Control of the chemostat: Some Results

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Quelques Résultats sur la Commande du Chemostat

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A la mémoire de mon grand–père Ricardo Veloso G.

Préface

Cette thèse s'attache à la commande de certains systèmes écologiques dans le chemostat, notamment un modèle de compétition entre plusieurs espèces pour un substrat et un modèle d'une chaîne trophique avec un substrat, une proie (espèce qui croît sur le substrat) et des prédateurs à différents niveaux. Ces modèles sont décrits par des systèmes d'équations différentielles ordinaires et la mise en œuvre des stratégies de commande sur ceux–ci doit prendre en compte certaines restrictions de positivité et bornitude de la commande ainsi que l'imprécision propre aux variables biologiques.

Avant d'entrer dans le détail, nous commençons par un aperçu des modèles de compétition et de chaîne trophique dans le chemostat ainsi qu'un rappel des concepts basiques de la théorie de la commande adaptée aux équations du chemostat. Ceci nous permet tout d'abord de montrer quelques applications pratiques et aussi de mettre en évidence la complexité mathématique de la commande.

Dans la première partie de ce travail nous considérons la commande robuste du chemostat simple (un substrat et une espèce). Nous étudions un chemostat qui présente des imprécisions déterministes tant dans le modèle que dans la sortie et nous proposons une famille de boucles de rétroaction qui stabilisent le modèle dans un polytope déterminé par la grandeur des imprécisions. Nous construisons aussi une famille de boucles de rétroaction qui stabilise un modèle bien connu tout en considérant des retards bornés dans la sortie.

Dans la deuxième partie de ce travail nous étudions la commande en boucle fermée d'un modèle de compétition dans le chemostat (deux espèces et un substrat). Le comportement asymptotique du modèle sans commande est décrit par le *principe d'exclusion compétitive*, lequel postule la survie d'une seule espèce et la disparition des autres. Dans un récent article, P. De Leenheer et H. Smith construisent une loi de commande en boucle fermée qui permet (sous certaines conditions suffisantes) la coexistence de deux espèces. Nous généralisons le résultat précédent dans deux directions: considération de fonctions de croissance plus générales et prise en compte de la mortalité des espèces.

La troisième partie de ce travail est relativement indépendante et est consacrée à la commande en boucle ouverte d'une chaîne trophique dans un chemostat. Nous savons que le comportement asymptotique peut devenir très complexe (notamment dans le cas d'un super prédateur) et présenter (pour un ensemble de paramètres) des attracteurs chaotiques. Nous présentons une méthode de réduction de dimension qui permet de caractériser l'ensemble d'atteignabilité du système et d'obtenir un résultat sur la commandabilité partielle de la chaîne.

La commande du chemostat a déjà fait l'objet de nombreux travaux au sein du projet COMORE (dont quelques–uns sont rappelés ci–après). V.

Lemesle [**88**] a étudié le problème de l'observabilité pour quelques modèles du chemostat et L. Mailleret [**97**] a étudié la stabilisation asymptotique robuste de certaines types de bioréacteurs. Je vois cette thèse en continuité et dialogue avec ces travaux déjà mentionnés.

Notations employées

Nous avons choisi les notations suivantes:

Notations générales

– $\mathbb{R}$ désigne l'ensemble des nombres réels.
– $\mathbb{R}_+ = \{x \in \mathbb{R} \colon x \geq 0\}$.
– $\mathbb{R}_+^* = \{x \in \mathbb{R} \colon x > 0\}$.
– $\mathbb{N}$ désigne l'ensemble des entiers.
– Pour un entier $n \geq 1$, $\mathbb{N}_n = \{1, \ldots, n\}$.
– Pour un ensemble quelconque $\mathbb{E}$ et $n \in \mathbb{N}$, $\mathbb{E}^n$ est l'ensemble produit $\mathbb{E} \times \ldots \times \mathbb{E}$ (n fois).
– Pour A un ensemble quelconque, $\mathcal{P}(A)$ est l'ensemble des parties de A.
– $\operatorname{Int} A$ est l'intérieur d'un ensemble A.
– $\operatorname{Cl} A$ est la fermeture d'un ensemble A.
– ∂A est la frontière d'un ensemble A.

Ensembles des fonctions

– Pour $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}^n$ et $k \in \mathbb{N}$, $C^k(A, B)$ désigne l'ensemble des fonctions $f \colon A \mapsto B$, k–fois différentiables et continues.
– Pour $E \subseteq \mathbb{R}^n$, $F \subseteq \mathbb{R}^k$, $BC(E, F)$ désigne l'ensemble des fonctions $f \colon E \mapsto F$, continues et bornées.
– Pour $E \subseteq \mathbb{R}^n$, $F \subseteq \mathbb{R}^k$, $L^\infty(E, F)$ désigne l'ensemble des fonctions $f \colon E \mapsto F$, mesurables sur E et il existe C tel que $||f(x)|| \leq C$ p.p. sur E et $|| \cdot ||$ désigne une norme dans $\mathbb{R}^k$.
– Pour $I = [t_0, t_1] \subset \mathbb{R}$ et $B \subseteq \mathbb{R}^n$, $\operatorname{Car}(I, B)$ désigne l'ensemble des fonctions $f \colon I \times B \mapsto \mathbb{R}^n$ qui satisfont les conditions de Carathéodory. C'est–à–dire:

(C1) Pour tout $b \in B$ fixe, la fonction $f(\cdot, b) \colon I \mapsto \mathbb{R}^n$ est mesurable au sens de Lebesgue sur I.

(C2) Pour tout $t \in I$ fixe, la fonction $f(t, \cdot) \colon B \mapsto \mathbb{R}^n$ est continue sur B.

(C3) Pour chaque ensemble compact $K \subset B$, la fonction $h_K \colon I \mapsto \mathbb{R}$ définie par

$$h_K(t) = \sup \Big\{ ||f(t, b)||, b \in B \Big\}$$

est mesurable au sens de Lebesgue sur I.

Contents

CHAPTER 1

Introduction

L'Introduction est essentiellement consacrée à dresser un aperçu de la modélisation de quelques processus biologiques dans le chemostat, à introduire les notions de base de la théorie de la commande (commandabilité, stabilisation asymptotique, ensemble atteignable) et à montrer à quoi elles correspondent dans le cas spécifique du chemostat.

Pour cela nous avons partagé l'introduction en deux parties: la première partie décrit le modèle de compétition dans le chemostat (section 1.1) et ses propriétés asymptotiques (section 1.2) ainsi qu'un modèle de chaîne trophique (section 1.3). Bien évidemment, elle ne constitue pas une revue exhaustive, mais elle présente différents résultats déjà classiques qui ont servi de base à nos travaux.

La deuxième partie de notre introduction considère le modèle du chemostat sous l'approche *système entrée–sortie* issue de la théorie de la commande. Nous présentons quelques notions essentielles de cette théorie dans la section 1.4 et introduisons la problématique de la robustesse dans la section 1.5.

1.1. Modèle du chemostat

1.1.1. Le Chemostat. Le chemostat est un appareil de laboratoire qui permet la culture continue de micro–organismes dans un milieu contrôlé, c'est–à–dire qui permet de faire croître une population de micro–organismes (algues unicellulaires, bactéries, levures, moisissures) sur certains substrats, tout en conservant des conditions environnantes (température, luminosité, pH, aération). Il est utilisé pour la production de la masse cellulaire elle–même, pour l'extraction et la dégradation de certains polluants dans un milieu liquide, pour la production de substances organiques résultantes de l'activité métabolique, ou pour l'étude de procédés physiologiques et métaboliques de micro–organismes dans un milieu spécifique. Pour des références plus précises nous renvoyons le lecteur aux travaux [**31**],[**49**],[**107**],[**124**, Ch.5][**133**].

L'appareil est constitué de trois réservoirs reliés entre eux (voir Fig.1.1.1). Le premier contient des nutriments supposés être en abondance excepté un parmi eux, nommé *substrat limitant* à densité $s(t)$. Un débit F (provenant du premier réservoir) alimente le chemostat (deuxième réservoir) où interagissent et se mélangent la (ou les) biomasse(s) (une ou différentes espèces de micro–organismes à densité $x_i(t)$) avec des nutriments. De plus, il est supposé que le substrat limitant a une concentration constante $s_{in} > 0$.

On fera l'hypothèse suivante[1]:

[1]Au long de cette introduction, nous ferons quelques suppositions simplificatrices sur le modèle du chemostat, celles–ci seront écrites en italique et encadrées.

($\mathcal{A}1$) Homogénéité du milieu liquide
Le deuxième réservoir (chemostat) est supposé être parfaitement mélangé, afin que l'on puisse assurer que le milieu liquide est homogène et par conséquent qu'il n'y a pas de variation spatiale dans la concentration du substrat limitant et des espèces.

Le troisième réservoir (optionnel) réceptionne les échantillons prélevés dans le chemostat avec un débit F.

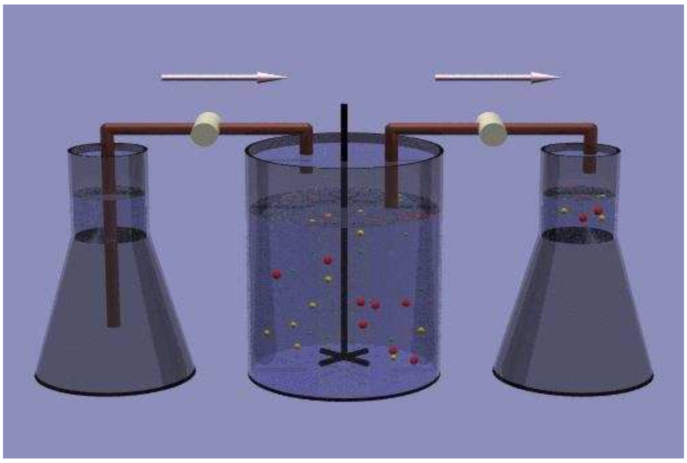

FIGURE 1.1.1. Schéma du chemostat, image disponible sur: *www.bio.vu.nl/thb/deb/deblab/chain/model.html*

C'est dans le deuxième réservoir qu'a lieu l'interaction entre le substrat limitant et les micro–organismes; parmi l'ensemble des procédés biologiques et chimiques susceptibles d'avoir lieu, on ne considérera que les suivants:

- La consommation du nutriment par les espèces de micro–organismes.
- La croissance microbienne.

1.1.2. Équations du chemostat. La variation de la masse du substrat limitant dans le deuxième réservoir est donnée par le bilan du substrat entrant, celui du substrat prélevé et celui du substrat consommé par les espèces de micro–organismes:

$$V s'(t) = \underbrace{F s_{in}}_{Entrant} - \underbrace{F s(t)}_{Sortant} - \underbrace{[c_1(\cdot)x_1(t) + \ldots c_n(\cdot)x_n(t)]}_{Consommation} \tag{1.1.1}$$

• La constante V est le volume du chemostat.
• $c_i\colon \mathbb{R}^k \mapsto \mathbb{R}$ $(k \leq n)$ est une *fonction de consommation* du substrat limitant par la i–ième espèce de micro–organismes.

La variation de la masse de la i–iéme espèce de micro–organisme est donnée par le bilan de la masse prélevé, celle de la biomasse morte et celle de la masse issue de la croissance microbienne:

$$Vx_i'(t) = \underbrace{\bar{f}_i(\cdot)x_i(t)}_{Croissance} - \underbrace{Fx_i(t)}_{Prelevement} - \underbrace{\bar{d}_i x_i(t)}_{Mortalite} . \tag{1.1.2}$$

• $\bar{f}_i \colon \mathbb{R}^k \mapsto \mathbb{R}$ $(k \leq n)$ est une *fonction de croissance* de la i–éme espèce de micro–organismes.
• $\bar{d}_i$ est le taux de mortalité de la i–ème espèce.

Il existe une vaste littérature consacrée à la modélisation des fonctions de croissance et de consommation. Tout d'abord nous ferons quelques hypothèses concernant la croissance des espèces et la consommation du substrat dans le chemostat:

($\mathcal{A}2$) Hypothèses de croissance et consommation

(i) *La vitesse de croissance est non–négative et nulle dans l'absence de substrat.*
(ii) *La relation entre la consommation et la croissance est linéaire.*
(iii) *La fonction de croissance dépend continument et uniquement de la concentration du substrat, $(\bar{f}_i = \bar{f}_i(s))$.*

($\mathcal{A}3$) Absence de recyclage de la biomasse
Nous supposerons que le taux de recyclage de biomasse morte en nouveau substrat limitant est négligeable.

Il découle des hypothèses (i)-(iii) que la fonction $\bar{f}_i$ est continue, nulle en 0 et prend des valeurs positives. De plus, d'après (ii)-(iii) on déduit:

$$c_i(s(t)) = \alpha_i \bar{f}_i(s(t)) \tag{1.1.3}$$

ou $\alpha_i > 0$ est un **coefficient de rendement**.

Soit $D = F/V$, $d_i = \bar{d}_i/V$ et $f_i = \bar{f}_i/V$, alors on déduit les équations du chemostat:

$$\begin{cases} \dot{s} \ = D(s_{in} - s) - \sum\limits_{i=1}^{n} \alpha_i f_i(s) x_i, \\ \dot{x}_i = x_i\big(f_i(s) - D - d_i\big), \quad i \in \mathbb{N}_n. \end{cases} \tag{1.1.4}$$

Remarque 1.1.1. *Nous voulons souligner la différence qualitative entre deux classes de paramètres du système* (1.1.4)*: Les paramètres α_i et d_i sont déterminés exclusivement par les propriétés biologiques du système. D'un autre côté, les paramètres s_{in} et D peuvent être modifiés par l'utilisateur du chemostat (voir quelques résultats expérimentaux dans* [**75**],[**74**] et [**34**] *où l'expérimentateur les modifie plusieurs fois tandis que les autres paramètres restent inchangés). Cette remarque peut apparaître comme une subtilité de notre part. Il n'est en rien, car cette distinction entre les paramètres deviendra très importante dans le contexte de la théorie de la commande.*

Pour l'étude du système (1.1.4) nous avons besoin d'une connaissance plus approfondie des fonctions f_i. Ci–après nous faisons un bref résumé de la modélisation de celles–ci.

1.1.3. Les fonctions de croissance et de consommation. La modélisation des fonctions de croissance et de consommation est une des principales difficultés dans cette modélisation du chemostat, car c'est une fonction complexe comprenant nombreux facteurs physico-chimiques et biologiques. De plus elle est fortement dépendente du substrat et des espèces de biomasse particulières.

C'est dans un contexte expérimental que plusieurs expressions analytiques pour la fonction $f(s)$ ont été obtenues, nous ne ferons ici qu'une rapide présentation de quelques unes. Pour une liste plus exhaustive, nous renvoyons le lecteur à [**10**].

En étudiant les aspects quantitatifs de la croissance des cultures bactériennes en fonction de la concentration du substrat limitant, J. Monod (voir [**105**, pags.61–78]) présente (1942) des résultats expérimentaux relatifs à la croissance de *E. coli* pour trois types différents de substrat limitant: glucose, mannite et lactose.

Pour ces résultats, Monod propose la courbe suivante d'interpolation donnée par la fonction:

$$f(s) = \frac{\mu_m s}{k_s + s}, \quad \mu_m > 0 \text{ et } k_s > 0 \tag{1.1.5}$$

où la constante μ_m représente le *taux de croissance maximum.* La constante k_s est appelée *constante de demi–saturation* car $f(k_s) = \mu_m/2$.

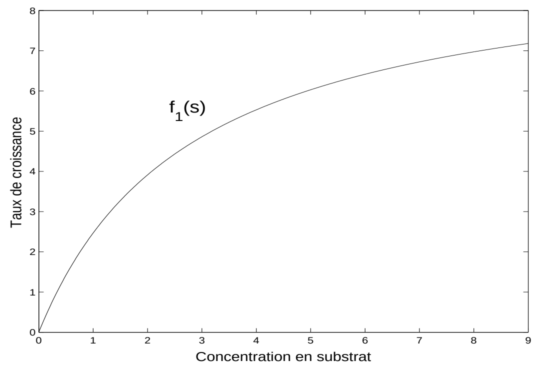

FIGURE 1.1.2. Graphe du fonction f définie par l'équation (1.1.5).

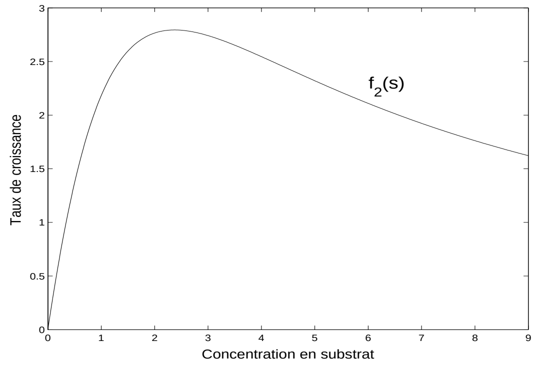

FIGURE 1.1.3. Graphe du fonction f définie par l'équation (1.1.6).

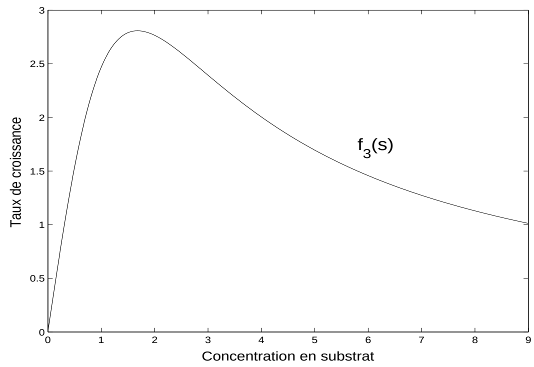

FIGURE 1.1.4. Graphe du fonction f définie par l'équation (1.1.7).

La fonction définie par l'équation (1.1.5) était à l'origine utilisée pour décrire la vitesse des réactions enzymatiques (voir par exemple [**101**]) et a été étendue par Monod pour décrire la croissance bactérienne et la consommation du substrat. C'est pour cette raison qu'elle est connue comme fonction de Monod ou fonction de Michaelis–Menten. Il y a eu un très grand effort de généralisation de celle–ci, par exemple par des fonctions qui ont des propriétés qualitatives similaires, notamment la monotonie et l'existence d'une asymptote horizontale.

D'autre part, divers travaux concernant des procédés biologiques de traitement biologique des eaux usées soulignent l'importance de considérer des fonctions non–monotones dans la modélisation de la croissance bactérienne.

En étudiant les données expérimentales sur la croissance de *Nitrobacter winogradski* avec du nitrate comme substrat limitant, B. Boon et H. Laudelot [**14**] (1962) suggèrent que la fonction proposée par Monod ne pourrait pas être valable pour certains substrats qui sont limitant à concentrations modérées mais qui à forte concentration s'avèrent inhibiteurs pour la croissance des espèces.

L'utilisation des fonctions monotones devrait être en conséquence un cas spécial d'une relation fonctionnelle entre le substrat limitant et la fonction de croissance. Pour le cas avec inhibition, ils proposent (consulter aussi [**2**] et [**35**]) une courbe d'interpolation définie par la fonction:

$$f(s) = \frac{\mu_m s}{k_s + s + \frac{s^2}{k_i}}, \quad \mu_m > 0, k_s > 0 \text{ and } k_i > 0 \tag{1.1.6}$$

comme dans l'équation (1.1.5), la constante μ_m représente le *taux de croissance maximum* et la constante k_s est la *constante de demi–saturation.* De plus, la constante k_i est la *constante d'inhibition*, il faut noter que si $k_i = 0$, l'équation (1.1.6) est équivalente à (1.1.5).

Enfin, nous voulons citer la fonction obtenue en 1981 par Sokol et Howell [**134**] en étudiant la croissance de *Pseudomonas putida* sur Phenol:

$$f(s) = \frac{k_1 s}{k_2 + s^{k_3}}, \quad k_1 > 0, k_2 > 0 \text{ and } k_3 > 1. \tag{1.1.7}$$

Dans la suite, nous allons étudier quelques propriétés qualitatives et asymptotiques du système (1.1.4).

1.2. Quelques propriétés asymptotiques

1.2.1. Un principe de conservation. Un cas spécial que nous rencontrerons à plusieurs reprises dans cette thèse, est vérifié quand les taux de mortalité sont négligeables; c'est–à–dire $d_i = 0$ pour chaque $i \in \mathbb{N}_n$. D'après la transformation:

$$v = s - s_{in} + \sum_{i=1}^{n} \alpha_i x_i$$

on vérifie aisément que le système (1.1.4) est équivalent au système:

$$\begin{cases} \dot{x}_1 = x_1\Big(f_1(v + s_{in} - \sum_{i=1}^{n} \alpha_i x_i) - D\Big), \\ \dot{x}_2 = x_2\Big(f_2(v + s_{in} - \sum_{i=1}^{n} \alpha_i x_i) - D\Big), \\ \quad\vdots \\ \dot{x}_n = x_n\Big(f_n(v + s_{in} - \sum_{i=1}^{n} \alpha_i x_i) - D\Big), \\ \dot{v} \ = -Dv. \end{cases} \tag{1.2.1}$$

Il en résulte que $\lim_{t\to+\infty} v(t) = 0$. Alors, quelques propriétés asymptotiques du système (1.2.1) peuvent être déduites en étudiant le système du type Kolmogorov:

$$
\begin{cases}
\dot{x}_1 = x_1\Big(f_1(s_{in} - \sum\limits_{i=1}^{n} \alpha_i x_i) - D\Big), \\
\dot{x}_2 = x_2\Big(f_2(s_{in} - \sum\limits_{i=1}^{n} \alpha_i x_i) - D\Big), \\
\qquad\vdots \\
\dot{x}_n = x_n\Big(f_n(s_{in} - \sum\limits_{i=1}^{n} \alpha_i x_i) - D\Big).
\end{cases} \tag{1.2.2}
$$

En effet, si les solutions du système (1.2.2) sont convergentes vers un point d'équilibre on peut utiliser quelques résultats de *réduction d'ordre* pour en déduire la convergence des solutions du système (1.2.1) vers un point critique. Nous renvoyons le lecteur aux références [**146**],[**147**] et l'Annexe B pour plus de précisions sur ces résultats.

En plus, étant donné que les solutions du système (1.2.1) sont bornées, celui–ci est *asymptotiquement autonome* (nous renvoyons le lecteur aux références [**100**],[**132**],[**139**],[**140**] et l'Annexe B pour plus de renseignements) et a une limite définie par le système (1.2.2). Tandis que la méthode précédente permet d'obtenir des résultats de stabilité globale des points critiques, l'utilisation de la théorie des systèmes dynamiques asymptotiquement autonomes permet une étude plus approfondie des propriétés asymptotiques des solutions du système (1.2.1). Par exemple, si $n = 2$, il existe une généralisation du théorème de Poincaré–Bendixson.

Nous utiliserons à plusieurs reprises ces deux techniques dans notre étude des équations du chemostat.

Dans le cas où les fonctions f_i du système (1.2.2) sont non–décroissantes, on vérifie aisément que les éléments en dehors de la diagonale de la matrice jacobienne sont négatifs. C'est–à–dire:

$$\frac{\partial F_i}{\partial x_j}(x_1, \ldots, x_n) \leq 0 \quad i \neq j$$

où F_i est définie par: $F_i(x_1, \ldots, x_n) = x_i f_i\Big(s_{in} - \sum\limits_{k=1}^{n} \alpha_k x_k\Big) - x_i D$.

D'une perspective mathématique, les systèmes avec cette propriété sont appelés *compétitifs* (voir par exemple [**51**],[**128**],[**130**],[**131**]). Dans une perspective écologique cette propriété a été résumée par S.Smale dans [**128**] avec l'expression *crowding inhibits growth*. C'est–à–dire que la croissance d'une espèce x_i inhibe la croissance de toute espèce x_j $(j \neq i)$.

Il faut souligner que les systèmes compétitifs ont été beaucoup employés en biologie mathématique; pour une présentation exhaustive de l'état de l'art nous renvoyons le lecteur aux travaux [**131**],[**153**].

1.2.2. L'exclusion compétitive. La compétition entre n espèces sur k ressources est un sujet très étudié en écologie théorique. Un postulat classique (voir par exemple [**7**],[**40**],[**58**] [**102**]) propose que si $n > k$, alors au moins $n-k$ espèces ne peuvent pas coexister à long terme: ce postulat est connu comme le *Principe d'exclusion compétitive.*

Dans le cas du système (1.1.4), le principe d'exclusion compétitive signifie que, au plus, une espèce x_j (avec $j \in \mathbb{N}_n$) de micro–organismes est capable de survivre à long terme, tandis que les autres $n-1$ doivent disparaître. Ce principe a été démontré mathématiquement (voir par exemple [**6**],[**19**],[**70**],[**92**],[**133**],[**149**]) et validé dans plusieurs expériences (voir la table ci–dessous).

Nous ferons une hypothèse complémentaire pour les fonctions de croissance f_i: pour chaque $i \in \mathbb{N}_n$, il existe au plus deux nombres réels étendus λ_i et μ_i tels que $f_i(\lambda_i) = f_i(\mu_i) = D + d_i$ et:

$$\begin{cases} f_i(s) < D + d_i, & \text{if} \quad s \notin [\lambda_i, \mu_i], \\ f_i(s) > D + d_i, & \text{if} \quad s \in (\lambda_i, \mu_i). \end{cases} \tag{1.2.3}$$

Le comportement asymptotique du système (1.1.4) est décrit plus en détail en [**92**],[**149**], nous rappelons leur résultat principal:

Proposition 1 (Wolkowicz et Lu (1992)[**149**]). *On suppose que l'ensemble* $\{\lambda_i\}_{i=1}^{n}$ *vérifie les inégalités* $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$ *et* $\lambda_1 < s_{in} < \mu_1$*. S'il est possible trouver des constantes* $\gamma_i > 0$ *pour chaque* $i \geq 2$ *satisfaisant* $\lambda_i < s_{in}$ *telles que*:

$$\max_{0<s<\lambda_1} g_i(s) \leq \gamma_i \leq \min_{\lambda_i<s<\rho_i} g_i(s), \tag{1.2.4}$$

où $\rho_i = \min\{s_{in}, \mu_i\}$ *et* g_i *est définie par*:

$$g_i(s) = \frac{f_i(s)(-[D+d_1] + f_1(s))(s_{in} - \lambda_1)}{[D+d_1](-[D+d_i] + f_i(s))(s_{in} - s)}, \tag{1.2.5}$$

alors toutes les solutions du système (1.1.4) *vérifient la propriété suivante*:

$$\lim_{t\to+\infty} (s(t), x_1(t), x_2(t), \ldots, x_n(t)) = \Big(\lambda_1, \frac{D(s_{in} - \lambda_1)}{\alpha_1[D+d_1]}, 0, \ldots, 0\Big)$$

ainsi, le point critique $(\lambda_1, D(s_{in} - \lambda_1)/\alpha_1(D + d_1), 0, \ldots, 0)$ *est globalement asymptotiquement stable.*

Nous présentons aussi quelques résultats expérimentaux qui vérifient l'exclusion compétitive pour deux espèces:

Compétiteur 1	Compétiteur 2	Substrat	Référence
E. coli	*A. vinelandii*	Glucose	[**74**] (1973)
E. coli	*P. aeruginosa*	Tryptophan	[**57**] (1980)
S. mutans	*S. sanguins*	Glucose	[**77**] (1983)
P. putida	*P. resinovorans*	Phenol	[**113**] (1989)
C. utilis	*S. cerevisiae*	Glucose	[**30**] (2004)

1.2.3. Persistance. On s'intéresse ici à un système biologique dans lequel les variables du modèle présentent un comportement asymptotique opposé à l'exclusion compétitive: en effet, dans ce système, toutes les variables x_i ($i \in \mathbb{N}_n$) qui représentent la concentration d'une espèce ne sont pas convergentes vers zéro. Cette notion capture une idée fondamentale dans l'écologie théorique qui est la survie des espèces dans un écosystème.

Definition 1.2.1 (Butler *et al.* [**17**], [**20**]). Le système (1.1.4) est

(a) *Faiblement Persistant* si pour chaque solution du système on a $\limsup_{t\to+\infty} x_i(t) > 0$, pour chaque $i \in \mathbb{N}_n$.
(b) *Persistant* si pour chaque solution du système on a $\liminf_{t\to+\infty} x_i(t) > 0$, $i \in \mathbb{N}_n$.
(c) *Uniformément Persistant* s'il existe un nombre $\delta_0 > 0$ (indépendant des conditions initiales) tel que $\liminf_{t\to+\infty} x_i(t) > \delta_0$, $i \in \mathbb{N}_n$.

Ces définitions peuvent être généralisées pour les solutions des équations différentielles à retard, équations différentielles stochastiques et équations aux dérivées partielles; pour plus de précisions nous renvoyons le lecteur à [**39**],[**56**],[**26**],[**38**].

Pour obtenir des résultats de persistance uniforme il existe deux types de techniques très employées: la première est l'analyse du semiflot défini par l'équation différentielle (1.1.4) dans $\partial\mathbb{R}^n_+$ (résultats du type théorème de Butler–McGehee [**133**],[**140**]) et la deuxième est la construction des *fonctionnelles de Lyapunov moyennisantes* (voir [**62**],[**63**],[**64**]), nous renvoyons le lecteur à l'Annexe A pour une explication plus approfondie de cette dernière technique.

1.3. Chaîne trophique dans un chemostat

Dans cette section on étudiera une chaîne n–trophique dans le chemostat, décrite par le système suivant:

$$
\begin{cases}
\dot{s} &= D(s_{in} - s) - \alpha_1 f_1(s) x_1, \\
\dot{x}_1 &= x_1 f_1(s) - (D + d_1) x_1 - \alpha_2 x_2 p_2(x_1), \\
\dot{x}_i &= x_i p_i(x_{i-1}) - (D + d_i) x_i - \alpha_{i+1} x_{i+1} p_{i+1}(x_i), \\
\dot{x}_n &= x_n p_n(x_{n-1}) - (D + d_n) x_n.
\end{cases}
\tag{1.3.1}
$$

pour chaque nombre entier $i \in \mathbb{N}_{n-1}$. La variable x_i désigne l'espèce du i–ième niveau trophique. Comme avant, les coefficients s_{in},D et d_i representent la concentration du substrat limitant, le taux de dilution et le taux de mortalité de la i–ième espèce. Les fonctions $p_i \colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ (avec $i = 2, \ldots, n$) décrivent la croissance du prédateur x_{i+1} sur la proie x_i.

L'espèce x_i est dans le i–ième niveau trophique et est en même temps le prédateur de l'espèce du $(i-1)$–ième niveau trophique et la proie pour l'espèce du $(i+1)$–ième niveau trophique.

En ce qui concerne la modélisation des fonctions de prédation p_i, dans le cadre de l'écologie théorique il y a une classification fournie par C.S. Holling dans une série de travaux (voir par exemple [**65**],[**66**],[**67**]) relatifs à la modélisation des systèmes du type proie–prédateur, dans lesquels on propose une classification des différents types de prédation. Néanmoins, par simplicité, nous allons suivre l'approche géométrique proposée dans [**27**, chap. 11] pour déduire ces trois types de prédation.

On fera les hypothèses suivantes sur la prédation:

- On considère le mouvement d'un prédateur comme la trajectoire d'un disque dans le plan. Ce disque a un rayon r dépendant du nombre V des proies existantes (*i.e.* $r = r(V)$) et est centré au prédateur. De plus, le prédateur a une vitesse constante s. L'aire parcourue par le disque dans un temps T_s est donnée par $2\pi r(V)sT_s$.
- On suppose que le prédateur n'est pas capable de repérer toutes les proies de la région parcourue: il existe un coefficient de détectabilité $k \in [0,1]$ qui correspond au quotient (Espèces repérées/Total d'espèces).
- Soit V le nombre de proies existantes dans l'aire parcourue par le prédateur. Alors, le nombre des proies rencontrées V_a est donné par:

$$V_a = a(V)T_sV \quad \text{avec } a(V) = 2\pi kr(V)s.$$

- Le prédateur essaie de capturer toutes les proies détectées. De plus, il existe un coefficient de capturabilité $\mu \in [0,1]$. Alors, le nombre des proies capturées est donné par l'équation:

$$V_c = \mu V_a = \mu a(V)T_sV. \tag{1.3.2}$$

- Le temps T_s parcouru par le prédateur est décomposé de la façon suivante:

$$T_s = \underbrace{T_t}_{Consommation} - \underbrace{T_cV_a}_{Chasse} - \underbrace{\mu T_mV_a}_{Manipulation} \tag{1.3.3}$$

Si on suppose que le rayon du cercle est constant (*i.e.* $a(V) = a$) et $T_c = T_m = 0$, c'est–à–dire, le temps de chasse et manipulation de la proie est négligeable par rapport à la consommation, on en déduit d'après les équations (1.3.2)–(1.3.3) que le nombre des proies capturées est donné par la fonction du **type I**:

$$p(V) = aT_tV.$$

Néanmoins, Holling suggère [**65**, pages 315–316] que cette fonction ne fournit un modèle valable que dans un intervalle des proies $v \in [0, V^*]$ qui détermine la *phase de croissance* de la population des prédateurs. En s'appuyant sur des données expérimentales, il suggère (voir Figure 8 dans

[**65**]) une fonction du **type I généralisée**:

$$p(V) = \begin{cases} aT_tV & \text{si } V \in [0, V^*] \\ aT_tV^* & \text{si } V \geq V^*. \end{cases}$$

Si on suppose que le rayon du cercle est constant (*i.e.* $a(V) = a$), mais $T_c > 0$ et $T_m > 0$, on en déduit d'après les équations (1.3.2)–(1.3.3) que le nombre des proies est donné par la fonction du **type II**:

$$p(V) = \frac{aT_tV}{aV + [T_c + \mu T_m]}.$$

Si on suppose que le rayon du cercle est une fonction du type $r(V) = rV^n$ avec $n > 0$ (*i.e.* $a(v) = aV^n$), mais $T_c > 0$ et $T_m > 0$, on en déduit d'après les équations (1.3.2)–(1.3.3) que le nombre des proies est donné par la fonction du **type III**:

$$p(V) = \frac{aT_tV^n}{aV^n + [T_c + \mu T_m]}.$$

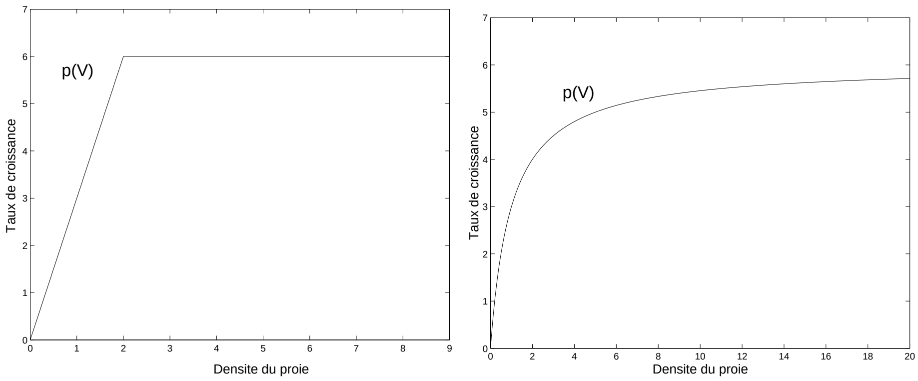

FIGURE 1.3.1. Graphique des fonctions du Holling type I (Gauche) et Holling type II (Droite)

La plupart des résultats sur le système (1.3.1) n'ont considéré que deux ou trois niveaux trophiques (c'est–à–dire $n = 2$ ou $n = 3$). D'abord nous présentons une liste de quelques travaux représentatifs concernant les chaînes bi–trophiques (*i.e.* $n = 2$):

Fonction f_1	Fonction p_2	Référence
Monod	Type II	[**23**] (1969)
Monod	Type II	[**24**] (1970)
Monod	Type II	[**75**] (1973)
Monod	Type II et Type III	[**74**] (1973)
Inconnue	Inconnue	[**34**] (1976)
Monod	Type II et Type III	[**137**] (1981)
Monod	Type II	[**83**] (1989)
Monod	Type II	[**93**] (2000)

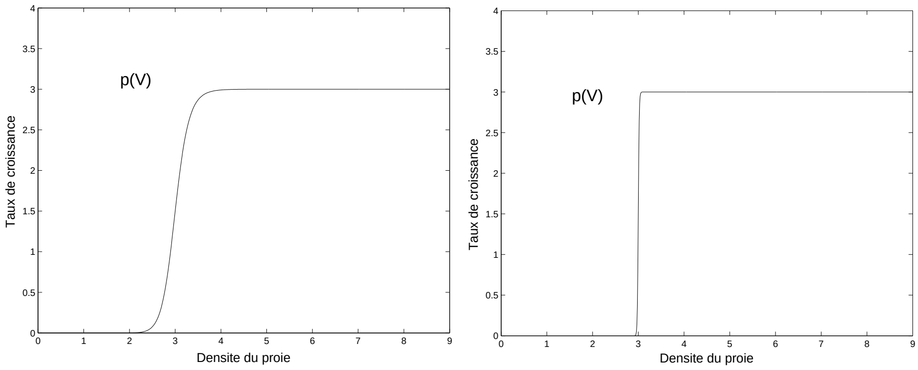

FIGURE 1.3.2. Graphique des fonctions du type Holling III pour $n = 3$ (Gauche) et $n = 60$ (Droite)

La plupart des résultats obtenus dans [**23**],[**24**],[**75**],[**74**],[**34**],[**137**] et [**83**] sont présentés dans [**133**, chap.3]. La mortalité n'est pas prise en compte (sauf dans la référence [**93**]) et, en utilisant la théorie des systèmes dynamiques asymptotiquement autonomes, on démontre que l'ensemble ω–limite du système (1.3.1) est défini par la trichotomie de Poincaré–Bendixson (voir Annexe B).

Alors, l'ensemble ω–limite d'une condition initiale du système est soit un cycle limite soit un des points d'équilibre suivantes:

$E_0 = (s_{in}, 0, 0)$ (lessivage du chemostat),
$E_1 = (s_1^*, x_1^*, 0)$ (extinction du prédateur),
$E^c = (s^c, x_1^c, x_2^c)$ (coexistence de la proie et le prédateur).

Les propriétés de stabilité de ces points peuvent être explicitées avec l'aide d'un diagramme de bifurcation (s_{in}, D).

Le même comportement asymptotique est retrouvé dans [**93**] où l'hypothèse $d_i = 0$ est abandonnée. On obtient des conditions suffisantes pour la stabilité globale des points critiques à l'aide des fonctions de Lyapunov et on obtient des conditions pour l'existence (unicité dans [**83**]) des cycles limites avec le théorème de bifurcation de Hopf.

Les résultats expérimentaux présentés dans [**75**],[**74**] et [**34**] montrent quelques limitations du modèle décrit par le système (1.3.1): la première objection a été levée dans [**34**] où on observe l'adhérence de l'espèce x_1 (*E. coli*) aux parois du chemostat quand la concentration de substrat devient trop élevée. D'autre part, dans [**137**], on remarque l'inconsistance entre quelques résultats expérimentaux qui montrent comment le lessivage du chemostat peut être atteint dans certains régimes oscillatoires tandis que théoriquement il serait impossible car E_0 est répulsif dans un régime oscillatoire. Ces types de problèmes suggèrent l'utilisation d'autres outils comme des modèles spatiaux ou stochastiques.

Maintenant, nous présentons une liste de quelques travaux représentatifs concernant les chaînes tri-trophiques (*i.e.* $n = 3$):

Fonction f_1	Fonction p_2	Fonction p_3	Référence
Monod	Type II	Type II	[**47**] (1998)
Monod	Type II	Type II	[**13**] (2000)
Quelconque	Quelconque	Quelconque	[**36**] (2005)

Les éventuels points critiques du système (1.3.1) sont:

$E_0 = (s_{in}, 0, 0, 0)$ (lessivage du chemostat),
$E_1 = (s_1^*, x_1^*, 0, 0)$ (extinction des prédateurs),
$E_2 = (s_2^*, y_1^*, y_2^*, 0)$ (extinction du super–prédateur),
$E^c = (s^c, x_1^c, x_2^c, x_3^c)$ (coexistence de la proie et les prédateurs).

Dans [**36**] El–Sheik et Mahrouf établissent des conditions nécessaires et suffisantes pour l'existence et la stabilité (locale et globale) des points critiques enoncées ci–dessus. De plus, en utilisant le théorème de bifurcation de Hopf on établit des conditions suffisantes pour l'existence des solutions périodiques.

Dans [**47**] Gragnani, De Feo et Rinaldi considèrent s_{in} et D comme des paramètres de bifurcation et font une étude des régions du plan s_{in}–D. Une différence remarquable par rapport au comportement asymptotique de la chaîne bi–trophique est l'existence de certaines couples (s_{in}, D) pour lesquelles on démontre l'existence d'attracteurs chaotiques (voir figures 3 et 4 dans [**47**]).

En ce qui concerne la recherche sur les chaînes n–trophiques ($n \geq 3$) il y a des résultats préliminaires qui démontrent l'existence d'attracteurs chaotiques. Nous renvoyons le lecteur intéressé à [**50**].

1.4. La commande du chemostat

Dans la section précédente on a considéré les modèles de compétition et la chaîne trophique dans un chemostat comme des systèmes d'équations différentielles (1.1.4) et (1.3.1) dans lesquelles on pouvait étudier les propriétés qualitatives (transitoire, stabilité, positivité, comportement asymptotique) mais où on ne pouvait pas agir sur l'évolution des variables du système car elles étaient déterminées par les conditions initiales et la configuration des paramètres.

Dorénavant nous nous plaçons dans un cadre plus général car les systèmes d'équations (1.1.4) et (1.3.1) contiennent de termes représentant des actions faites par l'homme et qui permettent de modifier le comportement du système (par exemple dans [**75**],[**74**] et [**34**] l'expérimentateur modifie s_{in}

et/ou D). Nous désignerons ce termes par u. Nous ferons la distinction entre deux types de variables: les concentrations de substrat et biomasse $(s, x_1, \ldots, x_n)$ seront appelées *variables d'état* et les termes dénotés par u seront appelés *variables de commande*.

Le but de la théorie de contrôle est alors, de déterminer la variable de commande u adéquate pour que (s, x_i) aient un comportement acceptable en fonctions des objectifs établis *a priori*.

Dans un premier temps, nous allons formaliser l'idée énoncée ci–dessus et pour cela on va se familiariser avec la théorie de la commande en présentant quelques notions de base (pour plus de précisions, nous renvoyons le lecteur intéressé par ce sujet aux références [**135**],[**9**],[**60**]). Puis, on fera une révision de l'état de l'art sur la commande d'un chemostat. Ensuite, on s'intéressera à la robustesse de la commande, compte tenu des particularités des systèmes biologiques.

Dorénavant, nous considérons le chemostat comme un système entrée–sortie; c'est–à–dire, un système composé de trois éléments:

(i) Une *entrée* ou *variable de commande*, définie dans notre cas par la variable $u \in \mathbb{R}^k$ (avec $k < n$).

(ii) Un *système* défini par les équations du chemostat, décrites de la façon suivante:

$$\begin{cases} \dot{z} = F(z, u), \\ z = (s, x_1, \ldots, x_n)^T, \quad u = (u_1, \ldots, u_k) \in \mathbb{R}^k. \end{cases} \tag{1.4.1}$$

où z est la *variable d'état*.

(iii) Une *sortie* $y(t) = h(t, z)$, décrite par la fonction $h\colon \mathbb{R}_+ \times \mathbb{R}^n_+ \mapsto \mathbb{R}^m_+$ (avec $m < n$), cette fonction est mesurable au sens de Lebesgue par rapport à t et continue par rapport aux variables (s, x_i) $(i \in \mathbb{N}_n)$ et correspond aux mesures qui ont été prises.

Un problème majeur dans l'étude des systèmes entrée/sortie est le développement des stratégies de commande de manière à élaborer des entrées et sorties convenables en fonction des objectifs. Il existe deux stratégies de commande: *commande en boucle ouverte* et *commande en boucle fermée*:

Definition 1.4.1. Le système (1.4.1) a une:

- *Commande en boucle ouverte* si l'entrée $u(\cdot)$ est une fonction mesurable dépendant du temps, *i.e.* $u = u(t)$ (c'est–à–dire $u(\cdot)$ ne dépend pas de la variable z) et la fonction $(t, z) \in \mathbb{R}_+ \times \mathbb{R}^n \mapsto F(z, u(t)) \in Car(\mathbb{R}_+, \mathbb{R}^n)$.

- *Commande en boucle fermée* si l'entrée $u(\cdot)$ est une fonction continue dépendant de la sortie, *i.e.* $u = u(y(t)) = u(h(t, z))$ appelée *boucle de rétroaction* et la fonction $(t, z) \in \mathbb{R}_+ \times \mathbb{R}^n \mapsto F(z, u(y(t))) \in Car(\mathbb{R}_+, \mathbb{R}^n)$.

Il faut souligner qu'en remplaçant u par $u(t)$ ou $u(y)$ dans le système (1.4.1), l'existence des solutions du système contrôlé est assurée par le conditions de Carathéodory (voir *e.g.* [**28**, Chap.2]). Par ailleurs, la continuité de $u(y)$ n'est pas une condition essentielle; dans ce cas il s'agirait d'un commande discontinu en boucle fermée. Nous renvoyons le lecteur intéressé à [**37**].

($\mathcal{A}4$) Unicité des solutions du système contrôle (1.4.1)
La fonction F est telle que la solution est unique et définie sur un intervalle maximal $[0, +\infty)$. Des précisions (e.g. conditions de Lipschitz) seront données par la suite.

Definition 1.4.2. Le système (1.4.1) est:

(a) *Commandable* ou *contrôlable* si pour toute couple de vecteurs z_0 et z_f de $\mathbb{R}^n_+$, il existe un temps T fini et une fonction mesurable et bornée $u\colon [0, T] \mapsto \mathbb{R}^k$ tels que la solution du problème de Cauchy suivante:
$$\begin{cases} \dot{z} = F(z, u(t)), \\ z(0) = z_0 \end{cases}$$
vérifie $z(T) = z_f$.

(b) *Asymptotiquement stabilisable* au point $z^* \in \mathbb{R}^n$, s'il existe une commande u telle que la solution du problème de Cauchy suivante:
$$\begin{cases} \dot{z} = F(z, u(t, z)), \\ z(0) \in \mathbb{R}^n \end{cases}$$
vérifie la propriété asymptotique:
$$\lim_{t\to+\infty} z(t) = z^*.$$

Maintenant on fera quelques hypothèses sur les entrées:

Hypothèse sur les entrées:
Le taux de dilution D et/ou la concentration initiale de substrat limitant s_{in} seront considérés comme des variables de commande éventuelles.

Jusqu'à la section précédente, on avait supposé un taux de dilution D et une concentration initiale dans le débit d'entrée du substrat limitant s_{in} constants dans le système (1.1.4). Le comportement asymptotique de celle–ci est en général bien connu. On avait remarqué dans la section précédente (voir Remarque 1.1.1) que s_{in} et D pouvaient être manipulés par l'utilisateur.

Il faut remarquer que dans le cas du système (1.4.1), pour des raisons physiques, on est contraint d'imposer la bornitude et la non–négativité de la commande u (car $s_{in} \geq 0$ et $D \geq 0$).

Ainsi, on admet uniquement des lois de commande non-négatives et bornées supérieurement tout en ajoutant la positivité des variables d'état

(s, x_i). Ces restrictions accroissent la difficulté et peuvent limiter la commandabilité comme on peut remarquer dans le résultat suivant:

LEMME 1.4.1. *Si la variable de commande pour le système* (1.1.4) *est le taux de dilution D et si on impose que $D = u(t)$ est une fonction non négative, bornée et mesurable au sens de Lebesgue, alors le système* (1.1.4) *n'est pas commandable.*

PREUVE. On fait la transformation:

$$v = s - s_{in} + \sum_{i=1}^{n} \alpha_i x_i$$

et on définit les ensembles Σ_- et Σ_+:

- $\Sigma_- = \Big\{(s, x_1, \dots, x_n) \in \mathbb{R}_+^n : v < 0\Big\}$,
- $\Sigma_+ = \Big\{(s, x_1, \dots, x_n) \in \mathbb{R}_+^n : v \geq 0\Big\}$.

Soit $z_0 \in \Sigma_-$ une condition initiale du système (1.1.4) avec D remplacée par $u(t) > 0$. Soit $z_f \in \Sigma_+$, si le système est commandable, il existe une commande $u^*(t)$ en boucle ouverte et un temps fini $T > 0$ tel que $v(T) = z_f$.

D'un autre côté, nous voyons que v satisfait:

$$v' = -u^*(t)v - \sum_{i=1}^{n} d_i x_i < -u^*(t)v, \quad \text{et} \quad v(0) = z_0 < 0.$$

On vérifie sans peine l'inégalité:

$$v(t) \leq z_0 \exp\Big(-\int_0^t u^*(r)\,dr\Big) \quad \text{pour tout } t \geq 0.$$

En faisant $t = T$, il en résulte que

$$z_f = v(T) \leq z_0 \exp\Big(-\int_0^T u^*(r)\,dr\Big) < 0 \quad \text{pour tout } t \geq 0$$

et on obtient une contradiction car $z_f \in \Sigma_+$. $\square$

Il peut arriver qu'un système entrée–sortie soit commandable mais que, par contre, il ne soit pas asymptotiquement stabilisable par une boucle de rétroaction, voir par exemple [**135**, exemple 6.2.1].

A cause de ces problèmes de contrainte sur la commande, on a intérêt à introduire des définitions moins restrictives que la commandabilité.

DEFINITION 1.4.3. ([**9**, Pag.77],[**60**, Pag.106]) Soit $z_0 \in \mathbb{R}_+^n$, on définit:

(i) Une *cible* de z_0 est un vecteur $z_f \in \mathbb{R}_+^n$ tel qu'il existe un temps T fini et une loi de commande u définie sur $[0, T]$ tels que, lorsqu'on applique la commande u, la solution du système (1.4.1) partant de $z(0) = z_0$ vérifie $z(T) = z_f$.

(ii) L'ensemble des cibles de z_0 avec un temps fixe T est noté par $\mathbf{A}(T, z_0, U) \subset \mathbb{R}_+^n$ et est appelé l'*ensemble atteignable* du vecteur z_0.

(iii) l'*espace d'atteignabilité* de z_0 est l' ensemble:

$$\mathcal{R}(z_0, U) = \bigcup_{t \geq 0} \mathbf{A}(t, z_0, U).$$

1.4.1. L'entrée et la commande en boucle ouverte. Le développement des lois de commande en boucle ouverte pour les systèmes (1.1.4) et (1.3.1) a été une branche très active de recherche. En général, on a considéré les paramètres s_{in} et/ou D du système (1.1.4) comme des variables de commande.

***Commande du système* (1.1.4)**

D'abord nous présentons une liste des travaux considérant s_{in} comme variable de commande du système (1.1.4):

Type de fonctions	Type de commande	Espèces	Référence
Monod	Périodique	2	[**69**] (1980)
Monod	Périodique	2	[**129**] (1981)
Monod	Périodique	n	[**55**] (1983)
Continue	Positif	n	[**114**] (1990)

Dans [**69**],[**129**] et [**55**] on suppose que s_{in} est une variable de commande du type:

$$s_{in}(t) = s^0 + be(\omega t), \quad s^0, b, \omega > 0$$

où $e\colon \mathbb{R} \mapsto \mathbb{R}$ est une fonction périodique, continue et bornée.

Dans [**69**], Hsu considère des fonctions du type Monod et on suggère (avec l'appui d'un exemple numérique) la possibilité de coexistence des espèces. Cette idée est reprise par H. Smith dans [**129**] où on considère b comme paramètre de bifurcation et on démontre l'existence d'une famille des solutions périodiques tout en obtenant des conditions de stabilité locale pour celles–ci.

Ces résultats sont généralisés pour un nombre quelconque d'espèces par Hale et Somolinos dans [**55**], en plus don obtient des conditions suffisantes de stabilité globale des solutions périodiques pour $n = 2$.

Il faut souligner que, bien que les travaux énoncés ci–dessus peuvent être considérés comme des exemples de commande en boucle ouverte pour le modèle de compétition du chemostat, leur motivation principale est biologique: tandis qu'au laboratoire le principe d'exclusion compétitive est vérifié, dans la nature il y a beaucoup d'exemples qui montrent la coexistence de plusieurs espèces. L'introduction d'une concentration du substrat limitant $s_{in}(t)$ périodique vise à modéliser la coexistence comme un résultat induit par des variations saisonnières de la concentration du substrat limitant.

L'approche suivie dans [**114**] utilise des lois de commande plus générales et permet de démontrer la commandabilité du système (1.1.4) pour $n = 1$ et donne une caractérisation de l'espace d'atteignabilité pour $n \geq 2$.

Maintenant, nous présentons une liste de travaux considérant D comme variable de commande du système (1.1.4):

Type de fonctions	Type de commande	Espèces	Référence
Monod	Périodique	2	[**18**] (1985)
Non Monotone	Périodique	2	[**89**] (1994)
Non Monotone	Périodique	3	[**90**] (1995)

Dans [**18**],[**89**] et [**90**] on suppose que D est une variable de commande du type:

$$D(t) = D^0 + be(\omega t), \quad \text{avec } D^0, b, \omega > 0$$

ou e est une fonction périodique. Dans ces trois travaux on suppose $d_i = 0$ et en utilisant le principe de conservation (voir sous–section 1.2.1) on présente des conditions suffisantes de persistance uniforme des espèces.

