Méthodes de couplage pour des équations stochastiques de type Navier-Stokes et Schrödinger

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N° d'Ordre : 3297

THÈSE

Présentée

DEVANT L'UNIVERSITÉ DE RENNES I

pour obtenir

le grade de DOCTEUR DE L'UNIVERSITÉ DE RENNES I

Mention Mathématiques et Applications

par

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U.F.R. de Mathématiques

TITRE DE LA THÈSE :

Méthodes de couplage pour des équations stochastiques de type Navier-Stokes et Schrödinger

Soutenue le 1er décembre 2005 devant la Commission d'Examen

COMPOSITION DU JURY :

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Remerciements

Tout d'abord, je voudrais exprimer toute ma gratitude à Arnaud Debussche qui a assuré la direction de ma thèse. Il a été un directeur de Thèse patient et disponible, et ce, malgré le nombre de théories fumantes ou incompréhensibles (voir les deux) que je lui ai soumis tout au long de ces deux années.

Messieurs Franco FLANDOLI et Etienne PARDOUX ont acceptés de rapporter cette thèse bien qu'ils soient très sollicités. Je tient à les en remercier chaleureusement.

Je voudrais aussi remercier Messieurs Marco FUHRMAN, Ying HU et Armen SHIRIKYAN de m'avoir fait l'honneur de participer à mon jury de thèse.

Je tient aussi à remercier les équipes de processus stochastiques de l'université de Rennes 1 et celle de mathématiques de l'ENS Cachan, antenne de Bretagne pour m'avoir accueilli chaleureusement en leur sein. Ces deux équipes m'ont fourni un environnement de travail exceptionnel, d'une part pour l'ambiance amicale qui y règne, mais aussi pour la compétence et la bonne entente des gens qui y travaillent.

Je voudrais enfin tout particulièrement remercier mes proches qui m'ont soutenu et qui ont supporté mon caractère "lunaire". Toute ma gratitude à mes parents et à Caroline, mais aussi à mes amis.

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Introduction

0.1 Les équations aux dérivées partielles stochastiques

Les équations aux dérivées partielles ont été introduites pour modéliser l'évolution de nombreux phénomènes physiques tels que le comportement des fluides ou les phénomènes de propagation. Il n'est néanmoins pas toujours possible d'avoir une approche déterministe de tout les facteurs influents sur le phénomène physique considéré. On est donc amené à bruiter les équations qui deviennent de fait des équations aux dérivées partielles stochastiques.

Par exemple, la propagation d'ondes non linéaires dispersives est modélisée par l'équation de Schrödinger non linéaire et apparaît dans divers domaines de la physique tels que l'hydrodynamique [48], [49], l'optique, la physique des plasmas, les réactions chimiques [32]...

Lorsqu'on étudie la propagation dans un milieu aléatoire, il convient d'introduire un bruit. Par exemple dans [19], ils considèrent l'équation de Schrödinger non linéaire bruitée et amortie dans le cas cubique. La propagation des ondes sur une longue distance y est étudiée. L'amortissement ne peut y être négligé mais il est compensé par la présence d'amplifications aléatoires. Le bruit blanc offre une bonne modélisation des amplifications aléatoires dans la mesure où la distance entre les amplificateurs est négligeable devant la distance de propagation.

L'équation de Schrödinger non linéaire amortie et bruitée (NLS) se met sous la forme

$$
\left\{
\begin{array}{rcll}
\multicolumn{4}{l}{du + \alpha u\, dt + \mathrm{i}\,(-\Delta)\, u\, dt + \lambda \mathrm{i}\, |u|^{2\sigma}\, u\, dt = \phi(u) dW,} \\
u(t,x) & = & 0, & \text{pour } x \in \partial D,\ t > 0 \\
u(0,x) & = & u_0(x), & \text{pour } x \in D,
\end{array}
\right.
\tag{0.1.1}
$$

où $\alpha > 0$ est le terme d'amortissement. L'inconnue u est une fonction à valeur complexe des variables temporelles $t \geq 0$ et spatiale $x \in D$. La constante λ vaut 1 dans le cas défocalisant et -1 dans le cas focalisant. Nous avons ici considéré les conditions de bord de Dirichlet sur un ouvert borné D à bordure régulière ∂D ou $D = (0,1)^d$, mais il est aussi possible d'imposer les conditions de Neumann ou de périodicités des solutions. Le paramètre $\sigma > 0$ est supposé vérifier la condition sous critique $\sigma d < 2$.

Il arrive que les ondes considérées subissent un amortissement plus conséquent qui s'assimile à un effet de viscosité. Dans ce cas, on introduit donc des termes visqueux dans (0.1.1). L'équation ainsi obtenue est appelée équation de Ginzburg-Landau Complexe bruitée (CGL). Il s'agit en fait d'une version dissipative de NLS qui apparaît dans les mêmes domaines et se met sous la forme

$$
\left\{
\begin{array}{rcll}
\multicolumn{4}{l}{du + (\varepsilon + \mathrm{i})(-\Delta) u\, dt + (\eta + \lambda \mathrm{i})\, |u|^{2\sigma}\, u\, dt = \phi(u) dW + f dt,} \\
u(t,x) & = & 0, & \text{pour } x \in \partial D, t > 0, \\
u(0,x) & = & u_0(x), & \text{pour } x \in D,
\end{array}
\right.
\tag{0.1.2}
$$

où D un ouvert borné D à bordure régulière ∂D ou $D = (0,1)^d$, où $\varepsilon > 0$, $\eta > 0$ et $\lambda \in \{-1, 1\}$. Dans le cas focalisant ($\lambda = -1$), on impose toujours la condition sous critique $\sigma \in (0, \frac{2}{d})$. On parle alors de condition sous critique L^2. Dans le cas défocalisant ($\lambda = 1$), le paramètre $\sigma > 0$ est autorisé à vérifier la condition sous critique plus large $\sigma \in (0, \infty)$ pour $d = 1, 2$ et $\sigma < \frac{2}{d-2}$ pour $d > 2$. Cette condition est dite condition sous critique H^1.

Un autre exemple de phénomène physique grandement étudié est le comportement des fluides incompressibles. Leur évolution est décrite par les équations de Navier-Stokes (Voir [47]). Comme pour NLS et CGL, on est souvent amené à bruiter ces équations. On obtient ainsi les équations de Navier-Stokes stochastiques ci dessous.

$$\begin{cases} du + \nu(-\Delta)u\,dt + (u,\nabla)u\,dt + \nabla p\,dt = \phi(u)dW + f dt, \\ \qquad (\text{div } u)\,(t,x) \;=\; 0, \qquad \text{pour } x \in D, \;\; t>0, \\ \qquad u(t,x) \;=\; 0, \qquad \text{pour } x \in \delta D, t>0, \\ \qquad u(0,x) \;=\; u_0(x), \quad \text{pour } x \in D, \end{cases} \tag{0.1.3}$$

où $u(t,x) \in \mathbb{R}^d$ est la vitesse au temps t et au lieu x, $p(t,x)$ la pression, $\nu > 0$ la viscosité et D un ouvert borné de $\mathbb{R}^d$ à bordure régulière δD ou $D = (0,1)^d$. Le champ de force extérieur $\phi(u)dW + f dt$ agissant sur le fluide est la somme d'une composante déterministe f et d'un champ aléatoire $\phi(u)dW$.

Du fait de la différence profonde des propriétés connues des équations de Navier-Stokes stochastiques bidimensionnelles (NS) et tridimensionnelles (NS3D), nous traiterons NS et NS3D comme des systèmes d'équations fondamentalement différents.

Nous n'avons pas seulement choisi d'étudier ces quatre équations pour leur intérêt physique mais aussi parce que chacune de ces équations est représentative d'une famille plus générale d'équations.

Plus précisément, NS représentent les équations aux dérivées partielles stochastiques (EDPS) fortements dissipatives "sympathiques". Cette famille à laquelle appartient aussi l'équation de Burger unidimensionnelle se caractérise par la propriété suivante. On peut contrôler l'incidence de la non linéarité sur la distance de deux solutions de façon Lipschitz si on ne contrôle qu'une seule des ces deux solutions. Dans le cas de NS, cela s'exprime de la façon suivante

$$\begin{aligned} \left|(u_2-u_1,(u_2,\nabla)u_2-(u_1,\nabla)u_1)\right| &= \left|(u_2-u_1,(u_2-u_1,\nabla)u_1)\right|, \\ &\le \tfrac{\nu}{2}\left|u_2-u_1\right|_{H^1}^2 + c\left|u_2-u_1\right|_{L^2}^2\left|u_1\right|_{H^1}^2, \end{aligned}$$

pour tout (u_1,u_2) suffisamment régulier. Cette propriété est très intéressante et simplifie grandement les preuves, ce qui à l'avantage de les rendre plus pédagogiques.

La généralisation des preuves pour NS vers les équations fortement dissipatives requiert certaines adaptations. L'étude de CGL fournit un exemple d'EDPS fortement dissipative n'ayant pas la propriété ci dessus. Plus précisément, dans le cas de CGL, on peut montrer une formule analogue. Néanmoins, elle n'induit les mêmes simplifications pour CGL que pour NS. En réalité, pour CGL, la formule véritablement analogue serait la suivante

$$|(\Delta(u_2-u_1),(u_2,\nabla)u_2-(u_1,\nabla)u_1)| \leq \frac{\varepsilon}{2}\,|u_2-u_1|_{H^2}^2 + c\,|u_2-u_1|_{H^1}^2\,|u_1|_{H^2}^2\,,$$

ce que nous n'avons pas.

Ces équations sont donc plus difficiles à traiter pour les questions étudiées ici, mais les preuves se généralisent assez bien modulo certains raffinements.

Une classe d'équations où les choses se passent nettement moins bien est la famille des équations faiblement amorties et dispersives. Cette famille trouve en NLS un de ses représentants les plus illustres. Comme nous le verrons, ce sont des équations beaucoup plus complexe à traiter. En effet, pour NS et CGL, nos preuves s'appuient sur la notion de forte dissipativité et ainsi, si le passage de NS à CGL a requis certains aménagements, l'adaptation de ces méthodes à NLS apparaît beaucoup plus difficile. Le principal problème est que, dans l'espace L^2, le terme αu n'offre qu'un amortissement très faible en comparaison de la dissipativité induite par $\nu(-\Delta)u$ pour NS et par $\varepsilon(-\Delta)u$ pour CGL. Il faudra montrer que, dans l'espace de Sobolev H^1, la dispersivité de NLS et le terme αu permettent de contrôler la non linéarité

Néanmoins, les classes d'équations considérées précédemment ont un point communs. Elles sont bien posées. En clair, les propriétés d'existence et d'unicité des solutions se démontrent assez facilement. Ce qui est rassurant puisque ces équations sont censées régirent un phénomène physique. Il serait désagréable qu'une de ces propriétés nous fasse défaut. Un autre avantage des équations précédemment citées est que l'on contrôle certaines quantités nécessaires à nos preuves (car elles permettent de faire fonctionner la plupart de nos outils). Malheureusement, toutes ces questions restent des problèmes ouverts en ce qui concerne NS3D. Ainsi, les problèmes que l'on rencontre lorsque l'on travaille avec NS3D sont de nature beaucoup plus basique. En clair, quasiment toutes les opérations que l'on fait de manière plus ou moins implicite pour les autres équations posent problème. Par exemple, on est souvent amené à construire des solutions par induction sur l'intervalle de temps $(0,n)$. Malheureusement, pour NS3D, se pose le problème de la diversité du choix des solutions. Plus grave : Est-il seulement possible de choisir une solution mesurablement par rapport à la condition initiale ? Un autre exemple est l'absence de contrôle énergétique qui rend caduc tout nos outils. Plus gênant encore, la propriété la plus utilisée dans nos preuves pour NS, CGL et NLS est la propriété de Markov forte. Or, du fait du manque d'unicité des solutions de NS3D, nous ignorons si les solutions de NS3D vérifie ne serait-ce que la propriété de Markov faible.

Pour toutes ces raisons nous démontrerons des résultats plus modestes en ce qui concerne NS3D. Néanmoins, comme nous l'avons dit, la difficulté est ailleurs. La question se pose de savoir si on peut généraliser ce type de résultat à d'autres équations mal posée tels que CGL H^1–surcritique. Nous le pensons, mais cela requiererait une étude spécifique.

0.2 Ergodicité des systèmes physiques

On s'intéressera plus particulièrement aux mesures invariantes et aux propriétés ergodiques des quatre EDPS précédemment citées lorsque celles-ci sont excitées par un bruit blanc en temps et régulier pour la variable d'espace.

La convergence rapide à l'équilibre des phénomènes physiques et l'étude des états stationnaires sont des questions de première importance.

Pour le comprendre, considérons la thermodynamique qui nous apparaît comme un exemple particulièrement éclairant de l'importance de telles notions. Cette discipline a d'abord fonctionnée sur des principes plus ou moins heuristiques dont la justification théorique n'est apparue que plus tard avec l'émergence de la mécanique statistique. Une des questions les plus marquantes a concerné le sens de l'écoulement du temps. L'irréversibilité du temps a longtemps été un phénomène postulé sous la forme du second principe de la thermodynamique. Ce principe stipule qu'il existe une fonction S de l'état du système $X(t)$ au temps t qui est additive (i.e. l'entropie de deux systèmes disjoints est la somme des entropies) et telle que l'entropie de l'univers tout entier est croissante. C'est cette croissance postulée qui signifie l'irréversibilité du temps.

Pour mieux comprendre tout cela, considérons un système physique donné. Pour simplifier nous traiterons l'exemple un gaz. L'état physique $X(t)$ du système au temps t est censé le déterminer entièrement. De plus la connaissance des conditions extérieures, des aléas et de l'état du système doit permettre de prédire son comportement à venir. Du point de vue de la mécanique Newtonienne, l'état $X(t)$ est l'ensemble des positions et des vitesses de l'ensemble des molécules constituant ce gaz. Rappelons qu'une mole de gaz est constituée d'à peu près 6.10^{23} molécules et qu'un volume de gaz à échelle humaine contient au moins plusieurs moles de gaz. Ajoutons à cela que parmis les connaissances requises pour pouvoir appliquer la mécanique Newtonienne, il faut une connaissance précise (de l'ordre du nanomètre) de la cartographie des parois du récipient où vit ledit gaz. L'approche Newtonienne apparaît donc limitée dans sa capacité à prédire le comportement à venir de notre gaz. Pire, elle apparaît même incapable de décrire de manière satisfaisante l'état $X(t)$ du système à l'instant t.

Plus grave encore, même si elle est inexploitable d'un point de vue calculatoire, ses principes théoriques devraient être vérifiés. Or l'un de ces principes est l'invariance par renversement temporel. En clair, la mécanique Newtonienne est indifférente au sens d'écoulement du temps. Pour le voir, prenons l'exemple simple d'un système soumis à la pesanteur (une pomme tombant d'un arbre par exemple). Sa courbe de position $t \to x(t)$ vérifie l'équation

$$\frac{d^2}{dt^2}x = g,$$

où g est le champ de pesanteur. Cette équation est aussi vérifiée par $t \to x(-t)$. En clair, si un habitant de Proxima du Centaure regardait un film du mouvement des planètes de notre système solaire, il n'aurait aucun moyen de savoir si ce film est passé en marche avant ou en marche arrière car ces deux comportements sont plausibles. Dés lors, le concept d'irréversibilité du temps est en apparence incompatible avec la mécanique Newtonienne.

Revenons à notre système gazeux. Malgré son apparente extrême complexité, on a très bien réussi à décrire l'état des gaz et à prédire leur comportement à partir de seulement trois paramètres réels positifs. Plus précisément, un gaz est uniquement déterminé par sa température T, sa pression P et son volume V, sachant que ces trois paramètres sont reliés par la relation

$$PV = nKT,$$

où K est une constante universelle (dite de Boltzman) et où n est le nombre de molécules de gaz.

Du point de vue de la mécanique Newtonienne, une telle simplification peut paraître difficile à croire, pour ne pas dire aberrante. Pourtant, considérons un gaz à un temps t_0 avec une pression P_0, une température T_0 et un volume V_0 fixés et imposons une condition au récipient qui le contient. Par exemple, on peut imposer un récipient adiabatique (i.e. thermiquement isolant : une glacière par exemple) ou, au contraire qui soit très bon conducteur thermique avec une température extérieure fixée. Il peut aussi être rigide, ou bien avec un piston qui impose une pression externe constante, etc... On sait qu'après un laps de temps assez court, le gaz se stabilise à un état d'équilibre uniquement caractérisé par sa température T, sa pression P et son volume V, eux-mêmes uniquement déterminés par le triplet (P_0, V_0, T_0) et par la condition imposée au récipient. La question qui se pose est donc : Comment un système d'une telle complexité peut il se simplifier à ce point ? De plus, comment la stabilisation du système, qui marque l'irréversibilité du temps, peut-elle être compatible avec la mécanique Newtonienne ?

Les fondateurs de la thermodynamique ont plus ou moins postulés des principes heuristiques amenant à ce genre de résultats, mais ils n'ont pas établi comment déduire la thermodynamique de la mécanique Newtonienne. Le comportement microscopique (d'abord régi par la mécanique Newtonienne puis plus tard par la mécanique quantique) est extrêmement complexe et possède la particularité d'être insensible à un renversement temporel. A l'opposé, le comportement macroscopique est simple et est fortement sensible au sens du temps. Il a fallu attendre l'émergence de la mécanique statistique et l'introduction des probabilités dans le sacro-saint antre du déterminisme Newtonien pour comprendre que c'est la notion d'ergodicité qui fait muter le comportement microscopique Newtonien en un comportement macroscopique thermodynamique.

Dans [17], sont expliqués les "fondements rationnels de la thermodynamique" (d'après l'éditeur). L'idée fut, comme nous l'avons écrit, de remplacer le cadre déterministe par un cadre probabiliste plus adéquate s'appuyant sur deux principes fondamentaux. Le premier principe postulé par la mécanique statistique est la convergence très rapide de la loi des systèmes physiques étudiés vers l'équilibre. En langage mathématique, cela signifie en particulier que le système est ergodique. Nous noterons μ la loi de l'état d'équilibre. Les physiciens considèrent donc ces systèmes comme stationnaires au bout d'un court laps de temps. En clair, pour tout laps de temps $(t, t+\Delta t)$, on obtient, si Δt est assez élevé, alors

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} F(X(s))ds \sim \int F(x)\,\mu(dx) \quad p.s., \tag{0.2.1}$$

où $X(s)$ est l'état microscopique du système au temps s et où F est une application mesurable.

Le second principe est la très rapide décorélation du système. En clair, notre système devient très vite indépendant de son passé. Cela signifie que le temps nécessaire Δt pour que la loi forte des grands nombres (0.2.1) soit effective est très court.

Les grandeurs physiques Newtoniennes du système (le volume par exemple) à un instant t sont données par $F(X(t))$ où F est une application mesurable associée à la grandeur physique. Il n'est cependant pas possible de mesurer un facteur du système à un instant donné. En effet, par exemple, comment mesurer la vitesse de façon instantanée ? En pratique, donc, lorsque l'on veut mesurer une grandeur physique, la grandeur mesurée $F_{\text{mesuré}}(X)(t)$ à l'instant t est en fait une moyenne en temps sur un court laps de temps

$$F_{\text{mesuré}}(X)(t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} F(X(s))ds.$$

On déduit donc de (0.2.1) que les grandeurs mesurées du système sont proches à la moyenne de F par rapport à μ

$$F_{\text{mesuré}}(X)(t) \sim \int F(x)\,\mu(dx).$$

En clair, on peut approcher les valeurs des grandeurs physiques Newtonienne par des espérances. Ce concept a été poussé plus avant. On a remplacé l'état Newtonien du système $X(t)$ par un état Thermodynamique $\mu(t)$ qui en fait est la loi du système. Les grandeurs physiques Newtoniennes associées à un état μ étant maintenant définies par leur moyenne

$$F(\mu) = \int F(x)\,\mu(dx).$$

Puis, de nouvelles grandeurs physiques purement thermodynamiques ont été définies. Ces grandeurs ne sont plus associées à des applications mesurables de l'état Newtonien $X(t)$ dont on prend ensuite la moyenne. Ce sont des fonctions $F(\mu)$ de l'état Thermodynamique μ. C'est ainsi, par exemple, qu'est définie l'entropie $S(\mu)$ de μ lorsqu le nombre état $\{x_n\}_n$ est dénombrable

$$S(\mu) = -\sum_n \mu(\{x_n\})\ln\left(\mu(\{x_n\})\right),$$

ou lorsqu μ est absolument continue

$$S(\mu) = -\int_{\mathbb{R}^d} \ln\left(\left(\frac{d\mu}{dx}\right)(x)\right)\mu(dx),$$

où $\left(\frac{d\mu}{dx}\right)$ est la dérivée de Radon-Nykodym de μ.

Revenons à l'étude de notre gaz. Le fait que l'état du gaz soit entièrement déterminé par (P, V, T) signifie que l'on peut paramétrer les mesures invariantes de notre système par ces trois paramètres et que ce système se rapproche de l'équilibre très vite. Un cas particulier est le cas d'un gaz défini dans un volume infini et stable par translation. Dans ce cas on fait disparaître le volume et on dénote par n le nombre de molécule par unité de gaz. On a alors

$$P = nKT.$$

Il reste donc à définir la température. Dans ce cas, la modélisation classique est que chaque molécule est indépendante et identiquement distribuée et que la loi de sa vitesse $v \in \mathbb{R}^3$ est donnée par

$$\mu_T(dv) = \frac{1}{Z(T)} e^{-\frac{mv^2}{KT}} dv,$$

où m est la masse d'une molécule.

Il résulte ainsi du comportement indépendant des molécules et de la loi forte des grands nombres que la pression P est l'énergique cinétique par unité de volume du gaz. De plus la température mesure l'agitation des molécules.

Pour conclure, on remarque donc que c'est l'ergodicité qui transforme le comportement Newtonien en comportement Thermodynamique, donnant de fait une direction au temps. C'est cette convergence vers l'équilibre qui accroît l'entropie de l'univers et qui implique le second principe de la thermodynamique. Une expression souvent employée est que, si au niveau microscopique, le temps n'a pas de direction, au niveau macroscopique, il va de "l'improbable vers le probable". La notion de "probable" étant comprise au sens de la mesure invariante caractérisant l'état d'équilibre.

Cet exemple a, nous l'espérons, permit de comprendre l'importance de l'étude de la convergence rapide à l'équilibre des phénomènes physiques et l'étude des états stationnaires. Dans cette thèse nous montrerons que sous certaines conditions, les solutions de NS, CGL et NS3D (resp NLS) convergent exponentiellement (resp polynomialement) vite vers l'équilibre et que leurs solutions stationnaires sont régulières.

