Particle Creation from Non-topological Solitons

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Thèse

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Pour Doris ma meilleure moitié

Resumé

Dans la plupart des domaines de la physique nous rencontrons des solitons, solutions classiques et localisées d'énergie finie des équations du mouvement. Ces objets donnent naissance à d'autres objets plus complexes comme les particules, les "domain walls" (murs de séparation entre deux zones). La plupart des solitons résultent d'un défaut topologique, l'état de plus basse énergie défini avec une multplicité plus grande que 1 (potentiel défini avec un "chapeau mexicain"). Dans ces types de théories les solitons sont les solutions reliant un "vide" à un autre. L'autre type de solitons que nous étudierons ici, sont les solitons non-topologiques. Leur existance et leur stabilité sont assurées cette fois par la conservation d'une charge,le nombre fermionique dans notre cas. Il existe un grand nombre de théories contenant ce type d'objet. Il suffit entre autre qu'il existe une charge conservée Q, associée à une symétrie interne non brisée [1].

Les solutions de ce type sont stables dans le sens qu'elles ne se désintégrent pas en particules scalaires, leur masse est plus petite que celle d'un ensemble de particules. L'ajout d'une interaction entre les Q balls et les fermions a pour conséquence de les rendre instables vis-à-vis de la désintégration en fermions. Ce sont ces instabilités que nous proposons d'étudier ici. Nous allons pour ce faire étudier les intéractions d'un Q ball avec des fermions de masse nulle en construisant une description quantique exacte d'un Q ball s'évaporant. Cette construction se base sur une construction inédite pour ce problème, supposant qu'aucun fermion ne se déplace vers le Q ball. Avec cette nouvelle construction nous avons prouvé que le Q ball s'évapore et nous avons même calculé la valeur du taux d'évaporation.

Nous avons ensuite étudié les interactions avec des fermions massifs. Pour ce faire nous avons utilisé une méthode consistant à calculer les amplitudes de réflection et de transmission. Cette methode nous a permis en plus de trouver le taux d'évaporation en fermions massifs et de résoudre le problème de l'intéraction entre un Q ball et un fermion extérieur.

Comme résultat principal nous pouvons dire que le taux d'évaporation dépend de la probabilité qu'un anti-fermion devienne un fermion. Car comme nous l'avons démontré le Q ball change l'énergie des particules qui interagissent avec lui d'un facteur dépendant de son énergie interne. De ce fait les fermions produits par le Q ball ont leur énergie contenue dans un intervalle fini. Ce résultat a pour conséquence le traitement différent des fermions et des anti fermions arrivant sur le Q ball.

Abstract

Non topological solitons, Q balls can arise in many particle theories with $U(1)$ global symmetries. As was shown by Cohen et al. [2], if the corresponding scalar field couples to massless fermions, large Q-balls are unstable and evaporate, producing a fermion flux proportional to the Q ball's surface. In this work we analyse Q-ball instabilities as a function of Q-ball size and fermion mass. In particular, we construct an exact quantum-mechanical description of the evaporating Q-ball. This new construction provides an alternative method to compute Q-Ball's evaporation rates. We shall also find a new expression for the upper bound on evaporation as a function of the produced fermion mass and study the effects of the size of the Q ball on particle production.

We also analyse what happens if external fermion is scattered on a Q ball and demonstrate that it can be converted into antiparticle with a probability of the order of one. This result has important implications for astrophysical applications of dark matter Q balls.

Mots clés, keywords

Solitons, Q balls, evaporation, production de particules, diffusion, matrice S.
Solitons, Q balls, particle production, evaporation, diffusion, S Matrix.

Contents

0.1 Foreword

Before we start with the big work I would like to share with you some of my feelings about physics. When I was a child my objective in life was to travel through space in a huge spaceship and live fabulous adventures like Captain Kirk or Mr. Spock. Obviously my life turned out to be different. Instead of travelling through space I have not moved a lot but my life still gives me lots of satisfaction. Especially, when I discovered physics a few years ago, that day I discovered far more than one universe, I discovered the path I wanted to follow.

More than just a sequence of calculations I discovered that physics was, I am sorry but I have to cite a famous French singer, "les champs du possible". A translation would be "the fields of possibility", which is, I think, a good definition of quantum field theory. Of course you will find in this work lots of strange calculations but I have to invite you to think what ideas can be behind these strange signs.

Another thing I found with physics is a new "family" of passionate people that share the will to undersand or to try understanding the world we live in. In the present work I tried the best I could to answer the mystery of particle creation. I am sure an enourmous amout of work still has to be done, but I am proud of the fact that I put my little stone on the building of physics.

Since I cited a famous French singer to describe the way I feel about quantum field theory, I give here the complete version of his song.

Champs du possible

Devant nous l'an 2000, quelques heures nous séparent.
Quel ordre se profile, quel rapport de pouvoir?
Devant nous l'an 2000. Qui aurait pu prévoir
qu'il ne restait qu'un fil accroché a l'histoire

Les champs magnétiques de ma déraison.
Les voix qui me dictent les cris d'alarme les mots d'amour,
qui me hantent, qui m'entourent, qui me hantent, qui m'entourent.

Devant nous l'an 2000 et le monde en furie,
le grand ordre mondial et les nations unies.
Pas de contre pouvoir aux dollars qui défilent.
Et puis dans les couloirs les banquiers qui se faufilent.
Derriere nous 2000 ans et des années lumiere,
quelque tyrans sanglants, quelques beaux militaires.

Les champs magnétiques de ma déraison.
Les voix qui me dictent les cris d'alarme les mots d'amour,

qui me hantent, qui m'entourent, qui me hantent, qui m'entourent.

Devant nous l'an 2000 et le champ du possible.
Il faut revisiter en partant du sensible.
Aux artistes il échoit des choses a inventer
et prendre quelque reves pour la réalité.

Les champs magnétiques de ma déraison.
Les voix qui me dictent les cris d'alarme les mots d'amour,
qui me hantent, qui m'entourent, qui me hantent, qui m'entourent.

B. Laviliers

Chapter 1

Introduction

1.1 General Motivation

A scalar field theory with an unbroken continuous global symmetry admits a remarkable class of solutions, non-topological solitons or Q balls. These solutions are spherically symmetric non-dissipative solutions to the classical field equations [1, 2]. In a certain way they can be viewed as a sort of Bose-Einstein condensate of "classical" scalars. The construction of these solutions uses the fact that they are absolute minima of the energy for a fixed value of the conserved $U(1)$-charge Q. So in the sector of fixed charge, the Q ball solution is the ground state and all its stability properties are due to charge conservation. An important amount of work has been done on Q ball dynamics and on their stability versus decay into scalars [1]. Apart from some existence theorems that depend on the type of symmetry and the potentials involved, the stability of Q balls is due to the fact that their mass is smaller than the mass of a collection of scalars.

Stable Q balls appear naturally in supersymmetric theories. Supersymmetry was introduced to solve the "hierarchy problem", the problem of the enormous ratio of the Plank mass to the 300 GeV energy scale of electroweak symmetry breaking. Supersymmetry unites particles of half-integer with integer spin in common multiplets. Some of the new scalar fields introduced by supersymmetry admit Q ball type solutions [3]. These supersymmetric Q balls will have a mass proportional to $Q^{3/4}$ and a size proportional to $Q^{1/4}$ [4, 5]. These supersymmetric Q balls can have an enormous charge, $Q > 10^{15}$ [6, 7], this fact makes them good candidates for dark matter [9, 4, 5].

It is known that the addition of a coupling with fermions leads to Q ball evaporation [2], so if we still want these supersymmetric Q ball to be dark matter candidates they will need to live for a sufficiently large time. Therefor we need to know their evaporation rate giving their life time. This problem has been considered in [2] for the production of massless fermions by a large Q ball, this leaves an opened questions : does the mass of produced fermions has an incidence on the Q ball's life time? This leads to another question : can a Q ball produce any type of fermions? The study of Q ball evaporation into both massless and massive fermions will answer these questions.

The long living Q balls will start interacting with matter[5], these interactions might have consequences on their life time. On the other hand if we know the way Q balls interact with matter we can imagine some experiments to observe them and measure some of their properties [10]. Once more only the study of interaction between Q balls and matter will answer these questions.

All these reasons pushed us to study the physics of Q balls, the choice we made was to use the simplest possible Q ball model. Most of the features will be the same if we work with large Q balls. In fact only large Q balls have an incidence on cosmological models. We also know that Q balls have spherical symmetry so we studied one dimensional Q ball models. Having all this in mind we constructed a model of fermions interacting with Q balls having the same features as more complicated models. The study of our model will give us all the physics concerning Q balls and their properties.

Apart form these cosmological motivations the methods used to study Q balls properties can be applied to many other problems where we have a scalar field interacting with a fermion. In order to be able to use the techniques we developed the scalar field needs to have a classical behaviour, admit large expectation value. We shall introduce two different methods to solve this problem, so we can be sure to have the widest application range.

1.2 Problems linked to particle creation

Quantum particle creation in external classical fields (gravitational, electro-magnetic or scalar) plays an important role in cosmology and particle physics. Examples include e^+e^- pair creation in external electric fields [12, 13], black hole evaporation [16], particle creation during early stages of the universe expansion [17, 18], Q ball evaporation [2], creation of gravitational waves [21] and many more. The problem we shall study is particle creation from a "lump" of matter (a "lump" of scalars producing fermions). This problem includes Q ball evaporation and many other problems. Before we give a brief review of the different formalisms we can use to solve the problem of particle creation, let us introduce the concepts and definitions we need.

One of the problems related to particle production is associated with the *very definition* of the particle in time-dependent backgrounds. In the static situation, the particle state is related to the positive frequency part, $e^{-i\omega t}$ with $\omega > 0$, of the solution to the free field equation, and this identification cannot be used when external fields depend on time. However, in a number of setups commonly considered in the literature, the problem can be posed and solved in a consistent way. Namely, one assumes that in the distant past and in the far future the classical field is static. This allows us to introduce the asymptotic notions of particles, which are unique, to define in and out vacua, and to construct the S-matrix that relates in - and out - states (see, e.g. books [19, 20]). We shall refer to this method as the S-matrix formalism.

In the present work we consider a different situation, when the particle creation problem can also be well posed and the notion of particles can be well defined, whereas the S-matrix formalism in its canonical form is not applicable. Suppose that we have a finite region of space V near the origin $x = 0$ filled by some time and space - dependent classical field (one can call it a "source") which is equal to zero outside this region. A distant observer, located at some point x, has a natural definition of the vacuum state and of the particle states in the vicinity of x (to identify the vacuum and particle states one can imagine an experiment where a completely absorbing screen is placed between x and 0.) Thus, one can address well-defined questions on the particle flux, its time and space-dependence, the spectrum of emitted particles, etc. Obviously, the in- and out- states necessary for the S-matrix formalism cannot be defined for an arbitrary time dependence of the classical field : after all, the flux of particles is, in general, time-dependent.

Of course, a slight modification of an S-matrix formalism can be used here. One can write the total Hamiltonian H as $H = H_0 + H_{int}(t)$ where H_0 is the static part of the Hamiltonian associated with particles outside the volume V and $H_{int}(t)$ is the part localised in V. Now, the evolution operator in the interaction picture $S(-\infty, t) = T\exp\left(-i\int_{-\infty}^{t} dt' H_{int}(t')\right)$ can be constructed, and a state of the system at the moment t can be defined as

$$|\Psi(t)\rangle = S(-\infty, t)|0\rangle \ , \tag{1.2.1}$$

where $|0\rangle$ is the vacuum state of the free Hamiltonian H_0. Obviously, the state $|\Psi(t)\rangle$ contains all the information about the particle fluxes and any other parameters a distant observer can measure.

A good example of this modification was provided in the fifties [13] when studying the interaction of an electron with a time dependent electromagnetic field. Assumptions where made such as that the electromagnetic field would vanish on two boundary surfaces. Decomposition of the Dirac field into positive and negative frequency components is made so that the notion of a particle may exist. Finally it is shown that the function describing the transformation of the field between the boundary surface, gives all of the important results.

If the time-dependent part of the classical field can be considered as small, the solution to the problem can be derived through perturbation theory : simply expand the evolution operator with respect to $H_{int}(t')$ and compute all the necessary matrix elements. Our aim is to discuss how to proceed if the classical field is not weak and an exact result is required.

The main idea is similar to the one used in the exact computation of the particle production in the S-matrix formalism [19, 20]. Let us use the Heisenberg picture of quantum mechanics. The first step is to define "local" vacuum states $|0\rangle$ and "local" particle states, specific to a distant observer. This is possible since the background classical field is static far away from the source. The next step is to single out an exact "global" state $|\Psi\rangle$ which satisfies the physical requirement that in every point of space outside the source there are no "local" particles coming from infinity. Once this state is found in terms of the "local" vacuum and particle states, the problem of particle creation is completely solved: to find any required observable $O(x_1, \ldots, x_n; t_1, \ldots, t_n)$ one must simply find an expectation value $\langle\Psi|O(x_1, \ldots, x_n; t_1, \ldots, t_n)|\Psi\rangle$. The actual realisation of this program may appear to be quite non-trivial. This method should be compared with the problem of the potential barrier in standard quantum mechanics where we select the waves outside the barrier.

The problem we shall study is the interaction of a fermion with a time-dependent localised scalar field. As a first model we shall use a $1 \oplus 1$ dimensional version of a Q ball model [2]. This problem was solved in $3 \oplus 1$ dimensions using the S-matrix formalism [2]. The main interest in the study of Q balls is that, unlike solitons whose properties depend on topological effects, all the properties of a Q ball are due to the conservation of a global $U(1)$ charge. Q balls are sometime, referred to in the literature as non-topological solitons. The reason we shall work in $1 \oplus 1$ dimensions is for simplicity: the helicity and spin problems are reduced to trivial expressions and only the essential properties and effects will be of interest.The next chapter gives all the standard properties of Q ball particles and a description of simple models describing them.

To avoid the problems linked to the time dependence of the Q ball we shall propose an alternative method. Using Heisenberg's picture of quantum mechanics. We shall construct the time independent state representing particle production, by solving the condition that no fluxes are coming from infinity (no particles are moving towards the Q ball). We can then build the Heisenberg field operator containing all the relevant information and time dependence. Particle creation is then computed by using the number operator, but any other operator valued quantity can also be calculated. The use of this method needs no limit calculations on the Q ball's size, so it can be used to study small Q balls as well. The difficulty now lies in solving the production condition. This condition can be solved by considering asymptotics of the fields far away from the Q ball.

The work is organised as follows: chapter 2 will describe the properties of Q balls and review earlier work. Chapters 3 and 4 will give the description of the two models used to compute the evaporation rate. The last two chapter will investigate the interactions of Q balls and matter to finally propose the conclusions.

1.3 Solving the problem of particle creation

In this section we shall give a quick review of all the methods one can use to solve the particle production problem. We divide this section into three parts, each one giving a "different" approach of the solution. All these methods are well known and widely described in the literature, so this section can be omitted in a first reading. The example we shall use is that of a fermion field having a Yukawa interaction with a scalar field. This choice is motivated by the fact that Q ball evaporation is a problem of fermions coupled with scalars.

1.3.1 Review of the standard S-matrix formalism

A standard method to compute particle production in external classical fields is based on the S-matrix formalism (see books [19, 20] and references therein) which is valid if the field is static for $t \to \pm\infty$. We will briefly review it in order to introduce notation. For concreteness, we will discuss spin 1/2 fermions, though the general strategy is applicable to any local fields.

Consider a Dirac fermion field coupled by a Yukawa interaction to the time- and space-dependent real scalar field $\phi(x,t)$. The quantised Heisenberg fermionic field ψ satisfies the Dirac equation:

$$[i\gamma^\mu\partial_\mu - m(t,x)]\psi(t,x) = 0 \ , \tag{1.3.1}$$

where $m(t,x) = m + f\phi(t,x)$, and f is the Yukawa coupling constant. The field ψ obeys the equal-time commutation relations

$$[\psi(t,x), \psi(t,x')]_+ = 0,$$

$$[\psi(t,x), \psi^\dagger(t,x')]_+ = \delta(x-x').$$

A general solution to the classical field equations (1.3.1) can be expanded over a complete set of classical solutions $\psi_\alpha^{(\pm)}(t,x)$:

$$\psi(x,t) = \sum_\alpha [\psi_\alpha^{(-)}(t,x)A_\alpha + \psi_\alpha^{(+)}(t,x)B_\alpha^\dagger] \ , \tag{1.3.2}$$

where the (continuous, in general) index α numerates the solutions. The functions $\psi_\alpha^{(\pm)}(t,x)$ obey the equal-time orthogonality conditions

$$\begin{aligned}(\psi_\alpha^{(+)}, \psi_\beta^{(+)}) = (\psi_\alpha^-, \psi_\beta^-) &= \delta_{\alpha\beta}, \\ (\psi_\alpha^+, \psi_\beta^-) &= 0 \ .\end{aligned} \tag{1.3.3}$$

where the scalar product is defined as

$$(\xi,\chi) \equiv \int d^3x [\xi(t,x)]^\dagger \chi(t,x) \ . \tag{1.3.4}$$

When we quantise the solution the A_α, B_α coefficients will become operators that are identified with the creation and annihilation operators (we shall call them Heisenberg operators since they define a solution to the Heisenberg equation of motion for ψ) that obey the standard (anti) commutation relations

$$[A_\alpha, A_\beta]_+ = [B_\alpha, B_\beta]_+ = [A_\alpha, B_\beta]_+ = [A_\alpha, B^\dagger_\beta]_+ = 0,$$

$$[A_\alpha, A^\dagger_\beta]_+ = [B_\alpha, B^\dagger_\beta]_+ = \delta_{\alpha\beta},$$

which follow from the orthogonality conditions (1.3.3) and the equal time anti-commutators (1.3.2).

For an arbitrary time-dependent background, the choice of the basis functions can be made in an arbitrary way, so the notion of particle is not well defined : each change of basis induces the redefinition of creation and annihilation operators.

Now, if the background scalar field has a well-defined past asymptotic, $\phi(x,t)|_{t\to-\infty} = \phi_-(x)$, the *in* set of basis functions can be chosen in such a way that their asymptotics have exactly positive and negative frequency behaviour,

$$\psi^{(\pm)}_{in,\alpha}(t,x)|_{t\to-\infty} \propto e^{\pm i\omega_\alpha t} \ , \tag{1.3.5}$$

with $\omega_\alpha > 0$. The operators A_α and B_α can then be unambiguously identified with the annihilation operators of particles $a_{in,\alpha}$ and antiparticles $b^\dagger_{out,\alpha}$, and the initial vacuum state can be constructed. The reason why we use upper case and lower case letters to identify the operators is motivated by the fact that upper case operators are the operators associated with the general solution whereas lower case operators are the free operators. The use of asymptotic relations identifies the upper case and the lower case operators.

If the scalar field has a well-defined future asymptotic $\phi(x,t)|_{t\to+\infty} = \phi_+(x)$, another, *out* set of basis functions can be constructed with the same property as above,

$$\psi^{(\pm)}_{out,\alpha}(t,x)|_{t\to+\infty} = e^{\pm i\omega_\alpha t} \ , \tag{1.3.6}$$

and the corresponding set of *out* creation and annihilation operators $a_{out,\alpha}$, $b^\dagger_{out,\alpha}$ can be introduced.

The relation between the *in* and *out* sets of operators is given by the Bogolubov transformation

$$a_{in,\alpha} = \sum_\beta [A_{\alpha\beta} a_{out,\beta} + B_{\alpha\beta} b^\dagger_{out,\beta}] \ , b^\dagger_{in,\alpha} = \sum_\beta [C_{\alpha\beta} a_{out,\beta} + D_{\alpha\beta} b^\dagger_{out,\beta}] \ . \tag{1.3.7}$$

where the Bogolubov coefficients can be found from the orthogonality relations (1.3.3):

$$\begin{aligned} A_{\alpha\beta} &= (\psi^{(-)}_{in,\alpha}, \psi^{(-)}_{out,\beta}), \\ B_{\alpha\beta} &= (\psi^{(-)}_{in,\alpha}, \psi^{(+)}_{out,\beta}), \end{aligned} \tag{1.3.8}$$

$$C_{\alpha\beta} = (\psi^{(+)}_{in,\alpha}, \psi^{(-)}_{out,\beta}),$$
$$D_{\alpha\beta} = (\psi^{(+)}_{in,\alpha}, \psi^{(+)}_{out,\beta}). \quad (1.3.9)$$

The knowledge of these coefficients solves completely the problem of the particle creation for a future ($t \to \infty$) observer.

The S-matrix is defined as the unitary operator that converts *in*-operators to *out*-operators, $a_{out} = S a_{in} S^\dagger$ and follows from (1.3.8). Before going any further, let us take a look at the variety of problems one can solve using S-matrix formalism. In two papers of the early fifties [22] the problem of pair production in bremsstrahlung is studied using the S-Matrix formalism. This problem is well known, but it gives a very good review of the main techniques one needs to use when working with the S-matrix. Consider a system defined by the Hamiltonian: $H = H_0 + H_r + H'$, where H_0 is the Hamiltonian of the electrons and their Coulomb interaction, H_r that of the photon field and H' that describing the interaction of electrons and photons. The solution to the equations of motion is found. Then defining the S-matrix as the scalar product of in- and -out states the cross section is calculated. There are two different cases. The first one is for standard bremsstrahlung where the in-state is an electron and the out-state an electron and a photon. This case has a matrix element given by : $\int \psi_2^\star \epsilon_k \mathrm{e}^{-ikr} \psi_1 d\tau$, where ψ_2 is the out-electron of momentum k, the photon contributes the exponential, the ϵ_k polarisation factor is the out-state, and ψ_1 is the in-electron. The other case is the case of pair production. This time the out state is an electron positron pair with matrix element: $\int \psi_-^\star \epsilon_k \psi_+^\star \mathrm{e}^{ikr} d\tau$, with the $\psi_\pm$ the wave functions of the electron and positron in the final state, the initial state being just the photon field. In both relations the integration variable is labelled τ as it is a special space variable. In these expressions we can recognise a scalar product (eq. 1.3.4).

Recent work [23] gives a survey of problems linked to pair production in heavy ion collisions. The approach used here is the link between the S-Matrix and the evolution operator. In these two problems the method used is the S-matrix formalism, but in one case it is defined as the scalar product of the in- and out-states and in the other it is defined as the evolution operator. Both definitions were shown to be equivalent [24, 25]. In most work done on particle production the electron positron pair is considered, which epitomises the general fermion anti-fermion pair.

1.3.2 The Green's function method

One can find the Bogolubov coefficients for the problem discussed above by another equivalent method, using the retarded and advanced Green's functions. This method is more general since it allows us to determine not only the total number of produced particles and their spectrum, but also the different time and space correlations of arbitrary operators.

Let us write the fermion mass in the form

$$m(t,\vec{x}) = m_{in,out} + \Delta m_{in,out}(t,\vec{x}) \ , \tag{1.3.10}$$

where $\Delta m(t,x)_{in} \to 0$ for $t \to +\infty$ and $\Delta m(t,x)_{out} \to 0$ for $t \to -\infty$.

The basic Dirac equation can be rewritten as

$$D_{in,out}\psi(t,\vec{x}) \equiv [i\gamma^{\mu}\partial_{\mu} - m_{in,out}]\psi(t,\vec{x}) = \Delta m_{in,out}(t,\vec{x})\psi(t,\vec{x}) \ , \tag{1.3.11}$$

or in the form of two equivalent integral equations

$$\psi(t,\vec{x}) = \psi_{in}(t,\vec{x}) + \int dt'd^3x' S^R_{in}(t,\vec{x};t',\vec{x'})\Delta m_{in}\psi(t',\vec{x'}) \ , \tag{1.3.12}$$

and

$$\psi(t,\vec{x}) = \psi_{out}(t,\vec{x}) + \int dt'd^3x' S^A_{out}(t,\vec{x};t',\vec{x'})\Delta m_{out}\psi(t',\vec{x'}) \ , \tag{1.3.13}$$

where $S^R_{in}(t,\vec{x};t',\vec{x'})$ is the retarded Green's function of the free Dirac operator D_{in} and $S^A_{out}(t,\vec{x};t',\vec{x'})$ is the advanced Green's function of the free Dirac operator D_{out}. This difference in the Green's functions come from the different integration contour we use. The functions $\psi_{in,out}(t,\vec{x})$ satisfy the free Dirac equation and are nothing but asymptotic free fields: $\psi(t,\vec{x})|_{t\to\mp\infty} \to \psi_{in,out}(t,\vec{x})$. The Heisenberg state, time independent and containing no particles at $t \to -\infty$, satisfies the conditions

$$\psi_{in}|0\rangle = (\psi^+_{in})^{\dagger}|0\rangle = 0 \tag{1.3.14}$$

Now, the general solution (1.3.2) can be inserted in (1.3.12) to express ψ_{in} through $A_{\alpha}, \ B_{\beta}$. The inverse transformation gives $A_{\alpha}, \ B_{\beta}$ in terms of creation and annihilation operators associated with ψ_{in}, and, therefore, any Heisenberg operator O_H can be expressed via $a_{in,\alpha}, b_{in,\beta}$ in this way. To find physical observables, one simply computes averages of the type

$$\langle 0|O_H|0\rangle \ . \tag{1.3.15}$$

To find the Bogolubov coefficients discussed in the previous section, one can use (1.3.13) to express ψ_{out} through $A_{\alpha}, \ B_{\beta}$, and then use, in turn, the relations between them and $a_{in,\alpha}, b_{in,\beta}$. This method is a variation of the S-matrix since we also consider transformations between past and future observers.

1.3.3 Localised sources

Assume now that the scalar field depends on $\vec{x}$ and t inside some finite spatial volume V, while outside this volume $\phi = \phi_0 = const$. In this case, one should expect that particle production takes place inside V which results in a flux of fermions coming out of the region V. To define the Heisenberg vacuum state, let us proceed in exact analogy with the previous section and write

$$m(t,\vec{x}) = m + \Delta m(t,\vec{x}) \ , \tag{1.3.16}$$

where $\Delta m(t,\vec{x}) = 0$ for $\vec{x} \notin V$, and rewrite the Dirac equation in the integral form (1.3.12) :

$$\psi_0(t,\vec{x}) = \psi(t,\vec{x}) - \int_{-\infty}^{t} dt' \int_V d^3x' S^R(t,\vec{x};t',\vec{x'})\Delta m(t',\vec{x'})\psi(t',\vec{x'}) \ , \tag{1.3.17}$$

where ψ_0 satisfies the free Dirac equation and S^R is the retarded Green's function for this equation. The construction of the Heisenberg vacuum state proceeds in the same way as above, with the replacement $\psi_{in} \to \psi_0$. If the general solution of the basic Dirac equation (1.3.1) is known, inverting it into (1.3.17) gives ψ_0 in terms of Heisenberg creation and annihilation operators, and through (1.3.14) singles out the vacuum state. This solves, in principle, the problem of particle creation.

Now we will show that, in fact, the equations for the vacuum state can be derived in a simpler way, without performing the integrations in eq. (1.3.17). Consider the region far outside the volume V. The second term in eq. (1.3.17), due to the properties of the retarded Green's function, contains only expanding waves, $\propto \exp(\pm i(\epsilon_k t - kr))$. Here $\epsilon_k = +\sqrt{k^2+m^2}$, $k > 0, r = |\vec{x}| > 0$, and the origin of the coordinate system is assumed to be somewhere inside the volume V. Therefore, the contracting waves $\propto \exp(\pm i(\epsilon_k t + kr))$ are the same for the Heisenberg field ψ and for the free field ψ_0.

The free operator $\psi_0 = \psi_0^A(t,x) + \psi_0^R(t,x)$ can be expanded over creation and annihilation operators $b^{\sigma\dagger}(\vec{k})$, $a^\sigma(\vec{k})$ as usual

$$\psi_0(x,t) = \int d^3k [a^\sigma(\vec{k}) u_\sigma(\vec{k}) \mathrm{e}^{-i(\epsilon_k t - \vec{k}\vec{x})} + b^{\sigma\dagger}(\vec{k}) v_\sigma(\vec{k}) \mathrm{e}^{i(\epsilon_k t - \vec{k}\vec{x})}] \ , \tag{1.3.18}$$

and the vacuum state satisfies the standard conditions

$$a^\sigma(\vec{k})|0\rangle = 0, \quad b^\sigma(\vec{k})|0\rangle = 0 \tag{1.3.19}$$

for all $\vec{k}$ and σ. As these equations must be valid for all $\vec{k}$, they can be equally rewritten as

$$\psi_0^-|_{contracting}|0\rangle = 0, \quad (\psi_0^+)^\dagger|_{contracting}|0\rangle = 0 \tag{1.3.20}$$

or as

$$\psi^-|_{contracting}|0\rangle = 0, \quad (\psi^+)^\dagger|_{contracting}|0\rangle = 0 \tag{1.3.21}$$

directly for the Heisenberg fields (we stress that the separation into negative and positive frequencies and into contracting and expanding waves is possible since the time-dependent source term is local in space). Very naturally, this corresponds to the absence of waves coming from infinity.

To find the physical observables, it is convenient to introduce the Bogolubov coefficients, relating the sets of operators $A_\alpha, B^\dagger_\alpha$ and $a^\sigma(\vec{k}), [b^\sigma(\vec{k})]^\dagger$. They follow from eq. (1.3.17).

In order to clarify the use of the formalism, we consider in the following chapters a simple $1+1$ dimensional model, exhibiting particle production from a local source. These considerations will be done using two different methods so we can easily extend our results to $3 \oplus 1$ dimensions.

1.4 Summary

We have shown how to compute particle production for a situation when the background field has an arbitrary time dependence but is localised in some spacial volume. For this case, the S-matrix formalism in its traditional form cannot be applied as the asymptotic states in the past and the future cannot be defined. To clarify these ideas, we can say that the S-matrix formalism builds a transformation between the vacuum in the past and the vacuum in the future. Particle creation is then linked to the probability of the vacuum remaining a vacuum. The third formalism does something different : it creates a transformation between vacua at the same time but at different places in space, using the fact that the solutions are free solutions far away from the volume. We can link the operators A, B of the general solution to the free operators a and b. With this third formalism we shall obtain "local" current densities. This method might be difficult to apply (see chap. 3) but it allows us to compute more general results.

This kind of approach was used in the late sixties to solve, without any use of S-matrix, the problem of particle production in an expanding universe. Here the interaction is between the fields and the varying metric. An interesting solution was proposed by Parker [17, 18] where no use is made of S-Matrix. The idea is to work with quantised Heisenberg fields where the operators (that were before quantisation the coefficients of the Fourier expansion) have equal time commutation relations $[a_k(t), a^\dagger_k(t)] = \delta_{kk'}$. New time independent operators are introduced by the definition $A_k = a_k(t_1)$, it is then shown that there is a Bogolubov transformation between the time independent operators and time dependent ones given by : $a_k(t) = \alpha(k,t)^\star A_k + \beta(k,t) A^\dagger_{-k}$. The particle production can be computed directly using the α and β coefficients. This problem gives a second alternative method to that of S-matrix. With the Bogolubov transformation the number operator $N = a^\dagger_k a_k$ expressed in terms of A_k operators no longer gives zero when applied on the vacuum. This work gives us a starting point to compute particle production rates without the use of S-matrix formalism.

We shall use adaptations of these methods to solve the problem of Q ball evaporation. We shall start first using a very simple model then we will improve it in order to obtain a full analytical solution for every type of Q ball. In fact we shall not need to use a complicated Q ball profile, we shall work with either very big Q balls or small ones. In these limits the Q ball's shape has only little importance.

Chapter 2

Q Balls

2.1 Non-Topological Solitons.

Unlike topological solitons, non-topological solitons arise from the existence of locally conserved charges rather than topological effects. The best way to understand their properties is to study a simple model. We provide here two simple models. The first one is a model with one scalar field, while the other one contains two scalar fields and can be linked to supersymmetry. The common point of these two models and of all Q-Balls models is that we minimise energy with a fixed charge constraint.

2.1.1 Introduction and a first simple model

The existence and stability of non-topological solitons or Q-Balls is due to charge conservation and to dynamics. A scalar field theory with a spontaneously broken $U(1)$-symmetry may contain stable non-topological solitons, Q-Balls: the Q-Ball is a coherent state of a complex scalar field carrying a global $U(1)$ charge (the leptonic or baryonic number in our case).

We review here the basic properties of a 3-dimensional Q-Ball using the simplest possible model. As we mentioned in the introduction, the Q-Ball is a sort of ground state of a scalar theory containing a global symmetry. We can now build the simplest model in $3 \oplus 1$ dimensions having a Q-Ball solution: it is a $SO(2)$ invariant theory of two real scalar fields (or a is the $U(1)$ theory of one complex scalar field) [1]. We start by writing down the Lagrangian and the equations of motion for the scalar field, to obtain the conserved charge and current. The Lagrangian of the scalar sector is given by :

$$\mathcal{L} = \partial_\mu \phi^\star \partial^\mu \phi - U(|\phi|). \tag{2.1.1}$$

The $U(1)$ symmetry is

$$\phi \rightarrow \mathrm{e}^{i\alpha} \phi.$$

The conserved current is

$$j_\mu = i(\phi^\star \partial_\mu \phi - (\partial_\mu \phi^\star)\phi), \tag{2.1.2}$$

and the conserved charge is

$$Q = \int d^3x j_0. \tag{2.1.3}$$

We build a solution with the minimal energy : if $U(0) = 0$ is the absolute minimum of the potential, $\phi = 0$ is the ground state and the $U(1)$ symmetry is unbroken. It was shown in [1] that new particles (Q-Balls) appear in the spectrum, if the potential is such that the minimum of $\frac{U}{|\phi|^2}$ is at some value $\phi_0 \neq 0$.

$$Min[2U/|\phi|^2] = 2U_0/|\phi_0|^2 < \mu^2 = U''(0). \tag{2.1.4}$$

The charge and energy of a given ϕ field configuration are :

$$\begin{array}{c} Q = \frac{1}{2i}\int(-\partial_t\phi^\star\phi + c.c.)d^3x, \\ E = \int\left[\frac{1}{2}|\dot{\phi}|^2 + \frac{1}{2}|\nabla\phi|^2 + U(\phi)\right]d^3x. \end{array} \tag{2.1.5}$$

The Q ball solution is a solution with minimum energy for a fixed charge, we thus introduce the following Lagrange multiplier

$$\varepsilon_\omega = E + \omega[Q - \frac{1}{2i}\int(\phi^\star\partial_t\phi + c.c.)d^3x]. \tag{2.1.6}$$

Minimising this functional with the standard Q ball ansatz :

$$\phi = \phi(\vec{x})\mathrm{e}^{i\omega t}, \tag{2.1.7}$$

where $\phi(r)$ is a monotonically decreasing function of distance to the origin which reaches zero at infinity. Inserting the Q ball ansatz in the equations of motion gives in spherical coordinates

$$\frac{d^2\phi}{dr^2} = -\frac{2}{r}\frac{d\phi}{dr} - \omega^2\phi + U'(\phi). \tag{2.1.8}$$

If we interpret ϕ as a particle position and r as time this equation is similar to a Newtonian equation of motion for a particle of unit mass subject to viscous damping moving in the potential $\frac{1}{2}\omega^2\phi^2 - U$. We are searching for a solution in which the particle starts at $t = 0$ at some position $\phi(0)$, at rest, $\frac{d\phi}{dr} = 0$, and comes to rest at infinite time at $\phi = 0$. Solving this problem is not difficult (see [1] for details). One of the solutions can be the localised step function. Although we can solve exactly the equation of motion, we will not do it in this work.

