Formules de monotonie appliquées à des problèmes à frontière libre et de modélisation en biologie

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THÈSE

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Résumé : Ce mémoire présente des résultats de régularité pour des problèmes d'équations aux dérivées partielles paraboliques. Dans la première partie nous nous intéressons à des problèmes à frontière libre issus du problème de l'obstacle parabolique à coefficients variables. Nous montrons des résultats de régularité de la solution et de la frontière libre. Cette étude utilise des méthodes d'explosion et des formules de monotonie. La seconde partie est consacrée à l'étude d'un problème issu de la modélisation de l'agrégation en biologie : le système de Keller-Segel. En utilisant une énergie libre, nous montrons l'existence d'une masse critique en deçà de laquelle les solutions existent et au delà de laquelle elles explosent en temps fini. Nous précisons leur comportement asymptotique, dans le cas où les solutions existent en temps long.

Mots clés : frontière libre, problème de l'obstacle, options américaines, modèles mathématiques en biologie, Keller-Segel, méthode de blow-up, formule de monotonie, théorème de Liouville, énergie libre, méthode d'entropie, comportement asymptotique, hypercontractivité.

Monotonicity formula applied to free boundary problems and modelling in biology

Abstract : This thesis presents regularity results for partial differential equations of parabolic type. In the first part we concentrate on free boundary problems which stem from the parabolic obstacle problem with variable coefficients. We prove regularity results for the solution and for the free boundary. This study uses blow-up and monotonicity formulae methods. The second part is dedicated to a problem which comes from aggregation modelling in biology : the Keller-Segel system. Making use of a free energy, we prove the existence of a threshold which governs the initial data, below which the solutions exist, and above which they blow-up in finite time. We specify their long-time behaviour in the case where the solutions exist.

Keywords : free boundary, obstacle problem, American options, biology modelling, Keller-Segel, blow-up method, monotonicity formula, Liouville's theorem, free energy, entropy methods, long-time behaviour, hypercontractivity

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"C'est dans la monotonie que s'embusque le bonheur" (Amin Maalouf, Samarcande).

Table des matières

Ce mémoire est découpé en deux parties précédées d'une introduction. La première partie est référencée comme Partie A et la seconde comme Partie B. La Section i de la Partie A est appelée Section A.i. Le résultat i de la Partie A est noté Résultat A.i. Les mêmes conventions sont adoptées pour la Partie B. Les références citées dans chaque chapitre le sont par un numéro entre crochets : [ ]. Ils renvoient aux références placées en fin de chapitre. La bibliographie générale se trouve en fin de mémoire. Les chapitres dont le titre est en anglais sont composés des articles de journaux suivant :

(i) Chapitre A.II : *On the continuity of the time derivative of the solution to the parabolic obstacle problem with variable coefficients*, accepté pour publication dans *Journal de Mathématiques Pures et Appliquées* en collaboration avec J. Dolbeault et R. Monneau,

(ii) Chapitre A.III : *On the regularity of the free boundary in the parabolic obstacle problem. Application to American options*, accepté pour publication dans *Nonlinear Analysis Series A : Theory, Methods & Applications*,

(iii) Chapitre A.IV : *On the singular set in the parabolic obstacle problem*, prétirage,

(iv) Chapitre B.II : *Two dimensional Keller-Segel model : Optimal critical mass and qualitative properties of the solutions*, en collaboration avec J. Dolbeault et B. Perthame, prétirage.

Introduction

Cette introduction est divisée en deux sections. La Section 1 présente les résultats et les méthodes utilisées pour l'étude du problème de l'obstacle parabolique. La Section 2 introduit les résultats et les méthodes de la Partie B sur les propriétés qualitatives des solutions du système de Keller-Segel. Nous nous appliquerons, dans cette introduction, à présenter les méthodes de preuves utilisées. Lorsque cela permet une plus simple compréhension, en s'affranchissant des détails plus techniques, nous ferons explicitement des hypothèses simplificatrices. Le lecteur intéressé pourra se reporter, pour une démonstration plus complète, aux articles de journaux des Parties A et B de ce manuscrit.

1 Le problème de l'obstacle parabolique

Le problème de l'obstacle parabolique est, typiquement, l'étude des fonctions continues qui sont solutions au sens presque partout de

$$\begin{cases} \Delta u - \dfrac{\partial u}{\partial t} = \mathbb{1}_{\{u>0\}} \\ u \geq 0 \end{cases} \tag{1}$$

dans un domaine $D \subset \mathbb{R}^{d+1}$. Nous ne précisons pas ici les conditions au bord ni les données initiales parce que nous nous intéressons à des propriétés locales, c'est-à-dire à l'intérieur de l'ensemble D.

Ce problème apparaît dans le problème de Stefan à une phase. Le problème de Stefan modélise l'interface eau-glaçon lorsqu'un glaçon est plongé dans l'eau (voir Section A.1 pour une description plus complète). De façon plus inattendue un problème similaire, dans lequel on remplace l'opérateur de la chaleur par un opérateur uniformément parabolique à coefficients variables, apparaît dans un modèle issu des mathématiques financières (voir Section A.2).

Il y a, ici, deux inconnues qui sont : la solution, et le domaine dans lequel l'équation est vérifiée. On appelle *frontière libre* le bord de l'ensemble $\{u = 0\}$. Les deux questions qui sont étudiées dans cette section sont :

(i) la continuité de la dérivée en temps de la solution,

(ii) la régularité de la frontière.

Les points de $\mathbb{R}^d \times \mathbb{R}$ sont notés (x, t), où la variable d'espace $x = (x_1, \ldots, x_d)$ appartient à $\mathbb{R}^d$ et la variable de temps, t, appartient à $\mathbb{R}$. Si une fonction de l'espace et du temps est suffisamment régulière on notera $\frac{\partial u}{\partial x_i}$, $D_{x_i}u$, et $\frac{\partial^2 u}{\partial x_i \partial x_i}$, $D_{x_i x_j}u$, de même $\frac{\partial u}{\partial t}$, sera noté $\partial_t u$ ou u_t. On prend la convention de sommer les indices répétés.

À x_0 dans $\mathbb{R}^d$, $P_0 = (x_0, t_0)$ dans $\mathbb{R}^d \times \mathbb{R}$ et R positif, on associe la boule ouverte $B_R(x_0) := \{x \in \mathbb{R}^d \,:\, |x - x_0|^2 < R^2\}$ et le cylindre parabolique

$$Q_R(P_0) := B_R(x_0) \times \{\, t \in \mathbb{R} \,:\, |t - t_0| < R^2 \,\}\,.$$

On définit l'espace

$$\mathcal{H}(Q_R(P_0)) := \{u \in L^\infty(Q_R(P_0)) \;:\; \sup_{Q_R(P_0)} \left[\, \left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right| + \left|\frac{\partial u}{\partial t}\right| + \frac{1}{R}\left|\frac{\partial u}{\partial x_i}\right| \,\right] < \infty\}\,.$$

Par convention on note

$$\mathcal{H}(\mathbb{R}^{d+1}) := \{u \in L^\infty(\mathbb{R}^{d+1}) \;:\; \sup_{\mathbb{R}^{d+1}} \left[\, \left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right| + \left|\frac{\partial u}{\partial t}\right| \,\right] < \infty\}\,.$$

Pour a, b et c des fonctions de x et de t continues sur $Q_R(P_0)$ données, on définit l'opérateur parabolique à coefficients variables, L, pour toute fonction u de $\mathcal{H}(Q_R(P_0))$ par :

$$Lu := a_{ij}(\cdot,\cdot)\, \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(\cdot,\cdot)\, \frac{\partial u}{\partial x_i} + c(\cdot,\cdot)\, u - \frac{\partial u}{\partial t}\,.$$

Ces définitions sont classiques dans la théorie des équations paraboliques (voir [Lie96, Fri64, LSU67]). L'opérateur modèle dans les équations parabolique est l'opérateur de la chaleur, $\Delta - \partial_t$, qui donne naissance à des échelles différentes dans les variables d'espace et de temps.

On s'intéresse aux propriétés qualitatives locales du problème de l'obstacle parabolique à coefficients variables suivant : pour f une fonction de x et de t continue sur $Q_R(P_0)$ donné, on considère des solutions u de

$$\begin{cases} u \in \mathcal{H}(Q_R(P_0)) \\ Lu(x,t) = f(x,t)\, \mathbb{1}_{\{u>0\}}(x,t)\,, \qquad \text{p.p. } (x,t) \in Q_R(P_0) \\ u(x,t) \geq 0\,, \end{cases} \tag{2}$$

où la fonction $\mathbb{1}_{\{u>0\}}$ est la fonction caractéristique de l'ensemble $\{u > 0\} := \{(x,t) \in Q_R(P_0) : u(x,t) > 0\}$:

$$\mathbb{1}_{\{u>0\}}(x,t) = \begin{cases} 1 & \text{si} \quad u(x,t) > 0\,, \\ 0 & \text{si} \quad u(x,t) = 0\,. \end{cases}$$

Dans le cas de l'opérateur de la chaleur, il est montré dans [CPS04] que les solutions au sens des distributions de (2) sont, en fait, à priori dans $\mathcal{H}(Q_R(P_0))$. L'ensemble fermé $\{u = 0\}$ et sa frontière $\Gamma := \partial\{u = 0\} \cap Q_R(P_0)$ sont respectivement appelés *ensemble de coïncidence* et *frontière libre* du problème de l'obstacle parabolique (2).

Afin d'énoncer les hypothèses on définit, pour α dans $(0,1]$, l'espace, $\mathcal{C}^\alpha_{x,t}(D)$, des fonctions de x et de t qui sont höldériennes d'exposant α dans un domaine D de $\mathbb{R}^{d+1}$ par

$$\mathcal{C}^\alpha_{x,t}(D) := \{\, f \in \mathcal{C}^0_{x,t}(D) \;:\; \sup_{\substack{(x,t),(y,s)\in D \\ (x,t)\neq(y,s)}} \frac{|f(x,t) - f(y,s)|}{(\,|x-y|^2 + |t-s|\,)^{\alpha/2}} < \infty \,\}\,.$$

On fait des hypothèses d'uniforme ellipticité et de non-dégénérescence de l'opérateur, et de régularité des coefficients de l'opérateur et de la fonction f : il existe une constante $\delta_0 > 0$ telle que

$$\begin{cases} \textit{pour tout } (x,t) \in Q_R(P_0)\,,\ f(x,t) \geq \delta_0\,, \\ \textit{pour tout } (x,t) \in Q_R(P_0)\,, \forall (\xi_i, \xi_j) \in \mathbb{R}^d\,,\ a_{ij}(x,t)\,\xi_i\,\xi_j \geq \delta_0\,|\xi|^2\,, \\ a,\, b,\, c\,\textit{et } f \textit{ appartiennent à } \mathcal{C}^{\alpha}_{x,t}(Q_R(P_0))\,\textit{pour un certain } \alpha \in (0,1)\,. \end{cases} \tag{3}$$

Dans [VM74], P. Van Moerbeke a montré, dans le cadre des problèmes de temps d'arrêt, que u_t est continue à l'exception d'un point dit “point d'extinction”, dans un cas particulier de fonction f et en dimension un d'espace. Il donnait des asymptotiques de la frontière libre en ce point. Dans [Fri75], A. Friedman a montré que lorsque la frontière est monotone elle est de classe $\mathcal{C}^1_t$. Cependant, dans le cas à coefficients variables, il est, en pratique, très difficile de déterminer la monotonie de la frontière.

Récemment dans [CPS04], L. Caffarelli, A. Petrosyan et H. Shahgholian considèrent le problème de l'obstacle parabolique sans condition de signe sur la solution. Ils ont classifié les points de la frontière libre en deux classes : les points “réguliers” et les points “singuliers” (voir Définition 3). Ils démontrent que u_t est continue aux points réguliers et que la frontière libre est une surface de classe $\mathcal{C}^\infty_{x,t}$ au voisinage des points réguliers. Cependant leur travail a été fait pour l'opérateur de la chaleur et ne s'applique pas dans notre cadre. Ils n'ont pas fait l'étude des points singuliers. Leur travaux pourraient s'adapter au cas à coefficient variables si les coefficients sont Lipschitz, la frontière serait alors Lipschitz.

Dans la Section 1.1 nous énonçons les principaux résultats sur la continuité de u_t et sur la régularité de la frontière libre, en dimension un d'espace. La Section 1.2 donne des résultats de régularité et des propriétés géométriques de l'ensemble des points singuliers en dimension quelconque d'espace. La Section 1.3 est consacrée à la description des méthodes utilisées pour prouver ces résultats.

1.1 Principaux résultats en dimension un d'espace

Dans l'ensemble $\{u = 0\}$, la régularité de u est évidente. À l'intérieur de $\{u > 0\}$, u est solution de $Lu = f$. Les estimations intérieures de Schauder (Théorème 4.9 p. 59, [Lie96]) assurent que u_t est bornée dans $\mathcal{C}^\alpha_{x,t}$. La question de la continuité de la dérivée en temps se pose donc, uniquement, au passage de la frontière libre.

Théorème 1 (Continuité de u_t pour presque tout temps, [BDM05]). *Sous les hypothèses (3), on considère une solution u de (2) en dimension un d'espace. Pour presque tout temps t_1 de $(t_0 - R, t_0 + R)$, si P_1 est un point de la frontière libre, alors*

$$\lim_{\substack{P \to P_1 \\ P \in \{u > 0\}}} \frac{\partial u}{\partial t}(P) = 0\,.$$

On ne peut pas espérer que ce résultat soit vrai pour tout temps sans condition supplémentaire. En effet, dans le cas à coefficients constants (*i.e.* $a \equiv 1$, $b \equiv 0$, $c \equiv 0$ et $f \equiv 1$) la fonction $\max\{-t, 0\}$ est une solution dans $\mathbb{R}^2$. Sa dérivée en temps vaut -1 lorsque t est négatif et 0 lorsque t est positif. Donc la dérivée en temps de la solution $\max\{-t, 0\}$ n'est pas continue au temps $t = 0$.

Cependant si on impose l'hypothèse supplémentaire que u_t est positif on obtient le résultat suivant :

Théorème 2 (Continuité partout de u_t lorsque u_t est positif, [BDM05]). *Sous les hypothèses (3), on considère une solution u de (2) en dimension un d'espace. Si u_t est positive, alors u_t est continue partout et satisfait*

$$\frac{\partial u}{\partial t} = 0 \quad sur \quad \Gamma\ .$$

Ce résultat avait été prouvé par L. Caffarelli dans [Caf77].

On peut donner des critères pour caractériser les points de la frontière pour lesquels u_t est continue. Ce critère est basé sur la densité $\theta(P_1)$ de l'ensemble de coïncidence $\{u=0\}$ en un point P_1 de la frontière libre :

$$\theta(P_1) := \liminf_{r\to 0} \frac{|\{u=0\}\cap Q_r(P_1)|}{|Q_r(P_1)|}$$

et sur la densité inférieure $\theta^-(P_1)$ de $\{u=0\}$ en P_1 de la frontière libre, définie par

$$\theta^-(P_1) := \liminf_{r\to 0} \frac{|\{u=0\}\cap Q_r^-(P_1)|}{|Q_r^-(P_1)|}\ ,$$

où $Q_r^-(P_1)$ est le cylindre parabolique inférieur définit par

$$Q_r^-(P_1) := \{\,(x,t)\in\mathbb{R}^2\ :\ |x-x_1|<r \text{ et } 0<t_1-t<r^2\,\}\ .$$

Définition 3. *Sous les hypothèses (3), on considère une solution u de (2) en dimension un d'espace.*

(i) *Si $\theta^-(P_1)\neq 0$ on dit que P_1 est un* point régulier*. On note $\mathcal{R}$ l'ensemble des points réguliers.*

(ii) *Si $\theta^-(P_1)=0$ on dit que P_1 est un* point singulier*. On note $\mathcal{S}$ l'ensemble des points singuliers. On définit aussi l'ensemble $\mathcal{S}_0$ des points singuliers tels que $\theta(P_1)=0$.*

Nous donnerons, dans le Théorème 13, un critère d'énergie pour caractériser les points réguliers et les points singuliers de la frontière. Cette caractérisation des points de la frontière permet de préciser le Théorème 1 : si P_1 est un point régulier ou de l'ensemble $\mathcal{S}_0$ alors

$$\lim_{\substack{P\to P_1\\ P\in\{u>0\}}} \frac{\partial u}{\partial t}(P) = 0\ .$$

De plus pour presque tout temps le frontière libre n'est constituée que de points réguliers ou de points de l'ensemble $\mathcal{S}_0$. En effet

Théorème 4 (Projection en temps de $\Gamma\backslash(\mathcal{R}\cup\mathcal{S}_0)$, [BDM05]). *Sous les hypothèses (3), on considère une solution u de (2) en dimension un d'espace. L'ensemble*

$$\{t\in[-R^2,R^2]\ :\ \exists\, x\in[-R,R], (x,t)\in\Gamma\setminus(\mathcal{R}\cup\mathcal{S}_0)\}$$

a une mesure de Lebesgue nulle.

Pour la régularité de la frontière libre, on doit définir une notion de régularité locale d'une courbe de $\mathbb{R}^2$ définit par $x = g(t)$, où g est donné. Rappelons que pour tout $t_1 < t_2$ l'ensemble, $\mathcal{C}_t^{1/2}(t_1, t_2)$, des fonctions qui sont continues au sens de Hölder d'exposant 1/2 est défini par

$$\mathcal{C}_t^{1/2}(t_1, t_2) := \left\{ g \in \mathcal{C}_t^0(t_1, t_2) \ : \ \sup_{\substack{t,s\in(t_1,t_2)\\ t\neq s}} \frac{|g(t) - g(s)|}{|t-s|^{\frac{1}{2}}} < \infty \right\}.$$

Nous utiliserons aussi la notion de sous-graphe et de sur-graphe (voir Figures 1 et 2) définis de la manière suivante : soit P_0 dans $\mathbb{R}^2$ et R positif. Soient A inclus dans $Q_R(P_0)$ et $P_1 = (x_1, t_1)$ dans A.

(i) On dit que A est localement un *graphe de classe* $\mathcal{C}_t^{1/2}$ au voisinage de P_1 s'il existe un rayon ρ strictement positif et une fonction g de classe $\mathcal{C}_t^{1/2}(t_1 - \rho^2, t_1 + \rho^2)$ tels que $Q_\rho(P_1) \subset Q_R(P_0)$ et

$$A \cap Q_\rho(P_1) = \{(x,t) \in Q_\rho(P_1) \ : \ x = g(t)\} .$$

(ii) On dit que A est localement au voisinage de P_1 un *sous-graphe* (respectivement un *sur-graphe* de classe $\mathcal{C}_t^{1/2}$) s'il existe un rayon ρ strictement positif et une fonction g de classe $\mathcal{C}_t^{1/2}(t_1 - \rho^2, t_1 + \rho^2)$ tels que $Q_\rho(P_1) \subset Q_R(P_0)$ et

$$A \cap Q_\rho(P_1) = \{(x,t) \in Q_\rho(P_1) \ : \ x \leq g(t)\}$$
$$(\text{resp. } A \cap Q_\rho(P_1) = \{(x,t) \in Q_\rho(P_1) \ : \ x \geq g(t)\}) .$$

Rappelons que, par le Théorème 4, pour presque tout temps, la frontière libre n'est constituée que de points de $\mathcal{R} \cup \mathcal{S}_0$. Nous pouvons maintenant énoncer notre théorème sur la régularité de ces ensembles $\mathcal{R}$ et $\mathcal{S}_0$:

Théorème 5 (Régularité de la frontière libre, [Bla05a]). *Sous les hypothèses (3), on considère une solution u de (2) en dimension un d'espace.*

(i) *L'ensemble $\mathcal{R}$ des points réguliers est localement un sur-graphe ou un sous-graphe de classe $\mathcal{C}_t^{1/2}$.*

(ii) *L'ensemble $\mathcal{S}_0$ est localement contenu dans un graphe $\mathcal{C}_t^{1/2}$.*

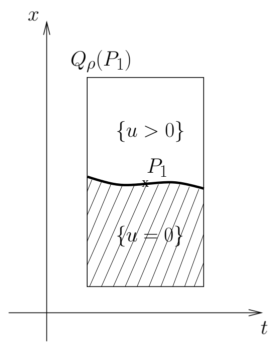


FIG. 1 – sous-graphe

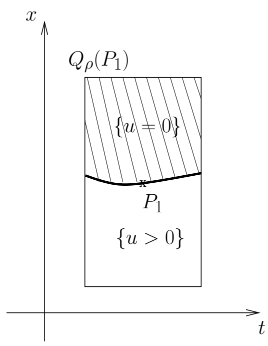


FIG. 2 – sur-graphe

Remarquons que la solution $\max\{-t, 0\}$ est strictement positive lorsque t est négatif et est nulle lorsque t est positif. La frontière libre est, ici, décrite par $\{t = 0\}$ et n'est pas un graphe en temps en $t = 0$.

1.2 Étude des points singuliers en dimension supérieure

D. G. Schaeffer, en 1977, dans [Sch77], a initié l'étude des points singuliers dans les problèmes à frontière libre. H. Alt, L. Caffarelli et A. Friedman ont introduit, en 1984, dans [ACF84], une formule de monotonie qui permet de contrôler les pentes des ensembles de coïncidence autour des points singuliers dans le problème de l'obstacle elliptique sans condition de signe sur la solution. Dans le cas du problème de l'obstacle elliptique avec condition de signe sur la solution, R. Monneau s'est appuyé sur la formule de monotonie de G. Weiss dans [Wei99a] pour introduire une formule de monotonie pour les points singuliers. Cette méthode lui a permis d'obtenir des résultats sur les propriétés géométriques de l'ensemble des points singuliers.

Le critère de densité introduit dans la section précédente pour les points réguliers et pour les points singuliers est aussi valable en dimension supérieures. On définit l'ensemble des points réguliers et des points singuliers comme dans la Définition 3. Nous donnerons une caractérisation énergétique des poins singuliers dans le Théorème 13.

Sous les hypothèses (3), considérons une solution u de (2), P_1 de la frontière libre et $(\varepsilon_n)_{n\in\mathbb{N}}$ une suite convergeant vers 0. La *suite d'explosion* $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ associée à u est la suite de terme générique

$$u^{\varepsilon_n}_{P_1}(x,t) = \varepsilon_n^{-2}\, u\left(x_1 + \varepsilon_n\, x, t_1 + \varepsilon_n^2\, t\right) \quad \forall (x,t) \in Q_{r/2\varepsilon_n}(0)\,. \tag{4}$$

Nous reviendrons en détail dans la Section 1.3 sur cette notion de suite d'explosion.

On montre

Théorème 6 (Unicité des limites d'explosion en un point singulier, [Bla05b]). *Sous les hypothèses (3), on considère une solution u de (2) et P_1 un point singulier. Si $[a_{ij}(P_1)] = \mathrm{Id}_d$ et $f(P_1) = 1$ alors toute la suite $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ converge vers une fonction $m\,t + \frac{1}{2}X^T\cdot A\cdot X$, où X^T désigne la transposée du vecteur $X\in\mathbb{R}^d$, $m\in[-1,0]$ et A est une matrice symétrique positive de trace $m+1$.*

On peut se ramener au cas où $[a_{ij}(P_1)] = \mathrm{Id}_d$ et $f(P_1) = 1$ par un changement de variable.

On peut donc définir

Définition 7 (m_{P_1}, A_{P_1}, $\mathcal{S}(k)$ et $\mathcal{S}_m$). *Soit P_1 est point singulier. On définit :*

(i) *m_{P_1} et A_{P_1} comme l'unique couple (m_{P_1}, A_{P_1}) tel que la limite de $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ soit $m_{P_1}\,t + \frac{1}{2}X^T\cdot A_{P_1}\cdot X$,*

(ii) *pour $k\in\{0,\dots,d-1\}$, l'ensemble $\mathcal{S}(k)$ des points P de $\mathcal{S}$ tels que* $\dim\mathrm{Ker}\,A_P = k$ *et dont la valeur absolue des $d-k$ valeurs propres non-nulles et bornée inférieurement par un réel a strictement positif fixé.*

(iii) *L'ensemble, $\mathcal{S}(d)$, des points singuliers tels que toutes les valeurs propres sont nulles.*

On a le théorème de régularité suivant :

Théorème 8 (Régularité de $\mathcal{S}(k)$, [Bla05b]). *Sous les hypothèses (3), considérons une solution u de (2).*

(i) *Si P_1 appartient à $\mathcal{S}(d)$ alors il existe un graphe $\tilde{\Gamma}$ de classe $\mathcal{C}^2_x$ tel que*

$$\mathcal{S}(d) \cap Q_r(P_1) \subset \tilde{\Gamma}\,,$$

pour un certain $r > 0$ suffisamment petit.

(ii) *Soit k dans $\{0, \ldots, d-1\}$. Si P_1 appartient à $\mathcal{S}(k)$ alors il existe une variété Γ de dimension k et de classe $\mathcal{C}^{1/2}_{x,t}$ telle que*

$$\mathcal{S}(k) \cap Q_r(P_1) \subset \Gamma\,,$$

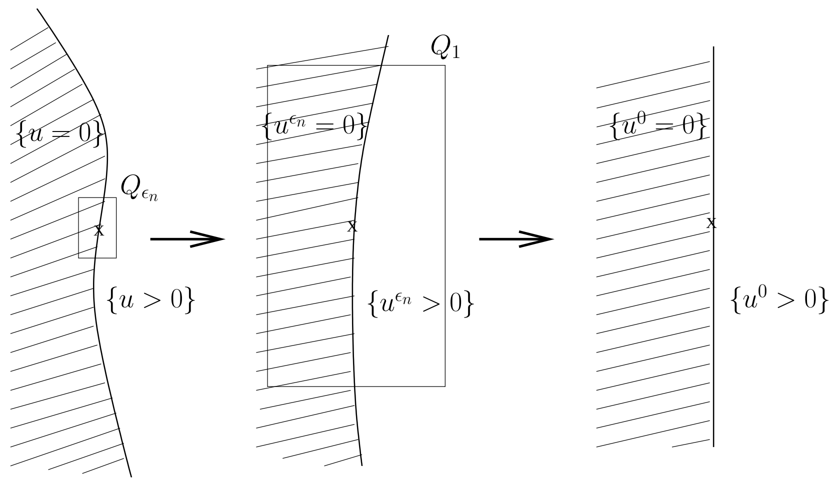
pour un certain $r > 0$ suffisamment petit.

1.3 Schéma de la preuve et résultats intermédiaires

1.3.1 Explosion et de limite d'explosion

L'idée fondatrice de notre méthode est due à L. Caffarelli. Dans [Caf80], il caractérise les points de la frontière libre du problème de l'obstacle elliptique par les *limites d'explosions*. Pour le problème de l'obstacle parabolique on définit une suite d'explosion par (4). Cette méthode permet d'obtenir par un argument indirect des informations sur le comportement de la solution en un point de la frontière libre sans avoir aucune connaissance directe sur la frontière. Notons que le changement d'échelle parabolique $(x,t) \mapsto (X := \lambda x, T := \lambda^2 t)$ transforme le cylindre parabolique $Q_\lambda(0)$ en le cylindre parabolique $Q_1(0)$. Après changement d'échelle, les points (X,T) qui sont dans $Q_1(0)$ sont les images par cette transformation des points (x,t) qui étaient dans $Q_{\varepsilon_n}(P_1)$ avant le changement d'échelle. Remarquons aussi que l'opérateur de la chaleur est conservé au sens que

$$\left[\Delta u_0^{\varepsilon_n} - \partial_t u_0^{\varepsilon_n}\right](x,t) = \left[\Delta u - \partial_t u\right](\varepsilon_n x, \varepsilon_n^2 t)\,.$$



Effet du changement d'échelle et passage à la limite en 0

Grâce à la régularité $\mathcal{H}(Q_R(P_0))$ de u, on peut appliquer le théorème d'Ascoli-Arzela à la suite d'explosion $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$: il existe une fonction $u^0_{P_1}$ de $\mathcal{H}(\mathbb{R}^{d+1})$ dite *limite d'explosion* et une sous-suite $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ telle que $(u^{\varepsilon_{n_k}}_{P_1})_{k\in\mathbb{N}}$ converge uniformément sur tout compact de $\mathbb{R}^{d+1}$ vers $u^0_{P_1}$. Mais par passage à la limite on pourrait imaginer que la frontière libre disparaisse, ou simplement que le point P_1 de la frontière libre ne reste pas sur la frontière. Ceci n'est pas possible grâce à l'estimation de non-dégénérescence suivante

Lemme 9 (lemme de non-dégénérescence, [Bla05b]). *Sous les hypothèses (3), on considère une solution u de (2) dans $Q_R(P_0)$. Soit P_1 dans $\overline{\{u>0\}}$ tel que $Q_r^-(P_1)$ soit contenu dans $Q_R(P_0)$ pour un certain rayon r suffisamment petit. Alors il existe des entiers positifs $\bar{C}$ et $\bar{r}$, qui ne dépendent que de R et de l'opérateur parabolique L, tels que*

$$r \leq \bar{r} \quad \Longrightarrow \quad \sup_{Q_r^-(P_1)} u \geq \bar{C}\, r^2 \,.$$

La démonstration de ce lemme utilise le principe du maximum fort (voir la démonstration du Lemme A.II.6). Ce type de lemme avait déjà été démontré pour la première fois dans [Caf80] pour le problème de l'obstacle elliptique. Ce lemme a été étendu dans [CPS04] pour le problème de l'obstacle parabolique à coefficients constants. Ce lemme de non-dégénérescence permet de montrer que, après passage à la limite, 0 est sur la frontière libre $\partial\{u^0_{P_1}=0\}$ où $u^0_{P_1}$ est une limite d'explosion en P_1 de la frontière libre. En effet, si on considère 0 sur la frontière libre, $(u^{\varepsilon_n})_{n\in\mathbb{N}}$ une suite d'explosion en 0 qui tend vers u^0, on a

$$\bar{C}\, r^2 \leq \sup_{(x,t)\in Q_r^-(0)} u(x,t) = \sup_{(x,t)\in Q_r^-(0)} \epsilon_n^2\, u^{\epsilon_n}\left(\frac{x}{\epsilon_n}, \frac{t}{\epsilon_n^2}\right) \,,$$

$$\bar{C}\left(\frac{r}{\epsilon_n}\right)^2 \leq \sup_{(x,t)\in Q^-_{r/\epsilon_n}(0)} u^{\epsilon_n}(x,t) \,.$$

Si on remplace $\epsilon_n\, r$ par $\tilde{r}$, on obtient

$$\bar{C}\, \tilde{r}^2 \leq \sup_{Q_{\tilde{r}}^-(0)} u^{\epsilon_n} \to \sup_{Q_{\tilde{r}}^-(0)} u^0 \quad \text{quand} \quad n\to\infty \,,$$

ce qui prouve que 0 est sur la frontière libre de $\partial\{u^0=0\}$. Finalement, on prouve (voir Proposition A.II.13) que $u^0_{P_1}$ est solution de

$$\left\{\begin{array}{ll} [a_{ij}](P_1)\, \dfrac{\partial^2 u^0_{P_1}}{\partial x_i \partial x_j}(x,t) - \dfrac{\partial u^0_{P_1}}{\partial t}(x,t) = f(P_1)\, 1\!\!1_{\{u^0_{P_1}>0\}}(x,t) & \\ & \text{p.p. } (x,t)\in\mathbb{R}^2 \\ u^0_{P_1}(x,t) \geq 0 & \end{array}\right.$$

et 0 est sur la frontière libre de $\partial\{u^0_{P_1}=0\}$. Si on suppose que $[a_{ij}(P_1)] = \mathrm{Id}_d$ et $f(P_1)=1$ (sinon il faut juste faire un changement d'échelle qui nous ramène à ce cas), on obtient que $u^0_{P_1}$ est solution de

$$\left\{\begin{array}{ll} \Delta u^0_{P_1}(x,t) - \partial_t u^0_{P_1}(x,t) = 1\!\!1_{\{u^0_{P_1}>0\}}(x,t) & \\ & \text{p.p. } (x,t)\in\mathbb{R}^2 \,. \\ u^0_{P_1}(x,t)\geq 0 & \end{array}\right. \tag{5}$$

1.4 Théorème de type Liouville

Supposons un instant que les limites d'explosions sont invariantes par changement d'échelle au sens où pour tout λ positif

$$u^0(\lambda x, \lambda^2 t) = \lambda^2 u^0(x,t) \quad \text{pour tout } (x,t)\in\mathbb{R}\times(-\infty,0) \,. \tag{6}$$

L. Caffarelli, A. Petrosyan et H. Shahgholian ont montré que

Théorème 10 (Théorème de type Liouville pour t négatif. Lemme 6.3, [CPS04]). *Les solutions de (5) dans $\mathcal{H}(\mathbb{R}^{d+1})$ sous la contrainte (6) pour t négatif sont*

$$\begin{aligned} u^0_+(x,t) &:= \frac{1}{2}\,(\max\{0,x\})^2 \\ u^0_-(x,t) &:= \frac{1}{2}\,(\max\{0,-x\})^2 \end{aligned} \qquad \textit{et} \quad u^0_m(x,t) = m\,t + \frac{1}{2}X^T \cdot A \cdot X$$

où A est une matrice symétrique de trace $m{+}1$.

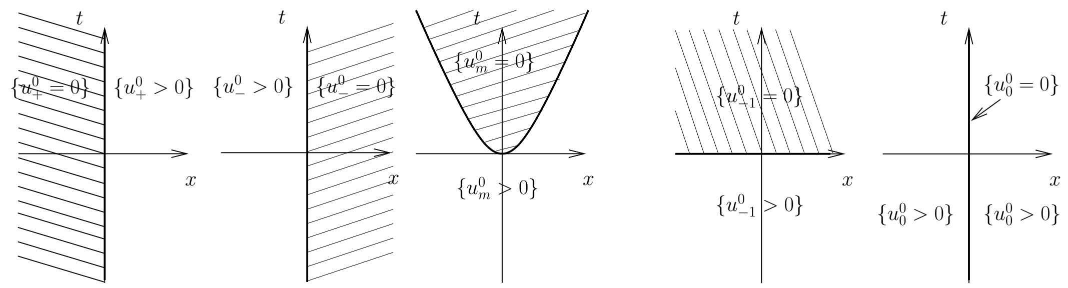
Dans le cas de la dimension un d'espace on peut affiner le résultat.

Théorème 11 (Théorème de type Liouville dans $\mathbb{R}^2$, [CPS04], [BDM05]). *Les solutions de (5) dans $\mathcal{H}(\mathbb{R}^2)$ sous la contrainte (6) dans $\mathbb{R}^2$ sont u^0_+ ou u^0_- où u^0_+ et u^0_- sont définies dans le Théorème 10 ou*

$$u^0_m(x,t) = \begin{cases} m\,t + \dfrac{1+m}{2}\,x^2 & \textit{si } t \le 0\ , \\ t\,V_m\left(\dfrac{|x|}{t}\right) > 0 & \textit{si } 0 < t < C_m \cdot x^2\ , \\ 0 & \textit{si } t \ge C_m \cdot x^2\ , \end{cases}$$

où l'expression précise de V_m et de C_m est donnée dans le Théorème A.II.19. De plus C_m est une fonction croissante de m.

En particulier on a $u^0_{-1}(x,t) = \max\{0,-t\}$ et $u^0_0(x,t) = x^2/2$.



u^0_+ $\quad$ u^0_- $\quad$ u^0_m, $m \in (-1,0)$ $\quad$ u^0_{-1} $\quad$ u^0_0

1.4.1 Formule de monotonie pour l'énergie

L'objet de cette section est de prouver l'invariance par changement d'échelle des limites d'explosion. G. Weiss avait démontré, dans [Wei99a], une formule de monotonie pour le problème de l'obstacle elliptique. Cette formule de monotonie est le coeur de notre approche et le lecteur intéressé pourra se référer à la lecture de la Section 1 de l'appendice pour de plus amples développements. Afin de simplifier la présentation, nous considérerons, dans la suite de cette section, que u est solution du problème de l'obstacle parabolique suivant

$$\begin{cases} u \in \mathcal{H}(\mathbb{R}^{d+1}) \\ \Delta u - \dfrac{\partial u}{\partial t} = \mathbb{1}_{\{u>0\}} \qquad \text{p.p dans } \mathbb{R}^{d+1}\ . \\ u \ge 0 \end{cases} \tag{7}$$

Sans cette hypothèse il faut faire une localisation (en multipliant par une fonction "cut-off") et corriger la formule de monotonie par des termes qu'il faut alors contrôler (voir Proposition A.II.15). On a vu Section 1.3.1 que, même dans le cas à coefficients variables, les limites d'explosion sont solutions du problème de l'obstacle parabolique à coefficients constants.

Pour tout temps t négatif, on définit la fonction

$$\mathcal{E}(t;u) = \int_{\mathbb{R}^d} \left\{ \frac{1}{-t} \left(|\nabla u(x,t)|^2 + 2\,u(x,t) \right) - \frac{1}{t^2}\, u^2(x,t) \right\} \mathcal{G}(x,t)\ dx$$

où $\mathcal{G}$ est donné par

$$\mathcal{G}(x,t) = (2\,\pi\,(-t))^{-d/2} \exp\left(\frac{-|x|^2}{4(-t)} \right) .$$

La fonction $\mathcal{G}$ satisfait l'équation de la chaleur rétrograde $\mathcal{G}_{xx} + \mathcal{G}_t = 0$ pour t négatif, avec condition initiale $\mathcal{G}(0) = \delta$. Cette énergie est invariante par changement d'échelle au sens où elle vérifie $\mathcal{E}(\varepsilon_n^2 t; u) = \mathcal{E}(t; u_0^{\varepsilon_n})$, de plus

Théorème 12 (Formule de monotonie pour l'énergie, [Bla05b]). *On considère une solution u de (7). La fonction $\mathcal{E}$ est décroissante en temps pour tout temps t négatif, et elle satisfait*

$$\frac{d}{dt}\mathcal{E}(t;u) = -\frac{1}{2(-t)^3} \int_{\mathbb{R}^d} |\mathcal{L}u(x,t)|^2 \mathcal{G}(x,t)\ dx$$

où

$$\mathcal{L}u(x,t) = -2\,u(x,t) + x \cdot \nabla u(x,t) + 2\,t \cdot u_t(x,t) .$$

G. Weiss avait démontré, dans [Wei99b], une formule de monotonie similaire. Cette formule de monotonie permet de prouver l'invariance par changement d'échelle des limites d'explosion. Plaçons nous ici au point 0 de la frontière libre. En passant à la limite dans $\mathcal{E}(\varepsilon_n^2 t; u) = \mathcal{E}(t; u_0^{\varepsilon_n})$, lorsque ε_n tend vers 0, on obtient $\lim_{\tau\to 0} \mathcal{E}(\tau; u) = \mathcal{E}(t; u_0^0)$ pour tout t négatif. Par la formule de monotonie Théorème 12, on a $\mathcal{L}u_0^0(x,t) = 0$ pour t négatif. Ce qui implique l'invariance par changement d'échelle des limites d'explosion. Pour tout point P_1 de la frontière, on peut définir l'énergie $\mathcal{E}_{P_1}$ par $\mathcal{E}_{P_1}(t;u) = \mathcal{E}(t; u(\cdot + x_1, \cdot + t_1))$.

On peut classifier l'ensemble des limites d'explosions par leur énergie

Théorème 13 (Classification des limites d'explosions, [Bla05b]). *On considère une solution u de (7) et P_1 un point de la frontière libre. On a*

$$\lim_{\tau\to 0} \mathcal{E}_{P_1}(\tau; u) \in \{K, 2\,K\} ,$$

où K est une constante positive.

(i) *Si $\lim_{\tau\to 0} \mathcal{E}_{P_1}(\tau; u) = K$ alors P_1 est un point régulier.*

(ii) *Si $\lim_{\tau\to 0} \mathcal{E}_{P_1}(\tau; u) = K$ alors P_1 est un point singulier.*

En effet, les Théorèmes 10 et 12 permettent de classifier les limites d'explosions. De plus $\lim_{\tau\to 0} \mathcal{E}(\tau; u) = \mathcal{E}(t; u_0^0)$. On calcule $\mathcal{E}(t; u_+^0) = \mathcal{E}(t; u_-^0) = K$ et, pour tout $m \in [-1, 0]$, $\mathcal{E}(t; u_m^0) = 2\,K$.

1.4.2 Étude des points réguliers en dimension un

Dans le cas spécifique de la dimension un d'espace on prouve

Lemme 14 (Unicité des limites d'explosion aux points réguliers, [Bla05a]). *On considère une solution u de (7). Si P_1 est un point régulier alors il existe un unique $\gamma \in \{+,-\}$ tel que la suite d'explosion $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ converge uniformément vers $u^0_\gamma(\cdot + x_1, \cdot + t_1)$, où u^0_+ et u^0_- sont définie dans le Théorème 10.*

En effet, supposons qu'il y ait deux sous-suites de $(u^{\varepsilon_n}_0)_{n\in\mathbb{N}}$ qui converge respectivement vers u^0_+ et u^0_-. On a $u^0_+(1,0) = 1/2$ et $u^0_-(1,0) = 0$. Par continuité de $\varepsilon \mapsto u^\varepsilon_{P_1}$, il existe une sous-suite $(\tilde{\varepsilon}_n)_{n\in\mathbb{N}}$ telle que $\lim_{n\to\infty} u^{\tilde{\varepsilon}_n}_0(1,0) = 1/4$. Mais cette propriété n'est satisfaite ni par u^0_+, ni par u^0_-. On obtient donc une contradiction. On définit $\mathcal{R}^+$ (respectivement $\mathcal{R}^-$) comme l'ensemble des points P_1 de la frontière libre tels que l'explosion limite en P_1 est $u^0_+(\cdot + x_1, \cdot + t_1)$ (resp. $u^0_-(\cdot + x_1, \cdot + t_1)$).

1.4.3 Étude des points singuliers

R. Monneau a introduit, dans [Mon03], une formule de monotonie pour l'études des points singuliers dans le problème de l'obstacle elliptique. Il en existe une analogue pour le problème de l'obstacle parabolique :

Proposition 15 (Formule de monotonie pour les points singuliers, [Bla05b]). *On considère une solution u de (7). Si 0 est un point singulier et u^0_m est définit dans le Théorème 10, la fonction*

$$t \mapsto \Phi_m(t;u) := \frac{1}{t^2}\int_{\mathbb{R}^d} \left|u - u^0_m\right|^2 \mathcal{G}\, dx \,.$$

est décroissante et bornée dans $W^{1,1}(-1,0)$.

Soit $(u^{\varepsilon_n})_{n\in\mathbb{N}}$ une suite d'explosion en 0 et $u^0 = u^0_m$, on a

$$\lim_{\tau\to 0} \Phi(\tau;u,u^0) = \lim_{n\to\infty} \Phi(\varepsilon_n^2 t;u,u^0) = \lim_{n\to\infty} \Phi(t;u^{\varepsilon_n},u^0)\,.$$

La limite est indépendante de la suite $(\varepsilon_n)_{n\in\mathbb{N}}$ choisie. Ce qui montre l'unicité des limites d'explosions en un point singulier. Ceci permet de justifier la Définition 7.

1.4.4 Démonstration des Théorèmes 1, 2 et 4

En dimension un d'espace, sous l'hypothèse $a(P_1) = f(P_1) = 1$, on définit, pour tout m de $[-1,0]$, l'ensemble

$$\mathcal{S}_m = \{P_1 \in \Gamma \;:\; \lim_{n\to\infty} u^{\varepsilon_n}_{P_1} = u^0_m(\cdot + x_1, \cdot + t_1)\}\,,$$

où u^0_m est défini dans le Théorème 11. La définition de l'ensemble $\mathcal{S}_m$ peut être faite sans l'hypothèse $a(P_1) = f(P_1) = 1$ mais on s'y ramène par changement de variable.

Démonstration du Théorème 1 : dans le cas où 0 est dans le fermé $\mathcal{R}^+$ on sait que $(u^{\varepsilon_n})_{n\in\mathbb{N}}$ converge uniformément vers $u^0_+ = (1/2)\,(\max\{x,0\})^2$. Or $\partial_t u^0_+ = 0$. On utilise ces deux propriétés pour prouver que u est à dérivée nulle au passage de la frontière libre. Ceci s'applique aux points de $\mathcal{R}^-$ et $\mathcal{S}_0$.

Démonstration du Théorème 4 : les points de $\mathcal{S}_m$, $m \neq 0$ sont des points pour lesquels la frontière libre de la limite d'explosion est une parabole (voir Théorème 11). L'ensemble des temps où il existe de tels points est de mesure nulle.

Démonstration du Théorème 2 : au moins pour t négatif, $\partial_t u_m^0 = m$ est négatif. Ceci permet de montrer que lorsque u_t est positive, il n'y a pas de points de $\mathcal{S}_m$. Donc la frontière libre est la réunion des points réguliers et des points de $\mathcal{S}_0$ pour lesquels u_t est continue.

1.4.5 Démonstration du Théorème 5

Le facteur 1/2 pour la régularité höldérienne apparaît naturellement grâce au changement d'échelle. Il faut en effet voir que par changement d'échelle et passage à la limite, les points de $\{(x,t) : x^2 = |t|\}$ restent sur $\{(X,T) : X^2 = |T|\}$, les points de $\{(x,t) : x^2 < |t|\}$ sont envoyés sur $\{(X,T) : X = 0\}$, les points de $\{(x,t) : x^2 > |t|\}$ sont envoyés sur $\{(X,T) : T = 0\}$. C'est pourquoi localement il ne peut pas y avoir d'autres points de la frontière dans $\{(x,t) : x^2 < |t|\}$ si 0 est un point régulier ou un point de $\mathcal{S}_0$. En effet, supposons, par contradiction, qu'il existe une suite $(P_n)_{n\in\mathbb{N}}$ de points de la frontière libre dans l'ensemble $\{(x,t) : x^2 < |t|\}$ qui converge vers 0. On définit une suite d'explosion en 0 avec ε_n la distance parabolique de P_n à 0. Alors la limite de $(P_n)_{n\in\mathbb{N}}$ est sur $\{(X,T) : X = 0\}$. Ce qui contredit la forme explicite de la frontière libre pour les points réguliers et les points de $\mathcal{S}_0$ (voir Théorème 10). Un raffinement de cet argument permet de montrer que la frontière est un sous-graphe de classe $C_t^{1/2}$ au voisinage des points de $\mathcal{R}^+$, un sur-graphe de classe $C_t^{1/2}$ au voisinage des points de $\mathcal{R}^-$ et que les points de $\mathcal{S}_0$ sont inclus dans un graphe de classe $C_t^{1/2}$.

1.4.6 Démonstration du Théorème 8

S'ils s'accumulent, on définit la régularité des ensembles singuliers en contrôlant la régularité des normales au noyau de la matrice A_{P_n} qui apparaît dans la Définition 7. Ensuite on étend la régularité à une variété qui la contient grâce au théorème de Whitney. Rappelons que le théorème de Whitney permet d'étendre une fonction de régularité $\mathcal{C}^k$ (ou $\mathcal{C}^{k,\alpha}$ pour la généralisation due à Stein dans [Ste70]) sur n'importe quel ensemble fermé en une fonction de même régularité sur l'espace tout entier. La principale difficulté est que noyau de A_{P_n} peut être de dimension quelconque. Ceci pose problème lorsque une suite $(A_{P_n})_{n\in\mathbb{N}}$ converge vers une matrice A_{P_1} de dimension différente. Puisqu'il se pourrait que $\operatorname{Ker} A_n$ ne converge pas vers $\operatorname{Ker} A_{P_1}$. On contourne ce problème en considérant les ensembles fermé $\mathcal{S}(k)$ dont la dimension du noyau est fixée et ne peut pas dégénérer par passage à la limite. Ainsi on a bien $\lim_{n\to\infty} \operatorname{Ker} A_n = \operatorname{Ker} A_{P_1}$.

1.5 Questions ouvertes

L'étude du problème de l'obstacle parabolique s'applique à l'étude des options américaines en finance (voir Chapitre A.I pour la terminologie relative aux mathématiques financières). La détermination de la frontière libre donne la meilleure stratégie pour l'exercice de l'option. C'est pourquoi de nombreux schémas numériques sont développés pour approcher la frontière libre et la solution. Cette analyse numérique nécessite une bonne compréhension du comportement de la frontière libre. On peut clarifier le comportement de la frontières dans plusieurs directions :
- On peut vérifier les critères de régularité sur des exemples aussi simples que les options

vanilles dans le cadre de Black-Scholes à volatilité locale.
- On peut espérer montrer que, si les coefficients sont de régularité $\mathcal{C}^{k,\alpha}_{x,t}$, alors la solution de (2) est $\mathcal{C}^{k+2,\alpha}_{x,t}$ autour des points réguliers.
- Il y a un gros effort à faire dans la direction de l'analyse numérique. Il serait intéressant de trouver et d'implémenter numériquement des fonctions f, réalistes d'un point de vue financier, pour lesquels la frontière libre n'est pas un graphe (voir [BD97]).
- L'étude des points singuliers n'a pas été faite dans la cadre du problème de l'obstacle parabolique sans condition de signe sur la solution. Ce problème apparaît en théorie du potentiel (voir [CPS04]). Dans le cas où les coefficients sont Lipschitz, la formule de monotonie de L. Caffarelli ([Caf93]) devrait permettre de contrôler les pentes des noyaux aux points singuliers. Nous obtiendrions les mêmes résultats que ceux du Théorème 5.
- Récemment les financiers s'intéressent à des modèles plus complexes dits "à volatilité stochastique" (dans lesquels la volatilité est elle-même régie par une équation différentielle stochastique : voir [FPS00]). Cependant des travaux de traitement du signal ont montré que la spectre des variations d'une volatilité présente des sauts. Ceci est en en contradiction avec le spectre constant du mouvement brownien. C'est pourquoi il semble que les "processus de Lévy" sont bien mieux adaptés à la modélisation de la variation des prix sur les marchés financiers (voir [Ber96]). Dans ce modèle l'équation de Black-Scholes est remplacée par une équation intégrale. Nos méthodes peuvent probablement s'adapter à ces équations.

2 Le modèle de Keller-Segel

La deuxième partie de ce mémoire est consacrée à l'étude d'un problème issu de la modélisation d'agrégation en biologie : le modèle de Keller-Segel. Nous nous intéressons dans cette section à des solutions au sens des distributions du système de Keller-Segel suivant :

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \chi \nabla \cdot (n \nabla c) & x \in \mathbb{R}^2 \,,\ t > 0 \,, \\ -\Delta c = n & x \in \mathbb{R}^2 \,,\ t > 0 \,, \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \,. \end{cases} \tag{8}$$

où n est une densité de bactérie ou d'amibes (typiquement de Dictyostelium discoïdeum) et c la concentration de chemo-attractant. Il s'agit de la version parabolique-elliptique simplifiée du système introduit par Evelyn Fox Keller et Lee A. Segel dans [KS70] et précédemment par C. Patlak dans [Pat53]. Ce système se réduit, en fait, à une équation non-linéaire non-locale si on convient que la solution de l'équation de Poisson est décrite par

$$c(x,t) = -\frac{1}{2\,\pi} \left(\log |\cdot|\right) * n(\cdot, t) \tag{9}$$

où $*$ désigne l'opérateur de convolution en espace. La norme $L^1(\mathbb{R}^2)$ de n est conservée. On note $M := \int_{\mathbb{R}^2} n(x,t)\, dx$. On introduit les hypothèses :

$$\begin{cases} n_0 \in L^1(\mathbb{R}^2, (1 + |x|^2)\, dx) \,, \\ n_0 \log n_0 \in L^1(\mathbb{R}^2, dx) \,. \end{cases} \tag{10}$$

Ce système modélise un phénomène d'agrégation d'amibes. Les Dictyostelium discoïdeum, par exemple, se déplacent en suivant un mouvement brownien dans une boîte de Pétri (voir Section 3 de l'appendice). En situation de pénurie de nourriture une ou plusieurs amibes se mettent à émettre une substance chimique : le chemo-attractant. Les autres amibes sont attirées par un gradient plus fort de concentration de chemo-attractant et en émettent à leur tour. Dans le modèle le plus simple, la diffusion du chemo-attractant se fait en suivant une équation de Poisson : $\Delta c = -n$ (voir Chapitre B.I pour plus de détails). Le comportement de ce système est donc le résultat d'une compétition entre le terme diffusif : Δu, et le terme de dérive : $-\nabla \cdot (n \nabla c)$. On peut imaginer que s'il y a peu d'amibes et qu'elles sont très étalées, la propension à diffuser l'emportera. Par contre si la densité est suffisamment forte, on verra apparaître des agrégats. L'objectif du modèle de Keller-Segel est de décrire l'apparition de ces agrégats. Sur le plan mathématique, il s'agit de montrer que les solutions explosent. Cette section est consacrée à la question de savoir s'il y a une masse initiale au delà de laquelle le système explose et en deçà de laquelle le système existe pour tout temps. Dans ce dernier cas on cherche à décrire le comportement en temps long des solutions. Ce modèle pourtant rudimentaire de biologie modélise de façon simple un phénomène d'agrégation en réponse à un stimulus chimique que l'on s'attend à retrouver dans de nombreux phénomènes biologiques. Un exemple important d'agrégation en réponse à un stimulus chimique apparaît en angiogénèse (dans le développement de tumeurs cancéreuses). Lorsqu'un cancer se développe il puise ses nutriments directement dans le réseau sanguin du malade. Mais lorsque la tumeur atteint une taille importante elle a besoin, pour continuer à se développer, d'un apport direct de nutriments. Elle secrète alors une substance chimique qui attire les cellules qui constituent la paroi des réseaux sanguins pour créer son propre réseau sanguin. La description d'un tel phénomène est évidemment beaucoup plus complexe que le problème décrit par (8).

Cette section se divise en trois parties : la Section 2.1, est un historique des principaux résultats relatifs aux propriétés qualitatives des solutions de (8), la Section 2.2 est consacrée à l'énoncé des principaux résultats nouveaux et la Section 2.3 donne une idée de la preuve de ces résultats. La régularité des solutions est obtenue par hypercontractivité et le comportement en temps long par des méthodes d'énergie libre.

2.1 Existence et non-existence

2.1.1 Explosion

Si on considère les solutions régulières de (8) on peut calculer

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\, n(x,t)\,dx &= \int_{\mathbb{R}^2} |x|^2\,\left(\Delta n - \chi \nabla\cdot(n\nabla c)\right) \\
&= 4\,M - \frac{\chi}{2\,\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 2\,x\,n(x,t)\,n(y,t)\,\frac{x-y}{|x-y|^2} \\
&= 4M - \frac{\chi}{2\,\pi}\,M^2 \\
&= 4M\left(1-\frac{\chi\,M}{8\,\pi}\right)\,.
\end{aligned}$$

Ce qui permet de prouver que

Lemme 16. *On considère une solution positive n de (8) sur un intervalle $[0,T]$ et dont le moment d'ordre deux est borné, qui vérifie (10) et tel que $\int_{\mathbb{R}^2} \frac{1+|x|}{|x-y|}\,n(y,t)\,dy$ est bornée* *dans* $L^\infty\left((0,T)\times\mathbb{R}^2\right)$. *Alors on a*

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2\,n(x,t)\,dx = 4M\left(1-\frac{\chi\,M}{8\pi}\right)\,.$$

Si $\chi\,M>8\pi$, le moment d'ordre deux est strictement décroissant donc la solution explose avant que le moment d'ordre deux ne devienne négatif. Ce résultat avait été conjecturé par V. Nanjundiah [NS92], et S. Childress et J. K. Percus ([CP81]). Si $\chi\,M<8\pi$, le moment d'ordre deux est borné dans $L^\infty(0,T;L^1(\mathbb{R}^2))$, pour autant que la solution soit bien définie et régulière.

2.1.2 Estimation à priori par l'inégalité de Gagliardo-Nirenberg-Sobolev

L'entropie (physique) est la quantité $\int_{\mathbb{R}^2} n\log n$. Cette notion est à différencier de celle d'énergie libre (en suivant la terminologie de [ACD+04]) que nous introduirons dans la Section 2.1.3.

Un calcul formel et une intégration par partie conduit à

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} n\log n\,dx &= -4\textstyle\int_{\mathbb{R}^2}\left|\nabla\sqrt{n}\right|^2\,dx+\chi\textstyle\int_{\mathbb{R}^2}\nabla n\cdot\nabla c\,dx\\ &= -4\textstyle\int_{\mathbb{R}^2}\left|\nabla\sqrt{n}\right|^2\,dx+\chi\textstyle\int_{\mathbb{R}^2} n^2\,dx\,.\end{aligned}\tag{11}$$

Ceci met en évidence la compétition entre le terme de diffusion, $-4\int_{\mathbb{R}^2}\left|\nabla\sqrt{n}\right|^2\,dx$, qui décroît l'entropie et le terme de dérive qui fait le contraire. L'entropie est décroissante si $\chi M\leq 4C_{\mathrm{GNS}}^{-2}$, où $C_{\mathrm{GNS}}=C_{\mathrm{GNS}}^{(4)}$ est la meilleure constante avec $p=4$ dans l'inégalité de Gagliardo-Nirenberg-Sobolev :

$$\|u\|^2_{L^p(\mathbb{R}^2)}\leq C_{\mathrm{GNS}}^{(p)}\,\|\nabla u\|^{2-4/p}_{L^2(\mathbb{R}^2)}\,\|u\|^{4/p}_{L^2(\mathbb{R}^2)}\quad\forall\,u\in H^1(\mathbb{R}^2)\,,\quad\forall\,p\in[2,\infty)\,.\tag{12}$$

La valeur explicite de C_{GNS} n'est pas connue mais elle peut être approximée numériquement (voir [Wei83]). L'entropie est décroissante si $\chi M\leq 4C_{\mathrm{GNS}}^{-2}\approx 1.862...\times(4\pi)<8\pi$. Ce résultat est due à W. Jäger et S. Luckhaus (voir [JL92]).

2.1.3 Estimation à priori par la méthode d'énergie libre

L'énergie libre est définie par

$$F[n(\cdot,t)]:=\int_{\mathbb{R}^2} n\log n\,dx-\frac{\chi}{2}\int_{\mathbb{R}^2} n\,c\,dx\,.$$

Lemme 17 (Décroissance de l'énergie libre, [DP04]). *On considère une solution positive n continues à valeur* $L^1(\mathbb{R}^2)$ *de (8). Si* $n(1+|x|^2)$ *et* $n\log n$ *sont bornés dans* $L^\infty_{\mathrm{loc}}(\mathbb{R}^+,L^1(\mathbb{R}^2))$, *et si* $\nabla\sqrt{n}$ *est borné dans* $L^1_{\mathrm{loc}}(\mathbb{R}^+,L^2(\mathbb{R}^2))$ *et* ∇c *dans* $L^\infty_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^2)$, *alors*

$$\frac{d}{dt}F[n(\cdot,t)]=-\int_{\mathbb{R}^2} n\left|\nabla\left(\log n\right)-\chi\nabla c\right|^2\,dx\,.$$

Rappelons l'inégalité de Hardy-Littlewood-Sobolev logarithmique :

Lemme 18 (Inégalité de Hardy-Littlewood-Sobolev logarithmique, [CL92, Bec93]). *Si f est une fonction positive dans $L^1(\mathbb{R}^2)$ telle que $f\log f$ et $f\log(1+|x|^2)$ sont dans $L^1(\mathbb{R}^2)$. Si $\int_{\mathbb{R}^2} f\,dx = M$, alors*

$$\int_{\mathbb{R}^2} f\log f\ dx + \frac{2}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)f(y)\log|x-y|\ dx\,dy \geq -C(M)\ , \tag{13}$$

avec $C(M) := M(1+\log\pi - \log M)$.

Cette inégalité permet d'obtenir une borne inférieure sur l'énergie libre de la façon suivante :

$$\begin{aligned}(1-\theta)\int_{\mathbb{R}^2} n(x,t)\log n(x,t)\ dx + \theta\Big[\int_{\mathbb{R}^2} n(x,t)\log n(x,t)\ dx \\ + \frac{\chi}{4\pi\theta}\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x,t)\,n(y,t)\ \log|x-y|\ dx\,dy\Big]\end{aligned} \tag{14}$$

est bornée par $F[n_0]$. On choisit $\theta = \chi\,M/(8\pi)$ et on applique l'inégalité de Hardy-Littlewood-Sobolev logarithmique (Lemme 18) :

$$(1-\theta)\int_{\mathbb{R}^2} n(x,t)\log n(x,t)\ dx - \theta\,C(M) \leq F[n_0]\ .$$

Si $\chi\,M < 8\pi$, alors $\theta < 1$ et

$$\int_{\mathbb{R}^2} n(x,t)\log n(x,t)\ dx \leq \frac{F[n_0] + \theta\,C(M)}{1-\theta}\ .$$

Ceci démontre que $\int n\log n$ est bornée supérieurement. Pour la borne inférieure on procède de la façon suivante,

$$\frac{1}{1+t}\int_{\mathbb{R}^2} |x|^2\,n(x,t)\ dx \leq K \quad \forall\,t>0\ .$$

On en déduit que

$$\int_{\mathbb{R}^2} n(x,t)\log n(x,t) \geq \int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)}\log\left(\frac{n(x,t)}{\mu(x,t)}\right)\mu(x,t)\,dx - M\ \log[\pi(1+t)] - K$$

avec $\mu(x,t) := \frac{1}{\pi(1+t)}\ \exp\left(-\frac{|x|^2}{1+t}\right)$. Par l'inégalité de Jensen on obtient,

$$\int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)}\log\left(\frac{n(x,t)}{\mu(x,t)}\right)\mu(x,t)\,dx \geq X\ \log X \quad \text{où} \quad X = \int_{\mathbb{R}^2}\frac{n(x,t)}{\mu(x,t)}\,\mu(x,t)\,dx = M\ .$$

Par conséquent, l'entropie, $\int_{\mathbb{R}^2} n\log n$, est bornée inférieurement. Ces estimations sont toutefois formelles dans la mesure où l'on suppose que les solutions de (8) sont régulières.

2.2 Principaux résultats

Le premier résultat est un résultat d'existence et de régularité des solutions lorsque la condition $\chi\,M < 8\,\pi$ est vérifiée.

Théorème 19 (Existence pour $\chi M < 8\pi$, [BDP05]). *Sous l'hypothèse (10), si $\chi M < 8\pi$, alors le système de Keller-Segel (8) a une solution n globale en temps correspondant à une donnée initiale n_0. La solution n est telle que pour tout t positif*

$$(1 + |x|^2 + |\log n|)\, n \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+, L^1(\mathbb{R}^2)) \quad et \quad \int_0^t \int_{\mathbb{R}^2} n|\nabla \log n - \chi \nabla c|\, dx\, dt < \infty\,.$$

De plus n est borné dans $L^\infty_{\mathrm{loc}}((\varepsilon, \infty), L^p(\mathbb{R}^2))$ pour tout p dans $(1, \infty)$ et tout ε positif. Et pour tout t positif on a :

$$F[n(\cdot,t)] + \int_0^t \left(\int_{\mathbb{R}^2} n\, |\nabla (\log n) - \chi \nabla c|^2 \; dx \right) ds \leq F[n_0]\,. \tag{15}$$

Les estimations à priori qui permettent de montrer le résultat d'existence était dans [DP04]. Il s'agit de montrer qu'elles permettent de régulariser le problème et de passer à la limite. Les estimations de régularité sont nouvelles et reposent sur une méthode d'hypercontractivité.