0.3 Convergence des solutions des EDPS vers l'équilibre

Dans cette thèse, nous développerons deux approches basées sur la notion de couplage pour établir la convergence à l'équilibre des solutions. La première pour NLS, CGL et NS munies de bruits pouvant être dégénérés, la seconde pour NS3D muni de bruits suffisamment non dégénérés.

Ces équations sont des systèmes difficiles à traiter avec des arguments standards parce que l'espace des phases est de dimension infinie. Pour une meilleure compréhension de NS, CGL et NLS, nous considérerons ici le cas de conditions périodiques avec moyenne nulles sur le tore au lieu des conditions de Dirichlet sur un ouvert borné de $\mathbb{R}^d$, ce qui a l'avantage de nous permettre de travailler avec les coefficients de Fourier des solutions. On notera $(\hat{u}(k))_k$ les coefficients de Fourrier d'une fonction u de x. Les systèmes d'équations aux dérivées partielles stochastiques NS et NS3D deviennent le système infini dimensionnel d'équations différentielles stochastiques suivant.

$$\begin{cases} d\hat{u}(k) + \nu \left|k\right|^2 \hat{u}(k)\, dt + F_k\left(u\right)\, dt = \phi_k\left(u\right) dW + f_k dt, \\ \qquad (\hat{u}(k)(t), k) = 0, \qquad \text{pour } k \in \mathbb{Z}^d \backslash \{0\}, \;\; t > 0, \\ \qquad \hat{u}(0)(t) = 0, \qquad \text{pour } t > 0, \\ \qquad \hat{u}(k)(0) = \hat{u}_0(k), \qquad \text{pour } k \in \mathbb{Z}^d \backslash \{0\}, \end{cases} \tag{0.3.1}$$

où

$$F_k\left(u\right) = \mathrm{i}\, \pi_k \left(\sum_h (\hat{u}(k-h), h)\hat{u}(h) \right),$$

et où $\pi_k : \mathbb{R}^d \to \mathbb{R}^d$ est le projecteur orthogonal de noyau engendré par le vecteur k.

Pour sa part, CGL devient

$$\begin{cases} d\hat{u}(k) + (\varepsilon + \mathrm{i}) \left|k\right|^2 \hat{u}(k)\, dt + F'_k\left(u\right)\, dt = \phi_k\left(u\right) dW + f_k dt, \\ \qquad \hat{u}(0)(t) = 0, \qquad \text{pour } t > 0, \\ \qquad \hat{u}(k)(0) = \hat{u}_0(k), \qquad \text{pour } k \in \mathbb{Z}^d \backslash \{0\}, \end{cases} \tag{0.3.2}$$

où on a dans le cas cubique ($\sigma = 1$)

$$F'_k(u) = (\eta + \lambda \mathrm{i}) \left(\sum_{h_1,h_2} \hat{u}(h_1)\hat{u}(h_2)\hat{u}(k-h_1-h_2) \right).$$

Le caractère fortement dissipatif de NS (resp CGL) provient du terme $\nu\,|k|^2\,\hat{u}(k)$ (resp $\varepsilon\,|k|^2\,\hat{u}(k)$) qui permet de contrôler la non-linéarité lorsque $|k|$ est grand. Ainsi, pour simplifier nos explications, on est donc naturellement amené à modéliser les EDPS fortement dissipatives comme NS et CGL par le système bidimensionnel ci-dessous

$$\begin{cases} dX \;+\; Xdt \;+\; f(X,Y)dt \;=\; \sigma_l(X,Y)\,dW, \quad X(0) = x_0, \\ dY \;+\; NYdt \;+\; g(X,Y)dt \;=\; \sigma_h(X,Y)\,dW, \quad Y(0) = y_0, \end{cases} \tag{0.3.3}$$

où toutes les fonctions considérées seront supposées Lipschitz bornées et où N sera autorisé à être aussi grand qu'on le désire. Les variables X et Y représentent respectivement les bas et les hauts modes des solutions $u = (X,Y)$ de l'équation.

Pour une meilleure compréhension, nous traiterons ce modèle dans les chapitres 1 et 3, puis nous reporterons aux annexes A et C les preuves associées à NS et CGL.

Pour sa part, NLS devient

$$\begin{cases} d\hat{u}(k) + \alpha\hat{u}(k)\,dt + \mathrm{i}\,|k|^2\,\hat{u}(k)\,dt + F_k"(u)\;dt = \phi_k(u)\,dW, \\ \hat{u}(0)(t) \;=\; 0, \qquad \text{pour } t > 0, \\ \hat{u}(k)(0) \;=\; \hat{u}_0(k), \quad \text{pour } k \in \mathbb{Z}^d \backslash \{0\}, \end{cases} \tag{0.3.4}$$

où, dans le cas cubique $\sigma = 1$, on a

$$F_k"(u) = \lambda \mathrm{i} \left(\sum_{h_1,h_2} \hat{u}(h_1)\hat{u}(h_2)\hat{u}(k-h_1-h_2) \right).$$

Considérons NLS du point de vue L^2. Si $\alpha = 0$, c'est une équation qui conserve la norme L^2. On est donc naturellement amené à la modéliser par le système suivant

$$\begin{cases} dX + Xdt + f(X,Y)dt = \sigma_l(X,Y)\,dW, & X(0) = x_0, \\ dY + Ydt + g(X,Y)dt = \sigma_h(X,Y)\,dW, & Y(0) = y_0. \end{cases} \tag{0.3.5}$$

Malheureusement, comme nous le verrons plus tard, un tel modèle est inexploitable. Dans le chapitre 2, nous seront amenés à raffiner ce modèle pour qu'il prenne en compte les propriétés de dispersivité de NLS.

La modélisation de NS3D fera l'objet d'un traitement particulier au chapitre 5 du fait du caractère pathologique des propriétés (ou plutôt de l'absence des dites propriétés).

Considérons d'abord les trois premières équations (NS, CGL et NLS). Pour ces équations, on s'autorise à traiter des bruits fortement dégénérés. Pour nos modèles bidimensionnels, cela signifie que la matrice $\Sigma(X,Y)$ est autorisée à ne pas être inversible, ce qui rend difficile l'application du théorème de Doob.

L'idée est de compenser la dégénérescence du bruit sur certains sous-espaces par des arguments de dissipativités. Cette idée a été introduite dans [5], [18] et [37]. Ce genre de raisonnement fonctionne bien lorsque le système considéré à un nombre fini de directions instables, ce qui est évidemment le cas de NS et CGL (mais cela semble beaucoup moins évident pour NLS sans l'introduction de divers fonctionnelles).

Pour nos modèles bidimensionnels, cette hypothèse se reformule de la façon suivante. L'opérateur de covariance du bruit $\Sigma(X,Y)$ est supposé non dégénéré dans les bas modes. En clair, cette hypothèse signifie que la droite $\mathbb{R}\times\{0\}$ est toujours dans l'image de $\Sigma(X,Y)$, ce qui équivaut à l'existence d'une fonction Σ_l^{-1} vérifiant pour tout réel h

$$\Sigma(X,Y)\cdot\Sigma_l^{-1}(X,Y)\cdot h = (h,\ 0)\,.$$

Il apparaît naturel que cette hypothèse implique la convergence à vitesse exponentielle à l'équilibre, du moins dans le cas des équations fortement dissipatives tels que NS et CGL. En fait nous verrons que c'est aussi suffisant pour que prouver que NLS converge à vitesse polynomiale vers l'équilibre.

Pour prouver ce résultat dans le cas d'équations fortement dissipatives, des méthodes de couplages ont été introduites (voir [29], [39], [40], [41], [45] et [56]). A l'exception de [45], ces articles traitent des bruits diagonaux constants avec force déterministe nulle. Pour nos models bidimensionnel, cela donne des opérateurs du type

$$\Sigma(X,Y) = \begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_h \end{pmatrix} \qquad \text{avec} \qquad \sigma_l \neq 0.$$

Dans [45], le bruit est toujours diagonal, mais est autorisé à dépendre des bas modes X.

$$\Sigma(X,Y) = \begin{pmatrix} \sigma_l(X) & 0 \\ 0 & \sigma_h(X) \end{pmatrix} \qquad \text{avec} \qquad \inf \sigma_l > 0.$$

De plus, la force déterministe peut ne pas être nulle. Ces méthodes quoique très intéressantes semble limitées à ce type particulier de bruit et aux équations fortement dissipatives, ce qui semble exclure NLS. Nous commencerons, dans le chapitre 1 et l'annexe A, par adapter ces méthodes à CGL qui a le désavantage d'être moins "sympathique" que NS, ce qui requiera certains aménagements. Puis, dans le chapitre 2 et l'annexe B, nous verrons comment ces méthodes peuvent s'étendre à NLS cubique unidimensionnelle. Finalement, dans le chapitre 3 et l'annexe C, nous verrons qu'à condition de repenser ces méthodes en profondeur, il est possible de dépasser les hypothèses de diagonalisabilité et de non-dépendance par rapport à Y. Cette évolution sera l'occasion de simplifier ces méthodes. En fait nous développerons un critère général qui permet d'établir qu'un processus de Markov converge exponentiellement vite vers l'équilibre. Ce critère à l'avantage d'être facile à appliquer et de ne pas requérir de connaissance de la preuve, ni même de la notion de couplage (pourtant abondamment utilisée dans la preuve).

Notons que, dans la lignée de ce genre de travaux mais en utilisant des outils complêtements différents, une étude fine de la non linéarité de NS a permit d'établir que si on excite seulement 4 modes bien précis par 4 mouvements browniens indépendants, alors NS admet une unique mesure invariante (Voir [30], [46]). On peut espérer que leur méthode, basée sur le calcul de Malliavin, permettra d'obtenir la convergence à vitesse exponentielle dans ce cas.

Du fait des problèmes inhérents à NS3D, on se restreindra au cadre plus modeste d'un bruit suffisamment non dégénéré. Pour le modèle (0.3.3), cela signifie que l'on suppose que la matrice $\Sigma(X,Y) = (\sigma_l(X,Y), \sigma_h(X,Y))$ est inversible d'inverse borné. Ceci est un cadre plus simple car c'est le cadre où le Théorème de Doob s'applique le plus naturellement. Néanmoins, pour NS3D, la difficulté est ailleurs. Pour cette raison, nous étudierons d'abord, au chapitre 4 et dans l'annexe D, la régularité spatiale des solutions stationnaires de NS3D et nous en déduirons des informations sur le coefficient de dissipation de Kolmogorov (K41). Puis, au chapitre 5 et dans l'annexe E, nous établirons le caractère exponentiellement mélangeant des solutions de NS3D.

Chapitre 1

Ergodicité des équations de Ginzburg–Landau Complexes

Introduction

Rappelons la formulation de CGL

$$\left\{\begin{array}{rcl} du+(\varepsilon+\mathrm{i})(-\Delta)u\,dt+(\eta+\lambda\mathrm{i})\,|u|^{2\sigma}\,u\,dt &=& \phi(u)dW+fdt,\\ u(t,x) &=& 0,\quad \text{pour } x\in\delta D, t>0,\\ u(0,x) &=& u_0(x),\ \text{pour } x\in D. \end{array}\right.$$

Le but de ce chapitre est d'établir l'ergodicité de CGL dans le cadre des hypothèses de [45]. En clair, ϕ est supposé recouvrir un nombre suffisant de bas modes et être diagonal. De plus, contrairement à [41], f est autorisé à être non nul et ϕ à dépendre des bas modes des solutions. En clair, on fait l'hypothèse suivante

Comme nous l'avons déjà dit, l'idée principale est de compenser la dégénérescence du bruit sur certains sous espace par des arguments de disssipativité, en l'occurrence en utilisant une inégalité de type Foias-Prodi. Nous avons donc adapté les méthodes de couplage pour établir le caractère exponentiellement mélangeant des solutions de CGL. Ce qui a demandé certains aménagements car la non linéarité de CGL est moins sympathique que celle de NS. Il faudra, par exemple, développé une inégalité de type Foias-Prodi H^1 dépendant des deux solutions et une inégalité de type Foias-Prodi L^2 ne dépendant que d'une seule solution. Il faudra de plus introduire une énergie H^1. L'esprit de la preuve reste néanmoins la même.

1.1 Notions de convergence

Avant d'énoncer notre résultat, rappelons quelques définitions reliées à la notion de convergence.

La norme variationnelle $\|\cdot\|_{var}$ d'une mesure signée sur un espace polonais E est définie par

$$\|\mu\|_{var} = \sup\left\{|\mu(\Gamma)| \ \mid \Gamma \in \mathcal{B}(E)\right\},$$

où $\mathcal{B}(E)$ sont les boréliens de E.

Il est facile de voir que si μ_1, μ_2 sont deux mesures de probabilité mutuellement équivalentes, on a

$$\|\mu_2 - \mu_1\|_{var} = \frac{1}{2}\int_E \left|\frac{d\mu_1}{d\mu_2} - 1\right| d\mu_2. \tag{1.1.1}$$

De plus, il est notoire que $\|\cdot\|_{var}$ est la norme duale de $|\cdot|_\infty$. Cela signifie que

$$\|\mu_2 - \mu_1\|_{var} = \sup_{|\psi|_\infty \le 1}\left|\int_E \psi(x)\,(\mu_2 - \mu_1)(dx)\right|, \tag{1.1.2}$$

où le supremum peut être indifféremment pris sur les fonctions $\psi : E \to \mathbb{R}$ mesurables ou uniformément continues.

Posons $E = \mathbb{R}^d$. On dénotera par δ_x la masse de Dirac en x. Il est facile de voir que

$$\|\delta_x - \delta_0\|_{var} = 1_{\{0\}}(x).$$

Donc δ_x ne tend pas vers δ_0 en variation totale quand $x \to 0$.

En fait, dire que μ_t tend en variation totale vers δ_0 en variation totale signifie que $\mu_t(\mathbb{R}^d \backslash \{0\}) \to 0$. Donc, si on pose $d = 1$ et

$$\mu_t = t\delta_1 + (1-t)\delta_0,$$

on obtient que

$$\|\mu_t - \delta_0\|_{var} = t \to 0,$$

quand t tend vers 0.

On voit que ce type de convergence est très fort, mais qu'on peut raisonnablement envisager une notion de convergence plus faible.

On introduit donc la norme de Wasserstein d'une mesure signée comme la norme duale des fonctions Lipschitz bornées

$$\|\mu\|_* = \sup_{|\psi|_{Lip_b(R)}\leq 1} \left| \int_E \psi(x)\,\mu(dx) \right|, \tag{1.1.3}$$

où

$$|\psi|_{Lip_b(E)} = \operatorname{Lip}\psi + |\psi|_\infty\,.$$

Dans ce cas, si $E=\mathbb{R}^d$, on a

$$\|\delta_x - \delta_0\|_{var} = |x|\,,$$

pourvu que $|x| \leq 1$.

Dans ce chapitre, nous allons établir que les solutions de (0.1.2) convergent exponentiellement vite à l'équilibre en variation totale pour les bas modes et au sens de Wasserstein pour les hauts modes. Pour mesurer cette convergence, on introduit la distance $\|\cdot\|_{var-W}$ définie sur l'ensemble des mesures finies du produit d'espace polonais $\mathcal{X}\times\mathcal{Y}$

$$\|\mu_2 - \mu_1\|_{var-W} = \sup_{|\psi|_{Lip_Y(\mathcal{X}\times\mathcal{Y})}\leq 1} \left| \int_{\mathcal{X}\times\mathcal{Y}} \psi(u)\,(\mu_2-\mu_1)(du) \right|, \tag{1.1.4}$$

où

$$|\psi|_{Lip_Y(\mathcal{X}\times\mathcal{Y})} = |\psi|_\infty + \sup_{(x,y_1,y_2)\in\mathcal{X}\times\mathcal{Y}\times\mathcal{Y}} \left(\frac{|\psi(x,y_2)-\psi(x,y_1)|}{|y_2-y_1|} 1_{y_1\neq y_2} \right).$$

1.2 Résultats principaux

On pose

$$A = -\Delta, \quad D(A) = H_0^1(D)\cap H^2(D).$$

On peut maintenant réécrire (0.1.2) sous la forme

$$\begin{cases} du + (\varepsilon + \mathrm{i})Au\,dt + (\eta + \lambda\mathrm{i})\,|u|^{2\sigma}\,u\,dt = \phi(u)dW + f, \\ u(0) = u_0, \end{cases} \tag{1.2.1}$$

où W est un processus de Wiener cylindrique sur $L^2(D)$.

On notera $(e_n, \mu_n)_n$ la suite des vecteurs et valeurs propres associés à A tel que $(\mu_n)_n$ soit une suite croissante. De plus, on notera P_N (resp Q_N) le projecteur orthogonal de $L^2(D)$ sur l'espace engendré par $\{e_1, \dots, e_N\}$ (resp $\{e_{N+1}, \dots\}$).

On fera l'hypothèse suivante.

Hypothèse 1.2.1 *On a*

$$\sigma \in \left(0, \frac{2}{d} \wedge \frac{3}{2}\right).$$

L'application $\phi : H \to \mathcal{L}_2(L^2(D); L^2(D))$ est Lipschitz bornée et $f \in L^2(D)$. De plus, il existe $N \in \mathbb{N}$ tel que pour tout $u \in L^2(D)$

$$\phi(u) = \phi(P_N u), \; P_N\phi(u)Q_N = 0, \; Q_N\phi(u)P_N = 0$$

et tel que $P_N\phi(u)P_N$ soit uniformément inversible

$$\sup_{u \in L^2(D)} \left|(P_N\phi(u)P_N)^{-1}\right|_{\mathcal{L}(P_N L^2(D); P_N L^2(D))} < \infty.$$

Ici $\mathcal{L}(K_1; K_2)$ (resp $\mathcal{L}_2(K_1; K_2)$) désigne l'espace des applications linéaires (resp Hilbert-Schmidt) allant de l'espace de Hilbert K_1 vers l'espace de Hilbert K_2.

Remarquons que, sous l'hypothèse 1.2.1, on a existence et unicité des solutions de que l'on dénotera $u(t, W, (x_0, y_0))$. De plus, les solutions vérifient la propriété de Markov forte. On note $(\mathcal{P}_t)_t$ le semi-groupe de Markov associé sur H.

Le but de ce chapitre est d'établir le résultat suivant.

Théorème 1.2.2 *Sous l'hypothèse 1.2.1, il existe $N_0(\sup|\phi|, |f|)$ tel que $N \geq N_0$ implique l'existence et unicité de la mesure invariante μ. De plus, il existe C et $\gamma > 0$ tels que, pour toute mesure de probabilité λ de $\mathbb{R}^2$, on ait*

$$\|\mathcal{P}_t^*\lambda - \mu\|_{var-W} \leq Ce^{-\gamma t}\left(1 + \int_{L^2(D)} |u|^2_{L^2(D)} \, \lambda(du)\right),$$

où $\|\cdot\|_{var-W}$ est associée à l'espace produit $L^2(D) = P_N H \times Q_N H$.

Pour décrire l'idée de la preuve, nous allons concentrer notre attention sur le modèle bidimensionnel (0.3.3) précédemment introduit. La preuve rigoureuse pour CGL étant repoussé à l'annexe A. La traduction de l'hypothèse 1.2.1 donne l'opérateur de covariance diagonal et ne dépendant que des bas modes

$$\Sigma(X,Y) = \begin{pmatrix} \sigma_l(X) & 0 \\ 0 & \sigma_h(X) \end{pmatrix},$$

ce qui donne

$$\begin{cases} dX & + & Xdt & + & f(X,Y)dt & = & \sigma_l(X)d\beta, & X(0) = x_0, \\ dY & + & NYdt & + & g(X,Y)dt & = & \sigma_h(X)\,d\eta, & Y(0) = y_0, \end{cases} \tag{1.2.2}$$

où f, g, σ_l et σ_h sont supposés Lipschitz bornés.

On a supposé de plus que le bruit recouvrait les bas modes, ce qui signifie ici que σ_l est uniformément minoré

$$\sigma_0 = \inf_x \sigma_l(x) > 0.$$

Remarquons que, sous ces hypothèses, on a existence et unicité des solutions de que l'on dénotera $u(t,W,(x_0,y_0)) = (X,Y)$. De plus, les solutions vérifient la propriété de Markov forte. On note $(\mathcal{P}_t)_t$ le semi-groupe de Markov associé sur $\mathbb{R}^2$.

Pour ce modèle, le Théorème 1.2.2 devient

Théorème 1.2.3 *Sous les hypothèses précédentes, il existe C et $\gamma > 0$ tels que, pour toute mesure de probabilité λ de $\mathbb{R}^2$, on ait*

$$\|\mathcal{P}_t^*\lambda - \mu\|_{var-W} \leq Ce^{-\gamma t}\left(1 + \int_{\mathbb{R}^2} |u|^2\, \lambda(du)\right), \tag{1.2.3}$$

où μ est une mesure invariante et où $\|\cdot\|_{var-W}$ est associée à l'espace produit $\mathbb{R}^2$.

Cela signifie que, pour tout ψ mesurable bornée par 1 et 1-Lipschitz par rapport à sa seconde variable, on a

$$\left|\mathbb{E}\psi(u(t,W,u_0)) - \int_{\mathbb{R}^2} \psi(u)\,\mu(du)\right| \leq Ce^{-\alpha t}\left(1 + \left|u_0^1\right|^2\right), \tag{1.2.4}$$

pour tout $u_0 \in \mathbb{R}^2$.

Pour une meilleure compréhension, nous allons d'abord traiter diverses versions simplifiées de (1.2.2) que raffinerons progressivement.

1.3 La notion de couplage

Pour résoudre ces problèmes, on est donc amené à introduire la notion de couplage. Un couple de variables aléatoires (Z_1, Z_2) définies sur le même espace est un couplage des lois (μ_1, μ_2) si ces dernières sont les lois marginales du couple.

Il s'agit d'une notion à la fois très simple et très utile. Considérons l'exemple suivant. On pose

$$\mu = \frac{1}{2}\left(\delta_1 + \delta_{-1}\right),$$

et on construit (Z_1, Z_2) un couple de variable aléatoire de loi $\mu \otimes \mu$. On pose ensuite

$$(Z_1', Z_2') = (Z_1, Z_1) \text{ et } (Z_1", Z_2") = (Z_1, -Z_1).$$

On a donc fabriqué trois couplages (Z_1, Z_2), (Z_1', Z_2') et $(Z_1", Z_2")$ de (μ, μ). Pourtant, même si les lois marginales sont fixées, les lois jointes sont très différentes. En particulier, on a

$$\left\{\begin{array}{rcl} \mathbb{P}\left(Z_1 = Z_2\right) & = & \frac{1}{2}, \\ \mathbb{P}\left(Z_1' = Z_2'\right) & = & 1, \\ \mathbb{P}\left(Z_1" = Z_2"\right) & = & 0. \end{array}\right. \tag{1.3.1}$$

Pour un couple de lois fixé, on s'intéressera plus particulièrement au couplage maximisant la probabilité d'être couplé (i.e. que les deux variables soient égalent). Dans l'exemple précédent, il s'agit du couplage (Z_1', Z_2'). Le Lemme suivant assure qu'un tel couplage existe et donne une propriété le caractérisant.

