Three studies on fragmentation and coalescent processes

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Chapitre I

Présentation générale

Les phénomènes de fragmentation dans lesquels des particules se brisent récursivement en des particules de plus en plus petites, apparaissent dans de nombreux domaines des sciences et des techniques, que ce soit dans l'industrie (par exemple le concassage des roches ou minerais, la dégradation des polymères), dans les sciences de la Terre (fragmentation du magma dans les éruptions volcaniques), dans les sciences de la vie (fragmentation moléculaire) ou en physique des particules. La théorie mathématique des processus de fragmentation s'inspire de ces problématiques pour dégager un modèle apte à décrire les propriétés génériques d'un tel phénomène. Symétriques des phénomènes de fragmentation, les phénomènes de coalescence (ou coagulation) dans lesquels des particules s'agglomèrent récursivement créant des particules de tailles de plus en plus grandes sont aussi présents dans de nombreux domaines des sciences et des techniques. Étant donnée cette symétrie entre la fragmentation et la coalescence, il n'est pas surprenant que les théories mathématiques des processus de fragmentation et des processus de coalescence soient liées.

Au début du XX$^{\text{ème}}$ siècle, Smoluchowski [61] fut le premier à s'intéresser au problème de la coalescence. Ses travaux portaient alors sur l'étude d'un gaz soumis à un phénomène de coagulation et la dynamique de ce procédé était déterminée par les solutions d'une équation aux dérivées partielles. De nombreux physiciens et chimistes développèrent son modèle en l'élargissant et en incorporant éventuellement des phénomènes de fragmentation (voir [28] pour plus de références). Les premiers à envisager une étude probabiliste du phénomène de coagulation furent, dans les années 1970, Marcus [45] et Lushnikov [42] ; puis, en 1998, Evans et Pitman [30] donnèrent une construction générale de ces processus (voir [1] pour une revue de la théorie mathématique de la coalescence). Ces processus décrivaient des situations où des particules peuvent s'agglomérer deux à deux avec un taux dépendant de leur masse. Peu de temps après Pitman [54] et Sagitov [58] donnèrent une construction d'un nouveau processus de coalescence qui permettait alors à plusieurs (voire une infinité) de particules de se regrouper au même instant, mais leur taux de coagulation devenait alors indépendant de leur masse. Enfin cette approche fut généralisée par Schweinsberg [60] et Möhle & Sagitov [48].

En ce qui concerne la théorie de la fragmentation, le premier à avoir posé le problème de façon probabiliste fut Kolmogoroff [40] en 1941 et les premiers résultats sur le sujet sont dus à son élève Filippov [32]. À la fin des années 80, Brennan et Durrett [24, 25] s'intéressèrent de nouveau à ce sujet en étudiant des cas de fragmentations binaires. Puis, dans une série de papiers, Bertoin [10, 11, 14] développa une théorie de la fragmentation comparable à la théorie existante pour la coalescence et permettant des dislocations créant un nombre quelconque (voire infini) de fragments. Naïvement on pourrait penser qu'en retournant en temps un processus de coalescence, l'on obtient un processus de fragmentation. En fait, la théorie de la fragmentation impose une indépendance d'évolution entre les différents fragments, pour cette raison, en retournant un processus de coalescence, on n'obtient pas, en général, une fragmentation. Cependant il existe deux cas où une telle dualité a été mise en évidence : la première fut exhibée par Pitman [54] dans le cas de la coalescence de Bolthausen et Sznitman et la seconde, dans le cas de la coalescence additive, par Aldous et Pitman [2] puis précisée par Bertoin [8] et Miermont [46] . Une partie de ce travail est ainsi consacrée à l'étude de ces deux processus en tant que fragmentation. Après un rappel, dans la section I.1, de définitions et résultats existants concernant fragmentation et coalescence, les principaux résultats de ce travail sont présentés dans les sections I.2 à I.4 avant d'être développés dans les chapitres II à IV.

Les chapitres II et III de cette thèse étudient les deux exemples de processus de fragmentation qui interviennent naturellement par dualité dans l'analyse de certains processus de coalescence. Quant au chapitre IV, il définit un processus de fragmentation dans lequel un ordre sur les différents blocs est pris en compte et montre dans quelle mesure la théorie des fragmentations peut s'étendre à ce cas là.

I.1 Résultats préliminaires

I.1.1 Les fragmentations de masse

Les processus de fragmentation sont donc des modèles qui décrivent l'évolution d'un objet qui se disloque de façon aléatoire et répétée au cours du temps. On s'intéresse à la masse des différents fragments créés. Dans toute la suite on supposera toujours que l'on part initialement d'une seule particule de masse 1. Pour étudier les fragmentations, il est donc naturel d'introduire l'espace des partitions de masse

$$\mathcal{S}^{\downarrow} = \{\mathbf{s} = (s_1, s_2, \ldots),\ s_1 \geq s_2 \geq \ldots \geq 0, \sum_{i\geq 1} s_i \leq 1\},$$

qu'on munit de la topologie de la convergence uniforme. Chacun des termes d'une suite $\mathbf{s} \in \mathcal{S}^{\downarrow}$ représente ainsi la masse d'un des fragments. Un processus de fragmentation est défini de la manière suivante :

Définition I.1

Soit $(F(t), t \geq 0)$ un processus de Markov à valeurs dans $\mathcal{S}^{\downarrow}$, continu en probabilité et issu de $(1, 0, 0, \ldots)$. Pour tout réel α, on dira que F est une fragmentation de masse auto-similaire d'indice α si pour tous $t, t' \geq 0$, la loi de $F(t + t')$ conditionnellement à $F(t) = (s_1, s_2, \ldots)$ est celle du réarrangement décroissant des termes des suites $s_i F^i(s_i^{\alpha} t')$, où les F^i sont des copies de F indépendantes.

Cette définition impose plusieurs restrictions sur les modèles que l'on étudie : d'une part ils sont sans mémoire, ils ne peuvent donc prendre en compte une quelconque usure qui rendrait une fragmentation plus probable ; d'autre part les différents fragments évoluent de façons indépendantes et leurs lois ne dépendent que de leurs tailles initiales. Cette propriété est appelée propriété de branchement. Lorsque le paramètre α sera égal à 0, on parlera alors de fragmentation homogène (en espace). À quelques exceptions près, on s'intéressera ici à des fragmentations homogènes, bien que la plupart des résultats puisse s'étendre au cas des fragmentations auto-similaires.

I.1.2 Caractéristiques d'une fragmentation

Une façon de construire une fragmentation consiste, par exemple, à se donner une mesure ν sur $\mathcal{S}^{\downarrow}$ de masse finie. On part alors initialement d'une particule de masse 1, et au bout d'un temps exponentiel de paramètre $\nu(\mathcal{S}^{\downarrow})$, cette particule se scinde en un ensemble de particules de masses $s_1, s_2, \ldots$ avec une probabilité $\nu(d\mathbf{s})/\nu(\mathcal{S}^{\downarrow})$ où $\mathbf{s} = (s_1, s_2, \ldots)$. Puis les particules obtenues suivent à leur tour la même évolution.

Bertoin [14] a montré que plus généralement, la loi d'une fragmentation homogène est entièrement déterminée par un couple (c, ν) où c est un nombre positif ou nul et ν est une mesure sur $\mathcal{S}^{\downarrow}$ satisfaisant $\int_{\mathcal{S}^{\downarrow}}(1 - s_1)\nu(d\mathbf{s}) < \infty$ et $\nu(1, 0, 0, \ldots) = 0$. Le nombre c est appelé coefficient d'érosion et correspond à la perte de masse déterministe qui se produit au cours du temps. La mesure ν est appelée mesure de dislocation et décrit la façon dont les blocs se cassent comme dans l'exemple ci-dessus. En ce qui concerne les fragmentations auto-similaires, Bertoin a démontré qu'elles se déduisent des fragmentations homogènes par changement de temps. La loi d'une telle fragmentation est donc caractérisée par un triplet (c, ν, α) où α est l'indice d'auto-similarité.

I.1.3 Les partitions échangeables

Un résultat important dans la théorie des fragmentations a été de remarquer que l'on peut aussi définir des fragmentations à valeurs dans les partitions de $\mathbb{N}$ et que les lois de celles-ci sont en bijection avec les lois des fragmentations de masse. Ce résultat s'appuie sur la théorie des partitions échangeables de Kingman, que nous allons ici rappeler.

Soit $\mathcal{P}_\infty$ l'ensemble des partitions de $\mathbb{N}$. Dans toute la suite, pour $\pi \in \mathcal{P}_\infty$, on notera $\pi_1, \pi_2, \ldots$ les blocs de π indexés par ordre croissant de leur plus petit élément. Soit $\pi \in \mathcal{P}_\infty$ et soit σ une permutation de $\mathbb{N}$. On définit alors la partition $\sigma\pi$ par

$$i \overset{\sigma\pi}{\sim} j \Longleftrightarrow \sigma i \overset{\pi}{\sim} \sigma j,$$

où $i \overset{\pi}{\sim} j$ signifie que les entiers i et j sont dans le même bloc de π. Si π est une variable aléatoire à valeurs dans $\mathcal{P}_\infty$, on dira que π est échangeable si pour toute permutation σ de $\mathbb{N}$, $\sigma\pi$ et π ont la même loi.

Une façon de créer une loi échangeable sur les partitions est donnée par la méthode des "boîtes de peinture" de Kingman [37] : on se donne $\mathbf{s} \in \mathcal{S}^{\downarrow}$ et on note $s_0 = 1 - \sum_{i \geq 1} s_i$. Puis on se donne une suite de variables aléatoires $(X_i)_{i \geq 1}$ indépendantes de même loi donnée par $\mathbb{P}(X_1 = i) = s_i$ pour $i \geq 0$. On définit la partition π par

$$i \overset{\pi}{\sim} j \Longleftrightarrow X_i = X_j \neq 0,$$

pour $i \neq j$. Alors π est clairement une partition aléatoire échangeable. La loi de cette partition sera désormais notée $\rho_{\mathbf{s}}$. Le théorème de Kingman énonce que toute loi d'une partition aléatoire échangeable est un mélange de boîtes de peinture. Plus précisément, si μ est la loi d'une partition aléatoire échangeable alors il existe une mesure de probabilité ν sur $\mathcal{S}^{\downarrow}$ telle que

$$\mu(\cdot) = \int_{\mathcal{S}^{\downarrow}} \rho_{\mathbf{s}}(\cdot)\nu(d\mathbf{s}).$$

On en déduit, par la loi des grands nombres, que si $\pi = (\pi_1, \pi_2, \ldots)$ est une partition aléatoire échangeable, la limite

$$|\pi_i| = \lim_{n\to\infty} \frac{Card(\pi_i \cap \{1, 2, \ldots, n\})}{n}$$

existe presque sûrement pour tout $i \in \mathbb{N}$. La quantité $|\pi_i|$ est appelée la fréquence asymptotique du bloc π_i et on notera $|\pi|^{\downarrow}$ l'élément de $\mathcal{S}^{\downarrow}$ qui correspond à la suite des fréquences asymptotiques des blocs de π rangées par ordre décroissant. Ce théorème établit donc une bijection entre les lois sur $\mathcal{S}^{\downarrow}$ et les lois de partitions aléatoires échangeables.

On peut alors définir la notion de fragmentation de partition :

Définition I.2

Soit $(\Pi(t), t \geq 0)$ un processus de Markov à valeurs dans les partitions, continu en probabilité et issu de la partition $(\mathbb{N}, \emptyset, \emptyset, \ldots)$. On dit que $(\Pi(t), t \geq 0)$ est un processus de fragmentation auto-similaire d'indice α si, pour tout $t \geq 0$, la loi de $\Pi(t)$ est échangeable et si, pour tous $t, t' \geq 0$, la loi de $\Pi(t + t')$ sachant $\Pi(t) = (A_1, A_2, \ldots)$ est la loi d'une partition dont les blocs sont les $A_i \cap \Pi^i(|A_i|^{\alpha} t)$, où les $(\Pi^i(t), t \geq 0)$ sont une suite de copies indépendantes de $(\Pi(t), t \geq 0)$.

Autrement dit, si on introduit l'opérateur $FRAG$ défini de la manière suivante

$$\begin{array}{rccc} FRAG: & \mathcal{P}_\infty \times (\mathcal{P}_\infty)^{\mathbb{N}} & \mapsto & \mathcal{P}_\infty \\ & (\pi, (\pi^{(i)})_{i\geq 1}) & \rightarrow & \pi', \end{array}$$

où π' est la partition dont les blocs sont ceux de $\pi_i \cap \pi^{(i)}$ pour $i \in \mathbb{N}$, alors la probabilité de transition d'une fragmentation homogène entre t et $t + t'$ a pour loi $FRAG(\Pi(t), (\pi^{(i)})_{i\geq 1})$ où $\pi^{(i)}$ est une suite de variables aléatoires indépendantes et identiquement distribuées dont la loi est échangeable et ne dépend que de t'.

Berestycki [6] a montré qu'il y a une correspondance bijective entre les lois des fragmentations de partition et les lois des fragmentations de masse. La loi d'une fragmentation auto-similaire de partition est donc aussi déterminée par un triplet (c, ν, α), avec c coefficient d'érosion, ν mesure de dislocation et α indice d'auto-similarité. Cette approche sera très utile dans la suite car elle permet de "discrétiser l'espace" en considérant la fragmentation de partition restreinte aux n premiers entiers.

I.1.4 La coalescence

Un processus de coalescence peut naïvement être vu comme l'inverse d'une fragmentation. Ainsi, on considère au départ un nuage de particules, qui au cours du temps vont coaguler pour former des particules plus grosses. Mathématiquement, il existe en fait deux définitions distinctes de processus de coalescence. La première consiste à se donner un noyau de coagulation, c'est-à-dire une fonction $K : [0,1]^2 \mapsto \mathbb{R}_+$ symétrique. Deux particules de masses respectives x et y coaguleront alors avec un taux $K(x,y)$ pour former une particule de masse $x+y$ et ceci indépendamment des autres particules. Si l'on part d'un nombre fini de particules et que K est bornée, ceci définit bien de manière unique la loi d'un processus. Evans et Pitman [30] ont cherché à voir si cette définition peut s'étendre lorsque le nombre initial de particules tend vers l'infini (mais on impose toujours que la masse totale soit finie). Un exemple classique pour lequel un tel prolongement est possible, est le coalescent de Kingman pour lequel $K \equiv 1$. La coalescence additive (i.e. $K(x,y) = x+y$) peut elle aussi se définir dans un certain sens pour un nombre initial de particules infini. On remarque que ce type de coalescence ne permet, à chaque instant, qu'à seulement deux particules de coaguler. Un tel processus de coalescence sera dans la suite nommé "processus de coalescence de type I".

La seconde approche dûe à Pitman [54] et Sagitov [58] puis généralisée par Möhle & Sagitov [48] et Schweinsberg [60], permet de considérer directement un nombre initial de particules infini et permet aussi à plusieurs blocs de se former simultanément, chaque regroupement pouvant concerner un nombre infini de particules. Cependant, dans ce modèle, le taux de coagulation de k blocs ne dépend que de k et du nombre total de blocs et est indépendant de la masse de chaque bloc. Ainsi, si le coalescent de Kingman est encore couvert par cette théorie, ce n'est plus le cas de la coalescence additive. Pour définir un tel processus, il est en fait plus simple de le définir sur les partitions. Soit $\pi = (\pi_1, \pi_2, \ldots)$ et $\pi' = (\pi'_1, \pi'_2, \ldots)$ deux partitions de $\mathbb{N}$ où les blocs de chaque partition sont indexés par ordre croissant de leur plus petit élément. La partition obtenue en regroupant les blocs de π dont les indices sont dans un même bloc de π' sera notée $COAG(\pi, \pi')$. On définit alors un "processus de coalescence de type II" de la manière suivante :

Définition I.3

Soit $(\Pi(t), t \geq 0)$ *un processus de Markov à valeurs dans les partitions, continu en probabilité et issu de la partition* $(\{1\}, \{2\}, \ldots)$. *On dit que* $(\Pi(t), t \geq 0)$ *est un processus de coalescence de type II si loi de* $\Pi(t+t')$ *sachant* $\Pi(t) = \pi$ *est la* *loi de $COAG(\pi, \pi')$ ou π' est une partition échangeable indépendante de $\Pi(t)$ et de loi $\Pi(t')$.*

En considérant le processus des fréquences asymptotiques de Π, $(|\Pi(t)|^{\downarrow}, t \geq 0)$, on définit donc aussi un processus de coalescence sur l'espace $\mathcal{S}^{\downarrow}$. Comme dans le cas d'une fragmentation, Möhle & Sagitov [48] et Schweinsberg [60] ont montré que la loi d'un tel processus est caractérisée par un couple (c, ν) où c est un nombre positif et est appelé coefficient de coagulation binaire (qui correspond à la coalescence de Kingman) et ν est une mesure sur $\mathcal{S}^{\downarrow}$ vérifiant $\int_{\mathcal{S}^{\downarrow}} \sum_{i=1}^{\infty} s_i^2 \nu(d\mathbf{s}) < \infty$ et $\nu(0, 0, \ldots) = 0$ et est appelée mesure de coagulation multiple. Par exemple, dans le cas où la mesure ν est de masse finie, un processus de coalescence de caractéristique $(0, \nu)$ se construit de la manière suivante : conditionnellement à $\Pi(t) = \pi$, au bout d'un temps exponentiel de paramètre $\nu(\mathcal{S}^{\downarrow})$, le processus coagule pour former la partition $COAG(\pi, \pi')$ où π' est une partition aléatoire de loi $\frac{1}{\nu(\mathcal{S}^{\downarrow})} \int_{\mathcal{S}^{\downarrow}} \rho_{\mathbf{s}}(\cdot)\nu(d\mathbf{s})$ (on rappelle que la mesure de probabilité $\rho_{\mathbf{s}}$ est celle obtenue via la méthode des boîtes de peinture de Kingman à partir de la suite $\mathbf{s}$).

I.1.5 Les lois de Poisson-Dirichlet

Les lois de Poisson-Dirichlet sont des lois sur $\mathcal{S}^{\downarrow}$ qui interviennent dans de nombreux domaines (voir [56] et les références citées dans ce travail), et jouent un rôle important dans l'étude des certains processus de coalescence ou de fragmentation. Elles dépendent de deux paramètres α et θ vérifiant $\alpha \in]0, 1[$ et $\theta > -\alpha$ et sont notées $PD(\alpha, \theta)$. Les lois de Poisson Dirichlet de paramètre $(\alpha, 0)$ sont définies à partir des sauts d'un subordinateur stable d'indice α : soit Z la fermeture de l'image d'un tel subordinateur, alors la suite décroissante des longueurs des composantes connexes de $[0, 1]\backslash Z$ a pour distribution $PD(\alpha, 0)$. Pour $\mathbf{x} = (x_n)_{n\geq 1}$ une variable aléatoire de loi $PD(\alpha, 0)$, on peut montrer que la limite

$$L_\alpha = \lim_{n\to\infty} n x_n^\alpha$$

existe presque sûrement et vérifie pour tout $\beta > -1$, $\mathbb{E}(L_\alpha^\beta) < \infty$ et $\mathbb{E}(L_\alpha^{-1}) = \infty$ (voir [56], proposition 9). Quant à la loi $PD(\alpha, \theta)$, elle est définie comme la mesure de probabilité absolument continue par rapport à la mesure $PD(\alpha, 0)$ et de densité

$$\frac{L_\alpha^{\theta/\alpha}}{\mathbb{E}(L_\alpha^{\theta/\alpha})}.$$

Il existe en fait d'autres façons de construire une variable aléatoire de loi $PD(\alpha, \theta)$. Une construction qui nous servira par la suite est donnée par le méthode du "stick breaking" [56]. Rappelons d'abord que pour $a > 0$ et $b > 0$, la loi $beta(a, b)$ est définie comme la distribution sur $]0, 1[$ de densité

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}dx.$$

Soit $(Y_n)_{n\geq 1}$ une suite de variables aléatoires indépendantes et de lois respectives $beta(1-\alpha, \theta + n\alpha)$. On pose

$$f_1 = Y_1 \quad \text{et} \quad f_n = (1-Y_1)\dots(1-Y_{n-1})Y_n \qquad \text{pour } n \geq 2.$$

Alors le réarrangement décroissant de la suite $(f_n)_{n\geq 1}$ a pour distribution la loi de Poisson-Dirichlet de paramètre (α, θ). Cette construction porte le nom de "stick breaking" car les variables $(f_n)_{n\geq 1}$ correspondraient aux longueurs des morceaux d'un bâton de longueur 1 qui aurait été cassé en deux de façon répétée et où le morceau le plus à droite serait à chaque fois cassé selon une loi *beta*. Cette approche est intéressante car la suite $(f_n)_{n\geq 1}$ obtenue avant réordonnement des termes est une suite ordonnée avec un biais par la taille. C'est-à-dire, connaissant l'image $(f_n^{\downarrow})_{n\geq 1}$ de la suite $(f_n)_{n\geq 1}$ après réordonnement décroissant, on a $(f_n)_{n\geq 1} = (f_{\sigma(n)}^{\downarrow})_{n\geq 1}$ avec σ permutation aléatoire de $\mathbb{N}$ vérifiant pour tout $i \geq 1$

$$\mathbb{P}(\sigma(1) = i \mid (f_n^{\downarrow})_{n\geq 1}) = f_i^{\downarrow},$$

et pour tout $n \geq 2$

$$\mathbb{P}(\sigma(n) = i \mid (f_n^{\downarrow})_{n\geq 1}, \sigma(1) = i_1, \dots, \sigma(n-1) = i_{n-1}) = \frac{f_i^{\downarrow}}{1 - \sum_{k=1}^{n-1} f_{i_k}^{\downarrow}} \mathbb{1}_{i \notin \{i_1, \dots, i_{n-1}\}}.$$

Ainsi si $\Pi = (\Pi_1, \Pi_2, \dots)$ est une partition aléatoire échangeable dont les blocs sont indexés par ordre croissant de leur plus petit élément et dont la suite des fréquences asymptotiques a pour loi $PD(\alpha, \theta)$, alors la suite $(|\Pi_1|, |\Pi_2|, \dots)$ constituée des fréquences des blocs de Π a même loi que la suite $(f_n)_{n\geq 1}$ définie ci-dessus. On notera désormais $p_{\alpha,\theta}$ la loi de cette partition Π. En calculant de manière explicite les marginales de la loi $p_{\alpha,\theta}$, Pitman [54] a montré que pour $\alpha, \beta \in]0, 1[$ et $\theta > -\alpha\beta$, il y a équivalence entre les deux assertions suivantes :

- π est une partition aléatoire de loi $p_{\alpha,\theta}$ et π' est la coagulation de π par π'', où π'' est une partition aléatoire indépendante de π et de loi $p_{\beta,\theta/\alpha}$.
- π' est une partition aléatoire de loi $p_{\alpha\beta,\theta}$ et π est la fragmentation de π' par $\pi^{(\cdot)}$, où $\pi^{(\cdot)}$ est une suite de partitions aléatoires indépendantes de π et i.i.d. de loi $p_{\alpha,-\alpha\beta}$.

Cette propriété est remarquable car elle permet d'établir, dans le cas des lois de Poisson-Dirichlet, une dualité entre l'opérateur $FRAG$ et l'opérateur $COAG$, lien qui n'est pas vérifié dans le cas général.

La théorie de la coalescence et celle de la fragmentation semblent donc proches mais il n'y a cependant pas de dualité générale entre ces deux théories. C'est-à-dire qu'un processus de coalescence retourné dans le temps ne satisfait pas en général la propriété de branchement d'un processus de fragmentation. Il existe cependant deux cas importants où l'on obtient "presque" un processus de fragmentation, il s'agit de la coalescence additive et du coalescent de Bolthausen et Sznitman [20], c'est-à-dire du processus de coalescence de type II de coagulation binaire nulle et de mesure de coagulation multiple ν caractérisée par $\nu(s_2 > 0) = 0$ et $\nu(s_1 \in dx) = \frac{dx}{x^2}$.

Dans la suite de cette introduction seront brièvement présentés les principaux résultats obtenus durant cette thèse. Dans deux premières sections, nous étudions les deux processus de coalescence cités ci-dessus en tant que fragmentations, et dans la dernière partie, nous étendons le lien entre fragmentation de partition et fragmentation de masse à des situations où un ordre sur les différents blocs est pris en compte.

I.2 Les cascades de Ruelle en tant que fragmentation[1]

Le point de départ de cet article est l'étude des cascades de Ruelle, processus introduit par Ruelle [57] afin d'étudier le GREM de Derrida. Bolthausen et Sznitman [20] ont montré qu'un retournement de temps exponentiel transformait ce processus en un processus de coalescence de type II de mesure de coagulation binaire nulle et de mesure de coagulation multiple ν caractérisée par $\nu(s_2 > 0) = 0$ et $\nu(s_1 \in dx) = \frac{dx}{x^2}$. Il est alors possible de donner explicitement le semi-groupe de transition de ce processus $(\overline{\Pi}(t), t \geq 0)$. Celui-ci s'exprime en fonction de lois de Poisson-Dirichlet : pour tous $t, s \geq 0$, la loi de $\overline{\Pi}(t+s)$ est celle de $COAG(\overline{\Pi}(t), \pi)$, où π est une partition aléatoire indépendante de $\overline{\Pi}(t)$ et de loi $p_{e^{-s},0}$. Grâce aux propriétés des lois de Poisson-Dirichlet, Pitman [54] en déduit les probabilités de transition du processus initial (avant retournement de temps). Si on note $(\Pi(t), t \in [0,1[)$ ce processus à valeurs dans les partitions, sa probabilité de transition du temps t au temps $t+s$

[1] A-L. Basdevant, à paraître dans Markov Processes and Related Fields.

s'exprime sous la forme $FRAG(\Pi(t), \pi^{(\cdot)})$ où $\pi^{(\cdot)}$ est une suite i.i.d. de partitions indépendantes de $\Pi(t)$ et de loi $p_{t+s,-t}$. Ainsi, ce processus possède la propriété de branchement propre aux fragmentations. En revanche il est inhomogène en temps. Dans une première partie, on s'est donc intéressé à étendre la théorie existante sur les fragmentations homogènes en temps au cas des fragmentations inhomogènes en temps, afin notamment de préciser la structure de ces dernières.

Théorème I.4

Soit $(\Pi(t), t \in (0,1[)$ une fragmentation de partition inhomogène en temps. Sous certaines hypothèses techniques, la loi de Π est caractérisée par une famille de couples $(c_t, \nu_t)_{t\in[0,1[}$ où $c_t \in \mathbb{R}_+$ est le coefficient d'érosion instantané au temps t et ν_t, dite mesure de dislocation instantanée au temps t, est une mesure sur $\mathcal{S}^{\downarrow}$ vérifiant $\nu_t(1,0,0,\ldots) = 0$ et $\int_{\mathcal{S}^{\downarrow}}(1-s_1)\nu_t(d\mathbf{s}) < \infty$. De plus on a

$$\forall u \in [0,1[, \int_0^u c_t dt < \infty \quad \text{et} \quad \int_0^u \left(\int_{\mathcal{S}^{\downarrow}} (1-s_1)\nu_t(d\mathbf{s}) \right) dt < \infty.$$

En ce qui concerne la fragmentation de Ruelle, on a établi le résultat suivant

Théorème I.5

1. *Pour tout $t \in [0,1[$, le coefficient d'érosion instantané de la fragmentation de Ruelle est nul.*
2. *Soit $PD(t,0)$ la loi de Poisson-Dirichlet de paramètre $(t,0)$. On note L_t son temps local, c'est-à-dire $L_t = \lim_{n\to\infty} n x_n^t$ avec $(x_n)_{n\geq 1}$ de loi $PD(t,0)$. Alors la mesure ν_t de dislocation instantanée au temps t de la fragmentation de Ruelle est absolument continue par rapport à $PD(t,0)$ et de densité $\frac{1}{t}L_t^{-1}$.*

La preuve consiste à montrer que ν_t s'exprime comme un cas dégénéré des lois de Poisson-Dirichlet. On a en fait la représentation suivante : soit η_t la mesure sur l'ensemble des suites de somme 1 telle que

$$\eta_t(s_1 \in dx) = t x^{-t}(1-x)^{-1} dx \quad \text{pour } x \in]0,1[,$$

et conditionnellement à $s_1 = x$, la suite $(s_{i+1}/(1-x), i \geq 1)$ a pour distribution l'image de la loi $PD(t,0)$ après un réarrangement biaisé par la taille. Par analogie avec la construction par "stick breaking" des lois de Poisson-Dirichlet, on note $PD(t,-t)$ l'image de η_t après réordonnement décroissant des termes de la suite. Alors la mesure de dislocation ν_t de la fragmentation de Ruelle est égale à $\frac{1}{t}PD(t,-t)$. Pour démontrer cela, on considère la fragmentation de Ruelle en tant que fragmentation de partition.

En se restreignant aux entiers $\{1, 2, \dots, n\}$, on est ramené à étudier un processus de Markov à espace d'état fini dont les taux de saut peuvent être explicitement calculés grâce aux travaux de Pitman [54]. On montre alors que ces taux de saut s'expriment en fonction de la mesure sur les partitions associée à la mesure $PD(t, -t)$. De plus, la définition de la mesure η_t donne une construction de la mesure $PD(t, -t)$ avec un biais par la taille. Enfin, l'absolue continuité de $PD(t, -t)$ par rapport à $PD(t, 0)$ est prouvée par un argument de martingale et prolonge l'absolue continuité des mesures $PD(t, s)$, $s > -t$ par rapport à la mesure $PD(t, 0)$. Comme l'espérance de L_t^{-1} est infinie, la mesure $PD(t, -t)$ est par contre de masse totale infinie.

En appliquant ce théorème, on en déduit les comportements asymptotiques de cette fragmentation en temps petits ou en temps grands. On obtient par exemple les résultats suivants :

Proposition I.6

Soit $(F(t), t \in [0, 1[)$ la fragmentation de Ruelle à valeurs dans $\mathcal{S}^{\downarrow}$. On définit la mesure aléatoire

$$\rho_t = \sum_{i=1}^{\infty} F_i(t) \delta_{(t-1) \ln F_i(t)}.$$

Alors pour toute fonction f continue bornée sur $\mathbb{R}_+$, on a

$$\lim_{t \to 1} \int f(y) \rho_t(dy) = \int_0^{\infty} f(y) e^{-y} dy \quad \text{dans } L^2$$

Informellement, ceci montre que la taille $X(t)$ d'un fragment typique au temps t lorsque t tend vers 1 vérifie

$$\ln X(t) \sim -\frac{C}{1-t},$$

où C est une variable aléatoire indépendante de t, finie et strictement positive.

En ce qui concerne les temps proches de 0, on établit un encadrement du comportement de la taille des deux plus gros blocs :

Proposition I.7

Soit $(F(t), t \in [0, 1[)$ une fragmentation de Ruelle à valeurs dans $\mathcal{S}^{\downarrow}$. On a :

1. *Pour t assez petit, $F_1(t) = \exp(-\xi_t)$ presque sûrement où ξ_t est un processus croissant à accroissements indépendants tel que $\exp(-\xi_t)$ ait pour loi $beta(1-t, t)$.*

2. Il existe une constante $\delta > 0$ telle que presque sûrement

$$\begin{cases} \liminf_{t\to 0} |\ln t|^{\gamma/t} F_2(t) = 0 & \text{si } \gamma < \delta \\ \liminf_{t\to 0} |\ln t|^{\gamma/t} F_2(t) = \infty & \text{si } \gamma > \delta. \end{cases}$$

On peut aussi obtenir un encadrement similaire de la limite supérieure de $F_2(t)$. La preuve de cette proposition s'appuie sur un résultat de Berestycki [6] qui exprime la taille des deux plus gros blocs d'une fragmentation en fonction de la mesure de dislocation.

I.3 Fragmentations liées à la coalescence additive[2]

Dans le paragraphe I.1.4, nous avons défini un processus de coalescence additive comme un processus de coalescence de type I de noyau de coagulation $K(x, y) = x+y$. Ce processus est au départ uniquement défini pour un nombre initial de particules fini. Cependant Evans et Pitman [30] ont montré que si l'on note $(C^n(t), t \geq 0)$ le coalescent additif issu de la configuration $(1/n, 1/n, \dots, 1/n)$, alors la suite de processus $(C^n(t + \frac{1}{2}\ln n), t \geq -\frac{1}{2}\ln n)$ converge en loi vers un processus $(C^\infty(t), t \in \mathbb{R})$ lorsque n tend vers l'infini. Ce processus est appelé processus de coalescence additive standard. De plus, une propriété remarquable de ce processus est que, si on pose $F(t) = C^\infty(-\ln t)$, le processus $(F(t), t \geq 0)$ est une fragmentation autosimilaire d'indice 1/2. On peut en fait construire explicitement ce processus F : soit $\varepsilon = (\varepsilon_s, s \in [0, 1])$ une excursion brownienne positive. Pour tout $t \geq 0$, on considère

$$\varepsilon_s^{(t)} = ts - \varepsilon_s \quad \text{et} \quad S_s^{(t)} = \sup_{0 \leq u \leq s} \varepsilon_u^{(t)}.$$

On note $G(t)$ la suite décroissante composée des longueurs des intervalles de constance de $S^{(t)}$. Alors $(G(t), t \geq 0)$ a même loi que $(F(t), t \geq 0)$ (cf. [8]).

D'autre part, Aldous et Pitman [3] ont montré que l'on peut définir d'autres coalescents éternels (i.e. définis pour $t \in \mathbb{R}$) : étant donnée pour chaque $n \in \mathbb{N}$ une suite décroissante $r_{n,1} \geq \ldots \geq r_{n,n} \geq 0$ de somme 1, on note $M^n = (M^n(t), t \geq 0)$ le coalescent additif issu de n particules de masses $r_{n,1} \geq \ldots \geq r_{n,n}$. Si l'on a

$$\lim_{n\to\infty} \sigma_n = 0 \text{ et } \lim_{n\to\infty} \frac{r_{n,i}}{\sigma_n} = \theta_i \text{ pour tout } i \in \mathbb{N},$$

[2] A-L. Basdevant, article soumis.

avec $\sigma_n^2 = \sum_{i=1}^n r_{n,i}^2$, et $\sum_i \theta_i^2 < 1$ ou $\sum_i \theta_i = \infty$, alors la suite de processus $(M^{(n)}(t - \ln \sigma_n), t \geq \ln \sigma_n)$ admet une distribution limite quand n tend vers l'infini. De plus, ceci donne tous les coalescents éternels extrêmaux.

Après retournement de temps, un coalescent additif éternel n'est pas forcément une fragmentation. Cependant, Miermont [46], en s'inspirant de la construction via l'excursion brownienne du coalescent additif standard, a exhibé une classe de coalescents éternels qui, après retournement du temps exponentiel, possèdent la propriété d'indépendance d'évolution entre les différents fragments. Pour cela, il reprend la construction via l'excursion brownienne en remplaçant juste le mouvement brownien par un processus de Lévy sans saut positif, d'espérance négative ou nulle et de variation infinie. Cette construction permet ainsi d'obtenir des processus de fragmentation inhomogènes qui retournés dans le temps, ont le semi-groupe de transition d'un coalescent additif. Il faut tout de même remarquer que l'on perd l'homogénéité en temps et l'auto-similarité de la fragmentation brownienne qui provenaient de la propriété de scaling du mouvement brownien.

Dans cet article, on a montré que la loi $\mathbb{P}^{(X)}$ de ce processus de fragmentation créé à partir de l'excursion d'un processus de Lévy X est, dans certains cas, absolument continue par rapport à la loi $\mathbb{P}^{(B)}$ de la fragmentation brownienne $F = (F(t), t \geq 0)$.

Théorème I.8

Soit $(\Gamma(t), t \geq 0)$ un subordinateur sans drift. On suppose que $\mathbb{E}(\Gamma_1) < \infty$ et on se fixe $c \geq \mathbb{E}(\Gamma_1)$. On définit alors $X_t = B_t - \Gamma_t + ct$, où B désigne un mouvement brownien indépendant de Γ. On note $(p_t(u), u \in \mathbb{R})$ et $(q_t(u), u \in \mathbb{R})$ les densités respectives de B_t et X_t. Soit $\mathcal{S}_1$ l'ensemble des suites positives de somme 1. On considère la fonction $\mathbf{h} : \mathbb{R}_+ \times \mathcal{S}_1$ définie par

$$\mathbf{h}(t, \mathbf{x}) = e^{tc} \frac{p_1(0)}{q_1(0)} \prod_{i=1}^{\infty} \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)} \qquad \text{avec } \mathbf{x} = (x_i)_{i \geq 1}.$$

Alors pour tout $t \geq 0$, la fonction $\mathbf{h}(t, \cdot)$ est bornée et possède les propriétés suivantes

- *$\mathbf{h}(t, F(t))$ est une $\mathbb{P}^{(B)}$-martingale,*
- *pour tout $t \geq 0$, la loi du processus $(F^X(s), 0 \leq s \leq t)$ est absolument continue par rapport à la loi de $(F^B(s), 0 \leq s \leq t)$ et a pour densité $\mathbf{h}(t, F^B(t))$.*

La preuve repose sur un résultat de Miermont [46] qui donne la loi de la fragmentation d'un Lévy à un temps t en fonction des sauts d'un subordinateur. En utilisant le fait que retournés dans le temps, ces deux processus ont même semi-groupe de transition, on en déduit l'absolue continuité en tant que processus.

On remarque aussi que la fonction **h** s'exprime comme un produit de fonctions ne dépendant chacune que de la taille d'un seul fragment. On parle alors de martingale multiplicative. Cette forme n'est pas anodine, en effet, grâce à cette propriété, on déduit aisément que la propriété de branchement vérifiée par la fragmentation construite via l'excursion brownienne, se transmet à la fragmentation construite à partir d'un processus de Lévy (résultat déjà démontré par Miermont [46]).

On peut aussi montrer que dans certains cas, il y a équivalence entre les lois $\mathbb{P}^{(B)}$ et $\mathbb{P}^{(X)}$:

Proposition I.9

Soit ϕ l'exposant de Laplace du subordinateur Γ. S'il existe $\delta > 0$ tel que

$$\lim_{x\to\infty} \phi(x)x^{\delta-1} = 0,$$

alors les lois $\mathbb{P}^{(X)}$ et $\mathbb{P}^{(B)}$ sont équivalentes.