Dans [**18**] on utilise le coefficient cinétique $\mu_{\max}$ (correspondant à la fonction f_2) comme paramètre de bifurcation, on démontre l'existence d'une famille des solutions ω–périodiques et on obtient aussi des critères de stabilité pour celles–ci.

Dans [**89**],[**90**] on suppose que les fonctions f_i sont décrites par l'équation (1.1.6). Comme dans les travaux précédents, on démontre l'existence des orbites périodiques attractives mais on vérifie aussi (en considérant D^0,s_{in},ω et b comme paramètres de bifurcation) l'existence de solutions chaotiques et presque–périodiques.

Une approche mixte développée dans [**150**],[**153**] considère D et s_{in} comme variables de commande continues, positives et ω–périodiques. De plus, on suppose que les fonctions de croissance et consommation sont de type $f_i\colon \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ et $f_i(t,s)$ est croissante et localement lipschitzienne par rapport à s et ω–périodique par rapport à t.

En utilisant la théorie des systèmes dynamiques asymptotiquement périodiques et la théorie des systèmes dynamiques monotones, on déduit des conditions suffisantes de persistance uniforme et l'existence des solutions périodiques.

***Commande du système* (1.3.1)**

Maintenant, nous présentons une liste des travaux sur la commande d'une chaîne bi–trophique:

Variable de commande	Type de commande	Référence
D	Périodique	[**109**] (1992)
s_{in}	Périodique	[**80**] (1992)
Autre	Périodique	[**48**] (1995)

Ces travaux ont mis en évidence le rôle essentiel joué par les paramètres D et s_{in}. Dans toutes les références énoncées ci–dessus on suppose que $d_i = 0$, f_1 est une fonction de Monod et p_2 est une fonction du type II. Dans [**109**] et [**80**] on considère respectivement des entrées du type:

$$D(t) = D^0 + be(\omega t) \quad \text{et} \quad s_{in}(t) = s^0 + be(\omega t) \quad \text{avec } D^0, s^0, b, \omega > 0.$$

Dans [**48**] on suppose que quelques paramètres de la fonction p_2 comme le temps de chasse T_c (voir section 1.3) peuvent être du type périodique:

$$T_c(t) = T_c^0 + be(\omega t) \quad T_c^0, b, \omega > 0.$$

En général, on fait une analyse de bifurcation du système forcé, létude d'une application de Poincaré et on conclut que le comportement asymptotique du système peut être périodique, presque périodique ou chaotique.

Comme dans le cas du système (1.1.4), il faut souligner que, bien que les articles [**109**],[**80**] et [**48**] puissent être considérés comme des exemples de commande en boucle ouverte pour une chaîne bi–trophique dans le chemostat, leur motivation principale est la mise au point d'un modèle plus réaliste: tandis que dans le cas précédent (absence de commande) le comportement asymptotique ne pouvait être que la convergence vers un point d'équilibre ou vers une solution périodique, on observe des phénomènes plus complexes dans la réalité. L'introduction de quelques perturbations périodiques vise à modéliser une chaîne trophique sous l'influence de plusieurs variations saisonnières.

1.4.2. L'entrée et la commande en boucle fermée. Elle est utilisée notamment pour des problèmes de stabilisation asymptotique. Nous ne retenons que quelques travaux dans la liste suivante:

Variable de commande	Espèces	Référence
s_{in}	1	[**85**] (1989)
s_{in} et D	1	[**85**] (1989)
s_{in}	1	[**10**] (1990)
D	1	[**31**] (2001)
D	2	[**87**] (2003)

Dans [**85**] (section III.2.5) on construit des familles de lois de commande du type:

$$s_{in} = \mu_1(s, x), \quad s_{in} = \mu_2(s) \quad \text{et} \quad s_{in} = \mu_3(x)$$

où les fonctions $\mu_1 : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ et $\mu_i : \mathbb{R}_+ \mapsto \mathbb{R}$ $(i = 2, 3)$ sont respectivement du type $C^1(\mathbb{R}_+^2, \mathbb{R})$ et $C^1(\mathbb{R}_+, \mathbb{R})$ avec dérivées bornées. Ces lois sont introduites pour obtenir la stabilité locale asymptotique d'un point $(s^*, x^*) \in \mathbb{R}_+^2$ et on compare plusieurs types de commande.

Une loi de commande mixte est proposée aussi dans [**85**] (section III.2.6), où l'on construit des familles de lois du type:

$$s_{in} = \mu_1(s) \quad \text{et} \quad D = \mu_2(x)$$

où les fonctions $\mu_i\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ sont du type $C^1(\mathbb{R}, \mathbb{R})$ avec dérivée bornée. Ces lois sont introduites pour obtenir la stabilité locale asymptotique d'un point $(s^*, x^*) \in \mathbb{R}^*$ et on compare plusieurs types de commande.

Dans [**10**, Chap.5] on propose une loi du type $s_{in}(x) = a + bx$ ($a > 0$ et $b > 0$) pour maximiser la production de biomasse.

Dans [**31**] on suggère l'utilisation de plusieurs types de commande en boucle fermée: régulateurs proportionnels, régulateurs PI, PID et on compare ses performances par rapport à la robustesse.

Dans [**87**] on construit une loi de commande linéaire qui a pour but d'obtenir la convergence des espèces vers un point critique dans $\mathbb{R}^3_+$.

1.5. La commande robuste du chemostat

En général, le modèle du chemostat considéré comme système entrée–sortie est un modèle incertain. Nous parlerons d'incertitude dans deux sens:

(1) Un sens *interne*, où les incertitudes sont liées à la connaissance du système: les fonctions f_i et les paramètres du système (1.5.1) ne sont que *partiellement connus*, de plus les phénomènes décrits sont complexes, non linéaires et souvent non stationnaires. De plus, la reproductibilité des expériences est souvent incertaine.
(2) Un sens *externe*, où les incertitudes sont liées à la sortie: les données fournis par $y(t)$ sont incomplètes et bruitées.

Alors, compte tenu de ces incertitudes, le chemostat peut être considéré comme un système entrée/sortie du type suivant:

$$\begin{cases} \dot{z} = F(z, u, w), \\ y(t) = h(t, z(t - \tau))[1 + \Delta_1] + \Delta_2, \quad z = (s, x_1, \dots, x_n)^T \end{cases} \tag{1.5.1}$$

où $h\colon \mathbb{R}_+ \times \mathbb{R}^{n+1}_+ \mapsto \mathbb{R}^k_+$ (avec $k < n+1$) est une fonction mesurable au sens de Lebesgue par rapport à t et continue par rapport aux variables (s, x_i) ($i = 1, \dots, n$). La constante $\tau > 0$ représente un éventuel retard dans la sortie, $u \in \mathbb{R}^k_+$ est un ensemble d'entrées du système.

L' expression w représente l'incertitude du modèle tandis que les expressions $[1+\Delta_1]$ et Δ_2 représentent les incertitudes de la sortie respectivement.

REMARQUE 1.5.1. *Dans le système* (1.5.1)*, l'expression* $[1+\Delta_1]$ *est appelée* ***perturbation multiplicative*** *et* Δ_2 *est appelée* ***perturbation additive****.*

Il faut souligner que la sortie du système (1.5.1) *montre une différence qualitative entre ces deux types de perturbations*: *la perturbation multiplicative donne une erreur sur la sortie dépendante de la fonction* h (*et par* *conséquent, des variables à mesurer) tandis que la perturbation additive dépend uniquement de la méthode de mesure.*

1.5.1. Le système. En général, la modélisation mathématique du chemostat se heurte à deux difficultés majeures: le problème d'estimation des fonctions de croissance f_i et l'estimation des paramètres du modèle (α_i,d_i et s_{in}).

L'estimation des fonctions de croissance et consommation

En général, la modélisation des cinétiques microbiennes a réussi à établir des relations fonctionnelles entre les paramètres et les variables du processus de croissance et consommation, à partir des résultats expérimentaux (voir subsection 1.1.3). Étant donné que dans la croissance des espèces et la consommation du substrat limitant se font des échanges de matière et d'énergie, des transformations chimiques s'opèrent dans chaque cellule et entre les cellules. Ainsi, la cinétique enzymatique, et –en particulier– le concept d'enzyme–substrat (voir [**101**]) ont joué un rôle clé dans la réalisation des expériences et la modélisation des fonctions f_i.

Comme les réactions et les échanges de matière et d'énergie sont dépendants des substrats et espèces particulières, il n'existe pas de modèle général de fonction de consommation. Nous renvoyons le lecteur à une liste avec quelques travaux de modélisation des fonctions de croissance:

Espèce	Substrats	Source
Escherichia coli	Glucose, Mannite, Lactose	[**105**],[**74**],[**118**]
Acetogenium kivui	Glucose	[**112**]
Zymomonas mobilis	Glucose	[**112**]
Azotobacter vinelandii	Glucose	[**74**]
Clostridium butyricum	Glycerol	[**112**]
Pseudomonas capacia	Phenol, Oxygène	[**112**]
Pseudomonas putida	Phenol	[**134**]
Nitrobacter winogradskyi	Nitrite	[**2**],[**14**],[**35**]
Rhodopseudomonas capsulata	Azote	[**144**]
Nitrosomonas	Ammoniaque	[**2**]
Saccharomyces cerevisiae	Glucose, Éthane	[**22**],[**112**]
Candida utilis	Acétate de sodium	[**35**]
Trichosporon cutaneum	Glucose, Oxygène	[**112**]
Trichoderma reesei	Glucose, Cellulose	[**112**]
Pennicilinum chrysogenum	Glucose	[**112**]
Dunaniella tertiolecta	Nitrate	[**11**],[**143**]
Cryptomonas	Nitrate	[**4**]

En général, l'estimation des paramètres cinétiques des fonctions présentées dans les équations (1.1.5),(1.1.6) et (1.1.7) est fortement dépendante des conditions de culture (*e.g.* luminosité, température, pH, aération, substrat).

Il existe une vaste littérature consacrée à l'estimation de ces coefficients cinétiques et il existe aussi une approche qui considère quelques uns de ces coefficients comme des fonctions dépendantes du substrat. Nous renvoyons le lecteur intéressé aux références [**73**],[**1**],[**134**],[**120**],[**68**],[**32**],[**144**] et [**76**].

Malgré tous ces inconvénients, nous supposerons que l'on dispose d'une connaissance *a priori* qualitative des incertitudes qui affectent les fonctions de croissance et par conséquence, on peut faire l'hypothèse que la fonction f_i est *partiellement connue* mais qu'elle vérifie l'égalité:

$$f_i(s) = f_{i_1}(s) + f_{i_2}(s)w, \quad \text{pour } s \in \mathbb{R}_+ \tag{1.5.2}$$

où f_{i_1} est une fonction bien connue et w représente une *perturbation* déterministe ou stochastique. Nous supposerons initialement que cette perturbation présente une borne connue.

L'équation (1.5.2) combinée avec la bornitude de la perturbation w implique l'existence de deux fonctions bien connues f^- et f^+ qui satisfont l'inégalité

$$f_i^-(s) \leq f_i(s) \leq f_i^+(s) \quad \text{pour } s \geq 0. \tag{1.5.3}$$

L'estimation des paramètres du système

Nous ferons une distinction entre les *paramètres biologiques* (α_i, d_i) et les *paramètres physiques* (D, s_{in}).

L'estimation des paramètres biologiques (α_i, d_i) présente des problèmes car ils peuvent être variables avec le temps: ils varient par exemple à cause des changements métaboliques de la biomasse ou de modifications génétiques plus ou moins aléatoires.

L'hypothèse d'un taux de mortalité d_i fixe suppose implicitement, que malgré la structuration de l'espèce (soit par exemple par taille, par age), la mortalité est constante. Pour plus de détails sur l'estimation des taux de mortalité, nous renvoyons le lecteur à [**151**] où l'estimation est considérée comme un problème inverse et [**137**] où le taux de mortalité est une variable aléatoire.

Dans le cas relatif à l'estimation des coefficients de rendement α_i, on s'aperçoit qu'ils peuvent évoluer au cours du temps ou être dépendants de la concentration du substrat. Pour des modèles qui prennent en compte ce cas, nous renvoyons le lecteur aux articles [**5**] et [**111**].

Les paramètres physiques (s_{in}, D) sont en général estimés sous la forme

$$s_{in} = s_{in}^* + \Delta_a \quad \text{et/ou} \quad D = D^* + \Delta_b$$

ou s_{in}^* et D^* sont bien connus et Δ_a et Δ_b modélisent les imprécisions. Nous renvoyons le lecteur interéssé aux articles [**12**] où on fait une présentation des aspects mécaniques du pompage dans le chemostat, [**74**] et [**34**] où on estime une borne supérieure pour Δ_a et [**136**] où on considère l'imprécision Δ_b comme un processus stochastique.

1.5.2. Le capteur et les sorties. Un obstacle considérable est le manque généralisé de capteurs permettant de mesurer la concentration des biomasses et du substrat limitant. La plupart des variables ne peuvent être mesurées qu'à l'aide d'analyseurs de laboratoire dont le coût et la durée des analyses limitent la fréquence des échantillonnages et dont la précision n'est pas toujours assurée.

La mesure des substrat(s) limitant(s)

Elle est effectuée avec des prélèvements du milieu liquide et l'étude des réactions chimiques sur ces échantillons permet une estimation de la concentration du substrat limitant. Le protocole expérimental et la nature de ces réactions chimiques sont fortement dépendants du type de substrat et de biomasse. Pour plus de précisions sur ce sujet, nous renvoyons le lecteur à [**122**] et ses références.

La mesure des espèces

Nous avions remarqué dans la section précédente que les incertitudes de la mesure ont deux sources: l'erreur liée à la méthode de mesure (source des perturbations additives) et l'erreur associée au substrat particulier (source des perturbations multiplicatives).

La mesure du substrat et des biomasses est réalisée avec des *senseurs photométriques* et/ou des *senseurs spectrophotométriques*; les premiers mesurent la turbidité du milieu liquide et les deuxièmes mesurent la densité optique à une longueur d'onde appropriée. Tandis que les premiers permettent une mesure de la biomasse totale $\left(i.e.\ \ y = \sum_{i=1}^{n} x_i\right)$, les deuxièmes permettent la mesure de certaines espèces de biomasse.

Comme dans le cas précédent, la méthode est dépendante du (des) biomasse(s) à mesurer. Pour un état de l'art des méthodes de mesure, nous renvoyons le lecteur à [**117**] et [**3**, Chap.1].

1.6. Robustesse

Les principales difficultés rencontrées pour rendre efficace la commande du chemostat sont générées par les sources d'incertitudes décrites ci–dessus.

D'après (1.5.2), le système (1.5.1) peut être considéré sous la forme:

$$\begin{cases} \dot{z} = F_1(z,u) + F_2(z,u)w, \\ y(t) = h(t, z(t-\tau))[1+\Delta_1] + \Delta_2, \quad z = (s, x_1, \dots, x_n)^T \end{cases} \tag{1.6.1}$$

L'idée de la commande robuste est que les objectifs de commandabilité et/ou de stabilisation asymptotique soient atteints malgré les incertitudes du modèle et la sortie. Une des premières définitions qualitatives de la robustesse a été proposée dans [**15**] par Box et Andrews, un système est robuste s'il est "*unsensitive to change, of a magnitude likely to ocurr in practice, in extraneous factors*".

Il existe plusieurs approches théoriques pour un traitement quantitatif de la robustesse et celui–ci est déterminé par la façon de considérer les incertitudes w et Δ_i: nous parlerons de l'approche *probabiliste* et l'approche *déterministe*.

1.6.1. L'approche probabiliste. Cette approche considère les quantités incertaines w et Δ_i comme des processus stochastiques réelles (W_t,Δ_{i_t}) dont la loi de probabilité respective P_{W_t},P_{Δ_i} peut être identifiée. Ceci implique que le système (1.6.1) peut être considéré comme l'équation différentielle stochastique:

$$d\mathrm{Z}_t = F_1(\mathrm{Z}_t, u)dt + F_2(\mathrm{Z}_t, u)d\mathrm{W}_t \tag{1.6.2}$$

et la sortie:

$$y(t) = h(t, \mathrm{Z_t})[1 + \Delta_{1_t}] + \Delta_{2_t}, \quad z = (s, x_1, \dots, x_n)^T. \tag{1.6.3}$$

Dans le contexte de la modélisation du chemostat, l'équation (1.6.2) a été étudiée dans [**33**] et [**72**]. En général, on suppose que W_t est un processus à bruit blanc; c'est–à–dire:

$$E[\mathrm{W}_t] = 0 \quad \text{et} \quad E[\mathrm{W}_t\mathrm{W}_s] = \delta(t - s) \quad (i = 1, 2)$$

où E est l'espérance mathématique.

Dans [**145**] and [**144**] on compare deux sorties, une avec perturbation multiplicative et l'autre avec perturbation additive. On suppose aussi que Δ_{i_t} est aussi un processus à bruit blanc.

Néanmoins, dans la modélisation biologique et autres situations avec important travail expérimental, il n'est pas évident de déterminer des propriétés statistiques pour W_t et Δ_{i_t}.

1.6.2. L'approche déterministe. Cette approche considère les quantités incertaines w et Δ_i comme des variables réelles incertaines qui peuvent appartenir à des intervalles et satisfaire quelques propriétés complémentaires déduites de notre connaissance *a priori* du modèle. Ceci implique que le système peut être considéré comme l'équation différentielle:

$$\begin{cases} \dot{z} = F_1(z, u) + F_2(z, u)w(t, z), \\ y = h(t, z(t - \tau))[1 + \Delta_1(t)] + \Delta_2(t), \quad z = (s, x_1, \dots, x_n)^T \end{cases} \tag{1.6.4}$$

où $w\colon \mathbb{R}_+ \times \mathbb{R}^{n+1} \mapsto \mathbb{R}$ satisfait les propriétés suivantes:

(W1) w est mesurable au sens de Lebesgue par rapport à t et continue par rapport à z.

(W2) $w \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^{n+1}, \mathbb{R})$ avec $||w||_\infty \leq \lambda_0$.

En fait, dans cette thèse, nous supposerons que la fonction $F_1 + F_2 w$ satisfait les inégalités

$$F^-(z, u) \leq_K F_1(z, u) + F_2(z, u)w(t, z) \leq_K F^+(z, u)$$

pour tout $z \in \mathbb{R}^{n+1}$ et $t \in \mathbb{R}_+$. Bien entendu, on doit considérer ces inégalités par rapport à un cône dans $\mathbb{R}^{n+1}$ (consulter Annexe C).

De plus, les fonctions Δ_i sont supposées être mesurables au sens de Lebesgue et bornées.

Une des perspectives envisageables suggère que le système (1.5.1) peut être considéré comme l'inclusion différentielle:

$$(1.6.5) \qquad \begin{cases} \dot{z} \in \mathcal{F}(z, u(t)), \\ y = h(t, z(t-\tau))[1 + \Delta_1(t)] + \Delta_2(t), \quad z = (s, x_1, \ldots, x_n)^T \end{cases}$$

où $\mathcal{F}\colon \mathbb{R}^{n+1} \times \mathbb{R}^k \times \mathbb{R} \rightrightarrows \mathcal{P}(\mathbb{R}^{n+1})$ est la multifonction:

$$\mathcal{F}(z, u) = \Big\{ F_1(z, u) + F_2(z, u)w\colon \text{où } w \text{ satisfait les propriétés } \mathbf{(W)} \Big\}.$$

En général, nous serons intéressés par les bornes supérieures et inférieures des sorties, lesquelles fournissent la marge d'erreur de notre loi de commande. Ce type d'étude est connu aussi comme l'approche du type *worst case*.

En utilisant la théorie de systèmes dynamiques monotones et quelques résultats de comparaison des inégalités différentielles, nous verrons qu'il suffira d'étudier l'inclusion

$$\dot{z} \in \partial\mathcal{F}(z, u(t))$$

qui est équivalente à un nombre fini d'équations différentielles.

1.7. Plan de la Thèse

Le sujet de cette thèse est de construire des stratégies de commande en boucle fermée $u(\cdot)$ pour le système (1.1.4) et boucle ouverte pour le système (1.3.1) telles que:

- Le système (1.1.4) (avec $n = 1$) est stabilisé asymptotiquement autour d'une valeur du substrat limitant $s^* < s_{in}$.
- Le système (1.1.4) (avec $n = 2$) devient uniformément persistant.
- On étudie l'atteignabilité du système (1.3.1).

Le plan est le suivant:

Partie I: Commande Robuste d'un chemostat.

Dans cette première partie, nous étudierons un problème de régulation de la sortie d'un chemostat incertain mais avec une seule espèce, décrit par le système suivant:

$$(1.7.1) \qquad \begin{cases} \dot{s} = D(s_{in} - s) - \alpha[f(s) + w(s)]x, \\ \dot{x} = x\big(f(s) + w(s) - D - d\big), \\ x(0) = x_0 > 0 \quad \text{et} \quad s(0) = s_0 > 0. \end{cases}$$

$$(1.7.2) \qquad y(t) = s(t - \tau)[1 + \Delta(t)]$$

où $\Delta\colon \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ est une fonction bornée et mesurable au sens de Lebesgue par rapport à t. La fonction $w\colon \mathbb{R}_+^2 \mapsto \mathbb{R}$ est bornée et continue.

Plus précisement on supposera l'existence de deux fonctions f^- et f^+ telles que $f^-(s) \leq f(s) + w(s) \leq f^+(s)$. De plus, les résultats pourraient s'étendre au cas $w(t,s)$.

En guise d'idée directrice, notre but est de construire des lois de commande en boucle fermée pour le système (1.7.1)–(1.7.2) qui permettent la stabilisation asymptotique du système (1.7.1) autour d'une consigne $s^* \in (0, s_{in})$, tout en supposant que le taux de dilution D est l'unique variable de commande du système. Éventuellement, on souhaite obtenir des estimations pour la variable non–mesurée $x(t)$, estimer les paramètres du modèle, etc.

Plus formellement, nous étudierons quelques cas particulières du problème suivant:

PROBLÈME 1. *Étant donnée une consigne $s^* < s_{in}$, trouver une famille des lois de commande en boucle fermée qui stabilisent le système* (1.7.1)–(1.7.2) *autour d'un voisinage de la consigne $s^* < s_{in}$. C'est–à–dire*:

$$s(t) \in (s^-, s^+) \quad \textit{pour tout } t > T$$

où T est un nombre fini et $s^* \in (s^-, s^+)$.

Pour illustrer les difficultés et motivations liées au Problème 1, nous présentons deux applications

- **Un modèle de dépollution utilisant le chemostat:** La gestion de la qualité de l'eau doit faire face au maintien des niveaux acceptables pour sa consommation et le tourisme. Dans cette direction, il y a un grand intérêt pour la gestion des contaminants toxiques et leur stabilisation en dessous d'un niveau acceptable, fixé par des autorités environnementales.

  Des contaminants toxiques comme le phénol et le toluène sont présents à la surface de l'eau et leur oxydation biologique est un important procédé de traitement de l'eau. Brièvement, ce procédé de dépollution utilise un chemostat dans lequel le contaminant est débité dans la cuve avec une concentration s_{in}, beaucoup plus grande que le niveau acceptable s^+ (relativement proche de zéro) et fixé par des dispositions environnementales. Le chemostat contient aussi une espèce de microorganismes x (par exemple *Pseudomonas putida* ou *Pseudomonas capacia*), lesquels peuvent résister à l'effet adverse des solvants organiques comme le phénol et sont capables de le dégrader en le consommant.

  Si on choisit une dilution fixe $D < f^+(s^+)$, la concentration des contaminants sera convergente vers un niveau admissible. Néanmoins, étant donné que le taux de dilution est relativement petit et la convergence vers le point d'équilibre est très lente. On introduit une loi de commande en boucle fermée pour augmenter la vitesse de convergence.
- **Simulation des environnements marins utilisant un chemostat:** Le Chemostat a été utilisé pour la modélisation de quelques

aspects des environnements marins (voir par exemple [**11**],[**12**],[**143**]). En fait, plusieurs caractéristiques de la mer (par exemple la température et l'intensité de la lumière) peuvent être reproduites en laboratoire. De plus, la commande du chemostat permet la reproduction de plusieurs niveaux s^* de substrat limitant et, en conséquence, permet aussi l'étude de la croissance des algues phytoplanctoniques unicellulaires dans un grand intervalle de niveaux de substrat limitant, de température et d'intensité de la lumière.

Nous sommes intéressés par la stabilisation du chemostat à niveau fixe $s^* < s_{in}$ de substrat limitant. Dans le chapitre 2, nous étudierons un modèle de croissance de l'algue phytoplanctonique *Dunaniella tertiolecta* considérant le nitrate comme substrat limitant [2]. Si –dans une première approximation– on suppose l'absence d'incertitude (c'est–à–dire $w(t,s) = 0$, $\Delta(t) = 0$ et $\tau = 0$), nous pouvons voir dans la Figure 1.7.1 que l'introduction d'une boucle de rétroaction appropriée permet d'améliorer la vitesse de convergence par rapport à l'usage d'un taux de dilution fixe.

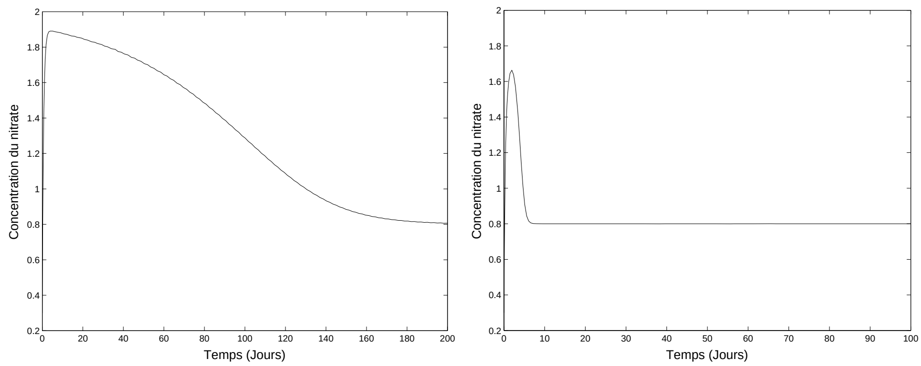

FIGURE 1.7.1. Concentration du nitrate dans une culture du phytoplancton: utilisation d'une dilution fixe (gauche) et utilisation d'une loi de commande (droite). Remarquer la différence entre les vitesses de convergence vers 0.8 μatg/L (microatomes grammes par litre).

Néanmoins, si l'on prend en compte l'existence d'un retard $\tau > 0$ dans la sortie, c'est–à–dire, si l'on suppose $y(t) = s(t-\tau)$, alors ce retard peut soit ralentir la convergence vers la consigne s^* soit induire des oscillations du substrat limitant ce qui empêche notre objectif de stabilisation.

Il faut souligner que si le modèle et la sortie ne présentent pas d'incertitudes, le système peut être stabilisé avec une loi de commande du type:

$$D(y(t)) = h(s^* - y(t)). \tag{1.7.3}$$

[2]Pour des renseignements plus détaillés voir le Chapitre 2.

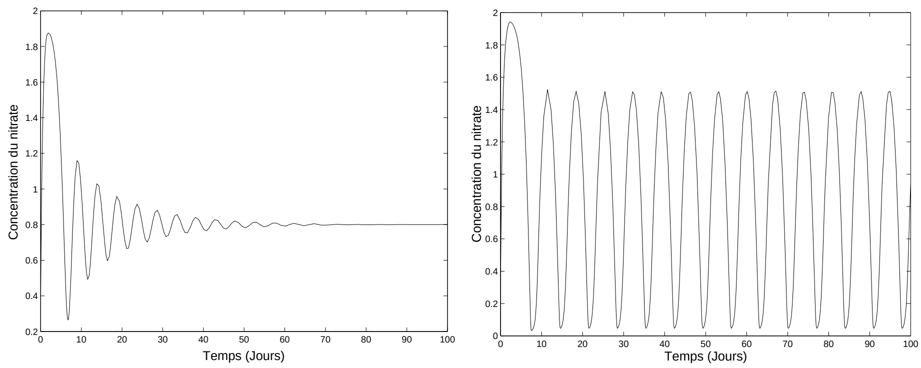

FIGURE 1.7.2. Utilisation d'une loi de commande avec deux sorties $y(t) = s(t-\tau_i)$ $(i = 1, 2)$. Remarquer le ralentissement de la vitesse de convergence vers 0.8 μatg/L (microatomes grammes par litre) et l'apparition d'une solution périodique pour τ assez grand.

où la fonction $h\colon \mathbb{R} \mapsto \mathbb{R}_+$ satisfait les propriétés suivantes:
(H1) h est une fonction lipschitzienne, bornée et strictement croissante.
(H2) $s^* \in (0, s_{in})$ est une racine unique de l'équation $f(s)-h(s^*-s)-d = 0$.

En effet, nous avons le résultat suivant (pour une démonstration voir Annexe E):

PROPOSITION 2. *On suppose que le taux de dilution est la variable de commande et que le système* (1.7.1) *vérifie* $w(t, s) = 0$. *De plus, la sortie* (1.7.2) *est du type* $y(t) = s(t)$ (*c'est–à–dire* $\Delta(t) = 0$ *et* $\tau = 0$).

Alors toute loi de commande (1.7.3) *avec les propriétés* **(H1)**–**(H2)** *stabilise globalement la sortie* $y(t)$ *du système* (1.7.1) *vers l'équilibre* $s^* > 0$.

REMARQUE 1.7.1. *La prise en compte de la mortalité impose quelques restrictions pour le taux de mortalité. En effet l'hypothèse* **(H2)** *suppose implicitement l'inégalité*:

$$d < f(s_{in}) - h(s^* - s_{in}).$$

COROLLAIRE 1.7.1. *On suppose que le taux de dilution est la variable de commande et le système* (1.7.1) *vérifie* $w(t, s) = 0$ *et la sortie* (1.7.2) *est du type* $y(t) = s(t)$ (*c'est–à–dire* $\Delta(t) = 0$ *et* $\tau = 0$).

Si $d = 0$, *alors toute loi de commande* (1.7.3) *avec les propriétés* **(H1)**–**(H2)** *stabilise globalement et asymptotiquement la sortie* $y(t)$ *du système* (1.7.1) *vers l'équilibre* $s^* > 0$.

REMARQUE 1.7.2. *Il faut souligner que, tout en conservant les hypothèses du Corollaire* (1.7.1), *la loi de commande* (1.7.3) *peut être considérée sous la forme suivante*:

$$D(y(t)) = D(s(t)) = D^* + \lambda g(s^* - s(t))$$

où $f(s^*) = D^*$,*g est une fonction lipschitzienne, bornée, strictement croissante,* $0 < \lambda||g||_\infty < D^*$ *et* $g(0) = 0$.

Comme g est une fonction croissante on en déduit les propriétés suivantes pour la loi de commande $D(y(t))$:

- $D(y(t))$ est une contre–réaction, c'est–à–dire réagit en sens inverse de l'écart $e(t) = s^* - s(t)$ car on vérifie que

$$\begin{cases} D(s(t)) \geq D^* & \text{si } s(t) \leq s^*, \\ D(s(t)) \leq D^* & \text{si } s(t) \geq s^*. \end{cases}$$

- Ceci implique les inégalités:

$$\dot{s}(t) = f(s(t)) - f(s^*) - \lambda g(s^* - s(t)) < 0, \quad \text{si } s(t) > s^*,$$

$$\dot{s}(t) = f(s(t)) - f(s^*) - \lambda g(s^* - s(t)) > 0, \quad \text{si } s(t) < s^*.$$

- Donc, quand $s > s^*$ on a que la mesure se trouve au–dessus de la consigne et la loi $D(y(t))$ fait diminuer le débit (car s est décroissant). D'un autre côté si $s < s^*$, alors la mesure se trouve en dessous de la consigne et la loi $D(y(t))$ fait augmenter le débit (car s est croissant).
- Si la convergence est assurée, le paramètre λ nommée **gain** dans la théorie de la commande) peut augmenter ou diminuer la vitesse de convergence.

Avec ces propriétés, on conclut que $D(y(t))$ est une version non–linéaire d'un *régulateur proportionnel*.

On sent intuitivement que la loi de commande (1.7.3) perd son efficacité dès qu'on prend en compte des incertitudes, mais si celles–ci sont bornées et que les bornes ne dépassent pas un certain seuil, on peut avoir encore quelques résultats de stabilisation asymptotique.

Dans les Chapitres 2 et 3, nous allons travailler sur quelques simplifications du Problème 1. Nous verrons que –en ajoutant des conditions complémentaires– les familles des lois de commande énoncées dans la Proposition 2 permettent l'étude du Problème 1.

Chapitre 2: Commande robuste d'un chemostat avec incertitudes

Dans ce chapitre, le système (1.7.1) est supposé mal connu, on suppose que la fonction f est incertaine et satisfait l'équation (1.5.3). On supposera aussi une sortie du type $y(t) = s(t)[1 + \Delta(t)]$ où $\Delta(t)$ est une perturbation déterministe.

Le Problème 1 devient un problème de régulation de la sortie d'un système non–linéaire incertain. On utilise la même loi de commande proposée par l'équation (1.7.3) et notre but est d'obtenir des conditions suffisantes sur la stabilisation asymptotique de la sortie dans un voisinage du point s^*.

La base du chapitre est l'article [**46**][3].

Chapitre 3: Commande d'un chemostat avec retard dans la sortie

Dans ce chapitre, le système (1.7.1) est supposé bien connu, mais on supposera une sortie du type:

$$y(t) = s(t-\tau), \quad \tau > 0.$$

On utilise la même loi de commande proposée par l'équation (1.7.3) et notre objectif est de trouver une borne supérieure pour le retard τ qui assure la convergence vers la consigne s^*. On fait aussi un analyse sur la vitesse de convergence et on pourra voir qu'il existe un seuil s^* à partir duquel les sorties sont convergentes vers une fonction périodique.

La base du chapitre est le rapport de recherche [**45**][4].

Partie II: Commande en boucle fermée d'un modèle de compétition dans le chemostat

La deuxième partie de cette thèse est consacrée à la compétition entre deux espèces pour un substrat limitant. Ce modèle est décrit par le système suivant:

$$\begin{cases} \dot{s} \ = D(s_{in}-s) - \alpha_1 f_1(s)x_1 - \alpha_2 f_2(s)x_2, \\ \dot{x}_1 = x_1\big(f_1(s) - D - d_1\big), \\ \dot{x}_2 = x_2\big(f_2(s) - D - d_2\big). \end{cases} \tag{1.7.4}$$

Le comportement asymptotique du système (5.1.1) a été décrit dans la section précédente (exclusion compétitive). L'objectif de cette partie est d'introduire des lois de commande en boucle fermée qui permettent obtenir la persistance uniforme du système (1.7.4). En 2003, P.De Leenheer et H.Smith [**87**] construisent une famille des lois de commande en boucle fermée pour le système qui permet la persistance uniforme du système.

THÉORÈME 1. (De Leenheer–Smith [**87**, Th.1]) *On suppose que la sortie du système est du type*:

$$y(t) = (x_1(t) \quad x_2(t))^T,$$

De plus:
(i) *Les fonctions f_i sont strictement croissantes.*
(ii) *Le taux de mortalité est négligeable* (*i.e.* $d_i = 0$).

[3] J.L. Gouzé and G. Robledo. Robust control for an uncertain chemostat model. *International Journal of Nonlinear and Robust Control*, 16 (2006) 133–155.

[4] J.L. Gouzé et G. Robledo, Feedback stabilization for a chemostat with delayed output. *Rapport de Recherche INRIA* 5844, 2006

En considérant la taux de dilution D comme variable de commande, la famille des lois de commande en boucle fermée:

$$D(x_1, x_2) = \frac{k_1}{\alpha_1} x_1 + \frac{k_2}{\alpha_2} x_2, \quad k_1 > k_2 > 0. \tag{1.7.5}$$

permet au système (1.7.4) *de devenir uniformément persistent.*

L'objectif de cette deuxième partie est d'étendre le Théorème 1 dans deux directions, d'abord en permettant l'usage des fonctions non–monotones et puis en considérant le taux de mortalité.

Chapitre 4: Commande d'un modèle de compétition avec des réponses fonctionnelles non–monotones.

Dans ce chapitre, on considère le système (1.7.4) avec les propriétés suivantes:
- Les fonctions f_i peuvent être non–monotones mais unimodales, c'est–à–dire qu'elles peuvent avoir –au plus– un maximum global.
- On suppose $d_i = 0$.
- La sortie du système est la biomasse totale $y(t) = x_1 + x_2$.

Comme dans l'article [**87**], nous construisons une loi de commande en boucle fermée pour rendre le système uniformément persistant (la persistance est obtenue sous la forme de convergence vers un point critique intérieur) La base du chapitre est l'article [**44**][5].

Chapitre 5: Commande d'un modèle de compétition avec prise en compte de la mortalité.

Dans ce chapitre, on considère le système (1.7.4) avec les propriétés suivantes:
- Les fonctions f_i sont monotones.
- On suppose $d_i > 0$.
- La sortie du système est la biomasse totale $y(t) = x_1 + x_2$.

Comme dans le chapitre précédent, nous construisons une loi de commande en boucle fermée pour rendre le système uniformément persistant. La prise en compte de la mortalité rend inappliquable le principe de conservation de la masse, et rend donc le problème beaucoup plus difficile.

La base du chapitre est le rapport de recherche [**43**][6].

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[5] J.L. Gouzé et G. Robledo, Feedback control for nonmonotone competition models in the chemostat. *Nonlinear Analysis, Real World Applications* 6 (2005) 671–690.

[6] J.L. Gouzé et G. Robledo, Feedback control for competition models with different removal rates in the chemostat. *Rapport de Recherche INRIA* 5555, 2005.

Partie III: Commande en boucle ouverte d'une chaîne trophique dans le chemostat.

La dernière partie de cette thèse est relativement indépendante et est consacrée à la commande d'une chaîne trophique, tout en considérant s_{in} comme variable de commande.

Chapitre 6: Résultats préliminaires sur l'atteignabilité

Nous proposons une idée (inspirée dans l'article [**114**] de N.Rao et E.Roxin) pour caractériser l'espace d'atteignabilité du système (1.3.1) tout en étudiant un système réduit qui ne considère pas l'équation du substrat.

Part 1

Robust control of chemostat with a single species

CHAPTER 2

Robust control for an uncertain chemostat model

In this chapter, we study a problem of asymptotic stabilization for an *uncertain* chemostat model; this problem immediately becomes nontrivial if we take into account that the model is inaccurate thus the real dynamic is relatively unknown and the output available is corrupted by noise.

Feedback stabilization of nonlinear uncertain systems [**21**],[**82**],[**91**],[**71**] deals with several approaches: *v.g.* deterministic, stochastic, adaptive control, etc. An interesting deterministic issue for this stabilization problem has been given in the framework of game-theoretical control theory (see *e.g.* [**81**]). Roughly speaking, it assumes that the real system is not known to us, but there exists a (relatively wide) class of admissible systems, including the real one. This assumption implies that, given a stabilization objective, it must be satisfied for any admissible system too. This approach has been complemented (see *e.g.* [**42**],[**52**],[**110**],[**115**],[**116**]) using the fact that (under some transformations of variables) a system Σ equivalent to the chemostat equations can be studied using monotone dynamical systems theory [**130**]. Moreover, supposing that the bounds of uncertainties defined before are known a priori, a couple of well–known dynamical systems (Σ^-,Σ^+) satisfying the inequality $\Sigma^- \leq \Sigma \leq \Sigma^+$ (in a sense that will be explained later on) is built. Hence, instead of satisfying a feedback stabilization objective for any admissible system, we will need only to satisfy this one for the "bounds" of the admissible systems.

Using these systems (Σ^-,Σ^+), interval observers for uncertain bioreactor models have been developed in [**42**],[**52**],[**115**],[**116**], allowing a partial estimation for the non–measured variables. Moreover, in [**116**] a feedback stabilization law is built using these interval observers. In almost all these articles it is supposed that the output available is unperturbed (except in [**115**]) and the mortality rate of the biomass(es) is (are) negligible. This last assumption is one key step in reducing many chemostat models to a monotone dynamic system where strong convergence properties are in evidence.

In this chapter, we assume that the output undergoes multiplicative disturbances. Moreover, we include the mortality rate. Nevertheless, we are able to apply the ideas stated above, arriving at stabilization of the uncertain chemostat without using interval observers.

This chapter is organized as follows: in section 2.1 we recall some facts about the chemostat model and state the assumptions about uncertainty. Section 2.2 presents the robust regulation problem in detail. The main result and its proof is given in section 2.3. The application and simulations are given in section 2.4.

2.1. Modeling of an uncertain chemostat

Let us recall the chemostat equations:

$$\begin{cases} \dot{s} = D(s_{in} - s) - \alpha x f(s), \\ \dot{x} = x(f(s) - D - d). \end{cases} \tag{2.1.1}$$

As we have seen in the introduction, sometimes in chemostat models, the mortality rate is neglected and a "conservation law" for total biomass implies that the weighted sum of microbial concentration and substrate concentration equilibrates. Hence, the substrate equation can be asymptotically eliminated.

As the chemostat has a finite volume, we can certainly assume that the weighted sum $s + \alpha x - s_{in}$ is bounded above by $v^* > 0$, where this constant is related to the chemostat volume.

We will assume that initial conditions of system (5.1.1) are in the box $[s_0^-, s_0^+] \times [x_0^-, x_0^+] \subset \Omega$, where the set Ω is defined by:

$$\Omega = \Big\{(s,x) \in \mathbb{R}_+^2 \colon 0 < s < s_{in}, x > 0 \quad \text{and} \quad s + \alpha x + s_{in} < v^*\Big\}$$

where v^* is a constant related to the volume of the chemostat.

Now, we make the assumptions **(H)** about the uncertainty of the chemostat model:

(H1) The function f is unknown but locally Lipschitz and functionally bounded, *i.e.* there exist a couple of well known continuous maps f^-,f^+ such that:
- f^- and f^+ have at most one local maximum in $(0, s_{in})$.
- f^- and f^+ satisfy the inequalities (see Fig.2.1.1):

$$f^-(s) \leq f(s) \leq f^+(s), \quad \text{for any } s \geq 0 \text{ and} \quad f^-(0) = f^+(0) = 0.$$

(H2) The only output available takes the form:

$$y(t) = s(t)[1 + \Delta(t)]$$

where the function $\Delta \colon \mathbb{R}_+ \mapsto \mathbb{R}$ is bounded and Lebesgue measurable. Moreover, there exist two bounds for $\Delta(t)$:

$$-1 < \Delta^- \leq \Delta(t) \leq \Delta^+ \quad \text{for any } t \geq 0.$$

It will be useful to define the lower and upper increasing envelope of the functions f^- and f^+ respectively, that means a couple of continuous and increasing functions l and u, satisfying $l(s) \leq f^-(s) \leq f^+(s) \leq u(s)$ for any $s \in [0, s_{in}]$ (see Fig.2.1).

Remark 2.1.1. *Notice that,* **(H2)** *gives a priori bounded estimates for the substrate*:

$$y^-(t) = \frac{y(t)}{1 + \Delta^+} \leq s(t) \leq \frac{y(t)}{1 + \Delta^-} = y^+(t). \tag{2.1.2}$$

Recall that **(H1)** and **(H2)** will be assumed for the remainder of this chapter.

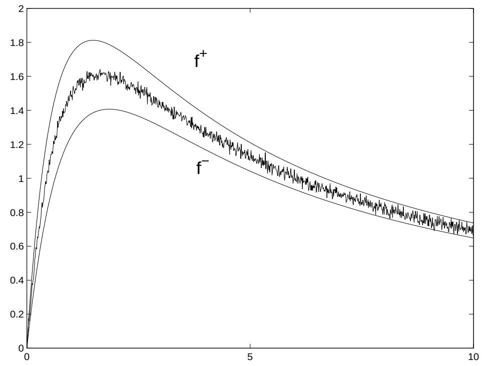


FIGURE 2.1.1. Geometrical interpretation of **(H1)**

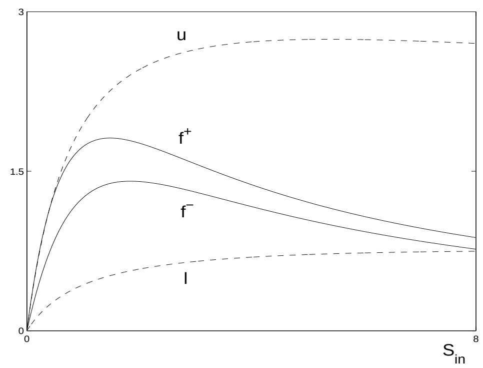


FIGURE 2.1.2. l is a lower envelope of f^- and u is an upper envelope of f^+

2.2. Motivation and formulation of stabilization problem

In several bioprocesses, the goal is to stabilize the substrate in a neighborhood of level s^*; moreover it is important to produce an estimation -even during the transient of systems- for the unmeasured variable x. Nevertheless, this task is made difficult by the uncertainties of the model summarized by **(H1)**–**(H2)**.

PROBLEM 2.2.1 (The robust regulation problem $\mathcal{P}$). *Given a reference value $s^* \in (0, s_{in})$; find a family of positive feedback control laws $D\colon \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}_+^*$ such that the closed–loop system* (5.1.1) *has the following properties*:

(a) *There are two bounded functions, an upper one and a lower one, for the unmeasured variable $x(t)$ and for the substrate $s(t)$, that improve the estimation given by Eq.*(2.1.2). *That means a set of well known functions s^-,s^+,x^- and $x^+\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that*:

$$x^-(t) \leq x(t) \leq x^+(t) \quad \textit{and} \quad s^-(t) \leq s(t) \leq s^+(t) \quad \textit{for any } t \geq 0.$$

(b) *There exists a compact set* $K = [s^-, s^+] \times [x^-, x^+] \in \Omega$ *and a finite time* $T \geq 0$ *such that* $(s(t), x(t)) \in K$ *for any* $t > T$; *moreover* $s^* \in (s^-, s^+)$.

There are some problems related to $(\mathcal{P})$:

In [**144**],[**145**], system (2.1.1) is studied under the following assumptions: $d = 0$, the function f is assumed to be of Monod type with unknown kinetic parameters (defined as time dependent unknown functions). Numerical simulations with additive and multiplicative perturbations are carried out (assumed to be white noise). In [**115**], the system (2.1.1) is studied under the following assumptions: $d = 0$, **(H1)** holds and the only output available is the substrate with additive perturbations *i.e.* $y(t) = s(t) + \Delta(t)$. Using monotone dynamical systems theory, an interval observer has been built, allowing the estimation of the non-measured variable (the biomass).

In [**116**], a more general bioreactor is studied under the following assumptions: $d = 0$,**(H1)** holds and the only output available is the substrate *i.e.* $y(t) = s(t)$. The non-measured variables are estimated with an interval observer and using these estimations, a feedback control law that stabilizes the substrate $s(t)$ in a neighborhood of the reference value s^* has been built.

2.3. Feedback control law

We introduce the following control hypothesis:

(H3) (Input hypothesis) The function dilution rated D is the feedback control variable.

Let us consider the following family of feedback control laws:

$$D(y(t)) = D^* + h(y(t)) \tag{2.3.1}$$

where D^* is a constant satisfying the inequality $f^-(s^*) < D^* + d < f^+(s^*)$ and the function $h\colon \mathbb{R} \mapsto \mathbb{R}$ satisfies the following assumptions **(G)**:
(G1) h is Lipschitz, decreasing such that:

$$h(s^*) = 0, \quad -D^* < h(s_{in}[1 + \Delta^+]) < h(s_{in}) < l(s_{in}) - (D^* + d).$$

(G2) The bounds Δ^- and Δ^+ are such that the equations:

$$D^* - u(s) + h(s[1 + \Delta^+]) + d\Big\{\frac{D^* + h(s[1 + \Delta^+])}{D^* + h(s[1 + \Delta^-])}\Big\} = 0,$$

$$D^* - l(s) + h(s[1 + \Delta^-]) + d\Big\{\frac{D^* + h(s[1 + \Delta^-])}{D^* + h(s[1 + \Delta^+])}\Big\} = 0$$

have one single root $s^l \in (0, s^*)$ and $s^u \in (s^*, s_{in})$ respectively.

(G3) The death rate satisfies the following inequalities:

$$d < \inf_{r\in(0,s_{in})} \left\{ l(r)\frac{h'(r[1+\Delta^+])}{h'(r[1+\Delta^-])}\frac{1+\Delta^+}{1+\Delta^-} - l'(r)\frac{D^*+h(r[1+\Delta^-])}{[1+\Delta^-]h'(r[1+\Delta^-])} \right\},$$

$$d < \inf_{r\in(0,s_{in})} \left\{ u(r)\frac{h'(r[1+\Delta^-])}{h'(r[1+\Delta^+])}\frac{1+\Delta^-}{1+\Delta^+} - u'(r)\frac{D^*+h(r[1+\Delta^+])}{[1+\Delta^+]h'(r[1+\Delta^+])} \right\}.$$

(G4) The death rate satisfies the following inequality:

$$d < \min\left\{ s_{in}\frac{D^*+h(0)}{v^*+s_{in}}, \left(\frac{s_{in}-s^u}{s_{in}-s^l}\right)[D^*+h(s^u[1+\Delta^+])] \right\}.$$

Remark 2.3.1. *Notice that assumption* **(G1)** *can always be satisfied with reasonable choices of* h*. Moreover, Eq.*(2.1.2) *implies that the output* $y(t)$ *is bounded by the interval* $[s_{in}(1+\Delta^-), s_{in}(1+\Delta^+)]$*. Hence, assumption* **(G1)** *implies that* $D(y(t))$ *is defined on a bounded interval, and we will be able to fulfill the physical constraints*:

$$0 < \underbrace{D^*+h(s_{in}[1+\Delta^+])}_{D_{\min}} < D(y(t)) < \underbrace{D^*+h(s_{in}[1+\Delta^-])}_{D_{\max}} \quad \text{for any } t \geq 0.$$

Assumption **(G2)** *can be satisfied if* $|\Delta^-|$ *and* $|\Delta^+|$ *are relatively small. Indeed, notice that if* $\Delta^- = \Delta^+ = 0$*, the equations stated in* **(G2)** *become*:

$$D^* - u(s) + h(s) + d = 0 \quad \text{and} \quad D^* - l(s) + h(s) + d = 0,$$

by using **(G1)**, *it is straightforward to prove that these equations have only one single root* $\tilde{s}^l \in (0, s^*)$ *and* $\tilde{s}^u \in (s^*, s_{in})$*. Finally, using the implicit function theorem we can prove the existence of a bound* $\Delta_0 > 0$ *such that the inequality* $\max\{|\Delta^-|, \Delta^+\} < \Delta_0$ *implies* **(G2)**.

Remark 2.3.2. *Assumptions* **(G3)**–**(G4)** *give an upper bound for the mortality rate. Notice that mortality rate* d *cannot exceed some threshold, otherwise the solution* $(s_{in}, 0)$ *of system* (5.1.1) *could become globally attractive. Moreover, notice that when* d *is relatively small with respect to* $D_{\min}$*, assumptions* **(G3)**–**(G4)** *can be satisfied with reasonable choices of* h.

Notice that by **(G1)** the feedback control law defined by Eq.(2.3.1) can be viewed as follows:

$$D(y(t)) = D^* + \lambda g(s^* - y(t))), \quad \lambda > -D^*/||g||_\infty$$

where g is an increasing function satisfying $g(0) = 0$.

So, if $y(t) < s^*$ it follows that $D(y(t)) > D^*$ and inversely if $y(t) > s^*$ it follows that $D(y(t)) > D^*$ and inversely, if $y(t) > s^*$ it follows that $D(y(t)) < D^*$ which implies that this control –under suitable assumptions– will stabilize the substrate concentration in a neighborhood of s^*.

2.3.1. Main result. In this section, we give sufficient conditions to solve the problem $(\mathcal{P})$ summarized in Theorem 2.3.1. The key idea of the proof is to transform the closed–loop system (2.1.1) into a system that can be compared with *cooperative* systems *i.e.* a system such that the off diagonal entries of the Jacobian matrix are nonnegative. Planar cooperative systems theory (see Appendix C) will be the main tool employed.

THEOREM 2.3.1. *The problem* $(\mathcal{P})$ *is solvable by a family of output feedback control defined by* (2.3.1) *satisfying* **(G1)**–**(G4)**.

PROOF. We will verify the properties **(a)** and **(b)** separately.

Step 1: Replacing D by $D(y(t))$, system (5.1.1) becomes:

$$\begin{cases} \dot{s} = \{D^* + h(s[1+\Delta(t)])\}(s_{in} - s) - \alpha x f(s), \\ \dot{x} = x\big[f(s) - D^* - h(s[1+\Delta(t)]) - d\big], \\ \big(s(0), x(0)\big) \in [s_0^-, s_0^+] \times [x_0^-, x_0^+] \subset \Omega. \end{cases} \tag{2.3.2}$$

Notice that, after **(H1)**–**(H2)**, system (2.3.2) satisfies Carathéodory conditions (see *e.g.* [**28**, Th 2.1.1]), that guarantee existence and uniqueness of solutions. Moreover, it is straightforward to verify that system (2.3.2) is positively invariant in Ω.