Le second résultat est consacré, dans le cas $\chi M < 8\pi$, au comportement en temps long des solutions, aux "asymptotiques intermédiaires" et à la convergence dans les variables auto-similaires vers des solutions auto-similaires définies par

$$u_\infty = M \, \frac{e^{\chi\, v_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{\chi\, v_\infty - |x|^2/2}\, dx} = -\Delta v_\infty\,, \quad \text{avec} \quad v_\infty = -\frac{1}{2\pi} (\log |\cdot|) * u_\infty\,, \tag{16}$$

Théorème 20 (Comportement en temps long, [BDP05]). *Sous l'hypothèse (10), supposons que $\chi M < 8\pi$ alors il existe une solution de (16) telle que*

$$\lim_{t\to\infty} \|n(\cdot,t) - n_\infty(\cdot,t)\|_{L^1(\mathbb{R}^2)} = 0 \quad et \quad \lim_{t\to\infty} \|\nabla c(\cdot,t) - \nabla c_\infty(\cdot,t)\|_{L^2(\mathbb{R}^2)} = 0$$

où n_∞ et c_∞ sont définie par

$$n_\infty(x,t) := \frac{1}{R^2(t)}\, u_\infty \left(\log(R(t)), \frac{x}{R(t)} \right) \qquad et \quad c_\infty(x,t) := v_\infty \left(\log(R(t)), \frac{x}{R(t)} \right)\,,$$

avec $R(t) := \sqrt{1+2t}$.

2.3 Méthode de preuve

2.3.1 Existence au sens des distributions

Le problème, lorsque l'on considère l'équation (8), est l'absence de borne et la singularité en 0 du logarithme dans (9). Le problème d'absence de borne sur le noyau de convolution n'est pas difficile à gérer puisque le moment d'ordre deux est borné dans $L^\infty(0,T; L^1(\mathbb{R}^2))$ (Lemme 16) donc

$$0 \leq \iint_{|x-y|>R} \log|x-y|\, n(x,t)\, n(y,t)\; dx\, dy \leq \frac{2\, \log\, R}{R^2}\, M \int_{\mathbb{R}^2} |x|^2\, n(x,t)\; dx < \infty\,.$$

Reste le problème de la singularité en 0. Il y a plusieurs façons de couper la singularité, L. Corrias, B. Perthame et H. Zaag, dans [CPZ04], avaient convolé le noyau pour le régulariser. Ici, on tronque la singularité de la façon suivante

$$\mathcal{K}^\varepsilon(x) := \mathcal{K}^1 \left(\frac{x}{\varepsilon} \right)$$

où $\mathcal{K}^1$ est une fonction radiale croissante qui satisfait

$$\begin{cases} \mathcal{K}^1(z) = -\dfrac{1}{2\pi} \log |z| & \text{if } |z| \geq 2\,, \\ \mathcal{K}^1(z) = 0 & \text{if } |z| \leq \frac{1}{2}\,. \end{cases}$$

Le système sur $(n^\varepsilon, c^\varepsilon)$ s'écrit

$$\begin{cases} \dfrac{\partial n^\varepsilon}{\partial t} = \Delta n^\varepsilon - \chi \nabla \cdot (n^\varepsilon \nabla c^\varepsilon) \\ c^\varepsilon = \mathcal{K}^\varepsilon * n^\varepsilon \end{cases} \qquad (x,t) \in \mathbb{R}^2 \times (0,\infty)\,. \tag{17}$$

Soit $V := \{v \in H^1(\mathbb{R}^2) \ :\ \sqrt{|x|}\, v \in L^2(\mathbb{R}^2)\}$ et V' son dual. Par des résultats classiques sur les opérateurs paraboliques on prouve que si n_0 est borné dans $L^2(\mathbb{R}^2)$ alors pour tout T positif il existe une fonction n^ε, bornée dans $C(0,T;L^2(\mathbb{R}^2)) \cap \{v \in L^2(0,T;V)\ :\ v_t \in L^2(0,T;V')\}$ solution de (17) dans $L^2([0,T] \times \mathbb{R}^2)$ (voir Proposition B.II.9).

Pour passer à la limite dans (17), dans le cas où $\chi\, M < 8\,\pi$, nous avons besoin d'une série d'estimations à priori.

(i) La décroissance de l'énergie libre (Lemme 17) est valable pour le système (17), donc les estimations à priori obtenues sur la solution de (8) présentées dans la Section 2.1 sont vraies pour (17). Donc l'entropie $\int_{\mathbb{R}^2} n^\varepsilon \log n^\varepsilon$, est bornée dans $[0,T]$. Le Lemme 16 assure une borne sur le moment d'ordre deux de n^ε. Il est classique alors d'obtenir une borne $L^\infty(0,T;L^1(\mathbb{R}^2))$ sur n^ε log n^ε (voir Lemme B.II.8).

(ii) L'argument (11) assure que

$$\frac{d}{dt}\int_{\mathbb{R}^2} n^\varepsilon \log n^\varepsilon \leq -4 \int_{\mathbb{R}^2} |\nabla \sqrt{n^\varepsilon}|^2 + \chi \int_{\mathbb{R}^2} (n^\varepsilon)^2 \leq \left(-4 + \chi\, M\, C^2_{\mathrm{GNS}}\right) \int_{\mathbb{R}^2} |\nabla \sqrt{n^\varepsilon}|^2\,.$$

Donc une borne $L^\infty(0,T;L^1(\mathbb{R}^2))$ sur n^ε log n^ε assure une borne dans $L^2(0,T;\mathbb{R}^2)$ de $\nabla\sqrt{n^\varepsilon}$ si M est assez petit. Sinon il faut découper les intégrales pour obtenir le résultat. En fait ici la principale difficulté technique réside dans le fait que l'on a pas $\Delta c = -n$ pour le problème approché. Cependant $-\Delta \mathcal{K}^\varepsilon$ tend vers un Dirac et on prouve que le résultat reste vrai pour ε suffisamment petit.

(iii) La dernière estimation porte sur une borne de $\sqrt{n^\varepsilon}\, \nabla c^\varepsilon$ dans $L^2(0,T,\mathbb{R}^2)$. Elle s'obtient grâce à

$$\frac{d}{dt}\int_{\mathbb{R}^2} \frac{1}{2}\, n^\varepsilon\, c^\varepsilon\, dx = \int_{\mathbb{R}^2} c^\varepsilon\, \nabla\left(\Delta n^\varepsilon - \chi\, \nabla \cdot (n^\varepsilon\, \nabla c^\varepsilon)\right) dx = \int_{\mathbb{R}^2} n^\varepsilon\, \Delta c^\varepsilon\, dx + \chi \int_{\mathbb{R}^2} n^\varepsilon\, |\nabla c^\varepsilon|^2\, dx\,.$$

Par l'inégalité (12) et l'estimation (ii), n^ε est bornée dans $L^2(0,T,\mathbb{R}^2)$. On peut contrôler Δc^ε parce que c^ε est la convolée de n^ε borné dans $L^2(0,T,\mathbb{R}^2)$, avec la fonction régulière $\mathcal{K}^\varepsilon$. Ceci permet en définitive de prouver que

$$\int_0^T \int_{\mathbb{R}^2} n^\varepsilon\, |\nabla c^\varepsilon|^2\, dx\, dt \leq \frac{1}{2\,\chi} \left| \int_{\mathbb{R}^2} n^\varepsilon\, c^\varepsilon\, dx - \int_{\mathbb{R}^2} n_0\, \mathcal{K}^\varepsilon * n_0\, dx \right| + \frac{1}{\chi} \int_0^T \int_{\mathbb{R}^2} n^\varepsilon\, |\Delta c^\varepsilon|\, dx\,.$$

Pour montrer que les solutions du système (17) sont solutions, au sens des distributions, de (8), il reste à prouver que $n^\varepsilon \nabla c^\varepsilon \rightharpoonup n\nabla c$ au sens des distributions. Remarquons que

$$\begin{aligned} \nabla c^{\varepsilon_k} - \nabla c = \ & -\ \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\, \left(n^{\varepsilon_k}(y,t) - n(y,t)\right)\, dy \\ & +\ \int_{|x-y|\leq 2\varepsilon_k} \left(\frac{1}{\varepsilon_k}\, \nabla \mathcal{K}^1 \left(\frac{x-y}{\varepsilon_k}\right) + \frac{x-y}{2\pi\, |x-y|^2}\right) n^{\varepsilon_k}(y,t)\, dy\,. \end{aligned}$$

Or $(n^\varepsilon(t))_{n\in\mathbb{N}}$ converge faiblement dans $L^p(\mathbb{R}^2)$ pour tout p dans $(1,\infty)$. Donc $n^\varepsilon\nabla c^\varepsilon$ converge au sens des distributions vers $n\nabla c$.

2.3.2 Hypercontractivité

Nous allons maintenant montrer que l'on a, en fait, une meilleure régularité des solutions grâce à l'hypercontractivité. Afin de rendre plus simple l'introduction de cette notion, nous allons appliquer la méthode du gain de régularité par hypercontractivité à l'opérateur de la chaleur. Il y a évidemment des méthodes bien plus directes pour obtenir la régularité des solutions de l'équation de la chaleur mais l'intérêt de cette méthode est qu'elle s'étend au cas non-linéaire dès que l'on a suffisamment d'estimations à priori.

On définit

$$F(t) := \int_{\mathbb{R}^2} n(x,t)^{p(t)}\,dx\ ,$$

où p' est positif. Et on calcule

$$F'\,F^{p-1} = \frac{p}{p'}\int_{\mathbb{R}^2} n^p \log\left(\frac{n^p}{\int_{\mathbb{R}^2} n^p}\right) + \int_{\mathbb{R}^2} n_t\, n^{p-1}\ .$$

Dans la cas où $n_t = \Delta n$ on obtient

$$F'\,F^{p-1} = \frac{p}{p'}\int_{\mathbb{R}^2} n^p \log\left(\frac{n^p}{\int_{\mathbb{R}^2} n^p}\right) - 4\frac{p-1}{p^2}\int_{\mathbb{R}^2} |\nabla n^{p/2}|^2\ .$$

On utilise alors l'inégalité de Sobolev logarithmique [Gro75]

$$\int_{\mathbb{R}^2} v^2\,dx \log\left(\frac{v^2}{\int_{\mathbb{R}^2} v^2\,dx}\right)\,dx \le 2\,\sigma\int_{-R^2} |\nabla v|^2\,dx - (2+\log(2\,\pi\,\sigma))\int_{\mathbb{R}^2} v^2\,dx\,,$$

pour tout σ positif, à $v = n^{p/2}$. On obtient

$$F'\,F^{p-1} = -\frac{p'}{p^2}\,(2+\log(2\,\pi\,\sigma))\int_{\mathbb{R}^2} n^p + \frac{-4\,(p-1)+2\,\sigma\,p'}{p^2}\int_{\mathbb{R}^2} |\nabla n^{p/2}|^2\ .$$

Pour le choix $p(t) := \frac{p-1}{t^*}\,t+1$ (qui n'est pas optimal) et $\sigma = 2\,t$ on obtient

$$F'\,F^{p-1} \le -\frac{p'}{p^2}\,[2+\log(4\,\pi\,t)]\,F^p\ .$$

Donc F est bornée. On obtient ainsi une borne sur la norme $L^p(\mathbb{R}^2)$ de $n(\cdot,t)$.

Dans le cas du système de Keller-Segel, un autre terme apparaît qu'il faut contrôler par l'inégalité de Gagliardo-Nirenberg-Sobolev. De plus, on est amené à diviser les intégrales pour aller au delà du cas qui est estimé par l'inégalité (12). On prouve ainsi que la solution n au sens des distributions du système (8) est bornée dans $L^\infty(t^*,T;L^p(\mathbb{R}^2))$ pour un certain t^* positif.

2.3.3 Régularité des solutions : preuve du Théorème 19

Il s'agit maintenant de montrer que la solution vérifie (15) Pour montrer cette inégalité d'énergie nous allons utiliser que l'ensemble $\{u \in H^1 \cap L^1_+(\mathbb{R}^2)\ :\ |x|^{\frac{\varepsilon}{1+\varepsilon}}\,u \in L^2(\mathbb{R}^2)\}$

s'injecte de façon compact dans $L^2(\mathbb{R}^2)$: par l'hypercontractivité, n est bornée dans $L^2([t^*,T]\times\mathbb{R}^2)$. La borne dans $L^2([t^*,T]\times\mathbb{R}^2)$ sur ∇n s'obtient grâce au calcul suivant :

$$\frac{d}{dt}\int_{\mathbb{R}^2} n^2\,dx = -2\int_{\mathbb{R}^2}|\nabla n|^2\,dx + 2\chi\int_{\mathbb{R}^2}\nabla n\cdot n\,\nabla c\,dx\,.$$

Ce qui prouve que $X := \|\nabla n\|_{L^2((t^*,T)\times\mathbb{R}^2)}$ vérifie

$$2\,X^2 - 2\,\chi\,\|n\,\nabla c\|_{L^\infty(t^*,T;L^2(\mathbb{R}^2))}\,X \le \|n\|^2_{L^\infty(t^*,T;L^2(\mathbb{R}^2))} + \|n(\cdot,t^*)\|^2_{L^2(\mathbb{R}^2)}\,.$$

Ceci implique que X est borné à condition de prouver que $\|n\,\nabla c\|_{L^\infty(t^*,T;L^2(\mathbb{R}^2))}$ est borné. Ceci se montre par l'inégalité de Hölder, ainsi pour tout t dans (t^*,T)

$$\|n(\cdot,t)\,\nabla c(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le \|n(\cdot,t)\|_{2+\varepsilon}\|\nabla c(\cdot,t)\|_{2+4/\varepsilon} \le C\,\|n(\cdot,t)\|_{2+\varepsilon}\|n(\cdot,t)\|_{2-\varepsilon/(1+\varepsilon)}\,.$$

La borne sur le moment d'ordre deux est obtenue grâce à

$$\int_{\mathbb{R}^2}|x|^{\frac{2\,\varepsilon}{1+\varepsilon}}\,n^2\,dx = \int_{\mathbb{R}^2}(n\,|x|^2)^{\frac{\varepsilon}{1+\varepsilon}}\cdot n^{\frac{2+\varepsilon}{1+\varepsilon}}\,dx \le \left(\int_{\mathbb{R}^2} n\,|x|^2\,dx\right)^{\frac{\varepsilon}{1+\varepsilon}}\left(\int_{\mathbb{R}^2} n^{2+\varepsilon}\,dx\right)^{\frac{1}{1+\varepsilon}}\,.$$

La fonction limite n est donc dans un compact de $L^2([t^*,T]\times\mathbb{R}^2)$.

Par semi-continuité inférieure on a

$$\int_{t^*}^T\int_{\mathbb{R}^2} n\,|\nabla c|^2\,dx\,dt \le \liminf_{\varepsilon\to 0}\int_{t^*}^T\int_{\mathbb{R}^2} n^\varepsilon\,|\nabla c^\varepsilon|^2\,dx\,dt\,.$$

La fonction $n \mapsto \int_{\mathbb{R}^2}|\nabla\sqrt{n}|^2\,dx$ est convexe donc

$$\int_{t^*}^T\int_{\mathbb{R}^2}|\nabla\sqrt{n}|^2\,dx\,dt \le \liminf_{\varepsilon\to 0}\int_{t^*}^T\int_{\mathbb{R}^2}|\nabla\sqrt{n^\varepsilon}|^2\,dx\,dt\,.$$

Enfin par le résultat ci-dessus n est dans un compact de $L^2([t^*,T]\times\mathbb{R}^2)$ donc

$$\int_{t^*}^T\int_{\mathbb{R}^2} n^2\,dx\,dt = \liminf_{\varepsilon\to 0}\int_{t^*}^T\int_{\mathbb{R}^2}|n^\varepsilon|^2\,dx\,dt\,.$$

Ce qui prouve l'estimation sur l'énergie libre en utilisant

$$\begin{aligned}&\int_{t^*}^T\int_{\mathbb{R}^2} n^\varepsilon\,|\nabla(\log n^\varepsilon) - \chi\nabla c^\varepsilon|^2\,dx\,dt\\ &\qquad = 4\int_{t^*}^T\int_{\mathbb{R}^2}|\nabla\sqrt{n^\varepsilon}|^2\,dx\,dt - 2\chi\int_{t^*}^T\int_{\mathbb{R}^2}|n^\varepsilon|^2\,dx\,dt + \chi^2\int_{t^*}^T\int_{\mathbb{R}^2} n^\varepsilon\,|\nabla c^\varepsilon|^2\,dx\,dt\,.\end{aligned}$$

On a alors

$$F[n(\cdot,t)] + \int_{t_0}^t\left(\int_{\mathbb{R}^2} n\,|\nabla\,(\log n) - \chi\nabla c|^2\,dx\right)ds \le F[n(\cdot,t_0)]\,.$$

Mais comme $s \mapsto \int_{\mathbb{R}^2} n^\varepsilon(\cdot,s)\,|\nabla(\log n^\varepsilon(\cdot,s)) - \chi\nabla c^\varepsilon(\cdot,s)|^2\,dx\,dt$ est intégrable sur $(0,T)$ on peut passer à la limite lorsque t^* tend vers 0. La convexité de $n\mapsto n\,\log n$ permet de vérifier que $\lim_{t_0\to 0_+} F[n(\cdot,t_0)] \le F[n_0]$. Ce qui prouve (15).

2.3.4 Asymptotiques intermédiaires

La méthode d'entropie-production d'entropie a été utilisée avec beaucoup de succès pour l'étude du comportement en temps long de solutions d'équations diffusives. Le lecteur intéressé pourra se référer à [ACD$^+$04] (voir [AGS05] pour des aspects liés au transport de masse et la distance de Wasserstein).

Afin de donner l'idée générale de la technique nous nous limiterons ici à l'exemple de l'équation de la chaleur sur $\mathbb{R}^{d+1}$ avec $u(\cdot,0) = u_0 \in L^2(\mathbb{R}^d)$.

On introduit R, τ et k tels que :

$$n(x,t) =: R(t)^{-k} u(y(t),\tau(t)) \quad \text{où} \quad y(t) := \frac{x}{R(t)} \,.$$

On a alors

$$n_t(x,t) = -k\, R'(t)\, R(t)^{-k-1}\, u(y,\tau) - R'(t)\, R(t)^{-k-1}\, y\, \nabla u(y,\tau) + R(t)^{-k}\, \tau'(t)\, u_\tau(y,\tau)$$
$$\Delta n(x,t) = R(t)^{-k-2}\, \Delta u(y,\tau)\,.$$

Pour obtenir une équation autonome on résout

$$\frac{R'(t)}{R(t)^{k+1}} = \frac{R'(t)}{R(t)^{k+1}} = \frac{\tau'(t)}{R(t)^{k+1}} = \frac{1}{R(t)^{k+2}}\,.$$

ce qui donne

$$R(t) = \pm\sqrt{t + R^2(0)} \quad \text{et} \quad \tau(t) = \log\sqrt{t} + \tau(0)\,.$$

Pour $k = d$, l'équation devient une équation de Fokker-Planck :

$$u_\tau(y,\tau) = \nabla\cdot\left[\nabla u(y,\tau) + y\, u(y,\tau)\right]\,.$$

On choisit $R(0) = 1$ et $\tau(0) = 0$. Ainsi $u(y,0) = n(0,x) = n_0$ la condition initiale est préservée. On définit :

$$u_\infty(y) = M\, \mu(y) \quad \text{où} \quad \mu(y) := \frac{e^{-|y|^2/2}}{(2\pi)^{d/2}}\,.$$

et l'énergie libre relative à u_∞ :

$$\Sigma(\tau) := \int_{\mathbb{R}^n} \sigma\left(\frac{u(y,\tau)}{u_\infty(y)}\right) d\mu(y)$$

où $d\mu(t) := u_\infty(y)\,dy$ et $\sigma(t) := t\log(t) - (t-1)$ par exemple (mais on peut prendre $(t-1)^2$ ou $t^p - 1 - p\,(t-1)$ avec $1 < p < 2$. Grâce à l'inégalité de Jensen, Σ est bornée inférieurement par 0. De plus Σ est décroissante : on calcule l'information de Fisher

$$-I\left(\frac{u(\tau)}{u_\infty}\right) := \frac{d}{d\tau}\Sigma(\tau) = -\int_{\mathbb{R}^n} \sigma''\left(\frac{u(y,\tau)}{u_\infty(y)}\right)\left|\nabla\left(\frac{u(y,\tau)}{u_\infty(y)}\right)\right|^2 u_\infty(y)\,dy \le 0\,.$$

On a même une décroissance exponentielle de Σ vers 0 : d'une part, on montre que $I_\tau(\tau) - \Sigma_\tau(\tau) \le 0$. On applique le lemme de Gronwall à $I_\tau(\tau) \le 2I(\tau)$ qui implique $0 \le I(\tau) \le I(0)\,e^{-2\tau}$. Donc Σ converge vers $\Sigma_{\lim}$ tel que $d_\tau\, \Sigma_{\lim} = I(\tau) \le I(0)\,e^{-2\tau}$. En conséquence $d_\tau\, \Sigma_{\lim} = 0$. Ce qui implique $\nabla(u/u_\infty) = 0$ et donc $u = \lambda u_\infty$. Mais la contrainte $\int u = \int u_\infty$ implique $\lambda = 1$. Donc $\lim_{\tau\to\infty} \Sigma(\tau) = 0$. De plus $I - 2\Sigma$ est décroissante en τ donc $I(\tau) - 2\Sigma(\tau) \ge \lim_{\tau\to\infty} I(\tau) - \Sigma(\tau) = 0$. Donc $I(\tau) \ge 2\Sigma(\tau)$. Cette décroissance donne la vitesse de convergence de u vers u_∞. Ceci permet d'obtenir une borne sur la norme L^1 grâce à l'inégalité de Csiszar-Kullback : Si σ est une de celles énoncées ci-dessus en (iii). Posons $K_1 := \inf[t\,\sigma''(t)] > 0$ et $K_2 := \inf_{t>0,\,\theta\in[0,1]} \sigma''(1+\theta t)(1+t)$. Si $K := \frac{1}{4}\min\{K_1, K_2\}$ alors

$$\frac{K}{\max\{\int_\Omega f\; d\mu, \int_\Omega g\; d\mu\}}\|f-g\|^2_{L^1(d\mu,\Omega)} \leq \int_\Omega \sigma\left(\frac{f}{g}\right) g\, d\mu\ .$$

En particulier si on a une vitesse de convergence de Σ vers 0 on a une vitesse de convergence de la solution vers la solution limite. On défait le changement de variable pour obtenir une vitesse de convergence de $n - n_\infty$ vers 0 en norme L^1.

Nous adaptons cette méthode à notre problème pour obtenir les asymptotiques intermédiaires ce qui est possible dès que (15) est vérifiée pour montrer que $\|n - n_\infty\|_{L^1}$ converge vers 0, sans toutefois obtenir le taux.

2.4 Questions ouvertes

Les méthodes utilisées ici devraient s'appliquer dans le cadre plus général où le chemo-attractant est donné par une équation du type

$$\Delta c = n - \beta\, c\ .$$

Ce modèle apparaît lorsque le chemo-attractant s'évapore. Le processus est plus stable que le processus décrit par (8). En conséquence, il est encore vrai que si $\chi\, M < 8\,\pi$ les solutions du système existent en temps long. Mais le Lemme 16 ne s'applique pas pour montrer l'explosion si $\chi\, M > 8\,\pi$. En fait, la méthode utilisées par L. Corrias, B. Perthame et H. Zaag dans [CPZ04] permet de montrer que si le moment d'ordre deux de la densité est suffisamment concentré au temps initial, le modèle explose en temps fini. Il est possible de prouver qu'il existe des densités initiales plus grandes que $8\,\pi$ telles que le système n'explose pas en temps fini. La situation ici est donc beaucoup moins claire que dans le système étudié dans ce mémoire. Il n'y a plus de masse critique en deçà de laquelle les solutions existent en temps long et au delà de laquelle les solutions explosent en temps fini.

Cette remarque s'applique aussi au système de Keller-Segel complet dans lequel la concentration de chemo-attractant vérifie une équation parabolique du type

$$c_t = \Delta c - n + \beta\, c\ .$$

Pour ce système l'énergie libre définie par H. Gajewski et K. Zacharias dans [GZ98] pour le cas d'un domaine borné est aussi valable dans tout l'espace. Cette énergie est bornée inférieurement par l'énergie libre du problème de Keller-Segel (8).

L'inégalité de Hardy-Littlewood-Sobolev logarithmique dans sa version optimale est utilisée à plusieurs reprises dans notre étude. Dans le cas d'un domaine borné on utilise plutôt l'inégalité de Moser-Trudinger (voir [Suz05]). Il serait intéressant de comprendre quel type de dualité il existe entre ses deux inégalités, dans le cas de $\mathbb{R}^2$ avec des poids correspondants aux moments d'ordre deux.

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Première partie

Problème de l'obstacle parabolique

A.I

Modélisation du problème de l'obstacle parabolique

Ce premier chapitre de la première partie, consacrée au problème de l'obstacle parabolique, est un chapitre introductif. Il ne présente pas de résultats destinés à la publication dans des journaux. La Section 1 est une description de modèles physiques qui conduisent aux problèmes de l'obstacle. Dans la Section 2 nous faisons un historique et une présentation d'une modélisation issue des mathématiques financières qui conduit à un problème de l'obstacle à coefficients variables.

1 Le problème de l'obstacle

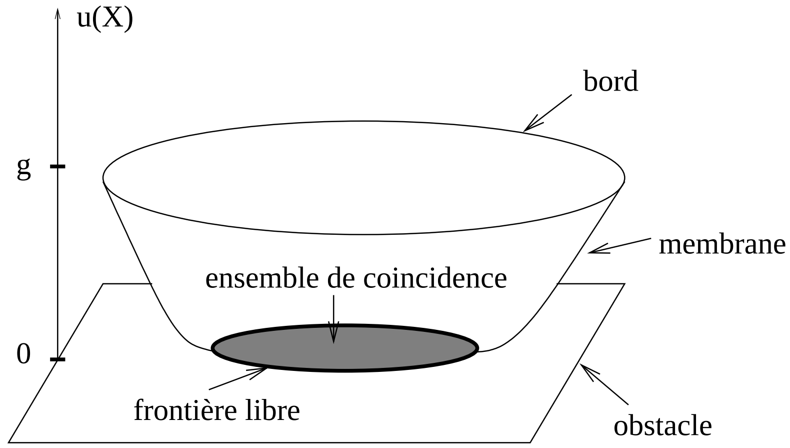
Considérons les fonctions continues solution au sens des distributions de

$$\begin{cases} \Delta u - \dfrac{\partial u}{\partial t} = \mathbb{1}_{\{u>0\}} \\ u \geq 0 \end{cases}$$

dans un domaine $D \subset \mathbb{R}^{d+1}$. C'est un problème de l'obstacle parabolique.

1.1 Le problème de l'obstacle stationnaire

Le cas le plus simple de problème de l'obstacle est celui qui modélise une membrane qui touche le sol lorsque l'on considère des fonctions qui ne dépendent que de l'espace. Lorsque la membrane est fixée en son bord, elle est soumise à l'équation $\Delta u = 1$ avec conditions de Dirichlet. Si elle touche le sol $\{x = 0\}$ elle est soumise à l'équation $\Delta u = \mathbb{1}_{\{u>0\}}$. On parle de *problème de l'obstacle elliptique*. On définit *l'ensemble de coïncidence* comme l'ensemble $\{u = 0\}$ et sa frontière $\partial\{u = 0\}$ est *la frontière libre*.



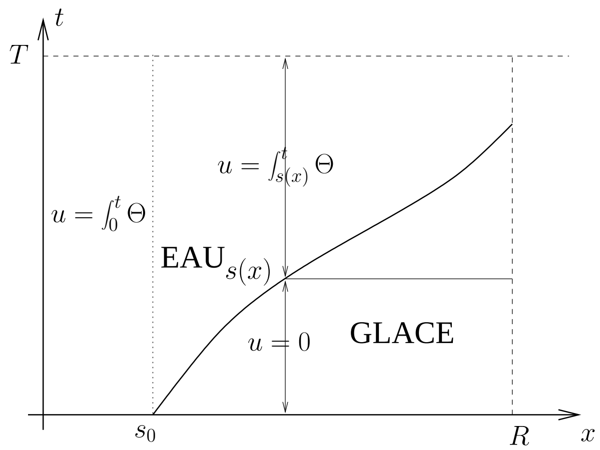
Pour plus de détails sur les problèmes de l'obstacle et les applications à la physique mathématiques on se référera à [KS80, Fri88, Rod87, Vis96]. Ces monographies contiennent de plus une importante bibliographie.

1.2 Le problème de Stefan

Le problème de Stefan à une phase est, typiquement, la description de la limite eau-glace dans un mélange où un morceau de glace, maintenu à zéro degré, est en contact avec de l'eau. C'est un *problème de l'obstacle parabolique.*

Pour donner une intuition du modèle en évitant les considérations techniques, plaçons nous en dimension un d'espace et considérons le cas très particulier où le bord du récipient est chauffé. Sous ces hypothèses la glace fond et l'interface eau-glace est une fonction strictement croissante $x \mapsto s(x)$. Il y a deux inconnues qui sont : la distribution de température de l'eau Θ (fonction de l'espace et du temps) et la frontière libre décrite, sous notre hypothèse, par $x \mapsto s(x)$.

On se donne $R > 0$, $T > 0$ et $s_0 \in [0, R]$. Au temps $t = 0$, la région occupée par la glace est décrite par $\{x : s_0 \leq x \leq R\}$.



Soit $h > 0 \in L^2([0, s_0])$ et $g \geq 0 \in L^2([0, T])$. La température initiale est donnée par :

$$\Theta(x, 0) = h(x) \text{ sur } 0 \leq x < s_0 \quad \text{et} \quad \Theta(x, 0) = 0 \text{ sur } s_0 \leq x \leq R\,,$$

et la température en $x = 0$ est donnée par :

$$\Theta(0,t) = g(t) > 0 \text{ sur } 0 \le t \le T \ ,$$

où g est la contribution positive de température. Le côté $\{x = R\}$ est laissé adiabatique. Le long de la frontière libre on suppose que la température est continue *i.e.* $\Theta(x, s(x)) = 0$ et que l'énergie est conservée au sens où $\Theta_x(x, s(x))s'(x) = -k$ où k est une constante positive dite *température de fusion*. La température Θ est soumise à l'équation de la chaleur dans l'eau (pour le lien entre le mouvement des particules et l'équation de la chaleur voir Section 3 de l'appendice).

On fait l'hypothèse que Θ et s sont des fonctions régulières. On définit u, pour tout (x,t) de $[0,R] \times [0,T]$ par

$$u(x,t) = \int_0^t \Theta(x,\tau)\, d\tau \ .$$

Avec cette définition, et en utilisant que Θ satisfait l'équation de la chaleur dans l'eau, on obtient dans $(0, s_0) \times (0,T)$:

$$u_{xx}(x,t) = \int_0^t \Theta_{xx}(x,\tau)\, d\tau = \int_0^t \Theta_\tau(x,\tau)\, d\tau = \Theta(x,t) - \Theta(x,0) = u_t(x,t) - h(x) \ .$$

De même dans $(s_0, R) \times (s(x), T)$ on a

$$u_x(x,t) = \int_{s(x)}^t \Theta_x(x,\tau)\, d\tau - s'(x)\,\Theta(x, s(x)) = \int_{s(x)}^t \Theta_x(x,\tau)\, d\tau$$

car par continuité de la température le long de la frontière libre. Et

$$u_{xx}(x,t) = \int_{s(x)}^t \Theta_{xx}(x,\tau)\, d\tau - s'(x)\Theta_x(x,s(x)) = \int_{s(x)}^t \Theta_{xx}(x,\tau)\, d\tau + k \ ,$$

par conservation de l'énergie le long de la frontière libre. Finalement dans $(s_0, R)\times(s(x), T)$ on a

$$u_{xx}(x,t) = \int_{s(x)}^t \Theta_\tau(x,\tau)\, d\tau + k = u_t(x,t) + k \ .$$

On définit

$$f(x) \quad := \quad \begin{cases} h(x) & \text{si } \ 0 \le x < s_0 \\ -k & \text{si } \ s_0 \le x \le R \ . \end{cases}$$

Il y a deux alternatives : ou bien $u(x,t) = 0$ et alors $\Theta(x,t) = 0$ sur $\{x\} \times [0,t]$ donc $u_{xx}(x,t) - u_t(x,t) = 0$; ou bien $u(x,t) > 0$ et alors le calcul ci-dessus montre que $u_{xx}(x,t) - u_t(x,t) = f(x,t)$. Finalement on peut écrire $u_{xx}(x,t) - u_t(x,t) = f(x,t)\, \mathbb{1}_{\{u>0\}}(x,t)$ et on obtient

$$\begin{cases} u_{xx}(x,t) - u_t(x,t) = f(x,t)\, \mathbb{1}_{\{u>0\}}(x,t) & \text{dans } [0,R] \times [0,T] \\ u(x,t) \ge 0 & \text{dans } [0,R] \times [0,T] \\ u_t(x,t) \ge 0 & \text{dans } [0,R] \times [0,T] \\ u(0,t) = \int_0^t g(\tau)\, d\tau & \text{dans } [0,T] \\ u(R,t) = 0 & \text{dans } [0,T] \\ u(x,0) = 0 & \text{dans } [0,R] \ . \end{cases}$$

Ce modèle est très simplifié. On peut, par exemple, se placer dans le cas où il n'y a pas de condition de signe sur u_t. C'est le *problème de Stefan classique à une phase.* On peut aussi considérer le cas où l'opérateur de la chaleur est remplacé par une opérateur parabolique à coefficients variables. On parle alors de *problème de l'obstacle parabolique à coefficients variables uni-dimensionnel.* Ce type de modèle apparaît d'ailleurs naturellement dans un modèle issu des mathématiques financières (voir Section 2). Enfin un modèle plus complexe, ne fait pas d'hypothèse d'uniformité de la température sur l'obstacle $\{u = 0\}$. On peut alors considérer que la température du glaçon est elle-même régit par une équation de la chaleur. On parle alors de *problème de Stefan à deux phases.* C'est un problème qui apparaît en théorie du potentiel.

2 Options américaines en finance

On considère un marché "complet" sans "opportunités d'arbitrage" (ces termes seront définis par la suite). La détermination du prix d'un outil commun sur les marchés financiers : l'option américaine, conduit à un problème de l'obstacle parabolique à coefficients variables. Nous allons donner, ici, une idée heuristique de la démarche à suivre (pour une introduction plus complète voir [LL97]).

2.1 Notion d'options et de portefeuilles

2.1.1 Le modèle de Black–Scholes

Dans sa thèse [Bac95], L. Bachelier modélise le prix des actifs sur un marché financier en faisant une analogie entre le mouvement désordonné des prix et le mouvement brownien (pour une description de mouvement brownien voir Section 3 de l'appendice). L'idée est très mal perçue par la communauté mathématiques d'alors. Seul Poincaré rapporteur de sa thèse le défendra. Il obtiendra finalement une mention honorable puis aura beaucoup de difficultés à intégrer le milieu universitaire avant de trouver un poste permanent à l'université de Besançon. Il fait pourtant la première étude mathématiques du mouvement brownien, cinq années avant celui que tout le monde considère comme le fondateur de la théorie du mouvement brownien : Albert Einstein. Aujourd'hui un séminaire parisien sur le thème des mathématiques financières porte son nom.

Il faut attendre 1973 (date de la première crise du pétrole) pour que l'idée soit acceptée par les milieux financiers grâce aux travaux de Fischer Black et Myron Scholes [BS73]. Ils supposent que le prix S_t d'un actif dit *risqué* est régi par l'équation aux dérivées stochastiques suivante :

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_t$$

où μ, le *retour moyen*, et σ, *la volatilité*, sont des réels donnés et dW_t est la mesure de Wiener (pour la construction de la mesure de Wiener, et de l'intégrale stochastique voir [LL97]). Ils obtiennent la formule de Black-Scholes (voir Section 2.2.2). En fait une année auparavant, A. James Boness, dans sa thèse de doctorat de l'université de Chicago démontrait déjà la formule dite de "Black-Scholes". Le papier [BS73] est refusé deux fois par *Journal of Political Economy* alors que c'est aujourd'hui le plus cité de la revue. En 1997, Myron Scholes et Robert Merton ont été récompensé du prix *Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel* dit "prix Nobel d'économie" pour leurs travaux relatifs à cette formule (F. Black décédé en 1995 était inéligible). Cette formule est utilisée dans les banques encore aujourd'hui[1]. Cependant ce modèle rencontre rapidement des incohérences dans sa confrontation avec la réalité des marchés. L'entreprise de spéculation financière fondée par M. Scholes et R. Merton en 1973 fera faillite en 1975. Dans [Dup97], S. Dupire se pose la question inverse de déterminer la valeur de la volatilité en lisant la variation des prix sur le marché. Il constate que la volatilité est une fonction convexe des prix. Cette constatation amène à généraliser le modèle de Black–Scholes en un modèle de Black–Scholes à *volatilité locale* où le processus des prix satisfait l'équation stochastique

$$dS_t = \mu(S_t, t)\, S_t\, dt + \sigma(S_t, t)\, S_t\, dW_t \tag{I.1}$$

où μ, le *retour moyen local*, et σ, la volatilité, sont des fonctions qui dépendent du temps et du prix de l'actif sous-jacent.

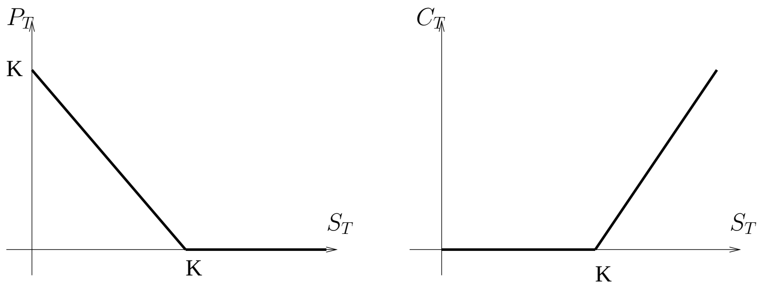
2.1.2 Définition d'une option

Une *option américaine* est un contrat passé entre un acheteur et un vendeur. Une *option d'achat* (respectivement *option de vente*) est le droit d'acheter (resp. de vendre) un bien financier à un prix, P^0, prescrit à l'avance (appelé *pay-off*) durant une période de temps et de réaliser ainsi, à tout temps t antérieur à une certaine date d'échéance appelé *maturité*, le profit suivant

$$\phi(S_t, t) := \max\{0, P^0(S_t, t) - S_t\}, \quad (\text{resp.} \quad \phi(S_t, t) := \max\{0, S_t - P^0(S_t, t)\}). \tag{I.2}$$

Nous conserverons, ici, la terminologie anglo-saxonne de pay-off puisque c'est celle invariablement adoptée par les financiers en français. De même pour les notions de "put", "call" et "strike" dans les définitions suivantes. Les exemples les plus communs d'options américaines sont les options vanilles : le *put* et le *call*. Soit K un réel appelé *strike*. Un put (respectivement call) américain sur un actif risqué dont le cours à la date t est donné par S_t est le droit d'acheter (resp. vendre) à toute date t jusqu'à la maturité T son option au prix $K - S_t$ (resp. $S_t - K$). Dans le cas du put, l'exercice de l'option permet de réaliser le bénéfice suivant :

- si $S_t > K$, le détenteur de l'option ne gagne rien en l'exerçant,
- si $S_t < K$, l'exercice de l'option permet à son détenteur de réaliser un profit égal à $S_t - K$ en vendant l'actif au prix K.



Profit au temps T pour le put : (P_T) et le call : (C_T)

[1] voir http ://www.diplomatie.gouv.fr/culture/expositions_scientifiques/maths_quot/pages/droite08.html ou www.spm.cnrs-dir.fr/actions/publications/idm/PDF2004/Gobet.pdf

2.2 Prix d'une option

2.2.1 Problème d'arrêt optimal

Le profit que l'on fera en exerçant au temps t doit être "actualisé". Le profit *actualisé* de l'exercice d'une option est :

$$\exp\left(\int_0^t r(S_\tau,\tau)\,d\tau\right)\phi(S_t,t)\ .$$

La première interprétation rigoureuse du prix de l'option Américaine en terme de "temps d'arrêt optimal" est due à A. Bensoussan ([Ben84]) et I. Karatzas ([Kar88]). Leur démarche repose sur l'introduction de la notion de "prix juste". Ils définissent le prix d'une option au temps 0 comme le supremum de ce que l'on peut espérer gagner avec son option sur l'ensemble des stratégies "admissibles". Mathématiquement cela se traduit par :

$$\sup_{\tau\in\Theta_{[0,T]}}\mathbb{E}\left[\exp\left(\int_0^t r(S_\tau,\tau)\,d\tau\right)\phi(S_t,t)\right]\ .$$

où $\Theta_{[0,T]}$ est l'ensemble des stratégies *admissibles* (les $\mathcal{F}_t$-temps d'arrêt τ à valeur dans $[0,T]$). Si on veut définir le prix de l'option en des temps différents de 0 on définit le flot stochastique $(S_\tau^{x,t})_\tau$ comme l'unique solution de

$$dS_\tau=\mu(S_\tau,\tau)\,S_\tau\,d\tau+\sigma(S_\tau,\tau)\,S_\tau\,dW_\tau$$

avec solution initiale $S_t=x$. Le prix de l'option de vente américain est alors donné par

$$\Pi(x,t):=\sup_{\tau\in\Theta_{[t,T]}}\mathbb{E}\left[\exp\left(-\int_t^\tau r(S_s^{x,t},s)ds\right)\phi(S_\tau^{x,t},\tau)\right] \tag{I.3}$$

2.2.2 Équation de Black-Scholes

Le *prix juste* de l'option est celui qui permet au vendeur de l'option de se *couvrir* c'est-à-dire d'obtenir en tout temps de quoi faire face au paiement de l'option si le possesseur de l'option venait à en réclamer l'exercice. Dans le cas d'un marché dit *complet* il existe une stratégie qui permette au vendeur de se couvrir.

On se place en dans le cas où la volatilité et le retour moyen sont constants. Dans l'ensemble où le prix de l'option est strictement plus grand que le pay-off, on forme un portefeuille, V_t, composé, au temps t, de α_t parts d'actif sans risque dont l'évolution est $dS_t^0=r\,S_t^0\,dt$ et de β_t actifs risqués S_t qui vérifient l'équation (I.1) :

$$V_t(S_t,t)=\alpha_t\,S_t^0+\beta_t\,S_t\ .$$

Par définition, le portefeuille V_t est *auto-financé* si

$$dV_t(S_t,t)=r\,V_t(S_t,t)\,dt+\beta_t\,dS_t\ .$$

Nous aurons besoin d'un lemme fondamental du calcul stochastique, qui donne la formule de différentiation. Acceptons de considérer que $\langle dW_t,dW_t\rangle=dt$. Cette simplification donne une bonne intuition (pour s'en convaincre penser que dW_s est une mesure de la distance parcourue par une particule soumise à un mouvement brownien : voir Section 3 de l'appendice).

Lemme 21 (lemme d'Itô). *Considérons X_t solution de (I.1) et f dans $C^2(\mathbb{R}\times]0,T[,)$. Si $Y_t = f(X_t,t)$ alors*

$$dY_t = \left(\frac{\partial f}{\partial t} + \mu\,\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\,\frac{\partial^2 f}{\partial x^2}\right) dt + \sigma\,\frac{\partial f}{\partial x}\,dW_t$$

Proof.

$$\begin{aligned}
Y_{t+dt} - Y_t &= f(dX_t + X_t, dt+t) - f(X_t,t) \quad \text{où} \quad dX_t = X_{t+dt} - X_t \\
&= \frac{\partial f}{\partial x}\,dX_t + \frac{\partial f}{\partial t}\,dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,dX_t{}^2 + o(dt) \\
&= \frac{\partial f}{\partial x}\,\mu\,dt + \frac{\partial f}{\partial x}\,\sigma\,dW_t + \frac{\partial f}{\partial t}\,dt + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\,\langle dW_t, dW_t\rangle + o(dt) \\
&= \left(\frac{\partial f}{\partial t} + \mu\,\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\,\frac{\partial^2 f}{\partial x^2}\right) dt + \sigma\frac{\partial f}{\partial x}\,dW_t + o(dt)\ .
\end{aligned}$$

□

On applique la formule d'Itô à V_t et on obtient

$$dV_t(S_t,t) = \left(\frac{\partial V_t}{\partial t}(S_t,t) + \mu\,S_t\,\frac{\partial V_t}{\partial x}(S_t,t) + \frac{\sigma^2}{2}\,S_t^2\frac{\partial^2 V_t}{\partial x^2}(S_t,t)\right) dt + \sigma\,S_t\,\frac{\partial V_t}{\partial x}(S_t,t)\,dW_t\ .$$

Si on pose

$$\sigma\,S_t\,\frac{\partial V_t}{\partial x}(S_t,t) = \beta_t\,\sigma\,S_t\ ,$$

on obtient

$$r\,V_t(S_t,t) = \frac{\partial V_t}{\partial t}(S_t,t) + r\,S_t\,\frac{\partial V_t}{\partial x}(S_t,t) + \frac{\sigma^2}{2}\,S_t^2\frac{\partial^2 V_t}{\partial x^2}(S_t,t)$$

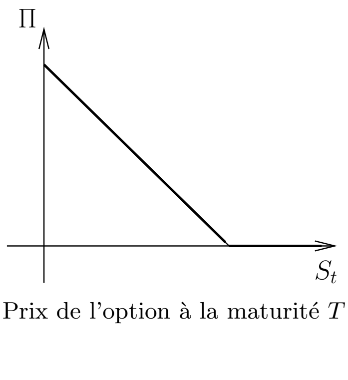
Le retour moyen μ n'apparaît pas dans le formule de Black-Scholes.

Le changement de variable $X_t = \log(S_t)$ conduit à

$$r\,V_t(X_t,t) = \frac{\partial V_t}{\partial t}(X_t,t) + \mu\,\frac{\partial V_t}{\partial x}(X_t,t) + \frac{\sigma^2}{2}\,\frac{\partial^2 V_t}{\partial x^2}(X_t,t)\ .$$

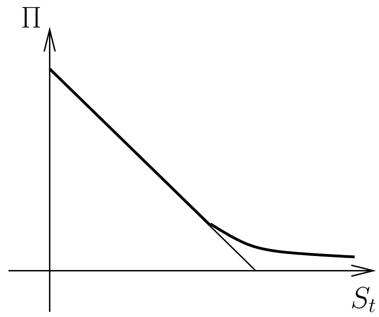
2.2.3 Interprétation heuristique

Considérons le cas de l'option de put américain sur un seul actif risqué dans le modèle de Black-Scholes à coefficients constants. Le prix de l'option à la maturité T est égal au pay-off.



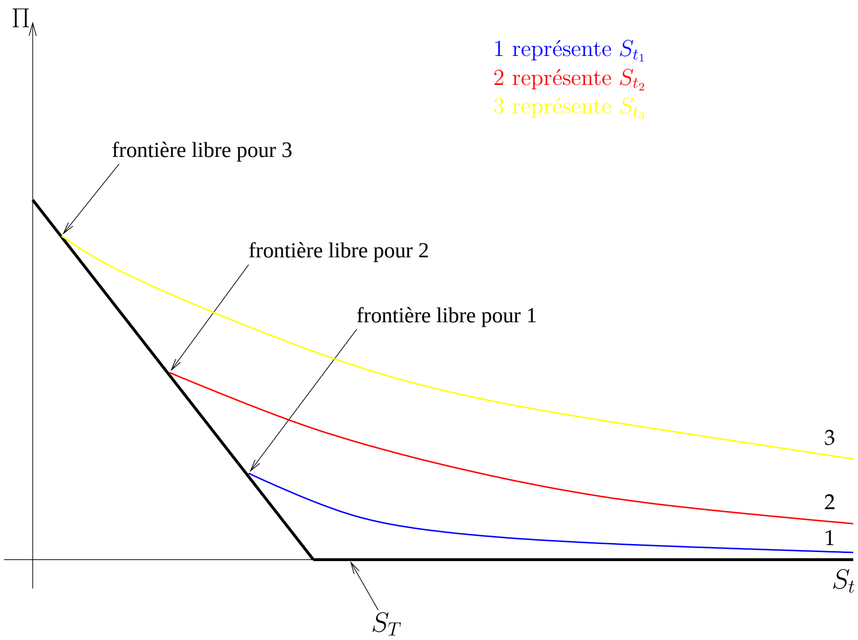
Prix de l'option à la maturité T

Si on s'intéresse au prix de l'option à un temps antérieur à la maturité T, le prix de l'option n'est pas nul. Sinon on pourrait acheter l'option au prix 0. Mais on peut toujours espérer faire un profit non-nul au temps T. Ce serait dont une contradiction avec l'hypothèse d'absence d'*opportunité d'arbitrage* (*i.e.* de faire un bénéfice sans prendre de risque). Au temps $T - \epsilon$ le prix de l'option est de la forme :



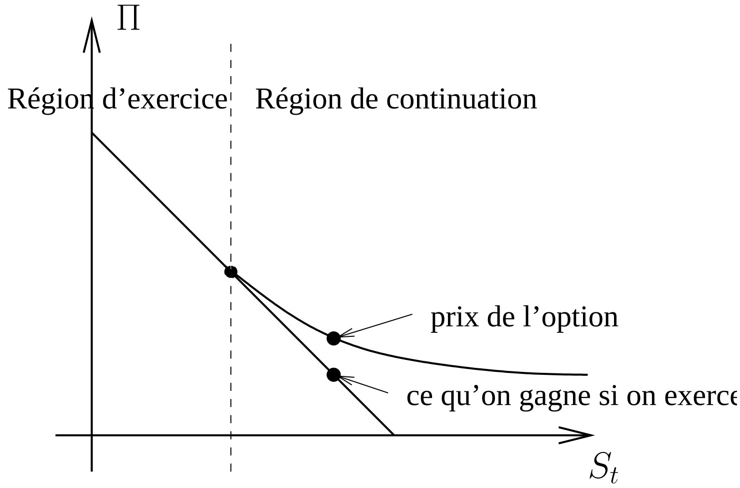
Profit au temps $T - \epsilon$ dans le cas de l'option d'achat (P_T)

En effet, à un temps fixé, le prix de l'option ne peut pas décoller du pay-off en deux prix différents par le principe du maximum. Si une stratégie est gagnante au temps t elle le sera à tous les temps supérieurs donc le prix de l'option est décroissant en temps. En particulier le prix de l'option est toujours supérieur au pay-off. On voit apparaître ici la notion d'obstacle. Si on dessine sur un même graphique le prix de l'option à différents temps on obtient :



Courbes représentatives du prix de l'option de put américain ($t_1 > t_2 > t_3$)

Le point où le prix de l'option américaine décolle du pay-off est appelé *prix critique d'exercice*. Et l'ensemble de ces point au cours du temps est *la frontière optimale d'exercice*. La Remarque 2.2.4 justifie cette terminologie car la frontière optimale d'exercice donne la stratégie à adopter pour le processeur de l'option. La frontière optimale d'exercice correspond, dans la terminologie du problème de l'obstacle, à la frontière libre. Dans la région où le prix de l'actif risqué est supérieur au prix critique d'exercice, le prix intrinsèque de l'option est supérieur à ce que l'on gagne si on l'exerce. Il vaut mieux donc continuer et ne pas exercer. On appelle cette région *région de continuation*. Nous avons vu que dans cette région de prix de l'option satisfait l'équation de Black-Scholes (voir Section 2.2.2). Si on se place maintenant en un point de la frontière optimale d'exercice il faut exercer. En effet, on ne peut pas espérer gagner plus que cette valeur. Autrement le supremum sur toutes les stratégies possibles de ce que l'on peut espérer gagner serait plus grand que ce que l'on gagne avec cette stratégie particulière. Ainsi le prix de l'option serait supérieur au pay-off.



La question de la stratégie à suivre lorsque l'on se retrouve dans la région où le prix de l'actif est plus petit que le prix critique d'exercice ne se pose pas en finance. Car si le possesseur de l'option se trouve dans cette situation il lui suffit d'exercer pour obtenir l'argent qu'il voulait.

2.2.4 Problème à frontière libre

On définit la fonction

$$u(x,t) = \Pi_t(e^x, t)$$

où Π est définit dans (I.3). Si u est suffisamment régulière alors elle est l'unique solution du système d'inéquations aux dérivées partielles suivant :

$$\begin{cases} -r\,u(x,t) + \partial_t u(x,t) + \dfrac{\sigma^2}{2}\partial^2_{xx}u(x,t) \leq 0 & \text{dans } \mathbb{R}\times[0,T] \\ u(x,t) \geq f(x) & \text{dans } \mathbb{R}\times[0,T] \\ \left(-r\,u(x,t) + \partial_t u(x,t) + \dfrac{\sigma^2}{2}\partial^2_{xx}u(x,t)\right)(f(x)-u(x,t)) = 0 & \text{dans } \mathbb{R}\times[0,T] \\ u(T.,) = f & \text{dans } \mathbb{R}\ . \end{cases} \tag{I.4}$$

Cette formulation s'inscrit, plus généralement, dans le cadre de la formulation variationnelle des problèmes d'arrêt optimal (on se référera au ouvrages de A. Friedman ([Fri88]), de A. Bensoussan et J.-L. Lions ([BL78])). Voir aussi [JLL90] pour un raffinement de ce lien entre le prix de l'option et le problème de l'obstacle parabolique. Plus récemment ce lien a été étudié lorsque le cadre variationnel est remplacé par le cadre des solutions de viscosité (voir [Vil99] et [Rap05]).

Pour une démonstration détaillée on se reportera à [BL78]. Faisons ici l'hypothèse que $r = 0$ et que u est suffisamment régulière. On définit

$$Au(X_t^x, t) := \frac{\partial u}{\partial x} X_t^x r + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t}$$

et

$$M_t := u(X_t^x, t) - \int_0^t Au(X_s^x, s)\, ds\ .$$

Par le formule d'Itô on a

$$du(X_t^x, t) = Au(X_t^x, t)\, dt + \frac{\partial u}{\partial x}(X_t^x, t)\, X_t^x\, \sigma(X_t^x, t)\, dW_t$$

ainsi

$$\frac{\partial u}{\partial t}(X_t^x, t) = Au(X_t^x, t) + \frac{\partial u}{\partial x}(X_t^x, t)\, X_t^x\, \sigma(X_t^x, t) \frac{dW_t}{dt}$$

Finalement

$$M_t = \int_0^t \frac{\partial u}{\partial x}(X_t^x, t)\, X_t^x\, \sigma(X_t^x, t) \frac{dW_t}{dt}\ .$$

On peut alors appliquer le théorème d'arrêt de Doob (voir [Kar88]) entre 0 et τ, où τ est un temps d'arrêt, qui assure que $\mathbb{E}(M_0) = \mathbb{E}(M_\tau)$ *i.e.*

$$u(x, 0) = \mathbb{E}\left[u(X_\tau^x, \tau) - \int_0^\tau Au(X_s^x, s)\, ds\right]$$

mais comme $Au(X_t^x, t) \le 0$ on a $u(x, 0) \ge \mathbb{E}\left[u(X_\tau^x, \tau)\right]$, et $u(x, t) \ge \phi(x, t)$ donc $u(x, 0) \ge \mathbb{E}\left[\phi(X_\tau^x, \tau)\right]$. À fortiori

$$u(x, 0) \ge \sup_{\tau \in \Theta_{[0,T]}} \mathbb{E}\left[\phi(X_\tau^x, \tau)\right]\ .$$

Maintenant si on considère $\tau_{\mathrm{opt}} = \inf\{0 \le s \le T, u(X_s^x, s) = \phi(X_s^x, s)\}$. C'est un temps d'arrêt pour tout $s \in [0, \tau_{\mathrm{opt}}]$ on a $Au(X_t^x, t) = 0$. Par le théorème d'arrêt de Doob on obtient $\mathbb{E}(M_0) = \mathbb{E}(M_{\tau_{\mathrm{opt}}})$ *i.e.*

$$u(x, 0) = \mathbb{E}\left[u(X_{\tau_{\mathrm{opt}}}^x, \tau_{\mathrm{opt}})\right]$$

mais $u(X_{\tau_{\mathrm{opt}}}^x, \tau_{\mathrm{opt}}) = \phi(X_{\tau_{\mathrm{opt}}}^x, \tau_{\mathrm{opt}})$, donc

$$u(x, 0) = \mathbb{E}\left[\phi(X_{\tau_{\mathrm{opt}}}^x, \tau_{\mathrm{opt}})\right].$$

Ce qui prouve que (I.3) est solution de (I.4).

Remarque. *On a montré que*

$$\Pi(x, t) = \mathbb{E}\left[\exp\left(-\int_t^{\tau^*} r(S_s^{x,t}, s) ds\right) \phi(S_{\tau^*}^{x,t}, \tau^*)\right]$$

où $\tau^* := \{\inf \tau \ge t\ :\ \Pi(S_\tau, \tau) = \phi(S_\tau, \tau)\}$. *La détermination de cette frontière permet donc de connaître la stratégie optimale pour l'exercice de l'option.*

Malheureusement il est impossible de trouver une expression analytique explicite de la solution, même dans le cas où l'opérateur parabolique est l'opérateur de la chaleur. Cependant dans le cas du put et d'option vanilles américaines sur un seul actif, et lorsque le taux d'intérêt et la volatilité sont constants, il existe des résultats partiels et de bonnes procédures d'approximation (voir [BBRS93]). Par exemple, il est connu dans ce cadre pour l'option de put, que l'ensemble de coïncidence est de la forme $\{(x_1, t) \in \mathbb{R}\times]0, T[, \, x_1 > g(t)\}$ pour une certaine fonction g qui reste à déterminer. Une importante littérature existe sur le comportement de la frontière en un temps proche de la maturité. Barles *et al* [BBRS93] ont prouvé dans le cas d'une option de put américain lorsque les coefficients sont constants que

$$\frac{g(t) - K}{-K\sigma\sqrt{(T-t)\log(T-t)}} \longrightarrow 1 \text{ quand } t \to T^- \, .$$

Ce dernier résultat a été étendu récemment par E. Chevalier ([Che05]) dans le modèle de Black-Scholes à volatilité locale lorsque le taux d'intérêt est constant pour les options de put et de call américain.

Ces indications permettent de penser que la frontière variable est de la forme ci-dessous.

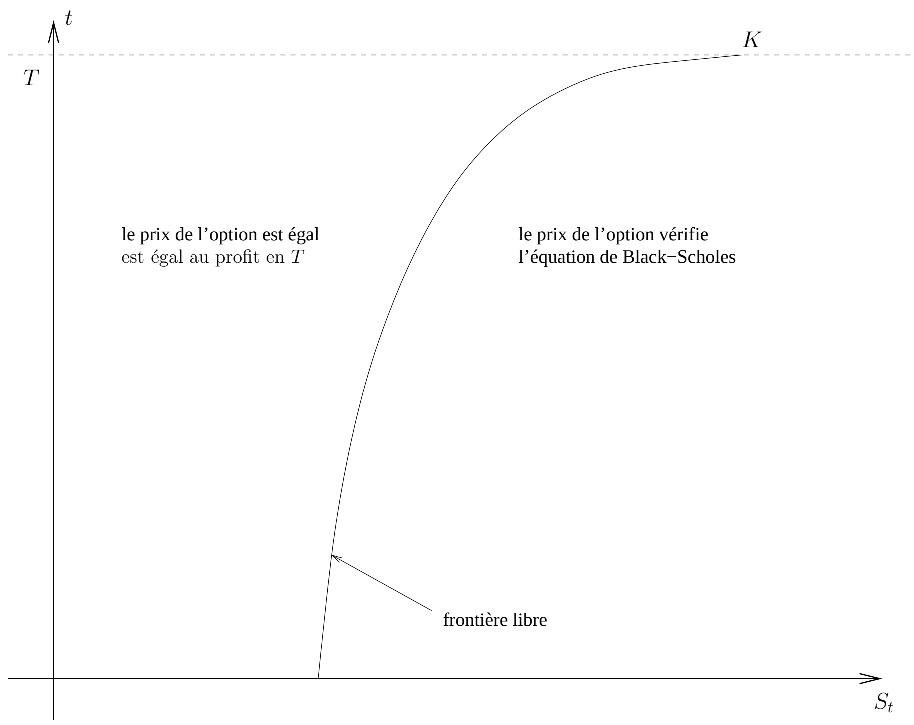


Schéma des stratégies financières

L'analyse numérique de ce problème nécessite une bonne compréhension du comportement de la frontière puisqu'il faut déterminer la solution approchée et la frontière approchée. Y. Achdou, dans [Ach05], considère le modèle de Black-Scholes à volatilité locale mais il s'intéresse au problème inverse qui est, connaissant la valeur des options américaines qui s'échangent sur le marché, de déterminer la volatilité locale qui régit le marché. Pour cette question il doit en particulier supposer que la frontière libre d'exercice est Lipschitz.

Pour une réflexion sur ces modèle voir aussi Section 1.5 de l'introduction.

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A.II

On the regularity of the solution of the parabolic obstacle problem

Ce chapitre reprend l'essentiel de [BDM05a] accepté pour publication dans *Journal de Mathématiques Pures et Appliquées* en collaboration avec J. Dolbeault et R. Monneau. Les résultats de ce chapitre ont donné lieu à une annonce de résultats dans [BDM05b].

Résumé

Cet article est consacré à des résultats de continuité de la dérivée en temps du problème de l'obstacle parabolique unidimensionnel à coefficients variables. Sous des hypothèses de régularité de l'obstacle et des coefficients, nous démontrons que la dérivée en temps est continue pour presque tout temps. Quand la solution est décroissante, le résultat a lieu pour tout temps. Nous donnons aussi un critère d'énergie qui caractérise la continuité en temps de la dérivée en un point de la frontière libre.

On the continuity of the time derivative of the solution to the parabolic obstacle problem with variable coefficients

A. Blanchet, J. Dolbeault et R. Monneau

Abstract

This paper is devoted to continuity results of the time derivative of the solution to the one-dimensional parabolic obstacle problem with variable coefficients. Under regularity assumptions on the obstacle and on the coefficients, we prove that the time derivative of the solution is continuous for almost every time. When the solution is nondecreasing in time this result holds for every time. We also give an energy criterion which characterizes the continuity of the time derivative of the solution at a point of the free boundary. Such a problem arises in the pricing of american options in generalized Black-Scholes models of finance. Our results apply in financial mathematics.

AMS Classification: 35R35.
Keywords: parabolic obstacle problem, free boundary, blow-up, Liouville's result, monotonicity formula.

1 Introduction

Let $\alpha \in (0,1)$ and consider a domain D of $\mathbb{R}^2$. We denote by $\mathcal{H}^\alpha$ the Banach space of Hölder functions

$$\mathcal{H}^\alpha(D) := \Big\{ f \in \mathcal{C}^0 \cap L^\infty(D) \ : \ \|f\|_{\alpha;D} < \infty \Big\}$$

where $\|f\|_{\alpha;D} = \|\cdot\|_{L^\infty(D)} + [f]_{\alpha;D}$,

$$[f]_{\alpha;D} := \sup_{\substack{(x,t),(y,s)\in D\\ (x,t)\neq(y,s)}} \frac{|f(x,t)-f(y,s)|}{(\,|x-y|^2+|t-s|\,)^{\alpha/2}}$$

(see [Fri64], Chap. 3, Sec. 2). For all $q \in [1,\infty]$ we also define the Sobolev space

$$W^{2,1;q}_{x,t}(D) := \left\{ u \in L^q(D) \ : \ \Big(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t}\Big) \in (L^q(D))^3 \right\} .$$

To $P_0 = (x_0,t_0) \in \mathbb{R}^2$ and $R \in (0,\infty)$, we associate the open parabolic cylinder

$$Q_R(P_0) := \{\, (x,t) \in \mathbb{R}^2 \ : \ |x-x_0| < R \text{ and } |t-t_0| < R^2 \,\} ,$$

and the lower half parabolic cylinder

$$Q^-_R(P_0) := \{\, (x,t) \in \mathbb{R}^2 \ : \ |x-x_0| < R \text{ and } 0 < t_0 - t < R^2 \,\} .$$

And for $M > 0$, we define $\mathcal{P}_M(Q_R(P_0))$ by

$$\mathcal{P}_M(Q_R(P_0)) := \{u \in L^\infty(Q_R(P_0)) \ : \ \sup_{Q_R(P_0)} \left[\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right| + \left|\frac{\partial u}{\partial t}\right| + \frac{1}{R}\left|\frac{\partial u}{\partial x_i}\right| \right] \le M < \infty\} .$$

Such notations for parabolic problems are standard. See [Lie96, Fri64, LSU67] for more details. On $W^{2,1;1}_{x,t}$, consider now the parabolic operator

$$Lu := a(x,t)\,\frac{\partial^2 u}{\partial x^2} + b(x,t)\,\frac{\partial u}{\partial x} + c(x,t)\,u - \frac{\partial u}{\partial t}\ ,$$

where a, b and c are variable coefficients which depend on x and t.