This construction is the Q ball we where looking for, in the sense that it is the ground state of the theory with constant charge. We used only one field to describe it but it is in fact made of a collection of scalars. Q balls rotate with constant angular velocity in internal space and are spherically symmetric in position space. As the charge Q grows to infinity, ω approaches

$$\omega_0 = \sqrt{2U_0/\phi^2}, \tag{2.1.9}$$

where U_0 is the value of the potential at the minimum. In this limit, ϕ resembles a smoothed-out step function. The two regions ($r < R$ and $r > R$) are separated by a transition zone of thickness μ^{-1}. This leads to the consideration of two approximations, namely the thick and thin wall regime (see [32] and [3] for details). The radius of the Q ball can easily be calculated using the definition of charge:

$$Q = \frac{4}{3}\pi R^3 \omega_0 \phi_0^2. \tag{2.1.10}$$

This calculation has been done for $\phi(r) = 0$ if $r > R$. All the properties of the Q ball are now known except the exact profile of the Q ball field. We shall now build up the Q ball solution to our problem. The energy is given by the integral (2.1.5) and using the previous properties and taking the limit $V \to \infty$, where V is the volume of the Q ball, the energy becomes

$$E = \frac{1}{2}\omega^2 |\phi|^2 V + UV. \tag{2.1.11}$$

The charge becomes

$$Q = \omega_0 |\phi|^2 V. \tag{2.1.12}$$

We wish to minimise the energy with fixed charge. Using eq. (2.1.12) to eliminate ω, we have in the limit of $Q \to \infty$,

$$E = \frac{1}{2}\frac{Q}{|\phi|^2 V} + UV. \tag{2.1.13}$$

As a function of V it has its minimum at

$$V = \frac{Q}{\sqrt{2|\phi|^2 U}}. \tag{2.1.14}$$

Here the energy is given now by

$$E = Q\sqrt{\frac{2U}{\phi^2}}, \tag{2.1.15}$$

This model can very easily be adapted to any dimensions since we have spherical symmetry, only one space dimension is relevant and the model of two real scalar fields is equivalent to models with one complex scalar field. The question if a Q ball is an absolute minimum of the energy at fixed charge was answered by S. Coleman [1] it is, as long as $E_{Qball} < mQ$, where m is the mass of the free scalar particle. The question of Q ball stability and dynamics is a very interesting question that has occupied many people through the past few years [33], [26], [3] and [27], but will not be considered here.

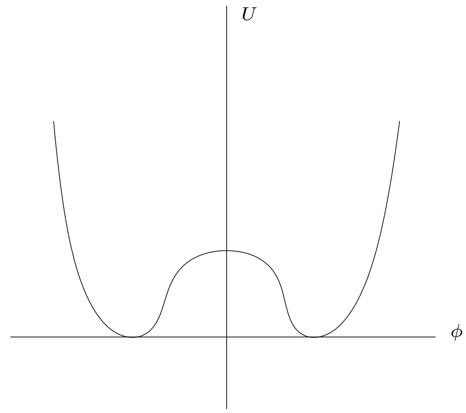

Figure 2.1: Sketch of the double well potential

2.1.2 A more complicated model

This type of model is used in a lot of work [43, 40, 41, 42] to show the main stability properties of Q balls. It can be applied also to most supersymmetric models where we have lots of scalar fields interacting together. We use now the theory of one real scalar field ϕ and one complex field χ in $d \oplus 1$ dimensions The action is given by

$$S = \int d^d x dt[\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi) + \partial_\mu(\chi^\star)\partial^\mu \chi - h\phi^2|\chi|^2], \tag{2.1.16}$$

where the scalar potential is the double well : $V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2$. The ground state, a classical vacuum, is given by

$$\phi = v, \quad \chi = 0. \tag{2.1.17}$$

The condition that χ equals zero decouples the equations of motion for the two scalar fields. There is a global symmetry given by $\chi \to \mathrm{e}^{i\alpha}\chi$ and the conserved current is

$$j_\mu = \frac{1}{i}(\chi^\star \partial_\mu \chi - \chi \partial_\mu \chi^\star), \tag{2.1.18}$$

with the conserved charge

$$Q = \int d^d x(\chi^\star \partial_t \chi - \chi \partial_t \chi^\star) \tag{2.1.19}$$

A state of charge Q is made out of Q χ-particles far away from each other, and the energy of such a state is simply $E = Qm_\chi$. Another state of charge Q can be constructed. Consider the ϕ field to be equal to zero inside a sphere of radius

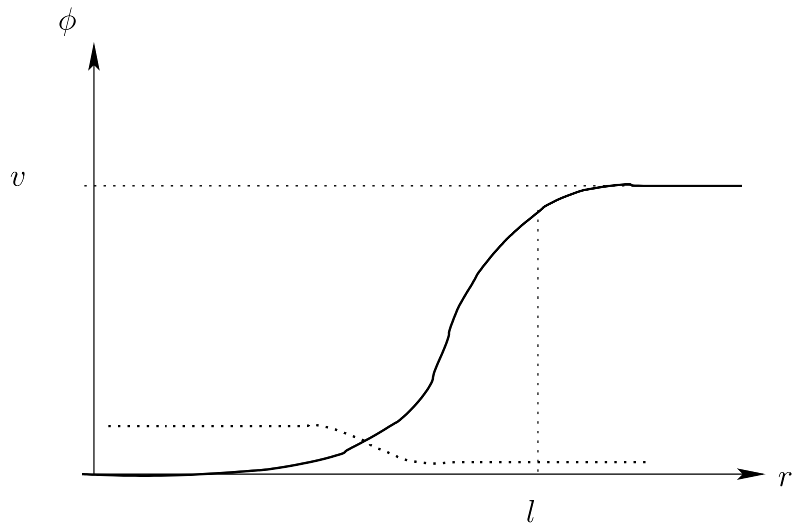

Figure 2.2: Sketch of the ϕ field. The dotted line is the wave function of the lowest energy state the of χ-field

l, in this sphere χ particles are massless. We now have to discuss Q χ-particles trapped in a potential of the type $h\phi^2$ (Fig 2.2). The energy of such a state is given by the sum of the classical energy of the ϕ field E_ϕ and the energy of Q χ-bosons E_χ trapped in the potential.

$$E_\phi = V(0)V_d + \text{surface term,} \tag{2.1.20}$$

dropping surface terms at large l

$$E_\phi = S_d V(0) l^d, \tag{2.1.21}$$

where S_d is the volume of the unit sphere in d dimensions. It is easy to check that the energy of χ-bosons inside the potential is proportional to $\frac{1}{l}$. So we have for E_χ

$$E_\chi = S_d \frac{1}{l} Q, \tag{2.1.22}$$

and the total energy is

$$E_{tot} = S_d(V(0)l^d + \frac{1}{l}Q) \tag{2.1.23}$$

$$l = S_d(V(0))^{-\frac{1}{1+d}} Q^{\frac{1}{1+d}} \tag{2.1.24}$$

for the second equation we minimise with respect to l. The energy of this state is now given by

$$E = S_d(V(0))^{\frac{1}{1+d}} Q^{\frac{1}{1+d}}. \tag{2.1.25}$$

For large Q this energy is smaller than $m_\chi Q$ so the Q ball is absolutely stable. With this example we see that at large enough Q, the Q ball is stable against decay into free χ-bosons. This model has a wide range of applications in various dimensions.

2.1.3 Q ball creation

As we have seen in the precedent sections Q balls are a type of Bose-Einstein classical scalar condensate. The minimal condition we need to build up Q ball solution is the existence of a self interacting scalar field. We can consider two levels for the existence of a scalar field, the first level would be the quantum phenomena, linked to the very small scale physics, while the second level would be classical, macroscopic level, of the model. One of the simplest construction that can be made to obtain classical quantities through quantum fields is the identification between the field and its expectation value [28], therefor fields with large expectation values are good candidates to Q ball creation. These considerations allows us to take the MSSM as first birth place of Q balls. In this picture the stability of Q balls is ensured by the condition that energy per charge unit in the Q ball is smaller than the free scalar field one.

The most popular model allowing "big" scalar field is the Affleck-Dine Baryogenesis process [11, 69, 29]. In this picture a large charged scalar condensate will break up into smaller objects that have Q ball properties. This scalar condensate is the one required for Affleck-Dine baryogenesis. This scalar field carries baryon (or lepton) number and has a very flat potential so it can be given a large expectation value. This breaking up is done through supersymmetry breakdown. We have two major ways for breaking up supersymmetry, the gauge or gravity mediated mechanism. Both mechanism will lead to different type of Q balls. The major difference is that the Q ball's size might not depend on its charge, while its charge will be linked to the one of the initial scalar field.

One other way we could create Q balls is by solitosynthesis a process of charge accretion around a Q ball seed [31]. This model has the advantage not to need any complicated symmetry breaking. The remaining question is of course, what is the original Q ball seed? These models are of course not the only ones. We shall mention to finish this discussion that a very wide range of physical problems need the presence of scalar field to be solved. So the role played by Q balls can be very important in various domains of physics.

2.2 Q balls and Particle production

In this section we give an overview of the method used in [2] and its application in $1 \oplus 1$ dimensions, we shall not compute any quantity we shall just develop the standard formalism used in literature. After the introduction of the very simple model we shall use, we give the general method used to solve the problem of particle production by a Q ball using an S-matrix based construction. This section will also point out some important results.

2.2.1 A simple toy model

We consider a very simple Q ball model, and introduce a Yukawa-type interaction with fermions and scalars. A simple Lagrangian leading to a Q ball solution is given by (see [1] and [32] for details):

$$\mathcal{L}_{scal.} = \partial_\mu \phi^\star \partial^\mu \phi - U(|\phi|). \tag{2.2.1}$$

The charge and energy of a given ϕ field configuration are (see section 2.1.1) :

$$\begin{aligned} Q &= \frac{1}{2i} \int (-\partial_t \phi^\star \phi + c.c.) d^3x \\ E &= \int \left[\frac{1}{2} |\dot{\phi}|^2 + \frac{1}{2} |\nabla \phi|^2 + U(\phi) \right] d^3x \end{aligned} \tag{2.2.2}$$

The Q ball solution is a solution with minimum energy for a fixed charge, we thus introduce the following Lagrange multiplier

$$\varepsilon_\omega = E + \omega[Q - \frac{1}{2i} \int (\phi^\star \partial_t \phi + c.c.) d^3x]. \tag{2.2.3}$$

Minimising this functional with the standard static Q ball ansatz :

$$\phi = \phi(\vec{x}) \mathrm{e}^{i\omega t}, \tag{2.2.4}$$

will give all the Q ball's properties. We now add an interaction with fermions [2] to compute the evaporation of Q balls into fermions. The simplest interaction is given by the following Lagrangian:

$$\mathcal{L}_{ferm.} = i \bar{\Psi} \sigma^\mu \partial_\mu \Psi + (g \phi \bar{\Psi}^C \Psi + h.c), \tag{2.2.5}$$

where the C superscript indicates the charge conjugated fermion, this means that the interaction with the Q ball field acts as a kind of Majorana mass coupling. A little work on this Lagrangian, using the properties of γ-matrices and of charge conjugation (see [86], [34], [35] for details) leads to the following equations of motion for the fermion fields:

$$i \sigma^\mu \partial_\mu \psi - g \phi \chi = 0, \tag{2.2.6}$$

$$i \bar{\sigma}^\mu \partial_\mu \chi - g \phi^\star \psi = 0. \tag{2.2.7}$$

with $\chi = i\sigma^2\psi^\star$ and ψ is a two component spinnor. In $(1 \oplus 1)$ dimensions we only have $\mu = 0, 3$ and $\sigma^0 = \mathbb{1}$. For simplicity we shall use the standard localised step-function solution for the Q ball.

$$\phi(z,t) = \phi_0 \mathrm{e}^{i\omega_0 t}\theta(l-z)\theta(l+z), \tag{2.2.8}$$

where the l parameter is the size of the Q ball. This solution was fully described in [2] and [32], in $(1 \oplus 1)$ dimensions (Fig 1.). This will give a "mass" to the fermions only in a localised zone in the z-direction. If we want to solve Dirac's

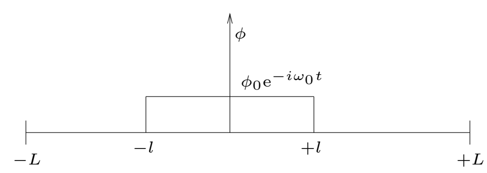

Figure 2.3: General shape of the Q ball field.

equation in all space, we shall need a continuous solution in the three space zones ($z < -l$, $z > +l$ and $z \in [-l, +l]$). After the solutions are built we will compute the particle production rate of the Q ball.

2.2.2 Particle production from Q balls

Following the same method as [2] we start with the equations of motion given by (2.2.6) and (2.2.7), where $\sigma^\mu = (\sigma^0, \sigma^3)$ and $\bar{\sigma}^\mu = (\sigma^0, -\sigma^3)$ for our $1 \oplus 1$ dimensional case. Since the Q ball is invariant under simultaneous time translations and Q rotations we can choose ψ to be proportional to $\mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}$ and χ to be proportional to $\mathrm{e}^{-i(\epsilon-\frac{\omega_0}{2})t}$. Pair production will occur in those modes mixing positive and negative frequencies, that is $\epsilon \in [-\frac{\omega_0}{2}, +\frac{\omega_0}{2}]$ (see chapter 3 for details and the way we can clearly identify this range). The first thing is to study the free solutions (at $t \to \pm\infty$), they are

$$-i\sigma^3\partial_3\psi = (\frac{\omega_0}{2} + \epsilon)\psi, \tag{2.2.9}$$

$$-i\sigma^3\partial_3\chi = (\frac{\omega_0}{2} - \epsilon)\chi, \tag{2.2.10}$$

the solutions are

$$\psi = u(k_+)\mathrm{e}^{-ik_+t}, \tag{2.2.11}$$

$$\chi = u(k_-)\mathrm{e}^{ik_-t}, \tag{2.2.12}$$

with $k_\pm = \frac{\omega_0}{2} \pm \epsilon$ and $u(k) = \frac{1}{4\pi}\begin{pmatrix} \mathrm{e}^{ikz} \\ \mathrm{e}^{-ikz} \end{pmatrix}$. The $\frac{1}{4\pi}$ factor ensures us the standard orthogonality relations for the u's. Constructing the free quantum fields is very simple, the energy range is only $\epsilon \in [-\frac{\omega_0}{2}, +\frac{\omega_0}{2}]$. Using this energy range ensures that the ψ field describes a particle while the χ field describes an anti-particle. In the ψ terms we do the change of variables $\epsilon = k_+ - \frac{\omega_0}{2}$ and in the χ terms we do $\epsilon = k_- + \frac{\omega_0}{2}$. The free quantum fields can be written as

$$\psi = \int_0^{\omega_0} u(k_+)\mathrm{e}^{-ik_+t}a(k_+)dk_+, \tag{2.2.13}$$

$$\chi = \int_{-\omega_0}^{0} u(k_-)\mathrm{e}^{ik_-t}a^\dagger(k_-)dk_-. \tag{2.2.14}$$

Using the orthogonality relations for the u's the a operators have the standard anti-commutation relations. In order to apply the same method we need to identify the positive and negative movers. In the u's it is done by separating them as

$$u(k) = u_1(k) + u_2(k), \tag{2.2.15}$$

with

$$u_1 = \begin{pmatrix} \mathrm{e}^{ikz} \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ \mathrm{e}^{-ikz} \end{pmatrix}. \tag{2.2.16}$$

Now the u_1 are the positive movers and the u_2 the negative ones. We can now construct the general solution outside the Q ball by choosing one of our solutions to have no incoming χ wave:

$$\psi = \mathrm{e}^{-ik_+t}[u_1(k_+) + R(k_+)u_2(k_+)], \tag{2.2.17}$$

$$\chi = \mathrm{e}^{ik_-t}[T(k_-)u_2(k_-)]. \tag{2.2.18}$$

In terms of scattering theory these coefficients mean that in the far past only the incoming wave will survive while in the far future both out-going waves will survive. All this construction was made considering an infinite Q ball that occupies half the space. Using the symmetry of the equations of motion, $\chi = i\sigma^2\psi^\star$ we can construct another solution using the fact that $i\sigma^2 u_1^\star = u_2$ and $i\sigma^2 u_2^\star = -u_1$

$$\psi = -\mathrm{e}^{-ik_-t}[T^\star(k_-)u_2(k_-)], \tag{2.2.19}$$

$$\chi = \mathrm{e}^{ik_+t}[u_2(k_+) + R^\star(k_+)u_1(k_+)]. \tag{2.2.20}$$

The two solutions we obtained are symmetric and ψ is a fermion while χ is the anti fermion. The last thing we need to be sure that the solutions are symmetric is that they need to come with adjoint coefficients, so we can write for the ψ solution

$$\psi = \int_0^{\omega_0} dk_+\mathrm{e}^{-ik_+t}[u_1(k_+) + R(k_+)u_2(k_+)]a_{in}(k_+) + \mathrm{e}^{-ik_+t}[T^\star(k_+)u_2(k_+)]a_{in}^\dagger(k_+), \tag{2.2.21}$$

where in the second term we resolved the change of variable $k_\pm \to k_\mp = \omega_0 - k_\pm$. The integral in the solution comes from the fact that we consider the quantised solution and not the solution in terms of modes. The physical meaning of the a_{in} operators is that in the far past only the incoming wave will survive and then become identical to the free solution so the a_{in} operator is a standard free operator. If we go to the far future only the outgoing wave will survive and we can build the Bogolubov transformation[1] between *in* and *out* operators:

$$a_{out}(k_+) = R(k_+)a_{in}(k_+) - T^\star(k_-)a_{in}^\dagger(k_-). \tag{2.2.22}$$

The average number of outgoing massless fermions in the far future is given by

$$< 0|a_{out}^\dagger(k)a_{out}(k')|0 >= |T|^2\delta(k-k'). \tag{2.2.23}$$

Due to the presence of the delta function this number is infinite, but the idea is to replace the delta function by a smooth function using

$$\delta(x) = \lim_{\tau\to\infty}\frac{1}{2\pi}\int_{-\tau/2}^{\tau/2} \mathrm{e}^{itx}dt. \tag{2.2.24}$$

The number of particles created is then

$$\frac{dN}{dt} = \frac{1}{2\pi}\int_0^{\omega_0} dk|T|^2 \leq \frac{1}{2\pi}\int_0^{\omega_0} dk = \frac{\omega_0}{2\pi}. \tag{2.2.25}$$

The problem is now completely solved, the T coefficient comes from the equations of motion inside the Q ball. If we have a solution inside the Q ball we just need to match it with the solution given by (2.2.21) and identify the coefficients we are searching for. The solution inside the Q ball is given by

$$\begin{aligned} \psi &= (\epsilon \pm \sqrt{\epsilon^2-(g\phi)^2})u(k'_\pm)\mathrm{e}^{-ik_+t} \\ &+ i\sigma^2[(\epsilon \pm \sqrt{\epsilon^2-(g\phi)^2})u(k'_\pm)\mathrm{e}^{-ik_+t}]^\star, && (2.2.26) \\ \chi &= g\phi u(k'_\pm)\mathrm{e}^{ik_-t} + i\sigma^2[g\phi u(k'_\pm)\mathrm{e}^{ik_-t}]^\star, && (2.2.27) \end{aligned}$$

with $k'_\pm = \frac{\omega_0}{2} \pm \sqrt{\epsilon^2-(g\phi)^2}$. The same variable changes and the separation in positive and negative movers gives the matching rules. This will be done in detail in the next chapter where we shall also compute the matching rules and the real evaporation rate.

[1] The evaporation range is the only range where the Bogolubov transformation are non trivial

2.3 Conclusions.

As we did for the previous section we shall clarify a bit these ideas. First a Q ball can be considered as a kind of "classical" Bose Einstein condensate, since it is made out of bosons that link together to form a stable object. This is the reason why in practise we only use one scalar field to describe a Q ball. The stability of Q balls versus decay into scalars depends on their charge, it was shown that the total mass of a Q ball is lower than the mass of a collection of scalars. The method we used to compute all the Q ball's properties is that of Lagrange multipliers. We minimise the energy with constant charge. This method can be applied to all possible Q ball models. The only reason why we decided to use a simple model is for simplicity of the fermion solutions that interact with the Q ball, but all the constructions we shall do can be applied to more complicated models. As we shall see in details in the following chapters one of the important computation that we need to do is matching the fermion solution through the Q ball, this needs a very good knowledge of the Q ball solution.

The decay of Q balls into scalars can be studied using the S-matrix approach to compute the value of the reflection and transmission coefficients. The main result we need to mention is that evaporation only occurs if the fermion energy is in the range $[-\frac{\omega_0}{2}; \frac{\omega_0}{2}]$. This is the only range where the Bogolubov transformations we can build are non trivial. Only fermions are produced not scalars as long as the Q ball is in its stable sector. This is the result we shall develop in a very precise way in the next two chapters. We shall also give in the next chapter a complete demonstration of the existence of the finite range for energy. We shall also show that it is really evaporation that occurs. To clearly demonstrate these results we shall need to construct the exact quantum-mechanical state describing an evaporating Q-ball.

One of the other important facts we need to mention is the way Q balls are created, we know they will produce fermions. One possibility to create Q balls is solitosyntesis, a process of gradual charge accretion, provided some primordial charge asymmetry and initial "seed" Q balls exist [31]. The other way they can be created is through breakup of the Affleck-Dine charge condensate. In fact it has been shown that Q balls exist in the MSSM [36], so we do not really need to be careful about Q ball creation.

We shall mention to finish this discussion that a large amount of work has been done on various type of Q balls. Non abelian Q balls [37, 38], or even Q balls with topological charge [39].

Chapter 3

Evaporation of Q balls into massless particles.

The solution to the problem of particle creation by a Q ball can be solved using two different pictures. The first one is based on the S-matrix formalism, using the idea that the field is free for $t \to \pm\infty$. This construction is done by finding the solution to the equations of motion for a fermion interacting with a Q ball in terms of a superposition of classical solutions. The quantisation is made by upgrading expansion coefficients to operators, this will give us the Heisenberg field operator. The S-matrix will then be constructed by identifying the fields in the far past and in the far future to fields having exact positive and negative frequency behaviour. This method was widely used to solve the problem for particle creation. This method was used to compute the evaporation rate of Q-balls ([2, 32]) where the expansion was made using rotational eigenfunctions. Once the total solution is known it is simple to build the transformation from the far past to the far future. In the far past only the incoming waves will survive and in the far future only the outgoing ones.

The construction we are going to use here is different, we shall in the first place solve the equations of motion and obtain the Heisenberg field operator representing a fermion interacting with a Q-ball. In one space dimension this solution will be expressed in the form,

$$\Psi_Q = \frac{1}{\sqrt{4\pi}} \int d\epsilon \left(\psi_Q^+(\epsilon, t, z) A(\epsilon) + \psi_Q^-(\epsilon, t, z) B(\epsilon) \right),$$

where the $\psi_Q^\pm(\epsilon, t, z)$ are a basis of the solution to the Dirac equation for fermions interacting with a Q ball of charge Q. $A(\epsilon)$ and $B(\epsilon)$ are operators depending on energy, their anti-commutation relations are the standard ones if the ψ solutions satisfy proper orthogonality conditions. The next is to consider the spatial asymptotics of this solution. Far away from the Q ball ($z = \pm\infty$ for one space dimension) the solution is the standard free field solution. This identification will give us a relation between the solution operators $A(\epsilon)$, $B(\epsilon)$ and the free asymptotic ones $a(p)$, $b(p)$. The only difficulty in this identification is that the quantisation of the solution was made using energy (due to the time dependence of the interaction) while the asymptotical operators depend on momentum. The next step will be to define and solve the particle production condition, saying that no particles are moving towards the Q ball. In terms of asymptotic operators it is

$$\begin{aligned} a_L(p)|\Psi>= b_L(p)|\Psi>=0 \quad \text{for } p>0, \text{ on the left} \\ a_R(p)|\Psi>= b_R(p)|\Psi>=0 \quad \text{for } p<0, \text{ on the right.} \end{aligned} \tag{3.0.1}$$

The last step of the resolution will consist in using the total Heisenberg operator Ψ, and the particle productive state to compute the fermionic flux giving evaporation rate.

3.1 Solutions to the equations of motion

Writing down the Lagrangian of a massless fermion having a Yukawa interaction with a scalar field gives in one spatial dimension,

$$\mathcal{L}_{ferm.} = i\bar{\psi}\sigma^\mu\partial_\mu\psi + (g\phi\bar{\psi}^C\psi + h.c), \tag{3.1.1}$$

where the C superscript indicates the charge conjugated fermion. The equations of motion and their solutions are fully described in literature on the subject ([1, 2, 32]). Instead of treating separately the fermion and the anti-fermion, we shall construct the exact global solution to this problem, this solution will be made of different parts first the solution inside the Q-Ball (for $z \in [-l, l]$). Equations of motion for the two components of the Ψ field are :

$$\begin{aligned} (i\partial_0 + i\partial_z)\psi_1 - g\phi\psi_2^\star = 0, \\ (i\partial_0 - i\partial_z)\psi_2^\star - g\phi^\star\psi_1 = 0. \end{aligned} \tag{3.1.2}$$

and $\phi = \phi_0 \mathrm{e}^{-i\omega_0 t}$ in the zone from $-l$ to $+l$ and zero everywhere else. Using the ansatz :

$$\begin{pmatrix} \psi_1 \\ \psi_2^\star \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-i\frac{\omega_0}{2}t} & 0 \\ 0 & \mathrm{e}^{i\frac{\omega_0}{2}t} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \mathrm{e}^{-i\epsilon t + i(k+\frac{\omega_0}{2})}, \tag{3.1.3}$$

the equations of motion are reduced to the following 2×2 linear system

$$\begin{pmatrix} k-\epsilon & M \\ M & -(k+\epsilon) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0.$$

The determinant of the system gives $k = \pm\sqrt{\epsilon^2 - M^2} \equiv \pm k_\epsilon$. Solving for the two cases $k = +k_\epsilon$ and $k = -k_\epsilon$, we obtain the solution inside the Q ball:

$$\Psi_Q = \begin{pmatrix} \psi_1 \\ \psi_2^\star \end{pmatrix} = A \begin{pmatrix} 1 \\ \frac{k_\epsilon+\epsilon}{M} \end{pmatrix} \mathrm{e}^{-ik_\epsilon z} + B \begin{pmatrix} \frac{k_\epsilon+\epsilon}{M} \\ 1 \end{pmatrix} \mathrm{e}^{ik_\epsilon z}, \tag{3.1.4}$$

where $M = g\phi_0$, g is the coupling constant and ϕ_0 the value of the scalar field. The second part is the solution when $\phi_0 = 0$ (outside the Q ball) it is,

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2^\star \end{pmatrix} = \mathrm{e}^{-i\epsilon t} \begin{pmatrix} C_1^{L,R}\mathrm{e}^{i\epsilon z} \\ C_2^{L,R}\mathrm{e}^{-i\epsilon z} \end{pmatrix}, \tag{3.1.5}$$

where superscripts L, R indicate the left and right side of the Q ball. In order to solve Dirac's equation everywhere the solution needs to be continuous in space. Space continuity gives at $z = -l$:

$$\begin{aligned} C_1^L &= A\mathrm{e}^{i(k_\epsilon+\epsilon)l} + B\alpha_\epsilon \mathrm{e}^{-i(k_\epsilon-\epsilon)l}, \\ C_2^L &= A\alpha_\epsilon \mathrm{e}^{i(k_\epsilon-\epsilon)l} + B\mathrm{e}^{-i(k_\epsilon+\epsilon)l}, \end{aligned}$$

and at $z = +l$,

$$\begin{aligned} C_1^R &= A\mathrm{e}^{-i(k_\epsilon+\epsilon)l} + B\alpha_\epsilon \mathrm{e}^{+i(k_\epsilon-\epsilon)l}, \\ C_2^R &= A\alpha_\epsilon \mathrm{e}^{-i(k_\epsilon-\epsilon)l} + B\mathrm{e}^{+i(k_\epsilon+\epsilon)l}. \end{aligned}$$

These matching relations are used to express the solution only using the parameters coming from the inner part of the solution. This construction will allow us to build a state where there is no incoming fermion, all the fermions are now produced inside the Q ball. Putting together all these parts gives the full solution continuous in space and time :

$$\begin{pmatrix} \psi_1 \\ \psi_2^\star \end{pmatrix} = \int d\epsilon \begin{pmatrix} \begin{pmatrix} \mathrm{e}^{i\epsilon l}(A\mathrm{e}^{ik_\epsilon l} + B\alpha_\epsilon \mathrm{e}^{-ik_\epsilon l})\mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon+\frac{\omega_0}{2})z} \\ \mathrm{e}^{-i\epsilon l}(A\alpha_\epsilon \mathrm{e}^{ik_\epsilon l} + B\mathrm{e}^{-ik_\epsilon l})\mathrm{e}^{-i(\epsilon-\frac{\omega_0}{2})t}\mathrm{e}^{-i(\epsilon-\frac{\omega_0}{2})z} \end{pmatrix} z < -l \\ \\ \begin{pmatrix} (A\mathrm{e}^{-ik_\epsilon z} + B\alpha_\epsilon \mathrm{e}^{ik_\epsilon z})\mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}\mathrm{e}^{i\frac{\omega_0}{2}z} \\ (A\alpha_\epsilon \mathrm{e}^{-ik_\epsilon z} + B\mathrm{e}^{ik_\epsilon z})\mathrm{e}^{-i(\epsilon-\frac{\omega_0}{2})t}\mathrm{e}^{i\frac{\omega_0}{2}z} \end{pmatrix} -l \geq z \geq +l \\ \\ \begin{pmatrix} \mathrm{e}^{-i\epsilon l}(A\mathrm{e}^{-ik_\epsilon l} + B\alpha_\epsilon \mathrm{e}^{ik_\epsilon l})\mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon+\frac{\omega_0}{2})z} \\ \mathrm{e}^{i\epsilon l}(A\alpha_\epsilon \mathrm{e}^{-ik_\epsilon l} + B\mathrm{e}^{ik_\epsilon l})\mathrm{e}^{-i(\epsilon-\frac{\omega_0}{2})t}\mathrm{e}^{-i(\epsilon-\frac{\omega_0}{2})z} \end{pmatrix} z > +l \end{pmatrix} \tag{3.1.6}$$

where

$$\alpha_\epsilon = \frac{k_\epsilon + \epsilon}{M}. \tag{3.1.7}$$

We would like the Ψ solution to have orthogonality properties, in order to be able to quantise it. To do this the first possibility is to impose the boundary condition $\Psi(-L) = \Psi(+L)$. This ensures the self-adjointness of operator $i\sigma^\mu\partial_\mu$. The boundary condition reads as follows:

$$\begin{aligned} A\sinh[i(k_\epsilon+\epsilon)l - i(\epsilon+\frac{\omega_0}{2})L] - B\sinh[i(k_\epsilon-\epsilon)l + i(\epsilon+\frac{\omega_0}{2})L]\alpha_\epsilon = 0, \\ A\sinh[i(k_\epsilon-\epsilon)l + i(\epsilon-\frac{\omega_0}{2})L]\alpha_\epsilon - B\sinh[i(k_\epsilon+\epsilon)l - i(\epsilon-\frac{\omega_0}{2})L] = 0. \end{aligned}$$

The determinant of the system gives :

$$\begin{aligned} &-\sinh[i(k_\epsilon+\epsilon)l-i(\epsilon+\tfrac{\omega_0}{2})L]\sinh[i(k_\epsilon+\epsilon)l-i(\epsilon-\tfrac{\omega_0}{2})L] \\ &+\sinh[i(k_\epsilon-\epsilon)l+i(\epsilon-\tfrac{\omega_0}{2})L]\sinh[i(k_\epsilon-\epsilon)l+i(\epsilon+\tfrac{\omega_0}{2})L]\alpha_\epsilon^2=0. \end{aligned} \tag{3.1.8}$$

The real and imaginary parts of this equation will always have the same zeros giving the energy spectrum of fermions. This relation leads to $A=\pm B$ and to a relation for $\frac{\omega_0}{2}$, that is $\frac{\omega_0}{2}L=2\pi n$. Taking the limit $L\to\infty$ leads to the same relation without any condition on $\frac{\omega_0}{2}$ and with a continuous spectrum. Finally the solution can be written in the form :

$$\Psi_Q=\begin{pmatrix}\psi_1\\ \psi_2^\star\end{pmatrix}=\int_{-\infty}^{+\infty}d\epsilon\left(\psi_Q^+A+\psi_Q^-B\right), \tag{3.1.9}$$

with

$$\psi_Q^\pm=\begin{pmatrix} \begin{pmatrix} \mathrm{e}^{i\epsilon l}(\pm\mathrm{e}^{ik_\epsilon l}+\alpha_\epsilon\mathrm{e}^{-ik_\epsilon l})\mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon+\frac{\omega_0}{2})z}\\ \mathrm{e}^{-i\epsilon l}(\pm\alpha_\epsilon\mathrm{e}^{ik_\epsilon l}+\mathrm{e}^{-ik_\epsilon l})\mathrm{e}^{-i(\epsilon-\frac{\omega_0}{2})t}\mathrm{e}^{-i(\epsilon-\frac{\omega_0}{2})z} \end{pmatrix} z<-l \\ \\ \begin{pmatrix} (\pm\mathrm{e}^{-ik_\epsilon z}+\alpha_\epsilon\mathrm{e}^{ik_\epsilon z})\mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}\mathrm{e}^{i\frac{\omega_0}{2}z}\\ (\pm\alpha_\epsilon\mathrm{e}^{-ik_\epsilon z}+\mathrm{e}^{ik_\epsilon z})\mathrm{e}^{-i(\epsilon-\frac{\omega_0}{2})t}\mathrm{e}^{i\frac{\omega_0}{2}z} \end{pmatrix} -l\geq z\geq +l \\ \\ \begin{pmatrix} \mathrm{e}^{-i\epsilon l}(\pm\mathrm{e}^{-ik_\epsilon l}+\alpha_\epsilon\mathrm{e}^{ik_\epsilon l})\mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon+\frac{\omega_0}{2})z}\\ \mathrm{e}^{i\epsilon l}(\pm\alpha_\epsilon\mathrm{e}^{-ik_\epsilon l}+\mathrm{e}^{ik_\epsilon l})\mathrm{e}^{-i(\epsilon-\frac{\omega_0}{2})t}\mathrm{e}^{-i(\epsilon-\frac{\omega_0}{2})z} \end{pmatrix} z>+l \end{pmatrix}. \tag{3.1.10}$$

This solution is for the moment a classical solution. We shall now verify the orthogonality properties of this solution in a more explicit form, to obtain the normalisation constant and to check the anticommutation relations of operators.

