_\pi(\theta)$, for every θ in $\mathcal{A}$. The conclusion follows easily.

□

An optimization problem

In this part of the paper F will be supposed to be a measurable function on $(\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}))$, almost surely in $C^1(\mathbb{R})$, strictly increasing, with F' being the derivative of F with respect to x, bounded on $\mathbb{R}$, uniformly in Ω. In the sequel ξ will be a continuous finite quadratic variation process with $\xi_0 = 0$.

The starting point of our construction is the following hypothesis.

Assumption 1.4.26. *1. If θ belongs to $\mathcal{A}$, then $\theta I_{[0,t]}$ belongs to $\mathcal{A}$ for every $0 \le t < 1$.*

2. Every θ in $\mathcal{A}$ ξ-improperly forward integrable, and

$$\mathbb{E}\left[\left|\int_0^1 \theta_t d^-\xi_t\right| + \left|\int_0^1 \theta_t^2 d[\xi]_t\right|\right] < +\infty.$$

Definition 1.4.27. *Let θ be in $\mathcal{A}$. We denote*

$$L^\theta = \int_0^1 \theta_t d^-\xi_t - \frac{1}{2}\int_0^1 \theta_t^2 d[\xi]_t, \quad dQ^\theta = \frac{F'(L^\theta)}{\mathbb{E}\left[F'(L^\theta)\right]}$$

and we set $f(\theta) = \mathbb{E}\left[F(L^\theta)\right]$.

We observe that point 2. of assumption 1.4.26 and the boundedness of F' implies that $\mathbb{E}\left[\left|F(L^\theta)\right|\right] < +\infty$. Therefore f is well defined.

Remark 1.4.28. *Point 2. of assumption 1.4.26 implies that $\mathbb{E}\left[|\xi_t| + [\xi]_t\right] \le +\infty$, for every $0 \le t \le 1$. This is due to the fact that $\mathcal{A}$ must contain real constants.*

We are interested in describing a link between the existence of an optimal process for f in $\mathcal{A}$ and the $\mathcal{A}$-semimartingale property for ξ under some probability measure equivalent to P, depending on the optimal process.

Lemma 1.4.29. *The function f is Gâteaux-differentiable on $\mathcal{A}$. Moreover for every π and θ in $\mathcal{A}$*

$$Df_\pi(\theta) = \mathbb{E}\left[F'(L^\pi)\int_0^1 \theta_t d^-\left(\xi_t - \int_0^t \pi_s d\left[\xi\right]_s\right)\right].$$

If F is concave, then f inherits the property.

Proof. Regarding the concavity of f, we recall that if F is increasing and concave, it is sufficient to verify that, for every θ and π in $\mathcal{A}$, it holds

$$L^{\pi+\lambda(\theta-\pi)} - L^{\pi} - \lambda\left(L^{\theta} - L^{\pi}\right) \geq 0, \quad 0 \leq \lambda \leq 1.$$

A short calculus shows that, for every $0 \leq \lambda \leq 1$,

$$L^{\pi+\lambda(\theta-\pi)} - L^{\pi} - \lambda\left(L^{\theta} - L^{\pi}\right) = \frac{1}{2}\lambda(1-\lambda)\int_0^1 (\theta_t - \pi_t)^2 d\left[\xi\right]_t \geq 0.$$

Using the differentiability of F we can write

$$a_\varepsilon = \frac{1}{\varepsilon}(f(\pi+\varepsilon\theta) - f(\pi)) = \mathbb{E}\left[H^{\varepsilon}_{\pi,\theta}\int_0^1 F'\left(L^{\pi} + \mu\varepsilon H^{\varepsilon}_{\pi,\theta}\right) d\mu\right],$$

with

$$H^{\varepsilon}_{\pi,\theta} = \int_0^1 \theta_t d^-\xi_t - \frac{1}{2}\int_0^1 (\theta_t^2\varepsilon + 2\theta_t\pi_t) d\left[\xi\right]_t.$$

The conclusion follows by Lebesgue dominated convergence theorem, which applies thanks to the boundedness of F' and point 2. in assumption 1.4.26.

□

Putting together lemma 1.4.29 and proposition 1.4.25 we can formulate the following.

Proposition 1.4.30. *If a process π in $\mathcal{A}$ is optimal for $\theta \mapsto \mathbb{E}\left[F\left(L^{\theta}\right)\right]$, then the process $\xi - \int_0^{\cdot} \pi_t d\left[\xi\right]_t$ is an $\mathcal{A}$-martingale under Q^{π}. If F is concave the converse holds.*

Proof. Thanks to lemma 1.4.29 and point 1. in assumption 1.4.26, for every θ in $\mathcal{A}$ and $0 \leq t \leq 1$

$$\begin{aligned} 0 &= Df_{\pi}(\theta I_{[0,t]}) = \mathbb{E}\left[F'(L^{\pi})\int_0^t \theta_s d^-\left(\xi_s - \int_0^s \pi_r d\left[\xi\right]_r\right)\right] \\ &= \mathbb{E}^{Q^{\pi}}\left[\int_0^t \theta_s d^-\left(\xi_s - \int_0^s \pi_r d\left[\xi\right]_r\right)\right]. \end{aligned}$$

□

The following proposition describes some sufficient conditions to recover the semimartingale property for ξ with respect to a filtration $\mathbb{G}$ on $(\Omega, \mathcal{F})$, when the set $\mathcal{A}$ is made up of $\mathbb{G}$-adapted processes. It can be proved using proposition 1.4.7.

Proposition 1.4.31. *Assume that ξ is adapted with respect to some filtration $\mathbb{G}$ and that $\mathcal{A}$ satisfies the hypothesis $\mathcal{E}$ with respect to $\mathbb{G}$. If a process π in $\mathcal{A}$ is optimal for $\theta \mapsto \mathbb{E}\left[F(L^{\theta})\right]$, then the process $\xi - \int_0^{\cdot} \beta_t d\left[\xi\right]_t$ is a $\mathbb{G}$-martingale under P, where $\beta = \pi + \frac{1}{p^{\pi}}\frac{d[p^{\pi},\xi]}{d[\xi,\xi]}$, and $p^{\pi} = \mathbb{E}\left[\frac{dP}{dQ^{\pi}} \mid \mathcal{G}_{\cdot}\right]$. If F is concave, then the converse holds.*

Proof. Thanks to point 2. of assumption 1.4.26, for every $0 \leq t < 1$, the random variable $\xi_t - \int_0^t \pi_t d[\xi]_t$ is in $L^1(\Omega)$ and so in $L^1(\Omega, Q^\pi)$ being $\frac{dQ^\pi}{dP}$ bounded. Then proposition 1.4.7 applies to state that $\xi - \int_0^\cdot \pi_t d[\xi]_t$ is a $\mathbb{G}$-martingale under Q^π. Using Girsanov theorem, chapter 6 of [48], we get the necessity condition. As far as the converse is concerned, we observe that, thanks to the hypotheses on $\mathcal{A}$, if $\xi - \int_0^\cdot \pi_t d[\xi]_t$ is a $\mathbb{G}$-martingale, then for every θ in $\mathcal{A}$, the process $\int_0^\cdot \theta_t d^-\left(\xi_t - \int_0^t \pi_s d[\xi]_s\right)$ is a $\mathbb{G}$-martingale starting at zero with zero expectation. This concludes the proof.

□

Proposition 1.4.32. *Suppose that there exists a measurable process* $(\gamma_t, 0 \leq t \leq 1)$ *such that the process* $\xi - \int_0^\cdot \gamma_t d[\xi]_t$ *is an* $\mathcal{A}$*-martingale. Assume, furthermore, the existence of a sequence of processes* $(\theta^n)_{n\in\mathbb{N}} \subset \mathcal{A}$ *with*

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^1 |\theta_t^n - \gamma_t|^2 \, d[\xi]_t\right] = 0.$$

If γ *belongs to* $\mathcal{A}$ *then* γ *is optimal for* $\theta \mapsto \mathbb{E}\left[L^\theta\right]$*. Moreover if there exists an optimal process* π*, then* $d\,|[\xi]|\,\{t \in [0,1), \gamma_t \neq \pi_t\} = 0$*, almost surely.*

Proof. Using proposition 1.4.30 we deduce that a process π is optimal for f if and only if the process $\int_0^\cdot (\gamma_t - \pi_t) d[\xi]_t$ is an $\mathcal{A}$-martingale under P. Then π is optimal if and only if for every θ is in $\mathcal{A}$ it holds : $\mathbb{E}\left[\int_0^\cdot \theta_t(\gamma_t - \pi_t) d[\xi]_t\right] = 0$. This permits to achieve immediately the end of the proof.

□

An example of $\mathcal{A}$-martingale and a related optimization problem

We illustrate a setting where proposition 1.4.32 applies. It will be deduced by [39]. In that paper the authors study a particular case of the optimization problem considered in proposition 1.4.32. As process ξ they take a Brownian motion W, and they find sufficient conditions in order to have existence of a process γ such that $W - \int_0^\cdot \gamma_t dt$ is an $\mathcal{A}$-martingale, being $\mathcal{A}$ some specific set we shall clarify later. To get their goal, they consider an anticipating setting and combine Malliavin calculus with substitution formulae, the anticipation being generated by a random variable possibly depending on the whole trajectory of W.

We work into the specific framework of subsection 1.3.2.

Assumption 1.4.33. *We suppose the existence of a random variable* L *in* $\mathbb{D}^{1,2}$*, satisfying the following assumption :*

1. $\int_{\mathbb{R}} \mathbb{E}\left[|L|^2 I_{\{0\leq x\leq L\}\cup\{0\geq x\geq L\}}\right] dx < +\infty$;
2. *for a.a.* t *in* $[0,1]$ *the process*

$$I(\cdot, t, L) := I_{[t,1]}(\cdot) I_{\left\{\int_t^1 (D_s L)^2 ds > 0\right\}} \left(\int_t^1 (D_s L)^2 ds\right)^{-1} (D_t L)(D.L)$$

belongs to $Dom\delta$ *and there exists a* $\mathcal{P}(\mathbb{F}) \times \mathcal{B}(\mathbb{R})$*-measurable random field*

$$(h(t,x), 0 \le t \le 1, x \in \mathbb{R})$$

such that $h(\cdot, L)$ *belongs to* $L^2(\Omega \times [0,1])$ *and*

$$\mathbb{E}\left[\int_0^1 I(u,t,L)dW_u \mid \mathcal{F}_t \vee \sigma(L)\right] = h(t,L), \quad 0 \le t \le 1.$$

Let $\Theta(L)$ be the set of processes $(\theta_t, 0 \le t < 1)$ such that there exists a random field $(u(t,x), 0 \le t \le 1, x \in \mathbb{R})$ with $\theta_t = u(t,L)$, $0 \le t < 1$ and

$$\begin{cases} u(t,\cdot) \in C^1(\mathbb{R}) \ \forall\ 0 \le t \le 1. \\ \int_{-n}^{n} \int_0^1 (\partial_x u(t,x))^2 dt dx < +\infty, \forall n \in \mathbb{N}\ a.s.. \\ \mathbb{E}\left[\int_{\mathbb{R}} \left(\int_0^1 (\partial_x u(t,x))^2 dt\right)^2 dx + \int_0^1 (u(t,0))^2 dt\right] < +\infty. \\ \mathbb{E}\left[\int_0^1 (\partial_x u(t,L))^2 (D_t L)^2 dt + \left(\int_0^1 (\partial_x u(t,L))^2 dt\right)\left(\int_0^1 (D_t L)^2 dt\right)\right] < +\infty. \end{cases}$$

Suppose that $\mathcal{A}$ equals $\Theta(L)$. With the specifications above we have the following.

Corollary 1.4.34. *Let* b *be a process in* $L^2(\Omega \times [0,1])$*, such that* $h(\cdot, L) + b$ *belongs to the closure of* $\mathcal{A}$ *in* $L^2(\Omega \times [0,1])$*. There exists an optimal process* π *in* $\mathcal{A}$ *for the function*

$$\theta \mapsto \mathbb{E}\left[\int_0^1 \theta_t d^- \left(W_t + \int_0^t b_s ds\right) - \frac{1}{2}\int_0^1 \theta_t^2 dt\right]$$

if and only if $h(\cdot, L) + b$ *belongs to* $\mathcal{A}$ *and* $h(\cdot, L) + b = \pi$*.*

Proof. It is clear that $\mathcal{A}$ is a real linear set of measurable and with bounded paths processes verifying condition 1. of assumption 1.4.26. Proposition 2.8 of [39] shows that every θ in $\mathcal{A}$ is in $L^2(\Omega \times [0,1])$, that it is W-improperly forward integrable and that the improper integral belongs to $L^2(\Omega)$. In particular, condition 2. of assumption 1.4.26 is verified. Furthermore, the proof of theorem 3.2 of [39] implicitly shows that the process $W - \int_0^{\cdot} h(t,L)dt$, is a $\mathcal{A}$-martingale. This implies that $W + \int_0^{\cdot} b_t dt - \int_0^{\cdot} \gamma_t dt$, with $\gamma = h(\cdot, L) + b$, is an $\mathcal{A}$-martingale. The end of the proof follows then by proposition 1.4.32.

□

1.5 The market model

We consider a market offering two investing possibilities in the time interval $[0,1]$. Prices of the two traded assets follow the evolution of two stochastic processes

$$\left(S_t^0, 0 \le t \le 1\right) \quad \text{and} \quad \left(S_t, 0 \le t \le 1\right).$$

We assume that

$$S^0 = (\exp(V_t), 0 \le t \le 1),$$

where $(V_t, 0 \le t \le 1)$ is a positive process starting at zero with bounded variation, and S is a continuous strictly positive process, with finite quadratic variation.

Remark 1.5.1. *If $V = \int_0^\cdot r_s ds$, being $(r_t, 0 \le t \le 1)$ the short interest rate, S^0 represents the price process of the so called money market account. Here we do not assume that V is a riskless asset, being that assumption not necessary to develop our calculus. We only suppose that S^0 is less risky then S.*

Assuming that S has a finite quadratic variation is not restrictive at least for two reasons.

Consider a market model involving an inside trader : that means an investor having additional informations with respect to the honest agent. Let $\mathbb{F}$ and $\mathbb{G}$ be the filtrations representing the information flow of the honest and the inside investor, respectively. Then it could be worthwhile to demand the absence of free lunches with vanishing risk (FLVR) among all simple $\mathbb{F}$-predictable strategies. Under the hypothesis of absence of (FLVR), by theorem 7.2, page 504 of [16], S is a semimartingale on the underlying probability space $(\Omega, P, \mathbb{F})$. On the other hand S could fail to be a $\mathbb{G}$-semimartingale, since (FLVR) possibly exist for the insider. Nevertheless, the inside investor is still allowed to suppose that S has finite quadratic variation thanks to proposition 1.2.7.

Secondly, as already specified in the introduction, if practical reasons induce to include S as a self-financing-portfolio, we have to require that $\int_0^\cdot S d^- S$ exists. This is equivalent to assume that S has finite quadratic variation, see proposition 4.1 of [56].

1.5.1 Portfolio strategies

We assume the point of view of an investor whose flow of information is modeled by a filtration $\mathbb{G}= (\mathcal{G}_t)_{t\in[0,1]}$ of $\mathcal{F}$, which satisfies the usual assumptions.

We denote with $C_b^-([0,1))$ the set of processes which have paths being left continuous and bounded on each compact set of $[0,1)$.

Definition 1.5.2. *A **portfolio strategy** is a couple of $\mathbb{G}$-adapted processes*

$$\phi = \left(\left(h_t^0, h_t\right), 0 \le t < 1\right).$$

*The market value X of the portfolio strategy ϕ is the so called **wealth process** $X = h^0 S^0 + hS$.*

We stress that there is no point in defining the portfolio strategy at the end of the trading period, that is for $t = 1$. Indeed, at time 1, the agent has to liquidate his portfolio.

Definition 1.5.3. *A portfolio strategy $\phi = (h^0, h)$ is **self-financing** if both h^0 and h belong to $C_b^-([0,1))$, the process h is locally S-forward integrable and its wealth process X verifies*

$$X = X_0 + \int_0^\cdot h_t^0 dS_t^0 + \int_0^\cdot h_t d^- S_t. \tag{1.11}$$

The interpretation of the first two items is straightforward : h^0 and h represent, respectively, the number of shares of S^0 and S held in the portfolio ; X is its market value. The self-financing condition (1.11) seems to be an appropriate formalization of the intuitive idea of trading strategy not involving exogenous sources of money. Among its justifications we can include the following ones.

As already explained in the introduction, the discrete time version of condition (1.11) reads as the classical self-financing condition. Furthermore, if S is a $\mathbb{G}$-semimartingale, forward integrals of $\mathbb{G}$-adapted processes with left continuous and bounded paths, agree with classical Itô integrals, see proposition 1.2.8 and 1.2.7.

In the sequel we will choose as *numéraire* the positive process S^0. That means that prices will be expressed in terms of S^0. We will denote with $\widetilde{Y}$ the value of a stochastic process $(Y_t, 0 \leq t \leq 1)$ *discounted* with respect to S^0 : $\widetilde{Y}_t = Y_t(S^0_t)^{-1}$, for every $0 \leq t \leq 1$.

The following lemma shows that, as well as in a semimartingale model, a portfolio strategy which is self-financing is uniquely determined by its initial value and the process representing the number of shares of S held in the portfolio. We remark that previous definitions and considerations can be made without supposing that the investor is able to observe prices of S and S^0. However, we need to make this hypothesis for the following characterization of self-financing portfolio strategies.

Assumption 1.5.4. *From now on we suppose that S and S^0 are $\mathbb{G}$-adapted processes.*

Lemma 1.5.5. *Let $\phi = (h^0, h)$ be a couple of $\mathbb{G}$-adapted processes in $C_b^-([0,1))$. Suppose that h is locally S-forward integrable. Then the portfolio strategy ϕ is self-financing if and only if its discounted wealth process $\widetilde{X}$ verifies*

$$\widetilde{X} = X_0 - \int_0^{\cdot} e^{-V_t} h_t S_t dV_t + \int_0^{\cdot} e^{-V_t} d^- \int_0^t h_s d^- S_s. \tag{1.12}$$

On the other hand, let $(h_t, 0 \leq t < 1)$ be a $\mathbb{G}$-adapted process in $C_b^-([0,1))$, which is locally S-forward integrable, and X_0 be a $\mathcal{G}_0$-random variable. Then the couple

$$\phi = \left(\left((X_t - h_t S_t)(S^0_t)^{-1}, h_t \right), 0 \leq t < 1 \right),$$

with X defined as in equality (1.12), is a self-financing portfolio strategy with wealth process X.

Proof. Regarding the first part of the statement we observe that corollary 1.2.9 and equality $X = h^0 B + hS$ imply the equivalence between equalities (1.12) and (1.11).

Let h, X_0 and X be as in the second part of the statement. It is clear that $h^0 = ((X_t - h_t S_t)(S^0_t)^{-1}, 0 \leq t < 1)$ is $\mathbb{G}$-adapted and belongs to $C_b^-([0,1))$. By construction, the wealth process corresponding to the strategy $\phi = (h^0, h)$ is equal to X. The conclusion follows by the first part of the statement.

□

Remark 1.5.6. *Suppose that (h^0, h) is a self-financing portfolio strategy with h locally forward integrable with respect to $\widetilde{S}$. Corollary 1.2.9 and previous lemma imply that its discounted wealth process $\widetilde{X}$ can be also be rewritten in the following way*

$$\widetilde{X} = X_0 + \int_0^{\cdot} h_t d^- \widetilde{S}_t + R,$$

with

$$R = \int_0^{\cdot} e^{-V_t} d^- \int_0^t h_s d^- S_s - \int_0^{\cdot} h_t d^- \int_0^t e^{-V_s} d^- S_s.$$

Lemma 1.5.5 leads to conceive the following definition.

Definition 1.5.7. *1. A **self-financing portfolio** is a couple (X_0, h) of a process h in $C_b^-([0,1))$, which is $\mathbb{G}$-adapted and locally S-forward integrable, and a $\mathcal{G}_0$-measurable random variable X_0.*

2. The discounted wealth process $\widetilde{X}$ of the self financing portfolio (X_0, h), and the number of shares h^0 of S^0 held in that portfolio are given by

$$\begin{cases} \widetilde{X} = X_0 - \int_0^{\cdot} e^{-V_t} h_t S_t dV_t + \int_0^{\cdot} e^{-V_t} d^- \int_0^t h_s d^- S_s \\ h^0 = (X - hS)(S^0)^{-1}. \end{cases}$$

*3. In the sequel we let us employ the term **portfolio** to denote the process h, in a self-financing portfolio, representing the number of shares of S held. Without further specifications the initial wealth of an investor will be assumed to be equal to zero.*

Lemma 1.5.5 and remark 1.5.6 immediately imply the following.

Corollary 1.5.8. *Let (X_0, h) be a self-financing portfolio such that h is locally $\widetilde{S}$-forward integrable and $\int_0^{\cdot} e^{-V_t} d^- \int_0^t h_s d^- S_s = \int_0^{\cdot} h_t d^- \int_0^t e^{-V_s} d^- S_s$. Then its discounted value $\widetilde{X}$ verifies the equality $\widetilde{X} = X_0 + \int_0^{\cdot} h d^- \widetilde{S}$.*

Remark 1.5.9. *If S is a $\mathbb{G}$-semimartingale, the hypothesis required on h in previous remark is always verified. Indeed, forward integrals coincide with classical Itô integrals for which the associative property holds true, see proposition 1.2.7.*

Some conditions to insure the existence of chain-rule formulae, when the semimartingale property of the integrator process fails to hold, can be found in section 1.3. For more informations about this topic we also refer to [24] and [21].

Assumption 1.5.10. *We assume the existence of a real linear space of portfolios $\mathcal{A}$, that is of $\mathbb{G}$-adapted processes h belonging to $C_b^-([0,1))$, which are locally S-forward integrable. The set $\mathcal{A}$ will represent the set of all **admissible strategies** for the investor.*

We proceed furnishing examples of sets behaving as the set $\mathcal{A}$ in assumption 1.5.10. They correspond to the examples discussed in section 1.3.

Admissible strategies via Itô fields

We refer the reader to subsection 1.3.1 for notations and definitions.

The following proposition is a straightforward consequence of proposition 1.3.7.

Proposition 1.5.11. *Let $\mathcal{A}$ be the set of processes $(h_t, 0 \le t < 1)$ such that for every $0 \le t < 1$ the process in $hI_{[0,t]}$ belongs to $\mathcal{S}(\mathcal{C}_S^1(\mathbb{G}))$. Then $\mathcal{A}$ is a real linear space satisfying the hypotheses of assumption 1.5.10.*

Admissible strategies via Malliavin calculus

For this example we refer to subsection 1.3.2. We recall that there, W was a real valued Wiener process defined on the canonical probability space $(\Omega, \mathbb{F}, \mathcal{F}, P)$. Regarding the price of S we make the following assumption.

Assumption 1.5.12. *We suppose that $S = S_0 \exp\left(\int_0^{\cdot} \sigma_t dW_t + \int_0^{\cdot} \left(\mu_t - \frac{1}{2}\sigma_t^2\right) dt\right)$, where μ and σ are $\mathbb{F}$-adapted, μ belongs to $L^{1,q}$ for some $q > 4$, σ has bounded and left continuous paths, it belongs $L_-^{1,2} \cap L^{2,2}$ and the random variable*

$$\sup_{t\in[0,1]} \left(|\sigma_t| + \sup_{s\in[0,1]} |D_s\sigma_t| \sup_{s,u\in[0,1]} |D_s D_u \sigma_t| \right)$$

is bounded.

Remark 1.5.13. *By remark of page 32, section 1.2 of [43] σ is in $L_-^{1,2}$ and $D^-\sigma = 0$.*

Using remarks 1.3.9 and 1.3.12, lemma 1.3.15 and lemma 1.3.16, it is not difficult to prove that the process $\log(S)$ belongs to $L_-^{1,q}$.

Proposition 1.5.14. *Let $\mathcal{A}$ be the set of all $\mathbb{G}$-adapted processes h in $C_b^-([0,1))$, such that for every $0 \le t < 1$, the process $hI_{[0,t]}$ belongs to $L_-^{1,p}$, for some $p > 4$. Then $\mathcal{A}$ is a real linear space satisfying the hypotheses of assumption 1.5.10.*

Proof. Let h be in $\mathcal{A}$. We set $A = \log(S) - \log(S_0) + \frac{1}{2}\int_0^{\cdot} \sigma_t^2 dt = \int_0^{\cdot} \sigma_t dW_t + \int_0^{\cdot} \mu_t dt$. We recall that, thanks to lemma 1.2.12, for every $0 \le t < 1$, $hI_{[0,t]}$ is S-forward integrable if and only if $hI_{[0,t]}S$ is forward integrable with respect to A. Let $0 \le t < 1$, be fixed. Each component of the vector process $u = \left(hI_{[0,t]}, \log(S)\right)$ belongs to $L_-^{1,p}$ for some $p > 4$ and it has left continuous and bounded paths. We can thus apply proposition 1.3.21 to state that $hI_{[0,t]}S$ is forward integrable with respect to $\int_0^{\cdot} \sigma_t dW_t$. This implies that $hI_{[0,t]}S$ is A-forward integrable. Letting t vary in $[0,1)$ we find that h is S-improperly integrable and we get the end of the proof.

□

Admissible strategies via substitution

We consider the setting of subsection 1.3.3. More precisely, we assume the existence of a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ on $(\Omega, \mathcal{F}, P)$, with $\mathcal{F}_1 = \mathcal{F}$, and of an $\mathcal{F}$-measurable random variable L with values in $\mathbb{R}^d$ such that $\mathcal{G}_t = (\mathcal{F}_t \vee \sigma(L))$, for every $0 \le t \le 1$. We suppose that $\mathbb{G}$ is right continuous. We assume that S and S^0 are $\mathbb{F}$-adapted, and that S is an $\mathbb{F}$-semimartingale.

We observe that this situation arises when the investor trades as an ***insider***, that is having an extra information about prices, at time 0, represented by the random variable L.

Proposition 1.5.15. *Let $\mathcal{A}$ be the set of processes h such that, for every $0 \le t < 1$, the process $hI_{[0,t]}$ belongs to $\mathcal{S}(\mathcal{A}^{p,\gamma}(L))$, for some $p > 1$ and $\gamma > d$. Then $\mathcal{A}$ satisfies the hypotheses of assumption 1.5.10.*

Proof. Processes in $\mathcal{A}$ are clearly $\mathbb{G}$-adapted and in $C_b^-([0,1))$. The end of the proof is a consequence of proposition 1.3.37 and remark 1.3.28.

□

The following lemma shows that is not so reductive to restrict the class of possible portfolio strategies to the collection of sets $(\mathcal{S}(\mathcal{A}^{p,\gamma}(L)), p > 1, \gamma > d)$.

Lemma 1.5.16. *Let $(\pi_t, 0 \leq t < 1)$ be a bounded $\mathcal{P}^{\mathbb{G}}$-measurable process. Then there exists a $\mathcal{P}^{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R})$-measurable function $\left(h(t,x), 0 \leq t < 1, x \in \mathbb{R}^d\right)$, such that $\pi = h(\cdot, L)$, almost surely.*

Proof. Define $L^{0,\mathcal{P}^{\mathbb{F}}}$ as the set of all functions $(h(t,x), 0 \leq t < 1, x \in \mathbb{R}^d)$ which are $\mathcal{P}^{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. Set

$$\mathcal{M} = \left\{ u : \Omega \times [0,1) \to \mathbb{R}, \mid \exists h \in L^{0,\mathcal{P}^{\mathbb{F}}}, \text{ s.t. } h(\cdot, L) = u \text{ a.s. } \right\}.$$

The set $\mathcal{M}$ is a *monotone vector space*, see definition in chapter 1 of [48]. Indeed, it is a linear vector space of bounded real functions containing all constants and, if $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence of positive random elements in $\mathcal{M}$, with $u = \sup_{n\in\mathbb{N}} u_n$ bounded, then u belongs to $\mathcal{M}$. In fact $h = sup_n h_n$ is still in $L^{0,\mathcal{P}^{\mathbb{F}}}$ and $u = h(\cdot, L)$. Consider the set $\mathcal{S}^{\mathbb{G}}$ of all $\mathcal{P}^{\mathbb{G}}$-measurable processes of the form $u = I_{\{0\}} h_0(L) f_0 + \sum_{i=0}^{k-2} I_{(t_i, t_{i+1}]} h_i(L) f_i + I_{(t_{k-1},1)} h_{k-1}(L) f_{k-1}$, where $0 = t_0 < t_1 < \ldots < t_k = 1$, and h_i is $\mathcal{B}(\mathbb{R})$-measurable and bounded, f_i is $\mathcal{F}_{t_i}$-measurable and bounded, for every $i = 0, \ldots, k$. It is clear that $\mathcal{S}^{\mathbb{G}}$ is stable with respect to multiplication. Moreover $\sigma\left(\mathcal{S}^{\mathbb{G}}\right)$ contains the σ-algebra generated by all bounded and $\mathcal{P}^{\mathbb{G}} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function. We can thus apply theorem 8 of [48] to get the result.

