Résolutions coniques des variétés discriminants e applications à la géométrie algébrique complexe et réelle

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Contents

Some conventions and notation

We shall denote by $\Pi_{d,n}$ the vector space of complex homogeneous polynomials of degree d in $n+1$ variables, and by $\Sigma_{d,n}$ the subset of $\Pi_{d,n}$ consisting of polynomials, whose differential vanishes somewhere in $\mathbb{C}^{n+1} \setminus \{0\}$. Set $\Pi_{d,n}(\mathbb{R})$ to be the subset of $\Pi_{d,n}$ that consists of polynomials with real coefficients.

Unless stated otherwise, we consider only homology and cohomology groups with real coefficients, and the fibers of all local systems are assumed to be real vector spaces of finite dimension.

We shall say that a topological space X is good if the one-point compactification of X can be provided with the structure of a finite CW-complex. Some statements below that are formulated for good spaces are true in a more general setting, but we shall not need this.

If X is a good topological space, and $\mathcal{L}$ is a local system on X, then we shall denote by $\bar{H}_*(X, \mathcal{L})$ the Borel-Moore homology groups of X with coefficients in $\mathcal{L}$; we set $P(X, \mathcal{L})$ to be the Poincaré polynomial of X "with coefficients in $\mathcal{L}$", i.e. the polynomial $\sum_i a_i t^i$, where $a_i = \dim(H^i(X, \mathcal{L}))$. In a similar way, we denote by $\bar{P}(X, \mathcal{L})$ the polynomial $\sum_i a_i t^i$, where $a_i = \dim(\bar{H}_i(X, \mathcal{L}))$.

The symbol "$\subset$" denotes an inclusion; a strict inclusion is denoted by "$\subsetneq$".

As usual, if X is a topological space, and $A \subset X$, we denote by $\bar{A}$ the closure of A in X.

If $f : X \to Y$ is a continuous map of topological spaces, and $\mathcal{A}$ and $\mathcal{B}$ are sheaves on X and Y respectively, then we denote by $f_*(\mathcal{A})$, respectively, by $f^{-1}(\mathcal{B})$ the direct image of $\mathcal{A}$, respectively the inverse image of $\mathcal{B}$, under f.

The symbol "#" denotes the cardinality of a finite set.

The symbol "\" denotes the difference of two sets and never the left quotient; we write X/G for the quotient of a manifold X by an action of a group G, regardless whether the action is right or left.

The symmetric group on k elements is denoted by S_k everywhere except the appendix, where it is denoted by $\mathfrak{S}_k$, and S_k means something else.

We systematically use the "topological" notation instead of the "algebraic" one, e.g., we write $\mathbb{C}P^n$ and $\mathbb{Z}_2$, and not $\mathbb{P}^n(\mathbb{C})$ and $\mathbb{Z}/2\mathbb{Z}$ etc.

"Random" symbols like X, A, x, a, f etc. may have different meanings in different chapters (and sometimes even within the same chapter); we hope however that this never leads to a confusion.

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Chapter 1

Introduction

Considérons la situation suivante : soit V un espace affine dont les éléments paramétrisent des fonctions définies sur une certaine variété. Souvent ces fonctions se partagent en génériques et singulières (la définition de ce que l'on appelle "singulier" varie selon le cas). Le sous-espace (topologique) Σ de V constitué des fonctions singulières est appelé un discriminant (généralisé).

Beaucoup d'espaces topologiques "célèbres" sont des complémentaires de discriminants (ou sont homotopiquement équivalents à des complémentaires de discriminants). On peut mentionner comme exemples les groupes de Lie classiques matriciels, les espaces de nœuds, les espaces d'applications lisses sans singularités compliquées, les espaces classifiants des groupes de tresses, les complémentaires d'arrangements de sous-espaces affines de dimensions quelconques, les espaces d'applications lisses entre des sphères, etc. Les questions concernant la géométrie et la topologie des complémentaires de discriminants se posent donc dans des situations très diverses.

Une stratégie générale pour calculer la cohomologie du complémentaire d'un discriminant a été introduite par V. A. Vassiliev il y a plus de 15 ans. Cette stratégie s'est avérée très efficace ; en plus de donner beaucoup de nouveaux résultats (comme les invariants de Vassiliev des nœuds), elle fournit une explication unifiée pour certains faits qui n'ont pas de liens apparents (comme la formule de Goresky-MacPherson qui calcule la cohomologie d'un arrangement de k-plans, la suite spectrale d'Adams pour la cohomologie des espaces de lacets, le théorème de Snaith sur le scindage de la cohomologie des groupes de Lie, etc.).

Expliquons brièvement comment cette stratégie marche dans le cas élémentaire suivant : soit V l'espace des polynômes $x^d + a_{d-1}x^{d-1} + \cdots + a_0$, où d est un entier fixé supérieur ou égal à 2, et les a_i sont des réels. Notons Σ le sous-ensemble de V formé des polynômes ayant une racine multiple. Par la dualité d'Alexander, le calcul des groupes do cohomologie $H^*(V \setminus \Sigma)$ est ramené au calcul des groupes d'homologie de Borel-Moore de Σ. La question qui se pose alors est le calcul de ces groupes d'homologie. Il existe une résolution lisse évidente Σ' de Σ : on pose

$$\Sigma' = \{(f, x) | f \in \Sigma, x \in \mathbb{R}, f(x) = f'(x) = 0\}.$$

Malheureusement, la projection $\Sigma' \to \Sigma$ n'induit presque jamais un isomorphisme au niveau de l'homologie de Borel-Moore, puisque, par exemple, si $d > 3$, il existe des polynômes ayant deux racines doubles, et on les compte alors deux fois en haut. Peu importe, se dit-on, et pour tout f ayant plus d'une racine multiple, et pour tous $x_1 \neq x_2$ deux racines multiples de f, on recolle formellement un segment reliant (f, x_1) et (f, x_2). L'espace Σ'' obtenu ainsi se projette lui aussi sur Σ, mais cette fois-ci ce sont les polynômes ayant au moins 3 racines multiples distinctes qui posent problème : si, par exemple, un polynôme a exactement trois racines multiples distinctes, sa préimage est un cercle (topologique). Pour tout f ayant au moins trois racines multiples distinctes, et tous x_1, x_2, x_3 racines multiples distinctes de f, on recolle donc comme tout à l'heure un 2-simplexe dont les sommets sont $(f, x_1), (f, x_2), (f, x_3)$. On reproduit cette procédure si nécessaire avec des simplexes de dimension supérieure à 2 jusqu'à ce que tous les générateurs indésirables de l'homologie en haut soient tués.

L'espace σ obtenu ainsi est appelé la *résolution simpliciale* de Σ. L'intérêt d'utiliser σ plutôt que Σ provient du fait que σ admet une filtration naturelle que l'on a en fait construite au cours de la bataille avec les générateurs d'homologie redondants :

$$\varnothing \subset \Sigma' \subset \Sigma'' \subset \cdots \subset \sigma.$$

La différence de deux termes consecutifs dans cette filtration est un fibré vectoriel au-dessus de l'espace des configurations non ordonnées de k points de $\mathbb{R}$. Cela permet d'écrire immédiatement le terme E^1 de la suite spectrale correspondant à cette filtration. Cette suite spectrale dégénère au premier terme et donne les groupes d'homologie de Borel-Moore de Σ.

Une source naturelle d'exemples de complémentaires de discriminants est fournie par la géométrie algébrique où les espaces d'hypersurfaces projectives lisses sont des objets d'étude classiques. Les espaces de modules de courbes de petit genre se décomposent canoniquement en morceaux dont chacun est le quotient du complémentaire d'un discriminant par l'action d'un groupe algébrique. Dans l'article [14], V. A. Vassiliev a expliqué comment sa stratégie s'applique au cas des hypersurfaces projectives lisses.

La majeure partie de la thèse est consacrée à une généralisation de cette méthode. En gros, nous présentons une méthode inspirée de [14] (plus précisément, des calculs faits dans [14]) qui nous permet dans certains cas d'aller un peu plus loin, par exemple, en augmentant le degré ou la dimension de l'espace ambiant, ou le genre etc. On présente également quelques applications. La liste des applications que l'on présente n'est pas exhaustive[1] ; elle ne contient pas non plus tous les cas intéressants (l'application la plus intéressante de la méthode présentée ici est probablement le calcul du polynôme de Poincaré de $\mathcal{M}_4$ par O. Tommasi dans [13]; l'auteur a fait le même calcul indépendamment, mais plus tard). Ces applications sont données dans le seul but de montrer les problèmes que l'on peut résoudre en utilisant notre méthode.

Par ailleurs, il faut remarquer que cette méthode n'a donné (au moins pour l'instant) aucun résultat sur la cohomologie des espaces d'hypersurfaces de degré

[1] Par exemple, l'auteur a récemment calculé les polynômes de Poincaré des quotients des espaces des quartiques nodales et GIT-stables par l'action évidente du groupe $PGL_3(\mathbb{C})$. Les réponses sont respectivement $1+t^2+t^4+t^6$ et $1+t^2+2t^4+2t^6+t^7+t^8$; malheureusement, ces calculs ont été terminés trop tard pour être inclus dans la thèse. Le premier de ces résultats permet de montrer que le polynôme de Poincaré de l'espace $\bar{\mathcal{M}}_3$ des courbes Deligne-Mumford-stables de genre 3 est égal à $1+3t^2+7t^4+10t^6+7t^8+3t^{10}+t^{12}$. Ce résultat n'est pas nouveau (cf. [4, page 19] ; je remercie J. Steenbrink et O. Tommasi pour cette référence), mais notre démonstration est indépendante. Le seul fait non trivial que l'on utilise est $H^5(\bar{\mathcal{M}}_3) = 0$, ce qui a été démontré par E. Arbarello et M. Cornalba [1] ; les détails seront présentés ailleurs.

quelconque, ou des espaces de modules de courbes de genre quelconque; elle ne donne pas non plus beaucoup d'information sur les groupes fondamentaux des complémentaires de discriminants (puisque notre premier pas consistera à remplacer le calcul des groupes de cohomologie du complémentaire à un discriminant par le calcul des groupes d'homologie de Borel-Moore du discriminant même).

La thèse est organisée de la manière suivante : dans le chapitre 3 on rappelle la construction de V. A. Vassiliev [14], on explique quel est l'intérêt de la généraliser, et on présente notre construction. Dans le chapitre 4 on donne quelques résultats homologiques dont on aura besoin dans la suite. Les chapitres suivants sont consacrés aux applications: dans le chapitre 5 on considère le cas des quintiques de $\mathbb{C}P^2$ (le discriminant est une hypersurface irréductible), dans le chapitre 6 on calcule les nombres de Betti de l'espace des cubiques lisses qui intersectent transversalement une conique lisse fixée (le discriminant est réductible, mais la méthode marche aussi bien), dans le chapitre 7 on considère l'espace des cubiques lisses réelles (cet espace peut sûrement être consideré en utilisant une technique plus standard, toutefois on présente nos calculs comme un exemple test dans le cas réel).

Certains résultats de la thèse ont été annoncés dans [5]; un texte contenant la construction de la résolution conique et l'une des applications mentionnées dessus (le cas des quintiques projectives planes) a été accepté [6].

La thèse contient un appendice reproduisant l'article [7]. On y démontre le théorème suivant : supposons que le cercle est muni d'un atlas où toutes les fonctions de changement de cartes sont des homographies ; alors ce cercle borde une surface orientable munie d'un atlas où toutes les fonctions de changement de cartes sont des homographies (à coefficients complexes cette fois-ci) compatibles dans le sens évident avec les applications de changement de cartes sur le bord. On y établit également une classification correcte des cercles projectifs (il s'avère que la classification trouvée il y a longtemps par N. Kuiper est incomplète).

Chapter 2

Introduction

Consider the following situation: suppose we are given an affine space V, whose elements parametrise functions defined on some manifold. Usually, these functions can be divided into generic and singular (the definition of "singular" varies from case to case). The subspace Σ of V that consists of the singular functions is called a (generalised) discrimimant.

Many "famous" topological spaces (such as classical matrix Lie groups, spaces of knots, spaces of smooth maps between spheres, classifying spaces of braid groups, complements of arrangements of planes of different dimensions, spaces of smooth maps without complicated singularities, just to mention a few examples) are discriminant complements (or are homotopy equivalent to discriminant complements), so questions concerning geometry and topology of discriminant complements arise in many different situations.

A general strategy of calculating cohomology groups of discriminant complements was introduced by V. A. Vassiliev over 15 years ago. This strategy turned out to be very effective; apart from giving many new results (such as Vassiliev knot invariants), it also provides a unified explanation for some seemingly unrelated facts (such as Goresky-MacPherson formula for the cohomology of the complement of a plane arrangement, Adams spectral sequence for the cohomology of loop spaces, the Snaith splitting theorem for the cohomology of classical Lie groups and some others).

Let us briefly explain how this strategy works in the following elementary example: let V be the space of polynomials of the form $x^d + a_{d-1}x^{d-1} + \cdots + a_0$, where d is some fixed integer > 1 , and the a_i's are real numbers; set Σ to be the subset of V that consists of the polynomials with multiple roots. Via the Alexander duality we can replace the calculation of the groups $H^*(V \setminus \Sigma)$ by the calculation of the Borel-Moore homology groups of Σ. But how can that be done? There is an obvious smooth resolution Σ' of Σ: we set

$$\Sigma' = \{(f, x) | f \in \Sigma, x \in \mathbb{R}, f(x) = f'(x) = 0\}.$$

Unfortunately, the projection $\Sigma' \to \Sigma$ almost never induces an isomorphism of the Borel-Moore homology groups, since, e.g., if $d > 3$, there are polynomials that have two multiple roots, and they are counted twice upstairs. "Never mind" – we say, and formally glue the segment that joins (f, x_1) and (f, x_2) for any f that has more than one multiple root, and any two multiple roots $x_1 \neq x_2$ of f. The resulting space Σ'' also projects onto Σ, but this time it is the polynomials with ≥ 3 distinct multiple roots that cause troubles: e.g., the preimage of a polynomial that has exactly three multiple roots is (topologically) a circle. So we proceed as above and glue in the triangle spanned by the points $(f, x_1), (f, x_2), (f, x_3)$ for any f that has three or more multiple roots, and any (unordered) triple x_1, x_2, x_3 of distinct multiple roots of f, and so on until all unwished homology generators upstairs are killed.

The resulting space σ is called the *simplicial resolution* of Σ. The point of using σ rather than Σ consists in the fact that σ admits a natural filtration, which we have already constructed while fighting the redundant homology classes:

$$\varnothing \subset \Sigma' \subset \Sigma'' \subset \cdots \subset \sigma.$$

The difference of any two consecutive terms of this filtration is a vector bundle over the space of unordered k-ples of points in $\mathbb{R}$, which enables us immediately to write down the term E^1 of the spectral sequence that corresponds to the filtration. This spectral sequence degenerates at the first term, and gives us immediately the Borel-Moore homology groups of Σ.

A natural source of examples of discriminant complements is provided by algebraic geometry, where spaces of smooth projective hypersurfaces are classical objects of study. Moduli spaces of curves of small genus can be canonically decomposed into several pieces, so that each piece is the quotient of a discriminant complement by the action of some algebraic group. In the article [14] V. A. Vassiliev explained how to apply his strategy to the case of smooth projective hypersurfaces.

The main part of the thesis is devoted to a generalisation of that method. Roughly speaking, we present a method inspired by [14] (more precisely, by the calculations performed in [14]), which enables one in some cases to go one step further, i.e., to increase by one the degree or the dimension of the ambient space or the genus etc. We also present some applications. The list of applications is not exhaustive [1] nor does it contain r all interesting cases (the most interesting application so far of the method presented here is probably the calculation of the Poincaré polynomial of $\mathcal{M}_4$ performed by O. Tommasi in [13]; the author has done the same calculation independently but later). The only point in presenting these applications is to show what problems can be handled using our method.

It should be noted by the way that (at least at present) this method does not give any results about the cohomology of spaces of hypersurfaces of arbitrary degree, or moduli spaces of curves of arbitrary genus; nor does it give much information on the fundamental groups of discriminant complements (since our first step will as above consist in replacing the calculation of the cohomology groups of a discriminant complement by the calculation of the Borel-Moore homology groups of the discriminant itself).

[1] For instance, the author has recently calculated the Poincaré polynomials of the quotients of the spaces of nodal and GIT-stable plane quartics by the obvious action of the group $PGL_3(\mathbb{C})$. The answers are respectively $1 + t^2 + t^4 + t^6$ and $1 + t^2 + 2t^4 + 2t^6 + t^7 + t^8$; unfortunately these calculations were completed too late to be included in the thesis. The first one of these results enables one to show that the Poincaré polynomial of the space $\bar{\mathcal{M}}_3$ of Deligne-Mumford-stable genus 3 curves is $1+3t^2+7t^4+10t^6+7t^8+3t^{10}+t^{12}$. This result is not new (see [4, page 19]; I am grateful to J. Steenbrink and O. Tommasi for this reference), but our proof is independent. The only nontrivial fact we use is $H^5(\bar{\mathcal{M}}_3) = 0$, as shown by E. Arbarello and M. Cornalba [1]; the details will be presented elsewhere.

The thesis is organised as follows: in chapter 3 we recall V. A. Vassiliev's construction from [14], give some argument why it should be generalised and then present our construction. In chapter 4 we give some homological results that we shall need in the sequel. The remaining chapters are devoted to applications: in chapter 5 we consider the case of smooth quintics in $\mathbb{C}P^2$ (the discriminant is an irreducible hypersurface), in chapter 6 we calculate the Betti numbers of the space of smooth plane cubics that intersect transversally a fixed smooth conic (the discriminant is reducible, but the method works just as well), in chapter 7 we consider the space of smooth real plane cubics (this space can surely be considered using some more standard technique, but we present our calculations as a test example in the real case).

Some of the results of the thesis were announced in [5]; a text containing the construction of the conical resolution and the first of the above-mentioned applications (the spase of smooth plane projective quintics) has been accepted [6].

The thesis contains an appendix, which reproduces the article [7]. Roughly speaking, we prove the following theorem: suppose the circle is equipped with an atlas, where all transition maps are fractional linear; then this circle bounds an orientable surface with an atlas where all transition maps are also fractional linear (only this time with complex coefficients) and are compatible in the obviuous way with the transition maps on the boundary. We also show that the classification of projective circles given long ago by N. Kuiper is not quite correct and give a correct one.

Chapter 3

The method of conical resolutions

3.1 Why generalise?

In this chapter we describe a general method of computing (at least additively) the cohomology groups of spaces like $\Pi_{d,n} \setminus \Sigma_{d,n}$. Our method is a generalisation of the one given in [14]. Everyone who proposes to generalise something has to face the question that is the title of this section. In order to answer this question, let us recall briefly V. A. Vassiliev's construction from [14, Section 2].

Suppose that we are interested in calculating the cohomology groups of $\Pi_{d,n} \setminus \Sigma_{d,n}$. The first remark is the following: via the Alexander duality we have

$$H^i(\Pi_{d,n} \setminus \Sigma_{d,n}) \cong \bar{H}_{2D-i-1}(\Sigma_{d,n}),$$

where $D = \dim_{\mathbb{C}}(\Pi_{d,n})$. This reduction was first used by V. I. Arnold in [2]. The variety $\Sigma_{d,n}$ is usually very singular, and there seems to be no immediate way to compute its Borel-Moore homology groups. In order to do this, we construct a space $\tilde{\sigma}_{d,n}$ called the *conical resolution* of $\Sigma_{d,n}$. We shall see that, on the one hand, there is a natural proper map $\pi : \tilde{\sigma}_{d,n} \to \Sigma_{d,n}$ such that the preimage of any point of $\Sigma_{d,n}$ is a cone (hence the term "conical"), and on the other hand, $\tilde{\sigma}_{d,n}$ admits a very nice filtration, which enables one to calculate the groups $\bar{H}_*(\tilde{\sigma}_{d,n})$ for some values of d and n. The space $\tilde{\sigma}_{d,n}$ is constructed as follows.

For any $K \subset \mathbb{C}P^n$ denote by $L(K)$ the vector subspace of $\Pi_{d,n}$ consisting of all polynomials that have singular points everywhere in K (and maybe elsewhere). For any $i = 1, \ldots, D$ denote by $G_i(\Pi_{d,n})$ the Grassmann manifold whose points are complex subspaces of $\Pi_{d,n}$ of codimension i. Denote by Ω_i the subspace of $\Pi_{d,n}$ consisting of all vector subspaces that have the form $L(K)$, where K is the set of singular points of some polynomial from $\Sigma_{d,n}$.

A simplex of the join

$$G_1(\Pi_{d,n}) * G_2(\Pi_{d,n}) * \cdots G_D(\Pi_{d,n})$$

is called *coherent*, if its vertices form a flag. The *main vertex* of a coherent simplex $\triangle$ is the vertex corresponding to the smallest subspace among the subspaces that correspond to the vertices of $\triangle$. The union of all coherent simplices with vertices in $\bar{\Omega}_i$ is denoted by $\tilde{\Lambda}_{d,n}$ (here $\bar{\Omega}_i$ stands for the closure of Ω_i in $G_i(\Pi_{d,n})$). For any $L \in \bar{\Omega}_i$ denote by $\tilde{\Lambda}_{d,n}(L)$ the union of all coherent simplices whose main vertex is L. Every space $\tilde{\Lambda}_{d,n}(L)$ is a cone with vertex L; denote the base of this cone by $\partial\tilde{\Lambda}(L)$.

Set $\tilde{\sigma}_{d,n}$ to be the subspace of $\Sigma_{d,n} \times \tilde{\Lambda}_{d,n}$ that consists of all couples (f, x) such that $f \in L$ and $x \in \tilde{\Lambda}_{d,n}(L)$, where $L \in \bar{\Omega}_i$ for some $i = 1, \dots, D$. Set $\tilde{\Phi}_i$ to be the union of all coherent simplices whose main vertices belong to $\bigcup_{j \leq i} G_j(\Pi_{d,n})$, and set

$$F_i = \{(f, x) \in \tilde{\sigma}_{d,n} | x \in \tilde{\Phi}_i\}.$$

We have the filtrations

$$\varnothing \subset \tilde{\Phi}_1 \subset \cdots \tilde{\Phi}_D = \tilde{\Lambda}_{d,n} \tag{3.1}$$

and

$$\varnothing \subset F_1 \subset \cdots F_D = \Sigma_{d,n}. \tag{3.2}$$

Theorem 3.1 1. *The obvious projection $\pi : \tilde{\sigma}_{d,n} \to \Sigma_{d,n}$ is a proper map and induces an isomorphism of Borel-Moore homology groups.*

2. *The space $F_i \setminus F_{i-1}$ is a $(D-i)$-dimensional complex vector bundle over $\tilde{\Phi}_i \setminus \tilde{\Phi}_{i-1}$.*

So via the Thom isomorphism the task of writing down the term E^1 of the spectral sequence corresponding to the above filtration on $\tilde{\sigma}_{d,n}$ is reduced to calculating the Borel-Moore homology groups of the spaces $\tilde{\Phi}_i \setminus \tilde{\Phi}_{i-1}$. There exists for any $i = 1, \dots, D$ the obvious map $\tilde{\Phi}_i \setminus \tilde{\Phi}_{i-1} \to \bar{\Omega}_i$ such that for any $L \in \bar{\Omega}_i$ the preimage of L is the open cone $\tilde{\Lambda}_{d,n}(L) \setminus \partial\tilde{\Lambda}_{d,n}(L)$. Usually one has to stratify $\bar{\Omega}_i$ so that the above map becomes the projection of a locally trivial bundle over each stratum.

It turns out that for any $L \in \bar{\Omega}_i \setminus \Omega_i$ the space $\partial\tilde{\Lambda}_{d,n}(L)$ is contractible, and for $L \in \Omega_i$ we have

$$\bar{H}_*(\tilde{\Lambda}_{d,n}(L) \setminus \partial\tilde{\Lambda}_{d,n}(L)) \cong \bar{H}_*(\Lambda_{d,n}(L) \setminus \partial\Lambda_{d,n}(L)),$$

where $\Lambda(L)$ and $\partial\Lambda(L)$ are obtained by taking the union of all coherent simplices (respectively from $\tilde{\Lambda}(L)$ and $\partial\tilde{\Lambda}(L)$), all of whose vertices belong to $\Omega_j, j \leq i$. So, in order to write down the term E^1 of the spectral sequence corresponding to (3.2), we have to know the Borel-Moore homology groups of certain fibre bundles over certain subvarieties of the Ω_i's.

However, sometimes the fibres or the bases of these bundles turn out to be quite complicated. Consider, e.g., the case of $\Pi_{5,2}$. We have then $D = 21$. The space Ω_{20} is stratified as follows: $\Omega_{20} = \Omega_{20}^1 \sqcup \Omega_{20}^2 \sqcup \Omega_{20}^3 \Omega_{20}^4$, where $\Omega_{20}^1, \dots, \Omega_{20}^4$ correspond respectively to the following singular curves: five lines in general position, three lines and a smooth conic in general position, two smooth conics and a line in general position, and two lines and an irreducible singular cubic in general position (in the last case "in general position" means that the cubic intersects each line at three distinct points, none of which is the intersection point of the lines).

Let us describe the space $\partial\Lambda_{5,2}(L)$ for $L \in \Omega_{20}^1$. The vector space L is spanned by a polynomial whose zero locus in $\mathbb{C}P^2$ consists of five lines in general position; denote by K the set of ten intersection points of those lines. Denote by $\triangle$ the 9-dimensional simplex, whose vertices formally correspond to the elements of K. Define a mapping $\partial\Lambda_{5,2}(L) \to \triangle$ as follows: take every $L(K'), K' \subset K$ to the barycentre of the face of $\triangle$ spanned by the elements of K', and extend linearly to all coherent simplices $\subset \partial\Lambda_{5,2}(L)$. This mapping is a homeomorphism on its image.

It can be proven using lemma 5.1 below that minimal vector spaces that contain L and have the form $L(K)$, where K is the singular locus of some quintic, are

1. L(four points on l + three points in general position outside l), and

2. L(three points on $l_1 \setminus l_2$ plus three points on $l_2 \setminus l_1$ plus $l_1 \cap l_2$),

where l, l_1, l_2 are lines, $l_1 \neq l_2$. One can show that the image of the above mapping can be contracted onto the union U of the images of $\Lambda_{5,2}(L')$, where L' is of the first type. U is the union of all 6-faces of $\triangle$ that are opposite to the 2-faces spanned by triples of points on a line.

There exists a natural action of the symmetric group S_5 on $H_*(U)$, and most irreducible representations of S_5 occur in the corresponding decomposition of $H_*(U)$. Hence, in order to find the contribution of Ω^1_{20}, it is necessary to calculate the Borel-Moore homology of Ω^1_{20} with coeficients in many local systems. The same holds for the spaces Ω^2_{20} etc.; calculating the contribution in these cases appears to be an even more difficult task, since the conditions that define these spaces are nonlinear.

Luckily, there is a way to overcome some of these difficulties. Below we present another version of the conical resolution. The main difference can be informally summarised as follows: we define coherent simplices using inclusions between singular loci themselves, rather then between the corresponding vector spaces. This gives us more flexibility in comparison with the above method, but forces us to introduce lots of "fake" singular loci, which we count once with a plus, and once with a minus.

3.2 The construction of the conical resolution

Let V be a vector space of $\mathbf{k}$-valued functions on a manifold $\mathcal{M}$ and $\Sigma \subset V$ is a closed subspace formed by the functions that are singular in some sence. (The space Σ is often called a *discriminant*; in all examples that we shall consider Σ will be an algebraic variety.) Suppose that $D = \dim_{\mathbf{k}} V < \infty$. We want to calculate the Borel-Moore homology of Σ. In order to do this, we preceed as above, i.e., we construct a resolution σ and a proper map $\pi : \sigma \to \Sigma$ such that the preimage of every point is contractible. We are going to describe a construction of σ via configuration spaces.

Remark. The method described below can be extended with obvious modifications to the case when V is an affine space. We assume V to be a vector space, since, on the one hand, the vector case is somewhat simpler, and on the other hand it is sufficient for all applications that we have in mind.

Suppose that with every function $f \in \Sigma$ a compact nonempty subset K_f of some compact CW-complex $\mathbf{M}$ is associated. For instance, if $\mathcal{M} = \mathbb{C}^{n+1} \setminus \{0\}, V = \Pi_{d,n}, \Sigma = \Sigma_{d,n}$, it is natural to set $\mathbf{M}$ equal to $\mathbb{C}P^n$ and K_f equal to the image of the set of singular points of f under the natural map $\mathcal{M} \to \mathbb{C}P^n$. We suppose that the following conditions are satisfied:

- If $f, g \in \Sigma$, and $K_f \cap K_g \neq \varnothing$, then $f + g \in \Sigma$ and $K_f \cap K_g \subset K_{f+g}$,
- If $f \in \Sigma$, then for any $\lambda \neq 0$ we have $K_{\lambda f} = K_f$,
- The zero function $0 \in \Sigma$, and $K_0 = M$.
- For any $K \subset \mathbf{M}$ set $L(K) \subset V$ to be the subset consisting of all f such that $K \subset K_f$. The previous three conditions imply that $L(K)$ is a $\mathbf{k}$-vector space. We suppose that there exists a positive integer d such that for any $x \in \mathbf{M}$ one can find a neighbourhood $U \ni x$ in $\mathbf{M}$ and continuous functions $l_1, \ldots, l_d$ from U to the Grassmannian $G_{D-1}(V)$ of $(D-1)$-dimensional $\mathbf{k}$-vector subspaces of V such that we have

$$L(\{x'\}) = \bigcap_{i=1}^{d} l_i(x')$$

  for any $x' \in U$.

Remark. One may ask a natural question: if we are dealing with functions on some manifold $\mathcal{M}$, why should we introduce some additional space $\mathbf{M}$? The problem is that for our construction it will be convenient to associate with a singular function a compact subset of a compact CW-complex. In the case when the manifold $\mathcal{M}$ itself is compact, we can assume, of course, that $\mathbf{M} = \mathcal{M}$, and K_f is the singular locus of f (the definition of the singular locus depends on the particular example we are considering).

By a *configuration* in a compact CW-complex $\mathbf{M}$ we shall mean a compact nonempty subset of $\mathbf{M}$. Denote by $2^{\mathbf{M}}$ the space of all configurations in $\mathbf{M}$. Suppose that ρ is the metric on $\mathbf{M}$. We introduce the Hausdorff metric on $2^{\mathbf{M}}$ by the usual rule:

$$\tilde{\rho}(K, L) = \max_{x \in K} \rho(x, L) + \max_{x \in L} \rho(x, K).$$

The resulting topology on $2^{\mathbf{M}}$ does not depend on the choice of the particular metric that induces the topology on $\mathbf{M}$.

It is easy to check that if $\mathbf{M}$ is compact, then the space $2^{\mathbf{M}}$ equipped with the metric $\tilde{\rho}$ is also compact.

Notation. We denote by $B(\mathbf{M}, k)$ the subspace of $2^{\mathbf{M}}$ that consists of all configurations that contain exactly k elements.

Note that we have $\overline{B(\mathbf{M}, k)} = \bigcup_{j \leq k} B(\mathbf{M}, j)$.

Proposition 3.1 *Let (K_j) be a Cauchy sequence in $2^{\mathbf{M}}$, and let K be the set consisting of the limits of all sequences (a_j) such that $a_j \in K_j$ for every j. Then K is nonempty and compact, and $\lim_{j \to \infty} \tilde{\rho}(K_j, K) = 0$.*

$\diamondsuit$

Proposition 3.2 *Let $(K_i), (L_i)$ be two sequences in $2^{\mathbf{M}}$. Suppose that there exist $\lim_{i \to \infty} K_i, \lim_{i \to \infty} L_i$, and denote these limits by K and L respectively. Suppose also that $K_i \subset L_i$ for every i. Then $K \subset L$.*

$\diamondsuit$

Suppose that $X_1, \dots, X_N$ is a finite collection of subspaces of $2^{\mathbf{M}}$ satisfying the following conditions:

1. For every $f \in \Sigma$ the set K_f belongs to some X_i.
2. Suppose that $K \in X_i, L \in X_j, K \subsetneq L$. Then $i < j$.
3. Recall that $L(K)$ is the space of all functions f such that $K \subset K_f$. Let us fix i; we assume that $\dim_{\mathbf{k}} L(K)$ is the same for all configurations $K \in X_i$. (We denote this dimension by d_i.)
4. $X_i \cap X_j = \varnothing$ if $i \neq j$.
5. For any i the space $\bar{X}_i \setminus X_i$ is included in $\bigcup_{j<i} X_j$.
6. For every i we denote by $\mathfrak{T}_i$ the subspace of $\mathbf{M} \times 2^{\mathbf{M}}$ consisting of pairs (x, K) such that $K \in X_i$ and $x \in K$. We assume that $\mathfrak{T}_i$ is the total space of a locally trivial bundle over X_i (the projection $pr_i : \mathfrak{T}_i \to X_i$ is obvious). This bundle will be called the *tautological bundle*[1] over X_i.
7. Note that any local trivialisation of $\mathfrak{T}_i$ has the following form:

$$(x, K') \mapsto (t(x, K'), K').$$

Here x is a point in some $K \in X_i$, K' belongs to some neighbourhood $U \ni K$ in X_i, and $t : K \times U \to \mathbf{M}$ is a continuous map such that if we fix $K' \in U$, then we obtain a homeomorphism $t_{K'} : K \to K'$. We require that for every $K \in X_i$ there exist a neighbourhood $U \ni K$ and a local trivialisation of $\mathfrak{T}_i$ over U such that every map $t_{K'} : K \to K'$ establishes a bijective correspondence between the subsets of K and K' that belong to $\bigcup_{j \leq i} X_j$.

Under these assumptions we are going to construct a resolution σ of Σ and a filtration on it such that the i-th term of the filtration is the total space of a fibre bundle over X_i.

Note that due to condition 3 for any $i = 1 \dots, N$ there exists an natural map $K \mapsto L(K)$ from X_i to the Grassmann manifold $G_{d_i}(V)$, which is continuous due to the last condition from the list on page 12.

Remark. The rather strange-looking condition 7 follows immediately from condition 6 in the following situation: suppose X_i consists of finite configurations, and for all K, L, such that $K \in X_i, L \subset K$ there is an index $j < i$ such that $L \in X_j$. In this case any trivialisation of $\mathfrak{T}_i$ fits.

Let us now recall the notion of the *k-th self-join* of a topological space. This notion was introduced by V.A. Vassiliev in [14] and will come out very useful on several occasions in the sequel.