Lemme 1.3.1 *Soit (μ_1, μ_2) un couple de mesures de probabilité sur un espace polonais E. Alors*

$$\|\mu_1 - \mu_2\|_{var} = \min \mathbb{P}(Z_1 \neq Z_2).$$

Le minimum est pris sur tout les couplages (Z_1, Z_2) de (μ_1, μ_2). Un tel couplage existe et est dit maximal.

On remarque que l'on peut raffiner ce Lemme sous la forme suivante contenue dans [45] (quoique non explicitement cité).

Lemme 1.3.2 *Soit $f_0 : E \to F$ une application mesurable entre deux espaces polonais et (Λ_1, Λ_2) deux mesures de probabilités sur E. On pose*

$$\mu_i = f_0^* \Lambda_i, \quad i = 1, 2.$$

Alors il existe un couplage (V_1, V_2) de (Λ_1, Λ_2) tel que $(f_0(V_1), f_0(V_2))$ est un couplage maximal de (μ_1, μ_2).

Pour une preuve des deux résultats précédents, voir l'Annexe A.

Considérons maintenant le cas particulier d'un mouvement brownien d-dimensionnel W défini sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$. Soit h une fonction mesurable de t et de W à valeur dans $\mathbb{R}^d$ qui soit non anticipative et telle que

$$\int_0^\infty |h(t, W)|^2 \, dt \leq M, \tag{1.3.2}$$

pour un $M \in \mathbb{R}^+$.

La condition de Novikov est donc vérifiée pour le drift h. On peut donc appliquer la transformation de Girsanov suivante

$$d\widetilde{\mathbb{P}} = \exp\left(\int_0^\infty h(t, W) \, dW(t) - \frac{1}{2} \int_0^\infty |h(t, W)|^2 \, dt \right) d\mathbb{P}.$$

On sait que, sous $\widetilde{\mathbb{P}}$, le processus drifté $W + \int_0^\cdot h(t, W) dt$ est un mouvement brownien. On peut donc se servir de cette transformation pour estimer la distance en variation totale des lois $\mu_2 = \mathcal{D}(W)$ et $\mu_1 = \mathcal{D}\left(W + \int_0^\cdot h(t, W) dt\right)$ sous $\mathbb{P}$. En clair, on a

$$\|\mu_2 - \mu_1\|_{var} = \left\| \widetilde{\mathbb{P}} - \mathbb{P} \right\|_{var}.$$

On déduit de (1.1.1) que

$$\|\mu_2 - \mu_1\|_{var} = \frac{1}{2} \mathbb{E} \left| e^{\int_0^\infty h(t,W) \, dW(t) - \frac{1}{2} \int_0^\infty |h(t,W)|^2 dt} - 1 \right|. \tag{1.3.3}$$

Traitons d'abord le cas M petit. Une inégalité de Hölder donne

$$\|\mu_2 - \mu_1\|_{var} \leq \frac{1}{2} \sqrt{\mathbb{E} \left(\left| e^{\int_0^\infty h(t,W) \, dW(t) - \frac{1}{2} \int_0^\infty |h(t,W)|^2 dt} - 1 \right|^2 \right)}.$$

On déduit du fait que $e^{\int_0^\cdot h(t,W)\,dW(t)-\frac{1}{2}\int_0^\cdot |h(t,W)|^2dt}$ est une martingale

$$\|\mu_2-\mu_1\|_{var}\leq\frac{1}{2}\sqrt{\mathbb{E}\left(e^{2\int_0^\infty h(t,W)\,dW(t)-\int_0^\infty |h(t,W)|^2dt}\right)-1}.$$

Ce qui donne, d'après (1.3.2),

$$\|\mu_2-\mu_1\|_{var}\leq\frac{1}{2}\sqrt{\mathbb{E}\left(e^{2\int_0^\infty h(t,W)\,dW(t)-\frac{4}{2}\int_0^\infty |h(t,W)|^2dt}\right)e^M-1}.$$

Comme $e^{2\int_0^\cdot h(t,W)\,dW(t)-\frac{4}{2}\int_0^\cdot |h(t,W)|^2dt}$ est une martingale, alors

$$\|\mu_2-\mu_1\|_{var}\leq\frac{1}{2}\sqrt{e^M-1}.$$

Donc, pour $M\leq 2\ln 2$, on a

$$\|\mu_2-\mu_1\|_{var}\leq\sqrt{M}. \tag{1.3.4}$$

Plus généralement, on peut prouver (Voir annexe A) l'existence d'une fonction décroissante p_0 strictement positive telle que pour tout M

$$\|\mu_2-\mu_1\|_{var}\leq 1-p_0(M). \tag{1.3.5}$$

Posons

$$\Lambda_1=\mathcal{D}\left(W,W+\int_0^\cdot h(t,W)dt\right),\ \Lambda_2=\mathcal{D}\left(W,W\right),\ f_0(W_1,W_2)=W_2,$$

et appliquons le Lemme 1.3.2. On obtient

$$V_1=\left(W_1,W_1+\int_0^\cdot h(t,W_1)dt\right),\ V_2=\left(W_2,W_2\right),$$

tel que (W_1,W_2) est un couple de mouvements browniens et tel que $(W_1+\int_0^\cdot h(t,W_1)dt,W_2)$ soit un couplage maximal de (μ_1,μ_2).

Appliquons maintenant le Lemme 1.3.1 et (1.3.4), (1.3.5), on obtient

$$\left\{\begin{array}{lcl}\mathbb{P}\left(W_2=W_1+\int_0^\cdot h(t,W_1)dt\right) & \geq & p_0\left(M\right),\\ \mathbb{P}\left(W_2\neq W_1+\int_0^\cdot h(t,W_1)dt\right) & \leq & \sqrt{M}\text{ si }M\leq 2\ln 2.\end{array}\right. \tag{1.3.6}$$

1.4 Convergence des bas modes

Nous allons d'abord nous intéresser à l'équation des bas modes de (1.2.2).

$$dX + Xdt + f(X)dt = d\beta,\ X(0) = x_0, \tag{1.4.1}$$

où f est supposé Lipschitz bornée.

Nous allons appliquer une méthode de couplage pour prouver le caractère exponentiellement mélangeant en variation totale de (1.4.1).

Tout les modèles successifs que nous considérons ont des solutions uniques et vérifiant la propriété de Markov forte. On dénotera à chaque fois par $(\mathcal{P}_t)_t$ le semi-groupe de Markov associé.

On note $X(\cdot,\beta,x_0)$ l'unique solution de (1.4.1) et on introduit un processus auxiliaire $\widetilde{X}$ qui interpole linéairement x_0^2 et $X(1,\beta,x_0^1)$:

$$\widetilde{X}(t,\beta,x_0^1,x_0^2) = X(t,\beta,x_0^1) + (1-t)^+(x_0^2 - x_0^1).$$

Il suit que $\widetilde{X}$ est solution de

$$d\widetilde{X} + \widetilde{X}dt + f(\widetilde{X})dt = d\beta + hdt, \quad X(0) = x_0^2,$$

où

$$h(t,\beta) = 1_{(0,1)}(t)\left(t(x_0^1 - x_0^2) + f(\widetilde{X}(t)) - f(X(t,\beta,x_0^1))\right).$$

On en déduit que

$$\widetilde{X}(\cdot,\beta,x_0^1,x_0^2) = X\left(\cdot,\beta + \int_0^{\cdot} h(t,\beta)dt, x_0^2\right).$$

On remarque que

$$\widetilde{X}(1,\beta,x_0^1,x_0^2) = X(1,\beta,x_0^1).$$

En combinant ces deux égalités, on obtient que

$$X(1,\beta,x_0^1) = X\left(1,\beta + \int_0^{\cdot} h(t,\beta)dt, x_0^2\right). \tag{1.4.2}$$

On remarque ensuite que

$$|h(t,\beta)| \le 1_{(0,1)}(t)\left|x_0^2 - x_0^1\right|(1 + \mathrm{Lip}f),$$

ce qui donne

$$\int_0^\infty |h(t,\beta)|^2 \, dt \leq c \left|x_0^2 - x_0^1\right|^2 .$$

Ainsi, en appliquant (1.3.6), on est à même de fabriquer un couple de mouvement brownien (β_1, β_2) vérifiant

$$\mathbb{P}\left(\beta_2 = \beta_1 + \int_0^\cdot h(t,\beta_1)dt\right) \geq p_0\left(c\left|x_0^2 - x_0^1\right|^2\right).$$

On déduit donc de (1.4.2) que

$$\mathbb{P}\left(X(1,\beta_1,x_0^1) = X(1,\beta_2,x_0^2)\right) \geq p_0\left(c\left|x_0^2 - x_0^1\right|^2\right). \tag{1.4.3}$$

En appliquant les Lemmes 1.3.1 et 1.3.2, on fabrique $(V_1(\cdot,x_0^1,x_0^2), V_2(\cdot,x_0^1,x_0^2))$ couplage de $(\mathcal{D}\left(X(\cdot,\beta,x_0^1)\right), \mathcal{D}\left(X(\cdot,\beta,x_0^2)\right))$ sur $(0,1)$ tel que $(V_1(1,x_0^1,x_0^2), V_2(1,x_0^1,x_0^2))$ soit un couplage maximal. On déduit de (1.4.3) et du caractère maximal du couplage que

$$\mathbb{P}\left(V_1(1,x_0^1,x_0^2) = V_2(1,x_0^1,x_0^2)\right) \geq p_0\left(c\left|x_0^2 - x_0^1\right|^2\right). \tag{1.4.4}$$

De plus, en raffinant (Voir l'appendice de l'Annexe A), on peut montrer que l'on peut fabriquer (V_1, V_2) dépendant mesurablement de (x_0^1, x_0^2).

Qui plus est, il est facile d'imposer à (V_1, V_2) la condition suivante

$$V_1(\cdot,x_0,x_0) = V_2(\cdot,x_0,x_0) \text{ pour tout } x_0 \in \mathbb{R}. \tag{1.4.5}$$

On fabrique maintenant un couplage (X_1, X_2) de $(\mathcal{D}\left(X(\cdot,\beta,x_0^1)\right), \mathcal{D}\left(X(\cdot,\beta,x_0^2)\right))$ par induction sur $(0,n)$. On pose d'abord

$$X_i(0) = x_0^i, \; i = 1,2.$$

Puis, une fois qu'on a fabriquer (X_1, X_2) sur $(0,n)$, on prend une version de (V_1, V_2) indépendante de (X_1, X_2) et on pose

$$X_i(n+\cdot) = V_i(\cdot, X_1(n), X_2(n)), \; i = 1,2.$$

On déduit de cette construction trois propriétés majeures. La première découle de (1.4.4) et signifie que la probabilité que X_1 et X_2 se couplent au temps 1 est minorée uniformément sur toute boule. La seconde découle de (1.4.5) et signifie que si $(X_1, X_2))$ sont couplés (i.e. égaux) au temps n, alors ils le restent pour tout les temps à venir. La troisième propriété est le caractère fortement Markov du processus discret $(X_1(n), X_2(n))_n$. De plus, il est facile de voir qu'il existe $\delta > 0$ tel que le temps de retour de (X_1, X_2) dans la boule de rayon δ admet un moment exponentiel. On peut donc itérer nos propriétés. En clair, on attend d'entrer dans la boule de rayon δ, puis on essaie de coupler (X_1, X_2). Si ça marche, alors ils restent couplés éternellement. Sinon, on attend de rentrer de nouveau dans la boule et on recommence. Au final, on tente un nombre fini de fois sa chance, ce qui assure que

$$\mathbb{P}\left(X_1(s) \neq X_2(s) \ \text{ pour un } s > t\right) \leq Ce^{-\alpha t}(1 + \left|x_0^1\right|^2 + \left|x_0^1\right|^2). \tag{1.4.6}$$

Pour une preuve rigoureuse de ce genre d'itération, voir la section 4 de l'Annexe E.

Appliquons le Lemme 1.3.1, on obtient

$$\left\|\mathcal{P}_t^*\delta_{x_0^2} - \mathcal{P}_t^*\delta_{x_0^1}\right\|_{var} \leq Ce^{-\alpha t}(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2). \tag{1.4.7}$$

En intégrant (x_0^1, x_0^2) par $\lambda \otimes \mu$ où μ est une mesure invariante, on obtient

$$\left\|\mathcal{P}_t^*\lambda - \mu\right\|_{var} \leq Ce^{-\alpha t}(1 + \left|x_0^1\right|^2). \tag{1.4.8}$$

1.5 Convergence d'un modèle décorélé

Nous allons maintenant considérer une version décorélée du modèle (1.2.2). En clair, on a retiré toute dépendance par rapport à Y dans l'équation des bas modes.

$$\begin{cases} dX \ + \ Xdt \ + \ f(X)dt \ = \ \sigma_l(X)d\beta, \quad X(0) = x_0, \\ dY \ + \ NYdt \ + \ g(X,Y)dt \ = \ \sigma_h(X)\,d\eta, \quad Y(0) = y_0, \end{cases} \tag{1.5.1}$$

où f, g, σ_l et σ_h sont Lipschitz bornées et où $W = (\beta, \eta)$ est un mouvement brownien dans $\mathbb{R}^2$. On dénote par $X(\cdot, \beta, x_0)$ l'unique solution de la première équation de ce système.

On rappelle l'hypothèse de non-dégénérescence du bruit sur les bas modes

$$\sigma_0 = \inf_x \sigma_l(x) > 0.$$

Il est possible, modulo certains aménagements, de fabriquer (X_1, X_2) comme précédemment tel que (1.4.6) soit vérifié même lorsque $\sigma_l \neq 1$. Ce n'est pas le point qui nous intéresse ici. Pour le voir, il suffit de combiner des idées de l'annexe A et la preuve de la partie précédente.

Traitons maintenant la variable Y. Soit η un mouvement brownien indépendant de (X_1, X_2). On dénotera par $\phi(X, \eta, y_0)$ l'unique solution de la deuxième équation du système (1.5.1). Il est facile de voir que le système (1.5.1) vérifie une inégalité de type Foias-Prodi. Soient X un processus adapté et $(y_0^1, y_0^2) \in \mathbb{R}^2$. Posons $r = \phi(X, \eta, y_0^2) - \phi(X, \eta, y_0^1)$, on obtient

$$\frac{d}{dt}|r|^2 + 2N|r|^2 = -2r\left(f(X, \phi(X, \eta, y_0^2)) - f(X, \phi(X, \eta, y_0^1))\right) \leq 2\mathrm{Lip} f\,|r|^2.$$

En intégrant, on obtient donc l'inégalité de type Foias-Prodi

$$\left|\phi(X, \eta, y_0^2)(t) - \phi(X, \eta, y_0^1)(t)\right|^2 \leq \left|y_0^2 - y_0^1\right|^2 e^{-2t(N - \mathrm{Lip} f)}. \tag{1.5.2}$$

On pose

$$Y_i \phi(X_i, \eta, y_0^i), \quad i = 1, 2.$$

Ainsi, si $N > 2\mathrm{Lip} f$ et si X_1 et X_2 sont couplés, alors la distance entre Y_1 et Y_2 décroît exponentiellement vite.

En combinant (1.4.6) et (1.5.2), on déduit l'existence et l'unicité de la mesure invariante μ et la convergence à vitesse exponentielle des solutions pour le modèle (1.5.1). Plus précisément, si on considère ψ mesurable bornée par 1 et 1-Lipschitz par rapport à sa seconde variable, on obtient

$$\left|\mathbb{E}\psi(u(t, W, u_0)) - \int_{\mathbb{R}^2} \psi(u)\,\mu(du)\right| \leq Ce^{-\alpha t}\left(1 + \left|u_0^1\right|^2\right),$$

pour tout $u_0 \in \mathbb{R}^2$. Ce qui assure (1.2.4).

1.6 Convergence du modèle

Nous allons maintenant établir (1.2.4) pour notre modèle (1.2.2). Rappelons qu'il se met sous la forme

$$\begin{cases} dX + Xdt + f(X,Y)dt = \sigma_l(X)d\beta, & X(0) = x_0, \\ dY + NYdt + g(X,Y)dt = \sigma_h(X)\,d\eta, & Y(0) = y_0, \end{cases}$$

et que l'on a fait l'hypothèse

$$\sigma_0 = \inf_x \sigma_l(x) > 0.$$

Ce cadre est représentatif des papiers qui nous ont précédés ([29], [39], [40], [41], [45] et [56]), la méthode employée y est la même.

L'introduction de la dépendance en Y de f n'as pas incidence majeure sur trois des quatre propriétés que nous avons établies pour le modèle précédent et qui assuraient la convergence.

Premièrement, le temps de retour des solutions dans une boule de rayon suffisamment élevée admet toujours un moment exponentiel. Deuxièmement, il est possible de fabriquer un couplage (u_1, u_2) des lois des solutions dont on peut minorer la probabilité pour (X_1, X_2) d'être couplé au temps 1. Troisièmement, l'inégalité de type Foias-Prodi (1.5.2) reste vraie si on dénote par $\phi(X, \eta, y_0)$ l'unique solution de la deuxième équation du système (1.2.2).

Malheureusement, la dépendance en Y de f rend *a priori* fausse l'affirmation selon laquelle on peut imposer à (X_1, X_2) de rester couplé indéfiniment si $x_0^1 = x_0^2$ et $y_0^1 \neq y_0^2$.

L'idée pour contourner cette difficulté est que pour que notre preuve marche, il suffit de fabriquer un couplage des lois des solutions vérifiant une propriété plus faible quand $x_0^1 = x_0^2$. D'une part, on veut pouvoir minorer (uniformément sur tout borné) la probabilité pour (X_1, X_2) de rester couplé indéfiniment. D'autre part, lorsque (X_1, X_2) découplent, on veut que le temps de découplage admette un moment exponentiel. Il est naturel de s'attendre à ce que, comme dans le cas précédent, cette propriété associée aux trois premières assure la convergence à vitesse exponentielle des solutions. La preuve est néanmoins plus complexe. En bref, on attend d'entrer dans la grande boule et on tente de coupler une unité de temps plus tard. Si ça marche, on tente de le maintenir coupler éternellement. Sinon on attend de retourner dans la grande boule pour retenter notre chance. Si ça marche, on a gagné. Si ça échoue, on sait que le temps de découplage admet un moment exponentiel. On peut donc recommencer depuis le début. Il est facile de voir qu'au bout d'un nombre fini de tentative, ça marche et que le moment à partir duquel les bas modes des solutions sont couplées éternellement admet un moment exponentiel, ce qui permet de conclure comme précédemment en utilisant le fait que si les bas modes sont égaux, alors la distance entre les hauts modes tend vers zéro exponentiellement vite.

Il reste donc à prouver cette dernière propriété. Notons x_0 la valeur de $x_0^1 = x_0^2$. Comme f est borné, on peut, via une transformation de Girsanov, fabriquer un couple (W_1, W_2) de mouvements browniens dont les solutions associées (u_1, u_2) aient même bas modes $X_1 = X_2$ et même bruit des haut modes $\eta_1 = \eta_2$ sur intervalle borné quelconque avec une probabilité minorée. Le problème, c'est de prouver que cette probabilité ne tend pas vers 0 quand l'intervalle devient $\mathbb{R}^+$. Moralement, l'idée est la suivante. Supposons qu'on ait $(X_1, \eta_1) = (X_2, \eta_2)$ pendant un long laps de temps. Alors Y_1 et Y_2 se rapprochent exponentiellement vite. Donc les fonctions $f(\cdot, Y_1)$ et $f(\cdot, Y_2)$ deviennent de plus en plus similaires. Ainsi les lois de (X_1, X_2) se rapprochent, ce qui accroît les chances pour (X_1, X_2) de rester couplé après ce long laps de temps. En conséquence Y_1 et Y_2 peuvent continuer à se rapprocher... et ainsi de suite. La vitesse de convergence de la distance entre (Y_1, Y_2) est exponentielle et donc en particulier sommable, ce qui assure que la quantité de (X_1, X_2) couplé ne tende pas vers 0 à l'infini.

Pour justifier rigoureusement ce qui vient d'être dit, on note $\phi(X, \eta, y_0)$ l'unique solution de la deuxième équation de (1.2.2). On note $u_1 = (X_1, Y_1)$ la solution du système (1.2.2) avec $y_0 = y_0^1$. On remarque que X_1 est l'unique solution de l'équation suivante

$$dX_1 + X_1 dt + f(X_1, \phi(X_1, \eta, \mathbf{y_0^1}))dt = \sigma_l(X_1) d\beta, \quad X(0) = x_0. \tag{1.6.1}$$

Ainsi, si l'on pose

$$h(t) = \sigma_l(X_1(t))^{-1} \left(f(X_1, \phi(X_1, \eta, y_0^2) - f(X_1, \phi(X_1, \eta, y_0^1)) \right), \tag{1.6.2}$$

on obtient que ce même X_1 vérifie

$$dX_1 + X_1 dt + f(X_1, \phi(X_1, \eta, \mathbf{y_0^2}))dt = \sigma_l(X_1) \left(d\beta + \mathbf{hdt} \right), \quad X(0) = x_0.$$

Ainsi donc la solution de (1.6.1) avec $y_0 = y_0^1$ est solution de (1.6.1) avec $y_0 = y_0^2$ et $d\beta = d\beta + h(t)dt$. Une transformation de Girsanov du mouvement brownien β associée au drift h ferait donc muter la loi d'une solution associée à $y_0 = y_0^1$ en celle d'une solution associée à $y_0 = y_0^2$. Pour utiliser ce résultat dans le but de fabriquer un couplage des lois des X qui reste couplé indéfiniment avec une probabilité minorée, il reste à contrôler le drift h.

Pour ce faire, on intègre en temps l'inégalité de type Foias-Prodi (1.5.2). Ce qui donne que pour $N > 2\text{Lip} f$, on a

$$\int_t^\infty \left| \phi(X, \eta, y_0^2)(s) - \phi(X, \eta, y_0^1)(s) \right|^2 ds \leq \frac{1}{N} \left| y_0^2 - y_0^1 \right|^2 e^{-Nt}.$$

On déduit du caractère Lipschitz de f et du fait que $\sigma_0 = \inf_x \sigma_l(x) > 0$ que

$$\int_t^\infty |h(s)|^2 \, ds \leq \frac{1}{N} \left(\frac{\mathrm{Lip} f}{\sigma_0} \right)^2 \left| y_0^2 - y_0^1 \right|^2 e^{-Nt}.$$

En procédant par induction sur l'intervalle de temps $(0, n)$, on peut ainsi appliquer (1.3.6) et obtenir une minoration de la probabilité de couplage de (X_1, X_2) sur $\mathbb{R}^+$ et le moment exponentiel du temps de découplage quand celui-ci est fini. Ce qui permet de conclure

Ainsi, les solutions de (1.2.2) convergent à vitesse exponentielle de la même manière que pour le modèle précédent : Pour tout ψ mesurable bornée par 1 et 1-Lipschitz par rapport à sa seconde variable, on a

$$\left| \mathbb{E}\psi(u(t, W, u_0)) - \int_{\mathbb{R}^2} \psi(u) \, \mu(du) \right| \leq C e^{-\alpha t} \left(1 + \left| u_0^1 \right|^2 \right),$$

pour tout $u_0 \in \mathbb{R}^2$. Ce qui assure (1.2.4).