□

1.5.2 Completeness and arbitrage : $\mathcal{A}$-martingale measures

Definition 1.5.17. *Let h be a self financing portfolio in $\mathcal{A}$, which is S-improperly forward integrable and X its wealth process. Then h is an **$\mathcal{A}$-arbitrage** if $X_1 = \lim_{t\to 1} X_t$ exists almost surely, $P(\{X_1 \geq 0\}) = 1$ and $P(\{X_1 > 0\}) > 0$.*

Definition 1.5.18. *If there are no $\mathcal{A}$-arbitrages we say that the market is **$\mathcal{A}$-arbitrage free**.*

Definition 1.5.19. *A probability measure $Q \sim P$ is said **$\mathcal{A}$-martingale measure** if under Q the process $\widetilde{S}$ is an $\mathcal{A}$-martingale according to definition 1.4.1.*

We will need the following assumption.

Assumption 1.5.20. *Suppose that for all h in $\mathcal{A}$ the following conditions hold.*

1. *The process $e^V h$ belongs to $\mathcal{A}$.*
2. *h is $\widetilde{S}$-improperly forward integrable and*

$$\int_0^\cdot e^{-V_t} d^- \int_0^t h_s d^- S_s = \int_0^\cdot h_t e^{-V_t} d^- S_t = \int_0^\cdot h_t d^- \int_0^t e^{-V_s} d^- S_s. \tag{1.13}$$

For the following proposition the reader should keep in mind the notation in equality (1.6). We omit its proof which is a direct application of corollary 1.4.20.

Proposition 1.5.21. *Let $\mathcal{A} = \mathcal{A}_S$. Suppose that $d\,[S]_t = \sigma^2 S_t^2 dt$, for some real σ. If there exists a $\mathcal{A}$-martingale measure then the law of $\widetilde{S}_t$ is absolutely continuous with respect to Lebesgue measure, for every $0 < t \leq 1$.*

Proposition 1.5.22. *Under assumption 1.5.20, if there exists an $\mathcal{A}$-martingale measure Q, the market is $\mathcal{A}$-arbitrage free.*

Proof. Suppose that h is an $\mathcal{A}$-arbitrage. Since $\widetilde{S}$ is an $\mathcal{A}$-martingale under Q, using corollary 1.5.8 we find $\mathbb{E}^Q[\widetilde{X}_1] = \mathbb{E}^Q[\int_0^1 h_t d^- \widetilde{S}_t] = 0$. This contradicts the arbitrage condition $Q(\{X_1 > 0\}) > 0$.

□

The proposition which follows characterizes the set of all $\mathcal{A}$-martingale measures.

Proposition 1.5.23. *Under assumption 1.5.20 the process $\widetilde{S}$ is an $\mathcal{A}$-martingale under Q, if and only if the process $S - \int_0^\cdot S_t dV_t$ is an $\mathcal{A}$-martingale under Q.*

Proof. If h is in $\mathcal{A}$ by assumption 1.5.20 we have

$$\begin{aligned}
\mathbb{E}^Q\left[\int_0^\cdot h_t d^-\left(S_t - \int_0^t S_s dV_s\right)\right] &= \mathbb{E}^Q\left[\int_0^\cdot (h_t e^{V_t}) e^{-V_t} d^-\left(S - \int_0^t S_s dV_s\right)\right] \\
&= \mathbb{E}^Q\left[\int_0^\cdot (h_t e^{V_t}) d^- \int_0^t e^{-V_s} d^- S_s\right] \\
&+ \mathbb{E}^Q\left[\int_0^\cdot (h_t e^{V_t}) d^- \int_0^t S_s d e^{-V_s}\right] \\
&= \mathbb{E}^Q\left[\int_0^\cdot (h_t e^{V_t}) d^- \widetilde{S}_t\right] = 0.
\end{aligned}$$

□

We proceed discussing *completeness.*

Definition 1.5.24. *A **contingent claim** is an $\mathcal{F}$-measurable random variable. $\mathcal{L}$ will be a set of $\mathcal{F}$-measurable random variables; it will represent all the contingent claims the investor is interested in.*

Definition 1.5.25.

1. *A contingent claim C is said **$\mathcal{A}$-attainable** if there exists a self financing portfolio (X_0, h) with h in $\mathcal{A}$, which is S-improperly forward integrable, such that the corresponding wealth process X verifies $\lim_{t\to 1} X_t = C$, almost surely. The portfolio h is said the **replicating** or **hedging** portfolio for C, X_0 is said the **replication price** for C.*
2. *The market is said to be **$(\mathcal{A}, \mathcal{L})$-complete** if every contingent claim in $\mathcal{L}$ is attainable trough a portfolio in $\mathcal{A}$.*

Assumption 1.5.26. *For every $\mathcal{G}_0$-measurable random variable η, and h in $\mathcal{A}$ the process $u = h\eta$, belongs to $\mathcal{A}$.*

Proposition 1.5.27. *Suppose that the market is $\mathcal{A}$-arbitrage free, and that assumption* 1.5.26 *is realized. Then the replication price of an attainable contingent claim is unique.*

Proof. Let (X_0, h) and (Y_0, k) be two replicating portfolios for a contingent claim C, with h and k in $\mathcal{A}$, and wealth processes X and Y, respectively. We have to prove that

$$P\left(\{X_0 - Y_0 \neq 0\}\right) = 0.$$

Suppose, for instance, that $P\left(X_0 - Y_0 > 0\right) \neq 0$. We set $A = \{X_0 - Y_0 > 0\}$. By assumption 1.5.26, $I_A(k-h)$ is a portfolio in $\mathcal{A}$ with wealth process $I_A(Y_t - Y_0 - X_t + X_0)$. Since both (X_0, h) and (Y_0, k) replicate C, $\lim_{t\to 1} I_A(Y_t - X_t) = I_A(X_0 - Y_0)$, with $P(\{I_A(X_0 - Y_0 > 0)\}) > 0$. Then $I_A(k-h)$ is an $\mathcal{A}$-arbitrage and this contradicts the hypotheses.

□

Proposition 1.5.28. *Suppose that there exists an $\mathcal{A}$-martingale measure Q and that V is bounded. Then the following statements are true.*

1. *Under assumptions 1.5.20 and 1.5.26, the replication price of an $\mathcal{A}$-attainable and Q-integrable contingent claim C is unique and equal to $\mathbb{E}^Q\left[\widetilde{C} \mid \mathcal{G}_0\right]$.*
2. *Let $\mathcal{G}_0$ be trivial. If Q and Q_1 are two $\mathcal{A}$-martingale measures, then $\mathbb{E}^Q[\widetilde{C}] = \mathbb{E}^{Q_1}[\widetilde{C}]$, for every $\mathcal{A}$-attainable and Q-integrable contingent claim C. In particular, if the market is $(\mathcal{A}, \mathcal{L})$-complete and $\mathcal{L}$ is an algebra, all $\mathcal{A}$-martingale measures coincide on the σ-algebra generated by all bounded discounted contingent claims in $\mathcal{L}$.*

Proof. Let (X_0, h) be a replicating $\mathcal{A}$-portfolio for C. By corollary 1.5.8

$$\mathbb{E}^Q\left[\widetilde{C} \mid \mathcal{G}_0\right] = X_0 + \mathbb{E}^Q\left[\int_0^1 h_t d^- \widetilde{S}_t \mid \mathcal{G}_0\right].$$

We observe that $\mathbb{E}^Q\left[\int_0^1 h_t d^- \widetilde{S}_t \mid \mathcal{G}_0\right] = 0$. In fact, if η is a $\mathcal{G}_0$-measurable random variable, then, thanks to assumption 1.5.26, ηh belongs to $\mathcal{A}$, so as to have $\mathbb{E}^Q\left[\left(\int_0^1 h_t d^- \widetilde{S}_t\right)\eta\right] = \mathbb{E}^Q\left[\int_0^1 \eta h_t d^- \widetilde{S}_t\right] = 0$. This implies point 1.

If $\mathcal{G}_0$ is trivial, we deduce that, if Q and Q_1 are two $\mathcal{A}$-martingale measures, $\mathbb{E}^Q[\widetilde{C}] = \mathbb{E}^{Q_1}[\widetilde{C}]$, for every $\mathcal{A}$-attainable contingent claim. The proof of the last point is then an application of theorem 8, chapter 1 of [48].

□

1.5.3 Hedging

In this part of the paper we price contingent claims via partial differential equations. In particular we show robustness of Black-Scholes formula for *European* and *Asian* contingent claims within a non-semimartingale model.

The following proposition generalizes a result obtained in a slight different form in [64], when the process S is supposed to be the sum of a Wiener process and a continuous process with zero quadratic variation.

We suppose here that $d[S]_t = \sigma(t,S_t)^2 S_t^2 dt$ and $dV_t = rdt$, with $r > 0$ and $\sigma : [0,1] \times (0,+\infty) \to \mathbb{R}$.

Proposition 1.5.29. *Let ψ be a function in $C^0(\mathbb{R})$. Suppose that there exists a function $(v(t,x), 0 \leq t \leq 1, x \in \mathbb{R})$ of class $C^{1,2}([0,1) \times \mathbb{R}) \cap C^0([0,1] \times \mathbb{R})$, which is a solution of the following Cauchy problem*

$$\begin{cases} \partial_t v(t,y) + \frac{1}{2}(\widetilde{\sigma}(t,y))^2 y^2 \partial^{(2)}_{yy} v(t,y) & = & 0 \quad on\ [0,1) \times \mathbb{R} \\ v(1,y) & = & \widetilde{\psi}(y), \end{cases} \tag{1.14}$$

where

$$\begin{cases} \widetilde{\sigma}(t,y) = \sigma(t, ye^{rt}) & \forall (t,y) \in [0,1] \times \mathbb{R}, \\ \widetilde{\psi}(y) = \psi(ye^r)e^{-r} & \forall y \in \mathbb{R}. \end{cases}$$

Set

$$h_t = \partial_y v(t, \widetilde{S}_t), \quad 0 \leq t < 1, \quad X_0 = v(0, S_0).$$

Then (X_0, h) is a self-financing portfolio replicating the contingent claim $\psi(S_1)$.

Proof. Assumption 1.5.4 tells us that h is a $\mathbb{G}$-adapted process in $C_b^-([0,1))$. By proposition 1.2.11, h is locally $\widetilde{S}$-forward integrable. Combining lemma 1.2.12 and proposition 1.2.11, it is possible to prove that

$$\int_0^\cdot e^{-V_t} d^- \int_0^t h_s d^- S_s = \int_0^\cdot h_t e^{-V_t} d^- S_t = \int_0^\cdot h_t d^- \int_0^t e^{-V_s} d^- S_s.$$

Similar arguments were used in [24], corollary 23. Corollary 1.5.8 implies then that its discounted wealth process verifies

$$\widetilde{X} = X_0 + \int_0^\cdot h d^- \widetilde{S}. \tag{1.15}$$

On the other hand by point 2. of proposition 1.3.7

$$[\tilde{S}] = \int_0^\cdot \widetilde{S}_s^2 \widetilde{\sigma}(s, \widetilde{S}_s)^2 ds. \tag{1.16}$$

Applying proposition 1.2.11, recalling equation (1.14), equalities (1.15) and (1.16) we find that

$$\widetilde{X}_t = v(t, \widetilde{S}_t), \quad \forall 0 \leq t < 1.$$

In particular $X_0 + \lim_{t\to 1} \int_0^t h_s d^- \widetilde{S}_s$ exists finite and coincides with $v(1, \widetilde{S}_1) = \widetilde{\psi}(\widetilde{S}_1) = \psi(S_1)e^{-r}$.

□

Remark 1.5.30. *In particular, under some minimal regularity assumptions on σ and no degeneracy, the market is $(\mathcal{A}_S, \mathcal{L})$-complete, if $\mathcal{L}$ equals the set of all contingent claims of type $\psi(S_1)$ with ψ in $C^0(\mathbb{R})$ with linear growth.*

The result of proposition 1.5.29 can also be adapted to hedge Asian contingent claims, that is contingent claims depending on the mean of S over the traded period : $\int_0^1 S_t dt$.

Proposition 1.5.31. *Suppose that $\sigma(t,x) = \sigma$, for every (t,x) in $[0,1] \times \mathbb{R}$, for some $\sigma > 0$. Let ψ be a function in $C^0(\mathbb{R})$ and $v(t,y)$ a continuous solution of class $C^{1,2}([0,1) \times \mathbb{R}) \cap C^0([0,1] \times \mathbb{R})$ of the following Cauchy problem*

$$\begin{cases} \frac{1}{2}\sigma^2 y^2 \partial^{(2)}_{yy} v(t,y) + (1 - ry)\partial_y v(t,y) + \partial_t v(t,y) &= 0, \quad \text{on } [0,1) \times \mathbb{R} \\ v(1,y) &= \psi(y). \end{cases}$$

Set $Z_t = \int_0^t S_s ds - K$, for some $K > 0$, $X_0 = v(0, \frac{K}{S_0})S_0$ and $h_t = v(t, \frac{Z_t}{S_t}) - \partial_y v(t, \frac{Z_t}{S_t})\frac{Z_t}{S_t}$, for all $0 \le t \le 1$. Then (X_0, h) is a self-financing portfolio which replicates the contingent claim $\psi\left(\frac{1}{S_1}\left(\int_0^1 S_t dt - K\right)\right) S_1$.

Proof. We set $\xi_t = \frac{Z_t}{S_t}, 0 \le t \le 1$. Applying proposition 1.2.11 to the function $u(t,z,s) = v(t, \frac{z}{s}e^{-rt})s$ and using the equation fulfilled by v we can expand the process $(e^{-rt}v(t,\xi_t)S_t, 0 \le t < 1)$ as follows :

$$u(t, Z_t, \widetilde{S}_t) = v(t, \xi_t)\,\tilde{S}_t = v(0, \xi_0)\,S_0 + \int_0^t h_t d^- \widetilde{S}_t. \tag{1.17}$$

By arguments which are similar to those used in the proof of previous proposition, it is possible to show that h is a self-financing portfolio and that (1.17) implies that $u(t, Z_t, \widetilde{S}_t) = \widetilde{X}_t$ for every $0 \le t < 1$. Therefore $\lim_{t \to 1} \widetilde{X}_t$ is finite and equal to $\psi(\xi_1)\,S_1 e^{-r}$. This concludes the proof.

□

1.5.4 Utility maximization

Formulation of the problem

We consider the problem of maximization of expected utility from terminal wealth starting from initial capital $X_0 > 0$, being X_0 a $\mathcal{G}_0$-measurable random variable. We define the function $U(x)$ modeling the utility of an agent with wealth x at the end of the trading period. The function U is supposed to be of class $C^2((0, +\infty))$, strictly increasing, with $U'(x)x$ bounded.

We will need the following assumption.

Assumption 1.5.32. *The utility function U verifies $\frac{U''(x)x}{U'(x)} \le -1, \quad \forall x > 0.$*

A typical example of function U verifying assumption 1.5.32 is $U(x) = \log(x)$.

We will focus on portfolios with strictly positive value. As a consequence of this, before starting analyzing the problem of maximization, we show how it is possible to construct portfolio strategies when only positive wealth is allowed.

Definition 1.5.33. *For simplicity of calculation we introduce the process*

$$A = \log(S) - \log(S_0) + \frac{1}{2}\int_0^{\cdot} \frac{1}{S_t^2} d[S]_t .$$

Lemma 1.5.34. *Let $\theta = (\theta_t, 0 \leq t < 1)$ be a $\mathbb{G}$-adapted process in $C_b^-([0,1))$ such that*

1. *θ is A-improperly forward integrable.*
2. *The process $A^\theta = \int_0^{\cdot} \theta_s d^- A_s$ has finite quadratic variation.*
3. *If X^θ is the process defined by*

$$X^\theta = X_0 \exp\left(\int_0^{\cdot} \theta_t d^- A_t + \int_0^{\cdot} (1-\theta_t)\, dV_t - \frac{1}{2}\left[A^\theta\right]\right),$$

then $\int_0^{\cdot} X_t^\theta \theta_t d^- A_t$ and $\int_0^{\cdot} X_t^\theta d^- \int_0^t \theta_s d^- A_s$ improperly exist and

$$\int_0^{\cdot} X_t^\theta \theta_t d^- A_t = \int_0^{\cdot} X_t^\theta d^- \int_0^t \theta_s d^- A_s. \tag{1.18}$$

Then the couple (X_0, h), with $h_t = \frac{\theta X_t^\theta}{S_t}$, $0 \leq t < 1$, is a self-financing portfolio with strictly positive wealth X^θ. In particular, $\lim_{t\to 1} X_t^\theta = X_1^\theta$ exists and it is strictly positive.

Proof. Thanks to lemma 1.2.12 h is locally S-forward integrable and $\int_0^{\cdot} h_t d^- S_t = \int_0^{\cdot} \theta_t X_t^\theta d^- A_t$. Applying corollary 1.2.9, proposition 1.2.11, and using hypothesis 3., $\widetilde{X^\theta}$ can be rewritten in the following way :

$$\begin{aligned} \widetilde{X_t^\theta} &= X_0 + \int_0^{\cdot} \widetilde{X_t^\theta} d^- \int_0^t \theta_s d^- A_s - \int_0^{\cdot} \widetilde{X_t^\theta} \theta_t dV_t \\ &= X_0 + \int_0^{\cdot} e^{-V_t} d^- \int_0^t h_s d^- S_s - \int_0^{\cdot} e^{-V_t} h_t S_t dV_t. \end{aligned} \tag{1.19}$$

Remark 1.5.6 tells us that X^θ is the wealth of the self-financing portfolio (X_0, h).

□

Remark 1.5.35. *The process θ in previous lemma represents the* ***proportion*** *of wealth invested in S.*

Remark 1.5.36. *Let θ be as in lemma 1.5.34. Then, for every $0 \leq t < 1$, X is, indeed, the unique solution, on $[0,t]$, of equation*

$$X^\theta = X_0 + \int_0^{\cdot} X_t^\theta d^- \left(\int_0^t \theta_s d^- A_s + \int_0^t (1-\theta_s) dV_s - \frac{1}{2}\left[A^\theta\right]_t\right).$$

In fact, uniqueness is insured by corollary 5.5 of [56]. It is important to highlight that, without the assumption on θ regarding the chain rule in equality (1.18), we cannot conclude that X^θ solves equation (1.19). However we need to require that X^θ solves the latter equation to interpret it as the value of a portfolio whose proportion invested in S is constituted *by θ. In the sequel we will construct, in some specific settings, classes of processes defining proportions of wealth as in lemma 1.5.34. We will consider, in particular, two cases already contemplated in [5] and [39]. Our definitions of those sets will result more complicated than the ones defined in the above cited papers. This happens because, in those works, the chain rule problem arising when the forward integral replaces the classical Itô integral is not clarified.*

Assumption 1.5.37. *We assume the existence of a real linear space $\mathcal{A}^+$ of $\mathbb{G}$-adapted processes $(\theta_t, 0 \leq t < 1)$ in $C_b^-([0,1))$, such that*

1. *θ verifies condition 1., 2. and 3. of lemma 1.5.34, and $\left[A^\theta\right] = \int_0^\cdot \theta_t^2 d\left[A\right]_t$.*
2. *$\theta I_{[0,t]}$ belongs to $\mathcal{A}^+$ for every $0 \leq t < 1$.*

For every θ in $\mathcal{A}^+$ we denote with Q^θ the probability measure defined by :

$$\frac{dQ^\theta}{dP} = \frac{U'(X_1^\theta)X_1^\theta}{\mathbb{E}\left[U'(X_1^\theta)X_1^\theta\right]}.$$

The utility maximization problem consists in finding a process π in $\mathcal{A}^+$ maximizing expected utility from terminal wealth, i.e. :

$$\pi = \arg\max_{\theta \in \mathcal{A}^+} \mathbb{E}\left[U(X_1^\theta)\right]. \tag{1.20}$$

Problem (1.20) is not trivial because of the uncertain nature of the processes A and V and the non zero quadratic variation of A. Indeed, let us suppose that $[A] = 0$ and that both A and V are deterministic. Then, it is sufficient to consider

$$\sup_{\lambda \in \mathbb{R}} \mathbb{E}\left[U(X_1^\lambda)\right] = \lim_{x \to +\infty} U(x),$$

and remind that U is strictly increasing, to see that a maximum can not be realized. The problem is less clear when the term $-\frac{1}{2}\int_0^\cdot \theta_t^2 d\left[A\right]_t$ and a source of randomness are added.

In the sequel, we will always assume the following.

Assumption 1.5.38. *For every θ in $\mathcal{A}^+$,*

$$\mathbb{E}\left[\left|\int_0^1 \theta_t d^-(A_t - V_t)\right| + \frac{1}{2}\int_0^1 \theta_t^2 \left[A\right]_t\right] < +\infty.$$

Definition 1.5.39. *A process π is said **optimal portfolio** in $\mathcal{A}^+$, if it is optimal for the function $\theta \mapsto \mathbb{E}\left[U(X_1^\theta)\right]$ in $\mathcal{A}^+$, according to definition 1.4.23.*

Remark 1.5.40. *Set $\xi = A - V$, $\mathcal{A} = \mathcal{A}^+$, and*

$$F(\omega, x) = U\left(X_0(\omega)e^{x+V_1(\omega)}\right), \quad (\omega, x) \in \Omega \times \mathbb{R}.$$

According to definitions of section 1.4.3, $\mathcal{A}$ satisfies assumption 1.4.26, the function F is measurable, almost surely in $C^1(\mathbb{R})$, strictly increasing and with bounded first derivative. If U satisfies assumption 1.5.32 then F is also concave. Moreover $F(L^\theta) = U(X_1^\theta)$ for every θ in $\mathcal{A}^+$.

Before stating some results about the existence of an optimal portfolio, we provide examples of sets of admissible strategies with positive wealth.

Admissible strategies via Itô fields

For this example the reader should keep in mind subsection 1.3.1.

Proposition 1.5.41. *Let $\mathcal{A}^+$ be the set of all processes $(\theta_t, 0 \leq t < 1)$ such that θ is the restriction to $[0,1)$ of a process h belonging to $\mathcal{S}(\mathcal{C}_A^1(\mathbb{G}))$. Then $\mathcal{A}^+$ satisfies the hypotheses of assumption 1.5.37.*

Proof. Let h be in $\mathcal{S}(\mathcal{C}_A^1(\mathbb{G}))$ and θ its restriction on $[0,1)$. It is clear that θ is in $C_b^-([0,1))$ and $\mathbb{G}$-adapted. Thanks to proposition 1.3.7, h is A-forward integrable, $\int_0^\cdot h_t d^- A_t$ belongs to $\mathcal{S}(\mathcal{C}_A^2(\mathbb{G}))$ and the process $\int_0^\cdot h_t d^- A_t$ has finite quadratic variation equal to $\int_0^\cdot h_t^2 d\,[A]_t$. By remark 1.2.5, $\int_0^\cdot h_t d^- A_t = \int_0^\cdot \theta_t d^- A_t$, and conditions 1. and 2. of lemma 1.5.34 are thus satisfied. Remark 1.3.6 implies that the process

$$\exp\left(\int_0^\cdot \theta_t d^- A_t + \int_0^\cdot (1-\theta_s) dV_s - \frac{1}{2}\int_0^\cdot \theta_t^2 d\,[A]_t\right)$$

belongs to $\mathcal{S}(\mathcal{C}_A^1(\mathbb{G}))$. Then, by proposition 1.3.7, again, θ fulfills also condition 3. of lemma 1.5.34. By construction, $\theta I_{[0,t]}$ is an element of $\mathcal{A}^+$ for every $0 \leq t < 1$ and this concludes the proof.

□

Admissible strategies via Malliavin calculus

We restrict ourselves to the setting of section 1.5.1. We recall that in that case $A = \int_0^\cdot \sigma_t dW_t + \int_0^\cdot \mu_t dt$. We make the following additional assumption :

$$S^0 = e^{\int_0^\cdot r_t dt},$$

with r in $L^{1,z}$ for some $z > 4$ and $\mathbb{F}$-adapted.

Proposition 1.5.42. *Let $\mathcal{A}^+$ be the set of all $\mathbb{G}$-adapted processes in $C_b^-([0,1))$ being the restriction on $[0,1)$ of processes h in $L_-^{1,2} \cap L^{2,2}$, such that $D^- h$ is in $L_-^{1,2}$, and the random variable*

$$\sup_{t\in[0,1]}\left(|h_t| + \sup_{s\in[0,1]} |D_s h_t| + \sup_{s,u\in[0,1]} |D_s D_u h_t|\right)$$

is bounded.

Then $\mathcal{A}^+$ satisfies the hypotheses of assumption 1.5.37.

Proof. Let h be as in the hypotheses. Proposition 1.3.18 applies to to state that h is A-forward integrable and

$$\begin{aligned}\int_0^\cdot h_t d^- A_t &= \int_0^\cdot h_t \sigma_t d^- W_t + \int_0^\cdot h_t \mu_t dt \\ &= \int_0^\cdot h_t \sigma_t \delta W_t + \int_0^\cdot \left(h_t \mu_t + \sigma_t D_t^- h_t\right) dt.\end{aligned}$$

On the other hand, proposition 1.3.17 applies to obtain

$$\left[\int_0^\cdot h_t d^- A_t\right] = \left[\int_0^\cdot h_t \sigma_t \delta W_t\right] = \int_0^\cdot h_t^2 \sigma_t^2 dt.$$

In particular, if θ is the restriction of h on $[0,1)$, then θ fulfills point 1. and 2. of lemma 1.5.34.

Consider the vector process $\left(\int_0^\cdot h_t d^- A_t, \int_0^\cdot (1-h_t) dV_t, \int_0^\cdot h_t^2 d\,[A]\right)_t$. We affirm that each of its components belongs to $L_-^{1,v}$ for some $v > 4$. In fact, the first component is equal to the sum of $\int_0^\cdot h_t \sigma_t \delta W_t$ and $\int_0^\cdot \left(\sigma_t D_t^- h_t + h_t \mu_t\right) dt$; the first term of the sum belongs to $L_-^{1,p}$ by lemma 1.3.16, which applies thanks to remark 1.3.9 ; the second term is in $L_-^{1,q\wedge p}$ thanks to lemma 1.3.15 ; remark 1.3.9 and lemma 1.3.15 again imply that both $\int_0^\cdot (1-h_t) r_t dt$ and $\int_0^\cdot h_t^2 \sigma_t^2 dt$ belong to $L_-^{1,z}$. We can thus apply proposition 1.3.21 to find that

$$\int_0^\cdot X_t^h d^- \int_0^t h_s \sigma_s dW_s = \int_0^\cdot X_t^h h_t d^- \int_0^t \sigma_s dW_s,$$

with $X^h = \exp\left(\int_0^\cdot h_t d^- A_t - \int_0^\cdot (1-h_s) dV_s - \frac{1}{2}\int_0^\cdot h^2 d\,[A]_t\right)$. This permits to conclude the proof.

□

Admissible strategies via substitution

We return here to the framework of subsection 1.5.1.

Proposition 1.5.43. *Let $\mathcal{A}^+$ be the set of all processes which are the restriction to $[0,1)$ of processes in $\mathcal{S}(\mathcal{A}^{p,\gamma}(L))$ for some $p > 3$ and $\gamma > 3d$. Then $\mathcal{A}^+$ satisfies the hypotheses of assumption 1.5.37.*

Proof. Let h be in $\mathcal{S}(\mathcal{A}^{p,\gamma}(L))$ for some $p > 3$ and $\gamma > 3d$. Proposition 1.3.37 insures that h is A-forward integrable, and that $\int_0^\cdot h_t d^- A_t$ has finite quadratic variation equal to $\int_0^\cdot h_t^2 d\,[A]_t$. The process

$$X^h = \exp\left(\int_0^\cdot h_t d^- A_t - \int_0^\cdot (1-h_t) dV_t - \frac{1}{2}\int_0^\cdot h_t^2 d\,[A]_t\right)$$

has bounded paths. Then, thanks to point 1. of remark 1.2.3, to prove that

$$\int_0^\cdot X_t^h d^- \int_0^t h_s d^- A_s = \int_0^\cdot X_t^h h_t d^- A_t, \tag{1.21}$$

we are allowed to replace X^h by $\psi(\log(X^h))$, being ψ a function of class $C^\infty(\mathbb{R})$ with bounded derivative. Using lemma 1.3.27 it is possible to show that the process $\psi\left(\log(X^h)\right)$ belongs to $\mathcal{S}(\mathcal{A}^{\frac{p}{2},\frac{\gamma}{2}}(L))$. Proposition 1.3.37 again let us get equality (1.21).

□

Optimal portfolios and $\mathcal{A}^+$-martingale property

Adapting results contained in section 1.4.3 to the utility maximization problem, we can formulate the following propositions. We omit their proofs, being particular cases of the ones contained in that section.

Proposition 1.5.44. *If a process π in $\mathcal{A}^+$ is an optimal portfolio, then the process $A - V - \int_0^\cdot \pi_t d[A]_t$ is an $\mathcal{A}^+$-martingale under Q^π. If U fulfills assumption 1.5.32, then the converse holds.*

Proposition 1.5.45. *Suppose that $\mathcal{A}^+$ satisfies assumption $\mathcal{E}$ with respect to $\mathbb{G}$. If a process π in $\mathcal{A}^+$ is an optimal portfolio, then the process $A - V - \int_0^\cdot \beta_t d[A]_t$ is a $\mathbb{G}$-martingale under P, with*

$$\beta = \pi + \frac{1}{p^\pi}\frac{d[p^\pi, A]}{d[A]}, \qquad \text{and} \qquad p^\pi = \mathbb{E}^{Q^\pi}\left[\frac{dP}{dQ^\pi} \mid \mathcal{G}.\right].$$

If U fulfills assumption 1.5.32, then the converse holds.