Definition. Let X be a topological space that can be embedded into a finite dimensional Euclidean space, and let k be a positive integer. We shall say that a proper embedding $\imath : X \to \mathbb{R}^\Omega, \Omega < \infty$, is *k-generic*, if the intersection of any

[1] For instance, if $\mathbf{M} = \mathbb{C}P^n$ and X_i consists of projective subspaces of $\mathbf{M}$ of the same dimension, then this is just the projectivisation of the usual tautological bundle over the corresponding subvariety of some Grassmann manifold of $\mathbb{C}^{n+1}$.

two $(k-1)$-simplices with vertices on $\imath(X)$ is their common face (in particular, the intersection is empty, if the sets of the vertices are disjoint). We set the k-th self-join X^{*k} of X to be the union of all $(k-1)$-simplices with vertices on the image $\imath(X)$ of any k-generic embedding $\imath$. For good spaces this definition does not depend on the choice of $\imath$.

Consider the space $Y = \bigcup_{i=1}^{N} \bar{X}_i = \bigcup_{i=1}^{N} X_i$. Denote by X the N-th self-join Y^{*N} of Y Note that the spaces Y, Y^{*N} are compact. Call a simplex $\triangle \subset X$ *coherent* if the configurations corresponding to its vertices form an ascending sequence. Note that then its vertices belong to different X_i (condition 2). Let $\triangle$ be a coherent simplex. Among the vertices of $\triangle$ there is a vertex such that the corresponding configuration contains the configurations that correspond to all other vertices of $\triangle$. Such vertex will be called the *main vertex* of $\triangle$. Denote by Λ the union of all coherent simplices. For any $K \in X_i$ denote by $\Lambda(K)$ the union of all coherent simplices, whose main vertices coincide with K. Note that the space $\Lambda(K)$ is contractible.

Denote by Φ_i the union $\bigcup_{j\le i} \bigcup_{K\in X_j} \Lambda(K)$. There is a filtration on Λ: $\varnothing \subset \Phi_1 \subset \cdots \subset \Phi_N = \Lambda$.

For any simplex $\triangle \subset X$ denote by $\overset{\circ}{\triangle}$ its interior, i.e. the union of its points that do not belong to the faces of lower dimension. Note that for every $x \in X$ there exists a unique simplex $\triangle$ such that $x \in \overset{\circ}{\triangle}$.

Proposition 3.3 *Let (x_i) be a sequence in X such that $\lim_{i\to\infty} x_i = x$. Suppose $x_i \in \triangle_i, x \in \overset{\circ}{\triangle}$, where $\triangle_i$ are coherent simplices, and suppose K is a vertex of $\triangle$. Then there exists a sequence (K_i) such that K_i is a vertex of $\triangle_i$ and $\lim_{i\to\infty} K_i = K$.*

$\diamondsuit$

Proposition 3.4 *All spaces $\Lambda, \Lambda(K), \Phi_i$ are compact.*

Proof. This follows immediately from propositions 3.3 and 3.2. $\diamondsuit$

For any $K \in X_i$ denote by $\partial\Lambda(K)$ the union $\bigcup_\kappa \Lambda(\kappa)$ over all maximal subconfigurations $\kappa \in \cup_{j<i} X_j, \kappa \subsetneq K$. The space $\Lambda(K)$ is the union of all segments that join points of $\partial\Lambda(K)$ with K, and hence it is homeomorphic to the cone over $\partial\Lambda(K)$.

Define the conical resolution σ as the subspace of $\Sigma \times \Lambda$ consisting of pairs (f, x) such that $f \in \Sigma, x \in \Lambda(K_f)$.

Proposition 3.5 *The space σ is closed in $\Sigma \times \Lambda$.*

Proof. Let (f_i, x_i) be a sequence such that all $f_i \in \Sigma$, every $x_i \in \Lambda(K_{f_i})$, and we have $\lim_{i\to\infty} f_i = f \in \Sigma, \lim_{i\to\infty} x_i = x \in \Lambda$. Let us prove that $x \in \Lambda(K_f)$. Let $\triangle$ be the coherent simplex such that $x \in \overset{\circ}{\triangle}$. Due to proposition 3.3, there exists a sequence (K_i) such that all $K_i \in Y$ and $K_i \subset K_{f_i}$. Take an element $a \in K$. By proposition 3.1, there exists a sequence (a_i) of elements of $\mathbf{M}$ such that $\lim_{i\to\infty} a_i = a$. For any i, we have $F_i \in L(K_i) \subset L(\{a_i\})$. Due to the last condition from the list on page 12, $f \in L(\{a\})$. So, for any $a \in K$ we have $a \in K_f$, which means that $K \subset K_f$. The proposition is proven.$\diamondsuit$

There exist obvious projections $\pi : \sigma \to \Sigma$ and $p : \sigma \to \Lambda$. We introduce a filtration on σ putting $F_i = p^{-1}(\Phi_i)$. The map π is proper; indeed, by proposition 3.5, the preimage of each compact set $C \subset \Sigma$ is a closed subspace of $C \times \Lambda$, which is compact.

Theorem 3.2 *Suppose $X_1, \ldots, X_N$ are subspaces of $2^{\mathbf{M}}$ that satisfy Conditions 1–7 of page 13. Then*

1. *π induces an isomorphism of Borel-Moore homology groups of the spaces σ and Σ.*
2. *Every space $F_i \setminus F_{i-1}$ is a $\mathbf{k}$-vector bundle over $\Phi_i \setminus \Phi_{i-1}$ of dimension d_i.*
3. *The space $\Phi_i \setminus \Phi_{i-1}$ is a fibre bundle over X_i, the fibre being homeomorphic to $\Lambda(K) \setminus \partial\Lambda(K)$.*

Proof. The first assertion of the theorem follows from the fact that π is proper and $\pi^{-1}(f) = \Lambda(K_f)$, the latter space being contractible. To prove the second assertion, let us study the preimage of a point $x \in \Phi_i \setminus \Phi_{i-1}$ under the map $p : F_i \setminus F_{i-1} \to \Phi_i \setminus \Phi_{i-1}$. We claim that $p^{-1}(x) = L(K)$ for some $K \in X_i$.

Recall that each point $x \in \Phi_i \setminus \Phi_{i-1}$ belongs to the interior of some coherent simplex $\triangle$, whose main vertex lies in X_i. Denote this vertex by K and denote the map $\Phi_i \setminus \Phi_{i-1} \ni x \mapsto K \in X_i$ by f_i.

Now suppose $f \in L(K)$, or, which is the same, $K_f \supset K$. This implies $\Lambda(K) \subset \Lambda(K_f)$. So we have $x \in \Lambda(K) \subset \Lambda(K_f)$ and $(f, x) \in \sigma, p(f, x) = x$, hence $f \in p^{-1}(x)$.

Suppose now that $f \in p^{-1}(x)$. We have $(f, x) \in \sigma$, hence $x \in \Lambda(K_f)$. This means that x belongs to some coherent simplex $\triangle'$, whose main vertex is K_f. But x belongs to the interior of $\triangle$, hence $\triangle$ is a face of $\triangle'$ and K is a vertex of $\triangle'$. But K_f is the main vertex of $\triangle'$, hence $K \subset K_f$ and $f \in L(K)$.

So, we see that $p^{-1}(x) = L(K), K \in X_i$, and the second assertion of theorem 3.2 follows immediately from the fact that the dimension of $L(K)$ is the same for all $K \in X_i$ (condition 3). In fact the bundle $p : F_i \setminus F_{i-1} \to \Phi_i \setminus \Phi_{i-1}$ is the inverse image of the tautological bundle over $G_{d_i}(V)$ under the composite map $x \overset{F_i}{\mapsto} K \mapsto L(K)$.

We shall show now that the map F_i is a locally-trivial fibration with fibre $\Lambda(K) \setminus \partial\Lambda(K)$. This will prove the third assertion of theorem 3.2.

Denote by $\mathfrak{X}_i$ the space consisting of all pairs (L, K) such that $L \subset K, L \in \bigcup_{j \leq i} X_j$. Let $P : \mathfrak{X}_i \to X_i$ be the natural projection. We shall construct a trivialisation of $F_i : \Phi_i \setminus \Phi_{i-1} \to X_i$ from a particular trivialisation of $\mathfrak{T}_i$ that exists due to condition 7.

For any $K \in X_i$ let U and t be the neighbourhood of K and the trivialisation of $\mathfrak{T}_i$ over U that exist due to condition 7. For any $K, K' \in U, x \in K$, put $t_{K'}(x) = t(x, K')$. Define a map $T : P^{-1}(K) \times U \to 2^{\mathbf{M}}$ by the following rule: take any $L \subset K, L \in \bigcup_{j \leq i} X_j$ and put $T(L, K')$ equal to the image of L under $t_{K'}$. Due to condition 7, $T(\bullet, K')$ is a bijection between the set of $L \in 2^{\mathbf{M}}$ such that $L \subset K$, $L \in \bigcup_{j \leq i} X_j$, and the set of $L' \in 2^{\mathbf{M}}$ such that $L' \subset K', L' \in \bigcup_{j \leq i} X_j$.

Proposition 3.6 *1. The map $T_1 : P^{-1}(K) \times U \to 2^{\mathbf{M}} \times U$ defined by the formula $T_1(L, K') = (T(L, K'), K')$ maps $P^{-1}(K) \times U$ homeomorphically onto $P^{-1}(U)$.*

2. *For any $K' \in U$ we have $L \subset M \subset K$ if and only if $T(L, K') \subset T(M, K') \subset T(K, K') = K'$.*

This implies that $\mathfrak{X}_i$ is a locally trivial bundle over X_i.

Proof. The second assertion is obvious. We have already seen that the map T_1 is a bijection. The verification of the fact that T_1 and its inverse are continuous is straightforward but quite boring. ♢

Now we can construct a trivialisation of the fibre bundle $F_i : \Phi_i \setminus \Phi_{i-1} \to X_i$ over the same neighbourhood $U \ni K$ as above: take any $x \in F_i^{-1}(K)$ and any $K' \in U$. x can be written in the form $x = \sum_j \alpha_j L_j$, where all $\alpha_j > 0, \sum \alpha_j = 1, L_j \in \bigcup_{k \leq i} X_k$, and exactly one L_j belongs to X_i. Put $F(x, K') = \sum_k \alpha_k T(L_k, K')$. Again a straightforward argument shows that F is a homeomorphism $F_i^{-1}(K) \times U \to F_i^{-1}(U)$. Theorem 3.2 is proven. ♢

Since every $\Lambda(K)$ is compact in the topology of Λ, we have

$$\bar{H}_*(\Lambda(K) \setminus \partial\Lambda(K)) = H_*(\Lambda(K), \partial\Lambda(K)) = H_{*-1}(\partial\Lambda(K), \text{point}).$$

The abstract axiomatic approach we have taken pays off when it comes to performing actual calculations. Indeed, we have the freedom to choose the X_i-s as we like, provided they satisfy Conditions 1-7 on page 13. For instance, in all examples we shall consider most singular loci are discrete (i.e., they consist of finitely many points). To make the fibres of the bundle $\Phi_i \setminus \Phi_{i-1} \to X_i$ as simple as possible, we introduce one more condition:

8. if K is a finite configuration from X_i, then every subset $L \subset K$ is contained in some X_j with $j < i$.

Note that if we are given spaces $X_1, \ldots, X_N$ that satisfy Conditions 1-7, we can construct another collection of spaces $X'_1, \ldots, X'_{N'}$ that satisfy Conditions 1-7 and Condition 8. Indeed, take the union $\bigcup_{i=1}^N X_i \subset 2^{\mathbf{M}}$ and add all subsets of all finite configurations $K \in \bigcup_{i=1}^N X_i$; then take an appropriate stratification of the resulting subspace of $2^{\mathbf{M}}$.

We have the following two lemmas.

Lemma 3.1 *If Condition 8 is satisfied, then for any i such that X_i consists of finite configurations the fibre of the bundle $F_i : (\Phi_i \setminus \Phi_{i-1}) \to X_i$ over a point $K \in X_i$ is an open simplex, whose vertices correspond to the points of K.*

The proof is by induction on the number of points in $K \in X_i$. ♢

Note that in this situation the simplicial complex $\Lambda(K)$ is (piecewise linearly) isomorphic to the first barycentric subdivision of the simplex $\triangle$ spanned by the vertices of K. The complex $\partial\Lambda(K)$ is isomorphic to the first barycentric subdivision of the boundary of $\triangle$.

Moreover, denote by Λ^{fin} the union of the spaces $\Lambda(K)$ over all finite $K \in \bigcup_{i=1}^N X_i$.

Lemma 3.2 *If Condition 8 is satisfied, then there exists a map $C : \Lambda^{fin} \to \mathbf{M}^{*N}$ that maps every $K \in \Lambda^{fin}$ to some element of the interior of the simplex $\triangle(K)$ spanned by the points of K. This map is a homeomorphism on its image, and it maps $\Lambda(K)$ (respectively, $\partial\Lambda(K)$) homeomorphically on $\triangle(K)$ (respectively, $\partial\triangle(K)$).*

$\diamondsuit$

It follows from lemmas 3.1, 3.2 that for any i such that X_i consists of finite configurations, the fibre bundle $\Phi_i \setminus \Phi_{i-1}$ is isomorphic to the restriction to X_i of the obvious bundle $\mathbf{M}^{*k} \setminus \mathbf{M}^{*(k-1)} \to B(\mathbf{M}, k)$, where k is the number of points in a configuration from X_i. So we have

$$\bar{H}_*(\Phi_i \setminus \Phi_{i-1}) = \bar{H}_{*-(k-1)}(X_i, \pm\mathbb{R}),$$

where $\pm\mathbb{R}$ is a coefficient system (which will be described explicitly a little later). Suppose now that $\mathbf{k} = \mathbb{C}$; since all complex vector bundles are orientable we obtain (using the second assertion of theorem 3.2) that

$$\bar{H}_*(F_i \setminus F_{i-1}) = \bar{H}_{*-2d_i}(\Phi_i \setminus \Phi_{i-1}) = \bar{H}_{*-2d_i-(k-1)}(X_i, \pm\mathbb{R}),$$

where $d_i = \dim_{\mathbb{C}} L(K), K \in X_i$.

In other words, we have reduced the calculation of most columns of the first term of the spectral sequence corresponding to the filtration $\varnothing \subset F_1 \subset \cdots \subset F_N$ to the calculation of the Borel-Moore homology groups of the spaces X_i with coefficients in some one-dimensional local systems. In the following chapter we shall explain how the latter groups can be calculated.

Chapter 4

Some further preliminaries

In this chapter we state and/or prove some homological results that will be used in the calculations that we have in mind.

Definition. For any topological space X, denote by $F(X,k)$ the space of all ordered k-ples from X, i.e. the space

$$X \times \cdots \times X \setminus \{(x_1,\ldots,x_k) | x_i = x_j \text{ for some } i \neq j\}.$$

The space $B(X,k)$ defined on page 12 is the quotient of $F(X,k)$ by the natural action of the symmetric group S_k. We shall denote by $\tilde{B}(\mathbb{C}P^2,k)$ the subspace of $B(\mathbb{C}P^2,k)$ consisting of generic k-configurations (i.e. the configurations that contain no three points that belong to a line, no 6 points that belong to a conic etc.). The spaces $\tilde{F}(\mathbb{C}P^2,k)$ $\tilde{B}(\mathbb{R}P^2,k)$ and $\tilde{F}(\mathbb{R}P^2,k)$ are defined in a similar way.

Definition. For any space X and integer $k > 0$ we denote by $\pm\mathbb{R}$ the local system on $B(X,k)$ with fibre $\mathbb{R}$ that changes the sign under the action of any loop defining an odd permutation in a configuration from $B(X,k)$. When this does not lead to a confusion, the restriction of $\pm\mathbb{R}$ to a subspace $Y \subset B(X,k)$ will be denoted by the same symbol $\pm\mathbb{R}$.

4.1 Some general theorems

The following version of the Leray theorem will be frequently used in the sequel:

Theorem 4.1 *Let $p : E \to B$ be a locally trivial fibre bundle with fibre F, and let $\mathcal{L}$ be a local system of groups or vector spaces on E. Then there exists a spectral sequence converging to the Borel-Moore homology groups of E with coefficients in $\mathcal{L}$; the term E^2 is given by the formula $E^2_{p,q} \cong \bar{H}_p(B, \bar{\mathcal{H}}_q)$, where $\bar{\mathcal{H}}_q$ is the local system with fibre $\bar{H}_q(F, \mathcal{L}|F)$ that corresponds to the natural action of $\pi_1(B)$ on $\bar{H}_q(F, \mathcal{L}|F)$.*

$\diamondsuit$

This is a version of the Leray theorem on the spectral sequence of a locally trivial fibration. Let us describe the action of $\pi_1(B, x_0)$ on the fibre $\bar{\mathcal{H}}_q|_{x_0}$, where x_0 is a distinguished point in B. Identify $\bar{\mathcal{H}}_q|_{x_0}$ with $\bar{H}_q(F, \mathcal{L}|F)$, and put $F_{x_0} = p^{-1}(x_0)$. A loop γ in B defines a map $\tilde{f} : \mathcal{L}|F_{x_0} \to \mathcal{L}|F_{x_0}$ covering some map $f : F_{x_0} \to F_{x_0}$. Recall the construction of f. Consider a family of curves $\gamma_x(t), x \in F$ that cover γ. Then we can put $f(x) = \gamma_x(1), x \in F_{x_0}$. The map $\tilde{f} : \mathcal{L}|F_{x_0} \to \mathcal{L}|F_{x_0}$ consists simply of the maps $\mathcal{L}_x \to \mathcal{L}_{f(x)}$ induced by γ_x.

The map $\tilde{f}$ induces for every q a map $F_* : \bar{H}_q(F_{x_0}, \mathcal{L}|F_{x_0}) \to \bar{H}_q(F_{x_0}, \mathcal{L}|F_{x_0})$, which is exactly the map $\bar{\mathcal{H}}_q|_{x_0} \to \bar{\mathcal{H}}_q|_{x_0}$ induced by γ.

Theorem 4.2 *Let $E_1 \to B_1, E_2 \to B_2$ be two bundles and $\mathcal{L}_1, \mathcal{L}_2$ be local coefficient systems of groups or vector spaces on E_1, E_2 respectively. Let $f : E_1 \to E_2$ be a proper map that covers some map $g : B_1 \to B_2$ (i.e., f is a proper bundle map). Suppose $\tilde{f} : \mathcal{L}_1 \to \mathcal{L}_2$ is a morphism of coefficient systems that covers f. Then the map $\tilde{f}$ induces a homomorphism of the spectral sequences of theorem 4.1.*

◊

The map of the terms E^2 of these spectral sequences can be described explicitly in the following way. Let $F_x^{(i)}$ be the fibre of E_i over $x \in B_i$, $\bar{\mathcal{H}}_q^{(i)}$ be the coefficient system on B_i from theorem 4.1, $i = 1, 2$. Since f is a bundle map, it maps $F_x^{(1)}$ into $F_{g(x)}^{(2)}$, and $\tilde{f}$ maps $\mathcal{L}|F_x^{(1)}$ into $\mathcal{L}|F_{g(x)}^{(2)}$. The latter map induces for any $x \in B_1$ a map $\bar{H}_q(F_x^{(1)}, \mathcal{L}|F_x^{(1)}) \to \bar{H}_q(F_{g(x)}^{(2)}, \mathcal{L}|F_{g(x)}^{(2)})$, which can be considered as restriction to $\bar{\mathcal{H}}_q^{(1)}|_x$ of a map $f'_q : \bar{\mathcal{H}}_q^{(1)} \to \bar{\mathcal{H}}_q^{(2)}$ that covers g. The desired map of the terms E^2 of spectral sequences is just the map $\bar{H}_*(B_1, \bar{\mathcal{H}}_q^{(1)}) \to \bar{H}_*(B_2, \bar{\mathcal{H}}_q^{(2)})$ induced by f'.

Theorem 4.1 has the following corollary:

Corollary 4.1 *Let $N, \tilde{N}$ be manifolds, let $p : \tilde{N} \to N$ be a finite-sheeted covering, and let $\mathcal{L}$ be a local system of groups on $\tilde{N}$. Then $H^*(\tilde{N}, \mathcal{L}) = H^*(N, p(\mathcal{L}))$, where $p(\mathcal{L})$ denotes the direct image of the system $\mathcal{L}$.*

◊

If $\mathcal{L}$ is the constant local system with fibre $\mathbb{R}$, then the representation of $\pi_1(N, x), x \in N$ in the fibre of $p(\mathcal{L})$ is isomorphic to the natural action of $\pi_1(N, x)$ on the vector space spanned by the elements of $p^{-1}(x)$. In particular, if $\tilde{N}$ is simply connected and p is the quotient by a free action of a group G, we get the regular representation of G (G acts on its group algebra by left shifts).

Recall that any irreducible real or complex representation of a finite group G is included into the regular representation. If the representation is complex, it is obvious. In the real case it follows from Schur's lemma and from the well-known fact that if $R : G \to GL(V)$ and $S : G \to GL(W)$ are real representations of G, and $R_\mathbb{C}, S_\mathbb{C}$ are their complexifications, then $\dim_\mathbb{R}(\mathrm{Hom}(R, S)) = \dim_\mathbb{C}(\mathrm{Hom}(R_\mathbb{C}, S_\mathbb{C}))$, where $\mathrm{Hom}(R, S)$ is the space of representation homomorphisms between R and S (i.e. operators $f : V \to W$ such that $f(R(g)x) = S(g)f(x)$ for any $g \in G, x \in V$).

The homological analogues of theorems 4.1 and 4.2 can be obtained by omitting all bars over H-es and $\mathcal{H}$-es. The cohomological versions of these theorems can be obtained as follows: in theorem 4.1 the action of a loop γ is the inverse of the cohomology map induced by $\tilde{f} : \mathcal{L}_x \to \mathcal{L}_{f(x)}$, and in theorem 4.2 we have to suppose that $\mathcal{L}_1$ is the inverse image of $\mathcal{L}_2$, i.e. that the restriction of $\tilde{f}$ over each point is bijective.

Neither in the homological, nor in the cohomological analogue of theorem 4.2 we have to require for f to be proper.

We shall also need the following version of Poincaré duality theorem:

Theorem 4.3 *Let M be a orbifold of dimension n, and let $\mathcal{L}$ be a local system on M, whose fibre is a real or complex vector space. Then we have*

$$H^*(M, \mathcal{L} \otimes Or(M)) \cong \bar{H}_{n-*}(M, \mathcal{L}),$$

where $Or(M)$ is the orienting sheaf of M.

$\diamondsuit$

The following lemma allows us to calculate real cohomology groups of the quotient of a semisimple connected Lie group by a finite subgroup with coefficients in arbitrary local systems. We shall use this lemma several times.

If X is a topological space, we shall denote by $\mathbb{R}^k(X)$ the constant sheaf on X with fibre $\mathbb{R}^k$. When this does not lead to a confusion, we shall write $\mathbb{R}^k$ instead of $\mathbb{R}^k(X)$.

Lemma 4.1 *Let G be a connected Lie group, let G_1 be a finite subgroup of G, and let $p : G \to G/G_1$ be the natural projection. Let $\mathcal{L}$ be a local system on G/G_1, and suppose that $p^{-1}(\mathcal{L}) = \mathbb{R}^k$ and that the action of $\pi_1(G/G_1)$ on the fibre of $\mathcal{L}$ is irreducible. Then the cohomology groups $H^*(G/G_1, \mathcal{L})$ are equal to the groups $H^*(G, \mathbb{R})$, if $\mathcal{L} = \mathbb{R}$, and are zero otherwise.*

Proof. The case $\mathcal{L} = \mathbb{R}$ is settled easily: the group G_1 acts identically on $H^*(G)$, hence $H^*(G, \mathbb{R}) = H^*(G/G_1, \mathbb{R})$.

Suppose now that $\mathcal{L} \neq \mathbb{R}$. If we apply Corollary 4.1 to the covering $G \to G/G_1$, we obtain $H^*(G, \mathbb{R}^k(G)) = H^*(G/G_1, p_*(\mathbb{R}^k(G)))$. The action of $\pi_1(G/G_1)$ on a fibre of $p_*(\mathbb{R}^k(G))$ is completely reducible, since this action factorises through the homomorphism $\pi_1(G/G_1) \to G_1$. The representation of G_1 in the fibre of $p_*(\mathbb{R}^k(G))$ is the tensor product of the regular representation and the trivial representation of dimension k, which implies that $p_*(\mathbb{R}^k(G))$ contains $\mathbb{R}^k(G/G_1)$. On the other hand, $p_*(\mathbb{R}^k(G)) = p_*(p^{-1}(\mathcal{L}))$ contains an isomorphic copy of $\mathcal{L}$ (indeed, since p is surjective, the canonical map $\mathcal{L} \to p_*(p^{-1}(\mathcal{L}))$ is injective, and the action of $\pi_1(G/G_1)$ on a fibre of $p_*(\mathbb{R}^k(G))$ is completely reducible).

Since $\mathcal{L} \neq \mathbb{R}$, and the action of $\pi_1(G/G_1)$ on a fibre of $\mathcal{L}$ is irreducible, we conclude that

$$H^*(G, \mathbb{R}^k) \cong H^*(G/G_1, \mathbb{R}^k) \oplus H^*(G/G_1, \mathcal{L}) \oplus \text{something}.$$

But $H^*(G, \mathbb{R}^k) \cong H^*(G/G_1, \mathbb{R}^k)$, since G_1 acts identically on $H^*(G)$. Therefore, the groups $H^*(G/G_1, \mathcal{L})$ are zero. $\diamondsuit$

After theorem 4.1 on page 18 we gave an explicit construction of local systems that appear in the Leray spectral sequence of a locally-trivial fibration. However, in many interesting cases we have a map that is "almost" a fibration, say the quotient of a smooth manifold by an almost free action of a compact group etc., and we would like to know, what the Leray sequence (which is defined for any continuous map) looks like in this case. It turns out that in the case of a quotient map the sheaves that occur in the Leray sequence can be explicitly described (at least for some actions of some groups on some spaces).

Let us fix a smooth left action of a Lie group G on a manifold M. A submanifold $S \subset M$ is called a *slice* at $x \in S$ for the action of G iff GS is open in M, and there is a G-equivariant map $GS \to G/Stab(x)$ such that the preimage of $Stab(x)$ under this map is S. Here GS is the union of the orbits of all points of S, and $Stab(x)$ is the stabiliser of x.

If G is compact, a slice exists for any action at every point $x \in M$: provide M with a G-invariant Riemannian metric and put S equal to the exponential of an ε-neighbourhood of zero in the orthogonal complement of $T_x(Gx)$ for any sufficiently small ε. (Here Gx is the orbit of x.)

Theorem 4.4 *Suppose that G is connected, and for any $x \in M$ the group $Stab(x)$ is finite. Let $\mathcal{L}$ be a local system on M. Suppose that for any $x \in M$ there exist a connected Lie group $\tilde{G}$ and a finite covering $\alpha : \tilde{G} \to G$ such that the inverse image of $\mathcal{L}|Gx$ under the map $\tilde{G} \ni g \mapsto \alpha(g)x$ is constant. If there exists a slice for the action of G at every point $x \in M$, then for any i the sheaf $\mathcal{H}^i_{\mathcal{L}}$ on M/G generated by the presheaf $U \mapsto H^i(p^{-1}(U), \mathcal{L})$ is isomorphic to $p(\mathcal{L}) \otimes H^i(G, \mathbb{R})$, where $p : M \to M/G$ is the natural projection, $p(\mathcal{L})$ is the direct image of $\mathcal{L}$, and $H^i(G, \mathbb{R})$ is the constant sheaf with fibre $H^i(G, \mathbb{R})$. Moreover, if $\mathcal{L} = \mathbb{R}$, then $p(\mathcal{L}) = \mathbb{R}$.*

Note that the sheaf $\mathcal{H}^0_{\mathcal{L}}$ is canonically isomorphic to $p(\mathcal{L})$ for any $\mathcal{L}$.

Proof. Let us first consider the case $\mathcal{L} = \mathbb{R}$. We have to show that for any i the sheaf $\mathcal{H}^i_{\mathbb{R}}$ is constant with fibre isomorphic to $H^i(G, \mathbb{R})$. Let S be a slice for the action of G at $x \in M$. It is easy to show that S is invariant with respect to $Stab(x)$ and GS is homeomorphic to $G \times_{Stab(x)} S$, which is the quotient of $G \times S$ by the following action of $Stab(x) : g(g_1, x_1) = (g_1 g^{-1}, gx_1)$ for any $g \in Stab(x), g_1 \in G, x_1 \in S$. This implies easily that for any $x' \in M/G$ a local basis at x' is formed by open sets $U \ni x'$ such that $p^{-1}(U)$ contracts to $p^{-1}(x')$. Hence the canonical map $\rho_{x'} : \mathcal{H}^i_{\mathbb{R}}(x') \to H^i(p^{-1}(x'), \mathbb{R})$, where $\mathcal{H}^i_{\mathbb{R}}(x)$ is the fibre of $\mathcal{H}^i_{\mathbb{R}}$ over x, is an isomorphism.

Note that for any $x \in M$ the action map $\tau_x : G \to Gx, \tau_x(g) = gx$ induces an isomorphism of real cohomology groups (the existence of a slice at x implies that Gx is homeomorphic to $G/Stab(x)$, and, since G is connected, and $Stab(x)$ is finite, the real cohomology map induced by $G \to G/Stab(x)$ is an isomorphism). Let $\sigma : M/G \to M$ be any map, such that $p \circ \sigma = Id_{M/G}$. Now for any $i, y \in H^i(G, \mathbb{R})$ define the section s_y of $\mathcal{H}^i_{\mathbb{R}}$ as follows: $s_y(x') = \rho^{-1}_{x'} \circ (\tau^*_{\sigma(x')})^{-1}(y)$.

Let U be a neighbourhood of x' such that $p^{-1}(U)$ contracts to $p^{-1}(x')$. It is easy to check that for any $y \in H^i(G, \mathbb{R})$ there is a $y' \in H^i(p^{-1}(U))$ such that s_y coincides on U with the canonical section of $\mathcal{H}^i_{\mathbb{R}}$ over U defined by y', so all maps $x' \mapsto s_y(x')$ are indeed sections. It follows immediately from the definition that the section s_y is nowhere zero if $y \neq 0$. Putting y equal to elements of some basis in $H^i(G, \mathbb{R})$, we obtain $\dim(H^i(G, \mathbb{R}))$ everywhere linearly independent sections of $\mathcal{H}^i_{\mathbb{R}}$, whose values span $\mathcal{H}^i_{\mathbb{R}}(x')$ for any x'. Hence the map $(x', y) \mapsto F_{x'}(y)$ establishes an isomorphism between $\mathcal{H}^i_{\mathbb{R}}$ and $H^i(G, \mathbb{R})$. The theorem is proven in the case, when $\mathcal{L} = \mathbb{R}$.

Now suppose that the system $\mathcal{L}$ is arbitrary. Note that for any open subset U of M/G there is a natural map (the $\smile$-product)

$$H^0(p^{-1}(U), \mathcal{L}) \otimes H^i(p^{-1}(U), \mathbb{R}) \to H^i(p^{-1}(U), \mathcal{L} \otimes \mathbb{R}) \cong H^i(p^{-1}(U), \mathcal{L})$$

This gives us a map

$$\mathcal{H}^0_{\mathcal{L}} \otimes \mathcal{H}^i_{\mathbb{R}} \to \mathcal{H}^i_{\mathcal{L}}. \tag{4.1}$$

Due to the existence of a slice at each point of M the groups $\mathcal{H}^0_{\mathcal{L}}(x'), \mathcal{H}^i_{\mathbb{R}}(x'), \mathcal{H}^i_{\mathcal{L}}(x')$ are canonically isomorphic to $H^0(p^{-1}(x'), \mathcal{L}), H^i(p^{-1}(x'), \mathbb{R}), H^i(p^{-1}(x'), \mathcal{L})$ respectively (for any $x' \in M/G$). Under this identification the restriction of the map (4.1) to the fibre over a point $x' \in M/G$ is the cup product map

$$H^0(p^{-1}(x'), \mathcal{L}) \otimes H^i(p^{-1}(x'), \mathbb{R}) \to H^i(p^{-1}(x'), \mathcal{L}). \tag{4.2}$$

Now take some $x \in p^{-1}(x')$. There exist a connected Lie group $\tilde{G}$, a finite subgroup $G_1 \subset G$ and a diffeomorphism $\beta : (\tilde{G}/G_1) \to p^{-1}(x')$ such that $(\beta \circ p_1)^{-1}(\mathcal{L}|p^{-1}(x'))$ is constant (here $p_1 : \tilde{G} \to \tilde{G}/G_1$ is the natural projection). Hence, the action of $\pi_1(p^{-1}(x'))$ in the fibre of $\mathcal{L}|p^{-1}(x')$ splits into a sum of irreducible representations, i.e., $\mathcal{L}|p^{-1}(x')$ can be decomposed into a sum $\mathcal{L} = \oplus \mathcal{L}_j$ such that the action of $\pi_1(p^{-1}(x'))$ on a fibre of each $\mathcal{L}_j$ is irreducible. The proof of theorem 4.4 is completed by applying lemma 4.1 to each $\mathcal{L}_j$. $\diamondsuit$

Note that there exists a natural action of the group $GL_{n+1}(\mathbb{C})$ on the space $\Pi_{d,n} \setminus \Sigma_{d,n}$. It is quite easy to show that the stabiliser of any point of $\Pi_{d,n} \setminus \Sigma_{d,n}$ is finite (cf. [9, Proposition 4.2]). The existence of a slice at each point follows either from the results of [8] or [10]. The following general result of C.A.M. Peters and J.H.M. Steenbrink (see [11]) gives a complete description of the Leray spectral sequence of the quotient map in this situation:

Theorem 4.5 *If $d > 2$, then the rational cohomological Leray sequence of the map $\Pi_{d,n} \setminus \Sigma_{d,n} \to (\Pi_{d,n} \setminus \Sigma_{d,n})/GL_{n+1}(\mathbb{C})$ degenerates in the term E_2.*

$\diamondsuit$

We shall not use this theorem in the sequel, but we would like to point out that it has the following interesting corollary in the real case:

Corollary 4.2 *For any even n and for any $d > 2$, the cohomological Poincaré polynomial (see page 1) of every connected component of the space $\Pi_{d,n}(\mathbb{R}) \setminus \Sigma_{d,n}$ is divisible by $(1+t^3)(1+t^7)\cdots(1+t^{2n-1})$.*

Proof of corollary 4.2. For any odd k, the cohomological Poincaré polynomial of U_k is the product of the Poincaré polynomials of $SO_k(\mathbb{R})$ and $U_k/SO_k(\mathbb{R})$ (see, e.g. [3, Chapter 4]). Hence, the restriction $H^*(U_k, \mathbb{R}) \to H^*(SO_k(\mathbb{R}), \mathbb{R})$ is surjective for odd k, and the corollary follows.$\diamondsuit$

4.2 Some technical lemmas

The following lemmas will be frequently used in our calculations:

Lemma 4.2 *The group $\bar{H}_*(B(\mathbb{C}^n, k), \pm\mathbb{R})$ is zero for any $k \geq 2, n \geq 1$.*

Lemma 4.2 is a classical result on the cohomology of configuration spaces (see, e.g., [15] for a proof; an alternative proof can be given using lemma 4.5 below). $\diamondsuit$

Remark. This lemma, which is modestly hidden somewhere in the middle of the text, is the key to most calculations in the complex case. The absence of the real analogue of this lemma is one of the reasons why in the real case the number of examples that have been considered using the method of chapter 3 is so limited; another reason may consist in the fact that all complex vector bundles, as opposed to real vector bundles, are orientable (and even canonically oriented).