Chapitre 2

Ergodicité d'une équation de Schrödinger non-linéaire amortie

Introduction

Nous nous sommes ensuite demander si la méthode développée dans le chapitre précédent pouvait se généraliser à des équations non fortement dissipatives. Nous avons considéré le cas de l'équation de Schrödinger non linéaire (NLS) cubique focalisante unidimensionnelle ($d = 1$, $\sigma = 1$ et $\lambda = -1$) avec opérateur ϕ diagonal constant. Sous ces hypothèses, (0.1.1) se réécrit

$$\begin{cases} du + \alpha u\, dt - \mathrm{i}\Delta u\, dt - \mathrm{i}\, |u|^2\, u\, dt = \phi dW, \\ \quad u(t,x) \quad = \quad 0, \qquad\quad \text{for } x \in \{0,1\},\ t > 0, \\ \quad u(0,x) \quad = \quad u_0(x), \quad \text{for } x \in [0,1], \end{cases} \tag{2.0.1}$$

où $\alpha > 0$.

2.1 Résultat principal

Rappelons que

$$A = -\Delta, \quad D(A) = H_0^1(D) \cap H^2(D),$$

et que $(e_n, \mu_n)_n$ est la suite des vecteurs et valeurs propres associés à A tel que $(\mu_n)_n$ soit une suite croissante. De plus, P_N (resp Q_N) est le projecteur orthogonal de $L^2(D)$ sur l'espace engendré par $\{e_1, \dots, e_N\}$ (resp $\{e_{N+1}, \dots\}$).

On peut maintenant réécrire (0.1.2) sous la forme

$$\begin{cases} du + \alpha u\, dt + \mathrm{i} A u\, dt - \mathrm{i}\, |u|^2\, u\, dt = \phi dW, \\ u(0) = u_0, \end{cases} \tag{2.1.1}$$

où W est un processus de Wiener cylindrique sur $L^2(D)$.

On fera l'hypothèse suivante.

Hypothèse 2.1.1 *L'opérateur $\phi \in \mathcal{L}_2(L^2(D); D(A^{\frac{3}{2}}))$ commute avec A. De plus, il existe $N \in \mathbb{N}$ tel que $P_N \phi P_N$ est inversible.*

Remarquons que, sous l'hypothèse 2.1.1, on a existence et unicité des solutions de que l'on dénotera $u(t, W, (x_0, y_0))$. De plus, les solutions vérifient la propriété de Markov forte. On note $(\mathcal{P}_t)_t$ le semi-groupe de Markov associé sur H.

Le but de ce chapitre est d'établir le résultat suivant.

Théorème 2.1.2 *Sous l'hypothèse 2.1.1, il existe $N_0(|\phi|)$ tel que $N \geq N_0$ implique l'existence et l'unicité de la mesure invariante μ. De plus, il existe C et $\gamma > 0$ tels que, pour toute mesure de probabilité λ de $\mathbb{R}^2$, on ait*

$$\|\mathcal{P}_t^* \lambda - \mu\|_{var-W} \leq C e^{-\gamma t} \left(1 + \int_{L^2(D)} |u|_{L^2(D)}^2\, \lambda(du) \right),$$

où $\|\cdot\|_{var-W}$ est associée à l'espace produit $L^2(D) = P_N H \times Q_N H$.

Nous reporterons la preuve rigoureuse du caractère mélangeant des solutions de (2.0.1) à l'annexe B et nous traiterons ici l'exemple d'un modèle bidimensionnel.

2.2 Un modèle bidimensionnel représentatif.

Rappelons (0.3.5). Si on considère NLS du point de vue L^2, on est naturellement amené à la modéliser par le système suivant

$$\begin{cases} dX + Xdt + f(X,Y)dt = \sigma_l(X,Y)dW, & X(0) = x_0, \\ dY + Ydt + g(X,Y)dt = \sigma_h(X,Y)dW, & Y(0) = y_0. \end{cases}$$

Malheureusement, l'inégalité de type Foias-Prodi pose problème. Il s'agit en effet d'une propriété des équations fortement dissipatives et il semble illusoire de vouloir la prolonger à NLS dans L^2 ou à ce modèle.

L'idée est de montrer que les propriétés de dispersivité de NLS rendent le terme d'amortissement αu dissipatif. Pour ce faire, on observe NLS du point de vue H^1 dans l'esprit de [28]. Plus précisément, dans l'annexe B, on introduit une fonctionnelle $J(u_1, u_2, u_2 - u_1)$ qui contrôle la norme de l'espace de Sobolev H^1 de la différence.

$$J(u_1, u_2, u_2 - u_1) \geq |u_2 - u_1|_{H^1}^2 . \tag{2.2.1}$$

La formule d'Ito de J donne

$$dJ + \alpha J dt \leq dM + C(u_1, u_2) \, |u_2 - u_1|_{H^1}^{2-\gamma_1} |u_2 - u_1|_{L^\infty}^{\gamma_1} \, dt,$$

où M est une martingale.

L'inégalité de Agmon donne donc

$$dJ + \alpha J dt \leq dM + C(u_1, u_2) \, |u_2 - u_1|_{H^1}^{2-\gamma} |u_2 - u_1|_{L^2}^{\gamma} .$$

Si l'on suppose maintenant que u_2 et u_1 ont même bas modes (i.e. mêmes coefficients de Fourier d'ordre inférieur à N), alors on a

$$|u_2 - u_1|_{L^2} \leq \frac{c}{N} \, \|u_2 - u_1\|_{H^1}$$

ce qui donne, d'après (2.2.1),

$$dJ + \alpha J dt \leq dM + \frac{1}{N^\gamma} C(u_1, u_2) J. \tag{2.2.2}$$

Nous sommes donc amené à raffiner notre modèle bidimensionnel (0.3.5) sous la forme

$$\left\{ \begin{array}{rcccccl} dX & + & X dt & + & f_N(X,Y) dt & = & \sigma_N^l(X,Y) \, dW, \quad X(0) = x_0, \\ dY & + & Y dt & + & g_N(X,Y) dt & = & \sigma_N^h(X,Y) \, dW, \quad Y(0) = y_0, \end{array} \right. \tag{2.2.3}$$

où l'on impose aux constantes de Lipschitz de f_N, g_N, σ_N^l et σ_N^h par rapport à la seconde variable Y d'être majorées par $\frac{L}{N}$.

2.3 Convergence du modèle bidimensionnel.

Rappelons maintenant que nous avons supposé l'opérateur ϕ diagonal constant. Dans notre cas, cela donne

$$\left\{\begin{array}{lllllll} dX & + & Xdt & + & f_N(X,Y)dt & = & d\beta, \qquad\qquad X(0)=x_0, \\ dY & + & Ydt & + & g_N(X,Y)dt & = & \sigma_N(X,Y)\,d\eta, \quad Y(0)=y_0, \end{array}\right. \tag{2.3.1}$$

où le terme $\sigma_N(X,Y)d\eta$ modélise les termes résiduels du à l'introduction de notre fonctionnel J. La constante de Lipschitz de σ_N par rapport à Y est, elle aussi, supposée bornée par $\frac{L}{N}$. On s'aperçoit que l'inégalité de type Foias-Prodi (1.5.2) ne peut être obtenu et qu'il va falloir l'affaiblir de deux manières.

On dénote par $\phi(X,\eta,y_0)$ la solution de la deuxième équation du système (2.3.1). Soient X un processus adapté et $(y_0^1,y_0^2)\in\mathbb{R}^2$. Posons $r=\phi(X,\eta,y_0^2)-\phi(X,\eta,y_0^1)$, la formule d'Ito de $|r|^2$ donne

$$d\,|r|^2+2\,|r|^2\,dt=dM+(I_1+I_2)dt,$$

où

$$\left\{\begin{array}{lll} M & = & \int_0^{\cdot}\left(\sigma_N(X,\phi(X,\eta,y_0^2))-\sigma_N(X,\phi(X,\eta,y_0^1))\right)d\eta, \\ I_1 & = & -2r\left(f_N(X,\phi(X,\eta,y_0^2))-f_N(X,\phi(X,\eta,y_0^1))\right), \\ I_2 & = & \left|\sigma_N(X,\phi(X,\eta,y_0^2))-\sigma_N(X,\phi(X,\eta,y_0^1))\right|^2. \end{array}\right.$$

Le caractère Lipschitz de σ_N et f_N donne

$$d\,|r|^2+2\left(1-\frac{L}{N}\right)|r|^2\,dt\le dM. \tag{2.3.2}$$

Considérons d'abord le cas déterministe ($\sigma_N=0$). Intégrons (2.3.2). Cela donne

$$\frac{d}{dt}\,|r|^2+2\,|r|^2\le 2\frac{L}{N}\,|r|^2.$$

On obtient donc l'inégalité de type Foias-Prodi "faible"

$$\left|\phi(X,\eta,y_0^2)(t)-\phi(X,\eta,y_0^1)(t)\right|^2\le\left|y_0^2-y_0^1\right|^2e^{-2t\left(1-\frac{L}{N}\right)}. \tag{2.3.3}$$

On obtient assez facilement que cette inégalité de type Foias-Prodi "faible" assure que, pour N assez grand, la distance entre deux solutions $\phi(X,\eta,y_0^1)$ et $\phi(X,\eta,y_0^2)$ de la deuxième équation du système (2.3.1) associées au même X tend vers 0 exponentiellement vite.

Considérons maintenant le cas général. Nous avons introduit le terme $\sigma_N(X,Y)d\eta$ car des termes stochastiques de ce type apparaissent naturellement du fait des fonctionnelles que nous utilisons pour travailler dans H^1. Ce terme modélise donc bien les ennuis que nous rencontrons, même si nous ne travaillons en réalité qu'avec des opérateurs constants. En particulier, l'intégrale stochastique M apparaît dans (2.3.2). Il devient alors difficile d'obtenir une inégalité de type Foias-Prodi trajectorielle comme (2.3.3). En clair, la distance entre deux solutions $\phi(X,\eta,y_0^1)$ et $\phi(X,\eta,y_0^2)$ de la deuxième équation du système (2.3.1) n'est plus *a priori* majorée par un terme du type Ce^{-t} quand N est assez grand. Et cela même avec C aléatoire. Il faut alors se contenter d'une inégalité de type Foias-Prodi en espérance. Intégrons (2.3.2). Pour tout processus X on a

$$\mathbb{E}\left|\phi(X,\eta,y_0^2)(t)-\phi(X,\eta,y_0^1)(t)\right|^2 \leq \left|y_0^2-y_0^1\right|^2 e^{-2t(1-\frac{L}{N})}. \tag{2.3.4}$$

Il faut alors montrer que la preuve de la convergence des solutions peut être adapté si on remplace l'inégalité de type Foias-Prodi trajectorielle (1.5.2) par l'inégalité de type Foias en espérance (2.3.4). Rappelons que cet ingrédient essentiel intervient en effet deux fois dans la preuve.

D'une part on s'en sert pour prouver que si un couplage (u_1,u_2) des lois des solutions a les même bas modes $X_1=X_2$ et est associé au même bruit dans les haut modes $\eta_1=\eta_2$, alors la distance entre les haut modes (Y_1,Y_2) des solutions tend vers 0 exponentiellement vite, pourvu que N soit assez grand. Ce résultat est toujours vrai. On doit juste remplacé la convergence presque sure par la convergence en probabilité.

D'autre part, on s'en sert pour contrôler la norme L^2 en temps du drift $h(t)$ défini en (1.6.2) (avec $\sigma_l(X)=1$). En intégrant (2.3.4), on obtient pour N assez grand

$$\mathbb{E}\left(\int_t^\infty |h(s)|^2\,ds\right) \leq \left(\frac{L}{N}\right)^2 \left|y_0^2-y_0^1\right|^2 e^{-t}.$$

Rappelons que ce drift h sert à transformer la loi de $X(\cdot,W,(x_0,y_0^1))$ en celle de $X(\cdot,W,(x_0,y_0^2))$. Le contrôle obtenu sur sa norme L^2 ne nous permet pas d'appliquer directement (1.3.6). Néanmoins, via une troncature, on peut faire le même raisonnement et ainsi de fabriquer un couplage (X_1, X_2) de lois des solutions dont on minore la probabilité d'être couplé éternellement.

La preuve du cas fortement dissipatif est donc adaptable à ce modèle et à NLS. D'une certaine manière, on peut dire que l'on n'a jamais quitté le cadre dissipatif. Certes, du point de vue L^2, le faible amortissement de NLS ne crée aucune dissipation et on ne peut donc rien faire (Comme dans notre premier modèle (0.3.5)). Néanmoins, du point de vue H^1, ce faible amortissement combiné avec les propriétés de dispersivités de NLS permet de faire apparaître une dissipativité dans la non-linéarité. C'est là le sens profond de la condition sur la constante de Lipschitz par rapport à Y de f_N, g_N et σ_N dans notre second modèle (2.3.1). En conclusion, pour NLS, la dissipativité est une question de point de vue et plus précisément d'espace de travail.

Néanmoins, dans l'annexe B, nous n'obtiendrons pas la convergence exponentielle pour NLS. En effet, pour appliquer cette méthode à NLS, nous avons du utiliser des troncatures énergétiques qui ont limité la vitesse de convergence. Nous avons établi que cette vitesse est plus rapide que n'importe quelle puissance négative du temps, mais nous ignorons si elle est exponentielle. Une autre difficulté a été l'introduction des bonnes fonctionnelles qui nous ont amenées à traiter la subtilité cruciale ci dessous. De plus, remplacer l'inégalité de type Foias-Prodi trajectorielle par celle en espérance s'est avéré plus complexe que pour notre modèle.

2.4 Une subtilité cruciale.

Nous allons maintenant mettre l'accent sur une subtilité qui à une importance cruciale, mais qui n'apparaît pas de manière évidente dans nos modèles. Dans les modèles que nous avons étudiés, (2.3.4) est vrai indépendamment du processus X considéré. Dans le cas de NLS, la situation est tout à fait différente. Ne pouvant contrôler directement le terme $\mathbb{E}\left|\phi(X,\eta,y_0^2)(t)-\phi(X,\eta,y_0^1)(t)\right|^2$, nous avons contourné la difficulté en introduisant une fonctionnelle $J(u_1,u_2,u_2-u_1)$ qui majore $|u_2-u_1|^2$. De fait, il ne semble pas possible d'obtenir (2.3.4) pour tout processus X.

On est alors tenté de vouloir contrôler l'espérance de cette fonctionnelle évaluée pour un couple de solutions (u_1,u_2) lorsqu'elles ont même bas modes X et même bruit dans les haut modes η. On obtiendrait ainsi une inégalité de type Foias-Prodi en espérance

$$\mathbb{E}\left(|Y_2(t)-Y_1(t)|^2\, 1_{(X_1,\eta_1)=(X_2,\eta_2)}\right) \leq e^{-2t(1-\frac{L}{N})}\left|y_0^2-y_0^1\right|^2.$$

Et l'on serait tenter de conclure en refaisant la preuve précédente.

Ce n'est malheureusement pas possible car l'inégalité précédente n'est pas celle dont nous avons besoin. L'inégalité (2.3.4) nous permet de contrôler le drift h défini en (1.6.2) de la transformation de Girsanov qui transforme la loi de la première variable X d'une solution en celle d'une autre solution.

On considère donc une solution $u=(X,Y)$ associé au processus de Wiener $W=(\beta,\eta)$ et à la condition initiale (x_0,y_0^1). Dans notre preuve, nous avons exactement besoin de démontrer que (2.3.4) est vrai pour ce processus X. Il permettra à la fois d'obtenir la majoration du drift, mais aussi de contrôler la distance entre deux solutions lorsque leurs bas modes sont couplés. Pour NLS on établit que

$$\mathbb{E}J(u,\widetilde{u},\widetilde{u}-u)(t) \leq \left|y_0^2-y_0^1\right|^2 e^{-2t(1-\frac{L}{N})}.$$

où on a définit le processus auxiliaire par

$$\widetilde{u}=(X(\cdot,W,(x_0,\mathbf{y_0^1})),\phi(X(\cdot,W,(x_0,\mathbf{y_0^1})),\eta,\mathbf{y_0^2})).$$

Il important de noter que $\widetilde{u}$ n'a pas la loi d'une solution. On peut considérer $\widetilde{u}$ de deux manières. La première, donnée par la définition ci dessus, est celle d'un processus hybride composé avec les bas modes d'une première solution associé au conditions initiales (x_0,y_0^1) et les hauts modes reconstruits comme solution de la deuxième équation de (2.3.1) avec condition initiale (x_0,y_0^2) mais associé à $X=X(\cdot,W,(x_0,\mathbf{y_0^1})$. En clair, on considère le système suivant

$$\left\{\begin{array}{lllllllll} d\widetilde{X} & + & \widetilde{X}dt & + & f_N(\widetilde{X},\mathbf{Y})dt & = & d\beta, & & \widetilde{X}(0)=x_0,\\ dY & + & Y dt & + & g_N(\widetilde{X},Y)dt & = & \sigma_N(\widetilde{X},Y)\,d\eta, & & Y(0)=\mathbf{y_0^1},\\ d\widetilde{Y} & + & \widetilde{Y}dt & + & g_N(\widetilde{X},\widetilde{Y})dt & = & \sigma_N(\widetilde{X},\widetilde{Y})\,d\eta, & & Y(0)=\mathbf{y_0^2}, \end{array}\right.$$

et on note $(\widetilde{X},Y,\widetilde{Y})$ son unique solution. On a alors

$$\widetilde{u}=(\widetilde{X},\widetilde{Y}).$$

Cette première approche à l'intérêt de donner l'inégalité de type Foias-Prodi dans le cas de NLS.

La seconde approche est de voir $\widetilde{u}$ comme une solution du modèle associé à la condition initiale (x_0, y_0^2) et à un processus de Wiener "drifté" $W = (\beta + \int_0^{\cdot} h(t)dt, \eta)$ dont on contrôle la norme L^2 du drift h :

$$\begin{cases} d\widetilde{X} & + & \widetilde{X}dt & + & f_N(\widetilde{X}, \widetilde{\mathbf{Y}})dt & = & d\beta + \mathbf{hdt}, & \widetilde{X}(0) = x_0, \\ d\widetilde{Y} & + & \widetilde{Y}dt & + & g_N(\widetilde{X}, \widetilde{Y})dt & = & \sigma_N(\widetilde{X}, \widetilde{Y})\, d\eta, & Y(0) = \mathbf{y_0^2}, \end{cases}$$

Cette seconde façon de voir à pour intérêt d'autoriser la Transformation de Girsanov et ainsi de permettre le couplage des bas modes.

La découverte de cette subtilité a eu des effets désastreux sur la preuve de la convergence des solutions de NLS. En effet, suite à des problèmes de troncatures énergétiques, cette subtilité nous a empêché de traiter le cas d'un opérateur de covariance diagonal dépendant des bas modes des solutions. Nous avons donc du nous contenter de prouver le cas diagonal constant.

Cette subtilité a eu néanmoins un effet positif. Elle nous a permis d'acquérir une meilleure compréhension de ce que nous faisions, ce qui a conduit à la généralisation qui va suivre.

Chapitre 3

Le cas non additif

Introduction

Se pose maintenant la question de généraliser cette méthode à des opérateurs de covariance plus généraux. Jusqu'à présent, nous n'avons considéré que des opérateurs diagonaux et ne dépendant que des bas modes des solutions et ce pour des raisons purement techniques (on ne savait pas traiter d'autre cas). Il n'y a en effet aucune raison physique de se restreindre à cette classe d'opérateur. Comme nous l'avons déjà écrit, l'hypothèse naturelle est le recouvrement des modes instables par le bruit.

3.1 Résultats principaux

Considérons d'abord NS. Soient H et V les fermetures de l'ensemble des fonctions régulières à support compact dans D et à divergence nulle par les normes de $L^2(D;\mathbb{R}^2)$ et $H_0^1(D;\mathbb{R}^2)$ respectivement. Soit Π le projecteur orthogonal de $L^2(D;\mathbb{R}^2)$ sur H. Posons

$$A = \Pi\,(-\Delta)\,, \quad D(A) = V \cap H^2(D;\mathbb{R}^2) \;\text{ et }\; B(u) = \Pi\,((u,\nabla)u)\,.$$

On peut réécrire (0.1.3) sous la forme

$$\left\{ \begin{array}{rcl} du + \nu Audt + B(u)dt & = & \phi(u)dW + fdt, \\ u(0) & = & u_0, \end{array} \right. \tag{3.1.1}$$

où W est un processus de Wiener cylindrique sur l'espace de Hilbert H.

Notre hypothèse de recouvrement des modes instables par le bruit s'écrit donc

Hypothèse 3.1.1 *La fonction $\phi : H \to \mathcal{L}_2(H;H)$ est Lipschitz bornée et $f \in H$. De plus, il existe $N \in \mathbb{N}$ et une fonction mesurable bornée $g : H \to \mathcal{L}(H;H)$ telle que, pour tout $u \in H$*

$$\phi(u)g(u) = P_N.$$

On pose

$$B_0 = 1 + \sup_{u\in H} \|\phi(u)\|^2_{\mathcal{L}_2(H;H)} + |f|^2_H, \quad L = (\mathrm{Lip}\,\phi)^2.$$

L'un des buts de ce chapitre est de prouver que le semi-groupe $(\mathcal{P}_t)_t$ associé aux solutions de (3.1.1) vérifie le résultat suivant.

Théorème 3.1.2 *Supposons l'Hypothèse 3.1.1 vérifiée. Alors il existe $N_0(B_0, L)$ tel que, si $N \geq N_0$, alors $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ admet une unique mesure de probabilité invariante μ sur H et il existe $C, \gamma > 0$, tels que pour $\lambda \in \mathcal{P}(H)$*

$$\|\mathcal{P}_t^*\lambda - \mu\|_* \leq Ce^{-\gamma t}\left(1 + \int_H |u|^2\, d\lambda(u)\right). \tag{3.1.2}$$

Considérons maintenant CGL. On reprend les notations du chapitre 1 et on fait l'hypothèse suivante.