En calculant le générateur d'une fragmentation et en utilisant que $h(t, F(t))$ est une $\mathbb{P}^{(B)}$-martingale, on en déduit de plus une équation intégro-différentielle dans le cas où la mesure d'intensité du subordinateur Γ est finie :

$$\text{Soit} \quad g(t,x) = e^{tcx}\frac{q_x(-tx)}{p_x(-tx)} \quad \text{pour } x \in]0,1], t \geq 0 \quad \text{et } g(t,0) = 1.$$

Alors g satisfait l'équation suivante :

$$\begin{cases} \partial_t g(t,x) + \sqrt{x}\displaystyle\int_0^1 \frac{dy}{\sqrt{8\pi y^3(1-y)^3}}\Big(g(t,xy)g(t,x(1-y)) - g(t,x)\Big) = 0 \\ g(0,x) = \frac{q_x(0)}{p_x(0)}. \end{cases}$$

I.4 Fragmentations de composition et d'intervalle[3]

Précédemment, nous avons travaillé avec deux types de fragmentations : les fragmentations de masse (définition I.1) et les fragmentations de partition (définition I.2). On a vu qu'il y avait bijection entre les lois de ces deux types de fragmentations. En fait, un troisième type de fragmentation a aussi été introduit, les fragmentations d'intervalle [11]. Elles sont définies de la manière suivante :

Définition I.10

Soit $\mathcal{U}$ l'ensemble des ouverts de]0,1[. Un processus $(U(t), t \geq 0)$ à valeurs dans $\mathcal{U}$ est une fragmentation homogène d'intervalle si c'est un processus de Markov

[3] A-L. Basdevant, Electron. J. Probab.,11 : no 16, 394-417, 2006.

tel que :

- *U est continu en probabilité et $U(0) =]0,1[$ p.s.*
- *les ouverts $(U(t))_{t\geq 0}$ sont emboîtés , i.e. pour tous $s > t$ on a $U(s) \subset U(t)$.*
- *Conditionnellement à $U(t) = \coprod_{i\geq 1}]a_i, b_i[$, les processus $(U(t+s) \cap]a_i, b_i[, s \geq 0)$ pour $i \geq 1$ sont indépendants et ont pour loi respective celle du processus $(a_i + (b_i - a_i)U(s), s \geq 0)$.*

Celles-ci sont ainsi l'analogue d'une fragmentation de masse où un ordre sur les blocs est pris en compte. Dans ce travail, on s'est donc intéressé à définir un processus de fragmentation sur les partitions avec un ordre sur les blocs. Pour cela on considère la notion de composition introduite par Gnedin [33] : pour $n \in \mathbb{N}$, une composition de l'ensemble $\{1, \ldots, n\}$ est une suite *ordonnée* de sous-ensembles disjoints et non vides de $\{1, \ldots, n\}$, $\gamma_n = (A_1, A_2, \ldots, A_k)$, tels que $\cup A_i = \{1, \ldots, n\}$. Soit γ une suite ordonnée de sous ensemble de $\mathbb{N}$, on dira que γ est une composition de $\mathbb{N}$ si pour tout n, sa restriction à $\{1, \ldots, n\}$ est une composition. On notera $\mathcal{C}$ l'ensemble des compositions de $\mathbb{N}$. On peut alors définir une notion de fragmentation de composition :

Définition I.11

Soit $n \in \mathbb{N}$ et $\gamma = (\gamma_1, \ldots, \gamma_k)$ une composition de $\{1, \ldots, n\}$. Soit $\gamma^{(\cdot)} = (\gamma^{(i)}, i \in \{1, \ldots, n\})$ une suite de compositions de $\{1, \ldots, n\}$. On note $m_i = \min \gamma_i$ et $\tilde{\gamma}^{(i)}$ la restriction de $\gamma^{(m_i)}$ à l'ensemble γ_i. Ainsi $\tilde{\gamma}^{(i)}$ est une composition de γ_i. On note alors $FRAG(\gamma, \gamma^{(\cdot)})$ la composition $\tilde{\gamma} = (\tilde{\gamma}^{(1)}, \ldots, \tilde{\gamma}^{(k)})$.

Un processus de Markov $(\Gamma_n(t), t \geq 0)$ à valeurs dans les compositions de l'ensemble $\{1, \ldots, n\}$ est une fragmentation de composition s'il est issu de la composition composée d'un seul bloc et si, pour tous $t, t' \geq 0$, conditionnellement à $\Gamma(t) = \gamma$, la loi de $\Gamma(t + t')$ est celle de $FRAG(\gamma, \gamma^{(\cdot)})$ où $\gamma^{(\cdot)}$ est une suite de compositions aléatoires échangeables indépendantes et identiquement distribuées, et dont la loi ne dépend que de t'.

Gnedin [33] a démontré un analogue du théorème de Kingman sur les partitions qui établit une bijection entre les lois de compositions aléatoires échangeables et les lois sur les ouverts de $]0,1[$: soit Γ une composition aléatoire échangeable de $\mathbb{N}$ et Γ_n sa restriction à l'ensemble $\{1, \ldots, n\}$. On note $(N_1, \ldots, N_k)$ le cardinal des blocs de Γ_n et on pose $N_0 = 0$. Pour $i \in \{0, \ldots, k\}$, soit $M_i = \sum_{j=0}^{i} N_j$ et U_n l'ouvert de $]0,1[$ défini par

$$U_n = \bigcup_{i=1}^{k} \left] \frac{M_{i-1}}{n}, \frac{M_i}{n} \right[.$$

Alors U_n converge presque sûrement vers un ouvert aléatoire U_Γ. Ainsi à toute loi d'une composition aléatoire échangeable, on peut associer une loi sur les ouverts de $]0,1[$. Réciproquement, à partir d'une loi sur les ouverts de $]0,1[$, on obtient la loi d'une composition aléatoire échangeable en reprenant l'idée des boîtes de peinture de Kingman : étant donné un ouvert de $]0,1[$, on tire des variables aléatoires $(X_i)_{i\geq 1}$ indépendantes et de loi uniforme sur $]0,1[$ et on crée une composition en décrétant que deux entiers distincts i et j sont dans le même bloc si et seulement si X_i et X_j tombent dans une même composante connexe de l'ouvert. Quant à l'ordre des blocs, il est donné par l'ordre des composantes connexes. Pour U ouvert de $]0,1[$, on notera Γ_U la composition obtenue par ce procédé.

On établit alors de même une correspondance bijective entre les fragmentations de composition et d'intervalle :

Théorème I.12

- *Si $(U(t), t \geq 0)$ est une fragmentation d'intervalle, alors le processus à valeurs dans les compositions $(\Gamma_U(t), t \geq 0)$ défini par le procédé ci-dessus est une fragmentation de composition et l'on a $U_{\Gamma_{U(t)}} = U(t)$ p.s. pour tout $t \geq 0$.*
- *Réciproquement, si $(\Gamma(t), t \geq 0)$ est une fragmentation de composition, alors le processus $(U_{\Gamma(t)}, t \geq 0)$ est une fragmentation d'intervalle.*

Ainsi ceci étend la correspondance déjà connue entre les fragmentations de partition et les fragmentations de masse. En introduisant la projection canonique $\wp_1$ de l'ensemble des compositions dans l'ensemble des partitions, et l'application $\wp_2$ qui à un ouvert de $]0,1[$ associe la suite décroissante composée des longueurs de ses composantes connexes, on peut donc résumer ces résultats dans le diagramme commutatif suivant :

$$\begin{array}{ccc}
\big(\mathcal{C}, (\Gamma(t), t \geq 0)\big) & \xleftrightarrow{\text{Theorem I.12}} & \big(\mathcal{U}, (U_{\Gamma(t)}, t \geq 0)\big) \\
\wp_1 \big\downarrow & & \wp_2 \big\downarrow \\
\big(\mathcal{P}_\infty, (\Pi(t), t \geq 0)\big) & \xleftrightarrow{\text{Berestycki}} & \big(\mathcal{S}^\downarrow, (U^\downarrow_{\Gamma(t)}, t \geq 0)\big).
\end{array}$$

Les résultats caractérisant la loi d'une fragmentation de masse s'étendent de la manière suivante pour les fragmentations d'intervalle :

Théorème I.13

La loi d'une fragmentation homogène d'intervalle est caractérisée par un triplet (c_l, c_r, ν), où c_l et c_r sont des nombres positifs ou nuls appelés coefficients d'érosion à gauche et à droite, et ν est une mesure sur $\mathcal{U}$ appelée mesure de dislocation.

De plus ν vérifie $\nu(]0,1[) = 0$ et $\int_{\mathcal{U}}(1 - |U_1|)\nu(dU) < \infty$, où $|U_1|$ est la taille de la plus grande composante connexe de U. Réciproquement, pour tout triplet (c_l, c_r, ν) de ce type, on peut construire une fragmentation d'intervalle ayant ce triplet caractéristique.

La preuve de ce théorème est en fait très proche de la démonstration dans le cas des fragmentations de masse [14]. La différence principale réside dans le fait que l'on a ici deux coefficients d'érosion, l'un caractérisant l'érosion à gauche et l'autre l'érosion à droite. Pour cette raison, on ne peut pas transposer au cas des fragmentations d'intervalle le résultat de Berestycki [6] qui dit que si $(F(t), t \geq 0)$ est une fragmentation de masse de caractéristique $(0, \nu)$, alors $(e^{-ct}F(t), t \geq 0)$ une fragmentation de masse de caractéristique (c, ν). D'autre part, si $(U(t), t \geq 0)$ est une fragmentation d'intervalle de caractéristique (c_l, c_r, ν), on montre que sa projection sur $\mathcal{S}^{\downarrow}$ est une fragmentation de masse de caractéristique $(c_l + c_r, \nu^{\downarrow})$, où $\nu^{\downarrow}$ est la mesure image de ν par la projection sur $\mathcal{S}^{\downarrow}$. Enfin, tous ces résultats peuvent être étendus au cas des fragmentations auto-similaires.

Cela permet de calculer par exemple la mesure de dislocation de la fragmentation de Ruelle considérée comme fragmentation d'intervalle, c'est-à-dire construite à partir de l'image d'une famille de subordinateur stable d'indice variant entre 0 et 1 (cf. [15]). On montre alors que celle ci donne un ordre uniforme sur les composantes connexes de l'ouvert.

Dans le cas de la fragmentation d'intervalle induite par la fragmentation liée au coalescent additif construite à partir de l'excursion brownienne (cf section I.3), on montre que la mesure de dislocation ν ne charge que les ouverts du type $]0, x[\cup]x, 1[$, et si on identifie un tel ouvert par le point de coupe x et que l'on note $\nu(dx)$ sa distribution, on a pour tout $x \in]0, 1[$,

$$\nu(dx) = (2\pi x(1 - x^3))^{-1/2} dx.$$

Pour une fragmentation d'intervalle $(U(t), t \geq 0)$ sans perte de masse, c'est-à-dire lorsque la mesure de Lebesgue de $U(t)$ vaut 1 presque sûrement pour tout $t \geq 0$, on peut aussi s'intéresser à la géométrie de cet ouvert en étudiant la dimension de Hausdorff de son complémentaire. La proposition suivante répond en partie à ce problème :

Proposition I.14

Soit $(U(t), t \geq 0)$ une fragmentation d'intervalle auto-similaire d'indice strictement positif, d'érosion nulle et de mesure de dislocation ν. On suppose que ν

vérifie les propriétés suivantes :

- ν *est conservatrice i.e* $\nu(\sum_i |U_i|^\downarrow < 1) = 0$.
- *Il existe un entier* k *tel que* $\nu(|U_k|^\downarrow > 0) = 0$, *i.e.* ν *ne charge que les ouverts qui ont au plus* $k-1$ *composantes connexes.*
- *Soit* $h(\varepsilon) = \int_{\mathcal{U}} (Card\{i, |U_i| \geq \varepsilon\} - 1)\nu(dU)$. *Alors* h *varie régulièrement d'indice* $-\beta$ *quand* $\varepsilon \to 0+$.
- *Soit* g *l'extrémité de gauche de la plus grand composante connexe d'un ouvert de]0,1[et* d *l'extrémité de droite. Alors, on a soit* $\liminf_{\varepsilon\to 0+} \frac{\nu(g\geq\varepsilon)}{\nu(d\leq 1-\varepsilon)} > 0$ *ou soit* $\limsup_{\varepsilon\to 0+} \frac{\nu(g\geq\varepsilon)}{\nu(d\leq 1-\varepsilon)} < \infty$.

Alors la dimension de Hausdorff du complémentaire de $U(t)$ *vaut* β *pour tout* $t > 0$ *presque sûrement.*

La preuve s'appuie en grande partie sur un article de Bertoin [13] qui donne une estimation du nombre de fragments de taille supérieure à un nombre fixé et de la masse totale de ces fragments. On peut par exemple appliquer ce résultat à la fragmentation construite à partir de l'excursion brownienne pour en déduire que la dimension de Hausdorff de son complémentaire vaut presque sûrement $1/2$.

Chapter II

Ruelle's probability cascades seen as a fragmentation process[1]

Abstract. In this paper, we study Ruelle's probability cascades [57] in the framework of time-inhomogeneous fragmentation processes. We describe Ruelle's cascades mechanism exhibiting a family of measures $(\nu_t, t \in [0,1[)$ that characterizes its infinitesimal evolution. To this end, we will first extend the time-homogeneous fragmentation theory to the inhomogeneous case. In the last section, we will study the behavior for small and large times of Ruelle's fragmentation process.

II.1 Introduction

Ruelle [57] introduced a cascade of random probability measures in order to study Derrida's GREM model in statistical mechanics. This approach was further developed by Bolthausen and Sznitman [20], who pointed out that an exponential time-reversal transforms Ruelle's probability cascades into a remarkable coalescent process. Previously Neveu [50] observed that Ruelle's probability cascades were also related to the genealogy of some continuous state branching process; we refer to [15] for precise statements and the connexion with Bolthausen-Sznitman coalescent. Furthermore, Pitman [54] obtained a number of explicit formulas on the law of Ruelle's cascades; in particular he showed that the latter can be viewed as a fragmentation process and specified its semi-group in terms of certain Poisson-Dirichlet distributions. Returning to applications to Derrida's GREM model, we mention the important works

[1]This chapter is an extended version of the article: A-L. Basdevant, *Ruelle's probability cascades seen as a fragmentation process*, 2005. To appear in Markov Processes and Related Fields.

by Bovier and Kurkova [21, 22, 23] who established in particular properties of the limiting Gibbs measure.

The purpose of this paper is to dwell on Pitman's observation that Ruelle's cascade can be viewed as a time-inhomogeneous fragmentation process. The theory of time-homogeneous fragmentation processes was developed recently (see eg [6, 14, 10]), and we shall briefly show how it can be extended to the time-inhomogeneous setting. Roughly the basic result is that the distribution of a time-inhomogeneous fragmentation can be characterized by a so-called instantaneous rate of erosion (which is a non-negative real number that depends on the time parameter), and an instantaneous dislocation measure (which specifies the rate of sudden dislocation). We shall establish that for Ruelle's probability cascades, the instantaneous erosion is zero, and we will provide several descriptions of the instantaneous dislocation measure. Specifically, the latter is related to the well-known Poisson-Dirichlet distributions, in particular we shall establish a stick-breaking construction, compute the corresponding exchangeable partition probability function, and derive some relations of absolute continuity. In this direction, we mention that related (but somewhat less precise) results have been proven independently by Marchal [44]. Finally, as examples of applications, we shall prove some asymptotic results for Ruelle's probability cascades at small and large times.

The rest of this work is organized as follows. The next section is devoted to preliminaries, then we briefly present the extension of the theory of fragmentation processes to the time-inhomogeneous setting. The main results on Ruelle's probability cascades are established in Section II.4, and finally section deals with applications to the asymptotic behavior.

II.2 Preliminaries

II.2.1 Mass fragmentations

A fragmentation process describes an object which splits as time goes on. In the whole paper we will assume that our initial object has mass 1. As our goal is to study the ordered sequence $(s_1, s_2, \ldots)$ of the fragments masses of this object, a fragmentation process will take values on

$$\mathcal{S}^{\downarrow} = \{s = (s_1, s_2, \ldots), s_1 \geq s_2 \ldots \geq 0, \sum_i s_i \leq 1\}.$$

Notice that we have $\sum_i s_i < 1$ if a part of the initial mass has been reduced to dust. In the following, we also denote $\mathcal{S}_1^\downarrow$ the subset of $\mathcal{S}^\downarrow$ where there is no loss of mass:

$$\mathcal{S}_1^\downarrow = \{s = (s_1, s_2, \ldots), s_1 \geq s_2 \ldots \geq 0, \sum_i s_i = 1\}.$$

The set $\mathcal{S}_1^\downarrow$ is endowed with the topology of pointwise convergence. Let us notice that $\mathcal{S}^\downarrow$ is the closure of $\mathcal{S}_1^\downarrow$ and this is a compact set. In order to define properly a fragmentation process, we must first define an operator on $\mathcal{S}^\downarrow$:

Definition II.1

Let $s = (s_i, i \in \mathbb{N})$ be an element of $\mathcal{S}^\downarrow$ and $s^{(\cdot)} = (s^{(i)}, i \in \mathbb{N})$ a sequence in $\mathcal{S}^\downarrow$. Consider the fragmentation of s_i by $s^{(i)}$, i.e. the sequence $\tilde{s}^{(i)} = (s_i s_j^{(i)}, j \in \mathbb{N})$. The decreasing rearrangement of all the terms of the sequences $\tilde{s}^{(i)}$ as i describes $\mathbb{N}$ is called fragmentation of s by $s^{(\cdot)}$. If $\mathbb{P}$ is a probability on $\mathcal{S}^\downarrow$, we define the transition kernel $\mathbb{P} - FRAG\,(s, .)$ as the distribution of a fragmentation of s by $s^{(\cdot)}$, where $s^{(\cdot)}$ is an iid sequence of random mass-partition with law $\mathbb{P}$.

Let $A \in]0, \infty]$. A Markov process $(F(t), t \in [0, A[)$ with values in $\mathcal{S}^\downarrow$ is called a fragmentation process if the following properties are fulfilled:

- $F(t)$ is continuous in probability.
- Its semi-group has the following form:
  for all $t, t' \in [0, A[$ such that $t + t' \in [0, A[$, the conditional law of $F(t+t')$ given $F(t) = s$ is the law of $\mathbb{P}_{t,t+t'} - FRAG(s, \cdot)$ where $\mathbb{P}_{t,t+t'}$ is a probability on $\mathcal{S}^\downarrow$.

We say that a fragmentation is homogeneous (in time) if $\mathbb{P}_{t,t+t'}$ depends only on t'. Besides, $(F(t), t \in [0, A[)$ is called a standard fragmentation process if $F(0)$ is almost surely equal to the sequence $\mathbf{1} = (1, 0, \ldots)$. In the sequel, it will be convenient to assume that the fragmentation process is defined on $[0, 1[$ (and it is the case of the fragmentation associated to Ruelle's cascades), but the results are obviously still true for fragmentation processes defined on $[0, A[$ even if A is equal to infinity.

II.2.2 Fragmentations of exchangeable partitions

A very useful tool when studying mass-fragmentations, is the theory of exchangeable partitions: Kingman [38] has established a correspondence between laws on $\mathcal{S}^\downarrow$ and laws of exchangeable partitions. This correspondence can be extended between mass fragmentation and fragmentation of exchangeable partitions. Let us be more precise: we denote by $\mathbb{N}$ the set of positive integers. For $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \dots, n\}$ and $\mathcal{P}_n$ denotes the set of partitions of $[n]$, $\mathcal{P}_\infty$ the set of partitions of $\mathbb{N}$. For all $n < m$, for all $\pi \in \mathcal{P}_m$, $\pi_{|n}$ denotes the restriction of π to $\mathcal{P}_n$. We endow $\mathcal{P}_\infty$ with the distance $d(\pi, \pi') = \frac{1}{\sup\{n \in \mathbb{N}\ \pi_{|n} = \pi'_{|n}\}}$. The partition with a single block is denoted by $\mathbf{1}$. We always label the blocks of a partition according to the increasing order of their smallest element.

A random partition of $\mathbb{N}$ is called exchangeable if its distribution is invariant by the action of the group of finite permutations of $\mathbb{N}$. Kingman [38] has proved that each block of an exchangeable random partition has a frequency, i.e. if $\pi = (\pi_1, \pi_2, \dots)$ is an exchangeable random partition, then

$$\forall i \in \mathbb{N} \qquad f_i = \lim_{n\to\infty} \frac{\sharp\{\pi_i \cap [n]\}}{n} \qquad \text{exists a.s.}$$

One calls f_i the frequency of the block π_i. Therefore, for every exchangeable random partitions, we can associate a probability on $\mathcal{S}^\downarrow$ which will be the law of the decreasing rearrangement of the sequence of the partition frequencies.

Conversely, given a law $\mathbb{P}$ on $\mathcal{S}^\downarrow$, we can construct an exchangeable random partition whose law of its frequency sequence is $\mathbb{P}$ (cf. [38]). Let us specify this construction: we pick $s \in \mathcal{S}^\downarrow$ with law $\mathbb{P}$ and we draw a sequence of independent random variables U_i with uniform law on $[0, 1]$. Conditionally on s, two integers i and j are in the same block of Π iff there exists an integer k such that $\sum_{l=1}^{k} s_l \le U_i < \sum_{l=1}^{k+1} s_l$ and $\sum_{l=1}^{k} s_l \le U_j < \sum_{l=1}^{k+1} s_l$. This construction of a law on the set of partitions from a law on $\mathcal{S}^\downarrow$ is often called "paint-box process".

Kingman's representation Theorem states that any exchangeable random partition can be constructed in this way. Therefore, we have a natural bijection between the laws on $\mathcal{S}^\downarrow$ and the laws on exchangeable random partitions. We also define an exchangeable measure ρ_ν on $\mathcal{P}_\infty$ from a measure ν on $\mathcal{S}^\downarrow$ by:

$$\rho_\nu(\cdot) = \int_{\mathcal{S}^\downarrow} \rho_u(\cdot)\nu(du),$$

where ρ_u is the law on $\mathcal{P}_\infty$ obtained by the paint-box process based on the mass-partition u.

We can also define a notion of fragmentation process of exchangeable partitions such that there is still a bijection with fragmentation processes of mass-partitions:

Set $A \subseteq B \subseteq \mathbb{N}$ and $\pi \in \mathcal{P}_A$ with $\#\pi = n$. Let $\pi^{(\cdot)} = (\pi^{(i)}, i \in \{1, \dots, n\})$, $\pi^{(i)} \in \mathcal{P}_B$ for all i. Consider the partition of the i-th block of π,π_i, induced by $\pi^{(i)}$, i.e. $\pi^{(i)}_{|\pi_i} = \tilde{\pi}^{(i)}$.

As i describes $\{1,\ldots,n\}$, the blocks of $\tilde{\pi}^{(i)}$ form the blocks of a partition $\tilde{\pi}$ of A. This partition is denoted $FRAG(\pi,\pi^{(\cdot)})$. This is the fragmentation of π by $\pi^{(\cdot)}$. If $\mathbb{P}$ is a probability on $\mathcal{P}_B$, define the transition kernel $\mathbb{P}-FRAG\,(\pi,.)$ as the distribution of a fragmentation of π by $\pi^{(\cdot)}$, where $\pi^{(\cdot)}$ is a sequence of iid partition with law $\mathbb{P}$.

Let $(\Pi(t),t\in[0,1[)$ be a Markov process on $\mathcal{P}_\infty$. We call $(\Pi(t),t\in[0,1[)$ an exchangeable fragmentation process if the following properties are fulfilled:

- $\Pi(t)$ is continuous in probability.
- Its semi-group has the following form:
  for all $t,t^{'}\geq 0$ such that $t+t'<1$, the conditional law of $\Pi(t+t^{'})$ given $\Pi(t)=\pi$ is $\mathbb{P}_{t,t'}-FRAG(\pi,\cdot)$, where $\mathbb{P}_{t,t'}$ is an exchangeable probability on $\mathcal{P}_\infty$.

The fragmentation is homogeneous if $\mathbb{P}_{t,t'}$ depends only on t'. Furthermore, $(\Pi(t),t\in[0,1[)$ is a standard fragmentation process if $\Pi(0)$ is equal to $\mathbf{1}$.

We can check that, with these definitions, if $(\Pi(t),t\in[0,1[)$ is a fragmentation process on partitions, then $(F(t),t\in[0,1[)$ the frequency process of Π, is a fragmentation process on mass-partitions. Furthermore, the converse is true, i.e., if one considers a fragmentation process on mass-partitions, then one can construct a fragmentation process on partitions Π such that the frequency process of Π is equal to the initial fragmentation process (cf. [6]).

We also remark that if we consider a fragmentation $(\Pi(t),t\in[0,1[)$ with semi-group $\mathbb{P}_{t,t'}-FRAG$, then its restriction to $\mathcal{P}_n$, $(\Pi_{|n}(t),t\in[0,1[)$, is a Markov process with semi-group $\mathbb{P}^n_{t,t'}-FRAG$ where $\mathbb{P}^n_{t,t'}$ is the image of $\mathbb{P}_{t,t'}$ by the canonical projection $\mathcal{P}_\infty\mapsto\mathcal{P}_n$ (cf. [14]).

Working with fragmentations of partitions is sometimes easier than working with mass fragmentations because a probability $\mathbb{P}$ on $\mathcal{P}_\infty$ is fully characterized by a symmetric function p on finite sequences of $\mathbb{N}$ with p is defined by:

$$\forall n\in\mathbb{N},\forall n_1,\ldots,n_k\in\mathbb{N}^k \text{ such that } n=n_1+\ldots+n_k,\ p(n_1,\ldots,n_k)=\mathbb{P}(\Pi_{|n}=\pi),$$

where π is a partition of $[n]$ with k blocks of size $n_1,\ldots,n_k$. The fact that $\mathbb{P}(\Pi_{|n}=\pi)$ depends only on $n_1,\ldots,n_k$ stems from the exchangeability of Π. One calls p the EPPF (exchangeable partition probability function) of Π.

II.2.3 Ruelle's cascades and their representation with stable subordinators

Let us briefly recall the construction of Ruelle's cascades [15, 20, 57]. Let $p > 1$ be an integer and let $0 < x_1 < \ldots < x_p < 1$ be a finite sequence of real numbers. For $k \in \{1, \ldots, p\}$, $(\eta_{i_1,\ldots,i_k}, i_1 \ldots i_k \in \mathbb{N})$ denotes a family of random variables such that:

- for $k \in \{1, \ldots, p\}$, $i_1, \ldots, i_{k-1} \geq 1$ fixed, the distribution of $(\eta_{i_1,\ldots,i_{k-1},j}, j \in \mathbb{N})$ is that of the sequence of atoms of a Poisson measure on $]0, \infty[$ with intensity $x_k r^{-1-x_k} dr$, arranged according to the decreasing order of their sizes,

- the families $(\eta_{i_1,\ldots,i_{k-1},j}, j \in \mathbb{N})$ for $k \in \{1, \ldots, p\}$, $i_1, \ldots, i_{k-1} \geq 1$ are independent.

Set $\theta_{i_1,\ldots,i_p} = \eta_{i_1} \ldots \eta_{i_1,\ldots,i_p}$. We can easily show that $C = \sum_{i_1 \ldots i_p} \theta_{i_1,\ldots,i_p}$ is almost surely finite. Next we define Ruelle's cascades:

$$\overline{\theta}_{i_1,\ldots,i_p} = \frac{\theta_{i_1,\ldots,i_p}}{C} \qquad \text{and recursively} \qquad \overline{\theta}_{i_1,\ldots,i_{k-1}} = \sum_{j=1}^{\infty} \overline{\theta}_{i_1,\ldots,i_{k-1},j}.$$

Bertoin and Le Gall [15] have proved we can relate this process to the genealogy of Neveu's CSBP (continuous-state branching process). Precisely, they have proved that there exists a process $(S^{(s,t)}(a), 0 \leq s < t, a \geq 0)$ such that:

- For all $t > s > 0$, the process $S^{(s,t)} = (S^{(s,t)}(a), a \geq 0)$ is a stable subordinator with index $e^{-(t-s)}$,

- For all $p \geq 2$ and for all $t_p \geq \ldots \geq t_1 \geq 0$, the processes $S^{(t_1,t_2)}, \ldots, S^{(t_{p-1},t_p)}$ are independent and $S^{(t_1,t_p)}(a) = S^{(t_{p-1},t_p)} \circ \ldots \circ S^{(t_1,t_2)}(a)$.

Set $0 < t_1 < \ldots < t_p$ such that

$$x_1 = e^{-t_p} \text{ and } x_k = e^{-(t_p - t_{k-1})}, k = 2, \ldots, p.$$

Let us fix $a > 0$. We define recursively, for $k = 1, \ldots, p$, random intervals $D^{(t_1,\ldots,t_k,a)}_{i_1,\ldots,i_k}$ in the following way:
$D^{(a)} =]0, a[$.
Let $k \geq 1$, $i_1, \ldots, i_{k-1} \in \mathbb{N}$. Let $(b_{i_1,\ldots,i_k}, i_k \in \mathbb{N})$ be the jump times of $S^{(t_{k-1},t_k)}$ on the interval $D^{(t_1,\ldots,t_{k-1},a)}_{i_1,\ldots,i_{k-1}}$ listed in the decreasing order of sizes. We set

$$D^{(t_1,\ldots,t_k,a)}_{i_1,\ldots,i_k} =]S^{(t_{k-1},t_k)}(b_{i_1,\ldots,i_k}-), S^{(t_{k-1},t_k)}(b_{i_1,\ldots,i_k})[\text{ and } \xi^{(t_1,\ldots,t_k,a)}_{i_1,\ldots,i_k} = |D^{(t_1,\ldots,t_k,a)}_{i_1,\ldots,i_k}|. \tag{II.1}$$

Bertoin and Le Gall have proved that the families

$$\left(\left(S^{(0,t_p)}(a)\right)^{-1}\xi_{i_1,\dots,i_p}; i_1,\dots,i_p\in\mathbb{N}\right) \text{ and } \left(\overline{\theta}_{i_1,\dots,i_p}; i_1,\dots,i_p\in\mathbb{N}\right)$$

have the same law.

II.2.4 Ruelle's cascades as fragmentation processes

Using this representation of Ruelle's cascades in terms of stable subordinators, we can exhibit a link with fragmentation processes. Recall that the Beta distribution $\beta(a,b)$ has density

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}\mathbf{1}_{[0,1]}dx,$$

and let us introduce some definition:

Definition II.2 ***[56]***

For $0\le\alpha<1$, $\theta>-\alpha$, let $(Y_n)_{n\ge1}$ be a sequence of independent random variables with respective laws $\beta(1-\alpha,\theta+n\alpha)$. Set

$$\widehat{f}_1=Y_1 \qquad \widehat{f}_n=(1-Y_1)\dots(1-Y_{n-1})Y_n \qquad \widehat{f}=\left(\widehat{f}_n\right)_{n\ge1}.$$

Then $\sum_i\widehat{f}_i=1$. Let $f=(f_n)_{n>0}$ be the decreasing rearrangement of the sequence $(\widehat{f}_n)_{n\ge1}$. We define the Poisson-Dirichlet law with parameter (α,θ), denoted $PD(\alpha,\theta)$, as the distribution of f.

In the case of Ruelle's cascades, using the work of Bertoin et Pitman [16] (Lemma 9), we know that for any integer $2\le k\le p$, $(\overline{\theta}_{i_1,\dots,i_k}, i_1,\dots,i_k\ge1)$ is a $PD(x_k,-x_{k-1})$-fragmentation of $(\overline{\theta}_{i_1,\dots,i_{k-1}}, i_1,\dots,i_{k-1}\ge1)$. More precisely we have:

Proposition II.3

There exists a time-inhomogeneous fragmentation $(F(t), t\in[0,1])$ with semi-group $\mathbb{P}_{t,t+t'}=PD(t+t',-t)$ such that

$$\left(\left(\overline{\theta}_{i_1}; i_1\in\mathbb{N}\right),\dots\left(\overline{\theta}_{i_1,\dots,i_p}; i_1,\dots,i_p\in\mathbb{N}\right)\right)\overset{law}{=}\left(F(x_1),\dots,F(x_p)\right).$$

In the sequel, we call F Ruelle's fragmentation. To study Ruelle's cascade, it should be possible to use the fragmentation process theory developed for example in [12], but first, we must extend this theory to time-inhomogeneous fragmentations.

As explained in Section II.2.2, we can also study Ruelle's fragmentation as a fragmentation of exchangeable partitions.

Proposition II.4 *[53, 54]*

Let $\widehat{f} = \left(\widehat{f}_n\right)_{n\in\mathbb{N}}$ be a sequence of random variables of $[0,1]$ defined as in Definition II.2 . Then there exists an exchangeable random partition with frequency distribution $\widehat{f}$, where $\widehat{f}_i$ is the i-th block frequency and where the blocks are listed in order of their smallest element. This partition will be denoted a (α,θ)-partition. Besides the EPPF of this partition is

$$p_{\alpha,\theta}(n_1,\ldots,n_k) = \frac{[\frac{\theta}{\alpha}]_k}{[\theta]_n}\prod_{i=1}^{k} -[-\alpha]_{n_i} \quad \text{for } \theta \neq 0 \qquad \text{(Ewens-Pitman's formula)} \tag{II.2}$$

where $[x]_n = \prod_{i=1}^{n}(x+i-1)$ and $n = \sum_{i=1}^{k} n_i$.
For $\theta = 0$, the formula is extended by continuity.

This proposition also proves that the law of the sequence $\widehat{f}$ is invariant by size-biased rearrangement, i.e., if we draw successively and without replacement terms of the sequence $\widehat{f}$ with a probability proportional to its size:

$$\mathbb{P}\left(\tilde{f}_1 = \widehat{f}_i \mid (\widehat{f}_j)_{j\in\mathbb{N}}\right) = \widehat{f}_i,$$

$$\text{and} \quad \forall n \geq 1,\ \mathbb{P}\left(\tilde{f}_{n+1} = \widehat{f}_i \mid (\widehat{f}_j)_{j\in\mathbb{N}}, \tilde{f}_1,\ldots,\tilde{f}_n\right) = \frac{\widehat{f}_i \mathbb{1}_{\{\widehat{f}_i \neq \tilde{f}_j, 1\leq j\leq n\}}}{1-\sum_{j=1}^{n}\tilde{f}_j},$$

then $(\tilde{f}_j)_{j\in\mathbb{N}}$ has the same law as $(\widehat{f}_j)_{j\in\mathbb{N}}$.

In the case of Ruelle's fragmentation, we know that, at time t, $F(t)$ has the $PD(t,0)$ law. So we have the following proposition:

Proposition II.5

The EPPF q_t of the random partition associated with Ruelle's fragmentation at time t, $F(t)$, is:

$$q_t(n_1,\ldots,n_k) = \frac{(k-1)!}{(n-1)!} t^{k-1} \prod_{i=1}^{k} [1-t]_{n_i-1}, \tag{II.3}$$

where $n = \sum_{i=1}^{k} n_i$.

Remark : We can also construct a random partition with distribution $p_{\alpha,\theta}$ recursively (Chinese restaurant construction):

First, the integer 1 necessarily belongs to the first block, denoted B_1. Suppose the n

first integers split up in b blocks: $\Pi_n = (B_1, \ldots, B_b)$, where block B_i has cardinal n_i. We now define Π_{n+1} with the following rule:
$\mathbb{P}(\Pi_{n+1} = (B_1, \ldots, B_i \cup \{n+1\}, \ldots, B_b)) = \frac{n_i - \alpha}{n+\theta}$
$\mathbb{P}(\Pi_{n+1} = (B_1, \ldots, B_b, \{n+1\})) = \frac{b\alpha+\theta}{n+\theta}$.
Then Π is a (α, θ)-partition (cf. [52]).

For Ruelle's fragmentation, we have an explicit construction of its corresponding fragmentation on partitions. Indeed, recall the representation of Ruelle's cascades with the jumps of a family of subordinators (cf. Section II.2.3). Let $(\sigma^*_t, t \in [0,1[)$ be a family of stable subordinators such that for every $0 \le t_p < \ldots < t_1 < 1$, the joint distribution of $\sigma^*_{t_1}, \ldots, \sigma^*_{t_p}$ is the same as that of $\sigma_{t_1}, \ldots, \sigma_{t_p}$ with $\sigma_{t_i} = \tau_{\beta_1} \circ \ldots \circ \tau_{\beta_i}$ where $t_i = \beta_1 \ldots \beta_i$ and $\tau_{\beta_1}, \ldots, \tau_{\beta_p}$ are independent stable subordinators with indices $\beta_1, \ldots, \beta_p$. For $t \in]0,1[$, let M_t be the closure of $\{\sigma^*_t(u), u \ge 0\}$. Consider then the family of open subsets of $[0,1[$: $G(t) = [0,1[\backslash M_t$, for $t \in [0,1[$. Then $(G(t), t \in [0,1[)$ is a nested family, i.e. $G(t) \subset G(s)$ for $0 < s < t < 1$ and furthermore, if $F(t)$ is the sequence of ranked lengths of the component intervals of $G(t)$, then $(F(t), t \in [0,1[)$ has the law of Ruelle's fragmentation; see [16].

Set $0 \le t_1 < \ldots < t_p < 1$. Let us now draw $(U_i)_{i \in \mathbb{N}}$, uniform and independent random variables on $]0,1[$. For $1 \le k \le p$, we construct a partition $\Pi(k)$ of $\mathbb{N}$ with the rule:

$$i \overset{\Pi(k)}{\sim} j \Leftrightarrow U_i \text{ and } U_j \text{ are in the same component interval of } G(t_i).$$

Then $(\Pi(1), \ldots, \Pi(k))$ has the law of a Ruelle's fragmentation on partitions at times $(t_1, \ldots, t_p)$.

II.2.5 Connection with Bolthausen-Sznitman's coalescent

Bolthausen et Sznitman [20] have shown that it is possible to formulate Ruelle's fragmentation as a coalescent process if we reverse time. Moreover, for a good choice for the time reversal, the coalescent process is time-homogeneous [20]. Let us first recall the definition of a coalescent process.