Lemma 4.3 *The group $\bar{H}_*(B(\mathbb{C}P^n, k), \pm\mathbb{R})$ for $n \geq 1$ is isomorphic to $H_{*-k(k-1)}(G_k(\mathbb{C}^{n+1}), \mathbb{R})$, where $G_k(\mathbb{C}^{n+1})$ is the Grassmann manifold of k-dimensional subspaces in $\mathbb{C}^{n+1}$.*

In particular, the group $\bar{H}_*(B(\mathbb{C}P^n, k), \pm\mathbb{R})$ is zero if $k > n+1$. This lemma follows easily from lemma 4.2 (see [14]). $\diamondsuit$

Lemma 4.4 *If $k \geq 2$, then the group $H_*((S^2)^{*k}, \mathbb{R})$ is zero in all positive dimensions, where $(S^2)^{*k}$ is the k-th self-join of S^2.*

Lemma 4.4 is proven in [14]. Alternatively, one can note that for any good topological space X, the "stable" autojoin $X^{*\infty}$, which is defined to be the inductive limit

$$X \subset X^{*2} \subset X^{*3} \subset \cdots,$$

is contractible, hence the terms of positive dimension of the spectral sequence that corresponds to this filtration kill each other. One can easily see that $X^{*n} \setminus X^{*(n-1)}, n \geq 2$, is fibered over $B(X, n)$, the fibre being the open $(n-1)$-simplex, and we have $\bar{P}(X^{*n} \setminus X^{*(n-1)}) = t^{n-1}\bar{P}(B(X, n), \pm\mathbb{R})$. So, by lemma 4.3, for $X = S^2$ the only two nonzero columns of the corresponding spectral sequence will be the 0-th and the 1-st ones; the differential $d_1 : E^1_{1,2} \to E^1_{0,2}$ is nonzero, and lemma 4.4 follows. $\diamondsuit$

Lemma 4.5 *Let X be a good topological space such that the symmetric group $S_k, k > 0$ acts identically on $\bar{H}_*(X^{\times k}, \mathbb{R})$. Then we have $\bar{H}_*(B(X, k), \pm\mathbb{R}) = 0$.*

Proof. Let us prove that $\bar{H}_*(F(X, k), \mathbb{R})$ does not contain nonzero totally antisymmetric elements. We have $F(X, k) = X^{\times k} \setminus Diag$, where $Diag = \{(x_1, \ldots, x_k) | x_i = x_j \text{ for some } i \neq j\}$. Since S_k acts identically on $\bar{H}_*(X^{\times k}, \mathbb{R})$, an antisymmetric element in $\bar{H}_*(F(X, k), \mathbb{R})$ can come only from $\bar{H}_*(Diag, \mathbb{R})$.

For any $i = 1, \ldots, k-1$ set

$$Diag_i = \{(x_1, \ldots, x_k) | \#\{x_1, \ldots, x_k\} = i\},$$

where # denotes the cardinalty of a finite set. The absence of totally antisymmetric elements in $\bar{H}_*(Diag, \mathbb{R})$ results from the following observations:

1. We have $Diag = Diag_1 \sqcup \ldots \sqcup Diag_{k-1}$, and the closure in $X^{\times k}$ of any $Diag_i$ is $\bigcup_{j \leq i} Diag_j$.
2. Every $Diag_i$ is the disjoint union of many copies of $F(X, i)$, hence $\bar{H}_*(Diag_i) \cong \bigoplus_{\text{many times}} \bar{H}_*(F(X, i))$.
3. For every copy of $F(X, i)$ in $Diag_i$ there is a transposition from S_k that acts identically on that copy.

The lemma is proven.$\diamondsuit$

Consider the space $\mathbb{C} \setminus \{1, -1\}$ and the coefficient system $\mathcal{L}$ on it that changes its sign under any loop based at 0 that passes once around 1 or -1. Let f be the map $z \mapsto -z$ and let $\tilde{f} : \mathcal{L} \to \mathcal{L}$ be the map that covers f and is identical over 0.

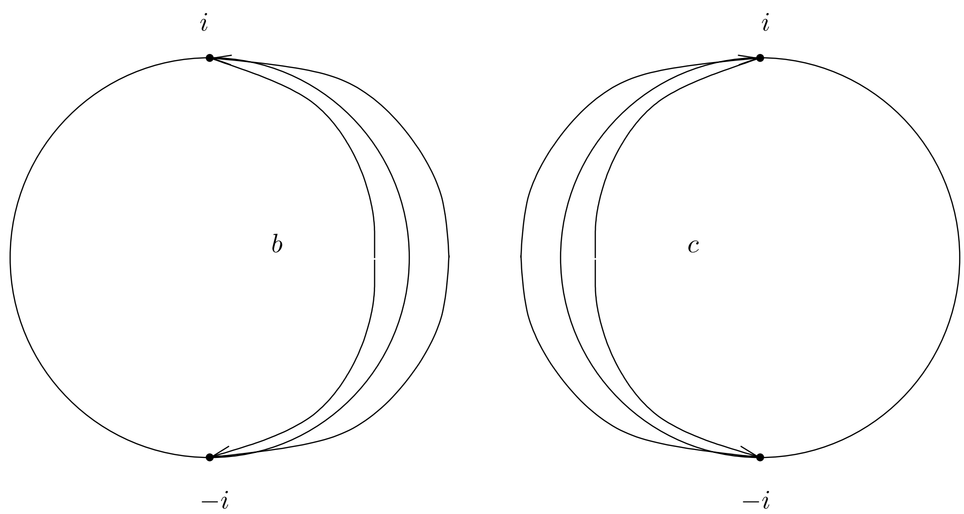

Figure 4.1:

Proposition 4.1 *The map $\tilde{f}$ acts on the groups $H_1(\mathbb{C} \setminus \{1,-1\}, \mathcal{L})$, $H^1(\mathbb{C} \setminus \{1,-1\}, \mathcal{L})$, and $\bar{H}_1(\mathbb{C} \setminus \{1,-1\}, \mathcal{L})$ as the multiplication by -1.*

$\diamondsuit$

Consider $B(\mathbb{C}^*, 2)$, i.e. the space of pairs of points in $\mathbb{C} \setminus \{0\}$. It is a fibre bundle over $\mathbb{C}^*$, the projection $p : B(\mathbb{C}^*, 2) \to \mathbb{C}^*$ being just the multiplication. The fibre is homeomorphic to $\mathbb{C} \setminus \{1, -1\}$, and the action of the generator of $\pi_1(\mathbb{C}^*)$ is $z \mapsto -z$. The fibre $p^{-1}(1)$ contracts to the character "8". Denote by b, c the loops in $p^{-1}(1)$ based at $\{\pm i\}$ and represented schematically on Figure 4.1; note that they correspond to the circles of the "8". Denote by a the loop $t \mapsto \{ie^{\pi it}, -ie^{\pi it}\}$.

Consider the following three local systems on $B(\mathbb{C}^*, 2)$ (the fibre of each of them is $\mathbb{R}$):

1. $\mathcal{A}_1$ changes its sign under a and does not change its sign under b and c.
2. $\mathcal{A}_2$ changes its sign under b and c and does not change its sign under a.
3. $\mathcal{A}_3$ changes its sign under all loops a, b, c.

Note that we have $\mathcal{A}_3 = \pm\mathbb{R}$.

Let f be the map $B(\mathbb{C}^*, 2) \to B(\mathbb{C}^*, 2)$ induced by the map $z \mapsto 1/z$ and let $f^i : \mathcal{A}_i \to \mathcal{A}_i, i = 1, 2, 3$ be the map that covers f and is identical over the fibre $p^{-1}(1)$.

Proposition 4.2 *1. $\bar{P}(B(\mathbb{C}^*, 2), \mathcal{A}_1) = \bar{P}(B(\mathbb{C}^*, 2), \mathcal{A}_3) = t^2(1 + t)$, $\bar{P}(B(\mathbb{C}^*, 2), \mathcal{A}_2) = 0$ (recall that we denote by $\bar{P}$ the "Borel-Moore-Poincaré" polynomial, see page 1).*

2. The map f^i_* acts as the identity on $\bar{H}_3(B(\mathbb{C}^*, 2), \mathcal{A}_i)$ and as minus identity on $\bar{H}_2(B(\mathbb{C}^*, 2), \mathcal{A}_i)$ (i=1,3).

Note that the lifting of the generator of $\pi_1(\mathbb{C}^*)$ is as follows: $\gamma_{\{a,b\}}(t) = \{ae^{\pi it}, be^{\pi it}\}$, where $\{a,b\} \in p^{-1}(1)$.

This assertion follows immediately from theorems 4.1 and 4.2. $\diamondsuit$

Chapter 5

Smooth plane quintics

In this chapter we use the method described in chapter 3 to calculate the real cohomology groups of the space of polynomials that define smooth complex plane projective quintics. Recall that on page 1 we defined the spaces $\Pi_{d,n}$ and $\Sigma_{d,n}$ for any integers $d, n \geq 1$.

Theorem 5.1 *The Poincaré polynomial of the space* $\Pi_{5,2} \setminus \Sigma_{5,2}$ *is equal to*

$$(1+t)(1+t^3)(1+t^5).$$

By the Alexander duality, the cohomology group of $\Pi_{5,2} \setminus \Sigma_{5,2}$ is isomorphic to the Borel-Moore homology of $\Sigma_{5,2}$:

$$H^i(\Pi_{5,2} \setminus \Sigma_{5,2}, \mathbb{R}) = \bar{H}_{2D-1-i}(\Sigma_{5,2}, \mathbb{R}),$$

where $D = \dim_{\mathbb{C}}(\Pi_5) = 21$ and $0 < i < 2D-1$. This reduction was used first by V. I. Arnold in [2]. To calculate the latter group $\bar{H}_*(\Sigma, \mathbb{R})$ we use the method constructed in chapter 3. It is described by the following theorem.

Theorem 5.2 *There exists a spectral sequence that converges to the real Borel-Moore homology of the space* $\Sigma_{5,2}$ *and is defined by the following conditions:*

1. Any its nontrivial term $E^1_{p,q}$ *belongs to the quadrilateral in the* (p,q)*-plane, defined by the conditions* $[1 \leq p \leq 3, 29 \leq q \leq 39]$.

2. In this quadrilateral all the nontrivial terms $E^1_{p,q}$ *look as is shown in (5.1).*

3. The spectral sequence stabilises in this term, i.e. $E^1 \equiv E^\infty$.

39	$\mathbb{R}$		
38			
37	$\mathbb{R}$		
36			
35	$\mathbb{R}$	$\mathbb{R}$	
34			
33		$\mathbb{R}$	
32			
31		$\mathbb{R}$	
30			
29			$\mathbb{R}$
	1	2	3

(5.1)

The rest of this chapter is devoted to the proof of theorem 5.2. Put $V = \Pi_{5,2}, \Sigma = \Sigma_{5,2}$. In the sequel, by a *conic* we shall mean any curve of degree 2 in $\mathbb{C}P^2$. We shall say that points $x_1, \ldots, x_k \in \mathbb{C}P^2$ are *in general position*, if among these points there are no 3 points that are on the same line, no 6 points on a conic, no 10 points on a cubic, etc. A line $l \subset \mathbb{C}P^2$ is said to be nontangential to an algebraic curve C, if $l \cap Q$ consists of $\deg C$ points. By definition, lines $l_1, \ldots, l_k \subset \mathbb{C}P^2$ are *in general position*, if the elements of $\mathbb{C}P^{2\vee}$ corresponding to $l_1, \ldots, l_k$ are in general position.

5.1 Configuration spaces

Proposition 5.1 *The configuration spaces $X_1, \ldots, X_{42}$ that consist of the following configurations satisfy Conditions 1–7 and Condition 8 (see pages 13 and 16). The number indicated in brackets equals the dimension of $L(K)$ for K lying in the corresponding X_i.*

1. *One point in $\mathbb{C}P^2$ (18).*
2. *2 points in $\mathbb{C}P^2$ (15).*
3. *3 points in $\mathbb{C}P^2$ (12).*
4. *4 points on a line (11).*
5. *5 points on a line (10).*
6. *6 points on a line (10).*
7. *7 points on a line (10).*
8. *8 points on a line (10).*
9. *9 points on a line (10).*
10. *10 points on a line (10).*
11. *A line in $\mathbb{C}P^2$ (10).*
12. *4 points in $\mathbb{C}P^2$ not on a line (9).* (*Any three of them may belong to a line though.*)
13. *4 points on a line + a point not belonging to the line (8).*
14. *5 points on a line + one point not belonging to the line (7).*
15. *6 points on a line + one point not belonging to the line (7).*
16. *7 points on a line + one point not belonging to the line (7).*
17. *A line in $\mathbb{C}P^2$+ a point not belonging to the line (7).*
18. *5 points in $\mathbb{C}P^2$ such that there is no line containing 4 of them (6).*
19. *4 points on a line + 2 points not belonging to the line (5).*
20. *5 points on a line + 2 points not belonging to the line (4).*

21. *6 points on a line + 2 points not belonging to the line (4).*

22. *A line in $\mathbb{C}P^2$+ 2 points not belonging to it (4).*

23. *3 points on each of two intersecting lines such that none of the points coincides with the point of intersection (4).*

24. *6 points on a nondegenerate conic (4).*

25. *A configuration of type 23 + the point of intersection of the lines (4).*

26. *6 points not belonging to a (possibly degenerate) conic such that there is no line containing 4 of those points (3).*

27. *4 points on a line + 3 points on another line such that none of the 7 points coincides with the point of intersection of the lines (3).*

28. *5 points on a line l_1 + 3 points on $l_2 \setminus l_1$, where l_2 is a line $\neq l_1$ (3).*

29. *A line + 3 points of some other line, none of which coincides with the point of intersection of the lines (3).*

30. *4 points on a line + 4 points on some other line such that none of the 8 points coincides with the point of intersection of the lines (3).*

31. *A union of two lines in $\mathbb{C}P^2$ (3).*

32. *7 points on a nondegenerate conic (3).*

33. *A nondegenerate conic (3).*

34. *4 points on a line + 3 points in general position not belonging to the line (2).*

35. *A configuration of type 23 + a point not belonging to the union of the lines (1).*

36. *6 points on a nondegenerate conic + a point not belonging to the conic (1).*

37. *A configuration of type 35 + the point of intersection of the lines (1).*

38. *4 points $A, B, C, D \in \mathbb{C}P^2$ in general position + 4 points of intersection of a line l not passing through A, B, C, D and two (possibly degenerate) conics passing through A, B, C, D and not tangential to l (1).*

39. *3 points $A, B, C \in \mathbb{C}P^2$ in general position + 6 points of intersection of 3 lines AB, BC, AC and a (possibly degenerate) conic not passing through A, B, C, and not tangential to the lines AB, BC, AC (1).*

40. *10 points of intersection of 5 lines in $\mathbb{C}P^2$ in general position (1).*

41. *A line in $\mathbb{C}P^2$+ 3 points in general position not belonging to the line (1).*

42. *The whole $\mathbb{C}P^2$ (0).*

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Proof. Condition 1 follows from the following observations: 1. the singular set of a curve defined by a product of two polynomials is the union of the singular sets of the curves defined by those polynomials and the intersection points of the curves; 2. the singular set of an irreducible curve of degree 5 consists of 1 to 6 points in general position; 3. all possible singular sets of curves of degree ≤ 4 are described in [14]. Note that some spaces X_i contain (or consist of) configurations that are not sets of singular points of any curve of degree 5. We introduce them to make sure that Conditions 5 and 8 are satisfied.

The verification of Conditions 2, 4 and 8 is straightforward.

Condition 3 will be deduced below from the following lemma.

Lemma 5.1 *1. Let $x_1, \ldots, x_k, k \leq 6$, be several points in general position in $\mathbb{C}P^2$. Then the complex dimension of the space $L(\{x_1, \ldots, x_k\})$ (which consists of polynomials of degree 5 that have singularities at all points $x_1, \ldots, x_k$ and, maybe, elsewhere) is equal to $21 - 3k$.*

2. Let l_1, l_2 be two distinct lines in $\mathbb{C}P^2$. Suppose that
a) $x_1, x_2, x_3 \in l_1 \setminus l_2$,
b) $y_1, y_2, y_3 \in l_2 \setminus l_1$,
c) $A \notin l_1 \cup l_2$.
Then there exists exactly one cubic passing through all the points $x_i, y_j, i, j = 1, 2, 3$ and having a singularity at A.

3. Let Q be a nondegenerate conic in $\mathbb{C}P^2$, and suppose that $x_1, \ldots, x_6 \in Q, A \notin Q$. Then there exists exactly one cubic passing through $x_1, \ldots, x_6$ and having a singularity at A.

4. If a curve of degree 5 has three singular points on a line, then it contains the line. If a curve of degree 5 has six singular points on a nondegenerate conic, then it contains the conic.

5. Consider a point $A \in \mathbb{C}P^2$. For any d define $L_d^x(A)$ (respectively, $L_d^y(A), L_d^z(A), M_d(A)$) as the linear space of homogeneous polynomials f of degree d such that $\frac{\partial f}{\partial x} = 0$ (respectively, $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial z} = 0$, $f = 0$) at every point of the preimage of A under the natural map $\mathbb{C}^3 \setminus \{0\} \to \mathbb{C}P^2$. Suppose that l is a line in $\mathbb{C}P^2, x_1, x_2, x_3, x_4 \in l, y_1, y_2, y_3 \notin l$, and suppose that y_1, y_2, y_3 are not on a line. Then

$$dim_{\mathbb{C}}((\cap_{i=1}^4 M_4(x_i)) \cap (\cap_{i=1}^3 L_4^x(y_i)) \cap (\cap_{i=1}^3 L_4^y(y_i)) \cap (\cap_{i=1}^3 L_4^z(y_i))) = 2.$$

Remark. Assertion 5 of lemma 5.1 implies that the thirteen hyperplanes $M_4(x_i), i = 1, \ldots, 4, L_4^x(y_i)$, $L_4^y(y_i), L_4^z(y_i), i = 1, 2, 3$ intersect transversally.

Let us prove the first assertion of the lemma. Suppose that $x_1, \ldots, x_6$ are points of $\mathbb{C}P^2$ in general position. It suffices to prove that 18 linear conditions on the space $\Pi_{5,2}$ that define the space $L(\{x_1, \ldots, x_6\})$ are independent. Suppose they are not, then $\dim_{\mathbb{C}} L(\{x_1, \ldots, x_6\}) \geq 4$. Choose a point x' such that $x_1, \ldots, x_6, x'$ are in general position. The space $L(\{x_1, \ldots, x_6, x'\})$ would be then of dimension ≥ 1, which is impossible, because no curve of degree 5 can have 7 singular points in general position (if the curve is irreducible, this follows from [12, exercise 3, §2 of Chapter III], otherwise this is trivial).

The assertions 2, 3, 5 can be proved in an analogous way. The assertions 4 follows from Bézout's theorem. ◇

Let us prove now that the spaces X_i introduced in proposition 5.1 satisfy Condition 3. The case of X_1 and X_2 is obvious. If we have a configuration K that consists of three points in $\mathbb{C}P^2$ that do not belong to any line, then using the first assertion of lemma 5.1, we get $\dim_{\mathbb{C}} L(K) = 21 - 9 = 12$. If $K \subset \mathbb{C}P^2$ consists of three points on a line $l \subset \mathbb{C}P^2$, then due to assertion 4 of lemma 5.1, any function $f \in L(K)$ has the form $f = gh$, where g is a fixed linear homogeneous function, and h is a polynomial that defines a curve that intersects l in every point of K and, maybe, elsewhere. Using assertion 5 of lemma 5.1, we obtain $\dim_{\mathbb{C}} L(K) = 15 - 3 = 12$. We have thus proven that $\dim_{\mathbb{C}} L(K) = 12$ for any $K \in X_3$.

The case of X_4 can be considered in an analogous way.

If a curve of degree 5 contains five singular points on a line, then this curve is defined by a polynomial of the form $f^2 g$, where f is a polynomial that defines the line, and g is a polynomial of degree 3. This gives the dimensions of all spaces $L(K), K \in X_5, \dots, X_{11}, X_{14}, \dots, X_{17}, X_{20}, \dots, X_{22}$.

Consider a configuration $K \in X_{12}$. If no three of the points of K are on a line, we have $\dim_{\mathbb{C}}(L(K)) = 21 - 12 = 9$ by the assertion 1 of lemma 5.1. If K contains 3 points on a line l, then, due to assertion 4 of lemma 5.1, every $f \in L(K)$ has the form $f = gh$, where g is a fixed polynomial of degree 1, and h is an arbitrary polynomial of degree 4 that defines a curve that has 3 fixed intersection points with l and a fixed singular point outside l. Using assertion 5 of the same lemma (the transversality of intersection), we see that the dimension of $L(K)$ is equal to $15 - 3 - 3 = 9$. The same argument gives the dimensions of $L(K), K \in X_{13}, X_{19}, X_{34}, X_{18}, X_{26}$.

Note that if l_1, l_2 are two distinct lines in $\mathbb{C}P^2$, $x_1, x_2, x_3 \in l_1 \setminus l_2, y_1, y_2, y_3 \in l_2 \setminus l_1, A \notin l_1 \cup l_2$, then assertion 2 of lemma 5.1 implies that

$$\dim_{\mathbb{C}}(\bigcap_{i=1}^{3} M_3(x_i)) \cap (\bigcap_{i=1}^{3} M_3(y_i)) \cap L_3^x(A) \cap L_3^y(A) \cap L_3^z(A) = 1,$$

which means that nine hyperplanes $M_3(x_i), M_3(y_i), i = 1, 2, 3, L_3^x(A)$, $L_3^y(A)$, and $L_3^z(A)$ intersect transversally. This gives the dimensions of $L(K), K \in X_{23}, X_{25}, X_{35}, X_{37}$. The same argument works for X_{24}, X_{36}, except that we apply assertion 3 of lemma 5.1 (instead of assertion 2).

It is easy to see that for any $K \in X_{27}, \dots, X_{31}$ the vector space $L(K)$ consists of polynomials of the form $f^2 g^2 h$, where f, g are some fixed polynomials of degree 1 that define two distinct lines, and h is an arbitrary polynomial of degree 1. Analogously, for any $K \in X_{32}, X_{33}$ the space $L(K)$ consists of polynomials of the form $f^2 g$, where f is a fixed polynomial of degree 2 that defines a nondegenerate conic, and h is an arbitrary polynomial of degree 1.

Consider a configuration $K \in X_{38}$ and $f \in L(K)$. It follows from assertion 4 of lemma 5.1 that $f = gh$, where g is a polynomial of degree 1, and h is a polynomial of degree 4 that has singularities at four points A, B, C, D in general position outside the line l defined by g. This implies that $h = h_1 h_2$, where h_1, h_2 are polynomials of degree 2 that define two conics Q_1, Q_2 passing through A, B, C, D. f must also have singularities at four points on l, hence each of these four points belongs to exactly one of the conics Q_1, Q_2. It follows that f is defined by K up to multiplication by a nonzero constant.

Analogously it can be proved that for any $K \in X_{39}, X_{40}$ and any $f \in L(K)$, f is defined by K up to nonzero constant. The cases X_{41}, X_{42} are trivial. Thus, we have proved that the spaces X_i satisfy Condition 3.

Let us prove now that these spaces satisfy Conditions 6 and 7. Recall that the spaces X_i satisfy Condition 8. This implies (see p. 13) that for the spaces X_i consisting of finite configurations Condition 7 follows from Condition 6. Consider some X_i that consists of finite configurations. It is immediate to check that the number of elements in all configurations from X_i is the same. Denote this number by k. Put $\mathcal{M}_k = \{(x,K) | x \in \mathbb{C}P^2, K \subset \mathbb{C}P^2, \#(K) = k, x \in K\}$ (here and below, for a finite set K we denote by $\#(K)$ the cardinality of K). It is easy to see that $\mathcal{M}_k$ is the total space of a fibre bundle over $B(\mathbb{C}P^2, k)$ (with the projection $(x,K) \mapsto K$). The triple $(\mathcal{T}_i, X_i, pr_i)$ is the restriction of this fibre bundle to X_i.

Thus, all spaces X_i that consist of finite configurations satisfy Conditions 6 and 7.

Now consider, for instance, the space X_{31}. Note that if G is a Lie group that acts smoothly on a smooth manifold M and $a \in M$, there exist submanifolds $S \subset M, S' \subset G$ such that

1. $a \in S, e \in S'$ (e is the unit element of G),
2. S' is transversal to $Stab(a)$ at e,
3. $S'a \subset M$ is a submanifold that intersects S transversally at a (here $S'x = \{gx | g \in S'\}$),
4. the map $S \times S' \to M$, defined by $(a', g) \mapsto ga', g \in S', a' \in S$ is a diffeomorphism onto an open neighbourhood of a in M.

Put $M = X_{31}, G = PGL(\mathbb{C}P^2)$. The action of G on M is transitive, so for any $K \in X_{31}$ the above remark gives us a neighbourhood $U \ni K$ and a diffeomorphism $r : U \to S', S' \subset PGL(\mathbb{C}P^2)$ such that for any $K' \in U$ we have $r(K')K = K'$. Now for any $x \in K$ and $K' \in U$, put $t(x,K') = r(K')x$. It is clear that the map $(x,K') \mapsto (t(x,K'), K')$ is a local trivialisation of $\mathcal{T}_{31}$ over U. This trivialisation satisfies Condition 7, since all spaces X_i are invariant under $PGL(\mathbb{C}P^2)$.

Local trivialisations of tautological bundles over the spaces $X_{11}, X_{17}, X_{22}, X_{41}$, and X_{33} can be constructed in the same way.

However, this method does not work for X_{29}, because the action of $PGL(\mathbb{C}P^2)$ on this space is no longer transitive. But we can proceed as follows. Consider $K \in X_{29}$. We have $K = K_1 \sqcup K_2$, where K_1 is a line, and K_2 consists of three points of another line. Denote by B' the space of all configurations of $\mathbb{C}P^2$ consisting of three points on a line. Let U_1 (respectively, U_2) be neighbourhoods of K_1 in X_{11} (respectively, of K_2 in B') such that the bundle $(\mathcal{T}_{11}, X_{11}, pr_{11})$ is trivial over U_1, $(\mathcal{T}_3, X_3, pr_3)$ is trivial over U_2, and for every $K_1' \in U_1', K_2' \in U_2'$ we have $K_1' \cap K_2' = \varnothing$. For $j = 1, 2$ let $t_j : K_j \times U_j \to \mathbb{C}P^2$ be a map such that the map $(x, K_j') \mapsto (t_j(x, K_j'), K_j'), x \in K_j, K_j' \in U_j$ is a trivialisation of the corresponding tautological bundle over U_j. Put $U = \{K_1' \sqcup K_2' | K_1' \in U_1', K_2' \in U_2'\}$, and for any $K' = K_1' \sqcup K_2' \in U$ put $t(x,K')$ equal to $t_j(x, K_j')$, if $x \in K_j, j = 1, 2$. It is clear that U is an open neighbourhood of K in X_{29} and that the map $(x,K') \mapsto (t(x,K'), K')$ is a trivialisation of $(\mathcal{T}_{29}, X_{29}, pr_{29})$ over U. It follows from the construction of t that for any fixed $K' \in U$, the map $x \mapsto t(x,K')$

establishes a bijective correspondence between the subsets of K and K' that belong to $\bigcup_{j\leq 29} X_j$. (Due to Condition 8, this needs to be checked only for maximal finite subconfigurations of K (which belong to $X_{28}, X_{21}, X_{16}, X_{10}$) and for nondiscrete subconfigurations (which belong to X_{11}, X_{17}, X_{22}).)

We have shown that the spaces X_i satisfy Conditions 6 and 7. It remains to verify Condition 5.

Let us begin with the following three lemmas.

Lemma 5.2 *Denote by Π_d the vector space of all homogeneous polynomials $\mathbb{C}^3 \to \mathbb{C}$ of degree d. The map $\Pi_d \setminus \{0\} \to 2^{\mathbb{C}P^2}$ that sends a polynomial into the projectivisation of the set of its zeroes is continuous.*

$\diamondsuit$

Corollary 5.1 *The subspace of $2^{\mathbb{C}P^2}$ consisting of all zero sets of homogeneous polynomials of some fixed degree is closed.*

Remark. In the case of real polynomials, the analogous map to the real projective plane is neither everywhere defined nor continuous on its domain of definition.

For any $f \in \Pi_d \setminus \{0\}$ denote by $[f]$ the image of f under the natural map $\Pi_d \setminus \{0\} \to (\Pi_d \setminus \{0\})/\mathbb{C}^*$.

Lemma 5.3 *Suppose we have a sequence $(K_i), K_i \in 2^{\mathbb{C}P^2}$, and a sequence $(F_i), F_i \in \Pi_d \setminus \{0\}$, and suppose that F_i has a singularity at every point of K_i. If $K \in 2^{\mathbb{C}P^2}, f \in \Pi_d \setminus \{0\}$ are such that $\lim_{i\to\infty} K_i = K, \lim_{i\to\infty} [f]_i = [f]$, then f has a singularity at every point of K.*

$\diamondsuit$

Lemma 5.4 *Suppose we have sequences (L_i) and (M_i) in $2^{\mathbb{C}P^2}$ and suppose that $K \in 2^{\mathbb{C}P^2}, K = \lim_{i\to\infty}(L_i \cup M_i)$. Then there exists a sequence of indices (i_j) such that $K = (\lim_{j\to\infty} L_{i_j}) \cup (\lim_{j\to\infty} M_{i_j})$.*

Proof of lemma 5.4. Choose a sequence (i_j) such that there exist $\lim_{i\to\infty} L_i, \lim_{i\to\infty} M_i$, and denote these limits by L, M respectively. Let ρ be a metric that induces the usual topology on $\mathbb{C}P^2$, and let $\tilde{\rho}$ be the corresponding Hausdorff metric on $2^{\mathbb{C}P^2}$. If $A, B, C, D \in 2^{\mathbb{C}P^2}$, then $\tilde{\rho}(A \cup B, C \cup D) \leq \tilde{\rho}(A, C) + \tilde{\rho}(B, D)$. This implies that $\tilde{\rho}(M_{i_j} \cup L_{i_j}, M \cup L) \leq \tilde{\rho}(M_{i_j}, M) + \tilde{\rho}(L_{i_j}, L)$. Hence $M \cup L = \lim_{j\to\infty}(M_{i_j} \cup L_{i_j}) = \lim_{i\to\infty}(M_i \cup L_i) = K.\diamondsuit$

Now the verification of Condition 5 becomes straightforward in all cases except X_{38}, X_{39}, X_{40}. Consider, for instance, $K \in \bar{X}_{30}$. We have $K = \lim_{i\to\infty} K_i$, since all $K_i \in X_{30}$, they can be represented as $K_i = (K_i \cap l_1^i) \cup (K_i \cap l_2^i)$, where l_1^i, l_2^i are lines. Due to lemma 5.4 we can suppose that $K = (\lim_{i\to\infty}(K_i \cap l_1^i)) \cup (\lim_{i\to\infty}(K_i \cap l_2^i))$. Using Corollary 5.1, we can suppose that the sequences $(l_1^i), (l_2^i)$ converge. Applying proposition 3.2, we see that K is a configuration of the form (≤ 4 points on a line l_1)$\cup$(≤ 4 points on a line l_2). All such configurations belong to $\bigcup_{i=1}^{30} X_i$.

However, this argument does not work for X_{38}, X_{39}, X_{40}. Let us see, what happens in these cases. Consider, for instance, $K \in X_{38}$. Using lemma 5.3, we see that K is included into the singular set of some polynomial f of degree 5. If this singular set is discrete, there is nothing to prove: due to Conditions 1 and 8, if a subset of a discrete singular set consists of ≤ 8 elements, then this subset belongs to $\bigcup_{i=1}^{38} X_i$. Otherwise we can do the following.

We have $K = \lim_{i\to\infty} K_i$, where all K_i are of the form $(Q_1^i \cap Q_2^i) \cup (Q_1^i \cap l^i) \cup (Q_2^i \cap l^i)$, Q_1^i and Q_2^i are conics, l^i are lines. Applying lemma 5.4 and Corollary 5.1, we can assume that

$$K = (\lim_{i\to\infty}(Q_1^i \cap Q_2^i)) \cup (\lim_{i\to\infty}(Q_1^i \cap l^i)) \cup (\lim_{i\to\infty}(Q_2^i \cap l^i))$$

and that the sequences $(Q_1^i), (Q_2^i)$ and (l^i) converge. Denote the limits of these sequences by Q_1, Q_2 and l respectively.

f can have the following nondiscrete singular sets: a line of multiplicity ≥ 2, two double lines, a double line + a triple line, a double nondegenerate conic. Let us consider all these cases.

A double line. In this case we have the following possibilities:

1. $Q_1 = m_1 \cup m_2, Q_2 = m_1 \cup m_3$, where m_1, m_2, m_3, l are 4 pairwise distinct lines. $\lim_{i\to\infty}(Q_1^i \cap Q_2^i)$ is included into a configuration of the form (3 points on m_1){the point $m_2 \cup m_3$}. Thus, K is included into a configuration of the form (the points $l \cap m_1, l \cap m_2, l \cap m_3$ and $m_2 \cap m_3$)$\cup$(3 points on m_1). Such a configuration is a subset of a configuration from X_{34} or X_{20}.

2. $Q_1 = l \cup m$, where $m \neq l$, and Q_2 contains neither l nor m. $\lim_{i\to\infty}(Q_1^i \cap l^i)$ consists of one or two points on l. Hence K is included into a configuration of the form $(Q_2 \cap l) \cup (Q_2 \cap m)$$\cup$(2 points on l), which contains ≤ 6 points.