This paper is devoted to regularity properties of the solutions to the one-dimensional parabolic obstacle problem

$$\begin{cases} u \in \mathcal{P}_M(Q_R(P_0)) & \\ Lu(x,t) = f(x,t)\,\mathbb{1}_{\{u>0\}}(x,t) & (x,t) \in Q_R(P_0) \text{ a.e.} \\ u(x,t) \geq 0 & \end{cases} \tag{II.1}$$

The function $\mathbb{1}_{\{u>0\}}$ denotes the characteristic function of the set $\{u > 0\} := \{(x,t) \in Q_R(P_0) : u(x,t) > 0\}$:

$$\mathbb{1}_{\{u>0\}}(x,t) = \begin{cases} 1 & \text{if} \quad u(x,t) > 0\ , \\ 0 & \text{if} \quad u(x,t) = 0\ . \end{cases}$$

The $\mathcal{P}_M$-regularity of u has been proved in [CPS04] for constant coefficient case for problem (II.1) with no sign assumption on u.

Our main assumption is the following assumption on uniform parabolicity and non degeneracy and regularity of the coefficients and of the function f:

$$\begin{cases} a,\ b,\ c \textit{ and } f \textit{ belong to } \mathcal{H}^\alpha(Q_R(P_0)) \quad \text{for some } \alpha \in (0,1)\ , \\ \textit{there exists } \delta_0 > 0 \textit{ such that for any } (x,t) \in Q_R(P_0)\,,\ a(x,t) \geq \delta_0 \textit{ and } f(x,t) \geq \delta_0\ . \end{cases} \tag{II.2}$$

By [Fri88], under Assumption (II.2), (II.1) has a unique solution for suitable initial datum and boundary conditions. From standard regularity theory for parabolic equations, [Lie96, Fri64, LSU67], it is known that any solution u belongs to $W^{2,1;q}_{x,t}(Q_r(P_0))$ for any $r < R$ and $q < +\infty$. As a consequence of Sobolev's embeddings, u is continuous. The set $\{u = 0\}$ is then closed in $Q_R(P_0)$.

Definition II.1. *The sets $\{u = 0\}$ and $\Gamma := Q_R(P_0) \cap \partial\{u = 0\}$ are respectively called the* coincidence set *and the* free boundary *of the parabolic obstacle problem (II.1).*

Notations. We will use u_t, u_x and u_{xx} respectively for $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$. By $|A|$ we denote the volume of the set $A \subset \mathbb{R}^2$ with respect to the Lebesgue measure, and by $\mathcal{D}(\mathbb{R})$ the set of smooth functions with compact support. The heat operator will be abbreviated to H, $Hu := u_{xx} - u_t$. We denote $W^{2,1;\infty}_{x,t;\mathrm{loc}}(\mathbb{R}^2)$ the set of function which are in $W^{2,1;\infty}_{x,t}(D)$ for all $D \subset \mathbb{R}^2$. The *parabolic boundary* of $Q^-_r(P_0)$ is the set $\partial^p Q^-_r(P_0) := [x_0 - r, x_0 + r] \times \{t_0 - r^2\} \cup \{x_0 - r, x_0 + r\} \times [t_0 - r^2, t_0]$. We define the parabolic distance dist_p between two points $P = (x,t)$ and $P' = (x',t')$ by $\mathrm{dist}_p(P,P') := \sqrt{(x - x')^2 + |t - t'|}$.

By standard parabolic estimates u_t is continuous in a neighborhood of any point P such that $u(P) > 0$. If P is in the interior of the region $\{u = 0\}$, u_t is obviously continuous. The key issue is therefore the regularity of u_t on the free boundary Γ. Our first result states that u is almost never discontinuous.

Theorem II.2 (Continuity of u_t for almost every t). *Let u be a solution of (II.1) and assume (II.2). For almost any $t_1 \in (t_0 - R, t_0 + R)$, if $P_1 = (x_1, t_1)$ is a point on the free boundary Γ, then*

$$\lim_{P \to P_1} \frac{\partial u}{\partial t}(P) = 0 \, .$$

As far as the authors know, this result is new, even in the case of constant coefficients. The continuity of u_t cannot be obtained everywhere in t, as shown by the following example. Let $u(x,t) = \max\{0, -t\}$. It satisfies $u_{xx} - u_t = \mathbb{1}_{\{u>0\}}$ and its time derivative is obviously discontinuous at $t = 0$. If we additionally assume that $u_t \geq 0$, we achieve a more precise result:

Theorem II.3 (Continuity of u_t for all t when $u_t \geq 0$). *Under the assumptions of Theorem II.2, if u_t is nonnegative, then u_t is continuous everywhere, and satisfies*

$$\frac{\partial u}{\partial t} = 0 \quad on \quad \Gamma \, .$$

The assumption that u_t is nonnegative can be established in some special cases (special initial conditions, boundary conditions, and time independent coefficients). See for example the results of Friedman [Fri75], for further results on the one-dimensional parabolic obstacle problem with particular initial conditions.

When we are not assuming that u is nondecreasing in time, it is useful to have some criteria to determine the points where the time derivative of the solution is continuous. We begin with a *density criterion* based on the density $\theta(P_1)$ of the coincidence set $\{u = 0\}$ at the point $P_1 \in Q_R(P_0)$:

$$\theta(P_1) := \liminf_{r \to 0} \frac{|\{u = 0\} \cap Q_r(P_1)|}{|Q_r(P_1)|}$$

and on the lower density $\theta^-(P_1)$ of $\{u = 0\}$ at P_1:

$$\theta^-(P_1) := \liminf_{r \to 0} \frac{|\{u = 0\} \cap Q_r^-(P_1)|}{|Q_r^-(P_1)|} \, .$$

Theorem II.4 (Density criterion: continuity of u_t). *Let u be a solution of (II.1), assume (II.2) and consider a point $P_1 \in Q_R(P_0)$. If either $\theta(P_1) = 0$, or $\theta(P_1) \neq 0$ and $\theta^-(P_1) \neq 0$, then we have*

$$\lim_{P \to P_1 \ P \in Q_R(P_0) \setminus \Gamma} \frac{\partial u}{\partial t}(P) = 0 \, .$$

Otherwise, if $\theta(P_1) \neq 0$ and $\theta^-(P_1) = 0$, then u_t is not continuous at P_1.

The second criterion is an *energy criterion* based on a monotonicity formula. Consider a nonnegative cut-off function $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi \equiv 1$ on $\left(-\frac{r}{2}\sqrt{\frac{f(P_1)}{a(P_1)}}, \frac{r}{2}\sqrt{\frac{f(P_1)}{a(P_1)}}\right)$ and $\psi \equiv 0$ on $\left(-\infty, -r\sqrt{\frac{f(P_1)}{a(P_1)}}\right] \cup \left[r\sqrt{\frac{f(P_1)}{a(P_1)}}, \infty\right)$. Let $Q_r(P_1) \subset Q_R(P_0) \subset \mathbb{R}^2$. With $P_1 =$

(x_1, t_1), and a, f the functions involved respectively in the definition of the operator L and in Equation (II.1), define the function v_{P_1} for all $(x,t) \in \mathbb{R} \times (-r^2 f(P_1), r^2 f(P_1))$ by

$$v_{P_1}(x,t) := u\left(x_1 + x\sqrt{\tfrac{a(P_1)}{f(P_1)}}, t_1 + \frac{t}{f(P_1)}\right) \cdot \psi(x) \text{ if } |x| \le r\sqrt{\tfrac{f(P_1)}{a(P_1)}}\,, \quad v_{P_1} \equiv 0 \text{ otherwise }. \tag{II.3}$$

For all $t \in (-r^2 f(P_1), 0)$, let

$$\mathcal{E}(t;v) := \int_{\mathbb{R}} \left[\frac{1}{-t}\left(\left|\frac{\partial v}{\partial x}\right|^2 + 2\,v\right) - \frac{v^2}{t^2}\right] G\,dx - \int_t^0 \frac{1}{s^2}\int_{\mathbb{R}} (Hv-1)\,(\mathcal{L}v)\,G\,dx\,ds\,,$$

with $Hv := v_{xx} - v_t$, $\mathcal{L}v := -2\,v + x\cdot v_x + 2\,t\,v_t$ and $G(x,t) := (2\pi(-t))^{-1/2}\exp\left(-x^2/(-4t)\right)$.

Theorem II.5 (Energy criterion: continuity of u_t). *Under the assumptions of Theorem II.2,*
(i) *either* $\lim_{t\to 0,\, t<0} \mathcal{E}(t; v_{P_1}) = \sqrt{2}$,
(ii) *or there exists some* $t \in (-r^2 f(P_1), 0)$ *such that* $\mathcal{E}(t; v_{P_1}) < \sqrt{2}$. *In that case,* $\lim_{t\to 0,\, t<0} \mathcal{E}(t; v_{P_1}) = \sqrt{2}/2$ *and* u_t *is continuous in a neighborhood of* P_1.

The one-dimensional parabolic obstacle problem for differential operators with variable coefficients is a generalisation to the case of an operator with variable coefficients of Stefan's problem (case where the parabolic operator is $Lu = u_{xx} - u_t$). Stefan's problem describes the interface of ice and water (see [KS80, Rod87, Fri88]). The problem with variable coefficients arises in the pricing of american options in mathematical finance (see [BS73, BL78, VM74, LL97, JLL90, Vil99, Ach05, Bjö97, Øks85, Pha98]). The regularity of u_t is a natural question to apply the "smooth-fit principle" which amounts to require the C^1 continuity of the solution at the free boundary. This principle is often assumed in numerical methods (see for instance [DW02]).

In [VM74] Van Moerbeke studied a special case where he proved that u_t is continuous except at one point and gave some asymptotics of the free boundary at this point. In [Fri75], Friedman specifically studied the case of an american option and proved that u_t is continuous on some subsets of the free boundary. Using the maximum principle, he also proved for a special class of initial data that the free boundary is piecewise monotone. Then until recently the theory of the obstacle problem has essentially been studied in the stationary case (see [KS80, Rod87, Fri88] and references therein). Variational inequalities have been related to probabilistic methods in [BL78, JLL90, LL97], and also to viscosity solutions methods [Vil99, Rap05]. Also see [Ach05] for a recent paper revisiting variational inequalities and raising questions on the regularity of the solution and of the free boundary.

Recently in [CPS04], Caffarelli, Petrosyan and Shahgholian considered the case with constant coefficients in any dimension and without any sign assumptions on the solution. They developed a nice theory of the regularity of the free boundary, based on Liouville type results and monotonicity formulas, like the one introduced by Weiss in [Wei00]. As we shall see below, such tools are extremely useful for our purpose.

This paper is organized as follows. In Section 2 we prove a non-degeneracy lemma. As a consequence the free boundary is a closed subset of zero measure. In Section 3 we introduce the notion of blow-up sequences which are a kind of zooming at a point of the free boundary. We will use them to study the regularity of the solution. These sequences converge, up to the extraction of sub-sequences, to the blow-up limit which is a solution in the whole space of the obstacle problem with constant coefficients. Using a monotonicity formula we prove in Section 3 that the blow-up limit is scale-invariant. This allows us to classify all possible blow-up limits in a Liouville theorem. The energy also gives a criterion to distinguish regular and singular points of the free boundary, see Section 4. In Section 5 we prove the uniqueness of the blow-up limit at each singular point. The last section is devoted to the completion of the proofs of all results stated in Section 1 and some additional results on the time derivative of the solution.

2 Regularity estimates and properties of the free boundary

Some of the results of the section will be true for the following problem:

$$\begin{cases} Lu \leq f \quad \text{a.e. } Q_R(P_0) \ , \\ Lu = f \quad \text{a.e} \quad \{u > 0\} \ , \\ u \geq 0 \quad \text{a.e. } Q_R(P_0) \ . \end{cases} \tag{II.4}$$

2.1 Non-degeneracy lemma

The non-degeneracy lemma is an important tool which has first been introduced by Caffarelli in [Caf77] for the elliptic obstacle problem. It can be interpreted as the fact that the free boundary can not appear or disappear suddenly, or is not "blurred". It has been for instance proved for the parabolic problem with constant coefficient in [CPS04]. Here we extend it to the case of variable coefficients.

Lemma II.6 (Non-degeneracy lemma). *Under Assumption (II.2), consider a solution u of (II.4) in $Q_R(P_0)$. Let $R' \in (0,R)$, $P_1 \in \overline{\{u > 0\}}$ be such that $Q_r^-(P_1) \subset Q_{R'}(P_0)$ for some $r > 0$ small enough. There exist two positive constants $\bar{C}$ and $\bar{r} > 0$ such that if $Q_{\bar{r}}(P_1) \cap \{u = 0\} \neq \emptyset$:*

$$r \leq \bar{r} \quad \Longrightarrow \quad \sup_{Q_r^-(P_1)} u \geq \bar{C}\, r^2 \, .$$

The constants $\bar{C}$ and $\bar{r}$ only depend on R' and L.

Proof – Consider first $P' = (x', t') \in \{u > 0\} \cap Q_r(P_1)$. For some positive constant $\bar{C}$ to be fixed later, we set for all $(x,t) \in Q_r(P') \subset Q_{R'}(P_0)$

$$w(x,t) := u(x,t) - u(P') - \bar{C}\left((x - x')^2 + |t - t'|\right) .$$

By Assumption (II.2), $Lu = f \geq \delta_0$ in $\{u > 0\}$. For all $(x,t) \in Q_r(P') \cap \{u > 0\}$, we have

$$\begin{aligned} Lw(x,t) - c(x,t)\, w(x,t) \quad &\geq Lu(x,t) - c\, u(x,t) - \bar{C}(2\, a(x,t) + 1) - 2\,\bar{C}\, b(x,t) \cdot (x - x') \\ &\geq \delta_0 - \tilde{C}\, |c(x,t)|\, (2r)^2 - \bar{C}\, (2\, a(x,t) + 1) - 2\,\bar{C}\, |b(x,t)|\, (2r) \end{aligned}$$

according to $\mathcal{P}_M$-estimates. With $\bar{C} := \frac{\delta_0}{4}\, \left(2\, \|a\|_{L^\infty(Q_R(P_0))} + 1\right)^{-1}$ and

$$\bar{r} := \frac{\delta_0}{8} \min\left\{ \left(4\,\bar{C}\, \|b\|_{L^\infty(Q_R(P_0))}\right)^{-1}, \left(4\,\tilde{C}\, \|c\|_{L^\infty(Q_R(P_0))}\right)^{-1/2} \right\} ,$$

we obtain

$$Lw(x,t) - c(x,t)\, w(x,t) \geq 0 \quad \text{in} \quad Q_{\bar{r}}(P') \cap \{u > 0\} \, .$$

Notice that $w(P') = 0$. Applying the parabolic maximum principle in $Q^-_\rho(P') \cap \{u > 0\}$ for $\rho \leq \bar{r}$ (*cf.* [Lie96] Theorem 2.9 (p. 13), or [Fri64] Theorem 1, Chap. 2, Sec. 1 (p. 34)) we get that the maximum of w is nonnegative and achieved in $\{(x,t) \in \overline{Q^-_\rho(P')} : u(x,t) > 0,\, t < t'\}$. On $\partial\{u = 0\} \cap \overline{Q^-_\rho(P')}$, $u = 0$ implies that w is negative then there exists $P_2 = (x_2, t_2) \in \partial^p Q^-_\rho(P') \cap \{u > 0\}$ such that

$$\sup_{Q^-_\rho(P')\cap\{u>0\}} w = w(P_2) = u(P_2) - u(P') - \bar{C}\left((x_2 - x')^2 + |t_2 - t'|\right) \geq 0\,.$$

This means that when there exists $P_1 \in \Gamma$ such that $P' \in Q_{\bar{r}}(P_1) \cap \{u > 0\}$, then for $\rho \leq \bar{r}$ we have

$$\sup_{Q^-_\rho(P')} u \geq u(P_2) \geq u(P') + \bar{C}\,\rho^2 \geq \bar{C}\,\rho^2$$

and by continuity of u, the estimate remains true when P' tends to $P_1 \in \Gamma$. □

2.2 Properties of the free boundary

Theorem II.7. *Under Assumption (II.2), the free boundary Γ associated to a solution u of (II.4) is a closed set of zero Lebesgue measure.*

The proof is a based on several results which are consequences of $\mathcal{P}_M$-estimates and Lemma II.6.

Lemma II.8 (Cube property of the free boundary). *Under Assumption (II.2), consider a solution u of (II.4) in $Q_R(P_0)$. There exists a constant $\lambda \in (0, \frac{1}{2})$ such that for any $r > 0$ small enough, for any $P_1 \in \Gamma \cap Q_{3R/4}(P_0)$ such that $Q_r(P_1) \subset Q_{3R/4}(P_0)$, there exists $P_2 \in Q^-_{r/2}(P_1)$ such that $Q_{\lambda r}(P_2) \subset \{u > 0\} \cap Q_r(P_1)$.*

Proof – By Lemma II.6, there exists $P_2 = (x_2, t_2) \in Q^-_{r/2}(P_1)$ such that

$$u(P_2) \geq \frac{1}{4}\,\bar{C}\,r^2\,.$$

On the other hand, assume by contradiction that for all λ there exists $P = (x,t) \in Q_{\lambda r}(P_2)$ such that $u(P) = 0$. We have

$$\begin{aligned} u(P_2) &\leq |u(P_2) - u(x_2,t)| + |u(x_2,t) - u(x,t)| \\ &\leq \sup_{Q_R(P_0)} |u_t| \cdot |t - t_2| + u_x(P)|x - x_2| + \frac{|x - x_2|^2}{2} \int_0^1\!\!\int_0^\alpha |u_{xx}(x + \beta(x - x_2), t)|\, d\beta\, d\alpha \\ &\qquad\qquad \leq \sup_{Q_R(P_0)} |u_t| \cdot |t - t_2| + \sup_{Q_R(P_0)} u_{xx} \cdot |x - x_2|^2\,. \end{aligned}$$

Hence

$$\frac{1}{4}\,\bar{C}\,r^2 \leq u(P_2) \leq \|u\|_{W^{2,1;\infty}(Q_R(P_0))}\,(\lambda\, r)^2\,.$$

Which is a contradiction for $\lambda < (1/2)\sqrt{\bar{C}/\,\|u\|_{W^{2,1;\infty}(Q_R(P_0))}}$. □

Recall now the following result on measurable sets.

Lemma II.9 (Density in a point of a measurable set). *Let A be a measurable subset in $\mathbb{R}^2$. If A has non-zero Lebesgue measure, then for almost every $P_1 = (x_1, t_1) \in A$, we have*

$$\limsup_{n\to\infty} \frac{|A \cap C_n(P_1)|}{|C_n(P_1)|} = 1\,,$$

where $C_n(P_1) := \left[x_1 - \frac{1}{n}, x_1 + \frac{1}{n}\right] \times \left[t_1 - \frac{1}{n}, t_1 + \frac{1}{n}\right]$.

See [Fed69], Theorem 2.9.11 (p. 158), Remark 2.9.12 (p. 158), Theorem 2.8.18 (p. 152) and Remark 2.8.9 (p. 145).

Proof of Theorem II.7. For the convenience of the reader, we recall here a proof that can be found in [CPS04]. Let us suppose by contradiction that the measure of Γ is non-zero. By Lemma II.9 there exists at least one point P_1 such that

$$\limsup_{n\to\infty} \frac{|\Gamma \cap C_n(P_1)|}{|C_n(P_1)|} = 1\,.$$

Divide the euclidean cylinder $C_n(P_1)$ into n parabolic cylinders $Q_{i,n} := Q_{\frac{1}{n}}(x_1, t_i)$, $t_i := t_1 - \frac{1}{n} + \frac{2i+1}{n^2}$, $i \in \{0, \ldots, n-1\}$. If $Q_{i,n} \cap \Gamma = \emptyset$, we set $E_{i,n} := Q_{i,n}$. Otherwise, by Lemma II.8 there exists $E_{i,n}$ in $Q_{i,n} \cap \{u > 0\}$ with $|E_{i,n}| \geq \lambda^3 |Q_{i,n}|$.

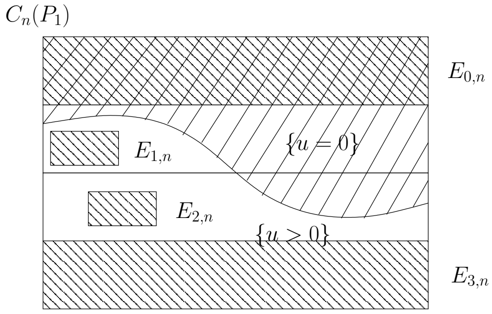

Figure II.1: *Construction of the sets $E_{i,n}$.*

Let us set $E_n := \cup_{i=0}^{n-1} E_{i,n}$. We have

$$\limsup_{n\to\infty} \frac{|\Gamma \cap C_n(P_1)|}{|C_n(P_1)|} \leq 1 - \liminf_{n\to\infty} \frac{|E_n|}{|C_n(P_1)|} \leq 1 - \lambda^3 < 1$$

which contradicts Lemma II.9. □

A straightforward consequence of $\mathcal{P}_M$-estimates and Theorem II.7 is the following result:

Proposition II.10. *Let D be a domain of $\mathbb{R}^2$. If $U^0 \in \mathcal{P}_M(\mathbb{R}^2)$ is a fonction satisfying*

$$\begin{cases} U^0_{xx} - U^0_t \leq 1 & a.e.\ D \\ U^0_{xx} - U^0_t = 1 & a.e. \quad \{U^0 > 0\} \\ U^0 \geq 0 & a.e.\ D \end{cases}$$

then

$$|\partial \{U^0 > 0\}| = 0 \quad and \quad U^0_{xx} - U^0_t = \mathbb{1}_{\{U^0>0\}}\,.$$

3 Properties of blow-up limits

3.1 Reduction to the constant coefficient case

The reduction of a general operator L to the heat operator H is done by a classical transformation which goes as follows. Assume (II.2) and consider a solution of (II.1). Let $P_1 = (x_1, t_1) \in \Gamma$ and take $r > 0$ such that $Q_r(P_1) \subset\subset Q_R(P_0)$. For all $P = (x,t) \in Q_r(P_1) \cap \{u > 0\}$, Equation (II.1) can be rewritten as

$$a(P_1)\,\frac{\partial^2 u}{\partial x^2}(P) - \frac{\partial u}{\partial t}(P) = f(P_1) + (f(P) - f(P_1)) - (a(P) - a(P_1))\,\frac{\partial^2 u}{\partial x^2}(P) - b(P)\,\frac{\partial u}{\partial x}(P) - c(P)\,u(P)\ .$$

Consider the affine change of variables

$$(x,t) \mapsto \Big(X := \sqrt{\tfrac{f(P_1)}{a(P_1)}}\ (x - x_1),\quad T := f(P_1)\ (t - t_1)\Big) \tag{II.5}$$

and define

$$U(X,T) := u(x,t)\ ,$$
$$g(X,T) := \frac{1}{f(P_1)}\left((f(P) - f(P_1)) - (a(P) - a(P_1))\,\frac{\partial^2 u}{\partial x^2}(P) - b(P)\,\frac{\partial u}{\partial x}(P) - c(P)\,u(P)\right).$$

In the (X,T) variables, the function U is a solution in $W^{2,1;1}_{x,t}(Q)$ of the parabolic obstacle problem

$$\frac{\partial^2 U}{\partial X^2} - \frac{\partial U}{\partial T} = (1+g)\,\mathbb{1}_{\{U>0\}}\ ,\quad U \geq 0 \quad \text{a.e. in } Q \tag{II.6}$$

such that $\partial\{U = 0\} \ni 0$ where

$$Q := \left(-r\sqrt{\frac{f(P_1)}{a(P_1)}}, r\sqrt{\frac{f(P_1)}{a(P_1)}}\right) \times \left(-r^2 f(P_1), r^2 f(P_1)\right)\ .$$

By construction, $g(0) = 1$.

Important remark *To avoid further tedious notations and up to make a previous reduction of the problem, we will assume (except when we will have to move the point P_1) from now on and in the whole paper that $f(P_1) = a(P_1) = 1$ and $r = 1$.*

Then U satisfies

$$\frac{\partial^2 U}{\partial X^2} - \frac{\partial U}{\partial T} = (1+g)\,\mathbb{1}_{\{U>0\}}\ ,\quad U \geq 0 \quad \text{a.e. in } Q_1(0) \tag{II.7}$$

From Assumption (II.2), we deduce that there exist an $\alpha \in (0,1)$ and a positive constant C such that for $r > 0$, small enough,

$$\frac{1}{2} \leq |g(X,T)| \leq C(X^2 + |T|)^{\alpha/2} \quad \forall\,(X,T) \in Q_1(0)\ . \tag{II.8}$$

Proposition II.11. *Under Assumption (II.2), consider a solution u of (II.1). With the above notations, $U \in \mathcal{P}_M(Q_1(0))$ and there exist a positive constant $\bar{C}$ such that for any $P \in \overline{\{U > 0\}} \cap Q_1(0)$,*

$$Q_r(P) \subset Q_1(0) \quad \Longrightarrow \quad \sup_{Q_r^-(0)} U \geq \bar{C}\,r^2\,.$$

Moreover, $\partial\{U = 0\}$ has zero Lebesgue measure.

Proof – This result is a straightforward consequence of Lemma II.6 and Theorem II.7 using the change of variables (II.5). □

3.2 Localisation, localised energy

Let us first rephrase in terms of U the energy which has been introduced in Section 1. We need to localize the solution first.

To a nonnegative cut-off function $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi \equiv 1$ on $(-1/2, 1/2)$ and $\psi \equiv 0$ on $(-\infty, -1] \cup [1, \infty)$, we associate the function

$$v(x,t) = v_{P_1}(x,t) := U(x,t)\,\psi(x) \;, \quad (x,t) \in \mathbb{R} \times (-1,0) \;.$$

To simplify the notations, we shall drop the index P_1 whenever there is no ambiguity. The energy now takes the form:

$$\mathcal{E}(v;t) := \int_{\mathbb{R}} \left[\frac{1}{-t} \left(\left| \frac{\partial v}{\partial x} \right|^2 + 2\,v \right) - \frac{v^2}{t^2} \right] G\; dx - \int_t^0 \frac{1}{s^2} \int_{\mathbb{R}} (Hv - 1)\, \mathcal{L}v\, G\; dx\, ds \;, \tag{II.9}$$

with

$$\begin{aligned} &Hv := v_{xx} - v_t \;, \\ &\mathcal{L}v := -2\,v + x \cdot v_x + 2\,t\,v_t \;, \\ &G(x,t) := \frac{e^{-\frac{|x|^2}{(-4t)}}}{\sqrt{2\pi(-t)}} \;. \end{aligned}$$

The function G satisfies the backward heat equation:

$$G_{xx} + G_t = 0 \quad \text{in} \quad \mathbb{R} \times (\infty, 0) \;.$$

The kernel of $\mathcal{L}$ is spanned by the space of scale-invariant functions:

$$\mathcal{L}v \equiv 0 \quad \Longleftrightarrow \quad v(x,t) = \lambda^{-2}\, v(\lambda\,x, \lambda^2\,t) =: v_\lambda(x,t) \;, \quad \forall\, (x,t) \in \mathbb{R}\times(-\infty,0) \;, \quad \forall\, \lambda > 0 \;. \tag{II.10}$$

This is easily proved by writing $v_\lambda(x,t) - v(x,t) = \int_1^\lambda \mu^{-3} (\mathcal{L}v)(\mu\,x, \mu^2\,t)\, d\mu$.

3.3 Notion of blow-up

In [Caf77] Caffarelli introduces the notion of blow-up sequences in order to study the free boundary of the elliptic obstacle problem. Such a tool is convenient as long as only $\mathcal{P}_M$ estimates of the solution is known. Here we adapt such a notion of blow-up sequences to the parabolic obstacle problem.

Definition II.12 (Blow-up sequence). *Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence which converges to 0. The blow-up sequence $(U^{\epsilon_n})_{n\in\mathbb{N}}$ associated to a function $U : Q_1(0) \longrightarrow \mathbb{R}$ around 0 is the sequence defined by*

$$U^{\epsilon_n}(x,t) := \epsilon_n^{-2}\, U\left(\epsilon_n\, x,\, \epsilon_n^2\, t\right) \quad \forall\, (x,t) \in Q_{1/\epsilon_n}(0)\,, \quad \forall\, n \in \mathbb{N} \;.$$

The parabolic scaling $(x,t) \mapsto (\epsilon x, \epsilon^2 t)$ transforms the parabolic cylinder $Q_\epsilon(0)$ into the parabolic cylinder $Q_1(0)$ and preserves the heat operator H, in the sense that, for any $\epsilon > 0$,

$$(HU^\epsilon)\,(x,t) = (HU)\,(\epsilon\,x, \epsilon^2\,t) \quad \forall\, (x,t) \in Q_{1/\epsilon}(0) \;.$$

Proposition II.13 (Blow-up limit). *Assume (II.8) and consider a blow-up sequence* $(U^{\epsilon_n})_{n\in\mathbb{N}}$ *associated to a solution* U *of (II.7). There exist a subsequence* $(\epsilon_{n_k})_{k\in\mathbb{N}}$ *and a function* $U^0 \in W^{2,1;\infty}_{x,t;\mathrm{loc}}(\mathbb{R}^2)$ *such that*
(i) *For any compact set* K *in* $\mathbb{R}^2$, $\lim_{k\to\infty} \|U^{\epsilon_{n_k}} - U^0\|_{L^\infty(K)} = 0$,
(ii) *The limit* U^0 *is nonnegative almost everywhere and it is a solution of*

$$\frac{\partial^2 U^0}{\partial x^2} - \frac{\partial U^0}{\partial t} = \mathbb{1}_{\{U^0>0\}}\ ,$$

(iii) 0 *belongs to the free boundary of the limit,* $\partial\{U^0 = 0\}$.

Proof – By Proposition II.11 and Ascoli-Arzela theorem (see for instance [Bre83], Theorem IV.24 p. 72), up to the extraction of a subsequence that we still denote by $(\epsilon_n)_{n\in\mathbb{N}}$, $(U^{\epsilon_n})_{n\in\mathbb{N}}$ uniformly converges to a nonnegative function $U^0 \in W^{2,1;\infty}_{x,t;\mathrm{loc}}(\mathbb{R}^2)$ in any compact set $K \subset\subset \mathbb{R}^2$. Let $P' \in \{U^0 > 0\}$. There exists $r > 0$ such that $U^0 > U^0(P')/2$ in $Q_r(P')$. Because of the uniform convergence, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n > N$ implies

$$U^{\epsilon_n}(P) \geq \frac{1}{4}\, U^0(P) > 0 \quad \forall\, P \in Q_r(P')\ .$$

In other words, $Q_r(P') \subset \{U^{\epsilon_n} > 0\}$ for $n > N$ and we can pass to the limit in the equation:

$$\frac{\partial^2 U^0}{\partial x^2} - \frac{\partial U^0}{\partial t} = 1 \quad \text{in} \quad Q_r(P')\ .$$

The equation $U^0_{xx} - U^0_t = \mathbb{1}_{\{U^0>0\}}$ holds both in $\{U^0 > 0\}$ and in the interior of $\{U^0 = 0\}$. Moreover $U^0_{xx} - U^0_t \leq 1$ in $\mathbb{R}^2$. Then from Proposition II.10, we deduce that $\partial\{U^0 = 0\}$ has zero Lebesgue measure which proves Assertion (ii).
To prove that $0 \in \partial\{U^0 = 0\}$ we first notice that $U^0(0) = 0$ by uniform convergence. Because of Proposition II.11 there exists a positive constant $\bar{C}$ such that for all $r > 0$ small enough,

$$\begin{aligned} \bar{C}\, r^2 &\leq \sup_{(x,t)\in Q_r^-(0)} U(x,t) = \sup_{(x,t)\in Q_r^-(0)} \epsilon_n^2\, U^{\epsilon_n}\left(\tfrac{x}{\epsilon_n}, \tfrac{t}{\epsilon_n^2}\right)\ , \\ \bar{C}\left(\tfrac{r}{\epsilon_n}\right)^2 &\leq \sup_{(x,t)\in Q^-_{r/\epsilon_n}(0)} U^{\epsilon_n}(x,t)\ . \end{aligned}$$

Replacing $\epsilon_n\, r$ by $\tilde{r}$, we obtain

$$\bar{C}\,\tilde{r}^2 \leq \sup_{Q^-_{\tilde{r}}(0)} U^{\epsilon_n} \to \sup_{Q^-_{\tilde{r}}(0)} U^0 \quad \text{as} \quad n\to\infty\ ,$$

which proves that $0 \in \partial\{U^0 = 0\}$. □

Lemma II.6 gives a much more detailed result than the statement of Proposition II.13, (iii).

Proposition II.14. *Under the assumptions of Proposition II.13,*

$$\mathbb{1}_{\{U^{\epsilon_n}>0\}} \to \mathbb{1}_{\{U^0>0\}} \quad \textit{in} \quad \mathbb{R}^2 \quad \textit{a.e.} \quad \textit{as} \quad n\to\infty\ ,$$

where $(U^{\epsilon_n})_{n\in\mathbb{N}}$ *is a convergent blow-up sequence associated to* U*, with blow-up limit* U^0.

Proof – From the proof of Proposition II.13 if $P \in \{U^0 > 0\}$ there exists N such that, if $\mathbb{N} \ni n > N$, then $P \in \{U^{\epsilon_n} > 0\}$. Assume now by contradiction that $P \in \mathrm{Int}\{U^0 = 0\}$ is such that $P \in \{U^{\epsilon_n} > 0\}$ for all $n \in \mathbb{N}$. By Proposition II.11, $\sup_{Q_r^-(P)} U^0 \geq \bar{C}\, r^2$, which means that $P \in \partial\{U^0 = 0\}$, and is a contradiction. To conclude we apply Proposition II.10 to U^0. □

3.4 A monotonicity formula and application to blow-up limits

Some monotonicity formulas have been introduced by G. Weiss in [Wei99] to study the elliptic obstacle problem and also by Giga and Kohn in [GK85], in a different context.

Proposition II.15 (Local monotonicity formula). *Under Assumption (II.8), if U is a solution of (II.7), then the function $t \mapsto \mathcal{E}(t;v)$ is a nonincreasing function, which is bounded from below and bounded in $W^{1,\infty}(-1,0)$, and such that for almost every $t \in (-1,0)$*

$$\frac{d}{dt}\mathcal{E}(t;v) = -\frac{1}{2\,(-t)^3}\int_{\mathbb{R}} |\mathcal{L}v(x,t)|^2\, G(x,t)\; dx\;.$$

Before to prove Proposition II.15, let us remark that a simple change of variable gives

$$\mathcal{E}(\lambda^2\, t; v) = \mathcal{E}(t; v^\lambda) \quad \forall t \in (-\lambda^{-2}, 0) \tag{II.11}$$

where $v^\lambda(x,t) := \lambda^{-2} v(\lambda\, x, \lambda^2\, t)$. Using (II.10), we obtain a characterization of the functions which are invariant under the scaling $v \mapsto v^\lambda$.

Corollary II.16 (Scale invariance of $\mathcal{E}$). *Let $v \in W^{2,1;\infty}_{x,t;\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_-)$. Then*

$$\mathcal{L}v \equiv 0 \Leftrightarrow \mathcal{E}(t;v) = \mathcal{E}(t;v^\lambda) \quad \forall\, t<0\,, \quad \forall\,\lambda>0\,.$$

Proof of Proposition II.15. We split it into two main steps.
First Step. Exactly as in [CPS04], we can evaluate the time derivative of the first term in the expression of $\mathcal{E}$. Assume that $v \in \mathcal{D}(\mathbb{R}\times[-1,0])$, let

$$\mathsf{e}(t;v) := \int_{\mathbb{R}} \left\{ \frac{1}{-t}\left(\left|\frac{\partial v}{\partial x}(x,t)\right|^2 + 2\,v(x,t)\right) - \frac{1}{t^2}\,v^2(x,t)\right\} G(x,t)\; dx$$

and compute $\frac{d}{d\lambda}\mathsf{e}(t;v_\lambda)$ at $\lambda=1$ using $\frac{d}{d\lambda}\,v_\lambda = \mathcal{L}v$ at $\lambda=1$, and $\mathsf{e}(\lambda^2\,t;v) = \mathsf{e}(t;v_\lambda)$:

$$\frac{d\mathsf{e}}{dt}(t;v) = \frac{1}{2t}D_v\mathsf{e}(t;v)\cdot\mathcal{L}v(x,t)\,,$$

where $D_v\mathsf{e}$ is defined for all ϕ in $\mathcal{C}^\infty(\mathbb{R}\times(-1,0\,))$ by

$$D_v\mathsf{e}(t;v)\cdot\phi := \int_{\mathbb{R}}\left\{\frac{1}{-t}\left(2\,\frac{\partial v}{\partial x}\cdot\frac{\partial \phi}{\partial x} + 2\,\phi\right)\right\} G\; dx - \int_{\mathbb{R}} \frac{2}{t^2}\,v\,\phi\,G\; dx\,.$$

To compute $D_v\mathsf{e}(t;v)\cdot\mathcal{L}v$, we integrate by parts.

$$D_v\mathsf{e}(t;v)\cdot\mathcal{L}v(x,t) = \int_{\mathbb{R}}\left\{\frac{2}{-t}\Big(1 - Hv(x,t)\Big) + \frac{1}{t^2}\,\mathcal{L}v(x,t)\right\}\mathcal{L}v(x,t)\,G(x,t)\; dx\;.$$

This proves

$$\frac{d}{dt}\,\mathsf{e}(t;v) = \int_{\mathbb{R}}\left\{\frac{1}{2t^3}\,|\mathcal{L}v(x,t)|^2 + \frac{1}{t^2}\mathcal{L}v(x,t)\Big(Hv(x,t)-1\Big)\right\} G(x,t)\; dx\;.$$

By density, the above expression also holds for a.e. time for any $v \in W^{2,1;\infty}_{x,t;\mathrm{loc}}$, with compact support, and the function $t \mapsto \mathsf{e}(t;v)$ is bounded from below and bounded in $W^{1,\infty}_{\mathrm{loc}}(-1,0)$.

Second Step. We prove that the function

$$s \mapsto \mathtt{r}(v;s) := \frac{1}{s^2}\int_{\mathbb{R}} (Hv(x,s)-1)\,\mathcal{L}v(x,s)\,G(x,s)\,dx$$

is integrable. The integral $\int_t^0 \frac{1}{s^2}\int_{\mathbb{R}} |(Hv(x,s)-1)\mathcal{L}v(x,s)\,G(x,s)|\;dx\,ds$ can indeed be bounded by (I) + (II), with

$$\begin{aligned}
\text{(I)} \;&:=\; \int_t^0 \frac{1}{s^2}\int_{\mathbb{R}} |(Hv(x,s)-(1+g(x,s)))\,\mathcal{L}v(x,s)\,G(x,s)|\;dx\,ds\;,\\
\text{(II)} \;&:=\; \int_t^0 \frac{1}{s^2}\int_{\mathbb{R}} |g(x,s)\,\mathcal{L}v(x,s)\,G(x,s)|\;dx\,ds\;.
\end{aligned}$$

By definition of $v(x,t) := U(x,t)\,\psi(x)$, $(Hv-(1+g))\,\mathcal{L}v$ vanishes on $(-1/2,1/2)$ because U is a solution of (II.7), and on $(-\infty,-1)\cup(1,+\infty)$ because of ψ. As a consequence of $\mathcal{P}_M$-estimates, there exists a constant $C>0$ such that $|\mathcal{L}v(x,t)|\le C(x^2+|t|)$. For $t\in(-1,0)$, with $c{:=}\,C'\,(\|\psi_{xx}\|_{L^\infty}\,\|U\|_{L^\infty(Q_1(0))}+\|\psi_x\|_{L^\infty}\,\|U_x\|_{L^\infty(Q_1(0))}\|+\|\psi\|_{L^\infty}(\|U_{xx}\|_{L^\infty(Q_1(0))}+\|U_t\|_{L^\infty(Q_1(0))})+\|1{+}g\|_{L^\infty(Q_1(0))})$, we get

$$\text{(I)} \le c\int_0^{|t|}\frac{ds}{s^2}\int_{1/2}^1 \frac{e^{-\rho^2/4s}}{\sqrt{2\pi\,s}}\,d\rho \le \frac{c}{2\sqrt{2\pi}}\int_0^{|t|}\frac{e^{-1/16s}}{s^{5/2}}\,ds\;.$$

With the change of variable

$$(s,x)\mapsto\left(\beta:=\sqrt{\frac{x^2-s}{-s}},\;\theta:=\sqrt{x^2-s}\right)$$

we get

$$\text{(II)} \le \text{Const}\int_1^{+\infty}\frac{\beta^3}{\sqrt{\beta^2-1}}\,e^{-\beta^2/4}\left(\int_0^{\theta_{\max}(\beta,|t|)}\frac{\sigma(\theta)}{\theta}\,d\theta\right)d\beta$$

where $\theta_{\max}(\beta,|t|):=\min(\beta\sqrt{|t|},\sqrt{1+|t|})$ and $\sigma(\theta):=\text{supess}_{\sqrt{x^2+|t|}\le\theta}\,g(x,t)$ is the modulus of continuity of g at the origin. By (II.8), σ is Dini-integrable, *i.e.* $\theta\mapsto\theta^{-1}\sigma(\theta)$ is integrable, which ends the proof. □

Remark II.1. *An inspection of the proof shows that Proposition II.15 holds under the following weaker conditions: $U\in\mathcal{P}_M(Q_1(0))$ is a solution of (II.7) and σ, defined as above, is Dini-integrable.*

Lemma II.17. *Under Assumption (II.2), consider a solution u of (II.1). Then for any $t_0<0$, $\Gamma\ni P\mapsto\mathcal{E}(v_P,t_0)$ is continuous.*

Proof – This is a straightforward consequence of the dominated convergence theorem of Lebesgue and the $\mathcal{P}_M$-bounds on the solution. □

Proposition II.15 applies to blow-up limits.

Proposition II.18 (Scale invariance of the blow-up limit for $t < 0$). *Under Assumption (II.8), consider a solution U of (II.7), and U^0 a blow-up limit corresponding to a blow-up sequence associated to U. Then U^0 is scale-invariant for $t < 0$:*

$$U^0(\lambda\, x, \lambda^2 t) = \lambda^2\, U^0(x,t) \quad \forall\, (x,t) \in \mathbb{R} \times (-\infty, 0)\ , \quad \forall\, \lambda > 0\ .$$

Proof – Consider as above $v(x,t) := U(x,t)\,\psi(x)$. Let $(v^{\epsilon_n})_{n\in\mathbb{N}}$ be a blow-up sequence associated to v, and v^0 a blow-up limit. By (II.11) we have

$$\mathcal{E}(\epsilon_n^2 t; v) = \mathcal{E}(t; v^{\epsilon_n}) \quad \forall\, t \in \left(-\epsilon_n^{-2}, 0\right)\ . \tag{II.12}$$

Since $\mathcal{E}$ is monotone nonincreasing and bounded from below by Proposition II.15, we may pass to the limit in (II.12) and obtain

$$\lim_{n\to\infty} \mathcal{E}(\epsilon_n^2 t; v) = \mathcal{E}(t; v^0) \quad \forall\, t < 0\ . \tag{II.13}$$

Note that because of the monotonicity of $\mathcal{E}$, $\lim_{n\to\infty} \mathcal{E}(\epsilon_n^2 t; v)$ does not depend on the subsequence. As a consequence,

$$0 = \frac{d}{dt}\mathcal{E}(t; v^0) = \frac{1}{2\,t^3} \int_{\mathbb{R}} |\mathcal{L}v^0(x,t)|^2\, G(x,t)\, dx \quad \forall\, t < 0$$

and v^0 is scale invariant by (II.10). Since $U^{\epsilon_n}(x,t) = v^{\epsilon_n}(x,t)$ for any x, t such that $|\epsilon_n\, x| \le 1/2$, $-1 < \epsilon_n^2\, t < 0$, we have: $U^0 \equiv v^0$, which ends the proof. □

3.5 Classification of the blow-up limits

According to Proposition II.13, blow-up limits are solutions in $\mathbb{R}^2$ of the parabolic obstacle problem with constant coefficients:

$$\begin{cases} Hu^0(x,t) = \mathbb{1}_{\{u^0>0\}}(x,t) \quad (x,t) \in \mathbb{R}^2 \text{ a.e.} \\ u^0(x,t) \ge 0 \quad (x,t) \in \mathbb{R}^2 \text{ a.e.} \\ 0 \in \partial\{u^0 > 0\} \end{cases} \tag{II.14}$$

which are scale-invariant in $\mathbb{R} \times (-\infty, 0)$ by Proposition II.18. For all $(x,t) \in \mathbb{R}^2$, define the functions:

$$u^0_+(x,t) := \tfrac{1}{2}\,(\max\{0, x\})^2\ ,$$

$$u^0_-(x,t) := \tfrac{1}{2}\,(\max\{0, -x\})^2\ ,$$

$$u^0_m(x,t) := \begin{cases} m\,t + \frac{1+m}{2}\,x^2 & \text{if } t < 0\ , \\ \max\left\{0, t\,V\left(\frac{|x|}{\sqrt{t}}\right)\right\} & \text{if } t \ge 0\ , \end{cases}$$

where $m \in [-1, 0]$ and $V(\xi) = -1 + C_1(a)\,(\xi^2+2) + C_2(a)\,\left(2\,\xi\, e^{-\xi^2/4} + (\xi^2+2)\int_0^\xi e^{-s^2/4}\, ds\right)$. The constants $C_1(a)$ and $C_2(a)$ are given by

$$C_1(a) = -\frac{1}{4}\left(2 + e^{a^2/4}\int_0^a e^{-s^2/4}\, ds\right)\ , \quad C_2(a) = \frac{a}{4}\, e^{a^2/4}\ .$$

where the parameter $a \in [0, +\infty]$ is uniquely determined in terms of m by the equation

$$1 + m = 2(C_1(a) + \sqrt{\pi}\, C_2(a)) \,. \tag{II.15}$$

The limiting cases correspond to

$$\begin{array}{lll} m = -1\,, & a = 0\,, & u^0_{-1}(x,t) = \max\{0, -t\}\,, \\ m = 0\,, & a = +\infty\,, & u^0_0(x,t) = \frac{1}{2}\, x^2\,. \end{array}$$

We have the following classification result.

Theorem II.19 (A Liouville type result). *Consider a solution* $u^0 \in W^{2,1;\infty}_{x,t;\mathrm{loc}}(\mathbb{R}^2)$ *of (II.14) with* u^0_{xx} *and* u^0_t *bounded. If* u^0 *is such that*

$$u^0(\lambda\, x, \lambda^2\, t) = \lambda^2 u^0(x,t) \quad \forall\, (x,t) \in \mathbb{R} \times (-\infty, 0)\,, \quad \forall\, \lambda \in (0, +\infty)\,,$$

then $u^0 = u^0_+$*,* $u^0 = u^0_-$ *or* $u^0 = u^0_m$ *for some* $m \in [-1, 0]$*.*

Proof – We first classify the solutions in $\mathbb{R} \times (-\infty, 0)$. Then we extend the solutions to $\mathbb{R}^2$.

First Step: Classification in $\mathbb{R} \times (-\infty, 0)$. This result is given in [CPS04]. We reproduce it for completeness.

(1) Assume first that the interior of $\{u^0 = 0\} \cap \{t < 0\}$ is non-empty. Because of the self-similarity property, the function $V(\xi) := u^0(\xi, -1)$ is such that $v(x,t) = |t|\, V(x/\sqrt{-t})$ and it is solution in $\{u > 0\}$ of

$$V''(\xi) + V(\xi) - \frac{\xi}{2}\, V'(\xi) = 1\,.$$

A direct computation gives $V(\xi) = 1 + C_1\, (\xi^2 - 2) + C_2\, \left(-2\,\xi\, e^{\xi^2/4} + (\xi^2 - 2) \int_0^\xi e^{s^2/4}\, ds\right)$. Because of the regularity of u^0, we have to choose $a \in \mathbb{R}$ such that $V(a) = V'(a) = 0$. The functions $\xi \mapsto \xi^2 - 2 =: V_1(\xi)$ and $\xi \mapsto -2\,\xi\, e^{\xi^2/4} + (\xi^2 - 2) \int_0^\xi e^{s^2/4}\, ds =: V_2(\xi)$ are respectively even and odd, so there is no restriction to take $a \geq 0$, up to a sign change of C_1 and C_2. This amounts to

$$C_1 = \frac{1}{2} - \frac{a}{4}\, e^{-a^2/4} \int_0^a e^{s^2/4}\, ds \quad \text{and} \quad C_2 = \frac{a}{4}\, e^{-a^2/4}\,.$$

Note that $V''(a) = 1$ and $V'''(\xi) = 2\, C_2\, e^{\xi^2/4}$. If $a \neq 0$, this clearly contradicts the nonnegativity of V and we have therefore $a = 0$, $C_1 = 1/2$: $V(\xi) = \xi^2/2$ in $\{u^0 > 0\}$, or, equivalently, $u^0 = u^0_\pm$ since

$$\text{either} \quad V(\xi) = \frac{1}{2}\, (\max\{0, \xi\})^2 \quad \text{or} \quad V(\xi) = \frac{1}{2}\, (\max\{0, -\xi\})^2\,.$$

(2) Assume now that $\{u^0 = 0\} \cap \{t < 0\}$ is of empty interior: by Theorem II.7, Γ has zero Lebesgue measure and for almost all $(x,t) \in \mathbb{R} \times (-\infty, 0)$, $Hu^0(x,t) = 1$. As a consequence, $Hu^0_t = 0$ in $\mathbb{R} \times (-\infty, 0)$. Since u^0_t is bounded, $m := u^0_t$ has to be a constant by Liouville's principle (see for instance [Wid75], Chapter XIV, Theorem 1.2). Integrating with respect to t, we get: $u^0(x,t) = m\, t + v^0(x,0)$ with $u^0_{xx}(x,0) = 1 + m$. Taking into account the conditions $u^0 \geq 0$ and $u^0(0) = 0$, an integration with respect to x gives

$u^0(x,0) = (1+m)\,x^2/2$. Therefore $v^0(x,t) = v^0_m(x,t) := mt + (1+m)x^2/2$ in $\mathbb{R}\times(-\infty,0)$. Since u^0 is nonnegative, this implies that $m \in [-1,0]$.

Second Step: Classification in $\mathbb{R}^2$. The solution of (II.14) is uniquely extended to the domain corresponding to $t>0$, once it is known for $t<0$.
(1) If $u^0 = u^0_\pm$ in $\mathbb{R}\times(-\infty,0)$ a.e., by unique continuation $u^0 = u^0_\pm$ in $\mathbb{R}^2$.
(2) If $u^0 = u^0_m$ for some $m\in[-1,0]$, in $\mathbb{R}\times(-\infty,0)$ a.e., as in the first step of the proof, we may use the scale invariance. In the interior of $\{u^0>0\}\cap\{t>0\}$, the function $V(\xi) := u^0(\xi,1)$ is such that $u^0(x,t) = t\,V(x/\sqrt{t})$ is solution of

$$V''(\xi) - V(\xi) + \frac{\xi}{2}\,V'(\xi) = 1\,.$$

A direct computation gives $V(\xi) = -1 + C_1\,(\xi^2+2) + C_2\left(2\,\xi\,e^{-\xi^2/4} + (\xi^2+2)\int_0^\xi e^{-s^2/4}\,ds\right)$. The free boundary condition $V(a) = V'(a) = 0$ allows to parametrize C_1 and C_2 in terms of a: $C_1(a) = -\frac{1}{4}\left(2 + e^{a^2/4}\int_0^a e^{-s^2/4}\,ds\right)$ and $C_2(a) = \frac{a}{4}\,e^{a^2/4}$. Taking the limit $t\to 0$, $t>0$, we get

$$u^0(x,0) = (C_1 + \sqrt{\pi}\,C_2)\,x^2$$

that we have to identify with $\lim_{t\to 0,\,t<0} u^0_m(x,t) = \frac{1}{2}(1+m)x^2$. The point $\xi = a$ corresponds to $t = x^2/a^2$ it remains to characterize the solution in $(-\infty,-a)$. As $V_1(\xi) := \xi^2+2$ is even and $V_2(\xi) := 2\,\xi\,e^{-\xi^2/4} + (\xi^2+2)\int_0^\xi e^{-s^2/4}\,ds$ is odd we can keep the same C_1 and C_2 by replacing V_2 by $-V_2$. This provides (II.15) and completes the proof of Theorem II.19. $\square$

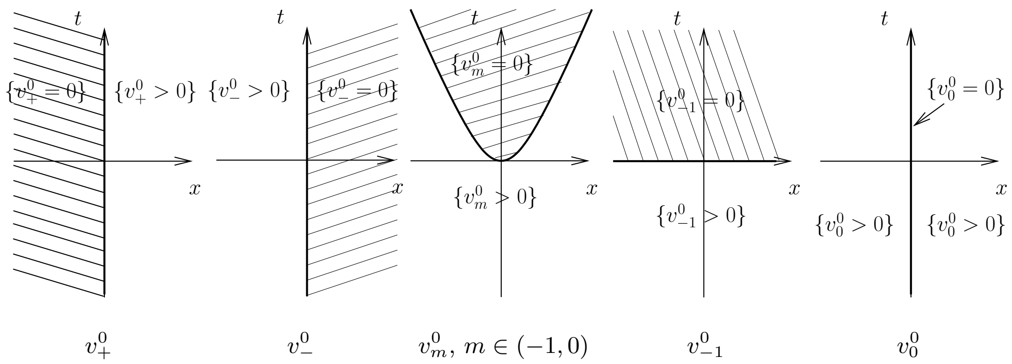

Figure II.2: *Solutions of Theorem II.19. The equation of the free boundary associated to* u^0_m *for* $m\in(-1,0)$ *is* $t(x) = a^{-2}\,x^2$*, where* a *and* m *are related by (II.15).*

4 Regular and singular points of the free boundary

4.1 An energy characterisation

As in Section 3.2, to a nonnegative cut-off function $\psi\in\mathcal{D}(\mathbb{R})$ such that $\psi\equiv 1$ on $(-1/2,1/2)$ and $\psi\equiv 0$ on $(-\infty,-1]\cup[1,\infty)$, we associate the function $v(x,t) := U(x,t)\,\psi(x)$, $(x,t)\in\mathbb{R}\times(-1,0)$ where U is given in terms of a solution of (II.1) as in Section 3.1 for some $P_1\in\Gamma$, and solves (II.7) (also see Equation (II.3)). The localized energy is defined by

(II.9). As in Section 3.2, we omit the index P_1 whenever there is no ambiguity. Otherwise, we write $v_{P_1} = v$. We refer to Section 3.5 for the definition of $u^0_\pm$ and u^0_m.

Proposition II.20 (Energy characterisation of the points of Γ). *Let u be a solution of (II.1) and consider $P_1 \in \Gamma$ such that $Q_r(P_1) \subset Q_R(P_0)$ for some $r > 0$. With the above notations and under Assumption (II.2), if u^0 is a blow-up limit associated to u, then*

$$\Lambda(u) := \lim_{\substack{\tau\to 0\\ \tau<0}} \mathcal{E}(\tau; u) \in \{\sqrt{2}, \sqrt{2}/2\} = \mathcal{E}(t; u^0) \quad \forall\, t < 0\,.$$

If $\Lambda(u) = \sqrt{2}/2$, then $u^0 = u^0_\pm$. If $\Lambda(u) = \sqrt{2}$, then $u^0 = u^0_m$ for some $m \in [-1, 0]$.

Proof – The uniqueness of the limit of the energy is a consequence of the monotone decay of $\mathcal{E}$, according to Proposition II.15, and of (II.13).

Since a blow-up limit is scale invariant by Proposition II.18, by (II.10) and Proposition II.15, $\mathcal{E}(t; u^0)$ does not depend on $t < 0$. By Theorem II.19, the only possible values of $\Lambda(u)$ are $\mathcal{E}(t; u^0_\pm)$ and $\mathcal{E}(t; u^0_m)$, $m \in [-1, 1]$. Using $\mathcal{L}u^0 = 0$ and integrating by parts with respect to x, we get

$$\begin{aligned}
\mathcal{E}(t; u^0) &= \int_{\mathbb{R}} \left\{ \frac{1}{-t}\left(\left|\frac{\partial u^0}{\partial x}\right|^2 + 2\,u^0 \right) - \frac{1}{t^2}\,(u^0)^2 \right\} G(x,t)\; dx \\
&= \int_{\mathbb{R}} \left\{ \frac{1}{-t}\left(-\frac{\partial^2 u^0}{\partial x^2} - \frac{x}{2\,t}\frac{\partial u^0}{\partial x} + 2 \right) u^0 - \frac{1}{t^2}(u^0)^2 \right\} G(x,t)\; dx \\
&= \int_{\mathbb{R}} \left\{ \frac{1}{-t}\left(-H u^0 + 2 \right) + \frac{1}{2\,t^2}\,\mathcal{L}u^0 \right\} u^0 G(x,t)\; dx
\end{aligned}$$

Using again $\mathcal{L}u^0 = 0$ and Equation (II.7), we get

$$\mathcal{E}(t; u^0) = \int_{\mathbb{R}} \frac{1}{-t}\left(-\mathbb{1}_{\{u^0>0\}} + 2 \right) u^0\, G(x,t)\; dx\,.$$

Taking into account that $\mathcal{E}(t; u^0) = \mathcal{E}(1; u^0)$, this amounts to

$$\mathcal{E}(t; u^0) = \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} \left(-\mathbb{1}_{\{u^0>0\}} + 2 \right) u^0(x,-1)\, e^{-x^2/4}\; dx\,.$$

We easily conclude that

$$\mathcal{E}(t; u^0_+) = \int_0^\infty \frac{x^2}{2}\,\frac{e^{-\frac{x^2}{4}}}{\sqrt{2\,\pi}}\; dx = \frac{\sqrt{2}}{2}\,,$$

and

$$\mathcal{E}(t; u^0_m) = \int_{\mathbb{R}} \left(-m + \frac{1+m}{2}\,x^2 \right) \frac{e^{-\frac{x^2}{4}}}{\sqrt{2\,\pi}}\; dx = \int_{\mathbb{R}} \frac{e^{-\frac{x^2}{4}}}{\sqrt{2\,\pi}}\; dx = \sqrt{2}\,.$$

□

Proposition II.20 allows to divide the free boundary in two sets, depending on the value of $\Lambda(u)$. Recall that according to the notations of Section 3.2, the function u depends on $P_1 \in \Gamma$. When there is no ambiguity on the blow-up point, we will denote the blow-up limit by $u^0_{m(P_1)}$, consistently with the notations of Section 3.5, and by $u^0_{P_1}$ when the point of blow-up P_1 is not fixed.

To emphasize the dependence of u on the point $P_1 \in \Gamma$, we will write explicitly the index and note u_{P_1} in the rest of this section.

Definition II.21 (Regular and singular points). *Under the assumptions of Proposition II.20, a point $P_1 \in \Gamma$ is said to be* regular *(respectively* singular*) if $\Lambda(u_{P_1}) = \sqrt{2}/2$ (respectively if $\Lambda(u_{P_1}) = \sqrt{2}$). We will denote by $\mathcal{R}$ the set of regular points, and by $\mathcal{S}$ the set of singular points.*

4.2 First topological properties of the regular and singular sets

Lemma II.22 (Topological properties of $\mathcal{R}$ and $\mathcal{S}$). *Under Assumption (II.2), $\mathcal{S}$ is a closed set, and $\mathcal{R} = \Gamma \setminus \mathcal{S}$ is open in Γ.*

Proof – Let P_1, $P_2 \in \Gamma$ and take $t_0 < t < 0$. We may write

$$\mathcal{E}(t, v_{P_2}) - \mathcal{E}(t, v_{P_1}) = \mathcal{E}(t_0, v_{P_2}) - \mathcal{E}(t_0, v_{P_1}) + \mathcal{E}(t, v_{P_2}) - \mathcal{E}(t_0, v_{P_2}) + \mathcal{E}(t_0, v_{P_1}) - \mathcal{E}(t, v_{P_1}) \; .$$

Since the function $t \mapsto \mathcal{E}(t, v_{P_1})$ is monotone nonincreasing, $\mathcal{E}(t, v_{P_2}) - \mathcal{E}(t_0, v_{P_2}) \leq 0$. Passing to the limit $t \to 0$, we get

$$\Lambda(v_{P_2}) - \Lambda(v_{P_1}) \leq \mathcal{E}(t_0, v_{P_2}) - \mathcal{E}(t_0, v_{P_1}) + \mathcal{E}(t_0, v_{P_1}) - \Lambda(v_{P_1}) \; .$$

We fix P_1 and will move P_2 close to P_1. For $|t_0|$ small enough, $\mathcal{E}(t_0, v_{P_1}) - \Lambda(v_{P_1})$ can be chosen arbitrarily small. Now, from Lemma II.17 for a fixed t_0, $P_2 \mapsto \mathcal{E}(t_0, v_{P_2})$ is continuous, so that $\mathcal{E}(t_0, v_{P_2}) - \mathcal{E}(t_0, v_{P_1})$ can also be chosen arbitrarily small for P_2 close enough to P_1. Then $\limsup_{P_2 \to P_1} \Lambda(v_{P_2}) \leq \Lambda(v_{P_1})$, i.e. the function $\Gamma \ni P \mapsto \Lambda(v_P)$ is upper semi-continuous. If $\Lambda(v_{P_1}) = \sqrt{2}/2$, then $\Lambda(v_{P_2}) = \sqrt{2}/2$ for P_2 in a neighborhood of P_1. This proves that $\mathcal{R}$ is an open set in Γ. □

5 Study of the singular points of the free boundary

5.1 A monotonicity formula for singular points

We adapt a monotonicity formula for the elliptic obstacle problem [Mon03] to the parabolic case. As in the second step of the proof of Proposition II.15, let

$$\mathtt{r}(v; s) := \frac{1}{s^2} \int_{\mathbb{R}} (Hv(x, s) - 1)\, \mathcal{L}v(x, s)\, G(x, s)\, dx \; .$$

With the notations of Sections 3.2 and 3.4 and 3.5 consider $v = v_{P_1}$ given by $v_{P_1}(x, t) := U(x, t)\, \psi(x)$, for some fixed point $P_1 \in \mathcal{S}$ and u^0_m one of the blow-up limit of v_{P_1}. We define the functional

$$\Phi_m(t; v) := \frac{1}{t^2} \int_{\mathbb{R}} \left| v - u^0_m \right|^2 G\, dx - \int_t^0 \frac{2}{s^2} \int_{\mathbb{R}} (Hv - 1)\, (v - u^0_m)\, G\, dx\, ds + \int_t^0 \frac{2}{s} \int_s^0 \mathtt{r}(\theta; v)\, d\theta\, ds \; .$$

Proposition II.23 (Local monotonicity formula for singular points). *Under Assumption (II.8), let U be a solution of (II.7). With the above notations the function $t \mapsto \Phi_m(t; v)$ is nonincreasing, bounded in $W^{1,1}(-1, 0)$.*

Proof – By density, it is sufficient to prove the result for a smooth function v as in the proof of Proposition II.15. Let $w := v - u^0_m$. Using the change of variable $x =$

$\sqrt{-t}\,y$, since $\frac{1}{t^2}\int_{\mathbb{R}} w^2(x,t)\,G(x,t)\,dx = \int_{\mathbb{R}} \frac{1}{t^2}\,w^2(\sqrt{-t}\,y,t)\,G(y,1)\,dx$ and $\frac{d}{dt}w(\sqrt{-t}\,y,t) = -\frac{y}{2\sqrt{-t}}\,\frac{\partial w}{\partial x}(\sqrt{-t}\,y,t) + \frac{\partial w}{\partial t}(\sqrt{-t}\,y,t)$, we get

$$\frac{d}{dt}\left[\frac{1}{t^2}\int_{\mathbb{R}} w^2(x,t)\,G(x,t)\;dx\right] = \frac{1}{t^3}\int_{\mathbb{R}} \mathcal{L}v(x,t)\,w(x,t)\,G(x,t)\;dx\;.$$

Let $\mathtt{e}(t;v) := -\int_{\mathbb{R}}\left\{\frac{1}{t}(|v_x|^2+2\,v)+\frac{1}{t^2}\,v^2\right\}G(x,t)\,dx$ be as in the first step of the proof of Proposition II.15.

$$\mathtt{e}(t;v)-\mathtt{e}(t;u^0_m) = -\int_{\mathbb{R}}\left[\frac{1}{t}\left(\frac{\partial}{\partial x}(v+u^0_m)\,\frac{\partial w}{\partial x}+2\,w\right)+\frac{1}{t^2}\,(v+u^0_m)\,w\right]G\;dx\;.$$

Integrating by parts with respect to x and using $Hu^0_m = 1$ and $\mathcal{L}u^0_m = 0$ for every $t<0$, we get

$$\begin{aligned}\mathtt{e}(t;v)-\mathtt{e}(t;u^0_m) &= \int_{\mathbb{R}}\left[\frac{1}{t}\left(\frac{\partial^2}{\partial x^2}(v+u^0_m)+\frac{x}{2\,t}\frac{\partial}{\partial x}(v+u^0_m)-2\right)w-\frac{1}{t^2}\,(v+u^0_m)\,w\right]G\\ &= \int_{\mathbb{R}}\left[\frac{1}{t}\,(Hv-1)+\frac{1}{2\,t^2}\mathcal{L}v\right]w\,G\;dx\;.\end{aligned}$$

Thus

$$\frac{d}{dt}\left[\frac{1}{t^2}\int_{\mathbb{R}} w^2(x,t)\,G(x,t)\;dx\right] = \frac{2}{t}\,\left[\mathtt{e}(t;v)-\mathtt{e}(t;u^0_m)\right]-\frac{2}{t^2}\int_{\mathbb{R}}(Hv-1)\;w\,G\;dx\;,$$

$$\frac{d}{dt}\Phi_m(t;v) = \frac{2}{t}\,\left[\mathtt{e}(t;v)-\mathtt{e}(t;u^0_m)\right]-\frac{2}{t}\int_t^0 \mathtt{r}(s;v)\,ds\;.$$

Recall that $\mathcal{E}(t;v) = \mathtt{e}(t;v)-\int_t^0 \mathtt{r}(v;s)\,ds$ by definition of $\mathcal{E}$, $\mathtt{e}$ and $\mathtt{r}$, and for any $t<0$, $\mathtt{e}(t;u^0_m) = \sqrt{2} = \lim_{\tau\to 0}\mathcal{E}(\tau;v)$ according to Proposition II.20. Thus

$$\frac{d}{dt}\Phi_m(t;v) = \frac{2}{t}\left[\mathcal{E}(t;v)-\lim_{\tau\to 0}\mathcal{E}(\tau;v)\right] \tag{II.16}$$

is nonpositive by Proposition II.15. It remains to prove that

$$(\mathrm{I}) = \frac{2}{t}\int_t^0 \mathtt{r}(s;v)\,ds \quad\text{and}\quad (\mathrm{II}) = \frac{-2}{t^2}\int_{\mathbb{R}}(Hv-1)\;w\,G\;dx\;,$$

are integrable. (I) can be evaluated as in the second step of the proof of Proposition II.15, using the integrability of

$$t\mapsto \frac{1}{|t|}\int_0^{|t|}\frac{e^{-1/16s}}{s^{5/2}}\;ds \;\text{ and }\; t\mapsto\frac{1}{|t|}\int_1^{+\infty}\frac{\beta^3}{\sqrt{\beta^2-1}}\,e^{-\beta^2/4}\left(\int_0^{\min\left(\beta\sqrt{|t|},\sqrt{1+|t|}\right)}\sigma(\theta)\,\frac{d\theta}{\theta}\right)d\beta\;,$$

where $\sigma(\theta) := \operatorname{supess}_{\sqrt{x^2+|t|}\le\theta}\, g(x,t) \le \mathrm{Const}\cdot\theta^\alpha$ by Assumption (II.8).