Set $s \in \mathcal{S}^\downarrow$ and let $\Pi = \{B_1, B_2, \ldots\}$ be a partition of $\mathbb{N}$. Set $\tilde{s}_i = \sum_{j \in B_i} s_j$. The Π-coagulation of s, denoted $COAG(s, \Pi)$ is the decreasing rearrangement of the sequence $(\tilde{s}_i, i \in \mathbb{N})$. If $\mathbb{P}$ is a probability on $\mathcal{S}^\downarrow$, we define the transition kernel $\mathbb{P} - COAG(s, .)$ as the distribution of a Π-coagulation of s, where Π has the law on $\mathcal{P}_\infty$ obtained from $\mathbb{P}$ by the paint-box construction.

Let $(C(t), t \geq 0)$ be a Markov process on $\mathcal{S}^{\downarrow}$. $(C(t), t \geq 0)$ is a time-homogeneous mass-coalescent process if the following properties are fulfilled:

- $C(t)$ is continuous in probability.
- Its semi-group has the following form:
for all $t, t' \geq 0$, the conditional law of $C(t+t')$ given $C(t) = s$ is the law of $\mathbb{P}_{t'} - COAG(s, \cdot)$ where $\mathbb{P}_{t'}$ is a probability on $\mathcal{S}^{\downarrow}$.

To see that Ruelle's fragmentation reversed in time is a time-homogeneous coalescent process, we use the following property:

Proposition II.6 *[54]*

Set $\alpha \in]0,1[$, $\beta \in [0,1[$ and $\theta > -\alpha\beta$. The following assertions are equivalent:

- *s has $PD(\alpha, \theta)$ distribution and s' is a $PD(\beta, \theta/\alpha)$-coagulation of s.*
- *s' has $PD(\alpha\beta, \theta)$ distribution and s is a $PD(\alpha, -\alpha\beta)$-fragmentation of s'.*

Thus, if we define $C(t) = F(e^{-t})$ where $(F(t), t \in [0,1[)$ is Ruelle's fragmentation, then $(C(t), t \geq 0)$ is a homogeneous coalescent process with semi-group $PD(e^{-t}, 0)$-COAG. This process is called the Bolthausen-Sznitman's coalescent.

Just like in the case of fragmentation processes, we can associate a coalescent process on exchangeable partitions to any mass-coalescent process. For the Bolthausen-Sznitman's coalescent process on partitions, we have an explicit construction [54]. It is a simple exchangeable coalescent process, i.e, at each jump-time of the process $\Pi_n(t)$, only one new block can be formed. The jump rates of this process can be explicitly written. If we start from a partition with b blocks, each k-uplet of blocks coagulates with rate $\lambda_{b,k}$ that depends only on b and k and that is equal to:

$$\lambda_{b,k} = \frac{(k-2)!\,(b-k)!}{(b-1)!} = \int_0^1 x^{k-2}\,(1-x)^{b-k}\,dx.$$

Remark : One can be surprised that a homogeneous Markov process becomes an inhomogeneous Markov process after time-reversal. In fact, Ruelle's fragmentation can also be seen as a homogeneous Markov process, but, if one takes this point of view, it is no longer a fragmentation process since the evolution of a particle depends on the other particles. Actually, it is known that if a random variable $x = (x_1, x_2, \ldots) \in \mathcal{S}^{\downarrow}$ has the PD$(\alpha, 0)$ law , then $\lim_{n\to\infty} \frac{\alpha \ln x_n}{\ln n} = -1$ (cf. [56]). In particular, in the case of Ruelle's fragmentation, $F(t)$ has law PD$(t, 0)$, therefore

$$t = -\lim_{n\to\infty} \frac{\ln n}{\ln x_n(t)}.$$

Let $(p_{t,t+s})_{t,s>0}$ be the transition probabilities of F. Suppose that the process is in state $x \in \mathcal{S}^{\downarrow}$. For all $t \in [0,1[$, the process F has a Poisson-Dirichlet law, so $T(x) = -\lim_{n\to\infty} \frac{\ln n}{\ln x_n}$ exists and $T(x)$ determines the considered time. For $y \in \mathcal{S}^{\downarrow}$. We define

$$q_s(x,y) = p_{T(x),T(x)+s}(x,y).$$

Then $(q_s)_{s\in[0,1[}$ is a homogeneous transition kernel for F. However, remark that to determine $T(x)$, we must know the other particles state and the branching property is lost.

II.3 General theory of time-inhomogeneous fragmentation processes

In this section, we extend the theory of time-homogeneous fragmentations to time-inhomogeneous fragmentations. For this, we will first work on fragmentations of partitions and next on mass-fragmentations.

II.3.1 Measure of an inhomogeneous fragmentation

Let us first define precisely the class of fragmentations we consider (which includes Ruelle's fragmentation). We denote $\mathcal{P}_n \setminus \{\mathbf{1}\}$ by $\mathcal{P}_n^*$.

Hypothesis II.7

In the sequel, we assume that $(\Pi(t), t \in [0,1[)$ is a standard time-inhomogeneous exchangeable fragmentation for which the following properties are fulfilled:

- *For all $n \in \mathbb{N}$, let τ_n be the time of the first jump of $\Pi_{|n}$ and λ_n be its law. We have*
$$\forall t \in [0,1[,\ \overline{\lambda}_n(t) := \lambda_n([t,1[) > 0$$
and λ_n is absolutely continuous with respect to Lebesgue measure with continuous density $g_n(t)$.

- *For all $\pi \in \mathcal{P}_n^*$, $h_\pi^n(t) = \mathbb{P}(\Pi_{|n}(t) = \pi \mid \tau_n = t)$ is a continuous function of t.*

Let us now define an instantaneous jump rate for a fragmentation fulfilling Hypothesis II.7. Let $\pi \in \mathcal{P}_n^*$ and set $h_\pi^n(t) = \mathbb{P}\left(\Pi_{|n}(t) = \pi \mid \tau_n = t\right)$. It is the law of the jump given τ_n. We set

$$f_n(t) = \lim_{s\to 0} \frac{1}{s} \mathbb{P}(\tau_n \in [t, t+s] \mid \tau_n \geq t) = \frac{g_n(t)}{\overline{\lambda}_n(t)},$$

and

$$q_{\pi,t} = h^n_\pi(t) f_n(t) = \lim_{s\to 0} \frac{1}{s} \mathbb{P}\left(\Pi_{|n}(\tau_n) = \pi \,\&\, \tau_n \in [t, t+s] \mid \tau_n \geq t\right).$$

It is the probability density that the process $\Pi_{|n}$ jumps at time t from the state $\mathbf{1}$ to the state π given that $\Pi_{|n}$ has not jumped before.

Proposition II.8

For $\pi \in \mathcal{P}_n$, $n' \geq n$, set $Q_{n',\pi} = \{\pi' \in \mathcal{P}_{n'}, \pi'_{|n} = \pi\}$. For each $t \in [0, 1[$, there exists a unique measure μ_t on $\mathcal{P}_\infty$ such that

$$\forall n \in \mathbb{N}\ \forall \pi \in \mathcal{P}^*_n\ \ \mu_t(Q_{\infty,\pi}) = q_{\pi,t} \text{ and } \mu_t(\mathbf{1}) = 0.$$

The family of measures $(\mu_t, t \geq 0)$ characterizes the law of the fragmentation.

Proof: We have

$$\forall n' > n, \forall \pi \in \mathcal{P}^*_n \sum_{\pi' \in Q_{n',\pi}} q_{\pi',t} = q_{\pi,t}. \tag{II.4}$$

In fact, at time t, if the process $\Pi_{|n'}$ has not jumped yet, it will jump between time t and time $t + dt$ to the state such that $\Pi_{|n} = \pi$ with probability $\sum_{\pi' \in Q_{n',\pi}} q_{\pi',t} dt$. Besides, we have the following equality:

$$\mathbb{P}\big(\Pi_{|n}(\tau_n) = \pi \,\&\, \tau_n \in [t, t+dt] \mid \tau_n \geq t\big) = \mathbb{P}\big(\Pi_{|n}(\tau_n) = \pi \,\&\, \tau_n \in [t, t+dt] \mid \tau_{n'} \geq t\big),$$

since the event that the block $[n']$ has already split, does not affect the process $\Pi_{|n}$. In fact, as $(\Pi_{|n}(t), t \in [0, 1[)$ is a Markov process, the law of the process $(\Pi_{|n}(t), t \in [t_0, 1[)$ depends only on $\Pi_{|n}(t_0)$. Therefore, we have Identity (II.4).

Let us now define $\mu_t(Q_{\infty,\pi}) = q_{\pi,t}$. By (II.4), this function can be extended to an additive function. By Caratheodory's Theorem, μ can be extended to a unique measure on $\mathcal{P}_\infty$.

We have now to prove that this family of measures determines the fragmentation law. To this end we just have to prove that the family of measures $(\mu_t, t \in [0, 1[)$ characterizes every jump rate of $\Pi_{|n}(t)$. Set $\pi, \pi' \in \mathcal{P}_n$, $t_0 \in [0, 1[$. Let τ'_n be the time of the first jump of $\Pi_{|n}(t)$ after t_0. We must express

$$\lim_{s\to 0} \frac{1}{s} \mathbb{P}\left(\Pi_{|n}(\tau'_n) = \pi' \,\&\, \tau'_n \in [t, t+s] \mid \Pi_{|n}(t_0) = \pi\right)$$

in terms of $(\mu_t, t \in [0,1[)$.
If π' can not be obtain from the fragmentation of one block of π, we clearly have:

$$\mathbb{P}\left(\Pi_{|n}\left(\tau'_n\right) = \pi' \ \&\ \tau'_n \in [t, t+dt] \mid \Pi_{|n}(t_0) = \pi\right) = 0.$$

Permuting the indices (which does not change the law by exchangeability), we can suppose $\pi = (A_1, \ldots, A_N)$ and $\pi' = (B_1, \ldots, B_k, A_2, \ldots, A_N)$ where $\pi'' = (B_1, \ldots, B_k)$ is a partition of the set A_1. Let $\tau'_{[A_i]}$ be the first jump time of $\Pi_{|A_i}$. Since distinct blocks evolve independently, we have:

$$\begin{aligned}
&\mathbb{P}\big(\Pi_{|n}\left(\tau'_n\right) = \pi' \ \&\ \tau'_n \in [t, t+dt] \mid \Pi_{|n}(t_0) = \pi\big) \\
&= \mathbb{P}\big(\Pi_{|A_1}\left(\tau_{[A_1]}\right) = \pi'' \ \&\ \tau_{[A_1]} \in [t, t+dt] \mid \Pi_{|A_1}(t_0) = \mathbf{1}\big) \prod_{i=2}^{N} \mathbb{P}\big(\tau_{[A_i]} > t \mid \tau_{[A_i]} > t_0\big) \\
&= \mathbb{P}\big(\Pi_{|A_1}\left(\tau_{[A_1]}\right) = \pi'' \ \&\ \tau_{[A_1]} \in [t, t+dt] \mid \tau_{[A_1]} > t_0\big) \prod_{i=2}^{N} \mathbb{P}\big(\tau_{[A_i]} > t \mid \tau_{[A_i]} > t_0\big) \\
&= \mathbb{P}\big(\Pi_{|A_1}\left(\tau_{[A_1]}\right) = \pi'' \ \&\ \tau_{[A_1]} \in [t, t+dt] \mid \tau_{[A_1]} > t_0\big) \prod_{i=1}^{N} \frac{\overline{\lambda}_{|A_i|}(t)}{\overline{\lambda}_{|A_i|}(t_0)}.
\end{aligned}$$

Thus we have:

$$\lim_{s \to 0} \frac{1}{s} \mathbb{P}\left(\Pi_{|n}\left(\tau'_n\right) = \pi' \ \&\ \tau'_n \in [t, t+s] \mid \Pi_{|n}(t_0) = \pi\right) = \mu_t\left(Q_{\infty, \pi''}\right) \prod_{i=1}^{N} \frac{\overline{\lambda}_{|A_i|}(t)}{\overline{\lambda}_{|A_i|}(t_0)}$$

and $\overline{\lambda}_n$ is easily expressed as a function of μ_t (cf. below). ■

Proposition II.9

The function from $[0,1[$ to the set of measures on $\mathcal{P}_\infty$ which at t associates μ_t constructed according to the proposition above, verifies:

- *μ_t is an exchangeable measure such that* $\mu_t\{\mathbf{1}\} = 0$ *and* $\forall n \in \mathbb{N}\ \mu_t\left(\{\pi \in \mathcal{P}_\infty, \pi_{|n} \neq \mathbf{1}\}\right) < \infty$,
- $\forall n \in \mathbb{N}\ \forall t \in [0,1[$ *we have* $\int_0^t \mu_u\left(\{\pi \in \mathcal{P}_\infty, \pi_{|n} \neq \mathbf{1}\}\right) du < \infty$.

Proof: The exchangeability is clear and we have

$$\mu_t\left(\{\pi \in \mathcal{P}_\infty, \pi_{|n} \neq \mathbf{1}\}\right) = f_n(t)$$

and

$$\int_0^t \mu_u\left(\{\pi \in \mathcal{P}_\infty, \pi_{|n} \neq \mathbf{1}\}\right) du = -\ln\left(\lambda_n\left(]t,1]\right)\right)$$

which is finite by Hypothesis II.7. ■

Set $\varepsilon_i = \{\{i\}, \{\mathbb{N} \setminus \{i\}\}\}$ and $\varepsilon = \sum_i \delta_{\varepsilon_i}$. So ε is a measure on $\mathcal{P}_\infty$. According to Bertoin [14], we know that for each exchangeable measure μ such that $\mu\{\mathbf{1}\} = 0$ and $\mu\left(\{\pi \in \mathcal{P}_\infty, \pi_{|n} \neq \mathbf{1}\}\right) < \infty$, we can find a unique measure ν on $\mathcal{S}^\downarrow$ (called dislocation measure) verifying $\nu(\mathbf{1}) = 0$ and $\int_{\mathcal{S}^\downarrow}(1-s_1)\nu(ds) < \infty$, and a unique constant $c \geq 0$ (called erosion coefficient) such that

$$\mu = \rho_\nu + c\varepsilon,$$

where ρ_ν denotes the measure on $\mathcal{P}_\infty$ associated to ν by the paint-box process.

So for $t \in [0,1[$ fixed, we can write $\mu_t = \rho_{\nu_t} + c_t\varepsilon$ where ν_t and c_t are the instantaneous dislocation and erosion rates of the fragmentation.

Proposition II.10

We have $\mu_t = \rho_{\nu_t} + c_t\varepsilon$ where ν_t and c_t fulfill the following properties:

- $$\forall t \in [0,1[\ \nu_t(\mathbf{1}) = 0 \text{ and } \int_{\mathcal{S}^\downarrow} (1-s_1)\,\nu_t(ds) < \infty, \tag{II.5}$$

- $$\forall u \in [0,1[\ \int_0^u \int_{\mathcal{S}^\downarrow} (1-s_1)\,\nu_t(ds)\,dt < \infty \text{ and } \int_0^u c_t dt < \infty. \tag{II.6}$$

Proof: The property (II.5) is clear. For the formula (II.6), we shall look at the proof of the theorem in the time-homogeneous case (cf. [14]). During the proof, we obtain the following upper bound:

$$\int_{\mathcal{S}^\downarrow} (1-s_1)\,\nu_t(ds) \leq \mu_t\left(\{\pi \in \mathcal{P}_\infty, \pi_{|2} \neq \mathbf{1}\}\right).$$

Then use Proposition II.9. For the upper bound concerning c_t, we write:

$$c_t = \mu_t(\{1\}, \mathbb{N} \setminus \{1\}) - \rho_{\nu_t}(\{1\}, \mathbb{N} \setminus \{1\}) = \mu_t(\{1\}, \mathbb{N} \setminus \{1\}).$$

■

Hence the law of a time-inhomogeneous fragmentation is characterized by a family $(\nu_t, c_t)_{0 \leq t < 1}$ where $(\nu_t)_{0 \leq t < 1}$ and $(c_t)_{0 \leq t < 1}$ fulfill (II.5) and (II.6). One calls ν_t the instantaneous dislocation rate and c_t the instantaneous erosion rate at time t of the fragmentation. We will next give a probabilistic interpretation of this family.

As for time-homogeneous fragmentations, we can construct a fragmentation with measure $(\mu_t, t \in [0,1[)$ considering a Poisson measure M on $[0,1[\times \mathcal{P}_\infty \times \mathbb{N}$ with intensity $\mu_t(d\pi)dt \otimes \sharp$ where $\sharp$ is the counting measure. Let M^n be the restriction of M to $[0,1[\times \mathcal{P}_n^* \times \{1,\ldots,n\}$. According to Proposition II.9, the intensity of the measure is finite on the interval $[0,t]$. Then, we are in a similar case as a time-homogeneous fragmentation (refer to [14] for a proof in the homogeneous case). Let us rearrange the atoms of M^n according to their first coordinate. For $n \in \mathbb{N}$, $(\pi, k) \in \mathcal{P}_n \times \mathbb{N}$, let $\Delta_n^{(\cdot)}(\pi,k)$ be the following sequence of partition of $[n]$:

$$\Delta_n^{(i)}(\pi,k) = \mathbf{1} \text{ if } i \neq k \qquad \text{and} \qquad \Delta_n^{(k)}(\pi,k) = \pi_{|n}.$$

We construct the process $(\Pi_{|n}(t), t \geq 0)$ in $\mathcal{P}_n$ with the following rules:

- $\Pi_{|n}(0) = \mathbf{1}$.
- $(\Pi_{|n}(t), t \geq 0)$ is a jump process which jumps at times s, atoms of M^n. More precisely, if (s, π, k) is an atom of M^n, we have $\Pi_{|n}(s) = FRAG(\Pi_{|n}(s^-), \Delta_n^{(\cdot)}(\pi,k))$.

We can then check that this construction is compatible with the restriction and the constructed process is a fragmentation with measure μ_t.

We also have a Poissonian construction of a mass-fragmentation (cf. [6]). First we use that if $F = (F(t), t \in [0,1[)$ is a mass-fragmentation with parameters $(\nu_t, 0)_{0\leq t<1}$, then $\tilde{F} = (e^{-\int_0^t c_s ds} F(t), t \in [0,1[)$ is a mass-fragmentation with parameters $(\nu_t, c_t)_{0\leq t<1}$. So, we remark that the family of instantaneous erosion coefficients plays only a deterministic role in the fragmentation. To find a Poissonian construction for the mass-fragmentations $(F(t), t \in [0,1[)$ with parameters $(\nu_t, 0)_{0\leq t<1}$, consider then a fragmentation of partitions $(\Pi(t), t \in [0,1[)$ such that $F = \Lambda(\Pi)$ where Λ is the function which associates to a partition its frequency sequence. So Π can be constructed from a Poisson measure M. Consider K, image of M by the mapping

$$\begin{array}{rcl} \mathcal{P}_\infty \times \mathbb{N} & \longrightarrow & \mathcal{S}^\downarrow \times \mathbb{N} \cup \infty \\ (\Delta(\cdot), k(\cdot)) & \longmapsto & (\Lambda(\Delta(\cdot)), f(\cdot, k(\cdot))), \end{array}$$

where f is the function which associates to k the frequency rank of the block $B_k(t^-)$. Berestycki [6] then proves that K is a Poisson measure on $[0,1[\times \mathcal{S}^\downarrow \times \mathbb{N}$ with intensity measure $\nu(ds)dt \otimes \sharp$.

Set

$$K = (t, S(t), k(t))_{t\in[0,1[} = (t, (s_1(t), s_2(t), \ldots), k(t))_{t\in[0,1[}.$$

Then, if $(t, S(t), k(t))$ is an atom of K, then at time t, the $k(t)$-th largest block of the fragmentation at time t^- will be fragmented according to $S(t)$.

Let us now determine the effects of a deterministic change-time on a fragmentation.

Proposition II.11

Let $(\Pi(t), t \in [0,1[)$ be a fragmentation with parameter $(c_t, \nu_t)_{0 \le t < 1}$. Set $\Pi'(t) = \Pi(\beta(t))$ where $\beta : [0,1[\to \mathbb{R}_+$ is a strictly increasing derivable function. Let J be the image of $[0,1[$ by β (J is thus an interval of $\mathbb{R}_+$).
Then $(\Pi'(t), t \in J)$ is a fragmentation with parameter $(c'_t; \nu'_t)_{t\in J}$ where

$$c'_t = \beta'(t) c_{\beta(t)} \qquad \nu'_t = \beta'(t)\nu_{\beta(t)}.$$

Proof: A Markov process remains a Markov process after a deterministic time-change. The law of $\Pi'(t+t')$ given $\Pi'(t) = \pi$, is $FRAG(\pi, \pi^{(\cdot)})$, where $\pi^{(\cdot)}$ is an iid sequence with law $\mathbb{P}_{\beta(t), \beta(t+t')-\beta(t)}$. Thus Π' is a fragmentation.
Let us calculate its jump rates $q'_{\pi,t}$.

$$\begin{aligned}
q'_{\pi,t}dt &= \mathbb{P}\left(\Pi'_{|n}\left(\tau'_n\right) = \pi \ \&\ \tau'_n \in [t, t+dt] \mid \tau'_n \ge t\right) \\
&= \mathbb{P}\left(\Pi_{|n}\left(\beta(\tau'_n)\right) = \pi \ \&\ \tau'_n \in [t, t+dt] \mid \tau'_n \ge t\right) \\
&= \mathbb{P}\left(\Pi_{|n}\left(\tau_n\right) = \pi \ \&\ \tau_n \in [\beta(t), \beta(t+dt)] \mid \tau_n \ge \beta(t)\right) \\
&\sim \mathbb{P}\left(\Pi_{|n}\left(\tau_n\right) = \pi \ \&\ \tau_n \in [\beta(t), \beta(t)+\beta'(t)dt)] \mid \tau_n \ge \beta(t)\right) \\
&\sim \beta'(t) q_{\pi,\beta(t)} dt.
\end{aligned}$$

So $q'_{\pi,t} = \beta'(t) q_{\pi,\beta(t)}$. We thus deduce similar relations between ν_t and ν'_t and between c_t and c'_t. ■

II.3.2 Law of the tagged fragment

An application of the above decomposition is for example to calculate the law of the frequency of the block containing the integer 1, $|\Pi_1(t)|$, for an exchangeable standard fragmentation. We have the following theorem:

Theorem II.12

There exists a process $(\xi(t), t \in [0,1[)$ with independent increments such that

$|\Pi_1(t)| = \exp(-\xi_t)$. *Its law is characterized by the identity:*

$$\mathbb{E}\Big(|\Pi_1(t)|^q\Big) = \mathbb{E}\Big(\exp(-q\xi_t)\Big) = \exp\left(-\int_0^t \phi_u(q)\,du\right), \qquad q > 0$$

$$\text{where } \phi_t(q) = c_t(q+1) + \int_{\mathcal{S}_1^\downarrow}\left(1 - \sum_{i=1}^{\infty} s_i^{q+1}\right)\nu_t(ds).$$

In the sequel, we will also use the notation $\psi(t,q) = \int_0^t \phi_u(q)\,du$.

Proof: This result is very close to the corresponding result in the homogeneous case. We just loose the stationarity of the increments of $\xi(t)$. The demonstration itself is similar to the homogeneous case and we just sketch the proof here. For more details, refer to [14].

We use the equality:

$$\mathbb{P}[\Pi_{|k+1}(t) = \mathbf{1}] = \mathbb{E}[|\Pi_1(t)|^k],$$

which we get by conditioning on $|\Pi_1(t)|$. Then remark that the event $\{\Pi_{|k+1}(t) = \mathbf{1}\}$ corresponds, looking at the Poissonian construction, to an absence of Poisson atom in the subset $[0,t] \times \{\pi \in \mathcal{P}_\infty, \pi_{|k+1}(t) \neq \mathbf{1}\} \times \{1\}$. So the formula is true for every positive integer. Besides, we remark that the law of $|\Pi_1(t)|$ is characterized by its moments, thanks to the independence of the increments (when you take the logarithm) and since the process takes values in $[0,1]$. By uniqueness of the analytic continuation, we deduce that the formula is true for every $q > 0$. And by the monotone convergence theorem, $\psi(t,q)$ is continuous in q at 0. ■

Thanks to this formula, we can characterize the processes which have proper frequencies, i.e. with $\sum_{i=1}^{\infty}|\pi_i| = 1$ a.s.

Proposition II.13

We have:

$$\mathbb{P}(\Pi(t) \text{ is proper}) = 1 \Leftrightarrow \left(c_u = 0 \text{ and } \nu_u\left(\sum_i s_i < 1\right) = 0 \text{ for } 0 \le u \le t \text{ a.e.}\right).$$

Proof: First remark

$$\lim_{k\to 0}\mathbb{E}[|\Pi_1(t)|^k] = \lim_{k\to 0}\mathbb{E}[|\Pi_1(t)|^k \mathbf{1}_{|\Pi_1(t)|\neq 0}] = \mathbb{E}[\mathbf{1}_{|\Pi_1(t)|\neq 0}] = 1 - \mathbb{P}(|\Pi_1(t)| = 0).$$

Then we have:

$$\begin{aligned}\mathbb{P}\left(\Pi\left(t\right) \text{ is proper }\right)=1 &\Leftrightarrow \mathbb{P}\left(\left|\Pi_1\left(t\right)\right|=0\right)=0\\ &\Leftrightarrow \exp\left(-\psi(t,0)\right)=1\\ &\Leftrightarrow \psi(t,0)=0\\ &\Leftrightarrow \phi_u\left(0\right)=0 \text{ for } 0\leq u\leq t \; a.e.\end{aligned}$$

■

Recall from [12] that if $(X\left(t\right), t \in [0,1[)$ is a time-homogeneous mass fragmentation, ϕ the Laplace exponent associated to the tagged fragment and $\mathcal{F}_t = \sigma\left(X\left(s\right), s \leq t\right)$, then

$$\exp\left(t\phi\left(p\right)\right)\sum_{i=1}^{\infty} X_i^{p+1}\left(t\right) \text{ is a } \mathcal{F}_t\text{-martingale.}$$

We can obtain a similar theorem in the time-inhomogeneous case.

Proposition II.14

Consider $(\Pi(t), t \in [0,1[)$ a time-inhomogeneous fragmentation of partitions. Let $X\left(t\right) = \left(X_i\left(t\right)\right) \in \mathcal{S}^{\downarrow}$ be its decreasing sequence of frequencies. Set $\mathcal{F}_t = \sigma\left(X\left(u\right); u \leq t\right)$. Let ϕ_u be its instantaneous Laplace exponent and $\psi(t,p) = \int_0^t \phi_u(p)du$. Then

$$M(t,p) = \exp\left(\psi(t,p)\right)\sum_{i=1}^{\infty} X_i^{p+1}\left(t\right) \text{ is a } \mathcal{F}_t\text{-martingale.}$$

Proof: It is the same idea as in the time-homogeneous case. Set $\mathcal{G}_t = \sigma\left(\Pi\left(u\right), u \leq t\right)$. Then $\mathcal{E}\left(t,p\right) = \exp(-p\xi_t + \psi(t,p))$ is an $\mathcal{G}_t$-martingale and we remark that $M(t,p)$ is the projection of $\mathcal{E}\left(t,p\right)$ on $\mathcal{F}_t$. ■

II.4 Application to Ruelle's cascades

II.4.1 Jump rates of Ruelle's fragmentation

Let $(\Pi(t), t \in [0,1[)$ be Ruelle's fragmentation with values in partitions. For each integer n, $(\Pi_{|n}, t \in [0,1[)$ is a Markov process in the finite space of partition of $[n]$. The law of such a process is entirely determined by its jump rates from one state to another.

Let us calculate its jump rates. Set $\pi = (\pi_1, \dots, \pi_k) \in \mathcal{P}_n^*$. Fix $t \in [0,1]$. Let $q_t(n_1, \dots, n_k)$ be the probability that $\Pi_{|n}(t)$ has blocks with size $(n_1, \dots, n_k)$. Recall (cf. Proposition II.5) that

$$q_t(n_1, \dots, n_k) = \frac{(k-1)!}{(n-1)!} t^{k-1} \prod_{i=1}^{k} [1-t]_{n_i - 1}.$$

So from Proposition II.3 and II.4

$$\begin{aligned}
\mathbb{P}\left(\tau_n \in [t, t+s], \Pi_{|n}(\tau_n) = \pi \mid \tau_n \geq t\right) &= \mathbb{P}\left(\Pi_{|n}(t+s) = \pi \mid \Pi_{|n}(t) = \mathbf{1}\right) \\
&= p_{t+s,-t}(n_1, \dots, n_k) \\
&= \frac{\left[\frac{-t}{t+s}\right]_k}{[-t]_n} \prod_{i=1}^{k} -[-t]_{n_i} \\
&\sim s \frac{(-1)^{k+1}(k-2)! \prod_{i=1}^{k} [-t]_{n_i}}{t[-t]_n}.
\end{aligned}$$

We point out that we could also have calculated this quantity using Proposition II.6 and Bayes' Formula. So we obtain the following proposition:

Proposition II.15

For $\pi = (\pi_1, \dots, \pi_k) \in \mathcal{P}_n^*$ and for $t \in [0,1[$ we have:

$$q_{\pi,t} = \frac{q_t(n_1, \dots, n_k)}{t(k-1)\, q_t(n)}.$$

II.4.2 Instantaneous erosion coefficient and dislocation measure

It is well known that the Bolthausen-Sznitman's coalescent is a process with proper frequencies (cf. Proposition II.13). So, the erosion coefficient c_t should be identically zero. We can check this with a short calculation. In fact, consider $\pi = \varepsilon_1 = \left\{\{1\}, \mathbb{N} \setminus \{1\}\right\}$ and $\pi_n = \pi_{|n}$. According to Proposition II.15, we have $q_{\pi_n,t} = \frac{q_t(1,n-1)}{t q_t(n)} = \frac{1}{n-1-t}$. And $c_t = \lim_{n\to\infty} q_{\pi_n,t} = 0$. Thus $c_t = 0$ for all $t \in [0,1[$.

Let us denote by $\mathcal{S}_1$ the set of positive sequences with sum 1. From a measure η on $\mathcal{S}_1$, we can define a measure p on $\mathcal{P}_\infty$ by the paint-box construction (cf. [55] p. 61):

Conditionally on a sequence $(s_i, i \geq 1)$ drawn with respect to the measure η, we

construct the following law on partitions:
1 is in the first block. Fix $n \geq 1$. Suppose Π_n has k blocks. The integer $n+1$ will be:

- in the block j with probability s_j (for $j \leq k$),
- in a new block with probability $1 - \sum_{i=1}^k s_i$.

So we have

$$p(\pi) = \mathbb{E}^\eta\Big(\prod_{i=1}^k s_i^{n_i-1} \prod_{i=1}^{k-1}(1 - \sum_{j=1}^i s_j)\Big), \tag{II.7}$$

where $\pi = (\pi_1, \dots, \pi_k)$ and $|\pi_i| = n_i$.

If the measure η is a dislocation measure (i.e verifies $\int_{\mathcal{S}^\downarrow}(1-s_1)\eta(ds) < \infty$), then p is finite on $\mathcal{P}_n^*$. In fact, for all $k \geq 2$, we have
$\prod_{i=1}^k s_i^{n_i-1} \prod_{i=1}^{k-1}(1 - \sum_{j=1}^i s_j) \leq 1 - s_1$.

Let us now look at the dislocation measure of Ruelle's fragmentation. In this direction, let us introduce the following measure:

Definition II.16

Fix $\alpha \in]0,1[$. Consider the measure η_α defined as follows on $\mathcal{S}_1$: first,

$$\eta_\alpha(s_1 \in dx) = \alpha x^{-\alpha}(1-x)^{-1}\mathbb{1}_{0<x<1}dx,$$

and second, conditionally on $s_1 = x$, the sequence $(s_{i+1}/(1-x), i \in \mathbb{N})$ has the law of a random variable with law $PD(\alpha, 0)$ of which the terms have been size-biased rearranged. We denote $PD(\alpha, -\alpha)$ the image of η_α by ranking the s_i in the decreasing order. $PD(\alpha, -\alpha)$ is then an infinite measure on $\mathcal{S}_1^\downarrow$.

Remark that the construction of the measure $PD(\alpha, -\alpha)$ is similar, except for the normalization, to the construction of a Poisson-Dirichlet measure with the forbidden parameter $\theta = -\alpha$.

Proposition II.17

Let us define p_α as the measure on $\mathcal{P}_\infty$ associated to η_α as above by the paint-box construction. Then p_α is an exchangeable measure on $\mathcal{P}_\infty$. Its EPPF for the partitions non-reduced to a single block is:

$$p_\alpha(n_1, \dots, n_k) = \frac{(k-2)!}{-[-\alpha]_n} \prod_{i=1}^k -[-\alpha]_{n_i} \quad \textit{for all } k \geq 2. \tag{II.8}$$

Proof: Let us first check that $\int_{\mathcal{S}^\downarrow}(1-s_1)\eta_\alpha(ds)<\infty$.

$$\int_{\mathcal{S}^\downarrow}(1-s_1)\eta_\alpha(ds)=\int_0^1(1-s_1)\alpha s_1^{-\alpha}(1-s_1)^{-1}ds_1=\frac{\alpha}{1-\alpha}. \quad \text{(II.9)}$$

Using formula (II.7) and the definition of η_α, we have:

$$\begin{aligned}
p_\alpha(\pi) &= \left(\int_0^1 x^{n_1-1}(1-x)^{\sum_{i=2}^k n_i}\eta_\alpha(s_1\in dx)\right)p_{\alpha,0}(n_2,\dots,n_k)\\
&= \alpha\left(\int_0^1 x^{n_1-1-\alpha}(1-x)^{n-n_1-1}dx\right)p_{\alpha,0}(n_2,\dots,n_k)\\
&= \alpha\frac{\Gamma(n_1-\alpha)\Gamma(n-n_1)}{\Gamma(n-\alpha)}\frac{(k-2)!}{\alpha(n-n_1-1)!}\prod_{i=2}^k-[-\alpha]_{n_i} \qquad \text{according to (II.2)}\\
&= \frac{[-\alpha]_{n_1}}{[-\alpha]_n}(k-2)!\prod_{i=2}^k-[-\alpha]_{n_i}\\
&= \frac{(k-2)!}{-[-\alpha]_n}\prod_{i=1}^k-[-\alpha]_{n_i}.
\end{aligned}$$

So, we find the foretold formula and this one is symmetric in the variables $(n_1,\dots,n_k)$, thus the measure is an exchangeable measure (cf. [55] Theorem 24). We also deduce that η_α is the image of $PD(\alpha,-\alpha)$ by a size-biased reordering and $p_\alpha=\rho_{PD(\alpha,-\alpha)}$ (where $\rho_{PD(\alpha,-\alpha)}$ is the measure on $\mathcal{P}_\infty$ obtained from $PD(\alpha,-\alpha)$ by the paint-box construction). ■

Next, we observe that for every partition π not reduced to one block, we have

$$q_{\pi,t}=\frac{1}{t}p_t(\pi).$$

Indeed, this follows from Proposition II.15 and Formula (II.3) of Pitman. In conclusion, we may now state the following theorem:

Theorem II.18

The instantaneous dislocation measure ν_t of Ruelle's fragmentation at time $t\in]0,1[$ is given by:

$$\nu_t=\frac{1}{t}PD(t,-t).$$

II.4.3 Absolute continuity of the dislocation measure with respect to $PD(\alpha,0)$

Let us recall that, if Π is a random partition with law $p_{\alpha,0}$ and K_n is the number of blocks of $\Pi_{|n}$, then the limit of K_n/n^α exists almost surely and has the Mittag-Leffler law with index α (cf. [55] Theorem 31).

Proposition II.19

For each $\alpha \in]0,1[$ the measure p_α is absolutely continuous with respect to the measure $p_{\alpha,0}$. More precisely, we have:

$$p_\alpha(d\pi) = \Gamma(1-\alpha) S_\alpha^{-1} p_{\alpha,0}(d\pi) \qquad \text{where } S_\alpha = \lim_{n\to\infty} \frac{K_n}{n^\alpha}.$$

Proof: Let $(\mathcal{F}_n)_{n\geq 1}$ be the filtration of $\Pi_{|n}$.

Fix $k \geq 2$. Set $p_\alpha^k = p_\alpha \mathbb{1}_{\mathcal{P}_k^*}$. We consider

$$M_{\alpha,n}^k = \frac{dp_\alpha^k}{dp_{\alpha,0}}\Big|\mathcal{F}_n.$$

Using formula (II.3) and (II.8), we have:

$$M_{\alpha,n}^k = \frac{\Gamma(1-\alpha)\Gamma(n)}{\Gamma(n-\alpha)(K_n-1)} \mathbb{1}_{\mathcal{P}_k^*} \qquad \text{for } n \geq k,$$

where K_n denotes the number of blocks of $\Pi_{|n}$. $M_{\alpha,n}^k$ is a positive martingale, thus it converges almost surely to a random variable M_α^k.

Let now use

$$\frac{K_n}{n^\alpha} \to S_\alpha \ \ \mathbb{P}_{\alpha,0} - \text{a.s.} \quad \text{and} \quad \frac{\Gamma(1-\alpha)\Gamma(n)}{\Gamma(n-\alpha)(K_n-1)} \sim \frac{\Gamma(1-\alpha)n^\alpha}{K_n}.$$

We deduce

$$M_\alpha^k = \frac{dp_\alpha^k}{dp_{\alpha,0}} = \Gamma(1-\alpha) S_\alpha^{-1} \mathbb{1}_{\mathcal{P}_k^*} \ \ \mathbb{P}_{\alpha,0} - \text{a.s.}$$

So, according to martingale theory (cf. [29] p.210), for all $A \subset \mathcal{P}_k^*$, we have :

$$p_\alpha(A) = \mathbb{E}_{\alpha,0}\left(\Gamma(1-\alpha) S_\alpha^{-1} \mathbb{1}_A\right) + p_\alpha(A \cap \{S=0\}),$$

where $S = \limsup \frac{K_n}{n^\alpha}$.

Set $x \in]0,1[$. Let us define $q_\alpha(\cdot) = c p_\alpha(\ \cdot \mid |\Pi_1| \in dx)$ where c is chosen such that q_α is a probability. Let $s = (s_1, \ldots) \in \mathcal{S}^\downarrow$ be the frequency sequence of a partition with law q_α. According to the construction of p_α, we have

$$(s_{i+1})_{i\in\mathbb{N}} \overset{law}{=} (1-x)(p_i)_{i\in\mathbb{N}},$$

where $(p_i)_{i\in\mathbb{N}}$ has the $PD(\alpha,0)$ law.