3. $Q_1 = m \neq l, Q_2$ contains neither l nor m. $K = (m \cap l) \cup (Q_2 \cap l) \cup (Q_2 \cap m)$, hence K contains ≤ 5 points.

A triple line. In this case K is included into a configuration of the form (a line)+(a point). Hence K is a subset of a configuration from X_8 or X_{16}.

A line of multiplicity ≥ 4. K is a subset of a configuration from X_8.

Two double lines or a double line + a triple line. K is a subset of the union of 2 lines and $\#(K) \leq 8$. All such configurations belong to $\bigcup_{i=1}^{30} X_i$.

A double nondegenerate conic. We have that $Q_1 = Q_2$ is nondegenerate. $\lim_{i\to\infty}(Q_1^i \cap Q_2^i)$ is included into a subconfiguration of the form (4 points on Q_1), $\lim_{i\to\infty}(Q_1^i \cap l^i) = \lim_{i\to\infty}(Q_2^i \cap l^i)$ contains 1 or 2 points on Q_1. Thus, K is included into a configuration from X_{24}.

We have checked Condition 5 for X_{38}. The spaces X_{39}, X_{40} can be considered in a similar way.

Proposition 5.1 is proven. $\diamondsuit$

Now we apply theorem 3.2 and lemmas 3.1,3.2 to construct a conical resolution σ and a filtration $\varnothing \subset F_1 \subset \cdots \subset F_{42} = \sigma$. The spectral sequence (5.1) is exactly the sequence corresponding to this filtration.

Most of the columns of the sequence (5.1) can be investigated in essentially the same way as in the case of nonsingular quartics considered in [14]. We shall only discuss the columns that need a somewhat different argument. We shall use the notations from the article [14] if not indicated otherwise.

5.2 Column 38

Let X_{38} be the space of all configurations of type 38 (see proposition 5.1). From lemma 3.1 we get

$$E^1_{38,i} = \bar{H}_{38+i-2-7}(X_{38}, \pm\mathbb{R}), \tag{5.2}$$

X_{38} is naturally fibered over the space $\tilde{B}(\mathbb{C}P^2, 4)$ of generic quadruples $\{A, B, C, D\} \subset \mathbb{C}P^2$. Let us denote by Y the fibre of this bundle, i.e. the space of all configurations from X_{38} such that the points of intersection of the conics are fixed.

Lemma 5.5 *The term E^2 of the spectral sequence of the bundle $X_{38} \to \tilde{B}(\mathbb{C}P^2, 4)$ is trivial.*

The proof will take the rest of the section.

Denote by L the space of all lines not passing through four points A, B, C, D in general position in $\mathbb{C}P^2$. For any such line l denote by Z the space of conics passing through A, B, C, D and not tangential to l. The space Z is homeomorphic to (S^2 minus 2 points)$= \mathbb{C}^*$.

Y is fibered over L with fibre $B(Z, 2) = B(\mathbb{C}^*, 2)$.

Lemma 5.6 *The Borel-Moore homology group of the fibre Y of the bundle $X_{38} \to \tilde{B}(\mathbb{C}P^2, 4)$ can be obtained from the spectral sequence of the bundle $Y \to L$, whose term E^2 is as follows:*

$$\begin{array}{c|ccc} 3 & \mathbb{R}^3 & \mathbb{R}^3 & \mathbb{R} \\ 2 & \mathbb{R} & & \\ \hline & 2 & 3 & 4 \end{array} \tag{5.3}$$

Proof. Recall that $B(\mathbb{C}^*, 2)$ is a fibre bundle with base $\mathbb{C}^*$ and fibre $\mathbb{C} \setminus \{1, -1\}$. Let us study the restriction of the coefficient system $\pm\mathbb{R}$ to the fibre $B(\mathbb{C}^*, 2)$ of the bundle $Y \to L$. This system changes its sign, when one of the points passes around zero (and the other stands still). This corresponds to the fact that if we fix all points in the configuration except the points of intersection of the line and one of the conics, we can transpose those points. On the contrary, a loop that transposes two conics, transposes two pairs of points and does not change the sign of the coefficient system. We see that the loops of the fibre do not change the sign of the coefficient system, and some loop that projects onto the generator of $\pi_1(\mathbb{C}^*)$ (and hence any other such loop) does. So $\pm\mathbb{R}|B(\mathbb{C}^*, 2)$ is the system $\mathcal{A}_1$ of proposition 4.2. We have $\bar{H}_2(B(\mathbb{C}^*, 2), \mathcal{A}_1) = \bar{H}_3(B(\mathbb{C}^*, 2), \mathcal{A}_1) = \mathbb{R}$.

The space L is homeomorphic to $\mathbb{C}^2$ minus the union of three complex lines in general position. We have $\bar{H}_i(L) = \mathbb{R}^3$ if $i = 2, 3$, $\bar{H}_i(L) = \mathbb{R}$ if $i = 4$ and $\bar{H}_i(L) = 0$ otherwise. We shall complete the proof of lemma 5.6 in the following two lemmas.

Lemma 5.7 *Let $l(t)$ be a loop in L that moves a line $l = l(0)$ around one of the points A, B, C, D. Let Z be the space of conics passing through A, B, C, D and not tangential to l. We can identify Z with $\mathbb{C}^*$ (choosing an appropriate coordinate map $z : Z \to \mathbb{C}^*$) in such a way that the map $Z \to Z$ induced by $l(t)$ can be written as $z \mapsto 1/z$. If moreover $A = (1, 0), B = (-1, 0), C = (0, 1), D = (0, -1), l(t) = \{x = \alpha(t)\}, \alpha(t) = 1 + \varepsilon e^{2\pi i t}$, where $\varepsilon = \frac{2}{\sqrt{3}} - 1$, then the conics $q_1 = \{xy = 0\}$ and $q_2 = \{x^2 + y^2 = 1\}$ are preserved, and the points* *of intersection of q_1 and l are preserved, while the points of intersection of q_2 and l are transposed.*

Proof. Denote by Q the space of conics passing through A, B, C, D. These conics can be written as follows:

$$ax^2 + ay^2 + bxy - a = 0.$$

Such a conic is tangential to $l(t)$ if and only if

$$(b\alpha(t))^2 - 4a^2\alpha(t)^2 + 4a^2 = 0. \tag{5.4}$$

Note that if $t = 0$, then $\alpha(0) = \frac{2}{\sqrt{3}}$, and the condition (5.4) becomes simply $a^2 = b^2$. The map $F_t : Q \to Q$ that carries the conics tangential to l to the conics tangential to $l(t)$ can be written as

$$(a, b) \mapsto (\frac{1}{2} \cdot a\sqrt{\alpha^2/(\alpha^2 - 1)}, b).$$

If $\alpha(t)$ is as above, $\sqrt{\alpha^2/(\alpha^2 - 1)}$ changes its sign, so the map $F_1(a, b) = (-a, b)$. The desired coordinate $z \in \mathbb{C}^*$ is $z = (b+a)/(b-a)$. Note that $z(q_1) = 1, z(q_2) = -1, z(-a, b) = 1/z(a, b)$. The conics q_1 and q_2 are preserved under any map F_t. The points of intersection of q_1 and l are clearly preserved. The assertion concerning the points of intersection of q_2 and l can be verified immediately. Lemma 5.7 is proven. $\diamondsuit$

Now we can describe the action of $\pi_1(L)$ on the Borel-Moore homology of the fibre $B(Z, 2) = B(\mathbb{C}^*, 2)$ of the bundle $Y \to L$.

Lemma 5.7 tells us that the points of intersection of exactly one conic of q_1, q_2 are transposed. Hence the covering map of coefficient systems $\pm\mathbb{R}|B(\mathbb{C}^*, 2) \to \pm\mathbb{R}|B(\mathbb{C}^*, 2)$ is minus identity over the configuration $\{1, -1\} \in B(\mathbb{C}^*, 2)$. This implies that the fibre of the coefficient system over the pair, say $\{i, -i\}$, is mapped identically.

Applying proposition 4.2, we obtain immediately that $\bar{H}_2(B(Z, 2), \pm\mathbb{R}|B(Z, 2))$ is multiplied by -1 and $\bar{H}_3(B(Z, 2), \pm\mathbb{R}|B(Z, 2))$ is preserved.

Thus, the 3-d line of the sequence (5.3) contains the Borel-Moore homology of L with constant coefficients. In order to obtain the 2-nd line we must calculate the Borel-Moore homology of L with coefficients in the system $\mathfrak{L}$ that changes its sign under the action of any loop in $\mathbb{C}P^{2\vee}$ that embraces exactly one of the lines corresponding to the points A, B, C, D.

Lemma 5.8 *Let L be the complement in $\mathbb{C}P^2$ of four complex lines in general position. Let $f : L \to L$ be the restriction to L of the projective linear map that transposes two of these lines and preserves the other two, and let $\tilde{f} : \mathfrak{L} \to \mathfrak{L}$ be the map that covers f and is identical over some point of L that is preserved under f. Then*

a) the Poincaré polynomial of L with coefficients in $\mathfrak{L}$ equals t^2, and

b) the map $\tilde{f}$ multiplies by -1 the groups $H^2(L, \mathfrak{L})$ and $\bar{H}_2(L, \mathfrak{L})$.

Note that if $\tilde{f}$ is the identity over some fixed point of f, then it is the identity over any other fixed point; this follows from the fact that the set of fixed points of f is connected.

Proof of lemma 5.8. Identify L with the space $\mathbb{C}^2 \setminus (\{z_1 = 0\} \cup \{z_2 = 0\} \cup \{z_1 + z_2 = 1\})$. Consider the map $p : L \to \mathbb{C} \setminus \{1\}, p(z_1, z_2) = z_1 + z_2$. Put $A_1 = p^{-1}(U_1(0)), A_2 = p^{-1}(\mathbb{C} \setminus \{0, 1\})$, where $U_1(0)$ is the open unit disc.

The space A_1 is homotopically equivalent to the torus $\{(z_1, z_2)||z_1| = \frac{1}{3}, |z_2| = \frac{1}{3}\}$. The loops in A_1 defined by the formulas $t \mapsto \frac{1}{3}(e^{2\pi i}, 1)$ and $t \mapsto \frac{1}{3}(1, e^{2\pi i})$ act non-trivially on the fibre of $\mathcal{L}$, which implies that the cohomology groups $H^*(A_1, \mathcal{L})$ are zero.

The restriction $p|A_2$ is a fibration. The restriction of $\mathcal{L}$ to the fibre $p^{-1}(\frac{1}{2})$ in nontrivial, hence we have that $H^i(p^{-1}(\frac{1}{2}), \mathcal{L})$ is isomorphic to $\mathbb{R}$ if $i = 1$ and is zero otherwise. Define the loops α and β in the space $\mathbb{C} \setminus \{0, 1\}$ as follows: $\alpha : t \mapsto 1 - \frac{1}{2}e^{2\pi it}, \beta : t \mapsto \frac{1}{2}e^{2\pi it}$. It is easy to check that both of them induce the identical mapping of $p^{-1}(\frac{1}{2})$ (hence the space A_2 is in fact homeomorphic to the direct product $\mathbb{C} \setminus \{0, 1\} \times p^{-1}(\frac{1}{2})$). Note, moreover, that a lifting of α into A_2 changes the sign of $\mathcal{L}$, while a lifting of β does not. Now it is clear that the Poincaré polynomial $P(A_2, \mathcal{L})$ is equal to t^2 and that the inclusion $A_1 \cap A_2 = p^{-1}(U_1(0) \setminus \{0\}) \subset A_2$ induces an isomorphism of 2-dimensional cohomology groups with coefficients in $\mathcal{L}$.

Now consider the cohomological Mayer-Vietoris sequence corresponding to $L = A_1 \cup A_2$. Its only nontrivial terms will be

$$H^1(A_1 \cap A_2, \mathcal{L}) \to H^2(L, \mathcal{L}) \to H^2(A_1, \mathcal{L}) \oplus H^2(A_2, \mathcal{L}) \to H^2(A_1 \cap A_2, \mathcal{L})$$

The map on the right is an isomorphism, hence so is the map on the left. So we have $P(L, \mathcal{L}) = t^2$.

The map f preserves each fibre of p. Moreover, using the Künneth formula and proposition 4.1, we obtain that $\tilde{f}$ acts on the groups $H^*(A_1 \cap A_2, \mathcal{L})$ as multiplication by -1. Since the boundary operator commutes with $\tilde{f}$, we obtain the assertion of the lemma concerning the group $H^2(L, \mathcal{L})$. The assertion about the Borel-Moore homology group follows from the Poincaré duality and the fact that f preserves the orientation. $\diamondsuit$

Lemma 5.6 follows immediately from lemma 5.8. $\diamondsuit$

To complete the proof of lemma 5.5 we must calculate the action of $\pi_1(\tilde{B}(\mathbb{C}P^2, 4))$ on the Borel-Moore homology groups of Y obtained from the spectral sequence (5.3). This will be done in the following three lemmas.

Lemma 5.9 *A loop $\gamma(t)$ in $\tilde{B}(\mathbb{C}P^2, 4)$ that belongs to the image of $\pi_1(\tilde{F}(\mathbb{C}P^2, 4))$ under the natural map $\tilde{F}(\mathbb{C}P^2, 4) \to \tilde{B}(\mathbb{C}P^2, 4)$ induces the identical map of the fibre Y and preserves the coefficient system $\pm\mathbb{R}|Y$ over it.*

Note that $\pi_1(\tilde{F}(\mathbb{C}P^2, 4)) \cong \mathbb{Z}_3$ (because $\tilde{F}(\mathbb{C}P^2, 4)$ is diffeomorphic to $PGL(\mathbb{C}P^2)$, which is the quotient of $SL_3(\mathbb{C})$ by its center).

Proof of lemma 5.9. We can represent every $\gamma \in \pi_1(\tilde{B}(\mathbb{C}P^2, 4))$ as follows: $\gamma(t) = \{A(t), B(t), C(t), D(t)\}$, where $A(t), \ldots, D(t)$ are some paths in $\mathbb{C}P^2$ such that for any t the points $A(t), B(t), C(t)$, and $D(t)$ are in general position. If we have a γ that comes from $\pi_1(\tilde{F}(\mathbb{C}P^2, 4))$, we have $A(0) = A(1), \ldots, D(0) = D(1)$. Denote by Y_t the fibre of the bundle $X_{38} \to \tilde{B}(\mathbb{C}P^2, 4)$ over $\gamma(t)$. Note that for any t there exists a unique projective linear map $M(t)$ that carries $A(0)$ into $A(t)$, $B(0)$ into $B(t)$, $C(0)$ into $C(t)$ and $D(0)$ into $D(t)$. This map induces the map $F_t : Y_0 \to Y_t$. The map F_1 is clearly identical. Moreover, if we have a configuration $K \in Y_0$, then the curve in X starting at K and covering γ is $t \mapsto M(t)K$. Since $M(1) = \mathrm{Id}_{\mathbb{C}P^2}$, γ does not transpose any pair of points from K. The lemma is proven. $\diamondsuit$

Lemma 5.10 *1. A loop $\gamma \in \pi_1(\tilde{B}(\mathbb{C}P^2, 4))$ transposing the points A and B induces a bundle map $F_1 : Y \to Y$. This map is covered by a map $F_1 : \pm\mathbb{R}|Y \to \pm\mathbb{R}|Y$.*

2. The corresponding map of L into itself is obtained from the projective linear map of $\mathbb{C}P^2$ that transposes the points A and B and preserves C and D.

3. Let l be a line that is preserved under the transposition of A and B. The restriction of F_1 to the fibre over l is the map $B(\mathbb{C}^*, 2) \to B(\mathbb{C}^*, 2)$ induced by $z \mapsto 1/z$. The restriction of F_1 to this fibre is minus identity over the pair $\{i, -i\}$.

4. The map $F_1|B(Z, 2)$ acts on the group $\bar{H}_3(B(Z, 2), \pm\mathbb{R}|B(Z, 2))$ as multiplication by -1 and acts on the group $\bar{H}_2(B(Z, 2), \pm\mathbb{R}|B(Z, 2))$ as the identical operathor; here $B(Z, 2)$ is the fibre of the bundle $Y \to L$ over some line from L that is preserved under the transposition of A and B.

Proof. Proceeding as in the proof of lemma 5.9 we obtain bundle maps $F_t : Y_0 \to Y_t$. The map $F_1 : Y_0 \to Y_1 = Y_0$ is induced by the projective linear map preserving C and D and transposing A and B. Note that the map $F_1 : \pm\mathbb{R}|x \to \pm\mathbb{R}|x$, where $x \in Y, F_1(x) = x$ is just the map induced by the loop $t \mapsto F_t(x)$. Now let A, B, C, D be the following points of the affine plane $\mathbb{C}^2 \subset \mathbb{C}P^2$: $A = (1, 0), B = (-1, 0), C = (0, 1), D = (0, -1)$. Put $l = \infty$. Then the map F_1 is induced by the linear map with the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

This map preserves the line $l = \infty$ and transposes the conics tangential to l. Identify Z with $\mathbb{C}^*$, and choose a coordinate $z \in \mathbb{C}^*$ such that the induced map can be written as $z \to 1/z$. Note that the conics in the pair that correspond to $\{i, -i\}$ are transposed, and the pair itself is preserved. Thus, the loop γ transposes 3 pairs of points over this pair, and hence the coefficient system $\pm\mathbb{R}|B(Z, 2)$ becomes multiplied by -1.

We have proved the first three assertions of lemma 5.10. The fourth assertion follows immediately from proposition 4.2. $\diamondsuit$

Now we can easily obtain the map of the sequence (5.3) induced by F_1. The third line of the sequence (5.3) contains the groups $\bar{H}_i(L, \bar{\mathcal{H}}_3)$, where $\bar{\mathcal{H}}_3$ is the constant local system on L with fibre $\bar{H}_3(B(Z, 2), \pm\mathbb{R}|B(Z, 2))$. Since the map F_1 multiplies the fibre of $\bar{\mathcal{H}}_3$ by -1, it multiplies $\bar{H}_4(L, \bar{\mathcal{H}}_3) = E^1_{4,3}$ by -1. The second line of the sequence (5.3) contains the groups $\bar{H}_i(L, \bar{\mathcal{H}}_2)$, where $\bar{\mathcal{H}}_2$ is the local nonconstant system on L considered in lemma 5.8. Due to lemma 5.10 F_1 preserves the system $\bar{\mathcal{H}}_2$ over some point of L. We obtain from lemma 5.8 that γ multiplies $\bar{H}_2(L, \bar{\mathcal{H}}_2) = E^1_{2,2}$ by -1.

It is easy to see that the action of $\pi_1(\tilde{B}(\mathbb{C}P^2, 4))$ on $E^1_{2,3}$ and on $E^1_{3,3}$ of the sequence (5.3) is nontrivial and irreducible.

Hence the action of $\pi_1(\tilde{B}(\mathbb{C}P^2,4))$ on $\bar{H}_*(Y,\pm\mathbb{R}|Y)$ is nontrivial and irreducible. Recall that the universal covering space of $\tilde{B}(\mathbb{C}P^2,4)$ is $SL_3(\mathbb{C})$, and the group $\pi_1(\tilde{B}(\mathbb{C}P^2,4))$ contains a normal subgroup isomorphic to $\mathbb{Z}_3 = \pi_1(\tilde{F}(\mathbb{C}P^2,4))$, the quotient being isomorphic to S_4. Lemma 5.5 follows immediately from lemma 4.1. In fact, putting $G = SL_3(\mathbb{C}), G_1 =$(the subgroup of $SL_3(\mathbb{C})$ generated by $e^{\frac{2}{3}\pi i}I$ (I is the identity matrix) and the (complexification of the) motions of a regular tetrahedron) in lemma 4.1 we obtain that the group $H^*(\tilde{B}(\mathbb{C}P^2,4),\mathcal{L}) = 0$ if the action of $\pi_1(\tilde{B}(\mathbb{C}P^2,4))$ on the fibre of $\mathcal{L}$ is nontrivial and irreducible. By Poincaré duality $\bar{H}_*(\tilde{B}(\mathbb{C}P^2,4),\mathcal{L})$ is also zero for any such $\mathcal{L}$.

5.3 Column 39

Recall that we denote by X_{39} the space of configurations of type 39. We have

$$E^1_{39,i} = \bar{H}_{39+i-2-8}(X_{39},\pm\mathbb{R}).$$

X_{39} is fibered over the space $\tilde{B}(\mathbb{C}P^2,3)$ of all generic triples of points $\{A,B,C\}$, $A,B,C\in\mathbb{C}P^2$.

Lemma 5.11 *The term E^2 of the spectral sequence of the bundle $X_{39}\to\tilde{B}(\mathbb{C}P^2,3)$ looks as follows:*

7	$\mathbb{R}$						
6			$\mathbb{R}$		$\mathbb{R}$		
5							$\mathbb{R}$
	6	7	8	9	10	11	12

(5.5)

and the differentials $E^2_{8,6}\to E^2_{6,7}$ and $E^2_{12,5}\to E^2_{10,6}$ are nontrivial. Hence, we have $E^3=\cdots=E^\infty=0$.

The proof will take the rest of the section.

If we fix three lines AB, BC, AC, then the intersection points of AB and BC with the conic can be chosen arbitrarily. The space of conics passing through these 4 points and not tangential to AC is homeomorphic to (S^2 minus three points): we have to exclude 2 tangential conics and the conic consisting of the lines AB and BC.

Thus, the fibre Y of the bundle $X_{39}\to\tilde{B}(\mathbb{C}P^2,3)$ is itself a fibre bundle over $B(\mathbb{C}^*,2)\times B(\mathbb{C}^*,2)$ with fibre (S^2 minus 3 points). Denote the latter fibre by Z. The space Z can be identified with $\mathbb{C}^*\setminus\{1\}$.

Lemma 5.12 *The term E^2 of the spectral sequence for the Borel-Moore homology of the bundle $Y\to B(\mathbb{C}^*,2)\times B(\mathbb{C}^*,2)$ looks as follows:*

1	$\mathbb{R}$	$\mathbb{R}^2$	$\mathbb{R}$
	4	5	6

(5.6)

Proof. If we fix all points in a configuration from X_{39} except the points of intersection of the conic and the line AC, then we can transpose these two points, hence the restriction of $\pm\mathbb{R}$ to (S^2 minus 3 points) is nontrivial. We have $\bar{H}_i(Z,\pm\mathbb{R})=\mathbb{R}$ if $i=1$, and 0 otherwise.

That is why the only nontrivial line in the spectral sequence of the bundle $Y \to B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2)$ is the first one; it contains the groups $\bar{H}_*(B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2), \bar{\mathcal{H}}_1)$, where $\bar{\mathcal{H}}_1$ is the system with fibre $\bar{H}_1(Z, \pm\mathbb{R}|Z)$ corresponding to the action of $\pi_1(B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2))$. The fibre of $\bar{\mathcal{H}}_1$ is $\mathbb{R}$, and, as we shall see, every element of $\pi_1(B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2))$ multiplies the fibre of $\bar{\mathcal{H}}_1$ by ± 1. So we can apply the Künneth formula, and we get

$$\bar{H}_*(B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2), \bar{\mathcal{H}}_1) = \bar{H}_*(B(\mathbb{C}^*, 2), \mathcal{B}_1) \otimes \bar{H}_*(B(\mathbb{C}^*, 2), \mathcal{B}_2), \quad (5.7)$$

where $\mathcal{B}_1, \mathcal{B}_2$ are the restrictions of $\bar{\mathcal{H}}_1$ on the first and the second factors of $B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2)$. To calculate $\mathcal{B}_1$ we fix an element in the second factor of the product $B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2)$ and study the action of the loops in the first factor on the group $\bar{H}_1(Z, \pm\mathbb{R}|Z)$.

We put $A = (0:1:0), B = (0:0:1), C = (1:0:0)$. Put $AC = \infty$, so that the points of the type $(z:w:1)$ belong to the affine plane $\mathbb{C}^2 \subset \mathbb{C}P^2$, and the spaces $B(\mathbb{C}^*, 2)$ consist of pairs of nonzero points on the coordinate axes. Now fix the points $\{(0,i),(0,-i)\}$ on the y-axis. Denote by Q the space of conics passing through $(i,0), (-i,0), (0,i)$, and $(0,-i)$. Note that the fibre Z over this quadruple is the subspace of Q that consists of the conics that are not tangential to $AC = \infty$ and are not equal to the union $AB \cup AC$. Note also that the conics from Q have the form

$$ax^2 + bxy + ay^2 + a.$$

Put $z = (2a-b)/2a+b)$. This identifies the space $Z \subset Q$ with $\mathbb{C} \setminus \{0, -1\}$.

In the following two lemmas we identify the space $B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2)$ with the space of configurations in $\mathbb{C}^2$ that consist of two nonzero points on the x-axis and two nonzero points on the y-axis.

Lemma 5.13 *Consider the following loop in the space* $B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2)$ $\gamma(t) = \{(\alpha(t), 0), (1/\alpha(t), 0), (0, i), (0, -i)\}$, *where* $\alpha(t)$ *is a curve in* $\mathbb{C} \setminus \{0\}$ *such that* $\alpha(0) = i, \alpha(1) = -i$. *Then* γ *induces the identical map of* Z *and preserves the points of intersection of the conics from* Z *with* ∞.

Lemma 5.14 *Consider the following loop in the space* $B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2)$ $\gamma(t) = \{(ie^{\pi it}, 0), (-ie^{\pi it}, 0), (0, i), (0, -i)\}$. *The map* $Z \to Z$ *induced by* γ *can be written as* $z \mapsto 1/z$. *This map preserves the conics* $q_1 = xy$ *and* $q_2 = x^2 + y^2 + 1$. *The points* $q_1 \cap \infty$ *are preserved, and the points* $q_2 \cap \infty$ *are transposed.*

The proof of these lemmas is an exercise in analytic geometry. $\diamondsuit$

Now we shall use lemmas 5.13 and 5.14 to calculate the action of $\pi_1(B(\mathbb{C}^*, 2))$ on the Borel-Moore homology of the fibre of the bundle $Y \to B(\mathbb{C}^*, 2)$.

Let us note that the loop γ considered in lemma 5.13 is the loop in the fibre of the bundle $B(\mathbb{C}^*, 2)$ over 1 (recall that on page 24 we defined a structure of a fibre bundle over $\mathbb{C}^*$ on the space $B(\mathbb{C}^*, 2)$, the projection being the multiplication of complex numbers). We obtain from lemma 5.13 that such γ induces the identical map of the space of Z, and the points of intersection of the conics are preserved. So γ transposes two points in a configuration from Y. Thus, the system $\pm\mathbb{R}|Z$ is multiplied by -1, and γ acts on the group $\bar{H}_1(Z, \pm\mathbb{R}|Z)$ as multiplication by -1.

Now let γ be the loop $t \mapsto \{ie^{\pi it}, -ie^{\pi it}\}$ in $B(\mathbb{C}^*, 2)$. Note that this is exactly the loop a from proposition 4.2. Applying lemma 5.14, we obtain that this loop transposes the tangential conics. Identify the space Z of nontangential conics with $\mathbb{C}^*$ taking the coordinate z as in lemma 5.14. The map $g : Z \to Z$ induced by γ is $z \mapsto 1/z$. Due to lemma 5.14 γ transposes the points of intersection of q_2 and AC. This implies that the map $\tilde{g} : \pm\mathbb{R}|Z \to \pm\mathbb{R}|Z$ induced by γ is identity over $1 = z(q_2)$ (it transposes two pairs of points).

Now introduce another coordinate w on Z, $w = (z-1)/(z+1)$. This identifies Z with $\mathbb{C} \setminus \{1, -1\}$. Since we have $w(1/z) = -w(z)$, the map $g : Z \to Z$ can be written as $w \mapsto -w$. The map $\tilde{g}$ is identity over $0 = w(1)$. Applying proposition 4.1, we obtain immediately that the map $\tilde{g}_* : \bar{H}_1(Z, \pm\mathbb{R}|Z) \to \bar{H}_1(Z, \pm\mathbb{R}|Z)$ is minus identity.

So we see that the restriction of the local system $\bar{\mathcal{H}}_1$ to $B(\mathbb{C}^*, 2)$ is in fact the system $\mathcal{A}_3$ of proposition 4.2 (it changes its sign both under the action of the loops of the fibre of $B(\mathbb{C}^*, 2) \to \mathbb{C}^*$ and under the "middle line"). Due to proposition 4.1 we have $\bar{P}(B(\mathbb{C}^*, 2), \mathcal{A}_3) = t^2(t + 1)$. Lemma 5.12 follows now from formula (5.7). $\diamondsuit$

Now we shall study the action of $\pi_1(\tilde{B}(\mathbb{C}P^2, 3)$ on the group $\bar{H}_*(Y, \pm\mathbb{R}|Y)$. The fundamental group of $\tilde{B}(\mathbb{C}P^2, 3)$ equals S_3 (since $\tilde{F}(\mathbb{C}P^2, 3)$ is simply-connected). We shall describe the map of the sequence (5.6) induced by the transposition of the points A and C.

Lemma 5.15 *1. Let γ be a loop in $\tilde{B}(\mathbb{C}P^2, 3)$ that transposes the points A and C. Above we represented Y as a fibre bundle over $B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2)$, the projection being defined as follows: $Q \mapsto (Q \cap AB, Q \cap BC)$, where Q is the conic that correspond to an element of Y[1]. The map $F_1 : Y \to Y$ induced by γ preserves this structure of a fibre bundle on Y_1.*

2. *The corresponding map $h : B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2) \to B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2)$ is the transposition of factors.* (*Recall that the first* (*respectively, the second*) *factor in this product is identified with the space of pairs of nonzero points on the x-* (*respectively, the y-*)*axis.*)

3. *Identify the space $B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2)$ with the space of configurations in $\mathbb{C}^2$ that consist of two nonzero points on the x-axis and two nonzero points on the y-axis. Let Z be the fibre of $Y \to B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2)$ over the point $\{(i, 0), (-i, 0), (0, i), (0, -i)\}$ (this point is clearly preserved under h). The map $F_1 : Z \to Z$ is identical, and the points of intersection of each conic $q \in Z$ with the line AC are transposed by γ. Hence a loop corresponding to the movement of any $q \in Z$ from this fibre transposes four pairs of points and preserves the coefficient system $\pm\mathbb{R}$ over this fibre.*

Proof. Since $\pi_1(\tilde{B}(\mathbb{C}P^2, 3)) = S_3$, any two loops that transpose A and C define the same map $Y \to Y$ and the same map $\pm\mathbb{R}|Y \to \pm\mathbb{R}|Y$. Recall that $A = (1 : 0 : 0), B = (0 : 0 : 1), C = (0 : 1 : 0) \in \mathbb{C}P^2$. Put $A(t) = (\frac{1}{2}(1 + e^{i\pi t}) : \frac{1}{2}(1 - e^{i\pi t}) : 0), C(t) = (\frac{1}{2}(1 - e^{i\pi t}) : \frac{1}{2}(1 + e^{i\pi t}) : 0)$. Put $\gamma(t) = \{A(t), B(t), C(t)\}$. Denote by Y_t the fibre of the bundle $X_{39} \to \tilde{B}(\mathbb{C}P^2, 4)$ over $\gamma(t)$. There exists a projective linear map that carries A into $A(t)$, C into

[1] Note that there are three ways to represent Y as a fibre bundle over $B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2)$; instead of the lines AB and BC we could have taken any other two of the lines AB, BC and AC.

$C(t)$ and preserves B. In the affine plane $\mathbb{C}^2 = \mathbb{C}P^2 \setminus AC$ it looks like

$$\begin{pmatrix} \frac{1}{2}(1+e^{i\pi t}) & \frac{1}{2}(1-e^{i\pi t}) \\ \frac{1}{2}(1-e^{i\pi t}) & \frac{1}{2}(1+e^{i\pi t}) \end{pmatrix}$$

This map induces a map $F_t : Y_0 \to Y_t$. In particular, the map F_1 is induced by the transposition of the axes in $\mathbb{C}^2$. The first and the second assertions of the lemma follow immediately. To prove the third assertion note that the fibre Z over the point $\{(i,0),(-i,0),(0,i),(0,-i)\}$ consists of conics of the type $ax^2+bxy+ay^2+a$. Such conics are preserved, if we change x and y, and their points of intersection with $AC = \infty$ are clearly transposed. $\diamondsuit$

The map F_1 preserves the system $\bar{\mathcal{H}}_1$ corresponding to the action of $\pi_1(B(\mathbb{C}^*,2)\times B(\mathbb{C}^*,2))$ on $\bar{H}_1(Z,\pm\mathbb{R}|Z)$, since F_1 induces the identical map of the fibre Z over some preserved point of $B(\mathbb{C}^*,2)\times B(\mathbb{C}^*,2)$ and preserves the restriction of the coefficient system $\pm\mathbb{R}$ to that fibre.

In general, suppose we have a space A and a local system $\mathcal{L}$ on $A\times A$, whose fibre is $\mathbb{R}$, and suppose that every element of $\pi_1(A\times A)$ multiplies the fibre of $\mathcal{L}$ by 1 or -1. Let $\mathcal{L}_1, \mathcal{L}_2$ be the restrictions of $\mathcal{L}$ to the first and the second factor, and let $f : A\times A \to A\times A$ be the transposition of factors. Suppose that $\tilde{f} : \mathcal{L} \to \mathcal{L}$ is a map that covers f and is identical over some point of the type $(x,x), x\in A$. Then, the map of $\bar{H}_*(A\times A,\mathcal{L}) = \bar{H}_*(A,\mathcal{L}_1)\otimes \bar{H}_*(A,\mathcal{L}_2)$ into itself can be written as $a\otimes b \mapsto (-1)^{deg(a)deg(b)} b\otimes a$.

Applying this to our situation, we obtain that the group $\bar{H}_4(B(\mathbb{C}^*,2)\times B(\mathbb{C}^*,2),\bar{\mathcal{H}}_1) = E^2_{4,1}$ is preserved by γ, and $\bar{H}_6(B(\mathbb{C}^*,2)\times B(\mathbb{C}^*,2),\bar{\mathcal{H}}_1) = E^2_{6,1}$ is multiplied by -1.

We have clearly $\bar{P}(\tilde{B}(\mathbb{C}P^2,3),\mathbb{R}) = t^{12}, \bar{P}(\tilde{B}(\mathbb{C}P^2,3),\pm\mathbb{R}) = t^6$. This gives us the 5-th and the 7-th lines of (5.5).