3.1.1 Orthogonality and Quantisation

We would like to check that $\int dz(\psi^{\sigma'}(\epsilon'))^\dagger\psi^\sigma(\epsilon)=\delta_{\sigma'\sigma}\delta(\epsilon'-\epsilon)$. To do so we separate the scalar product component by component and zone by zone :

$$\begin{aligned} \int dz\quad & (\psi^{\sigma'}(\epsilon'))^\dagger\psi^\sigma(\epsilon) \\ = \quad & \int_{-\infty}^{-l}dz(\psi^{\sigma'}(\epsilon'))^\dagger\psi^\sigma(\epsilon)+\int_{+l}^{+\infty}dz(\psi^{\sigma'}(\epsilon'))^\dagger\psi^\sigma(\epsilon)+\int_{-l}^{+l}dz(\psi^\sigma(\epsilon'))^\dagger\psi^\sigma(\epsilon)= \\ = \quad & \mathcal{D}_1^L\int_{-\infty}^{-l}\mathrm{e}^{-i(\epsilon'-\epsilon)z}dz+\mathcal{D}_2^L\int_{-\infty}^{-l}\mathrm{e}^{+i(\epsilon'-\epsilon)z}dz \\ + \quad & \mathcal{D}_1^R\int_{+l}^{+\infty}\mathrm{e}^{-i(\epsilon'-\epsilon)z}dz+\mathcal{D}_2^R\int_{+l}^{+\infty}\mathrm{e}^{+i(\epsilon'-\epsilon)z}dz \\ + \quad & \int_{-l}^{+l}dz(\psi^\sigma(\epsilon'))^\dagger\psi^\sigma(\epsilon), \end{aligned}$$

where the $\mathcal{D}$ coefficients are products of the $C_{1,2}^{L,R}$ coefficients defined in (3.1.5) :

$$\mathcal{D}_1^L = \mathcal{D}_1(l) = C_1^{L\star}(\epsilon')C_1^L(\epsilon)$$
$$= \mathrm{e}^{-i(\epsilon'-\epsilon)l}\Big(\sigma\sigma'\mathrm{e}^{-i(k^\star_{\epsilon'}-k_\epsilon)l} + \sigma'\alpha_\epsilon\mathrm{e}^{-i(k^\star_{\epsilon'}+k_\epsilon)l} +$$
$$\sigma\alpha^\star_{\epsilon'}\mathrm{e}^{i(k^\star_{\epsilon'}+k_\epsilon)l} + \alpha^\star_{\epsilon'}\alpha_\epsilon\mathrm{e}^{i(k^\star_{\epsilon'}-k_\epsilon)l}\Big),$$
$$\mathcal{D}_2^L = \mathcal{D}_2(l) = C_2^{L\star}(\epsilon')C_2^L(\epsilon)$$
$$= \mathrm{e}^{i(\epsilon'-\epsilon)l}\Big(\sigma'\sigma\alpha^\star_{\epsilon'}\alpha_\epsilon\mathrm{e}^{-i(k^\star_{\epsilon'}-k_\epsilon)l} + \sigma\alpha^\star_{\epsilon'}\mathrm{e}^{-i(k^\star_{\epsilon'}+k_\epsilon)l} +$$
$$\sigma'\alpha_\epsilon\mathrm{e}^{i(k^\star_{\epsilon'}+k_\epsilon)l} + \mathrm{e}^{-i(k^\star_{\epsilon'}-k_\epsilon)l}\Big),$$
$$\mathcal{D}_1^R = C_1^{R\star}(\epsilon')C_1^R(\epsilon) = \mathcal{D}_1(-l)$$
$$\mathcal{D}_2^R = C_2^{R\star}(\epsilon')C_2^R(\epsilon) = \mathcal{D}_2(-l)$$

We have to compute the middle terms, leading to hyperbolic sines:

$$\begin{aligned}\int_{-l}^{+l} dz\Psi^\dagger(\epsilon')\Psi(\epsilon) &= 2\frac{\sinh[i(k^\star_{\epsilon'}-k_\epsilon)l]}{i(k^\star_{\epsilon'}-k_\epsilon)}\left((\sigma'\sigma+1)(1+\alpha^\star_{\epsilon'}\alpha_\epsilon)\right)\\ &+ 2\frac{\sinh[i(k^\star_{\epsilon'}+k_\epsilon)l]}{i(k^\star_{\epsilon'}+k_\epsilon)}\left((\sigma'+\sigma)(\alpha^\star_{\epsilon'}+\alpha_\epsilon)\right). \quad (3.1.11)\end{aligned}$$

Before going any further we introduce the well known formula ([86]),

$$\begin{aligned}\int_{+l}^{+\infty} \mathrm{e}^{\pm i(\epsilon'-\epsilon)z}dz &= \mathrm{e}^{\pm i(\epsilon'-\epsilon)l}\lim_{\eta\to 0}\left(\frac{\eta}{(\epsilon'-\epsilon)^2+\eta^2} \pm i\frac{(\epsilon'-\epsilon)}{(\epsilon'-\epsilon)^2+\eta^2}\right)\\ &= \pi\delta(\epsilon'-\epsilon) \pm i\mathcal{P}\left(\frac{1}{(\epsilon'-\epsilon)}\right), \quad (3.1.12)\end{aligned}$$

and the following relations, $(\sigma'+\sigma) = 2\sigma\delta_{\sigma'\sigma}$, and $(1+\sigma'\sigma) = 2\delta_{\sigma'\sigma}$. After the integrals are done the coefficient in front of the δ-function is

$$\begin{aligned}N_\pm &= 4\pi\Big(\cosh[\mathrm{Im}[k_\epsilon]l](1+|\alpha_\epsilon|^2)\\ &\pm \cos[\mathrm{Re}[k_\epsilon]l]\mathrm{Re}[\alpha_\epsilon]\Big), \quad (3.1.13)\end{aligned}$$

and the coefficient in front of the principal value is,

$$\begin{aligned}R_{\sigma\sigma'} &= 4i\frac{(\epsilon'-\epsilon)}{(\epsilon'-\epsilon)^2+\eta^2}\left(\sinh[i(k^\star_{\epsilon'}-k_\epsilon)l](\alpha^\star_{\epsilon'}\alpha_\epsilon - 1)\right.\\ &+ \left.\sinh[i(k^\star_{\epsilon'}+k_\epsilon)l](\alpha^\star_{\epsilon'}-\alpha_\epsilon)\right). \quad (3.1.14)\end{aligned}$$

This coefficient will combine with the middle terms to give zero.Taking a precise look at the C coefficient in front of $\sinh[i(k^\star_{\epsilon'}-k_\epsilon)l]$ we write,

$$C = \frac{1+\alpha^\star_{\epsilon'}\alpha_\epsilon}{k^\star_{\epsilon'}-k_\epsilon} + \frac{(\epsilon'-\epsilon)(\alpha^\star_{\epsilon'}\alpha_\epsilon-1)}{(\epsilon'-\epsilon)^2+\eta^2}$$

if $\epsilon \neq \epsilon$ we can take directly the limit $\eta \to 0$ and after the simplification of $(\epsilon' - \epsilon)$ in the numerator and in the denominator we obtain,

$$\begin{aligned} C &= \frac{-(\epsilon' - \epsilon)(1 + \alpha^\star_{\epsilon'}\alpha_\epsilon) + (k^\star_{\epsilon'} - k_\epsilon)(\alpha^\star_{\epsilon'}\alpha_\epsilon - 1)}{(k^\star_{\epsilon'} - k_\epsilon)(\epsilon' - \epsilon)} \\ &= \frac{\frac{1}{M^2}(k^\star_{\epsilon'} + \epsilon')(k_\epsilon + \epsilon)[(k^\star_{\epsilon'} - k_\epsilon) - (\epsilon' - \epsilon)] - [(\epsilon' - \epsilon) - (k^\star_{\epsilon'} - k_\epsilon)]}{(k^\star_{\epsilon'} - k_\epsilon)(\epsilon' - \epsilon)}. \end{aligned}$$

We can work a little more on the first term of the numerator to obtain,

$$\begin{aligned} \tfrac{1}{M^2} &\quad (k^\star_{\epsilon'} + \epsilon')(k_\epsilon + \epsilon)[(k^\star_{\epsilon'} - k_\epsilon) - (\epsilon' - \epsilon)] \\ &= \frac{1}{M^2}(k^\star_{\epsilon'} + \epsilon')(k_\epsilon + \epsilon)[(k^\star_{\epsilon'} - \epsilon') - (k_\epsilon - \epsilon)] \\ &= [(\epsilon' - \epsilon) - (k^\star_{\epsilon'} - k_\epsilon)] \end{aligned}$$

in the last step we used the fact that $(k_\epsilon + \epsilon)(k_\epsilon - \epsilon) = M^2$, it proves that this coefficient is zero. The same type of calculations stands for all the other terms, if $\epsilon = \epsilon'$ the calculations are a little more complicated because we can not directly take the limit $\eta \to 0$. To quantise the solution Ψ_Q becomes an operator $\hat{\Psi}_Q$, with the equal time anti-commutation relations and the fact that we have :

$$\begin{aligned} A(\epsilon) &= \int_{-\infty}^{+\infty} dz (\psi_Q^+)^\dagger \hat{\Psi}_Q \\ B(\epsilon) &= \int_{-\infty}^{+\infty} dz (\Psi_Q^-)^\dagger \hat{\Psi}_Q. \end{aligned} \qquad (3.1.15)$$

We can calculate the anti-commutation relations for $A(\epsilon)$

$$\begin{aligned} \{A(\epsilon), A^\dagger(\epsilon')\} &= \int dz \int dz' \left(\psi_Q^+(z,\epsilon)\right)^\dagger \left(\psi_Q^+(z',\epsilon')\right) \times \underbrace{\{\hat{\Psi}_Q, \left(\hat{\Psi}'_Q\right)^\dagger\}}_{\delta(z'-z)} \\ &= \delta(\epsilon' - \epsilon), \end{aligned}$$

where we have used orthogonality for the last step. The same calculations with $B(\epsilon)$ lead to the same result so the solution we found is quantised with the standard anti-commutation relations. The only difference with the solution you can find in most textbooks, is that in this case the solution is quantised using energy and not momentum. This is valid only in $1 \oplus 1$ dimensions where we can treat separately the left- and right-movers.

3.2 Particle production

3.2.1 First calculations

Before going any further, we simplify a little the Q ball solution (3.1.9) by introducing the following matrix :

$$\Omega = \begin{pmatrix} e^{-i\frac{\omega_0}{2}t} & 0 \\ 0 & e^{+i\frac{\omega_0}{2}t} \end{pmatrix}, \tag{3.2.1}$$

using this matrix the solution (3.1.10) may be written in the form :

$$\Psi_Q = \frac{1}{\sqrt{4\pi}} \int d\epsilon e^{-i\epsilon t} \left(\psi_Q^+(\epsilon) A(\epsilon) + \psi_Q^-(\epsilon) B(\epsilon) \right) e^{i\frac{\omega_0}{2}z} \Omega(t), \tag{3.2.2}$$

with

$$\psi^\pm = \begin{pmatrix} \begin{pmatrix} f_1^\pm(\epsilon, l) e^{i\epsilon z} \\ (f_2^\pm(\epsilon, l))^\star e^{-i\epsilon z} \end{pmatrix} z < -l \\ \\ \frac{1}{\sqrt{N_\pm}} \begin{pmatrix} (\pm e^{-ik_\epsilon z} + \alpha_\epsilon e^{ik_\epsilon z}) \\ (\pm \alpha_\epsilon e^{-ik_\epsilon z} + e^{ik_\epsilon z}) \end{pmatrix} - l \geq z \geq +l \\ \\ \begin{pmatrix} f_1^\pm(\epsilon, -l) e^{i\epsilon z} \\ (f_2^\pm(\epsilon, -l))^\star e^{-i\epsilon z} \end{pmatrix} z > +l \end{pmatrix}, \tag{3.2.3}$$

and the functions $f^\pm$ having the form

$$f_1^\pm(\epsilon, l) = \frac{1}{\sqrt{4\pi N_\pm}} e^{i\epsilon l} (\pm e^{ik_\epsilon l} + \alpha_\epsilon e^{-ik_\epsilon l}), \tag{3.2.4}$$

$$f_2^\pm(\epsilon, l) = \frac{1}{\sqrt{4\pi N_\pm}} e^{i\epsilon l} (\pm \alpha_\epsilon^\star e^{-ik_\epsilon^\star l} + e^{ik_\epsilon l}), \tag{3.2.5}$$

$$\tag{3.2.6}$$

with α_ϵ given by(3.1.7). Obviously this simplification does not modify the orthogonality properties of the solution. We would like to build an particle productive state for the Q ball solution. First we conjugate the second component of the above solution, in order to compare it with the standard free solution for a massless fermion in $1 \oplus 1$ dimensions (see [44] for details). We look at the asymptotic behaviour of the Q ball solution (3.2.2). On the left and right-hand side of the Q ball, it has to be the standard free solution since the interaction is zero outside the Q ball's volume. After elimination of integrals and standard manipulations and variable changes, we obtain at $z \to -\infty$:

$$\frac{1}{\sqrt{2\pi}}\begin{pmatrix}\theta(p)\\ \theta(-p)\end{pmatrix}a(p) + \begin{pmatrix}\theta(-p)\\ \theta(p)\end{pmatrix}b^\dagger(-p) = \begin{pmatrix} f_1^+(\epsilon,l)A(\epsilon)\Big|_{\epsilon=p-\frac{\omega_0}{2}} \\ f_2^+(\epsilon,l)A^\dagger(\epsilon)\Big|_{\epsilon=p+\frac{\omega_0}{2}}\end{pmatrix} + \\ + \begin{pmatrix} f_1^-(\epsilon,l)B(\epsilon)\Big|_{\epsilon=p-\frac{\omega_0}{2}} \\ f_2^-(\epsilon,l)B^\dagger(\epsilon)\Big|_{\epsilon=p+\frac{\omega_0}{2}}\end{pmatrix}. \qquad (3.2.7)$$

The variable change in the right hand side of the above equation was introduced for elimination of exponentials. Multiplying eq. (3.2.7) by $\begin{pmatrix}\theta(p)\\ \theta(-p)\end{pmatrix}^\dagger$ and by $\begin{pmatrix}\theta(-p)\\ \theta(p)\end{pmatrix}^\dagger$ we obtain the two following equations :

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}a_l(p) &= [f_1^+(\epsilon,l)A(\epsilon)+f_1^-(\epsilon,l)B(\epsilon)]\Big|_{\epsilon=p-\frac{\omega_0}{2}}\theta(p) + \\ &+ [f_2^+(\epsilon,l)A^\dagger(\epsilon)+f_2^-(\epsilon,l)B^\dagger(\epsilon)]\Big|_{\epsilon=p+\frac{\omega_0}{2}}\theta(-p), \qquad (3.2.8)\\ \frac{1}{\sqrt{2\pi}}b_l^\dagger(-p) &= [f_1^+(\epsilon,l)A(\epsilon)+f_1^-(\epsilon,l)B(\epsilon)]\Big|_{\epsilon=p-\frac{\omega_0}{2}}\theta(-p) + \\ &+ [f_2^+(\epsilon,l)A^\dagger(\epsilon)+f_2^-(\epsilon,l)B^\dagger(\epsilon)]\Big|_{\epsilon=p+\frac{\omega_0}{2}}\theta(p). \qquad (3.2.9)\end{aligned}$$

In these two equations the l subscript indicates we are on the left-hand side of the Q ball, the same manipulations on the right-hand side lead to :

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}a_r(p) &= [f_1^+(\epsilon,-l)A(\epsilon)+f_1^-(\epsilon,-l)B(\epsilon)]\Big|_{\epsilon=p-\frac{\omega_0}{2}}\theta(p) + \\ &+ [f_2^+(\epsilon,-l)A^\dagger(\epsilon)+f_2^-(\epsilon,-l)B^\dagger(\epsilon)]\Big|_{\epsilon=p+\frac{\omega_0}{2}}\theta(-p), \qquad (3.2.10)\\ \frac{1}{\sqrt{2\pi}}b_r^\dagger(-p) &= [f_1^+(\epsilon,-l)A(\epsilon)+f_1^-(\epsilon,-l)B(\epsilon)]\Big|_{\epsilon=p-\frac{\omega_0}{2}}\theta(-p) + \\ &+ [f_2^+(\epsilon,-l)A^\dagger(\epsilon)+f_2^-(\epsilon,-l)B^\dagger(\epsilon)]\Big|_{\epsilon=p+\frac{\omega_0}{2}}\theta(p). \qquad (3.2.11)\end{aligned}$$

Checking the anticommutation relations of operators $a_{l,r}$ and $b_{l,r}$ is a difficult task but using the different energy ranges and orthogonality properties of the $f_{1,2}^{\pm}$ it can be done. These four equations will the basis of the construction of particle productive state, since they give a relation between free operators (lower case) and solution operators (upper case).

3.2.2 Construction of particle productive state

As mentioned before, the construction of this quantum state Ψ will be done using the fact that there are no particles moving towards the Q ball. These are negative momentum particles on the left and positive momentum particles on the right. In terms of $a_{L,R}$, and $b_{L,R}$ operators :

$$\begin{aligned} a_L(p)|\Psi>= b_L(p)|\Psi>=0 \quad \text{for } p>0, \text{ on the left} \\ a_R(p)|\Psi>= b_R(p)|\Psi>=0 \quad \text{for } p<0, \text{ on the right.} \end{aligned} \tag{3.2.12}$$

This construction will lead to the opposite sign of the fermionic current on the left and on the right hand side of Q ball using eqs. (3.2.8-3.2.11). We then obtain four equations. For positive p, we have :

$$\begin{aligned} (f_1^+(\epsilon,l)A(\epsilon)+f_1^-(\epsilon,l)B(\epsilon))\Big|_{\epsilon=p-\frac{\omega_0}{2}}|\Psi>=0, \\ (f_1^+(\epsilon,l))^\star A^\dagger(\epsilon)+(f_1^-(\epsilon,l))^\star B^\dagger(\epsilon)\Big|_{\epsilon=-p-\frac{\omega_0}{2}}|\Psi>=0, \end{aligned} \tag{3.2.13}$$

and for negative p

$$\begin{aligned} (f_2^+(\epsilon,-l)A^\dagger(\epsilon)+f_2^-(\epsilon,-l)B^\dagger(\epsilon))\Big|_{\epsilon=p+\frac{\omega_0}{2}}|\Psi>=0, \\ (f_2^+(\epsilon,-l))^\star A(\epsilon)+(f_2^-(\epsilon,-l))^\star B(\epsilon))\Big|_{\epsilon=-p+\frac{\omega_0}{2}}|\Psi>=0. \end{aligned} \tag{3.2.14}$$

Due to the relation between ϵ, p, $\frac{\omega_0}{2}$ given in the subindices of eqs. (3.2.13, 3.2.14) and the fact that p is either positive or negative, we can identify three ranges for ϵ :

- For $\epsilon>+\frac{\omega_0}{2}$ we only have the following two equations :

$$\begin{aligned} (f_1^+(\epsilon,l)A(\epsilon)+f_1^-(\epsilon,l)B(\epsilon))|\Psi>=0, \\ ((f_2^+(\epsilon,-l))^\star A(\epsilon)+(f_2^-(\epsilon,-l))^\star B(\epsilon))|\Psi>=0. \end{aligned} \tag{3.2.15}$$

- For the negative range $\epsilon<-\frac{\omega_0}{2}$ we have :

$$\begin{aligned} ((f_1^+(\epsilon,l))^\star A^\dagger(\epsilon)+(f_1^-(\epsilon,l))^\star B^\dagger(\epsilon))|\Psi>=0, \\ (f_2^+(\epsilon,-l)A^\dagger(\epsilon)+f_2^-(\epsilon,-l)B^\dagger(\epsilon))|\Psi>=0. \end{aligned} \tag{3.2.16}$$

- For the middle range $\epsilon\in[-\frac{\omega_0}{2},+\frac{\omega_0}{2}]$ we have :

$$\begin{aligned} (f_1^+(\epsilon,l)A(\epsilon)+f_1^-(\epsilon,l)B(\epsilon))|\Psi>=0, \\ (f_2^+(\epsilon,-l)A^\dagger(\epsilon)+f_2^-(\epsilon,-l)B^\dagger(\epsilon))|\Psi>=0. \end{aligned} \tag{3.2.17}$$

The range where $|\epsilon| > +\frac{\omega_0}{2}$ is easy to solve, since we expect the solution to be the vacuum and to lead to no evaporation. The determinant of the matrix :

$$\det[\begin{pmatrix} f_1^+(\epsilon,l) & f_1^-(\epsilon,l) \\ (f_2^+(\epsilon,-l))^\star & (f_2^-(\epsilon,-l))^\star \end{pmatrix}] = 2(1-\alpha_\epsilon^2), \tag{3.2.18}$$

is always different from zero. The only solution for an evaporating state in this range is the trivial solution given by :

$$\begin{array}{ll} A(\epsilon)|\Psi >= B(\epsilon)|\Psi >= 0 & \text{for} \quad \epsilon > \frac{\omega_0}{2}, \\ A^\dagger(\epsilon)|\Psi >= B^\dagger(\epsilon)|\Psi >= 0 & \text{for} \quad \epsilon < -\frac{\omega_0}{2}. \end{array} \tag{3.2.19}$$

In fact these two equations are the same, because we can always do the transformation $A(\epsilon) = A'(\epsilon)\theta(\epsilon) + B'^\dagger(\epsilon)\theta(-\epsilon)$, all equations will have vacuum solutions. Here the anti-commutation relations are trivial to check because of the two different energy ranges. For the middle range $\epsilon \in [-\frac{\omega_0}{2}, +\frac{\omega_0}{2}]$ things are a little more complicated, this being the range where particle production occurs as first shown in [2]. Taking a look at solution (3.1.10) in this range, only particles are created and changing the sign of ω_0 changes the particle type. Let us introduce :

$$\eta = f_1^+(\epsilon,l), \quad \zeta = f_1^-(\epsilon,l). \tag{3.2.20}$$

Using these definitions, the system (3.2.17), (3.2.17) can be written in the form :

$$(\eta A(\epsilon) + \zeta B(\epsilon))|\Psi >= 0, \tag{3.2.21}$$

$$(\eta^\star A^\dagger(\epsilon) - \zeta^\star B^\dagger(\epsilon))|\Psi >= 0. \tag{3.2.22}$$

These two relations will anti-commute if $|\eta|^2 = |\zeta|^2$, using the definitions we can write the square module :

$$\begin{aligned} |\eta|^2 &= \frac{1}{4\pi N_+}\left(\alpha_\epsilon^\star e^{i(k_\epsilon+k_\epsilon^\star)l} + e^{i(k_\epsilon-k_\epsilon^\star)l} + |\alpha_\epsilon|^2 e^{-i(k_\epsilon-k_\epsilon^\star)l} + \alpha_\epsilon e^{-i(k_\epsilon+k_\epsilon^\star)l}\right) \\ &= e^{-2\mathcal{I}m(k_\epsilon)l} + |\alpha_\epsilon|^2 e^{2\mathcal{I}m(k_\epsilon)l} + \mathcal{R}e(\alpha_\epsilon e^{2i\mathcal{R}e(k_\epsilon)l}) \end{aligned}$$

writing the exponentials in the form, $e^{\pm x} = \cosh[x] \pm \sinh[x]$, we obtain.

$$\begin{aligned} |\eta|^2 = \frac{1}{4\pi N_+}((1+|\alpha_\epsilon|^2)\cosh[2\mathcal{I}m[k_\epsilon]l] + (|\alpha_\epsilon|^2 - 1)\sinh[2\mathcal{I}m[k_\epsilon]l] + \\ 2(\mathcal{R}e[\alpha_\epsilon]\cos[2\mathcal{R}e[k_\epsilon]l] - \mathcal{I}m[\alpha_\epsilon]l\sin[2\mathcal{R}e[k_\epsilon]l]). \end{aligned}$$

Since k_ϵ is either purely real or purely imaginary we always have $|\eta|^2 = \frac{1}{4\pi}$. We introduce new evaporation operators in all the energy range defined by

$$a_e(\epsilon) = \left\{ \begin{array}{c} A^\dagger(\epsilon) \quad \epsilon < -\frac{\omega_0}{2} \\ \sqrt{8\pi}(\eta A(\epsilon) + \zeta B(\epsilon)) \quad \epsilon \in [-\frac{\omega_0}{2}, +\frac{\omega_0}{2}] \\ A(\epsilon) \quad \epsilon > +\frac{\omega_0}{2} \end{array} \right. , \tag{3.2.23}$$

and

$$b_e(\epsilon) = \left\{ \begin{array}{c} B^\dagger(\epsilon) \quad \epsilon < -\frac{\omega_0}{2} \\ \sqrt{8\pi}(\eta^\star A^\dagger(\epsilon) - \zeta^\star B^\dagger(\epsilon)) \quad \epsilon \in [-\frac{\omega_0}{2}, +\frac{\omega_0}{2}] \\ B(\epsilon) \quad \epsilon > +\frac{\omega_0}{2} \end{array} \right. , \tag{3.2.24}$$

where the $\sqrt{8\pi}$ factor is the normalisation $\frac{1}{\sqrt{|\eta|^2+|\zeta|^2}}$. The anticommutation relations of these operators are easy to check. The particle production state is now fully characterised by the simple relation :

$$a_e(\epsilon)|\Psi >= b_e(\epsilon)|\Psi >= 0. \tag{3.2.25}$$

3.2.3 Particle production rate

The particle production rate is given by the current operator $\vec{j}^\mu(x) = \bar{\psi}\gamma^\mu\psi$, which in our case is $\psi_1^\star\psi_1 - \psi_2^\star\psi_2$, that we shall apply on the evaporating state defined by the vacuum for a_e and b_e operators. First we invert the systems 3.2.23 and 3.2.24 to obtain :

- $\epsilon < -\dfrac{\omega_0}{2}$ $\quad \left\{ \begin{array}{l} A(\epsilon) = a_e^\dagger(\epsilon) \\ B(\epsilon) = b_e^\dagger(\epsilon) \end{array} \right.$ (3.2.26)
- $\epsilon > \dfrac{\omega_0}{2}$ $\quad \left\{ \begin{array}{l} A(\epsilon) = a_e(\epsilon) \\ B(\epsilon) = b_e(\epsilon) \end{array} \right.$ (3.2.27)
- $\epsilon \in [-\dfrac{\omega_0}{2}, +\dfrac{\omega_0}{2}]$ $\quad \left\{ \begin{array}{l} A(\epsilon) = \frac{1}{\sqrt{8\pi}2\eta}(a_e(\epsilon) + b_e^\dagger(\epsilon)) \\ B(\epsilon) = \frac{1}{\sqrt{8\pi}2\zeta}(a_e(\epsilon) - b_e^\dagger(\epsilon)) \end{array} \right.$ (3.2.28)

Now we can compute the first term of the current on the left hand side of the Q ball :

$$\begin{aligned}
< 0|\psi_1^\dagger\psi_1|0 >=< 0 \quad | \quad & \left(\int_{-\infty}^{-\frac{\omega_0}{2}} d\epsilon[(f_1^+(\epsilon,l))^\star \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})z} A^\dagger(\epsilon) + (f_1^-(\epsilon,l))^\star \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})z} B^\dagger(\epsilon)] \right. \\
+ \quad & \int_{-\frac{\omega_0}{2}}^{+\frac{\omega_0}{2}} d\epsilon[(f_1^+(\epsilon,l))^\star \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})z} A^\dagger(\epsilon) + (f_1^-(\epsilon,l))^\star \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})z} B^\dagger(\epsilon)] \\
+ \quad & \left. \int_{+\frac{\omega_0}{2}}^{+\infty} d\epsilon[(f_1^+(\epsilon,l))^\star \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})z} A^\dagger(\epsilon) + (f_1^-(\epsilon,l))^\star \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})z} B^\dagger(\epsilon)] \right) \\
\times \quad & \left(\int_{-\infty}^{-\frac{\omega_0}{2}} d\epsilon[f_1^+(\epsilon,l) \mathrm{e}^{+i(\epsilon+\frac{\omega_0}{2})z} A(\epsilon) + f_1^-(\epsilon,l) \mathrm{e}^{+i(\epsilon+\frac{\omega_0}{2})z} B(\epsilon)] \right. \\
+ \quad & \int_{-\frac{\omega_0}{2}}^{+\frac{\omega_0}{2}} d\epsilon[f_1^+(\epsilon,l) \mathrm{e}^{+i(\epsilon+\frac{\omega_0}{2})z} A(\epsilon) + f_1^-(\epsilon,l) \mathrm{e}^{+i(\epsilon+\frac{\omega_0}{2})z} B(\epsilon)] \\
+ \quad & \left. \int_{+\frac{\omega_0}{2}}^{+\infty} d\epsilon[f_1^+(\epsilon,l) \mathrm{e}^{+i(\epsilon+\frac{\omega_0}{2})z} A(\epsilon) + f_1^-(\epsilon,l) \mathrm{e}^{+i(\epsilon+\frac{\omega_0}{2})z} B(\epsilon)] \right) |0 >
\end{aligned}$$

Using anti-commutation relations and the separate range of integrals and the definition of $A(\epsilon)$ and $B(\epsilon)$ in terms of evaporation operators $a_e(\epsilon)$, $b_e(\epsilon)$ we obtain :

$$
\begin{aligned}
<0|\psi_1^\dagger\psi_1|0> &= \int_{-\infty}^{\frac{\omega_0}{2}} d\epsilon(|f_1^+(\epsilon,l)|^2 <0|a_e(\epsilon)a_e^\dagger(\epsilon)|0> +|f_1^-(\epsilon,l)|^2 <0|b_e(\epsilon)b_e^\dagger(\epsilon)|0>) \\
&+ \frac{1}{8\pi}\int_{-\frac{\omega_0}{2}}^{+\frac{\omega_0}{2}} d\epsilon\left|\frac{f_1^+(\epsilon,l)}{2\eta} - \frac{f_1^-(\epsilon,l)}{2\zeta}\right|^2 <0|b_e(\epsilon)b_e^\dagger(\epsilon)|0> \qquad (3.2.29)
\end{aligned}
$$

The second term is zero due to the definition of η and ζ The other term of the current, proportional to $\psi_2^\star\psi_2$, is very similar but we need to be careful with the fact that ψ_2 is proportional to $f_2(\epsilon,l)A^\dagger(\epsilon)+g_2(\epsilon,l)B^\dagger(\epsilon)$. Applying the same method we obtain :

$$
\begin{aligned}
<0|\psi_2^\dagger\psi_2|0> &= \frac{1}{8\pi}\int_{-\frac{\omega_0}{2}}^{+\frac{\omega_0}{2}} d\epsilon\left|\frac{(f_2^+(\epsilon,l))^\star}{2f_1^+(\epsilon,l)} + \frac{(f_2^-(\epsilon,l))^\star}{2f_1^-(\epsilon,l)}\right|^2 <0|a_e(\epsilon)a_e^\dagger(\epsilon)|0> \qquad (3.2.30) \\
&+ \int_{+\frac{\omega_0}{2}}^{+\infty} d\epsilon\left(|(f_2^+(\epsilon,l))^\star|^2 <0|a_e(\epsilon)a_e^\dagger(\epsilon)|0> +|(f_2^-(\epsilon,l))^\star|^2 <0|b_e(\epsilon)b_e^\dagger(\epsilon)|0>\right)
\end{aligned}
$$

It is easy to check that $|f_2(\epsilon,l)|^2 = |f_1(\epsilon,l)|^2$ so the two terms with infinite bounds will compensate. Finally the expression for the fermionic current on the left is :

$$
\begin{aligned}
\vec{j}_L(x) &= \int_{-\frac{\omega_0}{2}}^{+\frac{\omega_0}{2}} d\epsilon\left|\frac{(f_2^+(\epsilon,l))^\star}{2f_1^+(\epsilon,l)} + \frac{(f_2^-(\epsilon,l))^\star}{2f_1^-(\epsilon,l)}\right|^2 \\
&= \int_{-\frac{\omega_0}{2}}^{+\frac{\omega_0}{2}} d\epsilon\left|\frac{\alpha_\epsilon \sinh[2ik_\epsilon l]}{\mathrm{e}^{2ik_\epsilon l} - \alpha_\epsilon^2\mathrm{e}^{-2ik_\epsilon l}}\right|^2 \qquad (3.2.31)
\end{aligned}
$$

If the real part of k_ϵ equals zero ($\frac{\omega_0}{2} \leq M$), we use the definitions

$$
k_\epsilon = i\sqrt{M^2-\epsilon^2}, \quad \alpha_\epsilon = \frac{k_\epsilon+\epsilon}{M}, \quad |\alpha_\epsilon|^2 = 1
$$

and the current is then:

$$
j_L = \int_{-\frac{\omega_0}{2}}^{+\frac{\omega_0}{2}} d\epsilon\frac{\sinh^2[-2\sqrt{M^2-\epsilon^2}l]}{|\mathrm{e}^{-2\sqrt{M^2-\epsilon^2}l} - \alpha_\epsilon^2\mathrm{e}^{2\sqrt{M^2-\epsilon^2}l}|^2}. \qquad (3.2.32)
$$

In the limit $Ml \to \infty$ we can neglect the negative exponentials so the integrand is only one fourth, coming from the hyperbolic sine. On the right hand side the current is the same except for the sign, so the total current is equal to one half. Using the continuity equation we can write :

$$
\frac{\partial j_0}{\partial t} + \frac{\partial j_1}{\partial x} = 0 \Rightarrow \frac{dQ}{dt} = \int \partial_x j(x)dx = j_L - j_R = 2j_L. \qquad (3.2.33)
$$

The norm of the current is constant on both sides, so after integration over ϵ we obtain :

$$\frac{dQ}{dt} = \frac{1}{4\pi}\omega_0. \tag{3.2.34}$$

This expression gives the particle production rate as a function of ω_0 when ω_0 is smaller than M in the limit of big Ml. It is in fact [2] an evaporation rate since it does not depend on the Q ball's size. The case when the imaginary part of k_ϵ is equal to zero is a bit more complicated to solve, this time the current is given by after a few simple manipulations,

$$\begin{aligned} j_L &= \frac{1}{2\pi}M + \int_M^{+\frac{\omega_0}{2}} \frac{\sin^2[2k_\epsilon l]\alpha_\epsilon^2}{1+\alpha_\epsilon^4+2\alpha_\epsilon^2(\sin^2[2k_\epsilon l]-\cos^2[2k_\epsilon l])}d\epsilon + \\ &+ \int_{-\frac{\omega_0}{2}}^{-M} \frac{\sin^2[2k_\epsilon l]\alpha_\epsilon^2}{1+\alpha_\epsilon^4+2\alpha_\epsilon^2(\sin^2[2k_\epsilon l]-\cos^2[2k_\epsilon l])} \\ &= \frac{1}{2\pi}M + \int_M^{\frac{\omega_0}{2}} \frac{\sin^2[2k_\epsilon l]\alpha_\epsilon^2}{(1-\alpha_\epsilon^2)^2+4\sin^2[2k_\epsilon l]\alpha_\epsilon^2}d\epsilon + \\ &+ \int_{-\frac{\omega_0}{2}}^{-M} \frac{\sin^2[2k_\epsilon l]\alpha_\epsilon^2}{(1-\alpha_\epsilon^2)^2+4\sin^2[2k_\epsilon l]\alpha_\epsilon^2}d\epsilon. \end{aligned} \tag{3.2.35}$$

What we shall now do is construct the solutions in terms of dimensionless parameters starting with $k_\epsilon = \sqrt{\epsilon^2 - M^2} = M\sqrt{\frac{\epsilon^2}{M^2}-1} = M\bar{k}_\epsilon$, we can write,

$$\alpha_\epsilon = \bar{k}_\epsilon + \bar{\epsilon}, \quad 2ik_\epsilon l = 2i\bar{k}_{\bar{\epsilon}}\bar{l}, \tag{3.2.36}$$

with $\bar{\epsilon} = \frac{\epsilon}{M}$ and $\bar{l} = Ml$. We then obtain for the current

$$\vec{j}_L(x) = M\int_{-\frac{\omega_0}{2M}}^{+\frac{\omega_0}{2M}} d\bar{\epsilon}\left|\frac{\bar{\alpha}_{\bar{\epsilon}}\sinh[2i\bar{k}_{\bar{\epsilon}}\bar{l}]}{e^{2i\bar{k}_{\bar{\epsilon}}\bar{l}} - \bar{\alpha}_{\bar{\epsilon}}^2 e^{-2i\bar{k}_{\bar{\epsilon}}\bar{l}}}\right|^2 \tag{3.2.37}$$

Production rate in function of size

We shall first consider the limit where l is small. In this case we write

$$\begin{aligned} \sinh^2[2\sqrt{M^2-\epsilon^2}l] &= 4(M^2-\epsilon^2)l^2 = 4(Ml)^2(1-(\frac{\epsilon}{M})^2) \\ \left|e^{-2\sqrt{M^2-\epsilon^2}l} - \alpha_\epsilon^2 e^{2\sqrt{M^2-\epsilon^2}l}\right|^2 &= e^{-4\sqrt{M^2-\epsilon^2}l} + e^{+4\sqrt{M^2-\epsilon^2}l} - 2\mathcal{R}e[\alpha_\epsilon^2] \\ &= 2(1-\frac{2\epsilon-M^2}{M^2}) = 2(1-\frac{\epsilon^2}{M^2}). \end{aligned} \tag{3.2.38}$$

These two terms will simplify to give after integration over ϵ :

$$j_L = l^2M^2\frac{\omega_0}{8\pi}, \tag{3.2.39}$$

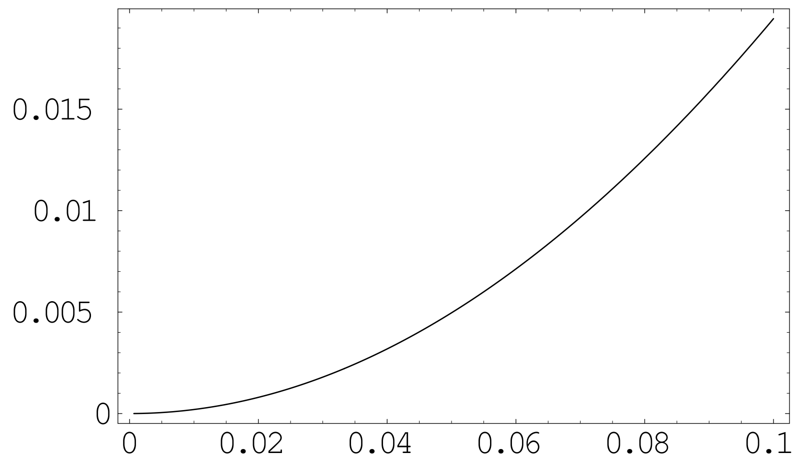

Figure 3.1: Particle production rate for small values of Ml and for a fixed value of $\frac{\omega_0}{2M} = 0.5$.

leading to the particle production rate:

$$\frac{dQ}{dt} = l^2 M^2 \frac{1}{4\pi} \omega_0. \tag{3.2.40}$$

This result ensures us the fact that when $Ml = 0$, the Q ball does not exist, the evaporation rate is zero. This behaviour is shown on figure 3.1. The next limit we shall study is the very large Q ball limit. To do so we take a look at the production rate for large values of the size parameter, Ml and observe that the production rate becomes constant for big values of the size parameter (see fig 3.2). These considerations also stand for all the possible values of the frequency parameter the only difference is when $\frac{\omega_0}{2M}$ gets bigger the stability of the evaporation rate comes for bigger values of Ml.