Let (s, x) be the solution of system (2.3.2). Using a standard argument, we build the function $v\colon \Omega \mapsto \mathbb{R}$ defined by:

$$v = s + \alpha x - s_{in}. \tag{2.3.3}$$

Clearly, it follows that (x, v) is a solution of the system:

$$\textbf{(S)} = \begin{cases} \dot{x} = x\Big[f(v + s_{in} - \alpha x) - \{D^* + h([v + s_{in} - \alpha x][1+\Delta(t)]\}) - d\Big], \\ \dot{v} = -\{D^* + h([v + s_{in} - \alpha x][1+\Delta(t)])\}v - \alpha d x, \\ x(0) > 0, \quad -s_{in} < v(0) < v^*. \end{cases}$$

We make the time transformation:

$$r = \int_0^t \Big\{D^* + h(s(\tau)[1+\Delta(\tau)])\Big\}\, d\tau$$

Notice that we have built a function $\mathbb{R}_+ \mapsto \mathbb{R}_+$, that can be proved be invertible using Remark 2.3.1. We see at once that

$$\frac{dx}{dr} = \dot{x}\{D^* + h(s[1+\Delta(t)])\}^{-1} \quad \text{and} \quad \frac{dv}{dr} = \dot{v}\{D^* + h(s[1+\Delta(t)])\}^{-1}.$$

As this function is injective, we can also define $t(r)$. Hence, the solutions of system **(S)** can be written as:

$$\begin{cases} \dfrac{dx}{dr} = x\Big[\dfrac{f(v + s_{in} - \alpha x) - d}{D^* + h([v + s_{in} - \alpha x][1+\Delta(r)])} - 1\Big] = F(r, x, v), \\ \dfrac{dv}{dr} = -v - \dfrac{\alpha d x}{D^* + h([v + s_{in} - \alpha x][1+\Delta(r)])} \quad = G(r, x, v). \end{cases} \tag{2.3.4}$$

Let us define the set:

$$\Omega_1 = \{(x,v) \in \mathbb{R}^2 \colon x < \alpha^{-1}[v + s_{in}], v \in (-s_{in}, v^*)\}.$$

System (2.3.4) is positively invariant in Ω_1, now we build the following comparison system in Ω_1:

$$\Sigma_0^- = \begin{cases} \dfrac{d\phi}{dr} = \phi\Big[\dfrac{u(z+s_{in}-\alpha\phi)}{D^+(z+s_{in}-\alpha\phi)} - \dfrac{d}{D^-(z+s_{in}-\alpha\phi)} - 1\Big] = F^+(\phi,z), \\ \dfrac{dz}{dr} = -z \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = G^+(\phi,z), \\ x(0) \leq \phi_0 \quad \text{and} \quad v(0) \leq z_0 \quad (\phi_0, z_0) \in \operatorname{Int}\Omega_1. \end{cases}$$

Where the functions D^+,$D^- \colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ are defined as follows:

$$D^+(s) = D^* + h(s[1+\Delta^+]) \quad \text{and} \quad D^-(s) = D^* + h(s[1+\Delta^-]) \quad \text{for any } s \geq 0.$$

System Σ_0^- is positively invariant in Ω_1. Indeed, notice that if there exists a finite time $r_0 > 0$ such that the function $L(r) = z(r) + s_{in} - \alpha\phi(r)$ –where (ϕ, z) are solutions of the system– verify $L(r_0) = 0$, it follows that $\frac{dL}{dr}|_{r=r_0} > 0$ and the invariance is verified. Moreover, by assumption **(G3)** it follows that system Σ_0^- is cooperative. Finally, by **(H1)**–**(H2)** and **(G1)** the inequalities:

$$F(r,x,v) \leq F^+(x,v) \quad \text{and} \quad G(r,x,v) \leq G^+(x,v)$$

follow for any $(r,x,v) \in \mathbb{R}_+ \times \Omega_1$.

Applying the comparison theorem for cooperative systems (see Th.C.2.1 in Appendix C) to the systems (2.3.4) and Σ_0^-, we see that:

$$x(r) \leq \phi(r) \quad \text{and} \quad v(r) \leq z(r) \quad \text{for any } r \geq 0. \tag{2.3.5}$$

Let $\big(\phi(r), z(r)\big)$ be the solution of system Σ_0^-. Now, we use this function $\phi(r)$ to build the following comparison system in Ω_1:

$$\Sigma_0^+ = \begin{cases} \dfrac{d\eta}{dr} = \eta\Big[\dfrac{l(\chi+s_{in}-\alpha\eta)}{D^-(\chi+s_{in}-\alpha\eta)} - \dfrac{d}{D^+(\chi+s_{in}-\alpha\eta)} - 1\Big] = F^-(\eta,\chi), \\ \dfrac{d\chi}{dr} = -\chi - \dfrac{\alpha d\phi(r)}{D^+(\chi+s_{in}-\alpha\eta)} \qquad\qquad\qquad\qquad = G^-(r,\eta,\chi), \\ 0 < \eta_0 \leq x(0) \quad \text{and} \quad \chi_0 \leq v(0) \quad \text{and} \quad (\eta_0, \chi_0) \in \operatorname{Int}\Omega_1. \end{cases}$$

Notice that the system Σ_0^+ is positively invariant in Ω_1. Indeed, notice that if there exists a finite time $r_0 > 0$ such that the function $L(r) = \chi(r) + s_{in} - \alpha\eta(r)$ verify $L(r_0) = 0$ it follows by **(G4)** that $\frac{dL}{dr}|_{r=r_0} > 0$ and the invariance is verified. Moreover, it follows from assumption (**G3**) that the system Σ_0^+ is cooperative. Finally, assumptions **(H1)**–**(H2)**,**(G1)** and Eq.(2.3.5) imply that:

$$F^-(x,v) \leq F(r,x,v) \quad \text{and} \quad G^-(r,x,v) \leq G(r,x,v)$$

follow for any $(r,x,v) \in \mathbb{R}_+ \times \Omega_1$.

Applying again Th.C.2.1 (see Appendix C) to systems (2.3.4) and Σ_0^+, we see that:

$$\eta(r) \le x(r) \quad \text{and} \quad \chi(r) \le v(r) \quad \text{for any } r \ge 0. \tag{2.3.6}$$

Using Eqs.(2.3.5) and (2.3.6), the functional bounds for the biomass are:

$$x^-(r) = \eta(r) \le x(r) \le \phi(r) = x^+(r).$$

Moreover, using Eq.(2.3.3), the estimation for the substrate given by Eq.(2.1.2) can be improved by $s^-(r) \le s(r) \le s^+(r)$, where $s^-(r)$ and $s^+(r)$ are defined by:

- $s^-(r) = \max\Big\{s_{in} - \alpha\phi(r) + z(r), y^-(r)\Big\}$,
- $s^+(r) = \min\Big\{s_{in} - \alpha\eta(r) + \chi(r), y^+(r)\Big\}$.

Indeed, to obtain the bounds on $s(t)$, we take the minimum of the bounds given by the output and of those given by the comparison systems and hence property **(a)** holds.
Step 2: In order to verify property **(b)**, notice that assumptions **(G2)** and **(G4)** imply that the critical points of system Σ_0^- are:

$$E_1^+ = (\alpha^{-1}[s_{in} - s^l], 0) \quad \text{and} \quad E_2^+ = (0, 0).$$

A linearization procedure combined with assumptions **(G2)**–**(G4)** shows that E_1^+ and E_2^+ are, respectively locally stable and unstable.

It can be proved that the critical point E_2^+ cannot be an ω–limit set for any initial condition (ϕ_0, z_0). We will sketch this proof:

- It is straightforward to show that the stable manifold of the critical point E_2^+ is defined by the set:
$$W^s(E_2^+) = \{(\phi, z) \in \Omega_1 \colon \phi = 0\}.$$
- We build the functional $P \colon \Omega_1 \mapsto \mathbb{R}$ defined by $P(\phi, z) = \phi$. Clearly $P = 0$ in $W^s(E_2^+)$ and $P > 0$ in $\Omega_1 \setminus W^s(E_2^+)$.
- It follows from system Σ_0^- that $\dot{P} = \Psi(\phi, z)P$ where $\Psi \colon \Omega_1 \mapsto \mathbb{R}$ is the continuous function:
$$\Psi(\phi, z) = \frac{u(z + s_{in} - \alpha\phi)}{D^+(z + s_{in} - \alpha\phi)} - \frac{d}{D^-(z + s_{in} - \alpha\phi)} - 1.$$
- It follows from **(G2)** that $\Psi(E_2^+) > 0$. This implies that P is an *average Lyapunov function* (see Appendix A and the references given there) and by using Corollary A.3.1 from appendix A it follows that E_2^+ cannot be attained from $\operatorname{Int}\Omega_1$.

Since the solutions of system Σ_0^+ are bounded, applying Th.C.2.1 from Appendix C, it follows that:

$$\lim_{r \to +\infty} \big(\phi(r), z(r)\big) = E_1^+.$$

Hence system Σ_0^+ is asymptotically autonomous (see Appendix B and the references given there) with limit system:

$$\widetilde{\Sigma}_0^+ = \begin{cases} \dfrac{d\eta}{dr} = \eta\Big[\dfrac{l(\chi + s_{in} - \alpha\eta)}{D^-(\chi + s_{in} - \alpha\eta)} - \dfrac{d}{D^+(\chi + s_{in} - \alpha\eta)} - 1\Big] = F^-(\eta, \chi), \\ \dfrac{d\chi}{dr} = -\chi - \dfrac{d[s_{in} - s^l]}{D^+(\chi + s_{in} - \alpha\eta)} \qquad\qquad\qquad\qquad = G^-(\eta, \chi), \\ 0 < \eta_0 \le x(0) \quad \text{and} \quad \chi_0 \le v(0) \quad \text{and} \quad (\eta_0, \chi_0) \in \operatorname{Int}\Omega_1. \end{cases}$$

Assumptions **(G3)**–**(G4)** imply that the system $\widetilde{\Sigma}_0^+$ is positively invariant and cooperative in Ω_1. Moreover, its critical points are:

$$E_1^- = \Big(\alpha^{-1}[s_{in} - s^u + \xi_0(\Delta^+)], \xi_0(\Delta^+)\Big) \quad \text{and} \quad E_2^- = (0, \bar{\chi}).$$

where

$$\xi_0(\Delta^+) = \frac{-d(s_{in} - s^l)}{D^* + h(s^u[1 + \Delta^+])}$$

and $\bar{\chi} \in (s^u - s_{in}, 0)$ is the unique root of the function $(-s_{in}, 0) \mapsto \mathbb{R}$ defined by:

$$r \mapsto -r - \frac{d(s_{in} - s^l)}{D^* + h([r + s_{in}][1 + \Delta^+])}.$$

By assumptions **(G2)**–**(G4)** combined with a linearization procedure, it follows that E_1^- and E_2^- are respectively locally stable and unstable. Moreover, following the lines of the proof given for the point E_2^+, we can prove that the critical point E_2^- cannot be an ω–limit set for any initial condition (η_0, χ_0).

Since the solutions of system Σ_0^- are bounded and Th.C.2.1 from Appendix C implies that E_1^- is a global attractor of system $\widetilde{\Sigma}_0^+$, it follows by the Poincaré–Bendixson trichotomy for asymptotically autonomous systems (see for example [**139**, Th.1.6],[**140**, Th.1.5] and Appendix B) that:

$$\lim_{r\to+\infty} \big(\eta(r), u(r)\big) = E_1^-.$$

Combining these estimates with Eq.(2.3.3) implies that property **(b)** holds with:

$$K = [s^l, s^u] \times [\alpha^{-1}(s_{in} - s^u + \xi_0(\Delta^+)), \alpha^{-1}(s_{in} - s^l)].$$

□

2.3.2. Some extensions. This subsection deals with some improvements of the main result. Notice that the area of the set K is $A(K) = \alpha^{-1}(s^u - s^l)[(s^u - s^l) - \xi_0(\Delta^+)]$ and using **(G2)** it follows that $A(K)$ is strongly determined by the feedback control law chosen. So, motivated by practical applications, it will be desirable to reduce the area of K by choosing an adequate feedback control law. In this sense, we have the following result:

COROLLARY 2.3.1. *Given $\varepsilon > 0$, there exists an appropriate function h such that*:

$$\left|\frac{s^l}{s^u} - \frac{1+\Delta^-}{1+\Delta^+}\right| < \varepsilon.$$

PROOF. We can choose a neighborhood V of s^* such that $(s^-, s^+) \subset V$. Hence, using **(G1)**–**(G2)** it follows that:

$$|h(s^l[1+\Delta^+]) - h(s^u[1+\Delta^-])| = \left|u(s^l) - l(s^u) + d\Big[\frac{D^+(s^l)}{D^-(s^l)} + \frac{D^-(s^u)}{D^+(s^u)}\Big]\right|.$$

By the mean value theorem, we have that for any $\rho \in V$, it follows that:

$$|h'(\rho)||s^l(1+\Delta^+) - s^u(1+\Delta^-)| \leq |u(s^l) - l(s^u)| + d\Big[\frac{D^+(s^l)}{D^-(s^l)} + \frac{D^-(s^u)}{D^+(s^u)}\Big]$$

and in consequence:

$$\left|\frac{s^u}{s^l} - \frac{1+\Delta^+}{1+\Delta^-}\right| \leq \frac{|u(s^l) - l(s^u)| + d\Big[\frac{D^+(s^l)}{D^-(s^l)} + \frac{D^-(s^u)}{D^+(s^u)}\Big]}{s^u|h'(\rho)|[1+\Delta^+]} \leq \frac{C}{|h'(\rho)|}$$

for some $\rho \in (s^l, s^u)$ and $C > 0$ a constant defined by:

$$C = (s^*[1+\Delta^+])^{-1}\Big[\max_{r\in[0,s_{in}]} |u(r) - l(r)| + 2d\frac{D_{\max}}{D_{\min}}\Big].$$

Now, choosing a control law such that $|h'(u)| > C\varepsilon^{-1}$ for any $u \in V$ completes the proof. ☐

Moreover, we can build a decreasing sequence of sets $\{K_j\}_j$ ($K_0 = K$) satisfying $A(K_{j+1}) \subset A(K_j)$ for any integer $j \geq 0$. This is the content of the following result:

COROLLARY 2.3.2. *There exist two sequences $\{\eta_j\}_j$,$\{\phi_j\}_j$ of functions in $C(\mathbb{R}_+, \mathbb{R}_+)$ satisfying the following properties*:

(i) *The sequence $\{\eta_j\}_j$ is nonnegative and the sequence $\{\phi_j\}_j$ is bounded above by x_0^+, and the components satisfy*:

$$0 \leq \eta_{j-1}(r) \leq \eta_j(r) \leq \ldots \leq x(r), \tag{2.3.7}$$

$$x(r) \leq \ldots \leq \phi_j(r) \leq \phi_{j-1}(r) \leq \alpha^{-1}(v^* + s_{in}) \tag{2.3.8}$$

for any integer $j \geq 1$ and $r \geq 0$.

(ii) *For any integer $j \geq 0$, the functions have the asymptotic behavior*:

$$\lim_{r\to+\infty} \Big(\phi_j(r), \eta_j(r)\Big) = \Big(\frac{s_{in} - s^l + \xi_j(\Delta^-)}{\alpha}, \frac{s_{in} - s^u + \xi_j(\Delta^+)}{\alpha}\Big) \tag{2.3.9}$$

where the sequences $\xi_j(\Delta^-)$ *and* $\xi_j(\Delta^+)$ *are recursively defined by*:

$$\xi_j(\Delta^-) = \begin{cases} 0 & \text{if } j = 0, \\ -\dfrac{d[s_{in} - s^u + \xi_{j-1}(\Delta^+)]}{D^* + h(s^l[1+\Delta^+])} & \text{if } j \in \{1, 2, \ldots\}. \end{cases}$$

$$\xi_j(\Delta^+) = -\frac{d[s_{in} - s^l + \xi_j(\Delta^-)]}{D^* + h(s^u[1+\Delta^-])} \quad j \in \{0, 1, \ldots\}.$$

(iii) *These sequences converge uniformly in* $BC([0,\infty), \mathbb{R})$ (*the Banach space of bounded continuous functions on* $\mathbb{R}_+$ *taking values in* $\mathbb{R}$) *to the functions* η_∞ *and* ϕ_∞ *respectively.*

Proof. (i) For any integer $j \geq 0$, let us build the systems:

$$\Sigma_j^+ = \begin{cases} \dfrac{d\phi}{dr} = \phi\Big[\dfrac{u(z + s_{in} - \alpha\phi)}{D^+(z + s_{in} - \alpha\phi)} - \dfrac{d}{D^-(z + s_{in} - \alpha\phi)} - 1\Big] = F_j^+(\phi, z), \\ \dfrac{dz}{dr} = -z - \dfrac{\alpha d \gamma_j(r)}{D^-(z + s_{in} - \alpha\phi)} \qquad\qquad\qquad\qquad = G_j^+(r, \phi, z), \\ x(0) \leq \phi_0 \quad \text{and} \quad v(0) \leq z_0 \quad \text{and} \quad (\phi(0), z_0) \in \operatorname{Int}\Omega_1. \end{cases}$$

$$\Sigma_j^- = \begin{cases} \dfrac{d\eta}{dr} = \eta\Big[\dfrac{l(\chi + s_{in} - \alpha\eta)}{D^-(z + s_{in} - \alpha\eta)} - \dfrac{d}{D^+(\chi + s_{in} - \alpha\eta)} - 1\Big] = F_j^-(\eta, \chi), \\ \dfrac{d\chi}{dr} = -\chi - \dfrac{\alpha d \lambda_j(r)}{D^+(\chi + s_{in} - \alpha\eta)} \qquad\qquad\qquad\qquad = G_j^-(r, \eta, \chi), \\ x(0) \geq \eta_0 \quad \text{and} \quad v(0) \geq \chi_0 \quad \text{and} \quad (\eta_0, \chi_0) \in \operatorname{Int}\Omega_1. \end{cases}$$

where the functions γ_j,$\lambda_j \colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfy:

$$0 \leq \gamma_j(r) \leq x(r) \leq \lambda_j(r) \leq \alpha^{-1}(v^+ + s_{in}).$$

Notice that these systems are cooperative in Ω_1. Moreover, given these bounds for $\gamma_j(r)$ and $\lambda_j(r)$, assumption **(G4)** implies that systems Σ_j^- and Σ_j^+ are positively invariant in Ω_1.

Let $\big(\phi_j(r), z_j(r)\big)$ and $\big(\eta_j(r), \chi_j(r)\big)$ be the solutions of systems Σ_j^+ and Σ_j^- respectively. Now, we define two sequences $\{\gamma_j\}_j$ and $\{\lambda_j\}_j$ recursively as follows:

$$\gamma_j(r) = \begin{cases} 0 & \text{if } j = 0 \\ \eta_{j-1}(r) & \text{if } j \in \{1, 2, \ldots\} \end{cases} \quad \text{and} \quad \lambda_j(r) = \phi_j(r) \quad j \in \{0, 1, \ldots\}.$$

Now, we will show (2.3.7),(2.3.8) and (2.3.9) hold using mathematical induction. Indeed, when $j = 0$, systems Σ_0^+ and Σ_0^- are described above and using Th.2.3.1 the inequalities (2.3.7) and (2.3.8) with the limits given by Eq.(2.3.9) hold for $j = 0$.

Now, we assume that inequalities (2.3.7),(2.3.8) and the limits given by Eq.(2.3.9) are satisfied for any integer $j \in \mathbb{N}_{k-1}$. We will prove that they are also satisfied for $j = k$.

Hence, it follows that the inequalities:

$$F_k^+(r, \phi, z) \leq F_{k-1}^+(r, \phi, z) \quad \text{and} \quad G_k^+(r, \phi, z) \leq G_{k-1}^+(r, \phi, z)$$

hold for any $(r, \phi, z) \in \mathbb{R}_+ \times \Omega_1$.

Using this inequality and applying Prop.C.2.1 to systems (2.3.4),Σ_{k-1}^+ and Σ_k^+, it follows that $x(r) \leq \phi_k(r) \leq \phi_{k-1}(r)$.

This last statement implies that:

$$F_{k-1}^-(r, \eta, \chi) \leq F_k^-(r, \eta, \chi) \quad \text{and} \quad G_{k-1}^-(r, \eta, \chi) \leq G_k^-(r, \eta, \chi)$$

hold for any $(r, \eta, \chi) \in \mathbb{R}_+ \times \Omega_1$.

Now, applying again Prop.C.2.1 to systems (2.3.4),Σ_{k-1}^- and Σ_k^-, it follows that $\eta_{k-1}(r) \leq \eta_k(r) \leq x(r)$ and the inequalities (2.3.7) and (2.3.8) are satisfied for $j = k$.

(ii) Notice that systems Σ_j^+ and Σ_j^- are asymptotically autonomous with limits:

$$\widetilde{\Sigma}_j^+ = \begin{cases} \dfrac{d\phi}{dr} = \phi\Big[\dfrac{u(z + s_{in} - \alpha\phi)}{D^+(z + s_{in} - \alpha\phi)} - \dfrac{d}{D^-(z + s_{in} - \alpha\phi)} - 1\Big] = \widetilde{F}_j^+(\phi, z), \\ \dfrac{dz}{dr} = -z - \dfrac{d[s_{in} - s^u - \xi_{j-1}(\Delta^+)]}{D^-(z + s_{in} - \alpha\phi)} \qquad\qquad\qquad = \widetilde{G}_j^+(\phi, z), \\ x(0) \leq \phi_0 \quad \text{and} \quad v(0) \leq z_0 \quad \text{and} \quad (\phi_0, z_0) \in \operatorname{Int}\Omega_1. \end{cases}$$

$$\widetilde{\Sigma}_j^- = \begin{cases} \dfrac{d\eta}{dr} = \eta\Big[\dfrac{l(\chi + s_{in} - \alpha\eta)}{D^-(\chi + s_{in} - \alpha\eta)} - \dfrac{d}{D^+(\chi + s_{in} - \alpha)} - 1\Big] = \widetilde{F}_j^-(\eta, \chi), \\ \dfrac{d\chi}{dr} = -\chi - \dfrac{d[s_{in} - s^l - \xi_j(\Delta^-)]}{D^+(\chi + s_{in} - \alpha\eta)} \qquad\qquad\qquad = \widetilde{G}_j^-(\eta, \chi), \\ x(0) \geq \eta_0 \quad \text{and} \quad v(0) \geq \chi_0 \quad \text{and} \quad (\eta_0, \chi_0) \in \operatorname{Int}\Omega_1. \end{cases}$$

Using the results given by the Poincaré–Bendixson trichotomy again and following the lines of step 2 in the proof of Theorem 2.3.1 we can prove that the solutions of systems Σ_j^+ and Σ_j^- are convergent to the points E_j^+ and E_j^- respectively defined as follows:

$$E_j^+ = \Big(\frac{s_{in} - s^l + \xi_j(\Delta^-)}{\alpha}, \xi_j(\Delta^-)\Big),$$

$$E_j^- = \Big(\frac{s_{in} - s^u + \xi_j(\Delta^+)}{\alpha}, \xi_j(\Delta^+)\Big).$$

(iii) Now, we will prove that the sequences $\{\eta_j\}_j$ and $\{\phi_j\}_j$ converge uniformly to the functions η_∞ and ϕ_∞ that satisfy the inequalities (2.3.7) and (2.3.8). We only prove the result for the sequence $\{\eta_j\}_j$. The other case can be proved analogously.

Firstly, given any $T > 0$, we study the properties of the sequence $\{\eta_j\}_j$ in the interval $[0, T]$. Notice that, by using the systems Σ_j^- and Σ_j^+, we can deduce that the Lipschitz constant for this sequence is given by a number $L \leq \alpha^{-1}v^*(l(s_{in}) + d)/D_{\min}$ and it is straightforward to verify that the sequence $\{\eta_j\}_j$ forms an equicontinuous and uniformly bounded set. By using the Arzelà–Ascoli theorem, we have the existence of a subsequence η_{j_k}, that converges uniformly to a continuous function $\eta \in C([0, T], \mathbb{R})$.

We can assume that this subsequence satisfies $\eta_{j_k}(r) \leq \eta_{j_{k+1}}(r)$ for any $r \in [0,T]$. Without loss of generality, we also can suppose that there exist an infinite number of indices j_k satisfying:

$$\eta_{j_k}(r) \leq \eta_j(r) \leq \eta_{j_{k+1}}(r). \tag{2.3.10}$$

Letting $k \to +\infty$, Eq.(2.3.10) implies that η_j is pointwise convergent to η in $[0,T]$. Now, Dini's theorem implies uniform convergence and consequently, η_j is a Cauchy sequence in $C([0,T],\mathbb{R})$. So, given any $\varepsilon > 0$, there exists a number $J_1(\varepsilon) > 0$ such that:

$$|\eta_j(r) - \eta_{j+l}(r)| < \varepsilon \quad \text{for any } j > J_1 \text{ and } r \in [0,T]. \tag{2.3.11}$$

Secondly, let $\{x_j^-\}_j$ be a sequence defined by:

$$x_j^- = \alpha^{-1}[s_{in} - s^l + \xi_j(\Delta^+)],$$

It is straightforward to check that $\{x_j^-\}_j$ converges to a number x_∞^-. Hence, given $\varepsilon > 0$ there exists $J_2(\varepsilon) > 0$ such that $|x_j^- - x_\infty^-| < \varepsilon/4$ for any $j > J_2$.

Thirdly, Eqs.(2.3.9) imply that given $\varepsilon > 0$, there exists a number $T_{J_2}(\varepsilon)$ such that $|\eta_j(r) - x_j^-| < \varepsilon/4$ for any $r > T_{J_2}$. Now for any $r > T_{J_2}$ and $j > J_2$, it follows that:

$$\begin{aligned} |\eta_j(r) - \eta_{j+l}(r)| &\leq |\eta_j(r) - x_j^-| + |x_j^- - x_{j+l}^-| + |x_{j+l}^- - \eta_{j+l}(r)| \\ &\leq \frac{\varepsilon}{2} + |x_j^- - x_\infty^-| + |x_\infty^- - x_{j+l}^-| \leq \varepsilon \end{aligned}$$

Now, let $J = \max\{J_1, J_2\}$. It follows that:

$$|\eta_j(r) - \eta_{j+l}(r)| < \varepsilon \quad \text{for any } j > J \text{ and } r > T_J. \tag{2.3.12}$$

Finally, putting $T = T_J$ in Eq.(2.3.11) and combining with Eq.(2.3.12) it follows that:

$$|\eta_j(r) - \eta_{j+l}(r)| < \varepsilon \quad \text{for any } j > J \text{ and } r \geq 0$$

and we conclude that $\eta_j(r)$ is a Cauchy sequence in $BC([0,\infty),\mathbb{R})$ and the lemma follows.

□

Remark 2.3.3. *Corollaries* 2.3.1 *and* 2.3.2 *improve our main result in several ways*:

(i) *Corollary* 2.3.1 *means that if* $\Delta^+ \approx \Delta^- \approx 0$ (*small noise in the output*), *then we are able to stabilize* s *nearly exactly around* s^* *and* $A(K)$ *is almost zero.*
(ii) *The area of the set* K *is reduced by Corollary* 2.3.1, *by choosing a feedback control law that minimize* $s^u - s^l$. *Moreover, using Corollary* 2.3.2, *we can build a decreasing sequence of sets* $K_{j+1} \subset K_j$ ($K_0 = K$) *satisfying* $A(K_{j+1}) \leq A(K_j)$.

(iii) *Corollary* 2.3.2 *improves our estimation of the functional bounds for the biomass* $x(r)$. *Indeed, notice that* η_0 *and* ϕ_0 *of the sequence defined in* (2.3.8) *are the functional bounds for the biomass given by Th.*2.3.1 *and as we can see, the two sequences of functions* $\{\eta_j\}$, $\{\phi_j\}$ *improve the initial estimation.*

Moreover, uniform convergence of the sequences, give us the best estimation for the functional bounds for the biomass $x(r)$.

REMARK 2.3.4. *The convergence velocity towards the equilibrium point is increased by the feedback law as is shown by Figure* 2.4.4 *in the next section. Although numerical solutions make it clear* (*see next section*), *it is rather difficult to prove this formally for the nonlinear system. The classical comparison tool consists in linearizing around the equilibria, and comparing the eigenvalues of the system with and without control. Notice that for our system* (*and taking for simplicity* $d = 0, \Delta(t) = 0$ *and* $f^- = f^+ = f$), *we obtain the two eigenvalues*:

- *Without control*: $-D^*, -(s_{in} - s^*)[f'(s^*)]$,
- *With control* : $-D^*, -(s_{in} - s^*)[h'(0) + f'(s^*)]$.

Therefore it can be seen that our feedback control law can (*locally*) *increase the convergence by giving an eigenvalue as negative as wanted by taking* $h'(0)$ *large.*

REMARK 2.3.5. *If the assumption* **(H1)** *is a consequence of the the uncertainty of the kinetic parameters, that means, there exists closed intervals* (*see Section 1.1.3 from Introduction*) I_i ($i = 1, 2, 3$) *defined by*

$$I_1 = [\mu_m - \varepsilon, \mu_m + \varepsilon], \quad I_2 = [k_s - \varepsilon, k_s + \varepsilon] \quad \text{and} \quad I_3 = [k_i - \varepsilon, k_i + \varepsilon].$$

The properties of the systems Σ_j^- *and* Σ_j^+ *make it possible to enhance our estimation of the parameters by combining our methods with some adaptive control techniques in the sense that the* ***static bounds*** $f^-(s)$ *and* $f^+(s)$ *could become* ***dynamic bounds*** $f^-(r, s)$ (*increasing with respect to* r) *and* $f^+(r, s)$ (*decreasing with respect to* r) *satisfying*:

$$f^-(s) \leq f^-(r, s) \leq f(s) \leq f^+(r, s) \leq f^+(s) \quad \text{for any } r, s \geq 0.$$

Indeed, using the fact that systems Σ_j^- *and* Σ_j^+ *are cooperative, we can carry out known algorithms* (*see* e.g. *SIVIA algorithm developed in* [**79**] *and the references given there*), *to obtain new dynamic intervals* $\widetilde{I}_i(r) \subset I_i$ ($i = 1, 2, 3$) *where* $|\widetilde{I}_i(r)|$ *is decreasing with respect to* r.

2.4. Examples: Depollution of water and simulation of marine environments

Let us come back to the problems of stabilization stated in the introduction:

2.4.1. A model of depollution of phenol in the water. We suppose that s is the phenol and the biomass x is *Pseudomonas putida* [**134**]. The objective is to stabilize the concentration of phenol below the level s^+ after a finite time T. Then, the contaminant concentration becomes bounded above:

$$s(t) < s^+ < s_{in} \quad \text{for any } t > T.$$

In the following table we present some maximal concentrations allowed by the U.S. environmental protection agency (EPA) and the Canadian evaluation of toxic residues service (SERT) [1].

s^+ (mg/L)	Prevention goals	Reference
0.30	Avoid pollution (water and organisms)	EPA, 1980
0.49	Avoid toxicity (aquatic life)	SERT, 1990
0.02	Avoid chronic effects (aquatic life)	SERT, 1990

This model is affected by several uncertainties: the data presented by Sokol and Howell [**134**], show that the function f that describes the growth of $x(t)$ is of type:

$$f(s) = \frac{k_1(s)s}{k_2 + s^2}$$

where $k_1 \colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ is an increasing and bounded function. Despite the evidence for variability of the coefficient k_1, its precise functional form is unknown. k_2 is a positive (uncertain) parameter. We summarize the experimental data presented in [**134**] in the following table (Liters are denoted by L and milligrams by mg):

Parameter / function	Uncertainty	Units
$k_1(s)$	$k_1(s) \in [9.43, 22.5]$	Day^{-1}
k_2	$k_2 \in [2.82, 3.09]$	mg/L

Hence, for any growth function f, we have that:

$$f^-(s) = \frac{9.43s}{2.82 + s^2} \leq f(s) \leq \frac{22.5s}{3.09 + s^2} = f^+(s).$$

We will work with the following function:

$$f(s) = \frac{[15.96 + \omega_1(t)]s}{2.955 + \omega_2(t) + s}$$

where the functions ω_1 and ω_2 have been constructed interpolating two sets of random data bounded by $[-6.535, 6.535]$ and $[-0.135, 0.135]$ respectively. In the same way, we build a function $\Delta(t)$ bounded by $[-0.2, 0.2]$.

The feedback control law is built taking $D^* = 1$ and using the function $h \colon \mathbb{R} \mapsto \mathbb{R}$, defined by:

$$h(y(t)) = 0.85 \tanh(s^* - y(t)).$$

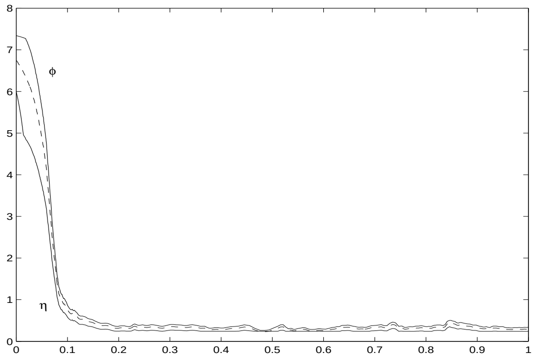


FIGURE 2.4.1. Concentration of phenol (dashed line) and its bounds ϕ and η given by the solutionsof systems Σ_0^+ and Σ_0^- respectively.

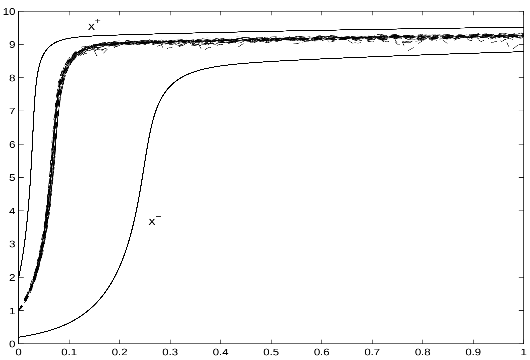


FIGURE 2.4.2. Concentration of *Pseudomonas putida* and its bounds $x^+ = \alpha^{-1}[s_{in} - \eta]$ and $x^- = \alpha^{-1}[s_{in} - \phi]$ where ϕ and η are given by the solutions of systems Σ_0^+ and Σ_0^- respectively.

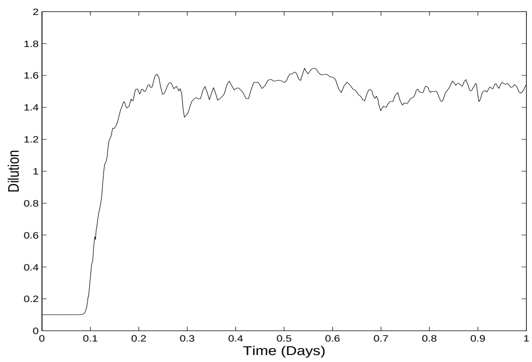


FIGURE 2.4.3. Dilution rate v/s time

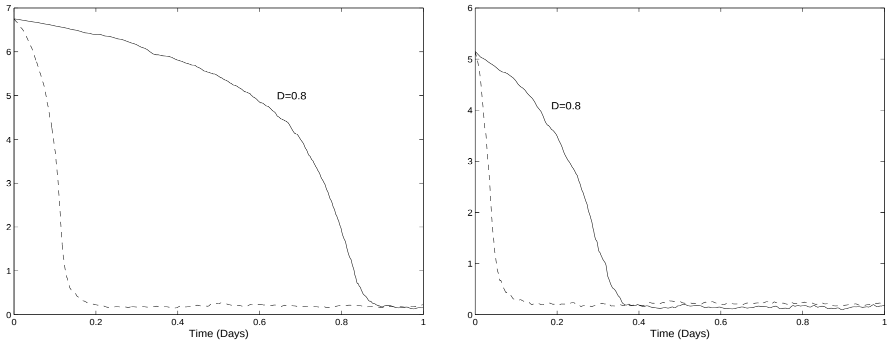


FIGURE 2.4.4. Substrate concentration taking initial conditions $s_0 = 6.8$mg/L (left) and $s_0 = 5.1$mg/L (right): Comparison between feedback control (dashed line) and fixed dilution (continuous line).

In the numerical simulation the following parameters are considered:

$$s_{in} = 8\text{mg/L} \quad s^+ = 0.49\mu\text{mg/L} \quad s^* = 0.25\text{mg/L} \quad d = 0.1\text{mg/L} \quad \alpha = 0.2.$$

We solve the systems using MATLAB ODE23. In order to apply Th.2.3.1, we easily check the conditions **(G)** using f^- and f^+. Figure 2.4.1 shows numerical results for the concentration of phenol graduated in mg/L and the time in days. Notice that the phenol concentration becomes lower than the value $s^+ = 0.49$ mg/L.

Figure 2.4.2 shows the numerical results for the concentration of *Pseudomonas putida* (graduated in mg/L) using ten classes of random data $(\omega_1, \omega_2, \Delta)$. Notice that this estimation could be enhanced in accuracy by using the results given in Corollary 2.3.2 and Remark 2.3.5.

We are also interested in comparing the depollution process described before with a process carried out using a fixed dilution rate (open–loop approach). Figure 2.4.4 shows simulations with two initial conditions s_0, we can see that the use of the feedback control law defined by Eq.(2.3.1) gives faster convergence.

2.4.2. Simulation of marine environments. We consider *Dunaniella tertiolecta* growth (a chlorophilian phytoplanktonic micro-algae) in a chemostat with nitrate as limiting substrate [**11**], and realistic values (from experiments [**12**]) for uncertainties and parameters.

For numerical simulations we take Michaelis Menten's function defined before with the kinetic parameters μ_m and k_s with experimental uncertainties given in [**143**] summarized in the following table (Liters are denoted by L, micro-atom grams by μatg and number of cells by Cell):

[1] See http://www.menv.gouv.qc.ca/eau

Parameter	Uncertainty	Units
μ_m	$\mu_m \in [1.2, 1.6]$	Day^{-1}
k_s	$k_s \in [0.01, 0.2]$	μatg/L
s_{in}	$s_{in} \in [80, 120]$	μatg/L
α^{-1}	$\alpha^{-1} \in [0.15, 0.6]$	non–dimensional

Hence, for any growth function $f(s)$ it follows that:

$$f^{-}(s) = \frac{1.2s}{0.2+s} \leq f(s) \leq \frac{1.6s}{0.01+s} = f^{+}(s).$$

We will work with the following function:

$$f(s) = \frac{[1.4+\omega_1(t)]s}{0.105+\omega_2(t)+s^2}$$

where $\mu_m = 1.42$,$k_s = 0.105$ and the functions ω_1 and ω_2 have been constructed interpolating two sets of random data bounded by $[-0.19, 0.19]$ and $[-0.07, 0.07]$ respectively. In the same way we build a function $\Delta(t)$ bounded by $[-0.03, 0.03]$.

We will work with the following realistic values for parameters s_{in},s^*,d and α:

$$s_{in} = 85\mu\text{gat/L} \quad s^* = 82\mu\text{gat/L} \quad d = 0.1\text{mg/L} \quad \alpha = 2.$$

The feedback control law is built taking $D^* = 1.3$ and using the function $h\colon \mathbb{R} \mapsto \mathbb{R}$ defined by:

$$h(y(t)) = 1.1\tanh(s^* - y(t)).$$

In order to apply Th.2.3.1, we verify easily the conditions **(G)**. Figure 2.4.5 shows numerical results for the concentration of Nitrate (graduated in $\mu\text{gat}L^{-1}$ and time in days. Figure 2.4.6 shows the numerical results for the concentration of *Dunaniella Tertiolecta* (graduated in $10^6\text{Cell}L^{-1}$) using ten classes of random data $(\omega_1, \omega_2, \Delta)$. Finally, Fig. 2.4.7 shows the dilution rate. Notice that washout is avoided, because it follows that the biomass is uniformly persistent. That means:

$$\liminf_{t\to+\infty} x(t) > x^{-} > 0 \quad \text{for any } x(0) > 0.$$

Uniform persistence of the biomass is essential because, as we stated in the Introduction, washout of the biomass must be avoided. Otherwise the chemostat would have to be re-inoculated. Notice that using a fixed dilution rate, uniform persistence of the biomass cannot be ensured for dilution rates $D > f^{-}(s_{in}) = 1.1972$. Furthermore, taking this restriction into account, it is not possible to stabilize the substrate in a neighborhood of s^* without control.

As before, we emphasize that the estimation given for the biomass could be enhanced in accuracy by using Corollary 2.3.2 and Remark 2.3.5.

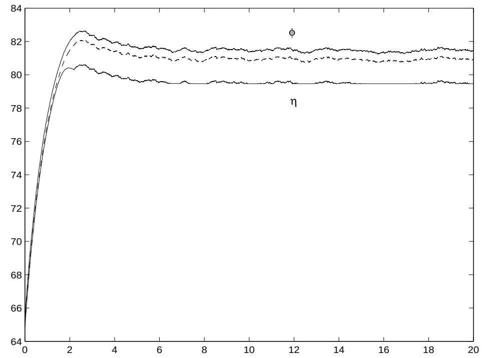


FIGURE 2.4.5. Concentration of nitrate (dashed line) and its bounds ϕ and η given by the solutionsof systems Σ_0^+ and Σ_0^- respectively.

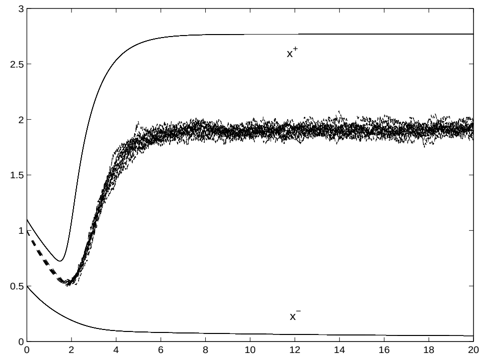


FIGURE 2.4.6. Concentration of *Dunaniella tertiolecta* and its bounds $x^+ = \alpha^{-1}[s_{in} - \eta]$ and $x^- = \alpha^{-1}[s_{in} - \phi]$ where ϕ and η are given by the solutions of systems Σ_0^+ and Σ_0^- respectively.

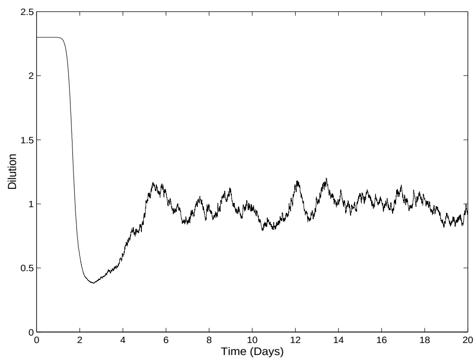


FIGURE 2.4.7. Dilution rate v/s time

2.5. Discussion

The problem $(\mathcal{P})$ of feedback stabilization for a chemostat with uncertainties concerning its output and internal structure has been discussed. It has been shown that, given known bounds for the uncertainties, a family of feedback control laws, which stabilize the system in a bounded set, can be constructed. Our approach is based on the theory of monotone dynamical systems.

We have extended the previous work discussed in the bibliography to allow mortality rates $d > 0$. This makes the problem far more difficult, because the "conservation principle" is not satisfied.

Many extensions are available in the spirit of **(H)** and/or **(G)**. We will distinguish between extensions with mathematical interest and feedback control interest

From a mathematical point of view, we could extend this chapter in the following ways:

(a) s_{in} can be viewed as an unknown function satisfying:

$$s_{in}^{-}(t) \leq s_{in}(t) \leq s_{in}^{+}(t), \quad \text{for any } t \geq 0$$

where s_{in}^{-} and s_{in}^{+} are bounded and positive measurable functions.

(b) Another natural extension of the present work would be to treat outputs more generally, by taking into account an *additive disturbance*, which means:

$$y(t) = s(t)[1 + \Delta_1(t)] + \Delta_2(t), \quad \text{for any } t \geq 0.$$

(c) In Remark 2.3.4 we show how a feedback control law can improve the speed of convergence toward a critical point. It will be desirable to study global results and their robustness.

The extensions (a) and (b) can certainly be handled by the methods presented in the proof of Th.2.3.1 combined with alternative/additional hypothesis.

From a point of view of feedback control theory, we pointed out in Remark 1.7.2 in the Introduction and Section 2.3, that the family of feedback control laws is a nonlinear version of a proportional regulator. It will be interesting and necessary to compare our results with other feedback control laws such as the following:

(a) The use of PI regulators (Proportional Integral) is suggested in [**31**, Chap.5] and this regulator is defined by:

$$D(s) = D^{*} + K_p(s^{*} - s) + \frac{1}{\tau_1}\int_0^t [s^{*} - s(r)]\,dr,$$

where $\tau_i > 0$ $(i = 1, 2)$ is a constant time.

(b) The application of L/A control strategies is used in [**85**] in the context of absence of perturbations.

CHAPTER 3

Feedback stabilization for a chemostat with delayed outputs

In this chapter, we study the problem of feedback asymptotic stabilization of a well known chemostat model (one single limiting substrate and one specie) by considering an output of type $y(t) = s(t-\tau)$.

We build a family of feedback control laws with the same properties as the family built in Chapter 2 (that means, a nonlinear version of a proportional regulator). Nevertheless, as we pointed out in the Introduction, the measures in bioprocesses are not available online from the plant and, in consequence, the presence of delays makes it more difficult to achieve our asymptotic stabilization objectives.

We state a control problem in a similar way as in the previous chapter, but in this case, we must deal with a system of two differential delay equations. For a rigorous presentation of the differential delay equations theory, we refer to Hale [**53**], Gopalsamy [**41**] and Diekmann *et.al.* [**29**]. By using some reduction techniques (see Section 1.2.1 and Appendix B) we prove that the asymptotic behavior can be studied by working only with the differential delay equation describing the biomass concentration.

By using an idea developed in [**94**] (and the references therein) we build a one–dimensional system of type $u_{n+1} = \chi(u_n)$ (with $\chi\colon \mathbb{R} \mapsto \mathbb{R}$) which inherits the asymptotic properties of the infinite–dimensional system related to the original control problem.

This chapter is organized as follows: In section 3.1 we recall some facts about the model and state the asymptotic stabilization problem. In section 3.2 we propose a family of feedback control laws. Some definitions and results related to dynamical systems are recalled in section 3.3. The main result is stated in section 3.4 and its proof is given in sections 3.5 and 3.6. Numerical examples are given in section 3.7.

3.1. Preliminaries

Let us recall the chemostat equations

$$\begin{cases} \dot{s} = D(s_{in} - s) - \alpha f(s)x, \\ \dot{x} = x\big(f(s) - D\big). \end{cases} \tag{3.1.1}$$

As in Chapter 2, we are interested in a problem of asymptotic stabilization of the variables of system (3.1.1); in this sense we state the control hypothesis:

(Input hypothesis) The function dilution rate D is the feedback control variable.

(Output hypothesis) The only output available is described by the equation:

$$y(t) = s(t-\tau), \quad \tau > 0. \tag{3.1.2}$$

The function $f(s)$ will be assumed to be of the following types:

$$f_1(s) = \frac{\mu_{\max} s}{k_s + s}, \quad \mu_{\max}, k_s > 0. \tag{3.1.3}$$

$$f_2(s) = \frac{\mu_{\max} s}{k_s + s + \frac{s^2}{k_i}}, \quad \mu_{\max}, k_s, k_i > 0. \tag{3.1.4}$$

$$f_3(s) = \frac{\mu_{\max} s}{k_s + s^2}, \quad \mu_{\max}, k_s > 0. \tag{3.1.5}$$

REMARK 3.1.1. *It is straightforward to verify that f_1 is strictly increasing, and concave and that the functions f_2* and *f_3 are unimodal, i.e. they have one critical point $s_{\max} > 0$ and moreover, $f_i''(s) < 0$ for any $s \in [0, s_c)$ with $s_c > s_{\max}$.*

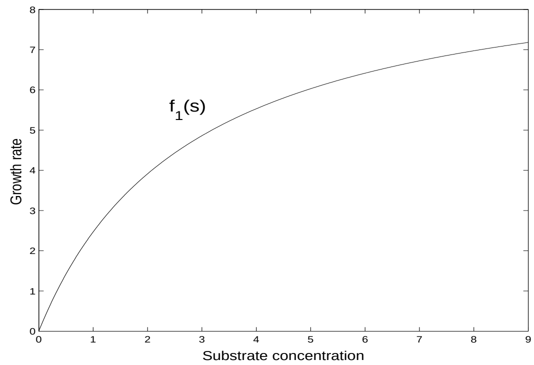


FIGURE 3.1.1. Graph of function f_1

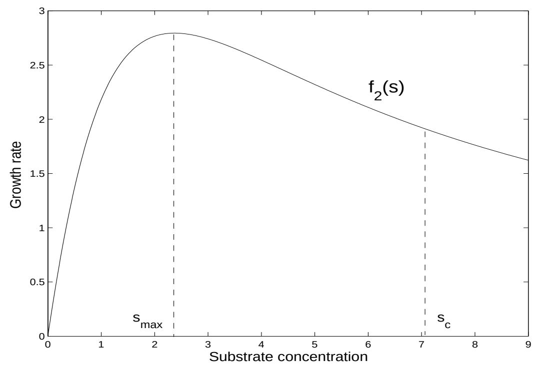


FIGURE 3.1.2. Graph of function f_2

In this chapter we consider the following problem:

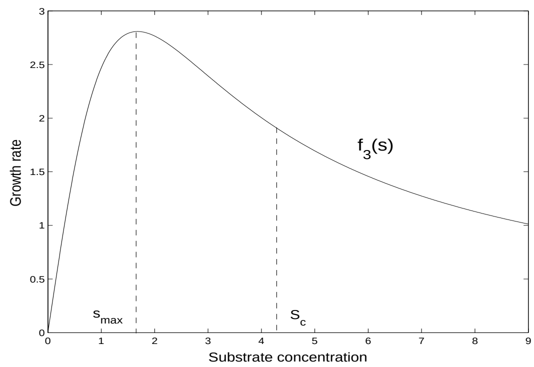


FIGURE 3.1.3. Graph of function f_3

PROBLEM 3.1.1. *Find a collection of feedback control laws that stabilize the system* (3.1.1)–(3.1.2) *with respect to a reference value* $s^* \in (0, \min\{s_{\max}, s_{in}\})$.

3.2. Feedback control law

Let us build the family of feedback control laws:

$$D(y(t)) = h(s^* - s(t - \tau)) \tag{3.2.1}$$

Where the function $h \in C^k(\mathbb{R}, \mathbb{R}_+)$ with $k \geq 1$ and satisfies the following properties:
(P1) h is increasing, nonnegative and $h(0) = f(s^*)$.
(P2) There exist two positive numbers $h_{\min}$ and $h_{\max}$ such that

$$h_{\min} \leq h(r) \leq h_{\max} \quad \text{for any } r \in \mathbb{R}.$$

(P3) The value s^* is the only root of the equation $h(s^* - s) - f(s) = 0$.

REMARK 3.2.1. *Properties* **(P1)**–**(P3)** *have been considered in the introduction and Chapter* 1. *Notice that, if* $\tau = 0$, *we can solve problem* (3.1.1). *Notice that if* f *is described by function* (3.1.3), *the property* **(P3)** *is automatically satisfied.*

3.3. Basic definitions and notations

We point out that if we replace D by the feedback control law (3.2.1), we deal with a system of differential delay equations; hence a way to solve Problem 3.1.1 is to find sufficient conditions for global attractivity of the critic point s^*.

In the following we shall make use of some results of dissipative dynamical systems theory and the Schwarz derivative of a real function that will be useful for the study of asymptotic properties of our control system. For the convenience of the reader we present some basic definitions adapted from [**53**],[**54**],[**126**].

Let (X, d) be a complete metric space. We define a continuous semiflow as a continuous function $\phi\colon \mathbb{R}_+ \times X \mapsto X$ satisfying the properties $\phi(0, \vec{\varphi}) = \vec{\varphi}$ and $\phi(t+s, \vec{\varphi}) = \phi(t, \phi(s, \vec{\varphi}))$. Moreover, we use the notation $\phi(t, \vec{\varphi}) = \phi_t(\vec{\varphi})$.

DEFINITION 3.3.1. ([**54**, chapt.3]) The semiflow ϕ_t is

(a) *Point dissipative* on X if there exist a bounded set B that attracts each point of X.
(b) *Conditionally completely continuous* for $t \geq t_1$ if, for each $t \geq t_1$ and each bounded set $B \subset X$ for which $\phi_s(B)$ (with $s \in [0, t]$) is bounded, we have that $\phi_t(B)$ is precompact for any $t > t_1$.
(c) *Completely continuous* for $t \geq t_1$ if it is conditionally completely continuous and, for each $t \geq 0$, the set $\phi_s(B)$ (with $s \in [0, t]$) is bounded.

PROPOSITION 3. [**54**, Th. 3.4.8] *If there is a number $t_1 \geq 0$ such that the semigroup ϕ_t is completely continuous for $t \geq t_1$ and point dissipative, then there is a global attractor A which is maximal, invariant and compact.*

Several times we shall use the concept of *Schwarz derivative* $(SF)(r)$ of a real valued function F having at least three continuous derivatives. The Schwarz derivative is defined by:

$$(SF)(r) = \frac{F'''(r)}{F'(r)} - \frac{3}{2}\left(\frac{F''(r)}{F'(r)}\right)^2.$$

We can also define the Schwarz derivative in terms of the *Pre-Schwarz derivative* which is defined as:

$$(PF)(r) = \frac{F''(r)}{F'(r)}$$

and in consequence we have that:

$$(SF)(r) = \frac{d}{dr}(PF)(r) - \frac{1}{2}\big[(PF)(r)\big]^2.$$

For more results about the Schwarz derivative see Appendix D.

REMARK 3.3.1. *We can easily verify that*:

$$(Sf_1)(s) = 0, \quad (Sf_2)(s) = \frac{-6k_s k_i}{(s^2 - k_s k_i)^2} \quad \textit{and} \quad (Sf_3)(s) = \frac{-6k_s}{(s^2 - k_s)^2}$$

and in consequence $(Sf_2)(s) < 0$ and $(Sf_3)(s) < 0$.