As for (II), in $\{v>0\}$, $Hv-1 = g$, and $v\equiv 0$ in $(\mathbb{R}\setminus(-1,1))\times(-1,0)$, so we have:

$$-\frac{1}{2}\times(\mathrm{II}) = \frac{1}{t^2}\int_{|x|<1/2} g\mathbb{1}_{\{v>0\}}\;w\,G\;dx+\frac{1}{t^2}\int_{1/2<|x|<1}(Hv-1)\,w\,G\;dx+\frac{1}{t^2}\int_{|x|>1}u^0_m\,G\;dx\;.$$

The last term is integrable: a trivial change of variable shows that the exponential decay is the dominant factor. The second term is integrable because of the gaussian weight, as in the second step of the proof of Proposition II.15: the function $t \mapsto |t|^{-5/2} \int_{1/2}^{1} e^{-s^2/(-4t)}\, ds$ is indeed integrable. The first term $|t|^{-2} \int_{|x|<r/2} g\, \mathbb{1}_{\{v>0\}}\, w\, G\, dx$ is also integrable: using the change of variables $(s,x) \mapsto (\beta,\theta)$ as in the second step of the proof of Proposition II.15 again, we can conclude as above. $\square$

Remark II.2. *As for the local monotonicity formula for $\mathcal{E}$ studied in Proposition II.15, an inspection of the proof shows that a sufficient condition for the proof of Proposition II.23 is that the map $\alpha \longmapsto \frac{1}{\alpha}\int_0^\alpha \frac{\sigma(\theta)}{\theta}\, d\theta$ is integrable, i.e. σ is twice Dini-integrable.*

As a consequence, we can state the following result.

Corollary II.24. *Under Assumption (II.2), consider a solution u of (II.1). Let us fix $P_1 \in \mathcal{S}$. Note $v^0_{m(P_1)}$ a blow-up limit in P_1. Then for any $r \in (0,R)$ there exists $t_r < 0$ and a continuous function $\mathtt{s} : (t_r, 0] \times Q_r(P_0) \cap \mathcal{S} \to \mathbb{R}$ with $\mathtt{s}(0,P_1) = 0$ such that for any $P_2 \in Q_r(P_0) \cap \mathcal{S}$ and $t \in (t_r, 0)$ we have*

$$\Phi_{m(P_1)}(t; v_{P_2}) \le \Phi_{m(P_1)}(t; v_{P_1}) + \mathtt{s}(t, P_2)\ .$$

Similarly there exists a continuous function $\tilde{\mathtt{s}}$ satisfying $\tilde{\mathtt{s}}(0,P_1) = 0$, such that

$$\int_{\mathbb{R}} \frac{1}{t^2} \left| v_{P_2} - v^0_{m(P_1)} \right|^2 G(x,t)\, dx \le \int_{\mathbb{R}} \frac{1}{t^2} \left| v_{P_1} - v^0_{m(P_1)} \right|^2 G(x,t)\, dx + \tilde{\mathtt{s}}(t, P_2)\ .$$

Proof – The point P_1 is fixed and we write for $t_0 < t < 0$,

$$\Phi(t; v_{P_2}) - \Phi(t; v_{P_1}) = \Phi(t; v_{P_2}) - \Phi(t_0; v_{P_2}) + \Phi(t_0; v_{P_2}) - \Phi(t_0; v_{P_1}) + \Phi(t_0; v_{P_1}) - \Phi(t; v_{P_1})\ .$$

By the monotonicity formula, the first term satisfies $\Phi(t; v_{P_2}) - \Phi(t_0; v_{P_2}) \le 0$. There exists a modulus of continuity $\omega_{t_0}(d)$, continuous in (t_0, d) such that $\omega_{t_0}(0) = 0$ and

$$|\Phi(t_0; v_{P_2}) - \Phi(t_0; v_{P_1})| \le \omega_{t_0}(|P_2 - P_1|)\ .$$

Finally there exists a monotone modulus of continuity ω such that

$$|\Phi(t; v_{P_1}) - \Phi(0; v_{P_1})| \le \omega(|t|)\ .$$

Therefore we get

$$\Phi(t; v_{P_2}) - \Phi(t; v_{P_1}) \le \mathtt{s}(t, P_2)$$

with

$$\mathtt{s}(t; P_2) = \inf_{t_0,\, t_r < t_0 < t} \left(\omega_{t_0}(|P_2 - P_1|) + 2\omega(|t_0|)\right)$$

We now prove the second inequality. A careful investigation of the proof of Proposition II.23 shows that the estimates on (I) and (II) are uniform with respect to the point $P_2 \in \mathcal{S}$. So there exists $t \mapsto \tilde{c}_1(t)$ which tends to zero when t tends to zero such that uniformly in $P_2 \in \mathcal{S}$, we have

$$\left| \Phi_{m(P_1)}(t; v_{P_2}) - \frac{1}{t^2}\int_{\mathbb{R}} \left| v_{P_2} - u^0_{m(P_1)} \right|^2 G\, dx \right| \le \tilde{c}_1(t)$$

This implies the result with $\tilde{\mathtt{s}}(t, P_2) = \mathtt{s}(t, P_2) + 2\tilde{c}_1(t)$. $\square$

5.2 Scale invariance and blow-up limits

A simple change of variable gives

$$\Phi_m(\lambda^2\, t; v) = \Phi_m(t; v^\lambda) \quad \forall\, t \in (-\lambda^{-2}, 0)\,, \quad \forall\, \lambda > 0\,, \tag{II.17}$$

where $v^\lambda(x,t) := \lambda^{-2} v(\lambda\, x, \lambda^2\, t)$. If we replace $\mathcal{E}$ by Φ_m, we have a result which is similar to Corollary II.16 and Proposition II.20.

Proposition II.25 (Scale invariance of Φ_m and consequences). *Under Assumption (II.2), consider a solution u of (II.1). For some $P_1 \in \mathcal{S}$ define u as in Section 3.2 and take $m \in [-1, 0]$. Consider a blow-up limit u^0 associated to u. Then*

$$\lim_{\substack{\tau\to 0\\ \tau<0}} \Phi_m(\tau; u) = \Phi_m(t; u^0) \quad \forall\, t < 0\,.$$

with

$$\Phi_m(t; u^0) = \frac{1}{t^2} \int_{\mathbb{R}} \left|u^0 - u^0_m\right|^2 G\, dx \quad \forall\, t < 0\,.$$

In the particular case where we choose $v^0_m = v^0$, we get $\lim_{\tau\to 0} \Phi_m(\tau; u) = \Phi_m(t; u^0_m) = 0$ *for all $t < 0$.*

5.3 Uniqueness of the blow-up limit at singular points

Proposition II.26. *Under Assumption (II.8) consider a solution U of (II.7) and $v = v_{P_1}$ given by $v_{P_1}(x,t) := U(x,t)\,\psi(x)$, for some fixed point $P_1 \in \mathcal{S}$. There exists a unique $m \in [-1, 0]$ such that for any sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converging to 0, the whole blow-up sequence $(U^{\epsilon_n})_{n\in\mathbb{N}}$ locally uniformly converges to u^0_m.*

Proof – Let $(v^{\epsilon_{n,1}})_{n\in\mathbb{N}}$ and $(v^{\epsilon_{n,2}})_{n\in\mathbb{N}}$ be two blow-up sequences associated to v, with blow-up limits $u^0_{(1)}$ and $u^0_{(2)}$. Assume that $u^0_{(1)} = u^0_m$. By (II.17),

$$\Phi_m(\epsilon^2_{n,1}\, t; v) = \Phi_m(t; v^{\epsilon_{n,1}}) \to \Phi_m(t; u^0_{(1)}) = 0 \quad \text{as } n \to \infty\,.$$

With no restriction, we may assume that $\epsilon_{n,2} \leq \epsilon_{n,1}$, so that by Proposition II.23,

$$\Phi_m(\epsilon^2_{n,1}\, t; v) \geq \Phi_m(\epsilon^2_{n,2}\, t; v) = \Phi_m(t; v^{\epsilon_{n,2}})\,.$$

Passing to the limit $n \to \infty$, we get

$$0 \geq \Phi_m(t; u^0_{(2)}) = \frac{1}{t^2} \int_{\mathbb{R}} \left|u^0_{(2)} - u^0_m\right|^2 G(x,t)\, dx \geq 0$$

since $\mathcal{L}u^0_{(2)} = 0$, $\mathtt{r}(u^0_{(2)}; t) = 0$ and $Hu^0_{(2)} \equiv 1$ for $t < 0$, since $\Lambda(u^0_{(2)}) = \sqrt{2}$ by Proposition II.20. This proves that $u^0_{(2)} = u^0_m = u^0_{(1)}$.

For any $(x,t) \in Q_{1/(2\epsilon_{n,i})}(0)$, $i = 1, 2$, $U^{\epsilon_{n,i}}$ coincides with $v^{\epsilon_{n,i}}$. This proves the uniqueness of the blow-up limit of U. □

To any $P_1 \in \Gamma$, we can therefore associate a unique $m(P_1) := m \in [-1, 0]$ such that the blow-up limit of a solution at this point is u^0_m. For any $m \in [-1, 0]$, we set

$$\mathcal{S}_m = \{P_1 \in \Gamma \;:\; m(P_1) = m\}\,.$$

5.4 Continuity properties of the singular set

Lemma II.27 (Continuity of the blow-up limit). *The function $P_1 \mapsto m(P_1)$ is continuous on $\mathcal{S}$.*

Proof – Let $P_1 \in \mathcal{S}$. From Corollary II.24 and the scale invariance of the monotonicity formula, we have with $v_{P_2}^{|t|}(y,\tau) = \frac{1}{t^2} v_{P_2}(|t|y, t^2\tau)$:

$$\Phi_{m(P_1)}(-1; v_{P_2}^{|t|}) \le \Phi_{m(P_1)}(-1; v_{P_1}^{|t|}) + \mathtt{s}(t, P_2) \ .$$

At the limit $t = 0$, we get

$$\Phi_{m(P_1)}(-1; v^0_{m(P_2)}) \le \Phi_{m(P_1)}(-1; v^0_{m(P_1)}) + \mathtt{s}(0, P_2)$$

i.e.

$$\int_{\mathbb{R}} \left| u^0_{m(P_2)} - u^0_{m(P_1)} \right|^2 G(x,-1)\, dx \le \mathtt{s}(0, P_2) \ .$$

The contiuity of $\mathtt{s}$ joint to the fact that $\mathtt{s}(0, P_1) = 0$ implies that

$$\lim_{P_2 \to P_1} m(P_2) = m(P_1)$$

□

Lemma II.28 (A uniform continuity result). *For any $r \in (0, R)$, there exists $t_r < 0$ such that for any $t \in (t_r, 0)$, if v_P is given in terms of U as in Section 3.2, where U is a solution of (II.7), and if (II.8) is satisfied, then*

$$\lim_{\epsilon \to 0} \quad \sup_{P \in \mathcal{S} \cap \overline{Q_r(P_0)}} \quad \frac{1}{t^2} \int_{\mathbb{R}} \left| v^{\epsilon}_P(x,t) - u^0_{m(P)}(x,t) \right|^2 G(x,t)\, dx = 0 \ .$$

Proof – Consider a monotone decreasing sequence $(\epsilon_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \epsilon_n = 0$ and a sequence $(P_n)_{n\in\mathbb{N}}$ of points in $\mathcal{S} \cap \overline{Q_r(P_0)}$, and assume by contradiction that

$$\lim_{n\to\infty} \quad \frac{1}{t^2} \int_{\mathbb{R}} \left| v^{\epsilon_n}_{P_n}(x,t) - u^0_{m(P_n)}(x,t) \right|^2 G(x,t)\, dx =: l > 0 \ .$$

We also assume that $P_n \to P_\infty \in \mathcal{S} \cap \overline{Q_r(P_0)}$. We first remark that by the scale invariance we have

$$\frac{1}{t^2} \int_{\mathbb{R}} \left| v^{\varepsilon_n}_{P_n} - v^0_{m(P_n)} \right|^2 G(x,t)\, dx = \frac{1}{(\varepsilon_n t)^2} \int_{\mathbb{R}} \left| v(x, \varepsilon_n^2 t) - v^0_{m(P_n)}(x, \varepsilon_n^2 t) \right|^2 G(x, \varepsilon_n^2 t)\, dx \ .$$

Next we estimate this expression by $2\left((\mathrm{I})_n + (\mathrm{II})_n\right)$ where

$$(\mathrm{I})_n = \frac{1}{(\varepsilon_n t)^2} \int_{\mathbb{R}} \left| v_{P_n}(x, \varepsilon_n^2 t) - v^0_{m(P_\infty)}(x, \varepsilon_n^2 t) \right|^2 G(x, \varepsilon_n^2 t)\, dx$$

$$(\mathrm{II})_n = \frac{1}{(\varepsilon_n t)^2} \int_{\mathbb{R}} \left| v^0_{m(P_\infty)}(x, \varepsilon_n^2 t) - v^0_{m(P_n)}(x, \varepsilon_n^2 t) \right|^2 G(x, \varepsilon_n^2 t)\, dx \ .$$

We also introduce the quantity

$$(\mathrm{III})_n = \frac{1}{(\varepsilon_n t)^2} \int_{\mathbb{R}} \left| v_{P_\infty}(x, \varepsilon_n^2 t) - v^0_{m(P_\infty)}(x, \varepsilon_n^2 t) \right|^2 G(x, \varepsilon_n^2 t)\, dx \ .$$

From Corollary II.24, we get

$$(\mathrm{I})_n \leq (\mathrm{III})_n + \tilde{\mathbf{s}}(\varepsilon_n^2 t, P_n)$$

with the particular choice $P_1 = P_\infty$. Moreover, still by scaling invariance, we have

$$(\mathrm{III})_n = \int_{\mathbb{R}} \left| v_{P_\infty}^{\varepsilon_n \sqrt{|t|}}(x,-1) - v_{m(P_\infty)}^0(x,-1) \right|^2 G(x,-1)\, dx \longrightarrow 0 \quad \text{as} \quad \varepsilon_n \to 0 \; .$$

This implies that

$$(\mathrm{I})_n \longrightarrow 0 \quad \text{as} \quad n \to +\infty \; .$$

Finally we remark that

$$\lim_{n\to\infty} (\mathrm{II})_n = \lim_{n\to\infty} \int_{\mathbb{R}} \left| v_{m(P_\infty)}^0 - v_{m(P_n)}^0 \right|^2 G(x,-1)\, dx \leq C \left| m(P_\infty) - m(P_n) \right|^2 = 0$$

This gives the contradiction with $l > 0$. □

5.5 Time projection of the singular set

Proposition II.29. *The set $I := \{t \in [-R^2, R^2] \,:\, \exists\, x \in [-R,R], (x,t) \in \mathcal{S} \setminus \mathcal{S}_0\}$ has zero Lebesgue measure.*

To prove Proposition II.29, we need several preliminary results.

Lemma II.30. *For any $m_0 \in (-1, 0)$, the set $\mathcal{S}_{[-1,m_0]} := \bigcup_{m \in [-1,m_0]} \mathcal{S}_m$ is locally a graph as a function of x.*

Proof – Consider two sequences $(P_n)_{n\in\mathbb{N}}$ and $(P'_n)_{n\in\mathbb{N}}$ of points in $\mathcal{S}_{[-1,m_0]}$ converging to some point $P_\infty \in \Gamma$. Since $\mathcal{S}$ is closed, $P_\infty \in \mathcal{S}$, and by Lemma II.27, $m(P_\infty) \in [-1, m_0]$. Assume by contradiction that $P_n = (x_n, t_n)$ and $P'_n = (x_n, t'_n)$, $t'_n > t_n$. Consistently with the previous notations, we consider the function $v = v_{P_n}$, which is associated to the change of coordinates (II.5) where now the point $P_1 = P_n$ is moving. In the new coordinates the image of P_n is the origin and the image of P'_n is a point $\overline{P}'_n = (0, \varepsilon_n^2)$ with $\varepsilon_n^2 = f(P_n)(t'_n - t_n)$. We then consider the sequence of functions

$$v_{P_n}^{\epsilon_n}(x,t) := \epsilon_n^{-2}\, v(\epsilon_n\, x, \epsilon_n^2\, t) \; .$$

But at time $t = -1$, we have

$$\int_{\mathbb{R}} \left| v_{P_n}^{\epsilon_n} - u_{m(P_\infty)}^0 \right|^2 G\, dx \leq 2 \int_{\mathbb{R}} \left| v_{P_n}^{\epsilon_n} - u_{m(P_n)}^0 \right|^2 G\, dx + 2 \int_{\mathbb{R}} \left| u_{m(P_n)}^0 - u_{m(P_\infty)}^0 \right|^2 G\, dx \; .$$

From lemma II.28, the sequence $(v_{P_n}^{\epsilon_n})_{n\in\mathbb{N}}$ is uniformly close to $u_{m(P_n)}^0$ and $m(P_n) \to m(P_\infty) \in [-1, m_0]$. Therefore on the one hand $v_{P_n}^{\epsilon_n}$ converges to $u_{m(P_\infty)}^0$. On the other hand, let us remark that by construction the point $\overline{P}' = (0,1)$ belongs to $\partial \left\{ v_{P_n}^{\epsilon_n} > 0 \right\}$ and from the non-degeneracy Proposition II.11 in the rescaled variables, we have for any $r \in (0,1)$:

$$\sup_{Q_r^-(\overline{P}')} v_{P_n}^{\epsilon_n} \geq \overline{C} r^2$$

At the limit we get for any $r \in (0,1)$:

$$\sup_{Q_r^-(\overline{P}')} v^0_{m(P_\infty)} \geq \overline{C} r^2$$

This is in contradiction (see Section 3.5) with the fact that $\overline{P}'$ is in the interior of the coincidence set of $v^0_{m(P_\infty)}$ when $m(P_\infty) \in [-1, m_0]$ with $m_0 < 0$. □

Although we will not use it later, we can state the following additional result.

Corollary II.31. *For any $m_0 \in (-1,0)$, $x_0 \in [-R,R]$, $t_0 \in [-R^2,R^2]$, the sets $\{(x,t) \in \mathcal{S}_{[-1,m_0]} \,:\, x = x_0\}$ and $\{(x,t) \in \mathcal{S}_0 \,:\, t = t_0\}$ are locally finite. Moreover $\mathcal{S}_0$ is locally contained in a graph, as a function of t.*

Proof – Taking into account Lemma II.30, we only have to prove that locally $\mathcal{S}_0$ is contained in a graph. Let us do it as in Lemma II.30, by contradiction. Consider two sequences $(P_n)_{n\in\mathbb{N}}$, and $(P'_n)_{n\in\mathbb{N}} \in \mathcal{S}_0^{\mathbb{N}}$ such that $\lim_{n\to\infty} P_n = \lim_{n\to\infty} P'_n = P_\infty$, $P_n = (x_n, t_n)$, $P'_n = (x'_n, t_n)$, $x'_n - x_n > 0$. By Lemma II.27, $P_\infty \in \mathcal{S}_0$. Consider the sequence $(v^{\epsilon_n}_{P_n})_{n\in\mathbb{N}}$ defined by $v^{\epsilon_n}_{P_n}(x,t) := \epsilon_n^{-2}\, v(\epsilon_n\, x, \epsilon_n^2\, t)$ for $v = v_{P_n}$ and $\varepsilon_n = \sqrt{\frac{f(P_n)}{a(P_n)}}(x'_n - x_n)$. The remainder of the proof is the same as above. We end up by noticing that the point $\overline{P}' = (1,0)$ needs to satisfy $v^0_0(\overline{P}') = 0$, while the limit of $v^{\varepsilon_n}_{P_n}$ is $v^0_0(x,t) = x^2/2$ when $m = 0$. □

By Lemma II.30, locally $\mathcal{S}_{[-1,m_0]}$ can be described as a graph: $x \mapsto (x, h(x))$. To the function $h : \mathbb{R} \to \mathbb{R}$, for any $\delta > 0$, we associate the quantity:

$$q_h(x,\delta) := \sup_{x';\ |x-x'|\leq\delta} \frac{|h(x') - h(x)|}{|x' - x|} \ .$$

Lemma II.32. *Let $m_0 \in (-1,0)$. With the above notations, $\lim_{\delta\to 0} q_h(x,\delta) = 0$, uniformly in x.*

Proof – If the Lemma is false, we can find two sequences of points $P_n = (x_n, h(x_n))$ and $P'_n = (x'_n, h(x'_n))$ such that $(P_n)_{n\in\mathbb{N}}$ and $(P'_n)_{n\in\mathbb{N}}$ converge to $P_\infty = (x, h(x))$, and such that

$$l_n := \frac{|h(x'_n) - h(x_n)|}{|x'_n - x_n|} \longrightarrow l \neq 0 \ .$$

Let us consider $v = v_{P_n}$ and the corresponding change of coordinates which transforms P_n in the origin and P'_n in a point $\overline{P}'_n = (\overline{x}'_n, \overline{t}'_n)$ with $\overline{x}'_n = \sqrt{\frac{f(P_n)}{a(P_n)}}(x'_n - x_n)$, $\overline{t}'_n = f(P_n)(h(x'_n) - h(x_n))$. We define $\varepsilon_n > 0$ such that

$$\overline{P}'_n \in \partial Q_{\varepsilon_n}(0)$$

and consider the blow-up sequence of functions

$$v^{\epsilon_n}_{P_n} := \epsilon_n^{-2}\, v(\epsilon_n\, x, \epsilon_n^2\, t)$$

and define

$$\tilde{P}'_n = (\tilde{x}'_n, \tilde{t}'_n) = \left(\frac{\overline{x}'_n}{\varepsilon_n}, \frac{\overline{t}'_n}{\varepsilon_n^2}\right) \in \partial Q_1(0) \ .$$

Up to the extraction of a subsequence, $(\tilde{P}'_n)_{n\in\mathbb{N}}$ converges to some $\tilde{P}' = (\tilde{x}', \tilde{t}') \in \partial Q_1(0)$. By construction, $\tilde{P}'_n$ belongs to $\partial\{v^{\epsilon_n}_{P_n} = 0\}$, hence by the non-degeneracy Proposition II.11, we have

$$\sup_{Q_r(\tilde{P}'_n)} v^{\epsilon_n}_{P_n} \geq \overline{C} r^2, \quad v^{\epsilon_n}_{P_n}(\tilde{P}'_n) = 0 \; .$$

Taking the limit as n goes to infinity, we identify $v^0_{m(P_\infty)}$ as the limit of $v^{\epsilon_n}_{P_n}$ as in the proof of Lemma II.30, and get

$$\sup_{Q_r(\tilde{P}')} v^0_{m(P_\infty)} \geq \overline{C} r^2, \quad v^0_{m(P_\infty)}(\tilde{P}') = 0 \; .$$

This implies that $\tilde{P}' \in \partial\left\{v^0_{m(P_\infty)} > 0\right\} = \left\{(x,t),\ t = x^2/a^2\right\}$ where a is related to m by (II.15). In particular there exists $a_0 > 0$ related to $m_0 \in (-1,0)$ by (II.15) such that $a \in [a_0, +\infty]$. Therefore we get that $\tilde{t}' = (\tilde{x}')^2/a^2$, which, joint to the fact that $\tilde{P}' = (\tilde{x}', \tilde{t}') \in \partial Q_1(0)$, implies that $\tilde{x}' \neq 0$. We now compute

$$\begin{aligned} l_n &= \frac{|h(x'_n) - h(x_n)|}{|x'_n - x_n|} \\ &= \left(\frac{\overline{t}'_n}{f(P_n)}\right) \Big/ \left(\overline{x}'_n \sqrt{\frac{a(P_n)}{f(P_n)}}\right) \\ &= \varepsilon_n \frac{\tilde{t}'_n}{\tilde{x}'_n} \frac{1}{\sqrt{a(P_n) f(P_n)}} \; . \end{aligned}$$

The fact that $\tilde{x}'_n \to \tilde{x}' \neq 0$ and $|\tilde{t}'_n| \leq 1$ implies that $l_n \to 0$. Contradiction. □

We will now use the Hausdorff area formula. According to [Fed69], Theorem 3.2.3 (p. 243), we have the following result (also see [Fed69], 2.8.9 (p. 145), Theorem 2.8.18 (p. 152), 2.9.12 (p. 158), 3.2.1 (p. 241), Theorem 3.1.8 (p. 217), Definition 2.8.16 (p. 161), 3.1.2 (p. 211), Theorem 2.10.35 (p. 197), for related results).

Lemma II.33 (Hausdorff area formula). *Let $\mathcal{A}$ be a measurable set of $\mathbb{R}$ and consider a function $h : \mathcal{A} \mapsto \mathbb{R}$ such that, with the above notations, for all $x \in \mathcal{A}$, $q_h(x,\delta) < \infty$ for some $\delta > 0$. If $N_h(y)$ is the number of elements of $h^{-1}(t)$, then*

$$\int_{\mathcal{A}} \left[\lim_{\delta\to 0} q_h(x,\delta)\right] dx = \int_{\mathbb{R}} N_h(t)\, dt \; .$$

Proof of Proposition II.29. Apply Lemmata II.32 and II.33 with $\mathcal{A} = \{x \in \mathbb{R}, \exists t \in \mathbb{R}, (x,t) \in \mathcal{S}_{[-1,m_0]}\}$: $\int_{\mathbb{R}} N_h(t)\, dt = 0$. This proves that for any $m_0 \in (-1,0)$, the measure of the set

$$I_{m_0} = \left\{t \in \mathbb{R} \;:\; \exists\, x \in \mathbb{R}\,,\ (x,t) \in \mathcal{S}_{[-1,m_0]}\right\}$$

is zero. Hence the measure of $I = \bigcup_{n\in\mathbb{N}} I_{-\frac{1}{n}}$ is also zero. □

Remark II.3. *An inspection of the proof of lemma II.32 shows that $\frac{h(x')-h(x)}{|x'-x|^2}$ is bounded. This ratio even goes to zero uniformly as $|x'-x| \to 0$ and $(x,h(x)), (x',h(x')) \in \mathcal{S}_{[-1,m_0]}$, because the two blow-up limits centered in P_n and in P'_n need to be the same which implies the limit a to be equal to $+\infty$.*

A simple consequence of the boundedness of the ratio $\frac{h(x')-h(x)}{|x'-x|^2}$ *is that the one-dimensional parabolic Hausdorff measure of* $\mathcal{S}_{[-1,m_0]}$*, i.e.* $\mathcal{H}^1_p\left(\mathcal{S}_{[-1,m_0]}\right)$ *is bounded. Let us recall that the parabolic Hausdorff measure is build on the parabolic distance* dist_p *defined for two points* $P=(x,t)$ *and* $P'=(x',t')$ *by* $\mathrm{dist}_p(P,P') := \sqrt{(x-x')^2+|t-t'|}$*. At this stage it can be seen that the time projection of* $\mathcal{S}_{[-1,m_0]}$ *defined by* $\Pi_{[-1,m_0]} = \left\{t,\ \exists (x,t)\in \mathcal{S}_{[-1,m_0]}\right\}$ *satisfies* $\mathcal{H}^{\frac{1}{2}}\left(\Pi_{[-1,m_0]}\right) < +\infty$ *for the classical euclidian Hausdorff measure. A further inspection shows that the convergence to zero of the ratio* $\frac{h(x')-h(x)}{|x'-x|^2}$ *implies that* $\mathcal{H}^{\frac{1}{2}}\left(\Pi_{[-1,m_0]}\right) = 0$ *As a consequence we get that*

$$\mathcal{H}^{\frac{1}{2}}\left(\Pi_{[-1,0)}\right) = 0$$

where

$$\Pi_{[-1,0)} = \{t,\ \exists (x,t)\in \mathcal{S}\backslash\mathcal{S}_0\}\ .$$

This last remark can be of particular interest in higher dimension, especially in space dimension 2.

Remark II.4. *Using a blow-up argument, it can be easily deduced from this section that any point in* $\mathcal{S}\backslash\left(\mathcal{S}_0\cup\mathcal{S}_{-1}\right)$ *is an isolated point in* $\mathcal{S}$ *and then is only surrounded by regular points from the free boundary.*

6 On the continuity of u_t: proof of the main results

In this section using the transformation of Section 3.1, we reduce the problem to the case $a\equiv 1$, $b\equiv 0$, $c\equiv 0$ and $f(P_1)=1$ where $P_1\in\Gamma$. After this transformation we have in the new coordinates $P_1=0$, but we will still keep the notation P_1 to avoid some possible confusions.

6.1 Proof of Theorem II.3: continuity of u_t

With direct estimates, we first prove the following result.

Lemma II.34 (Estimates on the limit of u_t at the boundary). *Under assumption (II.2), if* u *is a solution of (II.1) and* $P_1\in\Gamma$*, then we have*

$$\text{(i)}\quad \limsup_{P\to P_1,\, P\in\{u>0\}} \frac{\partial u}{\partial t}(P)\le 0 \quad \textit{and} \quad \text{(ii)}\quad \liminf_{P\to P_1,\, P\in\{u>0\}} \frac{\partial u}{\partial t}(P)\ge -1\ .$$

Theorem II.3 is a straightforward consequence of (i).

Proof – We first prove (i). Let $l := \limsup_{P\to P_1,\, P\in\{u>0\}} u_t(P)$. Assume by contradiction that there exists a sequence $(P_n=(x_n,t_n))_{n\in\mathbb{N}}$ such that

$$u(P_n)>0\ ,\quad \lim_{n\to\infty} P_n = P_1 \quad \text{and} \quad \lim_{n\to\infty}\frac{\partial u}{\partial t}(P_n) = l>0\ .$$

Define now $\Pi_n := (\overline{x}_n,\overline{t}_n)\in\Gamma$, $\eta_n>0$ such that

$$Q_{\eta_n}(P_n)\subset\{u>0\}\ ,\quad \Pi_n\in\partial\left(Q_{\eta_n}(P_n)\right)\cap\{u=0\}$$

and $\nu_n := \big(\eta_n^{-1}(x_n - \overline{x}_n), \eta_n^{-2}(t_n - \overline{t}_n)\big)$. Let $u^{\eta_n}_{\Pi_n}(x,t) := \eta_n^{-2}\, u(\Pi_n + (\eta_n x, \eta_n^2 t))$. Up to the extraction of a subsequence, $(u^{\eta_n}_{\Pi_n})_{n\in\mathbb{N}}$ converges locally uniformly on all compacts sets in $\mathbb{R}^2$ to a function $u^0 \in W^{2,1;\infty}_{x,t;\mathrm{loc}}(\mathbb{R}^2)$, and $(\nu_n)_{n\in\mathbb{N}}$ to some $\nu \in \partial Q_1(0)$, such that

$$\begin{cases} Hu^0 = 1 \text{ in } \big\{u^0 > 0\big\}\ , \quad u^0 \geq 0 \quad \text{a.e. in } \mathbb{R}^2\ , \\ \frac{\partial u^0}{\partial t}(\nu) = l \quad \text{and} \quad u^0(\nu) > 0\ . \end{cases}$$

Here $u^0(\nu) > 0$ is a consequence of the fact that $l \neq 0$.
By Schauder estimates we can pass to the limit in u_t because u_t is bounded in $\mathcal{H}^\alpha$, and the corresponding bound is uniform under zooming scaling. The function u^0_t achieves its maximum at ν. Otherwise, there would be a point $P' = (x', t') \in \{u^0 > 0\}$ such that $u^0_t(P') > l$ and then the point $T_n = \Pi_n + (\eta_n x', \eta_n^2 t')$ would satisfy

$$\lim_{n\to\infty} T_n = P_1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\partial u}{\partial t}(T_n) = \lim_{n\to\infty} \frac{\partial u^{\eta_n}_{\Pi_n}}{\partial t}(P') = \frac{\partial u^0}{\partial t}(P') > l\ ,$$

a contradiction. Thus $u^0_t \leq u^0_t(\nu) = l$. Moreover u^0_t satisfies the equation

$$H\left(\frac{\partial u^0}{\partial t}\right) = 0 \quad \text{a.e. in} \quad \big\{u^0 > 0\big\}\ .$$

By the strong maximum principle, $u^0_t \equiv l$ in $Q^-_r(\nu)$ for some $r > 0$, small enough, and as a consequence

$$\frac{\partial u^0}{\partial x^2} = 1 + l \quad \text{in} \quad Q^-_r(\nu)\ ,$$

which means that there exist $\overline{x}_0$ and $k \in \mathbb{R}$ such that

$$u^0(x,t) = l(t - t_\nu) + (1 + l)\,(x - \overline{x}_0)^2/2 + k \geq 0 \quad \text{in} \quad Q^-_r(\nu)\ .$$

Iterating the method, we may cover the parabolic connected component of $\{(x,t) \in \mathbb{R}^2 : u^0(x,t) > 0,\ t < t_\nu\}$ which contains ν. Its boundary is given by

$$x \mapsto \phi(x) := \min\big\{t_\nu,\ t_\nu - l^{-1}\left((1 + l)\,(x - \overline{x}_0)^2/2 + k\right)\big\}\ .$$

For any $x \in \mathbb{R}$ such that $\phi(x) < t_\nu$ and $x \neq \overline{x}_0$, $u^0_x(x, \phi(x)) = (1+l)(x - \overline{x}_0) \neq 0$ contradicts the continuity of $u^0_x(\cdot, t)$. Thus $l \leq 0$.

It remains to prove (ii). This is equivalent to prove that

$$q := \limsup_{P\to P_1,\, P\in\{u>0\}} \frac{\partial^2 u}{\partial x^2}(P) \geq 0\ .$$

Assume by contradiction that $q < 0$ and as for (i), define $P_n = (x_n, t_n)$ such that $\lim_{n\to\infty} u_{xx}(P_n) = q$, $\Pi_n = (\overline{x}_n, \overline{t}_n)$, η_n, ν_n and $u^{\eta_n}_{\Pi_n}$. Up to the extraction of a subsequence $(\nu_n)_{n\in\mathbb{N}}$ and $(u^{\eta_n}_{\Pi_n})_{n\in\mathbb{N}}$ respectively converge to $\nu \in \partial Q_1(0)$ and $u^0 \in W^{2,1;\infty}_{x,t;\mathrm{loc}}(\mathbb{R}^2)$, which satisfy

$$\begin{cases} Hu^0 = 1 \text{ in } \big\{u^0 > 0\big\}\ , \quad u^0 \geq 0 \quad \text{a.e. in } \mathbb{R}^2\ , \\ \frac{\partial^2 u^0}{\partial x^2}(\nu) = q \quad \text{and} \quad u^0(\nu) > 0\ . \end{cases}$$

As above, in the parabolic component of $\{(x,t) \in \mathbb{R}^2 : u^0(x,t) > 0,\ t < t_\nu\}$ which contains ν

$$u^0(x,t) = (q - 1)(t - t_\nu) + q\,(x - \overline{x}_0)^2/2 + k \geq 0\ .$$

This again contradicts the regularity of u^0_x on $\partial\{u^0 = 0\}$. □

6.2 A new characterization of some singular points and consequences

Lemma II.35. *Under Assumption (II.2) consider a solution u of (II.1). Let*

$$l := \liminf_{P \to P_1,\, P \in \{u>0\}} u_t(P)$$

be negative. Consider a minimizing sequence $(P_n = (x_n, t_n))_{n\in\mathbb{N}}$ for l. Define $\Pi_n = (\overline{x}_n, \overline{t}_n) \in \Gamma$, $\eta_n > 0$ such that

$$Q_{\eta_n}(P_n) \subset \{u > 0\}\ , \quad \Pi_n \in \partial\left(Q_{\eta_n}(P_n)\right) \cap \{u = 0\}$$

and $\nu_n := \left(\eta_n^{-1}(x_n - \overline{x}_n), \eta_n^{-2}(t_n - \overline{t}_n)\right)$. Up to the extraction of a subsequence, $(u_{\Pi_n}^{\eta_n} := \eta_n^{-2}\, u(\Pi_n + (\eta_n x, \eta_n^2 t)))_{n\in\mathbb{N}}$ converges locally uniformly on all compacts sets in $\mathbb{R}^2$ to a function $u^0 \in W^{2,1;\infty}_{x,t;\mathrm{loc}}(\mathbb{R}^2)$, and $(\nu_n)_{n\in\mathbb{N}}$ to some $\nu = (x_\nu, t_\nu) \in \partial Q_1(0)$. Moreover there exist $\overline{x}_0$ and $k \in \mathbb{R}$ such that

$$u^0(x,t) = l(t - t_\nu) + (1 + l)\,(x - \overline{x}_0)^2/2 + k \geq 0 \quad \forall\,(x,t) \in \mathbb{R} \times (-\infty, t_\nu)\ .$$

Proof – We proceed as in the proof of Lemma II.34. The function u^0 and ν are such that

$$\begin{cases} Hu^0 = 1 \text{ in } \left\{u^0 > 0\right\}\ , \quad u^0 \geq 0 \quad \text{a.e. in } \mathbb{R}^2\ , \\ \frac{\partial u^0}{\partial t}(\nu) = l \quad \text{and} \quad u^0(\nu) > 0\ . \end{cases}$$

The function u_t^0 achieves its minimum at ν: $u_t^0 \geq u_t^0(\nu) = l$. Moreover $Hu_t^0 = 0$ almost everywhere in $\left\{u^0 > 0\right\}$. By the strong maximum principle, $u_t^0 \equiv l$ in $Q_r^-(\nu)$ for some $r > 0$, small enough, and as a consequence

$$\frac{\partial^2 u^0}{\partial x^2} = 1 + l \quad \text{in} \quad Q_r^-(\nu)\ ,$$

which means that there exist x_0 and $k \in \mathbb{R}$ such that

$$u^0(x,t) = l(t - t_\nu) + (1 + l)\,(x - \overline{x}_0)^2/2 + k \geq 0 \quad \text{in} \quad Q_r^-(\nu)\ .$$

Iterating the method, we may cover the parabolic connected component of $\{(x,t) \in \mathbb{R}^2 : u^0(x,t) > 0,\ t < t_\nu\}$ which contains ν. This proves that its boundary is given by

$$x \mapsto \phi(x) := \max\left\{t_\nu,\ t_\nu - l^{-1}\left((1 + l)\,(x - \overline{x}_0)^2/2 + k\right)\right\}\ .$$

For any $x \in \mathbb{R}$ such that $\phi(x) < t_\nu$ and $x \neq \overline{x}_0$, $u_x^0(x, \phi(x)) = (1+l)(x - \overline{x}_0) \neq 0$ contradicts the continuity of $u_x^0(\cdot, t)$ if $l > -1$. Thus $\inf_{\mathbb{R}} \phi \geq t_\nu$ and u^0 is positive in $\{t < t_\nu\}$. By unique continuation, we establish the expression of u^0 in $\mathbb{R} \times (-\infty, t_\nu)$. □

As a consequence of this lemma we have

Lemma II.36. *Under Assumption (II.2) consider a solution u of (II.1) and take $P_1 \in \Gamma$. If*

$$\liminf_{P \to P_1,\, P \in \{u>0\}} \frac{\partial u}{\partial t}(P) < 0$$

then $P_1 \in \mathcal{S}$.

Proof – Consider a nonnegative cut-off function $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi = 1$ in a small neighborhood of $x = 0$ and with small enough compact support. Assume by contradiction that P_1 is regular. For any $P' = (x', t')$ we define

$$u_{P'}(x,t) = u(x + x', t + t') \ .$$

By Proposition II.20, $\lim_{\tau \to 0} \mathcal{E}(\tau; u_{P_1}\psi) = \sqrt{2}/2$. By Proposition II.15, for any $\delta > 0$, there exists a $\tau_0 < 0$ such that

$$\sqrt{2}/2 \leq \mathcal{E}(\tau_0; u_{P_1}\psi) < \sqrt{2}/2 + \delta/2 \ . \tag{II.18}$$

With the notations of Lemma II.35 and according to Lemma II.35 the sequence $(u^{\eta_n}_{\Pi_n})_{n\in\mathbb{N}}$ converges uniformly to $u^0(x,t) = l(t - t_\nu) + (1 + l)\,(x - \overline{x}_0)^2/2 + k \geq 0$ in $\mathbb{R} \times (-\infty, t_\nu)$. We compute

$$\lim_{t \to -\infty} \mathcal{E}(t; u^0) = \sqrt{2} \ .$$

Then for any $\delta > 0$, there exists $t_\infty < 0$ with $|t_\infty|$ large enough such that for $t < t_\infty$ we have (using the scaling invariance of the energy):

$$\sqrt{2} - \frac{\delta}{2} \leq \mathcal{E}(t; u^0) = \lim_{n\to\infty} \mathcal{E}(t; u^{\eta_n}_{\Pi_n}\psi(\eta_n \cdot)) = \lim_{n\to\infty} \mathcal{E}(\eta_n^2 t; u_{\Pi_n}\psi) \ .$$

So for τ_0 defined in (II.18) and $t < t_\infty$ fixed, there exists $N = N(t, \delta)$ such that

$$\forall n > N, \quad \mathcal{E}(t; u^{\eta_n}_{\Pi_n}\psi(\eta_n \cdot)) > \mathcal{E}(t; u^0) - \frac{\delta}{2} \text{ and } \eta_n^2 t > \tau_0 \ .$$

Proposition II.15 applies to u_{Π_n}:

$$\mathcal{E}(\eta_n^2 t; u_{\Pi_n}\psi) \leq \mathcal{E}(\tau_0; u_{\Pi_n}\psi) \ .$$

By continuity of the map $P' \longmapsto \mathcal{E}(\tau_0; u_{P'}\psi)$, we have

$$\mathcal{E}(\tau_0; u_{\Pi_n}\psi) \leq \mathcal{E}(\tau_0; u_{P_1}\psi) + \frac{\delta}{2} \ .$$

Collecting these estimates, we have for any $n > N$

$$\sqrt{2} - \delta \leq \mathcal{E}(t; u^0) - \frac{\delta}{2} < \mathcal{E}(\eta_n^2 t; u_{\Pi_n}\psi) \leq \mathcal{E}(\tau_0; u_{\Pi_n}\psi) \leq \mathcal{E}(\tau_0; u_{P_1}\psi) + \frac{\delta}{2} < \frac{\sqrt{2}}{2} + \delta \ ,$$

a contradiction for any $\delta < \sqrt{2}/4$. □

As a direct consequence of Lemmata II.34 and II.36 we obtain

Corollary II.37. *Under Assumption (II.2) consider a solution u of (II.1). If $P_1 \in \mathcal{R}$ then*

$$\lim_{P \to P_1,\, P \in \{u>0\}} \quad \frac{\partial u}{\partial t}(P) = 0 \ .$$

Lemma II.38. *Let u be a solution of (II.1) and assume that (II.2) holds. If there exist $r > 0$ and $P_1 = (x_1, t_1) \in Q_R(P_0)$ such that $Q_r(P_1) \subset Q_R(P_0)$ and $t' := \inf\{t \in (t_P - r^2, t_P + r^2] \ : \ \exists x' \in (x_P - r, x_P + r)$ such that $(x', t) \in \Gamma\}$ is achieved in $(x_P - r, x_P + r) \times (t_P - r^2, t_P + r^2]$ and u is positive in $\{(x,t) \in Q_r(P_1) \ : \ t < t'\}$ then (x', t') is a singular point.*

Proof – Assume by contradiction that $P' \in \mathcal{R}$. According to Theorem II.19 and Proposition II.26, the blow-up limit in P' corresponding to a blow-up sequence at scale ϵ_n is $u^0 = v^0_\pm$. There exists therefore some $\tilde{P} = (x,t)$ with $t < 0$ such that $\tilde{P} \in \mathrm{Int}\{u^0 = 0\}$. By Lemma II.6, this implies that $u(P' + \epsilon_n \tilde{P}) = 0$ for n large enough, a contradiction with the definition of P'. □

Theorem II.39. *Under Assumption (II.2) consider a solution u of (II.1). For any $m \in [-1,0]$, if $P_1 \in \mathcal{S}_m$ then*

$$\liminf_{P\to P_1,\, P\in\{u>0\}} \frac{\partial u}{\partial t}(P) = m\ .$$

Proof – Let $P_1 = (x_1,t_1)$ and $l := \liminf_{P\to P_1,\, P\in\{u>0\}} u_t(P)$. By considering a blow-up sequence $(\epsilon_n)_{n\in\mathbb{N}}$ and by computing $u_t(x_{P_1}, t_{P_1} - \epsilon_n/2) \to (v^0_{m(P_1)})_t(x_{P_1}, t_{P_1} - 1/2) = m(P_1)$ we get that $l \le m = m(P_1)$. Assume by contradiction that the inequality is strict.

Let $(P_n)_{n\in\mathbb{N}}$ be a sequence with $P_n = (x_n,t_n)$ such that $u(P_n) > 0$, $\lim_n P_n = P_1$ and $\lim_n u_t(P_n) = l$. For any $n \in \mathbb{N}$, define $\epsilon_n > 0$, such that $P_n \in \partial Q_{\varepsilon_n}(P_1)$. Let us consider a localized blow-up sequence $(u^{3\epsilon_n}_{P_1})_{n\in\mathbb{N}}$ which converges to v^0_m. Since $Q_{\epsilon_n}(P_n) \subset Q_{3\epsilon_n}(P_1)$, the sequence $(u^{\epsilon_n}_{P_n} := \epsilon_n^{-2} u(x_n + \epsilon_n\, x, t_n + \epsilon_n^2\, t))_{n\in\mathbb{N}}$ satisfies

$$u^{\epsilon_n}_{P_n} \longrightarrow v^0_m(\cdot + \overline{P}) \quad \text{with} \quad \overline{P} \in \partial Q_1(0)\,.$$

Here $\overline{P} = (\overline{x},\overline{t}) = \lim_{n\to+\infty} \overline{P}_n$ with $\overline{P}_n = \left(\frac{x_n - x_1}{\varepsilon_n}, \frac{t_n - t_1}{\varepsilon_n^2}\right)$. By Lemma II.35 and using the same notations, for some $\Pi_n := (\overline{x}_n, \overline{t}_n) \in \Gamma$, $(u^{\eta_n}_{\Pi_n})_{n\in\mathbb{N}}$ uniformly converges to $u^0(x,t) = l(t - t_\nu) + (1+l)\,(x - \overline{x}_0)^2/2 + k$ in $\mathbb{R}\times(-\infty, t_\nu)$. Let us define $\overline{t}_0$ such that $l(\overline{t}_0 - t_\nu) + k = 0$. Then for $\overline{P}_0 = (\overline{x}_0, \overline{t}_0)$, and by uniqueness of the limit solution u^0, we have $u^0 = v^0_l(\cdot + \overline{P}_0)$. Consequently we have

$$u^{\eta_n}_{\Pi_n} \longrightarrow v^0_l(\cdot + \overline{P}_0)\,.$$

Moreover we have

$$\eta_n \le \varepsilon_n\ .$$

Now let us consider the sequence $(u^{\varepsilon_n}_{\Pi_n})_{n\in\mathbb{N}}$ which satisfies $u^{\varepsilon_n}_{\Pi_n} = u^{\varepsilon_n}_{P_n}(\cdot + \overline{\Pi}_n)$ with $\overline{\Pi}_n = \left(\frac{\overline{x}_n - x_n}{\varepsilon_n}, \frac{\overline{t}_n - t_n}{\varepsilon_n^2}\right) \in \partial Q_{\eta_n/\varepsilon_n}(0)$. Up to extraction of a subsequence, we can assume that $\overline{\Pi}_n \to \overline{\Pi}$ with $\overline{\Pi} \in \overline{Q_1(0)}$ and then

$$u^{\varepsilon_n}_{\Pi_n} \longrightarrow v^0_m(\cdot + \overline{P} + \overline{\Pi})\ .$$

Because we assumed that $l \neq m$, this implies that $\eta_n/\varepsilon_n \to 0$ and then $\overline{\Pi} = 0$.
Given $\delta > 0$, we now consider $\mu > 0$ large enough such that

$$\begin{aligned} |v^0_m((0,-\mu) + \overline{P}) - m\mu| &\le \delta\mu \\ |v^0_l((0,-\mu) + \overline{P}_0) - l\mu| &\le \delta\mu\ . \end{aligned}$$

The function $\lambda \mapsto u^\lambda_{\Pi_n} := \lambda^{-2} u(\bar{x}_n + \lambda\, x, \bar{t}_n + \lambda^2\, t)$ is continuous: there exists a $\lambda_n \in (\eta_n, \epsilon_n)$ such that

$$\frac{1}{\mu} u^{\lambda_n}_{\Pi_n}(0,-\mu) = \frac{1}{2}\,[m + l\,]$$

for any n large enough. The sequence $(u^{\lambda_n}_{\Pi_n})_{n\in\mathbb{N}}$ converges to a function $\bar{u}$ in $W^{2,1;\infty}_{x,t;\mathrm{loc}}$ which satisfies

$$H\bar{u} = \mathbb{1}_{\{\bar{u}>0\}}\ , \quad \bar{u} \ge 0 \quad \text{and} \quad \bar{u}(0,-\mu) = \frac{\mu}{2}\,[\ m + l\ ]\ .$$

Consider a nonnegative cut-off function $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi \equiv 1$ on $(-1/2, +1/2)$, $\text{supp}(\psi) = [-1, 1]$. On the one hand, there exists $t_\infty = t_\infty(\delta) < 0$ such that

$$\sqrt{2} - \frac{\delta}{2} \leq \mathcal{E}(t; v_l^0(\cdot + \overline{P}_0)) = \lim_{n\to\infty} \mathcal{E}(t; u_{\Pi_n}^{\eta_n} \psi(\eta_n \cdot)) = \lim_{n\to\infty} \mathcal{E}(\eta_n^2 t; u_{\Pi_n} \psi) \quad \forall t < t_\infty \ .$$

On the other hand by definition of $\mathcal{S}$, $\lim_{\tau\to 0} \mathcal{E}(\tau; u_{P_1}\psi) = \sqrt{2}$. By Proposition II.15, for any $\delta > 0$, there exists $\tau_0 < 0$, with $|\tau_0|$ sufficiently small, such that

$$\mathcal{E}(\tau_0; u_{P_1}\psi) < \sqrt{2} + \delta/2 \ .$$

For any $\delta > 0$, $t > 0$, there exists a $N = N(t, \delta)$ such that

$$n > N \quad \Rightarrow \quad \mathcal{E}(t; u_{\Pi_n}^{\eta_n} \psi(\eta_n \cdot)) > \mathcal{E}(t; v_l^0(\cdot + \overline{P}_0)) - \frac{\delta}{2} \quad \text{and} \quad \eta_n^2 t > \tau_0 \ .$$

Moreover, for a fixed τ_0, by continuity of the energy, since $\Pi_n \in Q_{2\epsilon_n}(P_1)$,

$$\mathcal{E}(\tau_0; u_{\Pi_n}\psi) \leq \mathcal{E}(\tau_0; u_{P_1}\psi) + \frac{\delta}{2} \qquad \forall\, n > N$$

for N large enough. Using Proposition II.15, for all $t < t_\infty$, $n > N$, $s \in (\tau_0, \eta_n^2\, t) \subset (\tau_0, 0)$, we get

$$\sqrt{2} - \delta < \mathcal{E}(t; v_l^0(\cdot + \overline{P}_0)) - \frac{\delta}{2} < \mathcal{E}(\eta_n^2 t; u_{\Pi_n}\psi) \leq \mathcal{E}(s; u_{\Pi_n}\psi) \leq \mathcal{E}(\tau_0; u_{\Pi_n}\psi)$$
$$\leq \mathcal{E}(\tau_0; u_{P_1}\psi) + \frac{\delta}{2} < \sqrt{2} + \delta \ . \qquad \text{(II.19)}$$

For any given $t < t_\infty$ and $n > N(t, \delta)$ we define $s := \lambda_n^2\, \tau$, and $\tau \in (\tau_0/\lambda_n^2, (\eta_n/\lambda_n)^2\, t)$. As a consequence, the estimate

$$\sqrt{2} - \delta \leq \mathcal{E}(\lambda_n^2 \tau; u_{\Pi_n}\psi) = \mathcal{E}(\tau; u_{\Pi_n}^{\lambda_n} \psi(\eta_n \cdot)) \leq \sqrt{2} + \delta$$

holds true for any $t < t_\infty(\delta)$ and $n > N(t, \delta)$. From our construction we get that $\lambda_n \to 0$ and $\eta_n/\lambda_n \to 0$, so that $\left|\mathcal{E}(\tau; \bar{u}) - \sqrt{2}\right| \leq \delta$ for all $\tau < 0$ and for all $\delta > 0$. Therefore

$$\forall \tau \in (-\infty, 0), \quad \mathcal{E}(\tau; \bar{u}) = \sqrt{2} \ .$$

This means that $\bar{u}$ is scale-invariant by Corollary II.16. By Theorem II.19 there exists an $\bar{m}$ such that $\bar{u} = v_{\bar{m}}^0$. Because of the expression of $\bar{u}(0, -\mu)/\mu$ we obtain $\bar{m} = [m + l]/2 \in (-1, 0)$. From the convergence of $u_{\Pi_n}^{\lambda_n}$ to $v_{\bar{m}}^0$ with $\bar{m} \in (-1, 0)$ where the free boundary $\partial\left\{v_{\bar{m}}^0 > 0\right\}$ is a parabola oriented in the positive time direction, and from the fact that $P_n \in Q_{\lambda_n}(\Pi_n)$, we deduce that Lemma II.38 applies to u in $Q_{A\lambda_n}(P_n)$ for some $A > 0$ large enough, but independent of n. Then there exists a sequence of singular points $(Z_n)_{n\in\mathbb{N}}$ in $Q_{A\lambda_n}(P_n)$ such that $\lim_{n\to\infty} m(Z_n) = \bar{m}$, because $u_{\Pi_n}^{\lambda_n}$ converges to $v_{\bar{m}}^0$. Moreover the sequence Z_n converges to P_1 and then by Lemma II.27, we obtain $\bar{m} = m(P_1)$, which is impossible. □

As a very simple consequence, we obtain the following result.

Corollary II.40. *Under Assumption (II.2) consider a solution u of (II.1). If $P_1 \in \mathcal{S}_0$ then*

$$\lim_{P\to P_1,\, P\in\{u>0\}} \frac{\partial u}{\partial t}(P) = 0 \ .$$

6.3 Proofs of the results of Section 1

Proof of Theorem II.5. If for some $t < 0$, $\mathcal{E}(t; v_{P_1}) < \sqrt{2}$, then by Proposition II.15, $t \mapsto \mathcal{E}(t; v_{P_1})$ is monotone decreasing, and by Proposition II.20, $P_1 \in \mathcal{R}$. By Corollary II.37, u_t is continuous at P_1 and in a neighborhood of P_1 by Lemma II.22. □

Proof of Theorem II.4. By Proposition II.20, the limit of $\mathcal{E}$ is either $\sqrt{2}$ or $\sqrt{2}/2$. In the second case, Corollary II.37 applies at P_1 and the continuity of u_t holds because $\mathcal{R}$ is open in Γ according to Lemma II.22, which proves (ii). □

Proof of Theorem II.2. By Proposition II.29, the set I has zero Lebesgue measure. If $(x_1, t_1) = P_1 \in \Gamma$ is such that $t_1 \notin I$, then $P_1 \in \mathcal{S}_0 \cup \mathcal{R}$, and the result holds by Corollaries II.37 and II.40. □

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A.III

On the regularity of the free boundary of the parabolic obstacle problem

Ce chapitre reprend l'essentiel de [Bla05] accepté pour publication dans *Nonlinear Analysis Series A : Theory, Methods & Applications.*

Résumé

Ce papier est consacré à des résultats de régularité de la frontière d'exercice pour une option Américaine sur un actif sous-jacent. Nous donnons un critère d'énergie et un critère de densité pour caractérisé les sous-ensembles de la frontière d'exercice qui sont de continuité höldérienne d'exposant 1/2. Pour illustrer ces résultats, nous les appliquons au modèle de Black-Scoles généralisé où le volatilité et le taux d'intérêt ne dépendent pas du temps. Dans ce cas on prouve que la frontière d'exercice de l'option d'achat Américaine et de l'option de vente Américaine sont de continuité höldérienne d'exposant 1/2.

On the regularity of the free boundary of the parabolic obstacle problem. Application to American options

A. Blanchet

Abstract

This paper is devoted to local regularity results on the free boundary of the one-dimensional parabolic obstacle problem with variable coefficients. We give an energy and a density criterion to characterise the subsets of the free boundary which are Hölder continuous in time with exponent 1/2. Our results apply in theory of American option. As an illustration we apply these results to the generalised Black–Scholes model of a complete market which rules out arbitrage if the volatility and the interest rate do not depend on time. In this case we prove that the exercise boundary of the American put and call options are Hölder continuous with exponent 1/2 in time for every time.

AMS Classification: 35R35.
Keywords: Parabolic obstacle problem, American option, free boundary, exercise region, exercise boundary.

1 Introduction

For given $P_0 = (x_0, t_0) \in \mathbb{R}^2$ and $R > 0$ we consider the open parabolic cylinder,

$$Q_R(P_0) := \{\, (x,t) \in \mathbb{R}^2 \,:\, |x - x_0| < R \ \text{ and } \ |t - t_0| < R^2 \,\}\,,$$

the Sobolev space

$$W^{2,1;1}_{x,t}(Q_R(P_0)) := \left\{ u \in L^1(Q_R(P_0)) \,:\, \left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t}\right) \in (L^1(Q_R(P_0)))^3 \right\}$$

and for $M > 0$, the space $\mathcal{P}_{x,t}(Q_R(P_0))$ by

$$\mathcal{P}_M(Q_R(P_0)) := \{u \in L^\infty(Q_R(P_0)) \ : \ \sup_{Q_R(P_0)} \left[\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right| + \left|\frac{\partial u}{\partial t}\right| + \frac{1}{R}\left|\frac{\partial u}{\partial x_i}\right| \right] \le M < \infty\}\,.$$

and the parabolic operator L is defined for $u \in W^{2,1;1}_{x,t}(Q_R(P_0))$ by

$$Lu := a(\cdot,\cdot)\,\frac{\partial^2 u}{\partial x^2} + b(\cdot,\cdot)\,\frac{\partial u}{\partial x} + c(\cdot,\cdot)\,u - \frac{\partial u}{\partial t}\,.$$

In this paper we are interested in qualitative properties of the solution of the following one-dimensional parabolic obstacle problem with variable coefficients:

$$\begin{cases} u \in \mathcal{P}_{x,t}(Q_R(P_0)) \\ Lu = f\,\mathbb{1}_{\{u>0\}}, \\ u \ge 0, \end{cases} \tag{III.1}$$

where $\mathbb{1}_{\{u>0\}}$ denotes the characteristic function of the set $\{u > 0\} := \{(x,t) \in Q_R(P_0) : u(x,t) > 0\}$.

We introduce

$$\mathcal{H}^\alpha(Q_R(P_0)) := \Big\{ f \in \mathcal{C}^0 \cap L^\infty(Q_R(P_0)) \ : \ \sup_{\substack{(x,t),(y,s)\in Q_R(P_0)\\ (x,t)\neq(y,s)}} \frac{|f(x,t)-f(y,s)|}{(\,|x-y|^2+|t-s|\,)^{\alpha/2}} < \infty \Big\}.$$

Our main assumption is the following, concerning the uniform parabolicity and non-degeneracy of the operator and also the regularity of the coefficients and f:

$$\begin{cases} \textit{there exists } \delta_0 > 0 \textit{ such that for any } (x,t) \in Q_R(P_0)\,,\ a(x,t) \geq \delta_0 \textit{ and } f(x,t) \geq \delta_0\,, \\ a,\ b,\ c \textit{ and } f \textit{ belong to } \mathcal{H}^\alpha(Q_R(P_0)) \text{ for some } \alpha \in (0,1)\,. \end{cases} \tag{III.2}$$

By [Fri88], under Assumption (III.2), the equation (III.1) has a unique solution for suitable initial datum and boundary conditions. It is known that u is continuous. The sets $\{u = 0\}$ and $\Gamma := \partial\{u = 0\}$ are respectively called the *coincidence set* and the *free boundary* of the parabolic obstacle problem (III.1).