Energy flux far away form the Q ball

The next calculation we can do is the calculation of the energy flux far away from the Q-ball. In the case where we consider the observer very far from the Q-ball the only relevant coordinate is the distance to the Q-ball, we are in a one spacial dimension case. The energy flux a distant observer can measure is given after normalisation by M,

$$\frac{dE}{Mdtd\sigma} = \int_{-\frac{\omega_0}{2M}}^{+\frac{\omega_0}{2M}} d(\frac{\epsilon}{M}) \left| \frac{\alpha_{(\frac{\epsilon}{M})} \sinh[2i\bar{k}_{(\frac{\epsilon}{M})}\bar{l}]}{\mathrm{e}^{2ik_{(\frac{\epsilon}{M})}\bar{l}} - \bar{\alpha}^2_{(\frac{\epsilon}{M})} \mathrm{e}^{-2ik_{(\frac{\epsilon}{M})}\bar{l}}} \right|^2 (\frac{\epsilon}{M})^2, \tag{3.2.41}$$

this expression is obtained by computing the energy flux through a sphere containing the Q ball. When the real part of k_ϵ equals zero the fraction becomes equal to one. The result in this range will be proportional to ω_0^3 [2, 32].

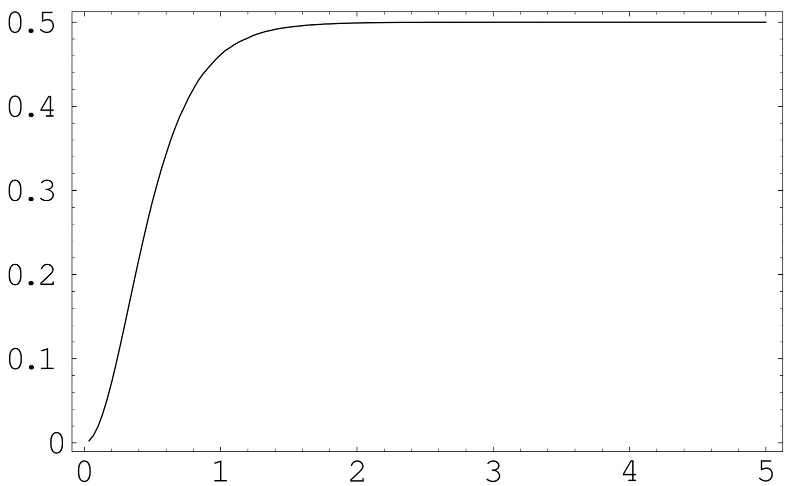

Figure 3.2: Particle production rate in function of Ml and for a fixed value of $\frac{\omega_0}{2M} = 0.5$.

Results of numerical integration

We can now give the evaporation rate of a Q ball into massless fermions in function of its internal frequency. In the first figure we can observe a limit in the evaporation rate. The absolute upper bound can be computed using

$$\frac{dN}{dt} \leq \int_{-\frac{\omega_0}{2}}^{+-\frac{\omega_0}{2}} d\epsilon = \omega_0,$$

this absolute upper bound will be used to normalise the evaporation rate.

3.3 Using a Scattering-like Formalism

We shall this time use a different method which consists of expressing the whole solution in function of the parameters on the left, instead of using the middle ones. This formalism is the one we described in the previous chapter when we studied the particle production from a Q ball using the formalism described in [2], but we shall this time compute the value of the transmission and reflection amplitudes. This time the matching conditions reads at $z = -l$:

$$\begin{pmatrix} C_1^L \\ C_2^L \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{i(k_\epsilon+\epsilon)l} & \alpha_\epsilon \mathrm{e}^{-i(k_\epsilon-\epsilon)l} \\ \alpha_\epsilon \mathrm{e}^{i(k_\epsilon+\epsilon)l} & \mathrm{e}^{-i(k_\epsilon-\epsilon)l} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix},$$
$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{1-\alpha^2} \begin{pmatrix} \mathrm{e}^{-i(k_\epsilon-\epsilon)l} & -\alpha_\epsilon \mathrm{e}^{-i(k_\epsilon-\epsilon)l} \\ -\alpha_\epsilon \mathrm{e}^{i(k_\epsilon+\epsilon)l} & \mathrm{e}^{i(k_\epsilon+\epsilon)l} \end{pmatrix} \begin{pmatrix} C_1^L \\ C_2^L \end{pmatrix}, \tag{3.3.1}$$

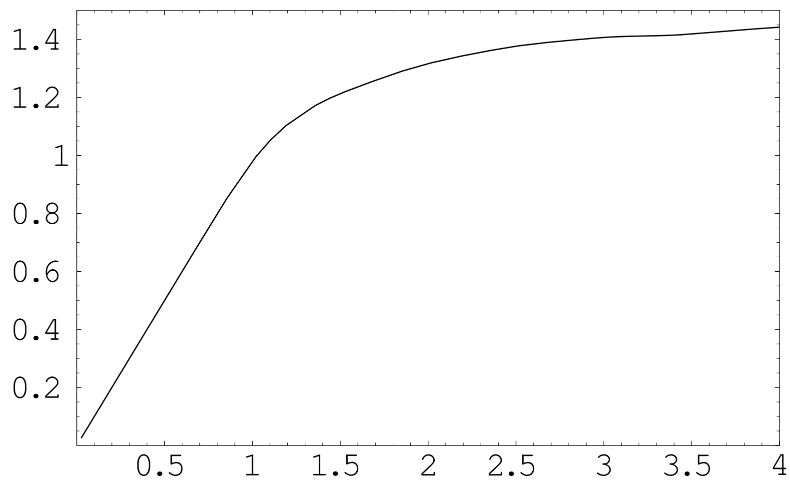

Figure 3.3: Particle production rate $\frac{2\pi dN}{Mdt}$ as a function of $\frac{\omega_0}{2M}$ in the limit of very big Ml parameter.

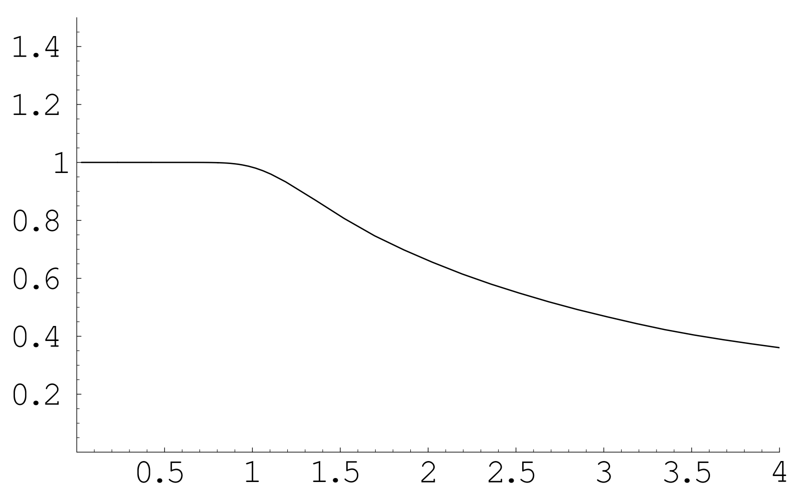

Figure 3.4: Normalised particle production rate $\frac{2\pi dN}{Mdt}\frac{1}{\text{upper bound}}$ as a function of $\frac{\omega_0}{2M}$ in the limit of a very big Ml parameter.

and at $z = +l$,

$$\begin{pmatrix} C_1^R \\ C_2^R \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-i(k_\epsilon+\epsilon)l} & \alpha_\epsilon \mathrm{e}^{i(k_\epsilon-\epsilon)l} \\ \alpha_\epsilon \mathrm{e}^{-i(k_\epsilon+\epsilon)l} & \mathrm{e}^{i(k_\epsilon-\epsilon)l} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \tag{3.3.2}$$

Mixing these three relations gives the coefficients on the right in function of those on the left,

$$\begin{pmatrix} C_1^R \\ C_2^R \end{pmatrix} = \frac{1}{1-\alpha^2} \begin{pmatrix} \mathrm{e}^{-2ik_\epsilon l} - \alpha_\epsilon^2 \mathrm{e}^{2ik_\epsilon l} & -\alpha_\epsilon \mathrm{e}^{-2ik_\epsilon l} + \alpha_\epsilon \mathrm{e}^{2ik_\epsilon l} \\ \alpha_\epsilon \mathrm{e}^{-2ik_\epsilon l} - \alpha_\epsilon \mathrm{e}^{2ik_\epsilon l} & \mathrm{e}^{2ik_\epsilon l} - \alpha_\epsilon^2 \mathrm{e}^{-2ik_\epsilon l} \end{pmatrix}$$

The first component corresponds to a particle moving towards the right, while the second component is a particle moving to the left. This is verified if the energy is in the particle productive range $\epsilon \in [-\frac{\omega_0}{2}; + -\frac{\omega_0}{2}]$. We shall now consider two different cases depending if the particle is incident from the left hand side of the Q ball or from the right hand side.

For the case where the particle is incident from the left hand side we have :

$$\begin{pmatrix} C_1^L \\ C_2^L \end{pmatrix} = \begin{pmatrix} A \\ R_L \end{pmatrix}, \quad \begin{pmatrix} C_1^R \\ C_2^R \end{pmatrix} = \begin{pmatrix} T_L \\ 0 \end{pmatrix}, \tag{3.3.3}$$

leading to the system,

$$T_L = \frac{\mathrm{e}^{-2ik_\epsilon l} - \alpha_\epsilon^2 \mathrm{e}^{2ik_\epsilon l}}{1-\alpha^2} A + \frac{-\alpha_\epsilon \mathrm{e}^{-2ik_\epsilon l} + \alpha_\epsilon \mathrm{e}^{2ik_\epsilon l}}{1-\alpha^2} R_L \tag{3.3.4}$$

$$0 = \frac{\alpha_\epsilon \mathrm{e}^{-2ik_\epsilon l} - \alpha_\epsilon \mathrm{e}^{2ik_\epsilon l}}{1-\alpha^2} A + \frac{-\alpha_\epsilon^2 \mathrm{e}^{-2ik_\epsilon l} + \mathrm{e}^{2ik_\epsilon l}}{1-\alpha^2} R_L \tag{3.3.5}$$

having for solution :

$$R_L = \frac{\alpha_\epsilon 2 \sinh[-2ik_\epsilon l]}{-\mathrm{e}^{2ik_\epsilon l} + \alpha_\epsilon^2 \mathrm{e}^{-2ik_\epsilon l}} \tag{3.3.6}$$

$$T_L = \frac{(1-\alpha_\epsilon^2)B}{\mathrm{e}^{2ik_\epsilon l} - \alpha_\epsilon^2 \mathrm{e}^{-2ik_\epsilon l}}. \tag{3.3.7}$$

For the second case where particle is incident from the right hand side we have :

$$\begin{pmatrix} C_1^L \\ C_2^L \end{pmatrix} = \begin{pmatrix} 0 \\ T_R \end{pmatrix}, \quad \begin{pmatrix} C_1^R \\ C_2^R \end{pmatrix} = \begin{pmatrix} B \\ R_R \end{pmatrix}, \tag{3.3.8}$$

leading to the system

$$R_R = \frac{\alpha_\epsilon \mathrm{e}^{-2ik_\epsilon l} + \alpha_\epsilon \mathrm{e}^{2ik_\epsilon l}}{1-\alpha^2} T_R \tag{3.3.9}$$

$$B = \frac{\mathrm{e}^{2ik_\epsilon l} - \alpha_\epsilon^2 \mathrm{e}^{-2ik_\epsilon l}}{1-\alpha^2} T_R \tag{3.3.10}$$

having for solution :

$$T_R = \frac{(1-\alpha_\epsilon^2)B}{\mathrm{e}^{2ik_\epsilon l} - \alpha_\epsilon^2 \mathrm{e}^{-2ik_\epsilon l}} \tag{3.3.11}$$

$$R_L = \frac{\alpha_\epsilon 2\sinh[-2ik_\epsilon l]}{-\mathrm{e}^{2ik_\epsilon l} + \alpha_\epsilon^2 \mathrm{e}^{-2ik_\epsilon l}}. \tag{3.3.12}$$

Before we start building the solutions we can remark that the transmission amplitudes and the reflection amplitudes are the same for particles incident from the left and the right hand side of the Q ball. A quick calculation shows that the transmission amplitudes vanish for big ml, they are proportional to e^{-ml}, all these properties are quite normal. Let us now take a look at the solutions they are :

$$\Psi_{left} = \begin{pmatrix} \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon+\frac{\omega_0}{2})z}C_1^L \\ \mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})z}(C_1^L)^\star \end{pmatrix} \tag{3.3.13}$$

$$\Psi_{right} = \begin{pmatrix} \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon+\frac{\omega_0}{2})z}C_1^R \\ \mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})z}(C_1^R)^\star \end{pmatrix} \tag{3.3.14}$$

The total solution is a superposition of both cases with operator valued expansion coefficients, on the left hand side the solution is,

$$\begin{aligned} \Psi_L &= \begin{pmatrix} \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon+\frac{\omega_0}{2})z} \\ 0 \end{pmatrix} A \\ &+ \begin{pmatrix} 0 \\ \mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})z} \end{pmatrix} (R_L A + T_R B)^\star \end{aligned} \tag{3.3.15}$$

and on the right hand side,

$$\begin{aligned} \Psi_{right} &= \begin{pmatrix} \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon+\frac{\omega_0}{2})z} \\ 0 \end{pmatrix} (T_L A + R_R B) \\ &+ \begin{pmatrix} 0 \\ \mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})z} \end{pmatrix} B^\star. \end{aligned} \tag{3.3.16}$$

Quantisation of these solutions is very simple since they have all the properties we need, they are continuous orthogonal and normalised. The calculations we did in the previous section shown that the reflection coefficient was equal to one in the case where $\frac{\omega_0}{2} \le M$, this case is the total reflection case since the reflected current is equal to the incident one. In order to have a orthogonal solution we need to set $A = B$ so the solution becomes continuous trough the Q ball and using the relation 3.1.12 we can show the orthogonality of solution, the terms with principal values will vanish, we can then use equal time anti-commutation relations for the Ψ-field and set A to be a standard particle operator. We shall now build a Bogoliubov transformation between the incoming operators, in the far past and the outgoing operators, in the far future. In the far past only the incoming wave survives, while in the far future only the outgoing wave survives, we find :

$$a_{out}(\epsilon - \frac{\omega_0}{2}) = R^\star a_{in}^\dagger(\epsilon + \frac{\omega_0}{2}) \tag{3.3.17}$$

We now compute the number of particles produced in the far future it is :

$$< 0|a_{out}^\dagger a_{out}|0 >= \frac{T}{2\pi}\int_{-\frac{\omega_0}{2}}^{+\frac{\omega_0}{2}} |R|^2 d\epsilon. \tag{3.3.18}$$

The $T/2\pi$ factor in front of the integral is due to the smoothing out of delta function (as in [2]), this result is the same as the one obtained in the previous section. Now we know that we we have two different formalisms leading to the same results.

3.4 Limit of current when the Q ball's size is infinite

In this section we shall compute exactly the limit of the current expression when the size of Q ball becomes infinite, i.e. $\lim_{l\to\infty} j_L$. Once it is done we can apply our results to the $3 \oplus 1$ dimensional case and obtain analytical results.

3.4.1 Direct limit in the current expression

Using the fact that :

$$\begin{aligned} 1-\alpha_\epsilon^2 &= 1 - \frac{\epsilon^2 - M^2 + \epsilon^2 + 2k_\epsilon \epsilon}{M^2} \\ &= \frac{2\epsilon(\varepsilon + k_\epsilon)}{M^2} = \frac{2\epsilon}{M}\alpha_\epsilon \end{aligned} \tag{3.4.1}$$

we can write for the current expression,

$$j_L = \frac{1}{4\pi}M + \frac{1}{2\pi}\int_M^{\frac{\omega_0}{2}} \frac{\sin^2[2k_\epsilon l]d\epsilon}{\frac{4\epsilon^2}{M^2} + 4\sin^2[2k_\epsilon l]}, \tag{3.4.2}$$

the only problem for taking the limit $l \to \infty$ is in the integral part. A few simple manipulations allows us to write the integral term in the form :

$$\frac{\sin^2[2k_\epsilon l]d\epsilon}{\frac{4\epsilon^2}{M^2} + 4\sin^2[2k_\epsilon l]} = \frac{1}{4}[1 - \frac{\epsilon^2/M^2}{\epsilon^2/M^2 + \sin^2[2k_\epsilon l]}] =,$$

$$= \frac{1}{4}[1 - \frac{1}{1 + \frac{M^2}{\epsilon^2}\sin^2[2k_\epsilon l]}]. \tag{3.4.3}$$

The problem we are left with is the calculation of the average of $\frac{1}{1+\frac{M^2}{\epsilon^2}\sin^2[2k_\epsilon l]}$ in the limit $l \to \infty$. Using the development into Taylor series gives

$$\frac{1}{1 + \alpha \sin^2[xl]} = \sum (\alpha \sin^2[xl])^n, \tag{3.4.4}$$

once we expended the sine into exponential we only keep the terms leading to non zero average that are the terms where the exponentials vanish we have

$$\begin{aligned} \sum (\sin^2[xl])^n &= \ldots + \frac{1}{2^{2n}}(-1)^n C_n^{2n} (\mathrm{e}^{ixl})^n (\mathrm{e}^{-ixl})^n + \ldots = \\ &= \ldots + \frac{1}{2^{2n}} \frac{2n!}{(n!)^2} + \ldots \end{aligned} \tag{3.4.5}$$

the dots sand for the terms having zero average. We have then

$$\begin{aligned} < \frac{1}{1 + \alpha \sin^2[xl]} > &= \sum \alpha^n \frac{1}{2^{2n}} \frac{2n!}{(n!)^2} = \\ \frac{1}{\sqrt{\pi}} \sum \alpha^n \frac{\Gamma(n + 1/2)}{n!} &= \frac{1}{\sqrt{1 + \alpha}}. \end{aligned} \tag{3.4.6}$$

Finally the expression for the current is only given by :

$$\begin{aligned} j_L &= \frac{1}{4\pi}\omega_0 + 2\int_M^{\frac{\omega_0}{2}} \frac{d\epsilon}{\sqrt{1 + \frac{M^2}{\epsilon^2}}} = \\ &= \frac{1}{4\pi}\omega_0 + 2\int_M^{\frac{\omega_0}{2}} \frac{\epsilon d\epsilon}{\sqrt{\epsilon^2 + M^2}} = \frac{1}{4\pi}\omega_0 + \frac{1}{2}\sqrt{\epsilon^2 + M^2}\Big|_M^{\frac{\omega_0}{2}} \end{aligned} \tag{3.4.7}$$

3.5 Summary of results

First of all, we found that particle production could only occur in a certain range i.e. $\epsilon \in [-\frac{\omega_0}{2}, +\frac{\omega_0}{2}]$. This range is the only one where we can build a non trivial Bogolubov transformation between operators. It is also the range where the ground state of theory is not the vacuum but a state where particles are produced. This a very important result since it gives us integration bounds and so we can nearly be sure that all kind of quantities we can compute are finite. We shall derive another version of this range in the next chapter while working with massive fermions. The reason for the existence of this range comes from the Ω Matrix that shifts the energy of some components up while it shifts the energy of the others down. The energy range is then the intersection of the two shifts (a kind of level crossing).

The second result we found and that has to be mentioned is that the particle production is in fact an evaporation since it does not depend on l at least in the limit where $\frac{\omega_0}{2} < M$ and $Ml \to \infty$ which is the case since we used a smooth step function as the Q ball profile. Finally we shall remember that the evaporation rate is constant and proportional to ω_0 for the case of an infinite Q ball in $1 \oplus 1$ dimensions. Except for the power this result is in full accordance with the literature on the subject [2, 32]. The presence of a power three in the current expression is due to the fact that the computation was done in $3 \oplus 1$ dimensions. This is the reason why we computed the energy flux far away from the Q ball.

In the last construction we made the computation of the evaporation rate using a scattering like formalism. This was done to show the equivalence of both formalisms. The main idea and difference of those two formalisms is that, in the first calculation we build the Heisenberg field operator using the fact that no particles move towards the Q ball. This first construction does not use any notion of past or future but only of far away from the Q ball and inside the Q ball so the interacting operators, inside the Q ball, can be linked to the free ones, far away from the Q ball. The second construction uses a free wave packet superposition as solution. In this last calculation the expansion coefficients are the matching coefficients that are identified to reflection and transmission amplitudes.

Chapter 4

Evaporation of Q balls into massive fermions.

The evaporation of a Q-ball into massive fermions is more complicated than the previous case. We can quite easily obtain the Heisenberg field operator but solving the evaporation condition is a difficult task, even with only one space dimension. So the method we are going to use is the same S-matrix based method used in [2, 32]. This picture will need as starting point the expression of the solution as a superposition of wave packets. It is done by expressing the motion equations in matrix form and then expanding the solutions over the eigenfunctions. This gives the separation into left and right movers. Choosing which wave is the incident one, we can write the solution as

$$\Psi_L = [B_1 \mathrm{e}^{i\bar{p}_1 x} u_{\bar{p}_1} + B_2 \mathrm{e}^{i\bar{p}_2 x} u_{\bar{p}_2} + r_1 \mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p}_1} + r_2 \mathrm{e}^{-i\bar{p}_2 x} u_{-\bar{p}_4}], \qquad (4.0.1)$$

$$\Psi_R = [t_1 \mathrm{e}^{i\bar{p}_1 x} u_{\bar{p}_1} + t_2 \mathrm{e}^{i\bar{p}_2 x} u_{\bar{p}_2}], \qquad (4.0.2)$$

where the u's and $\bar{p}$'s describe the solution away from the Q-ball and the L, R subscripts stand for the left- or right-hand side of the Q-ball. This solution has two incident waves associated with particles or anti-particles moving towards the Q-ball, giving two solutions. The reflected and transmitted waves are associated with particles moving away from the Q-ball. The same construction is done on the other side of the Q-ball, to give four solutions. Finally we obtain the total solution as a superposition of these four solutions with the expansion coefficients becoming operators. This canonical quantisation does not introduce any big problem and can be done in a straightforward way. The next step will be to consider that in the far past only the incoming wave survives, giving us a relation between the operators in the far past and in the far future (where only outgoing waves survive). The last step we shall do is to compute the number operator.

The only difficult task is the computation of reflection and transmission amplitudes appearing in the solutions. Will shall provide two methods for calculating these amplitudes. One method will consist in calculating all the scalar products of the motion eigenvectors, while the other one will consist in the diagonalisation of the motion matrices. The results are fully consistent, and the two methods serve to illustrate a variety of physical insights.

4.1 Solutions to the equations of motion

The last thing we want to do now is try and compute the particle production from a Q-Ball into massive fermions. To do so we shall apply a different method as before since we shall not use a solution in all space but rather use some matching rules. We shall need both solutions inside and outside of the Q-Ball, we shall this time quantify the free solutions (out side of Q-Ball) where their wave coefficients are given by the matching rules. This method seems complicated but it allows us to use a kind of S-Matrix formalism. The method we are going to use is exactly the one used in the last section of previous chapter.

4.1.1 Preliminaries

Using the same Lagrangian as for the massless case and adding a Dirac coupling with massive fermions, gives the Lagrangian :

$$\mathcal{L} = \bar{\psi} i\gamma^{\mu}\partial_{\mu}\psi + g(\bar{\psi}^{C}\psi\phi + h.c.) + M_D(x)\bar{\psi}\psi. \tag{4.1.1}$$

after a few manipulations we find for the equations of motion using two component ψ-field, still in $1 \oplus 1$-dimensions.

$$(i\partial_0 + i\partial_z)\psi_1 - M\mathrm{e}^{-i\frac{\omega_0}{2}t}\psi_2^{\star} + M_D\psi_2 = 0, \tag{4.1.2}$$

$$(i\partial_0 - i\partial_z)\psi_2 + M\mathrm{e}^{-i\frac{\omega_0}{2}t}\psi_1^{\star} + M_D\psi_1 = 0. \tag{4.1.3}$$

We have this time four degrees of freedom because of the double coupling. Solving this system will give us the solution inside the Q-Ball, which is the first step we need to do. To solve these equations of motion we use the following ansatz

$$\psi_1 = f_1(z)\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t} + f_2(z)\mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}, \tag{4.1.4}$$

$$\psi_2 = g_1(z)\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t} + g_2(z)\mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}. \tag{4.1.5}$$

Leading to the four equations :

$$\begin{aligned}
(-\epsilon + \frac{\omega_0}{2} + i\partial_z)f_1(z) - Mg_2^{\star}(z) + M_Dg_1(z) &= 0,\\
(\epsilon + \frac{\omega_0}{2} + i\partial_z)f_2(z) - Mg_1^{\star}(z) + M_Dg_2(z) &= 0,\\
(-\epsilon + \frac{\omega_0}{2} - i\partial_z)g_1(z) + Mf_2^{\star}(z) + M_Df_1(z) &= 0,\\
(\epsilon + \frac{\omega_0}{2} - i\partial_z)g_2(z) + Mf_1^{\star}(z) + M_Df_2(z) &= 0.
\end{aligned}$$

Conjugating the second and the fourth equation, to only have the complex conjugation in f_2 and in g_2, gives

$$(-\epsilon + \frac{\omega_0}{2} + i\partial_z)f_1(z) - Mg_2^\star(z) + M_D g_1(z) = 0, \tag{4.1.6}$$
$$(\epsilon + \frac{\omega_0}{2} - i\partial_z)f_2^\star(z) - Mg_1(z) + M_D g_2^\star(z) = 0, \tag{4.1.7}$$
$$(-\epsilon + \frac{\omega_0}{2} - i\partial_z)g_1(z) + Mf_2^\star(z) + M_D f_1(z) = 0, \tag{4.1.8}$$
$$(\epsilon + \frac{\omega_0}{2} + i\partial_z)g_2^\star(z) + Mf_1(z) + M_D f_2^\star(z) = 0. \tag{4.1.9}$$

These equations can easily be modified to reduce the numbers of parameters, we divide all equations by M. We shall now re-write these equations, taking $f_1(z) = A\mathrm{e}^{ipz}$, $f_2^\star(z) = B\mathrm{e}^{ipz}$, $g_1(z) = C\mathrm{e}^{ipz}$ and $g_2^\star(z) = D\mathrm{e}^{ipz}$. After some re-arrangement of the equations we obtain :

$$-\epsilon_- f_1 - Mg_2^\star + M_D g_1 = pf_1, \tag{4.1.10}$$
$$\epsilon_- g_1 - M^\star f_2^\star - M_D f_1 = pg_1, \tag{4.1.11}$$
$$-\epsilon_+ f_2^\star + Mg_1 - M_D g_2^\star = pf_2^\star, \tag{4.1.12}$$
$$\epsilon_+ g_2^\star + M^\star f_1 + M_D f_2^\star = pg_2^\star, \tag{4.1.13}$$

where $\epsilon_- = \epsilon - \frac{\omega_0}{2}$ and $\epsilon_+ = \epsilon + \frac{\omega_0}{2}$. This arrangement has the advantage that we can now write the ψ-field in terms of four component spinors in the way :

$$\Psi = \begin{pmatrix} \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \\ \begin{pmatrix} f_2^\star \\ g_2^\star \end{pmatrix} \end{pmatrix}. \tag{4.1.14}$$

The idea of having four component spinnors is that now the fermion field contains both energy components, just like the solution used in the previous chapter. The other advantage is that this rearrangement leads to the standard four component spinnor solution. The equations of motion become in matrix form,

$$\underbrace{\begin{pmatrix} -\epsilon_- & M_D & 0 & -M \\ -M_D & \epsilon_- & -M & 0 \\ 0 & M & -\epsilon_+ & -M_D \\ M & 0 & M_D & \epsilon_+ \end{pmatrix}}_{M_1} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = p \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}. \tag{4.1.15}$$

We shall now reduce the numbers of parameters by dividing by M we then have :

$$\begin{pmatrix} -\epsilon_- & M_D & 0 & -1 \\ -M_D & \epsilon_- & -1 & 0 \\ 0 & 1 & -\epsilon_+ & -M_D \\ 1 & 0 & M_D & \epsilon_+ \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = Mp \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}. \tag{4.1.16}$$

The fact that we have the M factor one the right hand side will allows us to simplify the space components and replace l by Ml. All the parameters we are left with now are all dimension less, we should read the matrix elements to be all divided by M and thus dimensionless. Solving this system is finding the four eigenvalues and eigenvectors of the $M1$ matrix. We can easily check that this matrix can be obtained using the standard chiral representation of Dirac matrices in $1 \oplus 1$ dimensions. When the $M \to 0$ limit is taken the spinnors just become the standard massive spinnors and the M_1 matrix gives the standard free solution.

4.1.2 Solution inside the Q ball

Before finding the eigenvectors of the M_1 matrix we can reduce the numbers of parameters by dividing everything by M. We are then left with dimension less parameters $M_D = M_D/M$ and so on for all of the parameters. This simplification is done so that in the end we shall have only two dimension less parameters M_D and ω_0. The four eigenvalues of the M_1 matrix are

$$\begin{aligned} p_{1,3} &= \pm\sqrt{\epsilon^2 + \omega^2 - (M_D^2 + 1) - 2k_\epsilon} \\ &\equiv \pm p_1 \end{aligned} \tag{4.1.17}$$

$$\begin{aligned} p_{2,4} &= \pm\sqrt{\epsilon^2 + \omega^2 - (M_D^2 + 1) + k_\epsilon} \\ &\equiv \pm p_2 \end{aligned} \tag{4.1.18}$$

with,

$$k_\epsilon = \sqrt{M_D^2 + \omega^2(\epsilon^2 - 1)} \tag{4.1.19}$$

The eigenvector coresponding to the first eigen value $+p_1$ is :

$$v_{p_1} = \begin{pmatrix} -\frac{(1+\epsilon\omega+k_\epsilon)}{\epsilon+p_1} \\ \\ \frac{\epsilon\omega^2 - k_\epsilon(\epsilon+p_1)+\omega(1+k_\epsilon-\epsilon(\epsilon+p_1))}{M_D(\epsilon+p_1)} \\ \\ -\frac{k_\epsilon+M_D^2+\omega(-\omega+p_1)}{M_D(\epsilon+p_1)} \\ \\ 1 \end{pmatrix}, \tag{4.1.20}$$

the second eigenvector corresponding to the second eigen value $+p_2$ is

$$v_{p_2} = \begin{pmatrix} -\frac{(1+\epsilon\omega-k_\epsilon)}{\epsilon+p_2} \\ \\ \frac{\epsilon\omega^2 + k_\epsilon(\epsilon+p_2)+\omega(1-k_\epsilon-\epsilon(\epsilon+p_2))}{M_D(\epsilon+p_2)} \\ \\ -\frac{-k_\epsilon+M_D^2+\omega(-\omega+p_2)}{M_D(\epsilon+p_2)} \\ \\ 1 \end{pmatrix}, \tag{4.1.21}$$

with $\omega = \frac{\omega_0}{2}$. The two last eigenvectors are given by:

$$v_{p_3} = v_{-p_1} \quad v_{p_4} = v_{-p_2}. \tag{4.1.22}$$

Inside the Q ball the static solution can be written in the form:

$$\Psi_Q = \sum_{j=1}^{4} C_j v_j \mathrm{e}^{ip_j z}, \tag{4.1.23}$$

For reasons that will become clear later on, the first two terms of this solution have positive momentum while the two last have negative momentum. This arrangement does not modify the shape or any properties of the solution. Inside the Q ball the time dependent solution is :

$$\Psi = \sum_{j=1}^{4} \mathrm{e}^{i(\epsilon-\omega)t} v_{p_i}^{up} \mathrm{e}^{i\bar{p}_i z} + \mathrm{e}^{-i(\epsilon+\omega)t} (v_{p_i}^{down} \mathrm{e}^{i\bar{p}_i z})^\star, \tag{4.1.24}$$

where the *up* superscript stands for the first two components of the eigenvectors, while the *down* one indicates we take the two last components.