3.4. Main Results

As we pointed out in the Introduction (see Corollary 1.7.1) and Remark 3.2.1, if $\tau = 0$, then the feedback control law stabilizes asymptotically the chemostat in s^*. Hence, it is reasonable to suppose that taking into account the influence of delays in the measurements, this nonlinear regulator could loose a big part of its effectiveness or became useless. On the other hand, there exists several results in the mathematical modeling of biological systems by using differential delay equations's literature summarized as *small delays are harmless* (see for example [**41**]). For these reasons we are interested in finding upper bounds for the delays in the measurements for which the feedback control law (3.2.1) is still effective.

We introduce more assumptions on h with the hope of solving Problem 3.1.1:

(P4) For any two functions $\psi_i \in C([0,\tau],\mathbb{R})$ ($i \in \mathbb{N}_2$), there exists a constant L_0 such that for any $t \geq 0$:

$$\sup_{\theta\in[-\tau,0]} |h(s^*-\psi_1(t+\theta))-h(s^*-\psi_2(t+\theta))| \leq L_0 \sup_{\theta\in[-\tau,0]} |\psi_1(t+\theta)-\psi_2(t+\theta)|.$$

(P5) $(Sh)(r) < 0$ for any $r \in \mathbb{R}$ and $h(0)$ and its derivatives in $r = 0$ satisfy the inequality:

$$\frac{[f'(s^*)+h'(0)]^2 - 3[(Ph)(0)+(Pf)(s^*)]^2 f'(s^*)h'(0)}{[f'(s^*)+h'(0)]\big\{(Sh)(0)h'(0)+(Sf)(s^*)f'(s^*)\big\}} < 2(s_{in}-s^*).$$

REMARK 3.4.1. *Property* **(P4)** *is a technical assumption which ensures the existence and uniqueness of the solutions of the I/O system defined by Eqs.*(3.1.1) *and* (3.1.2). *On the other hand* **(P5)** *can always be satisfied with reasonable choices of the control function h.*

Notice that if we replace D in the system (3.1.1) for any feedback control law satisfying properties **(P)**, we have that $s(t) > 0$ for any $t \in \mathbb{R}_+$. Nevertheless, in several articles which study chemostat models by using a dynamical systems approach, it is necessary to build a prolongation for the functions f_i for any $r < 0$. Constant (*i.e.* $f_i(r) = 0$) or odd prolongations $f_i(r) = -f_i(-r)$ have been used in some articles.

Notice that the functions f_i are still defined in an interval $[-a,0]$ (with $a > 0$). Using this fact we will define a prolongation $\mu_i\colon \mathbb{R} \mapsto \mathbb{R}$ for the functions f_i ($i \in \mathbb{N}_3$) as follows:

$$\mu_i(r) = \begin{cases} \rho_i(r) & \text{if } r \in (-\infty,-a), \\ f_i(r) & \text{if } r \in [-a,+\infty) \end{cases}$$

where $\rho_i\colon (-\infty,-a) \mapsto \mathbb{R}$ is a monotone function satisfying $(S\rho_i)(r) < 0$ and $\rho_i^{(k)}(-a) = f_i^{(k)}(-a)$ ($k = 1,2$). For example, we can consider:

$$\rho_i(r) = f_i(-a) + f_i'(-a)(r+a) + f_i''(-a)\frac{(r+a)^2}{2} \quad (i \in \mathbb{N}_3),$$

which is increasing in $(-\infty,-a)$ and

$$(S\rho_i)(r) = -\frac{3}{2}\left[\frac{f''(-a)}{f'(-a)+f''(-a)(r+a)}\right]^2 < 0.$$

Notice that the functions $\mu_i \in C^2(\mathbb{R},\mathbb{R})$ have continuous third derivative except at the point $r = -a$. Moreover, $(S\mu_i)(r) < 0$ for any $r \in \mathbb{R}$.

Let us now introduce some notation and make precise the mathematical setting: we will build a discrete dynamical system that inherits some asymptotic properties of the chemostat model. To build this discrete system we must introduce some auxiliary functions related to μ_i and h (we will make the distinction between the functions μ_1,μ_2 and μ_3) described by:

$$g_{1_j}(r) = \begin{cases} (\mu_j \circ \lambda_1)(r) & \text{if } r < 0, \\ \dfrac{2[\kappa(0)f'(s^*)]^2 r}{2\kappa(0)f'(s^*) + \kappa(0)[f'(s^*) - f''(s^*)]r} & \text{if } r > 0 \end{cases} \qquad (j = 2, 3),$$

$$g_{1_1}(r) = (\mu_1 \circ \lambda_1)(r) \quad \text{and} \quad g_2(r) = (h \circ \lambda_2)(r)$$

where $\kappa, \lambda_j \colon \mathbb{R} \to \mathbb{R}$ are defined by

$$\kappa(r) = [s_{in} - s^*]e^{-r}, \quad \lambda_1(r) = s_{in} - \kappa(r), \quad \text{and} \quad \lambda_2(r) = \kappa(r) - \kappa(0).$$

To simplify the notation, we will write g_1 instead of g_{1_j} $(j \in \mathbb{N}_3)$. We will give more details about the index j if it is necessary.

Let us define the interval $I_\tau = [\alpha\tau, \{g_2(\alpha) - g_1(\alpha)\}\tau]$ where:

$$\alpha = \begin{cases} h(s^* - s_{in}) - f_1(s_{in}) & \text{if } f = f_1, \\ h(s^* - s_{in}) - h(0) - \dfrac{2\kappa(0)f'(s^*)^2}{f'(s^*) - (s_{in} - s^*)f''(s^*)} & \text{if } f = f_2, f_3. \end{cases}$$

Notice that λ_1 is increasing and λ_2 is decreasing. As a consequence the function g_1 is increasing and g_2 is decreasing. Moreover, it is straightforward to verify that $\alpha < 0$ and $g_2(\alpha) - g_1(\alpha) > 0$. By using Remark 3.3.1 and property **(P5)**, we can see that the set of intervals I_τ where the inequality:

$$\frac{\left[g_2'(r) - g_1'(r)\right]\left[\sum_{i=1}^{2}(Sg_i)(r)g_i'(r)\right]}{\left[\sum_{i=1}^{2}(-1)^i(Pg_i)(r)\right]^2} < \frac{3}{2}\prod_{i=1}^{2} g_i'(r), \tag{3.4.1}$$

is satisfied is not empty and we can define the number:

$$\tau_a^* = \sup\Big\{\tau > 0 \colon \text{Inequality (3.4.1) is verified in } I_\tau\Big\}.$$

We are now in position to state our main results:

THEOREM 3.4.1. *Let f be a function defined by f_i $(i \in \mathbb{N}_3)$. If properties* **(P1)–(P5)** *hold and the delay τ satisfies*:

$$\tau < \min\left\{\frac{1}{\kappa(0)[h'(0) + f'(s^*)]}, \tau_a^*\right\} \tag{3.4.2}$$

then the feedback control law (3.2.1) *stabilizes asymptotically the output in s^*.*

THEOREM 3.4.2. *Let f be a function defined by f_i $(i \in \mathbb{N}_3)$. If properties* **(P1)–(P4)** *are fulfilled and the delay τ satisfies*:

$$\tau < \tau_b^* = \sup\Big\{\tau > 0\colon \tau\big|g_2'(r) - g_1'(r)\big| < 1 \quad \text{is verified in } I_\tau\Big\}. \tag{3.4.3}$$

then the feedback control law (3.2.1) *stabilizes asymptotically the output in s^*.*

3.5. Proof of Theorem 3.4.1

Firstly we will study the closed–loop system, replacing D in the system (3.1.1) by the feedback control law (3.2.1). The closed–loop system becomes:

$$\begin{cases} \dot{s} = h(s^* - s(t-\tau))(s_{in} - s) - \alpha f(s)x, \\ \dot{x} = x\big(f(s) - h(s^* - s(t-\tau))\big), \\ x(0) \geq 0 \quad 0 \leq s(\theta) = \varphi_1(\theta) \leq s_{in} \quad \text{for any } \theta \in [-\tau, 0], \end{cases} \tag{3.5.1}$$

where φ_1 is a nonnegative continuous function bound above on the interval $[-\tau, 0]$.

Let us define

$$C = C\big([-\tau, 0], \mathbb{R}^2\big) \quad \text{and} \quad C_+ = C\big([-\tau, 0], \mathbb{R}_+^2\big)$$

the Banach space of scalar continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^2$ and the cone of nonnegative continuous functions, respectively. C is equipped with the supremum norm and C_+ becomes a complete metric space (C_+, d) under the induced metric.

The initial conditions of the system (3.5.1) are in the space $C_+ \times \mathbb{R}$ and can be embedded in the space $X = C_+ \times C_+$. Using **(P1)** and **(P4)**, it can be easily proved (see *e.g.* Theorems 2.3 and 3.2 from [**53**]) global existence and uniqueness of the solutions of system (3.5.1) and consequently it defines a semiflow $\phi\colon \mathbb{R}_+ \times X \mapsto X$, where $\phi_t(\varphi_1, \varphi_2) = (s_t, x_t)$ with $s_t(\theta) = s(t+\theta)$ and $x_t(\theta) = x(t+\theta)$ for any $\theta \in [-\tau, 0]$ and $t \geq 0$.

By using **(P2)**, it is straightforward to prove that the equilibria of system (3.5.1) are given by $E_0 = (s_{in}, 0)$ and $E_1 = (s^*, \alpha^{-1}[s_{in} - s^*])$. We will prove that E_1 is a globally attractive equilibria for any nonnegative initial condition.

The proof will be divided into three steps. Firstly (in subsection 3.5.1), we will prove that the critical point E_0 cannot be attractive. Secondly (in subsection 3.5.2) we will prove that the asymptotic behavior of this system is –under some suitable assumptions– equivalent to the asymptotic behavior of a scalar differential delay equation. Finally (subsection 3.5.3), we will build a discrete dynamical system that inherits some asymptotic properties of the infinite dimensional dynamical system defined by the scalar delay equation constructed before.

3.5.1. Uniform persistence of system (3.5.1). The goal of this section is to prove that the critical point $E_0 = (s_{in}, 0)$ is a repeller. This is equivalent to proving that the biomass $x(t)$ is uniformly persistent, *i.e.* there exists a number $\delta_0 > 0$ (independent of the initial conditions) such that

$$\liminf_{t\to\infty} x_t > \delta_0.$$

In order to prove this property, we will present some compactness and invariance properties of the semiflow ϕ_t.

LEMMA 3.5.1. *There exists a global attractor set $A \subset X$ for the semiflow ϕ_t. That means, a set A maximal compact invariant which attracts each bounded set in X.*

PROOF. We will prove that the semiflow ϕ_t is point dissipative and completely continuous for $t > \tau$. Hence the Lemma is a consequence of Proposition 3.

Firstly we will prove that the semiflow is point dissipative: we take some initial condition (φ_1, φ_2) satisfying:

$$|\varphi_1(\theta) + \alpha\varphi_2(\theta) - s_{in}| \leq K \quad \text{for any } \theta \in [-\tau, 0].$$

Moreover, let us build the functional:

$$v(t) = s(t) + \alpha x(t) - s_{in}$$

where (s_t, x_t) is a solution of the system (3.5.1). It is straightforward to prove that $v(t)$ satisfies the following differential equation:

$$\begin{cases} \dot{v}(t) = -h\big(s^* - s(t-\tau)\big)v(t), \quad \text{for } t > 0 \\ v(\theta) = \eta(\theta) = \varphi_1(\theta) + \alpha\varphi_2(\theta) - s_{in}, \quad \theta \in [-\tau, 0]. \end{cases}$$

It is a simple exercise to prove that for any $t \geq 0$ it follows that:

$$|v(t)| = |\varphi_1(0) + \alpha\varphi_2(0) - s_{in}| \exp\Big(-\int_0^t h(s^* - s(r-\tau))\, dr\Big).$$

By using **(P1)**, we can prove that there exists $\rho > 0$ such that:

$$||s_t + \alpha x_t - s_{in}||_\infty \leq K e^{-\rho t} \quad \text{for any } t > 0 \text{ and } \theta \in [-\tau, 0]. \tag{3.5.2}$$

Now, letting $t \to \infty$ we have that that for any initial condition (φ_1, φ_2) it follows that $\lim_{t\to+\infty} d\big(\phi_t(\varphi_1, \varphi_2), K_0\big) = 0$, where the bounded set K_0 is defined by:

$$K_0 = \Big\{(\varphi_1, \varphi_2) \in C_+ \times C_+ \colon \varphi_1 + \alpha\varphi_2 = s_{in}\Big\}$$

which implies point dissipativity.

Secondly, we will prove that the semiflow ϕ_t is completely continuous for any $t > \tau$. Indeed, we take any initial condition (φ_1, φ_2) in a bounded set $B \subset X$. We will see that the orbits of system (3.5.1) form a precompact set for any $t \geq \tau$.

By using point dissipativity properties, we define the constants K_1 and K_2 as follows:

$$K_1 = \sup_{t \geq 0} \Big\{ ||s_t||_\infty \colon s_0 = \varphi_1 \in B \Big\} \quad \text{and} \quad K_2 = \sup_{t \geq 0} \Big\{ ||x_t||_\infty \colon x_0 = \varphi_2 \in B \Big\}.$$

Notice that, the set $\phi_t(B)$ is equicontinuous for any $t \geq \tau$. Indeed, there exists a number $\delta(\varepsilon) = \min \Big\{ \frac{\varepsilon}{L_1}, \frac{\varepsilon}{L_2} \Big\}$ where L_1,L_2 are defined by:

$$L_1 = \max_{|u| \leq K_1} [h(s^* - u)s_{in} + \alpha f(u) K_2] \quad \text{and} \quad L_2 = K_2 \max_{|u| \leq K_1} f(u) - h(s^* - u)$$

such that for any pair $\theta', \theta'' \in [-\tau, 0]$ satisfying $|\theta' - \theta''| < \delta$, we have $|s_t(\theta') - s_t(\theta'')| < \varepsilon$ and $|x_t(\theta') - x_t(\theta'')| < \varepsilon$.

By the Arzelà–Ascoli Theorem, it follows that the set $\phi_t(B)$ is precompact for any $t \geq \tau$, which implies that ϕ_t is completely continuous. □

LEMMA 3.5.2. *The biomass x is uniformly persistent.*

PROOF. Without loss of generality, we can assume in this proof that the initial conditions of the system (3.5.1) are in the compact set A.

Let us define the subset $A_0 = \big\{(\varphi_1, \varphi_2) \in A \colon \varphi_2 = 0\big\}$ and notice that the set A_0 is positively invariant under the semiflow ϕ_t. We will prove that A_0 is a repeller from which the uniform persistence follows.

Firstly, notice that for any initial condition in A_0, the semiflow (s_t, x_t) can be studied as a solution of the following integral equation:

$$\left\{ \begin{array}{l} s(t) = (\varphi_1(0) - s_{in}) \exp \Big(- \int_0^t h(s^* - s(r - \tau))\, dr \Big) + s_{in}, \\ s(\theta) = \varphi_1(\theta) \quad \text{for any } \theta \in [-\tau, 0]. \end{array} \right. \tag{3.5.3}$$

Secondly, let us build the functional $P \colon A \mapsto \mathbb{R}$ defined by $P(\phi_t(\vec{\varphi})) = x_t(0)$. This functional satisfies the following properties:

(a) $P(\phi_t(\vec{\varphi})) \equiv 0$ if $\vec{\varphi} \in A_0$ and $P(\phi_t(\vec{\varphi})) > 0$ if $\vec{\varphi} \in A \setminus A_0$.

(b) $\dot{P} = \Psi(\phi_t(\vec{\varphi}))P$ where $\Psi \colon A \mapsto \mathbb{R}$ is a continuous function defined by:
$$\Psi(\phi_t(\vec{\varphi})) = f(s_t(0)) - h(s^* - s_t(-\tau)).$$

(c) It follows from **(P1)**–**(P2)** and Eq.(3.5.3) that $\Psi(\phi_t(E_0)) = f(s_{in}) - h(s^* - s_{in}) > 0$ and for any initial condition in A_0 we have that there exists a number $\rho > 0$ such that:
$$||s_t - s_{in}||_\infty \leq |\varphi_0 - s_{in}| e^{-\rho t}$$
and in consequence it follows that
$$\lim_{t \to +\infty} (s_t, x_t) = E_0.$$

Notice that properties (a)-(b) imply that P is an average Lyapunov function (see *e.g.* [**63**] and Appendix A). Using the fact that ϕ_t is a semiflow defined on a compact metric space combined with property (c) and Corollary 2 from [**63**] it follows that the set A_0 is a repeller set and the lemma follows. $\square$

REMARK 3.5.1. *By using Eq.*(3.5.2) *combined with lemmas* 3.5.1 *and* 3.5.2 *we can prove that any solution satisfies* $s(t) \leq s_{in}$ *after a finite time and in consequence, we will consider only initial conditions satisfying the inequality* $||\varphi_1||_\infty \leq s_{in}$.

It is straightforward to prove that the system (3.5.1) is equivalent to the following system:

$$\begin{cases} \dot{s}(t) &= [h(s^* - s(t-\tau)) - f(s(t))](s_{in} - s(t)) - f(s(t))v(t), \\ \dot{v}(t) &= -h(s^* - s(t-\tau))v(t), \\ v(\theta) = \eta(\theta), & s(\theta) = \varphi_1(\theta) \leq s_{in} \quad \text{for any } \theta \in [-\tau, 0]. \end{cases} \tag{3.5.4}$$

3.5.2. Reduction of system. As we stated above, the asymptotic behavior of the systems (3.5.1) and (3.5.4) can be described by studying only the substrate equation. In this subsection we will formalize this idea.

Let us insert the solution $v(t)$ of system (3.5.4) into the equation $\dot{s}$. Then, for each initial condition η, we obtain the nonautonomous differential delay equation:

$$\begin{cases} \dot{s}(t) = [h(s^* - s(t-\tau)) - f(s(t))](s_{in} - s(t)) - f(s(t))v(t), \\ s(\theta) = \varphi_1(\theta) \in C_+ \quad \text{for any } \theta \in [-\tau, 0]. \end{cases} \tag{3.5.5}$$

where $v(t)$ is a solution of the system (3.5.4).

Let us define the set:

$$\Delta = \{(t, s) \colon 0 \leq s \leq t < +\infty\},$$

Using the results of asymptotically autonomous theory (see *e.g.* [**104**],[**139**]) it can be proved that the solutions of Eq.(3.5.5) define a nonautonomous continuous semiflow $\Phi \colon \Delta \times C_+ \mapsto A$ asymptotically autonomous to the semiflow defined by the scalar autonomous differential delay equation:

$$\begin{cases} \dot{s}(t) &= [h(s^* - s(t-\tau)) - f(s(t))](s_{in} - s(t)) = g(s_t), \\ s(\theta) = \varphi_1(\theta) \leq s_{in}. \end{cases} \tag{3.5.6}$$

LEMMA 3.5.3. *The solutions of system* (3.5.6) *are bounded above by* s_{in}.

PROOF. Notice that equation (3.5.6) is equivalent to the integral equation:

$$s_{in} - s(t) = \big(s_{in} - \varphi_1(0)\big) \exp\Big(\int_0^t f(s(r)) - h(s^* - s(r-\tau))\, dr\Big).$$

Using the fact that $||\varphi_1||_\infty < s_{in}$, it follows that $s(t) < s_{in}$ for any $t \geq 0$. $\square$

LEMMA 3.5.4. *If the critical point* s^* *is a globally attractive solution of Eq.*(3.5.6) *then it is also a globally attractive solution of Eq.*(3.5.5).

PROOF. Let us define by $\mu(t, s_t)$ the right hand side of system (3.5.5). Notice that using Eq.(3.5.2) the functionals $\mu(t, s_t)$ and $g(s_t)$ defined in system (3.5.6) satisfy the required properties combined with Proposition 2.2 from [**139**] we can show that the solution of Eq.(3.5.5) defines a nonautonomous semiflow $\Phi(t, t_0, \varphi_1)$ asymptotically autonomous with limit semiflow $\Theta(t, \varphi_1)$ (see Appendix B) defined by the solution of Eq.(3.5.6).

Notice that s_{in} and s^* are isolated and invariant subsets of A. Moreover, Lemma 3.5.2 implies that s_{in} is a repeller and as we have that s^* is a global attractor we can conclude that the existence of a Θ–cyclical chain of Θ–equilibria is not possible. Finally, using Theorem 4.2 from [**139**] (see also Proposition 7 from Appendix B) the Lemma follows. $\square$

The idea behind the proof is now clear: if we find sufficient conditions for the global attractivity of the critical point s^* in Eq.(3.5.6), then Lemma 3.5.4 implies that s^* is a globally attractive critical point of system (3.5.1). This reduction enables us to employ the extensive literature on differential delay equations of type

$$\dot{u}(t) = \mathcal{F}(u(t), u(t-1)),$$

where the functional $\mathcal{F}$ is decreasing with respect to $u(t-1)$. See for example [**99**] where it was proved that the Poincaré–Bendixson theorem holds and by consequence asymptotic periodicity is the "most complicated" type of behavior. See also [**84**, Chapter 4] for a more complete overview about the asymptotic behavior of this system.

3.5.3. End of proof. Making the transformations:

$$\begin{aligned} u(t) &= \ln\left(\frac{s_{in} - s^*}{s_{in} - s(t\tau)}\right), \\ F(r) &= \tau[h([s_{in} - s^*][e^{-r} - 1]) - h(0)] = \tau[g_2(r) - h(0)], \\ G(r) &= \tau[\mu(s_{in} - [s_{in} - s^*]e^{-r}) - f(s^*)] = \tau[g_1(r) - f(s^*)]. \end{aligned}$$

the system (3.5.6) becomes:

$$\begin{cases} \dot{u}(t) = -G(u(t)) + F(u(t-1)) & \text{for any } t \geq 0, \\ u(\theta) = \varphi(\theta) & \text{for any } \theta \in [-1, 0]. \end{cases} \tag{3.5.7}$$

Notice that $G \in C^2(\mathbb{R}, \mathbb{R})$ with third derivative continuous, except at the point $r_1 = \ln\left(\dfrac{s_{in} - s^*}{s_{in} + a}\right)$ and $F \in C^3(\mathbb{R}, \mathbb{R})$. Moreover the following properties are straightforward:

(a) $rF(r) < 0$ and $rG(r) > 0$ for any $r \in \mathbb{R} \setminus \{0\}$,
(b) F is decreasing and G can be increasing (when $f = f_1$) or unimodal (when $f = f_2, f_3$) with maximum in $r = \ln\left(\frac{s_{in} - s^*}{s_{in} - s_{\max}}\right) > 0$. Moreover Remark 3.1.1 implies that $G''(0) < 0$.
(c) $F(r) \to \tau[h(s^* - s_{in}) - h(0)]$ and $G(r) \to \tau[f(s_{in}) - f(s^*)]$ as $r \to +\infty$.

(d) By Lemma D.1.1 and Remark 3.3.1, it follows that $(SF)(r) < 0$ and $(SG)(r) < 0$.

By property (a) it follows that $u(t) \equiv 0$ is an equilibrium of Eq.(3.5.7). We will prove that this solution is globally attractive.

First of all, we notice that if the solution of (3.5.7) is nonoscillatory, then the following result stands

LEMMA 3.5.5. *If the solution $u(t)$ is non–oscillatory (that means, there exists a finite number $\tilde{t} > 0$ such that $u(t)$ has a constant sign), it follows that* $\lim_{t\to+\infty} u(t) = 0$.

PROOF. Without loss of generality, we suppose that $u(t) > 0$ for any $t > \tilde{t} + 1$. Hence, by the properties of F and G stated above, we have that $u'(t) < 0$ for any $t > \tilde{t} + 1$ and consequently:

$$\lim_{t\to+\infty} u(t) = l \geq 0.$$

We will prove that $l = 0$. To obtain a contradiction, let us suppose that $l > 0$. Integrating equation (3.5.7) between $T > \tilde{t} + 1$ and t we have that:

$$\begin{aligned} u(t) &= u(T) + \int_T^t F(u(r-1))\,dr - \int_T^t G(u(r))\,dr \\ &\leq u(T) + (t-T)\Big[\max_{r\in[u(T-1),l]} F(r)\Big] - (t-T)\Big[\min_{r\in[u(T),l]} G(r)\Big] \\ &\leq u(T) + (t-T)\Big[F(l) - \min_{r\in[u(T),l]} G(r)\Big]. \end{aligned}$$

By using the properties of F and G, it follows that $F(l) < 0$ and $\min_{r\in[u(T),l]} G(r) > 0$.

Letting $t \to +\infty$, it follows that $l < -\infty$ and we obtain a contradiction. Hence, $l = 0$ and the lemma follows. □

By virtue of Lemma 3.5.5, we have only to consider the case when solutions of Eq.(3.5.7) are oscillatory. This means, there exists a sequence $\{v_n\} \to +\infty$ when $n \to +\infty$ satisfying $u(v_n) = 0$ for any integer $n > 1$.

If the solution $u(t)$ is oscillatory, we can assume that

$$\liminf_{t\to+\infty} u(t) = m \leq 0 \leq M = \limsup_{t\to+\infty} u(t).$$

We will prove that $m = M = 0$. To obtain a contradiction, let us suppose that $m < 0$ and $M > 0$.

By the fluctuation lemma (see *e.g.* [**61**, Lemma 4.2][**92**, Lemma 1]) there exist two sequences of real numbers $\{t_n\}$,$\{s_n\} \to +\infty$ when $n \to +\infty$ such that for any integer $n \geq 1$ it follows that:

$$u'(t_n) = F(u(t_n - 1)) - G(u(t_n)) = 0, \tag{3.5.8}$$

$$u'(s_n) = F(u(s_n - 1)) - G(u(s_n)) = 0 \tag{3.5.9}$$

and

$$\lim_{n\to+\infty} u(t_n) = M \quad \text{and} \quad \lim_{n\to+\infty} u(s_n) = m.$$

Integrating Eq.(3.5.7) between $t_n - 1$ and t_n, it follows that:

$$M_n = u(t_n) = u(t_n - 1) + \int_{t_n-1}^{t_n} F(u(r-1))\,dr - \int_{t_n-1}^{t_n} G(u(r))\,dr. \tag{3.5.10}$$

Without loss of generality, we can suppose that $M_n > 0$ and $m_n < 0$ for any integer $n \geq 0$. Furthermore, by using Eq.(3.5.10) combined with properties of the sequence $u(t_n)$ and functions F and G for any integer $n \geq 1$ it follows that:

$$G(u(t_n)) = F(u(t_n - 1)) > 0 \quad \text{and} \quad G(u(s_n)) = F(u(s_n - 1)) < 0$$

and consequently $u(t_n - 1) < 0$ and $u(s_n - 1) > 0$. Using this fact combined with Eq.(3.5.8) we obtain:

$$M_n = u(t_n) \leq \int_{t_n-1}^{t_n} F(u(r-1))\,dr - \int_{t_n-1}^{t_n} G(u(r))\,dr.$$

Let us build the auxiliary function $R\colon \mathbb{R} \mapsto \mathbb{R}$ defined by

$$R(r) = \begin{cases} G(r) & \text{if } r \in [-\infty, 0], \\ H(r) & \text{if } r \geq 0, \end{cases}$$

where H is defined as

$$H(r) = \begin{cases} G(r) & \text{if } f = f_1, \\ \dfrac{2G'(0)^2 r}{2G'(0) - G''(0) r} & \text{if } f = f_2, f_3. \end{cases}$$

Notice that R satisfies the following properties:

(i) Proposition 10 implies that $G(r) \leq R(r)$ for any $r \geq 0$,
(ii) R is increasing. Moreover $R(0) = 0$,$G'(0) = R'(0)$ and $G''(0) = R''(0)$,
(iii) $(SR)(r) \leq 0$ for any $r \in \mathbb{R}$,
(iv) $R \in C^2(\mathbb{R}, \mathbb{R})$ with third derivative continuous, except at the points $r_1 = -a$ and 0. We can choose r_1 such that $F''(r_1) \neq R''(r_1)$.

Let us build the auxiliary function $\chi\colon \mathbb{R} \mapsto \mathbb{R}$ defined by

$$\chi(r) = F(r) - R(r).$$

Moreover, we let χ^n denote the n–fold composition of the function χ with itself ($n \in \mathbb{N}$).

By the definition of M and the properties of F,G and H stated above, it follows that for any $\varepsilon > 0$ there exist a number $T(\varepsilon) > 0$ such that for any $t > T + 2$ we have the following inequalities:

$$R(m-\varepsilon) \leq \min_{u\in[m-\varepsilon,0]} G(u) \leq G(u(t)) \leq \max_{u\in[0,M+\varepsilon]} G(u) \leq R(M+\varepsilon),$$

$$F(M+\varepsilon) \leq F(u(t)) \leq F(m-\varepsilon).$$

We thus deduce that for any $t_n > T(\varepsilon) + 2$ we have the inequality

$$M_n = u(t_n) \leq F(m-\varepsilon) - R(m+\varepsilon).$$

Letting $\varepsilon \to 0$ and $n \to +\infty$ we obtain:

$$M \leq F(m) - R(m) = \chi(m).$$

Analogously, using the sequence $\{s_n\}_n$ we have the inequality:

$$m \geq F(M) - R(M) = \chi(M).$$

Notice that $\chi(+\infty)$ is defined by:

$$\chi(+\infty) = \begin{cases} \tau[h(s^* - s_{in}) - f(s_{in})] & \text{if } f = f_1, \\ \tau\left[h(s^* - s_{in}) - h(0) - \dfrac{2(s_{in} - s^*)f'(s^*)^2}{f'(s^*) - (s_{in} - s^*)f''(s^*)}\right] & \text{if } f = f_2, f_3. \end{cases}$$

Let $I = [\chi(+\infty), \chi^2(+\infty)]$, it is straightforward to verify that $\chi(+\infty) < 0$ and $\chi^2(+\infty) > 0$, hence the map $\chi \colon I \mapsto I$ is well defined.

Using the fact $\chi(M) \leq m$ and $M \leq \chi(m)$ we obtain that $m, M \in I$ and it can be proved by mathematical induction that:

$$[m, M] \subset \chi([m, M]) \subset \ldots \subset \chi^k([m, M])$$

for any integer $k \geq 1$.

Notice that:

$$\chi^k([m, M]) = \begin{cases} [\chi^k(M), \chi^k(m)] & \text{if } k \text{ is odd}, \\ [\chi^k(m), \chi^k(M)] & \text{if } k \text{ is even}. \end{cases}$$

Moreover, by inequality (3.4.2) we have that $|\chi'(0)| < 1$ which implies that 0 is a locally stable (and unique) fixed point of χ.

To conclude, we will prove that $(S\chi)(r) < 0$. Firstly, we consider $f_1 = f$. Moreover, notice that statement (ii) of Lemma D.1.1 combined with $F_1 = F$ (*i.e.* $g_1 = g_{1_1}$) and $F_2 = R = G$ implies that:

$$(S\chi)\chi'^2 = \underbrace{(SF)F'\chi' - (SG)G'\chi'}_{K_1(r)} - \frac{3}{2}\underbrace{F'G'\Big[\frac{F''}{F'} - \frac{G''}{G'}\Big]^2}_{K_2(r)}$$

Moreover, by using the fact that $(S\lambda_1)(r) = (S\lambda_2)(r) = -1/2$ (see statement (i) of Lemma D.1.1) we have that:

$$\begin{aligned} (SF)(r)F'(r) &= -\tau\kappa(r)h'(\lambda_2(r))\left[(Sh)(\lambda_2(r))\kappa(r)^2 - \frac{1}{2}\right], \\ &= \tau(Sg_2)(r)g_2'(r). \\ (SG)(r)G'(r) &= \tau\kappa(r)f'(\lambda_1(r))\left[(Sf)(\lambda_1(r))\kappa(r)^2 - \frac{1}{2}\right], \\ &= \tau(Sg_1)(r)g_1'(r). \end{aligned}$$

It is easy to see that $\chi'(r)=\tau[g_2'(r)-g_1'(r)]$ and we obtain:

$$K_1(r)=\tau^2\Big[g_2'(r)-g_1'(r)\Big]\left(\sum_{i=1}^{2}(Sg_i)(r)g_i'(r)\right).$$

By using the definition of the Pre–Schwarz derivative we can deduce that $(P\lambda_1)(r)=-1$ and $(P\lambda_2)(r)=-1$. Moreover, by using statement (i) of Lemma D.1.1 we can see that:

$$K_2(r)=-\tau^2\kappa^2(r)h'(\lambda_2(r))\mu'(\lambda_1(r))\left[\frac{h''(\lambda_2(r))}{h'(\lambda_2(r))}+\frac{f''(\lambda_1(r))}{f'(\lambda_1(r))}\right]^2,$$

$$=\tau^2\left[\prod_{i=1}^{2}g_i'(r)\right]\left[\sum_{i=1}^{2}(-1)^i(Pg_i)(r)\right]^2.$$

In consequence $(S\chi)(r)<0$ for any $r\in I$ if and only if $K_1(r)<\frac{3}{2}K_2(r)$ for any $r\in I_\tau$ which is equivalent to inequality (3.4.1).

Now, we consider $f=f_2,f_3$ (which implies $g_1=g_{1_2},g_{1_3}$). Moreover, notice that statement (ii) of Lemma D.1.1 combined with $F_1=F$ and $F_2=R$ implies that:

$$(S\chi)\chi'^2=\underbrace{(SF)F'\chi'-(SR)R'\chi'}_{K_1(r)}-\frac{3}{2}\underbrace{F'R'\Big[\frac{F''}{F'}-\frac{R''}{R'}\Big]^2}_{K_2(r)}$$

As $R=G$ for any $r\leq 0$, we will consider only the case $r>0$. By using the fact that $(S\lambda_1)(r)=(S\lambda_2)(r)=-1/2$ (see statement (i) of Lemma D.1.1) we have that:

$$(SF)(r)F'(r)=-\tau\kappa(r)h'(\lambda_2(r))\left[(Sh)(\lambda_2(r))\kappa(r)^2-\frac{1}{2}\right],$$

$$=\tau(Sg_2)(r)g_2'(r).$$

$$(SR)(r)R'(r)=\tau(Sg_1)(r)g_1'(r)=0.$$

By using the fact that $\chi(r)=\tau[g_2(r)-g_1(r)]$, it is easy to see that:

$$K_1(r)=\tau^2\Big[g_2'(r)-g_1'(r)\Big]\left(\sum_{i=1}^{2}(Sg_i)(r)g_i'(r)\right).$$

By using the definition of the Pre-Schwarz derivative we can deduce that $(P\lambda_1)(r)=-1$ and $(P\lambda_2)(r)=-1$. Moreover by using statement (i) of Lemma D.1.1 we can see that:

$$\frac{F''(r)}{F'(r)}=-1-\kappa(r)\frac{h''(\lambda_2(r))}{h'(\lambda_2(r))}=(Pg_2)(r),$$

$$\frac{R''(r)}{R'(r)}=-\frac{2G''(0)}{2G'(0)-G''(0)r}=(Pg_1)(r).$$

$$
\begin{aligned}
F'(r)R'(r) &= -\tau\kappa(r)h'(\lambda_2(r))\frac{4G'(0)}{[2G'(0)-G''(0)r]^2},\\
&= -\tau^2\kappa(r)h'(\lambda_2(r))\frac{4\kappa(r)h'(\lambda_2(r))[\kappa(0)f'(s^*)^3]}{\kappa(0)^2[2f'(s^*)-\{f'(s^*)-f''(s^*)r)\}]^2},\\
&= \tau^2\prod_{i=1}^{2} g_i'(r).
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
K_2(r) &= \tau\left[\prod_{i=1}^{2} g_i'(r)\right]\left[1+\kappa(r)\frac{h''(\lambda_2(r))}{h'(\lambda_2(r))}-\frac{G''(0)}{2G'(0)-G''(0)r}\right]^2,\\
&= \tau\left[\prod_{i=1}^{2} g_i'(r)\right]\left[\sum_{i=1}^{2}(-1)^i(Pg_i)(r)\right]^2.
\end{aligned}
$$

In consequence $(S\chi)(r)<0$ for any $r\in I$ if and only if $K_1(r)<\frac{3}{2}K_2(r)$ for any $r\in I_\tau$ which is equivalent to inequality (3.4.1).

Applying Proposition 9 (see Appendix D) to map $\chi\colon I\mapsto I$ we conclude that 0 is a global attractor of χ. Hence as $[m,M]\subset\chi^k([m,M])\to\{0\}$ when $k\to+\infty$ it follows that $m=M=0$ and the Theorem follows.

3.6. Proof of Theorem 3.4.2

The proof is similar to the proof of Theorem 3.4.1 until the definition of the auxiliary decreasing function $\chi\colon\mathbb{R}\mapsto\mathbb{R}$. As before, it is straightforward to verify that $\chi(+\infty)<0$ and $\chi^2(+\infty)>0$. Hence the map $\chi\colon I\mapsto I$ (where $I=[\chi(+\infty),\chi^2(+\infty)]$) is well defined.

As before, it can be proved by mathematical induction that:

$$[m,M]\subset\chi([m,M])\subset\ldots\subset\chi^k([m,M])$$

for any integer $k\geq 1$.

By using the inequality $\tau<\tau_b^*$, it follows that $|\chi'(r)|<1$ for any $r\in I_\tau$ and consequently we have that

$$[m,M]\subset\lim_{k\to+\infty}\chi^k([m,M])=0$$

which implies $m=M=0$ and the theorem follows.

Remark 3.6.1. *A careful reading of our proof of Theorems* 3.4.1 *and* 3.4.2 *shows that we can generalize our result for any* $C^3(\mathbb{R},\mathbb{R})$ *function* f *satisfying the following properties*:

- $f(0)=0$, $f'(0)>0$, $f''(0)<0$,
- f *can have at most one maximum* $s_{\max}>0$ *and only one inflection point* $s_c>s_{\max}$,
- $(Sf)(r)<0$ *for any* $r\neq s_{\max}$.

3.7. Numerical Examples

Let us come back to the asymptotic stabilization problems stated in the Introduction.

3.7.1. Depollution of phenol in the water. We will consider biological degradation of Phenol in the water by using *Pseudomonas putida* whit growth described by the function:

$$f(s) = \mu_{max} \frac{s}{k_s + s^2}$$

where the parameters are defined in the Figure 3.7.1 (see also [**134**]):

Parameter	Value	Units
$\mu_{\max}$	15.96	Day^{-1}
k_s	1.82	mg/L
s_{in}	4	mg/L
α	1	non–dimensional

FIGURE 3.7.1. Parameters for depollution problem.

Our goal is to stabilize the Phenol concentration in a neighborhood of $s^* = 0.55$mg/L. For this task we build the feedback control law:

$$h(y(t)) = 4.1357 + 4.13 \tanh\big(\eta[s^* - s(t-\tau)]\big), \quad \eta > 0.$$

It is straightforward to verify that **(P1)**–**(P4)** are satisfied. Property **(P5)** is satisfied only for a small interval (η^-, η^+) and allows to use Theorem 3.4.1 only for small delays because $\tau_a^* < ([s_{in} - s^*][h'(0) + f'(s^*)])^{-1}$.

More explicit conditions can be obtained by using Theorem 3.4.2. Numerical simulations were carried out using DDE23 [**125**] (we only show the results for nitrate concentration). We give some results considering several delays τ and gains η, on the other hand we always consider initial conditions $(\varphi_1, \varphi_2) = (2.14, 0.14)$. Assumptions of Theorem 3.4.2 are verified in Figure 3.7.2.

Notice that this sufficient condition can be improved, for example fixing the gain $\eta = 0.17$ and increasing the delay the solution E_1 is still globally stable. Nevertheless, notice that when the size of the delay increases, the speed of convergence towards s^* is slower (see Figures 3.7.3 and 3.7.4).

Figure 3.7.5 show the solutions using a bigger gain but considering the same delay as in Figure 3.7.4. We point out that in this case the couple (τ, η) passes through some critical value and the point E_1 loses its stability and a periodic solution appears.

3.7.2. Culture of phytoplankton. We will consider *Dunaniella tertiolecta* growth in a chemostat with nitrate as the limiting substrate. We will work with a growth function given by the Michaelis–Menten function

$$f(s) = \mu_{\max} \frac{s}{k_s + s}$$

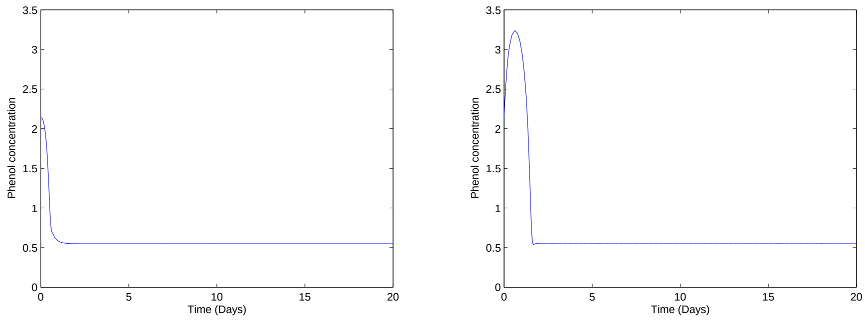


FIGURE 3.7.2. Output of system: $(\tau, \eta) = (0.05, 0.96)$ (left) and $(\tau, \eta) = (0.1, 0.17)$ (right)

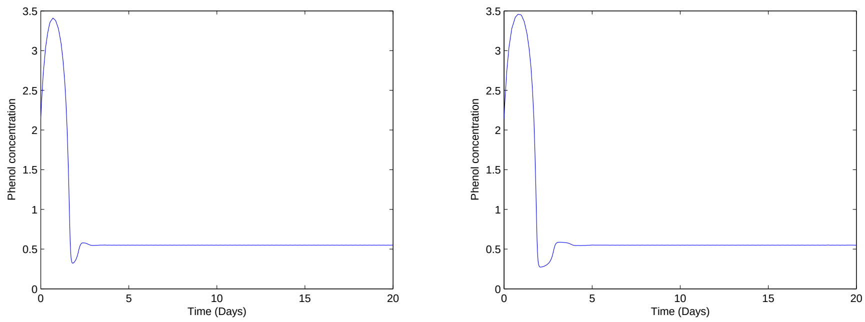


FIGURE 3.7.3. Output of system: $(\tau, \eta) = (0.5, 0.17)$ (left) and $(\tau, \eta) = (1.0, 0.17)$ (right)

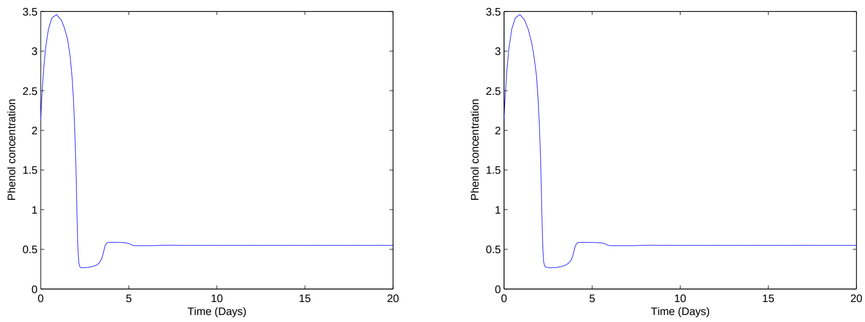


FIGURE 3.7.4. Output of system: $(\tau, \eta) = (1.5, 0.17)$ (left) and $(\tau, \eta) = (1.8, 0.17)$ (right)

where the parameters are shown in Figure 3.7.6 (see also [**11**],[**143**] for more details):

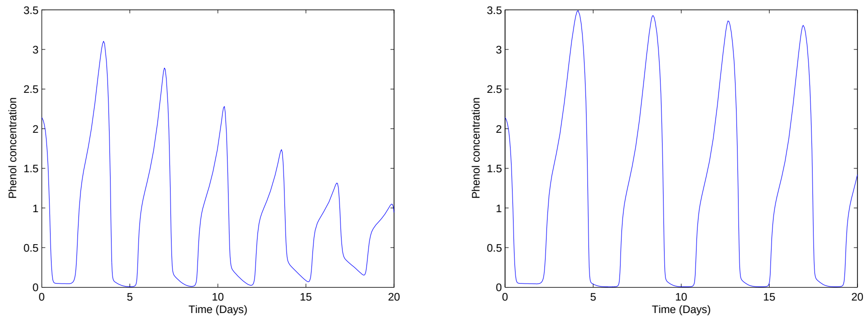


FIGURE 3.7.5. Output of system: $(\tau, \eta) = (1.5, 1.1)$ (left) and $(\tau, \eta) = (1.8, 1.1)$ (right)

Parameter	Value	Units
$\mu_{\max}$	1.6	Day^{-1}
k_s	0.02	μatg/L
s_{in}	2	μatg/L
α^{-1}	1	non–dimensional

FIGURE 3.7.6. Parameters for the culture of phytoplankton problem.

Our goal is to stabilize the nitrate concentration in a neighborhood of $s^* = 0.8\mu$atg/L, for this task we build the feedback control law:

$$h(y(t)) = 1.561 + \tanh\big(\eta[s^* - s(t-\tau)]\big), \quad \eta > 0.$$

It is straightforward to verify that **(P1)**–**(P5)** are satisfied. Indeed notice that $(Sh)(r) = -2$ (for any $r \in \mathbb{R}$) and using Lemma D.1.1 we have that

$$\lim_{\eta \to 0} (S\chi)(r, \eta) = -(SG)(r)G'(r)\chi'(r) < 0 \quad \text{for any } r$$

which implies **(P5)** and consequently, the existence of a number $\eta^* > 0$ and an interval $I_{\tau(\eta)}$ $(\eta < \eta^*)$ such that eq.(3.4.1) is satisfied.

We can see with the help of a computer that $\tau_a^* < ([s_{in} - s^*][h'(0) + f'(s^*)])^{-1}$. This allow us to solve Problem 3.1.1 for small delays satisfying $\tau < \tau^*$ by using Theorem 3.4.1

More explicit conditions can be obtained by using Theorem 3.4.2, we show a table relating the delay τ with the gain η (see Figure 3.7.7)

Numerical simulations were carried out using DDE23 [**125**] (we only show the results for nitrate concentration) considering initial conditions $(\varphi_1, \varphi_2) = (0.3, 0.1)$. Assumptions of Theorem 3.4.2 are verified in Figures 3.7.8 and 3.7.9.

As before, notice that this sufficient condition can be improved, for example fixing the gain $\eta = 0.17$ and increasing the delay the solution E_1 is still globally stable. Nevertheless, notice that when the size of the delay

Delay τ_b^*	η
0.5	0.796
0.8	0.36
0.9	0.324
1	0.281

FIGURE 3.7.7. Some critical delays $\tau_b^*(\eta)$ for the feedback control.

increases, the speed of convergence towards s^* is slower (see Figures 3.7.9 and 3.7.10).

Figure 3.7.11 show the solutions using a bigger gain but considering the same delay as in Figure 1.3. We point out that in this case the couple (τ, η) passes through some critical value and the point E_1 loses its stability and a periodic solution appears.

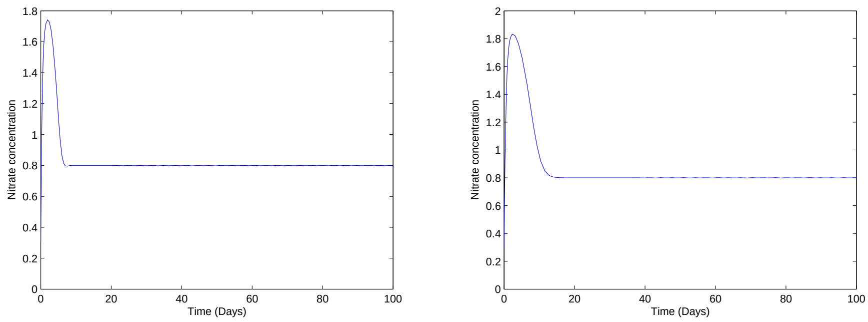


FIGURE 3.7.8. Output of system: $(\tau, \eta) = (0.5, 0.796)$ (left) and $(\tau, \eta) = (0.8, 0.36)$ (right)

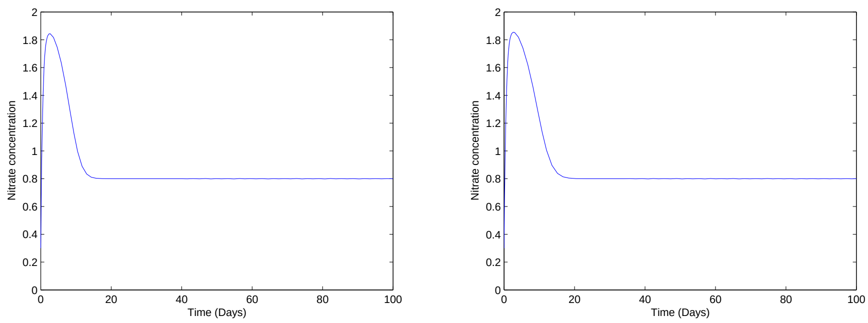


FIGURE 3.7.9. Output of system: $(\tau, \eta) = (0.9, 0.324)$ (left) and $(\tau, \eta) = (1.0, 0.281)$ (right)

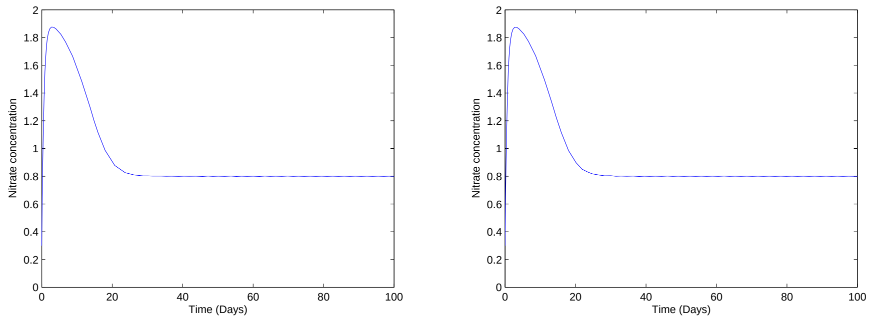


FIGURE 3.7.10. Output of system: $(\tau, \eta) = (1.32, 0.17)$ (left) and $(\tau, \eta) = (1.35, 0.17)$ (right)

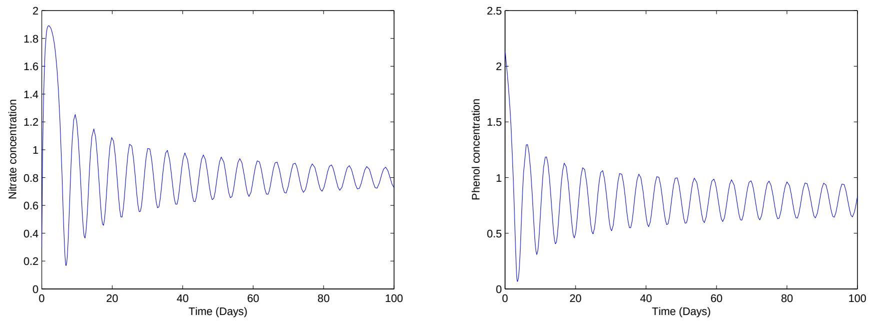


FIGURE 3.7.11. Output of system: $(\tau, \eta) = (1.32, 1)$ (left) and $(\tau, \eta) = (1.35, 1)$ (right)

3.8. Discussion

We have considered a problem of asymptotic stabilization for a chemostat model (with a delay in its output) described by Eq.(3.1.1). We obtained a sufficient condition that ensures the global asymptotic stability of a substrate concentration s^*. Nevertheless, there exist some interesting mathematical and control issues related to this work.

From a mathematical point of view, we want to emphasize that the key ideas and techniques employed are even important by themselves. Nevertheless much remains to be done. Conditions stated in Theorems 3.4.1 and 3.4.2 are restricted for relatively small delays and are too awkward to cope with them mathematically. We think that using Schwarz derivative properties (following the way shown in [**95**]) we can build more general discrete systems which ensures better estimations for the critical delay. Moreover, using the fact that the solutions of Eq.(3.5.7) satisfy the Poincaré–Bendixson theorem, we must rule out the existence of periodic solutions in order to prove the convergence of solutions to a critical point. In this direction, the study of the characteristic equation of Eq.(3.5.7):

$$\lambda + G'(0) - F'(0)e^{-\lambda} = 0 \tag{3.8.1}$$

suggests to us some interesting problems:

(a) Using τ as bifurcation parameter and considering that $G'(0)$ and $F'(0)$ are dependent of τ. We can obtain sufficient conditions ensuring the local asymptotic stability of the solution $u \equiv 0$ of Eq.(3.5.7). As we pointed out in section 3.6, the sufficient condition stated is not optimal. A possible optimal condition could be given by the study of roots of (3.8.1): if all roots have negative real part, are the solutions of Eq.(3.5.7) convergent to 0?

(b) As before, there exists a relationship between the size of the delay $\tau > 0$ and the existence of purely real roots of Eq.(3.8.1). Considering the practical applications of this control problem, we are interested in finding sufficient conditions to ensure a fast convergence toward s^*. In this sense the study of the roots of Eq.(3.8.1) could help us: if all roots are real, are the solutions non oscillatory? Are the solutions super exponential? (For more details about super exponential convergence, see for instance [**25**].)