Remark 1.5.46.
1. *We emphasize that if $U(x) = \log(x)$, then the probability measure Q^π appearing in propositions 1.5.44 and 1.5.45 is equal to P.*
2. *In [2] it is proved that if the maximum of expected logarithmic utility over all simple admissible strategies is finite, then S is a semimartingale with respect $\mathbb{G}$. This result does not imply proposition 1.5.45. Indeed, we do not need to assume that our set of portfolio strategies contains the set of simple predictable admissible ones. On the contrary, we want to point out that, as soon as the class of admissible strategies is not large enough, the semimartingale property of price processes could fail, even under finite expected utility.*

Proposition 1.5.47. *Suppose that $U(x) = \log(x)$, x in $(0, +\infty)$. Assume that there exists a measurable process γ such that $A - V - \int_0^\cdot \gamma_t d[A]_t$ is an $\mathcal{A}^+$-martingale and there exists a sequence $(\theta^n)_{n\in\mathbb{N}} \subset \mathcal{A}^+$ such that*

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^1 |\theta_t^n - \gamma_t|^2 \, d[A]_t\right] = 0.$$

Then if γ belongs to $\mathcal{A}^+$ it is an optimal portfolio. On the contrary, if an optimal portfolio π exists, then $d\,|[A]|\,\{t \in [0,1), \pi_t \neq \gamma_t\} = 0$ almost surely.

Example

We adopt the setting of section 1.5.4 and we further assume that σ is a strictly positive real.

Proposition 1.5.48. *If a process π is an optimal portfolio in $\mathcal{A}^+$, then the process $W - \int_0^\cdot \left(\frac{r_t - \mu_t}{\sigma} + \pi_t\sigma\right) dt$ is an $\mathcal{A}^+$-martingale under Q^π. If U fulfills assumption 1.5.32, then the converse holds.*

Proof. First of all we observe that it is not difficult to prove that $\mathcal{A}^+$ satisfies assumption 1.5.38. If a process π is an optimal portfolio in $\mathcal{A}^+$ then proposition 1.5.44 implies that the process M^π, with $M^\pi = \sigma\left(W - \int_0^\cdot \left(\frac{r_t-\mu_t}{\sigma} - \pi_t\sigma\right)dt\right)$, is an $\mathcal{A}^+$-martingale under Q^π. We observe that $\sigma^{-1}\mathcal{A}^+ = \mathcal{A}^+$. Therefore, $\sigma^{-1}M^\pi = W - \int_0^\cdot \left(\frac{r_t-\mu_t}{\sigma} + \pi_t\sigma\right)dt$ is an $\mathcal{A}^+$-martingale.

Similarly, if U satisfies assumption 1.5.32, the converse follows by proposition 1.5.44.

□

Corollary 1.5.49. *Let $\mathcal{A}^+$ satisfy assumption $\mathcal{E}$ with respect to $\mathbb{G}$. If a process π in $\mathcal{A}^+$ is an optimal portfolio then the process $\widetilde{W} = W - \int_0^\cdot \alpha_t dt$ with*

$$\alpha = \pi\sigma + \frac{r-\mu}{\sigma} + \frac{1}{p^\pi}\frac{d\left[p^\pi, W\right]}{d\left[W\right]}, \qquad and \qquad p^\pi = \mathbb{E}^{Q^\pi}\left[\frac{dP}{dQ^\pi} \mid \mathcal{G}_\cdot\right],$$

is a $\mathbb{G}$-Brownian motion under P. If U satisfies assumption 1.5.32, then the converse holds.

Proof. Let π be an optimal portfolio. By proposition 1.4.31, the process $\widetilde{W}$ is a $\mathbb{G}$-martingale and so a $\mathbb{G}$-Brownian motion under P.

□

The results concerning the example above were proved in [5]. We generalize them in two directions : we replace the geometric Brownian motion A by a finite quadratic variation process and we let the set of possible strategies vary in sets which can, a priori, exclude some simple predictable processes.

Example

We consider the example treated in section 1.4.3. We suppose, for simplicity, that

$$S_t = S_0 e^{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)t}, \quad S_t^0 = e^{rt} \quad 0 \le t \le 1,$$

being σ, μ and r positive constants. This implies $A_t = \sigma W_t + \mu t$, and $V_t = rt$ for $0 \le t \le 1$. We set $\mathcal{A}^+ = \Theta(L)$.

Proposition 1.5.50. *Suppose that $U(x) = \log(x)$, x in $(0, +\infty)$. Suppose that $h(\cdot, L)$ belongs to the closure of $\Theta(L)$ in $L^2(\Omega \times [0,1])$. Then an optimal portfolio π exists if and only if the process $h(\cdot, L) + \int_0^\cdot \frac{\mu-r}{\sigma}dt$ belongs to $\Theta(L)$ and $\pi = h(\cdot, L) + \frac{\mu-r}{\sigma}$.*

Proof. The result follows from corollary 1.4.34.

□

Sufficiency for the proposition above was shown, with more general σ, r and μ in theorem 3.2 of [39]. Nevertheless, in this paper we go further in the analysis of utility maximization problem. Indeed, besides observing that the converse of that theorem holds true, we find that the existence of an optimal strategy is strictly connected, even for different choices of the utility function, to the $\mathcal{A}^+$-semimartingale property of W. To be more precise, in that paper the authors show that an optimal process exists, under the given hypotheses, handling directly the expression of the expected utility, which has, in the logarithmic case, a nice expression. Here we reinterpret their techniques at a higher level which permits us to partially generalize those results.

Chapitre 2

Non-semimartingales : EDS et weak Dirichlet

Ce chapitre discute de l'existence et de l'unicité d'une solution à une EDS inhomogène unidimensionnelle dirigée par une semimartingale M et par un processus à variation cubique finie ξ. Le processus ξ est supposé avoir la structure $Q + R$, où Q est un processus à variation quadratique finie et R est *fortement prévisible* (*strongly predictable*) dans un sens technique : cette condition implique que R soit *weak Dirichlet*, et est vérifiée par exemple lorsque R est indépendant de M. Nous proposons une méthode basée sur une transformation réduisant le coefficient de *diffusion* à 1 basée sur des généralisations des formules de type Itô et Itô-Wentzell. Une approche similaire nous amène à traiter les problèmes d'existence et d'unicité lorsque ξ est un processus hölder continu et σ est une fonction hölderienne dans l'espace. En utilisant une formule d'Itô pour les semimartingales *réversibles* nous prouvons également l'existence d'une solution lorsque ξ est un mouvement brownien et σ est seulement continue.[1]

2.1 Introduction

This paper deals with the study of stochastic differential equations driven by a process which is not a semimartingale. We aim at illustrating how, using different types of Itô or Itô-Wentzell formulae, it is possible to establish existence and uniqueness results for a stochastic differential equation driven by a non-semimartingale ξ with a multiplication factor σ. When the paths of ξ have very few regularity, more regularity on σ is required. On the contrary, if the Hölder regularity of ξ is $\gamma > \frac{1}{2}$, σ only needs to fulfill a Hölder regularity.

As we said, one of the achievements of the paper is constituted by an Itô-Wentzell formula for processes having a finite cubic variation. There are today an incredible amount of generalized Itô formulae and it would be for us almost impossible to quote them all. The standard situation can be found in [25] and [56], see also [57]. Given a finite quadratic

[1] Ce chapitre fait l'objet d'un article en collaboration avec Francesco RUSSO qui va paraître à Annals of Probability.

variation process ξ, and $f \in C^{1,2}([0,1] \times \mathbb{R})$, one expands $f(t,\xi_t)$ as follows.

$$f(t,\xi_t) = f(0,\xi_0) + \int_0^t \partial_s f(s,\xi_s)ds + \int_0^t \partial_x f(s,\xi_s)d^\circ \xi_s,$$

where the integral with respect to ξ is a symmetric integral, see definition 2.2.6. In the literature, there are generalizations in several directions, among them the following :

1. the case that ξ is not of finite quadratic variation, for instance ξ is a finite cubic variation and f is of class $C^{1,3}$, see for instance [21], or ξ is a fractional Brownian motion with Hurst index $H > \frac{1}{6}$, and f is of class C^6, see e.g. [29, 11] ;
2. the case when ξ is a (*reversible*) semimartingale, so essentially a classical process but f is of class C^1, see in general [26, 55].

Itô formula for finite quadratic variation processes admits extensions of Itô-Wentzell type, as in [24], where the the dependence in time is of semimartingale type. More precisely, it is possible to expand the process $X_t(\xi_t)$, where $X_t(x)$ is a family of semimartingales depending on a parameter with respect to a given filtration $\mathbb{F} = (\mathcal{F}_t)$, if for every fixed parameter x, the semimartingale $X_t(x)$ admits a representation as a classical stochastic integral with respect to some vector of driving $\mathbb{F}$-semimartingales $(N^1, ..., N^n)$, ξ is $\mathbb{F}$-adapted, and the vector $(\xi, N^1, ..., N^n)$ has all its mutual brackets, see definition 2.2.3. We generalize this result, establishing an Itô-Wentzell formula for a finite cubic variation process ξ provided that some technical assumption on $(\xi, N^1, \ldots, N^n)$ is fulfilled, see hypothesis $(\mathcal{D})$ in definition 2.3.6 : we assume the existence of a filtration $\mathbb{H} \supseteq \mathbb{F}$, with respect to which the vector $(N^1, ..., N^n)$ is still a vector of semimartingales, such that ξ is decomposable into the sum of two $\mathbb{H}$-adapted processes Q and R, where $(Q, N^1, ..., N^n)$ has all its mutual brackets, and R is *strongly predictable* with respect to $\mathbb{H}$, see definition 2.3.5. In particular R is an $\mathbb{H}$-*weak Dirichlet* process in the sense of [21]. We recall that an $\mathbb{H}$-*weak Dirichlet* process is the sum of a continuous $\mathbb{H}$-local martingale and of an $\mathbb{H}$-adapted process Q such that $[Q, N] = 0$ for every $\mathbb{H}$-semimartingale N. Recent developments on that subject appeared in [28] and [12]. The mentioned hypothesis on R is verified in the following cases :

- R is $\mathcal{F}_0$ measurable ;
- R is independent from $(N^1, ..., N^n)$ and the filtration generated by $(N^1, ..., N^n)$ and the whole process R contains $\mathbb{F}$.

Among others, the calculus developed to perform Itô-Wentzell formula helps us to clarify the structure of $\mathbb{F}$-*weak Dirichlet* processes if $\mathbb{F}$ is the natural filtration associated with a Brownian motion W. If Q is an $\mathbb{F}$-adapted process and $[Q, W]$ has all its mutual brackets, the covariation $[Q, L]$ can be computed explicitly for every continuous $\mathbb{F}$-semimartingale L, see proposition 2.3.9. This allows us to prove that a process A is $\mathbb{F}$-*weak Dirichlet* if and only if it is the sum of an $\mathbb{F}$-local martingale and of an $\mathbb{F}$-adapted process Q, with $[Q, W] = 0$.

On the other hand a stochastic differential equation of the form

$$d^\circ X_t = \sigma(t, X_t) \left[d^\circ \xi_t + \beta(t, X_t) d^\circ M_t + \alpha(t, X_t) dV_t\right], \tag{2.1}$$

is considered where M is a local martingale, V a bounded variation process, and ξ is a finite cubic variation process with (ξ, M) verifying hypothesis $(\mathcal{D})$. We show, in different cases, how it is possible to apply Itô formula to reduce the *diffusion* coefficient σ to 1, and to formulate existence and uniqueness of equation (5) by studying equations where the process ξ appears only as an additive term. The improper terminology of diffusion coefficient will be indeed used in the whole paper. A particular case of that equation was considered in [21] when $\beta = 0$. There σ was of class C^3, and the notion of solution for a process X was somehow unnatural since it required that the couple (X, ξ) was a symmetric vector Itô process. In the case σ is bounded from below by a positive constant, that equation can be investigated with our techniques, weakening the assumptions on the coefficients, enlarging the class of uniqueness and improving the sense of solution avoiding the notion of symmetric vector Itô process.

In the literature, stochastic differential equations of forward type as

$$d^- X_t = \sigma(X_t) d^- \xi_t + b(t, X_t) dL_t,$$

were solved operating via classical transformations, in the case ξ has finite quadratic variation, see [54], for definition of forward integral. In [56] a first attempt was done when L has bounded variation. Similar independent results were established in [63]. In [24] existence and uniqueness were studied in a class of processes $(X(t, \xi_t))$ where $X(t, x)$ is a family of semimartingale depending on a parameter and L is a semimartingale. There the regularity of σ was of C^4 type with σ', σ'' being bounded. In that framework our result enlarges again the class of uniqueness, and we also require less regularity.

Equations looking similar to (5) were considered in the framework of T. Lyons and collaborators rough paths theory, see [41], even in the multidimensional case when σ is Lipschitz, $\alpha = 0$, for a process with deterministic p-variation strictly smaller than 3. Interesting reformulations of that integration theory and calculus with some applications to SDEs are given in [32], [23]. Rough path analysis is purely deterministic in contrast with ours which couples the *pathwise* techniques of stochastic calculus via regularization and probabilistic concepts, see hypothesis $(\mathcal{D})$.

Another topic of interest is the study of equation

$$d^\circ X_t = \sigma(t, X_t) \left[d^\circ \xi_t + \alpha(t, X_t) dt \right], \tag{2.2}$$

where σ is only locally Hölder continuous, α is locally Lipschitz with linear growth, and ξ is a γ-Hölder continuous process with $\gamma > \frac{1}{2}$. We apply the same method to this equation exploiting an Itô formula available for processes having Hölder continuous paths established in [64]. To this extent we need to show that the symmetric integral of a process f with respect to a process g being Hölder continuous respectively of order γ and δ, with $\gamma + \delta > 1$, is the type of integral studied in [64]. Indeed, we prove that this integral is a particular case of the so called *Young* integral introduced in a more general setting in [62]. Since the trajectories of the fractional Brownian motion are γ-Hölder continuous for every γ strictly smaller than the Hurst parameter H, we are naturally induced to treat equations driven by the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Moreover we combine our method with a recent result obtained in [44] with respect to an equation driven by a fractional Brownian motion with diffusion coefficient equal to 1. This permits us to improve our general result about existence and uniqueness of equation

(5) when $\xi = B^H$, and B^H is a fractional Brownian motion with Hurst index bigger than $\frac{1}{2}$, i.e.

$$d^\circ X_t = \sigma(t, X_t)\left[d^\circ B_t^H + \alpha(t, X_t)dt\right]. \tag{2.3}$$

If the fractional Brownian motion reduces to a Brownian motion ($H = \frac{1}{2}$), an Itô formula for C^1 functions of *reversible* semimartingales is taken into consideration to formulate an existence theorem for equation (2.3), when σ is only continuous and α is bounded measurable.

If H is smaller than $\frac{1}{2}$, Itô formula for Young type integral is no longer available. In spite of this, starting from our analysis, conditions to insure existence and uniqueness for equation (2.3) can still be deduced, treating the fractional Brownian motion with Hurst parameter $H \geq \frac{1}{3}$ as a strong finite cubic variation process. Essentially, in this case, the coefficient σ is required to admit second continuous derivative with respect to the space variable.

On the other hand, remaining in the pure pathwise spirit, the Hölder nature of the fractional Brownian motion can be exploited to study equations of type (2.3), even when the Hurst parameter H is smaller than $\frac{1}{2}$. The natural prolongation of Young integration and calculus, is indeed rough path analysis.

Recently, several efforts were made in this direction, see [14], [32], [13], to adapt results on rough paths theory to stochastic differential equations driven either by Hölder continuous processes with parameter $\gamma > \frac{1}{3}$ or by fractional Brownian motion with Hurst index $H > \frac{1}{4}$.

In [32] the author investigates existence and uniqueness of differential equations of type (2.2) with $\alpha = 0$, driven by irregular paths with Hölder exponent γ greater than $\frac{1}{3}$. The multiplicative non-linearity σ was required to be differentiable till order two with second derivative δ-Hölder continuous with $\delta > \frac{1}{\gamma} - 2$. At our knowledge, the first attempts to apply rough paths theory to the study of a stochastic differential equation driven by fractional Brownian motion of type (2.3) with H strictly smaller than $\frac{1}{2}$ is constituted by [14]. There the authors considered the case $\frac{1}{4} < H < \frac{1}{2}$, and again $\alpha = 0$. They presented a pathwise approach to the solution of stochastic differential equations based on the so called *universal limit theorem* established in [40]. To apply that result the multiplicative coefficient σ was assumed to be differentiable with bounded derivatives till order $\left[\frac{1}{H}\right] + 1$.

In both of the above-mentioned papers stochastic differential equations are solved in some specific setting and it is not obvious to see which kind of stochastic integral is involved.

A first result offering a link between the deterministic approach and the stochastic one can be found in [13]. There equation (2.3) is considered with α and σ time independent vector fields. Assuming σ differentiable and bounded till order $[\delta]$ with its $[\delta]$-derivative $(\delta - [\delta])$-Hölder, for some $\delta > \frac{1}{H}$, it is proved that the unique solution originated by the rough path method is actually a solution in some Stratonovich sense.

We come back to our paper. Our analysis of uniqueness, in the case of weak assumption on the *diffusion* coefficient, is inspired by classical ordinary differential equations of the type

$$\frac{dX(t)}{dt} = \sigma(X(t)), \tag{2.4}$$

with σ only continuous with linear growth. In that case, Peano theorem insures existence but not uniqueness. Suppose that x_0 is the only zero of σ. Then, if for some $\varepsilon > 0$,

$$\int_{x_0}^{x_0+\varepsilon} \frac{1}{|\sigma|}(y)dy = \int_{x_0-\varepsilon}^{x_0} \frac{1}{|\sigma|}(y)dy = +\infty, \tag{2.5}$$

for every initial condition, this equation admits a unique solution. If previous condition is not verified, then it is possible to show that at least two solutions for equation (2.4) exist, with initial condition $X_0 = x_0$. Suppose, for instance, that the second integral is finite. Setting $H(x) = \int_{x_0}^{x} \frac{1}{\sigma(y)}dy$, $x > x_0$, one can construct two solutions, i.e. $X(t) \equiv x_0$ and $X(t) = H^{-1}(t)$. This phenomenon will be illustrated in the stochastic case, even with σ inhomogeneous, see for instance proposition 2.4.30 and remark 2.4.31.

We observe that a similar condition as (2.5), appears in the study of one-dimensional stochastic differential equation of Itô type $dX(t) = \sigma(X(t))dW(t)$ where W is a classical Brownian motion. Uniqueness for every initial condition holds if and only if

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{\sigma^2}(t)dt = +\infty,$$

for every $x_0 \in \mathbb{R}$, see [20].

To summarize, towards the study of equation (5), we innovate along the following axes with respect to the literature.

- We suppose that ξ is a finite cubic variation process and σ is time inhomogeneous.
- The notion of solution is clarified and we do not need to introduce the notion of symmetric vector Itô process.
- One new tool that we establish is a Itô-Wentzell type formula where finite cubic variation processes are involved.
- We continue the analysis related to the structure of weak Dirichlet processes.
- When the paths of ξ are Hölder, with parameter greater than $\frac{1}{2}$ we require very weak regularity on the coefficients.
- In the case of classical Brownian motion a new existence theorem is established for the Stratonovich equation.
- We drastically weaken the classical assumptions on the coefficients for existence and uniqueness. Our regularity assumptions are generally weaker than those intervening in rough path theory.

The paper is organized as follows. In section 2 we recall some definitions and results about stochastic calculus with respect to finite cubic variation processes. We state Itô formula and a result of stability of finite cubic variation through C^1 transformations. We also show some technical properties of the symmetric integral regarding its behavior when it is restricted to some subspace of the reference probability space, *stopped* or *shifted* with respect to some random time.

Section 3 deals with the class $\mathcal{C}_\xi^k(\mathbb{H})$ of the processes Z so defined

$$Z_t = X_t(\xi_t),$$

being $X_t(x)$ an Itô field driven by a vector $(N^1, \ldots, N^n)$ of semimartingales such that hypothesis $(\mathcal{D})$ is verified for $(\xi, N^1, \ldots, N^n)$ with respect to the filtration $\mathbb{H}$, see definition

2.3.1, with regularity of order k in the space variable. We prove that, if ξ has a finite cubic variation, processes in $\mathcal{C}^1_\xi(\mathbb{H})$ still have finite cubic variation, and it is possible to establish an Itô-Wentzell formula to expand processes in $\mathcal{C}^3_\xi(\mathbb{H})$. In this section we also discuss connections with *weak Dirichlet* processes. We conclude this part proving the existence of the symmetric integral of a process in $\mathcal{C}^2_\xi(\mathbb{H})$ with respect to a process in $\mathcal{C}^2_\xi(\mathbb{H})$, and using this result to formulate a chain-rule formula.

Section 4 discusses uniqueness and existence of equation (2.1). It is divided into nine subsections. The first and the second parts specify the notion of solution and describe the framework : we restrict ourselves to the case where the support S of σ is time-independent and a non-integrability condition around its zeros of type (2.5) is fulfilled. The third part focuses on trajectories of solutions : if X is a solution of equation (2.1), it can be expressed as a function of ξ and a semimartingale. Moreover its trajectories are forced to live in some connected component of S, as soon as the initial condition does. In the case the coefficients driving the equation are autonomous, a solution starting in $D = \mathbb{R}\backslash S$, is identically equal to the initial condition. Putting things together, in the fourth part, we establish an equivalence between equation (2.1) and an equation of the same form but with diffusion coefficient equal to 1. We finally give some conditions for existence and uniqueness of this last equation. In the fifth subsection we use results of section 3 to show that, under additional assumptions on the regularity of σ and β, equation (2.1) admits a unique integral solution in the set $\mathcal{C}^2_\xi(\mathbb{H})$, provided that the vector (ξ, M) verifies the hypothesis $\mathcal{D}$ with respect to $\mathbb{H}$. In the sixth one we revisit our results in the case ξ has finite quadratic variation, and the symmetric integral is substituted by the forward integral. The seventh subsection is devoted to the application of the method when processes have Hölder trajectories. Subsection eight describes how it is possible to combine the result of [44] and ours to treat the specific case of an equation driven by fractional Brownian motion. Finally we discuss existence of solutions for a Stratonovich equation driven by a Brownian motion, with continuous diffusion coefficient and bounded measurable drift.

2.2 Definitions, notations and basic calculus

In this section we recall basic concepts and results about calculus with respect to finite cubic variation processes which will be useful later. For a more complete description of these arguments the reader may refers to [21] or [29]. Throughout the paper $(\Omega, \mathcal{F}, P)$ will be a fixed probability space. All processes are supposed to be continuous and indexed by the time variable t in $[0,1]$. We adopt the notation $X_t = X_{(t\vee 0)\wedge 1}$, for every t in $\mathbb{R}$. A sequence of continuous processes $(X^\varepsilon)_{\varepsilon>0}$ will be said to converge ***ucp*** (***uniformly convergence in probability***) to a process X, if $\sup_{0\le t\le 1}|X^\varepsilon_t - X_t|$ converges to zero in probability, when ε goes to 0.

In the paper $C^{h,k}$ will be the space of all continuous functions $f : [0,1]\times\mathbb{R} \to \mathbb{R}$, which are of class C^h in t, with derivatives in t up to order h continuous in (t,x), and of class C^h in x, with derivatives in x up to order k continuous in (t,x).

Let $n \geq 2$ and $(X^1, ..., X^n)$ be a vector of continuous processes. For any $\varepsilon > 0$ and t

in $[0,1]$ set

$$\left[X^1, X^2, ..., X^n\right]_\varepsilon (t) = \frac{1}{\varepsilon} \int_0^t \prod_{k=1}^n \left(X^k_{s+\varepsilon} - X^k_s\right) ds,$$

and

$$\left|\left|\left[X^1, X^2, ..., X^n\right]\right|\right|_\varepsilon = \frac{1}{\varepsilon} \int_0^1 \prod_{k=1}^n \left|X^k_{s+\varepsilon} - X^k_s\right| ds.$$

If $[X^1, X^2, ..., X^n]_\varepsilon (t)$ converges *ucp*, when $\varepsilon \to 0$, then the limiting process is called the n-***covariation*** process of the vector $(X^1, ..., X^n)$ and denoted $[X^1, X^2, ..., X^n]$. If, furthermore, every subsequence $(\varepsilon_k)_{k\geq 0}$ admits a subsequence $(\bar{\varepsilon}_k)_{k\geq 0}$ such that

$$\sup_{k\geq 0} \left|\left|\left[X^1, X^2, ..., X^n\right]\right|\right|_{\bar{\varepsilon}_k} < +\infty, \quad a.s., \tag{2.6}$$

then the n-covariation is said to exist in the ***strong sense***. If the processes $\left(X^k\right)_{k=1}^n$ are all equal to a real valued process X, then the n-covariation of the considered vector will be denoted by $[X; n]$ and called the n-**variation** process. If $n = 2$ this process is the ***quadratic variation*** and it is denoted by $[X]$ or $[X, X]$. If $n = 3$ we will speak about ***cubic variation***. If X has a quadratic (respectively, strong cubic) variation, X will be called ***finite quadratic variation*** (respectively ***strong cubic variation*** (**scv**)) process.

Remark 2.2.1. *In [21] a different version of the definition of the strong n-variation is given. However, results contained there and recalled in the sequel can be proved to hold even under our weaker assumption.*

Example 2.2.2. *We present several examples of strong finite cubic variation processes.*

1. *Let $\left(B^H_t, 0 \leq t \leq 1\right)$ be a fractional Brownian motion of Hurst index H, that is a Gaussian process with zero mean and covariance*

$$Cov(B^H_s, B^H_t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

   It follows from remark 2.8 of [21], that the fractional Brownian motion with Hurst parameter $H = \frac{1}{3}$ is a strong cubic variation process.

2. *Let $\left(B^{H,K}_t, 0 \leq t \leq 1\right)$ be a bifractional Brownian motion with parameters $H \in]0,1[, K \in]0,1]$. We recall, see [34], that $B^{H,K}$ is a Gaussian process with zero mean and covariance*

$$R(t,s) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right).$$

   In [51] is shown that $B^{H,K}$ is a strong finite cubic variation process if $HK \geq \frac{1}{3}$.

3. *Let $(X_t, 0 \leq t \leq 1)$ be a Gaussian mean zero process starting at zero, with stationary increments. Set $V(t)^2 := Var(X_t)$, for every t in $[0,1]$. Fubini theorem and the fact that the increments of X are stationary permit to perform the following evaluation :*

$$E\left[||X, X, X||_\varepsilon\right] = \frac{c}{\varepsilon}(V(\varepsilon))^3,$$

for some positive constant c. If furthermore $V(t) = O(t^{\frac{1}{3}})$, *condition (2.6) holds. Moreover, using similar methods as in [30] it is possible to prove that the sequences of processes* $[X, X, X]_\varepsilon$ *converges ucp. In particular* X *is a strong cubic variation process.*

4. *Using [21] it is possible to exhibit examples of non-Gaussian strong finite cubic variation processes. One such process is of the type* $X_t = \int_0^t G(t,s)dM_s$ *where* M *is a local martingale and* G *is a continuous random field independent from* M *essentially such that* $[G(\cdot, s_1), G(\cdot, s_2), G(\cdot, s_3)]$ *exist for any* s_1, s_2, s_3. *For example one may choose* $G(t,s) = B^H_{t-s}$, *where* B^H *is a fractional Brownian motion independent of* M, *with* $H \geq \frac{1}{3}$.

Definition 2.2.3. *A vector* $(X^1, X^2, ..., X^m)$ *of continuous processes is said to have all its* ***mutual*** *(respectively,* ***strong****)* n***-covariations*** *if* $[X^{i_1}, X^{i_2}, ..., X^{i_n}]$ *exists (respectively, exists in the strong sense) for any choice (even with repetition) of indices* $i_1, i_2, ..., i_n$ *in* $\{1, 2, ..., m\}$. *If* $n = 2$, *we will also say that the vector* $(X^1, X^2, ..., X^m)$ *has all its* ***mutual brackets****. In that case* $[X^1, ..., X^m]$ *has bounded variation.*

Proposition 2.2.4. *If condition (2.6) holds, then* $[X^1, X^2, ..., X^n]$ *has bounded variation whenever it exists.*

Remark 2.2.5.
1. *If the n-variation* $[X; n]$ *exists in the strong sense for some n, then* $[X; m] = 0$ *for all* $m > n$. *In particular, since the 2-covariation of two semimartigales exists strongly and agrees with their usual covariation (see [54]), for any semimartingale* S, $[S; n] = 0$ *for all* $n \geq 3$.
2. *Let* $(X^1, ..., X^n)$ *be a vector having a strong n-covariation, and* Y *a continuous process. Then*
$$\frac{1}{\varepsilon}\int_0^{\cdot} Y_s \prod_{k=1}^{n} \left(X^k_{s+\varepsilon} - X^k_s\right) ds$$
converges ucp to
$$\int_0^{\cdot} Y d\left[X^1, X^2, ..., X^n\right].$$
3. *If* $(X^1, ..., X^n)$ *has its strong n-covariation then for every vector of continuous processes* $(Y^1, Y^2, ..., Y^m)$, *the vector*
$$\left(X^1, ..., X^n, Y^1, ..., Y^m\right)$$
has its strong $(n+m)$*-covariation equal to zero.*
4. *If the n-variation* $[X; n]$ *exists in the strong sense, then for every continuous process* Y *and every* $m > n$ *such that* $[Y; m]$ *exists in the strong sense, we have*
$$[X, \overbrace{Y, Y, ..., Y}^{(m-1)times}] = 0.$$

Definition 2.2.6. *Let X and Y be two continuous processes. For any $\varepsilon > 0$ and t in $[0,1]$ set*

$$I_\varepsilon^\circ(t,X,Y) = \frac{1}{2\varepsilon}\int_0^t Y_s(X_{s+\varepsilon} - X_{s-\varepsilon})ds.$$

*If the process $I_\varepsilon^\circ(\cdot,X,Y)$ converges ucp when ε goes to zero, then the limiting process will be denoted by $\int_0^t Y d^\circ X$ and called the **symmetric integral**.*

Remark 2.2.7.