Hypothèse 3.1.3 *La fonction $\phi : H \to \mathcal{L}_2(L^2(D); L^2(D))$ est bornée et localement Lipschitz et $f \in L^2(D)$. Plus précisément*

$$\|\phi(u_2) - \phi(u_1)\|^2_{\mathcal{L}_2(L^2(D);L^2(D))} \leq L\,|u_2 - u_1|^2 \left(1 + |u_1|^{2\sigma} + |u_2|^{2\sigma}\right).$$

De plus, il existe $N \in \mathbb{N}$ et une fonction mesurable bornée $g : H \to \mathcal{L}(L^2(D); L^2(D))$ telle que, pour tout $u \in H$

$$\phi(u)g(u) = P_N.$$

On pose

$$B_0 = 1 + \sup_{u\in L^2(D)} \|\phi(u)\|^2_{\mathcal{L}_2(L^2(D);L^2(D))} + |f|_{L^2(D)}.$$

L'autre résultat important de ce chapitre est de prouver que le semi-groupe $(\mathcal{P}_t)_t$ associé aux solutions de (1.2.1) vérifie le résultat suivant.

Théorème 3.1.4 *Supposons l'Hypothèse 3.1.1 vérifiée. Alors il existe $N_0(B_0, L)$ tel que, si $N \geq N_0$, alors $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ admet une unique mesure de probabilité invariante μ sur H et il existe $C, \gamma > 0$, tels que pour $\lambda \in \mathcal{P}(H)$*

$$\|\mathcal{P}_t^*\lambda - \mu\|_* \leq Ce^{-\gamma t}\left(1 + \int_H |u|^2\, d\lambda(u)\right). \tag{3.1.3}$$

Nous reporterons la preuve rigoureuse des Théorèmes 3.1.2 et 3.1.4 à l'annexe C et nous traiterons ici l'exemple d'un modèle bidimensionnel.

Rappelons donc (0.3.3). On s'intéresse maintenant à un système modélisant les EDPS fortement dissipatives (telles NS ou CGL) dotées d'un opérateur de bruit général que l'on modélise ainsi

$$\begin{cases} dX & + & Xdt & + & f(X,Y)dt & = & \sigma_l(X,Y)\, dW, & X(0) = x_0, \\ dY & + & NYdt & + & g(X,Y)dt & = & \sigma_h(X,Y)\, dW, & Y(0) = y_0, \end{cases}$$

où l'opérateur $\Sigma(X,Y) = (\sigma_l(X,Y),\, \sigma_h(X,Y))$ est supposé recouvrir les modes instables. En clair on suppose qu'il existe un opérateur borné $\Sigma_l^{-1}(X,Y)$ vérifiant

$$\Sigma(X,Y)\cdot\Sigma_l^{-1}(X,Y)\cdot h = (h,\ 0)\,. \tag{3.1.4}$$

Il apparaît alors tout à fait naturel que les solutions d'un tel modèle convergent pour N assez grand. Nous verrons qu'un tel modèle est exponentiellement mélangeant.

De plus, nous verrons que ce résultat s'étend assez facilement au système (0.3.5) qui modélise NLS. Le principe de la preuve reste le même pour NLS. Néanmoins, cela requiert le développement et l'usage d'outils plus complexes. Ce résultat fera l'objet d'une publication ultérieure [54].

La méthode employée précédemment semble inadaptable. Et ce, à différents niveaux. En fait, nous renoncerons à vouloir prouver la convergence au sens de la norme $\|\cdot\|_{var-W}$ qui impliquerait la convergence en variation totale des bas modes. Nous nous contenterons de prouver la convergence pour la norme de Wasserstein. Le but de ce chapitre est donc d'établir pour nos deux modèles (0.3.3) et (0.3.5) l'existence de C et $\gamma > 0$ tels que, pour toute mesure de probabilité λ de $\mathbb{R}^2$, on ait

$$\|\mathcal{P}_t^*\lambda - \mu\|_* \leq Ce^{-\gamma t}\left(1 + \int_{\mathbb{R}^2} |u|^2\ \lambda(du)\right), \tag{3.1.5}$$

où μ est une mesure invariante. Cela signifie que, pour tout ψ Lipschitz bornée, on a

$$\left|\mathbb{E}\psi(u(t,W,u_0)) - \int_{\mathbb{R}^2} \psi(u)\,\mu(du)\right| \leq C\,|\psi|_{Lip_b(\mathbb{R}^2)}\, e^{-\alpha t}\left(1+\left|u_0^1\right|^2\right), \quad (3.1.6)$$

où $u(t,W,u_0)$ est l'unique solution du modèle et où $u_0 \in \mathbb{R}^2$.

Pour bien comprendre la nécessaire refondation de la méthode, essayons de coupler les lois des bas modes (X_1, X_2) de deux solutions (u_1, u_2). Lorsque Σ dépend de Y, on rencontre le problème suivant. Les opérateurs $\Sigma(u_1)$ et $\Sigma(u_2)$ n'ont *a priori* pas la même valeur si $X_1 = X_2$ et $Y_1 \neq Y_2$. Il n'apparaît donc pas possible de transformer la loi de X_1 en celle de X_2 en ajoutant un drift au processus de Wiener. Plus généralement, il faut renoncer à l'idée de coupler les lois de X_1 et de X_2, ce qui rend caduc toute la méthode.

Pour comprendre qu'il n'y a pas d'espoir de coupler efficacement X_1 et X_2, il suffit de remarquer que cela impliquerait de fait de coupler deux bruits avec opérateurs distincts. Pour le comprendre, considérons le modèle (0.3.3) avec $f = 0$, $g = 0$ et Σ de la forme

$$\Sigma(X,Y) = \begin{pmatrix} \sigma_l(X,Y) & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_l(X,Y) = 2 \wedge (1+|Y|)\,.$$

Le système devient

$$\begin{cases} dX + Xdt = 2 \wedge (1+|Y|)\; d\beta, \quad X(0) = x_0, \\ Y(t) = y_0 e^{-Nt}. \end{cases}$$

Fixons $x_0 = 0$, $y_0^1 = 0$ et $y_0^2 = 2$. Fabriquer un couplage (X_1, X_2) de $(\mathcal{D}(X(\cdot,\beta,(x_0,y_0^1))), \mathcal{D}(X(\cdot,\beta,(x_0,y_0^2))))$ sur $(0,1)$ tel que

$$\mathbb{P}\left(X_1 = X_2 \quad \text{sur} \;\; (0,1)\right) > 0, \quad (3.1.7)$$

équivaut donc à fabriquer deux mouvements browniens (β_1, β_2) tels que

$$\mathbb{P}\left(\beta_1 = \int_0^{\cdot} \left(1+e^{-Nt}\right) d\beta_2(t) \quad \text{sur} \;\; (0,1)\right) > 0 \quad (3.1.8)$$

Le problème est que les lois sur $(0,1)$ des deux processus $Z_1 = \beta_1$ et $Z_2 = \int_0^{\cdot} \left(1+e^{-Nt}\right) d\beta_2(t)$ sont à supports disjoints, ce qui rend (3.1.8) (et donc

(3.1.7)) impossible. Pour comprendre pourquoi les lois de Z_1 et Z_2 sur $(0,1)$ sont à support disjoints, considérons $V_n(Z)$ la variation quadratique discrétisée d'un processus Z

$$V_n(Z) = \sum_{k=1}^{2^n} \left(Z\left(\frac{k}{2^n}\right) - Z\left(\frac{k-1}{2^n}\right) \right)^2.$$

Il est classique que, pour nos deux processus Z_1 et Z_2, $V_n(Z)$ convergent presque sûrement vers $< Z > (1)$ la variation quadratique sur $(0,1)$ (Voir [33]). Or

$$< Z_2 > (1) = 1 + \frac{1}{N}\left(3 - 2e^{-2N} - e^{-4N}\right) > 1 =< Z_1 > (1),$$

Ce qui assure le caractère mutuellement singulier des lois de Z_1 et Z_2.

3.2 Un nouveau point de vue

Jusqu'à présent, les généralisations successives de la méthode se sont faites par raffinements progressifs. Mais nous venons de voir qu'un nouveau raffinement est à exclure. Si nous ne pouvons raffiner la méthode, il faut alors la simplifier en lui donnant un meilleur cadre. En bref la refondre entièrement. Un tel exercice requiert de prendre un certain recul. Pour ce faire nous allons décrire via un nouveau point de vue ce que nous avons déjà fait et nous en déduirons une nouvelle approche qui montrera que la précédente démarche était inutilement compliquée et que c'est cette complication qui limitait les résultats. Fort de cette nouvelle approche, nous verrons que le résultat général est à la fois assez facile à démontrer et que sa preuve est tout à fait naturelle.

Décrivons donc ce que nous avons fait pour traiter les deux modèles précédents des deux chapitres précédents.

La première chose que nous ayons prouvée est qu'il était possible de fabriquer un couplage (u_1, u_2) des lois des solutions tels que le temps où X_1 et X_2 se couplent pour la première fois admette un moment exponentiel. Appelons $\{((x_1, y_1), (x_2, y_2)) \mid x_1 = x_2,\ |x_0^1| + |x_0^2| \leq R_0\}$ "l'ensemble de démarrage". Nous avons donc fabriquer un couplage (u_1, u_2) dont le temps d'entrée dans l'ensemble de démarrage admet un moment exponentiel.

Ensuite, nous avons démontré l'inégalité de type Foias-Prodi. Plus précisément, rappelons la subtilité que nous avons mis en exergue plus tôt. Soient

$u(\cdot, W, u_0)$ la solution du système et $\phi(X, \eta, y_0)$ la solution de la seconde equation du système. Nous avons prouvé que si l'on pose $X_1 = X(\cdot, W, u_0^1)$, alors pour N assez grand la distance entre $\phi(X_1, \eta, y_0^2)$ et $\phi(X_1, \eta, y_0^1)$ tend exponentiellement vite vers 0 en espérance. Ce qui est équivalent à établir que lorsque les conditions initiales sont dans l'ensemble de démarrage, on a

$$\mathbb{E}\left|\widetilde{u}(t, W, u_0^1, u_0^2) - u(t, W, u_0^1)\right|^2 \leq Ce^{-t}, \tag{3.2.1}$$

où $\widetilde{u}(\cdot, W, u_0^1, u_0^2) = (X(\cdot, W, u_0^1), \phi(X(\cdot, W, u_0^1), \eta, y_0^2))$ est le processus auxiliaire décrit à la fin du chapitre précédent.

Le troisième point était la mutation de la loi de $X(\cdot, W, (x_0, y_0^1))$ en celle de $X(\cdot, W, (x_0, y_0^2))$ par le biais d'une transformation de Girsanov. En clair, il existe un drift h tel que drifter le processus de Wiener W par h équivaut à changer la condition initiale (x_0, y_0^2) en (x_0, y_0^1). Ecrivons ce fait en fonction du processus auxiliaire. Cela donne que pour tout (u_0^1, u_0^2) dans l'ensemble de démarrage

$$\widetilde{u}(\cdot, W, u_0^1, u_0^2) = u\left(\cdot, W + \int_0^{\cdot} h(t)dt, u_0^2\right). \tag{3.2.2}$$

Pour que cette transformation de Girsanov soit exploitable, nous avons eu besoin d'établir une majoration de la norme L^2 de h pour tout (u_0^1, u_0^2) dans l'ensemble de démarrage

$$\mathbb{E}\left(\int_T^\infty |h(t)|^2\, dt\right) \leq Ce^{-T}. \tag{3.2.3}$$

Ces trois points suffisent pour conclure. En effet, supposons que nous partions de l'ensemble de démarrage, (3.2.3) permet, en appliquant (1.3.6) via une troncature, la fabrication d'une paire de processus de Wiener (W_1, W_2) tels que la probabilité d'avoir $W_2(t) = W_1(t) + \int_0^t h(s)ds$ jusqu'à la fin des temps est minorée et tel que, lorsque ces processus découplent, le temps de découplage admet un moment exponentiel. On applique ensuite (3.2.2) qui donne que, partant de l'ensemble de démarrage, la probabilité que $\widetilde{u}(\cdot, W_1, u_0^1, u_0^2)$ et $u_2 = u(\cdot, W_2, u_0^2)$ soient indéfiniment couplés est minorée et que le temps de découplage admette un moment exponentiel.

On combine ce résultat avec le fait que l'on revienne exponentiellement souvent dans l'ensemble de démarrage. En clair, on attend d'être dans l'ensemble de démarrage, puis on tente notre chance en essayant de maintenir le processus auxiliaire $\widetilde{u}$ et la seconde solution u_2 couplée éternellement. Si ça marche, on a fini. Si ça échoue, on se rassure en se disant que le temps de découplage admet un moment exponentiel et on retente notre chance jusqu'à ce qu'on arrive à coupler éternellement $\widetilde{u}$ et u_2. Une subtilité importante : nous n'essayons nullement de coupler $u_2 = u(\cdot, W_2, u_0^2)$ et $\widetilde{u}(\cdot, W_1, u_0^1, u_0^2)$ sur (τ, ∞) où τ est un temps aléatoire. Par $\widetilde{u}$, nous entendons le processus auxiliaire démarrant au temps τ avec condition initiale $(u_1(\tau), u_2(\tau))$ et au processus de Wiener W_1.

On obtient ainsi qu'il existe un temps aléatoire τ admettant un moment exponentiel tel qu'au temps τ, on soit dans l'ensemble de démarrage et tel que $\widetilde{u}$, le processus auxiliaire associé à W_1 démarrant en τ avec conditions initiales $(u_1(\tau), u_2(\tau))$ reste indéfiniment couplé avec la seconde solution $u_2 = u(\cdot, W_2, u_0^2)$. On conclut la preuve en utilisant le fait que ledit processus auxiliaire se rapproche exponentiellement vite de la première solution $u(\cdot, W_1, u_0^1)$ (du fait de (3.2.1)), ce qui donne

$$\mathbb{P}\left(|u_2(t) - u_1(t)| \leq e^{\gamma t}\right) \leq C e^{\gamma t} \left(1 + \left|u_0^1\right|^2 + \left|u_0^2\right|^2\right).$$

On en déduit (3.1.6) puis (3.1.5).

Nous avons donc prouvé un critère très général pour qu'un processus de Markov converge exponentiellement vite à l'équilibre au sens de la norme de Wasserstein. En effet, il suffit de fabriquer un ensemble de démarrage et un processus auxiliaire vérifiant les trois propriétés précédentes pour conclure. Et on peut le faire sans avoir besoin de connaître des notions tel que le couplage, la transformation de Girsanov, où avoir la moindre connaissance de la preuve. En fait, seul une connaissance dudit processus de Markov est nécessaire. Dans le cas de solutions d'EDPS, seul une connaissance de l'EDP associée est requise.

Pour un énoncé et une preuve plus rigoureux dudit critère, voir l'Annexe C.

3.3 Résolution du cadre général

Nos choix passés en matière d'ensemble de démarrage et de processus auxiliaire se sont révélés particulièrement compliqués et difficilement généralisable du fait, et seulement du fait, de leur complexité. Nous allons donc maintenant nous appliquer faire le plus simple possible.

Appliquons notre critère à (0.3.3) sous la condition (3.1.4).

Quel est l'ensemble de démarrage réaliste le plus simple possible ? On serait tenté de penser à l'espace tout entier, mais cela pose évidemment des problèmes de majoration. Nous nous contenterons donc d'une boule. On remarque immédiatement que si son rayon est suffisamment élevé, alors le temps de retour dans cette boule admet un moment exponentiel.

Nous cherchons donc maintenant à construire un processus auxiliaire vérifiant (3.2.1), (3.2.2) et (3.2.3) pour toutes conditions initiales bornées. En clair, $\widetilde{u}$ devra vérifier (0.3.3) avec condition initiale u_0^2 et un processus de Wiener drifté par une fonction h. De plus le choix de h devra se faire de façon à ce que $\widetilde{u}$ et la solution avec condition initiale u_0^1 se rapprochent exponentiellement vite. Finalement il faudra pouvoir contrôler h. Un candidat vient tout naturellement à l'esprit. Il s'agit de $\widetilde{u} = (\widetilde{X}, \widetilde{Y})$ la solution du système suivant

$$\left\{\begin{array}{rcl} d\widetilde{X} + \widetilde{X}dt + (\mathbf{N-1})(\widetilde{\mathbf{X}} - \mathbf{X}(\mathbf{t}, \mathbf{W}, \mathbf{u_0^1})) + f(\widetilde{X}, \widetilde{Y})dt & = & \sigma_l(\widetilde{X}, \widetilde{Y})\, dW, \\ d\widetilde{Y} + N\widetilde{Y}dt + g(\widetilde{X}, \widetilde{Y})dt & = & \sigma_h(\widetilde{X}, \widetilde{Y})\, dW, \\ \widetilde{u}(0) & = & \mathbf{u_0^2}. \end{array}\right. \tag{3.3.1}$$

L'obtention de (3.2.2) pour

$$h(t) = -(N-1)\Sigma_l^{-1}(\widetilde{u}) \cdot (\widetilde{X} - X(t, W, u_0^1))$$

est immédiate.

De plus, il est assez facile d'obtenir (3.2.1) pour N assez grand. En effet, posons $r = \widetilde{u}(\cdot, W, u_0^1, u_0^2) - u(\cdot, W, u_0^1)$. Soustrayant (0.3.3) à (3.3.1), on obtient

$$dr + Nrdt + (\delta F)\, dt = (\delta\Sigma)\, dW,$$

où

$$\left\{\begin{array}{rcl} \delta F & = & \begin{pmatrix} f(\widetilde{u}) - f(u) \\ g(\widetilde{u}) - g(u) \end{pmatrix}, \\ \delta\Sigma & = & \Sigma(\widetilde{u}) - \Sigma(u). \end{array}\right.$$

La formule d'Ito de $|r|^2$ donne donc

$$d\,|r|^2 + 2N\,|r|^2\,dt = 2\left((\delta\Sigma)\,dW, r\right) - 2\left(\delta F, r\right)dt + |\delta\Sigma|^2\,dt.$$

On déduit du caractère Lipschitz des fonctions considérés qu'il existe $L > 0$ tel que

$$d|r|^2 + 2(N-L)|r|^2 dt \leq 2((\delta\Sigma)\, dW, r).$$

En intégrant et en prenant l'espérance, on obtient (3.2.1).

De plus, en intégrant (3.2.1) en temps et en utilisant le caractère borné de Σ_l^{-1}, on déduit (3.2.3).

Les conditions du critère sont donc vérifié par (0.3.3), ce qui achève la preuve de (3.1.5) et (3.1.6).

Comme nous l'avons écrit, c'est en simplifiant la méthode que nous l'avons raffinée. D'une compréhension plus en profondeur des preuves précédentes est née une nouvelle méthode plus simple, très facilement généralisable et beaucoup plus puissante.

Par exemple, qu'arrive-t-il lorsque l'on s'intéresse à une équation non fortement dissipative telle que NLS ? Rappelons (2.2.3). Pour modéliser NLS, nous avons considéré le système suivant

$$\left\{\begin{array}{lllllll} dX & + & X dt & + & f_N(X,Y)dt & = & \sigma_N^l(X,Y)\, dW, \quad X(0) = x_0, \\ dY & + & Y dt & + & g_N(X,Y)dt & = & \sigma_N^h(X,Y)\, dW, \quad Y(0) = y_0, \end{array}\right.$$

où on a imposé que les coefficients de Lipschitz par rapport à la variable Y des quatre fonctions f_N, g_N ,σ_N^l et σ_N^h soient majorées par $\frac{L}{N}$. De plus on conserve la condition d'inversibilité sur les bas modes (3.1.4).

On est naturellement amené à introduire le processus auxiliaire comme solution du système

$$\left\{\begin{array}{rcl} d\widetilde{X} + \widetilde{X} dt + (\mathbf{N}-\mathbf{1})(\widetilde{\mathbf{X}} - \mathbf{X}(\mathbf{t}, \mathbf{W}, \mathbf{u_0^1})) + f_N(\widetilde{X}, \widetilde{Y})dt & = & \sigma_N^l(\widetilde{X}, \widetilde{Y})\, dW, \\ d\widetilde{Y} + \widetilde{Y} dt + g_N(\widetilde{X}, \widetilde{Y})dt & = & \sigma_N^h(\widetilde{X}, \widetilde{Y})\, dW, \\ \widetilde{u}(0) & = & \mathbf{u_0^2}. \end{array}\right.$$

Comme précédemment, l'obtention de (3.2.2) pour

$$h(t) = -(N-1)\Sigma_l^{-1}(\widetilde{u}) \cdot (\widetilde{X} - X(t, W, u_0^1))$$

est immédiate.

Posons $r = \widetilde{u}(\cdot, W, u_0^1, u_0^2) - u(\cdot, W, u_0^1)$. Soustrayant (0.3.3) à (3.3.1), on obtient

$$dr + rdt + (N-1)(\delta X)dt + (\delta F_N)\, dt = (\delta \Sigma_N)\, dW,$$

où

$$\begin{cases} \delta F_N &= \begin{pmatrix} f_N(\widetilde{u}) - f_N(u) \\ g_N(\widetilde{u}) - g_N(u) \end{pmatrix}, \\ \delta \Sigma_N &= \Sigma_N(\widetilde{u}) - \Sigma_N(u), \\ \delta X &= \widetilde{X}(\cdot, W, u_0^1, u_0^2) - X(\cdot, W, u_0^1). \end{cases}$$

La formule d'Ito de $|r|^2$ donne donc

$$d\,|r|^2 + 2\,|r|^2\, dt + 2(N-1)\,|\delta X|^2\, dt = 2\,((\delta\Sigma_N)\, dW, r) - 2\,(\delta F_N, r)\, dt + |\delta \Sigma_N|^2\, dt.$$

Rappelons que les fonctions considérés sont globalement Lipschitz et que leur constante de Lipschitz par rapport à Y est du type $\frac{L}{N}$. Par exemple, cela donne

$$|\delta\Sigma_N| \le L\,|\delta X| + \frac{L}{N}\,|\delta Y| \le L\,|\delta X| + \frac{L}{N}\,|r|\,.$$

On obtient ainsi un L_0 tel que

$$d\,|r|^2 + 2\left(1 - \frac{L_0}{N}\right)|r|^2\, dt + 2\,(N - 1 - L_0)\,|\delta X|^2\, dt \le 2\,((\delta\Sigma)\, dW, r)\,.$$

Si on prend $N > 2L_0 + 1$, on obtient (3.2.1) en intégrant et en prenant l'espérance.

Comme précédemment, en intégrant (3.2.1) en temps et en utilisant le caractère borné de Σ_l^{-1}, on déduit (3.2.3).

Les conditions du critère sont donc vérifié par (0.3.5), ce qui achève la preuve de (3.1.5) et (3.1.6).

Dans l'annexe C, nous allons d'abord appliquer notre critère à NS, car du fait de sa sympathique non-linéarité, il est extrêmement facile d'appliquer toutes les méthodes citées ci avant à NS (L'équation de Burger 1D a elle aussi une non-linéarité très sympathique). En fait pour NS, on construit le processus auxiliaire exactement comme pour notre modèle (0.3.3). Les preuves de (3.2.1), (3.2.2) et (3.2.3) sont dans ce cas presque trivial. Le cas de NS a donc l'avantage d'être pédagogique.