4.1.3 Solution without Q ball background.

Outside the Q ball the solution is given by the eigenvalues and eigenvectors of the following matrix :

$$\underbrace{\begin{pmatrix} -\epsilon_- & M_D & 0 & 0 \\ -M_D & \epsilon_- & 0 & 0 \\ 0 & 0 & -\epsilon_+ & -M_D \\ 0 & 0 & M_D & \epsilon_+ \end{pmatrix}}_{M_0} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = p \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}. \tag{4.1.25}$$

The matrix and all its parameters are defined with respect to the division by M, so they are all dimension less. The eigenvalues are this time given by :

$$\bar{p}_{1,3} = \pm\sqrt{\epsilon_-^2 - M_D^2} \equiv \pm\bar{p}_1, \tag{4.1.26}$$

$$\bar{p}_{2,3} = \pm\sqrt{\epsilon_+^2 - M_D^2} \equiv \pm\bar{p}_2, \tag{4.1.27}$$

and the eigenvectors are this time,

$$u_{\bar{p}_1} = \begin{pmatrix} \frac{\epsilon_- - \bar{p}_1}{M_D} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_{\bar{p}_2} = \begin{pmatrix} 0 \\ 0 \\ \frac{-\epsilon_+ + \bar{p}_2}{M_D} \\ 1 \end{pmatrix}, \tag{4.1.28}$$

$$u_{\bar{p}_3} = u_{-\bar{p}_1}, \quad u_{\bar{p}_4} = u_{-\bar{p}_2} \tag{4.1.29}$$

Figure 4.1: Sketch of the possible ranges for ϵ

Once again all the parameters are dimensionless since we have to read them as being divided by M, the Majorana mass coupling. Here the $\bar{p}_{1,2}$ momentum can be complex or real. If we want some particle to propagate outside Q ball we need both $\bar{p}_{1,2}$ to be real, it gives for ϵ

$$|\epsilon_-| \geq M_D$$

to solve this we must identify two cases, the first case is,

$$\epsilon_- \geq 0 \Rightarrow \epsilon \geq \frac{\omega_0}{2}$$
$$\epsilon_- \geq M_D \Rightarrow \epsilon \geq M_D + \frac{\omega_0}{2} > \frac{\omega_0}{2}$$

the last inequality is verified if $\frac{\omega_0}{2} \geq M_D$ the second case is,

$$\epsilon_- \leq 0 \Rightarrow \epsilon \leq \frac{\omega_0}{2}$$
$$-\epsilon_- \geq M_D \Rightarrow \frac{\omega_0}{2} \geq \frac{\omega_0}{2} - M_D \geq \epsilon.$$

once more the last inequality is valid when $\frac{\omega_0}{2} \geq M_D$. A similar calculation for $|\epsilon_+| \geq M_D$ gives :

$$\epsilon \geq M_D - \frac{\omega_0}{2} \quad \text{and} \quad \epsilon \leq -M_D - \frac{\omega_0}{2} \tag{4.1.30}$$

The only way to avoid the gaps and have the two waves (both p_1 and p_2) is for ϵ to be in the range :

$$\epsilon \in [M_D - \frac{\omega_0}{2}, \frac{\omega_0}{2} - M_D]. \tag{4.1.31}$$

Where we also have :

$$\frac{\omega_0}{2} \geq M_D \tag{4.1.32}$$

This range is the equivalent as the range defined for the massless case. The solution outside the Q ball can also be written in the form

$$\Psi_0 = \sum_{j=1}^{4} (A_j, B_j) u_j \mathrm{e}^{i\bar{p}_j z}, \tag{4.1.33}$$

this time the B_j coefficients are on the left-hand side of Q ball while the A_j are on the right hand side. The time dependent solution is once more given by,

$$\Psi = \sum_{j=1}^{4} \mathrm{e}^{i(\epsilon-\omega)t} u_{p_i}^{up} \mathrm{e}^{i\bar{p}_i z} + \mathrm{e}^{-i(\epsilon+\omega)t} (u_{p_i}^{down} \mathrm{e}^{i\bar{p}_i z})^{\star}, \tag{4.1.34}$$

4.1.4 Symmetry and Normalisation

In order to have all the properties we need to perform quantisation, these eigenvectors need some orthogonality properties to find them we need to take a look at the symmetry of the matrices M_0 and M_1, a quick look and a few simple calculations show that :

$$\tau M_{0,1} \tau = M_{0,1}^T, \tag{4.1.35}$$

where

$$\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{4.1.36}$$

$$\tau^2 = \mathbb{1}, \tag{4.1.37}$$

and the T superscript stands for transposition of matrices. Starting from the standard relations for the eigenvectors we can write :

$$M\Psi_i = p_i \Psi_i$$
$$M\Psi_j = p_j \Psi_j$$

Taking the transpose of the first relation gives :

$$\Psi_i^T M^T = p_i \Psi_i^T,$$

and multiplying on the right by τ we obtain,

$$\Psi_i^T M^T \tau = p_i \Psi_i^T \tau$$
$$\Psi_i^T \tau M = p_i \Psi_i^T \tau$$

where we used the symmetry relation for the last step. We can now multiply by ψ_j on both sides to obtain the orthogonality relation of the eigenvectors we have :

$$\Psi_i^T \tau \underbrace{M \Psi_j}_{p_j \Psi_j} = p_i \Psi_i^T \tau \Psi_j,$$

this final relation leads to the orthogonality relation of the eigenvectors defined by :

$$u_i^T \tau u_j = v_i^T \tau v_j = \tau_{ii}, \tag{4.1.38}$$

Orthogonality of vectors is easy to verify, even if the calculations involved might seem difficult, every thing is straightforward. The normalisation constants needed to obtain the proper form containing the ones in the diagonal matrix τ are simply given by :

$$N_i = \tau_{ii}[v_{i1}^2 - v_{i2}^2 + v_{i3}^2 - v_{i4}^2]. \tag{4.1.39}$$

The first subscript stands for the eigenvector while the second one stands for the component. The same definition stands for the u_j eigenvectors:

$$\bar{N}_i = \tau_{ii}[u_{i1}^2 - u_{i2}^2 + u_{i3}^2 - u_{i4}^2]. \tag{4.1.40}$$

The solutions are now with the normalised vectors :

$$\Psi_Q = \sum_{j=1}^{4} C_j \frac{v_j}{\sqrt{N_j}} \mathrm{e}^{ip_j z}, \tag{4.1.41}$$

$$\Psi_0 = \sum_{j=1}^{4} (A_j, B_j) \frac{u_j}{\sqrt{\bar{N}_j}} \mathrm{e}^{i\bar{p}_j z}. \tag{4.1.42}$$

To simplify the writing we can redefine the eigenvectors in the way :

$$\tilde{u}_j = \frac{u_j}{\sqrt{\bar{N}_j}}, \tag{4.1.43}$$

$$\tilde{v}_j = \frac{v_j}{\sqrt{N_j}} \tag{4.1.44}$$

so the solutions are

$$\Psi_Q = \sum_{j=1}^{4} C_j \tilde{v}_j \mathrm{e}^{ip_j z}, \tag{4.1.45}$$

$$\Psi_0 = \sum_{j=1}^{4} (A_j, B_j) \tilde{u}_j \mathrm{e}^{\frac{i}{\sqrt{2}}\bar{p}_j z}. \tag{4.1.46}$$

The way we normalised the vectors is just a choice we made, one could easily imagine other ways to normalise the eigenvectors as long as we keep the orthogonality relations. We shall reconsider later on the choices we can make to normalise these vectors, the proper choice can be made in order to have orthogonality of the solutions. At this point with the solution we have we can construct a conserved current. Equations of motion 4.1.6-4.1.9 can be written in terms of M_1 matrix and Ψ-field, in the form :

$$i\partial_z \Psi = -M_1 \Psi, \tag{4.1.47}$$
$$-i\partial_z \Psi^\dagger = -\Psi^\dagger M_1^T. \tag{4.1.48}$$

Using these two relations we shall construct the conserved current we write,

$$\begin{aligned}
\partial_z[\Psi^\dagger o \Psi] &= \partial_z \Psi^\dagger o \Psi + \Psi^\dagger o \partial_z \Psi, \\
&= -i\Psi^\dagger M_1^T o \Psi + i\Psi^\dagger o M_1 \Psi, \\
&= i\Psi^\dagger [o M_1 - M_1^T o] \Psi,
\end{aligned}$$

Using now the relation 4.1.35 we can write :

$$\partial_z[\Psi^\dagger \tau \Psi] = 0. \tag{4.1.49}$$

This expression is the conserved current that will give the unitarity conditions for the S-Matrix. This current can be used to construct a normalisable solution, but as we shall see later on it is simpler to consider the anti-commutation relations of the fermionic operators.

4.2 Construction of diffusion matrix

We want to construct the matrix linking the solution at $z = -\infty$ to the solution at $z = +\infty$. We are searching for the matrix:

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = \mathcal{V} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \tag{4.2.1}$$

4.2.1 Matching in space

We first start by matching the solutions at $z = -l$ we have:

$$\begin{aligned}
B_1 u_{\bar{p}_1} \mathrm{e}^{-i\bar{p}_1 l} + B_2 u_{\bar{p}_2} \mathrm{e}^{-i\bar{p}_2 l} + B_3 u_{\bar{p}_3} \mathrm{e}^{-i\bar{p}_3 l} + B_4 u_{\bar{p}_4} \mathrm{e}^{-i\bar{p}_4 l} = \\
C_1 v_{p_1} \mathrm{e}^{-ip_1 l} + C_2 v_{p_2} \mathrm{e}^{-ip_2 l} + C_3 v_{p_3} \mathrm{e}^{-ip_3 l} + C_4 v_{p_4} \mathrm{e}^{-ip_4 l}
\end{aligned} \tag{4.2.2}$$

We redefine the B_i and the C_i in the way

$$\tilde{B}_i = B_i \mathrm{e}^{-i\bar{p}_i l}, \tag{4.2.3}$$
$$\tilde{C}_i = \frac{C_i}{\sqrt{N_i}} \tag{4.2.4}$$

Using the redefinition for the B's and the redefinition of the C's we can write, multiplying equation 4.2.11 by $\tilde{u}_i^T \tau$:

$$\tilde{B}_i u_{p_i}^T \tau u_{p_i} = \sum_{j=1}^{4} u_i^T \tau v_j \mathrm{e}^{-ip_j l} \tilde{C}_j, \tag{4.2.5}$$

Doing the same for all the B's and writing down all the relations in matrix form we obtain :

$$U \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \\ \tilde{B}_4 \end{pmatrix} = S\mathcal{E} \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \\ \tilde{C}_3 \\ \tilde{C}_4 \end{pmatrix}, \tag{4.2.6}$$

with,

$$S = \begin{pmatrix} u_1^T \tau v_1 & u_1^T \tau v_2 & u_1^T \tau v_3 & u_1^T \tau v_4 \\ u_2^T \tau v_1 & u_2^T \tau v_2 & u_2^T \tau v_3 & u_2^T \tau v_4 \\ u_3^T \tau v_1 & u_3^T \tau v_2 & u_3^T \tau v_3 & u_3^T \tau v_4 \\ u_4^T \tau v_1 & u_4^T \tau v_2 & u_4^T \tau v_3 & u_4^T \tau v_4 \end{pmatrix}, \tag{4.2.7}$$

$$\mathcal{E} = \begin{pmatrix} \mathrm{e}^{-ip_1 l} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-ip_2 l} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-ip_3 l} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-ip_4 l} \end{pmatrix}, \tag{4.2.8}$$

and

$$U = \begin{pmatrix} u_{\bar{p}_1}^T u_{\bar{p}_1} & 0 & 0 & 0 \\ 0 & u_{\bar{p}_2}^T u_{\bar{p}_2} & 0 & 0 \\ 0 & 0 & u_{\bar{p}_3}^T u_{\bar{p}_3} & 0 \\ 0 & 0 & 0 & u_{\bar{p}_4}^T u_{\bar{p}_4} \end{pmatrix}, \tag{4.2.9}$$

We can then write for the expression we obtain at $z =_l$

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = U^{-1} S\mathcal{E} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ {}_4 \end{pmatrix}. \tag{4.2.10}$$

At $z = +l$ we have:

$$A_1 u_{\bar{p}_1} \mathrm{e}^{-i\bar{p}_1 l} + A_2 u_{\bar{p}_2} \mathrm{e}^{-i\bar{p}_2 l} + A_3 u_{\bar{p}_3} \mathrm{e}^{-i\bar{p}_3 l} + A_4 u_{\bar{p}_4} \mathrm{e}^{-i\bar{p}_4 l} = \\ C_1 v_{p_1} \mathrm{e}^{-ip_1 l} + C_2 v_{p_2} \mathrm{e}^{-ip_2 l} + C_3 v_{p_3} \mathrm{e}^{-ip_3 l} + C_4 v_{p_4} \mathrm{e}^{-ip_4 l} \tag{4.2.11}$$

We use the same redefinition as for the B's, leads this time to :

$$U \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = S\mathcal{E}' \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}, \tag{4.2.12}$$

with

$$\mathcal{E} = \begin{pmatrix} e^{ip_1 l} & 0 & 0 & 0 \\ 0 & e^{ip_2 l} & 0 & 0 \\ 0 & 0 & e^{ip_3 l} & 0 \\ 0 & 0 & 0 & e^{ip_4 l} \end{pmatrix}, \tag{4.2.13}$$

Mixing up these two relations we obtain for the total transformation matrix $\mathcal{V}$ the following relation :

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} = U^{-1} S\mathcal{E}\mathcal{E}'^{-1} S^{-1} U \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}. \tag{4.2.14}$$

A little more calculation shows that,

$$\mathcal{E}\mathcal{E}'^{-1} = \begin{pmatrix} e^{-2ip_1 l} & 0 & 0 & 0 \\ 0 & e^{-2ip_2 l} & 0 & 0 \\ 0 & 0 & e^{-2ip_3 l} & 0 \\ 0 & 0 & 0 & e^{-2ip_4 l} \end{pmatrix} \equiv E \tag{4.2.15}$$

where we can easily show that :

$$[E, \tau]_- = 0. \tag{4.2.16}$$

As we shall find out later on the last form is a transformation that allows us to diagonalise the M_1 matrix. Using this matrix we shall construct all the reflection and diffusion coefficients for all the waves moving inside and outside of the Q ball. Before we continue we need to remember that $p_3 = -p_1$ and $p_4 = -p_2$ for both sets of p's (bared ones and no bar ones). We see here that the choice for normalisation of eigenvectors will just act on the U matrix that can be either the identity or the τ matrix or any other choice we can make. Finally the diffusion matrix V we where looking for is given by :

$$V = U^{-1} S E S^{-1} U. \tag{4.2.17}$$

4.2.2 Construction of reflection and transmission amplitudes

We shall first construct the reflection and transmission amplitudes from the *left side to the right hand side* of the Q ball. To do so we shall use the definition of the $\mathcal{V}$ matrix given by equation 4.2.1. As we already mentioned the first two coefficients are linked to positive moving waves while the last two coefficients are linked to negative moving waves. Due to the shape of the u spinnors and the Ω matrix in front the first and the third coefficients of the free solution have the same energy while the second and the fourth coefficients correspond to another energy wave. We shall identify these two energy ranges to the type one particles (1) and type two particles (2). Using equation 4.2.1 and separating the matrix into four two by two blocs we can write :

$$\begin{pmatrix} \rightarrow \\ \rightarrow \\ r_1 \\ r_2 \end{pmatrix} = \left(\begin{array}{c|c} \mathcal{V}_{11} & \mathcal{V}_{12} \\ \hline \mathcal{V}_{21} & \mathcal{V}_{22} \end{array}\right) \begin{pmatrix} t_1 \\ t_2 \\ 0 \\ 0 \end{pmatrix}, \tag{4.2.18}$$

where the two arrows stand for the incoming waves, the first two coefficients will be replaced by one. Using the bloc separation of the matrix we find :

$$\begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix} = \mathcal{V}_{11} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \tag{4.2.19}$$

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \mathcal{V}_{21} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \tag{4.2.20}$$

leading to

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \underbrace{\mathcal{V}_{11}^{-1}}_{\mathcal{T}} \begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix}, \tag{4.2.21}$$

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \underbrace{\mathcal{V}_{21}\mathcal{V}_{11}^{-1}}_{\mathcal{R}} \begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix}. \tag{4.2.22}$$

The $\mathcal{R}$ and $\mathcal{T}$ matrices give the reflection and transmission amplitudes when they are applied on the incoming wave coefficients. These two matrices are two by two the first line corresponding to transmission or reflection of particles with the two different incoming waves, while the second line gives the coefficients for anti-particles. We shall construct the transmission and reflection coefficients from the *right to the left hand side* of Q ball. Using the same method as before we have this time :

$$\begin{pmatrix} 0 \\ 0 \\ \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} = \left(\begin{array}{c|c} \mathcal{V}_{11} & \mathcal{V}_{12} \\ \hline \mathcal{V}_{21} & \mathcal{V}_{22} \end{array}\right) \begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \leftarrow \\ \leftarrow \end{pmatrix}, \tag{4.2.23}$$

leading this time to

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathcal{V}_{11} \begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{pmatrix} + \mathcal{V}_{12} \begin{pmatrix} \leftarrow \\ \leftarrow \end{pmatrix}, \tag{4.2.24}$$

$$\begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} = \mathcal{V}_{21} \begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{pmatrix} + \mathcal{V}_{22} \begin{pmatrix} \leftarrow \\ \leftarrow \end{pmatrix}. \tag{4.2.25}$$

Solving these two equations gives the reflection and transmission matrices for incoming particles from the left, they are :

$$\tilde{\mathcal{R}} = -\mathcal{V}_{12}\mathcal{V}_{11}^{-1}, \tag{4.2.26}$$

$$\tilde{\mathcal{T}} = \mathcal{V}_{22} - \mathcal{V}_{21}\mathcal{V}_{11}^{-1}\mathcal{V}_{12} \tag{4.2.27}$$

The next thing to do will be to compute the reflection and transmission coefficients from inside the Q ball.

4.2.3 Reflection and transmission amplitudes on the Q ball's surface

Using the results of matching at $x = -l$ we can write

$$\begin{pmatrix} \rightarrow \\ \rightarrow \\ r_1 \\ r_2 \end{pmatrix} = \underbrace{U^{-1}S\mathcal{E}}_{\mathcal{M}} \begin{pmatrix} t_1 \\ t_2 \\ 0 \\ 0 \end{pmatrix}, \tag{4.2.28}$$

leading to

$$\begin{pmatrix} \rightarrow \\ \rightarrow \\ r_1 \\ r_2 \end{pmatrix} = \left(\begin{array}{c|c} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \hline \mathcal{M}_{21} & \mathcal{M}_{22} \end{array} \right) \begin{pmatrix} t_1 \\ t_2 \\ 0 \\ 0 \end{pmatrix}, \tag{4.2.29}$$

giving for the reflection and transmission matrices :

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \mathcal{M}_{11}^{-1} \begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix} = T_l \begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix} \tag{4.2.30}$$

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \mathcal{M}_{21}\mathcal{M}_{11}^{-1} \begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix} = R_l \begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix}. \tag{4.2.31}$$

These R_l and T_l matrices represent the reflection and transmission amplitudes on the left boundary of the Q ball for incident particles coming from the left hand side of the Q ball. We shall now compute the amplitudes when incident particles come from inside the Q ball, we write :

$$\begin{pmatrix} 0 \\ 0 \\ \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} = \left(\begin{array}{c|c} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \hline \mathcal{M}_{21} & \mathcal{M}_{22} \end{array}\right) \begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \leftarrow \\ \leftarrow \end{pmatrix}, \tag{4.2.32}$$

leading to

$$\begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{pmatrix} = \underbrace{-\mathcal{M}_{11}^{-1}\mathcal{M}_{12}}_{\tilde{R}_l} \begin{pmatrix} \leftarrow \\ \leftarrow \end{pmatrix}, \tag{4.2.33}$$

and

$$\begin{aligned} \begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} &= \mathcal{M}_{21} \begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{pmatrix} + \mathcal{M}_{22} \begin{pmatrix} \leftarrow \\ \leftarrow \end{pmatrix} \\ &= \underbrace{(-\mathcal{M}_{21}\mathcal{M}_{11}^{-1}\mathcal{M}_{12} + \mathcal{M}_{22})}_{\tilde{T}_l} \begin{pmatrix} \leftarrow \\ \leftarrow \end{pmatrix} \end{aligned} \tag{4.2.34}$$

We now have the reflection and transmission amplitudes on the left boundary for particles coming from inside or outside of the Q ball. We now need these coefficients on the other boundary if we want to use them for later calculations, the same construction we stand on the other boundary, we write :

$$\begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \\ 0 \\ 0 \end{pmatrix} = \underbrace{U^{-1}S\mathcal{E}'}_{\bar{\mathcal{M}}} \begin{pmatrix} \rightarrow \\ \rightarrow \\ \tilde{r}_1 \\ \tilde{r}_2 \end{pmatrix}, \tag{4.2.35}$$

for particles coming from inside the Q ball, leading to the same development,

$$\begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \\ 0 \\ 0 \end{pmatrix} = \left(\begin{array}{c|c} \bar{\mathcal{M}}_{11} & \bar{\mathcal{M}}_{12} \\ \hline \bar{\mathcal{M}}_{21} & \bar{\mathcal{M}}_{22} \end{array}\right) \begin{pmatrix} \rightarrow \\ \rightarrow \\ \tilde{r}_1 \\ \tilde{r}_2 \end{pmatrix}, \tag{4.2.36}$$

giving for result

$$\begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} = \bar{\mathcal{M}}_{11}^{-1} \begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix} = \tilde{T}_r \begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix} \tag{4.2.37}$$

$$\begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{pmatrix} = \bar{\mathcal{M}}_{11}^{-1}\bar{\mathcal{M}}_{21} \begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix} = \tilde{R}_r \begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix}. \tag{4.2.38}$$

The tilde has been used to indicate that we consider the coefficients for particles travelling form inside to outside of the Q ball. We shall now compute the last pair of coefficients, those for particles moving towards the Q ball. We have this time :

$$\begin{pmatrix} 0 \\ 0 \\ t_1 \\ t_2 \end{pmatrix} = \left(\begin{array}{c|c} \bar{\mathcal{M}}_{11} & \bar{\mathcal{M}}_{12} \\ \hline \bar{\mathcal{M}}_{21} & \bar{\mathcal{M}}_{22} \end{array}\right) \begin{pmatrix} \leftarrow \\ \leftarrow \\ r_1 \\ r_2 \end{pmatrix}, \tag{4.2.39}$$

finally giving for the last coefficients :

$$\begin{aligned} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} &= \underbrace{\bar{\mathcal{M}}_{12}^{-1}\bar{\mathcal{M}}_{11}}_{R_r} \begin{pmatrix} \leftarrow \\ \leftarrow \end{pmatrix} \\ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} &= \underbrace{(\bar{\mathcal{M}}_{21} - \bar{\mathcal{M}}_{22}\bar{\mathcal{M}}_{12}^{-1}\bar{\mathcal{M}}_{11})}_{T_r} \begin{pmatrix} \leftarrow \\ \leftarrow \end{pmatrix}. \end{aligned} \tag{4.2.40}$$

All these calculations need some more explanations. We computed four pairs of reflection and transmission coefficients, two on the left boundary, with the l subscript, and two on the right boundary, the r subscript. We also had to consider the origin of the incident particle, if its origin was outside the Q ball or inside it, having the tilde expressions. All of these calculations where motivated the fact that the solution to the equations of motion depend on four parameters, the four column vectors. As we mentioned in the preceding section it was better to arrange the eigen-vectors, in the way that the first two correspond to positive movers.

Figure 4.2: Sketch of both cases used to build the solution : we have each time two incident particles, two reflected and two transmitted. It is an effect of massive particles because we can not identify any more the particles with the anit-particles.

4.3 Construction of solution

4.3.1 Standard solution

Using the transmission and reflection coefficients we can identify two different cases the first case is when incident particles are on the left hand side of the Q ball, while the other case stands for incident particles coming from the right hand side (see fig. 4.2). We shall treat separately the solution on the left and the solution on the right, the matching coefficients are those found in the previous section, they link the expansion coefficients on the right to those on the left. Writing down these two possibilities we have :

$$\Psi_L = [B_1 \mathrm{e}^{i\bar{p}_1 x} u_{\bar{p_1}} + B_2 \mathrm{e}^{i\bar{p}_2 x} u_{\bar{p_2}} + r_1 \mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p_1}} + r_2 \mathrm{e}^{-i\bar{p}_2 x} u_{-p_4}], \tag{4.3.1}$$

$$\Psi_R = [t_1 \mathrm{e}^{i\bar{p}_1 x} u_{\bar{p_1}} + t_2 \mathrm{e}^{i\bar{p}_2 x} u_{\bar{p_2}}], \tag{4.3.2}$$

for the first case, the two incident particles coming from the left hand side of the Q ball and

$$\Psi_L = [\tilde{t}_1 \mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p_1}} + \tilde{t}_2 \mathrm{e}^{-i\bar{p}_2 x} u_{-\bar{p_2}}], \tag{4.3.3}$$

$$\Psi_R = [\tilde{r}_1 \mathrm{e}^{i\bar{p}_1 x} u_{\bar{p_1}} + \tilde{r}_2 \mathrm{e}^{i\bar{p}_2 x} u_{\bar{p_2}} + A_1 \mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p_1}} + A_2 \mathrm{e}^{-i\bar{p}_2 x} u_{-\bar{p_2}}], \tag{4.3.4}$$

for the second case. In both of these definitions we have :

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \mathcal{R} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \mathcal{T} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{4.3.5}$$

$$\begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{pmatrix} = \tilde{\mathcal{R}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad \begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} = \tilde{\mathcal{T}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \tag{4.3.6}$$

To clearly understand the construction, the B's are the incident amplitudes from the left while the A's are the amplitudes from the right. The 1 and 2 subscript indicate the type of particle we are dealing with, we have two different exponentials in $\Omega(t)$. We then need to take the complex conjugate of the terms corresponding to the two last components of spinnors. To continue building the solution we still need to separate each of these two cases in two, considering only one type of incident particle at the time. This construction leads to the four following pieces, that will be identified to the four degrees of freedom that our solution has :

$$\Psi_L = [\mathrm{e}^{i\bar{p}_1 x} u_{\bar{p}_1} + r_{11}\mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p}_1}] + [r_{12}\mathrm{e}^{-i\bar{p}_2 x} u_{-\bar{p}_2}], \tag{4.3.7}$$
$$\Psi_R = [t_{11}\mathrm{e}^{i\bar{p}_1 x} u_{\bar{p}_1}] + [t_{12}\mathrm{e}^{i\bar{p}_2 x} u_{\bar{p}_2}], \tag{4.3.8}$$

for the one incident type one particle from the left and

$$\Psi_L = [r_{21}\mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p}_2}] + [\mathrm{e}^{i\bar{p}_2 x} u_{\bar{p}_2} + r_{22}\mathrm{e}^{-i\bar{p}_2 x} u_{-\bar{p}_2}], \tag{4.3.9}$$
$$\Psi_R = [t_{21}\mathrm{e}^{i\bar{p}_1 x} u_{\bar{p}_1}] + [t_{22}\mathrm{e}^{-i\bar{p}_2 x} u_{\bar{p}_2}], \tag{4.3.10}$$

for an incident type two particle. The coefficients are given by :

$$\begin{pmatrix} r_{11} \\ r_{12} \end{pmatrix} = \mathcal{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} t_{11} \\ t_{12} \end{pmatrix} = \mathcal{T}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.3.11}$$
$$\begin{pmatrix} r_{21} \\ r_{22} \end{pmatrix} = \mathcal{R}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} t_{21} \\ t_{22} \end{pmatrix} = \mathcal{T}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.3.12}$$

The two other pieces for particles incident from the right we have :

$$\Psi_L = [\tilde{t}_{11}\mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p}_1}] + [\tilde{t}_{12}\mathrm{e}^{-i\bar{p}_2 x} u_{-\bar{p}_2}], \tag{4.3.13}$$
$$\Psi_R = [\tilde{r}_{11}\mathrm{e}^{i\bar{p}_1 x} u_{\bar{p}_1} + \mathrm{e}^{-i\bar{p}_1 x} u_3] + [\tilde{r}_{12}\mathrm{e}^{-i\bar{p}_2 x} u_2], \tag{4.3.14}$$

for one incident type one particle from the right and

$$\Psi_L = [\tilde{t}_{21}\mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p}_1}] + [\tilde{t}_{22}\mathrm{e}^{-i\bar{p}_2 x} u_{-\bar{p}_2}], \tag{4.3.15}$$
$$\Psi_R = [\tilde{r}_{21}\mathrm{e}^{i\bar{p}_1 x} u_{\bar{p}_1}] + [\tilde{r}_{22}\mathrm{e}^{-i\bar{p}_2 x} u_{\bar{p}_2} + \mathrm{e}^{-i\bar{p}_2 x} u_{-\bar{p}_2}], \tag{4.3.16}$$

for an incident type two particle and finally the coefficients are given by :

$$\begin{pmatrix} \tilde{r}_{11} \\ \tilde{r}_{12} \end{pmatrix} = \tilde{\mathcal{R}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} \tilde{t}_{11} \\ \tilde{t}_{12} \end{pmatrix} = \tilde{\mathcal{T}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.3.17}$$
$$\begin{pmatrix} \tilde{r}_{21} \\ \tilde{r}_{22} \end{pmatrix} = \tilde{\mathcal{R}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} \tilde{t}_{21} \\ \tilde{t}_{22} \end{pmatrix} = \tilde{\mathcal{T}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.3.18}$$

If we want to easily remember the coefficients there is an easy trick. For the coefficients without the tilde we read the subscript from the left to the right, r_{21} is the coefficient for an incident type two particle being reflected as a type one particle and for the tilde ones the reading is the same except that this time the particle are incident from the left.

4.3.2 Quantisation and Bogoliubov transformations

Quantisation of solution is now easy, the total quantised solution will be a linear combination of all four parts, with expansion coefficients becoming operators after the normalisation is made. The solution is given by :

$$\Psi = \sum_{j=1}^{4} \mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t} u_{p_i}^{up} \mathrm{e}^{i\bar{p}_i z} + \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t} (u_{p_i}^{down} \mathrm{e}^{i\bar{p}_i z})^\star, \tag{4.3.19}$$

in our case only the eigenvectors corresponding to the $\bar{p}_1$ eigenvalue have *up* components, while only the eigenvectors corresponding to $\bar{p}_2$ have down components.

$$\Psi_L = \mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}[\mathrm{e}^{i\bar{p}_1 x} u_{\bar{p}_1}^{up} + r_{11}\mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p}_1}^{up}] + \mathrm{e}^{i(\epsilon+\frac{\omega_0}{2})t}[r_{12}\mathrm{e}^{-i\bar{p}_2 x} u_{\bar{p}_2}], \tag{4.3.20}$$

incident particle is the wave containing $u_{\bar{p}_1}$, all this superposition must be represented using the same operator so we are sure to have only four degrees of freedom. Quantisation will be done using energy, for the incident wave we have :

$$\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t} u_1 \to \epsilon - \frac{\omega_0}{2} \geq M_D, \tag{4.3.21}$$

$$\to \epsilon \geq M_D + \frac{\omega_0}{2}, \tag{4.3.22}$$

leading for this first wave to

$$\Psi_L = \int_{M_D+\frac{\omega_0}{2}}^{\infty} d\epsilon \{\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}[\mathrm{e}^{i\bar{p}_1 x} u_1^{up} + r_{11}\mathrm{e}^{-i\bar{p}_1 x} u_3^{up}] + \mathrm{e}^{i(\epsilon+\frac{\omega_0}{2})t}[r_{12}\mathrm{e}^{-i\bar{p}_2 x}(u_{-\bar{p}_2}^{down})^\star]\} b^\dagger(\bar{p}_1), \tag{4.3.23}$$

after the conjugation of the term proportional to $\mathrm{e}^{i(\epsilon+\frac{\omega_0}{2})t}$ we have,

$$\Psi_L = \int_{M_D+\frac{\omega_0}{2}}^{\infty} d\epsilon \{\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}[\mathrm{e}^{i\bar{p}_1 x} u_{\bar{p}_1}^{up} + r_{11}\mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p}_1}^{up}] b_{\bar{p}_1}^\dagger + \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}[r_{12}^\star \mathrm{e}^{i\bar{p}_2^\star x}(u_{-\bar{p}_2}^{down})^\star] b_{\bar{p}_1}\}. \tag{4.3.24}$$

Applying the same method to all the terms we finally obtain for the total solution having four degrees of freedom :

$$\begin{aligned}
\Psi = & \int_{M_D+\frac{\omega_0}{2}}^{\infty} d\epsilon \{\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}[\mathrm{e}^{i\bar{p}_1 x} u_{\bar{p}_1}^{up} + r_{11}\mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p}_1}^{up}] b_{\bar{p}_1}^\dagger + \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}[r_{12}^\star \mathrm{e}^{i\bar{p}_2^\star x}(u_{-\bar{p}_2}^{down})^\star] b_{\bar{p}_1}\}, \\
+ & \int_{M_D-\frac{\omega_0}{2}}^{\infty} d\epsilon \{\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}[r_{21}\mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p}_1}^{up}] a_{\bar{p}_2}^\dagger + \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}[\mathrm{e}^{-i\bar{p}_2^\star x} u_{\bar{p}_2}^\star + r_{22}^\star \mathrm{e}^{i\bar{p}_2^\star x}(u_{-\bar{p}_2}^{down})^\star] a_{\bar{p}_2}\}, \\
+ & \int_{M_D+\frac{\omega_0}{2}}^{\infty} d\epsilon \{\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}[\tilde{t}_{11}\mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p}_1}^{up}] b_{-\bar{p}_1}^\dagger + \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}[\tilde{t}_{12}^\star \mathrm{e}^{i\bar{p}_2^\star x}(u_{-\bar{p}_2}^{down})^\star] b_{-\bar{p}_1}\}, \\
+ & \int_{M_D-\frac{\omega_0}{2}}^{\infty} d\epsilon \{\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}[\tilde{t}_{21}\mathrm{e}^{-i\bar{p}_1 x} u_{-\bar{p}_1}^{up}] a_{-\bar{p}_2}^\dagger + \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}[\tilde{t}_{22}^\star \mathrm{e}^{i\bar{p}_2^\star x}(u_{-\bar{p}_2}^{down})^\star] a_{-\bar{p}_2}\}.
\end{aligned}$$

The upper integration bound is $\frac{\omega_0}{2} - M_D$ so we are only left with the terms containing the a operators,

$$\begin{aligned}\Psi_L &= \int_{M_D-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}-M_D} d\epsilon\{\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}[r_{21}\mathrm{e}^{-i\bar{p}_1 x}u^{up}_{-\bar{p}_1}]a^\dagger_{\bar{p}_2} + \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}[\mathrm{e}^{-i\bar{p}_2^\star x}(u^{down}_{\bar{p}_2})^\star + r^\star_{22}\mathrm{e}^{i\bar{p}_2^\star x}(u^{down}_{-\bar{p}_2})^\star]a_{\bar{p}_2}\}, \\ &+ \int_{M_D-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}-M_D} d\epsilon\{\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}[\tilde{t}_{21}\mathrm{e}^{-i\bar{p}_1 x}u^{up}_{-\bar{p}_1}]a^\dagger_{-\bar{p}_2} + \mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}[\tilde{t}^\star_{22}\mathrm{e}^{i\bar{p}_2^\star x}(u^{down}_{-\bar{p}_2})^\star]a_{-\bar{p}_2}\}.\end{aligned} \tag{4.3.25}$$