We first need to define some local qualitative properties of a curve. Consider a curve in $\mathbb{R}^2$ defined by the equation $x = g(t)$ for some function g. For every time $t_1 < t_2$ we define the Hölder space

$$\mathcal{C}^{1/2}(t_1,t_2) := \Big\{ g \in \mathcal{C}^0(t_1,t_2) \ : \ \sup_{t\in(t_1,t_2)} |g(t)| + \sup_{\substack{t,s\in(t_1,t_2)\\ t\neq s}} \frac{|g(t)-g(s)|}{|t-s|^{\frac{1}{2}}} < \infty \Big\}.$$

Note here that the corresponding definition of $\mathcal{C}^1$ would denote the set of Lipschitz functions.

We need the notion of local $\mathcal{C}^{1/2}$-graph, $\mathcal{C}^{1/2}$-subgraph and $\mathcal{C}^{1/2}$-uppergraph. Roughly speaking a "local graph" is a set which is locally contained in a graph and a "local uppergraph" is a set which is above a graph. Namely, let $P_0 \in \mathbb{R}^2$ and $R > 0$. Consider a subset $A \subset Q_R(P_0)$ and $P_1 = (x_1,t_1) \in A$.

(i) We say that A is locally a $\mathcal{C}^{1/2}$*-graph* near P_1 if there exists $\rho > 0$ and $g \in \mathcal{C}^{1/2}(t_1 - \rho^2, t_1 + \rho^2)$ such that $Q_\rho(P_1) \subset Q_R(P_0)$ and $A \cap Q_\rho(P_1) = \{(x,t) \in Q_\rho(P_1) : x = g(t)\}$.

(ii) We say that A is locally near P_1 a $\mathcal{C}^{1/2}$*-subgraph* (respectively a $\mathcal{C}^{1/2}$*-uppergraph*) if there exists $\rho > 0$ and $g \in \mathcal{C}^{1/2}(t_1 - \rho^2, t_1 + \rho^2)$ such that $Q_\rho(P_1) \subset Q_R(P_0)$ and $A \cap Q_\rho(P_1) = \{(x,t) \in Q_\rho(P_1) : x \leq g(t)\}$ (resp. $A \cap Q_\rho(P_1) = \{(x,t) \in Q_\rho(P_1) : x \geq g(t)\}$).

Under Assumption (III.2) if we consider a solution of (III.1) we can give a density characterisation of all point $P_1 = (x_1,t_1) \in \Gamma$. This criterion is based on the density $\theta(P_1)$ of the coincidence set $\{u = 0\}$ at the point $P_1 \in \Gamma$:

$$\theta(P_1) := \liminf_{r\to 0} \frac{|\{u=0\} \cap Q_r(P_1)|}{|Q_r(P_1)|}$$

and the lower density $\theta^-(P_1)$ of $\{u=0\}$ at $P_1 \in \Gamma$, defined by

$$\theta^-(P_1) := \liminf_{r\to 0} \frac{|\{u=0\} \cap Q_r^-(P_1)|}{|Q_r^-(P_1)|},$$

with $Q_r^-(P_1) := \{ (x,t) \in \mathbb{R}^2 \,:\, |x-x_1| < r \text{ and } 0 < t_1 - t < r^2 \}$.

(i) If $\theta^-(P_1) \neq 0$ we say that P_1 is a *regular point*. We denote by $\mathcal{R}$ the set of regular points.

(ii) If $\theta^-(P_1) = 0$ we say that P_1 is a *singular point*. We denote by $\mathcal{S}$ the set of singular points. Furthermore we define the set $\mathcal{S}_0$ of the singular points such that $\theta(P_1) = 0$.

For example, if $u(x,t) = (1/2)(\max\{0,x\})^2$ then $\theta(0) = 1/2$ and $\theta^-(0) = 1/2$. If $u(x,t) = \max\{-t,0\}$ then $\theta(0) = 1/2$ and $\theta^-(0) = 0$. If $u(x,t) = (1/2)x^2$ then $\theta(0) = 0$ and $\theta^-(0) = 0$. These functions are solutions in $\mathbb{R}^2$ of (III.1). The point 0 is a regular point in the first case, a point of $\mathcal{S}$ in the second and a point of $\mathcal{S}_0$ in the last case.

We give an energy criterion to characterise these points of Γ in Proposition III.3.

It is proved (see Proposition 5.8 in [BDM05]) that for almost every time there is no point of $\mathcal{S} \setminus \mathcal{S}_0$. Namely, the set $I := \{t \in [-R^2, R^2] \,:\, \exists\, x \in [-R,R], (x,t) \in \Gamma \setminus (\mathcal{R} \cup \mathcal{S}_0)\}$ has zero Lebesgue measure. It is also proved that if u is non-increasing then $\Gamma = \mathcal{R} \cup \mathcal{S}_0$ (Theorem 6.6 in [BDM05]). The main result of the paper deals with the regularity of $\mathcal{S}_0$ and $\mathcal{R}$.

Theorem III.1 (Regularity property of $\mathcal{R}$ and $\mathcal{S}_0$). *Under Assumption (III.2),*

(i) *the set of regular points, $\mathcal{R}$, is locally a $C^{1/2}$-graph. Furthermore, around points of $\mathcal{R}$ the coincidence set is locally described by a $C^{1/2}$-subgraph or by a $C^{1/2}$-uppergraph,*

(ii) *the set $\mathcal{S}_0$ is locally contained in a $C^{1/2}$-graph.*

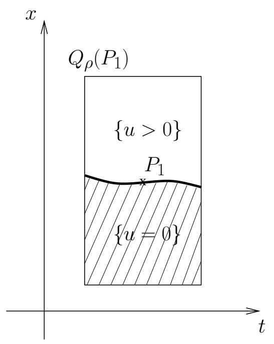


Figure III.1: $C^{1/2}$-subgraph

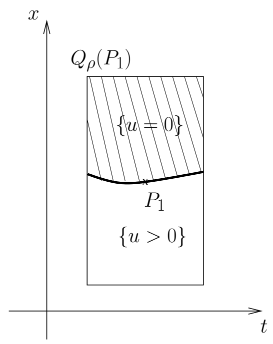


Figure III.2: $C^{1/2}$-uppergraph

The question of the regularity of the free boundary is crucial in financial mathematics and in particular in the numerical computations of Γ. For the derivation of this model from the theory of American option see Section 4. Moreover, the knowledge of the free boundary gives the best strategy for the owner of the option. Unfortunately we cannot give an explicit formula to describe the free boundary, even in the constant coefficients case. However, a better understanding of the behaviour of the free boundary is essential in financial markets. This question of the regularity of Γ is also raised in [Ach05], where Y. Achdou solves a calibration problem. It is an inverse problem in which he evaluates a parameter of the coefficient of the parabolic operator (the volatility see (III.15)) drawing on the price of American options on the financial market. He needs regularity of the exercise boundary to control his numerical computations.

P. Van Moerbeke studied in [VM74] the parabolic obstacle problem with constant coefficients for a special obstacle (call options in the classical Black–Scholes model). He subsequently proved that the free boundary has a continuous time derivative except at the maturity. In [Fri75], A. Friedman considered the case where the variable coefficients depend on time and space and are continuously differentiable. He proved under assumptions on the change of sign of the initial data that the free boundary consists of a finite number of curves $t \mapsto s_i(t)$ piecewise monotone and continuous. He also proved that $ds_i(t)/dt$ exists and is continuous in every t-interval where s_i is strictly monotone. The proof of these results needs to differentiate the equation with respect to x and to apply the maximum principle to u_x. This proof cannot apply to a parabolic operator with Hölder coefficients. Moreover hie monotonicity assumption on the free boundary need precisely a sharp knowledge of the free boundary.

Recently in [CPS04], L. Caffarelli, A. Petrosyan and H. Shahgholian consider the case of the parabolic potential problem (*i.e.* with constant coefficients in any dimension and without any sign assumptions on the solution). They give an energy and a density criterion to classify the points of the free boundary in two sets: the *regular points* and the *singular points*. Their density criterion is similar to the density characterisation given above and the energy criterion is similar to the one given in Proposition III.3. They prove that around regular points the free boundary is a graph $\mathcal{C}^\infty$. Their proof could apply to the variable coefficients case if the coefficients are Lipchitz, in this case the regular set should be a graph of derivative Hölder continuous of exponent α, for all $\alpha \in (0,1)$. They do not study the set of singular points. However, their method has been extended in [BDM05] by J. Dolbeault, R. Monneau and the author in the variable coefficients case with Hölder continuity in one dimension. They give an energy and a density criterion to characterise the region of the free boundary wherein the time derivative of the solution is continuous. They study the set of singular points in order to prove that the time derivative of the solution is continuous almost everywhere.

For the convenience of the reader all the known results and notations that we will use in the proofs are recalled in Section 2. Section 3 is devoted to the proof of the main theorem. We study the regular case in Section 3.1 and the singular case in Section 3.2. In Section 4 we apply our results to financial mathematics. We illustrate our results by classifying the possible shapes of the continuation region and the regularity of the optimal exercise boundary in the Black-Scholes model with homogeneous diffusion (case where the coefficients do not depend on time) with generic obstacle. We apply our results to the American put and call options in this model.

Notation. We will use u_t, u_x and u_{xx} respectively for $\partial u/\partial t$, $\partial u/\partial x$ and $\partial^2 u/\partial x^2$. For any domain $D \subset \mathbb{R}^2$, we define $W^{2,1;q}_{x,t}(D) := \left\{ u \in L^q(D) \,:\, (u_x,\, u_{xx},\, u_t) \in (L^q(D))^3 \right\}$. And we will write $u \in W^{2,1;q}_{x,t;\text{loc}}(D)$ if $u \in W^{2,1;q}_{x,t}(K)$ for all compact $K \subset\subset D$. The heat operator will be abbreviated to H, $Hu := u_{xx} - u_t$.

2 Known results

The theory we develop here and in [BDM05] lies in the founding idea of [Caf77]. The idea is to use blow-up sequences, which are kinds of zooms, and to look at the "infinite zoom". Namely consider $(P_n = (x_n, t_n))_{n\in\mathbb{N}}$ a sequence of point of Γ, $r > 0$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ a sequence converging to 0. The *blow-up sequence* $(u^{\varepsilon_n}_{P_n})_{n\in\mathbb{N}}$ associated to a function $u : Q_r(0) \longrightarrow \mathbb{R}$ is the sequence defined by

$$u^{\varepsilon_n}_{P_n}(x,t) = \varepsilon_n^{-2}\, u\left(x_n + \varepsilon_n x\sqrt{\frac{a(P_n)}{f(P_n)}}, t_n + \varepsilon_n^2 t\,\frac{1}{f(P_n)}\right) \quad \forall n \in \mathbb{N}, \quad \forall (x,t) \in Q_{r/\varepsilon_n}(0)\,.$$

Notice that the parabolic scaling $(x,t) \mapsto (\lambda x, \lambda^2 t)$ transforms the parabolic cylinder $Q_\lambda(0)$ into the parabolic cylinder $Q_1(0)$. Due to $\mathcal{P}_M$-regularity estimates, up to the extraction of a sub-sequence, blow-up sequences uniformly converge on every compact to a function $u^0 \in W^{2,1;\infty}_{x,t;\mathrm{loc}}(\mathbb{R}^2)$. We recall Lemma 2.6 in [BDM05] (see also Lemma 5.1 in [CPS04]):

Lemma III.2 (Non-degeneracy lemma). *Under Assumption (III.2), consider a solution u of (III.1) in $Q_R(P_0)$. Let $R' \in (0,R)$, $P_1 \in \overline{\{u > 0\}}$ be such that $Q^-_r(P_1) \subset Q_{R'}(P_0)$ for some $r > 0$ small enough. There exist two positive constants $\bar{C}$ and $\bar{r} > 0$ such that if $Q_{\bar{r}}(P_1) \cap \{u = 0\} \neq \emptyset$:*

$$r \le \bar{r} \quad \Longrightarrow \quad \sup_{Q^-_r(P_1)} u \ge \bar{C}\, r^2\,.$$

The constants $\bar{C}$ and $\bar{r}$ only depend on R' and the parabolic operator L.

The proof lies on the maximum principle. It is very useful as it prevents the exercise boundary from being flat with the exercise region below. It also gives interesting properties on the way in which the solution close to the exercise boundary "starts". As a consequence (see proof of Proposition 3.2, Proposition 3.3 and Proposition 2.9 in [BDM05]) the blow-up limit u^0 is solution in $\mathbb{R}^2$ of the following global parabolic obstacle problem:

$$\begin{cases} \dfrac{\partial^2 u^0}{\partial x^2}(x,t) - \dfrac{\partial u^0}{\partial t}(x,t) = \mathbb{1}_{\{u^0>0\}}(x,t) & \text{a.e. } (x,t) \in \mathbb{R}^2 \\ u^0(x,t) \ge 0 & \end{cases}$$

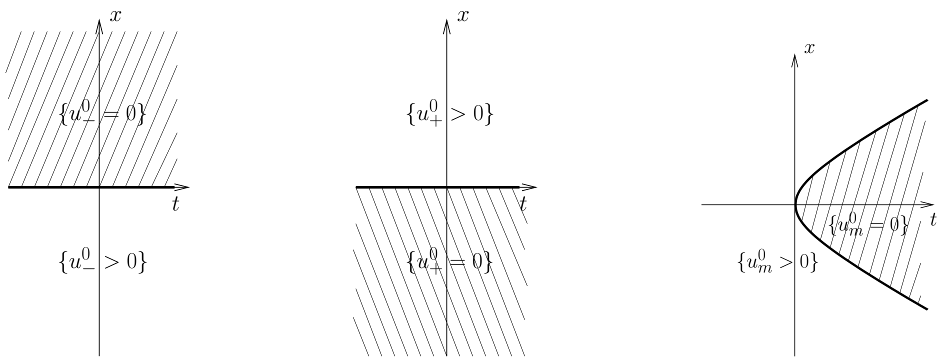
and $0 \in \partial\{u^0 > 0\}$. Furthermore

$$\lim_{n\to\infty} \mathbb{1}_{\{u^{\varepsilon_n}_{P_n}=0\}} = \mathbb{1}_{\{u^0=0\}} \quad \text{a.e. in } \mathbb{R}^2\,. \tag{III.3}$$

Moreover, if we consider a blow-up sequence at a fixed point P_1 (*i.e.* $P_n \equiv P_1$), the blow-up limit is one of the following functions

$$u^0_+(x,t) := \frac{1}{2}(\max\{0,x\})^2, \qquad u^0_-(x,t) := \frac{1}{2}(\max\{0,-x\})^2 \quad \text{and} \quad u^0_m(x,t) := \begin{cases} m\,t + \dfrac{1+m}{2}x^2 & \text{if}\, t < 0\,, \\ \max\left\{0, t\,U_m\left(\frac{|x|}{\sqrt{t}}\right)\right\} & \text{if}\, t \ge 0\,, \end{cases} \tag{III.4}$$

where $m \in [-1, 0]$ and U_m is explicitly given in Theorem 3.9 [BDM05] (see also Lemma 6.3 in [CPS04]). See the figure below.



The crucial difficulty in this classification of blow-up limits at fixed points is to prove their scale-invariance. For this purpose [Wei99] introduces a monotonicity formula for the elliptic obstacle problem. For the parabolic obstacle problem we define the energy as follows:

Let $Q_r(P_1) \subset Q_R(P_0) \subset \mathbb{R}^2$. With $P_1 = (x_1, t_1)$, and a, f the functions involved respectively in the definition of the operator L and in Equation (III.1). Consider a non-negative cut-off function $\psi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\psi \equiv 1$ on $\left(-(1/2)\sqrt{f(P_1)/a(P_1)}, (1/2)\sqrt{f(P_1)/a(P_1)}\right)$ and $\psi \equiv 0$ on $\left(-\infty, \sqrt{f(P_1)/a(P_1)}\right] \cup \left[\sqrt{f(P_1)/a(P_1)}, \infty\right)$ and define $\psi_r(x) := \psi(r\,x)$ and the function v (which depends on u, P_1 and r) for all $(x, t) \in \mathbb{R} \times (-r^2 f(P_1), r^2 f(P_1))$ by

$$v(x,t) := u\left(x_1 + x\sqrt{\tfrac{a(P_1)}{f(P_1)}}, t_1 + \frac{t}{f(P_1)}\right) \cdot \psi_r(x) \ \text{ if } |x| \le r\sqrt{\tfrac{f(P_1)}{a(P_1)}}, \quad v \equiv 0 \ \text{ otherwise}. \tag{III.5}$$

Heuristically, this definition of v brings us back to the case where the support of the solution is compact. If, to have just an idea of the method, the reader wants to assume, in a first read, that u is a solution in $\mathbb{R}^2$ of the parabolic obstacle problem (III.1) with $a \equiv 1$, $b \equiv 0$, $c \equiv 0$ and $f \equiv 1$ then this definition of v is superfluous, as is the second term in the following energy:

For all $t \in (-r^2 f(P_1), 0)$, define

$$\mathcal{E}_{u,P_1}(\tau, r) := \int_{\mathbb{R}} \left\{ \left[\frac{1}{-\tau}\left(\left|\frac{\partial v}{\partial x}\right|^2 + 2\,v \right) - \frac{v^2}{\tau^2} \right] G \right\}(x,\tau)\,dx$$
$$- \int_\tau^0 \frac{1}{s^2} \int_{\mathbb{R}} \left\{ (Hv - 1)\,(\mathcal{L}v)\,G \right\}(x,s)\,dx\,ds\,,$$

with $Hv := v_{xx} - v_t$, $\mathcal{L}v := -2\,v + x \cdot v_x + 2\,t\,v_t$ and $G(x,t) := (2\pi(-t))^{-1/2} \exp\left(-x^2/(-4t)\right)$. For this energy we have (Proposition 3.4, Lemma 3.7 and Proposition 4.1 in [BDM05]):

Proposition III.3 (Monotonicity formula). *Let $Q_r(P_1) \subset Q_R(P_0)$. Under Assumption (III.2), if u is a solution of (III.1) and v is defined in (III.5), then for a given $r > 0$ the function $\tau \mapsto \mathcal{E}_{u,P_1}(\tau, r)$ is a non-increasing function, which is bounded in $W^{1,\infty}(-1, 0)$. Furthermore for $r > 0$ and a given $\tau_0 < 0$, $P \mapsto \mathcal{E}_{u,P}(\tau_0, r)$ is continuous.*

Moreover, if $(u_{P_1}^{\varepsilon_n})_{n\in\mathbb{N}}$ is a blow-up sequence associated with u at a fixed point P_1 and

$u^0_{P_1}$ is a blow-up limit of $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ then

$$\forall r>0 \quad \mathcal{E}_{u,P_1}(\varepsilon_n^2\tau,r)=\mathcal{E}_{u^{\varepsilon_n}_{P_1},0}(\tau,\varepsilon_n r)\to\mathcal{E}_{u^0,0}(\tau,0)\in\{\frac{\sqrt{2}}{2},\sqrt{2}\}\,. \tag{III.6}$$

With these notations, if, for $r>0$, $\lim_{\tau\to 0}\mathcal{E}_{u,P_1}(\tau,r)=\sqrt{2}/2$, then P_1 is a *regular point*. And if, for $r>0$, $\lim_{\tau\to 0}\mathcal{E}_{u,P_1}(\tau,r)=\sqrt{2}$, then P_1 is a *singular point*. The blow-up limits in regular points are of the type u^0_+ or u^0_-, and the blow-up limits in singular points are of the type u^0_m, for a given $m\in[-1,0]$, where u^0_+, u^0_- and u^0_m are defined in (III.4).

As a consequence : $\mathcal{S}$ is a closed set, and $\mathcal{R}=\Gamma\setminus\mathcal{S}$ is open in Γ (Lemma 4.2 in [BDM05]).

Remark III.1. *This energy characterisation of the sets $\mathcal{R}$ and $\mathcal{S}$ gives a criterion to apply Theorem III.1 (i). It is sufficient to prove that $\lim_{\tau\to 0}\mathcal{E}_{u,P_1}(\tau,r)<\sqrt{2}$. It can be interesting for practical financial applications. The derivative with respect to the initial condition, u_x, is known as the Delta in Greeks formulae and can be numerically computed with Monte-Carlo methods. See also [FLLL01] for a recent approach of the calculus of Delta using Malliavin calculus.*

For a further inspection of the singular set, [BDM05] introduces a monotonicity formula for singular points. Consequently it proves the uniqueness of blow-up limits in singular points. Namely, under Assumption (III.2), if u is a solution of (III.1) and $(P_n)_{n\in\mathbb{N}}$ a sequence of singular points, then there exists a unique $m\in[-1,0]$ such that for any sequence, $(\varepsilon_n)_{n\in\mathbb{N}}$, converging to 0, the whole blow-up sequence, $(u^{\varepsilon_n}_{P_n})_{n\in\mathbb{N}}$, locally uniformly converges to u^0_m, where u^0_m is defined in (III.4) (see Proposition 5.5 and Lemmas 5.6 and 5.7 in [BDM05]).

The main theorem of [BDM05] (Corollaries 6.4 and 6.7) is the following:

Theorem III.4 (Continuity of the time derivative). *Under Assumption (III.2) consider a solution u of (III.1). If $P_1\in\mathcal{R}\cup\mathcal{S}_0$ then*

$$\lim_{P\to P_1,\, P\in\{u>0\}}\frac{\partial u}{\partial t}(P)=0\,.$$

3 Proof of the main theorems

In Section 3.1 we study the regularity of the regular set. We first prove the uniqueness of blow-up limits in regular points. Then we prove that $\mathcal{R}$ is a graph with uniform holderian bound. In Section 3.2, as the uniqueness of blow-up limit in singular points is already known (see Section 2), we simply have to prove a uniform holderian bound on singular points.

3.1 Proof of Theorem III.1 (i): the regular points

Lemma III.5 (Uniqueness of blow-up limits in regular points). *Under Assumption (III.2), consider a solution of (III.1). If $P_1\in\mathcal{R}$ then there exists a unique $\gamma\in\{+,-\}$ such that for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0, the whole blow-up sequence $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ in the fixed point P_1 locally uniformly converges to u^0_γ, where u^0_+ and u^0_- are defined in (III.4).*

Proof – By the blow-up limit characterisation of $\mathcal{R}$ (see the consequences of Proposition III.3), up to the extraction of a sub-sequence, $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ converges to u^0_+ or u^0_-. Assume by contradiction that there are two sub-sequences $(\varepsilon_{n'})_{n'\in\mathbb{N}}$ and $(\varepsilon_{n''})_{n''\in\mathbb{N}}$ such that $(u^{\varepsilon_{n'}}_{P_1})_{n'\in\mathbb{N}}$ converges to u^0_+ and $(u^{\varepsilon_{n''}}_{P_1})_{n''\in\mathbb{N}}$ converges to u^0_-. We have $u^0_+(1,0) = 1/2$ and $u^0_-(1,0) = 0$. By continuity of $\varepsilon \mapsto u^\varepsilon_{P_1}$, this implies that there exists another sub-sequence $(\tilde{\varepsilon}_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} u^{\tilde{\varepsilon}_n}_{P_1}(1,0) = 1/4$. But this property is satisfied neither by u^0_+ nor by u^0_-, however $P_1 \in \mathcal{R}$. We obtain thus a contradiction. □

Definition III.6 ($\mathcal{R}_+$ and $\mathcal{R}_-$). *Let $P_1 \in \mathcal{R}$. The set of points such as the blow-up limit in P_1 is u^0_+ is denoted $\mathcal{R}_+$. The set of points such as the blow-up limit in P_1 is u^0_- is denoted $\mathcal{R}_-$.*

For any $\delta > 0$, if not empty, let us define the closed set:

$$\mathcal{R}_\delta = \{P \in \mathcal{R}, \quad \mathrm{dist}(P, \mathcal{S}) \geq \delta\} \ .$$

Lemma III.7 (Regularity property of $\mathcal{R}_\delta$). *For any $\delta > 0$, $\mathcal{R}_\delta$ is locally contained in a $C^{1/2}$-graph. More precisely, for any $\delta > 0$, there exists a constant $M(\delta) > 0$ such that,*

$$\sup_{(x,t)\in\mathcal{R}_\delta} \ \sup_{\substack{(x',t')\in\mathcal{R}_\delta \\ (x',t')\neq(x,t)}} \frac{|x'-x|}{\sqrt{|t'-t|}} \leq M(\delta) \, .$$

Proof – Assume by contradiction that there are two sequences of points $(P_n = (x_n, t_n))_{n\in\mathbb{N}}$ and $(x'_n, t'_n)_{n\in\mathbb{N}}$ in $\mathcal{R}_\delta$ converging to a point $P_\infty \in \mathcal{R}_\delta$, such that

$$\lim_{n\to\infty} \frac{|x'_n - x_n|}{\sqrt{|t'_n - t_n|}} = +\infty \, .$$

The blow-up sequence $(u^{\varepsilon_n}_{P_n})_{n\in\mathbb{N}}$ with $\varepsilon_n := \sqrt{(x'_n - x_n)^2 + |t'_n - t_n|}$ converges, up to the extraction of a sub-sequence, to a function u^0. Let (x,t) be such that $u(x_n + \varepsilon_n x, t_n + \varepsilon_n^2 t) > 0$, be definition of $u^{\varepsilon_n}_{P_n}$ we have:

$$\frac{d}{dt} u^{\varepsilon_n}_{P_n}(x,t) = \frac{du}{dt}(x_n + \varepsilon_n x, t_n + \varepsilon_n^2 t) \ .$$

By Schauder interior estimates u_t is continuous in $\{u > 0\}$, and the corresponding bound is uniform under scaling so we can pass to the limit in u_t. The right term converges to 0 because of the continuity of the time derivative (Theorem III.4) applied to $(x_n + \varepsilon_n x, t_n + \varepsilon_n^2 t)$ which tends to $P_\infty \in \mathcal{R}$. Therefore

$$\lim_{n\to\infty} \frac{d}{dt} u^{\varepsilon_n}_{P_n} = \frac{d}{dt} u^0 \equiv 0 \ .$$

It is easy to compute that the sequence of generic term

$$\nu_n := \left(\frac{x_n - x'_n}{\varepsilon_n}, \frac{t_n - t'_n}{\varepsilon_n^2} \right)$$

is in $\partial Q_1(0) \cap \partial\{u^{\varepsilon_n}_{P_n} = 0\}$. So $(\nu_n)_{n\in\mathbb{N}}$ converges to $\nu = (x_\nu, t_\nu) \in \partial Q_1(0)$. By non-degeneracy lemma (Lemma III.2) there exists a positive constant $\bar{C}$ such that for all $r > 0$ small enough

$$\bar{C}\, r^2 \leq \sup_{Q_r(\nu_n)} u^{\varepsilon_n}_{P_n} \to \sup_{Q_r(\nu)} u^0 \,.$$

So ν belongs to $\partial\{u^0 = 0\}$.

To summarise $0 \in \partial\{u^0 = 0\}$ by non-degeneracy lemma (Lemma III.2), $\nu \in \partial\{u^0 = 0\}$ by the above demonstration, $u^0_x \equiv 0$ on Γ because u is non-negative and $u^0_t \equiv 0$ by Theorem III.4. It is easy to convince oneself that u^0 is one of the two following functions:

$$\begin{aligned} &\text{case } x_\nu > 0\text{:}\quad u^0(x,t) = \tfrac{1}{2}\left(\max\{0,-x\}\right)^2 + \tfrac{1}{2}\left(\max\{0, x - x_\nu\}\right)^2 \,, \\ &\text{case } x_\nu < 0\text{:}\quad u^0(x,t) = \tfrac{1}{2}\left(\max\{0,x\}\right)^2 + \tfrac{1}{2}\left(\max\{0, x + x_\nu\}\right)^2 \,. \end{aligned} \tag{III.7}$$

We will prove with an energy argument that this cannot be true.

$\mathcal{R}_\delta$ is closed so P_∞ is in $\mathcal{R}_\delta$. By the energy characterisation of the regular points (see the consequences of Proposition III.3)

$$\lim_{\substack{\tau\to 0\\ \tau<0}} \mathcal{E}_{u,P_\infty}(\tau, r) = \frac{\sqrt{2}}{2} \quad \forall r > 0\,.$$

Hence, for a given $r > 0$, as $\tau \mapsto \mathcal{E}_{u,P}(\tau, r)$ is continuous (Proposition III.3) for any $\delta > 0$, we can find $\tau_0 < 0$ such that

$$\mathcal{E}_{u,P_\infty}(\tau, r) \leq \frac{\sqrt{2}}{2} + \frac{\delta}{4} \quad \forall \tau \in (\tau_0, 0),\ \forall r > 0\,. \tag{III.8}$$

But by (III.7) we explicitly know u^0 and we can directly compute that for all $\tau < 0$

$$\mathcal{E}_{u^0,0}(\tau, 0) = \frac{\sqrt{2}}{2} + \gamma(\tau)$$

where $\gamma(\tau)$ is explicit, positive and only depends on τ. By scale-invariance of $\mathcal{E}_{u,P}$ (Proposition III.3)

$$\frac{\sqrt{2}}{2} + \gamma(\tau) = \mathcal{E}_{u^0,0}(\tau, 0) = \lim_{n\to\infty} \mathcal{E}_{u^{\varepsilon_n}_{P_n},0}(\tau, \varepsilon_n r) = \lim_{n\to\infty} \mathcal{E}_{u,P_n}(\varepsilon_n^2 \tau, r)\,.$$

So for any $\delta > 0$ and τ_0 given in (III.8) there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$\tau_0 < \varepsilon_n^2 \tau < 0 \quad \text{and} \quad \mathcal{E}_{u,P_n}(\varepsilon_n^2 \tau, r) \geq \frac{\sqrt{2}}{2} + \gamma(\tau) - \frac{\delta}{2} \quad \forall r > 0\,. \tag{III.9}$$

However, $P \mapsto \mathcal{E}_{u,P}(\tau_0, r)$ is continuous for given τ_0 and $r > 0$ (Proposition III.3). So for any $\delta > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$\mathcal{E}_{u,P_n}(\tau_0, r) \leq \mathcal{E}_{u,P_\infty}(\tau_0, r) + \frac{\delta}{4} \quad \forall r > 0\,. \tag{III.10}$$

Combining (III.9) and (III.10), and because $\tau \mapsto \mathcal{E}_{u,P_n}(\tau, r)$ is non-increasing (Proposition III.3), we have

$$\frac{\sqrt{2}}{2} + \gamma(\tau) - \frac{\delta}{2} \leq \mathcal{E}_{u,P_n}(\varepsilon_n^2 \tau, r) \leq \mathcal{E}_{u,P_n}(\tau_0, r) \leq \mathcal{E}_{u,P_\infty}(\tau_0, r) + \frac{\delta}{4} \leq \frac{\sqrt{2}}{2} + \frac{\delta}{2} \quad \forall r > 0\,.$$

Which is a contradiction if we choose $\delta < \gamma(\tau)$. □

We now prove

Lemma III.8. *$\mathcal{R}_+$ and $\mathcal{R}_-$ are open subsets of* Γ.

We first need

Lemma III.9. *If $P_1 = (x_1, t_1)$ belongs to $\mathcal{R}_+$ (resp. $\mathcal{R}_-$) then for any $\eta \in (0,1)$, there exists $\rho > 0$ such that $u \equiv 0$ (resp. $u > 0$) in $(x_1 - \rho, x_1 - \eta\rho) \times (t_1 - \rho^2, t_1 + \rho^2)$ and $u > 0$ (resp. $u \equiv 0$) in $(x_1 + \eta\rho, x_1 + \rho) \times (t_1 - \rho^2, t_1 + \rho^2)$.*

Proof – By symmetry, we can assume that $P_1 = (x_1, t_1) \in \mathcal{R}_+$. Assume by contradiction that there is $\eta \in (0,1)$ such that for any $\rho > 0$, $u > 0$ in $(x_1 - \rho, x_1 - \eta\rho) \times (t_1 - \rho^2, t_1 + \rho^2)$ and $u \equiv 0$ in $(x_1 + \eta\rho, x_1 + \rho) \times (t_1 - \rho^2, t_1 + \rho^2)$. This is impossible because by (III.3), $\mathbb{1}\{u^{\varepsilon_n}_{P_1} > 0\}$ converges to $\mathbb{1}\{x > 0\}$, so for all $\rho > 0$ and $\eta \in (0,1)$ there exists $N \in \mathbb{N}$ such that $n > N$ implies $\mathbb{1}_{\{u^{\varepsilon_n}_{P_1} > 0\}}(x_1 - \rho(1+\eta)/2, t_1) = 0$ and $\mathbb{1}_{\{u^{\varepsilon_n}_{P_1} > 0\}}(x_1 + \rho(1+\eta)/2, t_1) = 1$. $\square$

Proof of Lemma III.8. By symmetry, we can assume that $P_1 = (x_1, t_1) \in \mathcal{R}_+$. As $\mathcal{R}$ is open, for any $\delta > 0$ there exists $r > 0$ such that P_1 and $\Gamma \cap Q_r(P_1)$ are contained in $\mathcal{R}_\delta$. By Lemma III.9, for any $\eta \in (0,1)$, there exists ρ such that $u \equiv 0$ in $(x_1 - \rho, x_1 - \eta\rho) \times (t_1 - \rho^2, t_1 + \rho^2)$ and u is positive in $(x_1 + \eta\rho, x_1 + \rho) \times (t_1 - \rho^2, t_1 + \rho^2)$ (see Figure III.3 for a generic drawing).

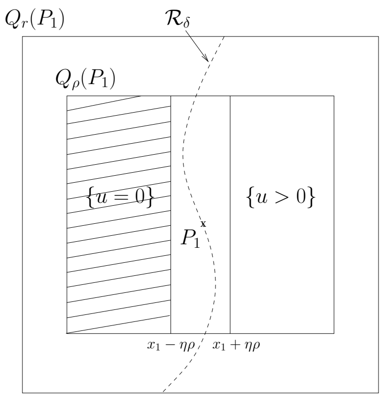


Figure III.3: Generic drawing for the proof of Lemma III.8

By the $\mathcal{C}^{1/2}$ regularity property of $\mathcal{R}_\delta$ (Lemma III.7), $\Gamma \cap Q_\rho(P_1) \subset \mathcal{R}_\delta$ is contained in a $C^{1/2}$-graph. This implies that for any $t \in (t_1 - \rho^2, t_1 + \rho^2)$, there exists a point $P = (g(t), t) \in \mathcal{R}$ such that $u(x,t) = 0$ in $\{(x,t) \in Q_\rho(P_1) \,:\, x < g(t)\}$ and $u(x,t) > 0$ in $\{(x,t) \in Q_\rho(P_1) \,:\, x > g(t)\}$.

To conclude consider $(u^{\varepsilon_n}_P)_{n\in\mathbb{N}}$ the blow-up sequence at the fixed point $P \in \mathcal{R}$ with ε_n in $\{(x,t) \,:\, x = g(t)\}$. As $u(x,t) = 0$ in $\{(x,t) \in Q_\rho(P_1) \,:\, x < g(t)\}$ and $u(x,t) > 0$ in $\{(x,t) \in Q_\rho(P_1) \,:\, x > g(t)\}$ the blow-up limit is $(1/2)(\max\{0,x\})^2$. So P belongs to $\mathcal{R}_+$. This is true for any $t \in (t_1 - \rho^2, t_1 + \rho^2)$. So all the points of $\Gamma \cap Q_\rho(P_1)$ are in $\mathcal{R}_+$. $\square$

Theorem III.1 (i) is a direct consequence of the $\mathcal{C}^{1/2}$ regularity property of $\mathcal{R}_\delta$ (Lemma III.7) and the topological property of $\mathcal{R}$. More precisely, locally around a point of $\mathcal{R}^+$ all the points are in $\mathcal{R}^+$ and the free boundary is locally $\mathcal{C}^{1/2}$. So around points of $\mathcal{R}^+$ the free boundary is locally a $\mathcal{C}^{1/2}$-subgraph. Respectively around points of $\mathcal{R}^-$, the free boundary is locally a $\mathcal{C}^{1/2}$-uppergraph.

3.2 Proof of Theorem III.1 (ii): the singular points

Assume by contradiction that there are two sequences of points $(P_n = (x_n, t_n))_{n\in\mathbb{N}}$ and $(x'_n, t'_n)_{n\in\mathbb{N}}$ in $\mathcal{S}_0$ converging to a point P_∞ such that

$$\lim_{n\to\infty} \frac{|x'_n - x_n|}{\sqrt{|t'_n - t_n|}} = +\infty\,. \tag{III.11}$$

Recall that as a consequence of the monotonicity formula (Theorem III.3), $\mathcal{S}_0$ is closed. So P_∞ belongs to $\mathcal{S}_0$. We define ε_n such that $P'_n \in Q_{\varepsilon_n}(P_n)$. By the uniqueness of the blow-up limit in singular points (see Section 2), and because $(P_n)_{n\in\mathbb{N}} \in \mathcal{S}_0^{\mathbb{N}}$, the blow-up sequence $(u^{\varepsilon_n}_{P_n})_{n\in\mathbb{N}}$ converges to $u^0(x,t) = (1/2)\,x^2$.

The sequence of generic term

$$\nu_n := \left(\frac{x'_n - x_n}{\varepsilon_n}, \frac{t'_n - t_n}{\varepsilon_n^2}\right)$$

is in $\partial\{u^{\varepsilon_n}_{P_n} = 0\} \cap \partial Q_1(0)$ and converges to $\nu = (x_\nu, t_\nu) \in \partial Q_1(0)$. Because of (III.11),

$$\lim_{n\to\infty} \frac{|x_{\nu_n}|}{\sqrt{|t_{\nu_n}|}} = \infty\,,$$

so ν belongs to $\{-1,0\} \cup \{1,0\}$. By non-degeneracy lemma (Lemma III.2) there exists a positive constant $\bar{C}$ such that for all $r > 0$ small enough

$$\bar{C}\,r^2 \le \sup_{Q_r(\nu_n)} u^{\varepsilon_n}_{P_n} \to \sup_{Q_r(\nu)} u^0\,.$$

So ν also belongs to $\partial\{u^0 = 0\}$. This is a contradiction with $u^0(x,t) = (1/2)\,x^2$.

4 Applications to American options

We consider a complete market which rules out arbitrage (*i.e.* the market rules out the possibility of making an instantaneous risk-free benefit).

In this market we can consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(W_t)_{t\ge 0}$ the standard Brownian motion under the risk-neutral probability $\mathbb{P}$, $(\mathcal{F}_t)_{t\ge 0}$ the $\mathbb{P}$-completion of the natural filtration of $(W_t)_{t\ge 0}$, $(S_t)_{t\ge 0}$ a one-dimensional price process and r the short rate of interest (to be quite general from a mathematical point of view, we assume here that r depends on the time t and also on the price S_t of the asset. See [Bjö97] for such kind of model). In the Black–Scholes model with local volatility (see [Dup97], [BBF02]), we assume that S_t satisfies the following stochastic differential equation

$$dS_t = r(S_t,t)\,S_t\,dt + \sigma(S_t,t)\,S_t\,dW_t\,, \quad \forall t \in [0,T]\,. \tag{III.12}$$

Here σ is called *the local volatility.*

An American option is the right to sell (or buy) during a period of time a share of a specific common stock, called *the underlying asset*, at a prescribed price P^0. Here P^0 depends on the price of this asset S_t and the time t. Given a positive time T, called *the maturity*, the American option allows the following *pay-off* if one sells (respectively buys) an asset at any time $t \in [0,T]$:

$$\phi(S_t,t) := \max\{0, P^0(S_t,t) - S_t\}\,, \quad (\text{resp.} \quad \phi(S_t,t) := \max\{0, S_t - P^0(S_t,t)\})\,. \tag{III.13}$$

Let us denote by $(S_s^{x,t})_{s\in[t,T]}$ the process solution of (III.12) with initial condition $S_t^{x,t} = x$. For a value of the risky asset equal to x at time t the price of the American option at time t is given by the following optimal stopping time problem:

$$\Pi(x,t) := \sup_{\tau\in\Theta_{[t,T]}} \mathbb{E}\left[\exp\left(-\int_t^\tau r(S_s^{x,t},s)ds\right)\phi(S_\tau^{x,t},\tau)\right] \tag{III.14}$$

where $\Theta_{[t,T]}$ is the set of all $\mathcal{F}_t$-stopping time τ with value in $[t,T]$. The rigorous financial interpretation of the American option pricing as an optimal stopping problem has been proved by [Ben84] and by [Kar88]. The first formulation of American option pricing in terms of optimal stopping problems is the prior work [MK65].

Let us define for $(x,t)\in\mathbb{R}\times[0,T]$

$$u(x,t) := \Pi(e^x, T-t) - \phi(e^x, T-t)$$

where Π is the price of the American option given by (III.14) and ϕ is the pay-off given by (III.13). In [BL78] (Théorème 4.1 p. 372) it is proved that the function u is the solution in $\mathbb{R}\times[0,T]$ in a variational sense of the parabolic obstacle problem (III.1) with initial condition $u(\cdot,0)=0$ where a, b c and f of the parabolic operator L are given from σ, r and ϕ by

$$\begin{array}{l} a(x,t) := \frac{\sigma^2}{2}(e^x,T-t),\ b(x,t) := r(e^x,T-t) - \frac{\sigma^2}{2}(e^x,T-t),\ c(x,t) := -r(e^x,T-t) \\ \text{and } \ -f(x,t) := (L\psi)(x,t) \ \text{ where } \ \psi(x,t) := \phi(e^x,T-t)\,. \end{array} \tag{III.15}$$

The assumptions of [BL78] are stronger than Assumption (III.2) but it rigorously provides the link between American option pricing and parabolic obstacle problems. This work was taken further by [JLL90], still in a variational interpretation but with stronger assumption than what we make here. More recently, this link has been justified in the framework of viscosity solutions under Assumption (III.2): by [Vil99] for the classical Black–Scholes model and by [Rap05] for the Black–Scholes model with local volatility. The proof that the stochastic formulation (III.14) is actually a solution almost everywhere of the problem (III.1) is still open.

The regularity assumptions (III.2) on the coefficients (*i.e.* on r and σ) are not too restrictive and they are usually admitted for local volatility models.

The set $\{u=0\}$ is then closed in $Q_R(P_0)$. The sets $\{u=0\}=\{\Pi=\phi\}$ and its boundary Γ are respectively called the *exercise region* and the *optimal exercise boundary*. The set $\{\Pi>\phi\}$ is called the *continuation region*.

Theorem III.1 applies in this framework. This result is local and is not true up to the maturity. There is a large literature on the study of the regularity of Γ close to the maturity (see [LV03], [BBRS93]). F. Charretour and R. Viswanathan were the first to notice that the exercise boundary cannot be $\mathcal{C}^{1/2}$ up to the maturity.

4.1 Application to homogeneous diffusion

Let us apply our results to the Black–Scholes model with homogeneous diffusion (case where σ and r do not depend on time). In this case the link between the stochastic formulation (III.14) and the obstacle problem has been proved in [Ach05] (Theorem 2.2).

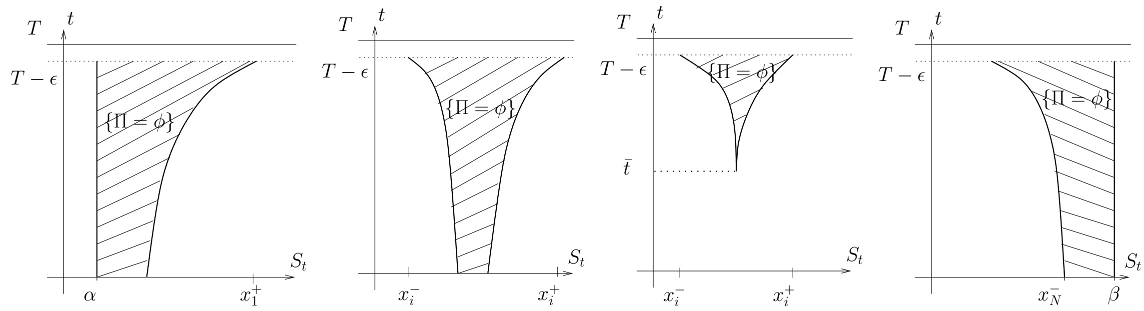
Theorem III.10 (Exercise boundary in the Black-Scholes model). *In the generalised Black–Scholes model where σ and r do not depend on time, consider the price Π of the American option given by (III.14) with a finite maturity $T > 0$. Assume $\sigma \in L^\infty(\mathbb{R})$ is such that $x(d\sigma/dx)$ is bounded in $L^\infty(\mathbb{R})$ and the pay-off ϕ depends only on S_t, satisfies $|\phi(x)| \leq M\, e^{M|x|}$, where $M > 0$. If*

$$\frac{\sigma^2}{2}x^2\frac{\partial^2\phi}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)x\frac{\partial\phi}{\partial x} + r\phi < 0$$

then for all $\varepsilon > 0$ and $[\alpha, \beta] \subset \mathbb{R}$ there exists $N \in \mathbb{N}$, $2N$ reels $x_1^- \leq x_1^+ < x_2^- \ldots < x_i^- \leq x_i^+ < \ldots < x_N^- \leq x_N^+$ in $[\alpha, \beta]$ and $2N$ graphs of class $\mathcal{C}^{1/2}([0, \tilde{t}\,])$, $(g_i^- \leq g_i^+)_{i\in\{1,\ldots,N\}}$ where g_i^- is non-increasing and g_i^+ is non-decreasing, such that for every $i \in \{1, \ldots, N\}$ there exists $\tilde{t}_i \in [0, T - \varepsilon]$,

$$\{\Pi = \phi\} \cap ([x_i^-, x_i^+] \times [0, \tilde{t}_i\,]) = \{(x, t) \in [x_i^-, x_i^+] \times [0, \tilde{t}_i\,] \,:\, g_i^-(t) \leq x \leq g_i^+(t)\}\,.$$

Moreover, if for a given $i \in \{1, \ldots, N\}$ there exists $\hat{t} \in [0, T - \varepsilon)$ such that $g_i^-(\,\hat{t}\,) = g_i^+(\,\hat{t}\,)$ then $\Pi > \phi$ in $[0,\, \hat{t}\,]$.



We begin to prove a slightly different lemma which brings us back to the framework of Section 3.

Lemma III.11. *Under Assumption (III.2), consider a solution of (III.1). If u is non-decreasing then for all $\varepsilon > 0$ and $[\alpha, \beta] \subset \mathbb{R}$ there exists $N \in \mathbb{N}$, $2N$ reels $x_1^- \leq x_1^+ < x_2^- \ldots < x_i^- \leq x_i^+ < \ldots < x_N^- \leq x_N^+$ in $[\alpha, \beta]$ and $2N$ graphs of class $\mathcal{C}^{1/2}([\varepsilon, \tilde{t}\,])$, $\{g_i^-, g_i^+\}_{i\in\{1,\ldots,N\}}$ where g_i^- is non-decreasing and g_i^+ is non-increasing, such that for every $i \in \{1, \ldots, N\}$ there exists $\tilde{t}_i \in [\varepsilon, T]$,*

$$\{u = 0\} \cap ([x_i^-, x_i^+] \times [\varepsilon, \tilde{t}_i]) = \{(x, t) \in [x_i^-, x_i^+] \times [\varepsilon, \tilde{t}_i] \,:\, g_i^-(t) \leq x \leq g_i^+(t)\}\ .$$

Moreover, if for a given $i \in \{1, \ldots, N\}$ there exists $\hat{t} \in (\varepsilon, T]$ such that $g_i^-(\,\hat{t}\,) = g_i^+(\,\hat{t}\,)$ then $u > 0$ in $[\,\hat{t}, T]$.

We first precise some properties of the exercise boundary around points of $\mathcal{S}_0$ and $\mathcal{R}$.

Lemma III.12 (Geometric properties of $\mathcal{R}$ and $\mathcal{S}_0$). *Under Assumption (III.2), consider a solution of (III.1).*

(i) *If* $P_1 = (x_1, t_1) \in \mathcal{R}^+$ *(resp.* $\mathcal{R}^-$*) then there exists* $\rho > 0$ *such that* $u \equiv 0$ *in* $(x_1-\rho, x_1)\times\{t = t_1\}$ *(resp.* $(x_1, x_1+\rho)\times\{t = t_1\}$*) and* $u > 0$ *in* $(x_1, x_1+\rho)\times\{t = t_1\}$ *(resp.* $(x_1 - \rho, x_1) \times \{t = t_1\}$*).*

(ii) *If* $P_1 = (x_1, t_1) \in \mathcal{S}_0$ *then there exists a positive radius* ρ *such that* u *is positive in* $((x_1 - \rho, x_1) \cup (x_1, x_1 + \rho)) \times \{t = t_1\}$.

Furthermore, for any time $t_0 \in (0, T)$ *there are a finite number of points of* $(\mathcal{R} \cup \mathcal{S}_0) \cap \{(x, t) \in Q_R(P_0) \,:\, t = t_0\}$.

Proof – (i) By symmetry we can assume $P_1 \in \mathcal{R}^+$. Assume by contradiction that there exists $(P_n = (x_n, t_1))_{n\in\mathbb{N}} \in \Gamma^{\mathbb{N}}$ converging to $P_1 \in \mathcal{R}^+$. Consider the sequence of generic term $\varepsilon_n := x_n$. By definition of $\varepsilon_n := x_n$, $u^{\varepsilon_n}_{P_1}(1, 0) = 0$. This implies that $u^0(1, 0) = 0$. But by the blow-up limit characterisation of $\mathcal{R}^+$ (Definition III.6), the blow-up sequence $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ converges to $(1/2)(\max\{0, x\})^2$. Which is a contradiction.
(ii) In the same manner, assume by contradiction that there exists $(P_n = (x_n, t_1))_{n\in\mathbb{N}}$ converging to $P_1 \in \mathcal{S}_0$ such that $u(P_n) = 0$. Consider the sequence of generic term $\varepsilon_n := x_n$. By definition of $\varepsilon_n := x_n$, $u^{\varepsilon_n}_{P_1}(1, 0) = 0$. This implies that $u^0(1, 0) = 0$. But by the blow-up limit characterisation of $\mathcal{S}$ (consequences of Proposition III.3), the blow-up sequence $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ converges to $(1/2)x^2$. Which is a contradiction. □

Proof of Lemma III.11. First of all let us recall that, if u is non-increasing, the free boundary only contains regular points and points of $\mathcal{S}_0$ (see Section 1). In $t = \varepsilon$, consider a point $P_i = (x_i, \varepsilon)$ of Γ.

First case. If P_i belongs to $\mathcal{S}_0$, by the geometric properties of $\mathcal{S}_0$ (Lemma III.12 (ii)), there exists r such that u is positive in $(x_i - r, x_i + r) \times \{\varepsilon\}$. As u is non-increasing u is positive in $(x_i - r, x_i + r) \times [\varepsilon, T]$. Assume by contradiction that there exists $\delta > 0$ such that $u = 0$ on $x_i \times [\varepsilon, \varepsilon + \delta]$. Then u would be positive almost everywhere in $(x_i - r, x_i + r) \times [\varepsilon, \varepsilon + \delta]$. So u_t is caloric in $(x_i - r, x_i + r) \times [\varepsilon, \varepsilon + \delta]$ and 0 on $x_i \times [\varepsilon, \varepsilon + \delta]$. By the maximum principle it implies that u_t is constant in $(x_i - r, x_i + r) \times [\varepsilon, \varepsilon + \delta]$. Hence f is zero in $(x_i - r, x_i + r) \times [\varepsilon, \varepsilon + \delta]$. As f does not depend on time it implies that f is zero in $(x_i - r, x_i + r) \times [\varepsilon, T]$. In this case we define $x_i^- = x_i - r$, $x_i^+ = x_i + r$.

Second case. If P_i belongs to $\mathcal{R}$, by symmetry consider $P_i \in \mathcal{R}^-$. By Theorem III.1, Γ is a $\mathcal{C}^{1/2}$-uppergraph locally around P_i. We can extend this property at $C_\Gamma(P_i)$, the connected component of Γ which contains P_i for all points in $\mathcal{R}^-$. Denote $P^* = (x^*, t^*) := \inf_{t\leq T}\{P \in C_\Gamma(P_i) \,:\, P \notin \mathcal{R}^-\}$. If $t^* = T$, the whole component $C_\Gamma(P_i)$ is a $\mathcal{C}^{1/2}$-uppergraph up to T. If not: thanks to Lemma III.9, P^* cannot belong to $\mathcal{R}^+$. So P^* belongs to $\mathcal{S}_0$. We set here $x_i = x_i^-$. The set $C_\Gamma(P_i)$ is a graph of class $\mathcal{C}^{1/2}([0, t^*])$, let us denote it g_i^-.

Remains in this case to deal with the other boundary of the connected component of $\{u = 0\}$. By the geometric properties of $\mathcal{R} \cup \mathcal{S}_0$ (Lemma III.12), there are only a finite number of points of $\mathcal{R} \cup \mathcal{S}_0$ in $\{t = \varepsilon\}$. Define $x_i^+ := \sup_{x > x_i^-}\{(x, t) \,:\, (x, t) \in \{u = 0\}\}$. By definition of x_i^+, $u = 0$ in $[x_i^-, x_i^+] \times \{t = \varepsilon\}$. By the geometric properties of $\mathcal{R}$ (Lemma III.12 (i)), (x_i^+, ε) is a point of $\mathcal{R}^+$. The previous argument gives the existence of a graph, g_i^+ of class $C^{1/2}([0, t_2^*])$ with $t_2^* \in [\varepsilon, T]$.

But if we cannot extend the $\mathcal{C}^{1/2}$-regularity of the graph up to $t = T$ it means that at least one of the two connected components of Γ defined above contains a point of $\mathcal{S}_0$. By symmetry we can assume that $t^* \leq t_2^*$. Assume by contradiction that the two curves do not meet in $(g_i^-(t^*), t^*)$. Then there exists $\bar{P} = (\bar{x}, t^*)$ in the connected component of

Γ which contains (x_i^+, ε) such that $u \equiv 0$ in $[\tilde{x}, \bar{x}]$. But this is a contradiction with the geometric properties of $\mathcal{S}_0$ (Lemma III.12 (ii)). □

Proof of Theorem III.10. In the generalised Black–Scholes model where σ and r do not depend on time, $u(x,t) = \Pi(e^x, T-t) - \phi(e^x)$ is non-decreasing in time on $[0,T]$ (see Proposition 5 in [Rap05]), and is the solution almost everywhere in $\mathbb{R} \times [0,T]$ of (III.1) with initial condition $u(\cdot, 0) = 0$. The Assumption (III.2) is satisfied with a, b, c and f given from σ, r and ϕ by (III.15). But [HP81] gives the equivalence between the completeness of the market and $\sigma > 0$. Thus Theorem III.10 is a direct consequence of Lemma III.11.□

4.2 Application to American vanilla options

The two most classical pay-off functions are the put (*i.e.* $\phi(S_t, t) = \max\{0, K - S_t\}$) and the call (*i.e.* $\phi(S_t, t) = \max\{0, S_t - K\}$), where the fixed price K is called *the strike*. Our results apply to these pay-off.

Theorem III.13 (The exercise boundaries for American vanilla options are regular). *In the generalised Black–Scholes model where σ and r do not depend on time. Assume $\sigma \in L^\infty(\mathbb{R})$ is such that $x(d\sigma/dx)$ is bounded in $L^\infty(\mathbb{R})$. Then*

(i) *If the underlying asset is the solution of (III.12) then the exercise boundary of an American put is a $\mathcal{C}^{1/2}$-subgraph for all $t < T$.*

(ii) *If the underlying asset is the solution of $dS_t = (r(S_t) - \delta)\, S_t\, dt + \sigma(S_t)\, S_t\, dW_t$, with δ constant then the exercise boundary of an American call is a $\mathcal{C}^{1/2}$-uppergraph for all $t < T$.*

Proof –
First step. We first need to verify the Assumption (III.2). The coefficients a, b and c are defined by r and σ as in (III.15) are in $\mathcal{H}^\alpha(\mathbb{R} \times [0,T])$. The market is complete, so $a > 0$.

The price of the option Π is non-increasing in time. We state that the free boundary takes off the set $\{\Pi = \phi\}$ which means that there is only one point of the exercise boundary at a fixed time. Indeed, assume by contradiction that at a time t_1, there are two different prices s_1 and s_2 such that $\Pi(s_1, t_1) = \phi(s_1)$ and $\Pi(s_2, t_1) = \phi(s_2)$. Then the maximum principle applied to the sub-caloric function $\Pi - \phi$ implies that $\Pi = \phi$ in $[(s_1, t_1), (s_2, t_1)]$. Due to the shape of the pay-off (see Figures III.4 and III.5) the exercise boundary is non-decreasing in time for the American put and non-increasing for the American call. In particular the exercise boundary is a graph in time.

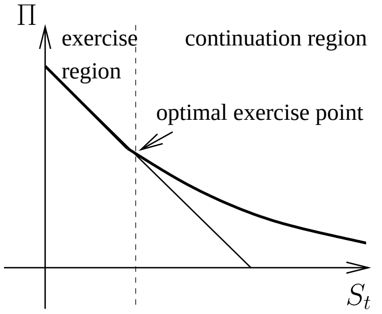


Figure III.4: American put

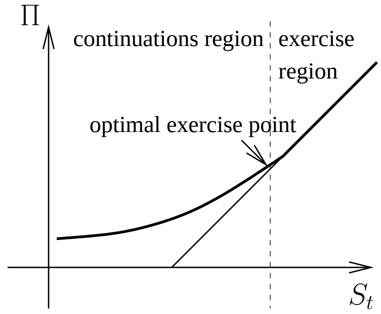


Figure III.5: American call

For the American put, from (III.15) we compute

$$f(x,t)\mathbb{1}_{\{\Pi>\phi\}}(x,t) = rK\mathbb{1}_{\{\Pi>\phi\}}\mathbb{1}_{\{t>s(t)\}} = rK\mathbb{1}_{\{\Pi>\phi\}}$$

because s is non-decreasing and $\lim_{t\to T} s(t) = K$. So we can consider the obstacle $\tilde{f} = rK$ in (III.1); $\tilde{f}$ is regular and non-degenerate.

For the American call, from (III.15) we compute

$$f(x,t) = r\min\{K, r\,K/\delta\}\mathbb{1}_{\{\Pi>\phi\}}\mathbb{1}_{\{x>K\}} = r\min\{K, rK/\delta\}\mathbb{1}_{\{\Pi>\phi\}}$$

for $t \le T-\varepsilon$ because s is non-decreasing and $\lim_{t\to T} s(t) = \min\{K, rK/\delta\}$ (see [Che05]). So we can consider the obstacle $\tilde{f} = r\min\{K, rK/\delta\}$ in (III.1) for $t \le T-\varepsilon$; $\tilde{f}$ is regular and non-degenerate.

Second step. Now let us verify one of the three criteria (energetic, blow-up and density) to prove that all the points of the exercise boundary are regular points. We decide here to use the density criterion.

For the American put: let $P_1 = (x_1, t_1) \in [0, T)$ be a point of the exercise boundary. There exists $r > 0$ such that $\Pi = \phi$ in $[x_1 - r, x_1] \times \{t = t_1\}$. But Π is non-increasing in time so $\Pi = \phi$ in $[x_1 - r, x_1] \times \{t_1, T\}$. So for r small enough

$$\frac{|\{\Pi = \phi\} \cap Q_r(P_1)|}{|Q_r(P_1)|} \supset \frac{|[x_1 - r, x_1] \times [t_1, t_1 + r^2]|}{|Q_r(P_1)|} > 0$$

So the density of the exercise region in P_1 is non-zero and so P_1 is not in $\mathcal{S}_0$. As u is non-increasing in time, there are no points of $\mathcal{S} \setminus \mathcal{S}_0$ (see remark before Theorem III.1) so the exercise boundary of the American put is only made up of points of $\mathcal{R}^+$. Hence the exercise boundary is a $\mathcal{C}^{1/2}$-subgraph for all $t < T$.

Similarly for the American call: for r small enough $\{\Pi = \phi\} \cap Q_r(P_1) \supset [x_1, x_1 + r] \times [t_1, t_1 + r^2]$. So there are no points of $\mathcal{S}_0$. Hence the exercise boundary of the American call is made up of points of $\mathcal{R}^-$, and is a $\mathcal{C}^{1/2}$-uppergraph in time for all $t < T$. □

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A.IV

On the singular set of the parabolic obstacle problem

Ce chapitre est la version préliminaire de [Bla05].

Résumé

Cet article est consacré à des résultats de régularité et à des propriétés géométriques de l'ensemble des points singuliers pour le problème de l'obstacle à second membre variable en dimension supérieure. Nous montrons, en particuliers, une formule de monotonie pour les points singuliers. On en déduit l'unicité des limites d'explosions aux points singuliers. Ces résultats s'appliquent au poblème de l'obstacle parabolique.

On the singular set of the parabolic obstacle problem

A. Blanchet

Abstract

This paper is devoted to regularity results and geometric properties of the singular set of the parabolic obstacle problem with variable right hand side. We introduce a monotonicity formula for singular points. We deduce the uniqueness of blow-up limits at singular points. These results apply to parabolic obstacle problem with variable coefficients.

AMS Classification: 35R35.
Keywords: parabolic obstacle problem, free boundary, singular points, singular set, monotonicity formula.

1 Introduction

Points in $\mathbb{R}^d \times \mathbb{R}$ are denoted (x,t), where the space variable $x = (x_1, \dots, x_d)$ belongs to $\mathbb{R}^d$ and the time variable, t, belongs to $\mathbb{R}$. To $x_0 \in \mathbb{R}^d$, $P_0 = (x_0, t_0) \in \mathbb{R}^d \times \mathbb{R}$ and $R > 0$, we associate the Eulerian open ball $B_R(x_0) := \{x \in \mathbb{R}^d \,:\, |x - x_0|^2 < R^2\}$ and the open parabolic cylinder

$$Q_R(P_0) := B_R(x_0) \times \{\, t \in \mathbb{R} \,:\, |t - t_0| < R^2 \,\} \,.$$

For $M > 0$ we define the sets

$$\mathcal{H}_M(Q_R(P_0)) := \{u \in L^\infty(Q_R(P_0)) \;:\; \sup_{Q_R(P_0)} \left[\left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right| + \left| \frac{\partial u}{\partial t} \right| + \frac{1}{R} \left| \frac{\partial u}{\partial x_i} \right| \right] \leq M < \infty\} \,,$$

and $\mathcal{H}_M(\mathbb{R}^{d+1})$ by

$$\mathcal{H}_M(\mathbb{R}^{d+1}) := \{u \in L^\infty(\mathbb{R}^{d+1}) \;:\; \sup_{\mathbb{R}^{d+1}} \left[\left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right| + \left| \frac{\partial u}{\partial t} \right| \right] \leq M < \infty\} \,.$$

We consider a solution u of the following parabolic obstacle problem:

$$\begin{cases} u \in \mathcal{H}_M(Q_R(P_0)) \,, \\ \Delta u(x,t) - \dfrac{\partial u}{\partial t}(x,t) = [1 + f(x,t)] \, \mathbb{1}_{\{u=0\}}(x,t) \,, \qquad \text{a.e. in } \; Q_R(P_0) \,, \\ u(x,t) \geq 0 \end{cases} \tag{IV.1}$$

where $\mathbb{1}_{\{u>0\}}$ denotes the characteristic function of the set $\{u = 0\} := \{(x,t) \in \mathbb{R}^{d+1} \;:\; u(x,t) = 0\}$. In the constant coefficients case it has been proved in [CPS04] that the solutions of (IV.1) in a distributional sense are actually in $\mathcal{H}_M$.

The set $\{u = 0\}$ and its boundary $\Gamma := Q_R(P_0) \cap \partial\{u = 0\}$ are respectively called the *coincidence set* and the *free boundary* of the parabolic obstacle problem (IV.1).

Up to a transformation the parabolic obstacle problem with variable coefficients reduces to this problem (see appendix). This model is the generalisation of the Stefan problem (case $f \equiv 0$) which describes the melting of an ice cube in a glass of water (see [Rod87, Vis96, Fri88, KS80] and reference therein). This problem also appears in the valuation of American option in the Black-Scholes model with local volatility (see [BL78, JLL90, Rap05]).