Dans l'annexe C, nous allons ensuite appliqués ce critère à CGL avec un bruit localement Lipschitz. Cela fournit un exemple d'une équation dissipative dont le traitement est plus complexe du fait du caractère moins sympathique de sa non-linéaritéé En effet, on ne peut pas construire le processus auxiliaire comme pour notre modèle (0.3.3). Il faut donc adopter un mode de construction plus complexe. Les preuves de (3.2.1), (3.2.2) et (3.2.3) y sont plus fines.

Le cas de NLS périodique 1D et 2D est plus problématique. C'est un cadre dans lequel il est beaucoup plus difficile de travailler. Cela requiert des fonctionnelles assez complexes et des contrôles énergétiques faisant intervenir les espaces de Bourgain dans l'esprit de [28]. Malheureusement les bruits blancs en temps s'accordent très mal avec ce genre d'espaces. Il faut de plus faire une hypothèse de structure supplémentaire sur l'opérateur et le critère doit être légèrement raffiné. En plus, comme pour NLS cubique focalisante 1D avec bruit diagonal constant, seule la convergence à vitesse polynomiale peut être obtenue avec cette méthode. Ce problème fera l'objet d'une publication ultérieure [54].

Chapitre 4

Régularité spatiale des solutions stationnaires des équations de Navier-Stokes tridimensionnelles

Introduction

On s'intéresse ici aux équations de Navier–Stokes stochastiques en dimension 3 (NS3D) avec conditions périodiques et moyenne nulle.

$$\left\{\begin{array}{rcll} \multicolumn{4}{l}{dX + \nu(-\Delta)X\,dt + (X,\nabla)X\,dt + \nabla p\,dt = \phi(X)dW + g(X)dt,} \\ (\mathrm{div}\ X)\,(t,\xi) & = & 0, & \text{for } \xi \in D,\ \ t>0, \\ \int_D X(t,\xi)d\xi & = & 0, & \text{for } t>0, \\ X(0,\xi) & = & x_0(\xi), & \text{for } \xi \in D, \end{array}\right. \tag{4.0.1}$$

où $D = (0,2\pi)^3$.

Rappelons qu'une solution est dite forte ou faible (au sens des EDPs) si elle vit dans H^1 ou dans L^2, respectivement.

Dans le cas déterministe ($\phi = 0$), il est possible d'établir l'existence globale de solutions faibles de NS3D en invoquant des arguments de compacité. Malheureusement le caractère unique de telles solutions est inconnu. En utilisant un argument de point fixe contractant, on peut néanmoins établir l'existence et l'unicité locale de solutions forte pourvue que la condition initiale soit dans H^1. Le problème est que la finitude du temps d'explosion des solutions dans H^1 reste un problème ouvert. (Voir par exemple [7], [10], [25], [34], [35], [43], [44], [57] et [62]).

Dans le cas stochastique, on ignore s'il est possible de fabriquer des solutions faibles globales (au sens des EDPs) associées à un processus de Wiener donné. Néanmoins, par des arguments de compacité, on peut prouver l'existence globale de solutions faibles (au sens des EDPs) du problème de martingale, l'unicité trajectorielle ou en loi restant inconnue. Il est néanmoins possible, étant donné un processus de Wiener et une condition initiale dans H^1 donnés, d'établir l'existence et l'unicité locale d'une solution forte, la finitude du temps d'explosion restant un problème ouvert (Voir par exemple [3], [4], [6], [8], [9], [21], [23], [47], [60] et [61]).

On considérera (X, W) une solution stationnaire faible (au sens des EDPs) du problème martingale (4.0.1) et on dénotera par μ la loi de $X(t)$. La loi μ pourra être dite invariante dés lors que l'on pourra prouver que (4.0.1) définit une évolution de Markov. Le but de ce chapitre est d'établir que μ admet des moments dans des espaces de fonctions régulières dés que la force extérieur est suffisamment régulière. Nous pensons qu'il s'agit là d'un problème très intéressant. Premièrement parce que si on peut prouver que μ admet un moment d'ordre suffisamment élevé dans un espace de Sobolev adéquate (ordre 4 dans H^1 ou 2 dans H^2 par exemple) alors on pourra en déduire l'existence global des solutions fortes pour μ presque toute condition initiales.

De plus, ce résultat est un ingrédient important si on désire suivre la méthode de [11] pour construire un semi-groupe de transition de Markov dans $H^p(D)$ lorsque ϕ et f vérifient certaines hypothèses. Comme l'unicité en loi reste un problème ouvert pour NS3D, un tel résultat est important.

On prouvera d'abord que si la force extérieur est dans $H^{p-1}(D)$ et que le bruit est à trajectoire dans $H^p(D)$ alors μ admet un moment dans l'espace de Sobolev $H^{p+1}(D)$.

Un tel résultat est classique pour les équation de Navier-Stokes en dimension 2 (NS). En fait, c'est un Corollaire d'un résultat beaucoup plus puissant. Soit x_0 dans L^2. L'unique solution de NS est continue de $(0, \infty)$ dans $H^p(D)$ et de carré intégrable de (t_0, t_1) dans $H^{p+1}(D)$. Il suit que μ admet des moments de tout ordre dans $H^p(D)$ et un moment d'ordre 2 dans $H^{p+1}(D)$. Ce résultat est une conséquence de l'existence global des solutions fortes pour

NS.

Cette technique ne peut bien entendu par être appliqué à NS3D. Nous généraliserons donc une idée utilisée dans [11] pour le cas $p = 1$. L'idée est d'utiliser les propriétés de l'opérateur de Kolmogorov appliquée à une fonctionnelle de Lyapunov adéquate. Ces fonctionnelles ont déjà été utilisé pour traiter le cas déterministe dans [58], chapitre 4.

Utilisant une méthode totalement différente, nous établirons que la mesure invariante μ admet un moment dans une classe de fonctions Gevrey si sa force extérieure est dans une telle classe. La régularité Gevrey a été étudiée dans le cas déterministe dans [26] et [31]. Notre méthode est basée sur des outils développés dans [26]. Dans [55], ces outils ont été utilisés pour établir que les mesures invariantes admettent un moment exponentiel pour une norme Gevrey dans le cas bidimensionnel. Malheureusement les arguments de [55] ne semblent pas se généraliser au cas tridimensionnel du fait du manque d'existence global des solutions fortes. Le cas tridimensionnel requiert donc quelques adaptations. Nous développerons donc une technique qui assure le contrôle de la norme Gevrey en n'utilisant uniquement une majoration de la norme $H^1(D)$ de $X(t)$ à temps fixé.

De cette manière, nous pourrons généralisés à NS3D les résultats de [55]. Cependant nous n'aurons pas des moments exponentiels. On en déduira que l'échelle de dissipation de Kolmogorov est plus grande que $\nu^{-(4+\delta)}$. Ce n'est certainement pas un résultat optimal car on s'attend à ce qu'elle soit de l'ordre de $\nu^{-\frac{3}{4}}$.

4.1 Notations

Pour $m \in \mathbb{N}$, $\mathbb{H}^m_{\text{per}}(D)$ est l'espace des fonctions restrictions de fonctions périodiques dans $H^m_{loc}(D;\mathbb{R}^3)$ et dont la moyenne sur D est nulle. On pose

$$H = \left\{ X \in \mathbb{H}^0_{\text{per}}(D) \mid \text{ div } X = 0 \text{ sur } D \right\},$$

et

$$V = H \cap \mathbb{H}^1_{\text{per}}(D).$$

Soit π le projecteur orthogonal de $L^2(D;\mathbb{R}^3)$ sur H. On pose

$$A = \pi\left(-\Delta\right), \quad D(A) = V \cap \mathbb{H}^2_{\text{per}}(D) \text{ and } B(u) = \pi\left((u,\nabla)u\right).$$

On note $|\cdot|$ la norme de L^2 et $\|\cdot\|_m = \left|A^{\frac{m}{2}}\cdot\right|$. Il est connu que la norme de $\mathbb{H}^m_{\text{per}}(D)$ est équivalente à $\|\cdot\|_m$. De plus, pour $m = 1$ on écrira $\|\cdot\| = \|\cdot\|_1$.

Pour deux espaces de Hilbert donnés K_1 et K_2, $\mathcal{L}_2(K_1; K_2)$ sera l'espace des opérateurs Hilbert-Schmidt de K_1 vers K_2 .

Le bruit est un processus de Wiener cylindrique W défini sur un espace de Hilbert U et ϕ est une fonction de H à valeur dans $\mathcal{L}_2(U; H)$. La force déterministe g est une fonction de H dans H. On précisera nos hypothèses sur ϕ et g plus tard.
On réécrit (4.0.1) sous la forme

$$\left\{ \begin{array}{rcl} dX + \nu AXdt + B(X)dt & = & \phi(X)dW + g(X)dt, \\ X(0) & = & x_0. \end{array} \right. \tag{4.1.1}$$

Dans ce chapitre, on considérera (X, W) solution stationnaire du problème martingale (4.1.1) à valeur dans H. L'existence de telles solutions a été établie dans [21]. On dénote par μ la loi de $X(t)$. En pratique, on ne considérera que des solutions limites d'approximations de Galerkin. De plus, les calculs que nous ferons seront formels. Pour obtenir nos résultats, il faut faire nos calculs avec les approximations de Galerkin et passer à la limite.
Suivant les définitions de [26] des classes de Gevrey, on pose pour tout $(\alpha, \beta) \in \mathbb{R}_*^+ \times (0, 1]$

$$\left\{ \begin{array}{l} (\cdot,\cdot)_{G(\alpha,\beta)} = \left(A^{\frac{1}{2}} e^{\alpha A^{\frac{\beta}{2}}} \cdot, A^{\frac{1}{2}} e^{\alpha A^{\frac{\beta}{2}}} \cdot \right), \qquad \|\cdot\|_{G(\alpha,\beta)} = \left| A^{\frac{1}{2}} e^{\alpha A^{\frac{\beta}{2}}} \cdot \right|, \\ G(\alpha,\beta) = \left\{ X \in H \;\middle|\; \|X\|_{G(\alpha,\beta)} < \infty \right\}. \end{array} \right.$$

Il est facile de voir que $\left(G(\alpha, \beta), (\cdot,\cdot)_{G(\alpha,\beta)}\right)$ est un espace de Hilbert.

On ne s'intéressera pas au comportement des solutions quand la viscosité ν est grande. Pour simplifier, nous supposerons donc que $\nu \leq 1$.

4.2 Régularité $\mathbb{H}^p_{\mathbf{per}}(D)$

Fixons $p \in \mathbb{N}$ et supposons

Hypothèse 4.2.1 *La fonction ϕ (resp. g) est à valeur dans $\mathcal{L}_2\left(U; H \cap \mathbb{H}^p_{per}(D)\right)$ (resp. $H \cap \mathbb{H}^{p-1}_{per}(D)$). De plus $\phi : H \to \mathcal{L}_2\left(U; H \cap \mathbb{H}^p_{per}(D)\right)$ et $g : H \to H \cap \mathbb{H}^{p-1}_{per}(D)$ sont bornées.*

On pose, lorsque l'Hypothèse 4.2.1 est vérifiée,

$$B_p = \sup_H \left(\|\phi\|^2_{\mathcal{L}_2(U;\mathbb{H}^p_{\text{per}}(D))} + \|g\|^2_{p-1} \right).$$

Le premier résultat de ce chapitre est le Théorème suivant.

Théorème 4.2.2 *Soit μ la loi invariante d'une solution stationnaire de NS3D. Sous l'Hypothèse 4.2.1 pour un $p \geq 1$, il existe $c_{p,\nu}$ dépendant de p, ν et B_p tel que pour $\nu \leq 1$*

$$\int_H \|x\|_{p+1}^{\frac{2}{2p+1}} \, d\mu(x) \leq c_{p,\nu}.$$

Avant d'établir ce résultat, faisons quelques commentaires.

Remarquons d'abord qu'il serait très intéressant d'obtenir une majoration sur $\int_H \|x\|_{p+1}^{\delta_p} \, d\mu(x)$ avec $p\delta_p > 3$. En effet, par l'inégalité de Agmon, on a

$$\int_H |x|_\infty^2 \, d\mu(x) \leq c \int_H |x|^{2-3/p} \, \|x\|_p^{\frac{3}{p}} \, d\mu(x),$$

et cela donnerait une majoration du terme de droite. Comme l'unicité est facile à établir pour des solutions dans $L^2(0,T;L^\infty(D;\mathbb{R}^3))$, un argument clasique peut être appliqué pour prouver que, pour μ presque toutes conditions initiales, il existe une unique solution faible. Combinant ces résultats avec [20], cela résoudrait partiellement la conjecture de Leray.

Considérons le cas $g = 0$, $U = H$ et $\phi = A^{-s-\frac{3}{2}}$. Alors l'Hypothèse 4.2.1 est vérifiée pour tout $p < s$ et l'unique mesure invariante de l'équation de Navier-Stokes linéarisée dans H est dans $\mathbb{H}^{r+1}_{\text{per}}(D)$ avec probabilité nulle si $r > s$. Il semble donc que $\|\cdot\|_{p+1}$ soit la norme la plus forte que l'on puisse contrôler sous l'Hypothèse 4.2.1.

Remarquons que dans le cas bidimensionnel, on a un résultat bien plus puissant. En fait, par des arguments standards, on déduit de l'Hypothèse 4.2.1 que pour toute mesure invariante μ et tout $q \in \mathbb{N}^*$

$$\int_H \|x\|_p^{2q} \, d\mu(x) < \infty, \quad \int_H \|x\|_{p+1}^2 \, d\mu(x) < \infty.$$

Dans la preuve, nous utilisons des idées développées dans [58]. Des techniques similaires mais plus raffinées ont été utilisées dans [31] pour déduire d'intéressantes propriétés sur la décroissance des coefficients de Fourier de solutions régulières des équations de Navier-Stokes déterministes. L'usage de tel techniques ne semble pas améliorer de façon significative notre résultat. Plus précisément, nous avons été capable de raffiner le Théorème 4.2.2 comme suit

$$\nu \int_H \|x\|_{p+1}^{\frac{c_*}{p}} \, d\mu(x) \leq \bar{B}_p + c2^p(1+\bar{B}_0),$$

où c et c_* sont des constantes positives et où c_* est proche de 1.02. Comme l'amélioration n'est pas significative et les calculs beaucoup plus complexes, nous avons préféré nous restreindre au résultat plus simple du Théorème 4.2.2.

Preuve du Théorème 4.2.2

La preuve est basée sur le fait que pour des fonctions f adéquates, on a

$$\int_H Lf(x)d\mu(x) = 0, \tag{4.2.1}$$

où L est l'opérateur de Kolmogorov associé à l'équation de Navier-Stokes (4.1.1)

$$Lf(x) = \frac{1}{2} tr\left(\phi(x)\phi^*(x)D^2f(x)\right) - (\nu Ax + B(x) - g(x), Df(x)).$$

Comme nous l'avons déjà dit, tout cela n'est pas complètement rigoureux. En fait, (4.2.1) est vrai seulement pour les approximations de Galerkin et pour des fonctions régulières. Pour une meilleur compréhension, nous nous restreindrons ici au cas $g = 0$. La généralisation est facile.

Etape 1 : $p = 0$

On applique (4.2.1) à $f = |\cdot|^2$.

$$\int_H (\nu Ax + B(x), x) \, d\mu(x) = \frac{1}{2} \int_H tr\left(\phi(x)\phi^*(x)\right) \, d\mu(x),$$

ce qui donne, puisque $(B(x), x) = 0$,

$$\nu \int_H \|x\|^2 \, d\mu(x) \quad \leq \quad \frac{1}{2}\bar{B}_0. \tag{4.2.2}$$

Etape 2 : Majoration d'un quotient de normes.

Pour tout ε_p, on pose

on pose

$$R'_p = \nu \int_H \frac{\|x\|_{p+1}^2}{\left(1+\|x\|_p^2\right)^{1+\varepsilon_p}} d\mu(x).$$

Puis, on applique (4.2.1) à

$$f = \frac{1}{\left(1+\|\cdot\|_p^2\right)^{\varepsilon_p}}.$$

Un calcul fait en annexe D donne

$$2R'_p \leq \bar{B}_p - 2\int_H \frac{(A^p x, B(x))}{\left(1+\|x\|_p^2\right)^{1+\varepsilon_p}} d\mu(x). \tag{4.2.3}$$

Il a été prouvé dans [59] (équation (4.8) chapitre 4) qu'il existe une constante $c_{p,\nu}$ telle que

$$-(A^p x, B(x)) \leq c_{p,\nu} \|x\|^2 \|x\|_p^{4p/2p-1} + \frac{\nu}{2} \|x\|_{p+1}^2. \tag{4.2.4}$$

Donc, si l'on pose

$$\varepsilon_p = \frac{1}{2p-1},$$

on obtient

$$R'_p \leq c'_{p,\nu}.$$

En posant

$$R_p = \nu \int_H \frac{1+\|x\|_{p+1}^2}{\left(1+\|x\|_p^2\right)^{1+\varepsilon_p}} d\mu(x),$$

on déduit

$$R'_p \leq \bar{c}_{p,\nu}. \tag{4.2.5}$$

Etape 3 : Majoration de $\int_H \|x\|_p^{2/2p-1} d\mu(x)$.

Nous allons majorer $A_p = \nu \int_H \left(1+\|x\|_p^2\right)^{1/2p-1} d\mu(x)$ par récurrence. Le cas $p = 1$ a été traité lors de l'étape 1.

On déduit d'inégalité de Hölder

$$\begin{aligned}
&\nu \int_H \left(1 + \|x\|_{p+1}^2\right)^{1/2p+1} d\mu(x) \\
&= \nu \int_H \left(\frac{\left(1 + \|x\|_{p+1}^2\right)}{\left(1 + \|x\|_{p+1}^2\right)^{1+\varepsilon_p}} \right)^{1/2p+1} \left(1 + \|x\|_{p+1}^2\right)^{(1+\varepsilon_p)/2p+1} d\mu(x) \\
&\leq R_p^{1/2p+1} \left(\nu \int_H \left(1 + \|x\|_{p+1}^2\right)^{(1+\varepsilon_p)/2p} d\mu(x) \right)^{2p/2p+1}.
\end{aligned}$$

Comme $(1+\varepsilon_p)/2p = 1/2p - 1$, on obtient

$$A_{p+1} \leq R_p^{1/2p+1} A_p^{2p/2p+1}.$$

Ainsi, on peut conclure.

4.3 Régularité Gevrey

On prouve dans cette section que si la force extérieure est bornée dans une classe de fonctions Gevrey, alors μ a son support dans une autre classe de fonctions Gevrey.
L'Hypothèse principale est la suivante.

Hypothèse 4.3.1 *Il existe $(\alpha, \beta) \in \mathbb{R}_+^* \times (0,1]$ tel que les fonctions $g : H \to H \cap G(\alpha, \beta)$ et $\phi : H \to \mathcal{L}_2(U; H \cap G(\alpha, \beta))$ sont bornées.*

On pose

$$B_0' = \sup_{x \in H} \|\phi(x)\|_{\mathcal{L}_2(U;G(\alpha,\beta))}^2 + \sup_{x \in H} \|g(x)\|_{G(\alpha,\beta)}^2.$$

Le second principal résultat de ce chapitre est le suivant.

Théorème 4.3.2 *Supposons l'Hypothèse 4.3.1. Il existe une famille $(C_\gamma)_\gamma$ seulement dépendant de (α, β, B_0') et une famille $(\alpha_\nu)_{\nu \in (0,1)}$ de fonctions mesurables $H \to (0, \alpha)$ tel que pour tout $\nu \in (0,1)$*

$$\int_H \|x\|_{G(\nu\alpha_\nu(x),\beta)}^{2\gamma} d\mu(x) \leq C_\gamma \nu^{-\frac{3}{4}}, \tag{4.3.1}$$

$$\int_H (\alpha_\nu(x))^{-\frac{\gamma}{4}} d\mu(x) \leq C_\gamma \nu^{-\frac{3}{4}}, \tag{4.3.2}$$

pour tout $\gamma \in (0,1)$.

Ce résultat donne des informations sur l'échelle de dissipation de Kolmogorov. Plus précisément, on déduit de (4.3.1), (4.3.2) que

$$|\hat{x}(k)| \leq \|x\|_{G(\nu\alpha_\nu(x),\beta)} |k| \, e^{-\nu\alpha_\nu(x)|k|^\beta},$$

où $(\hat{x}(k))_{k\in\mathbb{Z}^3}$ sont les coefficients de Fourier de x.
Donc, si l'Hypothèse 4.3.1 est vérifiée avec $\beta = 1$, alors $|\hat{x}(k)|$ décroît plus vite que toute puissance de $|k|$ pour $|k| >> (\nu\alpha_\nu(x))^{-1}$. On déduit de (4.3.2) que pour tout $\delta > 0$

$$\frac{1}{\alpha_\nu(x)} \leq c_{\delta,\nu}(x)\nu^{-3(1+\delta)} \quad \text{avec} \quad \int c_{\delta,\nu}(x)^{-\frac{1}{4(1+\delta)}} \, \mu(dx) \leq \Theta_\delta < \infty,$$

et que Θ_δ ne dépend pas de ν. Il suit que $|\hat{x}(k)|$ décroît plus vite que toute puissance de $|k|$ pour $|k| >> \nu^{-(4+3\delta)}$. Ce qui implique que l'échelle de dissipation de Kolmogorov est plus grand que $\nu^{-(4+3\delta)}$. Rappelons que par des arguments physiques, on s'attend à ce que ladite échelle soit de l'ordre de $\nu^{-\frac{3}{4}}$.
Dans l'annexe D, on verra que l'on peut en déduire un moment pour une norme de Gevrey fixée.

Corollaire 4.3.3 *Sous les même Hypothèses, il existe une famille* $(C_{\gamma,\alpha',\beta',\nu})_{\gamma,\alpha',\beta',\nu}$ *dépendant seulement de* $(\alpha, \beta, B_0', \nu)$ *telle que*

$$\int \left(\ln^+ \|x\|^2_{G(\alpha',\beta')}\right)^\gamma d\mu(x) \leq C_{\gamma,\alpha',\beta',\nu}, \tag{4.3.3}$$

où $\ln^+ r = \max\{0, \ln r\}$ *et pourvu que* $\alpha' > 0$ *et* $\beta', \gamma > 0$ *vérifient*

$$\beta' < \beta \quad \text{et} \quad 4\gamma < \frac{\beta}{\beta'} - 1.$$

Preuve du Théorème 4.3.2

Afin d'établir le Théorème 4.3.2, citons le résultat suivant qui dit que le temps d'explosion des solutions dans les espaces de Gevrey admet un moment négatif.