One interesting result can be found here is that like in the massless case if we change the sign of $\frac{\omega_0}{2}$ we change the particle type since we change the operator type. For the moment the a coefficients are only expansion coefficients since we have not quantised the wave yet. At $t = +\infty$ only the terms without any incident wave will survive we have then

$$r^\star_{22}a(\bar{p}_2) + \tilde{t}^\star_{22}a(-\bar{p}_2) + r_{21}a^\dagger(\bar{p}_2) + \tilde{t}_{12}a^\dagger(-\bar{p}_2) = a_{out}(\bar{p}_2), \tag{4.3.26}$$

this is the Bogoliubov transformation we where looking for. If we want the a_{out} coefficient to be a operator we need to check that it satisfies the same anti-commutation relations as a_{in}. We have :

$$\begin{aligned}\{(a^\dagger_{out})', a_{out}\} &= \left((r^\star_{21})'a_{\bar{p}_2'} + (r^\star_{22})'a^\dagger_{\bar{p}_2'} + (\tilde{t}^\star_{21})'a_{-\bar{p}_2'} + (\tilde{t}^\star_{22})'a^\dagger_{-\bar{p}_2'}\right) \\ &\times \left((r_{12})a^\dagger_{\bar{p}_2} + (r_{22})a_{\bar{p}_2} + (\tilde{t}_{21})a^\dagger_{-\bar{p}_2} + (\tilde{t}_{22})a_{-\bar{p}_2}\right) \\ &= \left((r^\star_{21})'r_{21} + (r^\star_{22})'r_{22} + (\tilde{t}^\star_{21})'\tilde{t}_{21} + (\tilde{t}^\star_{22})'\tilde{t}_{22}\right)\{(a^\dagger_{in})', a_{in}\}\end{aligned}$$

This relation can also be obtained if we set that the incident current is equal to the outgoing one, or even with the normalisation of wave packets. At this stage it can be important to use some normalised eigenvectors, as will shall show later on it is always the case if we diagonalise the matrix outside the Q ball. The number of created particles is now given by

$${}_{in} < 0|a^\dagger_{out}a_{out}|0 >_{in} = \left(\frac{|r_{21}|^2 + |\tilde{t}_{21}|^2}{|r_{21}|^2 + |r_{22}|^2 + |\tilde{t}_{21}|^2 + |\tilde{t}_{22}|^2}\right)\delta(\epsilon - \epsilon'). \tag{4.3.27}$$

We need to smooth out this result, to do so we shall use the same argument as [2] to finally obtain

$$\frac{dN}{dt} = \frac{1}{2\pi}\int_{M_D-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}-M_D}\left(\frac{|r_{21}|^2 + |\tilde{t}_{21}|^2}{|r_{21}|^2 + |r_{22}|^2 + |\tilde{t}_{21}|^2 + |\tilde{t}_{22}|^2}\right)d\epsilon. \tag{4.3.28}$$

Since we are dealing with a Bogoliubov transformation we have

$$\left(\frac{|r_{21}|^2 + |\tilde{t}_{21}|^2}{|r_{21}|^2 + |r_{22}|^2 + |\tilde{t}_{21}|^2 + |\tilde{t}_{22}|^2}\right) \leq 1 \tag{4.3.29}$$

$$\frac{dN}{dt} \leq \omega_0 - 2M_D \tag{4.3.30}$$

In fact the best thing to do is to consider the solution in all space on the left and on the right instead of considering only one side. To do so we just need to consider an incident particle on the left and build the solution without tilde factors. The rest of the procedure is the same we have,

$$\Psi_L = \int_{M_D-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}-M_D} d\epsilon\{e^{i(\epsilon-\frac{\omega_0}{2})t}[r_{21}e^{-i\bar{p}_1 x}u^{up}_{-\bar{p}_1}]a^{\dagger}_{\bar{p}_2} + e^{-i(\epsilon+\frac{\omega_0}{2})t}[e^{-i\bar{p}_2^{\star}x}(u^{down}_{\bar{p}_2})^{\star} + r_{22}^{\star}e^{i\bar{p}_2^{\star}x}(u^{down}_{-\bar{p}_2})^{\star}]a_{\bar{p}_2}\},$$
$$+ \int_{M_D-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}-M_D} d\epsilon\{e^{i(\epsilon-\frac{\omega_0}{2})t}[t_{21}e^{i\bar{p}_1 x}u^{up}_{\bar{p}_1}]a^{\dagger}_{-\bar{p}_2} + e^{-i(\epsilon+\frac{\omega_0}{2})t}[t_{22}^{\star}e^{-i\bar{p}_2^{\star}x}(u^{down}_{\bar{p}_2})^{\star}]a_{-\bar{p}_2}\}, \tag{4.3.31}$$

this time leading to

$$\frac{dN}{dt} = \frac{1}{2\pi}\int_{M_D-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}-M_D}\left(\frac{|r_{21}|^2+|t_{21}|^2}{|r_{21}|^2+|r_{22}|^2+|t_{21}|^2+|t_{22}|^2}\right)d\epsilon, \tag{4.3.32}$$

after normalisation of operators. These results seem to be correct because when the fermions become massless there is identification of both types of produced particles so the total coefficient becomes equal to one as in the previous chapter and there is total reflection. What we shall now as a last verification do a few variable changes[1], first we shall do the change $\epsilon - \frac{\omega_0}{2} = \epsilon'$ in the terms containing $e^{i(\epsilon-\frac{\omega_0}{2})t}$ and the change $\epsilon + \frac{\omega_0}{2} = \epsilon'$ in the ones containing $e^{i(\epsilon+\frac{\omega_0}{2})t}$ we obtain:

$$\Psi_1 = \int_{-M_D}^{-\omega_0+M_D} d\epsilon\{e^{i(\epsilon')t}[r_{21}(\epsilon'+\frac{\omega_0}{2})e^{-i\bar{p}_1(\epsilon'+\frac{\omega_0}{2})x}u^{up}_{-p_1(\epsilon'\bar{+}\frac{\omega_0}{2})}]a^{\dagger}_{\bar{p}_2(\epsilon'+\frac{\omega_0}{2})}$$
$$+ e^{i(\epsilon')t}[\tilde{t}_{12}(\epsilon'+\frac{\omega_0}{2})e^{-i\bar{p}_1(\epsilon'+\frac{\omega_0}{2})x}u^{up}_{-\bar{p}_1(\epsilon'+\frac{\omega_0}{2})}]a^{\dagger}_{-\bar{p}_2(\epsilon'+\frac{\omega_0}{2})}\} \tag{4.3.33}$$

$$\Psi_2 = \int_{M_D}^{\omega_0-M_D} d\epsilon\{e^{-i(\epsilon')t}[e^{-i\bar{p}_2^{\star}(\epsilon'-\frac{\omega_0}{2})x}(u^{down}_{\bar{p}_2(\epsilon'-\frac{\omega_0}{2})})^{\star}$$
$$+ r_{22}^{\star}(\epsilon'-\frac{\omega_0}{2})e^{i\bar{p}_2^{\star}(\epsilon'-\frac{\omega_0}{2})x}(u^{down}_{-\bar{p}_2(\epsilon'-\frac{\omega_0}{2})})^{\star}]a_{\bar{p}_2(\epsilon'-\frac{\omega_0}{2})}$$
$$+ e^{-i(\epsilon')t}[\tilde{t}_{22}^{\star}(\epsilon'-\frac{\omega_0}{2})e^{i\bar{p}_2^{\star}(\epsilon'-\frac{\omega_0}{2})x}(u^{down}_{-\bar{p}_2(\epsilon'-\frac{\omega_0}{2})})^{\star}]a_{-\bar{p}_2(\epsilon'-\frac{\omega_0}{2})}\}. \tag{4.3.34}$$

Remembering the expressions for $p_{1,2}$ we see that this variable change identifies p_1 and p_2 so we write,

$$\Psi_1 = \int_{-\omega_0+M_D}^{-M_D} d\epsilon\{e^{i(\epsilon')t}[r_{21}(\epsilon'+\frac{\omega_0}{2})e^{-i\bar{p}x}u^{up}_{-\bar{p}}]a^{\dagger}_{\bar{p}}$$
$$+ e^{i(\epsilon')t}[\tilde{t}_{12}(\epsilon'+\frac{\omega_0}{2})e^{-i\bar{p}x}u^{up}_{-\bar{p}}]a^{\dagger}_{-\bar{p}}\} \tag{4.3.35}$$

[1]These variable changes will help us the next chapter to study particle diffusion on a Q ball.

$$\Psi_2 = \int_{M_D}^{\omega_0 - M_D} d\epsilon\{e^{-i(\epsilon')t}[e^{-i\bar{p}^\star x}(u_{\bar{p}}^{down})^\star + r_{22}^\star(\epsilon' - \frac{\omega_0}{2})e^{i\bar{p}^\star x}(u_{-\bar{p}}^{down})^\star]a_{\bar{p}}$$
$$+ \ e^{-i(\epsilon')t}[\tilde{t}_{22}^\star(\epsilon' - \frac{\omega_0}{2})e^{i\bar{p}^\star x}(u_{-\bar{p}}^{down})^\star]a_{-\bar{p}}\}, \qquad (4.3.36)$$

with $\bar{p} = \sqrt{(\epsilon')^2 - M_D^2}$. We shall now change the sign of ϵ' in the first integral to obtain,

$$\Psi_1 = \int_{M_D}^{+\omega_0 - M_D} d\epsilon\{e^{-i(\epsilon')t}[r_{21}(-\epsilon' + \frac{\omega_0}{2})e^{-i\bar{p}x}u_{-\bar{p}}^{up}]a_{\bar{p}}^\dagger$$
$$+ \ e^{-i(\epsilon')t}[\tilde{t}_{12}(-\epsilon' + \frac{\omega_0}{2})e^{-i\bar{p}x}u_{-\bar{p}}^{up}]a_{-\bar{p}}^\dagger\} \qquad (4.3.37)$$

$$\Psi_2 = \int_{M_D}^{\omega_0 - M_D} d\epsilon\{e^{-i(\epsilon')t}[e^{-i\bar{p}^\star x}(u_{\bar{p}}^{down})^\star + r_{22}^\star(\epsilon' - \frac{\omega_0}{2})e^{i\bar{p}^\star x}(u_{-\bar{p}}^{down})^\star]a_{\bar{p}}$$
$$+ \ e^{-i(\epsilon')t}[\tilde{t}_{22}^\star(\epsilon' - \frac{\omega_0}{2})e^{i\bar{p}^\star x}(u_{-\bar{p}}^{down})^\star]a_{-\bar{p}}\}. \qquad (4.3.38)$$

The last thing we shall do is replace a_{-p} by $a_p^\dagger$ and $a_{-p}^\dagger$ by a_p, so we finally obtain,

$$\Psi_1 = \int_{M_D}^{+\omega_0 - M_D} d\epsilon\{e^{-i(\epsilon')t}[r_{21}(-\epsilon' + \frac{\omega_0}{2})e^{-i\bar{p}x}u_{-\bar{p}}^{up} + \tilde{t}_{22}^\star(\epsilon' - \frac{\omega_0}{2})e^{i\bar{p}^\star x}(u_{-\bar{p}}^{down})^\star]a_{\bar{p}}^\dagger$$
$$+ \ e^{-i(\epsilon')t}[\tilde{t}_{12}(-\epsilon' + \frac{\omega_0}{2})e^{-i\bar{p}x}u_{-\bar{p}}^{up} + e^{-i\bar{p}^\star x}(u_{\bar{p}}^{down})^\star + r_{22}^\star(\epsilon' - \frac{\omega_0}{2})e^{i\bar{p}^\star x}(u_{-\bar{p}}^{down})^\star]a_{\bar{p}}\}. \qquad (4.3.39)$$

In most cases after these variable changes are done the *up* and *down* components of eigenvectors identify, so the solution has an even simpler form. This work was done to link complicated solution we had with a simpler one. We now clearly see that particle production produces only one type of particles, fermions in our case, the energy range is bounded and starts on the mass shell. The thing we notice is that the incoming wave is proportional to u_{-p} while the outgoing one is proportional to u_p. These eigenvectors act like projectors on the incoming and outgoing operators, this the reason why we normalised both states separately, this orthogonality ensures us also the anticommutation of *in* and *out* operators. The last thing to do would be to pass from ϵ' to p for the quantisation. This last variable change will separate both energy ranges and double the solution, which means we shall have to find some selection rules to avoid double counting of the solutions. To keep the solution in a simple shape we shall continue working using the energy as integration variable, the only interesting component of the momentum will be in the radius direction, so the approximation of a one dimensional problem will be the most realistic one. When we shall to the extension to $3 \oplus 1$ dimensions since we will consider the case where we are far away from the Q ball this approximation will be quite satisfactory.

4.3.3 Simplifications

The important simplification we wish to make is the limit $l \to \infty$, if this limit can be done we are in a case where the Q ball becomes a wall. This limit has also the advantage that the extension to $3 \oplus 1$ dimension becomes a simple thing. The other thing we to need to care about while doing this limit is the place in our calculations where the l appears, it is only in the inside propagator ($\mathcal{E}$ eq. 4.2.8). Due to the shape of this propagator when the eigenvalues of the momentum are complex we only have increasing and decreasing exponentials, so the limit is a trivial thing to take we set the decreasing exponentials to zero and with a normalisation we eliminate the increasing ones. Taking a look at the definition of the eigenvalues inside the Q ball $p_{1,2,3,4}$,

$$\begin{aligned} p_{1,3} &= \pm\sqrt{\epsilon^2 + \Omega_0 - (M_D^2 + 1) - 2k_\epsilon} \\ &\equiv \pm p_1 \qquad (4.3.40) \\ p_{2,4} &= \pm\sqrt{\epsilon^2 + \Omega_0 - (M_D^2 + M^2) + 2k_\epsilon} \\ &\equiv \pm p_2 \qquad (4.3.41) \end{aligned}$$

with,

$$k_\epsilon = \sqrt{M_D^2 + \Omega_0^2(\epsilon^2 - 1)}, \qquad (4.3.42)$$

it is easy to find the range where they are complex. If k_ϵ is complex the eigenvalues become complex so the exponential in the propagator will become real and a part of it will go to zero when we the limit $l \to \infty$ is done. We have :

$$M_D^2 + \Omega_0^2(\epsilon^2 - M^2) \leq 0 \qquad (4.3.43)$$

$$|\epsilon| \leq \sqrt{1 - \frac{M_D^2}{\Omega_0^2}}, \qquad (4.3.44)$$

since we have $\Omega_0 \geq M_D$ the term on the right is always real, in this range we shall introduce

$$k_\varepsilon = i\sqrt{\Omega_0^2(1 - \epsilon^2) - M_D^2} = ik'_\epsilon, \qquad (4.3.45)$$

$$p_{1,3} = \pm(a - ik'_\epsilon)^{\frac{1}{2}} \quad a = \epsilon^2\Omega^2 - M_D^2 - 1, \qquad (4.3.46)$$

with k'_ϵ always positive. We can continue the simplifications by writing

$$a + ik'_\epsilon = r(\cos(\phi) + i\sin(\phi)) \qquad (4.3.47)$$

$$r = a^2 + \kappa'^2_\epsilon; \quad \phi = \arcsin(k'_\epsilon/r). \qquad (4.3.48)$$

where $\phi \in [0; -\pi]$ since k'_ϵ is positive ($p_{1,3}$ are in the lower half complex plane). We then obtain for $p_{1,3}$

$$p_{1,3} = \pm\sqrt{r}(\cos(\phi/2) - i\sin(\phi/2)), \qquad (4.3.49)$$

and for $p_{2,4}$

$$p_{2,4} = \pm\sqrt{r}(\cos(\phi/2) + i\sin(\phi/2)). \tag{4.3.50}$$

where $\sin(\phi/2) \in [0;1]$. With these simplifications and the limit $l \to \infty$ the propagator is just :

$$\mathcal{E} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-i\gamma l} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{i\gamma l} & 0 \\ 0 & 0 & 0 & \end{pmatrix}, \tag{4.3.51}$$

with $\gamma = \sqrt{r}\cos(\phi/2)$ and the L factor is still here but as we shall see later on it will disapear, the normalisation is done by multiplying the propagator by $\mathrm{e}^{\sqrt{r}\sin(\phi/2)l}$ was done to eliminate the increasing exponentials. When we shall use the symmetry of the different matrices this simplification will be of great help to obtain simple and easy to understand results. In fact we do not explicitly need l to become infinite but just big enough so the exponentials can be neglected, since we deal with square exponentials they decrease even faster then we need.

4.4 Direct construction of S Matrix

Using the shape of the different matrices we deal with we think there is a simpler way to construct the diffusion matrix. In fact all the matrices of motion equations can be diagonalised using simple transformations that preserve the symmetry of the problem. If the matrices can be diagonalised the eigenvectors will have automatic orthogonality and normalisation properties.

4.4.1 Diagonalisation of matrices

Taking a look at the M_0 matrix defined in eq. 4.1.25 we can diagonalise it using the Lorentz boost-transformation :

$$M_0' = \tau v_1^T \tau M_0 v_1, \tag{4.4.1}$$

with,

$$\begin{aligned} v_1 &= \begin{pmatrix} \cosh(x_1) & \sinh(x1) & 0 & 0 \\ \sinh(x_1) & \cosh(x_1) & 0 & 0 \\ 0 & 0 & \cosh(x_2) & -\sinh(x_2) \\ 0 & 0 & -\sinh(x_2) & \cosh(x_2) \end{pmatrix}, & (4.4.2) \\ v_1^T \tau v_1 &= \tau. & (4.4.3) \end{aligned}$$

The last equation ensures us the fact that $\tau v^T \tau = v^{-1}$ and that the symmetry of the problem is conserved. Setting x_1 and x_2 being solutions of :

$$\cosh(2x_1) = \frac{\epsilon_- \sinh(2x_1)}{M_D}, \tag{4.4.4}$$

$$\cosh(2x_2) = \frac{\epsilon_+ \sinh(2x_2)}{M_D}, \tag{4.4.5}$$

we find for the M_0' matrix the following diagonal form,

$$M_0' = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & -k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & -k_2 \end{pmatrix}, \tag{4.4.6}$$

where

$$k_1 = \frac{(M_D^2 - \epsilon_-^2)\sinh(x_1)}{M_D}, \tag{4.4.7}$$

$$k_2 = \frac{(M_D^2 - \epsilon_+^2)\sinh(x_2)}{M_D}. \tag{4.4.8}$$

All the parameters we find after this transformation are real, the k's that we find represent the momentum of the particles in this new base. The same transformation applied on the M_1 matrix defined by eq. 4.1.15 gives :

$$M_1' = \begin{pmatrix} k_1 & 0 & \sinh(x) & -\cosh(x) \\ 0 & -k_1 & -\cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) & k_2 & 0 \\ \cosh(x) & \sinh(x) & 0 & -k_2 \end{pmatrix}, \tag{4.4.9}$$

with $x = x_1 + x_2$. We can check that this new matrix has the same symmetry properties as M_1. This simple transformation allows us to eliminate the Dirac coupling of our equations but in presence of the Q ball it is replaced by a double Q ball coupling. Since this transformation is made every where, it will not change any of the properties. A solution of equations 4.4.4 and 4.4.5 is easy to construct it is :

$$\frac{\cosh(2x_1)}{\sinh(2x_1)} = \coth(2x_1) = \frac{\epsilon_-}{M_D}$$

$$2x_1 = \operatorname{argcoth}(\frac{\epsilon_-}{M_D}) \tag{4.4.10}$$

After this transformation is made inside and outside of the Q ball the eigenvectors of the solution outside the Q ball become :

$$u_{k_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_{-k_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_{k_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_{-k_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{4.4.11}$$

the important thing here is that the M_0' matrix is now self adjoint so its eigenvectors have the standard orthogonality properties without the τ matrix. The other important thing is that we do not need anymore to normalise the eigenvectors. When we do the matching in space eq. 4.2.11 instead of multiplying by $u_i^T\tau$ we multiply by $u_i^\dagger$ and the S matrix is made of the components of M_1' eigenvectors, so the only thing we did was to diagonalise the transformed matrix. To continue the Diagonalisation process we now transform M_1' in the way :

$$M_1'' = \tau(s_1 s_2)^T \tau M_1'(s_1 s_2) \tag{4.4.12}$$

with,

$$s_1 = \begin{pmatrix} \cosh[y/2] & 0 & 0 & \sinh[y/2] \\ 0 & \cosh[y/2] & -\sinh[y/2] & 0 \\ 0 & -\sinh[y/2] & \cosh[y/2] & 0 \\ \sinh[y/2] & 0 & 0 & \cosh[y/2] \end{pmatrix}, \tag{4.4.13}$$

$$s_2 = \begin{pmatrix} \cos[z/2] & 0 & \sin[z/2] & 0 \\ 0 & \cos[z/2] & 0 & -\sin[z/2] \\ -\sin[z/2] & 0 & \cos[z/2] & 0 \\ 0 & \sin[z/2] & 0 & \cos[z/2] \end{pmatrix}. \tag{4.4.14}$$

As before we have :

$$(s_1 s_2)^T \tau (s_1 s_2) = \tau, \tag{4.4.15}$$

to preserve the symmetry of the problem. This set of transformations looks more complicated then the simple boost we used to start, it is the case for the parameters we shall need to use but it is of great use for the final simplifications and results. Setting y and z to be solutions of :

$$\sin(z) = \frac{-2\cos(z)\cosh(y)\sinh(x)}{k_1 - k_2}, \tag{4.4.16}$$

$$\sinh(y) = \frac{2\cosh(y)\cosh(x)}{k_1 + k_2}, \tag{4.4.17}$$

$$\Rightarrow \tan(z) = \frac{-2\cosh(y)\sinh(x)}{k_1 - k_2} \tag{4.4.18}$$

$$\Rightarrow \tanh(y) = \frac{-2\cosh(x)}{k_1 + k_2} \tag{4.4.19}$$

we have

$$M_1'' = \begin{pmatrix} A & -\bar{M} & 0 & 0 \\ \bar{M} & -A & 0 & \\ 0 & 0 & B & \bar{M} \\ 0 & 0 & -\bar{M} & -B \end{pmatrix}, \tag{4.4.20}$$

with,

$$A = \frac{1}{2}\left((k_1 - k_2)\cos(z) + (k_1 + k_2 - \frac{4\cosh(x)^2}{k_1 + k_2})\cosh(y) + \frac{4\cos(z)\sinh^2(x)\cosh^2(y)}{k_1 - k_2}\right), \tag{4.4.21}$$

$$B = \frac{1}{2}\left((-k_1 + k_2)\cos(z) + (k_1 + k_2 - \frac{4\cosh(x)}{k_1 + k_2})\cosh(y) - \frac{4\cos(z)\sinh^2(x)\cosh^2(y)}{k_1 - k_2}\right), \tag{4.4.22}$$

$$\bar{M} = \frac{\cosh(y)\sinh(2x)}{k_1 + k_2}. \tag{4.4.23}$$

Taking a look at this M_1'' matrix we see it has the same form as the M_0 matrix so we shall diagonalise it using the same boost transformation. This time the transformation will not be a boost since some parameters can be complex. In fact before going any further we have to find the solution to equation 4.4.19 that night be complex, $k_1 + k_2$ is small so the fraction on the right hand side is always bigger than one. We shall have to set $y = i\frac{\pi}{2} + \eta$ it gives,

$$\begin{aligned}
\frac{e^y - e^{-y}}{e^y + e^{-y}} &= A \\
\frac{e^{i\frac{\pi}{2}\eta} - e^{-i\frac{\pi}{2}\eta}}{e^{i\frac{\pi}{2}\eta} + e^{-i\frac{\pi}{2}\eta}} &= A \\
\frac{e^{\eta} + e^{-\eta}}{e^{\eta} - e^{-\eta}} &= A \\
\eta &= \text{argcoth}[A]
\end{aligned} \tag{4.4.24}$$

Finally to finish the diagonalisation we transform using the v_1 matrix :

$$\begin{aligned}
M_1''' &= \tau v_1^T \tau M_1'' v_1 \\
&= \tau v_1^T \tau \tau v_2^T \tau M_1' v_2 v_1 \\
&= \tau v_3^T \tau M_1' v_3,
\end{aligned} \tag{4.4.25}$$

with $v_3 = v_2 v_1$. This last transformation can also be done using a slightly different matrix the v_1' matrix defined by :

$$v_1' = \begin{pmatrix} \cosh(a) & \sinh(a) & 0 & 0 \\ \sinh(a) & \cosh(a) & 0 & 0 \\ 0 & 0 & \cosh(b) & -\sinh(b) \\ 0 & 0 & -\sinh(b) & \cosh(b) \end{pmatrix}, \tag{4.4.26}$$

$$v_1'^T \tau v_1' = \tau. \tag{4.4.27}$$

We finally have for the M_1''' matrix the form

$$M_1''' = \begin{pmatrix} \xi_1 & 0 & 0 & 0 \\ 0 & -\xi_1 & 0 & 0 \\ 0 & 0 & \xi_2 & 0 \\ 0 & 0 & 0 & -\xi_2 \end{pmatrix}, \tag{4.4.28}$$

with

$$\xi_1 = \frac{(A^2 - \bar{M}^2)\sinh(2a)}{\bar{M}}, \tag{4.4.29}$$

$$\xi_2 = \frac{(B^2 - \bar{M}^2)\sinh(2b)}{\bar{M}}, \tag{4.4.30}$$

where

$$\cosh[2a] = \frac{A\sinh[2a]}{\bar{M}} \tag{4.4.31}$$

$$\cosh[2b] = \frac{B\sinh[2a]}{\bar{M}}. \tag{4.4.32}$$

Using these transformation the diffusion matrix $\mathcal{V}$ can be expressed in terms of the diagonalization matrices in the way :

$$\mathcal{V} = \tau(s_1 s_2 v_1')^T \tau E(s_1 s_2 v_1') \tag{4.4.33}$$

This form will be in fact far more simple then all the other possible ones, so this is the reason why we decided to use it rather then the form with the scalar products of the eigenvectors. What we do is exactly the same since we work in a base where the matrix of motion equations (4.1.15) is diagonal. If we keep the scalar products with the τ matrix we see that the only to vectors having negative values are the negative moving ones, with a simple calculation we could link the τ matrix to the helicity operator.

Small sized Q balls

Using the results of previous section we where able to compute all the amplitudes for small sized Q balls. The method used was to replace the exponentials in the E matrix by : $(1 - ip_{1,2,3,4}l)$. These amplitudes still have complicated expressions but all of them exept t_{22} are proportional to the size parameter $Ml \equiv l$. In the limit where l goes to zero all the amplitudes fall to zero except t_{22} going to one. The t_{22} amplitude representing the probability of a fermion remaining a fermion. This probability is obviously one if the Q ball disappears. If the size parameter is small the amplitudes will become proportional l^2 as for massless particle production. This quadratic behaviour is shown on figure 4.3.

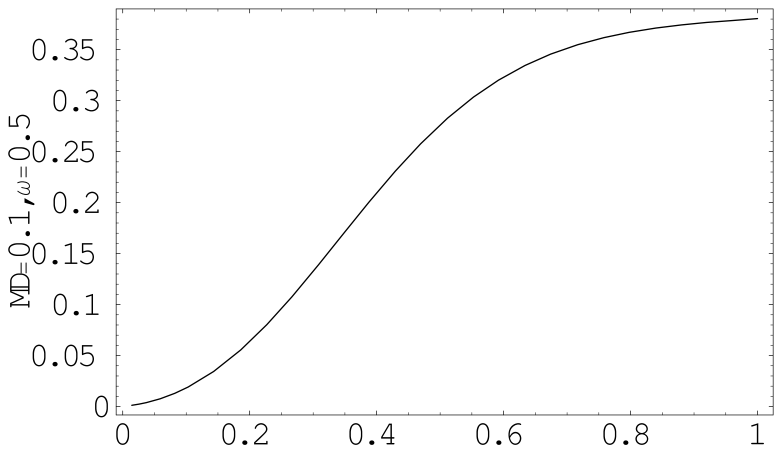

Figure 4.3: Particle production rate for small values of the size parameter.

4.5 Results

4.5.1 Approximations

A little more calculations shows :

$$|\epsilon| \leq \sqrt{1 - \frac{M_D^2}{\Omega_0^2}},$$

$$\epsilon \in [0; \frac{\omega_0}{2} - M_D] \Rightarrow 0 \leq \epsilon \leq \frac{\omega_0}{2} - M_D \leq \sqrt{1 - \frac{M_D^2}{\Omega_0^2}},$$

$$\epsilon \in [M_D - \frac{\omega_0}{2}; 0] \Rightarrow 0 \geq \epsilon \geq M_D - \frac{\omega_0}{2} \geq -\sqrt{1 - \frac{M_D^2}{\Omega_0^2}}.$$

With the 4.4.33 matrix and these simplifications we can compute the particle production coefficients we obtain :

$$r_{21} = -\frac{1}{\mathrm{cotg}(z)} \left(\cos(z)\sinh(A - B) + \sinh(A + B)\right) \tag{4.5.1}$$

$$r_{22} = \cosh(A - B)\mathrm{cotg}(z) - \frac{\cosh(A + B)}{\mathrm{cotg}(z)} \tag{4.5.2}$$

$$\tilde{t}_{12} = \tilde{t}_{22} = 0 \tag{4.5.3}$$

This result might seem strange but it is normal. If we remember that the tilde transmission coefficients where transmission from the left-hand side of Q ball, to the right-hand side. After the limit $l \to \infty$ is taken there is no more left- or right-hand side of the Q ball there is only a wall separating space into two domains. It was the major interest of taking the limit in the Q ball's size, the results are much simpler and we can do the extrapolation to three dimensions. In the case where we have :

$$1 \geq \frac{M_D - \frac{\omega_0}{2}}{\sqrt{1 - \frac{4M_D^2}{\omega_0^2}}} \tag{4.5.4}$$

the particle production rate which is an evaporation rate since there is no more l dependence, is given by :

$$\frac{dN}{dt} = \frac{1}{4\pi} \int_{M_D - \frac{\omega_0}{2}}^{\frac{\omega_0}{2} - M_D} \frac{|r_{21}|^2}{|r_{21}|^2 + |r_{22}|^2} d\epsilon \tag{4.5.5}$$

This integration has been done numerically with a mathematical software and the results we obtained are shown in figure 4.4. The result with $M_D = 0$ was done by replacing the integrand by one, since when the fermions are massless there is identification of both reflection amplitudes and it simplifies with the denominator. For the massive case we see that the heavier the fermions get the less are produced. The fact that heavy fermions are produced in lower quantities that light ones simply reflects the idea that they need more energy to be created and therefor appear in fewer number. The other main result is that there is no creation possible for values of $\frac{\omega_0}{2}$ lower than the values of the Dirac mass. The proportionality of evaporation rate and the Q-ball's internal frequency ω_0, reflects the fact that the fermions are produced from the desintegration of the scalar particles making the Q-ball.

4.5.2 Results of numerical integration

We first tested the stability of production rate in function of size to see if like in the previous case the particle production rate becomes constant and stable for big values of the size. If the production becomes constant above a certain size then we do not need to care about complex averaging processes. Figure 4.6 shows the stability of evaporation rate for large Q balls. The little oscillations are due to numerical instabilities that vanish for very big values of size. We can now compute the evaporation rate for values of the Dirac mass smaller then the Majorana mass (the coupling inside the Q ball). The next task we need to do is test the stability of our computations when the fermion mass parameter is bigger then the Majorana coupling inside the Q ball, it is the case when $M_D \geq 1$. This case shows exactly the same behaviour of the other one except for the fact that it takes more computer time to obtain the plot of evaporation. The results are on figure 4.8. A quick analysis of these results shows that there is a superior limit for all parameter sets, this limit does not depend on the mass parameter. Is seems to be normal since an infinite Q ball with an infinite internal frequency

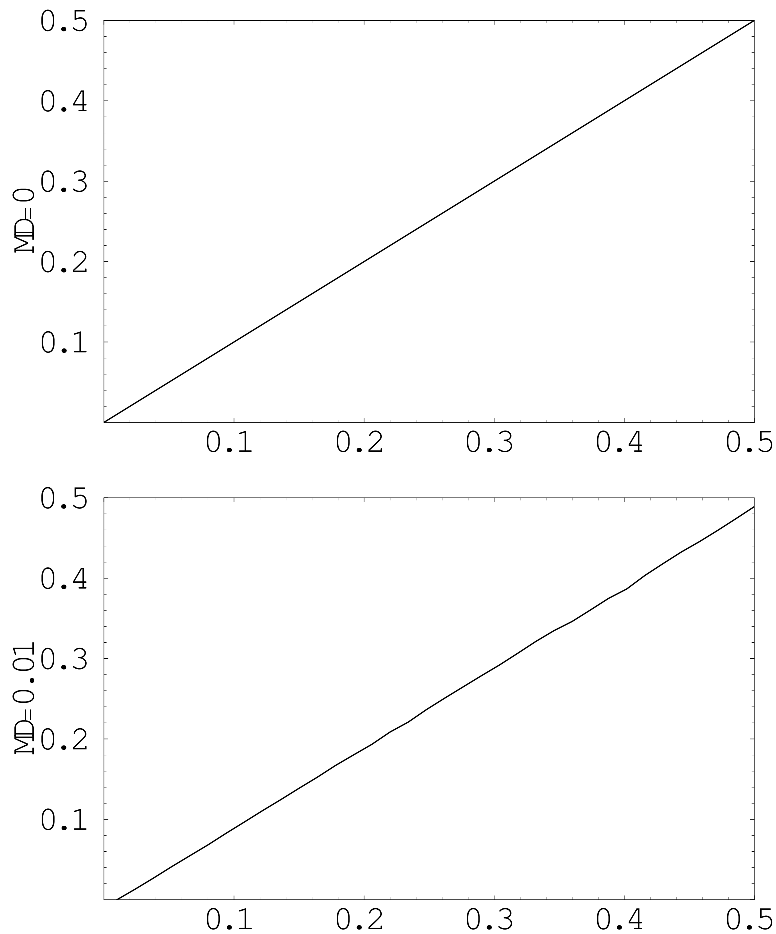

Figure 4.4: Evaporation rate of Q ball for different values of the parameter $\frac{M_D}{M}$ in function of $\frac{\omega_0}{2M}$, the first figure with $M_D = 0$ was obtained using the same program as the one for chapter three, while the $M_D = 0.01$ value was obtained using the massive solution.