According to Lemma 34 of Pitman's course [55], for a random partition, S exists and belongs almost surely to $]0,\infty[$ iff there exists Z random variable on $]0,\infty[$ such that $P_i \sim Zi^{-1/\alpha}$, where P_i is the decreasing sequence of the frequencies. Here we know the existence of such a random variable $Z \in]0, \infty[$ for a $PD(\alpha, 0)$ law. Set $Y = (1-x)Z$ then

$$s_i \sim Yi^{-1/\alpha}.$$

So we have

$$p_\alpha(S = 0 \mid |\Pi_1| \in dx) = 0.$$

Thus

$$p_\alpha(S = 0) = 0.$$

We conclude that

$$\forall A \in \mathcal{P}_\infty \text{ such that } \mathbf{1} \notin \overline{A} \quad p_\alpha(A) = \mathbb{E}_{\alpha,0}\left(\Gamma(1-\alpha)S_\alpha^{-1}\mathbb{1}_A\right).$$

■

Theorem II.20

The dislocation measure of Ruelle's fragmentation at time $t \in]0,1[$ is absolutely continuous with respect to the measure $PD(t,0)$. More precisely, we have for all continuous function f on $\mathcal{S}^\downarrow$:

$$\nu_t(f) = \frac{1}{t}\mathbb{E}_{(t,0)}\left(L_t^{-1} f(V)\right)$$

where $L_\alpha = \lim_{n\to\infty} nV_n^\alpha$.

Proof: We use that if $(s_i)_{i\geq 1} \in \mathcal{S}_1^\downarrow$ is the frequency sequence of an $(\alpha, 0)$-partition Π_∞, then $\Gamma(1-\alpha)L_\alpha$ exists almost surely and it is equal almost surely to $S_\alpha = \lim_{n\to\infty} \frac{K_n}{n^\alpha}$ (cf. [55] Theorem 36). Use Theorem II.18 to finish the proof. ■

Remark: L_α is not a continuous function on $\mathcal{S}_1^\downarrow$.

II.4.4 Law of the tagged fragment

In this section, we determine the law of the tagged fragment. Actually, its law has already been determined by Pitman [54]. He proves that $|\Pi_1(t)|$ has a $\beta(1-t, t)$ law. So we check that we find the same result.

Hence, according to Section II.3.2, we shall calculate

$$\phi_t(k) = \int_{\mathcal{S}^\downarrow} \left(1 - \sum_{i=1}^{\infty} s_i^{k+1}\right) \nu_t(ds).$$

Recall that p_t denotes the measure on $\mathcal{P}_\infty$ associated to the measure $PD(t,-t)$. We have

$$\mathbb{E}[|\Pi_1(t)|^k] = \exp\left(-\int_0^t \phi_u(k)\,du\right).$$

Thus

$$\begin{aligned}\phi_t(k) &= \mathbb{E}_{v_t}\left(\rho_s\left(\Pi_{|k+1} \neq \mathbf{1}\right)\ |s\right) \\ &= \frac{1}{t} p_t\left(\Pi_{|k+1} \neq \mathbf{1}\right).\end{aligned}$$

So we must calculate $p_t\left(\Pi_{|k+1} \neq \mathbf{1}\right)$. We will do this recursively. For $k=1$, we have

$$p_t\left(\Pi_{|2} \neq \mathbf{1}\right) = \frac{[-t]_1^2}{-[-t]_2} = \frac{t}{1-t},$$

and for $k \geq 2$

$$\begin{aligned}p_t\left(\Pi_{|k+1} \neq \mathbf{1}\right) &= p_t\left(\Pi_{|k} \neq \mathbf{1}\right) + p_t\left(\Pi_{|k+1} = \left\{\{1,\dots,k\},\{k+1\}\right\}\right) \\ &= p_t\left(\Pi_{|k} \neq \mathbf{1}\right) + \frac{t}{k-t}.\end{aligned}$$

Thus we have:

$$p_t\left(\Pi_{|k+1} \neq \mathbf{1}\right) = \sum_{i=1}^{k} \frac{t}{i-t} \qquad \text{and so} \qquad \int_0^t \phi_u(k)\,du = \ln\left(\prod_{i=1}^{k} \frac{i}{i-t}\right).$$

So we deduce

$$\mathbb{E}[|\Pi_1(t)|^k] = \prod_{i=1}^{k} \frac{i-t}{i}.$$

The right-hand side coincides with the k-th moment of a $\beta(1-t,t)$ law. So $|\Pi_1(t)|$ has a $\beta(1-t,t)$ law and we deduce:

$$\forall k>0,\ \mathbb{E}[|\Pi_1(t)|^k] = \frac{\Gamma(k+1-t)}{\Gamma(1-t)\,\Gamma(k+1)}.$$

More generally, we can determine the law of the process $(|\Pi_1(t)|, t\in[0,1[)$. By the property of homogeneity of the fragmentation in space, the process

$$\left(\frac{|\Pi_1(t+s)|}{|\Pi_1(t)|}, s\in[0,1-t[\right)$$

is independent of $|\Pi_1(t)|$ (cf. Theorem II.12). So we can calculate the finite dimensional law of the process $(|\Pi_1(t)|, t\in[0,1[)$ and we deduce that the process has the same law as the process $\left(\frac{\gamma(1-t)}{\gamma(1)}, t\in[0,1[\right)$ (result already proved by Pitman [54]).

Remark : We have also an expression for $\psi(t,k)$:

$$\psi(t,k)=\ln\left(\frac{\Gamma(1-t)\,\Gamma(k+1)}{\Gamma(k+1-t)}\right).$$

II.5 Behavior of the fragmentation at large and small times

II.5.1 Convergence of the empirical measure

Let $(\Pi(t), t\in[0,1[)$ be a Ruelle's fragmentation on partitions. Let $(X(t), t\in[0,1[)$, $X(t)=(X_i(t))_{i\geq 1}\in\mathcal{S}_1^{\downarrow}$ be its process of ranked frequencies. We are interested in the empirical measure ρ_t defined by :

$$\rho_t=\sum_{i=1}^{\infty}X_i(t)\delta_{(t-1)\ln X_i(t)}.$$

Proposition II.21

For every bounded continuous function f on $\mathbb{R}_+$:

$$\lim_{t\to 1}\int f(y)\rho_t(dy)=\int_0^{\infty}f(y)e^{-y}dy \quad \text{in } L^2(\mathbb{P}).$$

Proof : We split the proof in two parts. We will successively prove the following two points:

$$\lim_{t\to 1}\mathbb{E}\left(\int f(y)\rho_t(dy)\right)=\int_0^{\infty}f(y)e^{-y}dy, \tag{II.10}$$

$$\lim_{t\to 1}\mathbb{E}\left[\left(\int f(y)\rho_t(dy)\right)^2\right]=\left(\int_0^{\infty}f(y)e^{-y}dy\right)^2. \tag{II.11}$$

Set $\xi_t=-\ln|\Pi_1(t)|$. Let us recall

$$|\Pi_1(t)|\sim\beta(1-t,t),$$

and observe :

$$\mathbb{E}\left(\int f(y)\rho_t(dy)\right)=\mathbb{E}\Big(f((1-t)\xi_t)\Big).$$

The following lemma clearly implies (II.10).

Lemma II.22

Set $\xi_t=-\ln|\Pi_1(t)|$ where $\Pi(t)$ is Ruelle's fragmentation. Then

$$\lim_{t\to 1}(1-t)\xi_t=\mathbf{e} \quad \textit{in distribution,}$$

where $\mathbf{e}$ *denotes the exponential law with parameter* 1.

Proof: Let us calculate the Laplace transform of $(1-t)\xi_t$.

$$\begin{aligned}\mathbb{E}\left(e^{-q(1-t)\xi_t}\right) &= \mathbb{E}\left(|\Pi_1(t)|^{q(1-t)}\right) \\ &= \frac{\Gamma\left(q(1-t)+1-t\right)}{\Gamma\left(1-t\right)\Gamma\left(q(1-t)+1\right)} \\ &\underset{t\to 1}{\longrightarrow} \frac{1}{q+1}.\end{aligned}$$

Since $\frac{1}{q+1}$ is the Laplace transform of the exponential law, by Lévy's Theorem, $(1-t)\xi_t$ converges in law to $\mathbf{e}$. ■

To prove (II.11), we consider $\xi_t^{'} = -\ln|\Pi_2(t)|$ where $\Pi_2(t)$ is the block containing the integer 2. Observe that ξ_t and $\xi_t^{'}$ have the same law but are not independent, and that

$$\mathbb{E}\left[\left(\int f(y)\rho_t(dy)\right)^2\right] = \mathbb{E}\left[f\Big((1-t)\xi_t\Big)f\left((1-t)\xi_t^{'}\right)\right].$$

Set $T = \inf\{t > 0, \Pi_1(t) \neq \Pi_2(t)\}$, so T is almost surely finite and conditionally on T, ξ_T and $\xi_T^{'}$, the processes $(\xi_t, t \geq T)$ and $(\xi_t^{'}, t \geq T)$ are independent. From this, we deduce (II.11) and then the L^2-convergence of $\int f(y)\rho_t(dy)$ (refer to [12] for details). ■

So, informally, this proposition proves that, if we consider the size of a typical fragment $X(t)$, then, as t tends to 1, we have

$$|\ln X(t)| \sim \frac{C}{1-t},$$

where C is a random factor.

II.5.2 Additive martingale

In this section, we aim at studying the convergence of the martingale $M(t,p)$ defined in Section II.3.2 and we follow the ideas of Bertoin and Rouault [18] who introduce a new probability to prove the convergence.

Recall the following notation:
$\mathcal{F}_t = \sigma\left(X_i\left(u\right), u \leq t\right)$ is the filtration of the frequency sequence.
$\mathcal{G}_t = \sigma\left(\Pi\left(u\right), u \leq t\right)$ is the filtration of the fragmentation process on partitions.
So we have $\mathcal{F}_t \subseteq \mathcal{G}_t$.

Set $\xi_t = -\ln(|\Pi_1(t)|)$. It is an increasing process with independent increments.
Next $M(t,p) = \exp(\psi(t,p)) \sum_{i=1}^{\infty} |X_i(t)|^{p+1}$ is then a $\mathcal{F}_t$-martingale.
Further $\mathcal{E}(t,p) = \exp(\psi(t,p) - p\xi_t)$. $\mathcal{E}(\cdot,p)$ is a $\mathcal{G}_t$-martingale.
As $\mathbb{E}(|\Pi_1(t)|^p \mid X(t)) = \sum_i X_i(t)^{p+1}$, we have $\mathbb{E}(\mathcal{E}(t,p) \mid \mathcal{F}_t) = M(t,p)$.
We denote $\mathbb{Q}$ the probability on $\mathcal{G}$ defined by:

$$d\mathbb{Q}_{|\mathcal{G}_t} = \mathcal{E}(t,p)\, d\mathbb{P}_{|\mathcal{G}_t}. \text{ So we have also } d\mathbb{Q}_{|\mathcal{F}_t} = M(t,p)\, d\mathbb{P}_{|\mathcal{F}_t}.$$

Proposition II.23

Fix $p > 0$. We have:

$$\lim_{t \to 1} M(t,p) = 0 \quad \mathbb{P}\text{-a.s.}$$

Proof: A martingale theorem (cf. [29] p.210) asserts that if $\limsup M(t,p) = \infty$ $\mathbb{Q}$-a.s., then $\lim M(t,p) = 0$ $\mathbb{P}$-a.s.

We have

$$M(t,p) \geq \exp(\psi(t,p)) |\Pi_1(t)|^{p+1} = \exp(\psi(t,p) - (p+1)\xi_t).$$

Set $N_t = \psi(t,p) - (p+1)\xi_t$. We will prove that $\limsup N_t = \infty$ $\mathbb{Q}$-a.s.

Let us recall that, under $\mathbb{P}$, $|\Pi_1(t)|$ has $\beta(1-t,t)$ law. So for all $\lambda \geq 0$ we have:

$$\mathbb{Q}(\xi_t \geq \lambda) = \mathbb{E}^{\mathbb{P}}\left(\mathcal{E}(t,p)\, \mathbb{1}_{\{\xi_t \geq \lambda\}}\right) = \frac{\Gamma(p+1)}{\Gamma(p+1-t)\Gamma(t)} \int_0^{e^{-\lambda}} x^{p-t}(1-x)^{t-1}\, dx.$$

So for $A \leq \psi(t,p)$,

$$\mathbb{Q}(N_t \leq A) = \mathbb{Q}\left(\xi_t \geq \frac{\psi(t,p) - A}{p+1}\right) = \frac{\Gamma(p+1)}{\Gamma(p+1-t)\Gamma(t)} \int_0^{e^{-\frac{\psi(t,p)-A}{p+1}}} x^{p-t}(1-x)^{t-1}\, dx.$$

Recall $\psi(t,p) \sim -\ln(1-t)$ as $t \uparrow 1$. Choose $A(t) = -\frac{1}{3}\ln(1-t)$. So for t large enough, we have $\psi(t,p) - A(t) \geq -\frac{1}{3}\ln(1-t)$.
Set $g(t) = (1-t)^{\frac{1}{3(p+1)}}$. We have:

$$\begin{aligned}
\mathbb{Q}(N_t \leq A(t)) &\leq \frac{\Gamma(p+1)}{\Gamma(p+1-t)\Gamma(t)} \int_0^{g(t)} x^{p-t}(1-x)^{t-1}\, dx \\
&\leq \frac{\Gamma(p+1)}{\Gamma(p+1-t)\Gamma(t)} (1-g(t))^{t-1} \frac{1}{p+1-t} g(t)^{p+1-t} \\
&\leq \varepsilon_p(t),
\end{aligned}$$

where $\varepsilon_p(t)$ is a function with limit 0 at $t = 1$.

So $\lim_{t\to 1} \mathbb{Q}\left(N_t \geq A\left(t\right)\right) = 1$ and then $\mathbb{Q}\left(\limsup N_t < \infty\right) = 0$. We deduce:

$$\limsup_{t\to 1} M\left(t,p\right) = \limsup_{t\to 1} N\left(t,p\right) = \infty \ \mathbb{Q}\text{-a.s.} \quad \text{and so } \lim_{t\to 1} M\left(t,p\right) = 0 \quad \mathbb{P}\text{-a.s.}$$ ∎

Remark : In the case $p = 0$, as the process has proper frequencies, we have for all $t \in [0,1[$, $M\left(t,p\right) = 1 \quad \mathbb{P}$-a.s.

According to the value of $\psi(t,p)$, we deduce that,for all fixed $p > 0$, we deduce that, for t close enough to 1^-:

$$\sum_{i=1}^{\infty} X_i^{p+1}(t) = o(1-t) \ a.s.$$

whereas we have

$$\mathbb{E}\left(\sum_{i=1}^{\infty} X_i^{p+1}(t)\right) \sim \frac{1-t}{p}.$$

Let us notice that the behavior of the martingale $M(t,p)$ differs from its behavior in the time-homogeneous case. Actually, for time-homogeneous fragmentation, Bertoin and Rouault [18] proved that there exists a critical value $\overline{p} > 0$ such that $M(t,p)$ converges in $L^1(\mathbb{P})$ for all $0 \leq p < \overline{p}$, and converges to 0 $\mathbb{P}$-a.s. for all $p \geq \overline{p}$. In our settings, this result does not hold.

II.5.3 Small times behavior

In this section, we obtain information on the behavior of the two largest blocks of Ruelle's fragmentation at small times. In this direction, we use the following results due to Berestycki [6].

Let $X_k(t)$ be the frequency of the k-th largest block at time t of Ruelle's fragmentation. Recall that Ruelle's fragmentation can be constructed from a Poisson measure K on $[0,1[\times \mathcal{S}^{\downarrow} \times \mathbb{N}$ with intensity $(\nu_t(ds)dt) \otimes \sharp$. Set

$$K = (t, S(t), k(t))_{t\in[0,1[} = (t, (s_1(t), s_2(t), \ldots), k(t))_{t\in[0,1[}.$$

Let $(S^{(i)}(t), t \in [0,1[) = (s_1^{(i)}(t), s_2^{(i)}(t), \ldots, t \in [0,1[)$ be the Poisson measure obtained from K restricted to the atoms such that $k(t) = i$. So, it is a Poisson measure with intensity $\nu_t(ds)dt$.

Set

$$R(t) = \max_{s\leq t} s_2^{(1)}(s).$$

Lemma II.24

- *For t small enough, we have $X_1(t) = \exp(-\xi_t)$ a.s. where ξ_t is an increasing process with independent increments and such that:*

$$\forall k > 0,\ \mathbb{E}\left[\exp(-k\xi_t)\right] = \frac{\Gamma(k+1-t)}{\Gamma(1-t)\,\Gamma(k+1)}.$$

- $$X_2(t) \sim R(t) \text{ as } t \to 0^+ \text{ a.s.}$$

The proof is the same as in Berestycki [6], since there, time-homogeneity of the fragmentation plays no role.

Let us now determine the behavior of $R(t)$.

Proposition II.25

Fix $T_0 \in]0, 1/2[$. Then there exist three strictly positive constants C_1,C_2, C_3 such that for all $\lambda > 0$ and for all $t \in]0, T_0[$,

$$\exp(-C_1\lambda - C_3 t) \leq \mathbb{P}\left(R(t) \leq \exp\left(-\frac{\lambda}{t}\right)\right) \leq \exp(-C_2\lambda + C_3 t).$$

To estimate the distribution of $R(t)$, we study $\nu_t(s_2 \geq \varepsilon)$ for a fixed ε. Indeed,

$$\mathbb{P}(R(t) \leq \varepsilon) = \exp(-\int_0^t \nu_u(s_2 \geq \varepsilon)du),$$

and Proposition II.25 follows from the following lemma:

Lemma II.26

Fix $T_0 \in]0, 1/2[$. Then there exist three strictly positive constants C_1,C_2, C_3 such that for all $\varepsilon \in]0, 1[$ and for all $t \in]0, T_0[$,

$$-t(C_2 \ln \varepsilon + C_3) \leq \int_0^t \nu_u(s_2 \geq \varepsilon)du \leq -t(C_1 \ln \varepsilon - C_3).$$

Proof: We begin with the upper bound. If $(s_i)_{i\geq 1}$ is an element of $\mathcal{S}_1^{\downarrow}$, we denote $(\tilde{s}_i)_{i\geq 1}$ a size-biased rearrangement. We have:

$$s_2 \geq \varepsilon \Rightarrow s_1 \leq 1 - \varepsilon \Rightarrow \tilde{s}_1 \leq 1 - \varepsilon,$$

so

$$\nu_t(s_2 \geq \varepsilon) \leq \nu_t(s_1 \leq 1-\varepsilon) \leq \nu_t(\tilde{s_1} \leq 1-\varepsilon).$$

According to Theorem II.18, we know the law of $\tilde{s}_1$ under ν_t:

$$\begin{aligned} \nu_t(\tilde{s}_1 \leq 1-\varepsilon) &= \int_0^{1-\varepsilon} (1-y)^{-1} y^{-t} dy \\ &\leq \left(\int_0^{1/2} 2y^{-t} dy + \int_{1/2}^{1-\varepsilon} 2^t (1-y)^{-1} dy \right) \\ &\leq \left(-2^t \ln \varepsilon + \frac{2^t}{1-t} \right) \\ &\leq 2\left(-\ln \varepsilon + 2\right) \qquad \text{for } t \leq \frac{1}{2}. \end{aligned}$$

So we obtain

$$\int_0^t \nu_u(s_2 \geq \varepsilon) du \leq -t(2 \ln \varepsilon - 4).$$

Let us now prove the lower bound. First, we give a lower bound of $\int_0^t \nu_u(\tilde{s}_2 \geq \varepsilon) du$ and then we will deduce the lemma.

$$\begin{aligned} \nu_t(\tilde{s}_2 \in dx) &= \int_0^{1-x} \nu_t(\tilde{s}_1 \in dy) \nu_t(\tilde{s}_2 \in dx \mid \tilde{s}_1 \in dy) \\ &= \frac{1}{\Gamma(1-t)\Gamma(t)} \int_0^{1-x} (1-y)^{-1} y^{-t} \left(\frac{x}{1-y}\right)^{-t} \left(1 - \frac{x}{1-y}\right)^{t-1} \frac{dx}{1-y} dy \\ &= \frac{x^{-t} dx}{\Gamma(1-t)\Gamma(t)} \int_0^{1-x} (1-y)^{-1} y^{-t} \left(1-y-x\right)^{t-1} dy. \end{aligned}$$

Set

$$A = \int_\varepsilon^1 \int_0^{1-x} x^{-t} (1-y)^{-1} y^{-t} \left(1-y-x\right)^{t-1} dy dx,$$

so

$$\nu_t(\tilde{s}_2 \geq \varepsilon) = \frac{1}{\Gamma(1-t)\Gamma(t)} A.$$

We now calculate a lower bound for A:

$$\begin{aligned}
A &= \int_0^{1-\varepsilon}\int_\varepsilon^{1-y} x^{-t}(1-y)^{-1}y^{-t}\,(1-y-x)^{t-1}\,dxdy \\
&= \int_0^{1-\varepsilon}\left(\int_{\frac{\varepsilon}{1-y}}^1 z^{-t}\,(1-z)^{t-1}\,dz\right)(1-y)^{-1}y^{-t}dy \\
&= \int_\varepsilon^1\left(\int_{\frac{\varepsilon}{y}}^1 z^{-t}\,(1-z)^{t-1}\,dz\right)y^{-1}(1-y)^{-t}dy \\
&\geq \int_\varepsilon^1\left(\int_{\frac{\varepsilon}{y}}^1 (1-z)^{t-1}\,dz\right)y^{-1}(1-y)^{-t}dy \\
&\geq \frac{1}{t}\int_\varepsilon^1\left(1-\frac{\varepsilon}{y}\right)y^{-1}(1-y)^{-t}dy \\
&\geq \frac{1}{t}\int_\varepsilon^1\left(1-\frac{\varepsilon}{y}\right)y^{-1}dy \\
&\geq \frac{1}{t}\left(-\ln\varepsilon-1\right).
\end{aligned}$$

So

$$\nu_t(\tilde{s}_2\geq\varepsilon)\geq\frac{1}{\Gamma(1-t)\Gamma(t)t}\left(-\ln\varepsilon-1\right).$$

As $\Gamma(1-t)\Gamma(t)t=\frac{\pi t}{\sin(\pi t)}$ is a positive function which is bounded on $]0,T_0[$, let $1/C_2$ be its maximum. By integration, we obtain:

$$\int_0^t \nu_u(\tilde{s}_2\geq\varepsilon)du\geq tC_2\left(-\ln\varepsilon-1\right).$$

We would like now to deduce the lower bound for $\int_0^t\nu_u(s_2\geq\varepsilon)du$. We use

$$\nu_u(s_2\geq\varepsilon)\geq\nu_u(\tilde{s}_2\geq\varepsilon)-\nu_u(\tilde{s}_2>s_2),$$

and

$$\nu_u(\tilde{s}_2>s_2)=\nu_u(\tilde{s}_2=s_1)\leq\nu_u(\tilde{s}_1\neq s_1)=\int_{\mathcal{S}^\downarrow}(1-s_1)\nu_u(ds)\leq\int_{\mathcal{S}^\downarrow}(1-\tilde{s_1})\nu_u(ds).$$

We have already seen that

$$\int_{\mathcal{S}^\downarrow}(1-\tilde{s_1})\nu_u(ds)=\frac{1}{1-u}\qquad\text{(cf. Formula (II.9)).}$$

So, for all $t\leq T_0$, we have

$$\int_0^t\nu_u(\tilde{s}_2>s_2)du\leq-\ln(1-t)\leq\frac{1}{1-T_0}t.$$

Hence

$$\int_0^t\nu_u(s_2\geq\varepsilon)\geq t\left(-C_2\ln\varepsilon-C_3\right).$$

■

We can then deduce the lower-asymptotic behavior of $X_2(t)$ from this theorem.

Proposition II.27

There exists a constant $\delta > 0$ such that almost surely

$$\begin{cases} \liminf_{t\to 0} |\ln t|^{\gamma/t} X_2(t) = 0 & \textit{if } \gamma < \delta \\ \liminf_{t\to 0} |\ln t|^{\gamma/t} X_2(t) = \infty & \textit{if } \gamma > \delta. \end{cases}$$

Proof: According to Theorem II.24, we just have to prove the proposition replacing $X_2(t)$ by $R(t)$. Set $\gamma > \frac{1}{C_2}$. Choose $\beta > 0$ such that $\gamma > \frac{e^\beta}{C_2}$. For $i \in \mathbb{N}$, set $t_i = e^{-i\beta}$ and $f(t) = \gamma \ln(-\ln t)$. For $t \in [0, e^{-1}[$, $f(t)$ is a decreasing positive function. For $t \in [t_{i+1}, t_i]$, we have

$$R(t) \geq R(t_{i+1}) \text{ and } \exp\left(-\frac{f(t_i)}{t_i}\right) \geq \exp\left(-\frac{f(t)}{t}\right).$$

So if we prove

$$R(t_{i+1}) \geq \exp\left(-\frac{f(t_i)}{t_i}\right) \tag{II.12}$$

almost surely for i large enough, then we will deduce

$$\forall \gamma > \frac{1}{C_2},\ \liminf_{t\to 0} (\ln \frac{1}{t})^{\gamma/t} R(t) \geq 1 \text{ a.s.},$$

$$\text{and so} \quad \forall \gamma > \frac{1}{C_2},\ \liminf_{t\to 0} (\ln \frac{1}{t})^{\gamma/t} R(t) = \infty \text{ a.s.}$$

To prove (II.12), we apply Borel-Cantelli's Lemma. Using Proposition II.25, we obtain:

$$\mathbb{P}\left(R(t_{i+1}) \leq \exp\left(-\frac{f(t_i)}{t_i}\right)\right) \leq K(\beta i)^{-C_2\gamma e^{-\beta}}.$$

Thanks to the choice of γ and β, the series converges.

For the second part of the proposition, we use an extension of Borel-Cantelli's Lemma when the sum diverges but the events are not independent (cf. [39]): Let $(H_i)_{i\geq 1}$ be a sequence of events such that $\sum \mathbb{P}(H_i)$ diverges and

$$\forall N \geq 1,\ \frac{\sum_{i,j=1}^N \mathbb{P}(H_i \cap H_j)}{\left(\sum_{i=1}^N \mathbb{P}(H_i)\right)^2} \leq M. \tag{II.13}$$

Then the set $\{i, \omega \in H_i\}$ is infinite with a probability larger than $1/M$.

In our case, we fix a $\gamma < 1/C_1$ and a $\varepsilon > 0$ such that $(1+\varepsilon)\gamma C_1 < 1$. Set $t_i = e^{-i^{1+\varepsilon}}$ and $H_i = \{R(t_i) \leq (\ln(1/t_i))^{\gamma/t_i}\}$. Fix $i, j \geq 1$. Recall that $R(t)$ is the record process of a point Poisson process. So we have

$$\begin{aligned}\mathbb{P}(H_i \cap H_{j+i}) &= \mathbb{P}(H_i)\mathbb{P}(H_{i+j})\exp\left(\int_0^{t_{i+j}} \nu_u(s_2 \geq (\ln(1/t_i))^{\gamma/t_i})du\right) \\ &\leq K\mathbb{P}(H_i)\mathbb{P}(H_{i+j})\exp\left((1+\varepsilon)C_1\gamma \ln i e^{-(1+\varepsilon)i^{\varepsilon}}\right) \\ &\leq K'\mathbb{P}(H_i)\mathbb{P}(H_{i+j}).\end{aligned}$$

(We have used that $(i+j)^{1+\varepsilon} - i^{1+\varepsilon} \geq (1+\varepsilon)i^{\varepsilon}$ for all $i, j \geq 1$). With this upper bound, we deduce that the sequence H_i verifies (II.13). We now have to prove that the sum of probabilities diverges. Using Proposition II.25, we obtain:

$$\sum_i \mathbb{P}(H_i) \geq K \sum_i i^{-C_1\gamma(1+\varepsilon)}.$$

Thus this series diverges thanks to our choice of γ and ε. We now apply the 0-1 law to prove that the probability that the set $\{i, \omega \in H_i\}$ is infinite, is equal to 1.

So we have proved

$$\begin{cases} \liminf_{t\to 0}(\ln \frac{1}{t})^{\gamma/t}R(t) = 0 & \text{a.s.} \quad \forall \gamma < \frac{1}{C_1} \\ \liminf_{t\to 0}(\ln \frac{1}{t})^{\gamma/t}R(t) = \infty & \text{a.s.} \quad \forall \gamma > \frac{1}{C_2}. \end{cases}$$

Thus we deduce that there exists almost surely a (random) critical $\gamma_c \in]1/C_1, 1/C_2[$ such that

$$\begin{cases} \liminf_{t\to 0}(\ln \frac{1}{t})^{\gamma/t}R(t) = 0 & \forall \gamma < \gamma_c \\ \liminf_{t\to 0}(\ln \frac{1}{t})^{\gamma/t}R(t) = \infty & \forall \gamma > \gamma_c. \end{cases}$$

By the 0-1 law, the law of γ_c is trivial, i.e. it exists δ verifying Proposition II.27 ■

We can also determine the upper asymptotic behavior of $X_2(t)$:

Proposition II.28

We have almost surely

$$\begin{cases} \limsup_{t\to 0} \exp(\frac{1}{t}(-\ln(t))^{-\beta})X_2(t) = \infty & \text{if } \beta > 1 \\ \limsup_{t\to 0} \exp(\frac{1}{t}(-\ln(t))^{-\beta})X_2(t) = 0 & \text{if } \beta \leq 1. \end{cases}$$

Proof: We use the same approach as for the infimum. Fix $\beta > 1$. Set $t_i = e^{-i}$ and $f(t) = \exp(-\frac{1}{t}(-\ln(t))^{-\beta})$. We want to prove that $R(t) \leq f(t)$ almost surely for t small enough. As f is a decreasing function and R(t) an increasing process, we have $R(t) \leq R(t_i)$ and $f(t_{i+1}) \leq f(t)$. So we just have to prove that $R(t_i) \leq f(t_{i+1})$

almost surely for i large enough. We have

$$\begin{aligned}\mathbb{P}\left(R(t_i) \geq f(t_{i+1})\right) &\leq 1 - \exp\left(-C_3 e^{-i} - C_1 e(i+1)^{-\beta}\right) \\ &\leq C_1 e i^{-\beta} + o(i^{-\beta}).\end{aligned}$$

This series converges. Hence, thanks to Borel-Cantelli's Lemma, we can conclude.

Let us now prove the case $\beta \leq 1$. Set $t_i = e^{-i}$ and $f(t) = \exp(-\frac{1}{t}(-\ln(t))^{-\beta})$. Set $H_i = \{R(t_i) \geq f(t_i)\}$. Then we have

$$\sum_{i=1}^{N} \mathbb{P}(H_i) \geq \sum_{i=1}^{N} \left(1 - \exp\left(C_3 e^{-i} - C_2 i^{-\beta}\right)\right).$$

The right term is equivalent to $\sum_{i=1}^{N} C_2 i^{-\beta}$, so it diverges. We have now to check the condition (II.13) to apply the generalized Borel-Cantelli's Lemma.

$$\begin{aligned}\mathbb{P}(H_i \cap H_{i+j}) &= 1 - \mathbb{P}(\overline{H_i}) - \mathbb{P}(\overline{H_{i+j}}) + \mathbb{P}(\overline{H_i} \cap \overline{H_{i+j}}) \\ &= 1 - \mathbb{P}(\overline{H_i}) - \mathbb{P}(\overline{H_{i+j}}) + \mathbb{P}(\overline{H_i})\mathbb{P}(\overline{H_{i+j}}) \exp\left(\int_0^{t_{i+j}} \nu_u(s_2 \geq f(t_i))du\right) \\ &\leq \mathbb{P}(H_i)\mathbb{P}(H_{i+j}) + \exp\left(\int_0^{t_{i+j}} \nu_u(s_2 \geq f(t_i))du\right) - 1.\end{aligned}$$

Then notice that

$$\exp\left(\int_0^{t_{i+j}} \nu_u(s_2 \geq f(t_i))du\right) \leq \exp\left(C_3 e^{-i-j} + C_1 i^{-\beta} e^{-j}\right).$$

Hence we deduce

$$\sum_{i,j=1}^{N} \left(\exp\left(\int_0^{t_{i+j}} \nu_u(s_2 \geq f(t_i))du\right) - 1\right) \leq K \sum_{i=1}^{N} i^{-\beta}.$$

So

$$\frac{\sum_{i,j=1}^{N} \left(\exp\left(\int_0^{t_{i+j}} \nu_u(s_2 \geq f(t_i))du\right) - 1\right)}{\sum_{i=1}^{N} \mathbb{P}(H_i)}$$

is bounded and thus the condition (II.13) holds. This concludes the case $\beta < 1$. For $\beta = 1$, we just have

$$\limsup_{t \to 0} R(t) \exp\left(-\frac{1}{t \ln t}\right) \leq 1 \text{ a.s.}$$

Let us notice then that the same demonstration works with $\gamma f(t)$ instead of $f(t)$ with γ any strictly positive constant. So, we have

$$\limsup_{t \to 0} R(t) \exp\left(-\frac{1}{t \ln t}\right) \leq \gamma \text{ a.s. for all } \gamma > 0,$$

and thus

$$\limsup_{t \to 0} R(t) \exp\left(-\frac{1}{t \ln t}\right) = 0 \quad \text{a.s.}$$

■

Chapter III

On the equivalence of some eternal additive coalescents[1]

Abstract. In this paper, we study the additive coalescents. Using their representation as fragmentation processes, we prove that the law of a large class of eternal additive coalescents is absolutely continuous with respect to the law of the standard additive coalescent on any bounded time interval.

III.1 Introduction

The paper deals with additive coalescent processes, a class of Markov processes which have been introduced first by Evans and Pitman [30]. In the simple situation of a system initially composed of a finite number k of clusters with masses $m_1, m_2, \ldots, m_k$, the dynamics are such that each pair of clusters (m_i, m_j) merges into a unique cluster with mass $m_i + m_j$ at rate $m_i + m_j$, independently of the other pairs. In the sequel, we always assume that we start with a total mass equal to 1 (i.e. $m_1 + \ldots + m_k = 1$). This induces no loss of generality since we can then deduce the law of any additive coalescent process through a time renormalization. Hence, an additive coalescent lives on the compact set

$$\mathcal{S}^{\downarrow} = \{x = (x_i)_{i \geq 1}, x_1 \geq x_2 \geq \ldots \geq 0, \sum_i x_i \leq 1\},$$

endowed with the topology of uniform convergence.

[1] A-L. Basdevant, article available on Arxiv.

Evans and Pitman [30] proved that we can define an additive coalescent on the whole real line for a system starting at time $t = -\infty$ with an infinite number of infinitesimally small clusters. Such a process will be called an eternal coalescent process. More precisely, if we denote by $(C^n(t), t \geq 0)$ the additive coalescent starting from the configuration $(1/n, 1/n, \dots, 1/n)$, they proved that the sequence of processes $(C^n(t + \frac{1}{2}\ln n), t \geq -\frac{1}{2}\ln n)$ converges in distribution on the space of càdlàg paths with values in the set $\mathcal{S}^{\downarrow}$ toward some process $(C^{\infty}(t), t \in \mathbb{R})$, which is called the standard additive coalescent. We stress that this process is defined for all time $t \in \mathbb{R}$. A remarkable property of the standard additive coalescent is that, up to time-reversal, its becomes a fragmentation process. Namely, the process $(F(t), t \geq 0)$ defined by $F(t) = C^{\infty}(-\ln t)$ is a self-similar fragmentation process with index of self similarity $\alpha = 1/2$, with no erosion and with dislocation measure ν given by

$$\nu(x_1 \in dy) = (2\pi y^3(1-y)^3)^{-1/2}dy \quad \text{for } y \in]1/2, 1[, \quad \nu(x_3 > 0) = 0.$$

We refer to Bertoin [14] for the definition of erosion, dislocation measure, and index of self similarity of a fragmentation process and a proof. Just recall that in a fragmentation process, distinct fragments evolve independently of each others.

Aldous and Pitman [2] constructed this fragmentation process $(F(t), t \geq 0)$ by cutting the skeleton of the continuum Brownian random tree according to a Poisson point process. In another paper [3], they gave a generalization of this result: consider for each $n \in \mathbb{N}$ a decreasing sequence $r_{n,1} \geq \dots \geq r_{n,n} \geq 0$ with sum 1, set $\sigma_n^2 = \sum_{i=1}^n r_{n,i}^2$ and suppose that

$$\lim_{n\to\infty} \sigma_n = 0 \text{ and } \lim_{n\to\infty} \frac{r_{n,i}}{\sigma_n} = \theta_i \text{ for all } i \in \mathbb{N}.$$

Assume further that $\sum_i \theta_i^2 < 1$ or $\sum_i \theta_i = \infty$. Then, it is proved in [3] that if $M^n = (M^n(t), t \geq 0)$ denotes the additive coalescent process starting with n clusters with mass $r_{n,1} \geq \dots \geq r_{n,n}$, then $(M^{(n)}(t - \ln \sigma_n), t \geq \ln \sigma_n)$ has a limit distribution as $n \to \infty$, which can be obtained by cutting a specific inhomogeneous random tree with a point Poisson process. Furthermore, any extreme eternal additive coalescent can be obtained this way up to a deterministic time translation.