Lemme 4.3.4 *Supposons que l'Hypothèse 4.3.1 soit vérifiée. Pour toute solution stationnaire des équations de Navier-Stokes et tout $\nu \in (0,1)$, il existe K dépendant seulement de (α, β, B_0') et un temps d'arrêt $\tau > 0$ tel que*

$$\begin{aligned} \mathbb{E}\left(\sup_{t\in(0,\tau)} \|X(t)\|^2_{G(\nu t,\beta)}\right) &\leq 4(\bar{B}_0+1), &(4.3.4)\\ \mathbb{P}(\tau \leq t) &\leq K t^{\frac{1}{4}} \nu^{-\frac{3}{4}}. &(4.3.5) \end{aligned}$$

Ce résultat est un raffinement du résultat établi par Foias et Temam dans [26] et sa preuve s'appuie sur les outils développés dans cet article. Pour une preuve détaillée, voir l'annexe D.
Posons

$$\alpha_\nu(x) = \inf\left\{ s \geq 0 \;\middle|\; \|x\|^2_{G(\nu s,\beta)} > \frac{4}{s^{\frac{1}{4}}}\left(\bar{B}_0+1\right)\right\},$$

il suit que pour tout $\gamma \in (0,1)$

$$\int \|x\|^{2\gamma}_{G(\nu\alpha_\nu(x),\beta)}\, d\mu(x) \leq 4^\gamma \left(\bar{B}_0+1\right)^\gamma \int (\alpha_\nu(x))^{-\frac{\gamma}{4}}\, d\mu(x). \qquad (4.3.6)$$

Ainsi (4.3.1) est une conséquence de (4.3.2) et de $\bar{B}_0 \leq B_0'$. Alors pour établir le Théorème 4.3.2, il suffit de prouver (4.3.2).
Clairement

$$\mathbb{P}\left(\|X(t)\|^2_{G(\nu t,\beta)} > \frac{4}{t^{\frac{1}{4}}}\left(\bar{B}_0+1\right)\right) \leq \mathbb{P}\left(\sup_{s\in[0,\tau]} \|X(s)\|^2_{G(\nu s,\beta)} > \frac{4}{t^{\frac{1}{4}}}\left(\bar{B}_0+1\right)\right) + \mathbb{P}(\tau < t).$$

Appliquant une inégalité de Chebyshev, on déduit du Lemme 4.3.4 que pour tout $t > 0$

$$\mathbb{P}\left(\|X(t)\|^2_{G(\nu t,\beta)} > \frac{4}{t^{\frac{1}{4}}}\left(\bar{B}_0+1\right)\right) \leq (K+1) t^{\frac{1}{4}} \nu^{-\frac{3}{4}}. \qquad (4.3.7)$$

Comme μ est la loi de $X(t)$, on a

$$\mu\left(x \mid \alpha_\nu(x) \leq t\right) = \mathbb{P}\left(\alpha_\nu(X(t)) \leq t\right),$$

ce qui implique

$$\mu\left(x \mid \alpha_\nu(x) \leq t\right) = \mathbb{P}\left(\|X(t)\|^2_{G(\nu t,\beta)} > \frac{4}{t^{\frac{1}{4}}}\left(\bar{B}_0+1\right)\right).$$

Ainsi, on déduit de (4.3.7) que pour tout $t > 0$

$$\mu\,(x \mid \alpha_\nu(x) \leq t\,) \leq (K+1)t^{\frac{1}{4}}\nu^{-\frac{3}{4}-\frac{\kappa}{2}}. \tag{4.3.8}$$

Il est bien connu que (4.3.8) pour tout $t > 0$ implique (4.3.2), ce qui donne le Théorème 4.3.2.

Chapitre 5

Mélange exponentiel pour Navier-Stokes 3D

Introduction

On s'intéresse aux équations de Navier–Stokes stochastiques sur un domaine D borné tridimensionnel (NS3D) avec conditions aux bords de Dirichlet dont on rappelle la formulation (Voir (0.1.3))

$$\begin{cases} dX + \nu(-\Delta)X\,dt + (X,\nabla)X\,dt + \nabla p\,dt = \phi(X)dW + f\,dt, \\ (\text{div } X)\,(t,\xi) \;=\; 0, \qquad \text{for } \xi \in D, \quad t>0, \\ X(t,\xi) \;=\; 0, \qquad \text{for } \xi \in \partial D, \quad t>0 \\ X(0,\xi) \;=\; x_0(\xi), \quad \text{for } \xi \in D. \end{cases} \tag{5.0.1}$$

Ici D est un ouvert borné de $\mathbb{R}^3$ de bord ∂D régulière ou $D=(0,1)^3$.

Dans le cas déterministe ($\phi = 0$), il est possible d'établir l'existence globale de solutions faibles de NS3D en invoquant des arguments de compacité.

Dans ce chapitre, on établit que si ϕ est à la fois suffisamment régulier et non dégénéré, alors, les solutions converges exponentiellement vite vers l'équilibre. Précisons ce point. Pour une solution donnée, il existe une solution stationnaire tel que la distance variationnelle entre la solution donnée et la solution stationnaire décroît exponentiellement vite. Il est à noter un détail spécifique à NS3D. Dans les chapitres précédents, la convergence à l'équilibre impliquait de fait l'unicité dudit équilibre. Ce n'est pas le cas ici. Du fait du manque d'unicité, la question se pose de savoir si, partant d'une même condition initiale, deux solutions peuvent converger vers deux équilibres différents.

Du fait du manque d'unicité, il est difficile de définir un processus de Markov associé à (5.0.1). De récents progrès ont été fats dans cette direction. Dans [11], sous certaines conditions sur ϕ et f très similaires aux notres, ils ont pu fabriquer un semi-groupe de transition de Markov associé à (5.0.1). Notre résultat signifie que le semi-groupe qu'ils ont construit est exponentiellement mélangeant.

Dans [24], un argument de sélection Markovienne a permit d'établir l'existence d'une évolution Markovienne associée à (5.0.1). Notre résultat ne s'applique pas directement car nous ne travaillons qu'avec des solutions limites d'approximations de Galerkin. Néanmoins, nous pensons que notre preuve peut être adaptée pour prouver que, sous de bonnes hypothèses sur le bruit, le semi-groupe de Markov construit dans [24] est exponentiellement mélangeant.

Notre preuve s'appuie sur des arguments de couplage. Dans les chapitres précédents et chez [29], [36], [39], [40], [41], [45], [50], [51], [56]), le but était de prouver le caractère exponentiellement mélangeant pour des bruits dégénérés. Dans le cas bidimensionnel, il a même été prouvé dans [30], [46] qu'il suffit d'exciter 4 modes de NS pour avoir unicité de la mesure invariante.

Ici, la difficulté ne se situe pas dans la dégénérescence du bruit. Nous travaillons avec une équation dont nous ignorons si elle est bien posée ou non. Ce qui va induire des modifications substantielles de nos preuves. Pour cette raison, nous nous contentons de prouver un résultat plus modeste. En l'occurrence, nous nous restreignons à des bruits non dégénérés.

L'idée principale est de coupler les solutions lorsque les conditions initiales sont petites au sens d'une norme suffisamment régulière. Pour fabriquer ledit couplage, on utilisera une formule de type Bismuth-Elworthy-Li. Un autre ingrédient important de la preuve est le fait que le temps d'entrée des solutions faibles dans une petite boule admet un moment exponentiel. On compense le manque d'unicité en travaillant avec des approximations de Galerkin et passant à la limite.

5.1 Résultat principal

Soient H et V les fermetures de l'ensemble des fonctions régulières à support compact dans D et à divergence nulle par les normes de $L^2(D;\mathbb{R}^3)$ et $H_0^1(D;\mathbb{R}^3)$ respectivement.
Soit π le projecteur orthogonal de $L^2(D;\mathbb{R}^3)$ vers H. On pose

$$A = \pi\,(-\Delta)\,,\ D(A) = V \cap H^2(D;\mathbb{R}^3),\ B(u,v) = \pi\,((u,\nabla)v) \ \text{ et } \ B(u) = B(u,u).$$

Très classiquement, on peut réécrire le problème (0.1.3) sous la forme

$$\begin{cases} dX + \nu AX dt + B(X)dt &= \phi(X)dW + f\,dt, \\ X(0) &= x_0, \end{cases} \tag{5.1.1}$$

où W est un processus de Wiener cylindrique sur H.

L'opérateur $(A, \mathcal{D}(A))$ est autoadjoint à spectre discret. Voir [10], [57]. On note $(e_n)_n$ la base des vecteurs propres de $(A, \mathcal{D}(A))$ sur H et on note P_N la projection orthogonale dans H vers $Sp(e_k)_{1\leq N}$.

Les solutions considérées sont supposées faible au sens des probabilités et des EDPs. En clair, une solution est une loi $\mathbb{P}_{x_0}$ tel que, sous $\mathbb{P}_{x_0}$, le couple de processus canoniques (X_*, W_*) vérifie une forme faible de (5.1.1) (Pour de plus amples détails, voir l'annexe E). En fait, on se restreindra à l'étude des solutions limites en loi de solutions d'approximations de Galerkin de (5.1.1).

$$\begin{cases} dX_N + \nu AX_N dt + P_N B(X_N)dt &= P_N\phi(X_N)dW + P_N f dt, \\ X_N(0) &= P_N x_0. \end{cases} \tag{5.1.2}$$

L'opérateur de covariance ϕ est supposé à la fois suffisamment régulier et non dégénéré. Pour que notre résultat soit vrai, il faut faire une hypothèse supplémentaire (ϕ constant, ou bien ϕ diagonal, ou bien ν élevé). Pour des raisons de simplicité, on considérera ici le cas

$$\phi = A^{-\frac{s}{2}}, \quad s \in \left(\frac{5}{2}, 3\right].$$

Les hypothèses générales sont explicitées dans l'annexe E.

Rappelons qu'une solution faible $\mathbb{P}_\mu$ est dite stationnaire si la loi de $X_*(t)$ est indépendante de t. Celle-ci sera notée μ.

Théorème 5.1.1 *Il existe C et $\gamma > 0$ tel que pour toute solution faible $\mathbb{P}_\lambda$ limite de solutions de* (5.1.2)*, il existe une solution faible stationnaire $\mathbb{P}_\mu$ telle que*

$$\|\mathcal{D}_{\mathbb{P}_\lambda}(X_*(t)) - \mu\|_{var} \leq Ce^{-\gamma t}\left(1 + \int_H |x|^2 \lambda(dx)\right), \tag{5.1.3}$$

où λ est la loi de $X_*(0)$.

5.2 Méthodes de couplage

La preuve du Théorème 5.1.1 est basée sur une méthode de couplage. Pour comprendre les problèmes induit par NS3D, nous allons d'abord rapidement rappeler comment traiter le cas des solutions de (5.1.2).

Il est facile de voir que l'on a existence et unicité des solutions $X_N(\cdot, x_0)$ de (5.1.2). On remarque de plus que $X_N(\cdot, x_0)$ vérifie la propriété de Markov forte. On notera $\left(\mathcal{P}_t^N\right)_t$ le semigroupe de Markov associé.

Soient $N \in \mathbb{N}$ et $(x_0^1, x_0^2) \in H^2$. En combinant la méthode explicitée dans (1.5) et un argument de troncature, on obtient l'existence d'une fonction décroissante $p_N(\cdot) > 0$ telle que

$$\left\| \left(\mathcal{P}_1^N\right)^* \delta_{x_0^2} - \left(\mathcal{P}_1^N\right)^* \delta_{x_0^1} \right\|_{var} \leq 1 - p_N\left(\left|x_0^1\right| + \left|x_0^2\right|\right). \tag{5.2.1}$$

En appliquant le Lemme 1.3.1, on fabrique un couplage maximal $(Z_1, Z_2) = (Z_1(x_0^1, x_0^2), Z_2(x_0^1, x_0^2))$ de $(\left(\mathcal{P}_1^N\right)^* \delta_{x_0^1}, \left(\mathcal{P}_1^N\right)^* \delta_{x_0^2})$. Ce qui donne

$$\mathbb{P}\left(Z_1 = Z_2\right) \geq p_N\left(\left|x_0^1\right| + \left|x_0^2\right|\right) > 0. \tag{5.2.2}$$

Soient $(W, \widetilde{W})$ un couple de processus de Wiener cylindriques indépendants et $\delta > 0$. On notera $X_N(\cdot, x_0)$ et $\widetilde{X}_N(\cdot, x_0)$ les solutions de (5.1.2) associées à W et $\widetilde{W}$. On construit un couple de variables aléatoires $(V_1, V_2) = (V_1(x_0^1, x_0^2), V_2(x_0^1, x_0^2))$ sur $P_N H$ comme suit

$$(V_1, V_2) = \begin{cases} (X_N(\cdot, x_0), X_N(\cdot, x_0)) & \text{si } x_0^1 = x_0^2 = x_0, \\ (Z_1(x_0^1, x_0^2), Z_2(x_0^1, x_0^2)) & \text{si } (x_0^1, x_0^2) \in B_H(0, \delta) \backslash \{x_0^1 = x_0^2\}, \\ (X_N(\cdot, x_0^1), \widetilde{X}_N(\cdot, x_0^2)) & \text{sinon,} \end{cases} \tag{5.2.3}$$

où $B_H(0,\delta)$ est la boule de $H \times H$ de rayon δ.

Alors $(V_1(x_0^1,x_0^2), V_2(x_0^1,x_0^2))$ est un couplage de $(\left(\mathcal{P}_1^N\right)^* \delta_{x_0^1}, \left(\mathcal{P}_1^N\right)^* \delta_{x_0^2})$ qui dépend mesurablement de (x_0^1,x_0^2). On construit ensuite (X^1,X^2) couplage de $(\mathcal{D}(X_N(\cdot,x_0^1)), \mathcal{D}(X_N(\cdot,x_0^2)))$ par induction sur $\mathbb{N}$. On commence par poser $X^i(0) = x_0^i$ for $i=1,2$. Puis, après avoir construit (X^1,X^2) sur $\{0,1,\dots,k\}$, on prend une version de (V_1,V_2) indépendante de (X^1,X^2) et on pose

$$X^i(k+1) = V_i(X^1(k), X^2(k)) \qquad \text{pour } i=1,2.$$

Il est assez facile de démontrer que le temps de retour dans $B_H(0,\delta)$ admet un moment exponentiel pourvu que δ soit assez grand. On déduit donc de (5.2.2), (5.2.3) que, $(X^1(n),X^2(n)) \in B(0,\delta)$ implique que la probabilité de coupler (X^1,X^2) au temps $n+1$ est minorée par $p_N(\delta)>0$. Finalement, on remarque que si (X^1,X^2) est couplé au temps $n+1$, alors il le reste éternellement. En combinant ces trois propriétés avec le fait que $(X^1(n),X^2(n))_{n\in\mathbb{N}}$ est un processus de Markov fort discret, on montre facilement que

$$\mathbb{P}\left(X^1(n) \neq X^2(n)\right) \leq C_N e^{-\gamma_N n}\left(1+\left|x_0^1\right|^2+\left|x_0^2\right|^2\right). \tag{5.2.4}$$

Rappelons que (X^1,X^2) est un couplage de $(\mathcal{D}(X_N(\cdot,x_0^1)), \mathcal{D}(X_N(\cdot,x_0^2)))$ sur $\mathbb{N}$. On obtient ainsi que $(X^1(n),X^2(n))$ est un couplage de $((\mathcal{P}_n^N)^*\delta_{x_0^1}, (\mathcal{P}_n^N)^*\delta_{x_0^2})$. En combinant le Lemme 1.3.1 et (5.2.4), on déduit, pour $n\in\mathbb{N}$,

$$\left\|\left(\mathcal{P}_n^N\right)^*\delta_{x_0^2} - \left(\mathcal{P}_n^N\right)^*\delta_{x_0^1}\right\|_{var} \leq C_N e^{-\gamma_N n}\left(1+\left|x_0^1\right|^2+\left|x_0^2\right|^2\right).$$

On pose $n=\lfloor t\rfloor$ et on intègre (x_0^2,x_0^1) par $((\mathcal{P}_{t-n}^N)^*\lambda)\otimes\mu_N$ où μ_N est une mesure invariante. On obtient ainsi pour tout $\lambda\in P(P_NH)$

$$\left\|\left(\mathcal{P}_t^N\right)^*\lambda - \mu_N\right\|_{var} \leq C_N e^{-\gamma_N t}\left(1+\int_{P_NH}|x|^2\,\lambda(dx)\right). \tag{5.2.5}$$

Un tel résultat est inutilisable pour traiter (5.1.1) car C_N et γ_N dépendent fortement de N. Si on essaie directement d'appliquer la méthode précédente à (5.1.1), on doit faire face à un certain nombre de difficultés. Premièrement, on ignore si on peut utiliser la propriété de Markov. Vu que cette propriété est utilisée de façon plus ou moins implicite dans un très grand nombre d'arguments, il apparaît difficile de s'en passer. Un autre gros problème est que la transformation de Girsanov utilisée pour obtenir (5.2.1) ne semble pas pouvoir être appliquée car elle a lieu dans un espace de dimension infinie et que nous ne contrôlons pas de normes suffisamment régulières des solutions pour que les conditions de Novikov soient vérifiées. Nous verrons que nous sommes capables de prouver un résultat analogue à (5.2.1) par un argument totalement différent. Cependant, cela ne marchera que pour des conditions initiales petites dans $\mathbb{H}^2$. Un autre problème est que nous ignorons si, partant de $\mathbb{H}^2$, on reste dans $\mathbb{H}^2$.

On compense le manque de propriété de Markov en travaillant avec les approximations de Galerkin et en prouvant (5.2.5) avec des constantes uniformes par rapport à N. En particulier, nous prouverons que (5.2.1) est vrai pour x_0^1, x_0^2 dans une petite boule de $\mathbb{H}^2$ et uniformément en N. Puis, suivant le raisonnement précédent, il restera à prouver que le temps de retour dans cette boule admet un moment exponentiel. Dans l'annexe E, on prouvera que

Proposition 5.2.1 *Il existe $C > 0$ et $\gamma > 0$ tel que, pour tout $N \in \mathbb{N}$, il existe une unique mesure invariante μ_N pour $\left(\mathcal{P}_t^N\right)_{t\in\mathbb{R}^+}$. De plus, pour tout $\lambda \in P(P_N H)$, on a*

$$\left\|\left(\mathcal{P}_t^N\right)^* \lambda - \mu_N\right\|_{var} \leq Ce^{-\gamma t}\left(1 + \int_{P_N H} |x|^2 \ \lambda(dx)\right). \tag{5.2.6}$$

Puis on verra que ce résultat passe à la limite et implique le Théorème 5.1.1.

5.3 Convergence d'un modèle simplifié

Une preuve rigoureuse du Théorème 5.1.1 via la proposition 5.2.1 est donnée dans l'Annexe E. Du fait des différents espaces considérés et des problèmes de troncatures, nous préférons traiter ici un modèle unidimensionnel plus simple.

$$dX + Xdt + f(X)dt = d\beta,\ X(0) = x_0, \tag{5.3.1}$$

où β est un mouvement brownien unidimensionnel et où f modélise la nonlinéarité B de NS3D. Pour d'évidentes raisons de simplicité, nous imposerons à f d'être C^1 bornée à dérivée bornée. De plus, comme

$$(B(x), x) = 0 \text{ pour tout } \ x \ \text{ régulière},$$

nous imposerons à f la condition

$$xf(x) \geq 0. \tag{5.3.2}$$

Sous ces hypothèses, l'existence et l'unicité des solutions $(X(\cdot,x_0))_{x_0\in\mathbb{R}}$ de (5.3.1) est facile à établir. Ce qui implique la propriété de Markov forte. On dénotera par $(\mathcal{P}_t)_t$ le semi-groupe de transition de Markov associé à (5.3.1)

$$(\mathcal{P}_t g)(x_0) = \mathbb{E}\left(g(X(t,x_0))\right).$$

Appliquons donc le programme explicité ci-avant pour établir la proposition 5.2.1.

On considère l'équation suivante

$$\begin{cases} \frac{d}{dt}\eta + \eta + \eta f'(X(t,x_0)) &= 0, \\ \eta(0,x_0)\cdot h &= h, \end{cases} \tag{5.3.3}$$

où $(x_0,h)\in\mathbb{R}^2$ et $\eta(t)=\eta(t,x_0)\cdot h$.

L'existence et l'unicité des solutions de (5.3.3) est facile à établir. De plus, si $g\in C_b^1(\mathbb{R})$, alors, pour tout $T\geq 0$, on a

$$\left(\nabla\left(\mathcal{P}_T g\right)(x_0), h\right) = \mathbb{E}\left(\nabla g(X(T,x_0)), \eta(T,x_0)\cdot h\right).$$

Appliquant la formule de Bismuth-Elworthy-Li, on obtient

$$\left(\nabla\left(\mathcal{P}_T g\right)(x_0), h\right) = \frac{1}{T}\mathbb{E}\left(g(X(T,x_0))\int_0^T \eta(t,x_0)\cdot h\, d\beta\right). \tag{5.3.4}$$

On en déduit que

$$\left|\left(\nabla\left(\mathcal{P}_T g\right)(x_0), h\right)\right| \leq \frac{|g|_\infty}{T}\mathbb{E}\left|\int_0^T \eta(t,x_0)\cdot h\, d\beta\right|,$$

ce qui donne

$$\left|\left(\nabla\left(\mathcal{P}_T g\right)(x_0), h\right)\right| \leq \frac{|g|_\infty}{T}\sqrt{\mathbb{E}\left(\int_0^T \left|\eta(t,x_0)\cdot h\right|^2 dt\right)}. \tag{5.3.5}$$

Prenons le produit scalaire de (5.3.3) avec 2η. On obtient

$$\frac{d}{dt}|\eta|^2 + 2|\eta|^2 = -2|\eta|^2 f'(X(t,x_0)).$$

Comme f' est bornée, on obtient en intégrant que

$$|\eta(t,x_0)\cdot h|^2 \leq |h|^2 \exp\left(2\,|f'|_\infty\, t\right),$$

ce qui donne

$$\int_0^T \|\eta(t,x_0)\cdot h\|^2\, dt \leq \exp\left(2\,|f'|_\infty\, T\right) T\, |h|^2. \tag{5.3.6}$$

Combinons (5.3.5) et (5.3.6), on obtient

$$|(\nabla\left(\mathcal{P}_T g\right)(x_0), h)| \leq \frac{|g|_\infty}{\sqrt{T}} \exp\left(|f'|_\infty\, T\right)|h|. \tag{5.3.7}$$

On en déduit que pour tout $T>0$ et $(x_0^1,x_0^2)\in\mathbb{R}^2$, on a

$$\left|(\mathcal{P}_T g)(x_0^2) - (\mathcal{P}_T g)(x_0^1)\right| \leq \frac{|g|_\infty}{\sqrt{T}} \exp\left(|f'|_\infty\, T\right)\left|x_0^2 - x_0^1\right|.$$

Rappelons que $\|\cdot\|_{var}$ est la norme dual de $|\cdot|_\infty$, ce qui donne

$$\left\|\mathcal{P}_T^*\delta_{x_0^2} - \mathcal{P}_T^*\delta_{x_0^1}\right\|_{var} \leq \frac{1}{\sqrt{T}} \exp\left(|f'|_\infty\, T\right)\left|x_0^2 - x_0^1\right|. \tag{5.3.8}$$

Fixons $T=1$ et

$$\delta = \frac{1}{4\exp\left(|f'|_\infty\, T\right)}.$$

On déduit de (5.3.8) que $|x_0^2 - x_0^1| \leq \delta$ implique

$$\left\|\mathcal{P}_1^*\delta_{x_0^2} - \mathcal{P}_1^*\delta_{x_0^1}\right\|_{var} \leq \frac{1}{4}. \tag{5.3.9}$$

En appliquant le Lemme 1.3.1, on fabrique un couplage maximal $(Z_1,Z_2) = (Z_1(x_0^1,x_0^2), Z_2(x_0^1,x_0^2))$ de $((\mathcal{P}_1)^*\,\delta_{x_0^1}, (\mathcal{P}_1)^*\,\delta_{x_0^2})$. Il suit que

$$\mathbb{P}\left(Z_1 \neq Z_2\right) \leq \frac{1}{4}. \tag{5.3.10}$$

Soient $(\beta,\widetilde{\beta})$ un couple de mouvements browniens indépendants. On dénote par $X(\cdot,x_0)$ et $\widetilde{X}(\cdot,x_0)$ les solutions de (5.3.1) associées à β et $\widetilde{\beta}$. Puis on fabrique un couple de variables aléatoires réelles $(V_1,V_2) = (V_1(x_0^1,x_0^2), V_2(x_0^1,x_0^2))$

comme suit

$$(V_1, V_2) = \begin{cases} (X(\cdot, x_0), X(\cdot, x_0)) & \text{si } x_0^1 = x_0^2 = x_0, \\ (Z_1(x_0^1, x_0^2), Z_2(x_0^1, x_0^2)) & \text{si } (x_0^1, x_0^2) \in B(0,\delta) \backslash \{x_0^1 = x_0^2\}, \\ (X(\cdot, x_0^1), \widetilde{X}(\cdot, x_0^2)) & \text{sinon}, \end{cases} \tag{5.3.11}$$

où $B(0,\delta)$ est la boule de $\mathbb{R}^2$ de rayon δ.