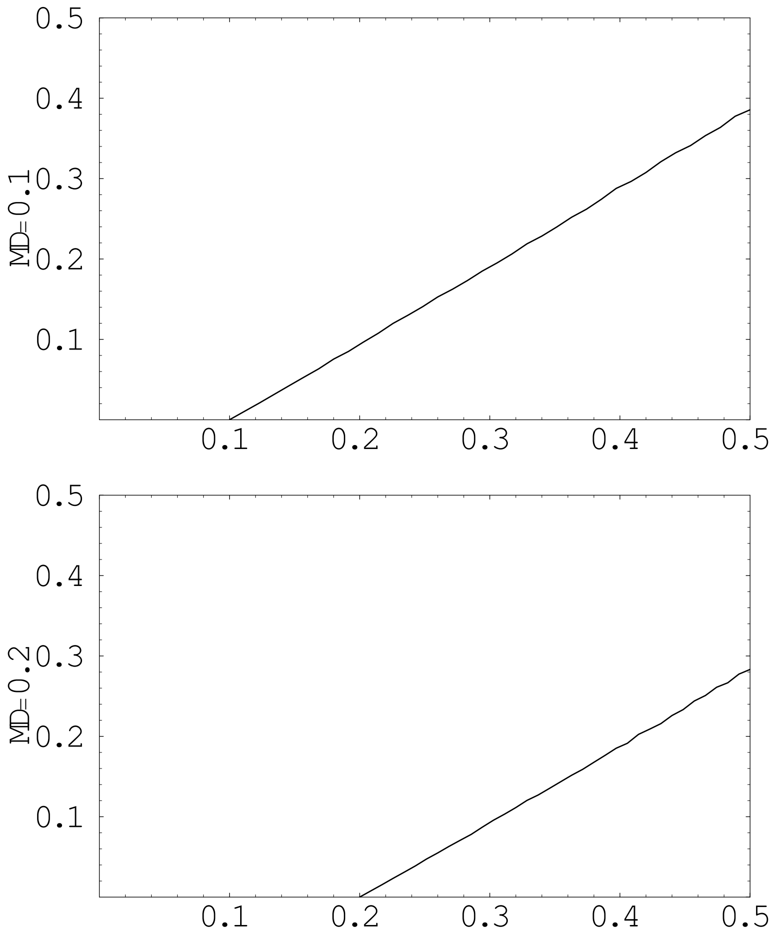

Figure 4.5: Evaporation rate of Q ball for different values of the parameter $\frac{M_D}{M}$ in function of $\frac{\omega_0}{2M}$.

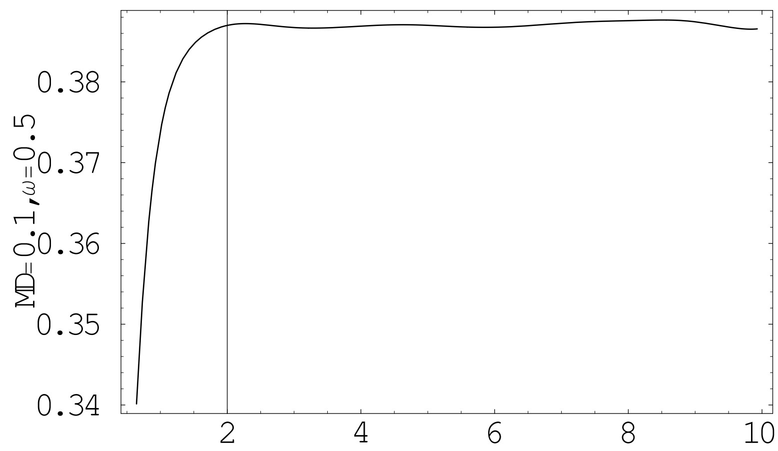

Figure 4.6: Particle production rate for small values of the size parameter.

can produce any mass fermions. The last words we shall say about these results is that the "angle" in the curve correspond to the value of the frequency where the imaginary part of the impulsion inside the Q ball becomes zero, it is the point where particles start to propagate inside the Q ball. The normalisation of evaporation by its upper bound will lead to the same shape as the massless case but there will be a gap from zero to the value of the fermion mass.

4.6 Energy flux far away from the Q ball

The last step we need to achieve is compute the energy flux far away from the Q ball it is done by considering the flux through a sphere surrounding the Q ball. As before if the observer is far away from the Q ball the only important dimension is the distance to the Q ball. We have[2],

$$\frac{dE}{M dt d\sigma} = \int_{-\frac{\omega_0}{2M}+\frac{M_D}{M}}^{+\frac{\omega_0}{2M}-\frac{M_D}{M}} \left(\frac{|r_{21}|^2 + |t_{21}|^2}{|r_{21}|^2 + |r_{22}|^2 + |t_{21}|^2 + |t_{22}|^2} \right) \bar{\epsilon}^2 d\bar{\epsilon}, \qquad (4.6.1)$$

the transmission amplitudes disappear when the Q ball's size is very big. This integration can be done numerically and we can also introduce the Q ball's size to see its influence on the energy flux. The only difference is that a small Q ball will produce less energy for until the value of the frequency parameter becomes big. We could normalise these figures with the absolute upper bound. This normalisation does not introduce any new features since the normalised curve for very big Q ball would start with a constant part to then fall down. For the small one we will not find any constant part in the normalised curve.

[2]We shall discuss more precisely this construction in chapter 5

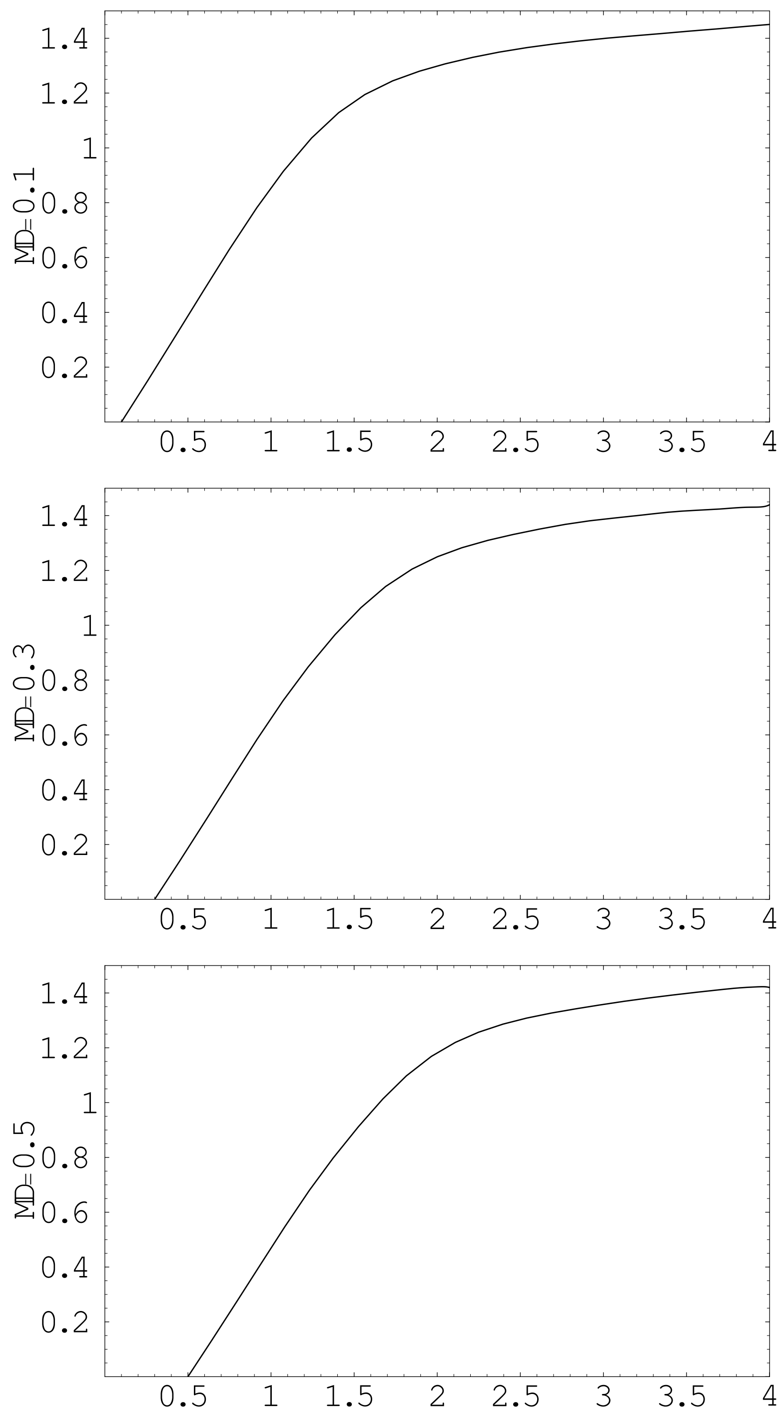

Figure 4.7: Evaporation rate, $\frac{2\pi dN}{Mdt}$ for infinite (very big) Q balls in function of the frequency parameter for different values of the fermion mass parameter.

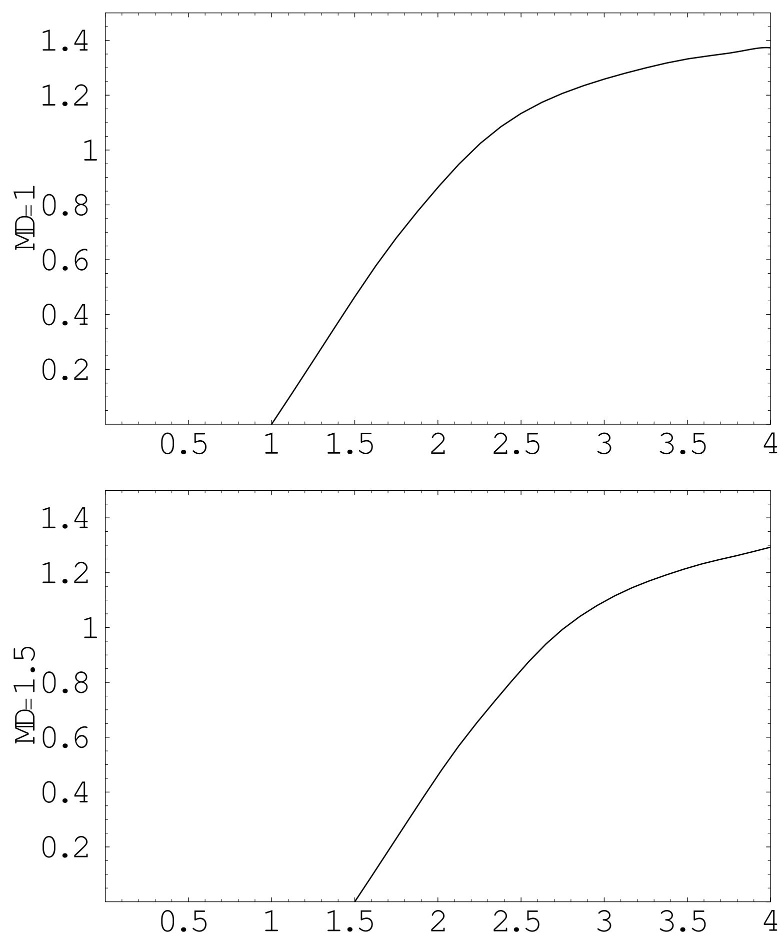

Figure 4.8: Evaporation rate, $\frac{2\pi dN}{Mdt}$ for infinite (very big) Q balls in function of the frequency parameter for different values of the fermion mass parameter bigger then one.

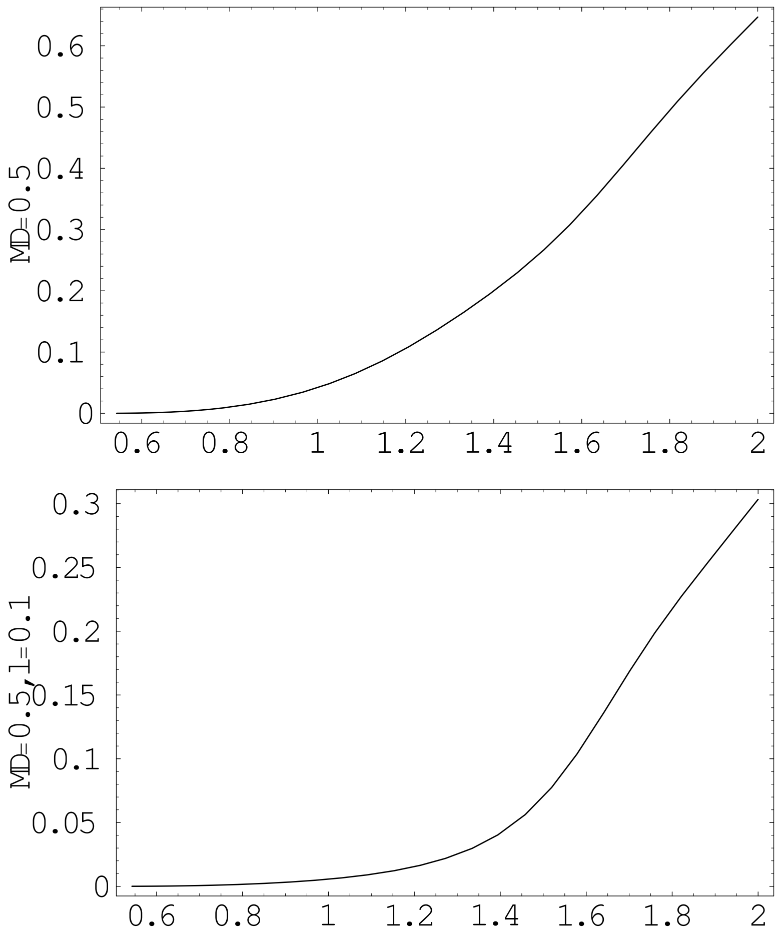

Figure 4.9: Energy spectrum far away from the Q ball in function of $\frac{\omega_0}{2M}$ once for a very big Q ball and once for a small one.

4.7 Final Remarks

In this range we have the two possibilities for the eigenvalues : either they are complex which is the same case as before or they are all real. The case where all four eigen values are real $|\epsilon| \geq \sqrt{1 - \frac{M_D^2}{\Omega_0^2}}$ is far more complicated to study since we are no more in presence of only increasing and decreasing exponentials but we have to face oscillatory functions. The bigger l gets the more oscillations we have in a period. When l is big enough we can replace the limit on l in the integrations by the mean value of the functions. So the calculation of the limit becomes the computation of a mean value and doing the limit on this mean value might be much simpler than doing it directly in the exponentials. A first thing to do is introduce the following relation

$$\lim_{l \to \infty} \overline{f(\mathrm{e}^{ip_1 l}, \mathrm{e}^{ip_2 l})} = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta f(\mathrm{e}^{i\alpha}, \mathrm{e}^{i\beta}) \tag{4.7.1}$$

Finally using the above lemma the part of the result where all eigen values are real will become a triple integral of a nasty function, but a part from the ugliness of the different functions we have to deal with the results are fairly simple. We tied to compute the value of the different parts of the amplitudes, the results obtained are quite complicated, we then applied the 4.7.1 relation to eliminate the terms containing l. We then tried to do the integration, it worked and gave the same results as before in the range where the imaginary part of the momentum is non zero but in the other range the results continued to be linear in ω_0, but we know that it is not the case since we just found there was limit for big ω_0. This limit is stable as ω_0 grows so we have to investigate a little more this problem to see why this analytical method does not work.

Before taking the limit on the size we need to solve the problem of waves being created inside the Q ball $C_i \neq 0$, these particles then moving and bouncing inside the Q ball. We would like to compute the total transmission rate, considering all possible bounces. We start with a wave appearing inside the Q ball and moving towards the left (C_3, C_4), when this wave hits the left boundary of the Q ball a part is transmitted $\tilde{T}_l$ another part is reflected $\tilde{R}_l$. When the reflected part hits the left boundary a piece is reflected $\tilde{R}_l \tilde{R}_r$ and another transmitted $\tilde{R}_l \tilde{T}_r$. The wave will then continue its bouncing and transmission process inside the Q ball, writing this we obtain :

$$\begin{pmatrix} \leftarrow \\ \leftarrow \end{pmatrix} = \tilde{T}_l (1 + \tilde{R}_r \tilde{R}_l + \tilde{R}_r \tilde{R}_l \tilde{R}_r \tilde{R}_l + \ldots) \begin{pmatrix} \leftarrow \\ \leftarrow \end{pmatrix}, \tag{4.7.2}$$

for the amplitude of particles crossing the left boundary, the amplitude of particles crossing the right boundary is ,

$$\begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix} = \tilde{T}_r (\tilde{R}_l + \tilde{R}_l \tilde{R}_r \tilde{R}_l + \ldots) \begin{pmatrix} \leftarrow \\ \leftarrow \end{pmatrix}. \tag{4.7.3}$$

The same considerations give for a wave moving towards the right :

$$\begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix} = \tilde{T}_r(1 + \tilde{R}_l\tilde{R}_r + \tilde{R}_l\tilde{R}_r\tilde{R}_l\tilde{R}_r + \ldots)\begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix}, \tag{4.7.4}$$

particles crossing the right boundary and

$$\begin{pmatrix} \leftarrow \\ \leftarrow \end{pmatrix} = \tilde{T}_l(\tilde{R}_r + \tilde{R}_r\tilde{R}_l\tilde{R}_r + \ldots)\begin{pmatrix} \rightarrow \\ \rightarrow \end{pmatrix}, \tag{4.7.5}$$

for particles crossing the left boundary of the Q ball. A rather long but straight forward calculation gives :

$$R_r = -R_l. \tag{4.7.6}$$

The right moving particles correspond to the first two constants while the left movers are associated with the two last constants. Using this construction the left movers are on the left hand side of Q ball, while the right movers are the right hand side of the Q ball. We can then write

$$\begin{aligned} \begin{pmatrix} B_3 \\ B_4 \end{pmatrix} &= \tilde{T}_l\{(1 - (\tilde{R}_l)^2 + (\tilde{R}_l)^4 + \ldots)\begin{pmatrix} C_3 \\ C_4 \end{pmatrix} + \\ &+ (-\tilde{R}_l + (\tilde{R}_l)^3 + \ldots)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\}, \end{aligned} \tag{4.7.7}$$

and

$$\begin{aligned} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} &= \tilde{T}_r\{(\tilde{R}_l - (\tilde{R}_l)^3 + \ldots)\begin{pmatrix} C_3 \\ C_4 \end{pmatrix} + \\ &+ (1 - (\tilde{R}_l)^2 + (\tilde{R}_l)^4 + \ldots)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \end{aligned} \tag{4.7.8}$$

This construction ensures us the fact that there is no particle moving towards the Q ball, so we really are in a particle producing state. This result is quite complicated and is very difficult to use with our other results. The reason of this incompatibility is due to method we used. We used here a method based on scattering, that works very well for the ranges where there is total reflection, waves are absorbed inside the Q ball and normalisation is simple. The other case is far more complicated because we have to consider all the waves moving inside the Q ball. We tried also to take the limit $l \rightarrow \infty$ before, this time the reflection and transmission coefficients where given by the results found in section 4.2.3, they gave the same results the other techniques so our problem is deeper than we suspected it to be. In fact as we shall explain with more detail in the next chapter the case where the momentum is real always happens for cases where $\frac{\omega_0}{2} \geq 1$, this case only happens for small Q balls.

Chapter 5

Further calculations.

5.1 Transmission coefficients

In the previous chapter we studied evaporation of a large Q ball, a good verification of our model would be to check if the transmitted waves really fall to zero when the Q ball's size gets big. This property seems very natural but it should be verified, the next two figures show the transmitted amplitudes in function of Q ball size.

The first figure shows the coefficient for a particle being transmitted into a particle, this coefficient has a regular decay as function of size. The other one, the transmission of a particle into an anti-particle shows a maximum which ensures the fact that the total transmitted amplitudes are equal to one. This explains why the second figure starts at zero. When the Q ball's size is equal to zero, no Q ball, the coefficient describing particle transformation t_{21} must be equal to zero and therefor the t_{22} coefficient equal to one. A small Q ball will only transform a little amount of particles, this is linked to the fact that a small Q ball has a small fermionic charge so it can only give or absorb a small fermionic number.

This kind of particle transformation property that Q balls have will be the starting point for the next investigations. In fact we have already noticed these transformation amplitudes since they are the coefficients we used to compute Q ball evaporation. Evaporation is in fact a side effect of particle transformation by a Q ball. Particle transformation seems to be an important property that we shall study in the next sections.

Both figures were obtained using the diagonalisation of motion matrix. The method we used to compute the evaporation rate and energy flux can be applied to compute any value or coefficient regarding this problem. As we shall see most calculations have been done in the last chapters.

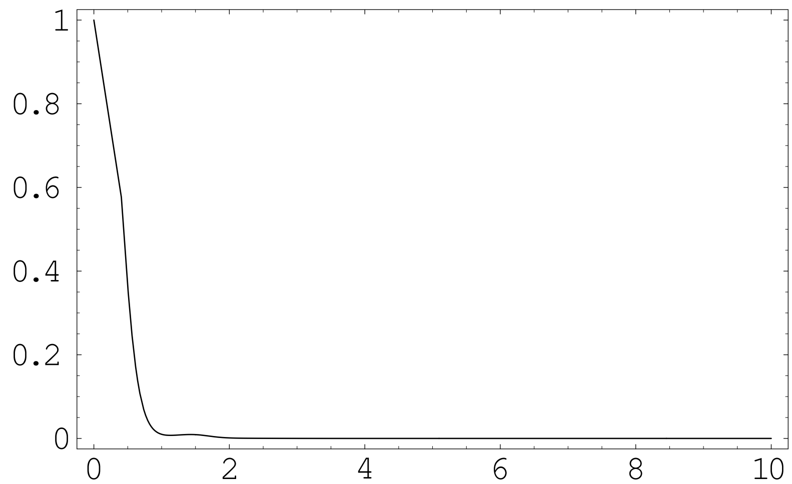

Figure 5.1: t_{22} transmission amplitude in function of Ml, for fixed values of other parameters.($\frac{\epsilon}{M} = 0.1$, $\frac{\omega_0}{2M} = 0.4$, $\frac{M_D}{M} = 0.15$)

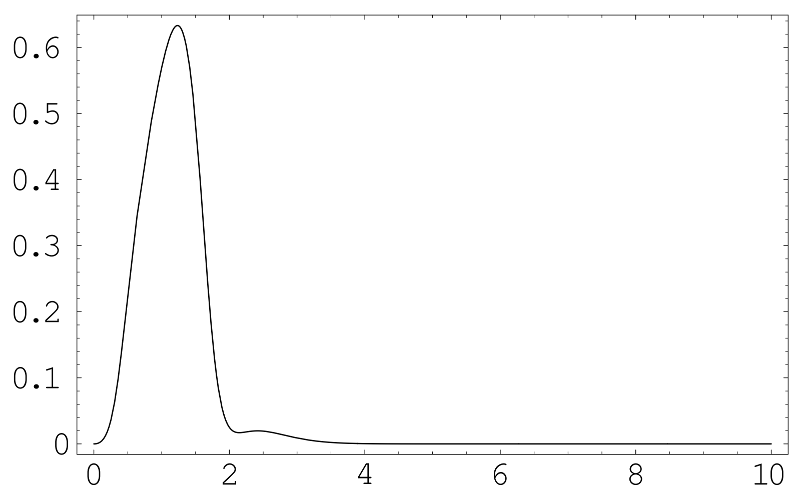

Figure 5.2: t_{21} transmission amplitude in function of Ml, for fixed values of other parameters.($\frac{\epsilon}{M} = 0.1$, $\frac{\omega_0}{2M} = 0.4$, $\frac{M_D}{M} = 0.15$)

5.2 Further study of Q balls properties.

We shall now study more properties of Q balls, because they do not only evaporate. They can also be used as particle transformators. This property comes directly from particle creation, if we look at the particles production coefficient it is in fact the transformation probability of an anti-particle becoming a particle. This property needs some development and further study, that will be the object of next section. The major motivation to this study is that we found out when we where verifying that the transmitted amplitudes would fall to zero for big values of the Q ball's size. Taking a look at the two transmission coefficients shows clearly that the ratio between them is not constant. This fact can have a lot of implications since it means that a Q ball can produce more particles then anti particles. The first thing we need to study is the behaviour of a Q ball when the fermion energy lies outside the evaporation rate.

5.2.1 Interaction of Q ball and matter, massless case.

The other property of Q balls we should study apart from evaporation, is the interaction of Q balls with matter. This study will be done by considering all the diffusion process on a Q ball. This construction will of course have to consider evaporation since we know from the previous chapters that evaporation can be solved using a scattering formalism. The most simple approach will be to consider all possible transmission through the Q ball, that is we might have a transmitted particle or anti-particle due to the energy shift (the $\frac{\omega_0}{2}$ factor in the time dependent solution). For simplicity we shall consider the construction where

Figure 5.3: General transmission through a Q ball.

there is no incoming anti-particle.

We start by using the solution outside the Q-Ball for the simple massles case we have:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} C_1^L e^{-i(\epsilon+\frac{\omega_0}{2})t} e^{i(\epsilon+\frac{\omega_0}{2})z} \\ (C_2^L)^\star e^{i(\epsilon-\frac{\omega_0}{2})t} e^{i(\epsilon-\frac{\omega_0}{2})z} \end{pmatrix}, \tag{5.2.1}$$

we have also a relation for coefficients on the right and on the left given by,

$$\begin{pmatrix} C_1^R \\ C_2^R \end{pmatrix} = \frac{1}{1-\alpha^2} \begin{pmatrix} e^{-2ik_\epsilon l} - \alpha_\epsilon^2 e^{2ik_\epsilon l} & -\alpha_\epsilon e^{-2ik_\epsilon l} + \alpha_\epsilon e^{2ik_\epsilon l} \\ \alpha_\epsilon e^{-2ik_\epsilon l} - \alpha_\epsilon e^{2ik_\epsilon l} & e^{2ik_\epsilon l} - \alpha_\epsilon^2 e^{-2ik_\epsilon l} \end{pmatrix} \begin{pmatrix} C_1^L \\ C_2^L \end{pmatrix}.$$

For the calculation of the evaporation rate we considered that both components were the same particles what we do now is consider both components to be different particles (doing so we can construct the normal field containing both particles and anti-particles). We shall consider as a first study :

$$\epsilon + \frac{\omega_0}{2} \geq 0 \to \epsilon \geq -\frac{\omega_0}{2}$$
$$\epsilon - \frac{\omega_0}{2} \geq 0 \to \epsilon \geq \frac{\omega_0}{2},$$

it means we have the same type of particles on the left hand side moving both of them towards the Q ball. Since we have chosen the parameters to be the incident amplitudes we can choose that there is no incident anti-particle, $C_2^L = 0$. Using the matrix relations we find,

$$C_1^R = \frac{\mathrm{e}^{-2ik_\epsilon l} - \alpha_\epsilon^2 \mathrm{e}^{2ik_\epsilon l}}{1-\alpha_\epsilon^2} = T_{pp},$$
$$C_2^R = \frac{\alpha_\epsilon \mathrm{e}^{-2ik_\epsilon l} - \alpha_\epsilon \mathrm{e}^{2ik_\epsilon l}}{1-\alpha_\epsilon^2} = T_{p\bar{p}},$$

The transformation amplitude we are looking for is the C_2^R coefficient. As long as k_ϵ is complex (see chap three), we can easily see that this amplitude decreases as the Q ball's size increases. Here we considered the range where $\epsilon \geq \frac{\omega_0}{2}$ and the solution is,

$$\Psi_L = \int_{\frac{\omega_0}{2}}^{\infty} d\epsilon \begin{pmatrix} A(\epsilon)\mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon+\frac{\omega_0}{2})t} \\ 0 \end{pmatrix}$$
$$\Psi_R = \int_{\frac{\omega_0}{2}}^{\infty} d\epsilon \begin{pmatrix} A(\epsilon)T_{pp}\mathrm{e}^{-i(\epsilon+\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon+\frac{\omega_0}{2})t} \\ A^\star(\epsilon)T_{p\bar{p}}^\star\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t}\mathrm{e}^{i(\epsilon-\frac{\omega_0}{2})t} \end{pmatrix}, \quad (5.2.2)$$

where the L and R subscripts indicate on which side of the Q ball we are. It is not the only range where transformation occurs we also have the other range $\epsilon \leq -\frac{\omega_0}{2}$, with this time incident particles coming from the right hand side of Q ball, we have

$$\begin{pmatrix} C_1^R \\ C_2^R \end{pmatrix} = \frac{1}{1-\alpha^2} \begin{pmatrix} \mathrm{e}^{-2ik_\epsilon l} - \alpha_\epsilon^2\mathrm{e}^{2ik_\epsilon l} & -\alpha_\epsilon\mathrm{e}^{-2ik_\epsilon l} + \alpha_\epsilon\mathrm{e}^{2ik_\epsilon l} \\ \alpha_\epsilon\mathrm{e}^{-2ik_\epsilon l} - \alpha_\epsilon\mathrm{e}^{2ik_\epsilon l} & \mathrm{e}^{2ik_\epsilon l} - \alpha_\epsilon^2\mathrm{e}^{-2ik_\epsilon l} \end{pmatrix} \begin{pmatrix} C_1^L \\ C_2^L \end{pmatrix}.$$

we have this time $C_R^1 = 0$, leading to the same solution for the transmission amplitudes,

$$\tilde{T}_{pp} = \frac{\mathrm{e}^{-2ik_\epsilon l} - \alpha_\epsilon^2 \mathrm{e}^{2ik_\epsilon l}}{1-\alpha_\epsilon^2} = T_{pp},$$
$$\tilde{T}_{p\bar{p}} = \frac{\alpha_\epsilon \mathrm{e}^{-2ik_\epsilon l} - \alpha_\epsilon \mathrm{e}^{2ik_\epsilon l}}{1-\alpha_\epsilon^2} = T_{p\bar{p}},$$

even if the amplitudes are the same, which is a effect of symmetry, we shall use tilde symbols to distinguish both contributions. In fact we would like to keep in mind that this second contribution comes from waves having negative energy. These two contributions will mix together to build the total solution. We have this time as solution,

$$\Psi_L = \int_{-\infty}^{-\frac{\omega_0}{2}} d\epsilon \begin{pmatrix} B(\epsilon)\tilde{T}_{p\bar{p}} e^{-i(\epsilon+\frac{\omega_0}{2})t} e^{i(\epsilon+\frac{\omega_0}{2})z} \\ B^\star(\epsilon)\tilde{T}^\star_{pp} e^{i(\epsilon-\frac{\omega_0}{2})t} e^{i(\epsilon-\frac{\omega_0}{2})z} \end{pmatrix}$$

$$\Psi_R = \int_{-\infty}^{-\frac{\omega_0}{2}} d\epsilon \begin{pmatrix} 0 \\ B^\star(\epsilon) e^{i(\epsilon-\frac{\omega_0}{2})t} e^{i(\epsilon-\frac{\omega_0}{2})t} \end{pmatrix}. \quad (5.2.3)$$

Let us take a look at the pieces on the left,

$$\begin{aligned} \Psi_L &= \int_{\frac{\omega_0}{2}}^{\infty} d\epsilon u_1 e^{-i(\epsilon+\frac{\omega_0}{2})t} e^{i(\epsilon+\frac{\omega_0}{2})z} A(\epsilon) + \int_{-\infty}^{-\frac{\omega_0}{2}} d\epsilon u_1 e^{-i(\epsilon+\frac{\omega_0}{2})t} e^{i(\epsilon+\frac{\omega_0}{2})z} \tilde{T}_{p\bar{p}} B(\epsilon) \\ &+ \int_{-\infty}^{-\frac{\omega_0}{2}} d\epsilon u_2 e^{i(\epsilon-\frac{\omega_0}{2})t} e^{i(\epsilon-\frac{\omega_0}{2})z} \tilde{T}_{pp} B^\star(\epsilon), \end{aligned}$$

where $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We change the variable $\epsilon \to -\epsilon$ in the integrals on the negative range to obtain,

$$\begin{aligned} \Psi_L &= \int_{\frac{\omega_0}{2}}^{\infty} d\epsilon u_1 e^{-i(\epsilon+\frac{\omega_0}{2})t} e^{i(\epsilon+\frac{\omega_0}{2})z} A(\epsilon) + \int_{\frac{\omega_0}{2}}^{\infty} d\epsilon u_1 e^{i(\epsilon-\frac{\omega_0}{2})t} e^{-i(\epsilon-\frac{\omega_0}{2})z} \tilde{T}_{p\bar{p}}(-\epsilon) B(-\epsilon) \\ &+ \int_{\frac{\omega_0}{2}}^{\infty} d\epsilon u_2 e^{-i(\epsilon+\frac{\omega_0}{2})t} e^{-i(\epsilon+\frac{\omega_0}{2})z} \tilde{T}_{pp}(-\epsilon) B^\star(-\epsilon). \end{aligned}$$

The next thing to do is change the variable $\epsilon + \frac{\omega_0}{2} = \epsilon'$ and $\epsilon - \frac{\omega_0}{2} = \epsilon'$ to obtain,

$$\begin{aligned} \Psi_L &= \int_{\omega_0}^{\infty} d\epsilon u_1 e^{-i\epsilon' t} e^{i\epsilon' z} A(\epsilon' - \frac{\omega_0}{2}) + \int_{0}^{\infty} d\epsilon u_1 e^{i\epsilon' t} e^{-i\epsilon' z} \tilde{T}_{p\bar{p}}(-\frac{\omega_0}{2} - \epsilon) B(-\frac{\omega_0}{2} - \epsilon) \\ &+ \int_{\omega_0}^{\infty} d\epsilon u_2 e^{-i\epsilon' t} e^{-i\epsilon' z} \tilde{T}_{pp}(-\epsilon' + \frac{\omega_0}{2}) B^\star(-\epsilon' + \frac{\omega_0}{2}). \end{aligned}$$

Due to the energy shift coming from the Q ball we have two different bounds on the integration $\epsilon \in [\omega_0, \infty[$ and $\epsilon \in [0, \infty[$. These two ranges will be of great interest, since each one can be identified to a different particle. Before we construct the quantised solution we need to do the same transformations to the solution on right-hand side.

$$\begin{aligned} \Psi_R &= \int_{\frac{\omega_0}{2}}^{\infty} d\epsilon u_1 e^{-i(\epsilon+\frac{\omega_0}{2})t} e^{i(\epsilon+\frac{\omega_0}{2})z} T_{pp} A(\epsilon) + \int_{\infty}^{\frac{\omega_0}{2}} d\epsilon u_2 e^{i(\epsilon-\frac{\omega_0}{2})t} e^{i(\epsilon-\frac{\omega_0}{2})z} T^\star_{p\bar{p}} A^\star(\epsilon) \\ &+ \int_{-\infty}^{-\frac{\omega_0}{2}} d\epsilon u_2 e^{i(\epsilon-\frac{\omega_0}{2})t} e^{i(\epsilon-\frac{\omega_0}{2})z} B^\star(\epsilon), \end{aligned}$$

after the variable changes and the sign change we obtain,

$$\begin{aligned}\Psi_R &= \int_{\omega_0}^{\infty} d\epsilon u_2 \mathrm{e}^{-i\epsilon' t}\mathrm{e}^{i\epsilon' z}B(-\epsilon'+\frac{\omega_0}{2}) + \int_0^{\infty} d\epsilon u_2 \mathrm{e}^{i\epsilon' t}\mathrm{e}^{-i\epsilon' z}T^{\star}_{p\bar{p}}(\frac{\omega_0}{2}+\epsilon)A(\frac{\omega_0}{2}+\epsilon)\\ &+ \int_{\omega_0}^{\infty} d\epsilon u_1 \mathrm{e}^{-i\epsilon' t}\mathrm{e}^{-i\epsilon' z}T_{pp}(\epsilon'-\frac{\omega_0}{2})A(\epsilon'-\frac{\omega_0}{2}).\end{aligned}$$

We see here that the $B(-\epsilon'+\frac{\omega_0}{2})$ and $A(\epsilon'-\frac{\omega_0}{2})$ are in the same integration range and therefore will correspond to the same particles. We now have both solutions one on each side of the Q ball, the last step of construction will be quantisation. Quantisation will be done identifying the parts corresponding to particles and the parts corresponding to anti-particles. It is easy to identify the particles and the anti-particles since the integration domain is well known and defined we have,

$$\begin{aligned}\Psi_L &= \int_{\omega_0}^{\infty} d\epsilon \mathrm{e}^{-i\epsilon t}\{\mathrm{e}^{i\epsilon z}u_1 + \mathrm{e}^{-i\epsilon z}\tilde{T}_{pp}(\frac{\omega_0}{2}-\epsilon)u_2\}a_l(\epsilon-\frac{\omega_0}{2})\\ &+ \int_0^{\infty} d\epsilon \mathrm{e}^{i\epsilon t}\{\mathrm{e}^{-i\epsilon z}\tilde{T}_{p\bar{p}}(\frac{\omega_0}{2}+\epsilon)u_1\}b_l^{\dagger}(\epsilon+\frac{\omega_0}{2})\end{aligned}$$

for the solution on the left and

$$\begin{aligned}\Psi_R &= \int_{\omega_0}^{\infty} d\epsilon \mathrm{e}^{-i\epsilon t}\{\mathrm{e}^{-i\epsilon z}u_2 a_r(\frac{\omega_0}{2}-\epsilon) + \mathrm{e}^{i\epsilon z}T_{pp}(\frac{\omega_0}{2}-\epsilon)u_1 a_r^{\dagger}(\frac{\omega_0}{2}-\epsilon)\}\\ &+ \int_0^{\infty} d\epsilon \mathrm{e}^{i\epsilon t}\{\mathrm{e}^{i\epsilon z}T_{p\bar{p}}(\frac{\omega_0}{2}+\epsilon)u_1\}b_r^{\dagger}(\epsilon+\frac{\omega_0}{2}),\end{aligned}$$

for the solution on the right hand side of the Q ball. During the construction we used,

$$B(k) = a(k)\theta(k) + b^{\dagger}(k)\theta(-k).$$

We keep for the moment the l and r subscript on the operators to identify the side of the Q ball they correspond.