Bertoin [9] gave another construction of the limit of the process $(M^{(n)}(t - \ln \sigma_n), t \geq \ln \sigma_n)$ in the following way. Let b_θ be the bridge with exchangeable increments defined for $s \in [0, 1]$ by

$$b_\theta(s) = \sigma b_s + \sum_{i=1}^{\infty} \theta_i(\mathbb{1}_{\{s \geq V_i\}} - s),$$

where $(b_s, s \in [0,1])$ is a standard Brownian bridge, $(V_i)_{i\geq 1}$ is an i.i.d. sequence of uniform random variable on [0,1] independent of b and $\sigma = 1 - \sum_i \theta_i^2$. Let $\varepsilon_\theta = (\varepsilon_\theta(s), s \in [0,1])$ be the excursion obtained from b_θ by Vervaat's transform, i.e. $\varepsilon_\theta(s) = b_\theta(s + m \mod 1) - b_\theta(m)$, where m is the point of [0,1] where b_θ reaches its minimum. For all $t \geq 0$, consider

$$\varepsilon_\theta^{(t)}(s) = ts - \varepsilon_\theta(s), \qquad S_\theta^{(t)}(s) = \sup_{0\leq u\leq s} \varepsilon_\theta^{(t)}(u),$$

and define $F^\theta(t)$ as the sequence of the lengths of the constancy intervals of the process $(S_\theta^{(t)}(s), 0 \leq s \leq 1)$. Then the limit of the process $(M^{(n)}(t - \ln \sigma_n), t \geq \ln \sigma_n)$ has the law of $(F^\theta(e^{-t}), t \in \mathbb{R})$. Miermont [46] studied the same process in the special case where ε_θ is the normalized excursion above the minimum of a spectrally negative Lévy process. More precisely let $(X_t, t \geq 0)$ be a Lévy process with no positive jump, with unbounded variation and with positive and finite mean. Let $\overline{X}(t) = \sup_{0\leq s\leq t} X_t$ and denote by $\varepsilon_X = (\varepsilon_X(s), s \in [0,1])$ the normalized excursion with duration 1 of the reflected process $\overline{X} - X$. We now define in the same way as for b_θ, the processes $\varepsilon_X^{(t)}(s)$, $S_X^{(t)}(s)$ and $F^X(t)$. Then, the process $(F^X(e^{-t}), t \in \mathbb{R})$ is a mixture of some eternal additive coalescents (see [46] for more details). Furthermore, $(F^X(t), t \geq 0)$ is a fragmentation process in the sense that distinct fragments evolve independently of each other (however, it is not necessarily homogeneous in time). It is quite remarkable that the Lévy property of X ensures the branching property of F^X. We stress that there exist other eternal additive coalescents for which this property fails. Notice that when the Lévy process X is the standard Brownian motion B, the process $(F^B(e^{-t}), t \in \mathbb{R})$ is then the standard additive coalescent and $(F^B(t), t \geq 0)$ is a self-similar and time-homogeneous fragmentation process.

In this paper, we study the relationship between the laws $\mathbb{P}^{(X)}$ of $(F^X(t), t \geq 0)$ and $\mathbb{P}^{(B)}$ of $(F^B(t), t \geq 0)$. We prove that, for certain Lévy processes $(X_t, t \geq 0)$, the law $\mathbb{P}^{(X)}$ is absolutely continuous with respect to $\mathbb{P}^{(B)}$ and we compute explicitly the density. Our main result is the following:

Theorem III.1

Let $(\Gamma(t), t \geq 0)$ be a subordinator with no drift. Assume that $\mathbb{E}(\Gamma_1) < \infty$ and take any $c \geq \mathbb{E}(\Gamma_1)$. We define $X_t = B_t - \Gamma_t + ct$, where B denotes a Brownian motion independent of Γ. Let $(p_t(u), u \in \mathbb{R})$ and $(q_t(u), u \in \mathbb{R})$ stand for the respective density of B_t and X_t. In particular $p_t(u) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{u^2}{2t})$. Let $\mathcal{S}_1$ be the space of positive sequences with sum 1. We consider the function $\mathbf{h} : \mathbb{R}_+ \times \mathcal{S}_1$

defined by

$$\mathbf{h}(t,\mathbf{x}) = e^{tc}\frac{p_1(0)}{q_1(0)}\prod_{i=1}^{\infty}\frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)} \qquad \text{with } \mathbf{x}=(x_i)_{i\geq 1}.$$

Then, for all $t \geq 0$, the function $\mathbf{h}(t,\cdot)$ is bounded on $\mathcal{S}_1$ and has the following properties:

- $\mathbf{h}(t,F(t))$ *is a* $\mathbb{P}^{(B)}$*-martingale,*
- *for every $t \geq 0$, the law of the process $(F^X(s), 0 \leq s \leq t)$ is absolutely continuous with respect to that of $(F^B(s), 0\leq s\leq t)$ with density $\mathbf{h}(t,F^B(t))$.*

Let us notice that $\mathbf{h}(t,\cdot)$ is a multiplicative function, i.e. it can be written as the product of functions, each of them depending only on the size of a single fragment. In the sequel we will use the notation

$$h(t,x) = e^{tcx}\left(\frac{p_1(0)}{q_1(0)}\right)^x\frac{q_x(-tx)}{p_x(-tx)} \qquad \text{for } x\in]0,1] \text{ and } t\geq 0,$$

so we have $\mathbf{h}(t,\mathbf{x}) = \prod_i h(t,x_i)$. This multiplicative form of $\mathbf{h}(t,\cdot)$ implies that the process F^X has the branching property (i.e. distinct fragments evolve independently of each other) since F^B has it. Indeed, for every multiplicative bounded continuous function $\mathbf{f} : \mathcal{S}^{\downarrow} \mapsto \mathbb{R}_+$, for all $t' > t > 0$ and $\mathbf{x}\in\mathcal{S}^{\downarrow}$, we have, since $\mathbf{h}(t,F^B(t))$ is a $\mathbb{P}^{(B)}$-martingale,

$$\mathbb{E}^{(X)}\Big(\mathbf{f}(F(t'))\,\big|\,F(t)=\mathbf{x}\Big) = \frac{1}{\mathbf{h}(t,\mathbf{x})}\mathbb{E}^{(B)}\Big(\mathbf{h}(t',F(t'))\mathbf{f}(F(t'))\,\big|\,F(t)=\mathbf{x}\Big).$$

Using the branching property of F^B and the multiplicative form of $\mathbf{h}(t,\cdot)$, we get

$$\mathbb{E}^{(X)}\Big(\mathbf{f}(F(t'))\,\big|\,F(t)=\mathbf{x}\Big) = \frac{1}{\mathbf{h}(t,\mathbf{x})}\prod_i\mathbb{E}^{(B)}\Big(\mathbf{h}(t',F(t'))\mathbf{f}(F(t'))\,\big|\,F(t)=(x_i,0,\ldots)\Big).$$

And finally we deduce

$$\begin{aligned}\mathbb{E}^{(X)}\Big(\mathbf{f}(F(t'))\,\big|\,F(t)=\mathbf{x}\Big) &= \frac{1}{\mathbf{h}(t,\mathbf{x})}\prod_i h(t,x_i)\mathbb{E}^{(X)}\Big(\mathbf{f}(F(t'))\,\big|\,F(t)=(x_i,0,\ldots)\Big)\\ &= \prod_i\mathbb{E}^{(X)}\Big(\mathbf{f}(F(t'))\,\big|\,F(t)=(x_i,0,\ldots)\Big).\end{aligned}$$

Let $M_{\mathbf{x}}$ (resp. M_{x_i}) be the random measure on]0,1[defined by $M_{\mathbf{x}} = \sum_i \delta_{s_i}$ where the sequence $(s_i)_{i\geq 1}$ has the law of $F(t')$ conditioned on $F(t)=\mathbf{x}$ (resp. $F(t) =$

$(x_i, 0, \ldots))$. Hence we have, for every bounded continuous function $g : \mathbb{R} \mapsto \mathbb{R}$,

$$\mathbb{E}\Big(\exp(- < g, M_{\mathbf{x}} >)\Big) = \prod_{i=1}^{\infty} \mathbb{E}\Big(\exp(- < g, M_{x_i} >)\Big),$$

which proves that $M_{\mathbf{x}}$ has the law of $\sum_i M_{x_i}$ where the random measures $(M_{x_i})_{i\geq 1}$ are independent. Hence the process F^X has the branching property. Notice also that other multiplicative martingales have already been studied in the case of branching random walks [19, 27, 49, 41].

This paper will be divided in two sections. The first section is devoted to the proof of this theorem and in the next one, we will use the fact that $\mathbf{h}(t, F^B(t))$ is a $\mathbb{P}^{(B)}$-martingale to describe an integro-differential equation solved by the function h.

III.2 Proof of Theorem III.1

The assumptions and notation in Theorem III.1 are implicitly enforced throughout this section.

III.2.1 Absolute continuity

In order to prove Theorem III.1, we will first prove the absolute continuity of the law $\mathbb{P}_t^{(X)}$ of $F^X(t)$ with respect to the law $\mathbb{P}_t^{(B)}$ of $F^X(t)$ for a fixed time $t > 0$ and for a finite number of fragments. We begin first by a definition:

Definition III.2

Let $x = (x_1, x_2, \ldots)$ be a sequence of positive numbers with sum 1. We call the random variable $y = (x_{j_1}, x_{j_2}, \ldots)$ a size biased rearrangement of x if we have:

$$\forall i \in \mathbb{N},\ \mathbb{P}(j_1 = i) = x_i,$$

and by induction

$$\forall i \in \mathbb{N}\backslash\{i_1, \ldots, i_k\},\ \mathbb{P}(j_{k+1} = i \mid j_1 = i_1, \ldots, j_k = i_k) = \frac{x_i}{1 - \sum_{l=1}^{k} x_{i_l}}.$$

Notice that for every Lévy process X satisfying hypotheses of Theorem III.1, we have $\sum_{i=1}^{\infty} F_i(t) = 1 \quad \mathbb{P}_t^{(X)}$-a.s. (it is clear by the construction from an excursion of X since X has unbounded variation, cf [46], Section 3.2). Hence the above definition can be applied to $F^X(t)$.

The following lemma gives the distribution of the first n fragments of $F^X(t)$, chosen with a size-biased pick:

Lemma III.3

Let $(\tilde{F}_1^X(t), \tilde{F}_2^X(t), \ldots)$ be a size biased rearrangement of $F^X(t)$. Then for all $n \in \mathbb{N}$, for all $x_1, \ldots, x_n \in \mathbb{R}_+$ such that $S = \sum_{i=1}^n x_i < 1$, we have

$$\mathbb{P}_t^{(X)}(\tilde{F}_1^X \in dx_1, \ldots, \tilde{F}_n^X \in dx_n) = \frac{t^n}{q_1(0)} q_{1-S}(St) \prod_{i=1}^n \frac{q_{x_i}(-tx_i)}{1 - \sum_{k=1}^i x_k} dx_1 \ldots dx_n.$$

Proof: On the one hand, Miermont [46] gave a description of the law of $F^X(t)$: let $T^{(t)}$ be a subordinator with Lévy measure $z^{-1} q_z(-tz) \mathbb{1}_{z>0} dz$. Then $F^X(t)$ has the law of the sequence of the jumps of $T^{(t)}$ before time t conditioned on $T_t^{(t)} = 1$.

One the other hand, consider a subordinator T on the time interval $[0, u]$ conditioned by $T_u = y$ and pick a jump of T by size-biased sampling. Then, its distribution has density

$$\frac{zuh(z) f_u(y-z)}{y f_u(y)} dz,$$

where h is the density of the Lévy measure of T and f_u is the density of T_u (see Theorem 2.1 of [51]). Then, in the present case, we have

$$u = t, \quad y = 1, \quad h(z) = z^{-1} q_z(-tz), \quad f_u(z) = \frac{u}{z} q_z(u - zt) \qquad \text{(cf. Lemma 9 of [46])}.$$

Hence we get

$$\mathbb{P}_t^{(X)}(\tilde{F}_1^X \in dz) = \frac{t q_z(-tz) q_{1-z}(zt)}{(1-z) q_1(0)} dz.$$

This proves the lemma in the case $n = 1$. The proof for $n \geq 2$ uses an induction. Assume that we have proved the case $n-1$ and let us prove the case n. We have

$$\mathbb{P}_t^{(X)}(\tilde{F}_1^X \in dx_1, \ldots, \tilde{F}_n^X \in dx_n) =$$
$$\mathbb{P}_t^{(X)}(\tilde{F}_1^X \in dx_1, \ldots, \tilde{F}_{n-1}^X \in dx_{n-1}) \mathbb{P}_t^{(X)}(\tilde{F}_n^X \in dx_n \mid \tilde{F}_1^X \in dx_1, \ldots, \tilde{F}_{n-1}^X \in dx_{n-1}).$$

Furthermore, Perman, Pitman and Yor [51] have proved that the n-th size biased picked jump Δ_n of a subordinator before time u conditioned by $T_u = y$ and $\Delta_1 = x_1, \ldots, \Delta_{n-1} = x_{n-1}$ has the law of a size biased picked jump of the subordinator T before time u conditioned by $T_u = y - x_1 - \ldots - x_{n-1}$. Hence we get:

$$\mathbb{P}_t^{(X)}(\tilde{F}_1^X \in dx_1, \ldots, \tilde{F}_n^X \in dx_n) =$$
$$\left(\frac{t^{n-1}}{q_1(0)} q_{1-S_{n-1}}(S_{n-1}t) \prod_{i=1}^{n-1} \frac{q_{x_i}(-tx_i)}{1 - S_i} \right) \frac{t q_{x_n}(-tx_n) q_{1-S_n}(S_n t)}{(1 - S_n) q_{1-S_{n-1}}(S_{n-1}t)} dx_1 \ldots dx_n,$$

where $S_i = \sum_{k=1}^{i} x_k$. And so the lemma is proved by induction. ■

Since the lemma is clearly also true for $\mathbb{P}^{(B)}$ (take $\Gamma = c = 0$), we get:

Corollary III.4

Let $(F(t), t \geq 0)$ be a fragmentation process. Let $(\tilde{F}_1(t), \tilde{F}_2(t), \ldots)$ be a size biased rearrangement of $F(t)$. Then for all $n \in \mathbb{N}$, for all $x_1, \ldots, x_n \in \mathbb{R}_+$ such that $S = \sum_{i=1}^{n} x_i < 1$, we have

$$\frac{\mathbb{P}_t^{(X)}(\tilde{F}_1 \in dx_1, \ldots, \tilde{F}_n \in dx_n)}{\mathbb{P}_t^{(B)}(\tilde{F}_1 \in dx_1, \ldots, \tilde{F}_n \in dx_n)} = h_n(t, x_1, \ldots, x_n),$$

$$\text{with } h_n(t, x_1, \ldots, x_n) = \frac{p_1(0)}{q_1(0)} \frac{q_{1-S}(St)}{p_{1-S}(St)} \prod_{i=1}^{n} \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)}.$$

To establish that the law of $F^X(t)$ is absolutely continuous with respect to the law of $F^B(t)$ with density $\mathbf{h}(t, \cdot)$, it remains to check that the function h_n converges as n tends to infinity to $\mathbf{h}$ $\mathbb{P}_t^{(B)}$-a.s. and in $L^1(\mathbb{P}_t^{(B)})$. In this direction, we first prove two lemmas:

Lemma III.5

We have $\frac{q_y(-ty)}{p_y(-ty)} < 1$ for all $y > 0$ sufficiently small. As a consequence, if $(x_i)_{i\geq 1}$ is a sequence of positive numbers with $\lim_{i\to\infty} x_i = 0$, then the product $\prod_{i=1}^{n} \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)}$ converges as n tends to infinity.

Proof: Since $X_t = B_t - \Gamma_t + tc$, notice that we have

$$\forall s > 0,\ \forall u \in \mathbb{R}, \quad q_s(u) = \mathbb{E}\Big(p_s(u + \Gamma_s - cs)\Big).$$

Hence if we replace $p_s(u)$ by its expression $\frac{1}{\sqrt{2\pi s}} \exp(-\frac{u^2}{2s})$, we get

$$\frac{q_s(u)}{p_s(u)} = \exp\left(cu - \frac{c^2 s}{2}\right) \mathbb{E}\left[\exp\left(-\frac{\Gamma_s^2}{2s} - \Gamma_s(\frac{u}{s} - c)\right)\right]. \tag{III.1}$$

i.e., for all $y > 0$, for all $t \geq 0$,

$$\frac{q_y(-ty)}{p_y(-ty)} = \exp\left(-y(ct + \frac{c^2}{2})\right) \mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2y} + \Gamma_y(t + c)\right)\right].$$

Using the inequality $(c-a)(c-b) \geq -\left(\frac{b-a}{2}\right)^2$, we have

$$-\frac{\Gamma_y^2}{2y} + \Gamma_y(t + c) \leq \frac{y(t+c)^2}{2}$$

and we deduce

$$\frac{q_y(-ty)}{p_y(-ty)} \leq e^{\frac{t^2 y}{2}}.$$

Fix $c' \in]0, c[$, let f be the function defined by $f(y) = \mathbb{P}(\Gamma_y \leq c'y)$. Since Γ_t is a subordinator with no drift, we have $\lim_{y\to 0} f(y) = 1$ (indeed, $\Gamma_y = o(y)$ a.s., see [7]). On the event $\{\Gamma_y \leq c'y\}$, we have

$$\begin{aligned} \exp\left(-y(ct+\frac{c^2}{2})\right)\exp\left(-\frac{\Gamma_y^2}{2y}+\Gamma_y(t+c)\right) &\leq \exp(-y(\frac{1}{2}(c-c')^2+t(c-c'))) \\ &\leq \exp(-\varepsilon y), \end{aligned}$$

with $\varepsilon = \frac{1}{2}(c-c')^2$. Hence, we get the upper bound

$$\frac{q_y(-ty)}{p_y(-ty)} \leq e^{-\varepsilon y} f(y) + (1-f(y))e^{\frac{yt^2}{2}}.$$

Since $f(y) \to 1$ as $y \to 0$, we deduce

$$e^{-\varepsilon y} f(y) + (1-f(y))e^{\frac{yt^2}{2}} = 1 - \varepsilon y + o(y).$$

Thus, we have $\frac{q_y(-ty)}{p_y(-ty)} < 1$ for y small enough, and so the product converges for every sequence $(x_i)_{i\geq 0}$ which tends to 0. ■

We prove now a second lemma:

Lemma III.6

We have

$$\lim_{s\to 1^-} \frac{q_{1-s}(st)}{p_{1-s}(st)} = e^{tc}.$$

Proof: We use again Identity (III.1) established in the proof of Lemma III.5. We get:

$$\frac{q_{1-s}(st)}{p_{1-s}(st)} = \exp\left(tsc - \frac{c^2}{2}(1-s)\right)\mathbb{E}\left[\exp\left(-\frac{\Gamma_{1-s}^2}{2(1-s)} - \Gamma_{1-s}(\frac{ts}{1-s}-c)\right)\right].$$

For s close enough to 1, $\frac{ts}{1-s} - c \geq 0$, hence we get

$$\mathbb{E}\left[\exp\left(-\frac{\Gamma_{1-s}^2}{2(1-s)} - \Gamma_{1-s}(\frac{ts}{1-s}-c)\right)\right] \leq 1$$

and we deduce

$$\limsup_{s\to 1^-} \frac{q_{1-s}(st)}{p_{1-s}(st)} \leq e^{tc}.$$

For the lower bound, we write

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\frac{\Gamma_{1-s}^2}{2(1-s)}-\Gamma_{1-s}(\frac{ts}{1-s}-c)\right)\right]\\
&\qquad\geq \mathbb{E}\left[\exp\left(-\frac{\Gamma_{1-s}}{2(1-s)}-\Gamma_{1-s}(\frac{ts}{1-s}-c)\right)\mathbb{1}_{\{\Gamma_{1-s}\leq 1\}}\right]\\
&\qquad\geq \mathbb{E}\left[\exp\left(-\Gamma_{1-s}\frac{1+2ts}{2(1-s)}\right)\mathbb{1}_{\{\Gamma_{1-s}\leq 1\}}\right]\\
&\qquad\geq \mathbb{E}\left[\exp\left(-\Gamma_{1-s}\frac{1+2ts}{2(1-s)}\right)\right]-\mathbb{P}(\Gamma_{1-s}\geq 1).
\end{aligned}$$

Since Γ_t is a subordinator with no drift, $\lim_{u\to 0}\frac{\Gamma_u}{u}=0$ a.s., and we have for all $K>0$,

$$\lim_{u\to 0^+}\mathbb{E}\left[\exp\left(-K\frac{\Gamma_u}{u}\right)\right]=1.$$

Hence, we get

$$\liminf_{s\to 1^-}\frac{q_{1-s}(st)}{p_{1-s}(st)}\geq e^{tc}.$$

■

We are now able to prove the absolute continuity of $\mathbb{P}_t^{(X)}$ with respect to $\mathbb{P}_t^{(B)}$. Since $S_n=\sum_{i=1}^n x_i$ converges $\mathbb{P}_t^{(B)}$-a.s. to 1, Lemma III.5 and III.6 imply that $H_n=h_n(t,\tilde{F}_1(t),\ldots,\tilde{F}_n(t))$ converges to $H=\mathbf{h}(t,F(t))$ $\mathbb{P}^{(B)}$-a.s.

Let us now prove that H_n is uniformly bounded, which implies the L^1 convergence. We have already proved that there exists $\varepsilon>0$ such that:

$$\forall x\in]0,\varepsilon[,\quad \frac{q_x(-tx)}{p_x(-tx)}\leq 1.$$

Besides, it is well known that, if $X_t=B_t-\Gamma_t+ct$, its density $(t,u)\to q_t(u)$ is continuous on $\mathbb{R}_+^*\times\mathbb{R}$. Hence, on $[\varepsilon,1]$, the function $x\to\frac{q_x(-tx)}{p_x(-tx)}$ is continuous and we can find an upper bound $A>0$ of this function . As there are at most $\frac{1}{\varepsilon}$ fragments of $F(t)$ larger than ε, we deduce the upper bound:

$$\prod_{i=1}^{\infty}\frac{q_{F_i}(-tF_i)}{p_{F_i}(-tF_i)}\leq A^{\frac{1}{\varepsilon}}.$$

Likewise, the function $S\to\frac{q_{1-S}(St)}{p_{1-S}(St)}$ is continuous on $[0,1[$ and has a limit at 1, so it is bounded by some $D>0$ on $[0,1]$. Hence we get

$$H_n\leq A^{\frac{1}{\varepsilon}}D\frac{p_1(0)}{q_1(0)}\quad \mathbb{P}^{(B)}\text{-a.s.}$$

So H_n converges to $H \quad \mathbb{P}^{(B)}$-a.s. and in $L^1(\mathbb{P}^{(B)})$. Furthermore, by construction, H_n is a $\mathbb{P}^{(B)}$-martingale, hence we get for all $n \in \mathbb{N}$,

$$\mathbb{E}^{(B)}(H \mid \tilde{F}_1, \dots, \tilde{F}_n) = H_n,$$

and so, for every bounded continuous function $f : \mathcal{S}_1 \to \mathbb{R}$, we have

$$\mathbb{E}^{(X)}\Big[f(F(t))\Big] = \mathbb{E}^{(B)}\Big[f(F(t))\mathbf{h}(t, F(t))\Big].$$

Hence, we have proved that, for a fixed time $t \geq 0$, the law of $F^X(t)$ is absolutely continuous with respect to that of $F^B(t)$ with density $\mathbf{h}(t, F^B(t))$. Furthermore, Miermont [46] has proved that the processes $(F^X(e^{-t}), t \in \mathbb{R})$ and $(F^B(e^{-t}), t \in \mathbb{R})$ are both eternal additive coalescents (with different entrance laws). Hence, they have the same semi-group of transition and we get the absolute continuity of the law of the process $(F^X(s), 0 \leq s \leq t)$ with respect to that of $(F^B(s), 0 \leq s \leq t)$ with density $\mathbf{h}(t, F^B(t))$.

III.2.2 Sufficient condition for equivalence

We can now wonder whether the measure $\mathbb{P}^{(X)}$ is equivalent to the measure $\mathbb{P}^{(B)}$, that is whether $\mathbf{h}(t, F(t))$ is strictly positive $\mathbb{P}^{(B)}$-a.s. A sufficient condition is given by the following proposition.

Proposition III.7

Let ϕ be the Laplace exponent of the subordinator Γ, i.e.

$$\forall s \geq 0, \forall q \geq 0, \quad \mathbb{E}(\exp(-q\Gamma_s)) = \exp(-s\phi(q)).$$

Assume that there exists $\delta > 0$ such that

$$\lim_{x \to \infty} \phi(x) x^{\delta - 1} = 0, \tag{III.2}$$

then the function $\mathbf{h}(t, F(t))$ defined in Theorem III.1 is strictly positive $\mathbb{P}^{(B)}$-a.s.

We stress that the condition III.2 is very weak. For instance, let π be the Lévy measure of the subordinator and $I(x) = \int_0^x \overline{\pi}(t)dt$ where $\overline{\pi}(t)$ denotes $\pi(]t, \infty[)$. It is well known that $\phi(x)$ behaves like $xI(1/x)$ as x tends to infinity (see [7] Section III). Thus, the condition III.2 is equivalent to $I(x) = o(x^\delta)$ as x tends to 0 (recall that we always have $I(x) = o(1)$).

Proof: Let $t > 0$. We must check that $\prod_{i=1}^{\infty} \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)}$ is $\mathbb{P}_t^{(B)}$-almost surely strictly positive. Using (III.1), we have:

$$\frac{q_y(-ty)}{p_y(-ty)} = \exp\left(-y(ct + \frac{c^2}{2})\right) \mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2y} + \Gamma_y(t+c)\right)\right].$$

Since we have $\sum_{i=1}^{\infty} x_i = 1 \quad \mathbb{P}_t^{(B)}$-a.s., we get

$$\prod_{i=1}^{\infty} \frac{q_{x_i}(-tx_i)}{p_{x_i}(-tx_i)} \geq \exp\left(-ct + \frac{c^2}{2}\right) \prod_{i=1}^{\infty} \mathbb{E}\left[\exp\left(-\frac{\Gamma_{x_i}^2}{2x_i} + c\Gamma_{x_i}\right)\right].$$

Hence we have to find a lower bound for $\mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2y} + c\Gamma_y\right)\right]$. Since $c \geq \mathbb{E}(\Gamma_1)$, we have

$$\mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2y} + c\Gamma_y\right)\right] \geq \mathbb{E}\left[\exp\left(\frac{\Gamma_y}{y}(\mathbb{E}(\Gamma_y) - \frac{\Gamma_y}{2})\right)\right].$$

Set $A = \mathbb{E}(\Gamma_1)$ and let us fix $K > 0$. Notice that the event $\mathbb{E}(\Gamma_y) - \frac{\Gamma_y}{2} \geq -Ky$ is equivalent to the event $\Gamma_y \leq (2A + K)y$ and by Markov inequality, we have

$$\mathbb{P}(\Gamma_y \geq (2A+K)y) \leq \frac{A}{2A+K}.$$

Hence we get

$$\begin{aligned}
\mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2y} + c\Gamma_y\right)\right] &\geq \mathbb{E}\left[\exp\left(\frac{\Gamma_y}{y}(\mathbb{E}(\Gamma_y) - \frac{\Gamma_y}{2})\mathbb{1}_{\{\Gamma_y \leq (2A+K)y\}}\right)\right] \\
&\geq \mathbb{E}\left(\exp(-K\Gamma_y)\mathbb{1}_{\{\Gamma_y \leq (2A+K)y\}}\right) \\
&\geq \mathbb{E}\left(\exp(-K\Gamma_y)\right) - \mathbb{E}\left(\exp(-K\Gamma_y)\mathbb{1}_{\{\Gamma_y > (2A+K)y\}}\right) \\
&\geq \exp(-\phi(K)y) - \frac{A}{2A+K}.
\end{aligned}$$

This inequality holds for all $K > 0$. Hence, with $\varepsilon > 0$ and $K = y^{-\frac{1}{2}-\varepsilon}$, we get

$$\mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2y} + c\Gamma_y\right)\right] \geq \exp\left(-\phi(y^{-\frac{1}{2}-\varepsilon})y\right) - Ay^{\frac{1}{2}+\varepsilon}.$$

Furthermore, the product $\prod_{i=1}^{\infty} \mathbb{E}\left[\exp\left(-\frac{\Gamma_{x_i}^2}{2x_i} + c\Gamma_{x_i}\right)\right]$ is strictly positive if the series

$$\sum_{i=1}^{\infty} 1 - \mathbb{E}\left[\exp\left(-\frac{\Gamma_{x_i}^2}{2x_i} + c\Gamma_{x_i}\right)\right]$$

converges. Hence, a sufficient condition is

$$\exists\, \varepsilon > 0 \text{ such that } \sum_{i=1}^{\infty}\left(1 - \exp\left(-\phi(x_i^{-\frac{1}{2}-\varepsilon})x_i\right) + x_i^{\frac{1}{2}+\varepsilon}\right) < \infty \quad \mathbb{P}_t^{(B)}\text{-a.s.}$$

Recall that the distribution of the Brownian fragmentation at time t is equal to the distribution of the jumps of a stable subordinator T with index $1/2$ before time t conditioned on $T_t = 1$ (see [2]). Hence, it is well known that we have for all $\varepsilon > 0$

$$\sum_{i=1}^{\infty} x_i^{\frac{1}{2}+\varepsilon} < \infty \quad \mathbb{P}_t^{(B)}\text{-a.s.} \quad \text{(see Formula (9) of [2]).}$$

Thus, we have equivalence between $\mathbb{P}_t^{(B)}$ and $\mathbb{P}_t^{(X)}$ as soon as there exist two strictly positive numbers $\varepsilon, \varepsilon'$ such that, for x small enough

$$\phi(x^{-\frac{1}{2}-\varepsilon})x \leq x^{\frac{1}{2}+\varepsilon'}.$$

One can easily check that this condition is equivalent to (III.2). ■

In Theorem III.1, we have supposed that X_t can be written as $B_t + \Gamma_t - ct$, with $c \geq \mathbb{E}(\Gamma_1)$ and Γ_t subordinator. We can wonder whether the theorem applies for a larger class of Lévy processes. Notice first that the process X must fulfill the conditions of Miermont's paper [46] recalled in the introduction, i.e. X has no positive jumps, unbounded variation and finite and positive mean. Hence, a possible extension of the Theorem would be for example for $X_t = \sigma^2 B_t + \Gamma_t - ct$, with $\sigma > 0$, $\sigma \neq 1$. In fact, it is clear that Theorem III.1 fails in this case. Let just consider for example $X_t = 2B_t$. Using Proposition 3 of [46], we get that

$$(F^X(2t), t \geq 0) \stackrel{\text{law}}{=} (F^B(t), t \geq 0).$$

But, it is well known that we have

$$\lim_{n\to\infty} n^2 F_n^{\downarrow}(t) = t\sqrt{2/\pi} \quad \mathbb{P}^{(B)}\text{-a.s.} \quad \text{(see [13])}$$

Hence, the laws $\mathbb{P}_t^{(B)}$ and $\mathbb{P}_{2t}^{(B)}$ are mutually singular.

III.3 An integro-differential equation

Since $\mathbf{h}(t, F(t))$ is the density of $\mathbb{P}^{(X)}$ with respect to $\mathbb{P}^{(B)}$ on the sigma-field $\mathcal{F}_t = \sigma(F(s), s \leq t)$, it is a $\mathbb{P}^{(B)}$-martingale. Hence, in this section, we will compute the infinitesimal generator of a fragmentation to deduce a remarkable integro-differential equation.

III.3.1 The infinitesimal generator of a fragmentation process

In this section, we recall a result obtained by Bertoin and Rouault in an unpublished paper [17].

We denote by $\mathcal{D}$ the space of functions $f : [0,1] \mapsto]0,1]$ of class $\mathcal{C}^1$ and with $f(0) = 1$. For $f \in \mathcal{D}$ and $\mathbf{x} \in \mathcal{S}^{\downarrow}$, we set

$$\mathbf{f}(\mathbf{x}) = \prod_{i=1}^{\infty} f(x_i).$$

For $\alpha \in \mathbb{R}_+$ and ν measure on $\mathcal{S}^{\downarrow}$ such that $\int_{\mathcal{S}^{\downarrow}} (1 - x_1)\nu(d\mathbf{x}) < \infty$, we define the operator

$$G_\alpha \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \sum_{i=1}^{\infty} x_i^\alpha \int \nu(d\mathbf{y}) \left(\frac{\mathbf{f}(x_i \mathbf{y})}{f(x_i)} - 1 \right) \qquad \text{for } f \in \mathcal{D} \text{ and } \mathbf{x} \in \mathcal{S}^{\downarrow}.$$

Proposition III.8

Let $(X(t), t \geq 0)$ be a self-similar fragmentation with index of self-similarity $\alpha > 0$, dislocation measure ν and no erosion. Then, for every function $f \in \mathcal{D}$, the process

$$\mathbf{f}(X(t)) - \int_0^t G_\alpha \mathbf{f}(X(s)) ds$$

is a martingale.

Proof: We will first prove the following lemma

Lemma III.9

For $f \in \mathcal{D}, \mathbf{y} \in \mathcal{S}^{\downarrow}, r \in [0,1]$, we have

$$\left| \frac{\mathbf{f}(r\mathbf{y})}{f(r)} - 1 \right| \leq 2 C_f e^{C_f} r (1 - y_1),$$

with $C_f = \left|\left| \frac{f'}{f^2} \right|\right|_\infty$.

Notice that, since f is $\mathcal{C}^1$ on $[0,1]$ and strictly positive, C_f is always finite.

Proof: First, we write

$$|\ln f(ry_1) - \ln f(r)| \leq \left|\left| \frac{f'}{f} \right|\right|_\infty (1 - y_1) r \leq C_f (1 - y_1) r.$$

We deduce then

$$\frac{\mathbf{f}(r\mathbf{y})}{f(r)} - 1 \leq \frac{f(ry_1)}{f(r)} - 1 \leq e^{C_f (1-y_1) r} - 1 \leq C_f e^{C_f} (1 - y_1) r.$$

Besides we have

$$\ln\frac{1}{f(x_1)}\leq\frac{1}{f(x_i)}-1\leq C_f x_i,\quad\text{which implies}\quad \mathbf{f}(\mathbf{x})\geq f(x_1)\exp(-C_f\sum_{i=2}^{\infty}x_i).$$

Hence we get

$$\frac{\mathbf{f}(r\mathbf{y})}{f(r)}\geq\frac{f(ry_1)}{f(r)}\exp(-C_f(1-y_1)r)\geq\exp(-2C_f(1-y_1)r),$$

and we deduce

$$1-\frac{\mathbf{f}(r\mathbf{y})}{f(r)}\leq 2C_f e^{C_f}(1-y_1)r.$$

■

We can now prove Proposition III.8. We denote by $\mathcal{T}$ the set of times where some dislocation occurs (which is a countable set). Hence we can write

$$\mathbf{f}(X(t))-\mathbf{f}(X(0))=\sum_{s\in[0,t]\cap\mathcal{T}}\Big(\mathbf{f}(X(s))-\mathbf{f}(X(s-))\Big),$$

as soon as

$$\sum_{s\in[0,t]\cap\mathcal{T}}\Big|\mathbf{f}(X(s))-\mathbf{f}(X(s-))\Big|<\infty$$

For $s\in\mathcal{T}$, if the i-th fragment $X_i(s-)$ is involved in the dislocation, we set $k_s=i$ and we denote by Δ_s the element of $\mathcal{S}^{\downarrow}$ according to $X(s-)$ has been broken. Hence, we have

$$\sum_{s\in[0,t]\cap\mathcal{T}}\Big|\mathbf{f}(X(s))-\mathbf{f}(X(s-))\Big|=\sum_{s\in\mathcal{T}\cap[0,t]}\mathbf{f}(X(s-))\left(\sum_{i=1}^{\infty}\mathbb{1}_{k_s=i}\Big|\frac{\mathbf{f}(X_i(s-)\Delta_s)}{f(X_i(s-))}-1\Big|\right).$$

Hence, since a fragment of mass r has a rate of dislocation $\nu_r(dx)=r^{\alpha}\nu(dx)$, the predictable compensator is

$$\begin{aligned}\int_0^t ds\,\mathbf{f}(X(s-))\int_{\mathcal{S}^{\downarrow}}\nu(d\mathbf{y})\sum_{i=1}^{\infty}X_i^{\alpha}(s-)\Big|\frac{\mathbf{f}(X_i(s-)\mathbf{y})}{f(X_i(s-))}-1\Big|&\\ \leq 2C_f e^{C_f}\int_0^t\sum_{i=1}^{\infty}X_i(s-)\int_{\mathcal{S}^{\downarrow}}(1-y_1)\nu(d\mathbf{y})ds.&\\ \leq 2C_f e^{C_f}t\int_{\mathcal{S}^{\downarrow}}(1-y_1)\nu(d\mathbf{y})&\end{aligned}$$

Hence

$$\sum_{s\in[0,t]\cap\mathcal{T}}\Big|\mathbf{f}(X(s))-\mathbf{f}(X(s-))\Big|<\infty\quad\text{a.s.},$$

and thus we have

$$\mathbf{f}(X(t)) - \mathbf{f}(X(0)) = \sum_{s\in[0,t]\cap\mathcal{T}} \Big(\mathbf{f}(X(s)) - \mathbf{f}(X(s-))\Big),$$

i.e.

$$\mathbf{f}(X(t)) - \mathbf{f}(X(0)) = \sum_{s\in\mathcal{T}\cap[0,t]} \mathbf{f}(X(s-)) \left(\sum_{i=1}^{\infty} \mathbb{1}_{k_s=i}\left(\frac{\mathbf{f}(X_i(s-)\Delta_s)}{f(X_i(s-))} - 1\right)\right),$$

whose predictable compensator is

$$\int_0^t ds\, \mathbf{f}(X(s-)) \int_{\mathcal{S}^{\downarrow}} \nu(d\mathbf{y}) \sum_{i=1}^{\infty} X_i^{\alpha}(s-) \left(\frac{\mathbf{f}(X_i(s-)\mathbf{y})}{f(X_i(s-))} - 1\right) = \int_0^t G_\alpha \mathbf{f}(X(s))ds. \quad \blacksquare$$

III.3.2 Application to $\mathbf{h}(t, F(t))$

Let $F(t)$ be a fragmentation process and $q_t(x)$ be the density of a Lévy process fulfilling the hypotheses of Theorem III.1. We have proved in the first section that the function

$$H_t = \mathbf{h}(t, F(t)) = e^{tc}\frac{p_1(0)}{q_1(0)} \prod_{i=1}^{\infty} \frac{q_{F_i(t)}(-tF_i(t))}{p_{F_i(t)}(-tF_i(t))}$$

is a $\mathbb{P}^{(B)}$-martingale (since it is equal to $\frac{d\mathbb{P}^{(X)}}{d\mathbb{P}^{(B)}}|\mathcal{F}_t$). We set

$$g(t,x) = e^{tcx}\frac{q_x(-tx)}{p_x(-tx)} \quad \text{for } x \in]0,1], t\geq 0 \quad \text{and } g(t,0)=1.$$

$$\text{Set now} \quad \mathbf{g}(t,\mathbf{x}) = \prod_{i=1}^{\infty} g(t, x_i(t)) \quad \text{for } \mathbf{x}\in\mathcal{S}^{\downarrow}, t\geq 0.$$

So we have, as $\sum_i F_i(t) = 1$ $\mathbb{P}^{(B)}$-a.s.,

$$H_t = \frac{p_1(0)}{q_1(0)}\mathbf{g}(t, F(t)) \quad \text{for all } t \geq 0.$$

It is well known that if $q_t(u)$ is the density of a Lévy process $X_t = B_t - \Gamma_t + ct$, the function $(t,u) \mapsto q_t(u)$ is $\mathcal{C}^\infty$ on $\mathbb{R}_+^* \times \mathbb{R}$. Hence $(t,x) \mapsto g(t,x)$ is also $\mathcal{C}^\infty$ on $\mathbb{R}_+\times]0,1]$ and in particular, for all $x\in[0,1]$, the function $t \to g(t,x)$ is $\mathcal{C}^1$ and so $\partial_t g(t,x)$ is well defined. The next proposition gives a integro-differential equation solved by the function g when g has some properties of regularity at points $(t,0)$, $t\in\mathbb{R}_+$.