(c) On the other hand, as we pointed out in section 1.5 of the Introduction, the estimation of parameters for the growth functions f_i is a difficult task; in general they are not well known. This motivates the study of delay equations of type

$$\dot{u}(t) = -G(u(t)) + F(u(t-1)) + w(t, u(t)),$$

where the function w reflects the uncertainty in the estimation of the parameters of functions f_i. This type of equation has been studied in several works (see *e.g.* [**106**],[**103**] and the references therein) and it will be extremely desirable to extend our result by working on robust stabilization problems.

From the point of view of control theory, we can see that such classical control strategies as proportional regulators are still effective –up to a threshold– against delays in the outputs. Nevertheless, we must take into account the following problems.

(a) It is necessary to find sufficient conditions relating speed of convergence to some delays.

(b) It is necessary to generalize our approach for outputs of type:

$$y(t) = s(t-\tau)[1 + \Delta_1] + \Delta_2$$

where Δ_i $(i = 1, 2)$ are perturbations (deterministic or stochastic).

(c) It is suggested in [**31**, Chapt. 6] that the implementation of proportional integral (PI) regulators could be employed in this problem. Moreover, there exist other approaches to solve this problem, mainly the use of Smith predictors (see *e.g.*[**108**] and the references given there). It will be interesting to compare the efficiency of these approaches.

Part 2

Feedback control for competition in a chemostat

CHAPTER 4

Feedback control for nonmonotone competition models in the chemostat

The model presented in this chapter concerns the competition and coexistence of two species in a chemostat with a single substrate.

De Leenheer and Smith [**87**] studied the linear feedback control for a well known model of competition between two species and one substrate in a chemostat with monotone uptake functions, considering the dilution rate as the feedback control variable and keeping the input substrate concentration at a fixed value. They build a feedback control law which makes it possible to avoid the competitive exclusion principle, obtaining the uniform persistence under the form of a globally asymptotically stable critical point in $\mathrm{Int}\,\mathbb{R}^3_+$.

The aim of the work presented in this chapter is to extend the results obtained in [**87**] to nonmonotone uptake functions. We have obtained sufficient conditions for the coexistence of two species which can be reached as convergence toward a critical point or a limit cycle (in some cases); to prove our main result, we will proceed in analogy to [**87**]. However, nonmonotony properties of uptake functions make the study more complex than the monotone case, mainly because there are several types of nonlinearities to consider.

This chapter is organized as follows: In Section 4.1 we have compiled some basic facts concerning the chemostat model with nonmonotone growth functions. In Section 4.2 we provide an exposition of the feedback control law and show the main result of coexistence. Section 4.3 presents some preliminaries results related to asymptotic behavior of the model with and without competition. The proof of the main result and some extensions is stated in Section 4.4, robustness of the model is studied in Section 4.5.

4.1. Model of competition in the chemostat

The chemostat model with competition [**133**] is described by the differential equations:

$$\left\{\begin{array}{l} \dot{s} \;= D(s_{in}-s)-\alpha_1 x_1 f_1(s)-\alpha_2 x_2 f_2(s),\\ \dot{x}_1 = x_1(f_1(s)-D),\\ \dot{x}_2 = x_2(f_2(s)-D). \end{array}\right. \tag{4.1.1}$$

We state the general assumptions on f_i $(i=1,2)$:

(F1) $f_i \in C^1(\mathbb{R}_+,\mathbb{R}_+)$.
(F2) $f_i(0)=0$.
(F3) f_i is unimodal (*i.e.* there exists a number $s_i^*>0$ such that f_i is increasing for $s\in[0,s_i^*)$ and decreasing for $s>s_i^*$) and $\lim\limits_{t\to+\infty} f_i(t)=c_i\geq 0$.
(F4) There is $s^*\in(0,s_{in})$ such that $f_1(s^*)=f_2(s^*)=D^*$, moreover

$$\left\{\begin{array}{lll} f_1(s)>f_2(s) & \text{if} & s\in(0,s^*),\\ f_1(s)<f_2(s) & \text{if} & s\in(s^*,+\infty). \end{array}\right.$$

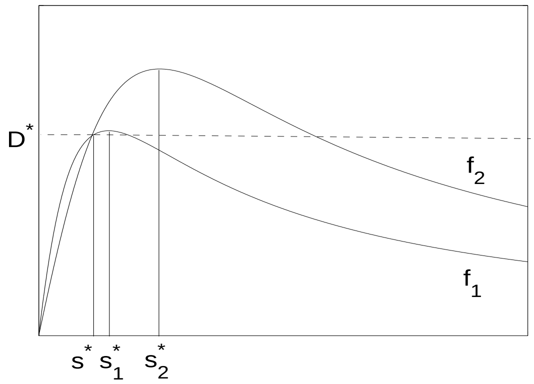


FIGURE 4.1.1. Graph of f_1 and f_2, Case (a): $f_2'(s^*) > f_1'(s^*) \geq 0$, that is equivalent to $s^* < \min\{s_1^*, s_2^*\}$.

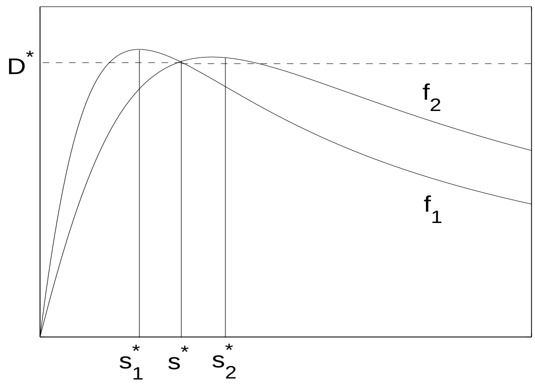


FIGURE 4.1.2. Graph of f_1 and f_2, Case (b): $f_1'(s^*) < 0 < f_2'(s^*)$, that is equivalent to $s_1^* < s^* < s_2^*$.

Assumptions **(F1)**–**(F2)** state general properties of population growth models, **(F3)** reflects the inhibition of growth of species x_1 and x_2 for high concentrations of substrate s.

Assumption **(F4)** involves a geometrical property on the graphs of f_1 and f_2. This implies several results about asymptotic behavior of solutions of (5.1.1) as we will see later on.

REMARK 4.1.1. *Clearly, $f_2'(s^*) \geq f_1'(s^*)$. Moreover we have three possibilities for the functions f_1 and f_2 satisfying* **(F1)-(F4)***, depending on the relative order of the intersection point s^* and the maximum points s_1^* and s_2^*. A graphical representation of all these cases is given in Figs.* 4.1.1,4.1.2 *and* 4.1.3*. Other graphs with more intersections between f_1 and f_2 have been considered by Lenas and Pavlou in* [**89**, Figure 1].

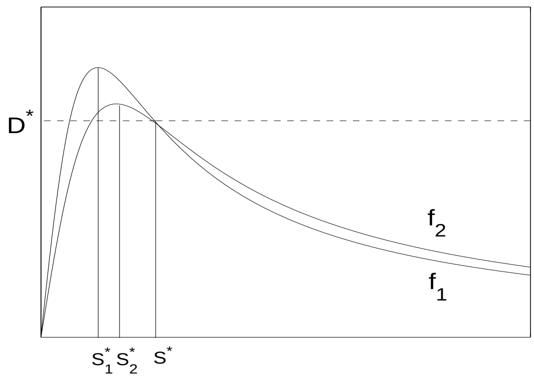


FIGURE 4.1.3. Graph of f_1 and f_2, Case (c): $f_1'(s^*) < f_2'(s^*) \leq 0$, that is equivalent to $s^* > \max\{s_1^*, s_2^*\}$.

This model has been studied in [**6**] and [**19**] for 2 and n species respectively. Next, we consider the main results for $n = 2$ and functions that satisfy **(F1)**–**(F4)**.

If $D \neq D^*$, there exist two uniquely defined positive real numbers η_i and μ_i such that $\eta_i < \mu_i \leq +\infty$ $(i = 1, 2)$ and

$$\begin{cases} f_i(s) < D & \text{if} \quad s \notin [\eta_i, \mu_i] \\ f_i(s) \geq D & \text{if} \quad s \in [\eta_i, \mu_i]. \end{cases}$$

Without loss of generality we will suppose that $\max\{\mu_1, \mu_2\} < s_{in}$. Other cases can be studied similarly.

Results in [**6**],[**19**] can be summarized coupling the relative order of numbers $\{s^*, s_1^*, s_2^*\}$ stated in Remark 4.1.1 and $\{D, D^*\}$:

PROPOSITION 4. [**6**, Aris and Humphrey],[**19**, Butler and Wolkowicz] *With the exception of a set of initial conditions of Lebesgue measure zero, all solutions of* (4.1.1) *are initial condition dependent and satisfy*:

If $D < D^*$ or $D > D^*$ and $s^* > \max\{s_1^*, s_2^*\}$:
$\lim\limits_{t \to +\infty} \big(s(t), x_1(t), x_2(t)\big) = (\eta_1, y_1[s_{in} - \eta_1], 0)$ or
$\lim\limits_{t \to +\infty} \big(s(t), x_1(t), x_2(t)\big) = (s_{in}, 0, 0)$.

If $D > D^*$ and $s_1^* < s^* < s_2^*$:
$\lim\limits_{t \to +\infty} \big(s(t), x_1(t), x_2(t)\big) = (\eta_1, y_1[s_{in} - \eta_1], 0)$,
$\lim\limits_{t \to +\infty} \big(s(t), x_1(t), x_2(t)\big) = (\eta_2, 0, y_2[s_{in} - \eta_2])$ or
$\lim\limits_{t \to +\infty} \big(s(t), x_1(t), x_2(t)\big) = (s_{in}, 0, 0)$.

If $D > D^*$ and $s^* < \min\{s_1^*, s_2^*\}$:

$\lim_{t\to+\infty} \big(s(t), x_1(t), x_2(t)\big) = (\eta_2, 0, y_2[s_{in} - \eta_2])$ or
$\lim_{t\to+\infty} \big(s(t), x_1(t), x_2(t)\big) = (s_{in}, 0, 0)$.

Notice that Proposition 4 is a result qualitatively different from the model with functions f_i strictly increasing: the novelty is that extinction of the two species can be expected because $(s_{in}, 0, 0)$ is a locally asymptotically stable solution (see *e.g.* [**19**],[**133**] for details).

In the remainder of this chapter we assume that $\alpha_1 \neq \alpha_2$. In the sequel $\alpha_{\min} = \min\{\alpha_1, \alpha_2\}$ and $\alpha_{\max} = \max\{\alpha_1, \alpha_2\}$ respectively.

4.2. The feedback control problem

Our goal is to obtain sufficient conditions for uniform persistence considering the following hypotheses:

(Input hypothesis) The dilution rate D is the feedback control variable.

(Output hypothesis) The only output available is given by the total biomass *i.e.*:

$$y = x_1 + x_2.$$

The output hypothesis is considered because in several cases, technical difficulties do not allow the measurements of x_1 and x_2 independently and it is necessary to consider total biomass. For example, the measurement is done often by photometric methods (see [**117**],[**122**][**134**] and the references given there) that do not allow us to distinguish between the two species.

We define the feedback control law $D\colon \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ by:

$$D(x_1, x_2) = g(x_1 + x_2). \tag{4.2.1}$$

We also make the following assumptions on the control function g:

- **(G1)** $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and is globally Lipschitz.
- **(G2)** $g(0) \in [0, f_1(s_{in}))$, g is strictly increasing and there is $s_c > 0$ such that $g(s_c) = D^*$.

Replacing D by the feedback control law (4.2.1), system (4.1.1) becomes:

$$\left\{\begin{aligned} \dot{s} &= g(x_1 + x_2)(s_{in} - s) - \alpha_1 x_1 f_1(s) - \alpha_2 x_2 f_2(s), \\ \dot{x}_1 &= x_1(f_1(s) - g(x_1 + x_2)), \\ \dot{x}_2 &= x_2(f_2(s) - g(x_1 + x_2)). \end{aligned}\right. \tag{4.2.2}$$

Remark 4.2.1. *Non negativity of the function g is assumed because dilution rate D cannot be negative. Assumption* **(G1)** *ensures existence and uniqueness of the initial value problem and* **(G2)** *implies existence of a new critical point.*

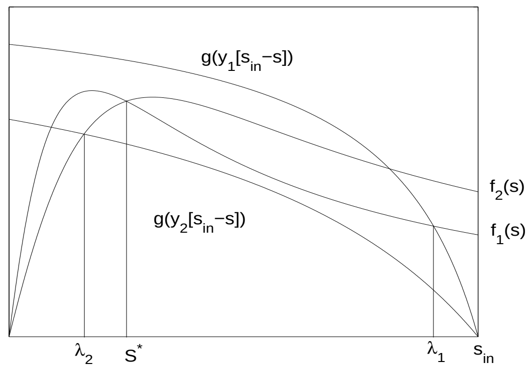


FIGURE 4.2.1. Geometrical interpretation of **(H1)**–**(H2)**.

4.2.1. Choice of the control. Our goal is to obtain sufficient conditions on the function g and its relations with f_1 and f_2 to have existence and global asymptotic stability of the interior critical point.

First, let us define the following equations that will be used to study the asymptotic behavior of system (5.2.5):

$$f_1(s) - g(\alpha_1^{-1}[s_{in} - s]) = 0. \tag{4.2.3}$$

$$f_2(s) - g(\alpha_2^{-1}[s_{in} - s]) = 0. \tag{4.2.4}$$

We make the assumptions

(H1) $g(\alpha_{\min}^{1}[s_{in} - s^*]) > D^* > g(\alpha_{\max}^{-1}[s_{in} - s^*])$.

(H2) Eqs. (4.2.3) and (4.2.4) have one positive solution λ_1 and λ_2 respectively. Moreover, if $\alpha_1 < \alpha_2$ we have that $\lambda_1 \in (s^*, s_{in})$ and $\lambda_2 \in (0, s^*)$.

(H3) $\alpha_{\max}^{-1} g'(x_1 + x_2) > -f_1'\big(s_{in} - \alpha_1 x_1 - \alpha_2 x_2\big)$ for all $(x_1, x_2) \in \mathcal{O}$.

(H4) $\alpha_{\max}^{-1} g'(x_1 + x_2) > -f_2'\big(s_{in} - \alpha_1 x_1 - \alpha_2 x_2\big)$ for all $(x_1, x_2) \in \mathcal{O}$, where the set $\mathcal{O}$ is defined by:

$$\mathcal{O} = \Big\{(x_1, x_2) \in \mathbb{R}_+^2 : 0 \leq \alpha_1 x_1 + \alpha_2 x_2 \leq s_{in}\Big\}.$$

REMARK 4.2.2. *As we can choose the strictly increasing function g, assumptions* **(H1)**–**(H2)** *are always satisfied with reasonable choices. In fact, those assumptions can be interpreted geometrically with the graph of functions defined in Eqs.* (4.2.3) *and* (4.2.4) (*see Fig.* 4.2.1).

Notice that, in some cases it can be difficult to find a function g satisfyng assumptions **(H3)**–**(H4)**. *Otherwise, if $s_1^* \geq s_{in}$ (respectively $s_2^* \geq s_{in}$) then assumption* **(H3)** (*respectively* **(H4)**) *is always satisfied.*

Inequality $y_1 \neq y_2$ implies that system (4.2.2) has a critical point (s^*, x_1^*, x_2^*) defined by

$$x_1^* = \frac{\alpha_2[\alpha_2^{-1}(s_{in} - s^*) - g^{-1}(D^*)]}{\alpha_1 - \alpha_2}, \quad x_2^* = \frac{\alpha_1[g^{-1}(D^*) - \alpha_1^{-1}(s_{in} - s^*)]}{\alpha_1 - \alpha_2}.$$

Assumption **(H1)** implies that $(s^*, x_1^*, x_2^*) \in \mathrm{int}\mathbb{R}^3_+$, **(H2)** implies that there are two hyperbolic critical points of system (5.2.5) on the boundary of $\mathbb{R}^3_+$ defined by:

$$E_1 = (\lambda_1, \alpha_1^{-1}[s_{in} - \lambda_1], 0) \quad \text{and} \quad E_2 = (\lambda_2, 0, \alpha_2^{-1}[s_{in} - \lambda_2]).$$

Finally, notice that if $g(0) = 0$, then

$$\Lambda = \Big\{(s, x_1, x_2) \in \mathbb{R}^3_+ : \quad s \geq 0, \quad x_1 = x_2 = 0\Big\}$$

is a set of non hyperbolic critical points of the system (5.2.5). In the remainder of this paper we assume that the initial conditions of system (5.2.5) are in $\mathbb{R}^3_+ \setminus \Lambda$.

4.2.2. Main Result. The main result of this chapter provides a sufficient condition for the global asymptotic stability of the critical point (s^*, x_1^*, x_2^*).

Theorem 4.2.1. *Let $\alpha_{\max} = \alpha_2$, if at least one of the following conditions hold*:

(i) *Assumptions* **(H1)–(H4)** *holds*
(ii) *Assumptions* **(H1)–(H3)** *holds and inequalities $s^* < \min\{s_1^*, s_2^*\}$ or $s^* \in (s_1^*, s_2^*)$ are satisfied*
(iii) *Assumptions* **(H1)–(H2)** *holds and $s^* < \min\{s_1^*, s_2^*\}$*

Then, the critical point (s^*, x_1^*, x_2^*) is a globally asymptotically stable solution of system (4.2.2) *for all initial conditions in* $\mathrm{int}\mathbb{R}^3_+$.

Notice that the relative order of points s^*,s_1^* and s_2^* summarized in Remark 4.1.1 implies different requirements on assumptions **(H1)**–**(H4)**; in fact, the functions depicted in Fig.4.1.1 –case (a)– satisfy **(H1)**–**(H2)**. Secondly, the functions depicted in Fig.4.1.2 –case (b)– satisfy **(H1)**–**(H3)**. Finally, the functions depicted in Fig.4.1.3 –case (c)– satisfy **(H1)**–**(H4)**. This is important, because assumption **(H4)** is unnecessarily restrictive for the case (ii) and assumptions **(H3)**–**(H4)** are unnecessarily restrictive for the case (iii). Furthermore, as we have pointed out in Remark 4.2.2, there are some cases where checking assumptions **(H3)**–**(H4)** can be rather complicated.

4.3. Preliminary results

In the following results, we establish some properties related to the asymptotic behavior of solutions which are needed in the proof of Theorem 4.2.1.

Lemma 4.3.1. *Let $(s(t), x_1(t), x_2(t))$ be a solution of system* (4.2.2) *with initial condition in* $\mathrm{int}\mathbb{R}^3_+$. *Then this solution is bounded and satisfies*:

$$\lim_{t\to+\infty} s + \alpha_1 x_1 + \alpha_2 x_2 = s_{in}. \tag{4.3.1}$$

PROOF. The main idea of the proof is taken from [**98**]. Let $V\colon \mathbb{R}_+ \mapsto \mathbb{R}$ defined by:

$$V(t) = s(t) + \alpha_1 x_1(t) + \alpha_2 x_2(t) - s_{in}.$$

Clearly, $V' = -g(x_1 + x_2)V$; the lemma follows if $V(t)$ is convergent to 0 when $t \to +\infty$.
Case (i): If $g(0) > 0$, the result is a consequence of the LaSalle invariance principle.
Case (ii): If $g(0) = 0$, clearly, Eq.(4.3.1) follows if and only if:

$$\lim_{t\to+\infty} \int_0^t g(x_1(s) + x_2(s))\,ds = +\infty.$$

Conversely, if we suppose that:

$$\lim_{t\to+\infty} \int_0^t g(x_1(s) + x_2(s))\,ds < +\infty,$$

it is easily seen that the function $t \mapsto g(x_1(t) + x_2(t))$ is nonnegative and integrable. Moreover, we can prove that every solution of system (4.2.2) is bounded: In fact, if $V(0) \le 0$ it follows that $V(t) \le 0$ for any $t \ge 0$ and every solution is bounded by the set

$$\Sigma = \Big\{(s, x_1, x_2) \in \mathbb{R}_+^3 \mid s + \alpha_1 x_1 + \alpha_2 x_2 = s_{in}\Big\};$$

if $V(0) > 0$ it follows that $V(t) > 0$ and $V'(t)$ is negative, hence the boundedness follows.

Using this fact, combined with the mean value theorem, implies that every solution of system (4.2.2) is uniformly continuous on $[0, \infty)$ and finally we conclude that the function $t \mapsto g(x_1(t) + x_2(t))$ is uniformly continuous. Therefore Barbălat's lemma (see *e.g.* [**41**, lemma 1.2.2],[**61**, lemma 3.1],[**78**, lemma 8.1]) yields:

$$\lim_{t\to+\infty} g(x_1(t) + x_2(t)) = 0.$$

As $g(0) = 0$ and g is strictly increasing, we obtain that $\lim\limits_{t\to+\infty} x_i(t) = 0$. On the other hand, by continuity of g we have that:

$$\lim_{t\to+\infty} \frac{\exp\Big[\int_0^t f_i(s(u))\,du\Big]}{\exp\Big[\int_0^t g(x_1(u) + x_2(u))\,du\Big]} = \lim_{t\to+\infty} x_i(t) = 0$$

and it follows that

$$\lim_{t\to+\infty} \exp\Big(\int_0^t f_i(s(u))\,du\Big) = 0.$$

But this is not possible. Hence (4.3.1) holds, which completes the proof. $\square$

If $g(0) = 0$, it follows by Lemma 4.3.1 that the non–hyperbolic critical points in $\Lambda \setminus \{(s_{in}, 0, 0)\}$ are not attractive.

We will denote by U_1 and U_2 the positively invariant sets:

$$U_1 = \Big\{(s, x_1, x_2) \in \mathbb{R}^3_+ : s \geq 0, x_1 > 0 \quad \text{and} \quad x_2 = 0\Big\},$$

$$U_2 = \Big\{(s, x_1, x_2) \in \mathbb{R}^3_+ : s \geq 0, x_2 > 0 \quad \text{and} \quad x_1 = 0\Big\}.$$

As we are interested in persistence of species x_1 and x_2, it is important to know if each species is persistent in the chemostat without competition. The following result gives an affirmative answer.

LEMMA 4.3.2. *Let* $(s(t), x_1(t), x_2(t))$ *be a solution of system* (4.2.2) *with initial condition in* U_i $(i = 1, 2)$. *Then this solution is bounded and satisfies*:

$$\lim_{t\to+\infty} s + \alpha_i x_i = s_{in}. \tag{4.3.2}$$

$$\lim_{t\to+\infty} x_i(t) = \alpha_i^{-1}[s_{in} - \lambda_i] \quad \text{and} \quad \lim_{t\to+\infty} s(t) = \lambda_i. \tag{4.3.3}$$

PROOF. We give the proof for the case $i = 1$; the other case is proved similarly.

Eq. (4.3.2) is an immediate consequence of Lemma 4.3.1. Clearly, $x_2(t) = 0$ for $t \geq 0$. Moreover, it is straightforward to prove that there exists a finite time $T > 0$ such that $s(t) < s_{in}$ for any $t > T$.

We make the transformation $v = s + \alpha_1 x_1 + \alpha_2 x_2 - s_{in}$ and the system becomes

$$\left\{\begin{array}{l} \dot{s} = [g(\alpha_1^{-1}[s_{in} - s]) - f_1(s)](s_{in} - s) - v f_1(s), \\ \dot{v} = -g(\alpha_1^{-1}[s_{in} - s])v, \\ (s_0, v(0)) \in U_1 \quad \text{and} \quad s_0 < s_{in} \end{array}\right. \tag{4.3.4}$$

Hypothesis **(H1)**–**(H3)** imply that $(\lambda_1, 0)$ is a locally asymptotically stable critical point of system (4.3.4). We will prove that it is globally asymptotically stable.

It is straightforward to verify that the critical point $(0, 0)$ of system (4.3.4) is not stable.

Now, we consider the equation

$$\dot{z} = [g(\alpha_1^{-1}[s_{in} - z]) - f_1(z)](s_{in} - z), \quad \text{and} \quad 0 < z(0) < s_{in}. \tag{4.3.5}$$

As the solutions of Eq.(4.3.5) satisfy

$$\lim_{t\to+\infty} z(t) = \lambda_1,$$

Eq.(4.3.3) is a direct consequence of Theorem (B.1.1) of Appendix B and Eq.(4.3.2). □

LEMMA 4.3.3. *If* $\alpha_{\max} = \alpha_1$, *then every component* $x_i(t)$ *of a solution of system* (4.2.2) *with initial condition in* $\mathrm{int}\mathbb{R}^3_+$ *is uniformly persistent.*

PROOF. Let $\mathrm{X} = \big\{(s, x_1, x_2) \in \mathbb{R}^3_+ : s \leq s_{in}, \quad x_1 + x_2 \leq L\big\}$, where $L > \alpha_1^{-1} s_{in}$ and $g(L) > \max\big\{f_1(s_1^*), f_2(s_2^*)\big\}$. Lemma 4.3.1 implies that X is positively invariant and every solution of system (4.2.2) reaches X in finite time and cannot leave it. Hence, we can consider only initial conditions in the set X.

Let $M = \mathrm{X} \cap \big(U_1 \cup U_2\big)$. Following the method developed in [**64**], we will prove that M is a repeller set, that is equivalent to proving uniform persistence. Next, we build the *average Lyapunov function* $P \colon \mathrm{X} \mapsto \mathbb{R}$, defined by

$$P(x_1, x_2) = x_1 x_2.$$

Clearly, $P(x_1, x_2) = 0$ for $(x_1, x_2) \in M$ and $P(x_1, x_2) > 0$ for $(x_1, x_2) \in \mathrm{X} \setminus M$. Moreover, $\dot{P}(x_1, x_2) = \Psi(s, x_1, x_2) P(x_1, x_2)$ where $\Psi \colon \mathrm{X} \mapsto \mathbb{R}$ is the continuous function:

$$\Psi(s, x_1, x_2) = f_1(s) + f_2(s) - 2g(x_1 + x_2).$$

Let $W^s(E_i)$ and $W^u(E_i)$ be the stable and unstable manifold respectively of the critical points E_i. By $\alpha_{\max} = \alpha_1$ and Lemma 4.3.2, we have that the sets U_i are included in $W^s(E_i)$. Moreover **(H2)** implies that the E_i are saddle–points and $\Psi(E_i) > 0$. Finally, Theorem 12.2.2 from [**64**] (see also Corollary A.3.1 in Appendix A) implies that M is a repeller set and the proof is complete. $\square$

4.4. Proof of main result

Let us return to system (4.2.2). Insert the solution $s(t)$ initiated at $s(0)$) into the equations $\dot{x}_1$ and $\dot{x}_2$. Then, for each initial condition $s(0)$ we obtain the nonautonomous system:

$$\left\{ \begin{array}{l} \dot{x}_1 = x_1(f_1\big(s(t)\big) - g(x_1 + x_2)), \\ \dot{x}_2 = x_2(f_2\big(s(t)\big) - g(x_1 + x_2)). \end{array} \right. \tag{4.4.1}$$

Notice that Lemma 4.3.1 implies that for each initial condition $s(0)$, the system (4.4.1) is asymptotically autonomous (see Appendix B) with limit system:

$$\left\{ \begin{array}{l} \dot{z}_1 = z_1(f_1\big(s_{in} - \alpha_1 z_1 - \alpha_2 z_2\big) - g(z_1 + z_2)), \\ \dot{z}_2 = z_2(f_2\big(s_{in} - \alpha_1 z_1 - \alpha_2 z_2\big) - g(z_1 + z_2)). \end{array} \right. \tag{4.4.2}$$

Moreover, system (4.4.2) defines a dynamical system in the set $\mathcal{O} \subset \mathbb{R}^2_+$. The relation between the asymptotic behavior of both systems is summarized by the reduction techniques stated in Appendix B. We will use mainly the Poincaré–Bendixson trichotomy stated by Theorem B.2.1.

Besides $(s_{in}, 0, 0) \in \Lambda$, the critical points of system (4.4.2) are the projections in the set $\mathcal{O}$ of the hyperbolic critical points stated in the previous section.

$$\begin{array}{l} E_0^p = (0, 0), \\ E_1^p = (\alpha_1^{-1}[s_{in} - \lambda_1], 0), \\ E_2^p = (0, \alpha_2^{-1}[s_{in} - \lambda_2]), \\ E_s^p = (x_1^*, x_2^*). \end{array}$$

The local properties of critical points of (4.4.2) are summarized in the following Lemma:

LEMMA 4.4.1. *Let assumptions* **(H1)**–**(H2)** *and* $\alpha_{\max} = \alpha_2$ *hold. Then all the critical points of* (4.4.2) *are hyperbolic. Moreover,*

(a) *Critical point* E_0^p *is a repeller.*

(b) *Critical points* E_i^p *are saddle–points.* $W^u(E_i^p)$ *are in* $\mathrm{int}\mathbb{R}_+^2$ $(i = 1, 2)$ *and*

$$W^s(E_1^p) = \{(z_1, z_2) \in \mathcal{O} \colon 0 < z_1 < \alpha_1^{-1} s_{in} \quad \text{and} \quad z_2 = 0\},$$

$$W^s(E_2^p) = \{(z_1, z_2) \in \mathcal{O} \colon 0 < z_2 < \alpha_2^{-1} s_{in} \quad \text{and} \quad z_1 = 0\}.$$

Moreover, E_1^p *and* E_2^p *cannot belong to* $\omega(\vec{z}(0))$ *when* $\vec{z}(0) \in \mathrm{int}\mathcal{O}$.

(c) *Local asymptotic stability of critical point* E_s^p *is always satisfied when* $s^* < \min\{s_1^*, s_2^*\}$, *is satisfied by assumption* **(H3)** *when* $s^* \in (s_1^*, s_2^*)$ *and by assumptions* **(H3)** *and* **(H4)** *when* $s^* > \max\{s_1^*, s_2^*\}$.

PROOF. Results are obtained from the standard linearization procedure. Indeed notice that the Jacobian matrix at the point E_0^p is defined by:

$$J(E_0^p) = \begin{bmatrix} f_1(s_{in}) - g(0) & 0 \\ 0 & f_2(s_{in}) - g(0) \end{bmatrix}.$$

By using **(G2)** it follows that the two eigenvalues of $J(E_0^p)$ are positive and (a) follows.

Result (b) is obtained following the lines of the proof of Lemma 4.3.3 and computing the eigenvalues of $J(E_i^p)$ $(i = 1, 2)$ which are defined by:

$$\sigma\Big(J(E_1^p)\Big) = \begin{cases} f_2(\lambda_1) - g(\alpha_1^{-1}[s_{in} - \lambda_1]) & > 0, \\ (\lambda_1 - s_{in})\Big(f_1'(\lambda_1) + g'(\alpha_1^{-1}[s_{in} - \lambda_1])\Big) & < 0. \end{cases}$$

$$\sigma\Big(J(E_2^p)\Big) = \begin{cases} f_1(\lambda_2) - g(\alpha_2^{-1}[s_{in} - \lambda_2]) & > 0, \\ (\lambda_2 - s_{in})\Big(f_2'(\lambda_2) + g'(\alpha_2^{-1}[s_{in} - \lambda_2])\Big) & < 0. \end{cases}$$

Finally, the Jacobian matrix of system (4.4.2) at the point E_s^p is defined by

$$J(E_s^p) = \begin{bmatrix} -x_1^*[\alpha_1 f_1'(s^*) + g'(x_1^* + x_2^*)] & -x_1^*[\alpha_2 f_1'(s^*) + g'(x_1^* + x_2^*)] \\ -x_2^*[\alpha_1 f_2'(s^*) + g'(x_1^* + x_2^*)] & -x_2^*[\alpha_2 f_1(s^*) + g'(x_1^* + x_2^*)] \end{bmatrix}.$$

By using the fact that $f_2'(s^*) > f_1'(s^*)$ (see Remark 4.1.1) and $\alpha_2 > \alpha_1$, it is straightforward to verify that:

$$\det J(E_s^p) = (\alpha_1 - \alpha_2) g'(x_1^* + x_2^*) x_1^* x_2^* [f_1'(s^*) - f_2'(s^*)] > 0.$$

Hence, a necessary and sufficient condition for local stability of E_s^p is that:

$$(x_1^* + x_2^*) g'(x_1^* + x_2^*) > -\Bigg[\alpha_1 x_1^* f_1'(s^*) + \alpha_2 x_2^* f_2'(s^*)\Bigg] \tag{4.4.3}$$

or the equivalent formulation:

$$g^{-1}(D^*)g'(g^{-1}(D^*)) > -\frac{s_{in} - s^*}{\alpha_1 - \alpha_2}\Big[\alpha_1 f_1'(s^*) - \alpha_2 f_2'(s^*)\Big]. \tag{4.4.4}$$

In the case $s^* < \min\{s_1^*, s_2^*\}$ this condition is satisfied immediately because $f_i'(s^*) > 0$ $(i = 1, 2)$. In the cases $s^* \in (s_1^*), s_2^*)$ and $s^* > \max\{s_1^*, s_2^*\}$ it is a direct consequence of **(H3)**–**(H4)**. Now, the proof of result (c) is straightforward. □

The proof of the Theorem will be divided into 3 steps:

(1) Let $\vec{x}(0) \in \mathrm{int}\mathbb{R}_+^2$ be an initial condition of system (4.4.1). We will prove that system (4.4.2) cannot have periodic orbits or a cycle of critical points. A consequence of the Poincaré–Bendixson trichotomy is that the set $\omega(\vec{x}(0))$ is a critical point of system (4.4.2).
(2) Lemma 4.3.3 implies that this critical point cannot be in $\partial\mathcal{O}$, hence $\omega(\vec{x}(0)) = (x_1^*, x_2^*)$.
(3) Finally, Eq. (4.3.1) makes it obvious that $\lim\limits_{t\to+\infty} s(t) = s^*$, which proves the Theorem.

We will prove all the cases i)-ii) and iii) in the statement of Th.5.3.1

4.4.1. Proof of case (i). Let $\vec{z}(0) \in \mathrm{int}\mathcal{O}$ be an initial condition of (4.4.2). Notice that system (4.4.2) is *competitive* on $\mathcal{O}$ (*i.e.* the off-diagonal entries of the Jacobian matrix on $\mathcal{O}$ are negative or zero) and its solutions are bounded and consequently, it follows that system (4.4.2) cannot have periodic orbits or a cycle of critical points (see Appendix B), which proves the Theorem.

Assume now that $s^* < \min\{s_1^*, s_2^*\}$ or $s^* \in (s_1^*, s_2^*)$ and that **(H4)** is not satisfied. Notice that in this case, system (4.4.2) is not necessary competitive and consequently, the competitive systems theory cannot be applied directly.

As before, let $\vec{z}(t)$ be a solution of system (4.4.2) with initial condition $\vec{z}(0) \in \mathrm{int}\mathcal{O}$. We will prove that $\vec{z}(t)$ cannot be a periodic orbit and that $\omega(\vec{z}(0))$ cannot be a cycle of critical points.

4.4.2. Proof of case (ii). Let $\hat{s} \in (s^*, s_2^*)$. We define an increasing $C^1(\mathbb{R}, \mathbb{R})$ function $e_2\colon [\hat{s}, +\infty) \mapsto \mathbb{R}$ such that $e_2^{(k)}(\hat{s}) = f_2^{(k)}(\hat{s})$ for $k = 0, 1$. Let us denote by m_2 the *increasing envelope* of f_2 as the function:

$$m_2(s) = \begin{cases} f_2(s) & \text{if} \quad s \in [0, \hat{s}], \\ e_2(s) & \text{if} \quad s \geq \hat{s}. \end{cases} \tag{4.4.5}$$

Let us consider the system:

$$\begin{cases} \dot{u}_1 = u_1(\ f_1\big(s_{in} - \alpha_1 u_1 - \alpha_2 u_2\big) - g(u_1 + u_2)), \\ \dot{u}_2 = u_2(m_2\big(s_{in} - \alpha_1 u_1 - \alpha_2 u_2\big) - g(u_1 + u_2)), \\ u_1(0) = z_1(0) > 0, u_2(0) = z_2(0) > 0. \end{cases} \tag{4.4.6}$$

Notice that system (4.4.6) has the same critical points as system (4.4.2) with the same local properties summarized by Lemma 4.4.1. Assumption

(H3) implies that the system (4.4.6) is competitive and replacing f_2 by m_2 in case (i) of Theorem 5.3.1 we have that:

$$\lim_{t\to+\infty}(u_1(t),u_2(t)) = (x_1^*,x_2^*).$$

Using the order $K_{(0,1)}$ and the comparison theorem for competitive systems (see Appendix C) we have the inequalities:

$$(z_1(0),z_2(0)) \geq_{K_{(0,1)}} (u_1(0),u_2(0)) \geq_{K_{(0,1)}} (0,u_2(0))$$

and

$$(z_1(t),z_2(t)) \geq_{K_{(0,1)}} (u_1(t),u_2(t)) \geq_{K_{(0,1)}} (0,u_2(t))$$

for all $t \geq 0$. Letting $t \to +\infty$, we have that:

$$\liminf_{t\to+\infty} z_1(t) \geq x_1^* \quad \text{and} \quad \limsup_{t\to+\infty} z_2(t) \leq x_2^* \leq y_2[s_{in}-\lambda_2]. \tag{4.4.7}$$

This gives that:

$$\omega(\vec{z}(0)) \subset \big\{(x_1,x_2)\in\mathcal{O} \mid x_1 \geq x_1^* \quad ,0 < x_2 \leq x_2^*\big\},$$

hence $\vec{z}(t)$ cannot be a periodic orbit. Indeed, otherwise we would have a periodic orbit parametrized by $\vec{\psi}$ and by the Poincaré–Bendixson theorem, the critical point (x_1^*,x_2^*) would be inside $\vec{\psi}$, yielding a contradiction.

It remains to prove that there is not a cycle of critical points. If we suppose the existence of one, Lemma 4.4.1 implies that E_0^p is a repeller and E_s^p is locally asymptotically stable, hence they cannot belong to this cycle. Moreover, Eq. (4.4.7) implies that E_2^p cannot belong to this cycle, so only E_1^p could possibly belong to it.

But Lemma 4.4.1 implies that $W^s(E_1^p)\cap W^u(E_1^p)\setminus E_1^p = \emptyset$, hence E_1^p cannot belong to this cycle, which proves the Theorem.

4.4.3. Proof of case (iii). Let $\hat{s} \in (s^*, \max\{s_1^*,s_2^*\})$. We define a couple of continuous increasing functions e_1,e_2: $[\hat{s},+\infty) \mapsto \mathbb{R}$ such that $e_i^{(k)}(\hat{s}) = f_i^{(k)}(\hat{s})$ for $k=0,1$ and $e_2(s) > e_1(s)$ for all $s > \hat{s}$. Let us denote by m_i the *increasing envelope* of f_i as the functions:

$$m_i(s) = \begin{cases} f_i(s) & \text{if} \quad s\in[0,\hat{s}], \\ e_i(s) & \text{if} \quad s \geq \hat{s}. \end{cases} \tag{4.4.8}$$

Let us consider the system:

$$\begin{cases} \dot{u}_1 = u_1(m_1\big(s_{in}-\alpha_1u_1-\alpha_2u_2\big) - g(u_1+u_2)), \\ \dot{u}_2 = u_2(m_2\big(s_{in}-\alpha_1u_1-\alpha_2u_2\big) - g(u_1+u_2)), \\ u_1(0)>0, u_2(0)>0. \end{cases} \tag{4.4.9}$$

Notice that system (4.4.9) is competitive and has the same interior critical point as (4.4.2). Now, we will prove that the system (4.4.2) can not have periodic orbits. Indeed, if we suppose that there is a solution of system that is a non trivial periodic orbit parametrized by $\vec{\psi}(t)$ with (x_1^*,x_2^*) inside, we shall arrive at a contradiction by considering the backward orbits of systems

(4.4.2) and (4.4.9). Note that this orbit is a solution of the reversed time cooperative system:

$$\text{(4.4.10)} \qquad \begin{cases} \dot{v}_1 = -v_1(m_1\big(s_{in} - \alpha_1 v_1 - \alpha_2 v_2\big) - g(v_1+v_2)), \\ \dot{v}_2 = -v_2(m_2\big(s_{in} - \alpha_1 v_1 - \alpha_2 v_2\big) - g(v_1+v_2)), \\ v_1(0) = u_1(0) > 0, v_2(0) = u_2(0) > 0. \end{cases}$$

We choose the initial conditions of systems such that

$$z_1(0) = \psi_1(0) > v_1(0) = x_1^*, \quad z_2(0) = \psi_2(0) > v_2(0) = x_2^*.$$

Applying Theorem C.2.1 of Appendix C, it follows that

$$\psi_1(t) > x_1^* \quad \text{and} \quad \psi_2(t) > x_2^* \quad \text{for all} \quad t < 0$$

it follows that the critical point (x_1^*, x_2^*) is not inside $\vec{\psi}$ obtaining a contradiction using the Poincaré–Bendixson theorem.

It remains to prove that there is not a cycle of critical points. If we suppose the existence of one, as in the proof of case (ii) Lemma 4.4.1 implies that E_0^p and E_s^p cannot belong to this cycle, so only E_1^s and/or E_2^s could possibly belong to it.

By Lemma 4.4.1 we have that $W^u(E_1^p) \cap W^s(E_2^p) = \emptyset$ and $W^u(E_2^p) \cap W^s(E_1^p) = \emptyset$, so there is no cycle connecting E_1^p and E_2^p.

Finally, as in the proof of case (ii), the existence of a cycle connecting E_i^p $(i = 1, 2)$ to itself is not possible, which proves the Theorem.

4.5. Robustness of model

We will suppose that the following properties are satisfied:

(R1) f_1 and f_2 are functionally bounded, *i.e.* there exist a couple of well known maps l_i and u_i (see Fig.4.5.1) , that satisfy assumptions **(F1)**–**(F4)** (with maximums noted by s_{i-}^* and s_{i+}^* respectively) and satisfy

$$\text{(4.5.1)} \qquad l_i(s) \le f_i(s) \le u_i(s), \quad s \ge 0, \quad i = 1, 2.$$

Let us denote by s^- and s^+ (see Fig.4.5.2) the points in $(0, s_{in})$ such that $s^- < s^+$ and

$$l_1(s^-) = u_2(s^-) = D^- > 0,$$
$$u_1(s^+) = l_2(s^+) = D^+ > 0.$$

(R2) $u_1(s) < l_2(s)$ for all $s \in (s^+, s_{in})$.

(R3) We have that $D^+ > D^-$ or $\alpha_{\max} > \dfrac{g^{-1}(D^-) - g^{-1}(D^+)}{s^+ - s^-}$ if $D^- > D^+$.

Let us build the system $(4.2.2)^-$ replacing f_1,f_2 by l_1,u_2 in system (4.2.2). Analogously, we build the system $(4.2.2)^+$ replacing f_1,f_2 by u_1,l_2 in system (4.2.2).

Let us denote by $(4.2.3^-)$ and $(4.2.3^+)$ the Eq. (4.2.3) with f_1 replaced by l_1 and u_1 respectively. Analogously we denote by $(4.2.4)^-$ and $(4.2.4)^+$ the Eq. (4.2.4) with f_2 replaced by l_2 and u_2 respectively.

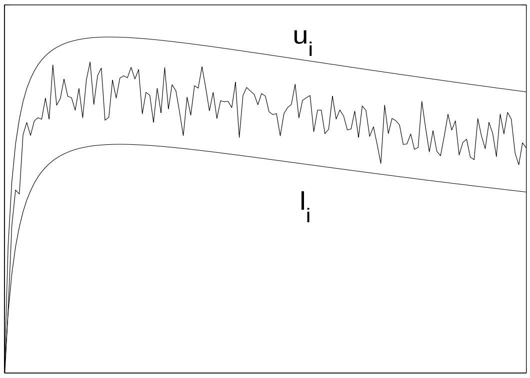


FIGURE 4.5.1. Geometrical interpretation of **(R1)**: Graphs of upper envelope u_i and lower envelope l_i for f_i.

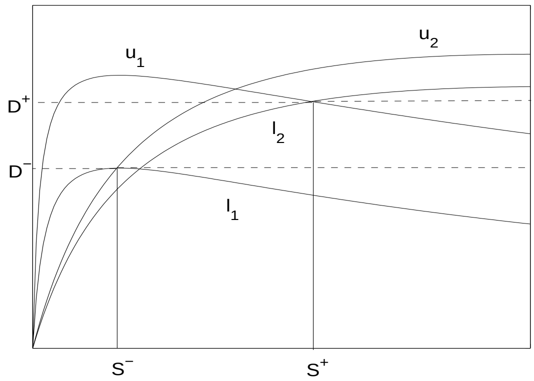


FIGURE 4.5.2. Location of points D^-,D^+,s^- and s^+.

We will make the following assumptions for systems $(4.2.2)^-$ and $(4.2.2)^+$:

(H1*) The following inequalities hold:

$$g(\alpha_1^{-1}[s_{in} - s^-]) > D^- > g(\alpha_2^{-1}[s_{in} - s^-]),$$
$$g(\alpha_1^{-1}[s_{in} - s^+]) > D^+ > g(\alpha_2^{-1}[s_{in} - s^+]).$$

(H2*) Eqs. $(4.2.3^-)$ and $(4.2.4^+)$ have one positive solution λ_1^- and λ_2^+ respectively. Eqs. $(4.2.3^+)$ and $(4.2.4^-)$ have one positive solution λ_1^+ and λ_2^- respectively. Moreover if $y_1 > y_2$, then λ_1^-,λ_1^+ are in (s^*, s_{in}) and λ_2^-,λ_2^+ are in $(0, s^*)$.

(H3*) $$\begin{cases} \alpha_{\min}^{-1} g'(x_1 + x_2) > -l_1' \Big(s_{in} - \alpha_1 x_1 - \alpha_2 x_2\Big) & \text{for all } (x_1, x_2) \in \mathcal{O}, \\ \alpha_{\min}^{-1} g'(x_1 + x_2) > -u_1' \Big(s_{in} - \alpha_1 x_1 - \alpha_2 x_2\Big) & \text{for all } (x_1, x_2) \in \mathcal{O}. \end{cases}$$

(H4*) $$\begin{cases} \alpha_{\max}^{-1} g'(x_1 + x_2) > -u_2' \Big(s_{in} - \alpha_1 x_1 - \alpha_2 x_2\Big) & \text{for all } (x_1, x_2) \in \mathcal{O}, \\ \alpha_{\max}^{-1} g'(x_1 + x_2) > -l_2' \Big(s_{in} - \alpha_1 x_1 - \alpha_2 x_2\Big) & \text{for all } (x_1, x_2) \in \mathcal{O}. \end{cases}$$

Theorem 5.3.1 implies that (s^-, x_1^-, x_2^-) and (s^+, x_1^+, x_2^+) are globally asymptotically stable solutions of $(4.2.2)^-$ and $(4.2.2)^+$ respectively. Moreover x_i^- and x_i^+ are defined by:

$$x_1^- = \frac{\alpha_2[\alpha_2^{-1}(s_{in} - s^-) - g^{-1}(D^-)]}{\alpha_1 - \alpha_2}, \quad x_2^- = \frac{\alpha_1[g^{-1}(D^-) - \alpha_1^{-1}(s_{in} - s^-)]}{\alpha_1 - \alpha_2}$$

$$x_1^+ = \frac{\alpha_2[\alpha_2^{-1}(s_{in} - s^+) - g^{-1}(D^+)]}{\alpha_1 - \alpha_2}, \quad x_2^+ = \frac{\alpha_1[g^{-1}(D^+) - \alpha_1^{-1}(s_{in} - s^+)]}{\alpha_1 - \alpha_2}.$$

By assumption **(R2)** we have that $x_1^- < x_1^+$ and $x_2^+ < x_2^-$.

Now, we can state the following result:

THEOREM 4.5.1. *Let $\alpha_{\max} = \alpha_1$. If the functions f_1 and f_2 are unknown but satisfy assumptions* **(R1)**–**(R3)** *and the functions g,u_i,l_i $(i = 1, 2)$ satisfy assumptions* **(H1*)**–**(H4*)**, *then the solutions of system* (4.2.2) *satisfy*:

$$\begin{aligned} x_1^- &\le \liminf_{t\to+\infty} x_1(t) \le \limsup_{t\to+\infty} x_1(t) \le x_1^+, \\ x_2^+ &\le \liminf_{t\to+\infty} x_2(t) \le \limsup_{t\to+\infty} x_2(t) \le x_2^-, \\ s^- &\le \liminf_{t\to+\infty} s(t) \le \limsup_{t\to+\infty} s(t) \le s^+. \end{aligned} \tag{4.5.2}$$

In particular, system (4.2.2) *is uniformly persistent.*

PROOF. Note that, even if f_1 and f_2 are unknown, asymptotic behavior stated by Lemma 4.3.1 is still valid. Then we can proceed as in the proof of Theorem 5.3.1 and we need study only the ω–limit set of the planar system (4.4.2). Moreover, we consider the restricted competitive systems associated with $(4.2.2)^-$ and $(4.2.2)^+$ respectively:

$$\left\{ \begin{aligned} \dot v_1 &= v_1(\, l_1\big(s_{in} - \alpha_1 v_1 - \alpha_2 v_2\big) - g(v_1 + v_2)), \\ \dot v_2 &= v_2(u_2\big(s_{in} - \alpha_1 v_1 - \alpha_2 v_2\big) - g(v_1 + v_2)). \end{aligned} \right. \tag{4.5.3}$$

$$\left\{ \begin{aligned} \dot w_1 &= w_1(u_1\big(s_{in} - \alpha_1 w_1 - \alpha_2 w_2\big) - g(w_1 + w_2)), \\ \dot w_2 &= w_2(\, l_2\big(s_{in} - \alpha_1 w_1 - \alpha_2 w_2\big) - g(w_1 + w_2)). \end{aligned} \right. \tag{4.5.4}$$

Replacing (4.2.2) by $(4.2.2)^-$ and $(4.2.2)^+$ in Theorem 5.3.1 we obtain that:

$$\lim_{t\to+\infty} (v_1(t), v_2(t)) = (x_1^-, x_2^-) \quad \text{and} \quad \lim_{t\to+\infty} (w_1(t), w_2(t)) = (x_1^+, x_2^+)$$

Let (z_1, z_2) be a solution of system (4.4.2) such that $z_i(0) = v_i(0) = w_i(0)$. By Theorem C.2.1 of Appendix C, we obtain

$$v_1(t) \le z_1(t) \le w_1(t) \quad \text{and} \quad w_2(t) \le z_2(t) \le v_2(t) \quad \text{for all} \quad t \ge 0.$$

Letting $t \to +\infty$, Theorem B.2.1 implies (4.5.2) and the proof is complete. $\square$

REMARK 4.5.1. *Let* s_{1l}^* *and* s_{1u}^* (s_{2l}^* *and* s_{2u}^*) *be the maximum of* l_1 *and* u_1 (l_2 *and* u_2) *respectively. In some cases, the relative order of those points allows us to drop some statements of assumptions* **(H3*)**–**(H4*)**:

If $s^+ < \min\{s_{2l}^*, s_{2u}^*\}$, *we can replace the function* u_2 *by an envelope* m_2 *as in the proof of case* (ii) *of Theorem* 5.3.1, *hence the first inequality in* **(H4*)** *is unnecessary and the proof of Theorem* 4.5.1 *runs as before.*

If $s^+ < \min\{s_{1l}^*, s_{1u}^*, s_{2l}^*, s_{2u}^*\}$ *we can replace the functions* u_i *by envelopes* m_i *as in the proof of case* (iii) *of Theorem* 5.3.1. *Moreover, if we can build an envelope* m_1 *that does not intersects* l_2 *before* s_{in}, *hence the second inequality in* **(H3*)** *and first inequality in* **(H4*)** *are unnecessary and the proof of Theorem* 4.5.1 *runs as before.*

4.6. Numerical example

System (4.2.2) is solved numerically using a couple of Haldane growth functions:

$$f_1(s) = \frac{2s}{2+s+s^2} \quad \text{and} \quad f_2(s) = \frac{3s}{6+s+s^2}.$$

Hence, it follows that $s_1^* = \sqrt{2}$,$s_2^* = \sqrt{6}$ and $s^* = 2$. The parameters of system were fixed at $s_{in} = 8$,$\alpha_1 = 1/3$ and $\alpha_2 = 1$. Moreover, we consider the family of feedback control laws:

$$g_\eta(x_1 + x_2) = \frac{\eta(x_1 + x_2)}{5 + (x_1 + x_2)}, \eta > 0.$$

Assumption **(H1)** is satisfied for $\eta \in I = (0.6389, 0.9167)$ and Eqs.(4.2.3)–(4.2.4) are equivalent to:

$$3\eta s^3 - (21\eta + 6)s^2 + (58 - 18\eta)s - 48\eta = 0,$$
$$\eta s^3 - (7\eta + 3)s^2 + (13 - 2\eta)s - 48\eta = 0.$$

Clearly, each polynomial has a positive real root $\lambda_1(\eta)$ and $\lambda_2(\eta)$ respectively and **(H2)** holds in a subset of I.

Numerical simulations were carried out in MATLAB using ODE45 and they are presented in the Figures 4.6.1, 4.6.2 and 4.6.3. We discuss briefly these results.

For parameters $\eta = 0.86$ and $\eta = 0.84$, we can see with the help of the computer that assumptions **(H3)**–**(H4)** are satisfied and Theorem 4.2.1 can be applied. Fig. 4.6.1 shows that (s^*, x_1^*, x_2^*) is globally asymptotically stable.

For parameters $\eta = 0.75$ and $\eta = 0.72$, we can see that assumption **(H3)** holds but not **(H4)**. Since $s^* \in (s_1^*, s_2^*)$ it follows that case (ii) of Theorem 4.2.1 can be applied. Fig.4.6.2 shows that (s^*, x_1^*, x_2^*) is globally asymptotically stable.