Let us calculate $\bar{P}(\tilde{B}(\mathbb{C}P^2,3),\mathcal{V}_{2,1})$, where $\mathcal{V}_{2,1}$ is the local system corresponding to the irreducible 2-dimensional representation $V_{2,1}$ of S_3. It is easy to show that $P(\tilde{F}(\mathbb{C}P^2,3)) = (1+t+t^2)(1+t)$. Applying Corollary 4.1 (see page 19) to the covering $\tilde{F}(\mathbb{C}P^2,3)\to\tilde{B}(\mathbb{C}P^2,3)$, we obtain

$$P(\tilde{F}(\mathbb{C}P^2,3)) = P(\tilde{B}(\mathbb{C}P^2,3),\mathbb{R}) + P(\tilde{B}(\mathbb{C}P^2,3),\pm\mathbb{R}) + 2P(\tilde{B}(\mathbb{C}P^2,3),\mathcal{V}_{2,1}),$$

since the regular representation of S_3 contains one trivial, one alternating representation and two copies of the 2-dimensional irreducible representation. Thus, $P(\tilde{B}(\mathbb{C}P^2,3),\mathcal{V}_{2,1}) = t^2(1+t^2)$, and by the Poincaré duality we obtain $\bar{P}(\tilde{B}(\mathbb{C}P^2,3),\mathcal{S}) = t^8(1+t^2)$.

It remains to prove that the 6-th line of the Leray sequence corresponding to the fibration $X_{39}\to\tilde{B}(\mathbb{C}P^2,3)$ is as in (5.5) (i.e., the action of S_3 in $\bar{H}_5(B(\mathbb{C}^*,2)\times B(\mathbb{C}^*,2),\bar{\mathcal{H}}_1) = \mathbb{R}^2$ is irreducible), and that the differentials $E^2_{8,6}\to E^2_{6,7}$ and $E^2_{12,5}\to E^2_{10,6}$ are nontrivial.

To this end note that the group SU_3 acts almost freely on X_{39} (via $SU_3 \to SU_3/\langle e^{\frac{2}{3}\pi i}I\rangle \hookrightarrow PGL(\mathbb{C}P^2)$, where I is the identity matrix). Apply theorem 4.4 putting $M = X_{39}, G = SU_3, \mathcal{L} = \pm\mathbb{R}|X_{39}$. From the Leray sequence of the map $M\to M/G$ (and from the fact that the cohomological dimension of M/G is clearly finite) we obtain that either $H^*(M,\mathcal{L}) = 0$ or $d_{max}-d_{min}\geq 8$ (here d_{max}, (respectively, d_{min}) is the greatest (respectively the smallest) i such that $H^i(M,\mathcal{L})\neq 0$).

By Poincaré duality, we have either $\bar{H}_*(M,\mathcal{L}) = 0$ or $d'_{max}-d'_{min}\geq 8$ (here d'_{max} (respectively, d'_{min}) is the greatest (respectively, the smallest) i such that

$\bar{H}_i(M, \mathcal{L}) \neq 0$. Obviously if the action of S_3 in $\bar{H}_5(B(\mathbb{C}^*, 2) \times B(\mathbb{C}^*, 2), \bar{\mathcal{H}}_1) = \mathbb{R}^2$ is reducible or any of the differentials $E^2_{8,6} \to E^2_{6,7}, E^2_{12,5} \to E^2_{10,6}$ is trivial, neither assertion holds. Lemma 5.11 is proven. $\diamondsuit$

5.4 Nondiscrete singular sets

We are going to show that the columns of the spectral sequence 5.1 corresponding to all nondiscrete singular sets are zero. The columns 11 and 33 are considered in exactly the same way as in [14].

Proposition 5.2 *Let l be a line in $\mathbb{C}P^2$, $A_1, \ldots, A_k, k \geq 0$ be points not on l, m be an integer > 1. Denote the union of all simplices in $(\mathbb{C}P^2)^{*(m+k)}$ with the vertices in $A_1, \ldots, A_k$ and m vertices on l by $\Lambda(l, m, A_1, \ldots, A_k)$. The space $\Lambda(l, m, A_1, \ldots, A_k)$ has zero real homology groups modulo a point.*

Proof of proposition 5.2. If $k = 0$, the assertion of the proposition follows from lemma 4.4. If $k > 0$, the space $\Lambda(l, m, A_1, \ldots, A_k)$ is contractible, since this space is a union of simplices that all contain the vertex A_1.$\diamondsuit$

Using this proposition, we can easily prove that $\partial\Lambda(K)$ has zero real homology groups modulo a point if $K \in X_i, i =$17, 22, 29, 41. Let us consider, for instance, the case $i =$41. Consider a configuration K consisting of a line l and 3 points A, B, C outside l such that the points A, B, C do not belong to any line.

Note that $\partial\Lambda(K) = L_1 \cup L_2 \cup L_3 \cup L_4$, where $L_1 = \Lambda(l + A + B), L_2 = \Lambda(l + B + C), L_3 = \Lambda(l + A + C), L_4 = \bigcup_\kappa \Lambda(\kappa)$, where κ runs through the set of all configurations of type "A, B, C+ 4 points on l". Using lemma 3.2 (see page 16), we conclude that L_4 is homeomorphic to the space $\Lambda(l, 4, A, B, C)$, which is contractible due to proposition 5.2.

The intersections $L_1 \cap L_2, L_2 \cap L_3, L_1 \cap L_3$ are all spaces of the type $\Lambda(l +$ a point not on $l)$. The intersection $L_1 \cap L_2 \cap L_3$ is just $\Lambda(l)$. All these spaces are contractible. Now, the intersections $L_i \cap L_4, i = 1, 2, 3$ are the unions of all $\Lambda(\kappa)$, κ running through the set of all configurations of type "the points (A, B) (resp., (B, C), (A, C)) outside l + 4 points on l". These spaces are homeomorphic to the space $\Lambda(l, 4, 2 \text{ points outside } l)$ and are contractible due to proposition 5.2. Analogously, the intersections $L_1 \cap L_2 \cap L_4, L_2 \cap L_3 \cap L_4, L_1 \cap L_3 \cap L_4$ are all homeomorphic to spaces of type $\Lambda(l, 4, \text{a point outside } l)$ and are also contractible. Finally, the quadruple intersection $L_1 \cap L_2 \cap L_3 \cap L_4$ is the union of all $\Lambda(\kappa)$, for all $\kappa =$"4 points in l", which is homeomorphic to $l^{*4} = \Lambda(l, 4, \varnothing)$.

So we see that the spaces $L_i, i = 1, \ldots, 4$ have zero real homology groups modulo a point, and so do all their intersections. This implies that real homology groups of their union $\partial\Lambda(K)$ modulo a point are also zero.

Let us now consider the case $K = l_1 \cup l_2$, where l_1, l_2 are two lines (column 31). Here $\partial\Lambda(K)$ is the union $L_1 \cup L_2 \cup L_3$, where L_i for $i = 1, 2$ is the union of the spaces $\Lambda(\kappa)$, where κ runs through the set of configurations of the type "l_i+3 points on the other line", and L_3 is the union of $\Lambda(K')$, for K' running through the set $\{K' \subset l_1 \cup l_2 | \#(K') = 8, \#(K' \cap l_i) \geq 4, i = 1, 2\}$.

First note that the intersection $L_1 \cap L_2$ is the union of the spaces $\Lambda(\kappa)$, where κ runs through the space of configurations of the type "3 points on $l_1 \setminus l_2$, 3 points on $l_2 \setminus l_1$, the point of intersection". It follows from lemma 3.2 that $L_1 \cap L_2$ is contractible.

The space L_1 admits the following filtration $\varnothing \subset \Lambda(l) \subset M_1 \subset M_2 \subset M_3 \subset M_4 \subset M_5 \subset M_6 = L_1$. Here $M_i, i = 1, \dots, 5$, is the union of all $\Lambda(\kappa)$, where $\kappa \subset K$ is a configuration of type 16, 17, 21, 22, 28 respectively.

The space $M_1 \setminus \Lambda(l_1)$ is fibered over $l_2 \setminus l_1$, the fibre over a point A being homeomorphic to $\Lambda(l_1, 7, A) \setminus l_1^{*7}$. This fibre has trivial real Borel-Moore homology. The space $M_2 \setminus M_1$ is fibered over $l_2 \setminus l_1$, the fibre over a point A being homeomorphic to $\Lambda(l_1 + A) \setminus \partial\Lambda(l_1 + A)$, whose Borel-Moore homology is also trivial.

The space $M_3 \setminus M_2$ is fibered over the space $B(\mathbb{C}, 2)$, the fibre over a pair $\{A, B\}$ being homeomorphic to $\Lambda(l_1, 6, A, B) \setminus (\Lambda(l_1, 6, A) \cup \Lambda(l_1, 6, B))$. This space also has trivial Borel-Moore homology.

The spaces $M_4 \setminus M_3$ and $M_6 \setminus M_5$ are considered in the same way as $M_2 \setminus M_1$. The space $M_5 \setminus M_4$ is fibered over $B(\mathbb{C}, 3)$. The fibre over $\{A, B, C\}$ is homeomorphic to the space $\Lambda(l_1, 5, A, B, C) \setminus (\Lambda(l_1, 5, A, B) \cup \Lambda(l_1, 5, B, C) \cup \Lambda(l_1, 5, A, C))$, whose Borel-Moore homology groups are zero. This implies that L_1 (and also L_2) have zero real homology groups modulo a point.

Now consider the space L_3. Let L_3' (respectively, L_3'') be the union of all $\Lambda(K')$ for K' running through the set $\{K' \subset l_1 \cup l_2 | \#(K') = 8, \#(K' \cap l_2) = 5, \#(K' \cap l_1) = 3\}$ (respectively, $\{K' \subset l_1 \cup l_2 | \#(K') = 8, \#(K' \cap l_1) = 5, \#(K' \cap l_2) = 3\}$). It is easy to see that all spaces $L_3', L_3'', L_3' \cap L_3'', L_3' \cup L_3''$ are contractible. The space $L_3 \setminus (L_3' \cup L_3'')$ is the union of $\Lambda(K') \setminus \partial\Lambda(K')$ for K' running through the set of configurations that consist of 4 points on $l_2 \setminus l_1$ and 4 points on $l_1 \setminus l_2$. Using lemma 3.2, we get $\bar{H}_*(L_3 \setminus (L_3' \cup L_3''), \mathbb{R}) = \bar{H}_{*-7}(B(\mathbb{C}, 4) \times B(\mathbb{C}, 4), \pm\mathbb{R}) = 0$. Hence the real homology groups of L_3 modulo a point are zero.

We have also $L_1 \cap L_3 = L_3'', L_2 \cap L_3 = L_3', L_1 \cap L_2 \cap L_3 = L_3' \cap L_3''$. These spaces are all contractible. This completes the proof that real homology groups of $\partial\Lambda(K)$ modulo a point are zero, when K is the union of two lines.

The fact that the last column is zero is proven exactly in the same way as in the case of plane cubics in $\mathbb{C}P^2$, see [14, Section 4].

5.5 End of the proof of theorem 5.2

We have now proved the first two assertions of theorem 5.2. In order to complete the proof of theorem 5.2, it remains to prove that the differential $E^1_{2,35} \to E^1_{1,35}$ of the spectral sequence (5.1) is zero. This can be done as follows (cf. [14, lemma 6]).

Let S be the image of $\Sigma_{5,2}$ under the natural map $\Pi_{5,2} \setminus \{0\} \to \mathbb{C}P^{20}$, and let $c_1 \in H^2(\mathbb{C}P^{20}, \mathbb{R})$ be the first Chern class of the tautological bundle over $\mathbb{C}P^{20}$. Since the fundamental class of any algebraic hypersurface is dual to a nonzero multiple of c_1, the restriction of c_1 to $\mathbb{C}P^{20} \setminus S$ is zero, which implies that $H^*(\Pi_{5,2} \setminus \Sigma_{5,2}, \mathbb{R}) = H^*(\mathbb{C}^*, \mathbb{R}) \otimes H^*(\mathbb{C}P^{20} \setminus S, \mathbb{R})$. Thus, the Poincaré polynomial of the space $\Pi_{5,2} \setminus \Sigma_{5,2}$ is divisible by $1 + t$, which implies easily that the differential $E^1_{2,35} \to E^1_{1,35}$ is zero. Theorem 5.2 is proven.$\diamondsuit$

Chapter 6

Smooth bielliptic genus 4 curves on a nondegenerate quadric in $\mathbb{C}P^3$

Denote by $\mathcal{Q}$ the quadric in $\mathbb{C}P^3$ defined by the equation

$$x_0^2 + \cdots + x_3^2 = 0.$$

Let V' be the vector space of homogeneous complex cubic polynomials $\mathbb{C}^4 \to \mathbb{C}$ that are invariant under the map

$$(x_0, x_1, x_2, x_3) \mapsto (-x_0, x_1, x_2, x_3). \tag{6.1}$$

Denote by τ the involution of $\mathbb{C}P^3$ obtained from (6.1). The set of fixed points of τ is $\mathcal{P} \cap \{\mathcal{A}\}$, where $\mathcal{P}$ is the plane $x_0 = 0$, and $\mathcal{A} = (1 : 0 : 0 : 0)$; set $D = \mathcal{P} \cap \mathcal{Q}$.

In this chapter we use the method described in chapter 3 to calculate the real cohomology groups of two isomorphic spaces: the space of complete intersections with $\mathcal{Q}$ of surfaces defined by elements of V' and the space of smooth plane cubics that intersect transversally a fixed smooth conic.

Recall that a smooth algebraic curve is called *bielliptic*, if it is a double cover of an elliptic curve. Any smooth curve on $\mathcal{Q}$ defined by an element of V' is bielliptic and is of bidegree $(3, 3)$, and hence, has genus 4. One can prove that conversely, each bielliptic smooth genus 4 curve on $\mathcal{Q}$ is ambient isomorphic to a curve defined by an element of $\mathcal{Q}$ ("ambient isomorphic" means here that the isomorphism between two curves is the restriction of an automorphism of $\mathcal{Q}$). This chapter is a part of a joint project A. V. Inshakov and the author; this project consists in calculating the rational cohomology groups of the moduli space of smooth bielliptic genus 4 curves.

Let V be the quotient of V' by the subspace consisting of polynomials of the form lq, where l is any linear function form V'. Denote by Σ' the subspace of V' that consists of all f such that the intersection of $\mathcal{Q}$ and the hypersurface defined by f is not complete; let Σ be the image of Σ' in V.

Theorem 6.1 *We have* $P(V \setminus \Sigma) = (1 + t)(1 + t^3)(1 + t + t^5 + 3t^6)$.

We shall apply theorem 3.2 to calculate the groups $\bar{H}_*(\Sigma, \mathbb{R})$.

For any $f \in \Sigma$ define $K_f \subset \mathcal{Q}$ as follows: let $f' \in \Sigma'$ be any preimage of f, and set K_f to be the set of all points of $\mathcal{Q}$, where the tangent space of $\mathcal{Q}$ and the tangent space of the hypersurface defined by f' intersect non-transversally; this definition does not depend on the choice of f' that projects into f.

Now we shall describe another interpretation of the space $V \setminus \Sigma$. Denote by $V^{(1)}$ the space of homogeneous cubic polynomials in variables x_1, x_2, x_3, and set $\Sigma^{(1)}$ to be the subset of $V^{(1)}$ that consists of polynomials such that the corresponding cubic curve is either singular or intersects nontransversally the conic $\mathcal{Q}'$ defined by

$$x_1^2 + x_2^2 + x_3^2 = 0. \tag{6.2}$$

If C is the cubic defined by an element $f \in \Sigma^{(1)}$, we set K_f to be the union of the singular locus of C and the set of points where C and the conic (6.2) intersect nontransversally. Denote by $\mathbf{p} : \mathcal{Q} \to \mathcal{P}$ the projection from $\mathcal{A}$ and let $\mathbf{P} : V \to V^{(1)}$ be the map that takes an element of V to its representative in V' that does not contain x_0.

Lemma 6.1 (A. V. Inshakov) *1. The map* $\mathbf{P}$ *is a homeomorphism of couples* (V, Σ) *and* $(V^{(1)}, \Sigma^{(1)})$.

2. The map $\mathbf{p}$ *induces a homeomorphism*

$$\{K_f \in 2^{\mathcal{Q}} | f \in \Sigma\} \leftrightarrow \{K_g \in 2^{\mathcal{P}} | g \in \Sigma^{(1)}\}$$

◊

For some purposes (such as the classification of all possible singular sets and the construction of a system of configuration spaces that verify all required conditions) this interpretation will be more convenient. However, the calculations of the some colums of the spectral sequence are easier, if we consider spaces of configurations on $\mathcal{Q}$.

For any $K \subset \mathcal{Q}$ (respectively, $K \subset \mathbb{C}P^2$) let $L(K)$ be the vector subspace of V (respectively, of $V^{(1)}$) that consists of all f such that $K_f \supset K$.

Lemma 6.2 *For the vector space $V^{(1)}$ and the discriminant $\Sigma^{(1)}$, the configuration spaces $Y_1, \ldots, Y_{29}$ described below satisfy Conditions 1–8 from chapter 3. The number indicated in square brackets equals the complex dimension of $L(K)$ for K lying in the corresponding Y_i.*

1. *A point on $\mathcal{Q}'$ [8].*
2. *A point in $\mathbb{C}P^2 \setminus \mathcal{Q}'$ [7].*
3. *A pair of points on $\mathcal{Q}'$ [6].*
4. *A point on $\mathcal{Q}'$ and a point in $\mathbb{C}P^2 \setminus \mathcal{Q}'$ [5].*
5. *Two points in $\mathbb{C}P^2 \setminus \mathcal{Q}'$ [4].*
6. *A point $x \in \mathcal{Q}'$ and two points in $\mathbb{C}P^2 \setminus \mathcal{Q}'$ that lie on the tangent line to $\mathcal{Q}'$ passing through x [4].*
7. *Three points on $\mathcal{Q}'$ [4].*
8. *A pair of points on $\mathcal{Q}'$ and a point in $\mathbb{C}P^2 \setminus \mathcal{Q}'$ [3].*

9. *Four points on* $\mathcal{Q}'$ *[3].*
10. *The whole* $\mathcal{Q}'$ *[3].*
11. *Three points in* $\mathbb{C}P^2 \setminus \mathcal{Q}'$ *that lie on a line tangent to* $\mathcal{Q}'$ *[3].*
12. *Three points in* $\mathbb{C}P^2 \setminus \mathcal{Q}'$ *that lie on a line* l *tangent to* $\mathcal{Q}'$ *plus the intersection point* $l \cap \mathcal{Q}'$ *[3].*
13. *A line tangent to* $\mathcal{Q}'$ *[3].*
14. *A configuration* K *of type 5 on a nontangent line* $l \subset \mathbb{C}P^2$ *plus one of the two points of* $l \cap \mathcal{Q}'$ *[3].*
15. *Three points in* $\mathbb{C}P^2 \setminus \mathcal{Q}'$ *that lie on a line nontangent to* $\mathcal{Q}'$ *[3].*
16. *A configuration* K *of type 5 on a line* l *nontangent to* $\mathcal{Q}'$ *plus both points of* $l \cap \mathcal{Q}'$ *[3].*
17. *A line nontangent to* $\mathcal{Q}'$ *[3].*
18. *A configuration* K *of type 5 plus a point* $x \in \mathcal{Q}'$ *that does not belong to the line that passes through the points of* K *[2].*
19. *A configuration* K *of type 6 plus a point in* $\mathcal{Q}' \setminus K$ *[2].*
20. *Three points in* $\mathcal{Q}'$ *and a point in* $\mathbb{C}P^2 \setminus \mathcal{Q}'$ *[1].*
21. *A line* l *tangent to* $\mathcal{Q}'$ *plus a point in* $\mathcal{Q}' \setminus l$ *[1].*
22. *A line* l *nontangent to* $\mathcal{Q}'$ *plus a point in* $\mathcal{Q}' \setminus l$ *[1].*
23. *Three points in* $\mathbb{C}P^2 \setminus \mathcal{Q}'$ *that are not on a line [1].*
24. *A point* $x \in \mathcal{Q}'$ *plus two points on* $l \setminus \{x\}$*, where* l *is the line that passes through* x *and is tangent to* $\mathcal{Q}'$*, plus a point in* $\mathbb{C}P^2 \setminus (l \cup \mathcal{Q}')$ *[1].*
25. *A pair of points* $\{x_1, x_2\} \subset \mathcal{Q}'$ *plus a pair of points* $\{y_1, y_2\} \subset \mathbb{C}P^2 \setminus \mathcal{Q}'$ *such that 1.* y_1, y_2 *lie on a conic that intersects* $\mathcal{Q}'$ *at exactly two points* x_1, x_2*, and 2. neither three of the points* x_1, x_2, y_1, y_2 *are on a line [1].*
26. *A point* $x \in \mathbb{C}P^2 \setminus \mathcal{Q}'$ *plus the points* $l_1 \cap \mathcal{Q}', l_2 \cap \mathcal{Q}'$ *(where* l_1 *and* l_2 *are the lines passing through* x *and tangent to* $\mathcal{Q}'$*) plus two points* y, z *such that* $y \in l_1 \setminus (\mathcal{Q}' \cup \{x\})$ *and* $z \in l_2 \setminus (\mathcal{Q}' \cup \{x\})$ *[1].*
27. *A point* $x \in \mathcal{Q}'$ *plus two points* $y, z \in \mathcal{Q}' \setminus \{x\}$ *plus two intersection points* $l \cap C$*, where* l *is the line tangent to* $\mathcal{Q}'$ *that passes through* x*, and* C *is a conic such that* $C \cap \mathcal{Q}' = \{y, z\}$ *and* $C \neq$ *a double line [1].*
28. *Three points* $x_1, x_2, x_3 \in \mathcal{Q}'$ *plus three intersection points* $l_1 \cap l_2, l_2 \cap l_3, l_1 \cap l_3$*, where for any* $i = 1, 2, 3$*,* l_i *is the line that passes through* x_i *and is tangent to* $\mathcal{Q}'$ *[1].*
29. *The whole* $\mathbb{C}P^2$ *[0].*

The proof is straightforward and pretty much analogous to the proof of lemma 5.1.$\diamondsuit$

Lemma 6.3 *For the vector space V and the discriminant Σ, the configuration spaces $X_1, \ldots, X_{29}$ described below satisfy Conditions 1–7 from chapter 3. The number indicated in square brackets equals the complex dimension of $L(K)$ for K lying in the corresponding X_i.*

1. *A point in D [8].*
2. *A τ-invariant pair of points in $\mathcal{Q} \setminus D$ [7].*
3. *A pair of points in D [6].*
4. *A point in D and a τ-invariant pair of points in $\mathcal{Q} \setminus D$ [5].*
5. *Two τ-invariant pairs of points in $\mathcal{Q} \setminus D$ [4].*
6. *A point $x \in D$ and two τ-invariant pairs of points in $\mathcal{Q} \setminus D$ that lie on the union of the two lines $\subset \mathcal{Q}$ that pass through x [4].*
7. *Three points on D [4].*
8. *A pair of points in D and a τ-invariant pair of points in $\mathcal{Q} \setminus D$ [3].*
9. *Four points on D [3].*
10. *The diagonal D [3].*
11. *Three τ-invariant pairs of points in $\mathcal{Q} \setminus D$ that lie on a τ-invariant pair of lines in $\mathcal{Q}$ [3].*
12. *Three τ-invariant pairs of points in $\mathcal{Q} \setminus D$ that lie on a τ-invariant pair of lines $(l_1 \cup l_2) \subset \mathcal{Q}$ plus the point $l_1 \cap \mathcal{Q}$ [3].*
13. *A τ-invariant pair of lines in $\mathcal{Q}$ [3].*
14. *A configuration K of type 5 on a nondegenerate τ-invariant conic $C \subset \mathcal{Q}$ plus one of the two points of $C \cap D$ [3].*
15. *Three τ-invariant pairs of points in $\mathcal{Q} \setminus D$ that lie on a nondegenerate τ-invariant conic $C \subset \mathcal{Q}$ [3].*
16. *A configuration K of type 5 on a nondegenerate τ-invariant conic $C \subset \mathcal{Q}$ plus both points of $C \cap D$ [3].*
17. *A nondegenerate τ-invariant conic in $\mathcal{Q}$ [3].*
18. *A configuration K of type 5 plus a point $x \in D$ that does not belong to the conic $\subset \mathcal{Q}$ that passes through the four points of K [2].*
19. *A configuration K of type 6 plus a point in $D \setminus K$ [2].*
20. *Three points in D and a τ-invariant pair of points in $\mathcal{Q} \setminus D$ [1].*
21. *A τ-invariant pair of lines $(l_1 \cup l_2) \subset \mathcal{Q}$ plus a point in $D \setminus (l_1 \cup l_2)$ [1].*
22. *A nondegenerate τ-invariant conic $C \subset \mathcal{Q}$ plus a point in $D \setminus C$ [1].*

23. *Six intersection points* $(l_1 \cup l_2 \cup l_3) \cap \mathcal{Q}$, *where* l_1, l_2, l_3 *are three noncomplanar lines in* $\mathbb{C}P^3$ *such that* $l_1 \cap l_2 \cap l_3 = \mathcal{A}$, $\tau(l_i) = l_i, i = 1, 2, 3$, *and none of the lines is tangent to* $\mathcal{Q}$ *[1].*

24. *Six intersection points* $(P_1 \cap P_2 \cap \mathcal{Q}) \cup (P_2 \cap P_3 \cap \mathcal{Q}) \cup (P_1 \cap P_3 \cap \mathcal{Q})$ *plus the point* $P_1 \cap D$, *where* P_1, P_2 *and* P_3 *are planes in* $\mathbb{C}P^3$ *such that* $P_1 \cap P_2 \cap P_3 = \mathcal{A}$, $\tau(P_i) = P_i, i = 1, 2, 3$, *neither one of the lines* P_1P_2, P_1P_3, P_2P_3 *is tangent to* $\mathcal{Q}$, *but the plane* P_1 *is [1].*

25. *Six intersection points* $(l_1 \cup l_2 \cup l_3) \cap \mathcal{Q}$, *where* l_1, l_2, l_3 *are three noncomplanar lines in* $\mathbb{C}P^3$ *such that* $l_1 \cap l_2 \cap l_3$ *is a point outside* $\mathcal{Q}$, $\tau(l_1) = l_2, \tau(l_3) = l_3$, *and none of the lines is tangent to* $\mathcal{Q}$ *[1].*

26. *A pair of points* $\{x, y\} \in D$ *plus six intersection points* $(P_x \cap P_y \cap \mathcal{Q}) \cup (P_x \cap \mathcal{P} \cap \mathcal{Q}) \cup (P_y \cap \mathcal{P} \cap \mathcal{Q})$, *where* P_x *and* P_y *are the hyperplanes tangent to* $\mathcal{Q}$ *at* x, *respectively,* y, *and* P *is a* τ*-invariant hyperplane that passes neither through* x, *nor through* y *[1].*

27. *Six intersection points* $(P_1 \cap P_2 \cap \mathcal{Q}) \cup (P_2 \cap P_3 \cap \mathcal{Q}) \cup (P_1 \cap P_3 \cap \mathcal{Q})$ *plus the point* $P_1 \cap D$, *where* P_1, P_2 *and* P_3 *are planes in* $\mathbb{C}P^3$ *such that* $P_1 \cap P_2 \cap P_3$ *is a point outside* $\mathcal{Q}$, $\tau(P_1) = P_1, \tau(P_2) = P_3$, *neither one of the lines* P_1P_2, P_1P_3, P_2P_3 *is tangent to* $\mathcal{Q}$, *but the plane* P_1 *is [1].*

28. *Nine intersection points* $\bigcup_{i,j=1,2,3}(l_i \cap \tau(l_j))$, *where* l_1, l_2, l_3 *are pairwise nonintersecting lines on* $\mathcal{Q}$ *[1].*

29. *The whole* $\mathcal{Q}$ *[0].*

This lemma follows immediately from lemmas 6.2 and 6.1. $\diamond$

Now we define for any $K \in X_i, i = 1, \ldots, 29$ the spaces $\Lambda(K)$ and $\partial\Lambda(K)$ according to the prescriptions of chapter 3. As shown in chapter 3, we can introduce the filtered spaces

$$\varnothing = \Phi_0 \subset \Phi_1 \subset \ldots \subset \Phi_{29} = \Lambda \tag{6.3}$$

and

$$\varnothing = F_0 \subset F_1 \subset \ldots \subset F_{29} = \sigma \tag{6.4}$$

such that $\bar{H}_*(\sigma) \cong \bar{H}_*(\Sigma)$, and for any $i = 1, \ldots, 29$ the space $F_i \setminus F_{i-1}$ is a complex vector bundle over $\Phi_i \setminus \Phi_{i-1}$, which in turn is a fibre bundle over X_i, the fibre over $K \in X_i$ being the space $\Lambda(K) \setminus \partial\Lambda(K)$. In the rest of the section, we denote by $E^r_{*,*} = (E^r_{p,q})$ the r-th term of the spectral sequence corresponding to the filtration (6.4).

6.1 First columns

Let us first make the following observations (we shall use some of them right now, and the rest a little later):

Lemma 6.4 *1. The spaces* $\mathcal{Q} \setminus D$ *and* $\mathbb{C}P^2 \setminus Q'$ *are homeomorphic to* $B(S^2, 2)$.

2. We have $P(B(S^2, 2)) = t^4$.

3. *Represent $\mathbb{C}P^1$ as $\mathbb{C} \cup \{\infty\}$, and denote by $f : B(\mathbb{C}P^1, 2) \to B(\mathbb{C}P^1, 2)$ the map defined as $\{a, b\} \mapsto \{1/a, 1/b\}$; let $\tilde{f}$ be the map of the local systems $\pm\mathbb{R}$ that covers f and is the identity in the fiber of $\pm\mathbb{R}$ over $\{0, \infty\}$. Then the map $\tilde{f}_* : \bar{H}_2(B(\mathbb{C}P^1, 2), \pm\mathbb{R}) \to \bar{H}_2(B(\mathbb{C}P^1, 2), \pm\mathbb{R})$ is the also identity.*

$\diamondsuit$

This implies that for any $k > 1$ we have $\bar{P}(B((\mathcal{Q} \setminus D)/\tau, k), \pm\mathbb{R}) = 0$ by lemma 4.5. Using this, and lemmas 4.2,4.3, we can recover $E^1_{p,q}$ for $p \leq 10$.

Lemma 6.5 *Nonzero $E^1_{p,q}$ for $p \leq 10$ look as follows.*

17	$\mathbb{R}$				
16		$\mathbb{R}$			
15	$\mathbb{R}$				
14					
13				$\mathbb{R}$	
12			$\mathbb{R}$		
11				$\mathbb{R}$	
10					
6					$\mathbb{R}$
	1	2	3	4	8

$\diamondsuit$

Lemma 6.6 *We have $\bar{H}_*(F_{15} \setminus F_{13}) = 0$.*

Proof. Let us note that though elements of X_{14} and X_{15} contain different amounts of points, the space $F_{15} \setminus F_{13}$ is nevertheless a fibre bundle over $X_{14} \sqcup X_{15}$. The fibre of this bundle over $K \in X_{14} \sqcup X_{15}$ is the product of the vector space $L(K)$ and the open 2-simplex such that the vertices of the simplex correspond to points in D or to τ-invariant pairs of points in $\mathcal{Q} \setminus D$.

So, we have $\bar{H}_*(F_{15} \setminus F_{13}) \cong \bar{H}_{*-6-2}(X, \pm\mathbb{R})$, where $X \subset B(\mathcal{Q}/\tau, 3)$ is the quotient $(X_{14} \sqcup X_{15})/\tau$. We have $X = Y \setminus Z$, where Y is the space of all $K \in B(\mathcal{Q}/\tau, 3)$ such that the preimage of K in $\mathcal{Q}$ is situated on a nondegenerate conic, and Z consists of all $K \in Y$ such that the preimage of K in $\mathcal{Q}$ is a configuration of type 8.

The space Y is a fibre bundle over something with generic fibre $B(S^2, 3)$. We have $\bar{H}_*(B(S^2, 3), \pm\mathbb{R}) = 0$, and hence, $\bar{H}_*(Y, \pm\mathbb{R}) = 0$. So, in order to prove the lemma it suffices to show that $\bar{H}_*(Z, \pm\mathbb{R}) = 0$. The space Z is a fibre bundle over $B(D, 2)$, the fibre being homeomorphic to $\mathbb{C}^*$. The restriction of $\pm\mathbb{R}$ to any fibre of the bundle $Z \to B(D, 2)$ is constant; the monodromy of the fibre induced by the nontrivial element of $\pi_1(B(D, 2))$ can be written as $\mathbb{C}^* \ni z \mapsto 1/z$. Moreover, the nontrivial element of $\pi_1(B(D, 2))$ acts nontrivially on the local system $\pm\mathbb{R}$ over any fixed point of the monodromy, and hence, the first term of Leray sequence for the Borel-Moore homology groups of the bundle $Z \to B(D, 2)$ looks as follows (see lemma 6.4):

2	$\mathbb{R}$		
1			$\mathbb{R}$
	2	3	4

(6.5)

Let us show that the differential $d^2_{4,1}$ of this sequence in nonzero. The cohomology of Z with coefficients in $\pm\mathbb{R}$ can also be obtained using another spectral sequence — the Leray sequence of the quotient map

$$Z \to Z/G', \tag{6.6}$$

where G' is any subgroup of the centraliser of τ in $\mathrm{Aut}(\mathcal{Q})$. Moreover, under some mild assumptions (which are always satisfied, when G' is connected and compact), the i-th line of the sequence of the map (6.6) is equal to (the 0-th line)$\otimes H^i(G', \mathbb{R})$, see theorem 4.4. Now take G' to be a maximal connected compact subgroup of the centraliser of τ in $\mathrm{Aut}(\mathcal{Q})$; it is easy to see that the only way the (cohomological) spectral sequence of the map (6.6) and the sequence (6.5) can fit together is for the differential $d^2_{4,1}$ of the sequence (6.5) to be nonzero.

So, we have $\bar{H}_*(Z, \pm\mathbb{R}) = 0$, and the lemma is proven.$\diamondsuit$

Lemma 6.7 *We have* $\bar{H}_*(F_{17} \setminus F_{15}) = 0$.

Proof. It is easy to see that the space $F_{17} \setminus F_{15}$ is a complex vector bundle over

$$\bigsqcup_C \left(\Lambda(C) \setminus \bigcup_{i \le 15} \bigcup_{\substack{K \in X_i, \\ K \subset C}} \Lambda(K) \right), \tag{6.7}$$

where C runs through the set of τ-invariant nondegenerate conics on $\mathcal{Q}$. For every such C, the space $\bigcup_{i \le 15} \bigcup_{K \in X_i} \Lambda(K)$ is homeomorphic to the third autojoin of $C/\tau \cong S^2$, and hence, the Poincaré polynomial of this space is trivial (see lemma 4.4). The space $\Lambda(C)$ is contractible, and hence the Borel-Moore homology groups of the space (6.7) are zero.$\diamondsuit$

Lemma 6.8 *We have* $\bar{P}(F_{18} \setminus F_{17}) = t^{11}(1 + t^3)$.