Proposition III.10

1. *Assume that for all $t \geq 0$, $\partial_x g(t,0)$ exists and the function $(t,x) \to \partial_x g(t,x)$ is continuous at $(t,0)$. Then g solves the equation:*

$$\begin{cases} \partial_t g(t,x) + \sqrt{x} \int_0^1 \frac{dy}{\sqrt{8\pi y^3(1-y)^3}} \Big(g(t,xy)g(t,x(1-y)) - g(t,x) \Big) = 0 \\ g(0,x) = \frac{q_x(0)}{p_x(0)}. \end{cases}$$

2. *If the Lévy measure of the subordinator Γ is finite, then the above conditions on g hold.*

Proof: Let us first notice that the hypotheses of the proposition imply that the integral

$$\int_0^1 \frac{dy}{\sqrt{8\pi y^3(1-y)^3}} \Big(g(t,xy)g(t,x(1-y)) - g(t,x) \Big)$$

is well defined and is continuous in x and in t. Indeed, this integral is equal to

$$2\int_0^{\frac{1}{2}} \frac{dy}{\sqrt{8\pi y^3(1-y)^3}} \Big(g(t,xy)g(t,x(1-y)) - g(t,x) \Big).$$

And for all $y \in]0,1/2[, x \in]0,1], t \in \mathbb{R}_+$, there exist $c, c' \in [0,x]$ such that

$$\frac{g(t,xy)g(t,x(1-y)) - g(t,x)}{y} = x(g(t,x)\partial_x g(t,c) - g(t,xy)\partial_x g(t,c')).$$

Thanks to the hypothesis that the function $(t,x) \to \partial_x g(t,x)$ is continuous on $\mathbb{R}_+ \times [0,1]$, $|x(g(t,x)\partial_x g(t,c) - g(t,xy)\partial_x g(t,c'))|$ is uniformly bounded on $[0,T] \times [0,1] \times [0,\frac{1}{2}]$ and so by application of the theorem of dominated convergence, the integral is continuous in t on $\mathbb{R}_+$ and in x on [0,1].

We begin by proving the first point of the proposition. Recall that, according to Proposition III.8, the generator of the Brownian fragmentation is

$$G_{\frac{1}{2}}\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \sum_{i=1}^{\infty} \sqrt{x_i} \int \nu(d\mathbf{y}) \left(\frac{\mathbf{f}(x_i\mathbf{y})}{f(x_i)} - 1 \right),$$

with

$$\nu(y_1 \in du) = (2\pi u^3(1-u)^3)^{-1/2} du \quad \text{for } u \in]1/2,1[, \quad \nu(y_1+y_2 \neq 1) = 0 \quad \text{(cf. [11])}.$$

Hence,

$$M_t = \mathbf{g}(t,F(t)) - \mathbf{g}(0,F(0)) - \int_0^t G_{\frac{1}{2}}\mathbf{g}(s,F(s)) + \partial_t \mathbf{g}(s,F(s))ds$$

is a $\mathbb{P}^{(B)}$-martingale. Since $\mathbf{g}(t, F(t))$ is already a $\mathbb{P}^{(B)}$-martingale, we get

$$G_{\frac{1}{2}}\mathbf{g}(s, F(s)) + \partial_t \mathbf{g}(s, F(s)) = 0 \quad \mathbb{P}^{(B)}\text{-a.s.} \quad \text{for almost every } s > 0,$$

i.e. for almost every $s > 0$

$$\mathbf{g}(s, F(s)) \sum_{i=1}^{\infty} \left[F_i^{1/2}(s) \int_{\mathcal{S}^{\downarrow}} \nu(d\mathbf{y}) \left(\frac{\mathbf{g}(s, F_i(s)\mathbf{y})}{g(s, F_i(s))} - 1 \right) + \frac{\partial_t g(s, F_i(s))}{g(s, F_i(s))} \right] = 0 \quad \mathbb{P}^{(B)}\text{-a.s.}$$

With $F(s) = (x_1, x_2, \ldots)$, we get

$$\sum_{i=1}^{\infty} \left[x_i^{1/2} \int_{\mathcal{S}^{\downarrow}} \nu(d\mathbf{y}) \left(\frac{\mathbf{g}(s, x_i\mathbf{y})}{g(s, x_i)} - 1 \right) + \frac{\partial_t g(s, x_i)}{g(s, x_i)} \right] = 0 \quad \mathbb{P}_s^{(B)}\text{-a.s.}$$

Notice also that this series is absolutely convergent. Indeed, thanks to Lemma III.9, we have

$$\left| x_i^{1/2} \int_{\mathcal{S}^{\downarrow}} \nu(d\mathbf{y}) \left(\frac{\mathbf{g}(s, x_i\mathbf{y})}{g(s, x_i)} - 1 \right) \right| \leq C_{g,s} x_i \int_{\mathcal{S}^{\downarrow}} (1 - y_1) \nu(d\mathbf{y}),$$

where $C_{g,s}$ is a positive constant (which depends on g and s), and, besides we have

$$g(t, x) = \exp\left(-x \frac{c^2}{2} \right) \mathbb{E}\left[\exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t + c) \right) \right].$$

Thus, by application of the theorem of dominated convergence, it is easy to prove that the function $t \to \mathbb{E}\left[\exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t+c)\right)\right]$ is derivable with derivative

$$\partial_t \mathbb{E}\left[\exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t + c) \right) \right] = \mathbb{E}\left[\Gamma_x \exp\left(-\frac{\Gamma_x^2}{2x} + \Gamma_x(t + c) \right) \right].$$

Notice also that this quantity is continuous in x on [0,1].

Hence we have

$$\forall x_i \in]0, 1[, \forall s > 0, \quad \frac{\partial_t g(s, x_i)}{g(s, x_i)} > 0.$$

Thus we deduce

$$\sum_{i=1}^{\infty} \frac{\partial_t g(s, x_i)}{g(s, x_i)} < \infty \quad \mathbb{P}_s^{(B)}\text{-a.s.}$$

Let define

$$k(t, x) = \partial_t g(t, x) + \sqrt{x} \int_0^1 \frac{dy}{\sqrt{8\pi y^3 (1 - y)^3}} \Big(g(t, xy) g(t, x(1 - y)) - g(t, x) \Big).$$

Hence we have

$$\sum_{i=1}^{\infty} k(s, x_i) = 0 \quad \mathbb{P}_s^{(B)}\text{-a.s.} \quad \text{for almost every } s > 0, \tag{III.3}$$

and

$$\sum_{i=1}^{\infty} |k(s,x_i)| < \infty \quad \mathbb{P}_s^{(B)}\text{-a.s.} \quad \text{for almost every } s > 0. \tag{III.4}$$

Furthermore, $x \to k(t,x)$ is continuous on $[0,1]$, hence, thanks to the following lemma, we get for almost every $s > 0$, $k(s,x) = 0$ for $x \in [0,1]$. And, since $s \to k(s,x)$ is continuous on $\mathbb{R}_+$, we deduce $k \equiv 0$ on $\mathbb{R}_+ \times [0,1]$. ∎

Lemma III.11

Fix $t > 0$. Let $\mathbb{P}_t^{(B)}$ denote the law of the Brownian fragmentation at time t. Let $k : [0,1] \mapsto \mathbb{R}$ be a continuous function, such that

$$\sum_{i=1}^{\infty} k(x_i) = 0 \quad \mathbb{P}_t^{(B)}\text{-a.s. and } \sum_{i=1}^{\infty} |k(x_i)| < \infty \quad \mathbb{P}_t^{(B)}\text{-a.s.}$$

Then $k \equiv 0$ on [0,1].

Proof: Let $F(t) = (F_1(t), F_2(t)\ldots)$ be a Brownian fragmentation at time t where the sequence $(F_i(t))_{i\geq 1}$ is ordered by a size-biased pick. We denote by $\mathcal{S}$ the set of positive sequence with sum less than 1. Since $F(t)$ has the law of the size biased reordering of the jumps of a stable subordinator T (with index $1/2$) before time t, conditioned by $T_t = 1$ (see [2]), it is obvious that we have

$$\forall x \in]0, 1-S[, \quad \mathbb{P}_t^{(B)}(F_1 \in dx \mid (F_i)_{i\geq 3}) > 0,$$

where $S = \sum_{i\geq 3} F_i$. Let $\mathbb{Q}_t$ be the measure on $\mathcal{S}$ defined by

$$\forall A \subset \mathcal{S}, \quad \mathbb{Q}_t(A) = \mathbb{P}_t^{(B)}((F_i)_{i\geq 3} \in A)$$

and λ the Lebesgue measure on $[0,1]$. Hence we have, for all $y \in \mathcal{S}$ - $\mathbb{Q}_t$-a.s.

$$\forall x \in]0, S[, \quad k(x) + k(1-S-x) + \sum_{i=1}^{\infty} k(y_i) = 0 \quad \lambda\text{-a.s.},$$

where $S = \sum_i y_i$. We choose now $y \in \mathcal{S}$ such that this equality holds for almost every $x \in]0, S[$. Thus, we get that there exists a constant $C = C(y)$ such that

$$k(x) + k(1-S-x) = C, \quad \text{for all } x \in]0, S[\quad \lambda\text{-a.s.}$$

Since k is continuous, this equality holds in fact for all $x \in [0, S]$. Furthermore, we have also

$$\forall s \in]0, 1[, \quad \mathbb{Q}_t(S \in ds) > 0.$$

Hence, this implies the existence for almost every $s\in]0,1[$ of a constant C_s such that

$$k(x)+k(1-s-x)=C_s \quad \text{for all } x\in]0,s[.$$

Thanks to the continuity of k, we can deduce that this property holds in fact for all $s\in[0,1]$. Hence we have

$$\forall x,y\in[0,1]^2,\ \text{such that } x+y\le 1,\ k(x+y)=k(x)+k(y).$$

So k is a linear function and since $\sum_{i=1}^{\infty}x_i=1\ \ \mathbb{P}_t^{(B)}$-a.s., we get $k\equiv 0$ on [0,1]. ■

We prove now the point 2 of Proposition III.10.

Proof: Assume that the Lévy measure of Γ is finite. It is obvious that g has the same regularity that the function $\frac{q_x(-tx)}{p_x(-tx)}$. Recall now that we have

$$\frac{q_x(-tx)}{p_x(-tx)}=\exp\left(-x(ct+\frac{c^2}{2})\right)\mathbb{E}\left[\exp\left(-\frac{\Gamma_x^2}{2x}+\Gamma_x(t+c)\right)\right].$$

Hence a sufficient condition for g to fulfill the hypotheses of Proposition III.10 is

- $u_t(x)=\mathbb{E}\left[\exp\left(-\frac{\Gamma_x^2}{2x}+\Gamma_x(t+c)\right)\right]$ is derivable at 0.
- $w(t,x)=u_t'(x)$ is continuous at $(t,0)$ for $t\in\mathbb{R}_+$.

We write $u_t(x)=a_t(x,x)$ with

$$a_t(y,z)=\mathbb{E}\left[\exp\left(-\frac{\Gamma_y^2}{2z}+\Gamma_y(t+c)\right)\right].$$

Since the function $(y,z)\to\frac{y^2}{2z^2}\exp\left(-\frac{y^2}{2z}+y(t+c)\right)$ is bounded on $\mathbb{R}_+\times[0,1]$, we get

$$\partial_z a_t(y,z)=\mathbb{E}\left[\frac{\Gamma_y^2}{2z^2}\exp\left(-\frac{\Gamma_y^2}{2z}+\Gamma_y(t+c)\right)\right] \qquad \text{for } z\in]0,1].$$

Recall that the generator of a subordinator with no drift and Lévy measure π is given for every bounded function f C^1 with bounded derivative by

$$\forall y\in\mathbb{R}_+,\ Lf(y)=\int_0^{\infty}(f(y+s)-f(y))\pi(ds), \quad \text{(c.f. Section 31 of [59])}.$$

Hence, we get for all $z_0>0$,

$$\begin{aligned}\partial_y a_t(y,z_0)&=\mathbb{E}(La_t(\Gamma_y,z_0))\\&=\mathbb{E}\left[\int_0^{\infty}\left(\exp\left(-\frac{(\Gamma_y+s)^2}{2z_0}+(\Gamma_y+s)(t+c)\right)-\exp\left(-\frac{\Gamma_y^2}{2z_0}+\Gamma_y(t+c)\right)\right)\pi(ds)\right].\end{aligned}$$

And we deduce

$$u'_t(x) = \mathbb{E}\left[\frac{\Gamma_x^2}{2x^2}\exp\left(-\frac{\Gamma_x^2}{2x}+\Gamma_x(t+c)\right)\right]$$
$$+\mathbb{E}\left[\int_0^\infty \left(\exp\left(-\frac{(\Gamma_x+y)^2}{2x}+(\Gamma_x+y)(t+c)\right)-\exp\left(-\frac{\Gamma_x^2}{2x}+\Gamma_x(t+c)\right)\right)\pi(dy)\right],$$

We must prove that $(t,x)\to u'_t(x)$ is continuous at $(t,0)$ for $t\geq 0$. For every Lévy measure π, the first term has limit 0 as (t',x) tends to $(t,0)$ (by dominated convergence). For the second term, notice that we have for all $x\in]0,1]$,

$$\left|\exp\left(-\frac{(\Gamma_x+y)^2}{2x}+(\Gamma_x+y)(t+c)\right)-\exp\left(-\frac{\Gamma_x^2}{2x}+\Gamma_x(t+c)\right)\right|\leq 2\exp\left(\frac{(t+c)^2x}{2}\right),$$

and for all $y>0$, $\exp\left(-\frac{(\Gamma_x+y)^2}{2x}+(\Gamma_x+y)(t+c)\right)-\exp\left(-\frac{\Gamma_x^2}{2x}+\Gamma_x(t+c)\right)$ converges almost surely to -1 as (t',x) tends to $(t,0)$. Hence, if $\pi(\mathbb{R}_+)<\infty$, we deduce that the $\lim_{(t',x)\to(t,0)} u'_t(x)$ exists (and is equal to $-\pi(\mathbb{R}_+)$). ■

Chapter IV

Fragmentations of ordered partitions and intervals[1]

Abstract. Fragmentation processes of exchangeable partitions have already been studied by several authors. This paper deals with fragmentations of exchangeable compositions, i.e. partitions of $\mathbb{N}$ in which the order of the blocks matters. We will prove that such a fragmentation is bijectively associated to an interval fragmentation. Using this correspondence, we then study two examples : Ruelle's interval fragmentation and the interval fragmentation derived from the standard additive coalescent. We also calculate the Hausdorff dimension of certain random closed sets that arise in interval fragmentations.

IV.1 Introduction

Random fragmentations describe objects that split as time goes on. Two types of fragmentation have received a special attention: fragmentations of partitions of $\mathbb{N}$ and mass-fragmentations, i.e. fragmentations on the space $\mathcal{S}^{\downarrow} = \{s_1 \geq s_2 \geq \ldots \geq 0, \sum_i s_i \leq 1\}$. Berestycki [6] has proved that to each homogeneous fragmentation process of exchangeable partitions, we can canonically associate a mass fragmentation. More precisely, let $\pi = (\pi_1, \pi_2, \ldots)$ be an exchangeable random partition of $\mathbb{N}$ (i.e. the distribution of π is invariant under finite permutations of $\mathbb{N}$) whose blocks $(\pi_i)_{i\geq 1}$ are ordered by increasing of their least elements. According to the

[1]This chapter is an extended version of the article: A-L. Basdevant, *Fragmentations of ordered partitions and intervals*, Electron. J. Probab.,11: no 16, 394-417, 2006.

work of Kingman and Pitman [38, 52], the asymptotic frequency of the i-th block π_i, $f_i = \lim_{n\to\infty} \frac{Card\{\pi_i \cap \{1,\ldots,n\}\}}{n}$, exists for every i a.s. We denote by $(|\pi_i|^{\downarrow})_{i\in\mathbb{N}}$ the sequence $(f_i)_{i\in\mathbb{N}}$ after a decreasing rearrangement. If $(\Pi(t), t \geq 0)$ is a fragmentation of exchangeable partitions, then $(|\Pi_i(t)|^{\downarrow}_{i\in\mathbb{N}}, t \geq 0)$ is a mass fragmentation. Conversely, a fragmentation of exchangeable partitions can be built from a mass fragmentation via a "paintbox process".

One goal of this paper is to develop a similar theory for fragmentations of exchangeable compositions and interval fragmentations. Let us recall that a composition of a natural number n is an ordered collection of natural numbers $(n_1, \ldots, n_k)$ with sum n. Here we will also use the definition of Gnedin [33]: a composition of the set $\{1, \ldots, n\}$ is an ordered collection of disjoint nonempty subsets $\gamma = (A_1, \ldots, A_k)$ with $\cup A_i = \{1, \ldots, n\}$. The vector of class size of γ, $(\sharp A_1, \ldots, \sharp A_k)$ is a composition of n and is called the shape of γ. Hence, there is a one to one correspondence between measures on compositions of n and measures on exchangeable compositions of the set $\{1, \ldots, n\}$. Gnedin proved a theorem analogous to Kingman's Theorem in the case of exchangeable compositions: for each probability measure P that describes the law of a random exchangeable composition, we can find a probability measure on the space of open subsets of [0,1], such that P can be recovered via a "paintbox process". This is why it seems very natural to look for a correspondence between fragmentations of compositions and interval fragmentations.

The first part of this paper develops the relation between probability laws of exchangeable compositions and laws of random open subsets, and its extension to infinite measures. Then we prove that there exists indeed a one to one correspondence between fragmentations of compositions and interval fragmentations. The next part gives some properties and characteristics of these processes and briefly presents how this theory can be extended to time-inhomogeneous fragmentations and self-similar fragmentations. Finally, as an application of this theory, Section IV.5 describes two well known interval fragmentations: first, the interval fragmentation introduced by Ruelle (cf. Chapter II) and second, the fragmentation derived from the standard additive coalescent [2, 8] and in the last section, we turn our attention to the estimation of the Hausdorff dimension of random closed sets which arise in an interval fragmentation.

IV.2 Exchangeable compositions and open subsets of $]0,1[$

IV.2.1 Probability measures

In this section, we define exchangeable compositions following Gnedin [33], and recall some useful properties. For $n \in \mathbb{N}$, let $[n]$ be the set of integers $\{1, \ldots, n\}$.

Definition IV.1

For $n \in \mathbb{N}$, a composition of the set $[n]$ is an ordered sequence of disjoint, non empty subsets of $[n]$, $\gamma = (A_1, \ldots, A_k)$, with $\cup A_i = [n]$. We denote by $\mathcal{C}_n$ the set of compositions of $[n]$.

Let $k_n : \mathcal{C}_n \to \mathcal{C}_{n-1}$ be the restriction mapping from compositions of the set $[n]$ to compositions of the set $[n-1]$ and let $\mathcal{C}$ be the projective limit of $(\mathcal{C}_n, k_n)$. We endow $\mathcal{C}$ with the product topology, it is then a compact set. The composition of $[n]$ (resp. $\mathbb{N}$) with a single nonempty block will be denoted by $\mathbb{1}_n$ (resp. $\mathbb{1}_{\mathbb{N}}$) and we will write $\mathcal{C}_n^*$ for $\mathcal{C}_n \backslash \{\mathbb{1}_n\}$. In the sequel, for $n \in \mathbb{N} \cup \{\infty\}$, $\gamma \in \mathcal{C}_n$ and $A \subset [n]$, γ_A will denote the restriction of γ to A. Hence, for $m \le n$, $\gamma_{[m]}$ will denote the restriction of γ to $[m]$. We say that a sequence $(P_n)_{n\in\mathbb{N}}$ of measures on $(\mathcal{C}_n)_{n\in\mathbb{N}}$ is consistent if, for all $n \ge 2$, P_{n-1} is the image of P_n by the projection k_n, i.e., for all $\gamma \in \mathcal{C}_{n-1}$, we have

$$P_{n-1}(\Gamma_{[n-1]} = \gamma) = \sum_{\gamma' \in \mathcal{C}_n : k_n(\gamma') = \gamma} P_n(\Gamma_{[n]} = \gamma').$$

By Kolmogorov's Theorem, such a sequence $(P_n)_{n\in\mathbb{N}}$ determines the law of a random composition of $\mathbb{N}$.

A random composition Γ of $\mathbb{N}$ is called exchangeable if for all $n \in \mathbb{N}$, for every permutation σ of $[n]$ and for all $\gamma \in \mathcal{C}_n$, we have

$$\mathbb{P}(\Gamma_{[n]} = \gamma) = \mathbb{P}(\sigma(\Gamma_{[n]}) = \gamma),$$

where $\sigma(\Gamma_{[n]})$ is the image of the composition $\Gamma_{[n]}$ by σ. Hence, given an exchangeable random composition Γ, we can associate a function defined on finite sequences of integers by

$$\forall k \in \mathbb{N}, \forall n_1, \ldots, n_k \in \mathbb{N}^k, p(n_1, \ldots, n_k) = \mathbb{P}(\Gamma_{[n]} = (B_1, \ldots, B_k)),$$

where $(B_1, \ldots, B_k)$ is a composition of the set $[n]$ with shape $(n_1, \ldots, n_k)$ and $n = n_1 + \ldots + n_k$. This function determines the law of Γ and is called the exchangeable composition probability function (ECPF) of Γ.

Notation IV.2

Let γ be a composition of $\mathbb{N}$. For $(i,j) \in \mathbb{N}^2$, we will use the following notation:

- *$i \sim j$, if i and j are in the same block.*
- *$i \prec j$, if the block containing i precedes the block containing j.*
- *$i \succ j$, if the block containing i follows the block containing j.*

Let $\mathcal{U}$ be the set of open subsets of $]0,1[$. For $u \in \mathcal{U}$, let

$$\chi_u(x) = \min\{|x-y|, y \in u^c\},\ x \in [0,1],$$

where $u^c = [0,1]\backslash u$. We also define a distance on $\mathcal{U}$ by:

$$d(u,v) = ||\chi_u - \chi_v||_\infty.$$

This makes $\mathcal{U}$ a compact metric space.

Definition IV.3

Let u be an open subset of $[0,1]$. We construct a random composition of $\mathbb{N}$ in the following way: we draw $(X_i)_{i\in\mathbb{N}}$ iid random variables with uniform law on $[0,1]$ and we use the following rules:

- *$i \sim j$, if $i = j$ or if X_i and X_j belong to the same component interval of u.*
- *$i \prec j$, if X_i and X_j do not belong to the same component interval of u and $X_i < X_j$.*
- *$i \succ j$, if X_i and X_j do not belong to the same component interval of u and $X_i > X_j$.*

This defines an exchangeable probability measure on $\mathcal{C}$ that we shall denote P^u; the projection of P^u on $\mathcal{C}_n$ will be denoted by P^u_n. If ν is a probability measure on $\mathcal{U}$, we denote by P^ν the law on $\mathcal{C}$ whose projections on $\mathcal{C}_n$ are:

$$P^\nu_n(\cdot) = \int_{\mathcal{U}} P^u_n(\cdot)\nu(du).$$

Let us recall here a useful theorem from Gnedin [33]:

Theorem IV.4

[33] Let Γ be an exchangeable random composition of $\mathbb{N}$, $\Gamma_{[n]}$ its restriction to $[n]$. Let $(N_1,\dots,N_k)$ be the shape of $\Gamma_{[n]}$ and $N_0 = 0$. For $i \in \{0,\dots,k\}$, we

write $M_i = \sum_{j=0}^{i} N_j$. *Define* $U_n \in \mathcal{U}$ *by:*

$$U_n = \bigcup_{i=1}^{k} \left] \frac{M_{i-1}}{n}, \frac{M_i}{n} \right[.$$

Then U_n *converges almost surely to a random element* $U \in \mathcal{U}$. *The conditional law of* Γ *given* U *is* P^U. *As a consequence, if* P *is an exchangeable probability measure on* $\mathcal{C}$, *then there exists a unique probability measure* ν *on* $\mathcal{U}$ *such that* $P = P^\nu$.

Hence, with each exchangeable composition Γ, we can associate a random open set that we will call asymptotic open set of Γ and denote U_Γ. We shall also write $|\Gamma|^\downarrow$ for the decreasing sequence of the lengths of the interval components of U_Γ. More generally, for $u \in \mathcal{U}$, $u^\downarrow$ will be the decreasing sequence of the interval component lengths of u.

Let us notice that this theorem is the analogue of Kingman's Theorem for the representation of exchangeable partitions. Actually, let $\Pi = (\Pi_1, \Pi_2, \ldots)$ be an exchangeable random partition of $\mathbb{N}$ (the blocks of Π are listed by increase of their least elements). Pitman [52] has proved that each block of Π has almost surely a frequency, i.e.

$$\forall i \in \mathbb{N} \qquad f_i = \lim_{n\to\infty} \frac{Card\{\Pi_i \cap [n]\}}{n} \qquad \text{exists almost surely.}$$

One calls f_i the frequency of the block Π_i. Therefore, for all exchangeable random partitions, we can associate a probability on $\mathcal{S}^\downarrow = \{s = (s_1, s_2, \ldots), s_1 \geq s_2 \geq \ldots \geq 0, \sum_i s_i \leq 1\}$ which will be the law of the decreasing rearrangement of the sequence of the partition frequencies.

Conversely, given a law $\tilde{\nu}$ on $\mathcal{S}^\downarrow$, we can construct an exchangeable random partition whose law of its frequency sequence is $\tilde{\nu}$ (cf. [38]): we pick $S \in \mathcal{S}^\downarrow$ with law $\tilde{\nu}$ and we draw a sequence of independent random variables V_i with uniform law on $[0,1]$. Conditionally on S, two integers i and j are in the same block of Π iff there exists an integer k such that $\sum_{l=1}^{k} S_l \leq V_i < \sum_{l=1}^{k+1} S_l$ and $\sum_{l=1}^{k} S_l \leq V_j < \sum_{l=1}^{k+1} S_l$. We denote by $\rho_{\tilde{\nu}}$ the law of this partition (and by a slight abuse of notation, ρ_s denotes the law of the partition obtained with $\tilde{\nu} = \delta_s$). Kingman's representation Theorem states that any exchangeable random partition can be constructed in this way.

Let $\wp_1$ be the canonical projection from the set of compositions $\mathcal{C}$ to the set of partitions $\mathcal{P}_\infty$ and $\wp_2$ the canonical projection from the set $\mathcal{U}$ to the set $\mathcal{S}^\downarrow$ that associates to an open set u the decreasing sequence $u^\downarrow$ of the lengths of its interval components. To sum up, we have the following commutative diagram between probability measures on $\mathcal{P}_\infty$, $\mathcal{C}$, $\mathcal{S}^\downarrow$, $\mathcal{U}$:

$$\begin{array}{ccc} (\mathcal{C}, P^\nu) & \xleftrightarrow{\text{Gnedin}} & (\mathcal{U}, \nu) \\ \wp_1 \downarrow & & \wp_2 \downarrow \\ (\mathcal{P}_\infty, \rho_{\tilde{\nu}}) & \xleftrightarrow{\text{Kingman}} & (\mathcal{S}^\downarrow, \tilde{\nu}). \end{array}$$

IV.2.2 Representation of infinite measures on $\mathcal{C}$

In this section, we show how Theorem IV.4 can be extended to a class of infinite measures on $\mathcal{C}$.

Definition IV.5

Let μ be a measure on $\mathcal{C}$. We call μ a fragmentation measure if the following conditions hold:

- *μ is exchangeable.*
- $\mu(\mathbb{1}_{\mathbb{N}}) = 0$.
- $\mu(\{\gamma \in \mathcal{C}, \gamma_{[2]} \neq \mathbb{1}_2\}) < \infty$.

Notice that by exchangeability, the last condition implies that, for all $n \geq 2$, we have $\mu(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbb{1}_n\}) < \infty$. We will see in the sequel that such a measure can always be associated to a fragmentation process and conversely.

Definition IV.6

A measure ν on $\mathcal{U}$ is called a dislocation measure if:

$$\nu(]0,1[) = 0, \qquad \int_{\mathcal{U}} (1 - s_1)\nu(du) < \infty,$$

where s_1 is the length of the largest interval component of u.

In the sequel, for any ν measure on $\mathcal{U}$, we define the measure P^ν on $\mathcal{C}$ by

$$P^\nu = \int_{\mathcal{U}} P^u \nu(du).$$

Notice that if ν is a dislocation measure, then P^ν is a fragmentation measure. In fact, the measure P^ν is exchangeable since P^u is an exchangeable measure. For $u \neq]0,1[$, we have $P^u(\mathbb{1}_{\mathbb{N}}) = 0$, and as $\nu(]0,1[) = 0$, we have also $P^\nu(\mathbb{1}_{\mathbb{N}}) = 0$. We now have to check that $P^\nu(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbb{1}_n\}) < \infty$ for all $n \in \mathbb{N}$. Let us fix $u \in \mathcal{U}$. Set $u^\downarrow = s = (s_1, s_2, \ldots)$.

$$P^u(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbb{1}_n\}) = 1 - \sum_{i=1}^{\infty} s_i^n \leq 1 - s_1^n \leq n(1-s_1)$$

and so $P^\nu(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbb{1}_n\}) < \infty$.

Let ϵ_l^i be the composition of $\mathbb{N}$ given by $(\{i\}, \mathbb{N}\setminus\{i\})$ and $\epsilon_l = \sum_i \delta_{\epsilon_l^i}$. Let ϵ_r^i be the composition of $\mathbb{N}$ given by $(\mathbb{N}\setminus\{i\}, \{i\})$ and $\epsilon_r = \sum_i \delta_{\epsilon_r^i}$. It is easy to check that ϵ_l and ϵ_r are also two fragmentation measures.

Theorem IV.7

If μ is a fragmentation measure, there exists two unique nonnegative numbers c_l and c_r, called coefficients of erosion, and a unique dislocation measure ν on $\mathcal{U}$ such that:

$$\mu = c_l\epsilon_l + c_r\epsilon_r + P^\nu.$$

Besides, we have $\mathbb{1}_{\{\gamma\in\mathcal{C},\, U_\gamma =]0,1[\}}\mu = c_l\epsilon_l + c_r\epsilon_r$ and $\mathbb{1}_{\{\gamma\in\mathcal{C},\, U_\Gamma \neq]0,1[\}}\mu = P^\nu$.

Recall that in the case of fragmentation measures on partitions, Bertoin [14] proved the following result: let $\tilde{\epsilon}^i$ be the partition of $\mathbb{N}$, $\big\{\{i\}, \mathbb{N}\setminus\{i\}\big\}$ and define the measure $\tilde{\epsilon} = \sum_i \delta_{\tilde{\epsilon}^i}$. Let $\tilde{\mu}$ be an exchangeable measure on $\mathcal{P}_\infty$ such that $\mu(\{\mathbb{N}\}) = 0$ and $\tilde{\mu}(\pi \in \mathcal{P}_\infty, \pi_n \neq \{[n]\})$ is finite for all $n \in \mathbb{N}$. Then there exists a measure $\tilde{\nu}$ on $\mathcal{S}^\downarrow$ such that $\tilde{\nu}((1,0,0,\ldots)) = 0$ and $\int_{\mathcal{S}^\downarrow}(1-s_1)\nu(ds) < \infty$, and a nonnegative number c such that:

$$\tilde{\mu} = \rho_{\tilde{\nu}} + c\tilde{\epsilon}.$$

Fragmentation measures on partitions fit in a more general framework of exchangeable semifinite measures on partitions as developed by Kerov (see [36], Chapter 1, Section 3).

Hence, Theorem IV.7 is an analogous decomposition in the case of fragmentation measures on compositions, except that, in this case, there are two coefficients of erosion, one characterizing the left side erosion and the other the right side erosion.

Proof: We adapt a proof due to Bertoin [14] for the exchangeable partitions to our case. Set $n \in \mathbb{N}$. Set $\mu_n = \mathbb{1}_{\{\Gamma_{[n]} \neq \mathbb{1}_n\}}\mu$, therefore μ_n is a finite measure. Let $\overrightarrow{\mu_n}$ be the image of μ_n by the n-shift, i.e.

$$i \overset{\overset{\to n}{\Gamma}}{\prec} j \Leftrightarrow i+n \overset{\Gamma}{\prec} j+n, \qquad i \overset{\overset{\to n}{\Gamma}}{\sim} j \Leftrightarrow i+n \overset{\Gamma}{\sim} j+n, \qquad i \overset{\overset{\to n}{\Gamma}}{\succ} j \Leftrightarrow i+n \overset{\Gamma}{\succ} j+n.$$

Then $\overrightarrow{\mu_n}$ is exchangeable since μ is, and furthermore it is a finite measure. So, we can apply Theorem IV.4:

$$\exists\,!\ \nu_n \text{ finite measure on } \mathcal{U} \text{ such that } \overrightarrow{\mu_n}(d\gamma) = \int_{\mathcal{U}} P^u(d\gamma)\nu_n(du).$$

According to Theorem IV.4, since $\overrightarrow{\mu_n}$ is an exchangeable finite measure, $\overrightarrow{\mu_n}$-almost every composition has an asymptotic open set and so μ_n-almost every composition has also an asymptotic open set, and as $\mu_n \uparrow \mu$, μ-almost every composition has also an asymptotic open set. Besides, we have

$$\forall A \subset \mathcal{U},\ \mu_n(|\gamma|^{\downarrow} \in A) = \overrightarrow{\mu_n}(|\gamma|^{\downarrow} \in A) = \nu_n(A).$$

Hence, since $\mu_n \leq \mu_{n+1}$, we deduce that $\nu_n \leq \nu_{n+1}$. Set $\nu = \lim_{n\to\infty} \uparrow \nu_n$. Furthermore, we have

$$\mu_n(n+1 \not\sim n+2 \mid U_\Gamma = u) = \overrightarrow{\mu_n}(1 \not\sim 2 \mid U_\Gamma = u) = P^u(1 \not\sim 2) = 1 - \sum {s_i}^2 \geq 1 - s_1.$$

So

$$\mu_n(n+1 \not\sim n+2) \geq \int (1-s_1)\nu_n(du).$$

Since $\mu_n(n+1 \not\sim n+2) \leq \mu(n+1 \not\sim n+2) = \mu(1 \not\sim 2) < \infty$, we deduce that $\int(1-s_1)\nu(du) < \infty$. Hence ν is a dislocation measure. Set $\gamma_k \in \mathcal{C}_k$.

$$\begin{aligned}
\mu(\Gamma_{[k]} = \gamma_k, U_\Gamma \neq]0,1[) &= \lim_{n\to\infty} \mu(\Gamma_{[k]} = \gamma_k, \Gamma_{\{k+1,\ldots,k+n\}} \neq \mathbb{1}_n, U_\Gamma \neq]0,1[) \\
&= \lim_{n\to\infty} \mu(\overset{\rightarrow n}{\Gamma}_{[k]} = \gamma_k, \Gamma_{[n]} \neq \mathbb{1}_n, U_\Gamma \neq]0,1[) \\
&= \lim_{n\to\infty} \overrightarrow{\mu_n}(\Gamma_{[k]} = \gamma_k, U_\Gamma \neq]0,1[) \\
&= \int_{\mathcal{U}^*} P^u(\Gamma_{[k]} = \gamma_k)\nu(du) \qquad \text{with } \mathcal{U}^* = \mathcal{U}\backslash\{]0,1[\}.
\end{aligned}$$

Thus we have

$$\mu(\,\cdot\,, U_\gamma \neq]0,1[) = \int P^u(\,\cdot\,)\nu(du).$$

We now have to study μ on the event $\{U_\gamma =]0,1[\}$. Let $\tilde{\mu} = \mathbb{1}_{\{1 \not\sim 2, U_\gamma =]0,1[\}}\mu$. Let $\overrightarrow{\tilde{\mu}}$ be the image of $\tilde{\mu}$ by the 2-shift. The measure $\overrightarrow{\tilde{\mu}}$ is finite and exchangeable and its asymptotic open set is almost surely $]0,1[$, so $\overrightarrow{\tilde{\mu}} = a\delta_{]0,1[}$ where a is a nonnegative number. So $\tilde{\mu} = c_1\delta_{\gamma_1} + \ldots + c_{10}\delta_{\gamma_{10}}$ where $\gamma_1, \ldots, \gamma_6$ are the six possible compositions build from the blocks $\{1\}$, $\{2\}$, $\mathbb{N}\backslash\{1,2\}$, $\gamma_7 = (\{1\}, \mathbb{N}\backslash\{1\})$, $\gamma_8 = (\{2\}, \mathbb{N}\backslash\{2\})$, $\gamma_9 = (\mathbb{N}\backslash\{1\}, \{1\})$, $\gamma_{10} = (\mathbb{N}\backslash\{2\}, \{2\})$. We must have $c_1 = \ldots = c_6 = 0$, otherwise, by exchangeability, we would have $\mu(\{1\}, \{n\}, \mathbb{N}\backslash\{1,n\}) = c > 0$ and this would yield $\mu(\mathcal{C}_2^*) = \infty$. By exchangeability, we also have $c_7 = c_8$ and $c_9 = c_{10}$ and so, by exchangeability,

$$\mu\mathbb{1}_{\{U_\gamma =]0,1[\}} = c_l \sum_i \delta_{\epsilon_l^i} + c_r \sum_i \delta_{\epsilon_r^i}.$$

■

As in Section IV.2.1, we can now establish connections among fragmentation measures on $\mathcal{C}$ and $\mathcal{P}_\infty$ and dislocation measures on $\mathcal{U}$ and $\mathcal{S}^\downarrow$. Let us recall that $\wp_1$ is the canonical projection from $\mathcal{C}$ to $\mathcal{P}_\infty$, and denote $q : (\mathcal{U}, \mathbb{R}_+, \mathbb{R}_+) \mapsto (\mathcal{S}^\downarrow, \mathbb{R}_+)$ the operation defined by $q(u, a, b) = q(u^\downarrow, a+b)$. Then we have the following commutative diagram:

$$\begin{array}{ccc} (\mathcal{C},\mu) & \overset{\text{Theorem IV.7}}{\longleftrightarrow} & \big(\mathcal{U},(\nu,c_l,c_r)\big) \\ \wp_1\big\downarrow & & \big\downarrow q \\ (\mathcal{P}_\infty,\tilde\mu) & \overset{\text{Bertoin}}{\longleftrightarrow} & \big(\mathcal{S}^\downarrow,(\tilde\nu,c_l+c_r)\big). \end{array}$$

Proof: It remains to prove that $\tilde\mu = \rho_{\tilde\nu} + (c_l + c_r)\tilde\epsilon$. Set $\tilde\mu = \rho_{\overline{\nu}} + c\tilde\epsilon$. Since $\tilde\mu$ is the image by $\wp_1$ of μ, we have

$$\tilde\mu(\tilde\epsilon^1) = \mu(\epsilon_l^1) + \mu(\epsilon_r^1) \text{ and then } c = c_r + c_l.$$

Let us fix $n \in \mathbb{N}$ and $\pi \in \mathcal{P}_n \backslash \{\mathbb{1}_n\}$. Set $A = \{\gamma \in \mathcal{C}_n, \wp_1(\gamma) = \pi\}$. Remark now that for all $u, v \in \mathcal{U}$ such that $u^\downarrow = v^\downarrow$, we have $P^u(A) = P^v(A)$. Moreover we have $P^u(A) = \rho_s(\pi)$ if $s = u^\downarrow$. So

$$P^\nu(A) = \int_{\mathcal{S}^\downarrow} P^u(A)\nu(u, u^\downarrow = ds) = \int_{\mathcal{S}^\downarrow} \rho_s(\pi)\tilde\nu(ds) = \rho_{\tilde\nu}(\pi).$$

We get

$$\mu(A) = P^\nu(A) + c_l\epsilon_l(A) + c_r\epsilon_r(A) = \rho_{\tilde\nu}(\pi) + (c_l+c_r)\tilde\epsilon(A) = \rho_{\overline{\nu}}(\pi) + (c_l+c_r)\tilde\epsilon(A) = \tilde\mu(\pi).$$

So we deduce that $\overline{\nu} = \tilde\nu$. ■

IV.3 Fragmentation of compositions and intervals

IV.3.1 Fragmentation of compositions

Definition IV.8

Let us fix $n \in \mathbb{N}$ and $\gamma \in \mathcal{C}_n$ with $\gamma = (\gamma_1, \ldots, \gamma_k)$. Let $\gamma^{(\cdot)} = (\gamma^{(i)}, i \in \{1, \ldots, n\})$ with $\gamma^{(i)} \in \mathcal{C}_n$ for all i. Set $m_i = \min \gamma_i$. We denote $\tilde\gamma^{(i)}$ the restriction of $\gamma^{(m_i)}$ to the set γ_i. So $\tilde\gamma^{(i)}$ is a composition of γ_i. We consider now $\tilde\gamma = (\tilde\gamma^{(1)}, \ldots, \tilde\gamma^{(k)}) \in \mathcal{C}_n$. We denote by $FRAG(\gamma, \gamma^{(\cdot)})$ the composition $\tilde\gamma$. If $\Gamma^{(\cdot)}$ is a sequence of i.i.d. random compositions with law p, p-$FRAG(\gamma, \cdot)$ will denote the law of $FRAG(\gamma, \Gamma^{(\cdot)})$.