Let $P_1 = (x_1, t_1) \in \Gamma$, we define $\sigma_{P_1} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ a non-decreasing function such that $\sigma_{P_1}(0) = 0$, $\lim_{t\to 0} \sigma_{P_1}(t) = 0$ and $|f(P) - f(P_1)| \leq \sigma_{P_1}\left(\sqrt{|x - x_1|^2 + |t - t_1|}\right)$ for all $P = (x, t) \in Q_R(P_0)$. We assume that:

$$\begin{array}{rcl} [0,1] & \longrightarrow & \mathbb{R} \\ \alpha & \longmapsto & \displaystyle\frac{1}{\alpha}\int_0^\alpha \frac{\sigma_{P_1}(\theta)}{\theta}\, d\theta \end{array} \quad \text{is integrable for all } P_1 \in \Gamma\,. \tag{IV.2}$$

As a consequence of (IV.2), for all $P_1 \in \Gamma$, $f(P_1) = 0$ and there exists r such that for all $P \in Q_r(P_1)$, $f(P) \geq -1/2$. We assume

$$\begin{cases} \text{(IV.2) is satisfied,} \\ f(P) \geq -\dfrac{1}{2} \text{ for all } P \in Q_r(P_1)\,. \end{cases} \tag{IV.3}$$

Under Assumption (IV.3), consider u solution of (IV.1), $P_1 \in \Gamma$ and $(P_n)_{n\in\mathbb{N}} \in \Gamma^{\mathbb{N}}$ converging to P_1. The *blow-up sequence* $(u_{P_n}^{\varepsilon_n})_{n\in\mathbb{N}}$ associated to $u \in \mathcal{H}_M(Q_r(P_n))$ is the sequence of generic term

$$u_{P_n}^{\varepsilon_n}(x,t) = \frac{1}{\varepsilon_n^2}\, u\left(x_n + x\,\frac{\varepsilon_n}{\sqrt{1+f(P_1)}}, t_n + t\,\frac{\varepsilon_n^2}{1+f(P_1)}\right) \quad \forall (x,t) \in Q_{r/2\varepsilon_n}(0)\,. \tag{IV.4}$$

Proposition IV.1 (Classification of blow-up limits in $\mathbb{R}^{d+1}$). *Under Assumption (IV.3), consider a solution u of (IV.1) and $P_1 \in \Gamma$. There exists a sub-sequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ of $(\varepsilon_n)_{n\in\mathbb{N}}$ such that the blow-up sequence at the fixed point P_1, $(u_{P_1}^{\varepsilon_{n_k}})_{k\in\mathbb{N}}$ converges to one of the following:*

(i) $u_e^0(x,t) := \frac{1}{2}\left(\langle x, e\rangle\right)^2$, *for a unit vector e, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^d$,*

(ii) $u_{m,A}^0$ *the unique non-negative solution in $\mathcal{H}_M(\mathbb{R}^{d+1})$ of $\Delta u - \frac{\partial u}{\partial t} = \mathbb{1}_{\{u=0\}}$ which coincides with $m\,t + \frac{1}{2}X^T \cdot A \cdot X$, in $\mathbb{R}^d \times (-\infty, 0)$, for $m \in [-1, 0]$ and $A \in \mathcal{M}_m := \{A$, homogeneous polynomial satisfying : $\mathrm{Tr}\,(A) = m + 1\}$.*

In the case (i) we say that P_1 is a *regular point*, L. Caffarelli, A. Petrosyan and H. Shagholian prove in [CPS04] that the free boundary is a $\mathcal{C}^\infty_{x,t}$-maniflod locally around regular points for the constant coefficients case. In the case (ii) we say that P_1 is a *singular point*. We denote $\mathcal{S}$ as the set of singular points. This paper is devoted to the study of the *singular set* $\mathcal{S}$.

For these points we have

Proposition IV.2 (Uniqueness of blow-up limits at singular points). *Under Assumption (IV.3), consider a solution u of (IV.1). Let $P_1 \in \mathcal{S}$ and $(P_n)_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ converging to P_1. There exists a unique $(m_{P_1}, A_{P_1}) \in [-1,0] \times \mathcal{M}_m$ such that for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0, the whole blow-up sequence $(u^{\varepsilon_n}_{P_n})_{n\in\mathbb{N}}$ locally uniformly converges to $u^0_{m_{P_1},A_{P_1}}$ where $u^0_{m,P}$ is defined in Proposition IV.1.*

To a point $P_1 \in \mathcal{S}$ we can hence associate a unique $(m_{P_1}, A_{P_1}) \in [-1,0] \times \mathcal{M}_m$.

Definition IV.3 (The sets $\mathcal{S}(k)$). *For $k \in \{0, d-1\}$ we define $\mathcal{S}(k)$ as the set of singular points P such that* $\dim \operatorname{Ker} A_P = k$ *and the smallest of the k non-zero eigenvalues is bounded from below by a positive constant δ fixed. The set $\mathcal{S}(d)$ is the set of singular points P such that* $\dim \operatorname{Ker} A_P = d$.

To state our results on the regularity of $\mathcal{S}(k)$ we need to define the set $\mathcal{C}^{1/2}_{x,t}$, of holderian function $D \subset \mathbb{R}^{d+1} \to \mathbb{R}$ of exponent $1/2$:

$$\mathcal{C}^{1/2}_{x,t}(D) := \{u \in C^0_{x,t}(D) \ : \ \sup_{\substack{(x,t),(y,s)\in D \\ (x,t)\neq(y,s)}} \frac{|u(x,t)-u(y,s)|}{\sqrt{|x-y|^2+|t-s|^2}} < \infty\} \, .$$

This leads to the definition of a $\mathcal{C}^{1/2}_{x,t}$-manifold.

For the sets $\mathcal{S}(k)$, $k \in \{0,\dots,d\}$ we state:

Theorem IV.4 (Regularity of $\mathcal{S}(k)$). *Under Assumption (IV.3), consider a solution u of (IV.1).*

(i) *If $P_1 \in \mathcal{S}(d)$ then there exists $\tilde\Gamma$, a $\mathcal{C}^2_x$-graph in space, such that*

$$\mathcal{S}(d) \cap Q_\rho(P_1) \subset \tilde\Gamma \, ,$$

for some $\rho = \rho(d,M) > 0$ small enough.

(ii) *If $P_1 \in \mathcal{S}(k)$, for $k \in \{0,\dots,d-1\}$, then there exists Γ, a k-manifold of class $\mathcal{C}^{1/2}_{x,t}$, such that*

$$\mathcal{S}(k) \cap Q_\rho(P_1) \subset \Gamma \, ,$$

for some $\rho = \rho(d,M) > 0$ small enough.

As a consequence we prove

Corollary IV.5 (Regularity of $\cup_{k=0}^d \mathcal{S}(k)$). *Under Assumption (IV.3), consider a solution u of (IV.1). If $P_1 \in \cup_{k=0}^d \mathcal{S}(k)$ then there exists $\hat\Gamma$, a d-manifold of class a $\mathcal{C}^{1/2}_{x,t}$ such that*

$$\bigcup_{k=0}^d \mathcal{S}(k) \cap Q_\rho(P_1) \subset \tilde\Gamma \, ,$$

for some $\rho = \rho(d,M) > 0$ small enough.

The study of the singular sets in obstacle problems has been through many developments over the past twenty years. Especially for the elliptic obstacle problem. In this area a lot of questions have been conjectured by the pioneering work of D. G. Schaeffer [Sch77]. L. Caffarelli developed in [Caf77] a new theory to study obstacle problems by introducing *the blow-up method*. This theory has been largely simplified by *the monotonicity method* of G. Weiss (see [Wei99a]). A further step in the study of the singular set of the parabolic obstacle problem is [Mon03] where R. Monneau takes the formula of G. Weiss further to obtain a *monotonicity formula for singular points*. In particular he proves the uniqueness of blow-up limits in singular points and gives sharp geometric results on the singular set. For the elliptic obstacle problem with no sign assumption on the solution, L. Caffarelli and H. Shahgholian prove in [CS] regularity properties on the singular set making use of the monotonicity formula of [ACF84] and [CJK02].

For the parabolic obstacle problem the analysis is quite recent. G. Weiss introduced in [Wei99b] a monotonicity formula for the parabolic obstacle problem. In a recent paper L. Caffarelli, A. Petrosyan and H. Shahgholian make an in-depth analysis of the parabolic obstacle problem with no sign assumption on the solution with constant coefficients. However they do not study the singular set. For the parabolic obstacle problem with assumption (IV.3), J. Dolbeault, R. Monneau and the author make a study of the singular set but in one-dimension, in [BDM05].

In Section 2 we prove preliminary results. In Section 3 we prove a monotonicity formula of Weiss' type. This energy gives an energy criterion to characterise the regular and singular points of the free boundary. Proposition IV.1 is a consequence of this characterisation. In Section 4 we prove a monotonicity formula for singular points and prove the uniform convergence of the whole blow-up sequence to the blow-up limit at singular points. In Section 5 we prove the geometric and regularity results of Theorem IV.4 and Corollary IV.5.

Notation 1. For u smooth enough, u_t denotes $\frac{\partial u}{\partial t}$, $D_i u$ denotes the derivative $\frac{\partial u}{\partial x_i}$ and D_{ij} denotes the derivative $\frac{\partial^2 u}{\partial x_i \partial x_j}$. We define the open parabolic lower half-cylinder $Q_R^-(P_0) := B_R(x_0) \times \{ t \in (-\infty, 0) \ : \ |t - t_0| < R^2 \}$ and its parabolic boundary $\partial^p Q_R^-(P_0)$ as the topological boundary minus $\{t = t_0\}$.

2 Preliminaries

This section is quite classic but it has not been proved in this framework. For the constant coefficients case the reader can refer to [CPS04]. For more detailed proofs the interested reader can refer to [BDM05], where these kinds of proofs are demonstrated in one dimension.

We need two preliminary results. The first one is the non-degeneracy lemma. This kind of lemma has been proved for the first time by L. Caffarelli for the elliptic obstacle problem in [Caf77].

Lemma IV.6 (Non-degeneracy lemma). *Let $f \in L^\infty(\mathbb{R}^{d+1})$ Consider a solution $u \in \mathcal{H}_M(Q_R(P_0))$ of*

$$\begin{cases} \Delta u(x,t) - u_t(x,t) \geq [1 + f(x,t)]\, \mathbb{1}_{\{u>0\}} > 0 \\ u(x,t) \geq 0 \end{cases} \tag{IV.5}$$

Let $P_1 \in \overline{\{u > 0\}}$. *If* $r > 0$ *is such that* $Q_r^-(P_1) \subset Q_R(P_0)$ *then*

$$\sup_{Q_r^-(P_1)} u \geq \bar{C}\, r^2 .$$

with $\bar{C} = \frac{1}{2d+1}\left(1 + \|f\|_{L^\infty(Q_R(P_0))}\right)$.

Proof – The proof lies on the maximum principle. Consider first $P' = (x', t') \in \{u > 0\} \cap Q_r^-(P_1)$. We set for all $(x,t) \in Q_r^-(P')$

$$w(x,t) := u(x,t) - u(P') - \bar{C}\left((x-x')^2 + |t-t'|\right).$$

By the maximum principle, for any $\rho \leq r$ the maximum of the sub-caloric function w in $\overline{Q_\rho^-(P')} \cap \{u > 0\}$ is achieved in the parabolic boundary of $\overline{Q_\rho^-(P')} \cap \{u > 0\}$. As w is negative in $\partial\{u = 0\} \cap \overline{Q_\rho^-(P')}$ and $w(P') = 0$, there exists $P_2 = (x_2, t_2) \in \partial^p Q_\rho^-(P') \cap \{u > 0\}$ such that

$$\sup_{\overline{Q_\rho^-(P')} \cap \{u>0\}} w = w(P_2) = u(P_2) - u(P') - \bar{C}\left((x_2-x')^2 + |t_2-t'|\right) \geq 0 .$$

So for any $\rho \leq r$

$$\sup_{\overline{Q_\rho^-(P')}} u \geq u(P_2) \geq u(P') + \bar{C}\rho^2 . \tag{IV.6}$$

By continuity of u we achieve the result when P' converges to P_1. □

This lemma is very useful and will be used several times throughout this paper. As a consequence we prove a second lemma

Lemma IV.7 (Measure of Γ). *Let* $f \in L^\infty(\mathbb{R}^{d+1})$. *Consider a solution* $u \in \mathcal{H}_M(Q_R(P_0))$ *of (IV.5). The set* $\partial\{u = 0\}$ *is a closed set of zero* $(d+1)$*-Lebesgue measure.*

For the proof of Lemma IV.7 we recall the following result on measurable sets (See [Fed69], Theorem 2.9.11 (p. 158), Remark 2.9.12 (p. 158), Theorem 2.8.18 (p. 152) and Remark 2.8.9 (p. 145)).

Lemma IV.8 (Density in a point of a measurable set). *Let* A *be a measurable subset in* $\mathbb{R}^{d+1}$. *If* A *has non-zero Lebesgue measure, then for almost every* $P_1 = (x_1, t_1) \in A$, *we have*

$$\limsup_{n\to\infty} \frac{|A \cap C_n(P_1)|}{|C_n(P_1)|} = 1 ,$$

where $C_n(P_1) := B_{\frac{1}{n}}(x_1) \times \left[t_1 - \frac{1}{n}, t_1 + \frac{1}{n}\right]$.

Proof of Lemma IV.7. The Lemma IV.8 is stated for Euclidean cylinder but we can divide $C_n(P_1)$ into n parabolic cylinders $Q_{\frac{1}{n}}(x_1, t_i)$ with $t_i := t_1 - \frac{1}{n} + \frac{2i+1}{n^2}$, $i \in \{1, \ldots, n-1\}$.

(i) If $Q_{\frac{1}{n}}(x_1, t_i) \cap \Gamma = \emptyset$, we set $E_{i,n} := Q_{\frac{1}{n}}(x_1, t_i)$,

(ii) If $Q_{\frac{1}{n}}(x_1, t_i) \cap \Gamma \neq \emptyset$: on the one hand, by (IV.6), if $P_1 \in \Gamma$ and $r > 0$ are such that $Q_r(P_1) \subset Q_{\frac{1}{n}}(x_1, t_i)$ then there exists $P_2 = (x_2, t_2) \in Q_{r/2}^-(P_1)$ such that $u(P_2) \geq \frac{1}{4}\bar{C}\, r^2$. On the other hand the $\mathcal{H}_M(Q_R(P_0))$ regularity of u implies that for all $P \in Q_{\lambda r}(P_2)$, $u(P) \geq C\, \lambda\, r^2$. Collecting these two estimates for λ small enough there exists $E_{i,n} := Q_{\lambda r}(P_2)$ in $Q_{\frac{1}{n}}(x_1, t_i) \cap \{u > 0\}$.

Hence for any $i \in \{1, \ldots, n-1\}$ there exists $E_{i,n}$ in $Q_{\frac{1}{n}}(x_1, t_i) \cap \{u > 0\}$ with $|E_{i,n}| \geq \lambda^3 \, |Q_{\frac{1}{n}}(x_1, t_i)|$. We finally have

$$\limsup_{n \to \infty} \frac{|\Gamma \cap C_n(P_1)|}{|C_n(P_1)|} \leq 1 - \liminf_{n \to \infty} \frac{|\bigcup_{i=0}^{n-1} E_{i,n}|}{|C_n(P_1)|} < 1 \, .$$

The lemma on the density in a point of a measurable set (Lemma IV.8) completes the proof. □

We can now state the main result of Section 2.

Proposition IV.9 (Blow-up limit). *Under Assumption (IV.3), consider a solution u of (IV.1), $P_1 \in \Gamma$, $(P_n)_{n \in \mathbb{N}} \in \Gamma^{\mathbb{N}}$ converging to P_1 and a blow-up sequence $(u_{P_n}^{\varepsilon_n})_{n \in \mathbb{N}}$. There exists a sub-sequence and a function $u_{P_1}^0 \in \mathcal{H}_M(\mathbb{R}^{d+1})$ such that the blow-up sequence $(u_{P_n}^{\varepsilon_n})_{n \in \mathbb{N}}$ uniformly converges in every compact $K \subset\subset \mathbb{R}^{d+1}$ to $u_{P_1}^0$. Furthemore $u_{P_1}^0$ is a solution of the following global parabolic obstacle problem with constant coefficients :*

$$\begin{cases} \Delta u_{P_1}^0(x,t) - \dfrac{\partial u_{P_1}^0}{\partial t}(x,t) = \mathbb{1}_{\{u_{P_1}^0 = 0\}}(x,t) \, , & a.e. \ in \ \mathbb{R}^{d+1} \, . \\ u_{P_1}^0(x,t) \geq 0 & \end{cases} \tag{IV.7}$$

Moreover, $0 \in \partial\{u_{P_1}^0 = 0\}$.

Proof – By Ascoli-Arzela's theorem, there exists a sub-sequence $(\varepsilon_{n_k})_{n \in \mathbb{N}}$ and a non-negative function $u_{P_1}^0 \in \mathcal{H}_M(\mathbb{R}^{d+1})$ such that $(u_{P_n}^{\varepsilon_n})_{n \in \mathbb{N}}$ uniformly converges to $u_{P_1}^0$ for any compact $K \subset \mathbb{R}^{d+1}$. By uniform convergence we can pass to the limit in the equation satisfied by $u_{P_n}^{\varepsilon_n}$ to obtain:

$$\frac{\partial^2 u_{P_1}^0}{\partial x^2} - \frac{\partial u_{P_1}^0}{\partial t} = 1 \quad \text{in} \quad \{u_{P_1}^0 > 0\} \, .$$

By Lemma IV.7, $\partial\{u_{P_1}^0 > 0\}$ has zero $(d+1)$-Lebesgue measure, so $u_{P_1}^0$ is a solution of (IV.7).

By non-degeneracy Lemma (Lemma IV.6)

$$\bar{C} \, r^2 \leq \sup_{Q_r^-(0)} u_{P_n}^{\varepsilon_n} \to \sup_{Q_r^-(0)} u_{P_1}^0 \quad \text{as} \quad n \to \infty \, ,$$

which proves that $0 \in \partial\{u_{P_1}^0 = 0\}$. □

3 Classification of blow-up limits

A crucial tool for our study is the monotonicity formula of Weiss' type. G. Weiss introduced this kind of tool in [Wei99a] to prove the scale-invariance of blow-up limits in the elliptic obstacle problem. This scale-invariance of blow-up limits is very interesting because of the following Liouville's type theorem for self-similar solutions of (IV.7) (see Lemma 6.3 and Theorems I and 8.1 in [CPS04]):

Proposition IV.10 (Liouville's type theorem for $t < 0$). *If $u^0 \in \mathcal{H}_M(\mathbb{R}^{d+1})$ for any compact $K \subset \mathbb{R}^{d+1}$ is a non-zero self-similar solution of (IV.7) i.e. solution of (IV.7) under the constrain*

$$u^0(\lambda x, \lambda^2 t) = \lambda^2 u^0(x,t) \quad \forall \lambda > 0 \quad \forall (x,t) \in \mathbb{R}^d \times (-\infty, 0)$$

then, either there exists a unit vector e such that $u^0 = u^0_e$ or there exists $(m, A) \in [-1,0] \times \mathcal{M}_m$ such that $u^0 = u^0_{m,A}$; where u^0_e and $u^0_{m,A}$ are defined in Proposition IV.1.

Furthermore $u^0_t \leq 0$ and $D_{\nu\nu} u^0 \geq 0$, for any spatial direction $\nu \in \mathbb{R}^d$.

In dimension 1, the proof of the first part of the theorem uses the self-similarity of the solutions to bring itself back to an ordinary differential equation. The reduction to the dimension 1 uses the monotonicity formula of Caffarelli ([Caf93]) (see Lemma 6.3 in [CPS04]). The second assertion is a direct consequence of Theorem I in [CPS04]. Indeed, the case (iii) of Theorem I cannot happen because non-negative solutions are unique. So we cannot truncate the solution.

Let $Q_r(P_1) \subset Q_R(P_0)$. Consider a non-negative cut-off function $\psi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that $\psi \equiv 1$ in $B_{1/\left(2\sqrt{1+f(P_1)}\right)}$ and $\psi \equiv 0$ in $\mathbb{R}^d \setminus B_{1/\left(\sqrt{1+f(P_1)}\right)}$. Define $\psi_r(x) := \psi(r\,x)$ and the function v_{P_1} (which depends on u, P_1 and r) for all $(x,t) \in \mathbb{R}^d \times (-r^2\,[1+f(P_1)], r^2\,[1+f(P_1)])$ by

$$v_{P_1}(x,t) := u\left(x_1 + \frac{x}{\sqrt{1+f(P_1)}}, t_1 + \frac{t}{1+f(P_1)}\right) \cdot \psi_r(x) \quad \mathbb{1}_{B_{1/\left(\sqrt{1+f(P_1)}\right)}}(x)\,. \tag{IV.8}$$

For all $t \in (-r^2\,[1+f(P_1)], 0)$, define

$$\mathcal{E}_{u,P_1}(t,r) := \int_{\mathbb{R}^d} \left[\frac{1}{-t}\Big(|\nabla v|^2 + 2\,v\Big) - \frac{v^2}{t^2}\right] \mathcal{G}(x,t)\;dx - \int_t^0 \frac{1}{s^2} \int_{\mathbb{R}^d} (\mathtt{H}v - 1)\,(\mathtt{L}v)\,\mathcal{G}(x,s)\;dx\,ds\;,$$

with $\mathtt{H}v := \Delta v - v_t$, $\mathtt{L}v := -2\,v + x\cdot\nabla v + 2\,t\,v_t$ and $\mathcal{G}(x,t) := (2\pi(-t))^{-\frac{d}{2}}\,e^{\left(-|x|^2/(-4t)\right)}$.

Proposition IV.11 (Monotonicity formula for energy). *Under Assumption (IV.3) consider a solution u of (IV.1) and v_{P_1} defined in (IV.8). The function $t \mapsto \mathcal{E}_{u,P_1}(t,r)$ is non-increasing, bounded in $W^{1,\infty}(-r^2\,[1+f(P_1)], 0)$ and such that for almost every $t \in (-r^2\,[1+f(P_1)], 0)$*

$$\frac{\partial}{\partial t}\mathcal{E}_{u,P_1}(t,r) = -\frac{1}{2\,(-t)^3}\int_{\mathbb{R}^d} |\mathtt{L}v_{P_1}(x,t)|^2\,\mathcal{G}(x,t)\;dx\;.$$

Proof – The first part of the proof follows the idea of [CPS04]. Assume first that $v_{P_1} =: v \in \mathcal{D}(\mathbb{R}^d \times (-r^2\,[1+f(P_1)], 0))$. We begin to compute the time derivative of

$$\mathtt{e}(t;v) := \int_{\mathbb{R}^d} \left\{\frac{1}{-t}\Big(|\nabla v(x,t)|^2 + 2\,v(x,t)\Big) - \frac{1}{t^2}\,v^2(x,t)\right\} \mathcal{G}(x,t)\;dx\;. \tag{IV.9}$$

A simple change of variable gives $\mathtt{e}(\lambda^2\,t;v) = \mathtt{e}(t;v^\lambda)$, for all t in $(-\lambda^{-2}, 0)$, where $v^\lambda(x,t) := \lambda^{-2} v(\lambda\,x, \lambda^2\,t)$. Because $\left(\frac{\partial}{\partial\lambda}\,v_\lambda\right)_{|\lambda=1} = \mathtt{L}v$, we obtain at $\lambda = 1$

$$\frac{d\mathtt{e}}{dt}(t;v) = \frac{1}{2t} D_v \mathtt{e}(t;v)\cdot\mathtt{L}v\,,$$

where $D_v\mathtt{e}$ is defined for all ϕ in $\mathcal{C}^\infty(\mathbb{R}^d \times (-r^2\,[1+f(P_1)],0))$ by

$$D_v\mathtt{e}(t;v)\cdot\phi := \int_{\mathbb{R}^d}\left\{\frac{1}{-t}\Big(2\,\nabla v(x,t)\cdot\nabla\phi(x,t) + 2\,\phi(x,t)\Big)\right\}\mathcal{G}(x,t)\;dx \\ - \int_{\mathbb{R}^d}\frac{2}{t^2}\,v(x,t)\,\phi(x,t)\,\mathcal{G}(x,t)\;dx\,.$$

Integration by parts and a reordering of the terms give

$$\frac{d}{dt}\,\mathtt{e}(t;v) = \int_{\mathbb{R}^d}\left\{\frac{1}{2t^3}\,|\mathtt{L}v(x,t)|^2 + \frac{1}{t^2}\mathtt{L}v(x,t)\Big(\mathtt{H}v(x,t)-1\Big)\right\}\mathcal{G}(x,t)\;dx\;.$$

This equality is still true for $v \in \mathcal{H}_M(\mathbb{R}^d \times (-r^2\,[1+f(P_1)],0))$ with compact support in space by a density argument. Note that $t \mapsto \mathtt{e}(t;v)$ is bounded in $W^{1,\infty}_{\mathrm{loc}}(-r^2\,[1+f(P_1)],0)$ by $\mathcal{H}_M$-regularity estimates on u.

The second part of the proof follows the idea of [BDM05]. We have to control the function $\mathtt{r}$ defined by

$$\mathtt{r}(t;v) := \int_t^0 \frac{1}{s^2}\int_{\mathbb{R}^d}[(\mathtt{H}v-1)\mathtt{L}v\,\mathcal{G}](x,s)\,dx\,ds\;. \tag{IV.10}$$

We write $|\mathtt{r}(t;v)| \le \mathtt{A}(t)+\mathtt{B}(t)$ with

$$\mathtt{A}(t) \;:=\; \int_t^0\frac{1}{s^2}\int_{\mathbb{R}^d}|[\mathtt{H}v-(1+g)]\,\mathtt{L}v\,\mathcal{G}|\,(x,s)\,dx\,ds\;, \tag{IV.11}$$

$$\mathtt{B}(t) \;:=\; \int_t^0\frac{1}{s^2}\int_{\mathbb{R}^d}|g\,\mathtt{L}v\,\mathcal{G}|\,(x,s)\,dx\,ds\;. \tag{IV.12}$$

By definition of v, $[\mathtt{H}v-(1+g)]\,\mathtt{L}v$ vanishes in $B_{1/2}(0)\cup[\mathbb{R}^{d+1}\setminus B_1(0)]$ and by $\mathcal{H}_M$-regularity,

$$\mathtt{A}(t) \le c\int_0^{|t|}\frac{ds}{s^2}\int_{B_1\setminus B_{1/2}}\frac{e^{-|\rho|^2/4s}}{(2\pi\,s)^{d/2}}\,d\rho \le \frac{c}{2(2\pi)^{d/2}}\int_0^{|t|}\frac{e^{-1/16s}}{s^{(d+4)/2}}\,ds\;,$$

which gives a control of $\mathtt{A}(t)$ for any $t\in(-r^2\,[1+f(P_1)],0)$.

Due to the $\mathcal{H}_M$-regularity of u and the Definition (IV.2) of σ we have

$$\mathtt{B}(t) \le \int_t^0\frac{1}{s^2}\int_{\mathbb{R}^d}\sigma\left(\sqrt{x^2+|s|}\right)\,(x^2+|s|)\,\mathcal{G}(x,s)\,dx\,ds\;.$$

The change of variable $(x,s)\mapsto(y:=\frac{x}{\sqrt{t}},-s)$ and Fubini-Tonelli's theorem give

$$\mathtt{B}(t) \le c\int_{\mathbb{R}^d}(y^2+1)e^{-y^2/4}\int_0^{-t}\frac{1}{s}\sigma\left(\sqrt{s}\sqrt{y^2+1}\right)\,ds\,dy\,.$$

A cylindric change of coordinates on y gives

$$\mathtt{B}(t) \le c\int_0^{+\infty}(r^2+1)e^{-r^2/4}\int_0^{-t}\frac{1}{s}\sigma\left(\sqrt{s}\sqrt{r^2+1}\right)\,ds\,r^{d-1}\,dr$$

and the change of variable $(r,s)\mapsto\Big(\beta:=\sqrt{r^2+1},\theta:=\sqrt{s(r^2+1)}\Big)$ gives

$$\mathtt{B}(t)\le c\int_1^{+\infty}(\beta^2-1)^{\frac{d-2}{2}}\beta^3e^{-\beta^2/4}\bigg(\int_0^{\min\big(\beta\sqrt{|t|},\sqrt{1+|t|}\big)}\frac{\sigma(\theta)}{\theta}\,d\theta\bigg)d\beta\ .$$

Thus $\mathtt{B}(t)$ is bounded by Assumption (IV.3). □

Proposition IV.1 is a corollary of

Proposition IV.12 (Energy characterisation of the points of Γ). *Under Assumption (IV.3), consider a solution u of (IV.1), $P_1\in\Gamma$ and $r>0$ such that $Q_r(P_1)\subset Q_R(P_0)$. If $u^0_{P_1}$ is a blow-up limit associated to u at the fixed point P_1, then*

$$\Lambda(P_1):=\lim_{\substack{t\to0\\t<0}}\mathcal{E}_{u,P_1}(t,r)=\mathcal{E}_{u^0_{P_1},0}(t,0)\in\{2\,K,K\}\quad\forall\,(t,r)\in(-\infty,0)\times\mathbb{R}^d\ ,$$

where K is a positive constant which only depends on the dimension d. If $\Lambda(P_1)=K$, then $u^0_{P_1}=u^0_e$, for a certain unit vector e. If $\Lambda(P_1)=2\,K$, then $u^0_{P_1}=u^0_{m,A}$ for some $(m,A)\in[-1,0]\times\mathcal{M}_m$, where u^0_e and $u^0_{m,A}$ are defined in Proposition IV.1.

Proof – By the monotonicity formula for the energy (Proposition IV.11), $\mathcal{E}$ is non-increasing in time and bounded from below, so $\lim_{t\to0}\mathcal{E}_{u,P_1}(t,r)$ is finite. A simple change of variable shows that $\mathcal{E}_{u^{\varepsilon_n}_{P_1},0}(t,\varepsilon_n\,r)=\mathcal{E}_{u,P_1}(\varepsilon_n^2t,r)$. So $\lim_{t\to0}\mathcal{E}_{u,P_1}(t,r)=\lim_{n\to\infty}\mathcal{E}_{u,P_1}(\varepsilon_n^2t,r)=\lim_{n\to\infty}\mathcal{E}_{u^{\varepsilon_n}_{P_1},0}(t,\varepsilon_n\,r)=\mathcal{E}_{u^0_{P_1},0}(t,0)$. Hence $\frac{\partial}{\partial t}\mathcal{E}_{u^0_{P_1},0}(t,0)=0$. And so $u^0_{P_1}$ is scale-invariant in $\{t<0\}$.

By the classification of the scale-invariant solutions of (IV.7) for $t<0$ (Proposition IV.10) we identify $u^0_{P_1}$ in $\{t<0\}$ as one of the functions u^0_e and $u^0_{m,A}$. By the uniqueness of non-negative solutions of (IV.7), $u^0_{P_1}$ is either u^0_e or $u^0_{m,A}$ in $\mathbb{R}^d\times\mathbb{R}$.

A direct computation gives

$$\mathcal{E}(t;u^0_e)=K\quad\text{and}\quad\mathcal{E}(t;u^0_{m,A})=2\,K\ .$$

□

Proposition IV.12 allows the division of the free boundary into two sets, depending on the value of Λ: the points P of the free boundary such that $\Lambda(P)=K$ are the regular points and the points P of the free boundary such that $\Lambda(P)=2\,K$ are the singular points.

Lemma IV.13 (Topological properties of $\mathcal{R}$ and $\mathcal{S}$). *Under Assumption (IV.3), $\mathcal{S}$ is a closed set, and $\mathcal{R}=\Gamma\setminus\mathcal{S}$ is open in Γ.*

Proof – Let $(P_1,P_2)\in\Gamma^2$. By the energy characterisation of the points of Γ (Proposition IV.12), for all $\delta>0$ there exists $t_0=t_0(\delta)$ such that $|\mathcal{E}_{u^0_{P_1},0}(t_0,0)-\Lambda(P_1)|<\delta/2$. By $\mathcal{H}_M$-regularity of u, for this t_0 there exists a continuous function $\omega_{t_0}:\mathbb{R}^+\to\mathbb{R}^+$ such that $\omega_{t_0}(0)=0$ and $|\mathcal{E}_{u^0_{P_1},0}(t_0,0)-\mathcal{E}_{u^0_{P_2},0}(t_0,0)|<\omega_{t_0}(|P_1-P_2|)$. We can choose P_2 close enough to P_1 such that $\omega_{t_0}(|P_1-P_2|)<\delta/2$. With these choices we compute

$$\Lambda(P_2)-\Lambda(P_1)\le\mathcal{E}_{u^0_{P_2},0}(t_0,0)-\mathcal{E}_{u^0_{P_1},0}(t_0,0)+\mathcal{E}_{u^0_{P_1},0}(t_0,0)-\Lambda(P_1)<\omega_{t_0}(|P_1-P_2|)+\delta/2<\delta\ .$$

Hence the function $\Gamma\ni P\mapsto\Lambda(P)$ is upper semi-continuous. If $\Lambda(P_1)=K$, then $\Lambda(P_2)=K$ for P_2 in a neighbourhood of P_1. This proves that $\mathcal{R}$ is an open set in Γ and its complementary, $\mathcal{S}$ is a closed set of Γ. □

4 Study of the singular points of the free boundary

4.1 A monotonicity formula for singular points

R. Monneau developed in [Mon03] a monotonicity formula to study the set of singular points in the elliptic obstacle problem. He used it to prove the uniqueness of blow-up limit in singular points. This tool has been extended to the parabolic obstacle problem in one-dimension in [BDM05]. We write it in higher dimensions.

For v defined as in (IV.8) and u^0 one of the functions of Proposition IV.1, we define, for $t \in (-r^2\,[1+f(P_1)], 0)$, the functional

$$\Phi^{u^0}_{u,P_1}(t;r) := \frac{1}{t^2}\int_{\mathbb{R}^d} \left|v-u^0\right|^2 \mathcal{G}\;dx - \int_t^0 \frac{2}{s^2}\int_{\mathbb{R}^d} (\mathtt{H}v-1)\;(v-u^0)\,\mathcal{G}\;dx\;ds + \int_t^0 \frac{2}{s}\mathtt{r}(s;v)\;ds\;.$$

where $\mathtt{r}$ is defined in (IV.10).

Proposition IV.14 (Monotonicity formula for singular points). *Under Assumption (IV.3), consider a solution u of (IV.1), $P_1 \in \mathcal{S}$, $r > 0$ such that $Q_r(P_1) \subset Q_R(P_0)$ and u^0 one of the functions of Proposition IV.1. The function $t \mapsto \Phi^{u^0}_{u,P_1}(t;r)$ is non-increasing and bounded in $W^{1,1}(-r^2\,[1+f(P_1)], 0)$.*

Proof – By density, it is sufficient to prove the result for a smooth function v.

Let $w := v - u^0$ and $y := x\,\sqrt{-t}$. Using $\mathcal{G}\left(\sqrt{-t}\,y, t\right)\;dx = \mathcal{G}(y,1)\,dy$, and $\mathtt{L}u^0_{P_2} = 0$ in $\{t < 0\}$ we have

$$\frac{d}{dt}\left[\frac{1}{t^2}\int_{\mathbb{R}^d} w^2(x,t)\,\mathcal{G}(x,t)\;dx\right] = \frac{1}{t^3}\int_{\mathbb{R}^d} \mathtt{L}v(x,t)\,w(x,t)\,\mathcal{G}(x,t)\;dx\;. \tag{IV.13}$$

On the one hand, by the monotonicity of $\mathcal{E}$ (Proposition IV.11)

$$\begin{aligned}\mathcal{E}_{u,P_1}(t,r) - \mathcal{E}_{u^0,0}(t,r) &= \mathcal{E}_{u,P_1}(t,r) - \mathcal{E}_{u^0_{P_1},0}(t,0) + \mathcal{E}_{u^0_{P_1},0}(t,0) - \mathcal{E}_{u^0,0}(t,r)\\ &= \mathcal{E}_{u,P_1}(t,r) - \mathcal{E}_{u^0_{P_1},0}(t,0) \le 0\;.\end{aligned}$$

On the other hand

$$\mathcal{E}_{u,P_1}(t,r) - \mathcal{E}_{u^0,0}(t,r) = \mathtt{e}(t;v) - \mathtt{e}(t;u^0) - \mathtt{r}(t,v)$$

where $\mathtt{e}$ is defined in (IV.9) and $\mathtt{r}$ in (IV.10). By integration by part and reordering we obtain

$$\mathtt{e}(t;v_{P_1}) - \mathtt{e}(t;u^0_{P_2}) = \int_{\mathbb{R}} \left[\frac{1}{-t}\left[1-\mathtt{H}v_{P_1}(x,t)\right] + \frac{1}{2\,t^2}\,\mathtt{L}v_{P_1}(x,t)\right] w(x,t)\,\mathcal{G}(x,t)\,dx\;. \tag{IV.14}$$

Here we use $\mathtt{H}u^0_{P_2} = 1$ and $\mathtt{L}u^0_{P_2} = 0$ in $\{t < 0\}$.

Finally combining (IV.13) and (IV.14) and adding and subtracting g give

$$\frac{\partial}{\partial t}\Phi^{u^0}_{u,P_1}(t;r) = \frac{2}{t}\left[\mathcal{E}_{u,P_1}(t,r) - \mathcal{E}_{u^0,0}(t,r)\right] \le 0\;. \tag{IV.15}$$

Remains to control

$$\mathtt{C}(t)=\int_t^0 \frac{2}{s^2}\int_{\mathbb{R}^d}(\mathtt{H}v_{P_1}-1)\; w\,\mathcal{G}\;dx\;ds \quad \text{and} \quad \mathtt{D}(t)=\int_t^0 \frac{2}{s}\mathtt{r}(s;v_{P_1})\;ds\;.$$

The term $\mathtt{C}(t)$ can be controlled in the same way as $\mathtt{B}(t)$ in the proof of Proposition IV.11 by replacing $|\mathtt{L}v|\leq C(|x|^2+|t|)$ by $|w|\leq C(|x|^2+|t|)$. The last term to control is $\mathtt{D}(t)$. With $\mathtt{B}(t)$ and $\mathtt{A}(t)$ defined in (IV.11) we have

$$\begin{aligned}
\mathtt{D}(t) &\leq \int_t^0 \frac{2}{s}\left[\mathtt{A}(s)+\mathtt{B}(s)\right]\;ds\\
&\leq c\int_t^0 2\int_0^{|t|}\frac{e^{-\frac{1}{16s}}}{s^{\frac{n+6}{2}}}\;ds\\
&\qquad +c\int_t^0\frac{1}{s}\int_1^{+\infty}(\beta^2-1)^{\frac{n-2}{2}}\beta^3 e^{-\beta^2/4}\left(\int_0^{\min\left(\beta\sqrt{|s|},\sqrt{1+|s|}\right)}\frac{\sigma(\theta)}{\theta}\;d\theta\right)d\beta\;.
\end{aligned}$$

Which is bounded by Assumption (IV.3). □

4.2 Uniqueness of blow-up limit in singular points

By the monotonicity formula for songular points (Proposition IV.14), $t\mapsto\Phi^{u^0}_{u,P_1}(t;r)$ is non-increasing and bounded from below, so $\lim_{t\to 0}\Phi^{u^0}_{u,P_1}(t;r)$ is finite. A simple change of variable shows that $\Phi^{u^0}_{u^{\varepsilon_n}_{P_1},0}(t;\varepsilon_n\,r)=\Phi^{u^0}_{u,P_1}(\varepsilon_n^2 t;r)$. Let $u^0_{P_1}$ be one of the blow-up limits at P_1. We have $\lim_{t\to 0}\Phi^{u^0}_{u,P_1}(t;r)=\lim_{n\to\infty}\Phi^{u^0}_{u,P_1}(\varepsilon_n^2 t;r)=\lim_{n\to\infty}\Phi^{u^0}_{u^{\varepsilon_n}_{P_1},0}(t;\varepsilon_n\,r)=\Phi^{u^0}_{u^0_{P_1},0}(t;0)$. We can apply this computation to two different limits of $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ to prove that the blow-up limit is unique at a fixed point P_1.

Hence $\lim_{t\to 0}\Phi^{u^0_{P_1}}_{u,P_1}(t;r)=0$. By the monotonicity formula for songular points (Proposition IV.14), for all $\delta>0$ there exists $t_0=t_0(\delta)$ such that $|\Phi^{u^0_{P_1}}_{u,P_1}(t;r)|<\delta/2$. By $\mathcal{H}_M$-regularity of u, for this t_0 there exists a continuous function $\omega_{t_0}:\mathbb{R}^+\to\mathbb{R}^+$ such that $\omega_{t_0}(0)=0$ and $|\Phi^{u^0_{P_1}}_{u,P_1}(t;r)-\Phi^{u^0_{P_1}}_{u,P_2}(t;r)|<\omega_{t_0}(|P_1-P_2|)$. So there exists $\eta=\eta(t_0(\delta))$ such that $|P_1-P_2|<\eta$ implies $\omega_{t_0}(|P_1-P_2|)<\delta/2$. With the above choice of δ, $t_0(\delta)$ and $\eta(t_0(\delta))$, take $N\in\mathbb{N}$ such that $\varepsilon_n^2<t_0$ and $|P_1-P_n|<\eta$. Let u^0 be one of the blow-up limits of $(u^{\varepsilon_n}_{P_n})_{n\in\mathbb{N}}$,

$$\begin{aligned}
\Phi^{u^0_{P_1}}_{u^0,0}(t;0)=\lim_{n\to\infty}\Phi^{u^0_{P_1}}_{u^{\varepsilon_n}_{P_n},0}(t;\varepsilon_n\,r)=\lim_{n\to\infty}\Phi^{u^0_{P_1}}_{u,P_n}({\varepsilon_n}^2 t;r) &\leq \lim_{n\to\infty}\Phi^{u^0_{P_1}}_{u,P_n}(t_0;r)\\
&\leq \Phi^{u^0_{P_1}}_{u,P_1}(t_0;r)+\omega_{t_0}(|P_1-P_n|)<\delta\;.
\end{aligned}$$

Hence $u^0=u^0_{P_1}$ in $\{t<0\}$. By the uniqueness of non-negative solutions of (IV.7), $u^0=u^0_{P_1}$ in $\mathbb{R}^{d+1}$. □

To any $P\in\Gamma$, we can therefore associate a unique $(m_P,A_P)\in[-1,0]\times\mathcal{M}_m$ such that the blow-up limit of a solution at this point is $u^0_{m_P,A_P}$ where $u^0_{m,A}$ is defined in Proposition IV.1.

5 Geometric properties of $\mathcal{S}$

In Section 5.1 we deduce some regularity properties on the set $\mathcal{S}(d)$ and in Section 5.2 we study the sets $\mathcal{S}(k)$, $k \in \{0, \ldots, d-1\}$.

5.1 Proof of Theorem IV.4 (i): the set $\mathcal{S}(d)$

Lemma IV.15 (Regularity property of $\mathcal{S}(d)$). *Under Assumption (IV.3), consider a solution u of (IV.1) and $P_1 \in \mathcal{S}(d)$.*

$$\sup_{\substack{(x,t),(y,s)\in\mathcal{S}(d)\cap Q_\rho(P_1)\\ (y,s)\neq(x,t)}} \frac{|s-t|}{|y-x|^2} = 0\,,$$

for some $\rho = \rho(M, d) > 0$.

We first state

Lemma IV.16. *Under Assumption (IV.3), consider a solution u of (IV.1). Let $(P_n = (x_n, t_n))_{n\in\mathbb{N}}$ and $(P'_n = (x'_n, t'_n))_{n\in\mathbb{N}}$ be two sequences of points of $\mathcal{S}$ which converge to P_∞. Introduce*

$$\varepsilon := \sqrt{|x_n - x'_n|^2 + |t_n - t'_n|} \quad \text{and} \quad \mu_n := \left(\frac{x_n - x'_n}{\varepsilon}, \frac{t_n - t'_n}{\varepsilon^2}\right)\,.$$

There exists a sub-sequence $(n_k)_{k\in\mathbb{N}}$ and a vector μ such that $(\mu_{n_k})_{k\in\mathbb{N}}$ converges to μ. Furthermore $\lim_{n\to\infty} u^\varepsilon_{P_n} = u^0_{P_\infty} \equiv 0$ in $\mathrm{Vect}(\mu)$.

Proof – By the uniqueness of blow-up limits at singular points (Proposition IV.2), $(u^\varepsilon_{P_n})_{n\in\mathbb{N}}$ converges to a $u^0_{P_\infty}$. The vector μ_n is in the parabolic ball $B^p_1(0) := \{(x,t) \in \mathbb{R}^{d+1} : |x|^2 + |t| \le 1\}$ and in $\partial\{u^\varepsilon_{P_n} = 0\}$. So $(\mu_{n_k})_{k\in\mathbb{N}}$ converges to a vector μ in $B^p_1(0)$. Furthermore, by non-degeneracy lemma (Lemma IV.6) μ belongs to $\partial\{u^0_{P_\infty} = 0\}$. By the convexity of $u^0_{P_\infty}$ (see Proposition IV.10), $u^0_{P_\infty} = 0$ in Vect (μ). □

Proof of Lemma IV.15. Assume by contradiction there are $(P_n = (x_n, t_n))_{n\in\mathbb{N}}$ and $(P'_n = (x'_n, t'_n))_{n\in\mathbb{N}}$ two sequences of points of $\mathcal{S}(d)$ converging to P_1 such that

$$\lim_{n\to\infty} \frac{|t'_n - t_n|}{|x'_n - x_n|^2} = \delta > 0\,. \tag{IV.16}$$

By lemma IV.16, $\lim_{n\to\infty} u^\varepsilon_{P_n} = u^0_{P_1} \equiv 0$ in Vect (μ). This is a contradiction because μ does not belong to $\mathrm{Ker}\, A_{P_1} = \{x = 0\}$ by (IV.16). □

Proof of Theorem IV.4 (i). Let $g : \mathcal{S}(d) \ni P = (x,t) \mapsto t \in \mathbb{R}$. By Lemma IV.15, there exists $\rho > 0$ such that $g_{|Q_\rho(P_1)}$ is one-to-one. We introduce the closed set $\mathcal{S}^*(d) := g_{|Q_\rho(P_1)}(\mathcal{S}(d))$ and define $g^{-1}_{|Q_\rho(P_1)} : \mathcal{S}^*(d) \to \mathbb{R}$ which associates to the projection the unique point of $\mathcal{S}(d)$. Thanks to Lemma IV.15, $g^{-1}_{|Q_\rho(P_1)}$ is a $\mathcal{C}^2_x(\mathcal{S}^*(d))$ function in space. By Whitney's extension theorem (see [Whi34]) there exists a $C^2_x(\mathbb{R}^d)$ function which extends $g^{-1}_{|Q_\rho(P_1)}$ in $\mathbb{R}$. The graph of this function contains $\mathcal{S}(d) \cap Q_\rho(P_1)$, for τ small enough. □

5.2 Proof of Theorem IV.4 (ii): the sets $\mathcal{S}(k)$

Lemma IV.17. *Under Assumption (IV.3), consider a solution u of (IV.1). For $k \in \{0, \ldots, d-1\}$, consider $P_1 \in \mathcal{S}(k)$. There exists $\nu \in (\operatorname{Ker} A_{P_1})^{\perp} \subset \mathbb{R}^d$ such that*

$$\sup_{\substack{P,P' \in \mathcal{S}(k) \cap Q_r(P_1) \\ P' \neq P}} \frac{|\langle P, (\nu,0)\rangle - \langle P', (\nu,0)\rangle|}{|\mathrm{Proj}_{(\nu,0)}(P) - \mathrm{Proj}_{(\nu,0)}(P')|^{1/2}} < \infty ,$$

for $r = r(M,d) > 0$ small enough, where $\mathrm{Proj}_{(\nu,0)}$ is the projection in $\nu^{\perp} \times \mathbb{R}$ defined by

$$\begin{array}{rccl} \mathrm{Proj}_{(\nu,0)} : \mathcal{S}(k) & \to & \nu^{\perp} \times \mathbb{R} \\ P & \mapsto & P - \langle\, P, (\nu,0)\,\rangle\, (\nu,0) \ . \end{array}$$

and $|\cdot|$ denotes the Eulerian distance in $\mathbb{R}^{d+1}$.

Proof – Consider $(P_n = (x_n,t_n))_{n\in\mathbb{N}}$ and $(P'_n = (x'_n,t'_n))_{n\in\mathbb{N}}$ two sequences of points of $\mathcal{S}(k)$ converging to P_1. Assume

$$\frac{|\langle P_n - P'_n, (\nu,0)\rangle|}{|\mathrm{Proj}_{(\nu,0)}(P_n - P'_n)|^{1/2}} = \infty \ . \tag{IV.17}$$

By Lemma IV.16, $\lim_{n\to\infty} u^{\varepsilon}_{P_n} = u^0_{P_1} \equiv 0$ in $\mathrm{Vect}\,(\mu)$. But $\mu \in B^p_1(0)$ implies $|x_n - x'_n|^2 + |t_n - t'_n|^2 \leq |x_n - x'_n|^2 + |t_n - t'_n| = 1$. Hence (IV.17) implies that μ actually belongs to $\mathrm{Vect}(\nu,0)$. So $u^0_{P_1}(\cdot,0) \equiv 0$ in $\mathrm{Vect}(\nu)$. This is a contradiction with ν in $[\operatorname{Ker} A_{P_1}]^{\perp}$. □

A further step toward the proof of Theorem IV.4 (ii) is

Lemma IV.18 ($\mathcal{C}^{1/2}_{x,t}$-regularity of $\mathcal{S}(k)$). *Under Assumption (IV.3), consider a solution u of (IV.1). For $k \in \{0, \ldots, d-1\}$, consider $P_1 \in \mathcal{S}(k)$ and $\nu_i \in \operatorname{Ker}(A_{P_1})^{\perp}$. There exists Γ_i, a d-dimensional manifold of class $\mathcal{C}^{1/2}_{x,t}$, such that*

$$\mathcal{S}(k) \cap Q_r(P_1) \subset \Gamma_i$$

for some $r = r(d,M) > 0$.

Proof – By Lemma IV.17, there exists $\rho > 0$ such that the restriction of $\mathrm{Proj}_{(\nu_i,0)}$ in $Q_\rho(P_1)$, $\mathrm{Proj}_{(\nu_i,0)|Q_\rho(P_1)}$ is one-to-one. We introduce the closed set $\mathcal{S}^*(k) := \mathrm{Proj}_{(\nu_i,0)|Q_\rho(P_1)}(\mathcal{S}(k))$ and define $\mathrm{Proj}_{(\nu_i,0)|Q_\rho(P_1)}^{-1} : \mathcal{S}^*(k) \to \mathbb{R}$ which associates to the projection the unique point of $\mathcal{S}(k)$. Thanks to Lemma IV.17, $\mathrm{Proj}_{(\nu_i,0)|Q_\rho(P_1)}^{-1}$ is a $\mathcal{C}^{1/2}_{x,t}(\mathcal{S}^*(k))$ function. By Whitney's extension theorem generalised to holderian functions by Stein (Theorem 3 p.174, [Ste70]) there exists a $\mathcal{C}^{1/2}_{x,t}(\nu_i^{\perp} \times \mathbb{R})$ function which extends $\mathrm{Proj}_{(\nu_i,0)|Q_\rho(P_1)}^{-1}$ in $\nu_i^{\perp} \times \mathbb{R}$. The d-manifold of this function is denoted Γ_i and contains $\mathcal{S}(k) \cap Q_\rho(P_1)$ for ρ small enough. □

Proof of Theorem IV.4 (ii). Let $P_1 \in \mathcal{S}(k)$, for $k < d$. By Lemma IV.18, for the k, independent $(\nu_i)_{\{1,\ldots,k\}}$ in $\operatorname{Ker}(A_m)^{\perp}$, there exists a $\mathcal{C}^{1/2}_{x,t}$-manifold, Γ_i such that $\mathcal{S}(k) \cap Q_r(P_1) \subset \Gamma_i$. Hence

$$\mathcal{S}(k) \cap Q_r(P_1) \subset \Gamma := \bigcap_{i=1}^{d-k} \Gamma_i \quad \text{and} \quad \dim \Gamma = d - k \ .$$

$\square$

Proof of Corollary IV.5. Points of $\mathcal{S}(d)$ are isolated. Let $P_1 \in \mathcal{S} \setminus \mathcal{S}(d) = \cup_{i=0}^{d-1} \mathcal{S}(k)$. There exists ν in $\operatorname{Ker} A_{P_1}^{\perp}$. By Lemma IV.18, there exists Γ_i, a d-dimensional manifold of class $\mathcal{C}_{x,t}^{1/2}$, such that

$$\mathcal{S}(k) \cap Q_r(P_1) \subset \Gamma_i$$

for some $r = r(d, M) > 0$. We achieve the result with $\hat{\Gamma} := \Gamma_i$. $\square$

Appendix: Transformation

Let D be a domain of D^{d+1}. Let a_{ij}, b_i, c and g be continuous function of space and time in D. Consider a solution v of the following parabolic obstacle problem with variable coefficients

$$\begin{cases} v \in \mathcal{H}_M(D) \\ a_{ij}(y,s)\, \dfrac{\partial^2 v}{\partial y_i \partial y_j} + b_i(y,s)\, \dfrac{\partial v}{\partial y_i} + c(y,s)\, v - \dfrac{\partial v}{\partial s} = g(y,s)\, \mathbb{1}_{\{y>0\}}(y,s) & \text{a.e. } (y,s) \in D \\ v(y,s) \geq 0\,. \end{cases} \tag{18}$$

The reduction of a general parabolic operator to the heat operator is done by a classical transformation which goes as follows. Let $P_1 = (y_1, s_1) \in \partial\{v = 0\}$ and take $r > 0$ such that $Q_r(P_1) \subset D$. For all $P \in Q_r(P_1) \cap \{v > 0\}$, Equation (18) can be rewritten as

$$\begin{aligned} &a_{ij}(P_1)\, \frac{\partial^2 v}{\partial y_i \partial y_j}(P) - \frac{\partial v}{\partial s}(P) \\ &= g(P_1) + (g(P) - g(P_1)) - (a_{ij}(P) - a_{ij}(P_1))\, \frac{\partial^2 v}{\partial y_i \partial y_j}(P) - b_i(P)\, \frac{\partial v}{\partial y_i}(P) - c(P)\, v(P)\,. \end{aligned}$$

Consider the affine change of variables

$$(y,s) \mapsto \left(x := \sqrt{\frac{f(P_1)}{a_{ij}(P_1)}}\, y, \quad t := f(P_1)\, s \right)\,.$$

The function $u(x,t) := v(y,s)$ is a solution of (IV.1) with

$$\begin{aligned} &1 + f(x,t) := \\ &\frac{1}{g(P_1)} \left((g(P) - g(P_1)) - (a_{ij}(P) - a_{ij}(P_1))\, \frac{\partial^2 v}{\partial y_i \partial y_j}(P) - b_i(P)\, \frac{\partial v}{\partial y_i}(P) - c(P)\, v(P) \right)\,. \end{aligned}$$

By construction, $P_1 \in \partial\{u = 0\}$ and $f(P_1) = 0$. Note that if a_{ij}, b_i, c and g are $\mathcal{C}_{x,t}$ function then (IV.2) is verified.

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Appendice

1 Démonstration de la formule de monotonie

La formule de monotonie est un élément crucial de notre méthode de régularité pour le problème de l'obstacle. Dans le cas du problème de l'obstacle elliptique, L. Caffarelli avait montré, dans [Caf77], en dimension un d'espace, que les limites d'explosion étaient soit $\frac{1}{2}x^2$, soit $\frac{1}{2}(\max\{0,x\})^2$ soit $\frac{1}{2}(\max\{0,-x\})^2$. G. Weiss, en comparant les ensembles de coïncidence des solutions à des cônes, a trouvé une formule de monotonie qui donne la convergence de la suite d'explosion vers l'ensemble des fonctions homogènes au sens que $u(\lambda x) = \lambda^2 u(x)$ (voir [Wei99a]). Il a étendu sa formule au cas parabolique dans [Wei99b]. Sa démarche est issue d'une culture des mathématiques orientée vers la géométrie et les questions de surfaces minimales.

Nous avons trouvé cette formule de monotonie en nous inspirant d'un travail de Y. Giga et R. Kohn ([GK85]) sur les solutions auto-similaires pour l'équation $u_t = \Delta u + |u|^{p-1}$. On considère une solution u de $\Delta u - u_t = 1$. Pour trouver les variables auto-similaires (voir aussi Section 2.3.4 de l'introduction) on pose

$$u(x,t) = R^k(t)\, v(y,\tau(t)) \quad \text{avec } y := \frac{x}{R(t)} \ .$$

On a alors

$$\begin{aligned} &\Delta_x u = R^{k-2}\,\Delta_y v \\ &u_t = k\,R^{k-1}\,\dot{R}\,v - R^k\,\nabla v\,\frac{x}{R}\,\frac{\dot{R}}{R} + R^k\,v_s\,\dot{\tau} \end{aligned}$$

d'où l'équation écrite pour v :

$$R^{k-2}\,\Delta_y v - k\,R^{k-1}\,\dot{R}\,v + R^k\,\nabla v\,\frac{x}{R}\,\frac{\dot{R}}{R} - R^k\,v_s\,\dot{\tau} = 1 \ .$$

Si on veut éliminer les dépendances en temps il faut choisir k et R et τ de sorte que :

$$R^{k-2} = -k\,R^{k-1}\,\dot{R} = -2\,\dot{R}\,R^{k-1} = R^k\,\dot{\tau} = 1 \ .$$

Ce qui impose

$$\begin{cases} k = 2 \\ R^2(t) = -t + R^2(0) \\ \tau(t) = -2\log R(t) + \tau(0) \end{cases}$$

On choisit $R(0) = 0$, et $\tau(0) = 0$ de sorte que

$$\begin{cases} R(t) = \sqrt{-t} \\ \tau(t) = -\log(-t) \\ u(x,t) = -t\, v\left(\dfrac{x}{\sqrt{-t}}, -\log(-t)\right) \end{cases}$$

L'équation autonome écrite sur v est :

$$-\rho v_s + \nabla.(\rho \nabla v) + \rho v = \rho \quad \text{où} \quad \rho(y) := e^{-\frac{y^2}{4}} \tag{19}$$

Et l'énergie locale $\mathcal{E}$ est l'équivalent pour u de la fonctionnelle de Lyapounov associée à l'équation (19) :

$$\frac{d}{ds}\int \rho\left(\frac{|\nabla v|^2}{2} - \frac{|v|^2}{2} + v\right) = -\int \rho |v_s|^2 .$$

Cette fonctionnelle devient, lorsque l'on défait le changement de variable sur v

$$\begin{cases} v(y,s) = e^s\, u\left(y\, e^{-\frac{s}{2}}, -e^{-s}\right) \\ \nabla_y v = \dfrac{\nabla_x u}{\sqrt{-t}} \\ v_s = -\dfrac{1}{2(-t)}\mathcal{L}u \end{cases}$$

la formule de monotonie : soit

$$\mathcal{E}(t;u) = \int_{\mathbb{R}^d} \left\{\frac{1}{-t}\left(|\nabla u(x,t)|^2 + 2\,u(x,t)\right) - \frac{1}{t^2}\,u^2(x,t)\right\} \mathcal{G}(x,t)\ dx$$

où $\mathcal{G}$ est donné par

$$\mathcal{G}(x,t) = (2\,\pi\,(-t))^{-d/2} \exp\left(\frac{-|x|^2}{4(-t)}\right) .$$

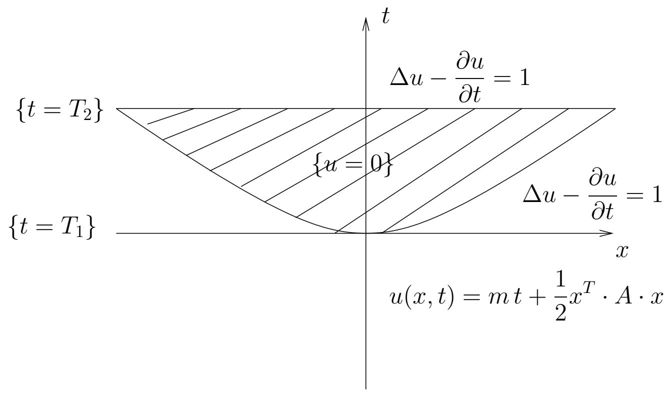
$$\frac{d}{dt}\mathcal{E}(t;u) = -\frac{1}{2(-t)^3}\int_{\mathbb{R}^d} |\mathcal{L}u(x,t)|^2 \mathcal{G}(x,t)\ dx$$

où

$$\mathcal{L}u(x,t) = -2\,u(x,t) + x \cdot \nabla u(x,t) + 2\,t \cdot u_t(x,t) .$$

2 Unicité de la solution

Lorsque l'on a une condition de signe sur u on peut prouver l'unicité de la solution d problème de l'obstacle parabolique pour des conditions initiales dans $L^2(\mathbb{R}^d)$. Cela n'est plus vrai lorsque l'on impose pas de condition de signe sur la solution. En effet on peut couper une solution pour tout temps $t = T_2$ et la prolonger pour les temps supérieurs par une solution de $\Delta u - u_t = 1$. cette solution est toujours solution de $\Delta u - u_t = 1$ lorsque u est non-nulle.



Dans le cas du problème de l'obstacle avec condition de signe sur la solution, on considère $\zeta \in \mathcal{D}(\mathbb{R}^d)$ une fonction cut-off en telle que $\psi \equiv 1$ dans $B_{\frac{1}{2}}(0)$ et $\psi \equiv 0$ dans $\mathbb{R}^d \setminus B_1(0)$, on impose de plus qu'il existe M telle que

$$\frac{2}{\zeta} \nabla \zeta \leq M \; . \tag{20}$$

Soit $w = (u - v)$

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^d} \frac{w^2}{2}\, \zeta^2\, dx = \int_{\mathbb{R}^d} \left[\Delta w - \left(\mathbb{1}_{\{u>0\}} - \mathbb{1}_{\{v>0\}} \right) \right] \, w\, \zeta^2$$

Mais $\left(\mathbb{1}_{\{u>0\}} - \mathbb{1}_{\{v>0\}} \right)$ w est égal à $\mathbb{1}_{\{v>0\}}\, v \geq 0$ si $u = 0$; est égal à $\mathbb{1}_{\{u>0\}}$ si $u > 0$ et $v = 0$; et $= 0$ si u et v sont positifs. Par intégration par parties on obtient

$$\begin{aligned}
\frac{\partial}{\partial t} \int_{\mathbb{R}^d} \frac{w^2}{2}\, \zeta^2\, dx &\leq \int_{\mathbb{R}^d} \Delta w\, w\, \zeta^2\, dx \\
&\leq \int_{\mathbb{R}^d} \Big[-\left(\nabla w\, \zeta \right)^2 - w \nabla w\, 2\, \zeta\, \nabla \zeta \Big]\, dx \\
&\leq \int_{\mathbb{R}^d} -\left(\nabla (w\, \zeta) \right)^2 - (w \nabla \zeta)^2 + 2\, \nabla (w\, \zeta)\, w \nabla \zeta + 2\, (w \nabla \zeta)^2 \\
&\leq \int_{\mathbb{R}^d} \left[-\nabla (w\, \zeta) + w \nabla \zeta \right]^2 \leq \int_{\mathbb{R}^d} (w \nabla \zeta)^2 \\
&\leq \int_{\mathbb{R}^d} \frac{w^2}{2}\, \zeta^2 \times \frac{2}{\zeta} \nabla \zeta \; .
\end{aligned}$$

Ainsi grâce à (20) on a

$$\int_{\mathbb{R}^d} \frac{w^2}{2}(x,t)\, \zeta^2(x)\, dx \leq e^{M\,t} \int_{\mathbb{R}^d} \frac{w^2}{2}(x,0)\, \zeta^2(x)\, dx \; .$$

Ce qui prouve l'unicité de la solution si elles ont même donnée initiale.