Proof. The space $F_{18} \setminus F_{17}$ is a fibre bundle over X_{18}, the fibre over $K \in X_{18}$ being the product of the vector space $L(K)$ and an open 2-simplex, so we have

$$\bar{H}_*(F_{18} \setminus F_{17}) \cong \bar{H}_{*-4-2}(X_{18}, \mathcal{L}), \tag{6.8}$$

where $\mathcal{L}$ is some local system on X_{18}. It is easy to see that if we view X_{18} as a subspace of $D \times B((\mathcal{Q} \setminus D)/\tau, 2)$, then $\mathcal{L}$ becomes the restriction of $\mathbb{R} \times \pm\mathbb{R}$.

Any configuration on type 5 lies on a unique τ-invariant conic $C \subset \mathcal{Q}$. The space X_{18} can be naturally filtered: $X_{18} = X' \sqcup X''$, where X' (respectively, X'') is the set of all $K \in X_{18}$ such that the τ-invariant conic corresponding to $K \setminus D$ is degenerate (reprectively, nondegenerate). It is easy to see that $\bar{H}_*(X', \mathcal{L}) = 0$, so it remains to calculate $\bar{H}_*(X'', \mathcal{L})$.

Set

$$Y = \{(x, y) \in D \times B(D, 2) | x \notin y\}.$$

The space X'' is a fibre bundle over Y, the projection of this bundle being

$$K \mapsto (K \cap D, D \cap (\text{the } \tau\text{-invariant conic containing } K \setminus D)).$$

Take $z = (x, y) \in Y$ (here $x \in D$, and y is a pair of points in $D \setminus \{x\}$). Let us identify fibre of the bundle $X'' \to Y$ over z with $B(\mathbb{C}^*, 2)$; the restriction of $\mathcal{L}$ to that fibre becomes then $\pm\mathbb{R}$. We have $\bar{P}(B(\mathbb{C}^*, 2), \pm\mathbb{R}) = t^2(1+t)$ (see proposition 4.2). The elements of $\pi_1(Y, z)$ that act nontrivially on $\bar{H}_*(B(\mathbb{C}^*, 2), \mathcal{L})$ are precisely the ones that transpose the elements of y. Such elements act on $B(\mathbb{C}^*, 2)$ as $\{a, b\} \to \{1/a, 1/b\}$; moreover, they act as minus identity on the fibre of $\mathcal{L} = \pm\mathbb{R}$ over each pair $\{a, 1/a\}$. Using proposition 4.2 we conclude that the action of $\pi_1(Y, z)$ on $\bar{H}_i(B(\mathbb{C}^*, 2), \mathcal{L})$ is the identity for $i = 2$ and minus identity for $i = 3$.

Finally, it is easy to see that Y is homeomorphic to $PSL_2(\mathbb{C})/\mathbb{Z}_2$, so $\bar{H}_*(Y, \text{any nontrivial local system}) = 0$, and $\bar{H}_*(Y, \mathbb{R}) = \bar{H}_*(SL_2(\mathbb{C}), \mathbb{R})$ (e.g., by lemma 4.1). Hence, $\bar{P}(X_{18}, \mathcal{L}) = t^5(1+t^3)$, and lemma 6.8 follows from (6.8).$\diamondsuit$

6.2 Column 23

Lemma 6.9 *We have* $\bar{H}_*(F_{23} \setminus F_{22}) = t^{10}(1+t^3)$.

Proof. The space $F_{23} \setminus F_{22}$ is the fibre bundle over X_{23}, the fibre being the product of a complex vector line and a 2-simplex. It is easy to show that the space X_{23} is homeomorphic to

$$\{\{x, y, z\} \in B(\mathcal{P} \setminus D, 3) | x, y, z \text{ are not on a line}\};$$

the local system on X_{23} corresponding to the bundle $(F_{23} \setminus F_{22}) \to X_{23}$ is $\pm\mathbb{R}$. By lemma 4.5, we have $\bar{P}(B(\mathcal{P} \setminus D, 3), \pm\mathbb{R}) = 0$, so, in order to prove the lemma, it is sufficient to calculate the Borel-Moore homology groups of the space

$$\{\{x, y, z\} \in B(\mathcal{P} \setminus D, 3) | x, y, z \text{ are on a line}\}, \tag{6.9}$$

which can be done as follows. The space

$$A = \{\{x, y, z\} \in B(\mathcal{P}, 3) | x, y, z \text{ are on a line nontangent to } D\}$$

can be filtered according to the number of intersection points with D: for $i = 1, 2, 3$ set

$$A^{(i)} = \{\{x, y, z\} \in A | \#(\{x, y, z\} \cap D) \geq 3 - i\}.$$

We have $\bar{P}(A, \pm\mathbb{R}) = 0$; the groups $\bar{H}_*(A^{(1)}, \pm\mathbb{R})$ are also zero, cf. the proof of lemma 6.6. The space $A^{(3)} \setminus A^{(2)}$ coincides with (6.9), so it remains to calculate the Borel-Moore homology groups of $A^{(2)} \setminus A^{(1)}$. This space if a fibre bundle over $F(D, 2)$, the fibre being homeomorphic to $B(\mathbb{C}^*, 2)$. So we obtain a spectral sequence converging to the Borel-Moore homology groups of $A^{(2)} \setminus A^{(1)}$ with the term E^2 equal to

3	$\mathbb{R}$		$\mathbb{R}$
2	$\mathbb{R}$		$\mathbb{R}$
	2	3	4

A group action agrument already used in the proof of lemma 6.6 shows that the differential $d^2_{4,2} : E^2_{4,2} \to E^2_{2,3}$ is nonzero, which implies that $\bar{P}(A^{(2)} \setminus A^{(1)}, \pm\mathbb{R}) = t^4(1+t^3)$. Hence, we have $\bar{P}(A^{(3)} \setminus A^{(2)}, \pm\mathbb{R}) = t^5(1+t^3)$, and the lemma follows.$\diamondsuit$

6.3 Column 24

Lemma 6.10 *We have* $\bar{H}_*(F_{24} \setminus F_{23}) = 0$.

Proof of lemma 24. Again, in order to prove the lemma it suffices to show that the Borel-Moore homology of X_{24} with coefficients in some system $\mathcal{L}$ is zero. Let us identify the space X_{24} with the space of configurations $(\{x, y\}, z)$, where x, y and z are noncollinear points of $\mathcal{P} \setminus D$ such that the line spanned by x and y is tangent to D (the points x, y and z are the intersecition points $P_1 \cap P_2 \cap \mathcal{P}$, $P_1 \cap P_3 \cap \mathcal{P}$, $P_2 \cap P_3 \cap \mathcal{P}$, where P_1, P_2, P_3 are planes that satisfy the conditions of the correspondind item of lemma 6.3). The system $\mathcal{L}$ is the inverse image of the system $\pm\mathbb{R}$ under the natural map $X_{24} \to B(\mathcal{P} \setminus D, 3)$. If we fix the point $z \in \mathcal{P} \setminus D$ and a line $l \subset \mathcal{P}$ tangent to D and not passing by z, then the space of $\{x, y\} \in B(l, 2)$ such that $(\{x, y\}, z) \in X_{24}$ is homeomorphic to $B(\mathbb{C}, 2)$. We have $\bar{P}(B(\mathbb{C}, 2), \pm\mathbb{R}) = 0$, and lemma 6.10 follows.$\diamondsuit$

6.4 Column 25

Lemma 6.11 *We have* $\bar{H}_*(F_{25} \setminus F_{24}) = t^{10}(1 + t^3)$.

A proof of this lemma will take the rest of the section.

The space $F_{25} \setminus F_{24}$ is a fibre bundle over X_{25}. Take $K \in X_{25}, K = (l_1 \cup l_2 \cup l_3) \cap \mathcal{Q}$, where l_1, l_2, l_3 are lines that satisfy the conditions from the corresponding item of lemma 6.3. The fibre of the bundle $F_{25} \setminus F_{24} \to X_{25}$ over K is the product of the vector space $L(K)$ and an open 3-simplex such that two vertices of the simplex correspond to the points of $K \cap D = l_3 \cap \mathcal{Q}$, and the other two vertices correspond to the τ-invariant pairs that make up $K \setminus D = (l_1 \cap \mathcal{Q}) \cup (l_2 \cap \mathcal{Q})$.

Let $\mathcal{L}$ be the local system on X_{25} with fibre $\mathbb{R}$ that corresponds to the fibre bundle $F_{25} \setminus F_{24} \to X_{25}$. We have $\bar{H}_*(F_{25} \setminus F_{24}) \cong \bar{H}_{*-2-3}(X_{25}, \mathcal{L})$.

It is easy to see that $l_3 \subset \mathcal{P}$ (recall that $\mathcal{P}$ is the plane that is pointwise fixed under τ), so the intersection point $l_1 \cap l_2 \cap l_3$ belongs to $\mathcal{P} \setminus D$. Hence, the space X_{25} is a fibre bundle over $\mathcal{P} \setminus D$. Denote by F the typical fibre of that bundle. Once we have fixed $x \in \mathcal{P} \setminus D$, all possible choices of the line l_3 are parametrised by the elements of $\mathbb{C}^*$ (we can take any line in $\mathcal{P}$ that passes through x except the lines tangent to D). Hence, F is a fibre bundle over $\mathbb{C}^*$; denote by F' the typical fibre of the bundle $F \to \mathbb{C}^*$.

Denote let x', l', l'' and C be respectively the point $(1 : 0 : 0) \in \mathbb{C}P^2$, the lines $\{(0 : x_1 : x_2)\}$ and $\{(x_0 : 0 : x_2)\}$ and the nondegenerate conic in $\mathbb{C}P^2$ defined by the equation $x_0^2 + x_1^2 + x_2^2 = 0$ (see Figure 6.1). Denote by τ', respectively, τ'', the involution $(x_0 : x_1 : x_2) \mapsto (-x_0 : x_1 : x_2)$, respectively, $(x_0 : x_1 : x_2) \mapsto (x_0 : x_1 : -x_2)$. Set $Y = l' \cup l'' \cup C$. Identify the space F' with $(\mathbb{C}P^2 \setminus Y)/\tau'$. The involution τ'' induces an involution of F', which will also be denoted by τ''. The system $\mathcal{L}|_{F'}$ changes the sign under any loop that transposes the τ-invariant pairs of points that make up $(l_2 \cup l_1) \cap \mathcal{Q}$. Denote by $\mathcal{M}$ the inverse image of $\mathcal{L}$ on $\mathbb{C}P^2 \setminus Y$. It is easy to see that $\mathcal{M}$ can be extended to a local system on $\mathbb{C}P^2 \setminus C$; this extension will be denoted by the $\mathcal{M}'$. Note that $\mathcal{M}$ is the nontrivial one-dimensional local system on $\mathbb{C}P^2 \setminus C$.

Lemma 6.12 1. *We have* $\bar{P}(F', \mathcal{L}) = t^2$.

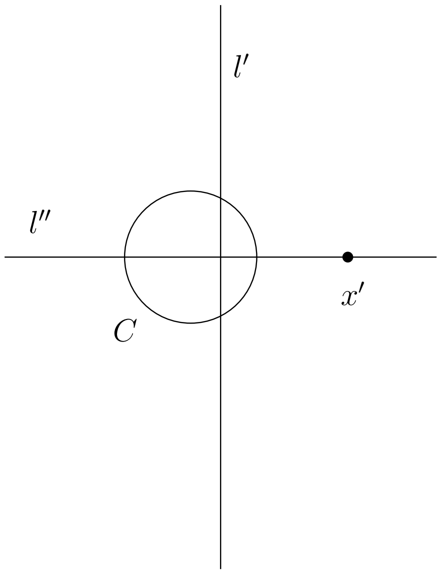
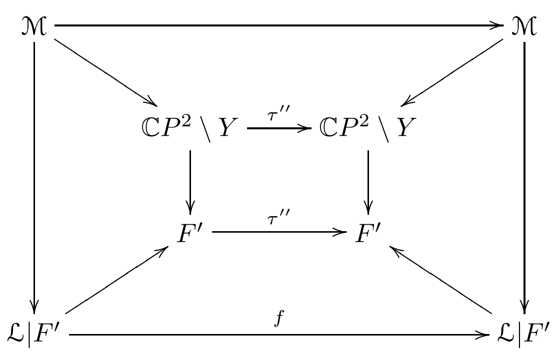


Figure 6.1:

2. *The map $f : \mathcal{L} \to \mathcal{L}$ that covers $\tau''|_{F'}$ and is minus identity over all pairs $\{(\pm x_0 : x_1 : 0)\}$ induces the identity map of $\bar{H}_2(F', \mathcal{L})$ into itself.*

Proof. Note that the actions of τ' and τ'' on $\mathbb{C}P^2 \setminus Y$ can be lifted into $\mathcal{M}$ in a natural way: indeed, τ' acts on $\mathcal{M}$ since $\mathcal{M}$ is the inverse image of $\mathcal{L}$ under $\mathbb{C}P^2 \setminus Y \to (\mathbb{C}P^2 \setminus Y)/\tau' = F'$, and the action of τ'' is included into the following commutative diagram:



Both these actions can be extended to $\mathcal{M}'$. Note that the extension of the action of τ' is the identity over x', and τ'' acts as minus identity over every point $(x_0 : x_1 : 0) \notin C$. Denote the extension of the action of τ'' to $\mathcal{M}'$ by f'.

We have $\bar{P}(\mathbb{C}P^2 \setminus C, \mathcal{M}') = t^2$. This implies easily that $\bar{P}(\mathbb{C}P^2 \setminus Y, \mathcal{M}) = 2t^2$. Since $\bar{H}_*(F', \mathcal{L})$ is isomorphic to the vector subspace of $\bar{H}_*(\mathbb{C}P^2 \setminus Y, \mathcal{M})$ that consists of τ'-invariant elements, we obtain (using a Euler characteristic argument) that $\bar{P}(F', \mathcal{L}) = t^2$.

Let us now prove the second assertion. As we noted above, $f' : \mathcal{M}' \to \mathcal{M}'$ is minus identity over the points of the line $\{(x_0 : x_1 : 0)\}$ that do not belong to C. It is easy to see that f' is then the identity over $(0:0:1)$, and hence, by lemma 6.4, the map $f'_*: \bar{H}_2(\mathbb{C}P^2 \setminus C, \mathcal{M}) \to \bar{H}_2(\mathbb{C}P^2 \setminus C, \mathcal{M})$ is the identity. We conclude from proposition 4.1 that the map of $f'_*: \bar{H}_1(l' \setminus (l'' \cup C), \mathcal{M}) \to \bar{H}_1(l' \setminus (l'' \cup C), \mathcal{M})$ is also the identity, which implies that f' induces the identical map of $\bar{H}_2(\mathbb{C}P^2 \setminus Y, \mathcal{M})$ into itself, which implies that $f_*: \bar{H}_2(F', \mathcal{L}) \to \bar{H}_2(F', \mathcal{L})$ is also the identity.$\diamondsuit$

Now let F be the fibre of the bundle $X_{25} \to \mathcal{P} \setminus D$ over $x = (0:0:0:1)$.

Lemma 6.13 *We have* $\bar{P}(F, \mathcal{L}) = t^3(1+t)$.

Proof. Let l_3 be the line defined by the equations $x_0 = x_1 = 0$. Let $A(t), t \in [0,1]$ be the linear map $\mathbb{C}P^3 \to \mathbb{C}P^3$ written in homogeneous coordinates as

$$(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 \cos \pi t + x_2 \sin \pi t : -x_1 \sin \pi t + x_2 \cos \pi t : x_3).$$

For any $t \in [0,1]$ the map $A(t)$ commutes with τ. It is easy to check that the loop $\gamma : t \mapsto A(t) l_3$ generates the π_1 of the space of all lines in $\mathcal{P}$ that pass through x and are not tangent to D. The corresponding monodromy map $F' \to F'$ is induced by $A(1)$; but $A(1) = \tau$ on the space of lines that pass through x, hence γ acts identically on F'.

The map $A(1)$ transposes the intersection points $l_3 \cap \mathcal{Q}$. Now let l_1 and l_2 be the lines defined by the conditions $x_0 - x_1 = 0, x_2 = 0$ and $x_0 + x_1 = 0, x_2 = 0$ respectively. The set $(l_1 \cup l_2) \cap \mathcal{Q}$ splits into two τ-invariant pairs, and the map $A(1)$ transposes these pairs. Hence, the loop γ acts identically on $\mathcal{L}|_{F'}$, and the lemma follows now from lemma 6.12.$\diamondsuit$

Let us now prove lemma 6.11. We shall use the notation from the proof of the previous lemma. In the same way as above, we conclude that the action of the nontrivial element $a' \in \pi_1(\mathcal{P} \setminus D, x)$ on F is induced by the linear map that can be written as

$$(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : -x_2 : -x_3). \tag{6.10}$$

This map transposes the lines passing through x and tangent to $D \cap \mathcal{Q}$, hence if we identify with $\mathbb{C}^*$ the space of lines in D that pass through x and are not tangent to $\mathcal{Q}$, then the action of a' may be written as $z \mapsto 1/z$.

The map (6.10) preserves the lines l_1, l_2, l_3; denote by F' the fibre of the bundle $F \to \mathbb{C}^*$ over l_3. By associating with a line l passing through x the intersection point of l and the plane defined by the condition $x_3 = 0$, we can identify F' with

$$\{(y_0 : y_1 : y_2) \in \mathbb{C}P^2 | y_0 \neq 0, y_1 \neq 0, y_0^2 + y_1^2 + y_2^2 \neq 0\}/\tau',$$

where $\tau'(y_0 : y_1 : y_2) = (-y_0 : y_1 : y_2)$. Under this identification, the map $F' \to F'$ induced by (6.10) is obtained from the involution τ'' of $\mathbb{C}P^2$, $\tau''(y_0 : y_1 : y_2) = (y_0 : y_1 : -y_2)$, and the τ-invariant pair of lines $\{l_1, l_2\}$ corresponds to the τ'-invariant pair $\{(1:1:0), (1:-1:0)\}$. Note that both points $(1:1:0), (1:-1:0)$ lie on the line of $\mathbb{C}P^2$ that is pointwise fixed under τ''; note also that the map (6.10) preserves the intersection points of l_3 and the conic D, while the intersection points of each one of the lines l_1, l_2 with $\mathcal{Q}$ are transposed under (6.10). Using the second assertion of lemma 6.12 we can write down the Leray spectral sequence of the bundle $X_{25} \to \mathcal{P} \setminus D$, and lemma 6.11 follows.$\diamondsuit$

6.5 Column 26

Lemma 6.14 *We have* $\bar{P}(F_{26} \setminus F_{25}) = t^{10}(1+t)(1+t^3)$.

Proof of lemma 26. As usual, we have $\bar{H}_*(F_{26} \setminus F_{25}) = \bar{H}_*(X_{26}, \mathcal{L})$, where $\mathcal{L}$ is a local system with fibre $\mathbb{R}$; we shall see below that $\mathcal{L}$ is constant.

The space X_{26} is a fibre bundle over $B(D,2)$ with fibre F, which is homeomorphic to the complex torus $(\mathbb{C}^*)^2$. Indeed, let us fix the intersection points x and y of $K \in X_{26}$ and D; let $l_x^i, l_y^i, i = 1, 2$ be the lines on $\mathcal{Q}$ that pass through x and y. We have $\sigma(l_x^1) = \sigma(l_x^2), \sigma(l_y^1) = \sigma(l_y^2)$, so, if $K \cup D$ is fixed, K is completely determined by the points $K \cup (l_x^1 \cap l_y^1)$; one of these points belongs to $l_x^1 \setminus (l_x^2 \cap l_y^2)$ and the other one to $l_y^1 \setminus (l_x^2 \cap l_y^2)$.

A loop $\gamma \in \pi_1(X_{26})$ induces transpositions of the points of $K \cap D$ and of the τ-invariant pairs of points of $K \setminus D$; the action of γ on the fibre of $\mathcal{L}$ is the product of the signs of these transpositions. It is easy to see that any loop in the fibre of the bundle $X_{26} \to B(D,2)$ does not transpose any points of K at all, and a loop that projects on the nontrivial loop in $B(D,2)$ transposes both the points $K \cap D$ and two τ-invariant pairs of points of $K \setminus D$. Hence, the action of $\pi_1(X_{26})$ on $\mathcal{L}$ is trivial.

Let us now calculate the action of $\pi_1(B(D,2))$ on the Borel-Moore homology groups of $F \cong (\mathbb{C}^*)^2$. We can introduce coordiantes z, w on F so that the map $F \to F$ induced by the nontrivial element of $\pi_1(B(D,2))$ will be $(z,w) \mapsto (w^{-1}, z^{-1})$. The eigenvalues of the corresponding map of $\bar{H}_i(F) \to \bar{H}_i(F)$ are 1 for $i = 4$, 1 and -1 for $i = 3$, and -1 for $i = 2$. Hence, the second term of the Leray spectral sequence of the bundle $X_{26} \to B(D,2)$ looks as follows:

$$\begin{array}{cccc} 4 & & & \mathbb{R} \\ 3 & \mathbb{R} & & \mathbb{R} \\ 2 & \mathbb{R} & & \\ & 2 & 3 & 4 \end{array} \tag{6.11}$$

Lemma 6.14 is proven.$\diamondsuit$

6.6 Column 27

Lemma 6.15 *We have* $\bar{H}_*(F_{27} \setminus F_{26}) = 0$.

Proof of lemma 6.15. As in the case of the previous columns, we see that $\bar{H}_*(F_{27} \setminus F_{26})$ is isomorphic (with a dimension shift) to $\bar{H}_*(X_{27}, \mathcal{L})$, where $\mathcal{L}$ is some 1-dimensional local system. The space X_{27} is a fibre bundle over $\mathcal{P} \setminus D$. The fibre of this bundle is homeomorphic to the product of the space lines in $\mathcal{P}$ that pass through x and are not tangent to D and the space of τ-invariant pairs of lines in one of the two τ-invariant planes that contain x and are tangent to $\mathcal{Q}$. The first factor in this product is homeomorphic to $\mathbb{C}^*$, and the restriction of $\mathcal{L}$ to it in nonconstant. This implies that the Borel-Moore homology of the fibre of this bundle $X_{27} \to \mathcal{P} \setminus D$ is zero, and lemma 6.15 follows.$\diamondsuit$

6.7 Column 28

Lemma 6.16 *We have* $\bar{P}(F_{28} \setminus F_{27}) = t^{10}(1+t^3)$.

Proof. The space $F_{28} \setminus F_{27}$ is a fibre bundle over$B(D,3)$, the fibre over $K \in X_{28}$ being the product of the vector space $L(K)$ and the open 5-simplex whose vertices correspond to the points of $K \cap D$ and to the τ-invariant pairs that make up $K \setminus D$. The bundle $F_{28} \setminus F_{27} \to B(D,3)$ is orientable, and the lemma follows.$\diamondsuit$

6.8 Nondiscrete singular sets

The case of X_{17} has already been taken care of (see lemma 6.7).

Lemma 6.17 *We have* $\bar{P}(F_{13} \setminus F_{12}) = 0$.

Proof. Let $K = l_1 \cup l_2$ be an element of X_{13} (here l_1 and l_2 are lines on $\mathcal{Q}$ such that $l_1 \cap l_2 = x \in D$). We have

$$\partial\Lambda(K) = \bigcup_{\substack{K' \in X_{12} \\ K' \subset K}} \Lambda(K'),$$

which is a contractible space.$\diamondsuit$

Lemma 6.18 *We have* $\bar{P}(F_{22} \setminus F_{21}) = t^{10}(1+t^3)$.

Proof. Let us first calculate the polynomial $\bar{P}(\Lambda(K) \setminus \partial\Lambda(K))$ for $K \in X_{22}$. Let $K = C \cup \{x\}$ be an element of X_{22} (here C is a τ-invariant conic, and $x \in D \setminus C$). Let us show that $P(\partial\Lambda(K)) = 1$. We have

$$\partial\Lambda(K) = \Lambda(C) \cup \bigcup_{\substack{K' \in X_{18} \cup X_{20}, \\ K' \subset K}} \Lambda(K').$$

The second term in this union can be contracted to $\Lambda(\{x\})$, and the space

$$\Lambda(C) \cap \bigcup_{\substack{K' \in X_{18} \cup X_{20}, \\ K' \subset K}} \Lambda(K') = \bigcup_{\substack{K' \in X_8 \cup X_5, \\ K' \subset C}} \Lambda(K')$$

is homeomorphic to

$$(S^2)^{*2} \bigcup \begin{array}{c} \text{the union of all 2-simplices of } (S^2)^{*3} \\ \text{that have vertices at two fixed points.} \end{array} \tag{6.12}$$

The difference of (6.12) and $(S^2)^{*2}$ is an orientable fibre bundle over $\mathbb{C}^*$, the fibre being the open 2-simplex. Hence, the group $\bar{H}_i$ of (6.12) is $\mathbb{R}$ if $i = 0, 3, 4$ and zero otherwise, which implies that $\bar{P}(\Lambda(K) \setminus \partial\Lambda(K)) = t^5(1+t)$.

Now, the space X_{22} is homeomorphic to the "mixed" configuration space

$$\{(\{x,y\}, z) \in B(D,2) \times D | z \notin \{x,y\}\}.$$

Any loop in this space has the form

$$((\{x,y\}, z), t) \mapsto A(t)(\{x,y\}, z),$$

where $t \mapsto A(t)$ is a loop in the automorphism group of D (based at the identity). Hence, the only nontrivial monodromy map of the fibre of the bundle $F_{22} \setminus F_{21} \to X_{22}$ over $(\{x,y\},z)$ is induced by the automorphism of D that transposes x and y, and preserves z. This monodromy map acts identically (respectively, as multiplication by -1) on the 3-d (respectively, 4-th) Borel-Moore homology group of (6.12). Hence, the 5-th (respectively, the 6-th) line of the term E^2 of the Leray sequence of the bundle $F_{22} \setminus F_{21} \to X_{22}$ contains the Borel-Moore homology groups of (6.12) with coefficients in the trivial (respectively, a nontrivial) local system. Since the space (6.12) is homeomorphic to $PGL_2(\mathbb{C})/\mathbb{Z}_2$, lemma 6.18 follows from lemma 4.1. $\diamondsuit$

Lemma 6.19 *We have* $\bar{P}(F_{21} \setminus F_{20}) = 0$.

Proof. We proceed as in the proof of the previous lemma and pick a $K = l_1 \cup l_2 \cup \{x\}$, where l_1 and l_2 are lines on $\mathfrak{Q}$, $\tau(l_1) = l_2$, and $x \in D \setminus (l_1 \cap l_2)$. We have

$$\partial\Lambda(K) = \Lambda(l_1 \cup l_2) \cup \bigcup_{\substack{K' \in X_{19} \\ K' \subset K}} \Lambda(K').$$

Both terms of this union are contractible, and their intersection is homeomorphic to the union of all simplices of $(S^2)^{*3}$ that have some fixed common vertice. The latter space is also contractible, which implies the lemma.$\diamondsuit$

6.9 The last column

The last column contains the Borel-Moore homology groups of the open cone over space $\partial\Lambda = \bigcup_{i<29} \bigcup_{K \in X_i} \Lambda(K)$.

Lemma 6.20 *We have* $P(\partial\Lambda) = 1 + (1+t^3)(t^7 + 5t^8 + t^9) - (1+t)(t^7 + t^{11} + at^{10} + bt^8)$*, where a and b a=re integers such that* $0 \leq a, b \leq 1$.

Proof of lemma 6.20. The homology groups of $\partial\Lambda$ can be calculated using the spectral sequence corresponding to the filtration $\varnothing \subset \Phi_1 \subset \cdots \subset \Phi_{28}$. We shall denote the terms of this spectral sequence by $e^r_{p,q}$. We already know all the groups $e^1_{p,q}$ (indeed, we have $t^{2d_i}\bar{P}(\Phi_i \setminus \Phi_{i-1}) = \bar{P}(F_i \setminus F_{i-1})$ for any $i = 1, \ldots, 28$, where d_i is the number indicated in square brackets in the corresponding item of the list of lemma 6.3.

The generators of dimension ≤ 7 that come from the first four columns of the spectral sequence $(e^r_{p,q})$ do not survive in the term e^∞, since $V \setminus \Sigma$ is isomorphic to a smooth affine hypersurface, and hence, has the homotopy type of a ≤ 10-dimensional CW-complex. The same argument proves that the 7-dimensional class that comes from column 18 of the sequence $(e^r_{p,q})$ does not survive.

So, in order to prove lemma 6.20, it remains to show that $\bar{H}_{12}(\partial\Lambda) = 0$.

Lemma 6.21 *We have* $\bar{H}_{12}(\partial\Lambda) = 0$.

Proof. For any $K \in X_{26}$ define $\hat{K} \in X_{25}$ as follows. Let x and y be the points such that $\{x,y\} = K \cap D$, and let $l^i_x, l^i_y, i = 1,2$ be the lines on $\mathfrak{Q}$ such that that $l^1_x \cap l^2_x = x, l^1_y \cap l^2_y = y$, and $l^i_x \cap l^i_y = \varnothing$ for $i = 1,2$; we set

$\hat{K} = K \setminus ((l_x^1 \cap l_y^2) \cup (l_y^1 \cap l_x^2))$. Denote by N the union of all configurations of the form $\hat{K}, K \in X_{26}$.

The space $\partial\Lambda$ can be filtered as follows:

$$\Phi_{24} \sqcup \bigsqcup_{K \in X_{26}} \left(\Lambda(K) \setminus \bigcup_{\substack{K' \subset K \\ K' \notin X_{25}}} \Lambda(K') \right) \sqcup \bigsqcup_{\substack{K \in X_{25} \\ K \notin N}} (\Lambda(K) \setminus \Lambda(K)) \sqcup (\partial\Lambda \setminus \Phi_{26}). \tag{6.13}$$

The the Borel-Moore homology of the second term in this filtration is zero, since the second term is a fibre bundle (over X_{26}) with fibre homeomorphic to the 4-simplex minus four faces of maximal dimension. We have seen before that no 12-dimensional homology class can come from the first and the fourth terms of (6.13). So the only place a 12-class can come from is the third term, which is fibered over $X_{25} \setminus N$ with fiber homeomorphic to an open 3-simplex, so in order to prove the lemma, it is sufficient to show that $\bar{H}_9(X_{25} \setminus N, \mathcal{L}) = 0$, where we denote by $\mathcal{L}$ the local system over X_{25} that was introduced in the proof of lemma 6.11. Note that the restriction of $\mathcal{L}$ to N is constant. Since $\bar{H}_9(X_{25}, \mathcal{L}) = 0$, it remains to show that the map $\bar{H}_8(N) \to \bar{H}_8(X_{25}, \mathcal{L})$ is injective.

The manifolds X_{25} and N are both fibre bundles over $\mathcal{P} \setminus D$, and the inclusion $N \subset X_{25}$ is a bundle map. If F is the fibre of the bundle $X_{25} \to \mathcal{P} \setminus D$ over some point of $\mathcal{P} \setminus D$, then both F and $F \cap N$ are fibre bundles over $\mathbb{C}^*$, and the inclusion $F \cap N \subset F$ is again a bundle map. Let F' be the fibre of the bundle $F \to \mathbb{C}^*$ some point of $\mathbb{C}^*$, and let us show that the homomorphism $\bar{H}_2(F' \cap N) \to \bar{H}_2(F', \mathcal{L})$ is injective.

Let x', l', l'', C and τ' be respectively the point, the lines, the conic and the involution of $\mathbb{C}P^2$ that were introduced in the proof of lemma 6.11. In addition, let m_1 and m_2 be the lines represented schematically on Figure 6.2. If we set $Y = l' \cup l'' \cup C$ and identify the space F' with $(\mathbb{C}P^2 \setminus Y)/\tau'$, then $F' \cap N$ will be identified with $((m_1 \cup m_2) \setminus Y)/\tau'$. Let $\mathcal{M}$ and $\mathcal{M}'$ be the local system respectively on $\mathbb{C}P^2 \setminus Y$ and $\mathbb{C}P^2 \setminus C$ that were defined in the proof of lemma 6.11. The image of $\bar{H}_2(m_i \setminus C, \mathcal{M}'), i = 1, 2$ in $\bar{H}_2(\mathbb{C}P^2 \setminus C, \mathcal{M}')$ is nonzero and coincides with the image of $\bar{H}_2(m \setminus C, \mathcal{M}')$ for any line m tangent to C. Choose m as shown on Figure 6.2.