1. *It is easy to show that the symmetric integral, if it exists, is the limit ucp of*

$$J_\varepsilon^\circ(t) = \frac{1}{2\varepsilon}\int_0^t (Y_{s+\varepsilon} + Y_s)(X_{s+\varepsilon} - X_s)ds.$$

2. *Let X be a continuous semimartingale with respect to some filtration $\mathbb{F}$ and Y an $\mathbb{F}$-adapted continuous process such that $[X,Y]$ exists. Then the symmetric integral $\int_0^\cdot Y_s d^\circ X_s$ exists,*

$$\int_0^\cdot Y_s d^\circ X_s = \int_0^\cdot Y_s dX_s + \frac{1}{2}[X,Y],$$

and it coincides with classical Stratonovich integral if Y is an $\mathbb{F}$-semimartingale.

We conclude this section by recalling a result about stability of the strong n-covariation through C^1 transformations, the Itô formula for *strong cubic variation* processes, and a *chain-rule* formula, all of them established in [21], propositions 2.7, 3.7, and lemma 3.18.

Proposition 2.2.8. *Let $F^1,...,F^n$ be n functions in $C^1(\mathbb{R}^n)$. Let $X = (X^1,...,X^n)$ be a vector of continuous processes having all its mutual strong n-covariations. Then the vector*

$$\left(F^1(X),...,F^n(X)\right)$$

has the same property and

$$\left[F^1(X),...,F^n(X)\right] = \sum_{1\le i_1,...,i_n\le n}\int_0^t \partial_{i_1}F^1(X)\cdots\partial_{i_n}F^n(X)d\left[X^{i_1},...,X^{i_n}\right].$$

Proposition 2.2.9. *Let $V = (V^1,...,V^m)$ be a vector of bounded variation processes and ξ be a strong cubic variation process. Then for every F belonging to the class $C^{1,3}(\mathbb{R}^m\times\mathbb{R})$ it holds*

$$\begin{aligned} F(V_t,\xi_t) &= F(V_0,\xi_0) + \sum_{i=1}^m\int_0^t \partial_{V^i}F(V_s,\xi_s)dV_s^i + \int_0^t \partial_\xi F(V_s,\xi_s)d^\circ\xi_s \\ &- \frac{1}{12}\int_0^t \partial_\xi^{(3)}F(V_s,\xi_s)d[\xi,\xi,\xi]_s. \end{aligned}$$

Lemma 2.2.10. *Let ξ be a strong cubic variation process. Suppose that ψ and ϕ are, respectively, in $C^{1,3}([0,1]\times\mathbb{R})$ and $C^{1,2}([0,1]\times\mathbb{R})$. Then*

$$X = \int_0^\cdot \phi(s,\xi_s)d^\circ\xi_s \quad \textit{and} \quad \int_0^\cdot \psi(s,\xi_s)d^\circ X_s$$

exist and

$$\int_0^{\cdot} \psi(s,\xi_s)d^\circ X_s = \int_0^{\cdot} \phi\psi(s,\xi_s)d^\circ \xi_s - \frac{1}{4}\int_0^{\cdot} \partial_\xi \psi \partial_\xi \phi(s,\xi_s)d\left[\xi,\xi,\xi\right]_s.$$

In the sequel of the paper we will need to deal with the restriction of symmetric integrals to subspaces of Ω, as well as with symmetric integrals *stopped* or *shifted* with respect to random times. We list some simple technical properties about these operations.

If B is an element of $\mathcal{F}$, with $P(B) > 0$, $\mathcal{F}^B$ will denote the restriction of $\mathcal{F}$ on $B : \mathcal{F}^B = \{F \cap B, F \in \mathcal{F}\}$, P^B the probability measure conditioned on B, and if f is a random variable on $(\Omega,\mathcal{F},P)$, f^B will denote the restriction of f to B.

Lemma 2.2.11. *Let B in $\mathcal{F}$ with $P(B) > 0$. Let X and Y be two continuous processes such that $\int_0^{\cdot} X d^\circ Y$ exists. Then $\int_0^{\cdot} X^B d^\circ Y^B$ exists and*

$$\int_0^{\cdot} X_t^B d^\circ Y_t^B = \left(\int_0^{\cdot} X_t d^\circ Y_t\right)^B \quad P^B \text{ a.s.}$$

Proof. The result follows immediately after having observed that for every $\delta > 0$,

$$\begin{aligned} &P^B\left(\left\{\sup_{t\in[0,1]}\left|I_\varepsilon^\circ(t,X^B,Y^B) - \left(\int_0^t X_s d^\circ Y_s\right)^B\right| > \delta\right\}\right) \\ &\leq \frac{1}{P(B)}P\left(\left\{\sup_{t\in[0,1]}\left|I_\varepsilon^\circ(t,X,Y) - \left(\int_0^t X_s d^\circ Y_s\right)\right| > \delta\right\}\right). \end{aligned}$$

□ If τ and X are, respectively, a random time and a stochastic process on $(\Omega,\mathcal{F},P)$, X^τ will denote the stochastic process X stopped to time $\tau : X_t^\tau = X_{t\wedge\tau}, 0 \leq t \leq 1$.

Lemma 2.2.12. *Let τ be a random time on $(\Omega,\mathcal{F},P)$, with $P(\tau \leq 1) = 1$, X and Y two continuous stochastic processes such that $\int_0^{\cdot} X d^\circ Y$ exists. Then it holds :*

$$\begin{cases} \int_0^{\cdot} X_s^\tau d^\circ Y_s^\tau = \left(\int_0^{\cdot} X_s d^\circ Y_s\right)^\tau; \\ \int_0^{\cdot} X_{\tau+s} d^\circ\left(Y_{\tau+s}\right) = \int_\tau^{\tau+\cdot} X_s d^\circ Y_s. \end{cases}$$

Proof. We clearly have

$$\sup_{t\in[0,1]}\left|I_\varepsilon^\circ(t\wedge\tau,X,Y) - \left(\int_0^{\cdot} X_s d^\circ Y_s\right)_{t\wedge\tau}\right| \leq \sup_{t\in[0,1]}\left|I_\varepsilon^\circ(t,X,Y) - \int_0^t X_s d^\circ Y_s\right|.$$

Therefore, for the first part of the statement we have to show that $\lim_{\varepsilon\to 0} a_\varepsilon = 0$, in probability, with

$$a_\varepsilon = \sup_{t\in[0,1]}\left|I_\varepsilon^\circ(t\wedge\tau,X,Y) - I_\varepsilon^\circ(t,X^\tau,Y^\tau)\right|.$$

We can write

$$\begin{aligned} a_\varepsilon &\leq \sup_{t\in[0,1]}\left|\frac{1}{\varepsilon}\int_{(\tau-\varepsilon)\wedge t}^{\tau\wedge t} X_s\left(Y_\tau - Y_{s+\varepsilon}\right)ds\right| \\ &+ \sup_{t\in[0,1]}\left|\frac{1}{\varepsilon}\int_{\tau\wedge t}^{(\tau+\varepsilon)\wedge t} X_\tau\left(Y_\tau - Y_{s-\varepsilon}\right)ds\right|. \end{aligned}$$

The convergence to zero almost surely, and so in probability, of the sequence of processes (a_ε) is due to the continuity of the processes X and Y.

The second statement is a straightforward consequence of a simple change of variables which let to obtain $I^\circ(t, X_{\tau+\cdot}, Y_{\tau+\cdot}) = I^\circ(\tau+\cdot, X, Y) - I^\circ(\tau, X, Y)$.

□

By similar arguments it is also possible to show the following lemma.

Lemma 2.2.13. *Let $(X^1, ..., X^n)$ be a vector of continuous processes having its n-covariation, τ a random time with $P(\tau \leq 1) = 1$, and B an element of $\mathcal{F}$. Then the vectors*

$$\left(\left(X^1\right)^B, ..., \left(X^n\right)^B\right), \left(\left(X^1\right)^\tau, ..., \left(X^n\right)^\tau\right) \quad \textit{and} \quad \left(X^1_{\tau+\cdot}, ..., X^n_{\tau+\cdot}\right)$$

have their n-covariation and

$$\begin{cases} \left[X^1, ..., X^n\right]^B = \left[\left(X^1\right)^B, ..., \left(X^n\right)^B\right] \quad P^B a.s.; \\ \left[X^1, ..., X^n\right]^\tau = \left[\left(X^1\right)^\tau, ..., \left(X^n\right)^\tau\right]; \\ \left[X^1_{\tau+\cdot}, ..., X^n_{\tau+\cdot}\right] = \left[X^1, ..., X^n\right]_{\tau+\cdot} - \left[X^1, ..., X^n\right]_\tau. \end{cases}$$

2.3 Itô-fields evaluated at scv processes

2.3.1 Stability of strong cubic variation

At this stage we introduce some definitions adapted from [24], which treated the finite quadratic variation case. From now on $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,1]}$ will denote a filtration on $(\Omega, \mathcal{F})$, satisfying the *usual assumptions*.

Definition 2.3.1. *A random field $(X(t,x), 0 \leq t \leq 1, x \in \mathbb{R})$ is a C^k $\mathbb{H}$-**Itô-martingale field** driven by the vector $N = (N^1, ..., N^n)$, if N is a vector of local martingales with respect to $\mathbb{H}$, and*

$$X(t,x) = f(x) + \sum_{i=1}^{n}\int_0^t a^i(s,x)dN^i_s, \tag{2.7}$$

where

$f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is, for every x, $\mathcal{H}_0$-measurable and belonging to $C^k(\mathbb{R})$ a.s.;*

X and $a^i : [0,1] \times \mathbb{R} \times \Omega \to \mathbb{R}$, $i = 1, ..., n$ are $\mathbb{H}$-adapted for every x, almost surely continuous with their partial derivatives with respect to x in (t,x) up to order k;

for every index $h \leq k$ it holds

$$\partial_x^{(h)} X(t,x) = \partial_x^{(h)} f(x) + \sum_{i=1}^{n} \int_0^t \partial_x^{(h)} a^i(s,x) dN_s^i.$$

Definition 2.3.2. *Let $p \geq 1$. A continuous random field $(Z(t,x), 0 \leq t \leq 1, x \in \mathbb{R})$, is called an $\mathbb{H}$-**strict zero p-variation process** if it is $\mathbb{H}$-adapted for every x, and*

$$\sup_{|x| \leq R} \frac{1}{\varepsilon} \int_0^1 |Z(t+\varepsilon, x) - Z(t,x)|^p \, dt \to 0 \quad \text{in probability}, \tag{2.8}$$

for all $R > 0$.

If $p = 2$ (respectively, $p = 3$) Z will be called an $\mathbb{H}$-***strict zero quadratic*** (respectively, ***cubic***) process.

Note that if

$$Z(t,x) = \sum_{j=1}^{m} \int_0^t b^j(s,x) dV_s^j, \tag{2.9}$$

where b^j are continuous fields, and $(V_t^j)_{0 \leq t \leq 1}$, $j = 1, ..., m$ are bounded variation processes, then (2.8) is verified for every $p > 1$.

Definition 2.3.3. *A random field X will be called a C^k $\mathbb{H}$-**Itô-semimartingale field** if it is the sum of a C^k $\mathbb{H}$-Itô-martingale field and an $\mathbb{H}$-strict zero quadratic variation process Z having the form (2.9) :*

$$X(t,x) = f(x) + \sum_{i=1}^{n} \int_0^t a^i(s,x) dN_s^i + \sum_{j=1}^{m} \int_0^t b^j(s,x) dV_s^j, \tag{2.10}$$

with coefficients $(b^j)_{j=1}^m$ continuous with their partial derivatives with respect to x in (t,x) up to order k.

Proposition 2.3.4. *Let $X = (X^i(t,x), 0 \leq t \leq 1, x \in \mathbb{R}, i = 1,2,3)$ be a vector of random fields being the sum of a vector of C^1 $\mathbb{H}$-Itô-martingale fields*

$$\left(Y^i(t,x), 0 \leq t \leq 1, x \in \mathbb{R}, i = 1,2,3\right),$$

driven by the vector of local martingales $(N^1, ..., N^n)$, and of a vector of $\mathbb{H}$-strict zero cubic variation processes $(Z^i(t,x), 0 \leq t \leq 1, x \in \mathbb{R}, i = 1,2,3)$ which are a.s. in $C^{0,1}([0,1] \times \mathbb{R})$:

$$X^i = Y^i + Z^i, \quad i = 1,2,3.$$

Let ξ be a strong cubic variation and $\mathbb{H}$-adapted process. Then the vector X has its strong mutual 3-covariations and

$$\left[X^{i_1}(\cdot,\xi), X^{i_2}(\cdot,\xi), X^{i_3}(\cdot,\xi)\right] = \int_0^{\cdot} \left(\partial_x X^{i_1}\right) \left(\partial_x X^{i_2}\right) \left(\partial_x X^{i_3}\right) (s, \xi_s) d\left[\xi,\xi,\xi\right]_s,$$

for every choice of indices (i_1, i_2, i_3) in $\{1,2,3\}$.

Proof. We first remark that it is not reductive to suppose that the vector of the driving local martingales is the same for all the Itô fields taken into consideration. We consider the case $X = X^1 = X^2 = X^3 = Y + Z$. The proof in the general case requires the same essential concepts. We suppose also, for simplicity of notations, that the C^1 $\mathbb{H}$-Itô-martingale field has the form (2.7) with $n = 1$, $N^1 = N$, $a^1 = a$. We have to prove that

$$C_\varepsilon = \frac{1}{\varepsilon} \int_0^\cdot (X(s+\varepsilon, \xi_{s+\varepsilon}) - X(s, \xi_s))^3 \, ds$$

converges *ucp* to $\int_0^\cdot (\partial_x X(s, \xi_s))^3 \, d[\xi,\xi,\xi]_s$, and that $X(\cdot, \xi)$ verifies condition (2.6).We can write

$$\begin{aligned} X(s+\varepsilon, \xi_{s+\varepsilon}) - X(s, \xi_s) &= (X(s+\varepsilon, \xi_{s+\varepsilon}) - X(s+\varepsilon, \xi_s)) \\ &+ (X(s+\varepsilon, \xi_s) - X(s, \xi_s)) \\ &= A(s,\varepsilon) + B(s,\varepsilon), \end{aligned}$$

so as to decompose C_ε as follows :

$$C_\varepsilon(t) = I^1_\varepsilon(t) + I^2_\varepsilon(t) + 3I^3_\varepsilon(t) + 3I^4_\varepsilon(t),$$

with

$$I^1_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t (A(s,\varepsilon))^3 \, ds, \quad I^2_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t (B(s,\varepsilon))^3 \, ds,$$

$$I^3_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t (A(s,\varepsilon))^2 (B(s,\varepsilon)) \, ds, \quad I^4_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t (A(s,\varepsilon)) (B(s,\varepsilon))^2 \, ds.$$

Since X is differentiable in ξ, $A(s,\varepsilon)$ may be rewritten as

$$A(s,\varepsilon) \quad = \quad \rho(s,\varepsilon) (\xi_{s+\varepsilon} - \xi_s),$$

with

$$\rho(s,\varepsilon) = \int_0^1 \partial_x X(s+\varepsilon, \xi_s + \lambda (\xi_{s+\varepsilon} - \xi_s)) d\lambda.$$

Then

$$\begin{aligned} I^1_\varepsilon(t) &= \frac{1}{\varepsilon} \int_0^t (\partial_x X(s, \xi_s))^3 (\xi_{s+\varepsilon} - \xi_s)^3 \, ds \\ &+ \frac{1}{\varepsilon} \int_0^t \left((\rho(s,\varepsilon))^3 - (\partial_x X(s, \xi_s))^3 \right) (\xi_{s+\varepsilon} - \xi_s)^3 \, ds. \end{aligned}$$

By remark 2.2.5.2 the first term of this sum converges *ucp* to

$$\int_0^\cdot (\partial_x X(s, \xi_s))^3 \, d[\xi,\xi,\xi]_s,$$

while the absolute value of the second term is bounded by

$$\sup_{s\in[0,1]} \left| (\rho(s,\varepsilon))^3 - (\partial_x X(s, \xi_s))^3 \right| \left(\frac{1}{\varepsilon} \int_0^1 |\xi_{s+\varepsilon} - \xi_s|^3 \, ds \right),$$

which converges to zero in probability since $\partial_x X$ is continuous, and ξ is a *strong cubic variation* process.

We show that $I^2_\varepsilon(t)$ converges to zero *ucp*. We observe that we can apply a substitution argument thanks to the Hölder continuity of a (see [56], proposition 2.1), and the adaptedness of the process ξ, and get

$$\begin{aligned} B(s,\varepsilon) &= \left(\int_s^{s+\varepsilon} a(r,x)dN_r\right)_{x=\xi_s} + (Z(s+\varepsilon,\xi_s) - Z(s,\xi_s)) \\ &= \int_s^{s+\varepsilon} a(r,\xi_s)dN_r + (Z(s+\varepsilon,\xi_s) - Z(s,\xi_s)). \end{aligned}$$

Then

$$\begin{aligned} \left|I^2_\varepsilon(t)\right| \le \frac{1}{\varepsilon}\int_0^1 |B(s,\varepsilon)|^3 ds &\le \frac{4}{\varepsilon}\int_0^1 \left|\int_s^{s+\varepsilon} a(r,\xi_s)dN_r\right|^3 ds \\ &+ \frac{4}{\varepsilon}\int_0^1 |Z(s+\varepsilon,\xi_s) - Z(s,\xi_s)|^3 ds. \end{aligned}$$

For every k in N^* we set

$$\Omega^k = \{[N]_1 \le k\} \cap \left\{\sup_{t\in[0,1]} |\xi_t| \le k\right\}, \quad \tau^k = \inf\{t|\,[N]_t \ge k\}, \quad N^k = N^{\tau^k}.$$

Then τ^k is a stopping time and by optional sampling theorem N^k is a local square integrable martingale. Since $\cup_{k=0}^\infty \Omega^k = \Omega$, almost surely, it is sufficient to verify that for every k in $\mathbb{N}^*$, the sequence of processes $(I_{\Omega_k} I^2_\varepsilon(t))$ converges to zero *ucp*. Since Z is an $\mathbb{H}$-*strict zero cubic variation* process and on Ω_k the process ξ is bounded by a constant,

$$\lim_{\varepsilon\to 0} I_{\Omega^k}\left(\frac{1}{\varepsilon}\int_0^\cdot (Z(s+\varepsilon,\xi_s) - Z(s,\xi))^3 ds\right) = 0 \quad ucp,$$

and so we get the desired convergence if

$$\lim_{\varepsilon\to 0}\int_0^1 \frac{1}{\varepsilon}\left|\int_s^{s+\varepsilon} a^k(r,\xi_s)dN^k_r\right|^3 ds = 0, \qquad \text{in probability},$$

where $a^k : [0,1]\times\mathbb{R}\to\mathbb{R}$ has the same regularity of a, it is bounded and it agrees with a on $[0,1]\times\{x\in\mathbb{R}|\,|x|\le k\}$. We can write

$$\begin{aligned} \int_0^1 \frac{1}{\varepsilon}\left|\int_s^{s+\varepsilon} a^k(r,\xi_s)dN^k_r\right|^3 ds &\le \frac{4}{\varepsilon}\int_0^1 \left|\int_s^{s+\varepsilon} a^k(r,\xi_r)dN^k_r\right|^3 ds \\ &+ \frac{4}{\varepsilon}\int_0^1 \left|\int_s^{s+\varepsilon} \left(a^k(r,\xi_s) - a^k(r,\xi_r)\right) dN^k_r\right|^3 ds. \end{aligned}$$

The process $\int_0^\cdot a^k(r,\xi_r)dN^k_r$ is a continuous semimartingale, then it has a finite quadratic variation by remark 2.2.5.1 and so the first term of the sum converges to zero in probability being bounded by

$$\left(\sup_{t\in[0,1]}\left|\int_s^{s+\varepsilon} a^k(r,\xi_r)dN^k_r\right|\right)\left(\int_0^1 \frac{1}{\varepsilon}\left|\int_s^{s+\varepsilon} a^k(r,\xi_r)dN^k_r\right|^2 ds\right).$$

Therefore, to conclude we only need to apply Burkholder inequality, and Lebesgue dominated convergence theorem to see that

$$\lim_{\varepsilon \to 0} E\left[\int_0^1 \frac{1}{\varepsilon}\left|\int_s^{s+\varepsilon} \left(a^k(r,\xi_s) - a^k(r,\xi_r)\right) dN_r^k\right|^3 ds\right] = 0.$$

Finally by Hölder inequality

$$\left|I_\varepsilon^3(t)\right| \;\leq\; \left(\frac{1}{\varepsilon}\int_0^1 |A(s,\varepsilon)|^3 \, ds\right)^{\frac{2}{3}} \left(\frac{1}{\varepsilon}\int_0^1 |B(s,\varepsilon)|^3 \, ds\right)^{\frac{1}{3}},$$

and

$$\left|I_\varepsilon^4(t)\right| \;\leq\; \left(\frac{1}{\varepsilon}\int_0^1 |A(s,\varepsilon)|^3 \, ds\right)^{\frac{1}{3}} \left(\frac{1}{\varepsilon}\int_0^1 |B(s,\varepsilon)|^3 \, ds\right)^{\frac{2}{3}},$$

then $I_\varepsilon^3(t)$, and $I_\varepsilon^4(t)$, converges to zero *ucp*, since, as already proved before, $\frac{1}{\varepsilon}\int_0^1 |B(s,\varepsilon)|^3 \, ds$ converges to zero in probability and

$$\frac{1}{\varepsilon}\int_0^1 |A(s,\varepsilon)|^3 \, ds \leq ||\xi,\xi,\xi||_\varepsilon \sup_{s\in[0,1]} |\rho(s,\varepsilon)|^3 . \tag{2.11}$$

We conclude observing that the cubic variation of X exists strongly thanks to inequality (2.11), the strong finite cubic variation of ξ and the convergence to zero in probability of $\frac{1}{\varepsilon}\int_0^1 |B(s,\varepsilon)|^3 \, ds$.

□

2.3.2 Strong predictability, covariations and weak Dirichlet processes

Given a vector of processes $(N^1,...,N^n)$, $\mathcal{S}(N^1,...,N^n)$, will denote the set of all filtrations on $(\Omega,\mathcal{F})$ with respect to which $(N^1,...,N^n)$ is a vector of semimartingales.

Definition 2.3.5. *A process R is* ***strongly predictable*** *with respect to $\mathbb{H}$ if*

$$\exists\ \delta > 0, \ \textit{such that } R_{\varepsilon+\cdot} \textit{ is } \mathbb{H}\textit{-adapted}, \quad \textit{for every } \varepsilon \leq \delta.$$

This notion constitutes in fact the direct generalization of the notion of predictability intervening in the discrete time case.

Definition 2.3.6. *We will say that the vector $(\xi, N^1,...,N^n)$ satisfies* ***hypothesis*** *$(\mathcal{D})$ with respect to $\mathbb{H}$, if $\mathbb{H}$ belongs to $\mathcal{S}(N^1,...,N^n)$, and there exist two continuous processes, adapted to $\mathbb{H}$, such that*

$$(\mathcal{D}) \quad \begin{cases} \xi = R + Q; \\ R \textit{ is strongly predictable with respect to } \mathbb{H}; \\ \textit{the vector } (Q, N^1,...,N^n) \textit{ has all its mutual brackets.} \end{cases}$$

We give two examples where there exists a filtration $\mathbb{H}$ with respect to which the decomposition $(\mathcal{D})$ occurs.

Example 2.3.7. *Let $(N^1, ..., N^n)$ be a vector of local martingales with respect to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$. Suppose that $\xi = R + Q$, where*

$$\begin{cases} R \text{ is } \mathbb{F}_0\text{-measurable;} \\ (Q, N^1, ..., N^n) \text{ has all its mutual brackets and } Q \text{ is } \mathbb{F}\text{-adapted.} \end{cases}$$

Then the hypothesis $(\mathcal{D})$ is satisfied with respect to the filtration $\mathbb{F}$.

Example 2.3.8. *Let $(N^1, ..., N^n)$ be a vector of semimartingales with respect to its natural filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,1]}$. Suppose that $\xi = R + Q$, where*

$$\begin{cases} R \text{ is independent from } (N^1, ..., N^n); \\ (Q, N^1, ..., N^n) \text{ has all its mutual brackets.} \end{cases}$$

Then, if Q is adapted to the filtration

$$\mathbb{H} = (\mathcal{G}_t \vee \sigma(R))_{t\in[0,1]},$$

the vector $(\xi, N^1, ..., N^n)$ satisfies the hypothesis $(\mathcal{D})$ with respect to $\mathbb{H}$.

For every $\mathbb{H}$-local martingale N we denote with $\mathcal{L}^2_N(\mathbb{H})$ the set of all progressively measurable processes h such that

$$||h||_{L^2(d[N])} = \int_0^1 h_s^2 d[N]_s < +\infty, \quad a.s..$$

$\mathcal{L}^2_N(\mathbb{H})$ endowed with the topology of the convergence in probability with respect to the norm $||\cdot||_{L^2(d[N])}$, is an F-space in the sense of [18]. The F-space of all continuous $\mathbb{H}$-adapted processes equipped with the uniform convergence in probability will be denoted by $\mathcal{A}(\mathbb{H})$.

Proposition 2.3.9. *Let Q be a continuous and $\mathbb{H}$-adapted process and N a continuous $\mathbb{H}$-local martingale such that (Q, N) has all its mutual brackets. Then for every h in $\mathcal{L}^2_N(\mathbb{H})$, and $Y = \int_0^\cdot h_s dN_s$, the bracket $[Q, Y]$ exists and*

$$[Q, Y] = \int_0^\cdot h_s d[Q, N]_s.$$

In particular (Q, Y) has all its mutual brackets and $[Q, Y]$ has bounded variation.

Proof. By localization arguments we do not loose generality if we suppose that Q is uniformly bounded and N is square integrable. We set $\Gamma(h) := \int_0^\cdot h_s d[Q, N]_s$, for every h in $\mathcal{L}^2_N(\mathbb{H})$, and for every $\varepsilon > 0$ we consider the map $\Gamma_\varepsilon : \mathcal{L}^2_N(\mathbb{H}) \to \mathcal{A}(\mathbb{H})$ so defined :

$$\Gamma_\varepsilon(h) = \frac{1}{\varepsilon} \int_0^\cdot (Q_{\varepsilon+s} - Q_s) \left(\int_s^{s+\varepsilon} h_r dN_r \right) ds.$$

Γ_ε is a linear and continuous operator from $\mathcal{L}^2_N(\mathbb{H})$ to $\mathcal{A}(\mathbb{H})$. Let h be continuous. We claim that $(\Gamma_\varepsilon(h))$ converges *ucp* to $\Gamma(h)$. Remark 2.2.5.2 implies

$$\lim_{\varepsilon\to 0}\int_0^\cdot h_s\left(Q_{s+\varepsilon}-Q_s\right)\left(N_{s+\varepsilon}-N_s\right)=\Gamma(h),\quad ucp.$$

We hence achieve the claim if

$$\lim_{\varepsilon\to 0} I^\varepsilon(t)=\lim_{\varepsilon\to 0}\left|\frac{1}{\varepsilon}\int_0^t\left(Q_{s+\varepsilon}-Q_s\right)\left(\int_s^{s+\varepsilon}\left(h_s-h_r\right)dN_r\right)ds\right|=0,\quad ucp.$$

Again by standard localization techniques we can suppose h uniformly bounded. We use Cauchy-Schwartz inequality to write

$$I^\varepsilon(t)\le\left(\frac{1}{\varepsilon}\int_0^\cdot\left(Q_{s+\varepsilon}-Q_s\right)^2 ds\right)^{\frac{1}{2}}\left(\int_0^\cdot\frac{1}{\varepsilon}\left|\int_s^{s+\varepsilon}\left(h_s-h_r\right)dN_r\right|^2 ds\right)^{\frac{1}{2}}.$$

The expectation of the second factor of the product is convergent to zero by Burkolder inequality, the continuity and the boundness of h.