We remark that the operator $FRAG$ has some useful properties. First, if $\mathbf{1}^{(.)}$ denotes the constant sequence equal to $\mathbb{1}_n$, we have $FRAG(\gamma, \mathbf{1}^{(.)}) = \gamma$. Furthermore, the fragmentation operator is compatible with the restriction i.e., for every $n' \le n$,

$$FRAG(\gamma, \gamma^{(.)})_{[n']} = FRAG(\gamma_{[n']}, \gamma^{(.)}).$$

This implies that $FRAG$ is a consistent operator and we can extend this definition to the compositions of $\mathbb{N}$. Notice that we have this equality since we take care of fragmenting the block γ_i by $\gamma^{(m_i)}$. Indeed, if we have fragmented the block γ_i by $\gamma^{(i)}$, the operator $FRAG$ would not be anymore compatible with the restriction. For example, take $n = 3$, $\gamma = (\{3\}, \{1, 2\})$, $\gamma^{(1)} = (\{1, 2, 3\})$ and $\gamma^{(2)} = (\{1\}, \{2\}, \{3\})$.

Besides, the operator $FRAG$ preserves the exchangeability. More precisely, let $(\Gamma^{(i)}, i \in \{1, \dots, n\})$ be a sequence of random compositions which is *doubly exchangeable*, i.e. for each i, $\Gamma^{(i)}$ is an exchangeable composition, and moreover, the sequence $(\Gamma^{(i)}, i \in \{1, \dots, n\})$ is also exchangeable. Let Γ be an exchangeable composition of $\mathcal{C}_n$ independent of $\Gamma^{(.)}$. Then $FRAG(\Gamma, \Gamma^{(.)})$ is an exchangeable composition. Let us prove this property. We fix a permutation σ of $[n]$ and we shall prove that

$$FRAG(\Gamma, \Gamma^{(.)}) \overset{law}{=} \sigma(FRAG(\Gamma, \Gamma^{(.)})).$$

Let k be the number of blocks of Γ and denote by $m_1, \dots, m_k$ the minima of $\Gamma_1, \dots, \Gamma_k$. Let us define now $m'_1, \dots, m'_k$ the minima of $\sigma(\Gamma_1), \dots, \sigma(\Gamma_k)$. and $\Gamma'^{(.)} = (\Gamma'^{(i)}, i \in \{1, \dots, n\})$ by

$$\Gamma'^{(m'_i)} = \sigma(\Gamma^{(m_i)}) \text{ for } 1 \le i \le k,$$
$$\Gamma'^{(j)} = \sigma(\Gamma^{(f(j))}) \text{ for } j \in \{1, \dots, n\} \setminus \{m'_i, 1 \le i \le k\},$$

where f is the increasing bijection from $\{1, \dots, n\} \setminus \{m'_i, 1 \le i \le k\}$ to $\{1, \dots, n\} \setminus \{m_i, 1 \le i \le k\}$. We get

$$\sigma(FRAG(\Gamma, \Gamma^{(.)})) = FRAG(\sigma(\Gamma), \Gamma'^{(.)}).$$

Since $\sigma(\Gamma) \overset{law}{=} \Gamma$ and $\Gamma'^{(.)} \overset{law}{=} \Gamma^{(.)}$ and $\Gamma'^{(.)}$ remains independent of Γ, we get

$$FRAG(\sigma(\Gamma), \Gamma'^{(.)}) \overset{law}{=} FRAG(\Gamma, \Gamma^{(.)}).$$

We can now define the notion of exchangeable fragmentation process of compositions.

Definition IV.9

Set $n \in \mathbb{N}$ and let $(\Gamma_n(t), t \ge 0)$ be a (possibly time-inhomogeneous) Markov process on $\mathcal{C}_n$ which is continuous in probability. We call Γ_n an exchangeable *fragmentation process of compositions if:*

- $\Gamma_n(0) = \mathbb{1}_n$ *a.s.*
- *Its semi-group is described in the following way: there exists a family of probability measures on exchangeable compositions* $(P_{t,s}, t \geq 0, s > t)$ *such that for all* $t \geq 0, s > t$ *the conditional law of* $\Gamma_n(s)$ *given* $\Gamma_n(t) = \gamma$ *is the law of* $P_{t,s}$*-FRAG*$(\gamma, \cdot)$.

The fragmentation is homogeneous in time if $P_{t,s}$ *depends only on* $s - t$. *A Markov process* $(\Gamma(t), t \geq 0)$ *on* $\mathcal{C}$ *is called an exchangeable fragmentation process of compositions if, for all* $n \in \mathbb{N}$, *the process* $(\Gamma_{[n]}(t), t \geq 0)$ *is an exchangeable fragmentation process of compositions on* $\mathcal{C}_n$.

Hence, in our definition we impose that the blocks split independently by the same rule (the "branching property"). This hypothesis is crucial for most of the following results (see however Section IV.4.5 where more general processes are considered).

In the sequel, a *c*-fragmentation will denote an exchangeable fragmentation process on compositions.

Proposition IV.10

The semi-group of transition of a time-homogeneous c-fragmentation has the Feller property.

Proof: Let $\phi : \mathcal{C} \to \mathbb{R}$ be a continuous function (recall that $\mathcal{C}$ is compact, so ϕ is bounded). Then the function $\gamma \to \mathbb{E}\Big(\phi(FRAG(\gamma, \Gamma^{(\cdot)}(t)))\Big)$ is also continuous on $\mathcal{C}$ since $FRAG$ is compatible with the restriction. Furthermore, for all $n \in \mathbb{N}$, $\lim_{t\to 0} \mathbb{P}(\Gamma_{[n]}(t) = \mathbb{1}_n) = 1$, so we have also

$$\lim_{t\to 0} \mathbb{E}\Big(\phi(FRAG(\gamma, \Gamma^{(\cdot)}(t)))\Big) = \phi(\gamma).$$

■

IV.3.2 Interval fragmentation

In this section we recall the definition of a homogeneous[2] interval fragmentation [11]. We consider a family of probability measures $(q_{t,s}, t \geq 0, s > t)$ on $\mathcal{U}$. For every interval $I =]a, b[\subset]0, 1[$, we define the affine transformation $g_I :]0, 1[\to I$ given by

[2] In [11], Bertoin defines more generally self-similar interval fragmentations with index α. Here, the term homogeneous means that we only consider the case $\alpha = 0$.

$g_I(x) = a + x(b-a)$. We still denote g_I the induced mapping on $\mathcal{U}$, so, for $V \in \mathcal{U}$, $g_I(V)$ is an open subset of I. We define then $q_{t,s}^I$ as the image of $q_{t,s}$ by g_I. Hence $q_{t,s}^I$ is a probability measure on the space of open subsets of I. Finally, for $W \in \mathcal{U}$ with interval decomposition $(I_i, i \in \mathbb{N})$, $q_{t,s}^W$ is the distribution of $\cup X_i$ where the X_i are independent random variables with respective laws $q_{t,s}^{I_i}$.

Definition IV.11

A process $(U(t), t \geq 0)$ on $\mathcal{U}$ is called a homogeneous interval fragmentation if it is a Markov process that fulfills the following properties:

- *U is continuous in probability and $U(0) =]0,1[$ a.s.*
- *U is nested i.e. for all $s > t$ we have $U(s) \subset U(t)$.*
- *There exists a family $(q_{t,s}, t \geq 0, s > t)$ of probability measures on $\mathcal{U}$ such that:*

$$\forall t \geq 0,\ \forall s > t,\ \forall A \subset \mathcal{U}, \quad \mathbb{P}(U(s) \in A |\, U(t)) = q_{t,s}^{U(t)}(A).$$

In the sequel, we abbreviate an interval fragmentation process as an i-fragmentation.

We remark that if we take the decreasing sequence of the sizes of the interval components of an i-fragmentation, we obtain a mass-fragmentation, denoted here as a m-fragmentation (see [14] for a definition of m-fragmentations).

IV.3.3 Link between i-fragmentation and c-fragmentation

From this point of the paper and until Section IV.4.4, the fragmentation processes we consider will always be homogeneous in time, i.e. $q_{t,s}$ depends only on $s-t$, hence we will just write q_{s-t} to denote $q_{t,s}$.

Let $(U(t), t \geq 0)$ be a process on $\mathcal{S}^{\downarrow}$. Let $(V_i)_{i\geq 0}$ be a sequence of independent random variables uniformly distributed on]0,1[. Using the same process as in Definition IV.3 with $U(t)$ and $(V_i)_{i\geq 1}$, we define a process $(\Gamma_U(t), t \geq 0)$ on $\mathcal{C}$.

Theorem IV.12

There is a one to one correspondence between laws of i-fragmentations and laws of c-fragmentations. More precisely:

- *If a process $(U(t), t \geq 0)$ is an i-fragmentation, then $(\Gamma_U(t), t \geq 0)$ defined as above is a c-fragmentation and we have $U_{\Gamma_U(t)} = U(t)$ a.s. for each $t \geq 0$.*

- *Let $(\Gamma(t), t \geq 0)$ be a c-fragmentation. Then $(U_{\Gamma(t)}, t \geq 0)$ is an i-fragmentation.*

Proof: We start by proving the first point. For the sake of clarity, we will write in the sequel $\Gamma(t)$ instead of $\Gamma_U(t)$. We have by Theorem IV.4, $U_{\Gamma(t)} = U(t)$ a.s. for each $t \geq 0$. Let us fix $n \in \mathbb{N}$ and $t \geq 0$. We are going to prove that, for $s > t$, the conditional law of $\Gamma_{[n]}(s)$ given $\Gamma_{[n]}(t) = (\Gamma_1, \ldots, \Gamma_k)$ is the law of $FRAG(\Gamma_{[n]}(t), \Gamma^{(\cdot)})$, where $\Gamma^{(\cdot)}$ is a sequence of i.i.d. exchangeable compositions with law $\Gamma_{[n]}(s-t)$. Since $(U(s), s \geq 0)$ is a fragmentation process, we have $U(t+s) \subset U(t)$. By construction of $\Gamma_{[n]}(t)$, it is then clear that $\Gamma_{[n]}(t+s)$ is a finer composition than $\Gamma_{[n]}(t)$. Hence, each singleton of $\Gamma_{[n]}(t)$ remains a singleton of $\Gamma_{[n]}(t+s)$. For $1 \leq i \leq k$, fix $l \in \Gamma_i$ and define

$$a_i = \sup\{a \leq V_l, a \notin U(t)\}, \qquad b_i = \inf\{b \geq V_l, b \notin U(t)\}.$$

Notice that a_i and b_i do not depend on the choice of $l \in \Gamma_i$. Furthermore, we have $a_i < b_i$ if Γ_i is not a singleton. We also define

$$Y_j^i = \left(\frac{V_j - a_i}{b_i - a_i}\right) \qquad j \in \Gamma_i,\ i \in J,$$

where $J = \{1 \leq i \leq k,\ \Gamma_i \text{ is not a singleton}\}$.

Conditionally on $\Gamma_{[n]}(t)$, the random variables $(Y_j^i)_{j \in \Gamma_i, i \in J}$ are independent and uniformly distributed on $]0,1[$. Besides, $(]a_i, b_i[)_{i \in J}$ are $\sharp J$ distinct interval components of $U(t)$. Since $U(t)$ is a fragmentation process, the processes

$$\left(U^i(s) = \frac{1}{b_i - a_i}(U_{]a_i,b_i[}(s) - a_i), s \geq t\right)_{i \in J}$$

are $\sharp J$ independent i-fragmentations with law $(U(s-t), s \geq t)$ and are also independent of the singletons of $\Gamma(t)$. For $i \in J$, let $\Gamma^{(i)}(s)$ be the composition of Γ_i obtained from $U^i(s)$ and $(Y_j^i)_{j \in \Gamma_i}$ using Definition IV.3; for $i \notin J$, we set $\Gamma^{(i)} = \mathbb{1}_{\Gamma_i}$. Hence, $\Gamma^{(i)}(s)$ has the law of $\Gamma_{\Gamma_i}(s-t)$ and the processes $(\Gamma^{(i)}(s), s \geq t)_{1 \leq i \leq k}$ are independent. Furthermore, by construction we have $\Gamma_{[n]}(t+s) = FRAG(\Gamma_{[n]}(t), \Gamma^{(\cdot)}(s))$. Hence, $(\Gamma_{[n]}(t), t \geq 0)$ has the expected transition probabilities.

Let us now prove the second point. In the sequel, we will write U_t to denote $U_{\Gamma(t)}$. First, we prove that for all $s > t$, $U_s \subset U_t$. Fix $x \notin U_t$, we shall prove $x \notin U_s$. We have $\chi_{U_t}(x) = \min\{|x-y|, y \in U_t^c\} = 0$. Let U_t^n be the open subset of $]0,1[$ corresponding to $\Gamma_{[n]}(t)$ as in Theorem IV.4. So we have $\lim_{n \to \infty} d(U_t^n, U_t) = 0$. Fix $\varepsilon > 0$. Hence, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $\chi_{U_t^n}(x) \leq \varepsilon$. This implies that:

$$\forall n \geq N, \exists y_n \notin U_t^n \text{ such that } |y_n - x| \leq \varepsilon.$$

Besides, as $(\Gamma(t), t \geq 0)$ is a fragmentation, we have for all $n \in \mathbb{N}$, $U_s^n \subset U_t^n$. Hence, we have also

$$\forall n \geq N, y_n \notin U_s^n,$$

and so $\chi_{U_s^n}(x) \leq \varepsilon$ for all $n \geq N$. We deduce that $\chi_{U_s}(x) = 0$ i.e. $x \notin U_s$.

We now have to prove the branching property. Fix $t > 0$. We consider the decomposition of U_t in disjoint intervals:

$$U_t = \coprod_{k \in \mathbb{N}} I_k(t).$$

Set $F_k(s) = U_{t+s} \cap I_k(t)$. We want to prove that, given U_t:

- $\forall l \in \mathbb{N},\ F_1, \ldots, F_l$ are independent processes.
- F_k has the following law: $\forall A$ open subset of $]a, b[$,

$$\mathbb{P}((F_k(s), s \geq 0) \in A\ |I_k(t) =]a, b[) = \mathbb{P}((U_s, s \geq 0) \in (b-a)A + a).$$

For all $k \in \mathbb{N}$, there exists $i_k \in \mathbb{N}$ such that, if $J_{i_k}^n(t)$ denotes the interval component of U_t^n containing the integer i_k, then $J_{i_k}^n(t) \overset{n \to \infty}{\longrightarrow} I_k(t)$. Let B_k be the block of $\Gamma(t)$ containing i_k. As B_k has a positive asymptotic frequency, it is isomorphic to $\mathbb{N}$. Let f be the increasing bijection from the set of elements of B_k to $\mathbb{N}$. Let us re-label the elements of B_k by their image by f. The process $(U_{\Gamma_{B_k}(t+s)}, s \geq 0)$ has then the same law as $(U_s, s \geq 0)$ and is independent of the rest of the fragmentation. Besides, given $I_k(t) =]a, b[$, $F_k(s) = a + (b-a)U_{\Gamma_{B_k}(t+s)}$, so the two points above are proved. ■

Hence, this result complements an analogous result due to Berestycki [6] in the case of m-fragmentations and p-fragmentation (i.e. fragmentations of exchangeable partitions). We can again draw a commutative diagram to represent the link between the four kinds of fragmentation:

$$\begin{array}{ccc}
\big(\mathcal{C}, (\Gamma(t), t \geq 0)\big) & \xleftrightarrow{\text{Theorem IV.12}} & \big(\mathcal{U}, (U_{\Gamma(t)}, t \geq 0)\big) \\
\wp_1 \big\downarrow & & \wp_2 \big\downarrow \\
\big(\mathcal{P}_\infty, (\Pi(t), t \geq 0)\big) & \xleftrightarrow{\text{Berestycki}} & \big(\mathcal{S}^\downarrow, (U_{\Gamma(t)}^\downarrow, t \geq 0)\big).
\end{array}$$

IV.4 Some general properties of fragmentations

In this section, we gather general properties of i and c-fragmentations. Since the proofs of these results are simple variations of those in the case of m and p-fragmentations [14], we will be a bit sketchy.

IV.4.1 Rate of a fragmentation process

Let $(\Gamma(t), t \geq 0)$ be a c-fragmentation. As in the case of p-fragmentation [14], for $n \in \mathbb{N}$ and $\gamma \in \mathcal{C}_n^*$, we define a jump rate from $\mathbb{1}_n$ to γ:

$$q_\gamma = \lim_{s \to 0} \frac{1}{s} \mathbb{P}\left(\Gamma_{[n]}(s) = \gamma\right).$$

With the same arguments as in the case of p-fragmentation, we can also prove that the family $(q_\gamma, \gamma \in \mathcal{C}_n^*, n \in \mathbb{N})$ characterizes the law of the fragmentation (you just have to use that distinct blocks evolve independently and with the same law). Furthermore, observing that we have

$$\forall n < m, \quad \forall \gamma' \in \mathcal{C}_n^*, \quad q_{\gamma'} = \sum_{\gamma \in \mathcal{C}_m, \gamma_{[n]} = \gamma'} q_\gamma,$$

and that

$$\forall n \in \mathbb{N}, \quad \forall \sigma \text{ permutation of } [n], \quad \forall \gamma \in \mathcal{C}_n^*, \quad q_\gamma = q_{\sigma(\gamma)},$$

we deduce that there exists a unique exchangeable measure μ on $\mathcal{C}$ such that $\mu(\mathbf{1}_\mathbb{N}) = 0$ and $\mu(\mathcal{Q}_{\infty,\gamma}) = q_\gamma$ for all $\gamma \in \mathcal{C}_n^*$ and $n \in \mathbb{N}$, where $\mathcal{Q}_{\infty,\gamma} = \{\gamma' \in \mathcal{C}, \gamma'_{[n]} = \gamma\}$. Furthermore, the measure μ characterizes the law of the fragmentation. We call μ the rate of the fragmentation.

We remark also that if a measure μ is the rate of a fragmentation process, we have for all $n \geq 2$,

$$\mu(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbb{1}_n\}) = \sum_{\gamma \in \mathcal{C}_n^*} q_\gamma < \infty.$$

So we can apply Theorem IV.7 to μ and we deduce the following result:

If μ is the rate of a c-fragmentation, then there exist a dislocation measure ν and two nonnegative numbers c_l and c_r such that:

- $\mu \mathbb{1}_{\{U_\gamma \neq]0,1[\}} = P^\nu$.
- $\mu \mathbb{1}_{\{U_\gamma =]0,1[\}} = c_l \epsilon_l + c_r \epsilon_r$.

With a slight abuse of notation, we will write sometimes in the sequel that $\mu = (\nu, c_l, c_r)$ when $\mu = P^\nu + c_l \epsilon_l + c_r \epsilon_r$.

IV.4.2 The Poissonian construction

We notice that if μ is the rate of a c-fragmentation, then μ is a fragmentation measure in the sense of Definition IV.5. Conversely, we now prove that, if we consider a fragmentation measure μ, we can construct a c-fragmentation with rate μ.

We consider a Poisson measure M on $\mathbb{R}_+ \times \mathcal{C} \times \mathbb{N}$ with intensity $dt \otimes \mu \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$. Let M^n be the restriction of M to $\mathbb{R}_+ \times \mathcal{C}_n^* \times \{1, \dots, n\}$. The intensity measure is then finite on the interval $[0, t]$, so we can rank the atoms of M^n according to their first coordinate. For $n \in \mathbb{N}$, $(\gamma, k) \in \mathcal{C} \times \mathbb{N}$, let $\Delta_n^{(.)}(\gamma, k)$ be the composition sequence of $\mathcal{C}_n$ defined by:

$$\Delta_n^{(i)}(\gamma, k) = \mathbb{1}_n \quad \text{if } i \neq k \qquad \text{and} \qquad \Delta_n^{(k)}(\gamma, k) = \gamma_{[n]}.$$

We construct then a process $(\Gamma_{[n]}(t), t \geq 0)$ on $\mathcal{C}_n$ in the following way:
$\Gamma_{[n]}(0) = \mathbb{1}_n$.
$(\Gamma_{[n]}(t), t \geq 0)$ is a pure jump process that only jumps at times when an atom of M^n appears. More precisely, if (s, γ, k) is an atom of M^n, set

$$\Gamma_{[n]}(s) = FRAG(\Gamma_{[n]}(s^-), \Delta_n^{(.)}(\gamma, k)).$$

We can check that this construction is compatible with the restriction; hence, this defines a process $(\Gamma(t), t \geq 0)$ on $\mathcal{C}$.

Proposition IV.13

Let μ be a fragmentation measure. The construction above of a process on compositions from a Poisson point process on $\mathbb{R}_+ \times \mathcal{C} \times \mathbb{N}$ with intensity $dt \otimes \mu \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$, yields a c-fragmentation with rate μ.

Proof: The proof is an easy adaptation of the Poissonian construction given by Bertoin [14] of p-fragmentations. As the sequence $\Delta_n^{(.)}(\gamma, k)$ is doubly exchangeable, we also have that $\Gamma_{[n]}(t)$ is an exchangeable composition for each $t \geq 0$. Looking at the jump rates of the process $\Gamma_{[n]}(t)$, it is then easy to check that the constructed process is a c-fragmentation with rate μ. ■

A Poissonian construction of an i-fragmentation with no erosion is also possible with a Poisson measure on $\mathbb{R}_+ \times \mathcal{U} \times \mathbb{N}$ with intensity $dt \otimes \nu \otimes \sharp$. The proof of this result is not as simple as for compositions because it cannot be reduced to a discrete case as above. In fact, to prove this proposition, we must take the image of the Poisson measure M above by an appropriate mapping. For more details, we refer to

Berestycki [6] who has already proved this result for m-fragmentation and the same approach works in our case.

To conclude this section, we turn our interest on how the two erosion coefficients affect the fragmentation. Let $(U(t), t \geq 0)$ be an i-fragmentation with parameter $(0, c_l, c_r)$. Set $c = c_l + c_r$. We have:

$$U(t) = \left] \frac{c_l}{c}(1 - e^{-tc}), 1 - \frac{c_r}{c}(1 - e^{-tc}) \right[\text{ a.s.}$$

Indeed, consider a c-fragmentation $(\Gamma(t), t \geq 0)$ such that $U_{\Gamma(t)} = U(t)$ a.s. We define $\mu_{c_l,c_r} = c_l \epsilon_l + c_r \epsilon_r$. Hence $(\Gamma(t), t \geq 0)$ is a fragmentation with rate μ_{c_l,c_r}. Recall that the process $(\Gamma(t), t \geq 0)$ can be constructed from a Poisson measure on $\mathbb{R}_+ \times \mathcal{C} \times \mathbb{N}$ with intensity $dt \otimes \mu_{c_l,c_r} \otimes \sharp$. By the form of μ_{c_l,c_r}, we remark that, for all $t \geq 0$, $\Gamma(t)$ has only one non-singleton block. Furthermore, for all $n \in \mathbb{N}$, the integer n is a singleton at time t with probability $1 - e^{-tc}$, and, given n is a singleton of $\Gamma(t)$, $\{n\}$ is before the infinite block of $\Gamma(t)$ with probability c_l/c and after with probability c_r/c. By the law of large numbers, we deduce that the proportion of singletons before the infinite block of $\Gamma(t)$ is almost surely $\frac{c_l}{c}(1 - e^{-tc})$ and the proportion of singletons after the infinite block of $\Gamma(t)$ is almost surely $\frac{c_r}{c}(1 - e^{-tc})$.

Remark : Berestycki [6] has proved a similar result for the m-fragmentation. He also proved that if $(F(t), t \geq 0)$ is a m-fragmentation with parameter $(\nu, 0)$, then $\tilde{F}(t) = e^{-ct}F(t)$ is a m-fragmentation with parameter (ν, c). There is no simple way to extend Berestycki's result to the case of an i-fragmentation since the Lebesgue measure of ${}^cU(t)$ squeezed between two successive interval components of $U(t)$ depends on the time where the two component intervals split.

IV.4.3 Projection from $\mathcal{U}$ to $\mathcal{S}^{\downarrow}$

We know that if $(U(t), t \geq 0)$ is an i-fragmentation, then its projection on $\mathcal{S}^{\downarrow}$, $(U^{\downarrow}(t), t \geq 0)$ is an m-fragmentation. More precisely, we can express the characteristics of the m-fragmentation in terms of the characteristics of the i-fragmentation.

Proposition IV.14

The ranked sequence of the lengths of an i-fragmentation with rate (ν, c_l, c_r) is a m-fragmentation with parameter $(\tilde{\nu}, c_l + c_r)$ where $\tilde{\nu}$ is the image of ν by the projection $U \to |U|^{\downarrow}$.

Proof: Let $(\Gamma(t), t \geq 0)$ be a c-fragmentation with rate $\mu = (\nu, c_l, c_r)$. Let $(\Pi(t), t \geq 0)$ be its image by $\wp_1$. The process $(\Pi(t), t \geq 0)$ is then a p-fragmentation. Set $n \in \mathbb{N}$ and $\pi \in \mathcal{P}_n^*$. We have

$$\begin{aligned} q_\pi &= \lim_{s \to 0} \frac{1}{s} \mathbb{P}(\Pi_{[n]}(s)) = \pi) \\ &= \lim_{s \to 0} \frac{1}{s} \mathbb{P}\left(\Gamma_{[n]}(s)) \in \wp^{-1}(\pi)\right) \\ &= \tilde{\mu}(\pi), \end{aligned}$$

where $\tilde{\mu}$ is the image of μ by $\wp_1$. Besides we have already proved that $\tilde{\mu} = (\tilde{\nu}, c_l + c_r)$. We consider now the i-fragmentation $(U_{\Gamma(t)}, t \geq 0)$ with rate (ν, c_l, c_r). We get that the process $(U^{\downarrow}_{\Gamma(t)}, t \geq 0)$ is a.s. equal to the m-fragmentation $(|\Pi(t)|^{\downarrow}, t \geq 0)$ with rate $(\tilde{\nu}, c_l + c_r)$. ■

According to Proposition IV.14 and using the theory of m-fragmentation (see [14]), we deduce then the following results:

- Let $(\Gamma(t), t \geq 0)$ be a c-fragmentation with parameter (ν, c_l, c_r). We denote by B_1 the block of $\Gamma(t)$ containing the integer 1. Set $\sigma(t) = -\ln |B_1(t)|$. Then $(\sigma(t), t \geq 0)$ is a subordinator. If we denote $\zeta = \sup\{t > 0, \sigma_t < \infty\}$, then there exists a non-negative function ϕ such that

  $$\forall q, t \geq 0, \;\; \mathbb{E}[\exp(-q\sigma_t), \zeta > t] = \exp(-t\phi(q)).$$

  We call ϕ the Laplace exponent of σ and we have:

  $$\phi(q) = (c_l + c_r)(q+1) + \int_{\mathcal{U}} (1 - \sum_{i=1}^{\infty} |U_i|^{q+1}) \nu(dU),$$

  where $(|U_i|)_{i \geq 0}$ is the sequence of the lengths of the component intervals of U.

- An (ν, c_r, c_l) i-fragmentation $(U(t), t \geq 0)$ is proper (i.e. for each t, $U(t)$ has almost surely a Lebesgue measure equal to 1) iff

  $$c_l = c_r = 0 \text{ and } \nu\left(\sum_i s_i < 1\right) = 0.$$

IV.4.4 Extension to the time-inhomogeneous case

We now briefly expose how the results of the preceding sections can be transposed in the case of time-inhomogeneous fragmentation. We will not always provide the details of the proofs since they are very similar to the homogeneous case. In the sequel, we shall focus on c-fragmentation $(\Gamma(t), t \geq 0)$ fulfilling the following properties:

- for all $n \in \mathbb{N}$, let τ_n be the time of the first jump of $\Gamma_{[n]}$ and λ_n be its law. Then λ_n is absolutely continuous with respect to Lebesgue measure with continuous and strictly positive density.

- for all $\gamma \in \mathcal{C}_n^*$, $\quad h_\gamma^n(t) = \mathbb{P}(\Gamma_{[n]}(t) = \gamma \mid \tau_n = t)$ is a continuous function of t.

Remark that a time homogeneous fragmentation always fulfills these two conditions. Indeed, in that case, λ_n is an exponential random variable and the function $h_\gamma^n(t)$ does not depend on t. As in the case of fragmentation of exchangeable partitions (cf. Chapter II), for $n \in \mathbb{N}$ and $\gamma \in \mathcal{C}_n^*$, we can define an instantaneous rate of jump from $\mathbb{1}_n$ to γ:

$$q_{\gamma,t} = \lim_{s \to 0} \frac{1}{s} \mathbb{P}\left(\Gamma_{[n]}(\tau_n) = \gamma \;\&\; \tau_n \in [t, t+s] \mid \tau_n \geq t\right).$$

With the same arguments as in the case of fragmentations of exchangeable partitions, we can prove that, for each $t > 0$, there exists a unique exchangeable measure μ_t on $\mathcal{C}$ such that $\mu_t(\mathbf{1}_\mathbb{N}) = 0$ and $\mu_t(\mathcal{Q}_{\infty,\gamma}) = q_{\gamma,t}$ for all $\gamma \in \mathcal{C}_n^*$ and $n \in \mathbb{N}$, where $\mathcal{Q}_{\infty,\gamma} = \{\gamma' \in \mathcal{C}, \gamma'_n = \gamma\}$. Furthermore, the family of measures $(\mu_t, t \geq 0)$ characterizes the law of the fragmentation. We call μ_t the instantaneous rate at time t of the fragmentation. We remark also that if $(\mu_t, t \geq 0)$ is the family of rates of a fragmentation process, we have for all $n \geq 2$,

$$\mu_t(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) = \sum_{\gamma \in \mathcal{C}_n^*} q_{\gamma,t} < \infty$$

$$\text{and} \qquad \int_0^t \mu_u(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\} = -\ln(\lambda_n(]t,\infty[)) < \infty.$$

So we can apply Theorem IV.7 to μ_t and we deduce the following proposition:

Corollary IV.15

Let $(\mu_t, t \geq 0)$ be the family of rates of a c-fragmentation. Then there exist a family of dislocation measures $(\nu_t, t \geq 0)$ and two families of nonnegative numbers $(c_{l,t}, t \geq 0)$, $(c_{r,t}, t \geq 0)$ such that:

- $\mu_t \mathbb{1}_{\{U_\pi \neq]0,1[\}} = P^{\nu_t}$.

- $\mu_t \mathbb{1}_{\{U_\pi =]0,1[\}} = c_{l,t}\epsilon_l + c_{r,t}\epsilon_r$.

Besides we have for all $T \geq 0$,

$$\int_0^T \int_{\mathcal{U}} (1 - s_1)\, \nu_t\,(dU)\, dt < \infty \text{ and } \int_0^T (c_{l,t} + c_{r,t}) dt < \infty.$$

Proof: The first part of the proposition comes from Theorem IV.7. For the second part, use that

$$\int_{\mathcal{U}} (1-s_1)\, \nu_t\,(dU) \leq \mu_t\left(\{\pi \in \mathcal{P}_\infty, \pi_{[2]} \neq \mathbf{1}_2\}\right).$$

For the upper bound concerning the erosion coefficients, we remark that:

$$c_t + c'_t = \mu_t\left(\{1\}, \mathbb{N}\setminus\{1\}\right) + \mu_t\left(\mathbb{N}\setminus\{1\}, \{1\}\right).$$

■

In the same way as for homogeneous fragmentation, we define a family of fragmentation measures as a family $(\mu_t, t \geq 0)$ of exchangeable measures on $\mathcal{C}$ such that, for each $t \in [0, \infty[$, we have:

- $\mu_t(\mathbb{1}_{\mathbb{N}}) = 0$.
- $\forall n \geq 2,\ \mu_t(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbb{1}_n\}) < \infty$ and $\int_0^t \mu_u(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbb{1}_n\})du < \infty$.
- $\forall n \in \mathbb{N},\ \forall A \subset \mathcal{C}_n^*,\ \mu_t(A)$ is a continuous function of t.

Proposition IV.16

Let $(\mu_t, t \geq 0)$ be a family of fragmentation measures. An inhomogeneous c-fragmentation with rate $(\mu_t, t \geq 0)$ can be constructed from a Poisson point process on $\mathbb{R}_+ \times \mathcal{C} \times \mathbb{N}$ with intensity $dt \otimes \mu_t \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$ in the same way as for time-homogeneous fragmentation.

It is very easy to check that the proof of the homogeneous case applies here too. Of course, a Poissonian construction of a time-inhomogeneous i-fragmentation with no erosion is also possible with a Poisson measure on $\mathbb{R}_+ \times \mathcal{U} \times \mathbb{N}$ with intensity $dt \otimes \nu_t \otimes \sharp$. Concerning the law of the tagged fragment, if one defines $\sigma(t) = -\ln|B_1(t)|$, with B_1 the block containing the integer 1, we have now that $\sigma(t)$ is a process with independent increments. And so, if we denote $\zeta = \sup\{t > 0, \sigma_t < \infty\}$, then there exists a family of non-negative functions $(\phi_t, t \geq 0)$ such that

$$\forall q, t \geq 0,\ \mathbb{E}[\exp(-q\sigma_t), \zeta > t] = \exp(-\int_0^t \phi_u(q)du).$$

We call ϕ_t the instantaneous Laplace exponent of σ at time t and we have

$$\phi_t(q) = (c_{l,t} + c_{r,t})(q+1) + \int_{\mathcal{U}} (1 - \sum_{i=1}^{\infty} |U_i|^{q+1})\nu_t(dU),$$

where $(|U_i|)_{i\geq 0}$ is the sequence of the lengths of the component intervals of U. Furthermore, an $(\nu_t, c_t, c'_t)_{t\geq 0}$ i-fragmentation $(U(t), t \geq 0)$ is proper iff

$$\forall t > 0, \quad c_{l,t} = c_{r,t} = 0 \quad \text{and} \quad \nu_t(\sum_i s_i < 1) = 0).$$

IV.4.5 Extension to the self-similar case

A notion of self-similar fragmentations has been also introduced [11]. We recall here the definition of a self-similar p-fragmentation, the reader can easily adapt this definition to the three other instances of fragmentations.