As we have seen in the proof of lemma 6.12, the natural action of τ' on $\mathcal{M}'$ is identical over x', which implies that τ'_* is the identity on the image of $\bar{H}_2((m_1 \cup m_2) \setminus C, \mathcal{M}')$ in $\bar{H}_2(\mathbb{C}P^2 \setminus C, \mathcal{M}')$. The commutative diagram

$$\begin{array}{ccc} \bar{H}_2((m_1 \cup m_1) \setminus C, \mathcal{M}') & \longrightarrow & \bar{H}_2((m_1 \cup m_2) \setminus Y, \mathcal{M}) \\ \downarrow & & \downarrow \\ \bar{H}_2(\mathbb{C}P^2 \setminus C, \mathcal{M}') & \longrightarrow & \bar{H}_2(\mathbb{C}P^2 \setminus Y, \mathcal{M}) \end{array}$$

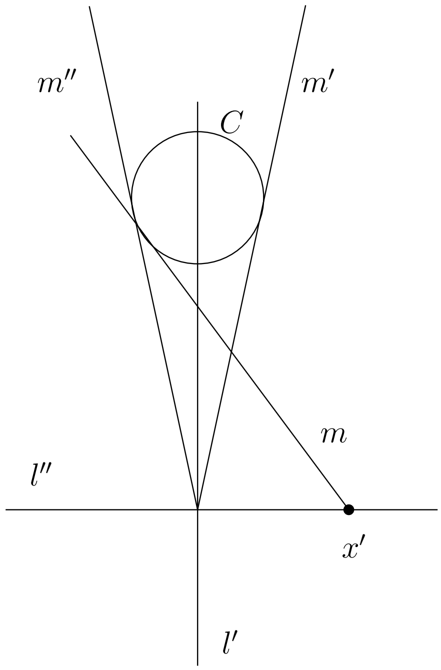


Figure 6.2:

shows that τ'_* is the identity on the image of $\bar{H}_2((m_1 \cup m_2) \setminus Y, \mathcal{M}')$ in $\bar{H}_2(\mathbb{C}P^2 \setminus Y, \mathcal{M}')$, and the diagram

$$\begin{array}{ccc} \bar{H}_2((m_1 \cup m_1) \setminus Y, \mathcal{M}) & \longrightarrow & \bar{H}_2(\mathbb{C}P^2 \setminus Y, \mathcal{M}) \\ \downarrow & & \downarrow \\ \bar{H}_2(F' \cap N, \mathcal{L}) & \longrightarrow & \bar{H}_2(F', \mathcal{L}) \end{array}$$

implies that the image of $\bar{H}_2(F' \cap N, \mathcal{L})$ in $\bar{H}_2(\mathbb{C}P^2 \setminus Y, \mathcal{M})$ is nonzero. Lemma 6.21 is proven.$\diamondsuit$

6.10 Proof of theorem 6.1

Lemma 6.22 *The nonzero element of $E^1_{26,14-26}$ does not survive in the term E^∞.*

Proof. Note that the union of the second and the third terms of (6.13) is $\Phi_{26} \setminus \Phi_{24}$. We have proven that $\bar{H}_{12}(\Phi_{26} \setminus \Phi_{24}) = 0$. We have $\dim_{\mathbb{C}} L(K) = 1$ for $K \in X_{25} \cup X_{26}$; one can check that $F_{26} \setminus F_{24}$ is a 1-dimensional complex vector bundle over $\Phi_{26} \setminus \Phi_{24}$. We have $\bar{H}_{14}(F_{26} \setminus F_{24}) = 0$, and hence, the differential d^1 is nonzero on the nonzero element of $E^1_{26,14-26}$. Lemma 6.22 is proven.$\diamondsuit$

Lemma 6.23 *The Poincaré polynomial of the space $V \setminus \Sigma$ is divisible by $(1 + t)(1 + t^3)$.*

Proof. Recall that the space $V \setminus \Sigma$ is homeomorphic to the space $V^{(1)} \setminus \Sigma^{(1)}$ of polynomials that define smooth cubics in $\mathbb{C}P^2$ that intersect transversally a fixed nondegenerate conic. The group $GL_3(\mathbb{C})$ acts almost freely on the space of $\Pi_{3,2} \setminus \Sigma_{3,2}$, and the Leray sequence of the corresponding quotient map degenerates at the second term. This is a consequence of a general result of J. Steenbrink and C. Peters ([11]); alternatively, this follows from the fact that the Poincaré polynomial of the space $\Pi_{3,2} \setminus \Sigma_{3,2}$ is equal to Poincaré polynomial of $GL_3(\mathbb{C})$ (see [14]). Hence, the restriction map from $H^*(\Pi_{3,2} \setminus \Sigma_{3,2}, \mathbb{R})$ to the real cohomology of any orbit of the action of $GL_3(\mathbb{C})$ is surjerctive.

Let G_1 be the subgroup of $GL_3(\mathbb{C})$ generated by scalar matrices and elements of $SO_3(\mathbb{C})$. The Poincaré polynomial of G_1 is $(1+t)(1+t^3)$. The group G_1 acts on $V^{(1)} \setminus \Sigma^{(1)}$, and the restriction map $H^*(GL_3(\mathbb{C}), \mathbb{R}) \to H^*(G_1, \mathbb{R})$ is surjective. The lemma follows now from the Leray-Hirsch principle.$\diamondsuit$.

Now the proof of theorem 6.1 becomes pretty straightforward. We have calculated the dimensions of all $E^1_{p,q}$ for $1 \leq p \leq 28$, and we have partial information on the last column due to lemma 6.20. An easy check shows that the Poincaré polynomial of the space $V \setminus \Sigma$ is equal to

$$(1+t)^2(1+t^3)+(1+t^3)(2t^5+6t^6+5t^7+t^8)-(1+t)(t^6+at^7+ct^8+bt^9+t^{10}),$$

where $a \in \{0,1,2,3\}$, and $b, c \in \{0,1\}$. The proof of theorem 6.1 is completed by checking all possibilites for a, b and c.$\diamondsuit$

Chapter 7

Real smooth plane cubics

In this chapter we apply the method developed in chapter 3 to calculate the real cohomology groups of the space $\Pi_{3,2}(\mathbb{R}) \setminus \Sigma_{3,2}$. We set $V = \Pi_{3,2}(\mathbb{R}), \Sigma = \Sigma_{3,2} \cap \Pi_{3,2}$, and we would like to apply theorem 3.2 to construct a conical resolution of $V \setminus \Sigma$. In this chapter a *line* means a conjugation-invariant line in $\mathbb{C}P^2$; we shall denote by $\mathbb{R}P^{2\vee}$ the space of all conjugation-invariant lines in $\mathbb{C}P^2$.

Lemma 7.1 *The configuration spaces X_1–X_8 that consist of the following configurations satisfy conditions 1–8 from chapter 3.*

1. *A point in $\mathbb{R}P^2$ [7].*
2. *Two complex conjugate points in $\mathbb{C}P^2 \setminus \mathbb{R}P^2$ [4].*
3. *A pair of points in $\mathbb{R}P^2$ [4].*
4. *Three points in $\mathbb{R}P^2$ on a line [3].*
5. *A configuration $\{x\}$ of type 1 plus a configuration $\{y, \bar{y}\}$ of type 2 such that x, y and $\bar{y}$ belong to some line $\in \mathbb{R}P^{2\vee}$ [3].*
6. *A line $\in \mathbb{R}P^{2\vee}$ [3].*
7. *Three points in $\mathbb{R}P^2$ not on a line [1].*
8. *A point in $\mathbb{R}P^2$ plus a pair of complex conjugate points in $\mathbb{C}P^2 \setminus \mathbb{R}P^2$ such that the three points are not on a line in $\mathbb{C}P^2$ [1].*
9. *The whole $\mathbb{C}P^2$ [0].*

$\diamondsuit$

Applying theorem 3.2, we construct a filtered space σ such that $\bar{H}_*(\sigma) \cong \bar{H}_*(\Sigma_{3,2} \cap \Pi_{3,2}(\mathbb{R}))$. As described in chapter 3, we construct the spaces $\Phi_i, F_i, i = 1, \ldots, 8$ and Λ, and for any $i = 1, \ldots, 8$ and $K \in X_i$, we introduce the spaces $\Lambda(K)$ and $\partial\Lambda(K)$. We denote the natural projections $\sigma \to \Sigma_{3,2}$ and $\sigma \to \Lambda$ respectively by π and p. Denote by *Geom* the union of all coherent simplices from Λ that do not have vertices on X_4 and X_5.

For our purposes it will be more convenient to omit the terms F_4 and F_5 in the filtration $\varnothing \subset F_1 \subset \ldots \subset F_8 = \sigma$: we set $\tilde{F}_i = F_i$ for $i \leq 3$ and $\tilde{F}_i = F_{i+2}$ otherwise.

Theorem 7.1 *1. There exists a filtration $\varnothing \subset \tilde{F}_1 \subset \cdots \tilde{F}_7 = \sigma$ on the space σ such that the term E^1 of corresponding spectral sequence is given by the following table.*

$$\begin{array}{r|ccccccc} 6 & \mathbb{R} & & & & & & \\ 5 & & & & & & & \\ 4 & & \mathbb{R} & & & & & \\ 3 & & & & & & & \\ 2 & & & & & & \mathbb{R} & \mathbb{R} \\ 1 & & & & & & & \\ 0 & & & & & & & \\ -1 & & & & & & & \mathbb{R} \\ \hline & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \qquad (7.1)$$

2. The differential $d^5_{6,2} : E^5_{6,2} \to E^5_{1,6}$ of this sequence is nonzero.

A proof of theorem 7.1 will take the rest of the chapter.

The space $\tilde{F}_1$ is homeomorphic to

$$\{(f, x) \in \Pi_{3,2}(\mathbb{R}) \times \mathbb{R}P^2 | \text{the curve defined by } f \text{ has a singularity at } x\}.$$

Set $x = (1 : 0 : 0) \in \mathbb{R}P^2$. The vector space $L(\{x\})$ is spanned by the monomials $x_1^3, x_2^3, x_1^2x_2, x_2^2x_1, x_0x_1^2, x_0x_2^2, x_0x_1x_2$. The action of the nontrivial element of $\pi_1(\mathbb{R}P^2, \{x\})$ on $L(\{x\})$ is induced by the map $x_0 \mapsto -x_0, x_1 \mapsto -x_1, x_2 \mapsto x_2$, hence, the vector bundle $\tilde{F}_1 \to \mathbb{R}P^2$ is orientable. We have $E^1_{1,q} = \bar{H}_{1+q}(\tilde{F}_1) = H_{q-6}(\mathbb{R}P^2)$.

Let A be the space of pairs $\{\{x, y\} \in B(\mathbb{C}P^2 \setminus \mathbb{R}P^2, 2) | x = \bar{y}\}$. The space $\tilde{F}_2 \setminus \tilde{F}_1$ is a vector fibre bundle over A. Note that any pair $\{x, \bar{x}\} \in A$ is situated on a unique real (i.e., invariant under the complex conjugation) line. Hence, the space A is a fibre bundle over the space $\mathbb{R}P^{2\vee}$ of all complex conjugation-invariant lines in $\mathbb{C}P^2$, the fibre being homeomorphic to $(\mathbb{C}P^1 \setminus \mathbb{R}P^1)$/(the complex conjugation), i.e., to the open disk. The bundle $A \to \mathbb{R}P^{2\vee}$ is nonorientable (i.e., the monodromy map induced by a nontrivial loop in $\mathbb{R}P^{2\vee}$ is orientation-reversing).

Set $K = \{(1 : i : 0), (1 : -i : 0)\}$. The space $L(K)$ is spanned by $x_2^2x_0, x_2^2x_1, x_2(x_0^2 + x_1^2), x_2^3$. The action of the nontrivial element of $\pi_1(A, K)$ is induced by $x_0 \mapsto x_0, x_1 \mapsto -x_1, x_2 \mapsto -x_2$, hence, the vector bundle $\tilde{F}_2 \setminus \tilde{F}_1 \to A$ is nonorientable. Hence, $\tilde{F}_2 \setminus \tilde{F}_1$ is a homologically trivial fibre bundle over $\mathbb{R}P^{2\vee}$, and we have have $E^1_{2,q} = \bar{H}_{2+q}(\tilde{F}_2 \setminus \tilde{F}_1) = H_{q-4}(\mathbb{R}P^2)$.

The space $\tilde{F}_3 \setminus \tilde{F}_2$ is a vector bundle over the space $\Phi_3 \setminus \Phi_2$. The latter space is a fibre bundle over $B(\mathbb{R}P^2, 2)$, the fibre over a pair $K = \{x, y\}$ being $\Lambda(K) \setminus \Phi_2$, which is homeomorphic to an open interval, whose endpoints correspond to x and y. Set $K = \{(1 : 0 : 0), (0 : 1 : 0)\}$. The group $\pi_1(B(\mathbb{R}P^2, 2), K)$ is generated by the loops $\gamma_1 : t \mapsto \{(\cos \pi t : 0 : \sin \pi t), (0 : 1 : 0)\}, \gamma_2 : t \mapsto \{((1 : 0 : 0), (0 : \cos \pi t : \sin \pi t)\}$ and $\gamma_3 : t \mapsto \{(\cos \pi t : \sin \pi t : 0), (-\sin \pi t : \cos \pi t : 0)\}, t \in [0, 1]$. Neither one of these loops changes the orientation of $L(K)$, but γ_3 transposes $(1 : 0 : 0)$ and $(0 : 1 : 0)$. Hence, we have $E^1_{3,q} = \bar{H}_{3+q}(\tilde{F}_2 \setminus \tilde{F}_3) = \bar{H}_{q-3}(B(\mathbb{R}P^2, 2), \pm\mathbb{R})$. Since the Borel-Moore homology groups of the open Möbius leaf are zero, we have $\bar{H}_*(F(\mathbb{R}P^2, 2), \mathbb{R}) = 0$, and hence, $\bar{H}_*(B(\mathbb{R}P^2, 2), \mathbb{R}) = \bar{H}_*(B(\mathbb{R}P^2, 2), \pm\mathbb{R}) = 0$, i.e., the third column of the spectral sequence (7.1) is zero.

The space $\tilde{F}_4 \setminus \tilde{F}_3$ is a vector bundle over $\Phi_6 \setminus \Phi_3$ (recall that we have omitted the terms F_4 and F_5 in the original filtration on the space σ). The space $\Phi_6 \setminus \Phi_3$ is a fibre bundle over $\mathbb{R}P^{2\vee}$.

For any $l \in \mathbb{R}P^{2\vee}$ set $l_{\mathbb{R}} = l \cap \mathbb{R}P^2$; the fibre of the bundle $\Phi_6 \setminus \Phi_3 \to \mathbb{R}P^{2\vee}$ over l is the cone over the space

$$\partial\Lambda(l) = \bigcup_{\substack{K = \text{three} \\ \text{of points in } l_{\mathbb{R}}}} \Lambda(K) \cup \bigcup_{\substack{K \text{ is a pair of complex} \\ \text{conjugate points in } l \setminus l_{\mathbb{R}} \\ \text{plus a point in } l_{\mathbb{R}}}} \Lambda(K) \qquad (7.2)$$

minus the space

$$\partial\Lambda(l) = \bigcup_{\substack{K \text{ is a pair} \\ \text{of points in } l_{\mathbb{R}}}} \Lambda(K) \cup \bigcup_{\substack{K \text{ is a pair of complex} \\ \text{conjugate points in } l \setminus l_{\mathbb{R}}}} \Lambda(K). \qquad (7.3)$$

Lemma 7.2 *The Borel-Moore homology groups of $\tilde{F}_4 \setminus \tilde{F}_3$ are equal to those of the restriction of the vector bundle $\tilde{F}_4 \setminus \tilde{F}_3 \to \Phi_6 \setminus \Phi_3$ to $(\Phi_5 \setminus \Phi_3) \cap Geom$.*

(The space *Geom* was defined on page 61.)

Proof. For any $l \in \mathbb{R}P^{2\vee}$ the vector bundle $\tilde{F}_4 \setminus \tilde{F}_3 \to \Phi_6 \setminus \Phi_3$ is constant over $\Lambda(l) \setminus \Phi_3$. The space $(\Phi_6 \setminus \Phi_3) \cap Geom$, as well as $\Phi_6 \setminus \Phi_3$, is a fibre bundle over $\mathbb{R}P^{2\vee}$, only this time the fibre over $l \in \mathbb{R}P^{2\vee}$ is the open cone over the space (7.3). The space quotient space

$$(\text{cone over } (7.2))/(\text{cone over } (7.3))$$

is contractible, hence for any $l \in \mathbb{R}P^{2\vee}$ the inclusion

$$((\Phi_6 \setminus \Phi_3) \cap Geom \cap \Lambda(l)) \to (\Phi_6 \setminus \Phi_3) \cup \Lambda(l)$$

induces an isomorphism of the Borel-Moore homology groups. The lemma is proven.$\diamondsuit$

Remark. This lemma is an analogue of V. A. Vassiliev's "geometrisation" construction [14, Section 2]. It can be immediately extended to the case of general conical resolutions constructed in chapter 3, which allows one to reduce somewhat long lists of configuration spaces (like the list of proposition 5.1). We did not need this in chapters 5 and 6, since most spaces of "fake" singular loci we were forced to introduce did not contribute anything to the spectral sequence; in the real case however (as well as, e.g., in the case of complex cubic threefolds) it is essential to keep the list of configuration spaces as short as posssible. We postpone the details to future work.

In (7.3), the first component is just the space $l_{\mathbb{R}}^{*2}$ (the second self-join of $l_{\mathbb{R}}$). Due to the Caratheodory theorem (see, e.g., [15, Chapter 7, §2]), $S^{1(*2)}$ is homeomorphic to S^3. The second component is $(l \setminus l_{\mathbb{R}})$/(the complex conjugation), which is homeomorphic to the open 2-disc. So the space $\partial\Lambda(l) \cap Geom$ is homeomorphic to S^3 with a 2-dimensional disc attached along some knot.

Let $m : \partial\Lambda(l) \cap Geom \to \partial\Lambda(l) \cap Geom$ be the monodromy induced by the generator of $\pi_1(\mathbb{R}P^2)$. The restriction $m|l_{\mathbb{R}}^{*2}$ is orientation-preserving, but $m|(l \setminus l_{\mathbb{R}})$/(the complex conjugation) is orientation-reversing. So the term E^2

of the Borel-Moore homology Leray spectral sequence of the bundle $(\Phi_6 \setminus \Phi_3) \cap Geom \to \mathbb{R}P^{2\vee}$ looks as follows:

$$\begin{array}{cccc} 4 & \mathbb{R} & & \\ 3 & & & \mathbb{R} \\ & 0 & 1 & 2 \end{array} \tag{7.4}$$

Lemma 7.3 *We have $\bar{H}_*((\Phi_6 \setminus \Phi_3) \cap Geom) = 0$.*

Proof of lemma 7.3. Recall that the group $SO_3(\mathbb{R})$ acts on the space $(\Phi_6 \setminus \Phi_3) \cap Geom$ in a natural way. If $G \subset SO_3(\mathbb{R})$ is the 1-parametric subgroup that preserves some line l, then there exist exactly two complex conjugate points $X(l), \bar{X}(l) \in l$ that preserved by any element of G; the group G acts freely on $l \setminus \{X(l), \bar{X}(l)\}$. Hence, the stabiliser of a point $x \in (\Phi_6 \setminus \Phi_3) \cap Geom$ is nondiscrete, iff x belongs to the semi-segment with vertices l and $\{X(l), \bar{X}(l)\}$ for some $l \in \mathbb{R}P^{2\vee}$. Since the Borel-Moore homology groups of a semi-segment are zero, we have $\bar{H}_*((\Phi_6 \setminus \Phi_3) \cap Geom) \cong \bar{H}_*(B)$, where B stands for the union of the points of $(\Phi_6 \setminus \Phi_3) \cap Geom$ whose stabilisers are discrete.

Using the Leray sequence for the cohomology groups with compact supports of the quotient map $B \to B/SO_3(\mathbb{R})$, we see that either the Borel-Moore homology groups of the space B are zero or there exist $0i, j$ such that $i - j \geq 3$, and $\bar{H}_i(B) \neq 0 \neq \bar{H}_j(B)$. The second possibility is excluded by the spectral sequence (7.4). The lemma is proven.$\diamondsuit$

Hence, we have $\bar{H}_*((\Phi_6 \setminus \Phi_3) \cap Geom, \mathbb{R}) = 0$. It is easily checked that the action of $\pi_1((\Phi_6 \setminus \Phi_3) \cap Geom) \cong \pi_1(\mathbb{R}P^{2\vee})$ on $L(l)$ (where l is a line $\in \mathbb{R}P^{2\vee}$) is trivial, hence $E^1_{4,p} = \bar{H}_{p+4}(\tilde{F}_4 \setminus \tilde{F}_3) = \bar{H}_{p+1}((\Phi_6 \setminus \Phi_3) \cap Geom), \mathbb{R}) = 0$.

Let us now consider the fifth column of the spectral sequence. We have $\bar{H}_*(\tilde{F}_5 \setminus \tilde{F}_4) = \bar{H}_{*-1-2}(\tilde{B}(\mathbb{R}P^2, 3), \pm\mathbb{R} \otimes \mathcal{L})$, where $\tilde{B}(\mathbb{R}P^2, 3)$ denotes the space of configurations of type "three points in $\mathbb{R}P^2$ not on a line", and $\mathcal{L}$ is the local system corresponding to the action of $\pi_1(\tilde{B}(\mathbb{R}P^2, 3))$ on the vector space $L(K)$ for $K \in \tilde{B}(\mathbb{R}P^2, 3)$. Let us calculate that action.

Denote by $\tilde{F}(\mathbb{R}P^2, 3)$ the space of ordered triples of points of $\mathbb{R}P^2$ in general position. Set $K = \{(1:0:0), (0:1:0), (0:0:1)\}$. The space $\tilde{F}(\mathbb{R}P^2, 3)$ can be contracted onto $F(\mathbb{R}P^2, 2)$; it is easy to see that any element of the image of $\pi_1(\tilde{F}(\mathbb{R}P^2, 3))$ under the natural projection $\tilde{F}(\mathbb{R}P^2, 3) \to \tilde{B}(\mathbb{R}P^2, 3)$ acts trivially on $\bar{H}_*(L(K))$. Consider the loop

$$\gamma : t \mapsto \{(\cos\frac{\pi t}{2} : \sin\frac{\pi t}{2} : 0), (-\sin\frac{\pi t}{2} : \cos\frac{\pi t}{2} : 0), (0:0:1)\}, t \in [0, 1]. \tag{7.5}$$

This loop transposes two elements of K and acts non-trivially on $\bar{H}_*(L(K))$, so we have $\mathcal{L} = \pm\mathbb{R}$.

Let us calculate the groups $\bar{H}_*(\tilde{B}(\mathbb{R}P^2, 3), \mathbb{R})$ and $\bar{H}_*(\tilde{B}(\mathbb{R}P^2, 3), \pm\mathbb{R})$.

Lemma 7.4 *The group $\bar{H}_i(\tilde{B}(\mathbb{R}P^2, 3), \pm\mathbb{R}) = \mathbb{R}$, if $i = 3$ or 6, and is zero otherwise. For any i, the group $\bar{H}_i(\tilde{B}(\mathbb{R}P^2, 3), \mathbb{R})$ is zero.*

Proof of lemma 7.4. The manifold $\tilde{F}(\mathbb{R}P^2, 3)$ can be contracted onto the space of ordered collections of triples of paiwise orthogonal lines in $\mathbb{R}^3$ that pass through zero; hence, $P(\tilde{F}(\mathbb{R}P^2, 3)) = P(SO_3(\mathbb{R})) = 1 + t^3$. On the other hand, $\tilde{F}(\mathbb{R}P^2, 3)$ is orientable (see Figure 7.1), and hence, $\bar{P}(\tilde{F}(\mathbb{R}P^2, 3)) = t^3(1 + t^3)$.

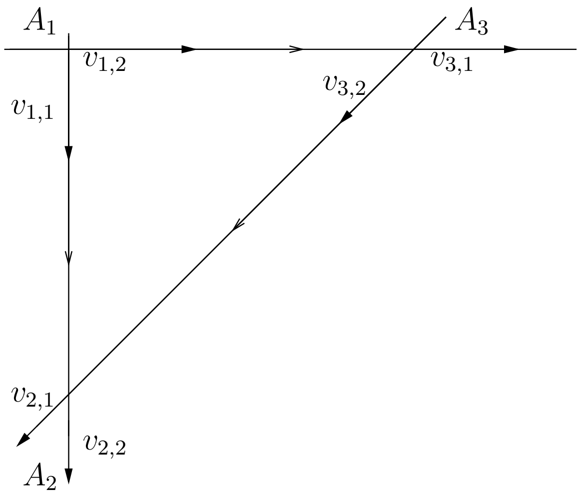


Figure 7.1: Here is how $\tilde{F}(\mathbb{R}P^2,3)$ can be oriented: take some $K = (A_1,A_2,A_3) \in \tilde{F}(\mathbb{R}P^2,3)$. Choose an orientation on each line A_1A_2, A_2A_3, A_3A_1. For any $i=1,2,3$ and $j=1,2$ let $v_{i,j} \in T_{A_i}A_iA_{i+j \mod 3}$ be a positive vector. Let $w_{i,j}$ be the element of $T_K(\tilde{F}(\mathbb{R}P^2,3))$ that corresponds to $v_{i,j}$. The orientation on $T_K\tilde{F}(\mathbb{R}P^2,3)$ defined by the frame $(w_{1,1},w_{1,2},\dots,w_{3,1},w_{3,2})$ does not depend on the choice of the orientations on the lines A_1A_2, A_2A_3, A_3A_1.

The Leray spectral sequence of the covering $\tilde{F}(\mathbb{R}P^2,3) \to \tilde{B}(\mathbb{R}P^2,3)$ imples that

$$\bar{P}(\tilde{F}(\mathbb{R}P^2,3)) = P(\tilde{B}(\mathbb{R}P^2,3)) + P(\tilde{B}(\mathbb{R}P^2,3),\pm\mathbb{R}) + 2P(\tilde{B}(\mathbb{R}P^2,3),\mathcal{V}_{2,1}), \tag{7.6}$$

where $\mathcal{V}_{2,1}$ is the 2-dimensional local system corresponding to the 2-dimensional representation $V_{2,1}$ of the group S_3. The manifold $\tilde{B}(\mathbb{R}P^2,3)$ on the other hand is nonorientable (the frame $(w_{1,1},w_{1,2},\dots,w_{3,1},w_{3,2})$ constructed on Figure 7.1 gives us a frame in the tangent space to $\tilde{B}(\mathbb{R}P^2,2)$ at $\{A_1,A_2,A_3\}$; the action of a loop like (7.5) on that frame consists of two blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and one block $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$), hence $\bar{H}_6(\tilde{B}(\mathbb{R}P^2,3)) = 0$. This, together with (7.6), easily implies the resting assertions of lemma 7.4.$\diamondsuit$

Applying lemma 7.4 we get $E^1_{p,5} = \bar{H}_{p+5}(\tilde{F}_5 \setminus \tilde{F}_4,\mathbb{R}) = 0$ for all p.

Now let us calculate the sixth column of the spectral sequence 7.1. We have $E^1_{6,p} = \bar{H}_{p+6}(\tilde{F}_6 \setminus \tilde{F}_5,\mathbb{R})$. The space $\tilde{F}_6 \setminus \tilde{F}_5$ ia a fibre bundle over $\Phi_8 \setminus \Phi_7$; the latter space is fibered over X_8, which itself is a fibre bundle over the space $\mathbb{R}P^{2\vee}$. The fibre of the bundle $\Phi_8 \setminus \Phi_7 \to X_8$ is an open interval, and the fibre of the bundle $X_8 \to \mathbb{R}P^{2\vee}$ over a conjugation-invariant complex line l is the product of two open 2-discs $(\mathbb{R}P^2 \setminus l_{\mathbb{R}}) \times ((l \setminus l_{\mathbb{R}})/\text{the complex conjugation})$. The action of $\pi_1(\mathbb{R}P^{2\vee},l)$ is orientation-reversing on both $(\mathbb{R}P^2 \setminus l_{\mathbb{R}})$ and $((l \setminus l_{\mathbb{R}})/\text{the complex conjugation})$, so we have $\bar{H}_i(\Phi_8 \setminus \Phi_7,\mathbb{R}) = \bar{H}_{i-1-4}(\mathbb{R}P^2,\mathbb{R}) = \mathbb{R}$, if $i=5$ and 0 otherwise.

Set $K = \{(0:0:1),(1:i:0),(1:-i:0)\}$. The vector space $L(K)$ is spanned by $x_2(x_0^2+x_1^2)$, and the action of $\pi_1(\Phi_8\setminus\Phi_7)$ on this space is nontrivial. So we get

$$\bar{H}_i(\tilde{F}_6\setminus\tilde{F}_5,\mathbb{R}) = \bar{H}_{i-1}(\Phi_8\setminus\Phi_7,\text{the nontrivial one-dimensional local system}) = \bar{H}_{i-1-4-1}(\mathbb{R}P^2,\text{the nontrivial one-dimensional local system}) = \mathbb{R},$$

if $i = 8$ and 0 otherwise. Hence, $E^1_{p,6} = \mathbb{R}$, if $p = 2$ and 0 for all other p.

The last column of the spectral sequence (7.1) contains the Borel-Moore homology groups of the space $\Lambda\setminus\Phi_8$, which is the open cone over Φ_8. We have $E_{8,q} = \bar{H}_{8+q}((\text{the cone over }\Phi_8)\setminus\Phi_8,\mathbb{R}) = H_{7+q}(\Phi_8,\text{point})$. We already know the Borel-Moore homology groups of the difference of any two consecutive terms of the filtration $\varnothing\subset\Phi_1\subset\Phi_2\subset\Phi_3\subset\Phi_6\subset\Phi_7\subset\Phi_8$, so we can write a spectral sequence that converges to $H_*(\Phi_8,\mathbb{R})$:

$$\begin{array}{rcccccc}
3 & & & & & \mathbb{R} & \\
2 & & \mathbb{R} & & & & \\
1 & & & & & & \\
0 & & & & & \mathbb{R} & \\
-1 & \mathbb{R} & & & & & \mathbb{R} \\
 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \tag{7.7}$$

Lemma 7.5 *We have $H_i(\Phi_8,\mathbb{R}) = \mathbb{R}$, if $i = 0,5,8$, and $H_i(\Phi_8,\mathbb{R}) = 0$ for all other i.*

Proof of lemma 7.5. Note that we have $H^i(\Pi_{3,2}\setminus\Sigma_{3,2},\mathbb{R}) = 0$ for $i > 4$ no matter what the differentials of the spectral sequences (7.1) and (7.7) are.

The group $SO_3(\mathbb{R})$ acts on the space $\Pi_{3,2}\setminus\Sigma_{3,2}$ in such a way that all stalilisers are finite. It follows from theorem 4.4 or corollary 4.2 that the term E_2 of the cohomological Leray spectral sequence of the projection $\Pi_{3,2}\setminus\Sigma_{3,2}\to((\Pi_{3,2}\setminus\Sigma_{3,2})/SO_3(\mathbb{R}))$ can be written as $E_2^{p,q} = H((\Pi_{3,2}\setminus\Sigma_{3,2})/SO_3(\mathbb{R}),\mathbb{R})\otimes H^q(SO_3(\mathbb{R}))$. Since the cohomology groups of $\Pi_{3,2}\setminus\Sigma_{3,2}$ are zero in dimensions > 4, the Poincaré polynomial of $\Pi_{3,2}\setminus\Sigma_{3,2}$ is equal to $(1+t^3)P(t)$, where P is a polynomial of degree 1.

It follows from the spectral sequence (7.7) that the Poincaré polynomial of Φ_8 is $1+at^4+(a+1)t^5+t^8$, where $a\in\{0,1\}$. It is easy to check using the sequence (7.1) that if $a = 1$, then the Poincaré polynomial of $\Pi_{3,2}\setminus\Sigma_{3,2}$ is not divisible by $1+t^3$, and lemma 7.5 follows.$\diamondsuit$

So far we have proven the first assertion of theorem 7.1. The second assertion follows from the divisibility argument already used in the proof of lemma 7.5.$\diamondsuit$

Bibliography

[1] E. Arbarello, M. Cornalba *Calculating cohomology groups of moduli spaces of curves via algebraic geometry*, Inst. Hautes Études Sci. Publ. Math., 1998, 88, 97–127.

[2] V. I. Arnold, *On some topological invariants of algebraic functions*, Transact. (Trudy) of Moscow Math. Society, 1970, 21, 27-46.

[3] D. B. Fuks, *Classical manifolds*, in: Current problems in mathematics: fundamental directions, 1986, vol. 12 (Russian), 253–314, VINITI, Moscow.

[4] E. Getzler, *Topological recursion relations in genus 2*, arXive:math.AG/9805114.

[5] A. G. Gorinov, *Conical resolutions of discriminant varieties and real cohomology groups of the space on nonsingular plane projective quintics*, Dokl. Akad. Nauk, 2003, 389, 6, 730-733; english transl. in: Doklady Mathematical Sciences, 2003, 67, 2, 259-262.

[6] A. G. Gorinov, *Real cohomology groups of the space of nonsingular curves of degree 5 in* $\mathbb{C}P^2$, arXiv:math.AT/0105108, to appear in Ann. Fac. Sci. Toulouse Math., 2005, 14, 2.

[7] A. G. Gorinov, *The cobordism group of Möbius manifolds in dimension 1 is zero*, Topology and its Applications, 2004, 143, 75-85.

[8] Luna, Domingo, *Slices étales,* in: Sur les groupes algébriques, Bull. Soc. Math. France, 1973, Memoire 33, Soc. Math. France, 81-105.

[9] D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory, Third edition*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 1994, 34, Springer-Verlag, Berlin.

[10] Palais, Richard S., *On the existence of slices for actions of non-compact Lie groups,* Ann. of Math. 2, 1961, 73, 295–323.

[11] C.A.M. Peters, J.H.M. Steenbrink, *Degeneration of the Leray spectral sequence for certain geometric quotients,* Moscow Mathematical Journal, 2003, 3, 3, 1085-1095.

[12] I. R. Shafarevich, *Basic Algebraic Geometry 1*, 1994, Springer-Verlag, Berlin.

[13] O. Tommasi, *Rational cohomology of the moduli space of genus 4 curves*, arXive:math.AG/0312055.

[14] V. A. Vassiliev, *How to calculate homology groups of spaces of nonsingular algebraic projective hypersurfaces in* $\mathbb{C}P^n$, Proc. Steklov Math. Inst., 1999, 225, 121-140.

[15] V. A. Vassiliev, *Topology of complements of discriminants*, 1997, Phasis, Moscow, (in Russian).

Appendix A

The Cobordism Group of Möbius Manifolds of Dimension 1 is trivial [1]

After recalling some general facts about Möbius structures, we define for any $n > 0$ the cobordism group of Möbius manifolds of dimension n. We use the classification of Möbius structures on the oriented S^1 to prove that the cobordism group of 1-dimensional Möbius manifolds is trivial. We also correct an inaccuracy in N. Kuiper's classification of projective circles.

A.1 Some definitions

Unless stated otherwise, all manifolds, diffeomorphisms and group actions under consideration are assumed to be of class C^∞.

Let M be a manifold, and let G be a Lie group acting on M. We shall suppose that if $g \in G$ acts identically on a nonempty open subset of M, then g is the unit element of G. Let N be a manifold of dimension $\dim M$. An (M, G)-*structure* on N is defined to be an atlas (U_i, ϕ_i), where $U_i \subset N$ are open sets, and $\phi_i : U_i \to M$ are coordinate maps such that

- Every ϕ_i is a diffeomorphism onto its image.
- If $U_i \cap U_j \neq \varnothing$, then $\phi_i \circ \phi_j^{-1}$ coincides on its domain of definition with the restriction of some element of G.
- The atlas (U_i, ϕ_i) is maximal with respect to the previous two conditions.

Such an atlas (U_i, ϕ_i) will be called (M, G)-*coordinates* on N.

If N_1 and N_2 are (M, G)-manifolds, then a local diffeomorphism $N_1 \to N_2$ is called a *local* (M, G)-*diffeomorphism*, if, when written in (M, G)-coordinates, it becomes a restriction of an element of G. If $p : N_1 \to N_2$ is a local diffeomorphism and N_2 is an (M, G)-manifold, then there exists a unique (M, G)-structure on N_1 such that p is a local (M, G)-diffeomorphism. We shall say that

[1] A. G. Gorinov, *The cobordism group of Möbius manifolds in dimension 1 is zero*, Topology and its Applications, 2004, 143, 75-85.

this structure is the *inverse image* of the (M,G)-structure on N_2 with respect to p. We shall say that two (M,G)-structures on N are equivalent, if one of them is the inverse image of the other with respect to some diffeomorphism $N \to N$ that is isotopic to the identity.