Moreover it is possible to show that for every h in $\mathcal{L}^2_N(\mathbb{H})$

$$\sup_{\varepsilon>0} d_1\left(\Gamma_\varepsilon(h),0\right)\le d_2\left(h,0\right),$$

being d_1 and d_2 two metrics inducing the given topologies of $\mathcal{A}(\mathbb{H})$ and $\mathcal{L}^2_N(\mathbb{H})$, respectively. We recall that $\mathbb{H}$-adapted continuous processes are dense in $\mathcal{L}^2_N(\mathbb{H})$, so that Banach-Steinhaus theorem for Fréchet spaces ([18] chapter 2.1) and the density of continuous processes permit to conclude. □

Proposition 2.3.10. *Let (Z^ε) be a sequence of continuous and $\mathbb{H}$-adapted processes, and N a continuous $\mathbb{H}$-local martingale. Suppose that (Z^ε) converges to zero in $\mathcal{A}(\mathbb{H})$. Then*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\cdot Z^\varepsilon_s\left(N_{s+\varepsilon}-N_s\right)ds=0,\quad ucp.$$

Proof. Since the convergence in probability is equivalent to existence of subsequences convergent to zero almost surely, it is not reductive to suppose that (Z^ε) converges uniformly to zero, almost surely. We perform the proof in the case N is a square integrable martingale. To reduce to this case localization techniques could be used. We set, for every k in $\mathbb{N}^*$,

$$\Omega_k=\left\{\omega\in\Omega,\ s.t.\ \sup_{0\le s\le 1}|Z^\varepsilon_s|\le k,\forall\varepsilon\le k^{-1}\right\},$$

and

$$Z^{\varepsilon,k}=Z^\varepsilon I_{\left\{\sup_{0\le u\le\cdot}|Z^\varepsilon_u|\le k\right\}}.$$

Then it is sufficient to show that

$$\lim_{\varepsilon\to 0} C^k_\varepsilon=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\cdot Z^{\varepsilon,k}_s\left(N_{s+\varepsilon}-N_s\right)ds=0,\quad ucp,\quad\forall k\in\mathbb{N}^*.$$

Let k be fixed. Thanks to adaptedness of the process $Z^{\varepsilon,k}$ we can write

$$C_\varepsilon^k = \frac{1}{\varepsilon}\int_0^\cdot \left(\int_s^{s+\varepsilon} Z_s^{\varepsilon,k} dN_r\right) ds. \tag{2.12}$$

Exercise 5.17, pag.165 of [49] applies to write

$$C_\varepsilon^k = \int_0^\cdot \left(\frac{1}{\varepsilon}\int_{r-\varepsilon}^r Z_s^{\varepsilon,k} ds\right) dN_r, \quad a.s..$$

Using Doob and Hölder inequalities we obtain

$$\begin{aligned} E\left[\sup_{t\in[0,1]} \left|C_\varepsilon^k(t)\right|^2\right] &\le cE\left[\int_0^1 \left(\frac{1}{\varepsilon}\int_{r-\varepsilon}^r Z_s^{\varepsilon,k} ds\right)^2 d[N]_r\right] \\ &\le cE\left[\sup_{s\in[0,1]} \left|Z_s^{\varepsilon,k}\right|^2 [N]_1\right], \end{aligned}$$

for some positive constant c. Lebesgue dominated convergence theorem permits to complete the proof.

□

Corollary 2.3.11. *Let R be an $\mathbb{H}$-strongly predictable continuous process. Then for every continuous $\mathbb{H}$-local martingale N, and h in $\mathcal{L}_N^2(\mathbb{H})$, $[R,Y]=0$.*

Proof. It has to be shown that $\left(\frac{1}{\varepsilon}\int_0^\cdot Z_s^\varepsilon \left(Y_{s+\varepsilon}-Y_s\right) ds\right)$, converges to zero *ucp*, with $Z^\varepsilon = R_{\varepsilon+\cdot} - R$. R is $\mathbb{H}$-strongly predictable, then ε small enough Z^ε is $\mathbb{H}$-adapted. Moreover the continuity of R insures the uniformly convergence to zero, almost surely, of Z^ε. Proposition 2.3.10 leads to the conclusion.

□

Remark 2.3.12. *$[R,Y]$ is zero either for pathwise regularity or for probabilistic reasons. The first situation arises if R has zero quadratic variation, when its paths are for instance Hölder continuous with parameter $\gamma > \frac{1}{2}$. The second (probabilistic) reason arises, for example, when R is strongly predictable as corollary 2.3.11 shows.*

We go on defining and discussing some properties of weak Dirichlet processes.

Definition 2.3.13. *An $\mathbb{H}$-**weak Dirichlet** process is the sum of a continuous $\mathbb{H}$-local martingale M and a continuous process Q such that $[Q,N]=0$, for every $\mathbb{H}$-local martingale N.*

Corollary 2.3.11 directly implies the following.

Corollary 2.3.14. *An $\mathbb{H}$-strongly predictable continuous process R is an $\mathbb{H}$-weak Dirichlet process.*

Proposition 2.3.9 permits to better specify the nature of such processes with respect to Brownian filtrations, as pointed out in the corollary below.

Corollary 2.3.15. *Suppose that W is a Brownian motion on $(\Omega, \mathcal{F}, P)$. Let $\mathbb{H}$ be its natural filtration augmented by the P null sets. An $\mathbb{H}$-adapted, with finite quadratic variation and continuous process D is an $\mathbb{H}$-Dirichlet process if and only if it is the sum of a continuous $\mathbb{H}$-local martingale M and a finite quadratic variation process Q, continuous, $\mathbb{H}$-adapted and such that $[Q, W] = 0$.*

Proof. Necessity is obvious. Suppose that D is the sum of an $\mathbb{H}$-local martingale M and a continuous process Q, with finite quadratic variation, $\mathbb{H}$-adapted and such that $[Q, W] = 0$. Let N be an $\mathbb{H}$-local martingale. Then there exists a process h in $\mathcal{L}^2_W(\mathbb{H})$ such that $N = N_0 + \int_0^\cdot h_s dW_s$. By proposition 2.3.9 $[Q, N] = \int_0^\cdot h_s d\,[Q, W]_s = 0$.

□

Theorem 2.3.16. *Let $(X(t,x), 0 \le t \le 1, x \in \mathbb{R})$ be the sum of a C^1 $\mathbb{H}$-Itô-martingale field of the form (2.7), and a $\mathbb{H}$-strict zero quadratic variation process Z in $C^{0,1}([0,1] \times \mathbb{R})$. Let ξ be such that the vector $(\xi, N^1, ..., N^n)$ satisfies the hypothesis $(\mathcal{D})$ with respect to the filtration $\mathbb{H}$. Then for any semimartingale of the form $Y = \sum_{i=1}^n \int_0^\cdot h^i_s dN^i_s$, with h^i in $\mathcal{L}^2_N(\mathbb{H})$ for every $i = 1, ..., n$, it holds :*

$$\begin{aligned} [X(\cdot,\xi), Y] &= \sum_{i=1}^n \int_0^\cdot \partial_x X(s,\xi_s) h^i_s d\left[\xi, N^i\right]_s \\ &+ \sum_{i,j=1}^n \int_0^\cdot a^j(s,\xi_s) h^i_s d\left[N^i, N^j\right]_s . \end{aligned}$$

In particular $[X(\cdot,\xi), Y]$ has bounded variation.

Remark 2.3.17. *In [24] the authors explore the existence of mutual brackets of Itô fields, and so it could appear natural to do the same in this context. However, it is clear that in this case such a bracket cannot exist unless R is a finite quadratic variation process.*

Proof. (of the theorem). We suppose for simplicity of notations that $n = 1$, and we denote with h the process h^1. We have to study the convergence *ucp* of

$$C_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t \left(X(s+\varepsilon, \xi_{s+\varepsilon}) - X(s,\xi_s)\right)(Y_{s+\varepsilon} - Y_s) ds.$$

We have

$$\begin{aligned} C_\varepsilon(t) &= \frac{1}{\varepsilon} \int_0^t \left(X(s+\varepsilon, \xi_{s+\varepsilon}) - X(s+\varepsilon, Q_s + R_{s+\varepsilon})\right)(Y_{s+\varepsilon} - Y_s) ds \\ &+ \frac{1}{\varepsilon} \int_0^t \left(X(s+\varepsilon, Q_s + R_{s+\varepsilon}) - X(s, Q_s + R_{s+\varepsilon})\right)(Y_{s+\varepsilon} - Y_s) ds \\ &+ \frac{1}{\varepsilon} \int_0^t \left(X(s, Q_s + R_{s+\varepsilon}) - X(s,\xi_s)\right)(Y_{s+\varepsilon} - Y_s) ds \\ &= J^1_\varepsilon(t) + J^2_\varepsilon(t) + J^3_\varepsilon(t) \end{aligned}$$

For $J_\varepsilon^1(t)$ we use Taylor type formula

$$\begin{aligned} X(s+\varepsilon,\xi_{s+\varepsilon}) - X(s+\varepsilon, Q_s + R_{s+\varepsilon}) &= \partial_x X(s,\xi_s)\,(Q_{s+\varepsilon} - Q_s) \\ &+ \rho(s,\varepsilon)\,(Q_{s+\varepsilon} - Q_s) \end{aligned}$$

with

$$\rho(s,\varepsilon) = \int_0^1 \left[\partial_x X(s+\varepsilon, \lambda(Q_{s+\varepsilon} - Q_s) + (Q_s + R_{s+\varepsilon})) - \partial_x X(s,\xi_s)\right] d\lambda,$$

to get

$$\begin{aligned} J_\varepsilon^1(t) &= \frac{1}{\varepsilon}\int_0^t \partial_x X(s,\xi_s)\,(Q_{s+\varepsilon} - Q_s)\,(Y_{s+\varepsilon} - Y_s) ds \\ &+ \frac{1}{\varepsilon}\int_0^t \rho(s,\varepsilon)\,(Q_{s+\varepsilon} - Q_s)\,(Y_{s+\varepsilon} - Y_s) ds. \end{aligned}$$

Since h is continuous and $\mathbb{H}$-adapted, it is progressively measurable and almost surely bounded. By proposition 2.3.9 $(Q, \int_0^\cdot h_s dN_s)$ has all its mutual brackets, and so by remark 2.2.5.2 the first term converges *ucp* to

$$\int_0^\cdot \partial_x X(s,\xi_s) h_s d\,[Q,N]_s\,,$$

while the second term has limit equal to zero *ucp* since both Q and Y have *finite quadratic variation*.

We consider the term $J^2(t)$. Thanks to the hypothesis $(\mathcal{D})$, the process

$$(Q_s + R_{s+\varepsilon}, 0 \le s \le 1)$$

is $\mathbb{H}$-adapted for every $\varepsilon \le \delta$. Then we can write for every $\varepsilon \le \delta$

$$\begin{aligned} J_\varepsilon^2(t) &= \frac{1}{\varepsilon}\int_0^t \left(\int_s^{s+\varepsilon} (a(r, Q_s + R_{s+\varepsilon}) - a(r,\xi_r))\, dN_r\right)(Y_{s+\varepsilon} - Y_s)\, ds \\ &+ \frac{1}{\varepsilon}\int_0^t \left(\int_s^{s+\varepsilon} a(r,\xi_r) dN_r\right)(Y_{s+\varepsilon} - Y_s)\, ds \\ &+ \frac{1}{\varepsilon}\int_0^t (Z(s+\varepsilon, Q_s + R_{s+\varepsilon}) - Z(s, Q_s + R_{s+\varepsilon}))\,(Y_{s+\varepsilon} - Y_s)\, ds. \end{aligned}$$

The second term converges *ucp* by definition to

$$\left[\int_0^\cdot a(s,\xi_s) dN_s, Y\right] = \int_0^t h_s a(s,\xi_s) d\,[N,N]_s\,,$$

while using Hölder inequality, and the fact that Z is a *strict zero quadratic variation* process it is possible to show that the last term converges to zero *ucp*. Again by Hölder inequality the first term converges to zero *ucp* if

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^1 \left(\int_s^{s+\varepsilon} (a(r, Q_s + R_{s+\varepsilon}) - a(r,\xi_r))\, dN_r\right)^2 ds = 0, \qquad \text{in probability.}$$

This can be proved with techniques already used for the convergence to zero of the term I_ε^2 in the proof of proposition 2.3.4. Regarding the term J^3, we apply proposition 2.3.10 to the sequence of processes $(X(\cdot, Q + R_{\cdot+\varepsilon}) - X(\cdot, \xi))$, the local martingale N, and the process h, which let us conclude that J^3 converges to zero *ucp*.

□

Using similar arguments to those of previous proposition one can prove the following.

Proposition 2.3.18. *Let β be in $C^{0,1}([0,1] \times \mathbb{R})$, and $(\xi, N^1, ..., N^n)$ be a vector of continuous processes satisfying the hypothesis $(\mathcal{D})$ with respect to $\mathbb{H}$. Then for every semimartingale of the form*

$$Y = \sum_{i=1}^{n} \int_0^{\cdot} h_s^i dN_s^i,$$

with h^i in $\mathcal{L}_{N^i}^2(\mathbb{H})$ for every $i = 1, ..., n$, $[\beta(\cdot, \xi), Y]$ exists and

$$[\beta(\cdot, \xi), Y] = \sum_{i=1}^{n} \int_0^{\cdot} h_s^i \partial_x \beta(s, \xi_s) d\left[\xi, N^i\right]_s. \tag{2.13}$$

In particular $[\beta(\cdot, \xi), Y]$ has bounded variation.

Corollary 2.3.19. *Let $(\xi, N^1, ..., N^n)$ be a vector of continuous processes satisfying the hypothesis $(\mathcal{D})$ with respect to $\mathbb{H}$. Let $X = (X(t,x), 0 \leq t \leq 1, x \in \mathbb{R})$ and*

$$Z = (Z(t,x), 0 \leq t \leq 1, x \in \mathbb{R})$$

be either functions in $C^{0,1}([0,1] \times \mathbb{R})$ or C^1 $\mathbb{H}$-Itô-semimartingale fields of the form (2.10). Then for every semimartingale of the form

$$Y = \sum_{i=1}^{n} \int_0^{\cdot} h_s^i dN_s^i$$

with h^i in $\mathcal{L}_{N^i}^2(\mathbb{H})$ for every $i = 1, ..., n$, it holds

$$\int_0^{\cdot} X(s, \xi_s) d^\circ \left(\int_0^s Z(r, \xi_r) d^\circ Y_r \right) = \int_0^{\cdot} (XZ)(s, \xi_s) d^\circ Y_s.$$

Proof. The corollary is a consequence of proposition 2.3.16 and the decomposition of the symmetric integral into a classical stochastic integral plus an half covariation as specified in remark 2.2.7.2.

□

2.3.3 Itô-Wentzell formula

Proposition 2.3.20. *Let $(X(t,x), 0 \leq t \leq 1, x \in \mathbb{R})$ be a C^3 $\mathbb{H}$-Itô-semimartingale field of the form (2.10). Let $(\xi, N^1, ..., N^n)$ be a vector of continuous processes satisfying the* *hypothesis* $(\mathcal{D})$ *with respect to* $\mathbb{H}$, *such that* ξ *is a strong cubic variation process. Then the symmetric integral* $\int_0^{\cdot} \partial_x X(s,\xi_s) d^\circ \xi_s$ *exists and*

$$\begin{aligned} X(\cdot,\xi) &= X(0,\xi_0) + \sum_{i=1}^{n} \int_0^{\cdot} a^i(s,\xi_s) dN_s^i + \sum_{j=1}^{m} \int_0^{\cdot} b^j(s,\xi_s) dV_s^j \\ &+ \int_0^{\cdot} \partial_x X(s,\xi_s) d^\circ \xi_s + \frac{1}{2} \sum_{i=1}^{n} \int_0^{\cdot} \partial_x a^i(s,\xi_s) d\left[N^i,\xi\right]_s \\ &- \frac{1}{12} \int_0^{\cdot} \partial_x^{(3)} X(s,\xi_s) d\left[\xi,\xi,\xi\right]_s . \end{aligned}$$

Proof. We suppose $n = m = 1$, and we make the usual simplification in the notation of the Itô field considered. By continuity of the process $X(\cdot,\xi)$ the sequence of processes

$$\frac{1}{\varepsilon} \int_0^t \left(X(s+\varepsilon,\xi_{s+\varepsilon}) - X(s,\xi_s) \right) ds$$

converges almost surely to $(X(t,\xi_t) - X(0,\xi_0))$. In particular

$$\begin{aligned} X(t,\xi_t) - X(0,\xi_0) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t \left(X(s+\varepsilon,\xi_{s+\varepsilon}) - X(s+\varepsilon,\xi_s) \right) ds \\ &+ \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t \left(X(s+\varepsilon,\xi_s) - X(s,\xi_s) \right) ds \\ &= \lim_{\varepsilon\to 0} I_\varepsilon^1(t) + \lim_{\varepsilon\to 0} I_\varepsilon^2(t), \end{aligned}$$

if the two limits on the right hand side of previous equality exist. Applying substitution arguments and interchanging the integrals with respect to time, the semimartingales N and V, $I_\varepsilon^2(t)$ converges *ucp* to

$$\int_0^{\cdot} a(s,\xi_s) dN_s + \int_0^{\cdot} b(s,\xi_s) dV_s.$$

Since $X(\cdot,x)$ is differentiable till order three with respect to x, we can write

$$\begin{aligned} X(s+\varepsilon,\xi_{s+\varepsilon}) &= X(s+\varepsilon,\xi_s) + \partial_x X(s+\varepsilon,\xi_s) \left(\xi_{s+\varepsilon} - \xi_s\right) \\ &+ \frac{1}{2} \partial_x^{(2)} X(s+\varepsilon,\xi_s) \left(\xi_{s+\varepsilon} - \xi_s\right)^2 \\ &+ \frac{1}{6} \partial_x^{(3)} X(s+\varepsilon,\xi_s) \left(\xi_{s+\varepsilon} - \xi_s\right)^3 + \rho(\xi_s,\xi_{s+\varepsilon}) \left(\xi_{s+\varepsilon} - \xi_s\right)^3, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} X(s+\varepsilon,\xi_s) &= X(s+\varepsilon,\xi_{s+\varepsilon}) + \partial_x X(s+\varepsilon,\xi_{s+\varepsilon}) \left(\xi_s - \xi_{s+\varepsilon}\right) \\ &+ \frac{1}{2} \partial_x^{(2)} X(s+\varepsilon,\xi_{s+\varepsilon}) \left(\xi_s - \xi_{s+\varepsilon}\right)^2 \\ &+ \frac{1}{6} \partial_x^{(3)} X(s+\varepsilon,\xi_{s+\varepsilon}) \left(\xi_s - \xi_{s+\varepsilon}\right)^3 + \rho(\xi_{s+\varepsilon},\xi_s) \left(\xi_s - \xi_{s+\varepsilon}\right)^3, \end{aligned} \tag{2.15}$$

with $\lim_{\varepsilon\to 0}\rho(\xi_s,\xi_{s+\varepsilon}) = \lim_{\varepsilon\to 0}\rho(\xi_{s+\varepsilon},\xi_s) = 0$, almost surely. By subtracting these two quantities and integrating over $[0,t]$ we get

$$\begin{aligned} I^1_\varepsilon(t) &= \frac{1}{2\varepsilon}\int_0^t \left(\partial_x X(s+\varepsilon,\xi_s) + \partial_x X(s+\varepsilon,\xi_{s+\varepsilon})\right)(\xi_{s+\varepsilon}-\xi_s)\, ds \\ &- \frac{1}{4\varepsilon}\int_0^t \left(\partial_x^{(2)} X(s+\varepsilon,\xi_{s+\varepsilon}) - \partial_x^{(2)} X(s+\varepsilon,\xi_s)\right)(\xi_{s+\varepsilon}-\xi_s)^2\, ds \\ &+ \frac{1}{12\varepsilon}\int_0^t \left(\partial_x^{(3)} X(s+\varepsilon,\xi_s) + \partial_x^{(3)} X(s+\varepsilon,\xi_{s+\varepsilon})\right)(\xi_{s+\varepsilon}-\xi_s)^3\, ds \\ &+ \frac{1}{2\varepsilon}\int_0^t \left(\rho(\xi_s,\xi_{s+\varepsilon}) + \rho(\xi_{s+\varepsilon},\xi_s)\right)(\xi_{s+\varepsilon}-\xi_s)^3\, ds \\ &= J^1_\varepsilon(t) + J^2_\varepsilon(t) + J^3_\varepsilon(t) + J^4_\varepsilon(t). \end{aligned}$$

Since ξ is a *strong cubic variation* process J^4_ε converges to zero *ucp*. J^2_ε converges *ucp* to

$$-\frac{1}{4}\left[\partial_x^{(2)} X(\cdot,\xi),\xi,\xi\right].$$

In fact,

$$\begin{aligned} J^2_\varepsilon(t) &= -\frac{1}{4\varepsilon}\int_0^t \left(\partial_x^{(2)} X(s+\varepsilon,\xi_{s+\varepsilon}) - \partial_x^{(2)} X(s,\xi_s)\right)(\xi_{s+\varepsilon}-\xi_s)^2\, ds \\ &+ \frac{1}{4\varepsilon}\int_0^t \left(\partial_x^{(2)} X(s+\varepsilon,\xi_s) - \partial_x^{(2)} X(s,\xi_s)\right)(\xi_{s+\varepsilon}-\xi_s)^2\, ds. \end{aligned}$$

The first term converges *ucp* to

$$-\frac{1}{4}\left[\partial_x^{(2)} X(\cdot,\xi),\xi,\xi\right] = -\frac{1}{4}\int_0^{\cdot} \partial_x^{(3)} X(s,\xi_s) d\left[\xi,\xi,\xi_\varepsilon\right]_s,$$

since $\partial_x^2 X(\cdot,x)$ is a C^1 $\mathbb{H}$-Itô-semimartingale field and proposition 2.3.4 can be applied. The second term converges to zero *ucp*. In fact, by Hölder inequality its absolute value is bounded by

$$\frac{1}{4}\left(\frac{1}{\varepsilon}\int_0^1 \left|\partial_x^{(2)} X(s+\varepsilon,\xi_s) - \partial_x^{(2)} X(s,\xi_s)\right|^3 ds\right)^{\frac{1}{3}} ds\, ||[\xi,\xi,\xi]||_\varepsilon^{\frac{2}{3}}.$$

Since $\partial_x^{(2)} X$ is a C^1 Itô-semimartingale field, the first factor of the product can be shown to converge to zero in probability, using tools already developed in the proof of proposition 2.3.4 for the term $\int_0^{\cdot} |B(s,\varepsilon)|^3\, ds$. The term J^3_ε can be written as

$$\begin{aligned} &\tfrac{1}{12\varepsilon}\int_0^t \left(\partial_x^{(3)} X(s+\varepsilon,\xi_{s+\varepsilon}) + \partial_x^{(3)} X(s+\varepsilon,\xi_s) - 2\partial_x^{(3)} X(s,\xi_s)\right)(\xi_{s+\varepsilon}-\xi_s)^3\, ds \\ &\qquad + \tfrac{1}{6\varepsilon}\int_0^t \partial_x^{(3)} X(s,\xi_s)\,(\xi_{s+\varepsilon}-\xi_s)^3\, ds. \end{aligned}$$

By remark 2.2.5.2, the second term converges *ucp* to $\frac{1}{6}\int_0^{\cdot}\partial_x^{(3)}X(s,\xi_s)d\left[\xi,\xi,\xi\right]_s$, while the first term converges to zero 0 *a.s.*, since ξ is has a finite strong cubic variation, and both $\partial_x^{(3)}X$ and ξ are continuous. Finally

$$\begin{aligned} J_\varepsilon^1 &= \frac{1}{2\varepsilon}\int_0^t \left(\partial_x X(s,\xi_s)+\partial_x X(s+\varepsilon,\xi_{s+\varepsilon})\right)\left(\xi_{s+\varepsilon}-\xi_s\right)ds \\ &+ \frac{1}{2\varepsilon}\int_0^t \left(\partial_x X(s+\varepsilon,\xi_s)-\partial_x X(s,\xi_s)\right)\left(\xi_{s+\varepsilon}-\xi_s\right)ds. \end{aligned}$$

The second term can be decomposed in the following way

$$\begin{aligned} &\frac{1}{2\varepsilon}\int_0^t \left(\partial_x X(s+\varepsilon,\xi_s)-\partial_x X(s,\xi_s)\right)\left(\xi_{s+\varepsilon}-\xi_s\right)ds \\ =& \frac{1}{2\varepsilon}\int_0^t \left(\int_s^{s+\varepsilon}\left(\partial_x a(r,\xi_s)-\partial_x a(r,\xi_r)\right)dN_r\right)\left(Q_{s+\varepsilon}-Q_s\right)ds \\ +& \frac{1}{2\varepsilon}\int_0^t \left(\int_s^{s+\varepsilon}\left(\partial_x a(r,\xi_s)-\partial_x a(r,\xi_r)\right)\left(R_{s+\varepsilon}-R_s\right)dN_r\right)ds \\ +& \frac{1}{2\varepsilon}\int_0^t \left(\int_s^{s+\varepsilon}\partial_x a(r,\xi_r)dN_r\right)\left(\xi_{s+\varepsilon}-\xi_s\right)ds \\ +& \frac{1}{2\varepsilon}\int_0^t \left(Z(s+\varepsilon,\xi_s)-Z(s,\xi_s)\right)\left(\xi_{s+\varepsilon}-\xi_s\right)ds, \end{aligned}$$

with $Z=\int_0^{\cdot}\partial_x b(s,\cdot)dV_s$. The first term of the sum converges to zero *ucp* by Hölder inequality, since Q is a finite quadratic variation process and

$$\lim_{\varepsilon\to 0}\int_0^1 \frac{1}{\varepsilon}\left(\int_s^{s+\varepsilon}\left(\partial_x a(r,\xi_s)-\partial_x a(r,\xi_r)\right)dN_r\right)^2 ds = 0, \quad \text{in probability .}$$

By proposition 2.3.16

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t\left(\int_s^{s+\varepsilon}\partial_x a(r,\xi_r)dN_r\right)\left(\xi_{s+\varepsilon}-\xi_s\right)ds = \frac{1}{2}\int_0^{\cdot}\partial_x a(s,\xi_s)d\left[N,\xi\right], \quad \text{ucp.}$$

The second term can be shown to converge to zero by arguments used in the proof of proposition 2.3.10, while the last term converges to zero *ucp* since Z is $\mathbb{H}$-strict zero p-variation process, for every $p>1$. As a consequence of this the first term of J_ε^1 is forced to converge to

$$\int_0^{\cdot}\partial_x X(s,\xi_s)d^\circ\xi_s,$$

and we get the result.

□

2.3.4 Existence of symmetric integrals and chain-rule formulae

In this subsection ξ is supposed to be a strong cubic variation process.

Definition 2.3.21. *We denote with $\mathcal{C}_\xi^k(\mathbb{H})$ the set of all processes of the form $Z_t = X(t,\xi_t)$, being X a C^k $\mathbb{H}$-Itô-semimartingale field driven by the vector of local martingales $(N^1,...,N^n)$, such that the vector $(\xi, N^1,...,N^n)$ satisfies the hypothesis $(\mathcal{D})$, with respect to the filtration $\mathbb{H}$.*

Remark 2.3.22. *The set $\mathcal{C}_\xi^k(\mathbb{H})$ is an algebra (apply classical Itô formula).*

Remark 2.3.23.

1. *A process Z belongs to $\mathcal{C}_\xi^3(\mathbb{H})$ if and only if there exist an $\mathcal{H}_0$-measurable random variable Z_0, a vector of $\mathbb{H}$-adapted processes $(N^1,...,N^n)$ such that $(\xi, N^1,...,N^n)$ satisfies the hypothesis $(\mathcal{D})$ with respect to $\mathbb{H}$, a vector of $\mathbb{H}$-adapted stochastic processes $(h^1,...,h^n)$, and a process γ in $\mathcal{C}_\xi^2(\mathbb{H})$, such that*

$$Z = Z_0 + \int_\circ^\cdot \gamma_s d^\circ \xi_s + \sum_{i=1}^n \int_0^\cdot h_s dN_s^i.$$

   The statement is a direct consequence of Itô-Wentzell formula.

2. *Combining remark 2.2.5, the reversed Itô-Wentzell formula, and proposition 2.3.4, it is possible to prove that if γ^1, γ^2, and γ^3 belong to $\mathcal{C}_\xi^2(\mathbb{H})$, then*

$$\left[\int_0^\cdot \gamma_s^1 d^\circ \xi_s, \int_0^\cdot \gamma_s^2 d^\circ \xi_s, \int_0^\cdot \gamma_s^3 d^\circ \xi_s\right] = \int_0^\cdot \gamma_s^1 \gamma_s^2 \gamma_s^3 \left[\xi,\xi,\xi\right]_s.$$

3. *A significant example of the class $\mathcal{C}_\xi^3(\mathbb{H})$ is given by the following. Let*

$$W = \left(W^1,...,W^n\right)$$

   be a n-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$ with respect to its natural filtration $\mathbb{H}$ augmented by the P null sets. Suppose that the vector $(\xi, W^1,...,W^n)$ satisfies the hypothesis $(\mathcal{D})$ with respect to $\mathbb{H}$. Then the set $\mathcal{C}_\xi^3(\mathbb{H})$ coincides with the processes of the form

$$Z = Z_0 + \int_0^\cdot \gamma_s d^\circ \xi_s + L$$

   where γ is in $\mathcal{C}_\xi^2(\mathbb{H})$ and L is an $\mathbb{H}$-semimartingale. This holds since every $\mathbb{H}$-local martingale, zero at $t=0$, admits a representation as a stochastic integral with respect to W.