For parameters $\eta = 0.687$ and $\eta = 0.685$, we can see that assumptions **(H3)**–**(H4)** are not satisfied and Theorem 4.2.1 cannot be applied. Fig.4.6.3 shows that (s^*, x_1^*, x_2^*) is unstable but there is a limit cycle and uniform persistence is achieved.

In the Fig.4.6.3 we see that using η as bifurcation parameter, for a critical value of η (determined by **(H1)**–**(H4)**), the positive equilibrium loses its stability and –under some conditions– a family of periodic solutions bifurcates from the positive equilibrium. Although the stability conditions of the bifurcating periodic solutions were not analytically given, numerical simulation was done to show that with suitable parameters, a stable limit cycle exists when the Hopf bifurcation parameter is near its critical value.

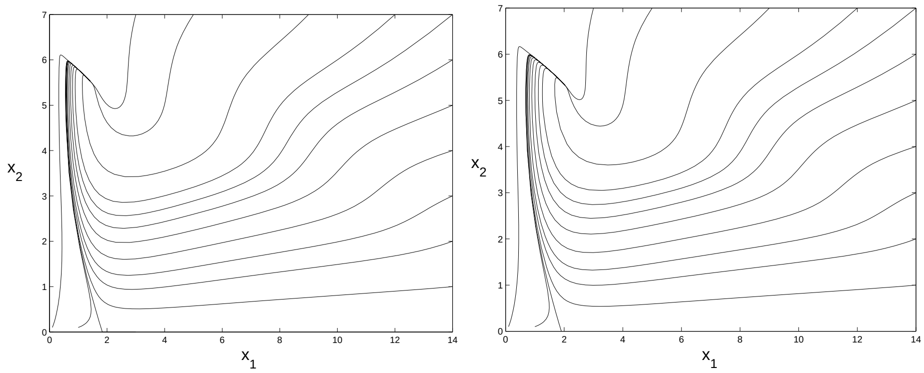


FIGURE 4.6.1. $\eta = 0.86$ and $\eta = 0.84$, case (i) of Theorem 4.2.1 is verified

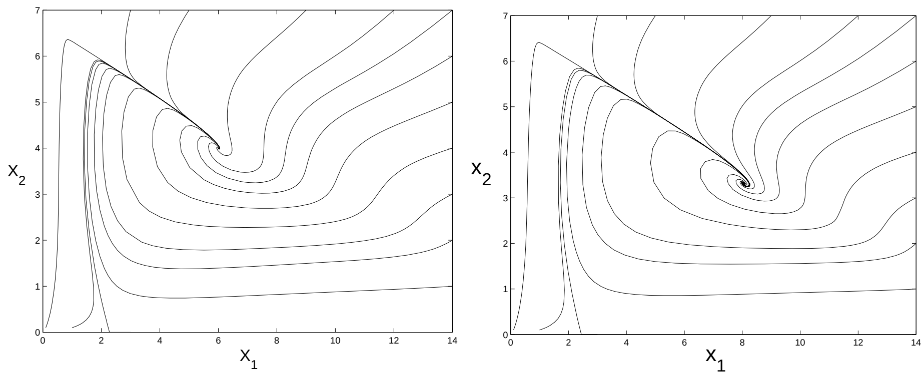
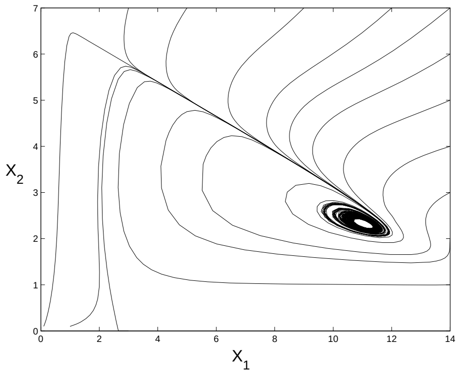


FIGURE 4.6.2. $\eta = 0.75$ and $\eta = 0.72$, case (ii) of Theorem 4.2.1 is verified

4.7. Discussion

We have analyzed a model of the chemostat with competition such that the only output available is the total biomass. The main result is that, considering the dilution rate D as a feedback control, one has –under some hypotheses– the uniform persistence of competing species in contrast to competitive exclusion in the classical chemostat. The novelty of this work is to



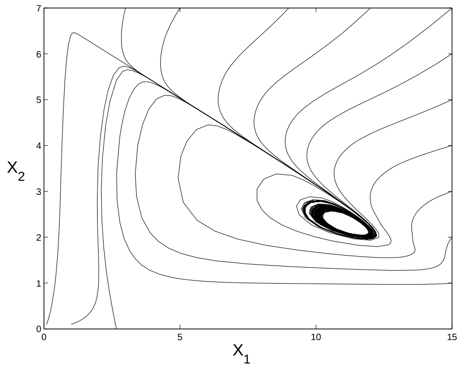


FIGURE 4.6.3. $\eta = 0.687$ and $\eta = 0.685$, Theorem 4.2.1 is not verified but uniform persistence is achieved

consider nonmonotone uptake functions, generalizing in some way the result presented in [**87**].

The model takes the form of a system of differential equations such that its asymptotic behavior is equivalent to a competitive planar differential system. Theory of asymptotically autonomous dynamical systems and theory of competitive dynamical systems played a prominent role.

If we consider d_i to be the specific death rate of species x_i and we substitute D by $D_i = D + d_i$ in Eq. (4.1.1), the tools mentioned above cannot be used because we can not eliminate one variable (the substrate) to study the asymptotic behavior of the model. In fact, the next chapter considers the case where f_i $(i = 1, 2)$ are increasing functions. Even tough the "conservation principle" is not satisfied, we will be able to obtain some sufficient conditions for uniform persistence. Handling different death rates (in the nonmonotone case) remains an open question, worth further study.

Moreover, from an experimental point of view, it would be very interesting to study the same problem considering s_{in} as the feedback control variable and the substrate s as the output available.

One of the strongest assumptions in our model is $\alpha_{\max} = \alpha_2$. It is clear that we must consider other feedback control laws for the cases $\alpha_{\max} = \alpha_1$ and $\alpha_1 = \alpha_2$.

CHAPTER 5

Feedback control for competition models with different removal rates in the chemostat

As we stated in the previous chapter, De Leenheer and Smith [**87**] built a family of feedback control laws (closed–loop control) for the model described above using the dilution rate (input flow nutrient) as the control variable. They prove the global asymptotic stability for an interior critical point of the closed–loop model, showing the possibility of coexistence between the two species. The underlying hypothesis of their model are the monotony of uptake functions and the assumption that the specific mortality rates of species are negligible in comparison with the dilution rate and hence can be ignored. In Chapter 4 we consider more general (nonmonotone) uptake functions, and we assumed $d_i = 0$ $(i = 1, 2)$.

In the present chapter, our goal is to extend the results given in [**87**] dropping the assumption $d_i = 0$ but assuming that the uptake functions are strictly increasing. We point out that the assumption $d_i > 0$ implies that asymptotically in time, none of the equations can be eliminated as in the case $d_i = 0$ and consequently, monotone dynamical systems theory cannot be applied directly to study the asymptotic behavior of chemostat model. Nevertheless, if we suppose that the mortality rates have suitable boundedness properties, we will be able to present a set of verifiable conditions that ensure the global asymptotic stability of a unique critical point in the positive orthant. We focus our attention on the upper bound of the mortality rates.

We show that when the mortality rates are relatively small, the asymptotic behavior of the closed–loop system is the convergence to an interior critical point. In fact, we show that the asymptotic behavior of the closed–loop model can be deduced from the study of a set of low-dimensional (planar) differential equations using differential inequalities. This will be useful in proving that there exists a globally attractive compact set K^*. Finally we shall build a Lyapunov-like function defined in K^*, proving the global asymptotic stability of a critical point in K^*.

This chapter is organized as follows. In Section 5.1 we compiled some facts concerning the two species competition chemostat model without control. In Section 5.2 we provide an exposition of the feedback control law, state some hypothesis related to mortality rates and expose the main result of coexistence. Section 5.3 presents some preliminary results related to asymptotic behavior of the model with control. The proof of the main result is given in Section 5.4.

5.1. Model of competition in the chemostat

The chemostat model with competition is described by the following equations[1]:

$$\left\{\begin{array}{ll}\dot{s} &= D(s_{in}-s) - \frac{x_1}{y_1}f_1(s) - \frac{x_2}{y_2}f_2(s),\\ \dot{x}_1 &= x_1(f_1(s)-D-d_1),\\ \dot{x}_2 &= x_2(f_2(s)-D-d_2).\end{array}\right. \tag{5.1.1}$$

Throughout this chapter, we assume that $f_i\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ (for $i=1,2$) satisfies the following properties **(F)**:

(F1) f_i is continuously differentiable, monotone increasing and $f_i(0)=0$.

(F2) There exists one root $u^* \in (0, s_{in})$ of $f_1(s) - f_2(s) = 0$. Moreover:

$$\left\{\begin{array}{lll} f_1(s) > f_2(s) & \text{if} & s \in (0,u^*),\\ f_1(s) < f_2(s) & \text{if} & s \in (u^*,+\infty).\end{array}\right.$$

(F3) The equation:

$$f_1(s) - f_2(s) - d_1 + d_2 = 0 \quad s \in (0, s_{in}) \tag{5.1.2}$$

has two roots w^* and s^* ($w^* < s^* < u^*$) when $d_1 > d_2$ and has one root $s^* \in (u^*, s_{in})$ when $d_1 \leq d_2$.

Properties **(F1)**–**(F3)** deserve some comments: **(F1)**–**(F2)** can be considered as "generic properties" of competition chemostat models. Otherwise, **(F3)** can be viewed as "small perturbations" of **(F2)** and supposes implicitly that mortality rates d_i are relatively small with respect to the parameters of functions f_i.

Properties **(F1)**–**(F2)** are usual in competition theory of chemostat; as it has been pointed out in [**133**, p.24], in the special case $d_i = 0$ ($i=1,2$) and $D = f_1(u^*) = f_2(u^*)$, assumption **(F2)** implies that there exists a positively invariant and globally attractive set:

$$\Sigma = \left\{(s,x_1,x_2) \in \mathbb{R}^3_+ \colon \frac{x_1}{y_1} + \frac{x_2}{y_2} = s_{in} - u^* \quad \text{and} \quad s = u^*\right\}.$$

Property **(F3)** is satisfied if and only if one of the following (see Fig.5.1) inequalities is holds:

$$0 < d_1 - d_2 < \max_{s\in(0,u^*)} f_1(s) - f_2(s),$$

$$0 \geq d_1 - d_2 > \min_{s\in(u^*,s_{in})} f_1(s) - f_2(s)$$

[1]For the sake of clarity, in this chapter we consider the yield constants as y_i^{-1} instead of α_i

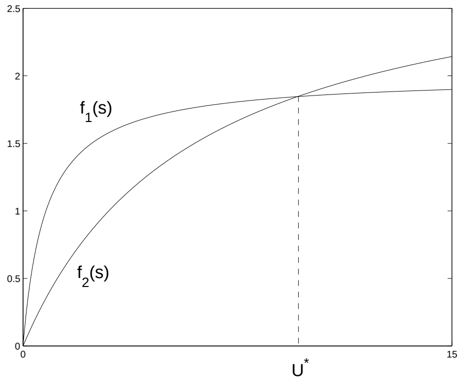


FIGURE 5.1.1. Geometric interpretation of **(F2)**: Graph of f_1 and f_2

REMARK 5.1.1. *A straightforward consequence of* **(F2)**–**(F3)** *is that there exists a constant* $\varepsilon_0 \in (0, s^*/2)$ such that $\hat{s}(\alpha,\beta) \in V_{\varepsilon_0} = (s^* - \varepsilon_0, s^* + \varepsilon_0)$ *is a root of the equation*:

$$f_1(s-\alpha) - f_2(s-\beta) - d_1 + d_2 = 0 \quad \text{with} \quad \alpha,\beta \in [0,\varepsilon_0). \tag{5.1.3}$$

Indeed, notice that if $\alpha = \beta = 0$ *Eq.*(5.1.3) *is equivalent to Eq.*(5.1.2), *hence* **(F3)** *gives the existence of a solution* $\hat{s}(0,0) = s^*$. *Finally, the implicit function theorem implies the existence of a number* ε_0 *such that Eq.*(5.1.3) *has a solution in a neighborhood* V_{ε_0} *of* s^*.

We give the last property for the functions f_i:

(F4) If Eq.(5.1.3) has another solution $w(\alpha,\beta) > 0$, it follows that:

$$w(\alpha,\beta) \in (\max\{\alpha,\beta\}, \max\{\alpha,\beta\} + \varepsilon_0).$$

In summary, these properties are easily satisfied by a couple of functions f_1 and f_2 provided that d_i and ε_0 are relatively small with upper bounds to be defined later in the text.

5.1.1. Competitive exclusion principle and persistence. It is well known that the system (5.1.1) satisfies the *competitive exclusion principle.* There are several experimental results demonstrating this principle (See section 1.2.2 in the Introduction).

The main theoretical features of this model are shown in [**92**], [**133**], [**149**] and are recalled by the following result for $n = 2$ and increasing functions f_i:

PROPOSITION 5 (Competitive exclusion principle [**92**],[**149**]). *Suppose that the properties* **(F1)**–**(F2)** *are satisfied and the equations* $f_i(s) = D + d_i$

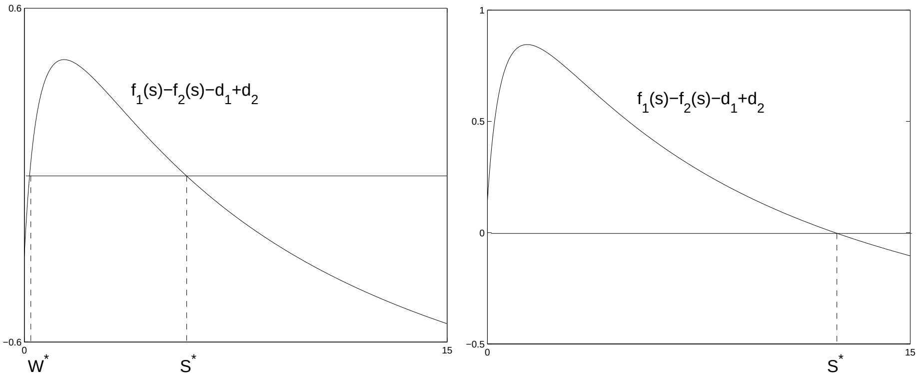


FIGURE 5.1.2. Geometric interpretation of **(F3)**: Graph of the function $f_1 - f_2 - d_1 + d_2$, with $d_1 > d_2$ (Left) and $d_1 \leq d_2$ (Right)

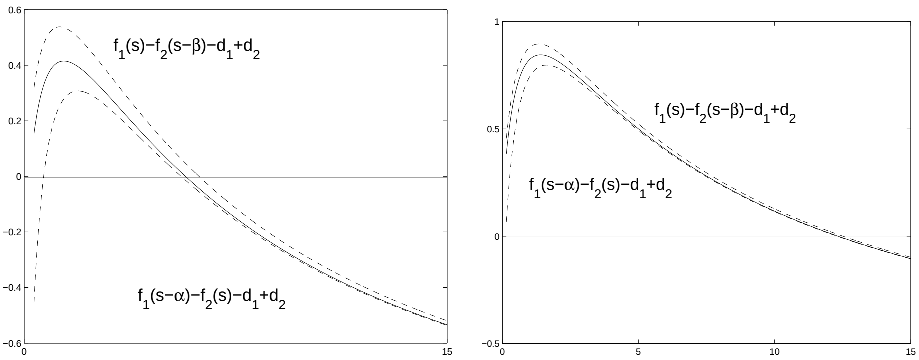


FIGURE 5.1.3. Geometric interpretation of Remark 5.1.1 and **(F4)**: The dashed graphs are the *worst perturbations* possible to Eq.(5.1.2), with $d_1 > d_2$ (Left) and $d_1 \leq d_2$ (Right)

have one solution $\lambda_i \in (0, s_{in})$. Let us build the functions $h_{i,j} \colon (0, \lambda_i) \cup (\lambda_j, s_{in}) \mapsto \mathbb{R}$ defined for any $i, j \in \{1, 2\}$ by :

$$h_{i,j}(s) = \frac{[f_i(s) - D - d_i](s_{in} - \lambda_i) f_j(s)}{D[f_j(s) - D - d_j](s_{in} - s)}.$$

(i) *If $\lambda_1 < \lambda_2$ and $\max\limits_{s \in (0,\lambda_1)} h_{1,2}(s) \leq \min\limits_{s \in (\lambda_2, s_{in})} h_{1,2}(s)$ then all solutions of Eq.(5.1.1) with $x_1(0) > 0$ satisfy:*

$$\lim_{t \to +\infty} (s(t), x_1(t), x_2(t)) = (\lambda_1, y_1 D(s_{in} - \lambda_1)/(D + d_1), 0).$$

(ii) *If* $\lambda_2 < \lambda_1$ *and* $\max_{s\in(0,\lambda_2)} h_{2,1}(s) \leq \min_{s\in(\lambda_1,s_{in})} h_{2,1}(s)$ *then all solutions of Eq.*(5.1.1) *with* $x_2(0) > 0$ *satisfy*:

$$\lim_{t\to+\infty}(s(t), x_1(t), x_2(t)) = (\lambda_2, 0, y_2D(s_{in} - \lambda_2)/(D + d_2)).$$

That means, at most one competitor population avoids extinction.

As in [**87**], we see the chemostat as a feedback control system; that means a system with three components: A *plant* (the chemostat to be controlled), a *sensor* to measure the output $y(t)$ of the plant, and a *controller* to generate the plant input. Our goal is to build a feedback control law for the system (5.1.1) that renders the closed–loop system uniformly persistent, and in addition establish existence of an interior point that is globally asymptotically stable.

5.2. Statement of the problem and main result

As in the previous chapter, our goal is to build a feedback control for the system (5.1.1) which ensures sufficient conditions for the uniform persistence of the model. We, thus consider the following hypothesis:

(Input hypothesis) The dilution rate D is the feedback control variable.

(Output hypothesis) The only output available is given by the total biomass *i.e.*: $$y = x_1 + x_2.$$

As explained in Chapter 4, in several cases there are technical difficulties which do not allow us to measure x_1 and x_2 independently. It is hence necessary to consider total biomass. For example, the measurement is made often by photometric methods (see for example [**123**] and the references given there) that do not allow us to distinguish between the two species.

5.2.1. Choice of the control. We define the feedback control law $D : \mathbb{R}^2_+ \mapsto \mathbb{R}_+$ by:

$$D(x_1, x_2) = g(x_1 + x_2). \tag{5.2.1}$$

where $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuously differentiable, increasing and locally Lipschitz function. Moreover we suppose the following properties **(G)** on the control law g:

(G1) The value $g(0)$ has the following restrictions:

$$f_i(\varepsilon_0) < g(0) < f_i(s^* - \varepsilon_0) - d_i \quad \text{for all } i = 1, 2. \tag{5.2.2}$$

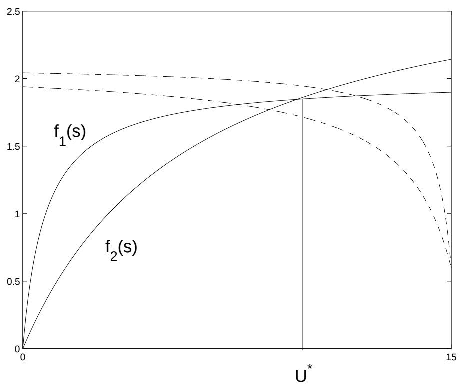


FIGURE 5.2.1. Geometric interpretation of **(F2)**: Graph of f_1 and f_2

(G2) For any α,$\beta \in [0, \varepsilon_0)$ the numbers $\hat{s}(\alpha, \beta) \in V_{\varepsilon_0}$ defined in **(F4)** satisfy the following inequalities:

$$g(y_{\max}[s_{in} - \hat{s}(\alpha, \beta)]) > f_1(\hat{s}(\alpha, \beta) - \alpha) - d_1 > g(y_{\min}[s_{in} - \hat{s}(\alpha, \beta)]), \tag{5.2.3}$$

$$g(y_{\max}[s_{in} - \hat{s}(\alpha, \beta)]) > f_2(\hat{s}(\alpha, \beta) - \beta) - d_2 > g(y_{\min}[s_{in} - \hat{s}(\alpha, \beta)]). \tag{5.2.4}$$

It is of interest to identify classes of functions g where properties **(G1)**–**(G2)** can always be found, and hence our control law applied. **(G1)** is always easy to verify. Notice that Eqs.(5.2.3)–(5.2.4) are always satisfied when $\alpha = \beta = 0$ (see Fig.4). Hence, by continuity of functions g and f_i it is possible to find a family of functions g satisfying **(G1)**–**(G2)** for a relatively small value of ε_0.

Replacing D by the feedback control law (5.2.1), system (5.1.1) becomes:

$$\begin{cases} \dot{s} &= g(x_1 + x_2)(s_{in} - s) - \dfrac{x_1}{y_1} f_1(s) - \dfrac{x_2}{y_2} f_2(s), \\ \dot{x}_1 &= x_1(f_1(s) - g(x_1 + x_2) - d_1), \\ \dot{x}_2 &= x_2(f_2(s) - g(x_1 + x_2) - d_2). \end{cases} \tag{5.2.5}$$

The case $d_1 = d_2 = 0$ has been studied by De Leenheer and Smith [**87**, Th.2]. We summarize their main result (using a slightly different notation):

PROPOSITION 6 (De Leenheer and Smith,[**87**]). *The system* (5.1.1) *with properties* **(F1)**–**(F2)**, *the output* $y = (x_1\ x_2)^T$ *and the control function*:

$$g(x_1, x_2) = \frac{k_1}{y_1} x_1 + \frac{k_2}{y_2} x_2 + \varepsilon \quad \text{with } k_1 > k_2 > 0 \text{ and } 0 \leq \varepsilon < f_1(s_{in})$$

has an interior critical point (u^*, e_1^*, e_2^*) *that is globally asymptotically stable. The constants* e_i^* *are defined as follows*:

$$e_1^* = \frac{y_1[s_{in} f_1(u^*) - k_2(s_{in} - u^*)]}{k_1 - k_2}, \quad e_2^* = \frac{y_2[k_1(s_{in} - u^*) - s_{in} f_2(u^*)]}{k_1 - k_2}.$$

REMARK 5.2.1. (a) *Although in* [**87**] *it is supposed explicitly that* $y = (x_1\, x_2)^T$, *this assumption can be dropped by using an output* $y = x_1 + x_2$ *and the above result is still valid whenever* $y_1 > y_2$.
(b) *Taking* $y = x_1 + x_2$ *and supposing* $y_1 > y_2$, *this result has been extended in* [**44**] *considering more general uptake functions.*

The assumption $d_i > 0$ makes the study of system (5.2.5) far more complex than the case $d_i = 0$; indeed under assumptions $d_i = 0$ the asymptotic behavior of system (5.2.5) can be analyzed by studying a reduced planar system. However, this assumption could be questionable because it would limit the use of the model to systems with relatively high dilution rates.

5.2.2. Main Result: Global asymptotic stability of a critical point. This subsection is devoted to the statement of our main result; we obtain sufficient conditions for the global asymptotic stability of the critical point (s^*, x_1^*, x_2^*) of system (5.2.5), where g^{-1} is the inverse function of g, s^* is the root of Eq.(5.1.2) and the numbers x_1^* and x_2^* are defined by:

$$x_1^* = \frac{y_1\big\{y_2[f_1(s^*) - d_1](s_{in} - s^*) - f_2(s^*)g^{-1}(f_1(s^*) - d_1)\big\}}{f_1(s^*)y_2 - f_2(s^*)y_1},$$

$$x_2^* = \frac{y_2\big\{f_1(s^*)g^{-1}(f_1(s^*) - d_1) - y_1[f_1(s^*) - d_1](s_{in} - s^*)\big\}}{f_1(s^*)y_2 - f_2(s^*)y_1}.$$

REMARK 5.2.2. *Notice that by* **(F4)** *we have that* $w^* < \varepsilon$ *and consequently* **(G1)** *implies that* $f_i(w^*) < g(0) + d_i$. *A consequence of this inequality is the fact that a critical point of type* $(w^*, x_1(w^*), x_2(w^*)) \in \mathbb{R}_+$ *cannot exist.*

Let us now introduce some notation and make precise the mathematical setting. We will work with the roots $\hat{s}(\alpha,\beta)$ of Eq.(5.1.3) and denote $\hat{s}(\alpha,\beta) = \hat{u}(\alpha,\beta)$ when $\alpha > \beta$ and $\hat{s}(\alpha,\beta) = \hat{v}(\alpha,\beta)$ when $\alpha \le \beta$.

Let us define a constant $k_0 \in (0, \varepsilon_0)$ (see **(F4)**) and let $\hat{u}_0$ and $\hat{v}_0$ be the solutions of Eq.(5.1.3) with $(\alpha,\beta) = (k_0, 0)$ and $(\alpha,\beta) = (0, k_0)$ respectively. We shall build two sequences $\{m_j\}_j$,$\{M_j\}_j$ of nonnegative terms bounded above by ε_0 as we will see later on the proof. We suppose that $m_0 = -k_0$ and $M_0 = 0$; the terms for any integer $j \ge 1$ are defined recursively by:

$$m_j = \frac{\left[\frac{d_1}{y_1}v^*_{1_{j-1}} + \frac{d_2}{y_2}u^*_{2_{j-1}}\right]}{g(u^*_{1_{j-1}} + v^*_{2_{j-1}})} \quad \text{and} \quad M_j = \frac{\left[\frac{d_1}{y_1}u^*_{1_{j-1}} + \frac{d_2}{y_2}v^*_{2_{j-1}}\right]}{g(v^*_{1_{j-1}} + u^*_{2_{j-1}})}. \tag{5.2.6}$$

Moreover, the numbers $u^*_{1_j}$,$u^*_{2_j}$,$v^*_{1_j}$ and $v^*_{2_j}$ $(j = 1, 2, \ldots)$ are defined by:

$$
\tag{5.2.7}
\begin{cases}
u^*_{1_j} = \frac{y_1\left\{y_2[s_{in}-\hat{u}_j]-g^{-1}(f_2(\hat{u}_j-m_j)-d_2)\right\}}{y_2-y_1}, \\
u^*_{2_j} = \frac{y_2\left\{g^{-1}(f_2(\hat{u}_j-m_j)-d_2)-y_1[s_{in}-\hat{u}_j]\right\}}{y_2-y_1}, \\
v^*_{1_j} = \frac{y_1\left\{y_2[s_{in}-\hat{v}_j]-g^{-1}(f_1(\hat{v}_j-M_j)-d_1)\right\}}{y_2-y_1}, \\
v^*_{2_j} = \frac{y_2\left\{g^{-1}(f_1(\hat{v}_j-M_j)-d_1)-y_1[s_{in}-\hat{v}_j]\right\}}{y_2-y_1}.
\end{cases}
$$

The constants $\hat{u}_j$ and $\hat{v}_j$ $(j = 1, 2, \ldots)$ are the solutions of Eq.(5.1.3) with $(\alpha, \beta) = (m_j, M_j)$ and $(\alpha, \beta) = (M_j, m_j)$ respectively.

We are now in a position to state our main result:

THEOREM 5.2.1 (Main Result). *Suppose that* $y_{\max} = y_1$ *and properties* **(F)** and **(G)** *are satisfied. If there exists an integer* $j \in \{1, 2, \ldots\}$ *such that the following inequalities hold*:

$$
\tag{5.2.8}
d_i < \frac{\varepsilon_0 g(0)}{2 s_{in}},
$$

$$
\tag{5.2.9}
d_i < \frac{M_j f_i(s_j^-)}{s_{in} - s_j^-} \quad \text{with } s_j^- = s_{in} - m_j - \frac{v^*_{1_j}}{y_1} - \frac{u^*_{2_j}}{y_2} \quad (i = 1, 2)
$$

then, for any initial condition on the interior of $\mathbb{R}^3_+$*, the critical point* (s^*, x_1^*, x_2^*) *is a globally asymptotically stable solution of system* (5.2.5).

In the proof of Theorem 5.2.1 we will see that the asymptotic behavior of system (5.2.5) can be determined by using lower dimensional (planar) systems. In section 4 we will find some estimates for the substrate $s(t)$ that are dependent on $x_1(t)$ and $x_2(t)$. Using the monotony of f_i, we shall build two planar Kolmogorov systems that inherit some properties of system (5.2.5) and will make it possible to prove that, in a finite time, the solutions of system (5.2.5) are in a convex, compact set $K \subset \operatorname{Int} \mathbb{R}^3_+$. In fact, an elementary analysis based on average Lyapunov functions (see for example Appendix A and references therein) and planar competitive systems theory (see for example Appendix C and references therein) are sufficient to study their boundedness properties. In section 5 we will improve these bounds and build a Lyapunov-like functional defined in a compact set $K_j \subset K$, that will make it possible to prove the main result.

5.3. Preliminary Results

We shall prove that the semiflow related to solutions of (5.2.5) is dissipative. In fact, we prove that there exists a compact, convex set $K \in \mathbb{R}^3_+$ globally attractive and positively invariant. The proof of this later result uses planar competitive dynamical systems and average Lyapunov functions. Several results of the next section will be drawn from this result.

5.3.1. A priori estimates. We introduce the constant θ^* defined by:

$$\theta^* = \frac{s_{in}}{g(0)}(d_1 + d_2).$$

It is not difficult to show that Eq.(5.2.8) implies that $\theta^* < \varepsilon_0$. Hence throughout the rest of this chapter we will consider $k_0 = \theta^*$.

Before stating the main result of this section, we need to introduce a useful estimation for $\hat{s}(\alpha, \beta)$ which will be used below in several steps of the proof of our main result:

LEMMA 5.3.1. *The following equations*:

$$h_1(s) = f_1(s - \alpha) - g\big(y_1[s_{in} - s]\big) - d_1 = 0, \tag{5.3.1}$$

$$h_2(s) = f_2(s - \beta) - g\big(y_2[s_{in} - s]\big) - d_2 = 0, \tag{5.3.2}$$

have one root $\eta(\alpha)$ and $\xi(\beta)$ on the intervals (α, s_{in}) and (β, s_{in}) respectively. Moreover the following inequality holds:

$$\beta < \xi(\beta) < \hat{s}(\alpha, \beta) < \eta(\alpha) < s_{in} \quad \text{for any } \alpha, \beta \in (0, \theta^*]. \tag{5.3.3}$$

PROOF. Notice that by monotony of functions f_i and g, it follows that the functions h_i are strictly increasing and they have at most one positive root. By **(G2)** it follows that $h_1(\hat{s}(\alpha, \beta)) < 0$ and Eq.(5.2.2) implies that $h_1(s_{in}) > 0$. Hence, the monotony of h_1 implies that Eq.(5.3.1) has one solution $\eta(\alpha) \in (\hat{s}(\alpha, \beta), s_{in})$.

Since $h_2(\beta) < 0$ and **(G2)** implies that $h_2(\hat{s}(\alpha, \beta) > 0$, we prove as before that Eq.(5.3.2) has one solution $\xi(\beta) \in (\beta, \hat{s}(\alpha, \beta))$. $\square$

We now turn to the main result of this section.

THEOREM 5.3.1. *Under the assumptions of Theorem* (5.2.1) *but excluding Eq.*(5.2.9) *there exists a positively invariant and globally attractive convex set*:

$$K = \left\{(s, x_1, x_2) \in \mathbb{R}^3_+ : \, -\theta^* \le s - s_{in} + \frac{x_1}{y_1} + \frac{x_2}{y_2} \le 0, \quad \text{and} \quad (x_1, x_2) \in B\right\}$$

*where $B = [u^*_{1_0}, v^*_{1_0}] \times [v^*_{2_0}, u^*_{2_0}]$ and the constants $u^*_{i_0}$ and $v^*_{i_0}$ (see Eq.*5.2.7) *are defined as follows*:

$$u^*_{1_0} = \frac{y_1\{y_2[s_{in} - \hat{u}_0] - g^{-1}(f_1(\hat{u}_0 - \theta^*) - d_1)\}}{y_2 - y_1},$$

$$u^*_{2_0} = \frac{y_2\{g^{-1}(f_2(\hat{u}_0) - d_2) - y_1[s_{in} - \hat{u}_0]\}}{y_2 - y_1},$$

$$v^*_{1_0} = \frac{y_1\{y_2[s_{in} - \hat{v}_0] - g^{-1}(f_1(\hat{v}_0) - d_1)\}}{y_2 - y_1},$$

$$v^*_{2_0} = \frac{y_2\{g^{-1}(f_2(\hat{v}_0 - \theta^*) - d_2) - y_1[s_{in} - \hat{v}_0]\}}{y_2 - y_1}.$$

PROOF. Notice that, by **(G1)-(G2)** with $k_0 = \theta^*$, it follows that the constants $u_{i_0}^*, v_{i_0}^*$ are well defined. Now, the proof will be divided into two steps:

Step (i) We will prove that the set:

$$\Gamma = \left\{(s, x_1, x_2) \in \mathbb{R}_+^3 \colon -\theta^* \leq s - s_{in} + \frac{x_1}{y_1} + \frac{x_2}{y_2} \leq 0\right\}$$

is positively invariant and globally attractive.

Step (ii) We will prove that the set B is positively invariant and globally attractive.

Step (i) Let $(s(t), x_1(t), x_2(t))$ be an arbitrary solution of system (5.2.5). We consider the function $V \colon \mathbb{R}_+ \mapsto \mathbb{R}$ defined by:

$$V(t) = s(t) - s_{in} + \frac{x_1(t)}{y_1} + \frac{x_2(t)}{y_2}.$$

Differentiating V, it follows from system (5.2.5) that:

$$V'(t) = -g(x_1 + x_2)V(t) - d_1\frac{x_1}{y_1} - d_2\frac{x_2}{y_2}. \tag{5.3.4}$$

Consider the sets:

$$\Gamma_0 = \left\{(s, x_1, x_2) \in \mathbb{R}_+^3 \colon V \leq 0\right\} \quad \text{and} \quad \tilde{\Gamma}_0 = \left\{(s, x_1, x_2) \in \mathbb{R}_+^3 \colon -\theta^* \leq V\right\}.$$

It is not difficult to verify that $\Gamma = \Gamma_0 \cap \tilde{\Gamma}_0$. Moreover, notice that if $V = 0$, then $V' < 0$ and the set Γ_0 is positively invariant. Now, we will prove that Γ_0 is globally attractive. If $V(0) \leq 0$, the proof is finished. Now, we suppose that $V(0) > 0$. This implies that $V(t) \leq V(0)\exp(-g(0)t)$. Finally, letting $t \to +\infty$ it, follows that:

$$\limsup_{t \to +\infty} V(t) \leq 0.$$

Now, there is no loss of generality when considering only initial conditions in the set Γ_0. Since $x_i < y_i s_{in}$, Eq.(5.3.4) implies that:

$$V' > -g(0)V - s_{in}(d_1 + d_2). \tag{5.3.5}$$

By (5.3.5) it follows that $V'(t) > 0$ when $V(t) = -\theta^*$. Hence, $\tilde{\Gamma}_0$ is positively invariant.

We also consider the following differential equation:

$$w' = -g(0)w - s_{in}(d_1 + d_2), \quad w(0) \leq V(0).$$

By the comparison Theorem for differential equations (see for example [**59**, III.4.1],[**138**, Th.9.5]), it follows that:

$$w(t) \leq V(t) \quad \text{for any } t \geq 0. \tag{5.3.6}$$

Now, we will prove that there exists a number $t_1 > 0$ such that $V(t) \in [-\theta^*, 0]$ for any $t \geq t_1$. Indeed, letting $t \to +\infty$, since $\lim_{t \to +\infty} w(t) = -\theta^*$, it follows from Eq.(5.3.6) that:

$$-\theta^* \leq \liminf_{t \to +\infty} V(t)$$

and, in consequence Γ is positively invariant and globally attractive.

Step (ii) We will need the following lemma:

LEMMA 5.3.2. *The set* $\Lambda = \Big\{(s, x_1, x_2) \in \Gamma \colon \frac{x_1}{y_1} + \frac{x_2}{y_2} \le s_{in} - \theta^*\Big\}$ *is positively invariant and globally attractive.*

The proof will be given later, we continue now with the proof of the theorem. By Lemma 5.3.2 there is no loss of generality in considering only initial conditions in the set Λ. Moreover it follows that:

$$s_{in} - \theta^* - \frac{x_1}{y_1} - \frac{x_2}{y_2} \le s \le s_{in} - \frac{x_1}{y_1} - \frac{x_2}{y_2}.$$

From these inequalities and Lemma 5.3.2, it follows that the solutions of system (5.2.5) satisfy the following differential inequalities in the set Λ:

$$\begin{cases} \dot{x}_1 \ge x_1\Big(f_1(s_{in} - \theta^* - \frac{x_1}{y_1} - \frac{x_2}{y_2}) - g(x_1 + x_2) - d_1\Big), \\ \dot{x}_2 \le x_2\Big(f_2(s_{in} - \frac{x_1}{y_1} - \frac{x_2}{y_2}) - g(x_1 + x_2) - d_2\Big). \end{cases} \tag{5.3.7}$$

$$\begin{cases} \dot{x}_1 \le x_1\Big(f_1(s_{in} - \frac{x_1}{y_1} - \frac{x_2}{y_2}) - g(x_1 + x_2) - d_1\Big), \\ \dot{x}_2 \ge x_2\Big(f_2(s_{in} - \theta^* - \frac{x_1}{y_1} - \frac{x_2}{y_2}) - g(x_1 + x_2) - d_2\Big). \end{cases} \tag{5.3.8}$$

In order to study these inequalities, we build the following comparison systems defined in Λ:

$$\begin{cases} \dot{u}_1 = u_1\Big(f_1(s_{in} - \theta^* - \frac{u_1}{y_1} - \frac{u_2}{y_2}) - g(u_1 + u_2) - d_1\Big), \\ \dot{u}_2 = u_2\Big(f_2(s_{in} - \frac{u_1}{y_1} - \frac{u_2}{y_2}) - g(u_1 + u_2) - d_2\Big), \\ 0 < u_1(0) \le x_1(0) \quad \text{and} \quad u_2(0) \ge x_2(0) > 0. \end{cases} \tag{5.3.9}$$

$$\begin{cases} \dot{v}_1 = v_1\Big(f_1(s_{in} - \frac{v_1}{y_1} - \frac{v_2}{y_2}) - g(v_1 + v_2) - d_1\Big), \\ \dot{v}_2 = v_2\Big(f_2(s_{in} - \theta^* - \frac{v_1}{y_1} - \frac{v_2}{y_2}) - g(v_1 + v_2) - d_2\Big), \\ v_1(0) \ge x_1(0) > 0 \quad \text{and} \quad 0 < v_2(0) \le x_2(0). \end{cases} \tag{5.3.10}$$

Since f_i and g are increasing, we see that the systems (5.3.9) and (5.3.10) are *competitive* in Λ (*i.e.*, the off–diagonal entries of the Jacobian matrix are negative or zero) and its solutions are bounded. Hence, Theorem C.2.1 (see Appendix C) shows that their solutions are convergent to a critical point.

Taking $(\alpha, \beta) = (\theta^*, 0)$ and $(\alpha, \beta) = (0, \theta^*)$, properties **(F)** and **(G)** combined with a simple algebraic calculation imply that systems (5.3.9) and (5.3.10) each have one interior critical point denoted by $(u_{1_0}^*, u_{2_0}^*)$ and $(v_{1_0}^*, v_{2_0}^*)$ respectively. We will prove that they are global attractors.

Using Eqs.(5.3.1) and (5.3.2) with $(\alpha, \beta) = (\theta^*, 0)$ and $(\alpha, \beta) = (0, \theta^*)$ respectively, it follows that system (5.3.9) has three critical points in $\Lambda \cap \partial \mathbb{R}_2^+$: $(0, 0)$,$(\eta_1, 0)$ and $(0, \eta_2)$ where $\eta_i = s_{in} - \frac{\bar{\eta}_i}{y_i}$ and $\bar{\eta}_i$ $(i = 1, 2)$ are solutions of Eqs.(5.3.1) and (5.3.2) respectively.

It can be proved that the set $\Lambda \cap \partial\mathbb{R}_2^+$ is not attractive. We will sketch this proof for the solution of system (5.3.9):

- We build the functional $P\colon \Lambda \mapsto \mathbb{R}$ defined by $P(u_1, u_2) = u_1 u_2$.
- Notice that $P = 0$ for any value in $\Lambda \cap \partial\mathbb{R}_+^2$ and $P > 0$ for any value in $\operatorname{Int}\Lambda$.
- It follows from system (5.3.9) that, $\dot{P} = \Psi(u_1, u_2)P$, where $\Psi\colon \Lambda \mapsto \mathbb{R}$ is the continuous function:

$$\Psi(u_1,u_2) = f_1\Big(s_{in} - \theta^* - \frac{u_1}{y_1} - \frac{u_2}{y_2}\Big) + f_2\Big(s_{in} - \frac{u_1}{y_1} - \frac{u_2}{y_2}\Big) - 2g(u_1+u_2) - (d_1+d_2).$$

- Eq.(5.3.3) implies that $\Psi(\eta_1, 0) > 0$ and $\Psi(0, \eta_2) > 0$. Moreover **(G2)** and the monotony of f_i and g imply that $f_1(s_{in} - \theta^*) - g(0) > 0$ and consequently $\Psi(0,0) > 0$. This implies that P is an *average Lyapunov function* (see Appendix A) and by using Corollary (A.3.1) it follows that any solution $(u_1(t), u_2(t))$ cannot converge to $\Lambda \cap \partial\mathbb{R}_+^2$ and following the statement (i) of Proposition C.2.1 (see Appendix C) must be convergent to $(u_{1_0}^*, u_{2_0}^*)$.

We proceed in a similar manner to prove that every solution of (5.3.10) converges to the interior critical point. Hence, it follows that:

(5.3.11)
$$\lim_{t\to+\infty}(u_1(t), u_2(t)) = (u_1^*, u_2^*) \quad \text{and} \quad \lim_{t\to+\infty}(v_1(t), v_2(t)) = (v_1^*, v_2^*).$$

Using (ii) and (iii) of Proposition C.2.1 (see Appendix) it follows that:

(5.3.12) $$u_1(t) \le x_1(t) \le v_1(t) \quad \text{and} \quad v_2(t) \le x_2(t) \le u_2(t) \quad \text{for any } t \ge 0.$$

Considering Eqs.(5.3.9)–(5.3.10) with initial conditions $(u_1(0), u_2(0)) = (u_{1_0}^*, u_{2_0}^*)$ and $(v_1(0), v_2(0)) = (v_{1_0}^*, v_{2_0}^*)$, and using Eqs.(5.3.11)–(5.3.12) we conclude that the box B is positively invariant.

Now, we will prove that B is globally attractive. We consider an initial condition for inequalities (5.3.7)–(5.3.8) such that $(x_1(0), x_2(0)) \notin B$. As above, letting $t \to +\infty$, Eqs.(5.3.11)-(5.3.12) imply that:

$$u_{1_0}^* \le \liminf_{t\to+\infty} x_1(t) \le \limsup_{t\to+\infty} x_1(t) \le v_{1_0}^*,$$

$$v_{2_0}^* \le \liminf_{t\to+\infty} x_2(t) \le \limsup_{t\to+\infty} x_2(t) \le u_{2_0}^*$$

and this completes the proof. □

REMARK 5.3.1. *Brower's fixed point theorem* (see *e.g.* [**16**, Th. 3.2],[**152**, Prop. 2.6]) *implies that there exists one fixed point* $(s^*, x_1^*, x_2^*) \in \operatorname{Int} K$.

5.3.2. Proof of Lemma 5.3.2. We complete the proof under the assumption that $-\theta^* \le V \le 0$. Let $(s(t), x_1(t), x_2(t))$ be an arbitrary solution of system (5.2.5). We build the function $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by:

$$h(t) = \frac{x_1(t)}{y_1} + \frac{x_2(t)}{y_2}.$$

(i) First, we will prove that if there exists $t_0 \geq 0$ such that $h(t_0) = s_{in} - \theta^*$, then $h'(t_0) < 0$ implying positive invariance.

By using $h(t_0) = s_{in} - \theta^*$ and $V < 0$, it is straightforward to show that $s(t_0) < \theta^*$. Moreover, differentiating $h(t)$, it follows from system (5.2.5) that:

$$h'(t_0) \leq h(t_0)\Big(f_{\max}(\theta^*) - g(x_1 + x_2) - d_{\min}\Big).$$

If $f_{\max}(\theta^*) \leq d_{\min}$, the proof is finished. Now, if we suppose that $f_{\max}(\theta^*) > d_{\min}$, since x_1+x_2 is a linear function in the plane $h(t_0) = s_{in}-\theta^*$, it follows that:

$$\min_{(x_1,x_2):\, h(t_0)=s_{in}-\theta^*} \Big\{x_1 + x_2\Big\} = y_2(s_{in} - \theta^*).$$

Combining these two estimates with the monotony of g, we obtain:

$$h'(t_0) \leq h(t_0)\Big(f_{\max}(\theta^*) - g(y_2[s_{in} - \theta^*]) - d_{\min}\Big).$$

By using Eq.(5.2.3), it follows that $f_i(\theta^*) < g(0) < g(y_2[s_{in} - \theta^*]) + d_{\min}$ for any $i = 1, 2$, hence $h'(t_0) < 0$.

(ii) We will prove that there exists $t_1 > 0$ such that $h(t_1) = s_{in} - \theta^*$ and $h(t) < s_{in} - \theta^*$ for any $t > t_1$, implying global attractivity.

Indeed, otherwise we would have $h(t) > s_{in} - \theta^*$ for any $t \geq 0$. As above, it is not difficult to show that $s(t) < \theta^*$ for any $t \geq 0$. Thus

$$h'(t) \leq h(t)\Big(f_{\max}(\theta^*) - g(x_1(t) + x_2(t)) - d_{\min}\Big) \quad \text{for any } t \geq 0.$$

Moreover, the solutions of system (5.2.5) are defined in the convex set:

$$\Theta = \Big\{(s, x_1, x_2) \in \Gamma \colon s_{in} - \theta^* \leq \frac{x_1}{y_1} + \frac{x_2}{y_2} \leq s_{in}\Big\}.$$

Since $x_1 + x_2$ is a linear function in Θ, it follows that:

$$\min_{(x_1,x_2)\in\Theta} \{x_1 + x_2\} = y_2(s_{in} - \theta^*).$$

This implies that $g(x_1(t) + x_2(t)) > g(y_2[s_{in} - \theta^*])$ for any $t \geq 0$ and by Eqs.(5.2.3)–(5.2.5) it follows that $h(t)$ satisfying the differential inequality

$$h'(t) \leq h(t)\Big\{f_{\max}(\theta^*) - g(y_2[s_{in} - \theta^*]) - d_{\min})\Big\}.$$

It is straightforward to prove that

$$\lim_{t\to+\infty} h(t) = 0$$

obtaining a contradiction. The lemma follows. $\square$

5.4. Proof of main result

Throughout this section, it is assumed that the initial conditions are in K. We have divided the proof into a sequence of lemmas. The key points of this proof are the generalization of estimations given for the attractor K:

- In Lemma 5.4.1 we prove –under the same assumptions as in Theorem 5.3.1– that there exists an attractor $K_1 \subset K$ for the solutions of Eq.(5.2.5). The proof is similar to the proof of Th.5.3.1. Nevertheless, we shall do a sketch of the proof to point out its recursive feature.

- In Lemma 5.4.2 we generalize the conclusions of Lemma 5.4.1 for a decreasing sequence of attractors $K_j \subset K_1$.

- In Lemma 5.4.3, we prove that if there exists an integer $j \in \{1, 2, \ldots\}$ such that the inequalities:
$$d_i < \frac{M_j f_i(s_j^-)}{s_{in} - s_j^-}, \quad i = 1, 2$$
are satisfied, then any solution in the set K_j converges to the critical point (s^*, x_1^*, x_2^*). The main tool is a Lyapunov–like functional $W : K_j \mapsto \mathbb{R}$.

Lemma 5.4.1. *Under the assumptions and conditions of Theorem* 5.3.1*, there exists a positively invariant and globally attractive set* $K_1 \subset K$ *defined by*:

$$K_1 = \Big\{(s, x_1, x_2) \in K\colon \; -m_1 \leq V \leq -M_1 \quad \text{and} \quad (x_1, x_2) \in B_1\Big\}$$

where $B_1 = [u_{1_1}^*, v_{1_1}^*] \times [v_{2_1}^*, u_{2_1}^*]$ *and the constants* $u_{1_1}^*$,$u_{2_1}^*$,$v_{1_1}^*$ *and* $v_{2_1}^*$ *are defined in* (4.5.1).

Proof. Following the lines of the proof of Theorem 5.2.1, it may be concluded that the set $\Gamma_1 = \Big\{(s, x_1, x_2) \in K\colon \; -m_1 \leq V \leq -M_1\Big\}$ is positively invariant and globally attractive. For the convenience of the reader, we will sketch the main ideas in the proof.

Differentiating the functional V with respect to t and using the estimations for x_1 and x_2 given by Theorem 5.2.1, it follows that:

$$V' > -g(u_{1_0}^* + v_{2_0}^*)V - d_1 \frac{v_{1_0}^*}{y_1} - d_2 \frac{u_{2_0}^*}{y_2}, \tag{5.4.1}$$

$$V' < -g(v_{1_0}^* + u_{2_0}^*)V - d_1 \frac{u_{1_0}^*}{y_1} - d_2 \frac{v_{2_0}^*}{y_2}. \tag{5.4.2}$$

We consider the following differential equations:

$$w' = -g(u_{1_0}^* + v_{2_0}^*)w - d_1\frac{v_{1_0}^*}{y_1} - d_2\frac{u_{2_0}^*}{y_2}, \quad w(0) \le V(0),$$

$$z' = -g(v_{1_0}^* + u_{2_0}^*)z - d_1\frac{u_{1_0}^*}{y_1} - d_2\frac{v_{2_0}^*}{y_2}, \quad z(0) \ge V(0).$$

By the Comparison Theorem for differential equations (see *e.g.* the references in [**59**, III.4.1],[**138**, Th.9.5]), it follows that:

$$w(t) \le V(t) \le z(t) \quad \text{for any } t \ge 0. \tag{5.4.3}$$

By using Eqs.(5.4.1)-(5.4.2), it follows that $V'(t) > 0$ ($V'(t) < 0$) when $V(t) = -m_1$ ($V(t) = -M_1$). Hence, Γ_1 is positively invariant.

Let us recall that:

$$m_1 = \frac{\frac{d_1}{y_1}v_{1_0}^* + \frac{d_2}{y_2}u_{2_0}^*}{g(u_{1_0}^* + v_{2_0}^*)} \quad \text{and} \quad M_1 = \frac{\frac{d_1}{y_1}u_{1_0}^* + \frac{d_2}{y_2}v_{2_0}^*}{g(v_{1_0}^* + u_{2_0}^*)}.$$

Letting $t \to +\infty$, Eq.(5.4.3) implies:

$$-m_1 \le \liminf_{t\to+\infty} V(t) \le \limsup_{t\to+\infty} V(t) \le -M_1,$$

hence Γ_1 is globally attractive.