3 Mouvement brownien

En 1827, le botaniste anglais Robert Brown note le mouvement désordonné et perpétuel de graines de pollen sur une goutte d'eau observée au microscope. Son papier sera rejeté. Au point de vue microscopique le mouvement de la population est vu comme une conséquence de mouvements irréguliers d'individus, isolés au sein de la population, qui induisent un mouvement macroscopique de l'ensemble de la population. Nous allons montrer comment décrire le mouvement continu d'une densité de particule, d'amibes ou d'actifs financiers qui n'ont pas de "mémoire".

On considère une marche aléatoire. Soit un espace probabilisé $(\Omega, \mathcal{F}, P)$. Soient $\Delta t > 0$ et X_n le déplacement de la particule entre les instants $(n-1)\Delta t$ et $n\,\Delta t$. La position de la particule à l'instant n est $S_n := \sum_{i=1}^n X_i$. On pose $S_0 = 0$. On fait l'hypothèse fondamentale que les $(X_i)_{i\in\mathbb{N}}$ sont indépendants. Pour le cas des particules cela signifie que le déplacement entre les instants n et $n+1$ est indépendant des déplacements antérieurs. Pour l'interprétation de cette hypothèse en finance voir Section A.I.2, et Chapitre B.I en biologie. On se place ici sous l'hypothèse que le mouvement se fait dans un espace uni-dimensionnel. Soit $\Delta x > 0$ la discrétisation en espace que nous caractériserons ultérieurement. On considère que la famille de variables aléatoires $(X_i)_{i\in\mathbb{N}}$ est indépendante et équi-distribuée de loi $P(X_i = \Delta x) = P(X_i = -\Delta x) = 1/2$. Alors $\mathbb{E}(S_n) = 0$ et $\mathrm{Var}(S_n) = (\Delta x)^2$.

Nous voulons maintenant étendre cette idée à un espace et un temps continu. Soit $S(\tau)$ la position de la particule à l'instant $\tau > 0$. Il existe n tel que $n\,\Delta t \le \tau < (n+1)\,\Delta t$. On veut obtenir $S(\tau)$ comme la limite de S_n lorsque $n \to \infty$ sachant que $n\,\Delta t \to \tau$. Le théorème de la limite centrale assure que

$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{n\,\mathrm{Var}(S_n)}} = \frac{S_n}{\sqrt{n}\,\Delta x} \to \mathcal{N}(0,1)$$

en loi, où $\mathcal{N}(0,1)$ est la loi normale centrée réduite. Comme $n\,\Delta t \to \tau$ si on veut que le comportement limite de soit pas trivial il faut prendre $\Delta x^2/\Delta t \to \sigma^2$ avec $\sigma > 0$. Sous cette condition sur Δx, $(S_n)_{n\in\mathbb{N}}$ converge en loi vers la gaussienne centrée de variance $\sigma\,\tau$, dont la densité est

$$u(x,\tau) = \frac{1}{\sqrt{2\,\pi\,\sigma\,\tau}} \exp\left(-\frac{x^2}{2\,\sigma\,\tau}\right) ,$$

qui est le noyau de l'équation de la chaleur.

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Deuxième partie

Modèle de Keller-Segel

B.I

Les modèles de Keller-Segel

Le modèle de Keller-Segel est un modèle de chemotaxis. Le terme "taxis" signifie "arrangement". Les êtres vivants correspondent avec leur environnement et y répondent par le biais de stimulus. La réponse à ce stimulus peut être un déplacement dans la direction de la source du stimulus (on parle de taxis positive) ou en s'en éloignant (on parle de taxis négative). Lorsque ce stimulus est chimique on parle de chemotaxis. Il existe aussi de la photo-taxis (réponse à un stimulus de lumière), etc...

L'exemple le plus étudié de chemotaxis est celui d'une espèce d'amibes appelés *dictyostelium discoïdeum*. Les dictyostelium discoïdeum ont été découvertes en 1935 par le biologiste K. B. Rapper. L'intérêt d'étudier ces organisme est que

> "*development in dictyostelium discoideum results only in two terminal cell types, but morphogenesis and pattern formation occur as in many higher organisms*" ([NS92]).

Ces organismes sont considérés comme des "organismes modèles" dans les recherches en biologie.

Une population de dictyostelium discoïdeum croît par division cellulaire tant qu'il y a de la nourriture en quantité suffisante. Les amibes se déplacent de façon aléatoire dans tout l'espace[1]. En condition de pénurie de nourriture, une amibe commence à émettre de la cyclique Adénosine mono-phosphate (cAMP) qui stimule les autres amibes à en émettre aussi. De plus les amibes se déplacent dans la direction de plus fort gradient de concentration de cAMP. Lorsque la densité d'amibe est suffisamment grande les amibes s'agrègent. À la fin de ce processus cet amas d'amibes forme un pseudoplasmoid. Le pseudoplasmoid se déplace par photo-taxis positive. Après un moment le corps forme un fruit et des spores. Les spores deviennent des amibes et le cycle recommence (voir [Hor04] et [Hor03b] ou [Hor03a] pour de plus amples détails).

1 Le point de vue macroscopique

Les travaux qui font référence dans la modélisation de ce phénomène d'agrégation des Distyostelium discoïdeum sont les travaux de C. Patlak [Pat53] et surtout de Evelyn Fox Keller et Lee A. Segel [KS70] qui donne leur nom au modèle. Soit n la densité d'amibes, et c la concentration de chemo-attractant. On fait les hypothèses suivantes

[1] voir http ://dictybase.org/Multimedia/chemotaxis/Chemotaxis.htm

(i) Les amibes et le chemo-attractant diffusent en suivant le loi de Fick,

(ii) Le chemo-attractant est produit par les amibes à un taux γ.

Si on suppose que la masse totale d'amibes est conservée dans un volume D au cours du processus, on a

$$\frac{d}{dt}\int_D n(t,x) = \int_{\partial D} J^{(n)}(x,t)\cdot\nu(x)\, dS$$

où $J^{(n)}$ désigne le flot de la densité d'amibes. Ce flot est constitué de deux parties : une partie qui est proportionnelle au gradient de densité (par la loi de Fick) et une partie qui est proportionnelle au gradient de chemo-attractant :

$$J^{(n)}(x,t) = k_1\nabla c - k_2\nabla n\ .$$

Si on suppose que le chemo-attractant diffuse, on obtient

$$\frac{d}{dt}\int_D c(t,x) = Q^{(c)}(t,D) - \int_{\partial D} J^{(c)}(x,t)\cdot\nu(x)\, dS$$

où $Q^{(c)}$ est la production de chemo-attractant c par unité de temps et de volume, et le flot $J^{(c)}(x,t)$ est donné par $J^{(c)}(x,t) = k_c\nabla c$. On suppose aussi que c est consommé au taux k_3. On obtient ainsi le système obtenu de E. F. Keller et L. A. Segel, [KS70], suivant :

$$\begin{cases} n_t = \nabla\cdot(k_1\nabla n - k_2\nabla c) \\ c_t = k_c\Delta c + \gamma n - k_3 c \\ \dfrac{\partial n}{\partial\nu} = \dfrac{\partial c}{\partial\nu} = 0 \\ n(x,0) = n_0(x) \\ c(x,0) = c_0(x) \end{cases}$$

2 Le point de vue microscopique

Les amibes n'ont pas de mémoire. Cette "absence de mémoire" permet de modéliser le mouvement des amibes par un mouvement brownien en l'absence de chemo-attractant (voir Section 3 de l'appendice). Le mouvement de la population d'amibes est vu comme la conséquence du mouvement d'individus isolés en supposant que les interactions entre amibes sont négligeables. À chaque pas de temps l'amibe tourne sur elle-même puis elle se déplace. Le temps passé à tourner et à positionner ses accroches est négligeable comparé au temps que met l'amibe à se déplacer. On suppose aussi que, durant une unité de temps, le nombre d'amibes et la distribution de chemo-attractant restent inchangés.

En dimension un, d'espace on considère un maillage de taille h de $[-N,N]$ et on pose $x = ih$, $i\in\mathbb{Z}$. Soit $n_i(t)$ la probabilité conditionnelle que l'individu soit en i au temps t sachant qu'il était en 0 au temps initial 0. Soit $\tau_i^{\pm}$ la probabilité de transition par unité de temps que l'individu en x_i bouge de $+h$ ou $-h$. Soit $\left(\tau_i^+(C)+\tau_i^-(C)\right)^{-1}$ le temps moyen d'attente au site i. On définit $C(c_0, c_{1/2},\ldots,c_{i-1/2},c_i,c_{i+1/2},\ldots)$ où c est une substance de contrôle donnée que nous supposons, pour la simplicité de l'exposé, donnée. En terme plus imagés, une amibe se déplace en se tirant sur le milieu qui l'entoure. Puis elle repositionne ses accroches pendant un temps $\left(\tau_i^+(C)+\tau_i^-(C)\right)^{-1}$ et se tire à nouveau. Elle n'a pas de mémoire, et la direction dans laquelle elle positionne ses accroches ne dépend pas des déplacements antérieurs. La probabilité conditionnelle que l'individu soit en i au temps t est égale à la somme de la probabilité qu'il était au temps précédent en $i-1$ et qu'il ait sauté de $+h$ et de la probabilité qu'il en $i+1$ et qu'il ait sauté de $-h$. Probabilités à laquelle il faut retirer la probabilité qu'il était en i et qu'il ait bougé. On obtient donc

$$\frac{\partial n_i(t)}{\partial t} = \tau_{i-1}^+(C)\, n_{i-1} + \tau_{i+1}^-(C)\, n_{i+1} - \left(\tau_i^+(C) + \tau_i^-(C)\right)\, n_i \tag{I.1}$$

On peut considérer plusieurs types de probabilité de transition $\tau_i^\pm$. Nous allons considérer le modèle dit *à gradient* où l'organisme teste l'environnement (et plus précisément la concentration de substance c) qui l'entoure immédiatement, avant de prendre une décision sur la façon de se déplacer :

$$\tau_{i-1}^+ = \alpha + \beta(\tau(c_i) - \tau(c_{i-1}) \ \text{ et } \ \tau_{i+1}^- = \alpha + \beta(\tau(c_i) - \tau(c_{i+1}) \ .$$

On étend c et n comme des fonctions paires sur $[-N, N]$. Si on passe formellement à la limite dans (I.1) on obtient

$$\frac{\partial n(t)}{\partial t} = D \left(\alpha \frac{\partial^2 n}{\partial x^2} - 2\,\beta \frac{\partial}{\partial x}\left[\tau'(c)\, n \frac{\partial c}{\partial x}\right]\right) \ .$$

où $\chi(C) := 2\,D\,\beta\,\tau'(C)$. Alors on a

$$\frac{\partial n}{\partial t} = \alpha D \frac{\partial^2 n}{\partial x^2} - \frac{\partial}{\partial x}\left[\chi(C)\frac{\partial c}{\partial x} n\right] \ .$$

Le coefficient $\alpha\, D/\chi$ détermine le régime dominant : diffusion ou agrégation. Pour plus de détails voir [OS97]. On peut aussi dériver le modèle de Keller-Segel comme limite d'une équation cinétique (voir [CMPS04]).

On peut considérer plusieurs types de comportement pour c. Dans le cas le plus élémentaire, c est solution de l'équation de Poisson $\Delta c = -n$. Nous parlerons de modèle de Keller-Segel parabolique-elliptique. Mais on peut aussi considérer des modèles où le chemo-attractant s'évapore au taux β l'équation sur c est alors $\Delta c = -n - \beta c$. Dans le cas plus complet c est solution de l'équation aux dérivées partielles $\alpha\, \frac{\partial c}{\partial t} = \Delta c + n - \beta\, c$. Dans ce mémoire l'étude a porté sur le cas où c est solution de l'équation de Poisson.

La principale question est celle de l'agrégation, D. Horstmann la formule dans [Hor03a] p.14 :

"*can one give the precise value for the threshold value which decides whether the solution might blow-up (in finite time) or not*".

Il a été calculé expérimentalement, dans [MHR71], que s'il y a plus de $5.10^4.\mathrm{cm}^{-2}$ dictyostelium discoïdeum, elles s'accumulent. V. Nanjundiah conjecturait dans [V.73], qu'il y a une densité d'amibes critique en deçà de laquelle les amibes diffusent et au delà de laquelle elles s'accumulent. Le Chapitre II est consacré à la question de cette densité limite et plus généralement au comportement qualitatif des solutions.

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B.II

Two-dimensional Keller-Segel model

Ce chapitre est la version préliminaire de [BDP05] en collaboration avec J. Dolbeault et B. Perthame.

Résumé

Le système de Keller-Segel décrit le mouvement collectif de cellules qui sont attirées par une substance chimique et qu'elles sont capables d'émettre. Dans sa forme la plus simple, il s'agit d'une équation de dérive-diffusion conservative pour la densité des cellules, couplée à une équation elliptique pour la concentration de chemo-attractant. On sait que, en dimension deux, il y a existence globale de solutions pour une masse initiale petite, alors que pour une masse initiale grande, la solution explose en temps fini. Dans cet article, nous précisons cette description et donnons une preuve détaillée de l'existence de solutions faibles avec masse inférieure à la masse critique, au-dessus de laquelle toute solution explose en temps fini. En utilisant des arguments d'hypercontractivité, nous établissons des résultats de régularité qui nous permettent de prouver une inégalité reliant l'énergie libre et sa dérivée en temps. Pour une solution avec masse sous-critique, cela nous permet de donner une description des "asymptotiques intermédiaires" en temps grand, qui décrivent la convergence locale des solutions vers zéro. Dans des coordonnées auto-similaires, nous montrons en fait un résultat de convergence vers une solution auto-similaire limite qui n'est pas simplement donnée par la diffusion.

Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions

A. Blanchet, J. Dolbeault et B. Perthame

Abstract

The Keller-Segel system describes the collective motion of cells that are attracted by a chemical substance and are able to emit it. In its simplest form it is a conservative drift-diffusion equation for the cell density coupled to an elliptic equation for the chemo-attractant concentration. It is known that, in two space dimensions, for small initial mass, there is global existence of solutions and for large initial mass blow-up occurs. In this paper we complete this picture and give a detailed proof of the existence of weak solutions below the critical mass, above which any solution blows-up in finite time. Using hypercontractivity methods, we establish regularity results which allow us to prove an inequality relating the free energy and its time derivative. For a solution with sub-critical mass, this allows us to give for large times an "intermediate asymptotics" description of the vanishing. In self-similar coordinates, we actually prove a convergence result to a limiting self-similar solution which is not a simple reflect of the diffusion.

MSC (2000).: Primary: 35B45, 35B30, 35D05, 35K15, 35B40; Secondary: 35D10, 35K60, 35B32.
Keywords: Keller-Segel, free energy, hypercontractivity, long time behaviour.

1 Introduction

Various versions of the Keller and Segel system for chemotaxis are available in the literature, depending on the phenomena and scales one is interested in. In this paper, we consider only the two-dimensional case and assume that the equations take the simplified form

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n - \chi \nabla\cdot(n\nabla c) & x\in\mathbb{R}^2\,,\ t>0\,,\\ -\Delta c = n & x\in\mathbb{R}^2\,,\ t>0\,,\\ n(\cdot,t=0)=n_0\geq 0 & x\in\mathbb{R}^2\,. \end{cases} \tag{II.1}$$

The Keller and Segel system for chemotaxis describes the collective motion of cells, usually bacteria or amoebae, that are attracted by a chemical substance and are able to emit it. There is a very large literature on this subject, among which we can quote [KS71a, Mur03, Mai01, Pat53, Hor02]. Here $n(x,t)$ represents the cell density, and $c(x,t)$ is the concentration of chemo-attractant which induces a drift force. An important parameter of the system is the *sensitivity* $\chi>0$ of the bacteria to the chemo-attractant which measures the nonlinearity in the system. Such a parameter can be removed by a scaling, to the price of a change of the total mass of the system

$$M := \int_{\mathbb{R}^2} n_0\,dx\,.$$

In bounded domains, it is usual to impose no-flux boundary conditions. For simplicity, we are going to consider the system in the full space $\mathbb{R}^2$, without boundary conditions. There are related models in gravitation which are defined on $\mathbb{R}^3$. The relevant case for chemotaxis is rather the two-dimensional space, although some three-dimensional versions of the model also make sense. Then, the L^1-norm is critical as established by [JL92]. More generally the critical space is $L^{d/2}(\mathbb{R}^d)$ for $d \geq 2$, see [CPZ04, HW05] and the references therein.

Historically the key papers for this family of models are the original contribution [KS71b] of Keller and Segel, and a work by Patlak, [Pat53]. We refer to the review papers [OS97, Hor03, Hor04] for additional references. Interesting results concerning (II.1) have been published by Gajewski and Zacharias in [GZ00, GZ98] and by Biler in, *e.g.,* [Bil98]. However, most of the results deal only with the bounded domain case.

More elaborate models sometimes called "'super-models" are now studied, involving for instance more than one chemo-attractant, [RH05], or volume effects [PH05, CC05]. Also see [HPS05, DS03b, DH03b, DH03a, DS03a, BDD05]. A very interesting justification of the Keller-Segel model as a diffusion limit of a kinetic model has recently been published, see [CMPS04]. Concerning blow-up phenomena, a key contribution has been brought by Rascle and Ziti in [RZ95]. Also see [Ste00, TSL00, HP01, CPZ03] for further contributions in this direction. Concerning global attractors and Lyapunov functionals, we refer to [LW04, Wrz04b, LW02] and to a more recent paper by Wrzosek, [Wrz04a]. The main interest of such a simplified model as (II.1) arises from the fact that some basic qualitative features of more elaborate models are already present in (II.1), at least concerning the existence *vs.* the blow-up in finite time, and can be completely classified.

As a technical tool we will use results on the bifurcation diagrams corresponding to the exponential nonlinearity in the last section: see [BM91, DP95, Ner04] for the study of the limiting point of the bifurcation diagram when approaching the critical explosion parameter. For a review of related results in connection with gravitation, see [BDE$^+$01]. An large series of results has been obtained by Suzuki and Senba. Most recent contributions of these two authors are [NSS98, SS04a, SS04b, SS03, SS02a].

The main tool in this paper is the *free energy* which provides useful *a priori* estimates for establishing an existence theory up the critical value of the parameter χ, or equivalently, up to the critical mass, above which all solutions blow-up in finite time. The limiting case has recently been studied in the radial case, see [BKL05a, BKL05b]. We will make a repeated use of the logarithmic Hardy-Littlewood-Sobolev inequality in its sharp form as established in [CL92, Bec93]. When dealing with bounded domains, most of the authors prefer to rely on various versions of the Moser-Trudinger inequality, see for instance [Suz05]. It is not yet fully understood which version of the Moser-Trudinger inequality would provide the same results in $\mathbb{R}^2$, although some kind of duality between the two inequalities can reasonably be expected, see [Bec93] for instance.

The literature on the Keller-Segel model is huge and it is out of reach to quote all related results. Some additional papers will be quoted in the text. Otherwise, we suggest to the interested reader to primarily refer to [OS97, Hor03, Hor04, Suz05].

Our first main result is the following existence and regularity statement.

Theorem II.1. *Assume that* $n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)$ *and* $n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)$*. If* $M < 8\pi/\chi$*, then the Keller-Segel system (II.1) has a global weak non-negative solution* n

with initial data n_0 such that

$$(1+|x|^2+|\log n|)\,n \in L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2)) \quad \textit{and} \quad \int_0^t \int_{\mathbb{R}^2} n|\nabla \log n - \chi\nabla c|\,dx\,dt < \infty$$

for any $t>0$. Moreover $n \in L^\infty_{\rm loc}((\varepsilon,\infty), L^p(\mathbb{R}^2))$ for any $p\in(1,\infty)$ and any $\varepsilon>0$, and the following inequality holds for any $t>0$:

$$F[n(\cdot,t)] + \int_0^t \left(\int_{\mathbb{R}^2} n\,|\nabla(\log n) - \chi\nabla c|^2\,dx\right) ds \le F[n_0]\,. \tag{II.2}$$

This result was partially announced in [DP04]. Compared to [DP04], the main novelty is that we prove the free energy inequality (II.2) and get the hypercontractive estimate: $n(\cdot,t)$ is bounded in $L^p(\mathbb{R}^2)$ for any $p\in(1,\infty)$ and almost any $t>0$.

Our second main result deals with large time behavior, intermediate asymptotics and convergence to asymptotically self-similar profiles given by the equation

$$u_\infty = M\,\frac{e^{\chi\,v_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{\chi\,v_\infty-|x|^2/2}\,dx} = -\Delta v_\infty\,, \quad \text{with} \quad v_\infty = -\frac{1}{2\pi}\log|\cdot| * u_\infty\,, \tag{II.3}$$

Theorem II.2. *Under the same assumptions as in Theorem II.1, there exists a solution of (II.3) such that*

$$\lim_{t\to\infty}\|n(\cdot,t)-n_\infty(\cdot,t)\|_{L^1(\mathbb{R}^2)} = 0 \quad \textit{and} \quad \lim_{t\to\infty}\|\nabla c(\cdot,t) - \nabla c_\infty(\cdot,t)\|_{L^2(\mathbb{R}^2)} = 0$$

with

$$\begin{aligned} n_\infty(x,t) &:= \frac{1}{R^2(t)}\,u_\infty\left(\log(R(t)), x/R(t)\right)\\ c_\infty(x,t) &:= v_\infty\left(\log(R(t)), x/R(t)\right)\end{aligned}$$

and $R(t) := \sqrt{1+2t}$.

This paper is organized as follows. Section 2 is devoted to the detailed proof of the existence of weak solutions with subcritical mass and without any symmetry assumption. *A priori* estimates have been derived in [DP04]. The point here is to establish the result with all necessary details: regularized problem, uniform estimates, passage to the limit in the regularization parameter. Compared to [DP04], we also establish Inequality (II.2). Proving such an inequality requires a detailed study of the regularity properties of the solutions, which is done in Section 3: By hypercontractivity methods, we prove that the solution is bounded in any L^p space for almost any positive t. Using the free energy we also study the asymptotic behavior of the solutions for large times, which corresponds to Theorem II.2. In the self-similar variables, the solution converges to a unique stationary solution, which proves a result on the "intermediate asymptotics" for the solution of (II.1).

2 Existence

We assume that the initial data satisfy the following asssumptions:

$$\begin{aligned} &n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)\,,\\ &n_0\log n_0 \in L^1(\mathbb{R}^2, dx)\,. \end{aligned} \tag{II.4}$$

Because of the divergence form of the right hand side of the equation for n, the total mass is conserved at least for smooth and sufficiently decay solutions

$$M := \int_{\mathbb{R}^2} n_0(x)\, dx = \int_{\mathbb{R}^2} n(x,t)\, dx\ . \tag{II.5}$$

Our purpose here is first to give a complete existence theory in the subcritical case, *i.e.* in the case

$$M < 8\pi/\chi\ .$$

This result has been announced in [DP04], which was dealing only with *a priori* estimates. Here, we give the proofs with all details. More precisely, we prove that under Assumption (II.4), there are only two cases:

1. Solutions to (II.1) blow-up in finite time when $M > 8\pi/\chi$,

2. There exists a global in time solution of (II.1) when $M < 8\pi/\chi$.

The case $M = 8\pi/\chi$ is delicate and for radial solutions, some results have been obtained recently, see [LW04, LW02].

Our existence theory completes the partial picture established in [JL92]. The solution of the Poisson equation $-\Delta c = n$ is given up to an harmonic function. From now on, we define the concentration of the chemo-attractant by

$$c(x,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\, n(y,t)\, dy\ . \tag{II.6}$$

There are other possible solutions, which may result in significantly different qualitative behaviors, as we shall see in Section 4.2. From now on, unless it is explicitly specified, we will only consider concentrations $c(x,t)$ given by (II.6). In the following sections, 2.1, 2.2 and 2.3, we closely follow the presentation given in [DP04].

2.1 Blow-up

The case $M > 8\pi/\chi$ (Case 1) is easy to understand using moments estimates. The methods is classical and has been repeatedly used for various similar problems. See for instance [Per90] in the similar context of the Euler-Poisson system, [HW05]. Concerning blow-up, we refer to [CPZ04, HW05] for recent references on the subject.

Lemma II.3. *Consider a non-negative distributional solution to (II.1) on an interval* $[0,T]$ *that satisfies (II.5),* $\int_{\mathbb{R}^2}|x|^2\, n(x,t)\, dx < \infty$ *and* $\int_{\mathbb{R}^2}\frac{1+|x|}{|x-y|}\, n(y,t)\, dy \in L^\infty\big((0,T)\times\mathbb{R}^2\big)$*. Then it also satisfies*

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2\, n(x,t)\, dx = 4M\left(1 - \frac{\chi M}{8\pi}\right)\ .$$

Proof – Consider a smooth function $\varphi_\varepsilon(|x|)$ with compact support that grows nicely to $|x|^2$ as $\varepsilon \to 0$. Then, we compute

$$\frac{d}{dt}\int_{\mathbb{R}^2}\varphi_\varepsilon\, n\, dx = \int_{\mathbb{R}^2}\Delta\varphi_\varepsilon\, n\, dx - \frac{\chi}{4\pi}\int_{\mathbb{R}^2}\frac{(\nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y))\cdot(x-y)}{|x-y|^2}\, n(x,t)\, n(y,t)\, dx\, dy\ .$$

As ε vanishes we may pass to the limit and obtain Lemma II.3. □

As a consequence, we obtain the statement of Case 1, namely there is a finite blow-up time T^*.

Corollary II.4. *Consider a non-negative distributional solution $n \in L^\infty(0,T^*;L^1(\mathbb{R}^2))$ to (II.1) and assume that $[0,T^*)$ is the maximal interval of existence, i.e;, that $n(\cdot,t)$ converges (up to extraction of sequences) as $t \to T^*$ to a measure which is not in $L^1(\mathbb{R}^2)$. We assume that the solution satisfies (II.5),*

$$I_0 := \int_{\mathbb{R}^2} |x|^2\, n_0(x)\, dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{1+|x|}{|x-y|}\, n(y,t)\, dy \in L^\infty\big((0,T)\times\mathbb{R}^2\big)$$

If $\chi\, M > 8\pi$, then

$$T^* \leq \frac{2\pi\, I_0}{M(\chi\, M - 8\pi)}\ .$$

As far as we know, it is an open question to decide whether the solutions of (II.1) with $\chi\, M > 8\pi$ and $I_0 = \infty$ also blow-up in finite time. Blow-up statements in bounded domains are available, see [NS98, BN93, Hor02, LW02] and the references therein. When the solution is radially symmetric in x, the second x-moment is not needed and the blow-up profile has been explicited, namely

$$n(x,t) \to \frac{8\pi}{\chi}\,\delta + \tilde{n}(|x|,t) \quad \text{as}\; t \nearrow T^*\ ,$$

where $\tilde{n}$ is a $L^1_{\text{loc}}(\mathbb{R}^2\times\mathbb{R}^+)$ function, see [HMV97, Vel02]. Except that solutions blow-up for large mass, in the general case very little is known on the blow-up profile (see [SS02b] for concentrations estimates, [Mar03] for numerical computations). Asymptotic expansions at blow-up and continuation of solutions after blow-up have have recently been studied by Velázquez in [Vel04b, Vel04a]. The case $\chi\, M = 8\pi$ has recently been investigated by Biler, Karch, Laurençot and Nadzieja in [BKL05a]. In a forthcoming paper, they prove that in the whole space case and $\chi\, M = 8\pi$, blow-up occurs only for infinite time, [BKL05b]. Here we will focus now on the subcritical regime and prove that solutions exist and are always asymptotically vanishing for large times.

If the problem is set in dimension $d \geq 3$, the critical norm is $L^p(\mathbb{R}^d)$ with $p = d/2$. In dimension $d = 2$, the value of the mass M is therefore natural to discriminate between super- and sub-critical regimes. However, the limit of the L^p-norm is rather $\int_{\mathbb{R}^2} n \log n\, dx$ than $\int_{\mathbb{R}^2} n\, dx$, which is preserved by the evolution. This explains why it is natural to introduce the entropy, or better, as we shall see below, the *free energy*.

2.2 The usual existence proof

The usual proof of existence is due to [JL92]. Here we follow the variant [CPZ04] which is based on the following computation. Consider the equation for n and compute $\frac{d}{dt}\int_{\mathbb{R}^2} n \log n\, dx$. Using an integration by parts and the equation for c, we obtain:

$$\begin{aligned} \tfrac{d}{dt}\textstyle\int_{\mathbb{R}^2} n \log n\, dx &= -4\textstyle\int_{\mathbb{R}^2} |\nabla\sqrt{n}|^2\, dx + \chi \int_{\mathbb{R}^2} \nabla n\cdot\nabla c\, dx \\ &= -4\textstyle\int_{\mathbb{R}^2} |\nabla\sqrt{n}|^2\, dx + \chi\int_{\mathbb{R}^2} n^2\, dx\ . \end{aligned}$$

This shows that two terms compete, namely the diffusion based entropy dissipation term $\int_{\mathbb{R}^2} |\nabla\sqrt{n}|^2\, dx$ and the hyperbolic production of entropy.

Thus the entropy is nonincreasing if $\chi M \leq 4C_{\text{GNS}}^{-2}$, where $C_{\text{GNS}} = C_{\text{GNS}}^{(4)}$ is the best constant for $p = 4$ in the Gagliardo-Nirenberg-Sobolev inequality:

$$\|u\|^2_{L^p(\mathbb{R}^2)} \leq C^{(p)}_{\text{GNS}}\, \|\nabla u\|^{2-4/p}_{L^2(\mathbb{R}^2)}\, \|u\|^{4/p}_{L^2(\mathbb{R}^2)} \quad \forall\, u \in H^1(\mathbb{R}^2)\ , \quad \forall\, p \in [2,\infty)\ . \tag{II.7}$$

The explicit value of $C_{\rm GNS}$ is not known but can be computed numerically (see [Wei83]) and one finds that the entropy is nonincreasing if $\chi M \leq 4C_{\rm GNS}^{-2} \approx 1.862... \times (4\pi) < 8\pi$. Such an estimate is therefore not sufficient to cover the whole range of M for global existence in the second case.

In [JL92] it is also shown that equiintegrability (deduced from the $n \log n$ estimate for instance) is enough to propagate any L^p initial norm. We will come back on this point in Section 2.7 and prove later that due to the regularizing effects, the solution is bounded in time with values in $L^p(\mathbb{R}^2)$ for all positive times.

2.3 Entropy method

To obtain sharper estimates and prove a global existence result (Case 2), we use the well known *free energy:*

$$F[n] := \int_{\mathbb{R}^2} n \log n \, dx - \frac{\chi}{2} \int_{\mathbb{R}^2} n \, c \, dx$$

See [GZ98] in the case of a bounded domain. The first term in F is the *entropy* and the second one a *potential energy term*. Such a free energy enters in the general notion of entropies, and this is why it is sometimes referred to the method as the "entropy method", although it the notion of *free energy* is physically more appropriate. See [ACD$^+$04] for an historical review on these notions. For any solution n of (II.1), $F[n(\cdot,t)]$ is monotone nonincreasing.

Lemma II.5. *Consider a non-negative $C^0(\mathbb{R}^+, L^1(\mathbb{R}^2))$ solution n of (II.1) such that $n(1+|x|^2)$, $n \log n$ are bounded in $L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$, $\nabla\sqrt{n} \in L^1_{\rm loc}(\mathbb{R}^+, L^2(\mathbb{R}^2))$ and $\nabla c \in L^\infty_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^2)$. Then*

$$\frac{d}{dt} F[n(\cdot,t)] = - \int_{\mathbb{R}^2} n \left| \nabla \left(\log n\right) - \chi \nabla c \right|^2 \, dx \; . \tag{II.8}$$

Following a usual denomination in the partial differential equation literature, we will call $\int_{\mathbb{R}^2} n \left| \nabla (\log n) - \chi \nabla c \right|^2 dx$ the *energy* although calling it an *entropy production term* or a *generalized relative Fisher information* would probably be more appropriate.

Proof – Because the potential energy term $\int_{\mathbb{R}^2} n \, c \; dx = \iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x,t)\, n(y,t) \log|x-y| \; dx\, dy$ is quadratic in n, using Equation (II.1), the time derivative of $F[n(\cdot,t)]$ is given by

$$\frac{d}{dt} F[n(\cdot,t)] = \int_{\mathbb{R}^2} \left[\Big(1 + \log n - \chi \, c\Big) \, \nabla \cdot \left(\frac{\nabla n}{n} - \chi \, \nabla c \right) \right] \, dx \; .$$

An integration by parts completes the proof. □

From the representation (II.6) of the solution to the Poisson equation, we deduce that

$$\frac{d}{dt} F[n(\cdot,t)] = \frac{d}{dt} \left[\int_{\mathbb{R}^2} n \log n \, dx + \frac{\chi}{4\pi} \iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x,t)\, n(y,t) \, \log|x-y| \, dx\, dy \right] \leq 0 \; .$$

On the other hand, we recall the logarithmic Hardy-Littlewood-Sobolev inequality.

Lemma II.6. [CL92, Bec93] *Let f be a non-negative function in $L^1(\mathbb{R}^2)$ such that $f \log f$ and $f \log(1+|x|^2)$ belong to $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f \, dx = M$, then*

$$\int_{\mathbb{R}^2} f \log f \; dx + \frac{2}{M} \iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x) f(y) \log|x-y| \; dx\, dy \geq - \, C(M) \; , \tag{II.9}$$

with $C(M) := M(1 + \log\pi - \log M)$.

This allows to prove *a priori* estimates on the two terms involved in the free energy.

Lemma II.7. *Consider a non-negative $C^0(\mathbb{R}^+, L^1(\mathbb{R}^2))$ solution n of (II.1) such that $n(1+|x|^2)$, $n\log n$ are bounded in $L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$, $\int_{\mathbb{R}^2} \frac{1+|x|}{|x-y|}\, n(y,t)\, dy \in L^\infty\big((0,T)\times\mathbb{R}^2\big)$, $\nabla\sqrt{n} \in L^1_{\rm loc}(\mathbb{R}^+, L^2(\mathbb{R}^2))$ and $\nabla c \in L^\infty_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^2)$. Then the following estimates hold:*

(i) Entropy:

$$M\,\log M - M\,\log[\pi(1+t)] - K \le \int_{\mathbb{R}^2} n\log n\; dx \le \frac{8\pi\, F_0 + \chi\, M\, C(M)}{8\pi - \chi\, M}$$

with $I_0 := \int_{\mathbb{R}^2} |x|^2\, n_0(x)\, dx$, $K := \max\left\{I_0, \frac{M}{2\pi}\,(8\pi - \chi\, M)\right\}$ *and* $F_0 := F[n_0]$.

(ii) Energy: *For all $t>0$, with $C_1 := F_0 + \frac{\chi\, M}{8\pi}\, C(M)$ and $C_2 := \frac{\chi\, M - 8\pi}{8\pi}$,*

$$0 \le \int_0^t ds \int_{\mathbb{R}^2} n(x,s)\, |\nabla\,(\log n(x,s)) - \chi\nabla c(x,s)|^2\, dx \le C_1 + C_2\left[M\,\log\left(\frac{\pi(1+t)}{M}\right) + K\right]$$

Proof – From (II.8), with $n(\cdot) = n(\cdot,t)$ for shortness, we get that the quantity

$$(1-\theta)\int_{\mathbb{R}^2} n\log n\; dx + \theta\left[\int_{\mathbb{R}^2} n\log n\; dx + \frac{\chi}{4\pi\theta}\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\, n(y)\,\log|x-y|\; dx\, dy\right]$$

is bounded from above by F_0. We choose

$$\frac{\chi}{4\pi\theta} = \frac{2}{M} \quad\Longleftrightarrow\quad \theta = \frac{\chi\, M}{8\pi}$$

and apply (II.9):

$$(1-\theta)\int_{\mathbb{R}^2} n(x,t)\log n(x,t)\; dx - \theta\, C(M) \le F_0\;.$$

If $\chi\, M < 8\pi$, then $\theta < 1$ and

$$\int_{\mathbb{R}^2} n(x,t)\log n(x,t)\; dx \le \frac{F_0 + \theta\, C(M)}{1-\theta}\;.$$

This estimate proves the upper bound for the entropy. We can also see that $\int_{\mathbb{R}^2} n\,\log n\, dx$ is bounded from below. By Lemma II.3,

$$\frac{1}{1+t}\int_{\mathbb{R}^2} |x|^2\, n(x,t)\; dx \le K \quad \forall\, t>0\;.$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}^2} n(x,t)\log n(x,t) \;&\ge\; \frac{1}{1+t}\int_{\mathbb{R}^2} |x|^2\, n(x,t)\; dx - K + \int_{\mathbb{R}^2} n(x,t)\log n(x,t)\; dx\\
&= \int_{\mathbb{R}^2} n(x,t)\log\left(\frac{n(x,t)}{e^{-\frac{|x|^2}{1+t}}}\right)\, dx - K\\
&= \int_{\mathbb{R}^2} n(x,t)\log\left(\frac{n(x,t)}{\mu(x,t)}\right)\, dx - M\,\log[\pi(1+t)] - K\\
&= \int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)}\log\left(\frac{n(x,t)}{\mu(x,t)}\right)\,\mu(x,t)\, dx - M\,\log[\pi(1+t)] - K
\end{aligned}$$

with $\mu(x,t) := \frac{1}{\pi(1+t)}\, e^{-\frac{|x|^2}{1+t}}$. By Jensen's inequality,

$$\int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)} \log\left(\frac{n(x,t)}{\mu(x,t)}\right) \mu(x,t)\, dx \geq X \,\log X \quad \text{where} \quad X = \int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)}\, \mu(x,t)\, dx = M \;.$$

This gives the lower estimate for the entropy term.

Now, from (II.8) and (II.9), we get

$$\begin{aligned}(1-\theta)\, \left[M\, \log\left(\frac{M}{\pi(1+t)}\right) - K\right] + \theta\, C(M)& \\ + \int_0^t ds \int_{\mathbb{R}^2} n(x,s)\, \left|\nabla\left(\log n(x,s)\right) - \chi\nabla c(x,s)\right|^2\, dx \leq F_0 \;.&\end{aligned}$$

This proves that $\sqrt{n}\; |\nabla\,(\log n) - \chi\nabla c|$ is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}^+, L^2(\mathbb{R}^2))$ and gives the estimate on the energy. □

The *a priori* upper bound on $\int_{\mathbb{R}^2} n\, \log n\, dx$ combined with the $|x|^2$ moment bound of Lemma II.3 shows that $n\, \log n$ is bounded in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+, L^1(\mathbb{R}^2))$.

Lemma II.8. *For any $u \in L^1_+(\mathbb{R}^2)$, if $\int_{\mathbb{R}^2} |x|^2\, u\, dx$ and $\int_{\mathbb{R}^2} u\, \log u\, dx$ are bounded from above, then $u\, \log u$ is uniformly bounded in $L^\infty(\mathbb{R}^+_{\mathrm{loc}}, L^1(\mathbb{R}^2))$ and*

$$\int_{\mathbb{R}^2} u\, |\log u|\; dx \leq \int_{\mathbb{R}^2} u\, \left(\,\log u + |x|^2\right)\, dx + 2\log(2\pi) \int_{\mathbb{R}^2} u\; dx + \frac{2}{e} \;.$$

Proof – The proof goes as follows. Let $\bar{u} := u\, \mathbb{1}_{\{u\leq 1\}}$ and $m = \int_{\mathbb{R}^2} \bar{u}\, dx \leq M$. Then

$$\int_{\mathbb{R}^2} \bar{u}\, (\log \bar{u} + \frac{1}{2}|x|^2)\; dx = \int_{\mathbb{R}^2} u \log u\; d\mu - m \log\,(2\pi)$$

where $U := \bar{u}/\mu$, $d\mu(x) = \mu(x)\, dx$ and $\mu(x) = (2\pi)^{-1} e^{-|x|^2/2}$. By Jensen's inequality,

$$\int_{\mathbb{R}^2} U\; \log U\; d\mu \geq \left(\int_{\mathbb{R}^2} U\; d\mu\right)\; \log\left(\int_{\mathbb{R}^2} U\; d\mu\right) = m\; \log m \;,$$

$$\int_{\mathbb{R}^2} \bar{u}\; \log \bar{u}\; dx \geq m \log\left(\frac{m}{2\pi}\right) - \frac{1}{2}\int_{\mathbb{R}^2} |x|^2\, \bar{u}\; dx \geq -\frac{1}{e} - M\log(2\pi) - \frac{1}{2}\int_{\mathbb{R}^2} |x|^2\, \bar{u}\; dx \;.$$

Using

$$\int_{\mathbb{R}^2} u\, |\log u|\; dx = \int_{\mathbb{R}^2} u\; \log u\; dx - 2\int_{\mathbb{R}^2} \bar{u}\; \log \bar{u}\; dx \;,$$

this completes the proof. □

2.4 Existence of weak solutions

Using the informations collected in Sections 2.1, 2.2 and 2.3, in the spirit of [CPZ03], we can now state, in the subcritical case $M < 8\pi/\chi$, the following existence result of weak solutions, which is essentially the one stated in [DP04].

Proposition II.9. *Under Assumption (II.4) and $M < 8\pi/\chi$, the Keller-Segel system (II.1) has a global weak non-negative solution such that, for any $T > 0$,*

$$(1 + |x|^2 + |\log n|)\, n \in L^\infty(0,T; L^1(\mathbb{R}^2)) \quad \textit{and} \quad \iint_{[0,T]\times\mathbb{R}^2} n|\nabla \log n - \chi\nabla c|\; dx\, dt < \infty \;.$$

In particular, the equation holds in the distributions sense. Indeed, writing

$$\Delta n - \chi \nabla \cdot (n \nabla c) = \nabla \cdot [n (\nabla \log n - \chi \nabla c)] \ ,$$

we can see that the flux is well defined in $L^1(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$ since

$$\iint_{[0,T]\times\mathbb{R}^2} n\, |\nabla \log n - \chi \nabla c| \; dx\, dt \leq \left(\iint_{[0,T]\times\mathbb{R}^2} n \; dx\, dt \right)^{1/2} \left(\iint_{[0,T]\times\mathbb{R}^2} n\, |\nabla \log n - \chi \nabla c|^2 \; dx\, dt \right)^{1/2}$$

is finite.

Propostion II.9 strongly relies on the estimates of Lemmata II.3 and II.7. To establish a complete proof, we need to regularize the problem (Section 2.5) and then prove that the above estimates hold uniformly with respect to the regularization procedure (Section 2.6). This allows to pass to the limit in the regularization parameter (Section 2.7) and proves the existence of a weak solution with a well defined flux. To prove Theorem II.1, we need additional regularity properties of the solutions. This is the purpose of Section 3. Hypercontractivity and the free energy inequality (II.2) will be dealt with in Sections 3.4 and 3.3 respectively.

2.5 A regularized model

The goal of this section is to establish the existence of solutions for a regularized version of the Keller-Segel model, for which the logarithmic singularity of the convolution kernel $\mathcal{K}^0(z) := -\frac{1}{2\pi} \log |z|$ is appropriately truncated.

There are indeed two difficulties when dealing with $\mathcal{K}^0$. It is unbounded and has a singularity at $z = 0$. First of all, the unboundedness from above of the kernel is not difficult to handle. For $R > \sqrt{e}$, $R \mapsto R^2/\log R$ is an increasing function, so that

$$0 \leq \iint_{|x-y|>R} \log |x-y|\, n(x,t)\, n(y,t) \; dx\, dy \leq \frac{2 \, \log\, R}{R^2}\, M \int_{\mathbb{R}^2} |x|^2\, n(x,t) \; dx \ .$$

Hence we only need to take care of a uniform bound on

$$\iint_{|x-y|<1} \log |x-y|\, n^\varepsilon(x,t)\, n^\varepsilon(y,t) \; dx\, dy \quad \text{and} \quad \int_{\mathbb{R}^2} n^\varepsilon(x,t) \; \log n^\varepsilon(x,t) \; dx \ .$$

for an approximating family $(n^\varepsilon)_{\varepsilon>0}$.

The other difficulty concerning the convolution kernel $\mathcal{K}^0$ is the singularity at $z = 0$. This is a much more serious difficulty that we are going to overcome by defining a truncated convolution kernel and deriving uniform estimates in Section 2.6. To do so, we first need to find solutions of the model with a truncated convolution kernel. Let $\mathcal{K}^\varepsilon$ be such that

$$\mathcal{K}^\varepsilon(z) := \mathcal{K}^1\left(\frac{z}{\varepsilon}\right)$$

where $\mathcal{K}^1$ is a radial monotone non-decreasing smooth function satisfying

$$\begin{cases} \mathcal{K}^1(z) = -\dfrac{1}{2\pi} \log |z| & \text{if } \ |z| \geq 2 \, , \\ \mathcal{K}^1(z) = 0 & \text{if } \ |z| \leq \frac{1}{2} \, . \end{cases}$$

Moreover, we can assume without restriction that

$$0 \leq -\nabla \mathcal{K}^1(z) \leq \frac{1}{2\pi\,|z|} \quad \text{and} \quad \mathcal{K}^1(z) \leq -\frac{1}{2\pi} \log |z| \tag{II.10}$$

for any $z \in \mathbb{R}^2$. Since $\mathcal{K}^\varepsilon(z) = \mathcal{K}^1(z/\varepsilon)$, we also have

$$0 \leq -\nabla \mathcal{K}^\varepsilon(z) \leq \frac{1}{2\pi\,|z|} \quad \forall\, z \in \mathbb{R}^2\,. \tag{II.11}$$

If we replace (II.1) by the following regularized version

$$\left\{ \begin{array}{l} \dfrac{\partial n^\varepsilon}{\partial t} = \Delta n^\varepsilon - \chi \nabla \cdot (n^\varepsilon \nabla c^\varepsilon) \\ c^\varepsilon = \mathcal{K}^\varepsilon * n^\varepsilon \end{array} \right. \qquad x \in \mathbb{R}^2\,,\ t > 0\,, \tag{II.12}$$

written in the distribution sense, then we can state the following existence result.

Proposition II.10. *For any fixed positive ε, under Assumptions (II.4), if $n_0 \in L^2(\mathbb{R}^2)$, then for any $T > 0$ there exists $n^\varepsilon \in L^2(0,T;H^1(\mathbb{R}^2)) \cap C(0,T;L^2(\mathbb{R}^2))$ which solves (II.12) with initial data n_0.*

To prove Proposition II.10, we will first fix a functional framework, then solve a linear problem before using it to make a fixed point argument in order to prove the existence of a solution to the regularized system (II.12).

2.5.1 Functional framework

We will use the Aubin-Lions *compactness method,* (see [Lio61], Ch. IV, §4 and [Aub63], and [Sim87, RS03] for more recent references). A simple statement goes as follows:

Lemma II.11 (Aubin Lemma). *Take $T > 0$, $p \in (1, \infty)$ and let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions in $L^p(0,T;V)$ where V is a Banach space. If $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^p(0,T;V)$, where V is compactly imbedded in H and $\partial f_n/\partial t$ is bounded in $L^p(0,T;V')$ uniformly with respect to $n \in \mathbb{N}$, then $(f_n)_{n\in\mathbb{N}}$ is relatively compact in $L^p(0,T;V)$.*

For our purpose, we fix $T > 0$, $p = 2$ and define $H := L^2(\mathbb{R}^2)$, $V := \{v \in H^1(\mathbb{R}^2) : \sqrt{|x|}\, v \in L^2(\mathbb{R}^2)\}$, V' its dual, $\mathcal{V} := L^2(0,T;V)$, $\mathcal{H} := L^2(0,T;H)$ and $\mathcal{W}(0,T) := \{v \in L^2(0,T;V) \,:\, \partial v/\partial t \in L^2(0,T;V')\}$. In this functional framework, the notion of solution we are looking for is actually more precise than in the distribution sense:

$$0 = \int_0^T \left\{ \langle n_t, \psi \rangle_{V' \times V} + \int_{\mathbb{R}^3} (\nabla n + \chi n\, \nabla c) \cdot \nabla \psi \, dx \right\} dt \quad \forall \psi \in L^2(0,T;V)\,.$$

Notice that V is relatively compact in H, since the bound on $|x|\,|v|^2$ in $L^1(\mathbb{R}^2)$ allows to consider only compact sets, on which compactness holds by Sobolev's imbeddings: Lemma II.11 applies.

2.5.2 Estimates for a linear drift-diffusion equation

We start with the derivation of some *a priori* estimates on the solution of the linear problem

$$\frac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\, f) \tag{II.13}$$

for some function $f \in (L^\infty((0,T) \times \mathbb{R}^2))^2$. We assume in this section that the initial data n_0 is in $L^2(\mathbb{R}^2)$. By a fixed-point method, this allows us to prove the

Lemma II.12. *Assume that (II.4) holds and consider $f \in L^\infty((0,T) \times \mathbb{R}^2)$. If $n_0 \in L^2(\mathbb{R}^2)$, for any $T > 0$, there exists $n \in \mathcal{W}(0,T)$ which solves (II.13) with initial data n_0.*

Proof – Consider the map $\mathcal{F} : L^\infty(0,T; L^1(\mathbb{R}^2)) \to L^\infty(0,T; L^1(\mathbb{R}^2))$ defined by

$$\mathcal{F}[n](\cdot,t) := G(\cdot,t) * n_0 + \int_0^t \nabla G(\cdot, t-s) * [n(\cdot,s)\, f(\cdot,s)]\ ds \quad \forall\,(x,t) \in [0,T] \times \mathbb{R}^2\ .$$

where $*$ denotes the space convolution. Here $G(x,t) := (4\pi t)^{-1} e^{-\frac{|x|^2}{4t}}$ is the Green function associated to the heat equation. Notice that $\|\nabla G(\cdot,s)\|_{L^1(\mathbb{R}^2)} \leq C\, s^{-1/2}$. We define the sequence $(n_k)_{k\in\mathbb{N}}$ by $n_{k+1} = \mathcal{F}(n_k)$ for $k \geq 1$. For any $t \in [0,T]$, we compute

$$\begin{aligned}
\|n_{k+1}(t) - n_k(t)\|_{L^1(\mathbb{R}^2)} &\leq \int_{\mathbb{R}^2} \left| \int_0^t \nabla G(\cdot, t-s) * [(n_k(\cdot,s) - n_{k-1}(\cdot,s))\ f(\cdot,s)]\ ds \right| dx \\
&\leq \|f\|_{L^\infty([0,T]\times\mathbb{R}^2)} \int_0^t \|\nabla G(\cdot,t-s) * (n_k(\cdot,s) - n_{k-1}(\cdot,s))\,\|_{L^1(\mathbb{R}^2)} \\
&\leq \|f\|_{L^\infty([0,T]\times\mathbb{R}^2)} \int_0^t \|\nabla G(\cdot,t-s)\|_{L^1(\mathbb{R}^2)}\, \|n_k(s) - n_{k-1}(s)\|_{L^1(\mathbb{R}^2)} \\
&\leq C\, \|f\|_{L^\infty([0,T]\times\mathbb{R}^2)} \sqrt{t}\, \|n_k - n_{k-1}\|_{L^\infty(0,t;L^1(\mathbb{R}^2))}\ .
\end{aligned}$$

For $T > 0$ small enough, $(n_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(0,T; L^1(\mathbb{R}^2))$, which converges to a fixed point of $\mathcal{F}$. Iterating the method, we prove the existence of a solution of (II.13) on an arbitrary time interval $[0,T]$. □

Next, let us establish some *a priori* estimates. The solution n is bounded in $L^\infty(0,T; L^2(\mathbb{R}^2))$ as a consequence of the following computation:

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^2} |n(x,t)|^2\ dx = -\int_{\mathbb{R}^2} |\nabla n(x,t)|^2\ dx + \int_{\mathbb{R}^2} \nabla n(x,t) \cdot n(x,t)\, f(x,t)\ dx\ .$$

The right hand side can be written $\int_{\mathbb{R}^2} a \cdot b\, f\ dx$ with $a := \sqrt{1/\lambda}\, \nabla n$ and $b := \sqrt{\lambda}\, n$. It is therefore bounded by $(\int_{\mathbb{R}^2} a^2\ dx + \frac{1}{4} \int_{\mathbb{R}^2} b^2\ dx)\, \|f\|_{L^\infty([0,T]\times\mathbb{R}^2)}$, which provides the estimate

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^2} |n|^2\ dx \leq \left(-1 + \frac{1}{\lambda} \|f\|_{L^\infty([0,T]\times\mathbb{R}^2)}\right) \int_{\mathbb{R}^2} |\nabla n|^2\ dx + \frac{\lambda}{4}\, \|f\|_{L^\infty([0,T]\times\mathbb{R}^2)} \int_{\mathbb{R}^2} |n|^2\ dx\ .$$

In case $\lambda = \|f\|_{L^\infty([0,T]\times\mathbb{R}^2)}$, we obtain

$$\int_{\mathbb{R}^2} |n|^2\ dx \leq \int_{\mathbb{R}^2} |n_0|^2\ dx\ e^{\|f\|^2_{L^\infty([0,T]\times\mathbb{R}^2)}\, T/2} \quad \forall\, t \in (0,T)\ .$$

Hence n is bounded in $L^\infty(0,T;L^2(\mathbb{R}^2)) \subset \mathcal{H}$. Similarly, in case $\lambda = \frac{3}{2}\,\|f\|_{L^\infty([0,T]\times\mathbb{R}^2)}$, we obtain

$$\frac{1}{2}\,\frac{d}{dt}\|n(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \le -\frac{1}{3}\|\nabla n\|^2_{L^2(\mathbb{R}^2)} + \frac{3}{8}\,\|f\|^2_{L^\infty([0,T]\times\mathbb{R}^2)}\|n(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}\;.$$

This also proves that ∇n is bounded in $L^2((0,T)\times\mathbb{R}^2)$, and n is therefore also bounded in $L^2(0,T;H^1(\mathbb{R}^2))$. Next, we need a moment estimate, which is achieved by

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2\,n(x,t)\;dx \le 4\int_{\mathbb{R}^2} n\;dx + 2\,\|f\|_{L^\infty_{x,t}}\left(\int_{\mathbb{R}^2} n\;dx\right)^{1/2}\left(\int_{\mathbb{R}^2}|x|^2\,n(x,t)\;dx\right)^{1/2}.$$

As a conclusion, this proves that $\int_{\mathbb{R}^2}|x|^2\,n(x,t)\;dx$ is bounded and therefore shows that n is bounded in $\mathcal{V}$. On the other hand, $\partial n/\partial t$ is bounded in V' as can be checked by an elementary computation. We can therefore apply Aubin's Lemma (Lemma II.11) to n:

If $(n_0^k)_{k\in\mathbb{N}}$ is a sequence of initial data with uniform bounds, then the corresponding sequence $(n^k)_{k\in\mathbb{N}}$ of solutions of (II.13) with f replaced by f_k, for a sequence $(f^k)_{k\in\mathbb{N}}$ uniformly bounded in $(L^\infty([0,T]\times\mathbb{R}^2))^2$, is contained in a relatively compact set in $L^2(0,T;V)$.

We will make use of this property in the next section.

2.5.3 Existence of a solution of the regularized problem

This section is devoted to the proof of Proposition II.10, using a fixed point method.

Define the truncation function $h(s) := \min\{1, h_0/s\}$, for some constant $h_0 > 1$ to be fixed later and consider the map $\mathcal{F} : L^2(0,T;H) \to L^2(0,T;H)$ such that

1. To a function $n \in L^2(0,T;H)$, we associate $\nabla c^\varepsilon := \nabla\mathcal{K}^\varepsilon * n$.

2. With ∇c^ε, we construct the truncated function

$$f := h\left(\|\nabla c^\varepsilon\|_{L^\infty((0,T)\times\mathbb{R}^2)}\right)\,\nabla c^\varepsilon\;.$$

3. The function f is bounded in $L^\infty((0,T)\times\mathbb{R}^2)$ by h_0, so we may apply Lemma II.12 and obtain a new function $\tilde{n} =: \mathcal{F}[n]$ which solves (II.13).

The continuity of $\mathcal{F}$ is straightforward. As noticed in Section 2.5.2, we may apply the Aubin-Lions *compactness method,* which gives enough compactness to apply Schauder's fixed point theorem (Theorem 8.1 p. 199 in [Lie96]) to a ball in $\mathcal{W}(0,T)$. Hence we obtain a solution of

$$\begin{cases} \dfrac{\partial n^\varepsilon}{\partial t} = \Delta n^\varepsilon - \chi\nabla\cdot(n^\varepsilon\,f^\varepsilon) \\[2mm] f^\varepsilon = h\left(\|\nabla c^\varepsilon\|_{L^\infty((0,T)\times\mathbb{R}^2)}\right)\,\nabla c^\varepsilon\;, \quad c^\varepsilon = \mathcal{K}^\varepsilon * n^\varepsilon\;. \end{cases}$$

Assuming that $h_0 > \|\nabla\mathcal{K}^\varepsilon\|_{L^\infty(\mathbb{R}^2)}\|n^\varepsilon\|_{L^1(\mathbb{R}^2)}$, we realize that n^ε is a solution of (II.12).
□

Notice that one can also prove easily a uniqueness result, using an appropriate Gronwall lemma. We refer for instance to [Ros01] for similar results in a ball.

2.6 Uniform a priori estimates

In this section, we prove *a priori* estimates for the regularized problem which are uniform with respect to the regularization parameter ε. These estimates correspond to the formal estimates of Section 2.3.

Lemma II.13. *Under Assumption (II.4), consider a solution n^ε of (II.12). If $\chi M < 8\pi$ then, uniformly as $\varepsilon \to 0$, we have:*

(i) The function $(t,x) \mapsto |x|^2 n^\varepsilon(x,t)$ is bounded in $L^\infty(\mathbb{R}^+_{\mathrm{loc}}; L^1(\mathbb{R}^2))$.

(ii) The functions $t \mapsto \int_{\mathbb{R}^2} n^\varepsilon(x,t) \log n^\varepsilon(x,t)\, dx$ and $t \mapsto \int_{\mathbb{R}^2} n^\varepsilon(x,t)\, c^\varepsilon(x,t)\, dx$ are bounded.

(iii) The function $(t,x) \mapsto n^\varepsilon(x,t) \log(n^\varepsilon(x,t))$ is bounded in $L^\infty(\mathbb{R}^+_{\mathrm{loc}}; L^1(\mathbb{R}^2))$.

(iv) The function $(t,x) \mapsto \nabla\sqrt{n^\varepsilon}(x,t)$ is bounded in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$.

(v) The function $(t,x) \mapsto n^\varepsilon(x,t)$ is bounded in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$.

(vi) The function $(t,x) \mapsto n^\varepsilon(x,t)\, \Delta c^\varepsilon(x,t)$ is bounded in $L^1(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$.

(vii) The function $(t,x) \mapsto \sqrt{n^\varepsilon}(x,t)\, \nabla c^\varepsilon(x,t)$ is bounded in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$.

Proof –
(i) The integral $\int_{\mathbb{R}^2} |x|^2\, n^\varepsilon(x,t)\, dx$ can be estimated as in the proof of Lemma II.3 because $\mathcal{K}^\varepsilon$ is radial and satisfies (II.11), so

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\, n^\varepsilon(x,t)\, dx &= 4M + 2\chi \iint_{\mathbb{R}^2\times\mathbb{R}^2} n^\varepsilon(x,t)\, n^\varepsilon(y,t)\, x \cdot \nabla\mathcal{K}^\varepsilon(x-y)\, dx\, dy \\
&= 4M + \chi \iint_{\mathbb{R}^2\times\mathbb{R}^2} n^\varepsilon(x,t)\, n^\varepsilon(y,t)\, (x-y)\nabla\mathcal{K}^\varepsilon(x-y)\, dx\, dy \\
&\le 4M - \frac{\chi}{2\pi} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{n^\varepsilon(x,t)\, n^\varepsilon(y,t)}{|x-y|}\, dx\, dy \le 4M\ .
\end{aligned}$$

(ii) On the one hand we compute

$$\frac{d}{dt}\left[\int_{\mathbb{R}^2} n^\varepsilon \log n^\varepsilon\, dx - \frac{\chi}{2}\int_{\mathbb{R}^2} n^\varepsilon\, c^\varepsilon\, dx\right] = -\int_{\mathbb{R}^2} n^\varepsilon\, |\nabla(\log n^\varepsilon) - \chi\nabla c^\varepsilon|^2\, dx\ .$$

On the other hand, by (II.10) and the logarithmic Hardy-Littlewood-Sobolev inequality, see Lemma II.6, it follows by Lemma II.7 that both terms of the right hand side are uniformly bounded.

(iii) It is a direct consequence of Lemma II.8.