Proposition 2.3.24. *For every Z in $\mathcal{C}_\xi^2(\mathbb{H})$ and U in $\mathcal{C}_\xi^3(\mathbb{H})$ the symmetric integral*

$$\int_0^\cdot Z_s d^\circ U_s,$$

exists and belongs to $\mathcal{C}_\xi^2(\mathbb{H})$. If $Z_t = Y(t,\xi_t)$, and $U_t = X(t,\xi_t)$, where $X(\cdot,x)$ and $Y(\cdot,x)$ have representations

$$X(\cdot,x) = X_0(x) + \sum_{i=1}^n \int_0^\cdot a^i(s,x) dN_s^i + \sum_{j=1}^m \int_0^\cdot b^j(s,x) dV_s^j, \tag{2.16}$$

and

$$Y(\cdot,x) = Y_0(x) + \sum_{i=1}^{n} \int_0^{\cdot} \bar{a}^i(s,x)dN_s^i + \sum_{j=1}^{m} \int_0^{\cdot} \bar{b}^j(s,x)dV_s^j, \tag{2.17}$$

then

$$\begin{aligned}\int_0^{\cdot} Z_s d^\circ U_s &= \sum_{i=1}^{n} \int_0^{\cdot} (Ya^i)(s,\xi_s)dN_s^i + \sum_{j=1}^{m} \int_0^{\cdot} (Yb^j)(s,\xi_s)dV_s^j \\ &+ \int_0^{\cdot} (Y\partial_x X)(s,\xi_s)d^\circ\xi_s + \frac{1}{2}\sum_{i=1}^{n} \int_0^{\cdot} \partial_x(Ya^i)(s,\xi_s)d\left[N^i,\xi\right]_s \\ &+ \frac{1}{2}\sum_{i,j=1}^{n} \int_0^{\cdot} (a^j\bar{a}^i)(s,\xi_s)d\left[N^i,N^j\right]_s \\ &- \frac{1}{12}\int_0^{\cdot} \left(\left(3\partial_x^{(2)}X\right)(\partial_x Y) + \left(\partial_x^{(3)}X\right)Y\right)(s,\xi_s)d\left[\xi,\xi,\xi\right]_s.\end{aligned}$$

Proof. We restrict ourselves to the case $n = m = 1$, and we denote $a^1 = a, \bar{a}^1 = \bar{a}$. We have to investigate the convergence of

$$\begin{aligned}C_\varepsilon(t) &= \frac{1}{2\varepsilon}\int_0^t (Z_{s+\varepsilon} + Z_s)(U_{s+\varepsilon} - U_s)\, ds \\ &= \frac{1}{2\varepsilon}\int_0^t (Z_{s+\varepsilon} + Z_s)(X(s+\varepsilon,\xi_{s+\varepsilon}) - X(s+\varepsilon,\xi_s))\, ds \\ &+ \frac{1}{2\varepsilon}\int_0^t (Z_{s+\varepsilon} + Z_s)(X(s+\varepsilon,\xi_s) - X(s,\xi_s))\, ds \\ &= I_\varepsilon^1(t) + I_\varepsilon^2(t).\end{aligned}$$

As concerns the second term we can write

$$\begin{aligned}I_\varepsilon^2(t) &= \frac{1}{2\varepsilon}\int_0^t (Z_{s+\varepsilon} - Z_s)(X(s+\varepsilon,\xi_s) - X(s,\xi_s))\, ds \\ &+ \frac{1}{\varepsilon}\int_0^t Z_s(X(s+\varepsilon,\xi_s) - X(s,\xi_s))\, ds.\end{aligned}$$

Using techniques already introduced in previous section and in proposition 2.3.16 one can show that these two terms converge, respectively, *ucp* to

$$\begin{aligned}\frac{1}{2}\left[Y(\cdot,\xi), \int_0^{\cdot} a(r,\xi_r)dN_r + \int_0^{\cdot} b(r,\xi_r)dV_r\right] &= \frac{1}{2}\int_0^{\cdot} ((\partial_x Y)a)(s,\xi_s)d\left[N,\xi\right]_s \\ &+ \frac{1}{2}\int_0^{\cdot} (\bar{a}a)(s,\xi_s)d\left[N,N\right]_s,\end{aligned}$$

and $\int_0^{\cdot} Z_s a(s,\xi_s)dN_s + \int_0^{\cdot} Z_s b(s,\xi_s)dV_s$.

We consider the first term. To this extent, for every s in $[0,1]$ we multiply equalities (2.14) and (2.15) respectively by Z_s and $Z_{s+\varepsilon}$ to get

$$\begin{aligned} I^1_\varepsilon(t) &= \frac{1}{2\varepsilon}\int_0^t \left(\partial_x X(s+\varepsilon,\xi_s)Z_s + \partial_x X(s+\varepsilon,\xi_{s+\varepsilon})Z_{s+\varepsilon}\right)(\xi_{s+\varepsilon}-\xi_s)\,ds \\ &- \frac{1}{4\varepsilon}\int_0^t \left(\partial_x^{(2)} X(s+\varepsilon,\xi_{s+\varepsilon})Z_{s+\varepsilon} - \partial_x^{(2)} X(s+\varepsilon,\xi_s)Z_s\right)(\xi_{s+\varepsilon}-\xi_s)^2\,ds \\ &+ \frac{1}{12\varepsilon}\int_0^t \left(\partial_x^{(3)} X(s+\varepsilon,\xi_s)Z_s + \partial_x^{(3)} X(s+\varepsilon,\xi_{s+\varepsilon})Z_{s+\varepsilon}\right)(\xi_{s+\varepsilon}-\xi_s)^3\,ds \\ &+ \frac{1}{2\varepsilon}\int_0^t \left(\rho(\xi_s,\xi_{s+\varepsilon})Z_s + \rho(\xi_{s+\varepsilon}Z_{s+\varepsilon},\xi_s)Z_s\right)(\xi_{s+\varepsilon}-\xi_s)^3\,ds. \end{aligned}$$

The proof follows the same outlines of the calculus already performed in the proof of the Itô-Wentzell formula for the term $I^1_\varepsilon(t)$. Itô-Wentzell formula is indeed a particular case of this result ($Z = 1$). The only difference, here, is that the symmetric integral $\int_0^\cdot \partial_x X(s,\xi_s)Z_s d^\circ\xi_s$ exists since $\partial_x X(\cdot,x)Z$ is still a C^2 $\mathbb{H}$-Itô-semimartingale field, and for such a field, the existence was already proved before. Then, similarly, we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} I^1_\varepsilon(t) &= \int_0^t Z_s\partial_x X(s,\xi_s)d^\circ\xi_s + \frac{1}{2}\int_0^t Z_s\partial_x a(s,\xi_s)d\,[N,\xi]_s \\ &- \frac{1}{4}\left[\partial_x^{(2)} X(\cdot,\xi)Z,\xi,\xi\right]_t + \frac{1}{6}\int_0^t Z_s\partial_x^{(3)} X(s,\xi_s)d\,[\xi,\xi,\xi]_s\,, \quad ucp. \end{aligned}$$

The conclusion follows applying proposition 2.3.4 to get the equality

$$\left[\partial_x^{(2)} X(\cdot,\xi)Z,\xi,\xi\right]_t = \int_0^t \left(Z_s\partial_x^{(3)} X(s,\xi_s) + \partial_x^{(2)} X\partial_x Y(s,\xi_s)\right)d\,[\xi,\xi,\xi]_s\,,$$

which leads to the result.

□

Proposition 2.3.25. *Let Z and U be in $\mathcal{C}^2_\xi(\mathbb{H})$, with $Z_t = Y(t,\xi_t)$, and $U_t = X(t,\xi_t)$, where $X(\cdot,x)$ and $Y(\cdot,x)$ have representations (2.16) and (2.17). Then the symmetric integral*

$$\int_0^\cdot Z_s d^\circ\left(\int_0^s U(r)d^\circ\xi_r\right)$$

exists and

$$\int_0^\cdot Z_s d^\circ\left(\int_0^s U_r d^\circ\xi_r\right) = \int_0^\cdot Z_s U_s d^\circ\xi_s - \frac{1}{4}\int_0^\cdot \left(\left(\partial_x X\right)\left(\partial_x Y\right)\right)(s,\xi_s)d\,[\xi,\xi,\xi]_s\,.$$

Proof. We consider the field $(X^*(t,x), 0\le t\le 1, x\in\mathbb{R})$ so defined

$$X^*(t,x) = \int_0^x X(t,z)dz.$$

Clearly X^* is a C^3 $\mathbb{H}$-Itô-semimartingale field, so Itô-Wentzell formula can be applied to write

$$\begin{aligned}\int_0^t X(s,\xi_s)d^\circ\xi_s &= X^*(t,\xi_t) - \sum_{i=1}^n \int_0^t a^{i,*}(s,\xi_s)dN_s^i - \sum_{j=1}^m \int_0^t b^{j,*}(s,\xi_s)dV_s^j \\ &- \frac{1}{2}\sum_{i=1}^n \int_0^t a^i(s,\xi_s)d\left[\xi,N^i\right]_s + \frac{1}{12}\int_0^t \partial_x^{(2)}X(s,\xi_s)d\left[\xi,\xi,\xi\right]_s,\end{aligned}$$

where

$$a^{i,*}(t,x) = \int_0^x a^i(t,z)dz, \quad b^{j,*}(t,x) = \int_0^x b^j(t,z)dz,$$

for $i = 1, ..., n$, and $j = 1, ..., m$, are the coefficients comparing in the representation of X^*. Since $Y(\cdot,\xi)$ and $X^*(\cdot,\xi)$ are in $\mathcal{C}_\xi^2(\mathbb{H})$, and $\mathcal{C}_\xi^3(\mathbb{H})$, respectively, we can use propositions 2.3.16 and 2.3.24 to conclude.

□

2.4 SDE driven by a scv process and semimartingales

2.4.1 The equation

On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, with $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$, $\mathcal{F}_1 = \mathcal{F}$, let ξ, M and V be adapted and respectively a *strong cubic variation* process a local martingale and a bounded variation process. We suppose $\xi_0 = 0$. Let $\sigma, \beta : [0,1]\times\mathbb{R} \to \mathbb{R}$ be continuous functions, $\alpha : [0,1]\times\mathbb{R}\times\Omega \to \mathbb{R}$ be progressively measurable and locally bounded in x, uniformly in t, almost surely, and η be a random variable $\mathcal{F}_0$-measurable.

Definition 2.4.1. *A continuous process $X : \Omega\times[0,1] \to \mathbb{R}$, is called **solution** to equation*

$$\begin{cases} d^\circ X_t = \sigma(t,X_t)\left[d^\circ\xi_t + \beta(t,X_t)d^\circ M_t + \alpha(t,X_t)dV_t\right], & 0 \le t \le 1 \\ X_0 = \eta \end{cases} \tag{2.18}$$

on $(\Omega, \mathcal{F}, P)$, if

1. *$X_0 = \eta$;*
2. *X is a strong cubic variation process;*
3. *$[\beta(\cdot,X), M]$ exists and it has bounded variation;*
4. *for every ψ in $C^{1,\infty}([0,1]\times\mathbb{R})$*

$$\begin{aligned}\int_0^\cdot \psi(t,X_t)d^\circ X_t &= \int_0^\cdot (\psi\sigma)(t,X_t)\left[d^\circ\xi_t + \beta(t,X_t)d^\circ M_t + \alpha(t,X_t)dV_t\right] \\ &- \frac{1}{4}\int_0^\cdot (\partial_x\sigma)\left(\sigma^2\right)(\partial_x\psi)(t,X_t)d\left[\xi,\xi,\xi\right]_t, \quad a.s..\end{aligned}$$

Remark 2.4.2. *1. A solution to equation (2.18) is a solution to the integral equation*

$$\begin{aligned} X_t &= \eta + \int_0^t \sigma(s, X_s) d^\circ \xi_s + \int_0^t (\sigma\beta)(s, X_s) d^\circ M_s \\ &+ \int_0^t (\sigma\alpha)(s, X_s) dV_s, \end{aligned} \tag{2.19}$$

(consider the case $\psi = 1$).

2. If X is a solution then property 4. is satisfied for every ψ in $C^{1,2}$ (see [21], remark 4.2, pag. 286).

2.4.2 Hypotheses on the coefficients

The construction here used to prove some results about uniqueness and existence of equation (2.18), is based on the following assumption

$$(\mathbf{H_1}) \quad \{(t,x) \in [0,1] \times \mathbb{R}, \text{ s.t. } \sigma(t,x) \neq 0\} = [0,1] \times S = \bigcup_{n=0}^{\infty} \left([0,1] \times S^n\right),$$

where S is an open set in $\mathbb{R}$, and thus the countable union of its connected components

$$\left(S^n = (a_n, b_n), -\infty \leq a_n < b_n \leq +\infty\right)_{n \in \mathbb{N}}.$$

For every n in $\mathbb{N}$ we define the function $H^n : [0,1] \times S^n$:

$$H^n(t,x) = \int_{c_n}^{x} \frac{1}{\sigma(t,z)} dz$$

being c_n in S^n, and we denote $H(t,x) = \sum_{n=0}^{+\infty} H^n(t,x) I_{[0,1]\times S^n}(t,x)$, for (t,x) in $[0,1] \times S$. We will also need to assume that for every t in $[0,1]$ and n in $\mathbb{N}$

$$(\mathbf{H_2}) \quad \begin{cases} \lim\limits_{(s,x)\to(t,a_n)} \displaystyle\int_x^{c_n} \frac{1}{|\sigma(s,z)|} dz = \lim\limits_{(s,x)\to(t,a_n)} |H^n(s,x)| = +\infty \\ \\ \lim\limits_{(s,x)\to(t,b_n)} \displaystyle\int_{c_n}^{x} \frac{1}{|\sigma(s,z)|} dz = \lim\limits_{(s,x)\to(t,b_n)} |H^n(s,x)| = +\infty. \end{cases}$$

Remark 2.4.3. *1. Assumption $(\mathbf{H_1})$ is always verified if σ is* ***autonomous****, that is if $\sigma(t,x) = \sigma(x)$, for every $0 \leq t \leq 1$.*

2. Suppose that σ is locally Lipschitz in space, then assumption $(\mathbf{H_2})$ is satisfied, for every n in $\mathbb{N}$ such that $-\infty < a_n < b_n < \infty$. In fact, since $\sigma(t,a_n) = \sigma(t,b_n) = 0$ for every t, there will be a constant $c > 0$, such that

$$\begin{cases} |H^n(t,a)| \geq c\left(\log(c_n - a_n) - \log(a - a_n)\right), & \forall a \in (a_n, c_n) \\ |H^n(t,b)| \geq c\left(\log(b_n - c_n) - \log(b_n - b)\right), & \forall b \in (c_n, b_n). \end{cases}$$

If σ is locally Lipschitz in space, assumption $(\mathbf{H_2})$ reduces to the non-integrability condition above only when a_n or b_n are infinity. Even in that case, $(\mathbf{H_2})$ is just there to avoid technicalities related to the possible explosion of the solution. As far as uniqueness is concerned, it is not needed.

Under assumption $(\mathbf{H_2})$, for every n in $\mathbb{N}$ and t in $[0,1]$, $H^n(t,\cdot) : S^n \to \mathbb{R}$, admits an inverse $K^n(t,\cdot) : \mathbb{R} \to S^n$. If σ never vanishes then we will simply denote K^n with K. Clearly, for every n, K^n is the solution of the following equation

$$\begin{cases} \partial_y K^n(t,y) = \sigma(t, K^n(t,y)), & (t,y) \in [0,1] \times \mathbb{R} \\ K^n(t,0) = c_n. \end{cases}$$

For every $g : [0,1] \times S \to \mathbb{R}$, we will denote

$$\widetilde{g}(t,y,\omega) = \sum_{n=0}^{+\infty} I_{\{\eta \in S^n\}}(\omega) g(t, K^n(t,y)), \quad (t,y,\omega) \in [0,1] \times \mathbb{R} \times \Omega.$$

2.4.3 Some properties on the trajectories of a solution

The key point of our construction relies on the following property about trajectories of solutions holding if σ never vanishes. As we will see, in this case, a solution to equation (2.18) can be represented in terms of the primitive of σ^{-1} which can be defined on $\mathbb{R}$ at every instant. When this is not the case this property will be still true only *locally*, the local character depending on the initial condition η, and for its description we will need to consider the primitives of σ^{-1} on each connected component of S.

Lemma 2.4.4. *Let σ be in $C^{1,2}$, never vanishing and satisfying $(\mathbf{H_2})$, β be in $C^{0,1}$. Suppose that X is a solution to equation (2.18) adapted to $\mathbb{F}$. Then*

$$H(\cdot, X) = \xi + N,$$

where N is the $\mathbb{F}$-semimartingale

$$\begin{aligned} N &= H(0,\eta) + \int_0^{\cdot} \beta(s,X_s) dM_s + \int_0^{\cdot} \alpha(s,X_s) dV_s + \int_0^{\cdot} \partial_s H(s,X_s) ds \\ &+ \frac{1}{2} [\beta(\cdot, X), M] + \frac{1}{12} \int_0^{\cdot} \left(\sigma \partial_x^{(2)} \sigma + (\partial_x \sigma)^2 \right)(s,X_s) d[\xi,\xi,\xi]_s . \end{aligned}$$

Furthermore if σ is autonomous, then the result still holds even if X fulfills property 4. of definition 2.4.1, only for autonomous functions ψ.

Proof. Considering the first part of the statement we set $Y = H(\cdot, X)$. By assumption X is a *strong cubic variation* process. Since σ is of class $C^{1,2}$ H is in $C^{1,3}$, and so by applying Itô formula for *strong cubic variation* processes (see proposition 2.2.9), property 4. of definition 2.4.1 and the decomposition of the symmetric integral into a classical integral and a covariation term (see remark 2.2.7.2), we deduce the following expression for Y :

$$\begin{aligned} Y &= H(0,\eta) + \xi + \int_0^{\cdot} \beta(s,X_s) dM_s + \frac{1}{2} [\beta(\cdot, X), M] + \int_0^{\cdot} \alpha(s,X_s) dV_s \\ &+ \int_0^{\cdot} \partial_s H(s,X_s) ds - \frac{1}{4} \int_0^{\cdot} \left(\sigma^2 (\partial_x \sigma) \left(\partial_x^{(2)} H \right) \right)(s,X_s) d[\xi,\xi,\xi]_s \\ &- \frac{1}{12} \int_0^{\cdot} \left(\partial_x^{(3)} H(s,X_s) \right) d[X,X,X]_s . \end{aligned}$$

By property 3., Y is a *strong cubic variation* process as sum of a *strong cubic variation* process and of an $\mathbb{F}$-semimartingale. Moreover by remarks 2.2.5.1 and 2.2.5.4, $[Y,Y,Y] = [\xi,\xi,\xi]$. Proposition 2.3.4 tells us that

$$\begin{aligned}[X,X,X] &= [K(\cdot,Y),K(\cdot,Y),K(\cdot,Y)] = \int_0^{\cdot} (\partial_y K(s,Y_s))^3 \, d[Y,Y,Y]_s \\ &= \int_0^{\cdot} (\sigma(s,X_s))^3 \, d[\xi,\xi,\xi]_s .\end{aligned}$$

Using previous equality and computing the partial derivative of H with respect to x we finally reach the result.

□

Before dealing with the case of a possibly vanishing diffusion coefficient σ, we state the lemma below which will be useful for it.

Lemma 2.4.5. *Let $(X_t, 0 \le t \le 1)$ be a solution of equation (2.18) on the probability space $(\Omega,\mathcal{F},P)$. Let $B \in \mathcal{F}$, and τ a random time. Then, according to the notations of section 2.2, the following statements are true :*

1. *the process X^B, fulfills properties 2., 3. and 4. of definition 2.4.1 with respect to ξ^B, M^B and V^B on the space $\left(B,\mathcal{F}^B,P^B\right)$;*
2. *the processes ξ^τ fulfills properties 2., 3. and 4. of definition 2.4.1 with respect to ξ^τ, M^τ, and V^τ;*
3. *if the coefficients of equation (2.18) are autonomous, and X fulfills property 4. only for autonomous functions, then the process $X_{\cdot+\tau}$ fulfills properties 2., 3. of definition 2.4.1, and property 4. only for autonomous functions, with respect to the processes $\xi_{\cdot+\tau}$, $M_{\cdot+\tau} - M_\tau$, and $V_{\cdot+\tau}$.*

Proof. The first and the last point are direct consequences of lemma 2.2.11, 2.2.12 and lemma 2.2.13. Concerning the second one we clearly have that X^τ is a *strong cubic variation* process by lemma 2.2.13. By lemma 2.2.12 :

$$[\beta(\cdot,X),M]^\tau = [\beta^\tau, M^\tau],$$

with $\beta^\tau = \beta(\cdot\wedge\tau, X_{\cdot\wedge\tau})$. Moreover, the continuity of M and β ensures the convergence to zero, almost surely, of the sequence of processes

$$\begin{aligned} & \frac{1}{\varepsilon}\int_0^{\cdot} \left(\beta((s+\varepsilon)\wedge\tau, X_{(s+\varepsilon)\wedge\tau}) - \beta(s\wedge\tau, X_{s\wedge\tau})\right)\left(M_{(s+\varepsilon)\wedge\tau} - M_{s\wedge\tau}\right) ds \\ - & \frac{1}{\varepsilon}\int_0^{\cdot} \left(\beta(s+\varepsilon, X_{(s+\varepsilon)\wedge\tau}) - \beta(s, X_{s\wedge\tau})\right)\left(M_{(s+\varepsilon)\wedge\tau} - M_{s\wedge\tau}\right) ds \\ = & \frac{1}{\varepsilon}\int_{(\tau-\varepsilon)\wedge\cdot}^{\tau\wedge\cdot} \left(\beta(\tau,X_\tau) - \beta(s+\varepsilon,X_\tau)\right)\left(M_\tau - M_s\right) ds. \end{aligned}$$

This implies that $[\beta(\cdot,X^\tau),M^\tau] = [\beta^\tau,M^\tau] = [\beta(\cdot,X),M]^\tau$ exists and it has bounded variation.

If ψ is in $C^{1,\infty}([0,1]\times\mathbb{R})$, at the same way we have

$$\begin{aligned} & \frac{1}{\varepsilon}\int_0^{\cdot}\left(\psi(s\wedge\tau, X_{s\wedge\tau})-\psi(s,X_{s\wedge\tau})\right)\left(X_{(s+\varepsilon)\wedge\tau}-X_{(s-\varepsilon)\wedge\tau}\right)ds \\ = & \frac{1}{\varepsilon}\int_{\tau\wedge\cdot}^{(\tau+\varepsilon)\wedge\cdot}\left(\psi(\tau,X_\tau)-\psi(s,X_\tau)\right)\left(X_\tau-X_{s-\varepsilon}\right)ds, \end{aligned}$$

and the right-hand side of the equality converges uniformly to zero almost surely. Then

$$\int_0^{\cdot}\psi(s,X_s^\tau)d^\circ X_s^\tau=\left(\int_0^{\cdot}\psi(s,X_s)d^\circ X_s\right)^\tau,$$

and so using successively lemma 2.2.11 and 2.2.12 we obtain that X^τ fulfills also property 4. of definition 2.4.1.

□

To treat the case when σ is possibly vanishing we define

$$\nu^\sigma := I_{\{\eta\in S\}}(\omega)H(0,\eta),\qquad \text{for every } \omega \text{ in } \Omega.$$

Proposition 2.4.6. *Let σ be in $C^{1,2}$ satisfying assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$, and β be in $C^{0,1}$. Then if $(X_t, 0\le t\le 1)$ is a solution to equation (2.18), adapted to $\mathbb{F}$, and*

$$P\left(\{\eta\in S^n\}\right)=1,$$

for some $n\ge 0$, it holds

$$P\left(\{X_t\in S^n,\forall t\in[0,1]\}\right)=1,$$

and

$$H(\cdot,X)=\xi+N,\qquad \textit{for all } t \textit{ in } [0,1],\quad a.s.,$$

where N is the $\mathbb{F}$-semimartingale

$$\begin{aligned} N &= \nu^\sigma+\int_0^{\cdot}\beta(s,X_s)dM_s+\int_0^{\cdot}\alpha(s,X_s)dV_s+\int_0^{\cdot}\partial_s H(s,X_s)ds \\ &+ \frac{1}{2}\left[\beta(\cdot,X),M\right]+\frac{1}{12}\int_0^{\cdot}\left(\sigma\partial_x^{(2)}\sigma+(\partial_x\sigma)^2\right)(s,X_s)d\left[\xi,\xi,\xi\right]_s. \end{aligned}$$

Furthermore, if σ is autonomous, the result still holds even if X fulfills property 4. of definition 2.4.1 only for autonomous functions.

Proof. Let $D=\mathbb{R}/S$. For every h in $\mathbb{N}^*$, let τ^h be the first instant the distance between the process X and D becomes smaller than h^{-1} :

$$\tau^h=\inf\left\{t\in[0,1],\ s.t.\ d(X_t,D)\le h^{-1}\right\}\wedge 1,$$

where for every C closed set of $\mathbb{R}$, $x\mapsto d(x,C)=\inf_{r\in C}|x-r|$, is continuous and its support is equal to C. We denote, according to the notations of section 2.2, $\Omega^h=\left\{\tau^h>0\right\}$, $\mathcal{F}_t^h=\mathcal{F}_t^{\Omega^h}$, $\mathbb{F}^h=\left(\mathcal{F}_t^h\right)_{0\le t\le 1}$, $P^h=P^{\Omega^h}$, and for every stochastic process Y on

Ω, we put $Y^h = (Y^{\Omega^h})^{\tau^h}$. Since $P(\eta \in S) = 1$ there exists $k > 0$ such that $P(\Omega^h) > 0$, for every $h \geq k$.

Let $h \geq k \vee (d(c_n, D))^{-1}$, be fixed. We observe that Ω^h is $\mathcal{F}_0$-measurable; hence $\mathbb{F}^h$ belongs to $\mathcal{S}(M^h)$. Suppose that X is a solution to equation (2.18). By lemma 2.4.5.1 and 2.4.5.2, X^h is a solution of

$$\begin{cases} d^\circ X_t^h = \sigma(t, X_t^h)\left[d^\circ \xi_t^h + \beta(t, X_t^h) d^\circ M_t^h + \alpha(t, X_t^h) dV_t^h\right], \quad 0 \leq t \leq 1 \\ X_0^h = \eta^h, \end{cases}$$

on the probability space $\left(\Omega^h, \mathcal{F}^h, P^h\right)$. Moreover, by construction,

$$P^h\left(\left\{X_t^h \in S^{n,h}, \forall t \in [0,1]\right\}\right) = 1,$$

with $S^{n,h} = \left\{x \in S^n, \text{ s.t. } d(x, D) \geq h^{-1}\right\}$. Let $\sigma^h : [0,1] \times \mathbb{R} \to \mathbb{R}$, be a function with the same regularity as σ, never vanishing, and agreeing with σ on $S^{n,h}$ together its first and second derivatives in x, and its first derivative in t. Then X^h is still a solution of

$$\begin{cases} d^\circ X_t^h = \sigma^h(t, X_t^h)\left[d^\circ \xi_t^h + \beta(t, X_t^h) d^\circ M_t^h + \alpha(t, X_t^h) dV_t^h\right], \quad 0 \leq t \leq 1 \\ X_0^h = \eta^h. \end{cases}$$

If X fulfills property 4. only for autonomous functions, then, by lemma 2.2.12, X^h carries on doing it with respect to the processes ξ^h, M^h, and V^h, even after having replaced σ by σ^h. In particular lemma 2.4.4 can be applied in both of these two cases. Consequently if

$$H^{n,h}(t, x) = \int_{c_n}^{x} \frac{1}{\sigma^h(t, z)} dz,$$

on Ω^h it holds P^h almost surely :

$$\begin{aligned} H^{n,h}(\cdot, X^h) &= H^{n,h}(0, \eta^h) + \xi^h + \int_0^{\cdot} \beta(s, X_s^h) dM_s^h + \int_0^{\cdot} \alpha(s, X_s^h) dV_s^h \\ &+ \int_0^{\cdot} \partial_s H^{n,h}(s, X_s^h) ds + \frac{1}{2}\left[\beta(\cdot, X^h), M^h\right] \\ &+ \frac{1}{12}\int_0^{\cdot} \left(\sigma^h \partial_x^{(2)} \sigma^h + (\partial_x \sigma^h)^2\right)(s, X_s^h) d\left[\xi^h, \xi^h, \xi^h\right]_s. \end{aligned}$$

We remark that $\left\{\tau^h > 0\right\} \subseteq \left\{\eta^h \in S^{n,h}\right\}$, and that $h \geq (d(c_n, D))^{-1}$ implies that c_n belongs to $S^{n,h}$. Furthermore, if x belongs to $S^{n,h}$, then $[c_n, x] \subseteq S^{n,h}$. Therefore $H^{n,h}(t, x) = H(t, x)$, and $\partial_t H^{n,h}(t, x) = \partial_t H(t, x)$, for every x in $S^{n,h}$. Then using lemma 2.2.11, lemma 2.2.12, and by similar reasonings to those already used in the proof of lemma 2.4.5, we obtain the following equality holding P^h almost surely on Ω^h :

$$H(t, X_t) = \xi_t + N_t, \quad t \leq \tau^h, \tag{2.20}$$

with

$$\begin{aligned} N &= \nu^\sigma + \int_0^{\cdot} \beta(s, X_s) dM_s + \int_0^{\cdot} \alpha(s, X_s) dV_s + \int_0^{\cdot} \partial_s H(s, X_s) ds \\ &+ \frac{1}{2}\left[\beta(\cdot, X), M\right] + \frac{1}{12}\int_0^{\cdot} \left(\sigma \partial_x^{(2)} \sigma + (\partial_x \sigma)^2\right)(s, X_s) d\left[\xi, \xi, \xi\right]_s. \end{aligned}$$

Let $\tau = \lim_{h\to+\infty} \tau^h$. Since $\bigcup_{h=0}^{+\infty} \Omega^h = \Omega$, almost surely, we get, for $t = \tau^h$

$$\lim_{h\to+\infty} H(\tau^h, X_{\tau^h}) = \xi_\tau + N_\tau, \quad a.s..$$

On the other hand, thanks to the continuity of X, $d(X_\tau, D) = 0$ on $\{\tau < 1\}$. This imply

$$\{\tau < 1\} \cup (\{\tau = 1\} \cap \{X_\tau \in D\}) \subseteq \{X_\tau \in \partial D\}.$$

Furthermore by assumption $(\mathbf{H_2})$

$$\{X_\tau \in \partial D\} \subseteq \left\{ \lim_{h\to+\infty} |H(\tau^h, X_{\tau^h})| = +\infty \right\} \subseteq \left\{ \lim_{h\to+\infty} H(\tau^h, X_{\tau^h}) = \xi_\tau + N_\tau \right\}^c.$$

Then it must hold $P(\{\tau < 1\} \cup (\{\tau = 1\} \cap \{X_1 \in D\})) = 0$. We thus have obtained the first part of our result since

$$(\{\tau < 1\} \cup (\{\tau = 1\} \cap \{X_1 \in D\}))^c = \{X_t \in S^n, \forall t \in [0,1]\}.$$

To complete the proof it is sufficient to take the limit for $h \to +\infty$ in (2.20).