An immediate consequence are the inequalities $0 < M_1 < m_1 < \theta^*$. In the rest of the proof we will suppose that the initial conditions are in Γ_1. Hence we have the inequalities:

$$s_{in} - m_1 - \frac{x_1}{y_1} - \frac{x_2}{y_2} \le s \le s_{in} - M_1 - \frac{x_1}{y_1} - \frac{x_2}{y_2}.$$

Let us define the set $\Lambda_1 : \Big\{(s, x_1, x_2) \in \Gamma_1 \colon \frac{x_1}{y_1} + \frac{x_2}{y_2} \le s_{in} - m_1\Big\}$. by the inequality $m_1 < \theta^*$, it follows that $\Lambda \subset \Lambda_1$. Hence, Lemma 5.3.2 implies that the set Λ_1 is positively invariant and globally attractive and that the solutions of system (5.2.5) satisfy the following differential inequalities in Λ_1:

$$\begin{cases} \dot{x}_1 \ge x_1\Big(f_1(s_{in} - m_1 - \frac{x_1}{y_1} - \frac{x_2}{y_2}) - g(x_1 + x_2) - d_1\Big), \\ \dot{x}_2 \le x_2\Big(f_2(s_{in} - M_1 - \frac{x_1}{y_1} - \frac{x_2}{y_2}) - g(x_1 + x_2) - d_2\Big). \end{cases} \tag{5.4.4}$$

$$\begin{cases} \dot{x}_1 \le x_1\Big(f_1(s_{in} - M_1 - \frac{x_1}{y_1} - \frac{x_2}{y_2}) - g(x_1 + x_2) - d_1\Big), \\ \dot{x}_2 \ge x_2\Big(f_2(s_{in} - m_1 - \frac{x_1}{y_1} - \frac{x_2}{y_2}) - g(x_1 + x_2) - d_2\Big). \end{cases} \tag{5.4.5}$$

We consider also the comparison systems defined in Λ_1:

$$\begin{cases} \dot{u}_1 = u_1\Big(f_1(s_{in} - m_1 - \frac{u_1}{y_1} - \frac{u_2}{y_2}) - g(u_1 + u_2) - d_1\Big), \\ \dot{u}_2 = u_2\Big(f_2(s_{in} - M_1 - \frac{u_1}{y_1} - \frac{u_2}{y_2}) - g(u_1 + u_2) - d_2\Big), \\ 0 < u_1(0) \le x_1(0) \quad \text{and} \quad u_2(0) \ge x_2(0) > 0. \end{cases} \tag{5.4.6}$$

$$
\left\{
\begin{array}{l}
\dot{v}_1 = v_1\Big(f_1(s_{in} - M_1 - \frac{v_1}{y_1} - \frac{v_2}{y_2}) - g(v_1+v_2) - d_1\Big),\\
\dot{v}_2 = v_2\Big(f_2(s_{in} - m_1 - \frac{v_1}{y_1} - \frac{v_2}{y_2}) - g(v_1+v_2) - d_2\Big),\\
v_1(0) \geq x_1(0) > 0 \quad \text{and} \quad 0 < v_2(0) \leq x_2(0).
\end{array}
\right.
\tag{5.4.7}
$$

Let (u_{1_1}, u_{2_1}) and (v_{1_1}, v_{1_2}) be the solutions of the competitive systems (5.4.6) and (5.4.7) respectively. Following the lines of the proof of Theorem 5.3.1, the comparison theorem for competitive system (see Th.C.2.1 in Appendix) implies:

(5.4.8)

$$u_{1_1}(t) \leq x_1(t) \leq v_{1_1}(t) \quad \text{and} \quad v_{2_1}(t) \leq x_2(t) \leq u_{2_1}(t) \quad \text{for any } t \geq 0.$$

As in the proof of Theorem 5.3.1, using the same average Lyapunov function $P(u_1, u_2)$, we can conclude that:

(5.4.9)

$$\lim_{t\to+\infty}(u_{1_1}(t), u_{2_1}(t)) = (u^*_{1_1}, u^*_{2_1}) \quad \text{and} \quad \lim_{t\to+\infty}(v_{1_1}(t), v_{2_1}(t)) = (v^*_{1_1}, v^*_{2_1}).$$

Putting $(u_1(0), u_2(0)) = (u^*_{1_1}, u^*_{2_1})$, $(v_1(0), v_2(0)) = (v^*_{1_1}, v^*_{2_1})$ and using Theorem C.2.1 and Eqs.(5.4.8)–(5.4.9), we conclude that the box B_1 is positively invariant. Now, if we suppose that $(x_1(0), x_2(0)) \notin B_1$. As in the proof of Theorem 5.3.1, we prove that B_1 is globally attractive. □

Let us emphasize that we can proceed recursively to improve the bounds and obtain a decreasing sequence of positively invariant and globally attractive sets K_i. This is the statement of the following lemma:

Lemma 5.4.2. *Under the assumptions and conditions of Theorem* 5.3.1, *there exists a decreasing sequence $K_j \subset K_{j-1} \subset \ldots \subset K_1$ of positively invariant and globally attractive sets defined by*:

$$K_j = \Big\{(s, x_1, x_2) \in K_{j-1} \colon -m_j \leq V \leq -M_j \quad \text{and} \quad (x_1, x_2) \in B_j\Big\}.$$

The sets B_j $(j = 2, 3, \ldots)$ are defined by:

$$B_j = [u^*_{1_j}, u^*_{2_j}] \times [v^*_{1_j}, v^*_{2_j}],$$

and the constants m_j and M_j are defined by:

$$m_j = \frac{\Big[\frac{d_1}{y_1}v^*_{1_{j-1}} + \frac{d_2}{y_2}u^*_{2_{j-1}}\Big]}{g(u^*_{1_{j-1}} + v^*_{2_{j-1}})} \quad \text{and} \quad M_j = \frac{\Big[\frac{d_1}{y_1}u^*_{1_{j-1}} + \frac{d_2}{y_2}v^*_{2_{j-1}}\Big]}{g(v^*_{1_{j-1}} + u^*_{2_{j-1}})}$$

where the numbers $u^*_{1_j}$,$u^*_{2_j}$,$v^*_{1_j}$ and $v^*_{2_j}$ are defined by Eqs.(5.2.6) *and* (4.5.1).

Proof. The proof is similar to the proof of Lemma 5.4.1. □

Remark 5.4.1. *An immediate consequence of Lemmas* 5.4.1 *and* 5.4.2 *is that the sequences $\{M_j\}$,$\{u^*_{1_j}\}$ and $\{v^*_{2_j}\}$ are monotone increasing; the sequences $\{m_j\}$,$\{v^*_{1_j}\}$ and $\{u^*_{2_j}\}$ are monotone decreasing; moreover the following inequalities are satisfied for any integer $j \in \{0, 1, \ldots\}$*:

$$M_j < m_j, \quad u^*_{1_j} < v^*_{1_j} \quad \text{and} \quad v^*_{2_j} < u^*_{2_j}.$$

LEMMA 5.4.3. *Under the same assumptions of Theorem* 5.3.1 *and in addition if there exists an integer* $j \in \{1, 2, \dots\}$ *such that the following inequalities hold for* $i = 1, 2$:

$$d_i \leq \frac{1}{\mu} \frac{f_i(s_j^-)}{s_{in} - s_j^-}, \quad \text{with} \quad \mu \geq 1/M_j \quad \text{and} \quad s_j^- = \min\{s \in K_j\} \tag{5.4.10}$$

then the critical point (s^*, x_1^*, x_2^*) *is a globally asymptotically stable solution of system* (5.2.5).

PROOF. We build the following Lyapunov–like functionals $W_\mu \colon K_j \mapsto \mathbb{R}$:

$$W_\mu = \underbrace{\int_{s^*}^{s} \frac{d\xi}{s_{in} - \xi}}_{W_1} - \mu \underbrace{\left(s - s^* + \sum_{i=1}^{2} \frac{x_i - x_i^*}{y_i}\right)}_{W_2} \quad \text{for any } \mu > 0. \tag{5.4.11}$$

It is clear that:

$$\dot{W}_\mu = g(x_1 + x_2)\Big(1 + \mu V\Big) + \sum_{i=1}^{2} \frac{x_i}{y_i}\Big(\mu d_i - \frac{f_i(s)}{s_{in} - s}\Big).$$

The following properties are elementary:

(i) W_μ and $\dot{W}_\mu$ are continuous and bounded in $[0, +\infty)$.

(ii) By inequality (5.4.10) and the bounds for V given by Lemma 5.4.2, it follows that $\dot{W}_\mu \leq 0$ for any $\mu \geq \frac{1}{M_j}$.

(iii) As $\lim_{t\to+\infty} W_\mu(t)$ exists and $\dot{W}_\mu(t)$ is uniformly continuous in $[0, +\infty)$, a result of Barbălat (see [**41**, Lemma 1.2.3]) implies that:

$$\lim_{t\to+\infty} \dot{W}_\mu(t) = 0 \quad \text{for any } \mu \geq \frac{1}{M_j}.$$

Let us define

$$E = \Big\{(s, x_1, x_2) \in \bar{K}_j \colon \dot{W}_\mu = 0\Big\}.$$

Let $\mathcal{M}$ be the union of all invariant solutions in E. Notice that $(s^*, x_1^*, x_2^*) \in \mathcal{M}$. By the LaSalle invariance principle, it follows that

$$\lim_{t\to+\infty} \big(s(t), x_1(t), x_2(t)\big) \in \mathcal{M}.$$

We will prove that $(s^*, x_1^*, x_2^*) = \mathcal{M}$. Firstly, we need to improve the characterisation of the sets E and $\mathcal{M}$. For this task we build the sets

$$E_1 = \Big\{(s, x_1, x_2) \in \bar{K}_j \colon \dot{W}_1 = 0\Big\} \quad \text{and} \quad E_2 = \Big\{(s, x_1, x_2) \in \bar{K}_j \colon \dot{W}_2 = 0\Big\}.$$

We shall prove that $E = E_1 \cup E_2$. Firstly, we suppose that $(s, x_1, x_2) \in E_1 \cup E_2$. Notice that the relation $\dot{W}_\mu = \dot{W}_1 - \mu \dot{W}_2$ implies that $E_1 \cup E_2 \subseteq E$. Now, to prove the opposite relation we suppose that $(s, x_1, x_2) \in E$. It is easy to see that for any couple of numbers $\lambda, \mu \geq \frac{1}{M_j}$ it follows that:

$$\lim_{t\to+\infty} \dot{W}_\lambda(t) - \dot{W}_\mu(t) = (\mu - \lambda) \lim_{t\to+\infty} \dot{W}_2(t) = 0$$

and this implies that $(s, x_1, x_2) \in E_2$. Combining this result with the property (iii) of the functional W_μ stated before, we have that $\lim_{t\to+\infty} \dot{W}_1(t) = 0$, hence $E \subseteq E_1 \cup E_2$ and consequently $E = E_1 \cup E_2$.

As a direct consequence of this equality, we have the following characterization for the set E:

$$E = \Big\{(s, x_1, x_2) \in \bar{K}_j \colon F_1(s, x_1, x_2) = F_2(s, x_1, x_2) = 0\Big\},$$

where F_1 and F_2 and are defined by:

$$\begin{aligned} F_1(s, x_1, x_2) &= \sum_{i=1}^{2} \frac{x_i}{y_i}[g(x_1 + x_2) - f_i(s)] - g(x_1 + x_2)V, \\ F_2(s, x_1, x_2) &= -g(x_1 + x_2)V - \sum_{i=1}^{2} \frac{x_i}{y_i} d_i. \end{aligned}$$

Notice that

$$\mathcal{M} \cap \Big\{(s, x_1, x_2) \in \bar{K}_j \colon s = s^*\Big\} = (s^*, x_1^*, x_2^*) \tag{5.4.12}$$

indeed, let $u_0 = \big(s^*, x_1(0), x_2(0)\big) \in \mathcal{M}$, the invariance of $\mathcal{M}$ combined with **(F3)** imply that $\big(x_1(0), x_2(0)\big)$ is a solution of the equations:

$$F_1(u_0) = g(x_1 + x_2)(s_{in} - s^*) - \sum_{i=1}^{2} \frac{x_i}{y_i} f_i(s^*) = 0,$$

$$F_1(u_0) - F_2(u_0) = [f_1(s^*) - d_1 - g(x_1 + x_2)] \underbrace{\sum_{i=1}^{2} \frac{x_i}{y_i}}_{>0} = 0.$$

Using **(F3)** combined with Remark 5.3.1 it is an easy exercise to show that the equality $(x_1(0), x_2(0)) = (x_1^*, x_2^*)$ holds and Eq.(5.4.12) follows.

To finish the proof, let $u_0 = \big(s(0), x_1(0), x_2(0)\big) \in \mathcal{M}$ with $s(0) \neq s^*$ and let ϕ_t be the solution of system (5.2.5) with initial condition u_0.

Without loss of generality we assume that $s(0) > s^*$. The invariance of the orbit implies that

$$\begin{aligned} F_1(\phi_t) &= g(x_1 + x_2)(s_{in} - s) - \sum_{i=1}^{2} \frac{x_i}{y_i} f_i(s) = 0, \\ F_1(\phi_t) - F_2(\phi_t) &= \sum_{i=1}^{2} \frac{x_i}{y_i} \underbrace{f_i(s) - g(x_1 + x_2) - d_i}_{=\lambda_i(\phi_t)} = 0 \end{aligned}$$

for any $t \geq 0$. Hence $F_1(\phi_t) = 0$ implies that $\dot{s}(t) = 0$ and consequently $s(t) = s(0)$ for any $t \geq 0$. Moreover $F_1(\phi_t) - F_2(\phi_t) = 0$ implies that

$$\sum_{i=1}^{2} \frac{x_i(t)}{y_i} \underbrace{[f_i(s_0) - d_i - g(x_1(t) + x_2(t))]}_{\lambda_i(\phi_t)} = 0.$$

Notice that $\lambda_i(t)$ $(i = 1, 2)$ cannot be equal to 0. Otherwise **(F3)** will imply that $s(0)$ is a solution of equation (5.1.2), obtaining a contradiction with $s(0) \neq s^*$.

By consequence, the functions λ_1 and λ_2 have a constant and opposite sign. Without loss of generality we assume that $\lambda_1 > 0 > \lambda_2$. Hence x_1 and x_2 are an increasing and a decreasing function respectively. This monotonicity of functions x_i combined with Lemma 5.4.2 imply that there exists a critical point $\tilde{E} \neq (s^*, x_1^*, x_2^*)$ such that $\lim_{t\to+\infty} \phi_t = \tilde{E}$ a contradiction with the uniqueness of (s^*, x_1^*, x_2^*). □

The end of the proof of Theorem 5.2.1 is now clear. Notice that Lemma 5.4.3 gives a sufficient condition for global asymptotic stability expressed in terms of an upper bound of mortality rate d_i. Secondly, it is straightforward to verify that:

$$s_j^- = s_{in} - m_j - \frac{v_{1_j}^*}{y_1} - \frac{u_{2_j}^*}{y_2}.$$

Finally, taking $\mu = 1/M_j$, Eq.(5.4.10) is equivalent to Eq.(5.2.9) and the Theorem is proved □.

5.5. Numerical Simulation

We consider the competition between two species of phytoplankton in a chemostat using nitrate as the limiting substrate. For numerical simulations we take Michaelis–Menten functions with realistic parameters (see *e.g.* [**11**],[**143**]) defined as follows

$$f_1(s) = \frac{0.9s}{0.01 + s} \quad \text{and} \quad f_2(s) = \frac{1.6s}{0.2 + s}.$$

We will consider the values $d_1 = 0.01$ and $d_2 = 0.05$ and initial condition $(s_0, x_{1_0}, x_{2_0}) = (0.4, 0.05, 0.45)$. Moreover, we consider the values $s_{in} = 0.6$, $y_1 = 2$ and $y_2 = 1$.

Notice that $s^* = 0.23495$ and let us build the feedback control law

$$g(x_1 + x_2) = c_1 + c_2 \frac{0.42(x_1 + x_2)}{0.2 + (x_1 + x_2)}, \quad c_i > 0.$$

Choosing parameters c_1 and c_2 so that assumptions **(G)** and the Eqs.(5.2.8)–(5.2.9) are satisfied, Theorem 5.2.1 implies that $(s(t), x_1(t), x_2(t))$ is convergent to (s^*, x_1^*, x_2^*) when $t \to +\infty$ (notice that x_1^* and x_2^* are dependent on c_1 and c_2). In the Figures 5.5.1,5.5.2 and 5.5.3 we show some numerical simulations obtained using the feedback control law defined above. Continuous lines are the concentration of biomass x_1 and dashed lines are the concentration of biomass x_2.

5.6. Discussion

In this chapter, we considered the model of two species competition in the chemostat –with a single growth limiting substrate– as a control system. We considered the total biomass as the only output available and choose the

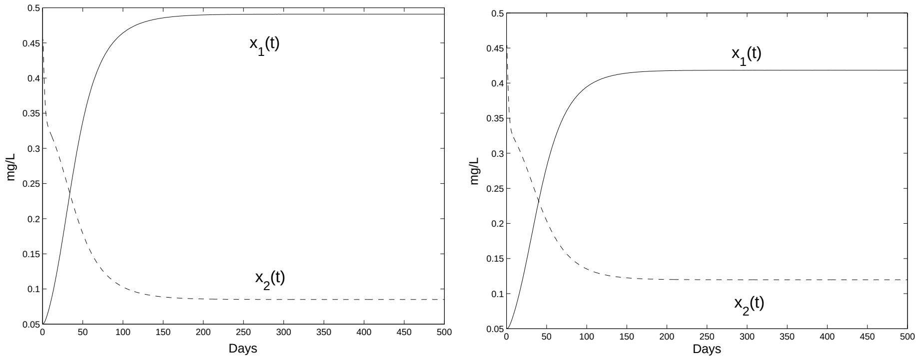


FIGURE 5.5.1. $(c_1, c_2) = (0.21, 2.075)$ (Left) and $(c_1, c_2) = (0.22, 2.08)$ (Right)

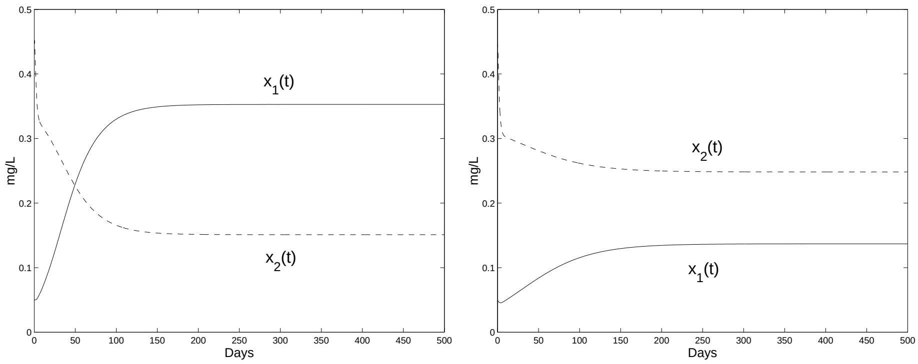


FIGURE 5.5.2. $(c_1, c_2) = (0.23, 2.085)$ (Left) and $(c_1, c_2) = (0.24, 2.09)$ (Right)

dilution rate as the feedback control variable. We built a feedback control law. Theorem 5.2.1 gave some sufficient conditions –summarized as upper bounds for d_i– that ensure (s^*, x_1^*, x_2^*) is a critical point that is globally asymptotically stable. This result contrasts with the model without control defined by Eq.(5.1.1) in the sense that control makes possible the coexistence between the two species.

Our result extends those in [**87**] in the sense that it deals with bounded mortality rates relaxing the assumption $d_i = 0$. This implies that the asymptotic behavior of the model can not be reduced to a two–dimensional system and therefore we must study the full system.

Despite this improvement, the bounds for mortality rates given by the Equations (5.2.8) and (5.2.9) are not optimal and this fact lead us to the question about the largest possible bound for d_i to avoid the competitive exclusion.

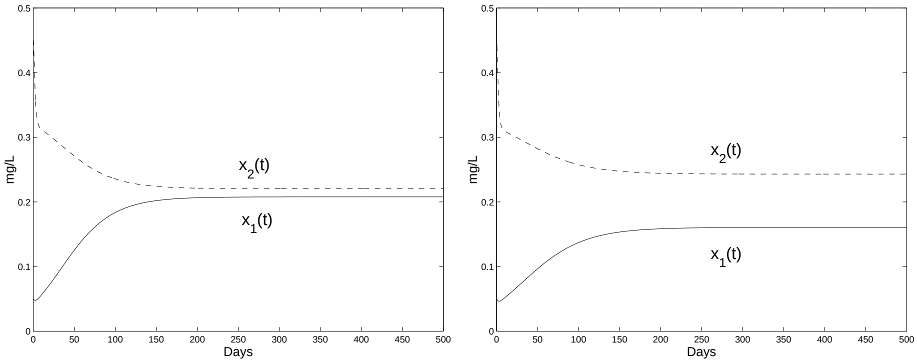


FIGURE 5.5.3. $(c_1, c_2) = (0.26, 2.085)$ (Left) and $(c_1, c_2) = (0.27, 2.09)$ (Right)

Several important issues and possibilities were left out of the present chapter. One possibility is the robustness: We wish to obtain the stability of a box $\tilde{K} \in \operatorname{Int} \mathbb{R}^3_+$ with error in the measurements and uncertainty in identification of growth functions f_i. This case can be solved by the same methods combined with alternative and more technical hypotheses.

Moreover, numerical simulations for the model studied in [**44**] suggest that this result can be extended to competition models with more general (nonmonotone) uptake functions. This however remains an unsolved problem, as our analysis makes essential use of monotony of f_i. We still do not know how to establish the results in the nonmonotone case by the methods of the present chapter.

Still another natural extension of our results would be to treat the case of delayed outputs $y(t) = x_1(t-\tau) + x_2(t-\tau)$ (with $\tau > 0$). In spite of the fact that delays in the measurements are generally small with respect to biological processes and consequently, they could not have impact on the stability of (s^*, x_1^*, x_2^*) (this idea is summarized as *small delays are harmless* [**41**],[**130**]), it would be extremely desirable to confirm these ideas by mathematical proofs.

Part 3

Open loop control for a trophic chain in the chemostat

CHAPTER 6

Preliminary results on attainable sets

This chapter is different from the previous ones, first because its nature is merely exploratory (we only have introductory results) and second because we shall focus our attention on the open–loop control for an n–trophic chain in the chemostat. For a complete presentation of the mathematical aspects of the open–loop control theory, we refer to Banks [**9**] and Hermes and Lasalle [**60**].

The study of trophic chains and other types of food webs (not from a point of view of control theory) has been a prolific research subject (see *e.g* section 1.3 and references therein). On the other hand, a rigorous study of open–loop control for a trophic chain has seldom been accomplished.

In this chapter we consider the trophic chain as a control system by regarding the input concentration of limiting substrate s_{in} as a control variable. We follow an approach based on ideas developed by Rao and Roxin in [**114**] who characterize the attainable and reachable sets of the control systems by constructing some *ideal systems* which don't take into account the equation corresponding to the substrate.

This chapter is organized as follows: In section 6.1 we recall some facts concerning an n–trophic chain. In section 6.2 we recall some results about the asymptotic behavior of a bi–trophic and tri–trophic chain. An introduction to the control problem is given in section 6.3. The main result, which characterizes the attainable and reachable set is given the section 6.4 and its proof in section 6.5.

6.1. Preliminaries

Let us recall the n–trophic chain equation in the chemostat:

$$
\begin{cases}
\dot{s} &= D(s_{in} - s) - \alpha_1 f_1(s) x_1, \\
\dot{x}_1 &= x_1 f_1(s) - (D + d_1) x_1 - \alpha_2 x_2 p_2(x_1), \\
\dot{x}_i &= x_i p_i(x_{i-1}) - (D + d_i) x_i - \alpha_{i+1} x_{i+1} p_{i+1}(x_i), \\
\dot{x}_n &= x_n p_n(x_{n-1}) - (D + d_n) x_n.
\end{cases}
\tag{6.1.1}
$$

where s denotes the concentration of nutrient, x_i denotes the biomass density of the species corresponding to the i–th trophic level ($i \in \mathbb{N}_n$). Also, x_1 is called the *prey* and x_n is called the *top–predator*. Variables x_i (for $i \in \{2, \ldots, n-1\}$) are called *intermediate predators*. The coefficients α_i, $d_i \geq 0$ for $i \in \mathbb{N}_n$ are yield constants and death rates respectively.

The continuous functions $f_1, p_i \colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ ($i \in \mathbb{N}_n$) satisfy the following properties:

(F1) $f_1(0) = 0$ and f_1 is bounded above by $||f_1||_\infty > D + d_1$.
(F2) f_1 is increasing.

An immediate consequence of properties **(F1)**–**(F2)** is that there exists a number $s^* > 0$ such that $f_1(s^*) = D + d_1$.

(P1) p_i are continuous, increasing, $p_i(0) = 0$ and $p_i'(0) > 0$.
(P2) p_i are bounded above and $||p_i||_\infty > D + d_i$.

For technical reasons, we will build the continuous auxiliary functions $q_i\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by:

$$q_i(x) = \frac{p_i(x)}{x}, \quad (i \in \mathbb{N}_n).$$

Besides, q_i satisfy the following properties:

(P3) $q_i(x) > 0$ for any $x > 0$.
(P4) q_i is unimodal or strictly decreasing, moreover $\lim\limits_{x\to+\infty} q_i(x) = 0$.

Notice that the Holling type functions described in the Introduction (see section 1.3) satisfy properties **(P1)**–**(P4)**. For instance, see Figures 6.1.1, 6.1.2 and 6.1.3.

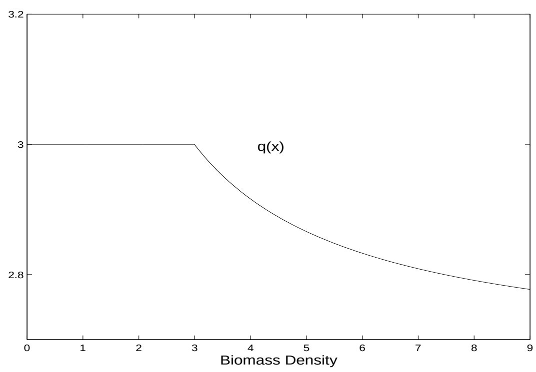


FIGURE 6.1.1. Graph of a function q related to a Holling type I function.

6.2. Asymptotic behavior of the model

For $n = 2$, the system (6.1.1) has been studied in [**23**],[**24**],[**74**],[**75**],[**83**] and [**93**] (see section 1.3.1 from the Introduction), considering f_1 as a Monod function and p_2 as a Holling type II or III function. In general for a bi–trophic chain, the asymptotic behavior can be described by "simple" limit sets that are equilibria and limit cycles. The stability of equilibria as well as the transition of an equilibrium to a limit cycle can be studied by considering

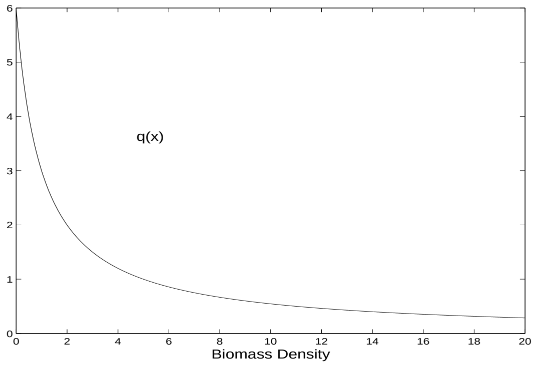


FIGURE 6.1.2. Graph of a function q related to a Holling type II function.

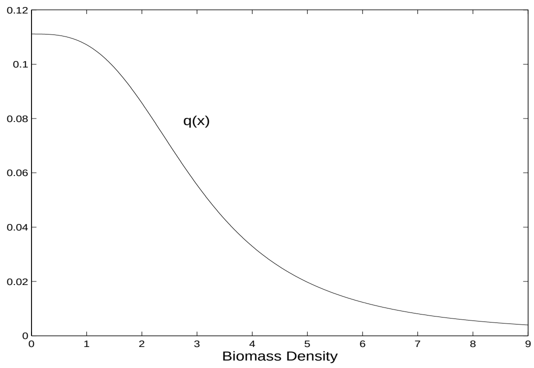


FIGURE 6.1.3. Graph of a function q related to a Holling type III function.

the parameters s_{in} and D. This implies that three regions in the plane s_{in}–D can be distinguished.

For $n = 3$, the system (6.1.1) has been studied in [**47**] and [**36**], and can be understood as the invasion of a bi–trophic chain by a top–predator. In general, the ω–limit set of the system is more complex than the previous case. In fact, by using a bifurcation analysis, it has been proved in [**47**] that for some pairs (s_{in}, D) there exist chaotic attractors (see section 1.4.1 from the Introduction).

For $n > 3$, a more general system has been studied in [**50**] with the suggestive title *Long chains are in general chaotic*. A very complete bifurcation analysis is carried out to prove the transition toward chaotic behavior.

6.3. Some control approaches

As we have seen before, the asymptotic behavior of system (6.1.1) is strongly determined by the parameters s_{in} and D. As a consequence, if we are interested in modifying the behavior the system (1.3.1), a natural idea could be to modify these parameters. Indeed, in the references [**74**], [**75**] and [**34**] some experimental results varying s_{in} and D are shown and it is proved that all types of asymptotic behaviors ($n = 2$) of the system (6.1.1) can be obtained.

Following this direction, Pavlou and Kevrekidis [**109**] and Kot, Sayler and Schultz [**80**] studied the asymptotic behavior of a forced bi–trophic chain by considering respectively D, s_{in} as periodic functions of type:

$$D(t) = \theta_0 + a\cos(\omega t), \quad \text{with } \theta_0 > 0,\ |a| < \theta_0 \text{ and } \omega > 0,$$

$$s_{in}(t) = s_{in}^0\Big[1 + \eta \sin\Big(\frac{2\pi}{T}t\Big)\Big], \quad \text{with } T > 0,\ s_{in}^0 > 0 \text{ and } |\eta| < 1.$$

The asymptotic behavior of these systems display almost–periodic functions, phase locking and chaotic attractors. Gragnani and Rinaldi [**48**] continue with this idea by considering a parameter of the function p_2 and obtaining results in the same direction. However, we must point out that the original motivation of these articles is to simulate the influence of seasonal variation rather than control theory.

Nevertheless, we also are interested in modifying the transient behavior of the systems and not only the asymptotic behavior. To emphasize this, it is shown in [**34**] that there exists a constant $M_{\mathfrak{S}} > 0$ such that the validity of the trophic chain systems is ensured only if $s(t) < M_{\mathfrak{S}}$ for any $t \geq 0$. Indeed, when this does not hold, the biomass begins to clump and adhere to the surface of the chemostat. A control strategy for theses systems –even in the transient– could be necessary and mathematically interesting. In this framework Lobry, Nival and Sciandra [**96**] (who work in a more general context) suggest taking s_{in} or D as a control variable of *Bang–Bang*[1] type and making a geometric study of the phase diagrams.

6.3.1. Some basic definitions. We collect some definitions and notations related to control theory to be used in our discussion.

DEFINITION 6.3.1. A system:

$$\begin{cases} \dot{z} = F(z, u(t)), \\ z \in \mathbb{R}^n, \quad u\colon \mathbb{R}_+ \mapsto \mathbb{R}^k \end{cases} \tag{6.3.1}$$

is *controllable* if for any pair z_0 and z_f, there exists a number $T > 0$ and a bounded and measurable function $u\colon [0,T] \mapsto \mathbb{R}^n$ such that the solutions of system (6.3.1) with initial condition $z(0) = z_0$ satisfy $z(T) = z_f$.

Moreover, we suppose that for any $t \geq 0$, the function $u\colon \mathbb{R}_+ \mapsto \mathbb{R}^k$ ($k \geq 1$) is in a set $\mathcal{U} \subset L^\infty([0,t], \mathbb{R})$ called the set of *admissible controls.*

Now, we give some definitions less restrictive than controllability:

[1] A *Bang-Bang* type control law is defined by a function $u\colon \mathbb{R} \mapsto [a_1, a_2]$ taking values only in the set $\{a_1, a_2\}$.

DEFINITION 6.3.2. Let z_0 and z_f be any points in $\mathbb{R}^n$. We say that:

(i) A point z_f is *reachable* from z_0 at a time $T > 0$ if there exists a control law $u\colon [0,T] \mapsto \mathbb{R}^n$ such that the solution of system (6.3.1) with initial condition $z(0) = z_0$ satisfy $z(T) = z_f$.
(ii) The *attainable set* $A(T, z_0, \mathcal{U}) \subset \mathbb{R}^n$ for system (6.3.1) is the set of points y satisfying $y = z(T)$.
(iii) The *reachable set* for the system (6.3.1) is given by:

$$R(z_0, \mathcal{U}) = \bigcup_{t \geq 0} A(t, z_0, \mathcal{U}).$$

REMARK 6.3.1. *Notice that the attainable sets of the system* (6.4.1) *satisfy the semigroup property:*

$$A\big(t_2, A(t_1, \tilde{z}_0, \mathcal{U}), \mathcal{U}\big) = A(t_1 + t_2, \tilde{z}_0, \mathcal{U})$$

for any t_1,$t_2 > 0$.

As we stated in the Introduction, attainable sets play a fundamental role in nonlinear control theory. They provide a starting point which enables the study of the behavior of a control system, also allowing us to focus on the important features without being distracted by details pertaining to a particular control function. For more details regarding properties of attainable sets we suggest [**8**],[**60**].

6.3.2. Open–loop control for a trophic chain. Now, we will study the system (6.1.1) in the framework given by Definitions (6.3.1) and (6.3.2). We state the following hypothesis:

Input hypothesis: s_{in} is the input variable. Moreover we suppose that $s_{in}(t) = u(t)$ where $u\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a positive, measurable and bounded function. Hence we can assume that $u \in L^\infty(\mathbb{R}_+, \mathbb{R}_+)$.

In consequence, we will study the following system:

$$\begin{cases} \dot{s} &= D(u(t) - s) - \alpha_1 f_1(s) x_1, \\ \dot{x}_1 &= x_1 f_1(s) - (D + d_1)x_1 - \alpha_2 x_2 p_2(x_1), \\ \dot{x}_i &= x_i p_i(x_{i-1}) - (D + d_i)x_i - \alpha_{i+1} x_{i+1} p_{i+1}(x_i), \\ \dot{x}_n &= x_n p_n(x_{n-1}) - (D + d_n)x_n. \end{cases} \tag{6.3.2}$$

For physical reasons (see the previous section and [**34**]), we will suppose that the set of admissible controls $u(t)$ for system (6.3.2) is given by the set of positive, bounded and measurable functions, that is

$$\mathcal{U} \subset \Big\{u \in L^\infty([0,t], \mathbb{R}_+)\colon 0 < u(r) \quad \text{for any } 0 \leq r \leq t\Big\}.$$

6.4. Main result

In this section we propose a characterization of the reachable set of system (6.3.2). We follow an idea developed by Rao and Roxin in [**114**] which consists in considering the *reduced* control system:

$$
\begin{cases}
\dot{\phi}_1 = \phi_1 f_1(\eta(t)) - (D + d_1)\phi_1 - \alpha_2\phi_2 p_2(\phi_1), \\
\dot{\phi}_i = \phi_i p_i(\phi_{i-1}) - (D + d_i)\phi_i - \alpha_{i+1}\phi_{i+1} p_{i+1}(\phi_i), \\
\dot{\phi}_n = \phi_n p_n(\phi_{n-1}) - (D + d_n)\phi_n, \\
\phi_i(0) = \phi_i^0 \geq 0 \quad i \in \mathbb{N}_n.
\end{cases} \tag{6.4.1}
$$

where $\eta\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a positive continuous function.

We will be able to deduce some properties of the attainable and reachable sets of control system (6.3.2) by studying the reduced system (6.4.1).

Before stating our main result, let us introduce some sets of functions which will be necessary to define admissible controls and attainable sets for system (6.4.1).

First, for any $t > 0$, we define the set $\mathcal{K}^t \subset C([0,t],\mathbb{R})$ as follows:

$$\mathcal{K}^t = \Big\{\eta \in C([0,t],\mathbb{R}_+)\colon \quad \frac{d}{dr}\Big(\eta(r)e^{Dr}\Big) \geq 0 \quad \text{for any } r \in [0,t]\Big\}.$$

Second, let s_0 and s_f be two points in $\mathbb{R}_+^2$. Now, we define the set

$$\mathcal{U}^t[s_0, s_f] = \Big\{\eta \in \mathcal{K}^t\colon \eta(0) = s_0 \quad \text{and} \quad \eta(t) = s_f\Big\}.$$

We are now in a position to state our main result:

THEOREM 6.4.1. *Let* $z_0 = (s_0, x_1^0, \ldots, x_n^0)$ *and* $\tilde{z}_0 = (x_1^0, \ldots, x_n^0)$ *be a pair of initial conditions of systems* (6.3.2) *and* (6.4.1) *respectively.*

(i) *For any* $s_f \in \mathbb{R}_+$ *there exists a number* $T(s_f) > 0$ *such that the attainable set* $A(T(s_f), z_0, \mathcal{U})$ *of system* (6.3.2) *satisfies the property*:

$$\{s_f\} \times A\big(T(s_f), \tilde{z}_0, \mathcal{U}^{T(s_f)}[s_0, s_f]\big) \subset A(T(s_f), z_0, \mathcal{U}) \tag{6.4.2}$$

where $A\big(T(s_f), \tilde{z}_0, \mathcal{U}^T[s_0, s_f]\big)$ *is the attainable set of system* (6.4.1).

(ii) *The reachable set* $R(z_0, \mathcal{U})$ *of system* (6.3.2) *statisfies the property*:

$$\bigcup_{s_f \in \mathbb{R}_+} \{s_f\} \times A\big(T(s_f), \tilde{z}_0, \mathcal{U}^{T(s_f)}[s_0, s_f]\big) \subset R(z_0, \mathcal{U}). \tag{6.4.3}$$

Theorem 6.4.1 deserves some comments: firstly, we can see that by using the admissible control family $\mathcal{U}$, the substrate can be controlled in the set $\mathbb{R}_+^*$.

Secondly, the reduced system (6.4.1) is a nonautonomous Kolmogorov system, positively invariant in $\mathbb{R}_+^n$. Hence a control strategy for system (6.4.1) must be considered in the framework given by nonlinear control for positive systems. Moreover, the positiveness and boundedness imposed on the admissible controls make our control problem far more difficult than in the general case.

The particular case $n = 1$ (equivalent to the simple chemostat studied in Part I) can be studied in–depth by using some ideas employed to prove Th.6.4.1, Indeed, we can prove that the simple chemostat (see Eq.(1.1.4) with $n = 1$ from Introduction) is controllable.

THEOREM 6.4.2. *For $n = 1$, system (6.3.2) is (positively) controllable.*

6.5. Proof of Main result

The plan for the proof of Theorem 6.4.1 is the following:

First, we will prove that there exists a number $T \geq 0$ such that $\mathcal{U}^T[s_0, s_f]$ is not empty. Second, we prove the Theorem 6.4.1 for $n = 2$. Finally, we generalize the result to the case of n–trophic levels.

We begin by the following lemma:

LEMMA 6.5.1. *For any pair $(s_0, s_f) \in \mathbb{R}_+^2$, there exists a number $T(s_f) > 0$ such that the set $\mathcal{U}^T[s_0, s_f]$ is not empty.*

PROOF. We choose a number $T > 0$ satisfying the inequality:

$$T > T^* = \max\left\{\frac{1}{D}\ln\left(\frac{s_0}{s_f}\right), 0\right\}. \tag{6.5.1}$$

Let $w > 0$ be a positive constant. Now, we build the continuous function $\eta\colon [0, T] \mapsto \mathbb{R}_+$ defined by:

$$\eta(t) = (s_0 - w)e^{-Dt} + w. \tag{6.5.2}$$

Notice that η satisfies two properties stated as follows:

$$\dot{\eta} = Dw - D\eta, \quad t \in [0, T], \tag{6.5.3}$$

$$\eta(T) = s_0 e^{-DT} + w\big(1 - e^{-DT}\big). \tag{6.5.4}$$

An immediate consequence of Eq.(6.5.3) is that

$$\frac{d}{dr}\Big(\eta(t)e^{Dt}\Big) \geq 0 \quad \text{for any } t \in [0, T]$$

which implies that $\eta \in \mathcal{K}^T$.

Notice that Eq.(6.5.1) implies that:

$$s_f - s_0 e^{-DT} > 0.$$

Now, we can choose $w = s_f - s_0 e^{-DT}$ and it follows that $\eta \in \mathcal{U}^T[s_0, s_f]$. $\square$

REMARK 6.5.1. *If we consider a piecewise constant function:*

$$w(t) = \left\{ \begin{array}{ll} w_1 & \text{if } t \in [t_1, t_2], \\ w_i & \text{if } t \in (t_i, t_{i+1}] \ (i = 2, \ldots, k-1), \end{array} \right. \tag{6.5.5}$$

where $w_j \in \mathbb{R}_+$ for any integer $j \in \mathbb{N}_{k-1}$ $(k \geq 2)$ and $t_k = T$, the proof follows as before.

Moreover, by using the fact that any measurable function can be considered as a limit of step functions (see for example [**119**]*), we can generalize the proof for positive and measurable functions $w(t)$.*

6.5.1. Proof of the main result for $n = 2$. We choose a function $\widetilde{w}(t) \in \mathcal{U}^{T(s_f)}$. Moreover, let (ϕ_1, ϕ_2) be a solution of system (6.4.1) with $n = 2$ an considering the function

$$\eta(t) = s_0 e^{-Dt} + De^{-Dt} \int_0^t e^{Dr} \tilde{w}(r)\, dr.$$

as input.

By using system (6.4.1) and Eq.(6.5.2), it can easily be proved that (η, ϕ_1, ϕ_2) is a solution of the system:

$$\begin{cases} \dot{\eta} \ = D\tilde{w}(t) - D\eta, \\ \dot{\psi}_1 = \psi_1 f_1(\eta) - (D + d_1)\psi_1 - \alpha_2 \psi_2 p_2(\psi_1), \\ \dot{\psi}_2 = \psi_2 p_2(\psi_1) - (D + d_2)\psi_2, \\ \eta(0) = s_0, \quad \psi_1(0) = \phi_1^0 \quad \text{and} \quad \psi_2(0) = \phi_2^0, \end{cases} \tag{6.5.6}$$

satisfying $\eta(T) = s_f$.

Now, let us build the control function $u \colon [0, T] \mapsto \mathbb{R}_+$ defined by:

$$u(t) = \widetilde{w}(t) + \frac{\alpha_1}{D} \phi_1(t) f(\eta(t)).$$

We can verify that $(u, \eta, \phi_1, \phi_2)$ is a solution of system (6.3.2).

Let us recall that $z_0 = (s_0, x_1^0, x_2^0)$ and $\tilde{z}_0 = (x_1^0, x_2^0)$. By using Lemma (6.5.1) we can easily verify that for any $s_f \in \mathbb{R}_+$ it follows that:

$$\{s_f\} \times A(T(s_f), \tilde{z}_0, \mathcal{U}^{T(s_f)}[s_0, s_f]) \subset A\big(T(s_f), z_0, \mathcal{U}\big)$$

and in consequence, statement (i) of Theorem is proved.

To prove statement (ii) of Theorem, we take the union for any $s_f \in \mathbb{R}_+$ obtaining the following property:

$$\bigcup_{s_f \in \mathbb{R}_+} \{s_f\} \times A\big(T(s_f), \tilde{z}_0, \mathcal{U}^{T(s_f)}[s_0, s_f]\big) \subset \bigcup_{s_f \in \mathbb{R}_+} A(T(s_f), z_0, \mathcal{U}).$$

Using Definition 6.3.2 we have that

$$\bigcup_{s_f \in \mathbb{R}_+} A(T(s_f), z_0, \mathcal{U}) \subseteq R(z_0, \mathcal{U}).$$

and Eq.(6.4.3) is verified and statement (ii) of the Theorem follows.

6.5.2. Proof for $n \geq 2$. Notice that in the proof for $n = 2$, we only work with some features of the function η defined in Eq.(6.5.2), mainly the fact that $\eta \in \mathcal{K}^t$ for any $t \geq 0$. Also the dimension of system (6.4.1) did not play any role in the proof.

Indeed, the generalization of the results given in section 6.5.1 can be proved almost word by word replacing system (6.5.6) by:

$$
\begin{cases}
\dot{\eta} &= Dw(t) - D\eta, \\
\dot{\psi}_1 &= \psi_1 f_1(\eta) - (D+d_1)\phi_1 - \alpha_2\psi_2 q_1(\psi_1), \\
\dot{\psi}_i &= \psi_i p(\psi_{i-1}) - (D+d_i)\psi_i - \alpha_{i+1}\psi_{i+1}p_{i+1}(\psi_i), \\
\dot{\psi}_n &= \psi_n p(\psi_{n-1}) - (D+d_n)\psi_n, \\
\eta(0) &= s_0, \quad \psi_i(0) = \phi_i^0 \quad i \in \mathbb{N}_n.
\end{cases} \tag{6.5.7}
$$

6.5.3. Proof of Theorem 6.4.2. Notice that for $n=1$, system (6.3.2) becomes system:

$$
\begin{cases}
\dot{s} &= D(u(t)-s) - \alpha_1 f_1(s)x_1, \\
\dot{x}_1 &= x_1 f_1(s) - (D+d_1)x_1.
\end{cases} \tag{6.5.8}
$$

Let $z_0 = (s_0, x_0)$ and $z_f = (s_f, x_f)$ be two points in $\mathbb{R}_+^2$. We will prove that there exist a finite time $T > 0$ and a control function $u \in \mathcal{U}$ such that the system (6.5.8) satisfies $\big(s(0), x(0)\big) = z_0$ and $\big(s(T), x(T)\big) = z_f$

Step 1: Following the lines of the proof of Theorem 6.4.1, we will prove that system:

$$
\begin{cases}
\dot{\eta} = D(\tilde{w}(t) - \eta), \\
\dot{\psi} = \psi f_1(\eta) - (D+d_1)\psi.
\end{cases} \tag{6.5.9}
$$

is controllable.

Indeed, given two points $z_0 = (s_0, x_0)$ and $z_f = (s_f, x_f)$ we will prove that there exists a finite time $T > 0$ and a measurable function $\tilde{w}\colon [0,T] \mapsto \mathbb{R}_+$ such that system (6.5.9) with initial condition $(\eta(0), \psi(0)) = z_0$ and $\tilde{w}$ as input satisfy $(\eta(T), \psi(T)) = x_f$.

Together with system (6.5.9), we will consider system:

$$
\begin{cases}
\dot{\chi} = -D(w - \chi), \\
\dot{\xi} = -\xi f_1(\chi) + (D+d_1)\xi, \\
\chi(0) = s_f \quad \text{and} \quad \xi(0) = x_f.
\end{cases} \tag{6.5.10}
$$

which describes the solution in reversed time of system (6.5.9) with positive constants inputs $w > 0$ and initial condition z_f.The solutions of this last system will be useful to build an explicit control law $\tilde{w}$ for system (6.5.9).

We will study the solutions of system (6.5.10) by considering the following cases on s_f:

Case (a) $f_1(s_f) > D + d_1$ (See Figure 6.5.1).
Case (b) $f_1(s_f) < D + d_1$ and $w > s_f$ (See Figure 6.5.2).

In **Case (a)**, properties **(F1)**–**(F2)** imply that $s^* < s^f$. Now, let us define by $Z(t, z_f, w^+)$ the solution of system (6.5.10) with $w^+ = s^f + \delta$ $(\delta > 0)$ as input.

Notice that:

$$\chi(t) = -\delta e^{Dt} + s^f + \delta, \quad \text{for any } t \geq 0,$$

which implies the existence of two numbers $0 < \tau_0 < \tau_1$ defined as follows:

$$\tau_0 = \frac{1}{D} \ln\Big(1 + \frac{s^f - s^*}{\delta}\Big) \quad \text{and} \quad \tau_1 = \frac{1}{D} \ln\Big(1 + \frac{s^f}{\delta}\Big)$$

satisfying $\chi(\tau_0) = s^*$ and $\chi(\tau_1) = 0$.

By using the fact that the line $\chi = s^*$ is a null–cline of system (6.5.10), it can be proved that the graph defined by $Z(t, z_f, w^+)$ in the plane (χ, ξ), lies inside the set

$$Z^+ = \big\{(\chi, \xi) \in \mathbb{R}^2_+ \colon \chi \in [0, s_f] \quad \text{and} \quad \xi \in [\xi(\tau_0), +\infty)\big\}$$

for any $t \in [0, \tau_1]$.

On the other hand, let us define by $Z(t, z_f, w^-)$ the solution of system (6.5.10) with $w^- = s^f - \delta$ $(0 < \delta < s^f)$ as input.

Notice that:

$$\chi(t) = \delta e^{Dt} + s^f - \delta$$

which implies that $\chi(t) > s^f$ for any $t \geq 0$ and $\lim\limits_{t\to+\infty} \chi(t) = +\infty$. By using these properties of $\chi(t)$ combined with $s^f > s^*$ we can deduce that ξ is strictly decreasing and $\lim\limits_{t\to+\infty} \xi(t) = 0$.

In consequence, for any finite time $t \geq 0$, the graph defined by $Z(t, z_f, w^-)$ lies inside the set:

$$Z^- = \big\{(\chi, \xi) \in \mathbb{R}^2_+ \colon \chi \in [s_f, +\infty) \quad \text{and} \quad \xi \in [0, x^f]\big\}.$$

We will denote by Γ_1 the curve described by $Z(t, z_f, w^+)$ for $t \in [0, \tau_1]$ and $Z(t, z_f, w^-)$ for any $t \geq 0$ and let $L = Z(\tau_0, z_f, w^+)$ be a point of Γ_1 (see Figure 6.5.1).

Now, we shall build a control function $\tilde{w}$ to prove that system (6.5.9) is controllable.

Without loss of generality we can suppose that $z_0 <_{K_{(0,0)}} L$ (see Appendix C). Indeed, otherwise we can use the function $\tilde{w} = 0$ and in a finite time T_0 we have that $\big(\eta(T_0), \xi(T_0)\big) <_{K_{(0,0)}} L$.

Let us define by $X\big(t, (\eta_0, \psi_0), w_1\big)$ the solution of system (6.5.9) with input $w_1 = s_f + \delta$ $(\delta > 0)$ and initial condition (η_0, ψ_0). It is easy to prove that there exists a finite time $t_1 > 0$ such that $X\big(t_1, (\eta_0, \psi_0), w_1\big) \in \Gamma_1$ (see Figure 6.5.1). Without loss of generality we will suppose that $X\big(t_1, (\eta_0, \psi_0), w_1\big) \in \partial Z^-$.

Now, it is straightforward to prove that there exists a finite time $T > t_1$ such that the solutions of system (6.5.9) with input:

$$w(t) = \begin{cases} s^f + \delta & \text{if } t \in [0, t_1], \\ s^f - \delta & \text{if } t \in (t_1, T], \end{cases}$$

satisfy $X\big(T, (\eta_0, \psi_0), w_1\big) = (s_f, x_f)$.

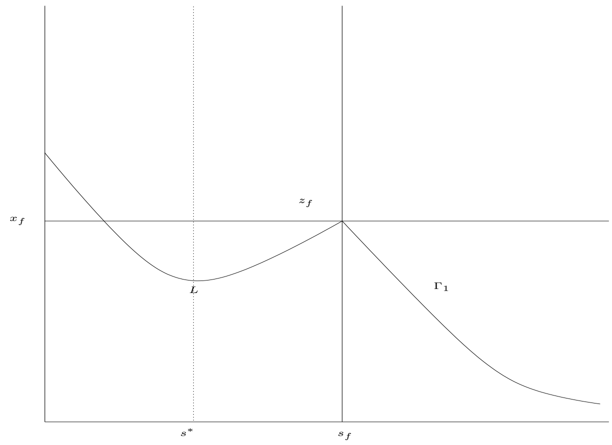


FIGURE 6.5.1. Case (a): From a point of view given by geometrical properties of solutions of systems (6.5.9) and (6.5.10), horizontal axis represents the values of variables η and χ and vertical axis represents the values of variables ξ and ψ. For any initial condition $z_0 <_{K_{(0,0)}} L$ of system (6.5.9), there exists a finite time $t_1 > 0$ such that $X(t_1, z_0, w_1) \in \Gamma_1$.

In **Case (b)**, properties **(F1)**–**(F2)** imply that $s^* > s^f$. Now, let us define by $Z(t, z_f, w^-)$ the solution of system (6.5.10) with $w^- = s^f - \delta$ $(0 < \delta < s^f)$ as input.

As before, we can see that:

$$\chi(t) = \delta e^{Dt} + s^f - \delta$$

and we can prove that $\chi(t) > s^f$ for any $t \geq 0$ and that there exist a finite time $\tau_0 > 0$ such that $\chi(\tau_0) = s^*$. Using the fact that the line $\chi = s^*$ is a null–cline of system (6.5.10) and $\lim_{t\to+\infty} \chi(t) = +\infty$ we obtain that $\lim_{t\to+\infty} \xi(t) = 0$.

In consequence, we have that the graph defined by $Z(t, z_f, w^-)$ in the plane (χ, ξ), lies inside the set

$$Z = \big\{(\chi, \xi) \in \mathbb{R}^2_+ : \chi \in [s_f, +\infty) \quad \text{and} \quad \xi \in [0, \xi(\tau_0))\big\}$$

for any $t \geq 0$.

We will denote by Γ_2 the curve described by $Z(t, z_f, w^-)$ for any $t > 0$ (see Figure 6.5.2). Moreover, let us define by $X\big(t, z_0, w_1\big)$ the solution of system (6.5.9) with input $w_1 = s^* + \delta$ $(\delta > 0)$ and initial condition (η_0, ψ_0).

Notice that the components of $X\big(t, z_0, w_1\big)$ satisfy:

$$\lim_{t\to+\infty} \eta(t) = s^* + \delta \quad \text{and} \quad \lim_{t\to+\infty} \psi(t) = +\infty.$$

In consequence, it is easy to prove that there exists a finite time $t_1 > 0$ such that $X\big(t_1, z_0, w_1\big) \in \Gamma_2$ (see Figure 6.5.2).

Now, we shall build a control function $\tilde{w}$ to prove that system (6.5.9) is controllable. Without loss of generality we suppose that $z_0 <_{K_{(0,0)}} z_f$.

Notice that there exist a finite time $T > t_1$ such that the solutions of system (6.5.9) with input:

$$w(t) = \begin{cases} s^* + \delta & \text{if } t \in [0, t_1], \\ s^f - \delta & \text{if } t \in (t_1, T], \end{cases}$$

verifies $X\big(T, (\eta_0, \psi_0), w_1\big) = (s_f, x_f)$ and in consequence, the system (6.5.9) is controllable.

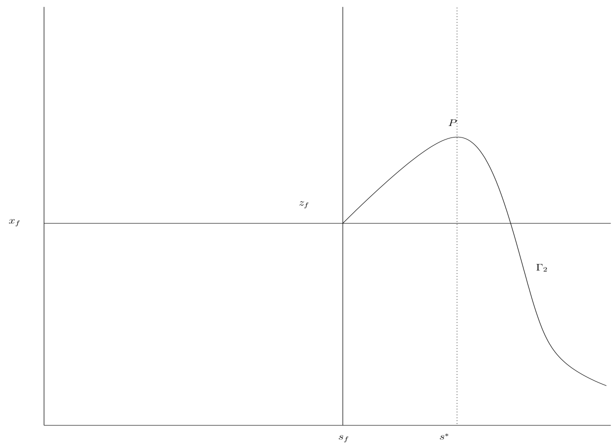


FIGURE 6.5.2. **Case (b)**: From a point of view given by geometrical properties of solutions of systems (6.5.9) and (6.5.10), horizontal axis represents the values of variables η and χ and vertical axis represents the values of variables ξ and ψ. For any initial condition $z_0 <_{K_{(0,0)}} z^f$ of system (6.5.9), there exists a finite time $t_1 > 0$ such that $X(t_1, z_0, w_1) \in \Gamma_2$.