Now that we have the total solution in terms of creation and annihilation operators we can compute anything we want, by applying the proper operator valued observable. We can see that the energy of the transmitted wave is the same as the energy of the incoming wave, while the energy of the reflected anti-particle is shifted by a factor of ω_0. This shift in energy, one component shifted downwards and the other one shifted upwards is the reason why we found finite evaporation ranges. This property is due to the Yukawa coupling inside the Q ball. We can choose a state where there is no incoming particles on the right so our transmission coefficients can be identified to reflection ones. In fact this construction can be used in the full energy range. The difficulty will be to separate the phenomena linked to diffusion and those linked to evaporation.

5.2.2 Diffusion on a Q ball

Now that we have the quantised solutions we can try to study diffusion on a Q ball. We need to use the total solution having no incident wave on the right-hand side of the Q ball. We write,

$$\begin{aligned}\Psi_L &= \int_{\omega_0}^{\infty} d\epsilon \mathrm{e}^{-i\epsilon t}\{\mathrm{e}^{i\epsilon z}u_1 + \mathrm{e}^{-i\epsilon z}\tilde{T}_{pp}(\frac{\omega_0}{2}-\epsilon)u_2\}a_l(\epsilon-\frac{\omega_0}{2})\\ &+ \int_0^{\infty} d\epsilon \mathrm{e}^{i\epsilon t}\{\mathrm{e}^{-i\epsilon z}\tilde{T}_{p\bar{p}}(\frac{\omega_0}{2}+\epsilon)u_1\}b_l^{\dagger}(\epsilon+\frac{\omega_0}{2})\\ &+ \int_{\omega_0}^{\infty} d\epsilon \mathrm{e}^{-i\epsilon t}\{\mathrm{e}^{i\epsilon z}T_{pp}(\frac{\omega_0}{2}-\epsilon)u_1 a_r^{\dagger}(\frac{\omega_0}{2}-\epsilon)\}\\ &+ \int_0^{\infty} d\epsilon \mathrm{e}^{i\epsilon t}\{\mathrm{e}^{i\epsilon z}T_{p\bar{p}}(\frac{\omega_0}{2}+\epsilon)u_1\}b_r^{\dagger}(\epsilon+\frac{\omega_0}{2}),\end{aligned}$$

the construction of the Bogolubov transformation linking the far past, the incident wave, to the far future reads :

$$\begin{aligned}c_{out} &= T_{pp}(\frac{\omega_0}{2}-\epsilon)a_l(\epsilon-\frac{\omega_0}{2}) + T_{p\bar{p}}(\frac{\omega_0}{2}+\epsilon)b_l^{\dagger}(\epsilon+\frac{\omega_0}{2})\\ &+ T_{pp}(\frac{\omega_0}{2}-\epsilon)a_r^{\dagger}(\frac{\omega_0}{2}-\epsilon) + T_{p\bar{p}}(\frac{\omega_0}{2}+\epsilon)b_r^{\dagger}(\epsilon+\frac{\omega_0}{2}).\end{aligned}$$

We here distinguish two energy ranges, if $\epsilon \geq \frac{\omega_0}{2}$ the $a_r^{\dagger}(\frac{\omega_0}{2}-\epsilon)$ operator becomes $a_r(\epsilon-\frac{\omega_0}{2})$, while if $\epsilon \leq \frac{\omega_0}{2}$ it is the $a_l(\epsilon-\frac{\omega_0}{2})$ becoming $a_l^{\dagger}(\frac{\omega_0}{2}-\epsilon)$. As we said before we had to separate both regimes in order to see what type of diffusion we have.

First we shall consider an infinite Q ball, so the right-hand side does not exist, and we have

$$c_{out} = T_{pp}(\frac{\omega_0}{2}-\epsilon)a_l(\epsilon-\frac{\omega_0}{2}) + T_{p\bar{p}}(\frac{\omega_0}{2}+\epsilon)b_l^{\dagger}(\epsilon+\frac{\omega_0}{2}), \tag{5.2.4}$$

when we constructed the solution the incident particle had energy bigger than ω_0. So in this range $\epsilon-\frac{\omega_0}{2}$ is always bigger than zero, and the a operator remains an annihilation operator. We have now in the final state (in the far future),

$$_{out}<0|c^{\dagger}c|0>_{out} = |T_{p\bar{p}}|^2 <\bar{p}|\bar{p}> . \tag{5.2.5}$$

This simply is a particle being reflected into an anti-particle to conserve helicity. To study the rest of the possibilities we shall use a small sized Q ball.

The first possibility stands for $\epsilon \geq \omega_0/2$ we need to change the type of the second a operator, $a_r^{\dagger}(\frac{\omega_0}{2}-\epsilon)$ becomes $a_r(\epsilon-\frac{\omega_0}{2})$:

$$\begin{aligned}c_{out} &= T_{pp}(\frac{\omega_0}{2}-\epsilon)a_l(\epsilon-\frac{\omega_0}{2}) + T_{p\bar{p}}(\frac{\omega_0}{2}+\epsilon)b_l^{\dagger}(\epsilon+\frac{\omega_0}{2})\\ &+ T_{pp}(\frac{\omega_0}{2}-\epsilon)a_r(\epsilon-\frac{\omega_0}{2}) + T_{p\bar{p}}(\frac{\omega_0}{2}+\epsilon)b_r^{\dagger}(\epsilon+\frac{\omega_0}{2}).\end{aligned}$$

Applying this to the vacuum state in the far future we have,

$$ {}_{out} < 0|c^\dagger c|0 >_{out} = |\tilde{T}_{p\bar{p}}|^2{}_l < \bar{p}|\bar{p} >_l + |T_{p\bar{p}}|^2{}_r < \bar{p}|\bar{p} >_r. \tag{5.2.6} $$

Figure 5.4: Diffusion on a Q ball for an incident particle having energy bigger than ω_0.

The second possibility will be for a particle having energy lower than $\omega_0/2$ we have to change the type of the first a, $a_l(\epsilon - \frac{\omega_0}{2})$ becomes $a_l^\dagger(\frac{\omega_0}{2} - \epsilon)$. We write :

$$ \begin{aligned} c_{out} &= T_{pp}(\frac{\omega_0}{2} - \epsilon)a_l^\dagger(\frac{\omega_0}{2} - \epsilon) + T_{p\bar{p}}(\frac{\omega_0}{2} + \epsilon)b_l^\dagger(\epsilon + \frac{\omega_0}{2}) \\ &+ T_{pp}(\frac{\omega_0}{2} - \epsilon)a_r^\dagger(\frac{\omega_0}{2} - \epsilon) + T_{p\bar{p}}(\frac{\omega_0}{2} + \epsilon)b_r^\dagger(\epsilon + \frac{\omega_0}{2}). \end{aligned} $$

This time we have four particles in the final state, it is the combination of evaporation and diffusion.

Figure 5.5: Diffusion on a Q ball for an incident particle having energy smaller than $\omega_0/2$.

The third possibility would be for incident fermions having their energy in the range $\epsilon \in [\omega_0/2; \omega_0]$, we shall not study this range since we only considered incident fermions with energy bigger than ω_0 or smaller than $\omega_0/2$. But this range does not introduce any new or difficult construction. These construction can also be used to compute the evaporation rate we just need to separate both constructions and then identify which contribution is diffusion and which one is evaporation.

5.2.3 Diffusion on a Q ball massive case.

Starting with the solution for a Q ball interacting with a fermion we have,

$$\begin{aligned}\Psi &= \int_{M_D+\frac{\omega_0}{2}}^{\infty} d\epsilon \{ e^{i(\epsilon-\frac{\omega_0}{2})t}[e^{i\bar{p}_1 x}u^{up}_{\bar{p}_1} + r_{11}e^{-i\bar{p}_1 x}u^{up}_{-\bar{p}_1}]b^\dagger_{\bar{p}_1} + e^{-i(\epsilon+\frac{\omega_0}{2})t}[r^\star_{12}e^{i\bar{p}^\star_2 x}(u^{down}_{-\bar{p}_2})^\star]b_{\bar{p}_1}\},\\ &+ \int_{M_D-\frac{\omega_0}{2}}^{\infty} d\epsilon \{ e^{i(\epsilon-\frac{\omega_0}{2})t}[r_{21}e^{-i\bar{p}_1 x}u^{up}_{-\bar{p}_1}]a^\dagger_{\bar{p}_2} + e^{-i(\epsilon+\frac{\omega_0}{2})t}[e^{-i\bar{p}^\star_2 x}u^\star_{\bar{p}_2} + r^\star_{22}e^{i\bar{p}^\star_2 x}(u^{down}_{-\bar{p}_2})^\star]a_{\bar{p}_2}\},\\ &+ \int_{M_D+\frac{\omega_0}{2}}^{\infty} d\epsilon \{ e^{i(\epsilon-\frac{\omega_0}{2})t}[\tilde{t}_{11}e^{-i\bar{p}_1 x}u^{up}_{-\bar{p}_1}]b^\dagger_{-\bar{p}_1} + e^{-i(\epsilon+\frac{\omega_0}{2})t}[\tilde{t}^\star_{12}e^{i\bar{p}^\star_2 x}(u^{down}_{-\bar{p}_2})^\star]b_{-\bar{p}_1}\},\\ &+ \int_{M_D-\frac{\omega_0}{2}}^{\infty} d\epsilon \{ e^{i(\epsilon-\frac{\omega_0}{2})t}[\tilde{t}_{21}e^{-i\bar{p}_1 x}u^{up}_{-\bar{p}_1}]a^\dagger_{-\bar{p}_2} + e^{-i(\epsilon+\frac{\omega_0}{2})t}[\tilde{t}^\star_{22}e^{i\bar{p}^\star_2 x}(u^{down}_{-\bar{p}_2})^\star]a_{-\bar{p}_2}\},\end{aligned}$$

as before this solution has two different integration ranges one for each type of particle. We shall now only consider the range where $\epsilon \geq M_D + \frac{\omega_0}{2}$ in this range we only have diffusion and to not need to study the interaction of both, diffusion and evaporation. We have ,

$$\begin{aligned}\Psi &= \int_{M_D+\frac{\omega_0}{2}}^{\infty} d\epsilon \{ e^{i(\epsilon-\frac{\omega_0}{2})t}[e^{i\bar{p}_1 x}u^{up}_{\bar{p}_1} + r_{11}e^{-i\bar{p}_1 x}u^{up}_{-\bar{p}_1}]b^\dagger_{\bar{p}_1} + e^{-i(\epsilon+\frac{\omega_0}{2})t}[r^\star_{12}e^{i\bar{p}^\star_2 x}(u^{down}_{-\bar{p}_2})^\star]b_{\bar{p}_1}\},\\ &+ \int_{M_D+\frac{\omega_0}{2}}^{\infty} d\epsilon \{ e^{i(\epsilon-\frac{\omega_0}{2})t}[r_{21}e^{-i\bar{p}_1 x}u^{up}_{-\bar{p}_1}]a^\dagger_{\bar{p}_2} + e^{-i(\epsilon+\frac{\omega_0}{2})t}[e^{-i\bar{p}^\star_2 x}u^\star_{\bar{p}_2} + r^\star_{22}e^{i\bar{p}^\star_2 x}(u^{down}_{-\bar{p}_2})^\star]a_{\bar{p}_2}\},\end{aligned}$$

the only reason why we used a very big Q ball is that we have less terms to deal with. With this solution we can construct a Bogolubov transformation as in the previous chapter considering that in the far future only out going waves survives. We have this time :

$$r_{11}b^\dagger_{\bar{p}_1} + r^\star_{12}b_{\bar{p}_1} + r_{21}a^\dagger_{\bar{p}_2} + r^\star_{22}a_{\bar{p}_2} = c_{out}. \tag{5.2.7}$$

If we want c_{out} to have the same anti-commutation relations as a and b we need to normalise the operators with :

$$a = \frac{a}{|r_{22}|^2 + |r_{21}|^2}, \tag{5.2.8}$$

$$b = \frac{b}{|r_{12}|^2 + |r_{11}|^2}, \tag{5.2.9}$$

There is no obvious reasons why $|r_{12}|^2 = |r_{21}|^2$ and $|r_{11}|^2 = |r_{22}|^2$. These equalities will depend on the type of Q ball we study, they depend on the way the Q ball mixes the particles and anti-particle energies[1]. We can use this c operator to compute the number of particles in the final state when the original one was the vacuum state,

$$\begin{aligned}{}_{out}<0|c^\dagger c|0>_{out} &= |r_{21}|^2_{in}<0|a^\dagger a|0>_{in} + |r_{11}|^2{}_{in}<0|b^\dagger b|0>_{in},\\ &= |r_{21}|^2 <p|p> + |r_{11}|^2 <ap|ap> . \end{aligned} \tag{5.2.10}$$

[1]In our case we have this symmetry since the Q ball shifts both particles and anti particles the same way. But we could have a Q ball having a different interaction with fermions.

We have this time, when fermions are massive, a particle and an anti-particle in the final state.

Verifications.

We shall in this section verify the results for massive fermions by doing all the variable changes. These variable changes will allow us to check the production and diffusion ranges.

For simplicity we shall do the variable change $\epsilon = \epsilon' + \frac{\omega_0}{2}$ in the terms containing exponentials wtih $\epsilon - +\frac{\omega_0}{2}$ and $\epsilon = \epsilon' - \frac{\omega_0}{2}$ for the others, we have then,

$$\begin{aligned}\int_{M_D}^{\infty} d\epsilon \quad [\quad & \mathrm{e}^{i(\epsilon')t}\mathrm{e}^{i\bar{p}_1(\epsilon'+\frac{\omega_0}{2})x}u^{up}_{\bar{p_1}}(\epsilon'+\frac{\omega_0}{2}) + r_{11}(\epsilon'+\frac{\omega_0}{2})\mathrm{e}^{-i\bar{p}_1(\epsilon'+\frac{\omega_0}{2})x}u^{up}_{-\bar{p_1}}(\epsilon'+\frac{\omega_0}{2})]b^{\dagger}_{\bar{p}_1(\epsilon'+\frac{\omega_0}{2})} \\ + \quad & \mathrm{e}^{i(\epsilon')t}[r_{21}(\epsilon'+\frac{\omega_0}{2})\mathrm{e}^{-i\bar{p}_1(\epsilon'+\frac{\omega_0}{2})x}u^{up}_{-\bar{p_1}}(\epsilon'+\frac{\omega_0}{2})]a^{\dagger}_{\bar{p}_2(\epsilon'+\frac{\omega_0}{2})}\end{aligned}$$

$$\begin{aligned}+\int_{M_D+\omega_0}^{\infty} d\epsilon \quad [\quad & \mathrm{e}^{-i(\epsilon')t}[r^{\star}_{12}(\epsilon'-\frac{\omega_0}{2})\mathrm{e}^{-i\bar{p}^{\star}_2(\epsilon'-\frac{\omega_0}{2})x}(u^{down}_{-\bar{p}_2}(\epsilon'-\frac{\omega_0}{2}))^{\star}]b_{\bar{p}_1(\epsilon'-\frac{\omega_0}{2})} \\ + \quad & \mathrm{e}^{-i(\epsilon')t}[\mathrm{e}^{-i\bar{p}^{\star}_2(\epsilon'-\frac{\omega_0}{2})x}u^{\star}_{\bar{p}_2}(\epsilon'-\frac{\omega_0}{2}) \\ + \quad & r^{\star}_{22}(\epsilon'-\frac{\omega_0}{2})\mathrm{e}^{i\bar{p}^{\star}_2(\epsilon'-\frac{\omega_0}{2})x}(u^{down}_{-\bar{p}_2})^{\star}(\epsilon'-\frac{\omega_0}{2})]a_{\bar{p}_2(\epsilon'-\frac{\omega_0}{2})}\end{aligned}$$

It is easy to check that $\bar{p}_1(\epsilon'+\frac{\omega_0}{2}) = \bar{p}^{\star}_2(\epsilon'-\frac{\omega_0}{2})$ so we have an identification of both momentum,

$$\begin{aligned}\int_{M_D}^{\infty} d\epsilon \quad [\quad & \mathrm{e}^{i(\epsilon')t}\mathrm{e}^{i\bar{p}x}u^{up}_{\bar{p}} + r_{11}(\epsilon'+\frac{\omega_0}{2})\mathrm{e}^{-i\bar{p}x}u^{up}_{-\bar{p}}]b^{\dagger}_{\bar{p}} \\ + \quad & \mathrm{e}^{i(\epsilon')t}[r_{21}(\epsilon'+\frac{\omega_0}{2})\mathrm{e}^{-i\bar{p}x}u^{up}_{-\bar{p}}]a^{\dagger}_{\bar{p}}\end{aligned}$$

$$\begin{aligned}+\int_{M_D+\omega_0}^{\infty} d\epsilon \quad [\quad & \mathrm{e}^{-i(\epsilon')t}[r^{\star}_{12}(\epsilon'-\frac{\omega_0}{2})\mathrm{e}^{-i\bar{p}x}(u^{down}_{-\bar{p}})^{\star}]b_{\bar{p}} \\ + \quad & \mathrm{e}^{-i(\epsilon')t}[\mathrm{e}^{-i\bar{p}x}u^{\star}_{\bar{p}} + r^{\star}_{22}(\epsilon'-\frac{\omega_0}{2})\mathrm{e}^{i\bar{p}x}(u^{down}_{-\bar{p}})]a_{\bar{p}}\end{aligned}$$

The first part of the integral corresponds to an incoming anti-particle and transformed particle, while the other term corresponds this time to an incoming particle. To study particle transformation we just need to study one of these to parts. We decide to study the part with the incoming particle,

$$\begin{aligned}\Psi = \int_{M_D+\omega_0}^{\infty} d\epsilon \quad [\quad & \mathrm{e}^{-i(\epsilon')t}[r^{\star}_{12}(\epsilon'-\frac{\omega_0}{2})\mathrm{e}^{-i\bar{p}x}(u^{down}_{-\bar{p}})^{\star}]b_{\bar{p}} \\ + \quad & \mathrm{e}^{-i(\epsilon')t}[\mathrm{e}^{-i\bar{p}x}u^{\star}_{\bar{p}} + r^{\star}_{22}(\epsilon'-\frac{\omega_0}{2})\mathrm{e}^{i\bar{p}x}(u^{down}_{-\bar{p}})]a_{\bar{p}}\end{aligned}$$

we can change the sign of the momentum using $a(-p) = a^{\dagger}(p)$.

$$\begin{aligned}\Psi = \int_{M_D+\omega_0}^{\infty} d\epsilon \quad [\quad & \mathrm{e}^{-i(\epsilon')t}[r^{\star}_{12}(\epsilon'-\frac{\omega_0}{2})\mathrm{e}^{i\bar{p}x}(u^{down}_{\bar{p}})^{\star}]b^{\dagger}_{\bar{p}} \\ + \quad & \mathrm{e}^{-i(\epsilon')t}[\mathrm{e}^{-i\bar{p}x}u^{\star}_{\bar{p}}a_{\bar{p}} + r^{\star}_{22}(\epsilon'-\frac{\omega_0}{2})\mathrm{e}^{-i\bar{p}x}(u^{down}_{\bar{p}})a^{\dagger}_{\bar{p}}]\end{aligned}$$

These two configurations have different energy ranges so if we need to use them both the range we consider is the same as in the previous section.

5.2.4 Towards a realistic model.

Before going any further we would like to try and find out what is the most realistic approximations possible. We know from chapter four that the problem is fairly simple in the case where the particle momentum inside the Q ball is complex. In this case the waves propagating inside the Q ball are only increasing or decreasing exponentials that greatly simplify the shape of solution. This situation mostly happens when $\frac{\omega_0}{2} \leq 1$. If we remember what was done in the chapter four, we know that all parameters we deal with are dimension less, in fact we can write : $\frac{\omega_0}{2} \leq M$. The other parameter we deal with is the size parameter given by Ml, and it only appears in exponentials. In the case where the Q ball produces massive fermions, the most simple case happens when $\frac{\omega_0}{2} \leq M$ and Ml very big. As we found out in the previous chapter when the Q ball is very big it can be considered as infinite, it becomes then a Q-Wall. This property allows us to reduce scattering of a Q ball to scattering on a domain wall. We know from most Q balls models that the Majorana mass parameter inside the Q ball is proportional to its size, so for simplicity we can consider only very big Q balls.

The other thing we need to sort out is the standard value of the Dirac mass of produced fermions. We know that most type of Q balls are big ones and that the Majorana coupling inside the Q ball is proportional to its size. This means that we are in the case where $M >> M_D$ so the mass parameter will be small. This is the case when we are in the limit of section 4.5.1 with small fermion mass parameter. The most realistic limit is finally quite simple to study and all its properties can be deduced from simple relations. The result we can use to have a quick idea of evaporation rate is just using the upper limit that is slightly bigger then one, so going back to dimensionful units it is bigger then M the Majorana mass of fermions inside the Q ball. This result is easy to understand, the Majorana mass of the Q ball is linked to the coupling and the numbers of scalars inside the Q ball. When the Q ball is infinite (very big) it is normal that the Dirac mass of fermions has no more role to play in the value of the evaporation rate. The Dirac mass is only the starting point of evaporation. This gap we shall observe in the evaporation spectrum can be used as the Q ball's signature to identify them.

The other very important simplification we used, was the fact that the Q ball is "thin walled". We know that we have two types of Q balls : the thin walled ones, that are approximated by a step function, and the thick walled ones where we use a gaussian profile to describe them. The results we obtain are very accurate only in the case of big Q balls. If we look at the standard Affleck-Dine mechanism leading to large Q balls we have to distinguish two scenarios. The first one, gauge mediated symmetry breaking, leading to Q balls having a very large charge and radius ($R \propto Q^{1/4}$). While for the second scenario, gravity mediated symmetry breaking, the Q balls are also very large but this time the radius is independent of charge [6, 7, 8]. Even if the models creating Q balls with very big charge are well known we still need the knowledge of the radius. The fact that we need a big sized Q ball will select the type of scenarios we can use for Q ball formation.

5.3 Extension to $3 \oplus 1$ dimensions.

5.3.1 Extension to $3 \oplus 1$ dimensions first considerations.

The first thing we can do is consider an observer situated far away from the Q ball, at this distance only the radial coordinate can be used, so the problem reduces to a $1 \oplus 1$ dimensional problem. At a large distance of a source we can only measure the flux emitted in the radial direction, while at short distances we have to consider all superpositions of waves. We shall now consider the impulsion flux through a sphere, this flux will give the energy spectrum of a Q ball for a far away observer. This can be shown using the fact that when we are far away from a spherical source all the waves that are not perpendicular to the surface will vanish, they have in fact only destructive interactions. As we mentioned before the momentum is perpendicular to the surface element we can write,

$$\Phi = \int_{sphere} \vec{E} d\vec{\sigma} = 4\pi r^2 E, \tag{5.3.1}$$

where E stands for the energy leak from the Q ball that can be computed using the evaporation rate. This energy flux will give the spectrum we are looking for. The question one could ask now is what happens if we are very close to the Q ball. This case is very complicated since we now need to study all the waves coming from all directions. The other thing we need to be careful with is the bounces of the wave, figure 5.6 shows a few examples of possible bounces. The major difficulty with this calculation is the fact that we need to check if there is a dependence in the angles for the transmitted and reflected coefficients. In most cases we know that the transmitted coefficients decay in an exponential way inside the Q ball, this will induce us to think that the corrections we need to use are of $\frac{1}{R}$ order.

All of this was to explain that if we are in presence of a huge Q ball and situated far away from it, we can use the 1-dimensional results. This finally comforts us in thinking that the only important quantity we need to measure is the energy flux a distant observer will see. The angle dependence in the transmitted coefficients will only modify the results for average size Q balls, for small ones the angles have very little influence. Finally the only problem with extension to three dimension will appear for medium sized Q ball. We guess that for this type of Q ball we can use interpolation.

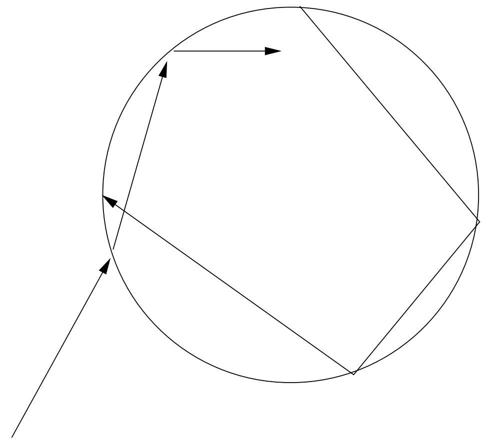

Figure 5.6: Bounces for waves appearing inside the Q ball and for waves incident from the outside.

5.3.2 Extension to $3 \oplus 1$ dimensions second method

Apart from computing the energy flux far away from the Q ball we can try to obtain an analytical solution to the three dimensional field motion equations. We shall try using the simplest possible construction whose major objective would be to directly obtain the previous results with maybe different values of parameters. The total Lagrangian is given by

$$\mathcal{L} = i\psi^\dagger \sigma^\mu \partial_\mu \psi + gM\bar{\psi}^c \psi, \tag{5.3.2}$$

leading to the following equations of motion

$$i\sigma^\mu \partial_\mu \psi + fM\varepsilon\psi^\star = 0. \tag{5.3.3}$$

Component by component we can write for $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$,

$$i(\partial_0 + \partial_z)\psi_1 + i(\partial_x - i\partial_y)\psi_2 - M\mathrm{e}^{i\omega_0 t}\psi_2^\star = 0 \tag{5.3.4}$$
$$i(\partial_0 - \partial_z)\psi_2 + i(\partial_x + i\partial_y)\psi_2 + M\mathrm{e}^{i\omega_0 t}\psi_1^\star = 0. \tag{5.3.5}$$

The first idea we can have is to do a Lorentz boost with a certain velocity in order to eliminate the x and y components. We tried to do so but we quickly saw that due to the coupling in time it was not possible. These two equations of motion are the same as the two equations for the massive case eqs. 4.1.2, 4.1.3. We shall solve the $3 \oplus 1$ dimensional case using the massive ansatz eqs. 4.1.4, 4.1.5. We shall introduce also the mass term for the fermions, since it has the same coupling as the x and y components. The equations of motion are then

$$(-\epsilon + \frac{\omega_0}{2} + i\partial_z)f_1 + M_{eff}g_1 - Mg_2^\star = 0,$$

$$(\epsilon + \frac{\omega_0}{2} - i\partial_z)f_2^\star + M_{eff}^\star g_2^\star - Mg_1 = 0,$$
$$(-\epsilon + \frac{\omega_0}{2} - i\partial_z)g_1 + M_{eff}^\star f_1 - Mf_2^\star = 0,$$
$$(\epsilon + \frac{\omega_0}{2} - i\partial_z)g_2^\star + M_{eff}^\star f_2^\star - Mf_1 = 0,$$

where $M_{eff} = (M_D - k_x) + ik_y$. These equations have the same type of solutions as the simple massive case in $1 \oplus 1$ dimensions. The interesting part for us is the solution on the left and on the right of the Q ball, it is in matrix form

$$\begin{pmatrix} -\epsilon + \frac{\omega_0}{2} & M_{eff} \\ M_{eff}^\star & \epsilon - \frac{\omega_0}{2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = p \begin{pmatrix} A \\ B \end{pmatrix}. \tag{5.3.6}$$

The eigenvectors are of the same type except for a few complex parts and the eigenvalues are $\pm\sqrt{-|M_{eff}|^2 + (\epsilon + \frac{\omega_0}{2})^2}$. The results will be the same but we replace M_D by M_{eff} in the integration bounds and we integrate with the d^3p measure instead of the $d\epsilon$ measure. We have to set this time $\epsilon = \sqrt{p_x^2 + p_y^2 + p_z^2 + M_D}$ and $\epsilon \in [M_{eff} - \frac{\omega_0}{2}, \frac{\omega_0}{2} - M_{eff}]$. This method is not the best one we can use, what we should do is use an expansion into Bessel functions rather then into Fourier series. The other difficulty using this method is the fact that evaporation occurs only when $\frac{\omega_0}{2} \geq M_D$ and here the range will be more difficult to establish. So this method with Fourier expansion has limits when we wish to study the exact 3-dimensional case.

Even if this method is not the best one we can apply it directly to the three dimension case, we just need now to integrate over the new components of momentum. In fact the big difficulty with this method will be variable changes we need to do in order to identify particles and anti-particles. The last point we have mention is the fact that using this method the complex part of the momentum inside the Q ball has no longer a simple limit.

Chapter 6

Conclusions.

In this work we wanted the properties of Q balls and their interactions with matter in order to determine if they can be good candidates for dark matter . We did this using different methods and models, and all the important results are summarised here.

As we have seen the coupling between the scalar field and fermionic field leads to particle production from the Q ball [2, 88]. To study this particle production we constructed the exact quantum-mechanical state describing the particle producing Q ball. We used Heisenberg's picture of quantum mechanics, when the state describing the produced fermions is fully characterised by the fact that no fluxes are moving towards the Q ball. This condition is solved considering the asymptotics of the fields far away from the Q ball. Using this state we constructed the Heisenberg field operator describing massless fermions produced by a Q ball. This construction allowed us to prove that for large Q balls in one space dimension the particle production does not depend on the Q ball's size. For small Q balls the particle production rate is proportional to l^2. The extension of these results to three space dimensions is simple. For large Q balls particle production is an evaporation, whereas for small Q balls it depends on the size. The other result we need to point out is that we can consider a variety of kinematical constructions to compute the evaporation rates. The first one is the standard one where we compute the reflection and transmission amplitudes for an incoming wave. The second one is based on the fact that no particles are moving towards the Q ball. We proved that these two pictures are equivalent.

The fact that fermions acquire a Dirac mass does not introduce many changes. In $1 \oplus 1$ dimensions the particle production rate does not depend on the Q ball's size once this is sufficiently large. This result is not very surprising, since taking the limit $m \to 0$ leads to the same results as the coupling with massless fermions. In this case the only difference is that evaporation occurs in a different range. The internal frequency ω_0, the energy of one single scalar forming the Q ball, must be bigger than the produced fermion mass. This result is also quite intuitive, since the scalars forming the Q ball desintegrate into fermions so their energy must be bigger than the created fermion mass. The second fact is that particle production occurs in the range mixing positive and negative frequency terms. In this range the Bogolubov transformations we build are non trivial. Using these two results we proved that evaporation can only take place in the range : $[M_D - \frac{\omega_0}{2}; \frac{\omega_0}{2} - M_D]$, with $\frac{\omega_0}{2} \geq M_D$. This result is in total accordance with the previous work done on the subject [2, 32], and extends it a significant way. This new definition for the evaporation range will introduce a new upper bound for the evaporation rate.

When the Q ball's size becomes small there is no more evaporation since the production rate depends on the size. For small sized Q balls the particle production rate is proportional to l^2. The size will also reduce the energy flux a distant observer can measure. Taking the limit $l \to \infty$ does not need any complex averaging processes since the evaporation rate is constant and l-independent for big values of the size.

We also computed all the transmission and reflection coefficients for a massive fermion being scattered by a large Q ball. This construction allowed us to compute the exact profile of the evaporation rate. Using these profiles we proved that both constructions are equivalent. The last result we have proved is that evaporation rate is proportional to $(\frac{\omega_0}{2} - M_D)^3$. These reflection and transmission amplitudes will be used to study the behaviour of Q balls in matter. In fact we obtained exact analytical results for the scattering of a massive fermion on a Q ball. This computation was not done before and allows many more applications.

As expected the interaction of Q balls and fermions can be separated into two different cases. The first one stands for interaction between the Q ball and fermions having their energy lying outside the evaporation rate. In this case we demonstrated that the interaction of Q balls and matter is nothing but standard diffusion. If we have an incident fermion we have a reflected and transmitted anti-fermion. The transmitted particles has its energy shifted by a $\omega_0/2$ factor due to interaction with the Q ball.

The second case happens when the fermion interacting with the Q ball has its energy lying inside the evaporation range. In this case we have a superposition of two phenomena the first one is diffusion while the second one is evaporation. It seems that both phenomena to not interfere and may be considered separately. This fact is important since it provides a new approach to compute Q ball evaporation rates. This new approach would be to study the diffusion of a particle on the Q ball's surface and find the two ranges where we have diffusion and both diffusion and evaporation. Then we just need to subtract both amplitudes to obtain the evaporation range.

The interaction with massive fermions does not introduce any new facts. The calculations become more complicated but the separation into two ranges remains the same, except this time the evaporation range is different since it depends on the fermion mass. In this case the other difference comes from the fact that we do not use helicity conservation to find the reflected waves.

All these calculations give a complete account of the properties of Q balls regarding their interaction with ordinary matter. The only type of interaction we did not study is the interaction of a Q ball with a scalar field. This type of interaction has no influence on our calculations since we studied stable Q balls with respect to decay into scalars. But we need this type of interaction to study the Q ball's last seconds, after evaporation into fermions the Q ball has a small enough charge to burst into scalars. This interaction would accelerate the destruction of Q balls.

It now seems that we discussed of all possible interactions of Q balls and matter, we shall mention to finish all discussions that Q ball also interact together [33]. In a real description of Q balls we should of course consider both interactions, Q balls interact with matter and with other Q balls. Studying these two interactions together would be a good way to extend this work.

The motivation for this work was the fact that Q balls can play the role of dark matter as long as they have a sufficiently long life time. Their life time will depend on their initial charge and on the evaporation rate. The evaporation rate is bounded so the Q ball's life time will only depend on its initial charge fixed by the cosmological model. In fact only the Q ball's charge fixes its fate.

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Ayant commencé par travailler en supra-conductivité j'ai gardé un grand intérêt pour ce domaine (Supercond. Sci. Technol **13**(2000) 1309-1314). Mais le domaine de la physique pour le quel j'ai le plus grand intérêt reste la théorie des champs et en particulier la creation de particules (ArXiv: hep-ph/0510078, hep-ph/0604137) sans oublier la cosmologie et l'astromomie.

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