□

Proposition 2.4.7. *Let σ, α and β be autonomous, σ in $C^{1,2}$, satisfying assumption $(\mathbf{H_2})$, and β in $C^{0,1}$. Let X be a solution to (2.18) adapted to $\mathbb{F}$. Then if $P(\{\eta \in D\}) = 1$*

$$P(\{X_t \in D, \forall t \in [0,1]\}) = 1,$$

and so $X_t = \eta, \forall t \in [0,1]$, almost surely.

Proof. For every $h \in \mathbb{N}^*$, we consider the first instant the distance between the process X and D becomes greater than h^{-1} :

$$\tau^h = \inf\left\{t \in [0,1] \ s.t. \ d(X_t, D) \geq h^{-1}\right\} \wedge 1,$$

and we put $Y_t^h = Y_{t+\tau^h}$, for $Y = X, \xi, V$, and $M_t^h = M_{t+\tau^h} - M_{\tau^h}$. We observe that X^h is adapted to $\mathbb{F}^h = \left(\mathcal{F}_t^h\right)_{t\in[0,1]}$, where

$$\mathcal{F}_t^h = \left\{A \in \mathcal{F} | A \cap \left\{\tau^h \leq s - t\right\} \in \mathcal{F}_s, \forall s \geq t\right\},$$

and that $\mathbb{F}^h$ belongs to $\mathcal{S}(M^h)$ (see problem 3.27 of [38]). Then combining lemma 2.4.5.3 and proposition 2.4.6 we find that

$$P\left(\{X_{\tau^h} \in S^m\} \cap \left\{X_t \in S^m, \forall t \geq \tau^h\right\}\right) = P(\{X_{\tau^h} \in S^m\}), \quad \forall h, m \in \mathbb{N}^*.$$

In particular, since $\tau^h \leq \tau^k$ when $h \geq k$,

$$P(\{X_{\tau^h} \in S^m\} \cap \{X_{\tau^k} \in S^n\}) = 0, \quad \forall n \neq m, h \geq k.$$

This implies

$$P(\{X_{\tau^k} \in S^n\}) = P\left(\bigcap_{h\geq k} \{X_{\tau^h} \in S^n\}\right), \quad \forall n \in \mathbb{N}, \quad \forall k \in \mathbb{N}^*. \tag{2.21}$$

Furthermore, again by proposition 2.4.6 we get

$$H(X_1) - H(X_{\tau^h}) - Y^h = 0, \quad a.s. \text{ on } \{X_{\tau^h} \in S^n\}, \quad \forall h \in \mathbb{N}^*, \tag{2.22}$$

with

$$\begin{aligned} Y^h &= \xi_1 + \int_{\tau^h}^1 \beta(X_s)dM_s + \int_{\tau^h}^1 \alpha(X_s)dV_s + \frac{1}{2}\left([\beta(X), M]_1 - [\beta(X), M]_{\tau^h}\right) \\ &+ \frac{1}{12}\int_{\tau^h}^1 \left(\sigma\partial_x^{(2)}\sigma + (\partial_x\sigma)^2\right)(X_s)d\,[\xi,\xi,\xi]_s\,. \end{aligned}$$

Using assumption ($\mathbf{H_2}$), and equality (2.21) we thus find

$$P\left(\{X_{\tau^k} \in S^n\}\right) = P\left(\bigcap_{h\geq k}\{X_{\tau^h} \in S^n\}\right) = 0, \quad \forall k, n \in \mathbb{N}.$$

since in the subspace $\bigcap_{h\geq k}\{X_{\tau^h} \in S^n\}$ we are allowed to take the limit in equality (2.22). This holds for every k and n in $\mathbb{N}^*$, so we get

$$P\left(\{X_t \in D, \forall t \in [0,1]\}^c\right) \leq P\left(\bigcup_{k>0}\{X_{\tau^k} \in S\}\right) = 0.$$

□

2.4.4 Existence and uniqueness

Proposition 2.4.8. *Suppose that there exists a filtration $\mathbb{H} \supseteq \mathbb{F}$, with respect to which the vector (ξ, M) satisfies the hypothesis $(\mathcal{D})$. Let σ be in $C^{1,2}$, satisfying assumptions ($\mathbf{H_1}$) and ($\mathbf{H_2}$), β be in $C^{0,1}$. If $(Y_t, 0 \leq t \leq 1)$ is an $\mathbb{F}$-adapted solution of the stochastic differential equation*

$$\begin{aligned} Y &= \nu^\sigma + \xi + \int_0^\cdot \widetilde{\beta}(s,Y_s)dM_s + \int_0^\cdot \widetilde{\alpha}(s,Y_s)dV_s + \int_0^\cdot \widetilde{\partial_s H}(s,Y_s)ds \\ &+ \frac{1}{2}\int_0^\cdot \widetilde{\partial_x\beta}\widetilde{\beta}\widetilde{\sigma}(s,Y_s)d\,[M,M]_s + \frac{1}{2}\int_0^\cdot \widetilde{\partial_x\beta}\widetilde{\sigma}(s,Y_s)d\,[M,\xi]_s \\ &+ \frac{1}{12}\int_0^\cdot (\widetilde{\sigma}\widetilde{\partial_x^{(2)}\sigma} + \widetilde{(\partial_x\sigma)}^2)(s,Y_s)d\,[\xi,\xi,\xi]_s\,, \end{aligned} \tag{2.23}$$

then the process

$$X = \sum_{n=0}^{\infty} I_{\{\eta\in S^n\}}K^n(\cdot, Y) + I_{\{\eta\in D\}}\eta \tag{2.24}$$

is a solution of equation (2.18) adapted to $\mathbb{F}$; Conversely, if $P(\{\eta \in S\}) = 1$, or σ, β, and α are autonomous and $(X_t, 0 \leq t \leq 1)$ is a solution to equation (2.18), adapted to $\mathbb{F}$, then the process

$$Y = I_{\{\eta\in S\}}H(\cdot, X) + I_{\{\eta\in D\}}\xi$$

solves equation (2.23), and it is $\mathbb{F}$-adapted.

Proof. Let $(Y_t, 0 \leq t \leq 1)$ be an $\mathbb{F}$-adapted solution of equation (2.23). Define the process $(X_t, 0 \leq t \leq 1)$ as in formula (2.24). X is a continuous process with $X_0 = \eta$. Furthermore Y is a *strong cubic variation* process as the sum of ξ and a semimartingale (recall remark 2.2.5.1), and so, by proposition 2.3.4, the process $K^n(\cdot, Y)$, for every n, has a finite strong cubic variation too. Then X has the same property and

$$\begin{aligned}
[X, X, X] &= \sum_{n=0}^{\infty} I_{\{\eta \in S^n\}} \left[K^n(\cdot, Y), K^n(\cdot, Y), K^n(\cdot, Y)\right] \\
&= \sum_{n=0}^{\infty} I_{\{\eta \in S^n\}} \int_0^{\cdot} (\sigma(s, K^n(s, Y_s))^3 \, d\,[\xi, \xi, \xi]_s \\
&= \int_0^{\cdot} (\sigma\,(s, X_s))^3 \,[\xi, \xi, \xi]_s \,,
\end{aligned}$$

where for the last equality we used the fact that $\sigma(t, X_t) I_{\{\eta \in D\}} = 0$, for every $0 \leq t \leq 1$. Thanks to hypothesis $(\mathcal{D})$, Y is the sum of R and the process $\widetilde{Q} = Y - R$, with $\widetilde{Q} = Q + \int_0^{\cdot} h_s dM_s + \widetilde{V}$, h continuous and $\mathbb{H}$-adapted, and $\widetilde{V}$ having bounded variation. Proposition 2.3.9 implies that $(\widetilde{Q}, M)$ has all its mutual brackets. Then the the vector (Y, M) verifies the hypothesis $(\mathcal{D})$, with respect to $\mathbb{H}$. By proposition 2.3.18 $[\beta(\cdot, X), M]$ has bounded variation since it is equal to

$$\sum_{n=0}^{\infty} I_{\{\eta \in S^n\}} \left[\beta(\cdot, K^n(\cdot, Y)), M\right],$$

with

$$\begin{aligned}
\left[\beta(\cdot, K^n(\cdot, Y)), M\right] &= \int_0^{\cdot} (\partial_x \beta \sigma)\,(s, K^n(s, Y_s)) d\,[Y, M]_s \\
&= \int_0^{\cdot} (\partial_x \beta \sigma)\,(s, K^n(s, Y_s)) d\,[\xi, M]_s \\
&+ \int_0^{\cdot} (\beta \partial_x \beta \sigma)\,(s, K^n(s, Y_s)) d\,[M, M]_s \,,
\end{aligned} \tag{2.25}$$

on $\{\eta \in S^n\}$. Let ψ of class $C^{1,\infty}$. We first remark that, since both classical and symmetric integral have a local character (see [48] for the classical integral and 2.2.11 for the symmetric one), for every n in $\mathbb{N}^*$ on $\{\eta \in S^n\}$ it holds :

$$\begin{aligned}
Y &= \nu^{\sigma} + \xi + \int_0^{\cdot} \beta(s, K^n(s, Y_s)) dM_s + \int_0^{\cdot} \alpha(s, K^n(s, Y_s)) dV_s \\
&+ \int_0^{\cdot} \partial_s H(s, K^n(s, Y_s)) ds + \frac{1}{2} \int_0^{\cdot} \partial_x \beta \beta \sigma(s, K^n(s, Y_s)) d\,[M, M]_s \\
&+ \frac{1}{2} \int_0^{\cdot} \partial_x \beta \sigma(s, K^n(s, Y_s)) d\,[M, \xi]_s \\
&+ \frac{1}{12} \int_0^{\cdot} (\sigma \partial_x^{(2)} \sigma + (\partial_x \sigma)^2)(s, K^n(s, Y_s)) d\,[\xi, \xi, \xi]_s \,.
\end{aligned}$$

We apply Itô formula for *strong cubic variation* processes to write

$$X = \sum_{n=0}^{\infty} I_{\{\eta \in S^n\}} X^n + I_{\{\eta \in D\}} \eta,$$

with

$$\begin{aligned} X^n &= \eta + \int_0^{\cdot} \partial_s K^n(s, Y_s) ds + \int_0^{\cdot} \partial_y K^n(s, Y_s) d^{\circ} Y_s \\ &- \frac{1}{12} \int_0^{\cdot} \partial_y^{(3)} K^n(s, Y_s) d\,[Y, Y, Y]_s \,. \end{aligned}$$

Using equality (2.25) we can write on $\{\eta \in S^n\}$,

$$\begin{aligned} Y &= \nu^{\sigma} + \xi + \int_0^{\cdot} \beta(s, K^n(s, Y_s)) d^{\circ} M_s + \int_0^{\cdot} \alpha(s, K^n(s, Y_s)) dV_s \\ &+ \int_0^{\cdot} \partial_s H(s, K^n(s, Y_s)) ds + \frac{1}{12} \int_0^{\cdot} (\sigma \partial_x^{(2)} \sigma + (\partial_x \sigma)^2)(s, K^n(s, Y_s)) d\,[\xi, \xi, \xi]_s \,. \end{aligned}$$

Deriving with respect to s the equality $H(s, K^n(s, y)) = y$, we obtain the relation

$$\partial_s K^n(s, y) = -\sigma(s, K^n(s, y)) \partial_s H(s, K^n(s, y)),$$

which combined with equation (2.23), the equalities

$$\partial_y K^n(s, y) = \sigma(s, K^n(s, y)), \quad \partial_y^{(2)} \left(\sigma(s, K^n(s, y))\right) = \partial_y^{(3)} K^n(s, y),$$

and corollary 2.3.19, gives the following expression for X^n on $\{\eta \in S^n\}$:

$$X^n = \eta + \int_0^{\cdot} \sigma(s, X_s^n) d^{\circ} \xi_s + \int_0^{\cdot} (\sigma\beta)(s, X_s^n) d^{\circ} M_s + \int_0^{\cdot} (\sigma\alpha)(s, X_s^n) dV_s.$$

Coefficients appearing in the last expression for X^n and function ψ are regular enough to use successively lemma 2.2.10 and corollary 2.3.19 to get on $\{\eta \in S^n\}$:

$$\begin{aligned} \int_0^{\cdot} \psi(t, X_t^n) d^{\circ} X_t^n &= \int_0^{\cdot} (\psi\sigma)(t, X_t^n) \left[d^{\circ} \xi_t + \beta(t, X_t^n) d^{\circ} M_t + \alpha(t, X_t^n) dV_t\right] \\ &- \frac{1}{4} \int_0^{\cdot} (\partial_x \sigma) \left(\sigma^2\right) (\partial_x \psi)(t, X_t^n) d\,[\xi, \xi, \xi]_t \,. \end{aligned}$$

The conclusion follows since $\int_0^{\cdot} \psi(t, X_t) d^{\circ} X_t = \sum_{n=0}^{\infty} I_{\{\eta \in S^n\}} \int_0^{\cdot} \psi(t, X_t^n) d^{\circ} X_t^n$, almost surely.

We consider the second part of the statement. By proposition 2.4.6

$$Y = H(\cdot, X) = \xi + N, \quad \text{on } \{\eta \in S\}.$$

The vector (ξ, N, M) fulfills the hypothesis $(\mathcal{D})$ with respect to $\mathbb{H}$. Indeed $N = \int_0^{\cdot} h_s dM_s + \widetilde{V}$, with h continuous and $\mathbb{H}$-adapted, and $\widetilde{V}$ with bounded variation. By proposition 2.3.18

$$I_{\{\eta \in S\}} \left[\beta(\cdot, X), M\right] = \sum_{n=0}^{\infty} I_{\{\eta \in S^n\}} \left[\beta(\cdot, K^n(\cdot, \xi + N)), M\right],$$

with

$$\begin{aligned}[\beta(\cdot,K^n(\cdot,\xi+N),M] &= \int_0^{\cdot}(\partial_x\beta\sigma(s,K^n(s,\xi_s+N_s)))\,d[\xi,M]_s \\ &+ \int_0^{\cdot}(\beta\partial_x\beta\sigma(s,K^n(s,\xi_s+N_s)))\,d[M,M]_s\,.\end{aligned}$$

Therefore, on $\{\eta\in S^n\}$, N is more explicitly given by the following expression

$$\begin{aligned}N &= \nu^\sigma+\int_0^{\cdot}\beta(s,K^n(s,Y_s))dM_s+\int_0^{\cdot}\alpha(s,K^n(s,Y_s))dV_s \qquad (2.26)\\ &+ \frac{1}{2}\int_0^{\cdot}(\sigma\beta\partial_x\beta)\,(s,K^n(s,Y_s))d[M,M]_s+\frac{1}{2}\int_0^{\cdot}(\sigma\partial_x\beta)\,(s,K^n(s,Y_s))d[M,\xi]_s \\ &+ \int_0^{\cdot}\partial_sH(s,K^n(s,Y_s))ds+\frac{1}{12}\int_0^{\cdot}\left(\sigma\partial_x^{(2)}\sigma+(\partial_x\sigma)^2\right)(s,K^n(s,Y_s))d[\xi,\xi,\xi]_s\,.\end{aligned}$$

Putting expression (2.26) in the equality

$$Y=I_{\{\eta\in S\}}(\xi+N)+I_{\{\eta\in D\}}\xi=\xi+\sum_{n=0}^{\infty}I_{\{\eta\in S^n\}}N$$

we achieve the proof of the proposition.

□

Theorem 2.4.9. *Suppose that there exists a filtration $\mathbb{H}\supseteq\mathbb{F}$, with respect to which the vector (ξ,M) satisfies the hypothesis $(\mathcal{D})$. Let σ satisfy assumptions $(\mathbf{H_1}),(\mathbf{H_2})$, and the following hypotheses*

$$(\mathbf{H_3})\begin{cases}(i) & \sigma \text{ is in } C^{1,2},\\ (ii) & \partial_x^{(2)}\sigma \text{ is locally Lipschitz in } x, \text{ uniformly in } t\,,\\ (iii) & \sup_{(t,x)\in[0,1]\times S^n}|\partial_t\log(|\sigma(t,x)|)|\le a^n,\forall n\in\mathbb{N}\\ (iv) & \left(|\partial_x\sigma|^2+\left|\sigma\partial_x^{(2)}\sigma\right|\right)(t,x)\le a_n\left(1+|H^n(t,x)|\right),\\ & (t,x)\in[0,1]\times S^n,\forall n\in\mathbb{N}\end{cases}$$

for some sequences $(a_n)_{n\in\mathbb{N}}$, in $\mathbb{N}$; let β and α verify

$$(\mathbf{H_4})\begin{cases}(i) & \beta \text{ is in } C^{0,1} \text{ and it is bounded},\\ (ii) & \partial_x\beta \text{ and } \alpha \text{ are locally Lipschitz in } x, \text{ uniformly in } t\\ (iii) & (|\sigma|\,|\partial_x\beta|+|\alpha|)\,(t,x)\le a_n\left(1+|H^n(t,x)|\right),(t,x)\in[0,1]\times S^n,\end{cases}$$

for all n in $\mathbb{N}$. Then if $P(\{\eta\in S\})=1$ or that σ, β, and α are autonomous, equation (2.18) has a unique $\mathbb{F}$-adapted solution given by

$$X=\sum_{n=0}^{\infty}I_{\{\eta\in S^n\}}K^n(\cdot,Y)+I_{\{\eta\in D\}}\eta,$$

where Y is the unique $\mathbb{F}$-adapted solution to equation (2.23).

Remark 2.4.10. *We emphasize that hypothesis* $(\mathbf{H_4})$ *has to be satisfied by* α *a.s.. In the sequel we will implicitly use this convention.*

Proof. The result follows from the existence and uniqueness of equation (2.23). The last holds since assumptions $(\mathbf{H_3})$ and $(\mathbf{H_4})$ imply the local Lipschitz continuity and the linear growth property of the coefficients of equation (2.23), which are sufficient conditions to ensure its existence and uniqueness (see [24], pag. 29, lemma 34). In fact, the functions

$$(t,y) \mapsto \beta(t, K^n(t,y)),\ \sigma\partial_x^{(2)}\sigma(t, K^n(t,y)),\ \alpha(t, K^n(t,y)),\ \left(\partial_x\sigma(t, K^n(t,y))\right)^2,$$

and

$$(t,y) \mapsto \sigma\partial_x\beta(t, K^n(t,y)),$$

have linear growth thanks to the boundness of β, (iv) of $(\mathbf{H_3})$ and (iii) of $(\mathbf{H_4})$; moreover they are locally Lipschitz being the composition of continuous functions differentiable with continuity or locally Lipschitz in y. The map $(t,y) \mapsto \partial_t H^n(t, K^n(t,y))$ is locally Lipschitz, being differentiable with continuity with respect to y. By (iii) of $(\mathbf{H_3})$, $|\partial_t H^n(t,x)| \leq a_n |H^n(t,x)|$, which implies the linear growth for $(t,y) \mapsto \partial_t H(t, K^n(t,y))$.

□

Recalling examples 2.3.8 and 2.3.7 one can prove the following results.

Corollary 2.4.11. *Suppose that there exist two adapted processes* Q *and* R*, such that* $\xi = R + Q$*,* R *is* $\mathcal{F}_0$*-measurable and* (Q, M) *has all its mutual brackets. Let* σ*,* β*, and* α *verify the regularity assumptions of proposition 2.4.9. Then if the* $P(\{\eta \in S\}) = 1$*, or the coefficients are autonomous, there exists a unique* $\mathbb{F}$*-adapted solution to equation (2.18).*

Corollary 2.4.12. *Suppose that there exist two adapted processes* Q *and* R*, such that* $\xi = R + Q$*, with* R *independent from* M*,* (Q, M) *having all its mutual brackets, and* $\mathbb{F} \subseteq \mathbb{H}$*, being* $\mathcal{H}_t = \sigma(M_s, 0 \leq s \leq t) \vee \sigma(R)$*, for every* $0 \leq t \leq 1$*. Let* σ*,* β*, and* α *verify the regularity assumptions of proposition 2.4.9. Then if the* $P(\{\eta \in S\}) = 1$*, or the coefficients are autonomous, there exists a unique* $\mathbb{F}$*-adapted solution to equation (2.18).*

If σ is bounded from below from a positive constant we can solve with our methods an equation already studied in [21], where the diffusion coefficients does not appear as multiplier factor. There the coefficient β was equal to zero, σ autonomous and of class $C^{1,3}$. The authors needed to introduce the notion of *strong cubic vector Itô processes* in the definition 2.4.1, requiring more that the finite cubic variation of a solution X. In particular existence and uniqueness were proved to hold in a smaller class than the ours, with more regularity on σ.

2.4.5 On the uniqueness of the integral equation

We aim here at adding hypotheses on the coefficients driving equation (2.18) to find a suitable class of processes among which its solution, in the sense described in definition 2.4.1 exists, and it is the unique solution to the the integral equation (2.19).

Remark 2.4.13. *1. Let* Z *be in* $\mathcal{C}^2_\xi(\mathbb{H})$ *and* ψ *in* $C^{1,4}$*, with* $\partial_t\psi$ *in* $C^{0,2}$*. Then the process* $(\psi(t, Z_t), 0 \leq t \leq 1)$ *is in* $\mathcal{C}^2_\xi(\mathbb{H})$*.*

2. *Let*
$$\left(X^k(t,x), 0 \le t \le 1, x \in \mathbb{R}\right)_{k\in\mathbb{N}}$$
be a sequence of C^2 $\mathbb{H}$-Itô-semimartingale fields, of this form
$$X^k(t,x) = f^k(x) + \sum_{i=1}^{n} \int_0^t a^{k,i}(s,x)dN_s^i + \sum_{j=1}^{m} \int_0^t b^{k,j}(s,x)dV_s^j,$$
and $(\Omega_k)_{k\in\mathbb{N}}$ be a sequence of subspaces of Ω in $\mathcal{H}_0$, with $\cup_{k=0}^{\infty}\Omega_k = \Omega$, a.s.. Then the random field
$$Y(t,x) = \sum_{k=0}^{\infty} I_{\Omega_k} X^k(t,x),$$
is a C^2 $\mathbb{H}$-Itô-semimartingale field of the form
$$Y(t,x) = f(x) + \sum_{i=1}^{n} \int_0^t a^i(s,x)dN_s^i + \sum_{j=1}^{m} \int_0^t b^j(s,x)dV_s^j,$$
with
$$f(x) = \sum_{k=0}^{\infty} I_{\Omega_k} f^k(x),\ \ a^i(t,x) = \sum_{k=0}^{\infty} I_{\Omega_k} a^{k,i}(t,x),\ \ b^j(t,x) = \sum_{k=0}^{\infty} I_{\Omega_k} b^{k,j}(t,x).$$

Proposition 2.4.14. *Suppose that there exists a filtration $\mathbb{H} \supseteq \mathbb{F}$, with respect to which the vector (ξ, M) satisfies the hypothesis $(\mathcal{D})$. Let σ β and α, satisfy hypotheses of proposition 2.4.9, with furthermore σ in $C^{1,4}$, $\partial_t\sigma$ in $C^{0,2}$, β in $C^{1,3}$, and $\partial_t\beta$ in $C^{0,1}$. Then if $P(\{\eta \in S\}) = 1$, or α, σ, and β are autonomous, there exists a unique $\mathbb{F}$-adapted solution to the integral equation (2.19) in the space $\mathcal{C}^2_{\xi,\eta}(\mathbb{H})$ of all processes in $\mathcal{C}^2_{\xi}(\mathbb{H})$, starting at η.*

Proof. The existence was proved in proposition 2.4.9. Consider, in fact, the process Y which is the unique solution of equation (2.23). Classical Itô formula for semimartingales applied to the function K^n and the semimartingale $N = Y - \xi$, shows that the random field $(K^n(t, x + N_t), t \in [0,1], x \in \mathbb{R})$ is a C^2 $\mathbb{H}$-Itô-semimartingale field driven by the local martingale M. Therefore by remark 2.4.13 X is in $\mathcal{C}^2_{\xi,\eta}(\mathbb{H})$.

Regarding uniqueness we show that an integral solution in $C^2_{\xi,\eta}(\mathbb{H})$ is a solution in the sense described in definition 2.4.1. Let Z be the random field in $\mathcal{C}^2_{\xi,\eta}(\mathbb{H})$ such that $X = Z(\cdot,\xi)$, where X is a solution to equation (2.19). Condition 1. is fulfilled by hypothesis. Since ξ is an $\mathbb{H}$-adapted *strong cubic variation* process and Z is a C^2 $\mathbb{H}$-Itô-semimartingale field, by proposition 2.3.4 X satisfies condition 2.. By classical Itô formula $(\beta(t, Z(t,x)), 0 \le t \le 1, x \in \mathbb{R})$ is a C^1 $\mathbb{H}$-Itô-semimartingale field driven by a vector of local martingales $(N^1, ..., N^n)$ such that the vector $(\xi, N^1, ..., N^n)$ satisfies the hypothesis $(\mathcal{D})$ with respect to $\mathbb{H}$. By definition there exist two $\mathbb{H}$-adapted processes $\bar{R}$ and $\bar{Q}$ such that $\xi = \bar{R} + \bar{Q}$, $\left(\bar{Q}, N^1, \cdots, N^n\right)$ has all its mutual brackets, and $\bar{R}_{\varepsilon+\cdot}$ is $\mathbb{H}$-adapted. By corollary 2.3.11, $\left[\bar{R}, M\right] = 0$. This implies the existence of $\left[M, \bar{Q}\right]$ which equals $[\xi, M]$. Then $(\xi, N^1, \cdots, N^n, M)$ verifies the hypothesis $(\mathcal{D})$ with respect to $\mathbb{H}$, and by proposition 2.3.16 condition 3. is established. Since $\partial_t\sigma$ belongs to $C^{0,2}$, it follows from the details of proofs that if condition 4. is fulfilled for functions ψ in $C^{1,\infty}$ with $\partial_t\psi$ in $C^{0,2}$, previous results about uniqueness remain true. Let then ψ be in $C^{1,\infty}$, with $\partial_t\psi$ in $C^{0,2}$. X is a solution of the integral equation, so we can write

$$\begin{aligned}\int_0^t \psi(s,X_s)d^\circ X_s &= \int_0^t \widehat{\psi}(s,\xi_s)d^\circ\left(\int_0^s \widehat{\sigma}(r,\xi_r)d^\circ\xi_r\right)\\ &+ \int_0^t \widehat{\psi}(s,\xi_s)d^\circ\left(\int_0^s \widehat{\beta\sigma}(r,\xi_r)d^\circ M_r\right)\\ &+ \int_0^t \widehat{\psi}(s,\xi_s)d^\circ\left(\int_0^s \widehat{\alpha\sigma}(r,\xi_r)dV_r\right),\end{aligned}$$

with the notation $\widehat{\psi}(t,x) = \psi(t,Z(t,x))$, for every function $\psi : [0,1]\times\mathbb{R}$. As already remarked before the processes $(\widehat{\psi}(t,\xi_t), 0\le t\le 1)$, as well as $(\widehat{\sigma}(t,\xi_t), 0\le t\le 1)$ are in $\mathcal{C}^2_{\xi,\eta}(\mathbb{H})$ so as to let us apply proposition 2.3.25. At the same way the random field $(\widehat{\beta\sigma}(t,x), 0\le t\le 1, x\in\mathbb{R})$ has the properties needed in corollary 2.3.19. Then we obtain

$$\begin{aligned}\int_0^\cdot \psi(s,X_s)d^\circ X_s &= \int_0^\cdot (\psi\sigma)(s,X_s)d^\circ\xi_s + \int_0^\cdot (\psi\beta\sigma)(s,X_s)d^\circ M_s\\ &+ \int_0^\cdot (\psi\alpha\sigma)(s,X_s)dV_s\\ &- \frac{1}{4}\int_0^\cdot (\partial_x\psi)(\partial_x\sigma)(s,X_s)\left(\partial_x Z(s,\xi_s)\right)^2 d\left[\xi,\xi,\xi\right]_s.\end{aligned}$$

By proposition 2.3.4

$$\int_0^\cdot \partial_x\psi\partial_x\sigma(s,X_s)\left(\partial_x Z(s,\xi_s)\right)^2 d\left[\xi,\xi,\xi\right]_s = \int_0^\cdot (\partial_x\psi\partial_x\sigma)(s,X_s)d\left[X,X,\xi\right]_s.$$

Finally, by multi-linearity of the 3-covariation application, and remarks 2.2.5.1 and 2.3.23

$$\begin{aligned}[X,X,\xi] &= \left[\int_0^\cdot \widehat{\sigma}(s,\xi_s)d^\circ\xi_s, \int_0^\cdot \widehat{\sigma}(s,\xi_s)d^\circ\xi_s, \xi\right]\\ &= \int_0^\cdot (\sigma(s,X_s))^2 d\left[\xi,\xi,\xi\right]_s.\end{aligned}$$

and so condition 4. is proved to hold. This leads to the conclusion of the proof.