Definition IV.17

Let $\Pi = (\Pi(t), t \geq 0)$ be an exchangeable process on $\mathcal{P}_\infty$. We order the blocks of Π by their least elements. We call Π a self-similar p-fragmentation with index $\alpha \in \mathbb{R}$ if

- $\Pi(0) = 1_{\mathbb{N}}$ *a.s.*
- Π *is continuous in probability*
- *For every $t \geq 0$, let $\Pi(t) = (\Pi_1, \Pi_2, \ldots)$ and denote by $|\Pi_i|$ the asymptotic frequency of the block Π_i. Then for every $s > 0$, the conditional distribution of $\Pi(t+s)$ given $\Pi(t)$ is the law of the random partition whose blocks are those of the partitions $\Pi^{(i)}(s_i) \cap \Pi_i$ for $i \in \mathbb{N}$, where $\Pi^{(1)}, \ldots$ is a sequence of independent copies of Π and $s_i = s|\Pi_i|^\alpha$.*

Notice that an homogeneous p-fragmentation corresponds to the case $\alpha = 0$.

We have still the same correspondence between the four types of fragmentation. In fact, a self-similar fragmentation can be constructed from a homogeneous fragmentation with a time change:

Proposition IV.18 ***[11]***

Let $(U(t), t \geq 0)$ be an homogeneous interval fragmentation with dislocation measure ν. For $x \in]0,1[$, we denote by $I_x(t)$ the interval component of $U(t)$ containing x. We define

$$T_t^\alpha(x) = \inf\{u \geq 0, \int_0^u |I_x(r)|^{-\alpha} dr > t\} \text{ and } U^\alpha(t) = U(T_t^\alpha) = \bigcup I_x(T_t^\alpha(x)).$$

Then $(U^\alpha(t), t \geq 0)$ is a self-similar interval fragmentation with index α.

A self-similar i-fragmentation (or c-fragmentation) is then characterized by a quadruple (ν, c_l, c_r, α) where ν is a dislocation measure on $\mathcal{U}$, c_l and c_r are two nonnegative numbers and $\alpha \in \mathbb{R}$ is the index of self-similarity.

IV.5 Examples

IV.5.1 Interval components in exchangeable random order

In this section we introduce the notion of random open set with interval components in exchangeable random order. In the next section, we will give an example of an i-fragmentation whose dislocation measure has its interval components in exchangeable random order.

Definition IV.19 *[35]*

Let $s \in \mathcal{S}^{\downarrow}$ such that $\sum s_i = 1$. Let $(V_i)_{i\in\mathbb{N}}$ be iid random variables uniform on $[0,1]$. We denote then U the random open subset of $]0,1[$ such that, if the decomposition of U in disjoint open intervals ranked by their length is $\coprod_{i=1}^{\infty} U_i$, we have

- *For all $i \in \mathbb{N}$, $|U_i| = s_i$.*
- *For all $i, j \in \mathbb{N}$, $U_i \prec U_j \Leftrightarrow V_i < V_j$.*

Since we have $\sum_i s_i = 1$, there exists almost surely a unique open subset of $]0,1[$ fulfilling these two conditions. We denote by $\mathcal{Q}_s$ the distribution of U.

Let $\tilde{\nu}$ be a measure on $\mathcal{S}^{\downarrow}$ such that $\tilde{\nu}(\sum_i s_i < 1) = 0$. We denote by $\widehat{\nu}$ the measure on $\mathcal{U}$ defined by:

$$\widehat{\nu} = \int_{\mathcal{S}^{\downarrow}} \mathcal{Q}_s \tilde{\nu}(ds).$$

A measure on $\mathcal{U}$ which can be written in that form is said to have interval components in exchangeable random order.

Proposition IV.20

Let $(U(t), t \geq 0)$ be an i-fragmentation with rate $(\nu, 0, 0)$ and such that for all $t \geq 0$, $U(t)$ has interval components in exchangeable random order. Then ν has also interval components in exchangeable random order.

Proof: Let $(F(t), t \geq 0)$ be the projection of $(U(t), t \geq 0)$ on $\mathcal{S}^{\downarrow}$. We know that F is then an m-fragmentation with rate $(\tilde{\nu}, 0)$ where $\tilde{\nu}$ is the image of ν by the canonical projection $\mathcal{U} \to \mathcal{S}^{\downarrow}$. Let $\gamma \in \mathcal{C}_n$. Let $\pi \in \mathcal{P}_n$ be the image of γ by the canonical projection $\wp_1$ from $\mathcal{C}$ to $\mathcal{P}_\infty$. Let us now remark that we have

$$q_\gamma = \frac{1}{s}\lim_{s\to 0}\mathbb{P}(\Gamma_{[n]}(s)=\gamma) = \frac{1}{k!}q_\pi,$$

where k is the number of blocks of γ and q_π the jump rate of the p-fragmentation. Let $\widehat{\nu}$ be the measure on $\mathcal{U}$ obtained in Definition IV.19 from ν. Let us recall that $\mathcal{Q}_{\infty,\gamma} = \{\gamma' \in \mathcal{C}, \gamma'_{[n]} = \gamma\}$ and define also $\mathcal{P}_{\infty,\pi} = \{\pi' \in \mathcal{P}_\infty, \pi'_{[n]} = \pi\}$. We then have

$$P^{\widehat{\nu}}(\mathcal{Q}_{\infty,\gamma}) = \frac{1}{k!}P^{\tilde{\nu}}(\mathcal{P}_{\infty,\pi}) = \frac{1}{k!}q_\pi = q_\gamma = P^{\nu}(\mathcal{Q}_{\infty,\gamma}).$$

So we get that $\nu = \widehat{\nu}$ and hence ν has interval components in exchangeable random order. ■

Let us notice that the proof uses the identity $q_\gamma = \frac{1}{k!}q_\pi$, so if we want to extend this proposition to the time-inhomogeneous case, then we must suppose not only that $U(t)$ has interval components in exchangeable random order, but more generally that for all $s > t \geq 0$, the probability measure $q_{t,s}^{]0,1[}$ governing the transition probabilities of U from time t to time s (see Definition IV.11), has interval components in exchangeable random order.

Conversely, we may wonder: if $(U(t), t \geq 0)$ is an i-fragmentation with rate $(\nu, 0, 0)$ and ν has interval components in exchangeable random order, does this imply that $U(t)$ has interval components in exchangeable random order? The answer is clearly negative. Indeed, let ν be the following measure:

$$\nu = \delta_{U_1} + \delta_{U_2} \text{ with } {}^cU_1 = \left\{\frac{1}{3}, \frac{2}{3}\right\} \text{ and } {}^cU_2 = \left\{\frac{1}{2}\right\}.$$

Then ν has interval components in exchangeable random order, but $U(t)$ does not have this property since we have

$$\mathbb{P}\left({}^cU(t) = \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right\}\right) > 0$$

(this is the probability that U has split two times before time t, the first time in tree fragments, and the second time, the middle fragment has split in two fragments); but we have

$$\mathbb{P}\left({}^cU(t) = \left\{\frac{1}{6}, \frac{1}{2}, \frac{5}{6}\right\}\right) = 0.$$

IV.5.2 Ruelle's fragmentation

In this section, we give the semi-group of Ruelle's fragmentation seen as an interval fragmentation. Let us recall the construction of this i-fragmentation [16].

Let $(\sigma_t^*, 0 < t < 1)$ be a family of stable subordinators such for every $0 < t_n < \ldots < t_1 < 1$, $(\sigma_{t_1}^*, \ldots, \sigma_{t_n}^*) \overset{law}{=} (\sigma_{t_1}, \ldots, \sigma_{t_n})$ where $\sigma_{t_i} = \tau_{\alpha_1} \circ \ldots \circ \tau_{\alpha_i}$ and $(\tau_{\alpha_i}, 1 \leq i \leq n)$ are n independent stable subordinators with indices $\alpha_1, \ldots, \alpha_n$ such that $t_i = \alpha_1 \ldots \alpha_i$. Fix $t_0 \in]0,1[$ and for $t \in]t_0, 1[$ define T_t by:

$$\sigma_t^*(T_t) = \sigma_{t_0}^*(1).$$

Then consider the open set:

$$U(t) = \Big]0,1\Big[\Big\backslash \left\{ \frac{\sigma_t^*(u)}{\sigma_{t_0}^*(1)}, 0 \leq u \leq T_t \right\}^{cl}.$$

Bertoin and Pitman [16] proved that $(U(t), t \in [t_0, 1[)$ is an i-fragmentation (with initial state $U(t_0) \neq]0,1[$ a.s.) and the transition probabilities of $U(t)$ from time t to time s of the m-fragmentation $(U^{\downarrow}(t), t \in [t_0, 1[)$ is $PD(s, -t)$-FRAG where $PD(s, -t)$ denotes the Poisson-Dirichlet law with parameter $(s, -t)$ (see [56] for more details about the Poisson-Dirichlet laws). Moreover, the instantaneous dislocation measure of this m-fragmentation at time t is $\frac{1}{t} PD(t, -t)$ (cf. Chapter II). We would like now to calculate the dislocation measure of the i-fragmentation $(U(t), t \in [t_0, 1[)$.

Lemma IV.21

Let us define $\widehat{PD}(t, 0)$ as the measure on $\mathcal{U}$ obtained from $PD(t, 0)$ by Definition IV.19. The distribution at time t of $U(t)$ is $\widehat{PD}(t, 0)$.

Proof: For $t \in]t_0, 1[$, we have $\sigma_{t_0}^* = \sigma_t^* \circ \tau_\alpha$ where $\alpha t = t_0$ and τ_α is a stable subordinator with index α and independent of σ_t^*. Hence we get

$$U(t) = \Big]0,1\Big[\Big\backslash \left\{ \frac{\sigma_t^*(u)}{\sigma_t^*(\tau_{\alpha(1)})}, 0 \leq u \leq \tau_\alpha(1) \right\}^{cl}.$$

We can thus write

$$U(t) = \Big]0,1\Big[\Big\backslash \left\{ \frac{\sigma_t(x)}{\sigma_t(a)}, x \in [0, a[\right\}^{cl},$$

where σ_t is a stable subordinator with index t and a is a random variable independent of σ_t. If we denote by $(t_i, s_i)_{i \geq 1}$ the time and size of the jump of σ_t in the interval $[0, a[$ ranked by decreasing order of the size of the jumps, this family has the same law of $(t_{\tau(i)}, s_i)_{i \geq 1}$ for any τ permutation of $\mathbb{N}$. ∎

Proposition IV.22

The semi-group of transition of Ruelle's interval fragmentation from time t to time s is $\widehat{PD}(s,-t)$-FRAG and the instantaneous dislocation measure at time t is $\frac{1}{t}\widehat{PD}(t,-t)$.

Proof: We would like now to apply Proposition IV.20 to determine the instantaneous measure of dislocation of Ruelle's fragmentation, but this proposition holds only for time-homogeneous fragmentation. If the fragmentation is inhomogeneous in time, we must also check that, for all $s > t \geq 0$, the probability measure $q_{t,s}^{]0,1[}$ on $\mathcal{U}$ governing the transition probabilities of U from time t to time s (see Definition IV.11), has interval components in exchangeable random order. Fix $t \geq 0$ and $s > t$. Fix $y \in]0,1[$ and denote by $I(t)$ the interval component of $U(t)$ containing y. We shall prove that $U(s) \cap I(t)$ has its interval components in exchangeable random order. By the construction of $U(t)$, there exists $x \in]0, T_t[$ such that

$$I(t) = \left] \frac{\sigma_t^*(x^-)}{\sigma_{t_0}^*(1)}, \frac{\sigma_t^*(x)}{\sigma_{t_0}^*(1)} \right[.$$

We have $\sigma_t^* = \sigma_s^* \circ \tau_{t/s}$ where $\tau_{t/s}$ is a stable subordinator with index t/s and is independent of σ_{t+s}^*. Hence, we get:

$$U(s) \cap I(t) = I(t) \Big\backslash \left\{ \frac{\sigma_s^*(y)}{\sigma_{t_0}^*(1)},\ \tau_{t/s}(x^-) \leq y \leq \tau_{t/s}(x) \right\}^{cl}.$$

Since $\tau_{t/s}$ is independent of σ_s^*, the jump of σ_s^* on the interval $]\tau_{t/s}(x^-), \tau_{t/s}(x)[$ are in exchangeable random order. Since, as m-fragmentation, the semi-group of transition is $PD(s,-t)$-FRAG, we deduce that, as i-fragmentation, the semi-group is $\widehat{PD}(s,-t)$-FRAG. To prove that the dislocation measure at time t is $\frac{1}{t}\widehat{PD}(t,-t)$, we just have to apply the Proposition IV.20. ■

We can also give the semi-group from time t to time s of the corresponding c-fragmentation. Indeed, the EPPF of a partition whose frequency law is a Poisson-Dirichlet law is well known (see [52, 54]), and since the blocks are in exchangeable random order, the semi-group from time t to time s of the c-fragmentation is $q_{t,s}$-FRAG$(\Gamma(t), \cdot)$ with

$$\forall n \in \mathbb{N}, \forall (A_1, \ldots, A_k) \in \mathcal{C}_n,\ q_{t,s}(\Gamma_{[n]} = (A_1, \ldots, A_k)) = \frac{[-t/s]_k}{k![-t]_n} \prod_{i=1}^{k} -[-s]_{n_i},$$

where $\sharp A_i = n_i$ and $[x]_n = \prod_{i=1}^{n}(x+i-1)$.

IV.5.3 Dislocation measure of the Brownian fragmentation

We consider the m-fragmentation introduced by Aldous and Pitman [2] to study the standard additive coalescent. Bertoin [8] gave a construction of an i-fragmentation $(U(t), t \geq 0)$ whose projection on $\mathcal{S}^{\downarrow}$ is this fragmentation. More precisely, let $\varepsilon = (\varepsilon_s, s \in [0,1])$ be a standard positive Brownian excursion. For every $t \geq 0$, we consider

$$\varepsilon_s^{(t)} = ts - \varepsilon_s, \qquad S_s^{(t)} = \sup_{0 \leq u \leq s} \varepsilon_u^{(t)}.$$

We define $U(t)$ as the constancy intervals of $(S_s^{(t)}, 0 \leq s \leq 1)$. Bertoin [11] proved also that $(U^{\downarrow}(t), t \geq 0)$ is an m-fragmentation with index of self-similarity $1/2$, with no erosion and its dislocation measure is carried by the subset of sequences

$$\{s = (s_1, s_2, \ldots) \in \mathcal{S}^{\downarrow}, s_1 = 1 - s_2 \text{ and } s_i = 0 \text{ for } i \geq 3\}$$

and is given by

$$\tilde{\nu}_{AP}(s_1 \in dx) = \left(2\pi x^3 (1-x)^3\right)^{-1/2} dx \quad \text{ for } x \geq 1/2.$$

Proposition IV.23

The i-fragmentation derived from a Brownian motion [8] has dislocation measure ν_{AP} such that:

- *ν_{AP} is supported by the sets of the form $]0, X[\cup]X, 1[$, so we shall identify each such set with X and write $\nu_{AP}(dx)$ for its distribution.*
- *For all $x \in]0,1[$, $\quad \nu_{AP}(dx) = (2\pi x(1-x^3))^{-1/2} dx$.*

Notice that we have $\nu_{AP}(dx) = x\tilde{\nu}_{AP}(s_1 \in dx \text{ or } s_2 \in dx)$ for all $x \in]0,1[$. Hence, given that the m-fragmentation splits in two blocks of size x and $1-x$, the left block of the i-fragmentation will be a size-biased pick from $\{x, 1-x\}$.

Proof: The first part of the proposition is straight forward since we have $\tilde{\nu}_{AP}(s_1 = 1 - s_2) = 1$. For the second part, let us use Theorem 9 of [8] which gives the distribution ρ_t of the leftmost fragment of $U(t)$:

$$\rho_t(dx) = t \frac{1}{\sqrt{2\pi x (1-x)^3}} \exp\left(-\frac{xt^2}{2(1-x)}\right) dx \quad \text{ for all } x \in]0,1[.$$

According to Proposition 3 of [47], we get

$$\nu_{AP}(dx) = \lim_{t \to 0} \frac{1}{t} \rho_t(dx) = \frac{dx}{\sqrt{2\pi x(1-x)^3}}.$$

■

We can also give a description of the distribution at time $t > 0$ of $U(t)$. Recall the result obtained by Chassaing and Janson [26]. For a random process X on $\mathbb{R}$ and $t \geq 0$, we define $\ell_t(X)$ as the local time of X at level 0 on the interval $[0, t]$, i.e.

$$\ell_t(X) = \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_0^t 1\!\!1_{\{|X_s| < \varepsilon\}} ds,$$

whenever the limit makes sense.

Let X^t be a reflected Brownian bridge conditioned on $\ell_1(X^t) = t$. We define $\theta \in]0, 1[$ such that

$$\ell_\theta(X^t) - t\theta = \max_{0 \leq u \leq 1} \ell_u(X^t) - tu.$$

It is well known that this equation has almost surely a unique solution. Let us define the process $(Z^t(s), 0 \leq s \leq 1)$ by

$$Z^t(s) = X^t(s + \theta\,[\text{mod } 1]).$$

Chassaing and Janson [26] have proved that for each $t \geq 0$

$$U(t) \stackrel{law}{=}]0, 1[\setminus \{x \in [0, 1], Z^t(x) = 0\}.$$

Besides, as the inverse of the local time of X^t defined by

$$T_x = \inf\{u \geq 0,\ \ell_u(X^t) > x\}$$

is a stable subordinator with Lévy measure $(2\pi x^3)^{-1/2} dx$ conditioned to $T_t = 1$, we deduce the following description of the distribution of $U(t)$:

Corollary IV.24

Let $t > 0$. Let T be a stable subordinator with Lévy measure $\frac{dx}{\sqrt{2\pi x^3}}$ conditioned to $T_t = 1$. Let us define m as the unique real number in $[0, t]$ such that

$$tT_{m^-} - m \leq tT_u - u \quad \text{for all } u \in [0, t],$$

where $T_{m^-} = \lim_{x \to m^-} T_x$. We set:

$$\begin{aligned} \tilde{T}_x &= T_{m+x} - T_{m^-} && \text{for } 0 < x < t - m, \\ & \ T_{m+x-t} - T_{m^-} + 1 && \text{for } t - m \leq x \leq t. \end{aligned}$$

Then

$$U(t) \stackrel{law}{=}]0, 1[\setminus \{\tilde{T}_x, x \in [0, t]\}^{cl}.$$

Proof: It is clear that $\{u, X^t(u) = 0\}$ coincides with $\{T_x, x \in [0,t]\}^{cl}$ when T is the inverse of the local time of X^t. Hence, we just have to check that if we set $m = \ell_\theta(X^t)$, then m verifies the equation $tT_{m^-} - m \leq tT_u - u$ for all $u \in [0,t]$. Since $X^t(\theta) = 0$, we have $T_{m^-} = \theta$, thus we get:

$$tT_{m^-} - m = t\theta - \ell_\theta(X^t) \leq tv - \ell_v(X^t) \text{ for all } v \in [0,1].$$

Let us fix $u \in [0,t]$. Since $\ell_v(X^t)$ is a continuous function, there exists $v \in [0,1]$ such that $\ell_v(X^t) = u$. Besides we have $T_u^- \leq v \leq T_u$, so we get

$$tT_{m^-} - m \leq tT_u - u.$$

■

Hence, the distribution of $[0,1] \setminus U(t)$ can be obtained as the closure of the shifted range of a stable subordinator $(T_s, 0 \leq s \leq t)$ with index $1/2$ and conditioned on $T_t = 1$ (recall also that Chassaing and Janson [26] have proved that the leftmost fragment of $U(t)$ is size-biased picked).

Remark: There exists an other way to construct an i-fragmentation from a Brownian excursion [11]. Let $\varepsilon = (\varepsilon(r), 0 \leq r \leq 1)$ be a Brownian excursion with unit duration. We consider $U(t) = \{r \in]0,1[, \varepsilon(r) > t\}$. Bertoin has proved that the process $(U(t), t \geq 0)$ is an i-fragmentation whose rate as m-fragmentation is $(0, 2\tilde{\nu}_{AP})$ and index of self-similarity $\alpha = -1/2$. Let us define the open set $V(t) = \{x \in]0,1[, (1-x) \in U(t)\}$. Since $(\varepsilon(1-r), 0 \leq r \leq 1)$ has also the law of a Brownian excursion with unit duration, we deduce that $(V^\downarrow(t), t \geq 0)$ is also an m-fragmentation with the same characteristics as $(U^\downarrow(t), t \geq 0)$. Besides, if ν_ε (resp. $\nu_{\varepsilon'}$)denotes the dislocation measure of the i-fragmentation U (resp. V), we must have $\nu_\varepsilon(dx) = \nu_{\varepsilon'}(1-dx)$ (recall that since $\tilde{\nu}_{AP}$ is binary, we write $\nu_\varepsilon(dx)$ to denote the distribution of $]0,x[\cup]x,1[$). Hence, we deduce that $\nu_\varepsilon(dx) = \nu_\varepsilon(1-dx)$ and using that $2\tilde{\nu}_{AP}(s_1 \in dx) = \nu_\varepsilon(dx) + \nu_\varepsilon(1-dx)$ for $x \in]1/2, 1[$, we get

$$\nu_\varepsilon(dx) = \frac{1}{\sqrt{2\pi x^3(1-x)^3}} dx \quad \text{for } x \in]0,1[,$$

and ν_ε has interval components in exchangeable random order.

IV.6 Hausdorff dimension of an interval fragmentation

Let $(U(t), t \geq 0)$ be a self-similar i-fragmentation with index $\alpha > 0$. Let $K(t) = [0,1] \backslash U(t)$. The set $K(t)$ is a closed set, and if the fragmentation is proper (i.e.

the fragmentation has with no erosion and its rate verifies $\nu(\sum_i |U_i| < 1) = 0)$, its Lebesgue measure is equal to 0. Hence, to evaluate the size of $K(t)$, we shall compute its Hausdorff measure. Here, we will just examine time-homogeneous fragmentation. First we recall the definition of the Hausdorff dimension of a subset of]0,1[.

Definition IV.25 *[31]*

Let $A \subset]0,1[$. Let $d \geq 0$ and $r > 0$. We set

$$J_d^r(A) = \inf\left\{\sum_{i=1}^{\infty} |b_i - a_i|^d, A \subset \bigcup_{i=1}^{\infty}[a_i, b_i], |b_i - a_i| \leq r\right\} \text{ and } H_d(A) = \lim_{r\to 0^+} J_d^r(A),$$

(this limit exists since $J_d^r(A)$ decreases with r). $H_d(A)$ is the d-Hausdorff measure of A. Furthermore, there exists a unique number D such that

$$\forall d > D, H_d(A) = 0 \text{ and } \forall d < D, H_d(A) = \infty.$$

This number is the Hausdorff dimension of A and is denoted by $\dim_{\mathcal{H}}(A)$.

We now give the Hausdorff dimension of the complement of a time-homogeneous i-fragmentation in the case where the rate of the fragmentation fulfils some conditions.

Hypothesis IV.26

Let ν be a dislocation measure. We assume that ν fulfills the following conditions:

(H1) ν is conservative i.e. $\nu(\sum_i |U_i|^{\downarrow} < 1) = 0$.

(H2) There exists an integer k such that $\nu(|U_k|^{\downarrow} > 0) = 0$, i.e. ν is carried by the open sets with at most $k-1$ interval components.

(H3) Let $h(\varepsilon) = \int_{\mathcal{U}}(Card\{i, |U_i| \geq \varepsilon\} - 1)\nu(dU)$. Then h is regularly varying with index $-\beta$ as $\varepsilon \to 0+$.

(H4) Let g be the left extremity of the largest interval component of a generic open set and d the right extremity. Then as $\varepsilon \to 0+$, we have either $\liminf \frac{\nu(g\geq\varepsilon)}{\nu(d\leq 1-\varepsilon)} > 0$ or $\limsup \frac{\nu(g\geq\varepsilon)}{\nu(d\leq 1-\varepsilon)} < \infty$.

We can now state the theorem:

Theorem IV.27

Let ν be a dislocation measure fulfilling Hypothesis IV.26. Let $(U(t), t \geq 0)$ be an i-fragmentation with characteristics $(\nu, 0, 0)$ and index of self-similarity α strictly positive. Let $K(t) = [0,1]\backslash U(t)$. Then the Hausdorff dimension of $K(t)$ is β for *all $t > 0$ simultaneously, a.s.*

In fact, if the index of self-similarity is zero, the lower bound of the Hausdorff dimension still holds. Besides, Hypothesis $(H4)$ is only needed to prove the lower bound and allows a large class of dislocation measure such as symmetric measures or, at the opposite, measures for which the largest fragment is always on the same side.

Proof: We will first prove the upper bound. Let us recall a lemma proved by Bertoin in [13] for m-fragmentation processes whose dislocation measure fulfills Hypothesis IV.26.

Lemma IV.28 *[13]*

Let $(U(t), t \geq 0)$ be a self-similar $(\nu, 0, 0, \alpha)$ i-fragmentation with index of self-similarity strictly positive and whose dislocation measure fulfills (H1), (H2), (H3). Let $(X(t) = (X_i(t))_{i\geq 1}, t \geq 0)$ the associated m-fragmentation. Let $N(\varepsilon, t) = Card\{i \geq 1, X_i(t) \geq \varepsilon\}$ and $M(\varepsilon, t) = \sum_i X_i(t)\mathbb{1}_{\{X_i \leq \varepsilon\}}$. Then the limits of $\frac{N(\varepsilon,t)}{h(\varepsilon)}$ and $\frac{M(\varepsilon,t)}{\varepsilon h(\varepsilon)}$ as $\varepsilon \to 0+$ exist almost surely and are strictly positive and finite.

Let us now fix $d \in]0, 1[$ and look for a upper bound of the d-Hausdorff measure of $K(t)$. Let $I_\varepsilon =]0, 1[\setminus\{$ interval components of $U(t)$ which size is larger than $\varepsilon\}$. So we have $K(t) \subset I_\varepsilon$ and $|I_\varepsilon| = M(\varepsilon, t)$ since ν is conservative. Furthermore, I_ε has at most $N(\varepsilon, t) + 1$ interval components. Using notation of Definition IV.25, we get:

$$J_d^\varepsilon(K(t)) \leq J_d^\varepsilon(I_\varepsilon) \leq \varepsilon^d \left(\frac{M(\varepsilon, t)}{\varepsilon} + N(\varepsilon, t) + 1\right) \leq h(\varepsilon)\varepsilon^d \left(\frac{M(\varepsilon, t)}{h(\varepsilon)\varepsilon} + \frac{N(\varepsilon, t) + 1}{h(\varepsilon)}\right).$$

As h is regularly varying as $\varepsilon \to 0+$ with index $-\beta$, we deduce that for $d > \beta, h(\varepsilon)\varepsilon^d \to 0$ as $\varepsilon \to 0+$ and so $H_d(K(t)) = 0$. This proves that $\dim_{\mathcal{H}} K(t) \leq \beta$.

Let us now prove the lower bound. We first prove the lower bound for a homogeneous i-fragmentation, i.e. we suppose here that $\alpha = 0$. Let us fix $T_0 > 0$ and search for a lower bound of the Hausdorff dimension of $K(T_0)$. The two conditions of Hypothesis (H4) are symmetric by the transformation $x \to 1 - x$, so, without loss of generality, we suppose here that $\liminf \frac{\nu(g\geq\varepsilon)}{\nu(d\leq 1-\varepsilon)} > 0$. Hence there exists a constant C such that for ε small enough we have $C\nu(g \geq \varepsilon) \geq \nu(d \leq 1 - \varepsilon)$. We denote by $]g_t, d_t[$ the largest interval of the fragmentation at time t and $T = \inf\{t \geq 0, d_t - g_t \leq 1/2\} \wedge T_0$. So, for $0 < s < t < T$, $]g_t, d_t[\subset]g_s, d_s[$. The idea is to prove that $\dim_{\mathcal{H}}\{g_t, 0 < t < T\} \geq \beta$ and as $\{g_t, 0 < t < T\} \subset K(T_0)$, we will conclude that lower bound holds for $\dim_{\mathcal{H}} K(T_0)$.

We know that $(U(t), t \geq 0)$ can be constructed from a PPP on $\mathbb{R} \times \mathcal{U} \times \mathbb{N}$ with intensity measure $dt \times \nu \times \sharp$. So we have

$$g_t = \sum_{s \in \mathcal{D} \cap [0,t]} \xi_s (d_{s^-} - g_{s^-}),$$

where $(s, \xi_s)_{s \in \mathcal{D}}$ are the atoms of a Poisson measure on $\mathbb{R} \times [0,1]$ with intensity $ds \times \nu(g \in \cdot)$. We introduce now

$$\sigma_t = \sum_{s \in \mathcal{D} \cap [0,t]} \xi_s.$$

Then σ is a subordinator with Levy measure $\Lambda(d\varepsilon) = \nu(g \in d\varepsilon)$ and we have:

$$\forall\, 0 < s < t < T,\ g_t - g_s \geq \frac{1}{2}(\sigma_t - \sigma_s),$$

since $d_s - g_s < 1/2$ for $s \leq T$.

It is then well known that, if we want to prove that $\dim_{\mathcal{H}}\{g_t, 0 < t < T\} \geq \gamma$, it is sufficient to prove that g^{-1} is Hölder-continuous with exponent γ. We have then the following lemma:

Lemma IV.29

Let $(f(t), 0 \leq t \leq T)$ and $(h(t), 0 \leq t \leq T)$ be two strictly increasing càdlàg functions such that for all $0 < s < t < T$, we have $h(t) - h(s) \geq \frac{1}{2}(f(t) - f(s))$. Define $f^{-1}(x) = \inf\{u \geq 0, f(u) > x\}$ and suppose that f^{-1} is Hölder-continuous with exponent γ. Then h^{-1} is also Hölder-continuous with exponent γ.

Proof: Let $s \geq t$ be two elements of the set $H = \{h(t), 0 \leq t \leq T\}$. Hence, there exist $x \geq y$ such that $h(x) = s$ and $h(y) = t$. Then, we have, for some constant K

$$h^{-1}(t) - h^{-1}(s) = y - x = f^{-1} \circ f(y) - f^{-1} \circ f(x) \leq K(f(y) - f(x))^{\gamma}.$$

Besides, we have $t - s = h(y) - h(x) \geq \frac{1}{2}(f(y) - f(x))$, so we get:

$$h^{-1}(t) - h^{-1}(s) \leq 2^{\gamma} K (t-s)^{\gamma}.$$

Furthermore, h^{-1} is constant on the interval components of H^c, and it follows then

$$h^{-1}(t) - h^{-1}(s) \leq 2^{\gamma} K (t-s)^{\gamma} \quad \text{for all } s < t.$$

■

Hence to prove that $\dim_{\mathcal{H}}\{g_t, 0 < t < T\} \geq \beta$, we just have to prove that σ^{-1} is Hölder-continuous with exponent γ for all $\gamma < \beta$. We use then the following lemma proved by Bertoin:

Lemma IV.30 *[7]*

Let $(\sigma_s, s \geq 0)$ be a subordinator with no drift and Lévy measure Λ. Let $\Phi(\lambda) = \int_0^\infty (1-e^{-\lambda x})\Lambda(dx)$ and $\gamma = \sup\{\alpha > 0, \lim_{\lambda\to\infty} \lambda^{-\alpha}\Phi(\lambda) = \infty\}$. Then, for every $\varepsilon > 0$, σ^{-1} is a.s. Hölder-continuous on compact intervals with exponent $\gamma - \varepsilon$.

To finish the proof of the homogeneous case, we have now to study $\Lambda(d\varepsilon) = \nu(g \in d\varepsilon)$. In the sequel, we denote by k an integer such that $\nu(s_k > 0) = 0$.

We remark that $\{g \geq \varepsilon\} \subset \{Card\{i, s_i > \varepsilon/k\} \geq 2\}$, so $h(\varepsilon/k) \geq \nu(g \geq \varepsilon)$. We notice also that $h(\varepsilon) \leq k\nu(g \geq \varepsilon \text{ or } d \leq 1-\varepsilon)$. As $\nu(d \leq 1-\varepsilon) \leq C\nu(g \geq \varepsilon)$ we get

$$\frac{h(\varepsilon)}{(C+1)k} \leq \nu(g \geq \varepsilon) \leq h(\varepsilon/k).$$

Using that h is regularly varying as $\varepsilon \to 0+$ with index $-\beta$, an easy calculus proves that $\sup\{\alpha > 0, \lim_{\lambda\to\infty} \lambda^{-\alpha}\Phi(\lambda) = \infty\} = \beta$ and so σ^{-1} is Holder-continuous with exponent $\beta - \varepsilon$ for all $\varepsilon > 0$. Hence we get that for each $t > 0$, $\dim_{\mathcal{H}} K(t) = \beta$ a.s. As for $t < s$, $K(t) \subset K(s)$, $\dim_{\mathcal{H}} K(t)$ increases with t, and so we have also $\dim_{\mathcal{H}} K(t) = \beta$ for all $t > 0$ simultaneously a.s.

It remains now to prove the lower bound for an i-fragmentation with strictly positive index of self-similarity. Let us use now Proposition IV.18 which changes the index of self-similarity of a fragmentation. Let $(U^\alpha(t), t \geq 0)$ be a self-similar fragmentation fulfilling (H). We write $U^\alpha(t) = U(T_t^\alpha)$ as in Proposition IV.18 where $(U(t), t \geq 0)$ is a homogeneous fragmentation. We denote by $(g_t, t \geq 0)$ (resp. $(g_t^\alpha, t \geq 0)$) the left bound of the largest interval component of $U(t)$ (resp. $U^\alpha(t)$). We know that for all $T > 0$, $\dim_{\mathcal{H}}\{g_t, 0 \leq t \leq T\} \geq \beta$. Or for t small enough, we have $g_t^\alpha = g_{f(t)}$ where f is a continuous increasing function, so for all $t > 0$, there exists $t' > 0$ such that

$$\dim_{\mathcal{H}}(K^\alpha(s), 0 \leq s \leq t) \geq \dim_{\mathcal{H}}(g_s^\alpha, 0 \leq s \leq t) \geq \dim_{\mathcal{H}}(g_s, 0 \leq s \leq t') \geq \beta. \qquad \blacksquare$$

Corollary IV.31

Let ν be a dislocation measure fulfilling Hypothesis IV.26. Let $(U(t), t \geq 0)$ be a self-similar i-fragmentation with characteristics $(\nu, 0, 0, \alpha)$ with $\alpha > 0$. Let $K(t) = [0,1]\backslash U(t)$. Then the packing dimension of $K(t)$ is β for all $t > 0$ simultaneously, a.s.

Proof: Let us first recall the definition of the packing dimension [62]. For a subset $E \subset \mathbb{R}$ and $\alpha > 0$, let us define

$$M_\alpha(E) = \lim_{\varepsilon\to 0+} \sup\left\{\sum_{i=1}^{\infty}(2r_i)^\alpha,\ [x_i - r_i, x_i + r_i] \text{ disjoint}, x_i \in E, r_i < \varepsilon\right\},$$

and

$$\widehat{M}_\alpha(E) = \inf\left\{\sum_{n=1}^{\infty} M_\alpha(E_n),\ E \subseteq \bigcup_{n=1}^{\infty} E_n\right\}.$$

The packing dimension of E is defined by

$$\dim_{\wp}(E) = \inf\{\alpha > 0,\ \widehat{M}_\alpha(E) = 0\} = \sup\{\alpha > 0,\ \widehat{M}_\alpha(E) = \infty\}.$$

For a subset $E \subset \mathbb{R}$ and $\varepsilon > 0$, let $Z(E, \varepsilon)$ be the smallest number of interval of lengths 2ε needed to cover E. We define

$$\Delta(E) = \limsup_{\varepsilon\to 0} \frac{\ln Z(E,\varepsilon)}{-\ln\varepsilon}.$$

Tricot [62] proved that we have:

$$dim_{\wp}(E) = \inf\left\{\sup_n \Delta(E_n),\ E \subset \cup_n E_n\right\}.$$

It is easy to see that for all $E \subset \mathbb{R}$, we have $\dim_{\mathcal{H}} E \leq \dim_{\wp} E$. Hence, to prove Corollary IV.31, we just have to get an upper bound of the packing dimension of $K(t)$. We use the same idea as for the Hausdorff dimension. Let $I_\varepsilon =]0,1[\backslash \{$ interval components of $U(t)$ which size is larger than $\varepsilon\}$. So we have $K(t) \subset I_\varepsilon$ and $|I_\varepsilon| = M(\varepsilon, t)$ since ν is conservative. Furthermore, I_ε has at most $N(\varepsilon, t) + 1$ interval components. We deduce that

$$Z(K(t), \varepsilon) \leq Z(I_\varepsilon, \varepsilon) \leq h(\varepsilon)\left(\frac{M(\varepsilon,t)}{2\varepsilon h(\varepsilon)} + \frac{N(\varepsilon,t)+1}{h(\varepsilon)}\right).$$

We get

$$\dim_{\wp}(K(t)) \leq \Delta(K(t)) \leq \limsup_{\varepsilon\to 0} \frac{\ln h(\varepsilon)}{-\ln\varepsilon} = \beta.$$

Hence, the packing dimension of the subset $K(t)$ coincides almost surely with its Hausdorff dimension (such subset is called "regular subset"). ■

To conclude this section, let us discuss an example. We consider the i-fragmentation $(U(t), t \geq 0)$ derived from standard additive coalescent (cf. Section IV.5.3). Recall that $(U^{\downarrow}(t), t \geq 0)$ is then an m-fragmentation with index of self-similarity 1/2 and dislocation measure carried by the subset of sequences

$$\{s = (s_1, s_2, \ldots) \in \mathcal{S}^{\downarrow}, s_1 = 1 - s_2 \text{ and } s_i = 0 \text{ for } i \geq 3\}$$

and given by

$$\tilde{\nu}_{AP}(s_1 \in dx) = \left(2\pi x^3(1-x)^3\right)^{-1/2} dx.$$

This proves that (H1), (H2) and (H3) hold with $\beta = 1/2$. Besides, we saw in Section IV.5.3 that the left most fragment of this fragmentation has almost surely a strictly positive length. This implies that $\nu_{AP}(g > 0)$ is finite. Hence we have $\limsup \frac{\nu(g \geq \varepsilon)}{\nu(d \leq 1-\varepsilon)} < \infty$ and Hypothesis (H4) holds. By Theorem IV.27, we deduce that the Hausdorff dimension of $[0,1] \backslash U(t)$ is $\frac{1}{2}$ a.s., a fact that can be checked directly using properties of Brownian motion.

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