Let us fix a point $x_0 \in N$. Denote by $\tilde{N}_{x_0}$ the universal covering space of N based at x_0, i.e., the space of homotopy classes of paths that start at x_0. Every (M,G)-structure on N gives rise to a local diffeomorphism $F : \tilde{N}_{x_0} \to M$ (the developing map, see, e.g., [3, Chapter 3]) and a homomorphism $f : \pi_1(N,x_0) \to G$ such that for any $a \in \pi_1(N,x_0), x \in \tilde{N}_{x_0}$ we have

$$F(a \cdot x) = f(a)F(x). \tag{A.1}$$

The set of equivalence classes of (M,G)-structures on N is in a one-to-one correspondence with the set of couples (F,f) (where $F : \tilde{N}_{x_0} \to M$ is a local diffeomorphism, $f : \pi_1(N,x_0) \to G$ is a homomorphism) that satisfy (A.1) modulo the following equivalence relation: $(F,f) \sim (gFh, g(f \circ \alpha_h)g^{-1})$ for all g,h such that $g \in G$, h is obtained by lifting some diffeomorphism of N that is isotopic to the identity [2], and α_h is the automorphism of $\pi_1(N)$ defined by the formula $\alpha_h(a) = hah^{-1}$.

If the manifold M is oriented, and the transformations from G are orientation-preserving, then we can define (M,G)-structures for oriented manifolds in an obvious way.

Let us now recall the definition of the Möbius groups $\text{Möb}_n, \text{Möb}_n^*$ and some of their properties.

Suppose that $S^n \subset \mathbb{R}^{n+1}$ is the unit sphere $\{(x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \cdots + x_{n+1}^2 = 1\}$, and suppose that $\mathrm{SO}_{n+1,1}$ acts on $\mathbb{R}^{n+2}$ via transformations that preserve the quadratic form $x_1^2 + \cdots + x_{n+1}^2 - x_{n+2}^2$. Set

$$Q = \{(x_1,\ldots,x_{n+2}) | x_1^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 = 0, x_{n+2} \neq 0\}.$$

The map $Q \to S^n$ defined by the formula $(x_1,\ldots,x_{n+2}) \mapsto (x_1,\ldots,x_{n+1})/x_{n+2}$ gives us an action of $\mathrm{SO}_{n+1,1}$ on S^n. The resulting group of transformations of S^n will be denoted Möb_n^*; we set Möb_n to be the connected component of the unit element of Möb_n^*.

A theorem of Liouville's (see, e.g., [1, §15]) states that if $n > 2$, then every diffeomorphism $V \to W$ (V and W being open subsets of S^n) that preserves the angles in the standard Riemannian metric is the restriction of an element of Möb_n^*.

Notation. An $(S^n, \text{Möb}_n)$-structure is called a *Möbius* structure. For any oriented manifold N, denote by $\mathcal{M}(N)$ the set of equivalence classes of Möbius structures on N.

Note that in dimension 1 a Möbius structure can be equivalently defined as an (M,G)-structure, where M is any circle C in $\mathbb{C}P^1$, and G is the group of linear-fractional transformations that preserve both disks bounded by C. Analogously, a Möbius structure on a 2-dimensional manifold is the same as a $(\mathbb{C}P^1, \mathrm{PSL}_2(\mathbb{C}))$-structure.

Notation. If N is a Möbius manifold, then denote by $-N$ the Möbius manifold "inverse to N", i.e., $-N = N$ as a smooth manifold, and the coordinate maps that define the Möbius structure on $-N$ are compositions of the corresponding maps for N and an element of $\text{Möb}_n^* \setminus \text{Möb}_n$.

[2] Note that any such h is a composition of an element of $\pi_1(N,x_0)$ and an element of the centraliser of $\pi_1(N,x_0)$ in $Diff(\tilde{N}_{x_0})$.

The definition of a Möbius structure can be naturally extended to manifolds with boundary. For any n, represent S^n as the unit sphere in $\mathbb{R}^{n+1}$. We shall say that $X \subset S^n$ is a *standard* $(n-1)$-*sphere*, if X is the intersection of S^n with a nontangent hyperplane. If N is a manifold of dimension n with boundary, then a Möbius structure on N is defined by an open covering (U_i) of N and coordinate maps $\phi_i : U_i \to S^n$ that satisfy the conditions on page 69 and the additional condition that for any i, $\phi_i(\partial(N) \cap U_i)$ should be a part of a standard $(n-1)$-sphere (or, equivalently: any developing map should send any component of $\partial\tilde{N}$ to a part of a standard $(n-1)$-sphere). In this case ∂N carries a Möbius structure, which can be defined as follows.

If $U_i \cap \partial N \neq \varnothing$, and $\phi : U_i \to S^n$ is the coordinate map, then we define $\phi'_i : U_i \cap \partial N \to S^{n-1}$ by $\phi'_i = \psi_i \circ \phi_i$, where ψ_i is a map that sends $\phi_i(U_i \cap \partial N)$ to the equator $S^n \cap \{(x_1, \ldots, x_n, 0) \in \mathbb{R}^{n+1}\}$ and transforms the internal normal to ∂N at any $x \in U_i \cap \partial N$ to a vector that looks into the northern hemisphere $\{(x_1, \ldots, x_{n+1}) \in S^n | x_{n+1} \geq 0\}$.

From now on, we suppose that the unit sphere $S^n \subset \mathbb{R}^{n+1}$ (and hence any Möbius manifold) is oriented in the standard way. We shall say that a diffeomorphism between two Möbius manifolds is *anti-Möbius*, if in Möbius coordinates it is a restriction of an element of $\text{Möb}_n^* \setminus \text{Möb}_n$.

Lemma A.1 *Let N be a Möbius manifold, and suppose that N_1 and N_2 are disjoint components of ∂N . If $p : N_1 \to N_2$ is an anti-Möbius diffeomorphism, then the manifold N' obtained from N by identifying x and $p(x)$ for all $x \in N_1$ can be equipped with a canonical Möbius structure.*

Proof of Lemma A.1. Note that if X is a standard $(n-1)$-sphere in $S^n, Y \subset S^n$ is a ball bounded by X, and g_1 and g_2 are elements of Möb_n that preserve Y and coincide on an open subset of X, then $g_1 = g_2$. This remark allows us to glue coordinate maps on N' from coordinate maps on N.$\diamondsuit$

Möbius structures are interesting, because, on the one hand, the class of Möbius structures is quite large (e.g, it includes all hyperbolic, elliptic or affine structures), and on the other hand, it has some remarkable properties (for example, by Lemma A.1, the connected sum of two Möbius manifolds can always be equipped with a Möbius structure; the analogous assertions for hyperbolic, affine or elliptic structures are not true).

Remark 1. We can define for any n the cobordism category $\mathcal{MC}_n$ of n-dimensional Möbius manifolds. The definition is as follows. Let the objects of $\mathcal{MC}_n$ be compact Möbius manifolds of dimension n. If N_1, N_2 are objects of $\mathcal{MC}_n$, then morphisms from N_1 to N_2 are $n+1$-dimensional compact Möbius manifolds N such that $\partial N = N_1 \sqcup -N_2$ (modulo Möbius diffeomorphisms that are identical on ∂N). Due to Lemma A.1, the composition of morphisms is well-defined.

A.2 Möbius Structures on the Oriented Circle

Suppose that $\mathbb{R}$ is canonically oriented. From now on, we shall denote by S^1 the (canonically oriented) $\mathbb{R}/\mathbb{Z}$. Recall the classification of Möbius structures on S^1. In this section we shall mainly use the projective model, i.e., any Möbius structure S constructed below will be defined by a local orientation-preserving diffeomorphism $F_S : \mathbb{R} \to \mathbb{R}P^1$ and a homomorphism $F_S : \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$ that satisfy the condition

$$F_S(x+1) = F_S(1)F_S(x). \tag{A.2}$$

Let A_S be a matrix that represents $F_S(1)$ (this matrix is defined up to a sign). We shall distinguish the following cases:

1. $|tr(A_S)| \leq 2$, and A_S is diagonalisable.

2. $|tr(A_S)| = 2$, A_S is nondiagonalisable.

3. $|tr(A_S)| > 2$.

The structures that verify (1), (2), (3), will be called respectively *elliptic, parabolic* and *hyperbolic*.

For any $\alpha > 0$ and $n > 0$ let $\psi_{n,\alpha}$ be a continuous map $[0,1] \to \mathbb{R}P^1$ such that 1. $\psi_{n,\alpha} = \mathrm{e}^{\alpha x} - 1$, when x is close to 0, $\psi_{n,\alpha} = \mathrm{e}^{\alpha}(\mathrm{e}^{\alpha x} - 1)$, when x is close to 1, 2. $\psi^{-1}_{n,\alpha}(0)$ consists of $n+1$ elements, and 3. $\psi_{n,\alpha}|(0,1)$ is an orientation-preserving local diffeomorphism. Analogously, for any couple (n, ε) (where $n > 0$ is an integer, and $\varepsilon = \pm 1$) let $\xi_{n,\varepsilon}$ be a continuous map $[0,1] \to \mathbb{R}P^1$ such that 1. $\xi_{n,\varepsilon} = -1/x$, when x is close to 0, $\xi_{n,\varepsilon} = \varepsilon + 1/(1-x)$, when x is close to 1, 2. $\xi^{-1}_{n,\varepsilon}(\infty)$ consists of $n+1$ elements, and 3. $\xi_{n,\varepsilon}|(0,1)$ is an orientation-preserving local diffeomorphism. The following theorem gives an explicit representative for each element of $\mathcal{M}(S^1)$ (cf. [2]).

Theorem A.1 *1. If two Möbius structures on S^1 belong to different types, they are not equivalent.*

2. Equivalence classes of hyperbolic Möbius structures are parametrised by couples (n, α), where α is a positive real number, and n is a nonnegative integer.

The corresponding maps $F_S : \mathbb{R} \to \mathbb{R}P^1, F_S : \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$ can be chosen as follows. Let $F_S(1)$ be the map $z \mapsto \mathrm{e}^{\alpha} z$. If $n = 0$, set $F_S(x) = \mathrm{e}^{\alpha x}$, otherwise set $F_S(x) = \psi_{n,\alpha}(x)$ for $x \in [0,1]$ and extend F_S to $\mathbb{R}$ using (A.2). Denote the equivalence class of hyperbolic Möbius structures corresponding to the couple (n, α) by $H_{n,\alpha}$.

3. Equivalence classes of parabolic Möbius structures are parametrised by elements of the set $(\{\text{nonnegative integers}\} \times \{\pm 1\}) \setminus \{(0, -1)\}$.

The equivalence class that corresponds to the couple $(0,1)$ is represented by the Möbius structure defined by the following maps $F_S : \mathbb{R} \to \mathbb{R}P^1, F_S : \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$: set $F_S(x) = x, F_S(1)(z) = z + 1$.

The equivalence class corresponding to the couple (n, ε) (where n is a positive integer, and ε is 1 or -1) is represented by the Möbius structure defined by the maps $F_S : \mathbb{R} \to \mathbb{R}P^1, F_S : \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$ that can be constructed as follows: set $F_S(1)(z) = z + \varepsilon$, set $F_S(x) = \xi_{n,\varepsilon}(x)$ for $x \in [0,1]$ and extend F_S to $\mathbb{R}$ using (A.2). Denote the equivalence class of parabolic Möbius structures corresponding to the couple (n, ε) by $P_{n,\varepsilon}$.

4. Equivalence classes of elliptic Möbius structures are parametrised by positive real numbers. The equivalence class corresponding to $\alpha >$

0 can be represented by the Möbius structure defined by the following maps $F_S : \mathbb{R} \to \mathbb{R}P^1, F_S : \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$*: set* $F_S(1)(z) = ((\cos\alpha + 1)z - \sin\alpha) / (z \sin\alpha + \cos\alpha + 1)$*, and set* $F_S(x) = \sin(\alpha x) / (1 - \cos(\alpha x))$*. Denote this equivalence class by* E_α*.*

If S is a Möbius structure on S^1, then denote by $[S]$ the equivalence class of S. The theorem follows from an argument analogous to the one used in [2] to classify projective circles modulo projective diffeomorphisms. However, that classification theorem is not quite correct, see Remark 2 below, so we shall give an outline of a proof of Theorem A.1.

Proof of Theorem A.1. First, let us show that any Möbius structure S on S^1 is equivalent to some structure described in Theorem A.1. Suppose, e.g., that S is hyperbolic. Choose a base point in S^1, and denote by F_S the corresponding developing map $\mathbb{R} \to \mathbb{R}P^1$; let $F_S : \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$ be the homomorphism such that F_S and F_S satisfy (A.2). Then, replacing S by an equivalent structure, we can assume that F_S is given by the formula $F_S(n)(z) = \mathrm{e}^{\alpha n} z$ for some $\alpha > 0$. The image of F_S is either the whole $\mathbb{R}P^1$ or not; we can assume (changing the base point, if necessary) that $F_S(0)$ is 1 in the first case and 0 in the second. Now it is easy to see that S is equivalent to some structure from the second assertion of Theorem A.1. The parabolic and elliptic cases are considered in an analogous way.

Now, let us prove that the structures introduced in Theorem A.1 are pairwise nonequivalent. Let S be a Möbius structure on S^1. Suppose that S is defined by an orientation-preserving local diffeomorphism $F_S : \mathbb{R} \to \mathbb{R}P^1$ and a homomorphism $F_S : \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$ that satisfy (A.2). We can associate the following invariants to S: the conjugacy class of $F_S(1)$ in $\mathrm{PSL}_2(\mathbb{R})$ and the number

$$\max_{x \in \mathbb{R}, y \in \mathbb{R}P^1} \#(F_S^{-1}(y) \cap [x, x+1]).$$

It can be easily checked that these invariants are sufficient to distinguish any two different structures defined in Theorem A.1.$\diamondsuit$

A *projective* structure on the circle is an $(\mathbb{R}P^1, \mathrm{PGL}_2(\mathbb{R}))$-structure. Every Möbius circle is a projective circle, and vise versa.

Lemma A.2 *Any two different Möbius structures on* S^1 *defined in Theorem A.1 are not equivalent as projective structures.*

Proof of Lemma A.2. We proceed as in the proof of Theorem A.1. Suppose that the projective structure S on S^1 is defined by a couple (F_S, F_S), where F_S is a local diffeomorphism $\mathbb{R} \to \mathbb{R}P^1$ and $F_S : \mathbb{Z} \mapsto \mathrm{PSL}_2(\mathbb{R})$ is a homomorphism that satisfy (A.2). It can be easily checked that the conjugacy class of $F_S(1)$ in $\mathrm{PGL}_2(\mathbb{R})$ and the number

$$\max_{x \in \mathbb{R}, y \in \mathbb{R}P^1} \#(F_S^{-1}(y) \cap [x, x+2])$$

depend only on the equivalence class of S. These invariants of projective structures distinguish any two nonequivalent Möbius structures on S^1.$\diamondsuit$

Remark 2. Due to Lemma A.2 there exists a bijection $\mathcal{M}(S^1) \leftrightarrow$ (projective circles modulo projective diffeomorphisms). Hence, a classification of projective circles can be obtained from Theorem A.1; the classification given in [2] in not quite correct: if $n > 0$, then S^1 provided with a Möbius structure of class $P_{n,1}$

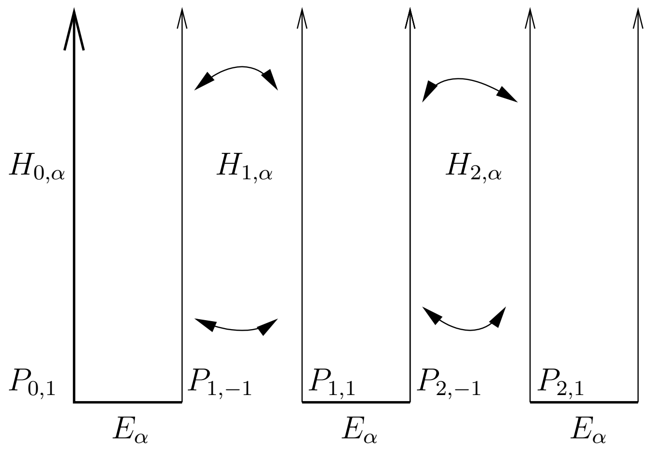


Figure A.1: $\mathcal{M}'(S^1)$

and S^1 provided with a Möbius structure of class $P_{n,-1}$ are not isomorphic as projective circles.

Remark 3. The interpretation in terms of developing maps allows us to introduce a topology on $\mathcal{M}(S^1)$. However, the resulting topological space is quite nasty. The space $\mathcal{M}'(S^1) = \mathcal{M}(S^1) \setminus \{E_{2\pi k} | k \text{ is an integer} > 0\}$ is the non-Hausdorff topological 1-manifold obtained by identifying the thin lines on Figure A.1; $\mathcal{M}'(S^1)$ is open in $\mathcal{M}(S^1)$, and for any $k > 0$ we can take the system of sets of the form $\{E_{2\pi k}\} \cup$ (a neighbourhood of $P_{k,1}$ in $\mathcal{M}'(S^1)$) $\cup$ (a neighbourhood of $P_{k,-1}$ in $\mathcal{M}'(S^1)$) as a local neighbourhood basis at $E_{2\pi k}$.

A.3 Main theorem

Let $\Omega_n^{\text{Möb}}$ be the group defined as follows. The elements of $\Omega_n^{\text{Möb}}$ are n-dimensional compact Möbius manifolds without boundary considered modulo the following equivalence relation: we set $N_1 \sim N_2$, iff there exists an $n+1$-dimensional compact Möbius manifold N such that ∂N is Möbius diffeomorphic to $N_1 \sqcup -N_2$. The group operation is induced by taking the disjoint union. The group $\Omega_n^{\text{Möb}}$ will be called the *cobordism group* of Möbius manifolds of dimension n.

Theorem A.2 $\Omega_1^{\text{Möb}} = 0$.

Proof of Theorem A.2. Note that an equivalence class of Möbius structures on any oriented circle can be canonically identified with an element of the space $\mathcal{M}(S^1)$. So, in order to prove the theorem, it is enough to show that any element of $\mathcal{M}(S^1)$ is equal to $[S]$, where S is the Möbius structure on the boundary of some 2-dimensional Möbius surface (recall that the boundary of any Möbius surface is canonically oriented). As in Section A.2, we shall use the projective model, i.e., we identify the unit sphere $S^2 \subset \mathbb{R}^3$ with $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ via some orientation-preserving stereographic projection so that the northern hemisphere is identified with the upper half-plane. Note that under this identi-

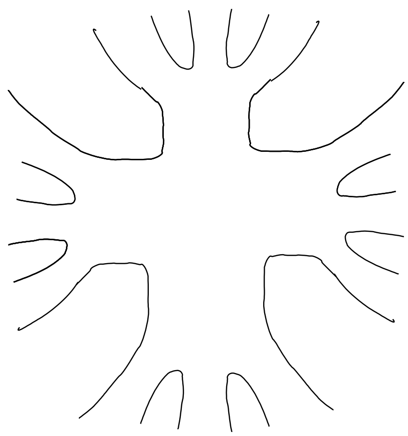


Figure A.2:

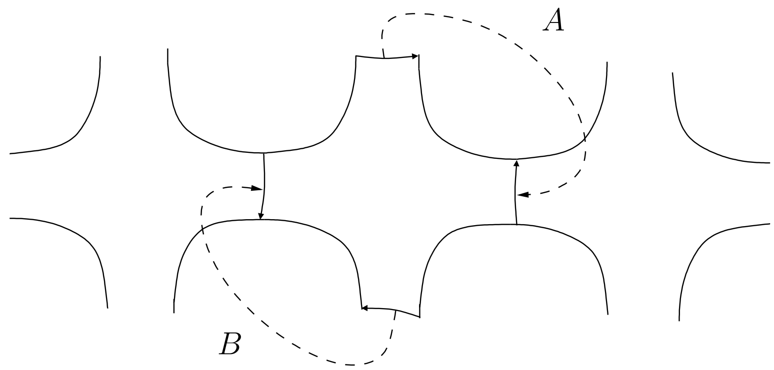


Figure A.3:

fication $\mathbb{R}P^1 =$ (the boundary of the upper half-plane) is oriented from left to right.

Let N be the sphere minus three disjoint open disks ("the pants"). The manifold $\tilde{N}$ is represented on Figure A.2.

Denote by $\mathbf{F}(A, B)$ the free group on the generators A, B. Suppose that $\mathbf{F}(A, B)$ acts on $\tilde{N}$ as shown on Figure A.3 (the octagon in the middle is the fundamental domain; the right and the left octagons are the images of the fundamental domain under A and B respectively).

In order to construct maps $F : \tilde{N} \to \mathbb{C}P^1, f : \mathbf{F}(A, B) \to \mathrm{PSL}_2(\mathbb{C})$ that satisfy (A.1), it is sufficient to choose two maps $a, b \in \mathrm{PSL}_2(\mathbb{C})$ and an embedded rectangular octagon in $\mathbb{C}$ with vertices $X_1, X_2, \ldots, X_8$ (see Figure A.4; arrows indicate the orientations of the boundary) such that the following conditions are satisfied:

(C1) All segments $X_1X_2, \ldots, X_8X_1$ are arcs of circles; denote the corresponding circles by $C_{X_1X_2}, \ldots, C_{X_8X_1}$.

(C2) $a(X_1) = X_4, a(X_2) = X_3, b(X_5) = X_8, b(X_6) = X_7$.

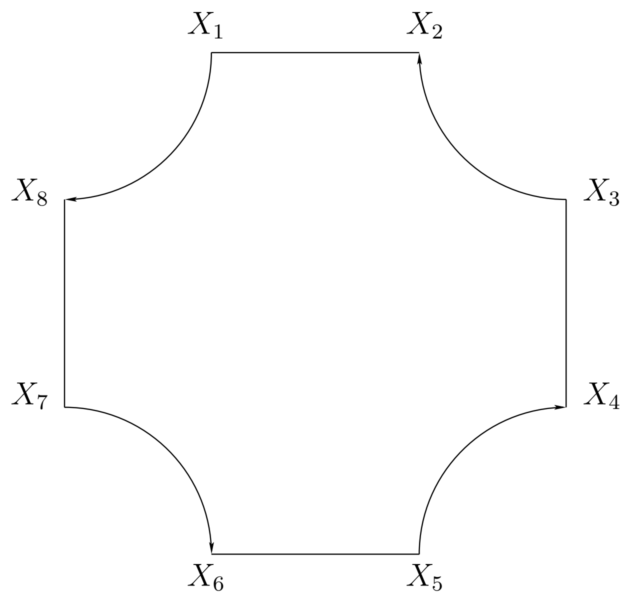


Figure A.4:

(C3) $a(C_{X_2X_3}) = C_{X_2X_3}, a(C_{X_8X_1}) = C_{X_4X_5}, b(C_{X_6X_7}) = C_{X_6X_7}, b(C_{X_4X_5}) = C_{X_8X_1}$.

If we manage to find such an octagon and such maps a, b, then we shall equip N and ∂N with Möbius structures.

Let x be a positive real number. Define the maps $a_x, b \in \mathrm{PSL}_2(\mathbb{C})$ respectively by $z \mapsto z + x, z \mapsto z/(1-z)$. The transformation $b \circ a_x$ is represented by a matrix with trace $2 - x$. Choose the octagon as shown in Figure A.5. The horizontal line at the bottom is the real axis, the circles C_1 and C_2 on Figure A.5 are defined respectively by the equations $(\mathrm{Re}\, z)^2 - 2\,\mathrm{Re}\, z + (\mathrm{Im}\, z)^2 = 0$ and $(\mathrm{Re}\, z)^2 + 2\,\mathrm{Re}\, z + (\mathrm{Im}\, z)^2 = 0$. The circles C_1' and C_2' are chosen in such a way that 1. $b(C_1') = C_2', a_x(C_2') = C_1'$, 2. C_1' is orthogonal to C_1 and depends continuously on x, 3. $C_1' \subset \{z |\, \mathrm{Re}\, z \geq 0\}$, and the imaginary part of the center of C_1' is > 0. Note that there are many ways to choose the circles C_1' and C_2'' that satisfy these conditions. The arc X_6X_7 is an arc of a sufficiently small circle in the upper half-plane that is tangent to the real axis at 0.

Note that we have $b(z) = -\bar{z}$ for $z = 1 + \mathrm{e}^{it}$, i.e., the restriction of b to C_1 is the symmetry with respect to the imaginary axis. This implies easily that for any $x > 0$, the octagon on Figure A.5 and the maps $a = a_x$ and b satisfy the conditions (C1)-(C3).

For any $x > 0$ we obtain Möbius structures on N and all components of ∂N. The components of ∂N that correspond to X_2X_3 and X_6X_7 will always have parabolic structures that belong to $P_{0,1}$, and the equivalence class of the structure on the third component changes as we change x. Denote this structure by S_x; S_x is elliptic for $x < 4$, parabolic for $x = 4$ and hyperbolic for $x > 4$. Denote by $N_1(x)$ the Möbius surface obtained by gluing together two parabolic components of ∂N (note that due to Lemma A.2, for any Möbius circle C there exists an anti-Möbius diffeomorphism $C \to C$).

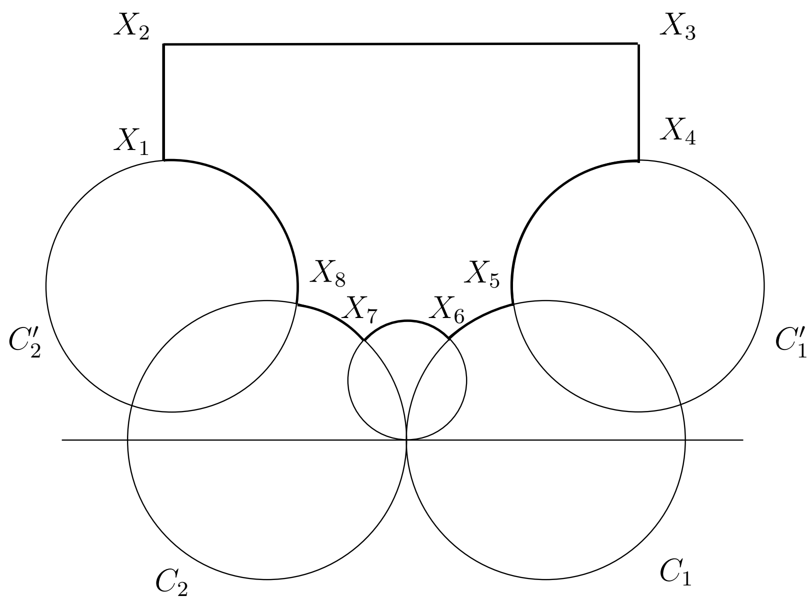


Figure A.5:

Let us determine the equivalence class of S_x. Let F_{S_x} be a developing map $\mathbb{R} \to \mathbb{R}P^1$ that corresponds to S_x.

Note that $[S_4] = P_{0,1}$, which implies that the image of F_{S_4} is not the whole $\mathbb{R}P^1$. Hence, if $D > 0$, then we have $F_{S_x}([0, D]) \neq \mathbb{R}P^1$ for any x sufficiently close to 4. This implies the following lemma:

Lemma A.3 *For any sufficiently small $\alpha > 0$ there exist x_1, x_2 such that the Möbius structure on $\partial N_1(x_1)$ (respectively, $\partial N_1(x_2)$) belongs to $H_{0,\alpha}$ (respectively, to E_α).*

Let us define the action of $\mathbb{N}$ on $\mathcal{M}(S^1)$ as follows: for any $k \in \mathbb{N}, \alpha > 0, n \geq 0$ and $\varepsilon \in \{\pm 1\}$ set $kH_{n,\alpha} = H_{kn,k\alpha}, kP_{n,\varepsilon} = P_{kn,\varepsilon}, kE_\alpha = E_{k\alpha}$.

Lemma A.4 *Let C' and C'' be oriented circles. Suppose that C'' is equipped with a Möbius structure, and denote this structure by S''. Let $p : C' \to C''$ be an orientation-preserving k-sheeted covering. Denote by S' the inverse image of S'' with respect to p. We have $[S'] = k[S'']$.*

♢

Lemma A.5 *Let N be a compact Möbius surface with one boundary component and nontrivial fundamental group, and let $k > 0$ be an integer. Denote by S the Möbius structure on ∂N. There exists a compact Möbius surface N' such that $\partial N'$ is a circle, and the Möbius structure on $\partial N'$ belongs to $k[S]$.*

Proof of Lemma A.5. Let $\mathfrak{S}_k$ be the symmetric group on k elements. If k is odd, then there exists a homomorphism $\pi_1(N) \to \mathfrak{S}_k$ that takes a generator of $\pi_1(\partial N)$ to some cycle of length k. Indeed, we can choose a system $v_1, w_1, \ldots, v_g, w_g$ of free generators of $\pi_1(N)$ so that $\pi_1(\partial N)$ will be spanned

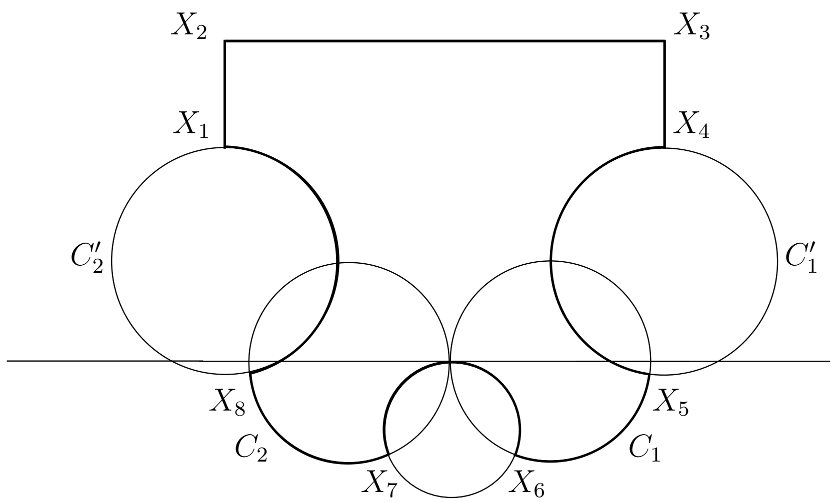


Figure A.6:

by $[v_1, w_1] \cdots [v_g, w_g]$. A required homomorphism can be constructed using the fact that in $\mathfrak{S}_k$ a cycle of length k is a commutator (e.g., we have

$$(1 \dots k) = (1\frac{k+3}{2} \dots k)(12 \dots \frac{k+1}{2}) = \sigma(12 \dots \frac{k+1}{2})^{-1}\sigma^{-1}(12 \dots \frac{k+1}{2})$$

for some $\sigma \in \mathfrak{S}_k$).

Hence, for k odd, there exists a surface N' with one boundary component and a k-sheeted covering $N' \to N$. This, together with Lemmas A.3 and A.4, allows us to construct for any $\alpha > 0$ a Möbius surface, whose boundary carries a Möbius structure that belongs to E_α.

Now cut out a small Möbius disk from N. The resulting Möbius surface has two boundary components; one of these components has the structure S, and the structure on the other one belongs to $E_{2\pi}$. Note that if M is a compact surface with two boundary components, then a generator of one of these components can be taken as a free generator of $\pi_1(M)$; hence, for any $k > 0$ there exists a k-sheeted covering $M' \to M$ that is cyclic over one of the components of ∂M. We have already constructed for any integer $l > 0$ a Möbius surface, whose boundary carries a structure from $E_{2\pi l}$, and the lemma follows. $\diamondsuit$

Lemmas A.3, A.4, A.5 imply that any of the classes $P_{0,1}, H_{0,\alpha}$ or $E_\alpha, \alpha > 0$ is the equivalence class of the Möbius structure of the boundary of some Möbius surface. In order to complete the proof of Theorem A.2, we can proceed as follows. Suppose that x is a real number > 2 and consider the rectangular octagon represented on Figure A.6.

Here the circles C_1 and C_2 are the same as in the proof of Lemma A.3. Suppose that the centers of C'_1 and C'_2 are the points $\pm x/2 + i$, and the radii of these circles are $x/2 - 1$. As above, the octagon and the maps a_x and b give us for any $x > 2$ a Möbius structure on N (the pants). The induced Möbius structure on the component of ∂N that corresponds to the arc X_2X_3 (respectively, the arc X_6X_7) belongs to the class $P_{0,1}$ (respectively, $P_{1,-1}$). Denote by S'_x the Möbius structure on the third component of ∂N. The structure S'_x is elliptic for $x < 4$;

by attaching Möbius surfaces to the elliptic component of ∂N and to the $P_{0,1}$-component, we obtain a Möbius surface, whose boundary has a Möbius structure of class $P_{1,-1}$, which means that we can eliminate the $P_{1,-1}$-component as well, i.e., for any $x > 2$ there exists a Möbius surface, whose boundary carries the Möbius structure S'_x.

Let us determine the equivalence class of $S'_x, x \geq 4$. Note that we can choose a developing map $F_{S'_x} : \mathbb{R} \to C'_1$ for the structure S'_x so that $F_{S'_x}$ sends the segment $[0, 1/2]$ to the arc X_5X_4 and the segment $[1/2, 1]$ to the arc $a_x(X_1X_8)$. It is easy to check that for $x \geq 4$, the fixed points of $a_x \circ b$ are exactly the intersection points of C'_1 and the real axis.

Hence, for any $x > 4$ the set $F^{-1}_{S'_x}$(the fixed points of $a_x \circ b) \cap [0, 1)$ consists of two elements, which implies that for any $x > 4$ we have $S'_x \in H_{1,\alpha}$ with $\alpha = 2\,\mathrm{arccosh}(x/2 - 1)$.

An analogous argument shows that $S'_4 \in P_{1,1}$ (note that $a_4 \circ b$ acts on C'_1 "clockwise"). The proof of Theorem A.2 is easily completed using Lemmas A.4 and A.5. $\diamondsuit$

Bibliography

[1] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov, *Modern Geometry – Methods and Applications. Part I, Second edition*, 1992, Springer-Verlag, New York.

[2] N. H. Kuiper, *Locally projective spaces of dimension one*, Michigan Math. J., 1954, 2, 95-97.

[3] W. Thurston, *The Geometry and Topology of 3-Manifolds*, 1978, Lecture Notes, Princeton University.