□

2.4.6 The finite quadratic variation case

In this section we suppose that the vector (ξ, M) has all its mutual brackets. In particular that ξ is a *finite quadratic variation* process. We observe that, under this assumption, the vector (ξ, M) satisfies the hypothesis $(\mathcal{D})$ with respect to the filtration $\mathbb{H}=\mathbb{F}$. Moreover $\mathcal{C}^k_\xi(\mathbb{F})$ reduces to the set of all the C^k $\mathbb{F}$-Itô-semimartingale fields driven by a vector of semimartingales $(N^1,\cdots,N^n)$ such that $(\xi,N^1,\cdots,N^n)$ has all its mutual brackets.

Results obtained in previous section can be improved regarding the regularity required for the *diffusion* coefficient σ, by using techniques which are similar to those already developed in [56] and [24] about stochastic calculus with respect to finite quadratic variation processes. More precisely Itô formula for *finite quadratic variation* processes holds for C^2 functions of the space variable, which allows us to reduce of one the degree of regularity of σ.

Definition 2.4.15. *A continuous stochastic process $(X_t, 0 \le t \le 1)$ will be said solution to equation (2.18) if $X_0 = \eta$, the vector (X, M) has all its mutual brackets, and for every ψ in $C^{1,\infty}$ it holds :*

$$\int_0^{\cdot} \psi(s, X_s) d^\circ X_s = \int_0^{\cdot} \psi\sigma(s, X_s) \left[d^\circ \xi_s + \beta(s, X_s) d^\circ M_s + \alpha(s, X_s) dV_s\right].$$

Remark 2.4.16. *Definition 2.4.1 and 2.4.15 are equivalent. It is sufficient to use proposition 2.3.18, and recall that $[\xi, \xi, \xi] = 0$.*

Similarly to the finite cubic variation case we state the following results.

Proposition 2.4.17. *Let σ be in $C^{1,1}$, satisfying assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$, β be in $C^{0,1}$. If $(Y_t, 0 \le t \le 1)$ is an $\mathbb{F}$-adapted solution of the stochastic differential equation*

$$\begin{aligned} Y &= \nu^\sigma + \xi + \int_0^{\cdot} \widetilde{\beta}(s, Y_s) dM_s + \int_0^{\cdot} \widetilde{\alpha}(s, Y_s) dV_s + \int_0^{\cdot} \widetilde{\partial_t H}(s, Y_s) ds \\ &+ \frac{1}{2} \int_0^{\cdot} \widetilde{\partial_x \beta} \widetilde{\beta} \widetilde{\sigma}(s, Y_s) d\left[M, M\right]_s + \frac{1}{2} \int_0^{\cdot} \widetilde{\partial_x \beta} \widetilde{\sigma}(s, Y_s) d\left[M, \xi\right]_s, \end{aligned} \quad (2.27)$$

then the process $X = \sum_{n=0}^{\infty} I_{\{\eta \in S^n\}} K^n(\cdot, Y) + I_{\{\eta \in D\}} \eta$ is a solution of equation (2.18) adapted to $\mathbb{F}$. Conversely, if $P(\{\eta \in S\}) = 1$, or σ, β, and α are autonomous and $(X_t, 0 \le t \le 1)$ is a solution to equation (2.18), adapted to $\mathbb{F}$, then the process $Y = I_{\{\eta \in S\}} H(\cdot, X) + I_{\{\eta \in D\}} \xi$ solves equation (2.27), and it is $\mathbb{F}$-adapted.

Proposition 2.4.18. *Let σ be in C^1, satisfy assumptions $(\mathbf{H_1}), (\mathbf{H_2})$, and such that*

$$\sup_{(t,x) \in [0,1] \times S^n} |\partial_t \log(|\sigma(t, x)|)| \le a^n, \forall n \in \mathbb{N}$$

for some sequences $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^+$; let β and α verify hypothesis $(\mathbf{H_4})$. Then if $P(\{\eta \in S\}) = 1$ or σ, β, and α are autonomous, equation (2.18) has a unique $\mathbb{F}$-adapted solution.

We aim at comparing the results obtained with our method with those achieved in [24], and [56]. There σ was not a multiplier coefficient. Then the comparison can be made if σ is bounded from below from a positive constant. In such a case equations studied by those authors are particular cases of equation (2.18), where the symmetric integral is replaced by the *forward* one, see [54], for definition.

We remember that, for two continuous stochastic processes X and Y, if the symmetric integral, $\int_0^{\cdot} X_s d^\circ Y_s$, and the forward integral, $\int_0^{\cdot} X_s d^- Y_s$, exist, then $\frac{1}{2}[X, Y]$ exists and

$$\int_0^{\cdot} X_s d^\circ Y_s = \int_0^{\cdot} X_s d^- Y_s + \frac{1}{2}[X, Y].$$

Using this relation, under assumptions of proposition 2.4.17, we can state this equivalence between the solution to equation (2.18) in the symmetric and the forward sense. This notion of solution in definition 2.4.15 has to be adapted replacing the symmetric integral with the forward one.

A process X is a solution of equation

$$\begin{cases} d^-X_t = \sigma(t,X_t)\left[d^-\xi_t + \beta(t,X_t)d^-M_t + \alpha(t,X_t)d^-V_t\right] \\ X_0 = \eta, \end{cases} \tag{2.28}$$

if and only if it solves

$$\begin{cases} d^\circ X_t = \sigma(t,X_t)\left[d^\circ\xi_t + \beta(t,X_t)d^\circ M_t + \alpha(t,X_t)dV_t\right] \\ -\frac{1}{2}\sigma(t,X_t)\left[\gamma^1(t,X_t))dV_t^1 + \gamma^2(t,X_t))dV_t^2 + \gamma^3(t,X_t))dV_t^3\right] \\ \\ X_0 = \eta, \end{cases} \tag{2.29}$$

with $\gamma^1 = \partial_x\sigma, \gamma^2 = 2\partial_x\sigma\beta + \sigma\partial_x\beta, \gamma^3(t,x) = \partial_x\sigma\beta^2 + \sigma\beta\partial_x\beta$, and $V^1 = [\xi,\xi], V^2 = [\xi,M], V^3 = [M,M]$.

This equivalence and proposition 2.4.18 imply the following.

Remark 2.4.19. *Suppose that, besides the hypotheses of proposition 2.4.18, $\partial_x\sigma$ is locally Lipschitz in x, uniformly in t, and*

$$|\partial_x\sigma|(t,x) \le a_n\left(1 + |H^n(t,x)|\right), (t,x) \in [0,1]\times S^n, \forall n \in \mathbb{N}.$$

Then equation (2.28) has a unique solution. Existence and uniqueness are ensured by equation (2.29). Moreover the solution is given by $X = \sum_{n=0}^{\infty} I_{\{\eta\in S^n\}}K^n(\cdot,Y) + I_{\{\eta\in D\}}\eta$ where $(Y_t, 0 \le t \le 1)$ is the unique solution of

$$\begin{aligned} Y &= \nu^\sigma + \xi + \int_0^\cdot \widetilde{\beta}(s,Y_s)dM_s + \int_0^\cdot \widetilde{\alpha}(s,Y_s)dV_s + \int_0^\cdot \widetilde{\partial_s H}(s,Y_s)ds \\ &- \frac{1}{2}\int_0^\cdot \widetilde{\partial_x\sigma}(s,Y_s)d[\xi,\xi]_s - \int_0^\cdot \widetilde{\partial_x\sigma}\widetilde{\beta}(s,Y_s)d[M,\xi]_s \\ &- \frac{1}{2}\int_0^\cdot \widetilde{\partial_x\sigma}\widetilde{\beta}^2(s,Y_s)d[M,M]_s. \end{aligned} \tag{2.30}$$

Remark 2.4.20. *If we assume β only continuous, bounded and locally Lipschitz, equation (2.30) still has a unique solution. Nevertheless X could fail to solve equation (2.29); indeed the bracket $[\beta(\cdot,X),M]$ may not exist under this weaker condition.*

In order to avoid this additional conditions on β, equation (2.28) has to be studied directly using stochastic calculus with respect to finite quadratic variation processes and forward integrals instead of symmetric ones. By these methods it is possible to show the following result.

Proposition 2.4.21. *Suppose that σ is in $C^{1,1}$ and it satisfies assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$, that β is continuous and bounded, $\beta, \alpha, \partial_x\sigma$ are locally Lipschitz in x uniformly in t, and moreover that*

$$\begin{cases} \sup\limits_{(t,x)\in[0,1]\times\mathbb{R}} |\partial_t \log(|\sigma(t,x)|)| < +\infty; \\ (|\partial_x\sigma| + |\alpha|)(t,x) \le a_n (1 + |H^n(t,x)|), (t,x) \in [0,1] \times S^n, \forall n \in \mathbb{N}. \end{cases}$$

Then equation (2.28) has a unique solution.

Moreover, as in the finite cubic variation case, we can also state the following.

Proposition 2.4.22. *Let σ β and α, satisfy hypotheses of proposition 2.4.21, with furthermore σ in $C^{1,3}$, and $\partial_t\sigma$ in $C^{0,1}$. Then, if $P(\{\eta \in S\}) = 1$, or α, σ, and β are autonomous, there exists a unique $\mathbb{F}$-adapted solution to the integral equation*

$$X = \eta + \int_0^{\cdot} \sigma(s, X_s) d^-\xi_s + \int_0^{\cdot} \sigma\beta(s, X_s) d^- M_s + \int_0^{\cdot} \sigma\alpha(s, X_s) dV_s,$$

in the space $\mathcal{C}^1_{\xi,\eta}(\mathbb{F})$ of all processes in $\mathcal{C}^1_{\xi}(\mathbb{F})$, starting at η.

In [24] the authors show the existence and uniqueness of the integral equation (2.28), supposing σ *autonomous* and in $C^{1,4}$, in the class $\mathcal{C}^2_{\xi,\eta} \subset \mathcal{C}^1_{\xi,\eta}$. In [56] an equation of type (2.28) is studied with semimartingale coefficient β equal to zero, and an *autonomous* diffusion coefficient. There σ is of class C^3, bounded with its partial derivative $\partial_x\sigma$. Moreover the sense of solution is more restrictive in that it involves the notion of *vector Itô processes* which are not necessary to introduce for the application our method.

2.4.7 The Hölder continuous case

We intend to apply the methods developed in previous sections to the study of the stochastic differential equation (2.18) when the processes ξ and V have γ-Hölder continuous paths, with $\frac{1}{2} < \gamma < 1$, the semimartingale coefficient is equal to zero, and $V_t = t$:

$$\begin{cases} d^\circ X_t = \sigma(t, X_t) [d^\circ \xi_t + \alpha(t, X_t) dt], \\ X_0 = \eta. \end{cases} \tag{2.31}$$

Remark 2.4.23. *This method could be extended to the case $V = \int_0^{\cdot} \psi_s ds$, with $\psi \in L^2([0,1])$. Indeed, this would imply V γ-Hölder continuous with $\gamma > \frac{1}{2}$.*

We will see that in this case the use of an Itô formula available for processes having Hölder continuous paths will let to reduce the regularity of σ. If $0 < \gamma < 1$, C^γ will denote the Banach space of all γ-Hölder continuous functions with the norm

$$||f||_\gamma = \sup_{s,t\in[0,1] s\neq t} \frac{|f(t) - f(s)|}{|t - s|^\gamma} + ||f||_\infty.$$

In this context we will look for existence and uniqueness of integral solutions with γ-Hölder continuous paths. We first recall some results about integral calculus with respect to Hölder functions contained in [22] and [64].

Lemma 2.4.24. *Let f and g be in C^1, with $f(0) = 0$, and $\alpha + \gamma > 1$. Then the following inequality holds :*

$$\left| \int_0^t f(r) dg(r) \right| \leq C \, ||f||_\alpha \, ||g||_\gamma \, t^{1+\varepsilon}$$

for some positive constant C and $0 < \varepsilon < \alpha + \gamma - 1$.

Corollary 2.4.25. *Let f and g be in C^1, and $\alpha + \gamma > 1$. Then the following inequality holds, for every t, s in $[0, 1]$:*

$$\left| \int_s^t f(r) dg(r) - f(s)(g(t) - g(s)) \right| \leq C \, ||f||_\alpha \, ||g||_\gamma \, |t - s|^{1+\varepsilon}$$

for some positive constant C and $0 < \varepsilon < \alpha + \gamma - 1$. In particular $\int_0^\cdot f dg$ is a γ-Hölder function.

Corollary 2.4.25 implies the following.

Proposition 2.4.26. *If $\alpha + \gamma > 1$, the map $F : (f, g) \mapsto \int_0^\cdot f dg$ defined on $C^1 \times C^1$, with values in C^γ, admits a unique continuous extension to $C^\alpha \times C^\gamma$.*

Proof. Let (f, g) and (h, k) in $C^\alpha \times C^\gamma$. The map F is bilinear, therefore

$$||F(f, g) - F(h, k)||_\gamma \leq ||F(f - h, k)||_\alpha + ||F(h, g - k)||_\gamma \, .$$

Let s, t be in $[0, 1]$. By corollary 2.4.25

$$|F(f - h, g)(t) - F(f - h, g)(s)| \leq C \, ||f - h||_\alpha \, ||g||_\gamma \, |t - s|^\gamma \, ,$$

and similarly

$$|F(h, g - k)(t) - F(h, g - k)(s)| \leq C \, ||h||_\alpha \, ||g - k||_\gamma \, |t - s|^\gamma \, .$$

This immediately implies

$$||F(f - h, g)||_\gamma + ||F(h, g - k)||_\gamma \leq 2C \left(||g||_\gamma \vee ||h||_\alpha \right) ||(f, g) - (h, k)||_{C^\alpha \times C^\gamma} \, .$$

□

The unique continuous extension of F will be called the ***Young*** integral and denoted with $\int_0^\cdot f d^y g$, for every f in C^α and g in C^γ.

Remark 2.4.27. *If f and h are in C^α and g in C^γ, with $\alpha + \gamma > 1$, we have*

$$\int_0^\cdot f d^y \left(\int_0^\cdot h d^y g \right) = \int_0^\cdot f h d^y g.$$

The equality holds for (f, g) in $C^1 \times C^1$, and it can be extended to $C^\alpha \times C^\gamma$ by density arguments.

L.C. Young [62] introduced that integral in a more general setting, *i.e.* for f, g having respectively p and q variation with $p^{-1} + q^{-1} = 1$. It can be proved that the Young integral $\int_0^\cdot f d^y g$ agrees with the symmetric integral $\int_0^\cdot f d^\circ g$, see [57], and that it is a *Riemann-Stieltjes* type integral as specified in the following proposition.

Proposition 2.4.28. *Let f be in C^{α} and g in C^{γ}, with $\alpha+\gamma>1$. Then for every $0\le t\le 1$*

$$\lim_{\delta\to 0}\sum_{i=0}^{n-1} f(t_i)\left(g(t_{i+1})-g(t_i)\right)$$

converges to $\int_0^{\cdot} f d^y g$ when the mesh δ of the partition

$$\pi=\{0=t_0<t_1<...<t_n=t\},$$

goes to zero.

Proposition 2.4.28 permits to identify the Young integral and the integral of [64], see Th.4.2.1. We thus are allowed to use the following Itô formula established in [64], Th. 4.3.1, pag.351.

Proposition 2.4.29. *Let f be in C^{γ} and F be in $C^1([0,1]\times\mathbb{R})$ such that $t\mapsto\partial_x F(t,f(t))$ belongs to C^{α} with $\alpha+\gamma>1$. Then*

$$F(t,f_t)=F(0,f_0)+\int_0^t \partial_x F(s,f_s)d^{\circ}f_s+\int_0^t \partial_s F(s,f_s)ds.$$

We will need the hypothesis

$$(\mathbf{H_1'})\quad\left\{\begin{array}{l}\sigma \text{ is in } C^{1,0};\\ |\sigma(t,x)-\sigma(t,y)|\le c_n|x-y|^{\delta},\quad \forall t\in[0,1],\quad |x|+|y|\le n,\end{array}\right.$$

for every n in $\mathbb{N}$, with $c,c_n>0$, $\delta>\frac{1}{\gamma}-1$.

We state the proposition, in the Hölder case, which is equivalent to proposition 2.4.8, in the finite cubic variation case.

Proposition 2.4.30. *Let σ satisfy $(\mathbf{H_1})$, $(\mathbf{H_1'})$, and $(\mathbf{H_2})$. Suppose that either $P(\eta\in S)=1$, or α and σ are autonomous. Then equation (2.31) has a unique solution with γ-Hölder continuous paths, if and only if the following stochastic differential equation has a unique solution*

$$Y=\nu^{\sigma}+\xi+\int_0^{\cdot}\left(\widetilde{\partial_s H}+\widetilde{\alpha}\right)(s,Y_s)ds. \tag{2.32}$$

We observe that since ξ is γ-Hölder with γ greater than $\frac{1}{2}$, its cubic variation its equal to zero, then equation (2.32) agrees with equation (2.23).

Remark 2.4.31. *Hypothesis $(\mathbf{H_2})$ on the the zeros of σ, is indeed necessary for uniqueness. Suppose $\alpha=0$, σ autonomous and vanishing only at some point x_0 with $\frac{1}{\sigma}$ being integrable around x_0. Then problem*

$$\left\{\begin{array}{lcl} d^{\circ}X_t &=& \sigma(X_t)d^{\circ}\xi_t\\ X_0 &=& x_0,\end{array}\right.$$

has at east two solutions $X_t^1\equiv x_0$ and $X_t^2=K(\xi_t)$, where $K=H^{-1}$ and $H(x)=\int_{x_0}^{x}\frac{1}{\sigma(z)}dz$.

Corollary 2.4.32. *Suppose that in addition to the assumptions of proposition (2.4.30), α is bounded and locally Lipschitz in x uniformly in t, and that σ verifies*

$$\sup_{(t,x)\in[0,1]\times S^n} |\partial_t \log(|\sigma(t,x)|)| \leq a_n$$

for some sequence of positive number $(a_n)_{n\in\mathbb{N}}$. Then equation (2.31) has a unique solution.

2.4.8 The case of the fractional Brownian motion

In this section we investigate a significant particular case. We suppose that $\xi = \left(B_t^H, 0 \leq t \leq 1\right)$ is a *fractional Brownian motion* on the given filtered probability space (the filtration $\mathbb{F}$ being generated by B^H and the sets of zero probability), with Hurst parameter H strictly larger than $\frac{1}{2}$. Furthermore, we assume that η is deterministic, and $\alpha : [0,1] \times \mathbb{R} \to \mathbb{R}$, is measurable and locally bounded in x, uniformly in t. It is well known that B^H has λ-Hölder continuous paths, for every $\lambda < H$, on $[0,1]$, almost surely. The information about the law B^H allows us to make use of some recent results about uniqueness and existence of a stochastic differential equation driven by a *fractional Brownian motion* with drift equal to 1, which can be found in [44]. More precisely, there the authors establish existence and uniqueness of the integral equation

$$Y_t = y + B_t^H + \int_0^t b(s, Y_s)ds, \quad 0 \leq t \leq 1, \quad y \in \mathbb{R},$$

under this regularity assumption on b :

$$(\mathbf{H_4'}) \quad |b(t,y) - b(s,x)| \leq C\left(|x-y|^\alpha + |t-s|^\beta\right), \tag{2.33}$$

for some positive constant C, with $1 > \alpha > 1 - \frac{1}{2H}$, $\beta > H - \frac{1}{2}$.

Imposing conditions ensuring that the assumption above is satisfied by the coefficients of equation (2.32) we get the following corollary.

Corollary 2.4.33. *Let σ be bounded, satisfying assumptions $(\mathbf{H_1})$, and $(\mathbf{H_2})$.*

1. *If $P(\eta \in D) = 1$ and both α and σ are autonomous, then the integral equation*

$$X = \eta + \int_0^\cdot \sigma(s, X_s)d^\circ B_s^H + \int_0^\cdot \sigma\alpha(s, X_s)ds, \tag{2.34}$$

has the unique solution $X \equiv \eta$.

2. *Suppose that $P(\eta \in S^n) = 1$, for some n in $\mathbb{N}$. Let α satisfy hypothesis $(\mathbf{H_4'})$, σ hypothesis $(\mathbf{H_1'})$ and*

$$(\mathbf{H_3'}) \quad \begin{cases} (i) \int_{S^n} |g(t,x) - g(s,x)|\, dx \leq a_n |t-s|^\beta, \\ (ii) \int_x^y \sup_{t\in[0,1]} |g(t,z)|\, dz \leq a_n |x-y|^\alpha, \quad x, y \in S^n, x \leq y \\ (iii) \int_{S^n} \sup_{t\in[0,1]} |g(t,z)|\, dz < +\infty, \end{cases}$$

with

$$g(t,x) = \frac{\partial_t \sigma(t,x)}{(\sigma(t,x))^2}, \quad (t,x) \in [0,1] \times S,$$

for some positive constant a_n. Then the integral equation (2.34) has a unique solution.

Proof. Suppose $\eta \in S^n$. Condition (iii) of $(\mathbf{H_3'})$ and the boundness of σ imply that $(t,y) \mapsto K^n(t,y)$ is Lipschitz in x, uniformly in t, and Lipschitz in t uniformly in x. Thanks to conditions (i) and (ii), $(t,x) \mapsto \partial_t H^n(t,x)$ fulfills assumption $(\mathbf{H_4'})$ for some C positive constant. Then equation (2.32) has a unique solution by the mentioned result of [44]. Proposition 2.4.30 permits to conclude. If $\eta \in D$ uniqueness follows by proposition 2.4.30.

□

2.4.9 Existence in the case of Brownian motion

If $H = \frac{1}{2}$, and $B^H = B$ is a Brownian motion, supposing σ only continuous, it is possible to find a solution to equation

$$\begin{cases} d^\circ X_t = \sigma(t, X_t)\,[d^\circ B_t + \alpha(t,X_t)dt] \\ X_0 = \eta. \end{cases} \tag{2.35}$$

This can be done using Itô formula permitting to expand C^1 functions of *reversible* semimartingales proved in [55]. We recall the result established by [55], see also [26], in the case of Brownian motion.

Definition 2.4.34. *A semimartingale X is a* **reversible semimartingale** *if the process $\hat{X} = (X_{1-t}, 0 \le t \le 1)$ is a semimartingale.*

Proposition 2.4.35. *Let $X = \left(X^1, ..., X^d\right)$ be a vector of continuous reversible semimartingales, and f in $C^1(\mathbb{R}^d)$. Then*

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \partial_i f(X_s) d^\circ X_s^i.$$

Then we can state the following.

Theorem 2.4.36. *Let σ satisfy $(\mathbf{H_1})$, $(\mathbf{H_2})$, $\alpha : [0,1]\times\mathbb{R} \to \mathbb{R}$ be measurable and bounded, and η deterministic. Suppose that for every n in $\mathbb{N}$, if η is in S^n*

$$\sup_{(t,x)\in[0,1]\times S^n} |\partial_t H^n(t,x)| < +\infty.$$

Then equation (2.35) has a solution.

Proof. If $\eta \in D$ $X_t \equiv \eta$, is a solution. Suppose $\eta \in S^n$, for some n in $\mathbb{N}$. Equation

$$Y = H^n(0,\eta) + B + \int_0^{\cdot} (\alpha(t, K^n(s, Y_s)) - \partial_s H(s, K^n(s, Y_s)))\, ds,$$

admits a solution since the function $(t,y) \mapsto \alpha(t, K^n(t,y)) - \partial_t H^n(t, K^n(t,y))$ is measurable and bounded, see Th.35 of [49]. Using Girsanov theorem we find that Y is a Brownian motion under a probability measure P^* equivalent to P. Therefore Y is a reversible semimartingale, see example of pag. 3 of [55]. Then Itô formula for reversible semimartingales provides a solution to equation (2.35) :

$$X = K^n(\cdot, Y) = \eta + \int_0^{\cdot} \sigma(t, X_t) d^\circ B_t + \int_0^{\cdot} \sigma\alpha(t, X_t) dt.$$

□

Remark 2.4.37. *We remark that for such a solution X, $\int_0^{\cdot} \sigma(s, X_s) d^\circ B_s$ is not a proper Stratonovich integral since $\sigma(\cdot, X)$ may not be a semimartingale.*

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RÉSUMÉ en français : Dans la première partie de cette thèse nous appliquons le calcul via régularisation à l'étude d'un marché où le processus des prix d'un actif risqué n'est pas une semimartingale mais simplement à variation quadratique finie. Cette condition est réalisée lorsque le prix de l'actif est admis dans la classe $\mathcal{A}$ de toutes les stratégies admissibles, et devient réaliste si la condition de non-arbitrage sur l'ensemble de toutes les stratégies simples prévisibles n'est pas plausible. Cette situation est vérifiée, par exemple, lorsque l'agent est un initié ou si $\mathcal{A}$ est restreinte. Nous fournissons des exemples de portefeuilles autofinancés et introduisons une notion de $\mathcal{A}$-martingale. Un calcul relatif à celle-ci est développé. La condition de non-arbitrage parmi toutes les stratégies dans $\mathcal{A}$ est récupérée si le processus des prix de l'actif risqué est une $\mathcal{A}$-martingale. Nous abordons le problème de la viabilité du marché, de la couverture et de la maximisation de l'utilité de la richesse terminale. La deuxième partie de la thèse est consacrée à l'étude d'une équation différentielle stochastique unidimensionnelle dirigée par une semimartingale mélangée à un processus à variation cubique finie. Le développement de la méthode utilisée nous conduit à des résultats significatifs dans l'analyse du calcul via régularisation. En particulier, une formule de type Itô-Wentzell relative aux processus à variation cubique finie est établie et la structure des processus weak-Dirichlet par rapport à la filtration brownienne est clarifiée. Nous démontrons, par une approche similaire, l'existence et l'unicité d'une équation dirigée par un processus hölder-continu dans l'espace. En utilisant une formule d'Itô pour les semimartingales réversibles nous prouvons l'existence d'une solution lorsque le processus dirigeant l'équation est le mouvement brownien et le coefficient de diffusion est juste continu.

TITRE en anglais : Calculus via regularization and financial applications

RÉSUMÉ en anglais : In the first part of this thesis we apply stochastic calculus via regularization to model financial markets when the price of the risky asset is not a semimartingale. The lack of the semimartingale property is justified if arbitrage is possible among simple predictable strategies. That is the case if the investor is an insider or the class $\mathcal{A}$ of admissible strategies is restricted. We assume that prices only have finite quadratic variation. That assumption is verified if the risky asset price itself has to be admitted in the class of all admissible strategies. We provide examples of self-financing strategies and we introduce the notion of $\mathcal{A}$-martingale process. A calculus with respect those processes is developed. We show that the no-arbitrage condition is recovered if the price process is an $\mathcal{A}$-martingale. We face some problems such as viability, hedging and utility maximization.
The second part of the thesis is devoted to the study of a one-dimensional stochastic differential equation driven by a strong cubic variation process and a semimartingale. The implementation of our method leads us to improve some results about stochastic calculus via regularization. In particular an Itô-Wentzell type formula related to finite cubic variation processes is established and the structure of weak Dirichlet processes is clarified when the underlying filtration is Brownian. Our approach applies to prove existence and uniqueness when the driven process is Hölder continuous in space. Using a Itô formula for reversible semimartingale we prove existence of a solution when the equation is driven by a Brownian motion and the diffusion coefficient is only continuous.

DISCIPLINE : Mathématiques (Paris 13), Matematica finanziaria e assicurativa (Pisa).

MOTS-CLÉS : calcul via régularisation, équations différentielles stochastiques, cours non-semimartingale d'un actif risqué, intégrale *forward* (*symetrique*), variation quadratique (cubique) finie, processus weak Dirichlet, mouvement brownien fractionnaire, arbitrage, valorisation, couverture, maximisation d'utilité.

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