Step 2: Notice that any solution (η, ψ) of system (6.5.10) with initial condition (s_0, x_0) is a solution of system (6.5.8) with input

$$u(t) = \tilde{w}(t) + \frac{\alpha}{D}\psi(t) f(\eta(t))$$

and we can conclude that the system (6.5.10) is controllable.

REMARK 6.5.2. *Notice that the open–loop control proposed for the simple chemostat model can have at most two switches.*

6.6. Discussion and future work

This chapter is the first step in our attempt at studying the controllability properties of a trophic chain by reducing the system (6.3.2) to (6.4.1).

In consequence, the study of attainability properties (with admissible controls in $\mathcal{U}$) for systems of type:

$$
\begin{cases}
\dot{\phi}_1 = \phi_1\big\{f_1(\eta(t)) - (D+d_1)\phi_1 - \alpha_2\phi_2 q_2(\phi_1)\big\}, \\
\dot{\phi}_i = \phi_i\big\{p_i(\phi_{i-1}) - (D+d_i)\phi_i - \alpha_{i+1}\phi_{i+1}q_{i+1}(\phi_i)\big\}, \\
\dot{\phi}_n = \phi_n\big\{p_n(\phi_{n-1}) - (D+d_n)\phi_n\big\}, \\
\phi_i(0) = \phi_i^0 \geq 0 \quad i \in \mathbb{N}_n.
\end{cases} \tag{6.6.1}
$$

will play a key role in the study of attainability properties for an n–trophic chain.

Much work has been done on nonlinear control theory for systems described by ordinary differential equations. In spite of this, the control for nonlinear systems like (6.6.1) which are positive (*i.e.* its state and control variables are positive in value at all times) remains a wide field of research.

Part 4

Conclusion et Perspectives

Dans cette thèse nous nous sommes intéressés à différents problèmes liés à la commande des processus biologiques dans le chemostat.

Dans la **Partie I** de cette thèse, nous avons étudié un problème de commande robuste en boucle fermée pour un chemostat simple (un substrat et une espèce) avec une connaissance imparfaite du modèle et avec la prise en compte des imprécisions et des retards dans la sortie.

Par ailleurs, dans la section 1.7 de l'introduction, nous considérons un modèle du chemostat parfaitement connu et une sortie du type $y = s(t)$ (c'est–à–dire, sans imprécisions et sans retards). Nous construisons une famille de boucles de rétroaction et dans le corollaire 1.7.1 on démontre que celle–ci stabilise asymptotiquement la sortie à la valeur s^*.

Dans les chapitres 2 et 3 nous démontrons que la famille de lois de commande décrite dans l'introduction s'avère encore efficace pour des imprécisions et des retards au-dessous d'une certaine borne supérieure. Néanmoins, il faut ajouter des propriétés supplémentaires à cette famille de lois de commande, notamment certaines propriétés sur sa dérivabilité.

D'un autre partie, bien que la motivation originale de cette partie était la commande robuste d'un chemostat simple, nous avons obtenu quelques résultats valables par eux–mêmes, notamment:

- Des conditions suffisantes pour la stabilité globale asymptotique de la solution $u(t) = 0$ de l'équation différentielle à retard:
$$u'(t) = -g(u(t-1)) + f(u(t)),$$
quand f et $g \in C(\mathbb{R}^3, \mathbb{R}^3)$ vérifient $f(0) = g(0) = 0$ et quelques propriétés de monotonie.
- Des conjectures concernant la stabilité globale asymptotique de la solution triviale et la vitesse de convergence vers celle-ci.
- Une définition explicite de la dérivée Schwarzienne d'une différence des fonctions $f_1, f_2 \in C^3(\mathbb{R}, \mathbb{R})$ définie par $\chi(r) = f_1(r) - f_2(r)$ et exprimée en fonction de Sf_1 et Sf_2 (voir Annexe D).

Dans la **Partie II** de cette thèse, nous avons étudié un problème de commande en boucle fermée pour un modèle de compétition entre deux espèces sur un substrat qui –sous certaines conditions suffisantes– permet la coexistence. Il faut souligner que en l'absence de commande, le comportement asymptotique de cette modèle est décrit par le principe d'exclusion compétitive.

Nous généralisons le Théorème 1 de P. De Leenheer et H. Smith dans deux directions:

Dans le Chapitre 4, nous considérons des fonctions de croissance et consommation f_i $(i = 1, 2)$ plus générales que le cas monotone croissant.

La prise en compte de la mortalité est considérée dans le Chapitre 5.

Dans ces deux chapitres, nous avons construit une famille de boucles de rétroaction qui stabilisent le système autour d'un point intérieur et globalement asymptotiquement stable. Néanmoins, il faut ajouter des propriétés additionnelles sur la dérivabilité de ces lois de commande.

La **Partie III** a un caractère exploratoire et est consacrée à la commande en boucle ouverte d'une chaîne trophique dans le chemostat. Nous avons proposé une caractérisation de l'ensemble atteignable de cette chaîne en suivant une idée proposé par N.Rao et E.Roxin (voir l'article [**114**]).

Nos résultats sont encore partiels et nous espérons qu'ils seront la base d'études ultérieures; la méthode employée pour caractériser l'ensemble atteignable, quant à elle, nous semble avoir un caractère de généralité que nous avons souligné et une souplesse qui lui permet d'appréhender sans grandes modifications des chaînes n–trophiques généralisées.

Conclusion générale: vers la commande des systèmes dissipatifs et positifs

Bien que dans le parcours de la thèse on a proposé diverses stratégies de commande, nous voulons souligner que celles–ci partagent quelques caractéristiques similaires que nous énoncerons maintenant:

(1) Les modèles sont décrits par des systèmes d'équations différentielles ordinaires ou à retard. Les systèmes dynamiques définies par les solutions de celles–ci sont (sauf dans la partie III) *dissipatifs*, c'est–à–dire, il existe un ensemble compact dans lequel toutes les orbites du système convergent à l'infini.

De plus, les systèmes dynamiques sont *positifs*, c'est–à–dire, toutes les variables du système présentent des valeurs positives.

Finalement, les variables de commande considérées sont des paramètres positifs de ces systèmes.

(2) Dans le cas de commande en boucle fermée, le problème de commande se traduit par la recherche de conditions suffisantes d'attractivité globale d'un polytope (Chapitre 2) ou d'un point d'équilibre du système résultant (Chapitre 3, Chapitre 4 et Chapitre 5). Dans tous les cas on construit des systèmes de comparaison et/ou systèmes limites (dans le contexte fourni par la théorie des systèmes dynamiques asymptotiquement autonomes) qui ne prennent pas en compte l'équation du substrat.

Les outils et techniques employées pour démontrer l'attractivité d'un ensemble (tant point critique que polytope) dans les chapitres 3 et 5 suivent la méthode suivante:

(i) On démontre des propriétés de dissipativité des systèmes dynamiques continus issus du problème de commande. C'est–à–dire, que l'on démontre l'existence d'un ensemble $\mathcal{K}$ compact et globalement attractif.

(ii) On construit un système dynamique discret qui hérite de certaines propriétés asymptotiques du système précédent. Le domaine de définition du système discret est construit en fonction des bornes de l'ensemble $\mathcal{K}$ désignée ci–dessus.

(iii) On étude le comportement asymptotique du système dynamique discret. Dans le chapitre 3 on utilise la dérivée Schwarzienne d'une différence de fonctions (nous proposons une définition explicite dans l'annexe D) et dans le chapitre 5 on utilise les propriétés de monotonie du système discret.

Perspectives

A partir du présent travail, plusieurs perspectives sont envisageables:

◦ **Sur la commande robuste d'un chemostat simple:** Les résultats présentées dans le Chapitre 2 ne considèrent que des imprécisions du type déterministe. Une étude comparative avec l'approche stochastique (c'est–à–dire, en supposant que les imprécisions satisfassent certaines propriétés statistiques) est souhaitable ainsi qui avec d'autres approches pour la commande des systèmes incertaines.

◦ **Sur la commande des systèmes à retard:**

1. Tout d'abord, il serait intéressant de faire un étude plus approfondie de l'équation différentielle à retard (3.5.7) présentée dans le Chapitre 3. Nous avons souligné que les conditions suffisantes de stabilité globale asymptotique de la solution nulle pouvaient être améliorées.
2. Une étude de l'équation (3.5.7) avec une perturbation bornée est envisagée. Des comparaisons avec les méthodes qui font l'utilisation des Théorèmes inverses de Lyapunov (cf. [**106**],[**103**]) sont envisagées.
3. Finalement, dans les applications aux problèmes de dépollution et culture de phytoplancton il apparaît l'importance d'obtenir des résultats sur la vitesse de convergence vers la solution nulle de l'équation (3.5.7).

◦ **Sur la commande d'un modèle de compétition:**

1. La méthode employée dans les chapitres 4 et 5 considère l'inégalité $\alpha_2 > \alpha_1$. Néanmoins, la méthode n'est pas généralisable pour le cas inverse. En effet, dans le cas $\alpha_2 \leq \alpha_1$, la famille des boucles de rétroaction ne permet que une stabilisation locale autour d'un point d'équilibre intérieur mais n'assure pas la persistance uniforme.

   Une première extension de cette partie pourrait donc aborder la commande d'un modèle de compétition dans le chemostat en considérant l'inégalité $\alpha_2 \leq \alpha_1$.
2. La méthode employée dans le Chapitre 5 ne considère que des fonctions f_i croissantes. Une deuxième extension de cette partie pourrait donc être d'étendre la méthode pour des fonctions non croissantes.
3. Finalement, le développement des lois de commande en boucle fermée pour un modèle de compétition entre $n \geq 3$ espèces reste un problème ouvert (cf.[**86**]).

○ **Sur la commande d'une chaîne trophique:** Certains aspects géométriques de la démonstration du Théorème 6.4.2 (notamment l'utilisation du système réduit en temps inversé) s'avèrent des outils très efficaces pour l'étude d'une chaîne bi–trophique. Une extension du résultat du Théorème 6.4.2 pour $n = 2$ est envisagée.

Enfin, il serait très intéressant de développer ce genre d'outils pour des réseaux trophiques plus complexes (voir par exemple [**148**]).

Part 5

Appendix

APPENDIX A

Uniform Persistence

A.1. Some results about flows

Let (X, d) be a complete metric space. A *flow* on X is an application $\phi\colon \mathbb{R} \times X \mapsto X$ that satisfies the following properties[1]:

ϕ is a continuous function,
$\phi(0, x) = x$ for any $x \in X$,
$\phi(t, \phi(s, x)) = \phi(t + s, x)$ for any $x \in X$ and $s, t \in \mathbb{R}_+$.

Remark A.1.1. Notice that if we replace $\mathbb{R}$ by $\mathbb{R}_+$, ϕ defines a *semiflow* on X.

Definition A.1.1. A set $M \subseteq X$ is:

- *Invariant* under the flow ϕ if $\phi_t(m) \in M$ for any $m \in M$ and $t \in \mathbb{R}$.
- *Positively invariant* under the semiflow ϕ if $\phi_t(m) \in M$ for any $m \in M$ and $t \in \mathbb{R}_+$.

Definition A.1.2. Let $x_0 \in X$. The ω–limit of x_0 is defined as:

$$\omega(x_0) = \Big\{ y\colon \exists t_n \to +\infty, \quad \phi_{t_n}(x_0) \to y, \quad \text{when } n \to +\infty \Big\}.$$

Definition A.1.3. Let $x_0 \in X$. The α–limit of x_0 is defined as:

$$\alpha(x_0) = \Big\{ z\colon \exists s_n \to -\infty, \quad \phi_{s_n}(x_0) \to z, \quad \text{when } n \to +\infty \Big\}.$$

A.2. Average Lyapunov functions

Definition A.2.1. Let M be a compact and positively invariant subset of a compact metric space (X, d). M is said to be a *repeller* if $\omega(x_0) \notin M$ for any $x_0 \in X$. That means that there exists a compact set $K \subset \operatorname{Int} X \setminus M$ such that for any $x_0 \in X$, it follows that $\phi_t(x_0) \in K$ for t sufficiently large.

Definition A.2.2. An *Average Lyapunov function* $P\colon X \mapsto \mathbb{R}$ is a continuous function on X, differentiable along orbits, satisfying $P(x) = 0$ for $x \in M$ and $P(x) > 0$ for $x \in X \setminus M$.

[1] We will use the notation $\phi_t(x) = \phi(t, x)$

A.3. Uniform persistence criteria

THEOREM A.3.1. *Let $P\colon X \mapsto \mathbb{R}$ be an average Lyapunov function. If there is a continuous function $\Psi\colon X \mapsto \mathbb{R}$, such that $\dot{P}(x) \geq P(x)\Psi(x)$ for all $x \in X$, and for each $x \in M$ there is a time $T > 0$ such that*

$$\int_0^T \Psi(\phi_t)\, dt > 0 \tag{A.3.1}$$

then M is a repeller.

PROOF. See for example [**62**, Th.1],[**63**, Corollary 2] □

COROLLARY A.3.1. *Let $P\colon X \mapsto \mathbb{R}$ be an average Lyapunov function. If there is a continuous function $\Psi\colon X \mapsto \mathbb{R}$, such that $\dot{P}(x) \geq P(x)\Psi(x)$ for all $x \in X$, then the condition* (A.3.1) *can be replaced by*

$$\Psi(E_i) > 0 \quad \text{for any } i \in \mathbb{N}_k. \tag{A.3.2}$$

APPENDIX B

Some reduction techniques

We show some results that relate the asymptotic behavior of a system Σ of ordinary (delay) differential equations with the asymptotic behavior of a simpler system Σ_∞ that *inherits* some asymptotic properties of Σ.

B.1. Global stability results

We present a theorem inspired by a result of Viel, Busvelle and Gauthier [**146**],[**147**].

THEOREM B.1.1. *Let us consider the nonlinear system*:

$$\begin{cases} \dot{z} = f(z,v), \\ \dot{v} = -h(z)v, \\ (z(0),v(0)) = (z_0,v_0) \in \Omega \subset \mathbb{R}_+^{n-1} \times \mathbb{R}_+. \end{cases} \tag{B.1.1}$$

where Ω is an open and bounded set ant the bounded functions $f\colon \mathbb{R}_+^{n-1} \times \mathbb{R}_+ \mapsto \mathbb{R}_+^{n-1}$ and $h : \mathbb{R}_+ \mapsto \mathbb{R}_+^*$ are such that global existence and uniqueness are ensured. Assume that:

- **(H1)** *The semiflow defined by the solutions of system* (B.1.1) *is positively invariant in Ω.*
- **(H2)** *There exists only one critical point $E^* = (z^*, 0)$ that is locally asymptotically stable and at most a finite number of other hyperbolic critical points.*
- **(H3)** *The point $z^* \in \mathbb{R}^{n-1}$ is a globally asymptotically stable equilibrium point for the nonlinear system*:

$$\begin{cases} \dot{z} = f(z,0), \\ z(0) = z_0 \in \operatorname{Int} \mathbb{R}_+^{n-1}. \end{cases} \tag{B.1.2}$$

Then, E^* is a globally attractive point for the system (B.1.1).

PROOF. Let $\vec{u}_0 = (z_0, v_0) \in \Omega$ be an initial condition of system (B.1.1). Let $\phi_t(\vec{u}_0)$ be the semiflow defined by the system.

Notice that boundedness and positiveness properties of h imply that there exist a positive constant ρ such that the component $v(t)$ of system (B.1.1) satisfy:

$$|v(t)| = |v(0)| \exp\Big(- \int_0^t h(z(r))\,dr \Big) \leq |v(0)| e^{-\rho t}.$$

Let $(\bar{z}, \bar{v}) \in \omega(\vec{u}_0)$. By using the last inequality, it is straightforward to verify that $\bar{v} = 0$.

Let us define $\vec{u}_1 = (\bar{z}, 0) \in \Omega$. Using the fact that $\omega(\vec{u}_0)$ is an invariant set, we obtain that $\phi_t(\vec{u}_1) \in \omega(\vec{u}_0)$. By using **(H3)**, it follows that $\phi_t(\vec{u}_1)$ converges towards $(z^*, 0)$ when $t \to +\infty$ and in consequence $E^* \in \omega(\vec{u}_0)$.

By property **(H2)**, we known that E^* is a locally asymptotically stable critical point of system (B.1.1). Now, from the above ω–limit study, it follows that $\phi_t(\vec{u}_0)$ enters the basin of attraction of E^* in a finite time and the result follows. □

REMARK B.1.1. The result given by Viel *et.al* assumes that the right part of the second equation in (B.1.1) is not dependent of z. Nevertheless, the result can be proved assuming the local stability of E^* using a similar proof.

B.2. Asymptotically autonomous dynamical systems

DEFINITION B.2.1 (Markus (1956), [**100**],[**104**],[**139**],[**140**],[**141**]). Consider the following systems of differential equations:

$$\dot{x} = f(t, x), \tag{B.2.1}$$

$$\dot{y} = g(y), \tag{B.2.2}$$

where the functions $f\colon \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ and $g\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ are such that global existence and uniqueness properties are satisfied.

The system (B.2.1) is called *asymptotically autonomous* with *limit system* (B.2.2) if

$$\lim_{t \to +\infty} f(t, x) = g(x)$$

for any compact subset of $\mathbb{R}^n$.

By Definition B.2.1, it follows that through each point $x_1 \in \mathbb{R}^n$ and initial time $t_1 > t_0$ there is a unique solution $x(t)$ of system (B.2.1) which is defined for a maximal interval $t_0 \leq t_1 \leq \tau_+ \leq +\infty$. Now, we can define the ω–limit for a solution of system (B.2.1):

$$\omega(x_1, t_1) = \bigcap_{r < \tau_+} \mathrm{Cl}\Big(\bigcup_{t_0 < r < t} x(t) \Big). \tag{B.2.3}$$

The following result relates the ω–limit of a forward bounded solution x of (B.2.1) with the ω–limit of a forward bounded solution y of (B.2.2) for $n = 2$.

THEOREM B.2.1 (Poincaré–Bendixson Trichotomy [**100**],[**139**],[**140**],[**141**]). *Let $n = 2$ and ω be the ω–limit of a forward bounded solution x of* (B.2.1). *Assume that there exists a neighborhood of ω which contains at most finitely many equilibria of* (B.2.2). *Then the following trichotomy holds:*

(i) *ω consists of an equilibrium of* (B.2.2).

(ii) *ω is the union of periodic orbits of* (B.2.2) *and possibly of centers of* (B.2.2) *surrounded by periodic orbits of* (B.2.2) *lying in ω.*
(iii) *ω contains equilibria of* (B.2.2) *that are cyclically chained to each other in ω by orbits of* (B.2.2).

PROOF. For statements (i) and (ii), see Theorem 7 from [**100**] and for statement (iii) see for example [**139**]. □

REMARK B.2.1. *The statement* (ii) *is about the possible existence of more than a unique limit cycle in the ω–limit of system* (B.2.1). *To illustrate this fact, we show an example given by Markus in* [**100**, Pag.22] *where the system* (*in polar coordinates*)

$$\begin{cases} \dot{r} = \frac{1}{t}\sin\big(\ln(t)\big), \\ \dot{\theta} = 1. \end{cases} \tag{B.2.4}$$

is asymptotically autonomous with limit

$$\begin{cases} \dot{r} = 0, \\ \dot{\theta} = 1. \end{cases} \tag{B.2.5}$$

It is straightforward to prove that system (B.2.5) *has solutions of concentric circles about the origin. On the other hand, the solution $\big(r(t),\theta(t)\big) = \big(2-\cos(\ln(t)),t\big)$ is a solution of system* (B.2.4) *with initial conditions* $r(t_0) = 1$,$\theta(t_0) = 1$ and $t_0 = 1$. Using Eq.(B.2.3), it can be proved that the ω–limit of system B.2.4 is the annulus $1 \le r \le 3$.

Let (X,d) be a metric space. We consider a mapping $\Phi\colon \Delta\times\mathcal{F}\mapsto\mathcal{F}$ where $\Delta = \{(t,s)\colon 0\le s\le t<+\infty\}$.

DEFINITION B.2.2. This map Φ is called a *nonautonomous semiflow* if it satisfies the the following properties:

(i) Φ is a continuous mapping,
(ii) $\Phi(t,s,\Phi(s,r,x)) = \Phi(t,r,x)$, for $t\ge s\ge r\ge 0$,
(iii) $\Phi(s,s,x) = x$, for $s\ge 0$.

For more information about nonautonomous dynamical systems see the references [**104**],[**132**],[**139**],[**142**] and [**153**, Ch.3].

DEFINITION B.2.3. A nonautonomous semiflow Φ on X is called *asymptotically autonomous* with *limit semiflow* Θ, if

(i) Θ is an autonomous semiflow on X,
(ii) For any three sequences $t_j\to t$,$s_j\to+\infty$, $x_j\to x$,$j\to+\infty$, with x,$x_j\in X$, $0\le t,t_j<+\infty$ and $s_j>t_0$ it follows that:

$$\Phi(t_j+s_j,s_j,x_j)\to\Theta(t,x),\quad j\to+\infty.$$

In the following we will assume that:
(B1) The semiflow Φ is asymptotically autonomous with limit semiflow Θ.
(B2) If the orbit (with respect to semiflow Θ) of $x\in X$ is a precompact set,

then the set $\omega_\Theta(x)$ (the ω–limit set –with respect to semiflow Θ– of $x \in X$) contains a Θ–equilibrium.

Definition B.2.4 (Thieme,[**139**]). Let E_i and E_j a be couple (not necessarily $i \neq j$) of critical points of the semiflow Θ.

(i) The point E_i is said to be Θ–*chained* to E_j ($E_i \mapsto E_j$), if there exists a point $y \notin X \setminus (E_i \cup E_j)$ such that $\alpha_\Theta(y) \subseteq E_i$ and $\omega_\Theta(y) \subseteq E_j$.

(ii) A finite number of critical points $E_1, \ldots, E_k$ ($k \geq 1$), is called Θ–*cyclical chain* in X if the following holds: In case that $k > 1$, $E_j \mapsto E_{j+1}$, $j \in \mathbb{N}_{k-1}$, and $E_k \mapsto E_1$ is chained. If $k = 1$, $E_1 \mapsto E_1$.

Proposition 7. (Thieme,[**139**, Theorem 4.2, and Corollary 4.3]) *Let* **(B1)–(B2)** *hold and assume that there is no Θ–cyclical chain of Θ–equilibria. Then any precompact forward Φ–orbit converges toward a Θ–equilibrium for $t \to +\infty$.*

Let $\tau \geq 0$ and consider the following differential delay equations

$$\dot{x}(t) = \mu(t, x_t), \quad x_{t_0} = \varphi \in C = C([-\tau, 0], \mathbb{R}), \tag{B.2.6}$$

$$\dot{y}(t) = g(y_t), \quad y_0 = \varphi \in C = C([-\tau, 0], \mathbb{R}). \tag{B.2.7}$$

We will assume that μ and g verify the following assumptions:
(B3) $\mu\colon \mathbb{R}_+ \times C \mapsto C$ is continuous and $\mu(t, \varphi)$ is Lipschitzian in φ in each compact subset in $\mathbb{R}_+ \times C$ and completely continuous; that is takes closed and bounded sets of $\mathbb{R} \times C$ into bounded set of $\mathbb{R}^n$.
(B4) $g\colon C \mapsto C$ is Lipschitzian in each compact set of C.

Proposition 8. *Let us denote by $\Phi(t, t_0, \varphi)$ a bounded solution $x(t, t_0, \varphi)$ of Eq.*(B.2.6) *and denote by $\Theta(t, \varphi)$ a bounded solution $y(t, \varphi)$ the solution of Eq.*(B.2.7)*.*

(a) *The map $\Phi\colon \Delta \times C \mapsto C$ defines a nonautonomous semiflow in the sense of Def.*(B.2.3)*.*

(b) *If for every compact subset $K \subset C$ there is a neighborhood V of K such that*
$$\mu(t, \varphi) \to g(\varphi), \quad t \to +\infty$$
uniformly for $\varphi \in V$, then the semiflow $\Phi(t, r, \varphi) = s_r(s, \varphi)$ is asymptotically autonomous with limit semiflow $\Theta(t, \varphi) = y_t(0, \varphi)$.

Proof. Notice that by **(B3)-(B4)** and Theorems 2.1 and 2.3 from [**53**], it follows that the solutions of Eq.(B.2.6) and (B.2.7) exist and are unique. Moreover, by Theorem 3.2 from [**53**] we have that the maximal solutions of these two equations are defined in $[t_0 - \tau, +\infty)$ and $[-\tau, +\infty)$ respectively.
Part (a): Properties (i) and (iii) of Def.(B.2.3) are straightforward. The property (ii) is a consequence of uniqueness of the solutions of Eq.(B.2.6).
Part (b): Suppose that there exists a three sequences $\varphi_j \to \varphi$, $t_j \to t_0$ and $s_j \to +\infty$ as $j \to +\infty$. Let $T > 0$ be such that $t_j \leq T$, $j \geq 0$, and define

$$u_j(t) = \Phi(t + s_j, s_j, \varphi_j) \quad \text{and} \quad u(t) = \Theta(t, \varphi)$$

for $0 \leq t \leq T$. Notice that $u_j(t)$ satisfies the equation

$$x'(t) = F_j(t, x_t) = \mu(t + s_j, x) - g(x_t) + g(x_t), \quad x_0(s_j) = \varphi_j.$$

By the assumptions stated before, we have that for any compact set $K \subset C$ there exists a neighborhood V of K such that $||F_j(t, x_t) - g(x_t)|| \to 0$ uniformly in V.

Using the theorem of continuous dependence (see [**53**, Th.2.2]), we obtain that $u_j(t) \to u(t)$ uniformly on $[0, T]$, as $u(t)$ is the solution of Eq.(B.2.7) this convergence is equivalent to

$$\Phi(t + s_j, s_j, \varphi_j) \to \Theta(t, \varphi), \quad \text{as } j \to +\infty \text{ and } t \in [0, T]$$

using the fact that $t_j \to t_0$ as $j \to +\infty$ and $t_j < T$, we have that $\Phi(t, r, \varphi) = s_r(s, \varphi)$ is asymptotically autonomous with limit semiflow $\Theta(t, \varphi) = y_t(0, \varphi)$.

□

APPENDIX C

Planar Monotone Systems

C.1. Cooperative systems and competitive systems

Let $F\colon \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}^n$, where Ω is a convex subset of $\mathbb{R}^n$, be a vector valued function and consider the system of differential equations:

$$\dot{x} = F(t, x). \tag{C.1.1}$$

Moreover, we will suppose that F satisfy properties ensuring global existence, and uniqueness of the solutions.

Definition C.1.1. The system (C.1.1) is:

- *Cooperative* if and only if the function F is such that:

$$\frac{\partial F_i}{\partial x_j} \geq 0 \quad \text{for any } i \neq j \in \mathbb{N}_n,\ t \in \mathbb{R}^n \text{ and } x \in \mathbb{R}^n.$$

- *Competitive* if and only if the function F is such that:

$$\frac{\partial F_i}{\partial x_j} \leq 0 \quad \text{for any } i \neq j \in \mathbb{N}_n,\ t \in \mathbb{R}^n \text{ and } x \in \mathbb{R}^n.$$

In some literature, cooperative systems are called as *Quasimonotone systems*, see for example [**121**],[**138**].

Denote by $K_{(0,0)}$ and $K_{(0,1)}$ the convex cones defined by:

$$K_{(0,0)} = \mathbb{R}^2_+ \quad \text{and} \quad K_{(0,1)} = \Big\{(u_1, u_2) \in \mathbb{R}^2 \colon u_1 \geq 0 \quad \text{and} \quad u_2 \leq 0\Big\}$$

and define an order in $\mathbb{R}^2$ by $\vec{y} \leq_K \leq \vec{x}$ if $\vec{x} - \vec{y} \in K$, for $K = K_{(0,0)}$ or $K = K_{(0,1)}$.

Definition C.1.2 (Kamke Condition). Let $F\colon \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}^2$ be a continuous function where Ω is an open and convex set in $\mathbb{R}^2$. $F = (F_1, F_2)$ is said to be of type K if for each i it follows that

$$(-1)^{m_i} F_i(\vec{a}) < (-1)^{m_i} F_i(\vec{b}) \quad \text{where } (m_1, m_2) = (0, 1) \quad \vec{a} \leq_K \vec{b} \quad \text{and} \quad a_i = b_i$$

Lemma C.1.1. *If a planar autonomous system is cooperative* (*competitive*)*, then it satisfies the Kamke condition with respect to the cone* $K_{(0,0)}$ ($K_{0,1}$).

C.2. Some comparison and asymptotic results

The goal is to study the asymptotic behavior of the cooperative (competitive) system (C.1.1 and to compare its solutions with these ones of the following systems of differential equations:

$$\dot{z} = G(t, z), \tag{C.2.1}$$

$$\dot{y} = H(t, y). \tag{C.2.2}$$

Such that the continuous functions G, $H\colon \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}^2$ verify $H \leq_K F \leq_K G$.

Theorem C.2.1. ([**130**, Th. 3.2.1],[**131**, Lemma 3.2]) *Assume that system* (C.1.1) *is cooperative* (*competitive*)*. Moreover, let $x(t)$ be a solution of* (C.1.1) *defined on $[a, b]$, hence*:

(i) *If the system* (C.1.1) *is autonomous and the orbit of each initial condition of it is bounded then $b = +\infty$ and every solution of* (C.1.1) *is convergent to a critical point and is monotone after a finite time.*
(ii) *If $z(t)$ is a continuous function on (a, b) satisfying* (C.2.1) *on (a, b) with $z(a) \leq_K x(a)$, then $z(t) \leq_K x(t)$ for all $t \in [a, b]$.*
(iii) *If $y(t)$ is a continuous function on (a, b) satisfying* (C.2.2) *on (a, b) with $y(a) \geq_K x(a)$, then $y(t) \geq_K x(t)$ for all $t \in [a, b]$.*

APPENDIX D

Schwarz Derivative

D.1. Some properties

DEFINITION D.1.1. The *Pre–Schwarz derivative* and the *Schwarz derivative* of a $C^3(\mathbb{R},\mathbb{R})$ function $F\colon \mathbb{R}\mapsto\mathbb{R}$ are defined respectively by:

$$(PF)(r)=\frac{F''(r)}{F'(r)} \quad\text{and}\quad (SF)(r)=\frac{F'''(r)}{F'(r)}-\frac{3}{2}\left(\frac{F''(r)}{F'(r)}\right)^2$$

for any $r\in\mathbb{R}$ such that $F'(r)\neq 0$

LEMMA D.1.1. *If F and G are $C(\mathbb{R},\mathbb{R})$ functions, then*
(i) *$P(F\circ G)$ and $S(F\circ G)(r)$ are defined by*:

$$P(F\circ G)(r)=(PF)(G(r))\{G'(r)\}+(PG)(r),$$

$$S(F\circ G)(r)=(SF)(G(r))\{G'(r)\}^2+(SG)(r).$$

(ii) *Let $\chi=F_1-F_2$, hence $(S\chi)(r)$ is defined by*:

$$(S\chi)=\frac{1}{(\chi')^2}\left\{(SF_1)F_1'\chi'-(SF_2)F_2'\chi'-\frac{3}{2}F_1'F_2'\left[\frac{F_1''}{F_1'}-\frac{F_2''}{F_2'}\right]^2\right\}.$$

PROOF. Property (i) can be checked by direct computation. Now, let $\chi(r)=F_1(r)-F_2(r)$, by definition we have that:

$$\begin{aligned}(S\chi)&=\frac{1}{(\chi')^2}\left\{\big(F_1'''-F_2'''\big)\big(F_1'-F_2'\big)-\frac{3}{2}\big(F_1''-F_2''\big)^2\right\}\\&=\frac{1}{(\chi')^2}\left\{F_1'''F_1'-F_1'''F_2'-F_2'''F_1'+F_2'''F_2'-\frac{3}{2}\Big[(F_1'')^2-2F_1''F_2''+(F_2'')^2\Big]\right\}\\&=\frac{1}{(\chi')^2}\left\{\underbrace{F_1'''F_1'-\frac{3}{2}(F_1'')^2}_{\text{(I)}}+\underbrace{F_2'''F_2'-\frac{3}{2}(F_2'')^2}_{\text{(II)}}-\underbrace{\big[F_1'''F_2'+F_2'''F_1'\big]}_{\text{(III)}}+3F_1''F_2''\right\}.\end{aligned}$$

We will study the expressions (I),(II) and (III). Notice that the expressions (I) and (II) are respectively $(i=1,2)$ equivalent to:

$$F_i'''(r)F_i'(r)-\frac{3}{2}\big(F_i''(r)\big)^2=\Big[F_i'''(r)F_i'(r)-\frac{3}{2}\big(F_i''(r)\big)^2\Big]\Big(\frac{F_i'(r)}{F_i'(r)}\Big)^2=(SF_i)(r)[F_i'(r)]^2,$$

Finally, notice that the expression (III) is equivalent to

$$F_1'''F_2' + F_2'''F_1' = F_1'''F_2'\Big[\frac{F_1'}{F_1'}\Big] + F_2'''F_1'\Big[\frac{F_2'}{F_2'}\Big] = F_1'F_2'\Big\{\frac{F_1'''}{F_1'} + \frac{F_2'''}{F_2'}\Big\}$$
$$= F_1'F_2'\Big\{SF_1 + SF_2 + \frac{3}{2}\Big[\Big(\frac{F_1''}{F_1'}\Big)^2 + \Big(\frac{F_2''}{F_2'}\Big)^2\Big]\Big\}$$

Hence, replacing (I),(II) and (III) we obtain that

$$(S\chi) = \frac{1}{(\chi')^2}\left\{(SF_1)[F_1']^2+(SF_2)[F_2']^2-F_1'F_2'(SF_1+SF_2)-\frac{3}{2}F_1'F_2'\Big[\Big(\frac{F_1''}{F_1'}\Big)^2+\Big(\frac{F_2''}{F_2'}\Big)^2\Big]+3F_1''F_2''\right\}$$
$$= \frac{1}{(\chi')^2}\left\{(SF_1)[F_1'](\chi')-(SF_2)[F_2'](\chi')-\underbrace{\left(\frac{3}{2}F_1'F_2'\Big[\Big(\frac{F_1''}{F_1'}\Big)^2 + \Big(\frac{F_2''}{F_2'}\Big)^2\Big] - 3F_1''F_2''\right)}_{\text{(IV)}}\right\}.$$

Notice that the expression (IV) is equivalent to

$$\frac{3}{2}F_1'F_2'\Big[\Big(\frac{F_1''}{F_1'}\Big)^2+\Big(\frac{F_2''}{F_2'}\Big)^2\Big]-3F_1''F_2'' = \frac{3}{2}F_1'F_2'\Big[\Big(\frac{F_1''}{F_1'}\Big)^2+\Big(\frac{F_2''}{F_1'}\Big)^2\Big]-2\frac{3}{2}\frac{F_1''F_2''}{F_1'F_2'}F_1'F_2'$$
$$= \frac{3}{2}F_1'F_2'\left[\Big(\frac{F_1''}{F_1'}\Big)^2 - 2\frac{F_1''}{F_1'}\frac{F_2''}{F_2'} + \Big(\frac{F_2''}{F_2'}\Big)^2\right]$$
$$= \frac{3}{2}F_1'F_2'\left[\Big(\frac{F_1''}{F_1'}\Big) - \Big(\frac{F_2''}{F_2'}\Big)\right]^2.$$

and property follows. □

The following results are a slight modification of Lemma 2.6 from [**127**] and Proposition 3.3 from [**126**] where it is assumed that $\chi \in C^3(\mathbb{R},\mathbb{R})$.

LEMMA D.1.2. *Let $\chi\colon [\alpha,\beta] \mapsto [\alpha,\beta]$ be a C^2 function with third derivative continuous, unless a finite set of points $I_n = \{a_1,\dots,a_n\} \subset (\alpha,\beta)$ such that $(S\chi)(r) < 0$ for any $r \in [\alpha,\beta] \setminus I_n$. If $\alpha < \gamma < \beta$ are consecutive fixed points of some iteration $\psi = \chi \circ \ldots \circ \chi$ of χ and $[\alpha,\beta]$ contains no critical point of ψ, then $\psi'(\gamma) > 1$.*

PROOF. By using the fact that ψ don't have critical points in $[\alpha,\beta]$ we conclude that ψ is monotone, this fact combined with the property that ψ has more than one fixed point in $[\alpha,\beta]$ implies that $\psi'(r) > 0$ on $[\alpha,\beta]$, moreover there exist u and v with $\alpha < u < \gamma < v < \beta$ such that $\psi'(u) = \psi'(v) = 1$ and $\psi'(r) \neq 1$ for any $r \in (u,v)$.

Now, we will prove that the inequality $\psi'(r) < 1$ for any $r \in (u,v)$ cannot be possible. Indeed, otherwise there exist a local minimum value $r_c \in (u,v)$ such that $\psi'(r_c) > 0$, $\psi''(r_c) = 0$ and $\psi'''(r_c) > 0$ wich implies $(S\psi)(r_c) > 0$.

Nevertheless, statement (i) of Lemma D.1.1 with $F = G = \chi$ implies that $(S\psi)(r_c) < 0$, a contradiction.

In consequence, we have that $\psi'(r) > 1$ for any $r \in (u,v)$ and the Lemma follows. □

Proposition 9. *Let* $\chi\colon [\alpha,\beta] \mapsto [\alpha,\beta]$ (*with* $0 \in [\alpha,\beta]$) *be a* C^2 *function with third derivative continuous, unless a finite set of points* $\{a_1,\dots,a_n\} \subset [\alpha,\beta]$ *and decreasing map with a unique fixed point* 0. *If* 0 *is locally asymptotically stable,* $g''(a_j) \neq 0$ *for any integer* $j \in \mathbb{N}_n$ *and the Schwarz derivative* $(S\chi)(r) < 0$ *for all* r*, then* 0 *is a global attractor of* χ.

Proof. Let W be connected component of the open set $S = \big\{r \in [\alpha,\beta]\colon \lim\limits_{k\to+\infty} \chi^k = 0\big\}$ which contains 0. Clearly $\chi(W) \subset W$. If $W \neq [\alpha,\beta]$, then we have $W = (l^-, l^+) \subset (\alpha,\beta)$; *i.e.* 0 is a local attractor.

As $l^- \notin W$, it is straightforward to prove that $\lim\limits_{\varepsilon\to 0^-} \chi(l^- + \varepsilon) \leq l^+ \leq \chi(l^-)$ which implies $\chi(l^-) = l^+$. By the same arguments, it follows $\chi(l^+) = l^-$. Thus $l^- < 0 < l^+$ are consecutive fixed points of $\psi = \chi\circ\chi$ and $\psi'(0) \leq 1$.

If χ does not have a critical point, we obtain a contradiction with Lemma D.1.2. If χ has a critical point r^*, Lemma D.1.2 implies that $r^* \in (l^-, l^+)$. Without loss of generality, we will suppose that χ has a local minimum at $r^* \in (0, l^+)$, hence it follows that $\chi(l^-) \leq \chi(r^*) > l^+$, a contradiction.

Hence $W = [\alpha,\beta]$, and therefore $\{0\}$ attracts each point of $[\alpha,\beta]$. This implies that 0 is the global attractor of χ.

□

Proposition 10. ([**94**, Lemma 2.1]) *Let* $g\colon \mathbb{R} \mapsto \mathbb{R}$ *be a* C^3 *function satisfying the properties*:
(a) $rg(r) > 0$ *for any* $r \neq 0$, $g'(0) > 0$ *and* $g''(0) < 0$,
(b) g *is bounded above and can have at most one critical point* r^* *which is a local maximum,*
(c) $(Sg)(r) < 0$ *for any* $r \in \mathbb{R}$ *such that* $g'(r) \neq 0$.

Hence there exists a function $R\colon [2g'(0)g''(0)^{-1}, +\infty] \mapsto \mathbb{R}$ *defined by*:

$$R(r) = \frac{2g'(0)^2 r}{2g'(0) - g''(0)r}$$

such that we have $R(r) > g(r)$ *for any* $r > 0$ *and* $R(r) < g(r)$ *for any* $r \in (2g'(0)g''(0)^{-1}, 0)$. *Moreover, it follows that* $R'(0) = g'(0)$ *and* $R''(0) = g''(0)$.

APPENDIX E

Some Lemmas

E.1. Proof of Theorem 2

Proof. We replace D in the system (1.7.1) by the control law (1.7.3). Hence the system becomes:

$$\begin{cases} \dot{s} = h(s^* - s)(s_{in} - s) - \alpha f(s)x, \\ \dot{x} = x\big[f(s) - h(s^* - s) - d\big], \\ (s(0), x(0)) = (s_0, x_0) \in \mathbb{R}^2_+. \end{cases} \tag{E.1.1}$$

It is only a straightforward task to verify that the critical points of the system (E.1.1) are:

$$E_0 = (s_{in}, 0) \quad \text{et} \quad E_1 = (s^*, x^*) \quad \text{avec } x^* = \alpha^{-1}[s_{in} - s^*]\frac{h(0)}{f(s^*)}.$$

By **(H2)** it follows that E_1 is a locally stable critical point and that E_0 is a saddle point with stable manifold:

$$W^s(E_0) = \Big\{(s, x) \in \mathbb{R}^2_+ : x = 0 \quad \text{et} \quad s > 0\Big\}.$$

Let point out an important property of the critical point E_0:

$$E_0 \notin \omega(\vec{\varphi}_0) \tag{E.1.2}$$

for any initial condition $\vec{\varphi}_0 = (s_0, x_0) \in \operatorname{Int} \mathbb{R}^2_+$.

Indeed, we build the functional $P \colon \mathbb{R}^2_+ \mapsto \mathbb{R}$ defined by $P(s, x) = x$.

It is straightforward to verify that P is an average Lyapunov functional (see Definition A.2.2 in Appendix A) and its derivative over the solutions of system (E.1.1) is:

$$\dot{P} = \Psi(s, x)P,$$

where the continuous function $\Psi \colon \mathbb{R}^2_+ \mapsto \mathbb{R}$ is defined by:

$$\Psi(s, x) = f(s) - h(s^* - s) - d.$$

By hypothesis **(H1)–(H2)** it follows that $\Psi(E_0) > 0$ and by using this fact combined with Theorem A.3.1 and Corollary A.3.1 from Appendix A we can deduce (E.1.2).

Now, by using the transformation $(s,x) \mapsto (s,v)$ where $v = s + \alpha x - s_{in}$ we can deduce that the system (E.1.1) is equivalent to system:

$$
\text{(E.1.3)} \qquad \begin{cases} \dot{s} = [h(s^* - s) - f(s)](s_{in} - s) - vf(s) = F_1(s,v), \\ \dot{v} = -h(s^* - s)v - d(v - s + s_{in}) \qquad = F_2(s,v), \\ (s(0), v(0)) = (s_0, v_0) \in \Omega. \end{cases}
$$

where the set Ω is defined by:

$$
\Omega = \Big\{ (s,v) \in \mathbb{R}_+ \times \mathbb{R} \colon v + s_{in} \geq s \Big\}.
$$

It is straightforward to verify that the system (E.1.3) is positively invariant in Ω and has the critical points:

$$
\widetilde{E}_0 = (s_{in}, 0) \quad \text{et} \quad \widetilde{E}_1 = (s^*, v^*) \quad \text{avec} \quad v^* = [s^* - s_{in}] \frac{d}{f(s^*)}.
$$

We will prove the asymptotic global stability of $\widetilde{E}_1$, which implies asymptotic global stability of critical point E_1 of system (E.1.1).

By **(H2)**, it follows that $\widetilde{E}_1$ is locally asymptotically stable and $\widetilde{E}_0$ is a saddle point with stable manifold:

$$
W^s(\widetilde{E}_0) = \Big\{ (s,v) \in \Omega \colon v + s_{in} - s = 0 \Big\} \subset \partial\Omega.
$$

Firstly , we will prove that solutions of system (E.1.3) are bounded; indeed boundedness of function h –ensured by **(H1)**– implies the existence of a constant $\rho > 0$ such that $v(t)$ satisfy the differential inequality:

$$
\dot{v}(t) \leq -\rho v(t),
$$

then we have the inequality: $v(t) \leq v(0)e^{-\rho t}$ for any $t \geq 0$ and consequently we have $\limsup\limits_{t \to +\infty} v(t) \leq 0$. Moreover, this last inequality implies $\limsup\limits_{t \to +\infty} s(t) \leq s_{in}$.

Notice that the set:

$$
\Omega_0 = \Big\{ (s,v) \in \Omega \colon v < 0 \quad \text{et} \quad s \leq s_{in} \Big\}
$$

is globally attractive and positively invariant. Hence without loss of generality, we will work only with initial conditions in the set Ω_0.

Applying the Poincaré–Bendixson's theorem we have that the ω–limit set of any initial condition in Ω may be:

(a) A critical point.

(b) A closed orbit in $\operatorname{Int} \Omega_0$ which contains the critical point $\widetilde{E}_1$ in its interior.

(c) A closed chain of critical points.

We will prove that the two last cases are not possible. Indeed, by property (E.1.2) combined with local stability of critical point $\widetilde{E}_1$ it follows that case (c) is not possible.

Now, we will prove that there does not exists any closed orbit:

Indeed, otherwise we can suppose the existence of an orbit parametrized by $\vec{\psi}(t)$ which contains the point $\widetilde{E}_1$

We build the comparison system:

$$\text{(E.1.4)} \qquad \begin{cases} \dot{\eta} = [h(s^* - \eta) - f(\eta)](s_{in} - \eta) - \rho f(\eta) = G_1(\eta, \rho), \\ \dot{\rho} = -h(s^* - \eta)\rho \qquad\qquad\qquad\qquad\qquad\quad = G_2(\eta, \rho), \\ (\eta(0), \rho(0)) = (s^*, 0) \in \Omega_0. \end{cases}$$

It is straightforward to verify that this system is positively invariant and competitive in Ω (see Appendix C).

Notice that for any $(s, v) \in \Omega_0$ we have:

$$F_1(s, v) = G_1(s, v) \quad \text{et} \quad F_2(s, v) \leq G_2(s, v).$$

Let $(\eta(t), \rho(t))$ be a solution of system (E.1.4) with initial condition $(s^*, 0)$; let $(s(t), v(t)) \in \vec{\psi}(t)$ be a closed orbit of system (E.1.3) with initial condition (s_0, v_0) such that $s_0 > s^*$ and $v(0) < 0$. By using the Theorem C.2.1 we conclude that:

$$s(t) < s^* \quad \text{et} \quad v(t) < 0 \quad \text{for any } t \geq 0.$$

Using the fact that $\vec{\psi}(t)$ is a closed orbit which contains the point $(s^*, 0)$, there exist time intervals I_n such that $s(t) < s^*$ for $t \in I_n$ and we obtain a contradiction. □

E.2. Proof of Corollary 1.7.1

Proof. We can prove that there exists a finite number $T \geq 0$ (which depends on initial conditions) such that any component s of the solution of system (E.1.3) satisfies the property $s(t) < s_{in}$ for any $t > T$. Hence, without loss of generality we will suppose that $s(0) < s_{in}$ in the remainder of the proof.

By using the transformation $(s, x) \mapsto (s, v)$ where $v = s + \alpha x - s_{in}$, we can deduce that the system (E.1.1) is equivalent to system:

$$\text{(E.2.1)} \qquad \begin{cases} \dot{s} = [h(s^* - s) - f(s)](s_{in} - s) - v f(s), \\ \dot{v} = -h(s^* - s)v, \\ (s(0), v(0)) = (s_0, v_0) \in \Omega \end{cases}$$

where the set Ω is defined by:

$$\Omega = \Big\{(s, v) \in (0, s_{in}) \times \mathbb{R} \colon \, : v + s > -s_{in}\Big\}.$$

It is straightforward to verify that the system (E.2.1) is positively invariant in Ω, the critical point $E^* = (s^*, 0)$ is locally asymptotically stable and the critical point $E_0 = (s_{in}, 0)$ is an unstable saddle point with stable manifold:

$$W^s(E_0) = \big\{(s, v) \in \Omega \colon v + s = s_{in}\big\} \subset \partial\Omega.$$

Notice that by boundedness of function h (ensured by **(H1)**), there exists a positive constant $\rho > 0$ such that the component $v(t)$ of the system (4.1.1) satisfies the inequality:

$$|v(t)| = |v(0)| \exp\Big(\int_0^t -h(s^* - s(r))\, dr\Big) \leq |v(0)|e^{-\rho t}.$$

Then we have that $\lim_{t\to+\infty} v(t) = 0$. Now, we consider the differential equation:

$$\dot{s} = [h(s^* - s) - f(s)](s_{in} - s), \quad s(0) = s_0 \in (0, s_{in}). \tag{E.2.2}$$

Notice that the solutions of equation (E.2.2) have the asymptotic behavior:

$$\lim_{t\to+\infty} s(t) = s^*.$$

As (E.2.1) and (E.2.2) satisfy the hypothesis of theorem B.1.1 (see Appendix B). It follows that the critical point E_1 is a global attractor of the system (E.2.1). $\square$

APPENDIX F

Publications

Articles in Journals:

[A1] J.L. Gouzé and G.Robledo. *Feedback control for nonmonotone competition models in the chemostats*. Nonlinear Analysis (Real World Applications) **6**:671–690 (2005).

[A2] G. Robledo and J.L. Gouzé. *Robust control for an uncertain chemostat model*. International Journal of Nonlinear and Robust Control **16**:133-155 (2006).

Proceedings:

[P1] J.L. Gouzé and G. Robledo. *Positive control for competition models with inhibition in the chemostat*. Proceedings Sixteen International Symposium on: Mathematical Theory of Networks and Systems (MTNS), Leuven, Belgium, July 5–9, 2004.

Technical Reports:

[TR1] G. Robledo *Feedback stabilization for a chemostat with delayed output*. Rapport de Recherche 5844 INRIA (2006).

[TR2] J.L. Gouzé and G. Robledo *Feedback control for nonmonotone competition models with different removal rates in the chemostat*. Rapport de Recherche INRIA, Rapport de recherche 5555 INRIA (2005).

[TR3] J.L. Gouzé and G. Robledo *Robust control for an uncertain chemostat*. Rapport de Recherche 5295 INRIA (2004).

[TR4] J.L. Gouzé and G. Robledo *Feedback control for nonmonotone competition models in the chemostat*. Rapport de Recherche 5033 (INRIA) (2003).

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Quelques Résultats sur la Commande du Chemostat

Cette thèse s'attache à la commande de certains systèmes écologiques en chemostat (appareil de culture de micro–organismes en laboratoire). Nous commençons par un aperçu des modèles de compétition et de chaîne trophique dans le chemostat ainsi qu'un rappel des concepts basiques de la théorie de la commande adaptée aux équations du chemostat. Ceci nous permet de montrer quelques applications pratiques et aussi de mettre en évidence la complexité mathématique de la commande.

La première partie considère la commande robuste d'un chemostat simple qui présente des imprécisions déterministes tant dans le modèle que dans la sortie, ainsi que des retards dans la sortie. Nous construisons une famille de boucles de rétroaction qui stabilise le modèle dans un polytope déterminé par la grandeur des imprécisions. Cette famille stabilise aussi la sortie autour d'une consigne en présence des retards, mais en l'absence d'imprécision sur le modèle et la sortie.

La deuxième partie considère la commande en boucle fermée d'un modèle de compétition entre espèces qui permet la coexistence de celles–ci. Nous généralisons un résultat proposé par P. De Leenher et H.Smith dans deux directions: considération de fonctions de croissance plus générales et prise en compte de la mortalité des espèces.

La troisième partie considère la commande en boucle ouverte d'une chaîne trophique dans un chemostat. Nous présentons une méthode de réduction de dimension qui permet de caractériser l'ensemble d'atteignabilité du système et d'obtenir un résultat sur la commandabilité partielle de la chaîne.

Mot–clés: chemostat, compétition, chaîne trophique, commande en boucle ouverte, commande en boucle fermée, commande robuste.

Control of the Chemostat: Some Results

This thesis deals with the control of some ecological systems in the chemostat (an experimental device used for microorganism growth in idealized conditions). We start with a brief description of the competition and trophic chain model in the chemostat and we recall some basic concepts of control theory adapted the the models stated above. This frame makes possible to show some applications and to point out the mathematical complexity of the problem.

The first part considers the robust control of a simple chemostat characterized by deterministic imprecisions in the model and the output and moreover with delays in the output. We build a family of feedback control laws which stabilizes the variables in a polytope determined by the bounds of imprecisions. Moreover, we stabilize the output around a reference value in the case of presence of delays but absence of imprecisions.

The second part considers the feedback control of a competition model between two species making possible the coexistence of them. We generalize a result given by P. De Leenheer and H. Smith in two ways: we consider more general uptake functions and we take into account the mortality of species.

The third part considers the open–loop control for a trophic chain in the chemostat. We give a method for reduction of dimension method which makes possible to characterize the attainable set of the system and to obtain a result related with the partial controllability of the chain.

Keywords: chemostat, competition, trophic chain, open–loop control, feedback control, robust control.