(iv) A simple computation shows that

$$\frac{d}{dt}\int_{\mathbb{R}^2} n^\varepsilon\, \log n^\varepsilon\, dx \le -4\int_{\mathbb{R}^2} \left|\nabla\sqrt{n^\varepsilon}\right|^2\, dx + \chi\int_{\mathbb{R}^2} n^\varepsilon \cdot (-\Delta c^\varepsilon)\, dx\ .$$

Up to the common factor χ, we can write the last term of the right hand side as

$$\int_{\mathbb{R}^2} n^\varepsilon \cdot (-\Delta c^\varepsilon)\, dx = \int_{\mathbb{R}^2} n^\varepsilon \cdot (-\Delta(\kappa^\varepsilon * n^\varepsilon))\, dx = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})$$

with

$$(\mathrm{I}) := \int_{n^\varepsilon < K} n^\varepsilon \cdot (-\Delta(\kappa^\varepsilon * n^\varepsilon)) \ , \quad (\mathrm{II}) := \int_{n^\varepsilon \geq K} n^\varepsilon \cdot (-\Delta(\kappa^\varepsilon * n^\varepsilon)) - (\mathrm{III}) \quad \text{and} \quad (\mathrm{III}) = \int_{n^\varepsilon \geq K} |n^\varepsilon|^2 \ .$$

We define ϕ_1 such that

$$\frac{1}{\varepsilon^2}\phi_1\Big(\frac{\cdot}{\varepsilon}\Big) = -\Delta\kappa^\varepsilon \ .$$

This gives an easy estimate of (I), namely

$$(\mathrm{I}) \leq \int_{n^\varepsilon < K} K \int_{\mathbb{R}^2} \frac{1}{\varepsilon^2}\phi_1\left(\frac{x-y}{\varepsilon}\right) n^\varepsilon(y)\, dy\, dx = M\, K \ .$$

Notice that

$$\frac{1}{\varepsilon^2}\phi_1\Big(\frac{\cdot}{\varepsilon}\Big) = -\Delta\kappa^\varepsilon \rightharpoonup \delta \quad \text{in } \mathcal{D}' \ , \tag{II.14}$$

which heuristically explain why (II) should be small. Let us prove that this is indeed the case: Using $\|\phi_1\|_1 = 1$, we get

$$\begin{aligned} (\mathrm{II}) \ &= \int_{n^\varepsilon \geq K} n^\varepsilon(x,t) \int_{\mathbb{R}^2} [n^\varepsilon(x-\varepsilon y,t) - n^\varepsilon(x,t)]\ \phi_1(y)\, dy\, dx \\ &\leq \int_{n^\varepsilon \geq K} n^\varepsilon(x,t) \int_{\mathbb{R}^2} \left[\sqrt{n^\varepsilon(x-\varepsilon y,t)} - \sqrt{n^\varepsilon(x,t)}\right]\ \sqrt{\phi_1(y)} \\ &\qquad \times \left[\sqrt{n^\varepsilon(x-\varepsilon y,t)} - \sqrt{n^\varepsilon(x,t)} + 2\sqrt{n^\varepsilon(x,t)}\right]\ \sqrt{\phi_1(y)}\, dy\, dx \ . \end{aligned}$$

By the Cauchy-Schwarz inequality and using $(a+2b)^2 \leq 2a^2 + 8b^2$ we obtain

$$\begin{aligned} (\mathrm{II}) \leq \int_{n^\varepsilon \geq K} n^\varepsilon(x,t) &\left[\|\phi_1\|_{L^\infty(\mathbb{R}^2)} \int_{1/2 \leq y \leq 2} \left|\sqrt{n^\varepsilon(x-\varepsilon y,t)} - \sqrt{n^\varepsilon(x,t)}\right|^2\ dy\right]^{1/2} \\ &\cdot \left[\int_{\mathbb{R}^2} \left[2\left|\sqrt{n^\varepsilon(x-\varepsilon y,t)} - \sqrt{n^\varepsilon(x,t)}\right|^2 + 8\,|n^\varepsilon(x,t)|\right]\ \phi_1(y)\ dy\right]^{1/2} dx \ . \end{aligned}$$

Using the Poincaré inequality we get

$$\begin{aligned} (\mathrm{II}) \leq \int_{n^\varepsilon \geq K} n^\varepsilon(x,t)\ &\|\phi_1\|^{1/2}_{L^\infty(\mathbb{R}^2)}\, C_P\, \|\nabla\sqrt{n^\varepsilon}\|_{L^2(\mathbb{R}^2)} \\ &\cdot \left[\sqrt{2}\,\|\phi_1\|^{1/2}_{L^\infty(\mathbb{R}^2)}\, C_P\, \|\nabla\sqrt{n^\varepsilon}\|_{L^2(\mathbb{R}^2)} + 2\sqrt{2}\sqrt{|n^\varepsilon(x,t)|}\ \|\phi_1\|^{1/2}_{L^1(\mathbb{R}^2)}\right]\ dx \ . \end{aligned}$$

Recall the Gagliardo-Nirenberg-Sobolev inequality (II.7):

$$\int_{n^\varepsilon \geq K} |n^\varepsilon|^2\, dx \leq C^2_{\mathrm{GNS}} \int_{n^\varepsilon \geq K} \left|\nabla\sqrt{n^\varepsilon}\right|^2\, dx \int_{n^\varepsilon \geq K} n^\varepsilon\ dx \ .$$

The left hand side can therefore be made as small as desired using:

$$\int_{n^\varepsilon \geq K} n^\varepsilon\ dx \leq \frac{1}{\log K}\int_{n^\varepsilon \geq K} n^\varepsilon\ \log n^\varepsilon\ dx \leq \frac{1}{\log K}\int_{\mathbb{R}^2} n^\varepsilon\,|\log n^\varepsilon|\ dx =: \eta(K) \ ,$$

for $K > 1$, large enough. Then

$$\int_{n^\varepsilon \geq K} |n^\varepsilon|^2\, dx \leq \eta(K)\, C^2_{\mathrm{GNS}} \left\|\nabla\sqrt{n^\varepsilon}\right\|^2_{L^2(\mathbb{R}^2)} \ . \tag{II.15}$$

By the Cauchy-Schwarz inequality

$$\int_{n^\varepsilon \geq K} |n^\varepsilon(x,t)|^{3/2}\, dx \leq \left(\int_{n^\varepsilon \geq K} |n^\varepsilon|\, dx\right)^{1/2} \cdot \left(\int_{n^\varepsilon \geq K} |n^\varepsilon|^2\, dx\right)^{1/2} \leq \eta(K)\, C_{\rm GNS}\, \|\nabla\sqrt{n^\varepsilon}\|_{L^2(\mathbb{R}^2)} \;.$$

From this, it follows that

$$({\rm II}) + ({\rm III}) \leq B\,\eta(K)\,\|\nabla\sqrt{n^\varepsilon}\|^2_{L^2(\mathbb{R}^2)}$$

with

$$B := C^2_{\rm GNS} + \sqrt{2}\,\|\phi_1\|_{L^\infty(\mathbb{R}^2)}\, C^2_P \; + 2\sqrt{2}\,\|\phi_1\|^{1/2}_{L^\infty(\mathbb{R}^2)}\,\|\phi_1\|^{1/2}_{L^1(\mathbb{R}^2)}\, C_P\, C_{\rm GNS} \;.$$

We can choose K large enough such that $\eta(K) < 4/B$. Collecting the estimates, we have shown that

$$\frac{d}{dt}\int_{\mathbb{R}^2} n^\varepsilon \,\log n^\varepsilon\, dx \leq M\,K + (-4 + B\,\eta(K))X(t)$$

with

$$X(t) := \|\nabla\sqrt{n^\varepsilon}(t)\|^2_{L^2(\mathbb{R}^2)}\;,$$

and so

$$(4 - B\,\eta)\int_0^T X(s)\,ds \leq M\,K\,T + \int_{\mathbb{R}^2} n_0\,\log n_0\, dx - \int_{\mathbb{R}^2} n^\varepsilon(x,T)\,\log n^\varepsilon(x,T)\, dx \;.$$

(v) The result follows from the Gagliardo-Nirenberg-Sobolev inequality (II.7).

(vi) is a straightforward consequence of *(iv)*. Notice that $-\Delta c^\varepsilon$ is non-negative as a convolution of two non-negative functions ϕ_1 and n^ε.

(vii) A computation shows that

$$\frac{d}{dt}\int_{\mathbb{R}^2} \frac{1}{2}\, n^\varepsilon\, c^\varepsilon\, dx = \int_{\mathbb{R}^2} c^\varepsilon\, \left(\Delta n^\varepsilon - \chi\,\nabla\cdot(n^\varepsilon\,\nabla c^\varepsilon)\right)\, dx = \int_{\mathbb{R}^2} n^\varepsilon\,\Delta c^\varepsilon\, dx + \chi\int_{\mathbb{R}^2} n^\varepsilon\,|\nabla c^\varepsilon|^2\, dx \;. \tag{II.16}$$

This proves that

$$\iint_{[0,T]\times\mathbb{R}^2} n^\varepsilon\,|\nabla c^\varepsilon|^2\, dx\,dt \leq \frac{1}{2\,\chi}\left|\int_{\mathbb{R}^2} n^\varepsilon\, c^\varepsilon\, dx - \int_{\mathbb{R}^2} n_0\,(\mathcal{K}^\varepsilon * n_0)\, dx\right| + \frac{1}{\chi}\int_0^T\int_{\mathbb{R}^2} n^\varepsilon\,(-\Delta c^\varepsilon)\, dx \;.$$

The last term of the right hand side is controlled by *(vi)*, while the previous one is bounded by *(ii)*. □

2.7 Passing to the limit

All estimates of Lemma II.13 are uniform in ε in the limit $\varepsilon \to 0$. The fact that n_0 is assumed to be bounded in $L^2(\mathbb{R}^2)$ in Lemma II.10 does not play any role. In this section, n_0 is assumed to satisfy Assumption (II.4) and we consider the solution n^ε of (II.12) with a non-negative initial data $n_0^\varepsilon = \min\{n_0, \varepsilon^{-1}\}$. We want to pass simultaneously to the limit as $\varepsilon \to 0$ in $n_0^\varepsilon \to n_0$ and in $\mathcal{K}^\varepsilon(z) \to \mathcal{K}^0(z) = -\frac{1}{2\pi}\log|z|$.

Lemma II.14. *Assume that n_0 satisfies Assumption (II.4) and consider the solution n^ε of (II.12) with a non-negative initial data $n_0^\varepsilon = \min\{n_0, \varepsilon^{-1}\}$. Then up to the extraction of a sequence ε_k of ε converging to 0, n^{ε_k} converges to a function n solution of (II.1) in the distribution sense. Furthermore the flux $n\,|\nabla(\log n) - \chi\nabla c|$ is bounded in $L^1([0,T)\times\mathbb{R}^2)$.*

Proof – Assertion *(vii)* of Lemma II.13 allows to give a sense to the equation in the limit $\varepsilon \searrow 0$. The term which is difficult to handle is $n^\varepsilon\, \nabla c^\varepsilon$. It is first of all bounded in $L^1((0,T)\times\mathbb{R}^2)$ uniformly with respect to ε, as shown by the Cauchy-Schwarz inequality:

$$\left(\iint_{[0,T]\times\mathbb{R}^2} n^\varepsilon\, |\nabla c^\varepsilon|\; dx\, dt\right)^2 \leq \iint_{[0,T]\times\mathbb{R}^2} n^\varepsilon\, dx\, dt \iint_{[0,T]\times\mathbb{R}^2} n^\varepsilon\, |\nabla c^\varepsilon|^2\; dx\, dt = M\, T \iint_{[0,T]\times\mathbb{R}^2} n^\varepsilon\, |\nabla c^\varepsilon|^2\; dx\, dt\;,$$

where the last term is controlled according to *(vii)* of Lemma II.13.

Actually, $n^\varepsilon\, \nabla c^\varepsilon$ converges to $n\, \nabla c$ in the sense of distributions. By the Gagliardo-Nirenberg-Sobolev inequality (II.7), for any $p>2$, for $t\in\mathbb{R}^+$ a.e.,

$$\int_{\mathbb{R}^2} |n^\varepsilon|^{p/2}\, dx \leq \left(C^{(p)}_{\rm GNS}\right)^{p/2}\, M \left(\int_{\mathbb{R}^2} \left|\nabla\sqrt{n^\varepsilon}\right|^2\, dx\right)^{\frac{p}{2}-1}\;,$$

which proves that n^ε is bounded in $L^q(\mathbb{R}^+_{\rm loc}\times\mathbb{R}^2)$ for any $p/2=q\in[1,+\infty)$, and that, up to the extraction of a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ which converges to 0, n^{ε_k} weakly converges to n in any $L^q_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^2)$, $q\geq 1$. Next,

$$\begin{aligned}\nabla c^{\varepsilon_k} - \nabla c = \; & -\; \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\; (n^{\varepsilon_k}(y,t) - n(y,t))\; dy \\ & +\; \int_{|x-y|\leq 2\varepsilon_k} \left(\frac{1}{\varepsilon_k}\,\nabla\mathcal{K}^1\left(\frac{x-y}{\varepsilon_k}\right) + \frac{x-y}{2\pi\,|x-y|^2}\right) n^{\varepsilon_k}(y,t)\; dy\;.\end{aligned}$$

Since $\frac{1}{\varepsilon_k}\,\nabla\mathcal{K}^1\left(\frac{z}{\varepsilon_k}\right) + \frac{z}{2\pi\,|z|^2}$ can be bounded by $\frac{1}{2\pi\,|z|}$, all terms converge to 0 for almost any $(x,t)\in\mathbb{R}^2\times\mathbb{R}^+$ and the convergence is strong in $L^q([0,T]\times\mathbb{R}^2_{\rm loc})$ for any $q\in(2,\infty)$, which is enough to prove that

$$n^{\varepsilon_k}\nabla c^{\varepsilon_k} \rightharpoonup n\nabla c \quad \text{in } \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^2)\;.$$

As a consequence, we also get by weak semi-continuity that

$$\begin{aligned}\iint_{[0,T]\times\mathbb{R}^2} n\,|\nabla c|^2\, dx\, dt &\leq \liminf_{\varepsilon_k\to 0} \iint_{[0,T]\times\mathbb{R}^2} n^{\varepsilon_k}\,|\nabla c^{\varepsilon_k}|^2\, dx\, dt\;,\\ \iint_{[0,T]\times\mathbb{R}^2} n^2\, dx\, dt &\leq \liminf_{\varepsilon_k\to 0} \iint_{[0,T]\times\mathbb{R}^2} |n^{\varepsilon_k}|^2\, dx\, dt\;.\end{aligned}$$

Since the functional $n\mapsto \int_{\mathbb{R}^2} |\nabla\sqrt{n}|^2\, dx$ is convex, we also get

$$\iint_{[0,T]\times\mathbb{R}^2} |\nabla\sqrt{n}|^2\, dx\, dt \leq \liminf_{\varepsilon_k\to 0} \iint_{[0,T]\times\mathbb{R}^2} |\nabla\sqrt{n^{\varepsilon_k}}|^2\, dx\, dt\;.$$

The proof of the convexity goes as follows. Let $n(\tau) = n_0 + \tau\,\nu$, $\tau>0$. Then

$$\frac{d^2}{d\tau^2}\int_{\mathbb{R}^2} \left|\nabla\sqrt{n(\tau)}\right|^2\, dx\,\Big|_{\tau=0} = \frac{1}{2n_0^3}\int_{\mathbb{R}^2} \left|\nu\nabla\sqrt{n_0} - n_0\nabla\sqrt{\nu}\right|^2 dx \geq 0\;.$$

See [Ben79, BBL81] for more details. Now, since

$$\begin{aligned}&\iint_{[0,T]\times\mathbb{R}^2} n^{\varepsilon_k}\,|\nabla(\log n^{\varepsilon_k}) - \chi\nabla c^{\varepsilon_k}|^2\, dx\, dt \\ &\quad = 4\iint_{[0,T]\times\mathbb{R}^2} |\nabla\sqrt{n^{\varepsilon_k}}|^2\, dx\, dt - 2\chi\iint_{[0,T]\times\mathbb{R}^2} |n^{\varepsilon_k}|^2\, dx\, dt + \chi^2\iint_{[0,T]\times\mathbb{R}^2} n^{\varepsilon_k}\,|\nabla c^{\varepsilon_k}|^2\, dx\, dt\end{aligned}$$

is bounded uniformly with respect to ε_k by (II.8), the free energy production

$$\iint_{[0,T]\times\mathbb{R}^2} n\,|\nabla(\log n) - \chi\nabla c|^2\,dx\,dt$$

is also finite. Notice that this is not enough to prove that (II.8) holds if n is a solution of (II.1), even with an inequality instead of the equality. This is however enough to prove that the flux $n\,|\nabla(\log n) - \chi\nabla c|$ is bounded in $L^1([0,T)\times\mathbb{R}^2)$, simply by using the Cauchy-Schwarz inequality. This concludes the proof of Lemma II.14. $\square$

As a consequence of the approximation procedure and of Lemma II.14, we have also proved Proposition II.9. To establish Inequality (II.2) in Theorem II.1, we only need to prove that

$$\iint_{[0,T]\times\mathbb{R}^2} n^2\,dx\,dt = \liminf_{\varepsilon_k\to 0}\iint_{[0,T]\times\mathbb{R}^2} |n^{\varepsilon_k}|^2\,dx\,dt\ ,$$

but this requires some additional work on the regularity properties of the solutions of (II.1).

3 Free energy inequality and regularity properties

In this section, we give some additional regularity properties of the solutions when $\chi\,M < 8\pi$.

3.1 Weak regularity results

The following result is due to Goudon, see [Gou05].

Theorem II.15. [Gou05] *Let $n^\varepsilon : (0,T)\times\mathbb{R}^N \to \mathbb{R}$ be such that for almost all $t\in(0,T)$, $n^\varepsilon(t)$ belongs to a weakly compact set in $L^1(\mathbb{R}^N)$ for almost any $t\in(0,T)$. If $\partial_t n^\varepsilon = \sum_{|\alpha|\le k}\partial_x^\alpha g_\varepsilon^{(\alpha)}$ where, for any compact set $K\subset\mathbb{R}^n$,*

$$\limsup_{\substack{|E|\to 0\\ E\subset\mathbb{R}\text{ is measurable}}}\left(\sup_{\varepsilon>0}\int\int_{E\times K} |g_\varepsilon^{(\alpha)}|\,dt\,dx\right) = 0\ ,$$

then $(n^\varepsilon)_{\varepsilon>0}$ is relatively compact in $C^0([0,T];L^1_{\rm weak}(\mathbb{R}^N))$.

This result immediately applies to the solution of (II.12).

Corollary II.16. *Let n^ε be a solution of (II.12) with initial data $n_0^\varepsilon = \min\{n_0,\varepsilon^{-1}\}$ such that $n_0\,(1+|x|^2+|\log n_0|)\in L^1(\mathbb{R}^2)$. If n is a solution of (II.1) with initial data n_0, such that, for a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty}\varepsilon_k = 0$, $n^{\varepsilon_k}\rightharpoonup n$ in $L^1((0,T)\times\mathbb{R}^2)$, then n belongs to $C^0(0,T;L^1_{\rm weak}(\mathbb{R}^2))$.*

Proof – We are able to apply Theorem II.15 to n^ε where $g_\varepsilon^{(1)} := -\chi\,n^\varepsilon\,\nabla c^\varepsilon = -\chi\,\sqrt{n^\varepsilon}\cdot\sqrt{n^\varepsilon}\,\nabla c^\varepsilon$ and $g_\varepsilon^{(2)} := n^\varepsilon$. Notice indeed that as a consequence of Lemma II.18, we have, uniformly with respect to ε,

$$\limsup_{t_1\to t_2}\sup_\varepsilon g_\varepsilon^{(1)} \le \chi\,\limsup_{t_1\to t_2} M(t_2-t_1)\int_{t_1}^{t_2}\int_{\mathbb{R}^2} n^\varepsilon|\nabla c^\varepsilon|^2\,dx\,ds = 0\ ,$$

$$\limsup_{t_1\to t_2}\sup_\varepsilon g_\varepsilon^{(2)} \le \limsup_{t_1\to t_2}\int_{t_1}^{t_2}\int_{\mathbb{R}^2} n^\varepsilon\,dx = 0\ .$$

$\square$

3.2 L^p uniform estimates

Here we prove that if the initial data n_0 is bounded in $L^p(\mathbb{R}^2)$, then it is also the case for the solution $n(\cdot,t)$ for any finite positive time t. By uniform, we mean estimates that hold up to $t=0$.

Proposition II.17. *Assume that (II.4) and $M < 8\pi/\chi$ hold. If n_0 is bounded in $L^p(\mathbb{R}^2)$ for some $p>1$, then any solution n of (II.1) is bounded in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^p(\mathbb{R}^2))$.*

Proof – We formally compute

$$\begin{aligned}
\frac{1}{2(p-1)}\frac{d}{dt}\int_{\mathbb{R}^2}|n(x,t)|^p\,dx &= -\frac{2}{p}\int_{\mathbb{R}^2}|\nabla(n^{p/2})|^2\,dx+\chi\int_{\mathbb{R}^2}\nabla(n^{p/2})\cdot n^{p/2}\cdot\nabla c\,dx\\
&= -\frac{2}{p}\int_{\mathbb{R}^2}|\nabla(n^{p/2})|^2\,dx+\chi\int_{\mathbb{R}^2}n^p\,(-\Delta c)\,dx\\
&= -\frac{2}{p}\int_{\mathbb{R}^2}|\nabla(n^{p/2})|^2\,dx+\chi\int_{\mathbb{R}^2}n^{p+1}\,dx\,. \qquad \text{(II.17)}
\end{aligned}$$

Using the following Gagliardo-Nirenberg-Sobolev inequality:

$$\int_{\mathbb{R}^2}|v|^{2(1+1/p)}\,dx \le K_p\int_{\mathbb{R}^2}|\nabla v|^2\,dx\int_{\mathbb{R}^2}|v|^{2/p}\,dx\,,$$

or equivalently, with $n=v^{2/p}$,

$$\int_{\mathbb{R}^2}|n|^{p+1}\,dx \le K_p\int_{\mathbb{R}^2}|\nabla(n^{p/2})|^2\,dx\int_{\mathbb{R}^2}|n|\,dx\,,$$

we get the estimate

$$\frac{1}{2(p-1)}\frac{d}{dt}\int_{\mathbb{R}^2}n^p\,dx \le \int_{\mathbb{R}^2}|\nabla(n^{p/2})|^2\,dx\left(-\frac{2}{p}+K_p\,\chi\,M\right),$$

which proves the decay of $\int_{\mathbb{R}^2}n^p\,dx$ if $M<\frac{2}{p\,K_p\,\chi}$. Otherwise, we can rely on the entropy estimate to get a bound: Let $K>1$ be a constant, to be chosen later.

$$\int_{\mathbb{R}^2}n^p\,dx=\int_{n\le K}n^p\,dx+\int_{n>K}n^p\,dx\,.$$

The first term of the right hand side is bounded by $K^{p-1}M$. Concerning the second one, define first

$$M(K):=\int_{n>K}n\,dx\,.$$

Using the fact that $|n\,\log n|$ is bounded in $L^\infty(\mathbb{R}^+_{\mathrm{loc}};L^1(\mathbb{R}^2))$, we can estimate $M(K)$ by

$$M(K)\le\frac{1}{\log K}\int_{n>K}n\,\log n\,dx\le\frac{1}{\log K}\int_{\mathbb{R}^2}|n\,\log n|\,dx$$

and choose it arbitrarily small on any given time interval $(0,T)$. Compute now

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}(n-K)_+^p\,dx+\frac{4}{p}\,(p-1)\int_{\mathbb{R}^2}|\nabla((n-K)_+^{p/2})|^2\,dx \qquad\qquad \text{(II.18)}\\
&= \;p\int_{\mathbb{R}^2}(n-K)_+^{p-1}\,[\Delta n-\chi\,\nabla(n\,\nabla c)]\;dx+\frac{4}{p}\,(p-1)\int_{\mathbb{R}^2}|\nabla((n-K)_+^{p/2})|^2\,dx\\
&= \;-p\,\chi\int_{\mathbb{R}^2}(n-K)_+^{p-1}\,[\nabla(n-K)\cdot\nabla c+n\,\Delta c]\;dx\\
&= \;-\chi\int_{\mathbb{R}^2}(n-K)_+^{p-1}\,[(n-K)_+\,(-\Delta c)-p\,n\,(-\Delta c)]\;dx\\
&= \;+(p-1)\,\chi\int_{\mathbb{R}^2}(n-K)_+^{p+1}\,dx+\;(2p-1)\,\chi\,K\int_{\mathbb{R}^2}(n-K)_+^p\,dx\\
&\qquad\qquad +p\,\chi\,K^2\int_{\mathbb{R}^2}(n-K)_+^{p-1}\,dx
\end{aligned}$$

The term involving $\int_{\mathbb{R}^2}(n-K)_+^{p-1}\,dx$ can be estimated as follows:

$$\int_{\mathbb{R}^2}(n-K)_+^{p-1}\,dx=\int_{K<n\le K+1}(n-K)_+^{p-1}\,dx+\int_{n>K+1}(n-K)_+^{p-1}\,dx\;,$$
$$\int_{K<n\le K+1}(n-K)_+^{p-1}\,dx\le\int_{K<n\le K+1}1\,dx\le\frac{1}{K}\int_{K<n\le K+1}n\,dx\le\frac{M}{K}\;,$$
$$\int_{n>K+1}(n-K)_+^{p-1}\,dx\le\int_{n>K+1}(n-K)_+^{p}\,dx\le\int_{\mathbb{R}^2}(n-K)_+^{p}\,dx\;.$$

By choosing K sufficiently large, we obtain

$$-\frac{4}{p}\,(p-1)\int_{\mathbb{R}^2}|\nabla((n-K)_+^{p/2})|^2\,dx+(p-1)\,\chi\int_{\mathbb{R}^2}(n-K)_+^{p+1}\,dx\le 0$$

using again the Gagliardo-Nirenberg-Sobolev inequality but with M replaced by $M(K)$, small. Summarizing, for a fixed interval $(0,T)$ with T arbitrarily large, we have found K such that

$$\frac{d}{dt}\int_{\mathbb{R}^2}(n-K)_+^p\,dx\le C_1\int_{\mathbb{R}^2}(n-K)_+^p\,dx+C_2$$

for some positive constants C_1 and C_2. A Gronwall estimate shows that $\int_{\mathbb{R}^2}(n-K)_+^p\,dx$ is finite on $(0,T)$.

To justify this estimate, one has as above to establish it for the regularized problem and then pass to the limit. This is purely technical but not difficult and we leave it to the reader.

To conclude, we still need to check that the bound on $\int_{\mathbb{R}^2}(n-K)_+^p\,dx$ is enough to control $\int_{n>K}n^p\,dx$. Using the estimate

$$x^p\le\Big(\frac{\lambda}{\lambda-1}\Big)^{p-1}(x-1)^p$$

for any $x\ge\lambda>1$, we get

$$\begin{aligned}
\int_{n>K}n^p\,dx &= \int_{K<n\le\lambda K}n^p\,dx+\int_{n>\lambda K}n^p\,dx\\
&\le (\lambda\,K)^{p-1}\,M+\Big(\frac{\lambda}{\lambda-1}\Big)^{p-1}K^p\int_{n>\lambda K}\Big(\frac{n}{K}-1\Big)^p\,dx\\
&\le (\lambda\,K)^{p-1}\,M+\Big(\frac{\lambda}{\lambda-1}\Big)^{p-1}\int_{\mathbb{R}^2}(n-K)_+^p\,dx\;.
\end{aligned}$$

□

Notice that very similar estimates have been derived, without the knowledge of the optimal bound $\chi M < 8\pi$, by Jäger and Luckhaus in [JL92] in $\mathbb{R}^d$, $d = 2$ (also see [CPZ04] if $d \geq 2$), by working directly in an L^p-framework, instead of the free energy framework.

3.3 Free energy inequality for n_0 in $L^{2+\varepsilon}(\mathbb{R}^2)$

Using the *a priori* estimates of the previous section for $p = 2 + \varepsilon$, we can prove that the free energy inequality (II.2) holds.

Lemma II.18. *Let n_0 be in a bounded set in $L^1_+(\mathbb{R}^2, (1+|x|^2)dx) \cap L^{2+\varepsilon}(\mathbb{R}^2, dx)$, for some $\varepsilon > 0$, eventually small. Then n_0 satisfies Assumption (II.4), the solution n of (II.1) found in Theorem II.1, with initial data n_0, is in a compact set in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$ and moreover the entropy production estimate (II.2) holds:*

$$F[n] + \int_0^t \left(\int_{\mathbb{R}^2} n \left| \nabla (\log n) - \chi \nabla c \right|^2 \, dx \right) ds \leq F[n_0] \, .$$

Proof – We will split the proof in several steps

First Step: n is bounded in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$. To apply Theorem II.1, we need to prove that $n_0 \log n_0$ is integrable. By Hölder's inequality we have

$$\|u\|_{L^q(\mathbb{R}^2)} \leq \|u\|^{\alpha}_{L^p(\mathbb{R}^2)} \, \|u\|^{1-\alpha}_{L^r(\mathbb{R}^2)} \tag{II.19}$$

with $\alpha = \frac{p}{q} \frac{r-q}{r-p}$, $p \leq q \leq r$. Take the logarithm of both sides:

$$\alpha \log \left(\frac{\|u\|_{L^q(\mathbb{R}^2)}}{\|u\|_{L^p(\mathbb{R}^2)}} \right) + (\alpha - 1) \log \left(\frac{\|u\|_{L^r(\mathbb{R}^2)}}{\|u\|_{L^q(\mathbb{R}^2)}} \right) \leq 0 \, .$$

Since this inequality trivializes to an equality when $q = p$, we may differentiate it with respect to q at $q = p$ and get that for any $u \in L^p(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$, $1 \leq p < r < +\infty$, we have

$$\int u^p \log \left(\frac{|u|}{\|u\|_{L^p(\mathbb{R}^2)}} \right) \, dx \leq \frac{r}{r-p} \, \|u\|^p_{L^p(\mathbb{R}^2)} \log \left(\frac{\|u\|_{L^r(\mathbb{R}^2)}}{\|u\|_{L^p(\mathbb{R}^2)}} \right) \, .$$

With $u = n_0$, $p = 1$ and $r = 2 + \varepsilon$, by applying Lemma II.8, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^2} n_0 \left| \log n_0 \right| \, dx \\ &\quad \leq \frac{M}{1+\varepsilon} \left[(2+\varepsilon) \log(\|n_0\|_{L^{2+\varepsilon}(\mathbb{R}^2)}) - \log M + 2 \log(2\pi) \right] + \int_{\mathbb{R}^2} |x|^2 \, n_0 \, dx + \frac{2}{e} < \infty \, . \end{aligned}$$

Since $n_0 \in L^1 \cap L^{2+\varepsilon}(\mathbb{R}^2)$, by Hölder's interpolation (II.19), n_0 is initially in any $L^q(\mathbb{R}^2)$ for all $q \in [1, 2+\varepsilon]$, and as a special case in $L^2(\mathbb{R}^2)$:

$$\|n_0\|^2_{L^2(\mathbb{R}^2)} \leq \|n_0\|^{\varepsilon/(1+\varepsilon)}_{L^1(\mathbb{R}^2)} \, \|n_0\|^{1/(1+\varepsilon)}_{L^{2+\varepsilon}(\mathbb{R}^2)} \, .$$

Hence by Theorem II.1, the solution n of (II.1) is bounded in $L^\infty(\mathbb{R}^+_{\mathrm{loc}}; L^1 \cap L^{2+\varepsilon}(\mathbb{R}^2))$. As a special case n is bounded in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$.

Second Step: ∇n is bounded in $L^2(\mathbb{R}^+_{\rm loc}\times\mathbb{R}^2)$. The following computation

$$\frac{d}{dt}\int_{\mathbb{R}^2} n^2\,dx = -2\int_{\mathbb{R}^2}|\nabla n|^2\,dx + 2\chi\int_{\mathbb{R}^2}\nabla n\cdot n\,\nabla c\,dx$$

shows that $X := \|\nabla n\|_{L^2((0,T)\times\mathbb{R}^2)}$ satisfies the estimate

$$2\,X^2 - 2\,\chi\,\|n\,\nabla c\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}\,X \le \|n\|^2_{L^\infty(0,T;L^2(\mathbb{R}^2))} + \|n_0\|^2_{L^2(\mathbb{R}^2)}\,.$$

This implies that X is bounded if $\|n\,\nabla c\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}$ is bounded. Let us prove that this is indeed the case. The drift force term takes the form

$$\nabla c(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}\,n(y,t)\,dy\,.$$

Since $n_0\in L^{2+\varepsilon}(\mathbb{R}^2)$, by Theorem II.1, the solution n of is bounded in $L^\infty(\mathbb{R}^+_{\rm loc};L^{2+\varepsilon}(\mathbb{R}^2))$. As a consequence of (II.20), for any $(p_1,q_1)\in(2,+\infty)\times(1,2)$ such that $\frac{1}{p_1}=\frac{1}{q_1}-\frac{1}{2}$, there exists a constant $C=C(p_1)>0$ such that for almost any $t>0$,

$$\|\nabla c(\cdot,t)\|_{L^{p_1}(\mathbb{R}^2)} \le C\,\|n(\cdot,t)\|_{L^{q_1}(\mathbb{R}^2)}\,.$$

We can indeed evaluate $\|f*|\cdot|^{-\lambda}\|_{L^{p_1}(\mathbb{R}^d)}$ by

$$\|f*|\cdot|^{-\lambda}\|_{L^{p_1}(\mathbb{R}^d)} = \sup_{\substack{g\in L^{q_1}(\mathbb{R}^d)\\ \|g\|_{L^{q_1}(\mathbb{R}^d)}\le 1}}\int_{\mathbb{R}^d}\left(f*|\cdot|^{-\lambda}\right)\,g\,dx$$

with $\frac{1}{p_1}+\frac{1}{q_1}=1$. The right hand side is bounded, up to a multiplicative constant, by $\|f\|_{L^p(\mathbb{R}^2)}$ according to the Hardy-Littlewood-Sobolev inequality, if $\frac{1}{p}+\frac{1}{q_1}+\frac{\lambda}{d}=2$ and $0<\lambda<d$. This inequality, see [HL28, HL30, Sob38, Sob63, dP55, Lie83], indeed states that: *For all $f\in L^p(\mathbb{R}^d)$, $g\in L^q(\mathbb{R}^d)$, $1<p,\,q<\infty$ such that $\frac{1}{p}+\frac{1}{q}+\frac{\lambda}{d}=2$ and $0<\lambda<d$, there exists a constant $C=C(p,q,\lambda)>0$ such that*

$$\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{|x-y|^\lambda}\,f(x)\,g(y)\,dx\,dy\right| \le C\|f\|_{L^p(\mathbb{R}^d)}\,\|g\|_{L^q(\mathbb{R}^d)}\,. \tag{II.20}$$

Applied with $\lambda=1$, $d=2$, this proves the claim.

Applying this estimate with $p_1=2(1+2/\varepsilon)$ and $q_1=2-\varepsilon/(1+\varepsilon)$, and using Hölder's inequality, we can write

$$\|n(\cdot,t)\,\nabla c(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le \|n(\cdot,t)\|_{2+\varepsilon}\|\nabla c(\cdot,t)\|_{p_1} \le C\,\|n(\cdot,t)\|_{2+\varepsilon}\|n(\cdot,t)\|_{q_1}\,.$$

which is bounded as $q_1\in[1,2+\varepsilon]$. Thus, if n is a solution of (II.1), $n\nabla c$ is bounded in $L^\infty(\mathbb{R}^+_{\rm loc};L^2(\mathbb{R}^2))$.

Third Step: Compactness. As a consequence of Hölder's inequality with $p:=(1+\varepsilon)/\varepsilon$, $q:=1+\varepsilon$:

$$\int_{\mathbb{R}^2}|x|^{\frac{2\,\varepsilon}{1+\varepsilon}}\,n^2\,dx = \int_{\mathbb{R}^2}(n\,|x|^2)^{\frac{\varepsilon}{1+\varepsilon}}\cdot n^{\frac{2+\varepsilon}{1+\varepsilon}}\,dx \le \left(\int_{\mathbb{R}^2}n\,|x|^2\,dx\right)^{\frac{\varepsilon}{1+\varepsilon}}\left(\int_{\mathbb{R}^2}n^{2+\varepsilon}\,dx\right)^{\frac{1}{1+\varepsilon}}\,,$$

the function $(x,t)\mapsto|x|^{\frac{\varepsilon}{1+\varepsilon}}\,n$ is bounded in $L^\infty(\mathbb{R}^+_{\rm loc};L^2(\mathbb{R}^2))$. The imbedding of the set $V:=\{u\in H^1\cap L^1_+(\mathbb{R}^2)\,:\,|x|^{\frac{\varepsilon}{1+\varepsilon}}\,u\in L^1(\mathbb{R}^2)\}$ into $L^2(\mathbb{R}^2)=:H$ is compact and by the

Aubin-Lions compactness method (see Lemma II.11) as in Section 2.5, it results that n belongs to a compact set of $L^2(\mathbb{R}^+_{\rm loc}\times\mathbb{R}^2)$.

Let $(n_k)_{k\in\mathbb{N}} := (n^{\varepsilon_k})_{k\in\mathbb{N}}$ be an approximating sequence defined as in the proof of Theorem II.1. Compared to the results of Lemma II.14, we have

$$\iint_{[0,T]\times\mathbb{R}^2} |\nabla n|^2\,dx\,dt \le \liminf_{k\to\infty} \iint_{[0,T]\times\mathbb{R}^2} |\nabla n_k|^2\,dx\,dt\;,$$
$$\iint_{[0,T]\times\mathbb{R}^2} n\,|\nabla c|^2\,dx\,dt \le \liminf_{k\to\infty} \iint_{[0,T]\times\mathbb{R}^2} n_k\,|\nabla c_k|^2\,dx\,dt\;,$$
$$\iint_{[0,T]\times\mathbb{R}^2} n^2\,dx\,dt = \liminf_{k\to\infty} \iint_{[0,T]\times\mathbb{R}^2} |n_k|^2\,dx\,dt\;,$$

where the only difference lies in the last equality, a consequence of the above compactness result. This proves the free energy estimate using

$$\iint_{[[0,T]\times\mathbb{R}^2} n\left|\nabla\left(\log n\right)-\chi\nabla c\right|^2\,dx\,dt = 4\iint_{[[0,T]\times\mathbb{R}^2}|\nabla\sqrt{n}|^2\,dx\,dt+\chi^2\iint_{[[0,T]\times\mathbb{R}^2} n\,|\nabla c|^2\,dx\,dt-2\chi\iint_{[[0,T]\times\mathbb{R}^2} n^2\,dx\,dt\;.$$

3.4 Hypercontractivity

Much more regularity can actually be achieved as follows. All computations are easy to justify for smooth solutions with sufficient decay at infinity. Up to a regularization step, the final estimates certainly hold if the initial data is bounded in $L^\infty(\mathbb{R}^2)$, which is the case for the regularized problem of Section 2.5 with truncated initial data $n_0^\varepsilon = \min\{n_0,\varepsilon^{-1}\}$. However, we will see that the $L^\infty(\mathbb{R}^+_{\rm loc};L^p(\mathbb{R}^2))$-estimates hold for any $p>1$ independently of ε, so that we may pass to the limit and get the result for any solution of (II.1) with initial data satisfying only (II.4) and $\chi M<8\pi$. To simplify the presentation of the method, we will therefore do the computations only at a formal level, for smooth solutions which behave well at infinity.

Theorem II.19. *Consider a solution n of (II.1) with initial data n_0 satisfying (II.4) and $\chi M<8\pi$. Then for any $p\in(1,\infty)$, there exists a continuous function h_p on $(0,\infty)$ such that for almost any $t>0$, $\|n(\cdot,t)\|_{L^p(\mathbb{R}^2)}\le h_p(t)$.*

Notice that unless n_0 is bounded in $L^p(\mathbb{R}^2)$, $\lim_{t\to0_+} h_p(t)=+\infty$. Such a result is called an *hypercontractivity* result since to an initial data which is originally in $L^1(\mathbb{R}^2)$ but not in $L^p(\mathbb{R}^2)$, we associate a solution which at almost any time $t>0$ is in $L^p(\mathbb{R}^2)$ with p arbitrarily large.

Proof – Fix $t>0$ and $p\in(1,\infty)$ and consider $q(s)=1+(p-1)\,\frac{s}{t}$, so that $q(0)=1$ and $q(t)=p$. Exactly as in the proof of Theorem II.1, for an arbitrarily small $\eta>0$ given in advance, we can find $K>1$ big enough such that $M(K):=\sup_{s\in(0,t)}\int_{n>K} n(\cdot,s)\,dx$ is smaller than η. It is indeed sufficient to notice that

$$\int_{n>K} n(\cdot,s)\,dx \le \frac{1}{\log K}\int_{n>K} n(\cdot,s)\,\log n(\cdot,s)\,dx \le \frac{1}{\log K}\int_{\mathbb{R}^2} |n(\cdot,s)\,\log n(\cdot,s)|\,dx\;.$$

Since $\chi M<8\pi$, $n\,|\log n|$ is bounded in $L^\infty(0,t;L^1(\mathbb{R}^2))$. This proves that for K big enough, we may assume

$$\int_{\mathbb{R}^2}(n-K)_+\,dx \le \eta\;,$$

for an arbitrarily small $\eta > 0$.

Next, we define

$$F(s) := \left[\int_{\mathbb{R}^2} (n-K)_+^{q(s)}(x,s)\,dx\right]^{1/q(s)}$$

for the function $s \mapsto q(s)$, with values in $[1,\infty)$. A derivation with respect to s gives

$$F'\,F^{q-1} = \frac{q'}{q^2}\int_{\mathbb{R}^2}(n-K)_+^q \log\left(\frac{(n-K)_+^q}{F^q}\right) + \int_{\mathbb{R}^2} n_t\,(n-K)_+^{q-1}\ .$$

If n is a solution to (II.1), then

$$\int_{\mathbb{R}^2}(n-K)_+^{q-1}\,n_t\,dx = -4\,\frac{q-1}{q^2}\int_{\mathbb{R}^2}|\nabla((n-K)_+^{q/2})|^2\,dx + \chi\,\frac{q-1}{q}\int_{\mathbb{R}^2}(n-K)_+^{q+1}\,dx\ ,$$

and we get

$$F'\,F^{q-1} = \frac{q'}{q^2}\int_{\mathbb{R}^2}(n-K)_+^q \log\left(\frac{(n-K)_+^q}{F^q}\right) \\ -4\,\frac{q-1}{q^2}\int_{\mathbb{R}^2}|\nabla((n-K)_+^{q/2})|^2 + \chi\frac{(q-1)}{q}\int_{\mathbb{R}^2}(n-K)_+^{q+1}\ .$$

Using the assumption $q' \geq 0$, we can apply the logarithmic Sobolev inequality [Gro75]

$$\int_{\mathbb{R}^2} v^2\,dx \log\left(\frac{v^2}{\int_{\mathbb{R}^2} v^2\,dx}\right)\,dx \leq 2\,\sigma\int_{\mathbb{R}^2}|\nabla v|^2\,dx - (2+\log(2\,\pi\,\sigma))\int_{\mathbb{R}^2} v^2\,dx$$

for any $\sigma > 0$, and the Gagliardo-Nirenberg-Sobolev inequality

$$\int_{\mathbb{R}^2}|v|^{2(1+1/q)}\,dx \leq \mathcal{K}(q)\,\|\nabla v\|^2_{L^2(\mathbb{R}^2)}\int_{\mathbb{R}^2}|v|^{2/q}\,dx\ , \quad \forall\, q\in[2,\infty)$$

to $v := (n-K)_+^{q/2}$, and get

$$F'\,F^{q-1} \leq \left(\frac{2\,\sigma\,q'}{q^2} - 4\,\frac{q-1}{q^2} + \chi\,\frac{q-1}{q}\mathcal{K}(q)\,\eta\right)\|\nabla v\|^2_{L^2(\mathbb{R}^2)} - \frac{q'}{q^2}\,(2+\log(2\,\pi\,\sigma))\,F^q\,.$$

With the specific choice of $\sigma := (q-1)/q'$ and provided η is chosen small enough in order that

$$-2\,\frac{q-1}{q^2} + \chi\,\frac{q-1}{q}\sup_{r\in(1,p)}[\mathcal{K}(r)]\,\eta \leq 0\ ,$$

this shows that

$$\frac{F'}{F} \leq -\frac{q'}{q^2}\,(2+\log(2\,\pi\,\sigma)) =: G(t)\ .$$

The function G is integrable on $(0,t)$, which proves that $F(t)$ can be bounded in terms of $F(0)$. $\square$

3.5 The free energy inequality for weak solutions

As a consequence of Lemma II.18 and Theorem II.19, we have the following result.

Corollary II.20. *Let $(n^k)_{k\in\mathbb{N}}$ be a sequence of solutions of (II.1) with initial data n_0^k satisfying Assumption (II.4) with uniform corresponding bounds. For any $t_0 > 0$, $T \in \mathbb{R}^+$ such that $0 < t_0 < T$, $(n^k)_{k\in\mathbb{N}}$ is relatively compact in $L^2((t_0,T)\times\mathbb{R}^2)$, and if n is the limit of $(n^k)_{k\in\mathbb{N}}$, then n is a solution of (II.1) such that the free energy inequality (II.2) holds.*

Proof – By Theorem II.19, for $t > t_0 > 0$, n^k is bounded in $L^\infty(t_0,t;L^{2+\varepsilon}(\mathbb{R}^2))$, for any $\varepsilon > 0$. We can therefore apply Lemma II.18 with initial data $n^k(\cdot,t_0)$ at $t = t_0$:

$$F[n^k(\cdot,t)] + \int_{t_0}^t \left(\int_{\mathbb{R}^2} n^k \left|\nabla\left(\log n^k\right) - \chi\nabla c^k\right|^2 \,dx\right) ds \le F[n^k(\cdot,t_0)] \,.$$

The compactness in $L^2([t_0,t]\times\mathbb{R}^2)$ follows from Lemma II.11. Passing to the limit as $k\to\infty$, we get

$$F[n(\cdot,t)] + \int_{t_0}^t \left(\int_{\mathbb{R}^2} n\left|\nabla\left(\log n\right) - \chi\nabla c\right|^2 \,dx\right) ds \le F[n(\cdot,t_0)] \,.$$

Since, as a function of s, $\int_{\mathbb{R}^2} n(\cdot,s)\left|\nabla\left(\log n(\cdot,s)\right) - \chi\nabla c(\cdot,s)\right|^2 dx$ is integrable on $(0,t)$, we can pass to the limit $t_0 \to 0$. By convexity of $n \mapsto n\,\log n$, it is easy to check that $\lim_{t_0\to 0_+} F[n(\cdot,t_0)] \le F[n_0]$. □

Apply Corollary II.20 with $n_0^k = \min\{n_0, {\varepsilon_k}^{-1}\}$ as in the regularization procedure of Section 2.5–2.7. This completes the proof of Theorem II.1.

4 Intermediate asymptotics and self-similar solutions

In this section, we investigate the behavior of the solutions as time t goes to infinity.

4.1 Self-similar variables

Assume that $\chi M < 8\pi$, consider a solution of (II.1) and define the rescaled functions u and v by:

$$n(x,t) = \frac{1}{R^2(t)}\,u\left(\frac{x}{R(t)},\tau(t)\right) \quad\text{and}\quad c(x,t) = v\left(\frac{x}{R(t)},\tau(t)\right) \tag{II.21}$$

with

$$R(t) = \sqrt{1+2t} \quad\text{and}\quad \tau(t) = \log R(t)\,.$$

The rescaled system is

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \nabla\cdot(u(x-\chi\nabla v)) & x\in\mathbb{R}^2\,,\ t>0\,,\\[2ex] -\Delta v = u & x\in\mathbb{R}^2\,,\ t>0\,,\\[2ex] u(\cdot,t=0) = n_0 \ge 0 & x\in\mathbb{R}^2\,. \end{cases} \tag{II.22}$$

The free energy now takes the form

$$F^R[u] := \int_{\mathbb{R}^2} u \log u \, dx - \frac{\chi}{2} \int_{\mathbb{R}^2} u\, v \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \, u \, dx \,.$$

If (u, v) is a smooth solution of (II.22) which decays sufficiently at infinity, then

$$\frac{d}{dt} F^R[u(\cdot,t)] = - \int_{\mathbb{R}^2} u \, \left|\nabla \log u - \chi \nabla v + x\right|^2 \, dx \,.$$

Because of the hypercontractivity, the above inequality holds as an inequality for the solution of Theorem II.1 after rescaling:

$$\frac{d}{dt} F^R[u(\cdot,t)] \leq - \int_{\mathbb{R}^2} u \, \left|\nabla \log u - \chi \nabla v + x\right|^2 \, dx \,.$$

For a rigorous proof, one has to redo the argument. Since there is no additional difficulty compared to Section 3.4, this is left to the reader.

4.2 The self-similar solution

System (II.22) has the interesting property that for $\chi\, M < 8\pi$, it has a stationary solution which minimizes the free energy.

Lemma II.21. *The functional F^R is bounded from below on the set*

$$\left\{ u \in L^1_+(\mathbb{R}^2) \; : \; |x|^2 \, u \in L^1(\mathbb{R}^2) \,, \; \int_{\mathbb{R}^2} u \log u \, dx < \infty \right\}$$

if and only if $\chi \, \|u\|_{L^1(\mathbb{R}^2)} \leq 8\pi$.

Proof – If $\chi \, \|u\|_{L^1(\mathbb{R}^2)} \leq 8\pi$, the result is a straightforward consequence of Lemma II.6. Notice that by Lemma II.13, (iii), $u \log u$ is then bounded in $L^1(\mathbb{R}^2)$.

The functional $F^R[u]$ has an interesting scaling property. For a given u, let $u_\lambda(x) = \lambda^{-2} u(\lambda^{-1} x)$. It is straightforward to check that $\|u_\lambda\|_{L^1(\mathbb{R}^2)} =: M$ does not depend on $\lambda > 0$ and

$$F^R[u_\lambda] = F^R[u] - 2M \left(1 - \frac{\chi\, M}{8\pi}\right) \log \lambda + \frac{\lambda - 1}{2} \int_{\mathbb{R}^2} |x|^2 \, u \, dx \,.$$

As a function of λ, $F^R[u_\lambda]$ is clearly bounded from below if $\chi\, M < 8\pi$, and not bounded from below if $\chi\, M > 8\pi$, which completes the proof. □

The free energy has a minimum which is a radial stationary solution of (II.22), see [CLMP92]. Such a solution is of course a natural candidate for the large time asymptotics of any solution of (II.22).

Lemma II.22. *Let $\chi\, M < 8\pi$. If u is a solution of (II.22), with initial data u_0 satisfying Assumptions (II.4), corresponding to a solution of (II.1) as given in Theorem II.1, then as $t \to \infty$, $u(x, t+s)$ converges almost everywhere with respect to $(x, s) \in \mathbb{R}^2 \times (0, 1)$ to a solution of (II.3) which is a stationary solution of (II.22) and moreover satisfies:*

$$\lim_{t \to \infty} \int_{\mathbb{R}^2} |x|^2 \, u(x,t) \, dx = \int_{\mathbb{R}^2} |x|^2 \, u_\infty \, dx = 2M \left(1 - \frac{\chi\, M}{8\pi}\right) \,. \tag{II.23}$$

Proof – We use the entropy production term:

$$F^R[u_0] - \liminf_{t\to\infty} F^R[u(\cdot,t)] = \lim_{t\to\infty} \int_0^t \left(\int_{\mathbb{R}^2} u \, |\nabla \log u - \chi \nabla v + x|^2 \, dx \right) ds \, .$$

As a consequence,

$$\lim_{t\to\infty} \int_t^\infty \left(\int_{\mathbb{R}^2} u \, |\nabla \log u - \chi \nabla v + x|^2 \, dx \right) ds = 0 \, ,$$

which shows that, up to the extraction of subsequences, the limit u_∞ of $u(\cdot, t+\cdot)$, which exists for the same reasons as in the proof of Theorem II.1, satisfies

$$\nabla \log u_\infty - \chi \nabla v_\infty + x = 0 \, , \quad v_\infty = -\frac{1}{2\pi} \log|\cdot| * u_\infty \, ,$$

where the first equation holds at least a.e. in the support of u_∞. This is equivalent to write that (u_∞, v_∞) solves (II.3).

As in the proof of Lemma II.3, consider a smooth function $\varphi_\varepsilon(|x|)$ with compact support that grows nicely to $|x|^2$ as $\varepsilon \to 0$. If (u, v) is a solution to (II.22), we compute

$$\frac{d}{dt} \int_{\mathbb{R}^2} \varphi_\varepsilon \, u \, dx = \int_{\mathbb{R}^2} \Delta \varphi_\varepsilon \, u \, dx - \frac{\chi}{4\pi} \int_{\mathbb{R}^2} \frac{(\nabla \varphi_\varepsilon(x) - \nabla \varphi_\varepsilon(y)) \cdot (x-y)}{|x-y|^2} \, u(x,t) \, u(y,t) \, dx \, dy - 2 \int_{\mathbb{R}^2} |x|^2 \, u \, dx \, .$$

As ε vanishes we may pass to the limit and obtain

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 \, u \, dx = 4M \left(1 - \frac{\chi M}{8\pi} \right) - 2 \int_{\mathbb{R}^2} |x|^2 \, u \, dx \, .$$

This proves that for any $t > 0$,

$$\int_{\mathbb{R}^2} |x|^2 \, u(x,t) \, dx = \int_{\mathbb{R}^2} |x|^2 \, n_0 \, dx \, e^{-2t} + 2M \left(1 - \frac{\chi M}{8\pi} \right) (1 - e^{-2t}) \, .$$

Passing to the limit $t \to \infty$, we get

$$\int_{\mathbb{R}^2} |x|^2 \, u_\infty \, dx \leq 2M \left(1 - \frac{\chi M}{8\pi} \right) \, .$$

However, u_∞ is a solution of Equation (II.22), which satisfies the same assumptions as n_0. Since u_∞ is also a solution of Equation (II.3), it satisfies: $\partial_t u_\infty = 0$, which means that u_∞ is a stationary solution with finite second moments, and, as a consequence,

$$\int_{\mathbb{R}^2} |x|^2 \, u_\infty \, dx = 2M \left(1 - \frac{\chi M}{8\pi} \right) \, .$$

□

Notice that under the constraint $\|u_\infty\|_{L^1(\mathbb{R}^2)} = M$, u_∞ is a critical point of the free energy but we did not prove that u_∞ is a minimizer of the free energy. As a first step in this direction, we can prove that it is radially symmetric. This can be done using the two following results, Lemma II.23 and II.24.

Lemma II.23. *Let $u \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)$ with $M := \int_{\mathbb{R}^2} u\,dx$, such that $\int_{\mathbb{R}^2} u\,\log u\,dx < \infty$, and define*

$$v(x) := -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\,u(y)\,dy\;.$$

Then there exists a positive constant C such that, for any $x \in \mathbb{R}^2$ with $|x| > 1$,

$$\left|\;v(x) + \frac{M}{2\pi}\,\log|x|\;\right| \le C\;.$$

Notice that as a straightforward consequence, v is non-positive outside of a ball.
Proof – We estimate

$$\left|v(x) + \frac{M}{2\,\pi}\,\log|x|\right| = \left|\frac{1}{2\,\pi}\int_{\mathbb{R}^2}\log\left(\frac{|x-y|}{|x|}\right)u(y)\;dy\right| \le \text{(I)} + \text{(II)} + \text{(III)}$$

by

$$\begin{aligned}
\text{(I)} &:= -\frac{1}{2\,\pi}\int_{\Omega_{\rm I}}\log\left(\frac{|x-y|}{|x|}\right)u(y)\;dy \quad \text{with } \Omega_{\rm I} := \left\{(x,y)\in\mathbb{R}^2\;:\;\frac{|x-y|}{|x|}\le\frac{1}{2}\right\}\\
\text{(II)} &:= \left|\frac{1}{2\,\pi}\int_{\Omega_{\rm II}}\log\left(\frac{|x-y|}{|x|}\right)u(y)\;dy\right| \quad \text{with } \Omega_{\rm II} := \left\{(x,y)\in\mathbb{R}^2\;:\;\frac{1}{2}<\frac{|x-y|}{|x|}\le 2\right\}\\
\text{(III)} &:= \frac{1}{2\,\pi}\int_{\Omega_{\rm III}}\log\left(\frac{|x-y|}{|x|}\right)u(y)\;dy \quad \text{with } \Omega_{\rm III} := \left\{(x,y)\in\mathbb{R}^2\;:\;\frac{|x-y|}{|x|}>2\right\}\;.
\end{aligned}$$

Using $|x-y|^2 \le 2(|x|^2+|y|^2)$ and $\log(1+t)\le t$, we get

$$4\,\pi\,\text{(III)} = \int_{\Omega_{\rm III}}\log\left(\frac{|x-y|^2}{|x|^2}\right)u(y)\;dy \le \int_{\Omega_{\rm III}}\log\left(2+2\,\frac{|y|^2}{|x|^2}\right)u(y)\;dy \le M + \frac{2}{|x|^2}\int_{\Omega_{\rm III}}|y|^2 u(y)\;dy\;.$$

On $\Omega_{\rm II}$, $|\log(|x-y|/|x|)|$ is bounded by $\log 2$: (II) $\le M\,\frac{\log 2}{2\,\pi}$. For the last term, denote $z_x(y) = \frac{|x|}{|x-y|}$:

$$\text{(I)} = \frac{1}{2\,\pi}\int_{\Omega_{\rm I}}\log\left(z_x(y)\right)u(y)\;dy\;.$$

By Jensen's inequality

$$\int_{\Omega_{\rm I}} u(y)\log\left(\frac{u(y)}{z_x(y)}\right)\;dy \ge \int_{\Omega_{\rm I}} u(y)\log\left(\frac{\int_{\Omega_{\rm I}} u(y)\;dy}{\int_{\Omega_{\rm I}} z_x(y)\;dy}\right)\;dy\;,$$

we get

$$2\,\pi\,\text{(I)} \le \int_{\Omega_{\rm I}} u(y)\log\left(u(y)\right)\;dy - \int_{\Omega_{\rm I}} u(y)\log\left(\frac{\int_{\Omega_{\rm I}} u(y)\;dy}{\int_{\Omega_{\rm I}} z_x(y)\;dy}\right)\;dy\;.$$

The right hand side is bounded since $u\log\,u$ is bounded in $L^1(\mathbb{R}^2)$ by Lemma II.8,

$$\int_{\Omega_{\rm I}} z_x(y)\;dy = \int_{\Omega_{\rm I}}\frac{|x|}{|x-y|}\;dy = \pi\,|x|^2\;,$$

and

$$\int_{\Omega_{\rm I}} u(y)\;dy \le \frac{4}{|x|^2}\int_{\Omega_{\rm I}}|y|^2 u(y)\;dy\;.$$

Hence we can control (I) because $\int_{\Omega_{\rm I}} u(y)\;dy\log\left(\int_{\Omega_{\rm I}} z_x(y)\;dy\right) \le \frac{4}{|x|^2}\log\left(\pi\,|x|^2\right)$. □

This is enough to prove that the solution is radially symmetric, see [Nai01, NS04].

Lemma II.24. [Nai01] *Assume that V is a non-negative non-trivial radial function on $\mathbb{R}^2$ such that $\lim_{|x|\to\infty} |x|^\alpha V(x) < \infty$ for some $\alpha \geq 0$. If u is a solution of*

$$\Delta u + V(x)\, e^u = 0 \quad x \in \mathbb{R}^2$$

such that $u_+ \in L^\infty(\mathbb{R}^2)$, then u is radially symmetric about the origin and $x \cdot \nabla u(x) < 0$ for any $x \in \mathbb{R}^2$.

Notice here that because of the asymptotic logarithmic behavior of v_∞, the result of Gidas, Ni and Nirenberg, [GNN79], does not directly apply. The boundedness from above is essential, otherwise non-radial solutions can be found, even with no singularity. Consider for instance the perturbation $\delta(x) = \frac{1}{2}\,\theta\,(x_1^2 - x_2^2)$ for any $x = (x_1, x_2)$, for some fixed $\theta \in (0,1)$, and define the potential $\phi(x) = \frac{1}{2}\,|x|^2 - \delta(x)$. By a fixed-point method we can find a solution of

$$w(x) = -\frac{1}{2\pi}\,\log|\cdot| * M\,\frac{e^{\chi w - \phi(x)}}{\int_{\mathbb{R}^2} e^{\chi w(y) - \phi(y)}\,dy}$$

since, as $|x| \to \infty$, $\phi(x) \sim \frac{1}{2}\left[(1-\theta)x_1^2 + (1+\theta)x_2^2\right] \to +\infty$. This solution is such that $w(x) \sim -\frac{M}{2\pi}\log|x|$ for reasons similar to the ones of Lemma II.23. Hence $v(x) := w(x) + \delta(x)/\chi$ is a non-radial solution of Equation (II.3), which behaves like $\delta(x)/\chi$ as $|x| \to \infty$ with $|x_1| \neq |x_2|$.

Lemma II.25. *If $\chi M > 8\pi$, Equation (II.22) has no stationary solution (u_∞, v_∞) such that $\|u_\infty\|_{L^1(\mathbb{R}^2)} = M$ and $\int_{\mathbb{R}^2} |x|^2\, u_\infty\, dx < \infty$. If $\chi M < 8\pi$, Equation (II.22) has at least one radial stationary solution given by (II.3). This solution is C^∞ and u_∞ is dominated as $|x| \to \infty$ by $e^{-(1-\varepsilon)|x|^2/2}$ for any $\varepsilon \in (0,1)$.*

Proof – The existence of a stationary solution if $\chi M < 8\pi$ is easy. It follows from Lemma II.22 but can also be achieved by minimizing the free energy, see [CLMP95, Ner04]. If the initial condition is radial or if the minimization is done among radial solutions, then the stationary solution is also radial. Direct approaches (fixed-point methods, ODE shooting methods) can also be used.

If $\chi M > 8\pi$ and if there was a stationary solution with finite second moment, we could write

$$0 = \frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\, u_\infty\, dx = 4M\left(1 - \frac{\chi M}{8\pi}\right) - 2\int_{\mathbb{R}^2} |x|^2\, u_\infty\, dx\ .$$

Since the right hand side is negative, this is simply impossible. □

In [CLMP92], it is conjectured that: *For any $M \in (0, 8\pi)$, (II.3) has a unique bounded radial solution, up to the addition of a constant,* and numerically there is no doubt about this. However, we are not aware of any complete proof. If the result is true, which is apparently the case, then the limiting stationary solution of (II.22) is unique. Define anyway by

$$\bar{n}(x,t) := \frac{1}{2t}\, u_\infty\left(\frac{1}{2}\log(2t), \frac{x}{\sqrt{2t}}\right)\ , \quad \bar{c}(x,t) := v_\infty\left(\frac{1}{2}\log(2t), \frac{x}{\sqrt{2t}}\right)$$

a self-similar solution of (II.1), with same mass as n, where u_∞ is the solution found in Lemma II.22. This self-similar solution is supposed to describe the large time asymptotics of (II.1), and this is what we are going to clarify in the last section.

4.3 Intermediate asymptotics

Lemma II.26. *Under the assumptions of Lemma II.22,*

$$\lim_{t\to\infty} F^R[u(\cdot,\cdot+t)] = F^R[u_\infty] \ .$$

Proof – By (II.23), we already know that $\lim_{t\to\infty}\int_{\mathbb{R}^2}|x|^2\,u(x,t)\,dx = \int_{\mathbb{R}^2}|x|^2\,u_\infty\,dx$. Using the estimates of Sections 2.5–2.7 and Lemma II.11, we know that $u(\cdot,\cdot+t)$ converges to u_∞ in $L^2((0,1)\times\mathbb{R}^2)$ and that $\int_{\mathbb{R}^2}u(\cdot,\cdot+t)\,v(\cdot,\cdot+t)\,dx$ converges to $\int_{\mathbb{R}^2}u_\infty\,v_\infty\,dx$. Concerning the entropy, it is sufficient to prove that $u(\cdot,\cdot+t)\,\log u(\cdot,\cdot+t)$ weakly converges in $L^1((0,1)\times\mathbb{R}^2)$ to $u_\infty\,\log u_\infty$. By Lemma II.8, there is a uniform L^1 bound. Concentration is prohibited by the convergence in $L^2((0,1)\times\mathbb{R}^2)$. Vanishing or dichotomy cannot occur either: Take indeed $R>0$, large, and compute $\int_{|x|>R}u\,|\log u| = \text{(I)}+\text{(II)}$, with

$$\begin{aligned}
\text{(I)} &= \int_{|x|>R,\;u\geq 1} u\,\log u\,dx \leq \frac{1}{2}\int_{|x|>R,\;u\geq 1}|u|^2\,dx\ ,\\
\text{(II)} &= -\int_{|x|>R,\;u<1} u\,\log u\,dx \leq \frac{1}{2}\int_{|x|>R,\;u<1}|x|^2\,u\,dx - m\log\left(\frac{m}{2\pi}\right)\ .
\end{aligned}$$

In the first case, we have used the inequality $u\log u\leq u^2/2$ for any $u\geq 1$, while the second estimate is based on Jensen's inequality in the spirit of the proof of Lemma II.8:

$$m := \int_{|x|>R,\;u<1} u\,dx \leq \frac{1}{R^2}\int_{|x|>R,\;u<1}|x|^2\,u\,dx\ .$$

Because of the convergence of the two quantities $\int_{|x|>R,\;u<1}|u|^2\,dx$ and $\int_{|x|>R,\;u<1}|x|^2\,u\,dx$ to 0 as $R\to\infty$, we have the uniform estimate

$$\lim_{R\to\infty}\int_{|x|>R} u\,|\log u| = 0\ ,$$

which completes the proof. □

The result we have shown above is actually slightly better, since it proves that all terms in the free energy, namely the entropy, the energy corresponding to the potential $\frac{1}{2}|x|^2$ and the self-consistent potential energy, converge to the corresponding values for the limiting stationary solution.

As noted above, u_∞ is a critical point of F^R under the constraint $\|u\|_{L^1(\mathbb{R}^2)}=M$. We can therefore rewrite $F^R[u]-F^R[u_\infty]$ as

$$F^R[u]-F^R[u_\infty] = \int_{\mathbb{R}^2} u\,\log\left(\frac{u}{u_\infty}\right)\,dx - \frac{\chi}{2}\int_{\mathbb{R}^2}|\nabla v-\nabla v_\infty|^2\,dx\ ,$$

and both terms in the above expression converge to 0 as $t\to\infty$, if u is a solution of (II.1). Since

$$\|u-u_\infty\|^2_{L^1(\mathbb{R}^2)} \leq \frac{1}{4\,M}\int_{\mathbb{R}^2} u\,\log\left(\frac{u}{u_\infty}\right)\,dx$$

by the Csiszár-Kullback inequality, [Csi67, Kul68], this proves the

Corollary II.27. *Under the assumptions of Lemma II.22,*

$$\lim_{t\to\infty}\|u(\cdot,\cdot+t)-u_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \textit{and} \quad \lim_{t\to\infty}\|\nabla v(\cdot,\cdot+t)-\nabla v_\infty\|_{L^2(\mathbb{R}^2)} = 0\ .$$

Undoing the change of variables (II.21), this proves Theorem II.2.

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