Alors $(V_1(x_0^1, x_0^2), V_2(x_0^1, x_0^2))$ est un couplage de $((\mathcal{P}_1)^* \delta_{x_0^1}, (\mathcal{P}_1)^* \delta_{x_0^2})$ qui dépend mesurablement de (x_0^1, x_0^2). On fabrique maintenant un couplage (X^1, X^2) de $(\mathcal{D}(X(\cdot, x_0^1)), \mathcal{D}(X(\cdot, x_0^2)))$ par induction sur $\mathbb{N}$. En fait, on pose d'abord $X^i(0) = x_0^i$ for $i = 1, 2$. Puis, supposant que l'on ait construit (X^1, X^2) sur $\{0, 1, \dots, k\}$, on prend une version de (V_1, V_2) indépendante de (X^1, X^2) et on pose

$$X^i(k+1) = V_i(X^1(k), X^2(k)) \quad \text{pour } i = 1, 2.$$

Supposons un instant que le temps de retour de (X^1, X^2) dans la boule $B(0,\delta)$ admette un moment exponentiel. Alors, on peut conclure. En effet, il est facile de déduire de sa construction, que (X^1, X^2) est un processus de Markov fort discret vérifiant deux propriétés très intéressantes. La première est que si $x_0^1 = x_0^2$, alors $X^1 = X^2$. La seconde est que si $(x_0^1, x_0^2) \in B(0,\delta)$, alors, on déduit de (5.3.10) que

$$\mathbb{P}\left(X^1(1) \neq X^2(1)\right) \leq \frac{1}{4}.$$

La stratégie est donc la suivante. On attend d'entrer dans la boule de rayon δ. Puis on tente de coupler X^1 et X^2. Si ça marche on sait que X^1 et X^2 resteront couplés éternellement. Si ça échoue, on retente notre chance. Comme le temps de retour admet un moment exponentiel et comme la probabilité d'échec de couplage est majorée, on sait qu'on tentera notre chance un nombre fini de fois. Ainsi, comme dans l'exemple unidimensionnel du chapitre 1, on obtient

$$\mathbb{P}\left(X^1(n) \neq X^2(n)\right) \leq Ce^{-\gamma n}\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right), \tag{5.3.12}$$

ce qui permet d'obtenir pour toute mesure λ de probabilité sur $\mathbb{R}$

$$\|(\mathcal{P}_t)^* \lambda - \mu\|_{var} \leq Ce^{-\gamma t}\left(1 + \int_{\mathbb{R}} |x|^2 \lambda(dx)\right). \tag{5.3.13}$$

Discutons maintenant de la validité d'une telle méthode pour NS3D. Les hypothèses que nous avons faites sur notre modèle unidimensionnel peuvent apparaître abusive. En particulier l'hypothèse f Lipschitz bornée. En effet c'est précisément parce qu'on ne controle pas la non-linéarité de NS3D que les problèmes d'existence et d'unicité surgissent. Or cette hypothèse est employée dans la formule (5.3.8). Dans l'annexe E, nous verrons qu'à condition d'imposer à T et aux conditions initiales d'être petits, on peut, via une troncature, obtenir un résultat analogue.

Plus précisément, on introduit

$$\left\{ \begin{array}{rcl} d\eta_N + A\eta_N\, dt + P_N \widetilde{B}(X_N, \eta_N)\, dt & = & P_N(\phi'(X_N) \cdot \eta_N)\, dW, \\ \eta_N(0, x_0) \cdot h & = & P_N h, \end{array} \right. \tag{5.3.14}$$

où $\widetilde{B}(X_N, \eta_N) = B(X_N, \eta_N) + B(\eta_N, X_N)$, $X_N = X_N(\cdot, x_0)$ est l'unique solution de l'approximation de Galerkin (5.1.2) et $\eta_N(t) = \eta_N(t, x_0) \cdot h$.

Il apparaît difficile de controler η_N. Néanmoins, si on tronque par la quantité $\int_0^T |X_N(t)|^2_{H^2(D)}\, dt$, on peut obtenir (Voir annexe E)

$$\mathbb{E}\left(\int_0^{\sigma(X_N(\cdot, x_0))} \|\eta_N(t, 0, x_0) \cdot h\|_3^2\, dt \right) \le c(M)\, \|h\|_2^2\,, \tag{5.3.15}$$

où, pour $M > 0$ et $T \le 1$,

$$\sigma_M(X) = \inf \left\{ t \in (0, T) \;\middle|\; \int_0^t \|X(s)\|_2^2\, ds \ge M \right\}. \tag{5.3.16}$$

L'idée pour adapter le raisonnement précédent est la suivante. On remarque que si δ et T sont petits, alors, pour un M ne dépendant que de D, on controle la probabilité de

$$\int_0^T \|X_N(s)\|_2^2\, ds \ge M,$$

uniformément en N.

Puis, en remplaçant la formule de Bismuth-Elworthy-Li par une variante tronquée, on est à même d'obtenir un résultat analogue à (5.3.8) avec des constantes indépendantes de N.

5.4 Temps de retour dans une petite boule

Il reste donc à prouver que le temps de retour τ de (X^1, X^2) dans la boule $B(0,\delta)$ admet un moment exponentiel. La formule d'Ito de $|X(t,x_0)|^2$ donne

$$d\,|X|^2 + 2\,|X|^2\,dt + 2Xf(X) = 2X d\beta + dt.$$

Rappelons (5.3.2) et intégrons, on obtient

$$\mathbb{E}\,|X(t,x_0)|^2 \leq |x_0|^2\,e^{-2t} + \frac{1}{2}.$$

Il suit que

$$\mathbb{E}\left(\left|X^1(n)\right|^2 + \left|X^2(n)\right|^2\right) \leq \left(\left|x_0^1\right|^2 + \left|x_0^2\right|^2\right) e^{-2t} + 1.$$

On en déduit que le temps de retour τ' dans la boule de rayon 2 admet un moment exponentiel. Il suffit donc de prouver qu'il exite $n \in \mathbb{N}$ et $p_0 > 0$ tel que si $(x_0^1, x_0^2) \in B(0,2)$, alors,

$$\mathbb{P}\left((X^1(k), X^2(k)) \in B(0,\delta) \ \text{ pour un } k \leq n\right) \geq p_0. \tag{5.4.1}$$

En effet, pour obtenir que le temps de retour τ de (X^1, X^2) dans la boule $B(0,\delta)$ admet un moment exponentiel, on attend d'entrer dans la boule de rayon 2. Puis, on tente de rentrer dans la boule de rayon δ. Si ça marche, c'est fini. Sinon, on attend de revenir dans la boule de rayon 2 et retente notre chance. Comme précédemment, on déduit du fait τ' admet un moment exponentiel et de (5.4.1) que τ admet un moment exponentiel.

Du fait de la construction de (X^1, X^2) et de (5.3.11), la preuve rigoureuse de (5.4.1) est fastidieuse car il faut disjoindre la preuve en trois cas à chaque temps $k \leq n$. Comme cette disjonction est faite dans l'annexe E et qu'elle n'apporte aucune compréhension à la preuve, nous traiterons ici uniquement le cas où (X^1, X^2) est le couplage indépendant. Dans ce cas, on a $(X^1, X^2) = (X(\cdot, x_0^1), \widetilde{X}(\cdot, x_0^2))$ sur $\mathbb{N}$.

Du fait de l'indépendance de $(X(\cdot, x_0^1), \widetilde{X}(\cdot, x_0^2))$, on a

$$\mathbb{P}\left((X^1(k), X^2(k)) \in B(0,\delta) \ \text{ pour un } k \leq n\right) \geq \prod_{i=1}^{2} \mathbb{P}\left(\left|X(n, x_0^i)\right| \leq \frac{\delta}{2}\right).$$

Donc, pour établir (5.4.1), il suffit de prouver qu'il existe $n \in \mathbb{N}$ tel que $|x_0| \leq 2$ implique

$$\mathbb{P}\left(|X(n, x_0)| \leq \frac{\delta}{2}\right) \geq \sqrt{p_0}. \tag{5.4.2}$$

L'idée est la suivante. On sait que pour tout couple $t, M > 0$, on a

$$\pi(T, M) = \mathbb{P}\left(\sup_{(0,T)} |\beta| \leq M\right) > 0. \tag{5.4.3}$$

Pour prouver (5.4.2) et ainsi conclure, il suffit donc de trouver $T_0(\delta)$ et $M(\delta)$ tel que pour tout $t \geq T_0(\delta)$ et tout $|x_0| \leq 2$

$$\sup_{(0,t)} |\beta| \leq M(\delta) \;\; \text{implique} \;\; |X(t, x_0)| \leq \frac{\delta}{2}. \tag{5.4.4}$$

Posons donc

$$M = \sup_{(0,t)} |\beta|\,, \; Y = X - \beta.$$

On a donc

$$\frac{d}{dt} Y + Y + f(Y + \beta) = -\beta, \; Y(0) = x_0.$$

Prenons le produit scalaire avec $2Y$, on obtient

$$\frac{d}{dt} |Y|^2 + 2\,|Y|^2 + 2Yf(Y + \beta) = -2Y\beta. \tag{5.4.5}$$

On remarque que

$$-2Y\beta \leq 2M\,|Y| \leq \frac{1}{2}\,|Y|^2 + 2M^2.$$

De plus

$$2Yf(Y + \beta) = 2Yf(Y) + 2Y(f(Y + \beta) - f(Y)).$$

Rappelons (5.3.2) et le fait que f est C^1 à dérivée bornée, on obtient

$$-2Yf(Y + \beta) \leq 2\,|f'|_\infty\, M\,|Y| \leq \frac{1}{2}\,|Y|^2 + 2\,|f'|_\infty\, M^2.$$

Finalement on a

$$\frac{d}{dt} |Y|^2 + |Y|^2 \leq 2(|f'|_\infty + 1)M^2,$$

ce qui donne en intégrant

$$|Y(t)|^2 \leq |x_0|^2 e^{-t} + 2(|f'|_\infty + 1)M^2. \tag{5.4.6}$$

Vu la définition de Y, il est maintenant aisée de conclure en prenant

$$T_0(\delta) = \ln^+\left(\frac{128}{\delta^2}\right), \quad M = \frac{\delta}{16}\left(1 \wedge \frac{1}{\sqrt{(|f'|_\infty + 1)}}\right).$$

L'idée d'attendre que les solutions deviennent petites a déjà été exploitée dans [41] pour traiter NS. C'est pour cette raison qu'ils se sont limité au cas où la partie déterministe du champs de force est nulle. Dans le cas de NS3D, les choses sont plus complexes. En effet, on a des solutions qui vivent dans L^2 et dont on ignore si elles vivent dans $\mathbb{H}^2$. Pourtant, on va prouver que les solutions deviennent petites dans $\mathbb{H}^2$ avec un temps d'arrivée admettant un moment exponentiel. L'affaire se complique grandement du fait de la non dégénérescence du bruit. En clair, le bruit est du type $\phi(X)dW$ avec ϕ qui n'est pas Hilbert-Schmidt à valeur dans $\mathbb{H}^2$. On pose donc

$$Z(t) = \int_0^t e^{-(t-s)A}\phi(X(s))dW(s).$$

Il faut prouver, d'une part que Z reste petit dans H^2 sur un intervalle de temps donné avec une probabilité minoré, et d'autre part que, dans ce cas, les solutions décroissent vers 0 dans $\mathbb{H}^2$, et ce, même lorsque l'on part de conditions initiales dans L^2.

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Annexe A

Preuve de l'ergodicité des équations de Ginzburg–Landau Complexes stochastiques

Résumé: Nous considérons l'équation de Ginzburg–Landau Complexe bruitée par un bruit blanc en temps et régulier par rapport aux variables spatiales et nous établissons le caractère exponentiellement mélangeant du semi-groupe de Markov vers une unique mesure de probabilité invariante. Comme le Théorème de Doob semble ne pas pouvoir être appliquer, nous utilisons une méthode dite de couplage. Pour une meilleur compréhension, nous focaliserons d'abord notre attention sur deux exemples qui bien que très simples contiennent l'essentiel des difficultés.

Mots clés : Équation de Ginzburg–Landau Complexe, semigroupe de transition de Markov, mesures invariantes, ergodicité, méthode de couplage, Formule de Girsanov, inégalité de type Foias–Prodi.

ERGODICITY FOR THE STOCHASTIC COMPLEX GINZBURG–LANDAU EQUATIONS

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Abstract: We study a stochastic complex Ginzburg–Landau (CGL) equation driven by a smooth noise in space and we establish exponential convergence of the Markov transition semi-group toward a unique invariant probability measure. Since Doob Theorem does not seem not to be useful in our situation, a coupling method is used. In order to make this method easier to understand, we first focus on two simple examples which contain most of the arguments and the essential difficulties.

Résumé: Nous considérons l'équation de Ginzburg–Landau Complexe bruitée par un bruit blanc en temps et régulier par rapport aux variables spatiales et nous établissons le caractère exponentiellement mélangeant du semi-groupe de Markov vers une unique mesure de probabilité invariante. Comme le Théorème de Doob semble ne pas pouvoir être appliquer, nous utilisons une méthode dite de couplage. Pour une meilleur compréhension, nous focaliserons d'abord notre attention sur deux exemples qui bien que très simples contiennent l'essentiel des difficultés.

MSC: $35Q60$; $37H99$; $37L99$; $60H10$; $60H15$.

Key words: Stochastic Complex Ginzburg–Landau equations, Markovian transition semigroup, invariant measure, ergodicity, coupling method, Girsanov's formula, Foias–Prodi estimate.

Introduction

Originally introduced to describe a phase transition in superconductivity [9], the Complex Ginzburg–Landau (CGL) equation also models the propagation of dispersive non-linear waves in various areas of physics such as hydrodynamics [20], [21], optics, plasma physics, chemical reaction [11]...

When working in non-homogenous or random media, a noise is often introduced and the stochastic CGL equation may be more representative than the deterministic one.

The CGL equation arises in the same areas of physics as the non-linear Schrödinger (NLS) equation. In fact, the CGL equation is obtained by adding two viscous terms to the NLS equation. The inviscid limits of the deterministic and stochastic CGL equation to the NLS equation are established in [2] and [17], respectively. The stochastic NLS equation is studied in [5] and [6].

Ergodicity of the stochastic CGL equation is established in [1] when the noise is invertible and in [10] for the one-dimensionnal cubic case when the noise is diagonal, does not depend on the solution and is smooth in space.

Our aim in this article is to study ergodicity for stochastic CGL equation under very general assumptions.

Let us recall that the stochastic CGL equation has the form

$$\begin{cases} \frac{du}{dt} - (\varepsilon + \mathrm{i})\Delta u + (\eta + \lambda \mathrm{i}) \left|u\right|^{2\sigma} u &= b(u)\frac{dW}{dt}, & \\ u(t,x) &= 0, & \text{for } x \in \delta D,\ t > 0 \\ u(0,x) &= u_0(x), & \text{for } x \in D. \end{cases}$$

The unknown u is a complex valued process depending on $x \in D$, $D \subset \mathbb{R}^d$ a bounded domain, and $t \geq 0$.

We want to consider noises which may be degenerate and our work is in the spirit of [3], [7], [10], [13], [14], [15], [16], [18] and [23]. Many ideas of this article are taken from these works. However, we develop several generalisations.

The main idea is to compensate the degeneracy of the noise on some subspaces by dissipativity arguments, the so-called Foias-Prodi estimates. A coupling method is developped in a sufficiently general framework to be applied and prove exponential convergence to equilibrium.

To describe the ideas, it is convenient to introduce $(e_k)_{k\in\mathbb{N}^*}$ the eigenbasis of the operator $-\Delta$ with Dirichlet boundary conditions (if periodic boundary conditions were considered, it would be the Fourier basis) and P_N the eigenprojector on the first N modes.

The main assumption of the papers cited above as well as in this work is that the noise is non-degenerate on the space spanned by $(e_k)_{1\leq k\leq N}$ for N sufficiently large. In [10], [16] and [23], the noise is also additive, i.e. $b(u)$ does not depend on u. The method developped in [18] allows to treat more general noises and, in [18], b is allowed to depend on $P_N u$. However, in this latter work, the author restricts his attention to the case when the high modes are not perturbed by noise. It is claimed that the method can be generalized to treat a noise which hits all components. Such a generalisation is contained in [19] in the purely additive case.

Here we develop also such a generalization and treat a noise which may hit all modes but depends only on $P_N u$. We have chosen to use ideas both from [18] and from [16], [23]. We hope that this makes our proof easier to understand. Moreover, we get rid of the assumption that b is diagonal in the basis $(e_k)_{k\in\mathbb{N}^*}$.

Also, if we work in the space $L^2(D)$, it is not difficult to get a Lyapunov structure and Foias-Prodi estimates. Thus, with an additive noise or with a noise as in [18], our results would be a rather easy applications of these methods.

However, this works only for small values of σ, namely $\sigma < \frac{2}{d}$. It is well known that the CGL equations are also well-posed for $\sigma \in \left[\frac{2}{d}, \frac{2}{d-2}\right)$ ($\sigma \in \left[\frac{2}{d}, \infty\right)$ for $d \in \{1,2\}$) provided we work with $H^1(D)$–valued solutions and the nonlinearity is defocusing ($\lambda = 1$). We also develop the coupling method in that context and show that it is possible to find a convenient Lyapunov structure and derive Foias-Prodi estimates. Thus we prove exponential convergence to equilibrium for the noises described above in all the cases when it is known that there exists a unique global solution and an invariant measure.

Moreover, using the smoothing effect of CGL and an interpolation argument, we are able to prove exponential convergence in the Wasserstein norm in $H^s(D)$ for any $s<2$. This give convergence to equilibrium for less regular funtionnal.

In order to make the understanding of the method easier, we start with two simple examples which motivate and introduce all arguments in a simpler context. The first example is particulary simple. It introduces the idea of coupling and the use of Girsanov transform to construct a coupling. The second example is similar to the one considered in [18]. However, it contains further difficulties and more details are given. We have tried to isolate every key argument. This is also the opportunity to state a very general result giving conditions implying exponential mixing (Theorem 1.8). It is a strong generalization of Theorem 3.1 of [16].

Then, in section 2 we deal with CGL equations. We state and prove the general ergodicity result described above

1. Preliminary results

The proof of our result is obtained by the combination of two main ideas: the coupling and the Foias-Prodi estimate. The first subsection is a simple example devoted to understand the use of the notion of coupling. The second subsection is a two dimensional example devoted to understand how we use the two main ideas. The third subsection is the statement of an abstract result which is both fundamental and technical. The other subsections are devoted to the proof of this abstract result. The understanding of the proof of the abstract result is not necessary to the understanding of the rest of the article. On the contrary the three first subsections contain the main ideas of this article.

1.1. **A simple example.**

In this subsection we introduce the notion of coupling and we motivate it on a simple example.

Let Π the one-dimensionnal torus. We consider the following example. We denote by $X(.,x_0)$ the unique solution in Π of

$$\frac{dX}{dt}+f(X)=\frac{dW}{dt},\quad X(0,x_0)=x_0, \tag{1.1}$$

where $f:\Pi\to\mathbb{R}$ is a Lipschitz function and W is a one-dimensionnal brownian motion. It is easy to prove that X is a Markovian process. We denote by $(\mathcal{P}_t)_t$ its Markovian transition semigroup.

We recall the definition of $\|\mu\|_{var}$, the total variation of a finite real measure μ:

$$\|\mu\|_{var}=\sup\left\{|\mu(\Gamma)|\ \mid \Gamma\in\mathcal{B}(\Pi)\right\},$$

where we denote by $\mathcal{B}(\Pi)$ the set of the Borelian subsets of Π. It is well known that $\|.\|_{var}$ is the dual norm of $|.|_\infty$. We prove that there exists a unique invariant measure ν and that for any probability measure μ

$$\|\mathcal{P}_t^*\mu-\nu\|_{var}\le ce^{-\beta t}.$$

Using a completeness argument and the markovian property of X, we obtain that it is sufficient to prove that for any $\psi:\Pi\to\mathbb{R}$ borelian bounded and for any $(t,x_1,x_2)\in\mathbb{R}^+\times\Pi^2$, we have

$$|\mathbb{E}\psi(X(t,x_1))-\mathbb{E}\psi(X(t,x_2))|\le c\,|\psi|_\infty\, e^{-\beta t}.$$

Clearly it is sufficient to find $(X_1(t), X_2(t))$ such that for any $(i,t) \in \{1,2\} \times \mathbb{R}^+$, we have $\mathcal{D}(X_i(t)) = \mathcal{D}(X(t,x_i))$, where $\mathcal{D}$ means distribution, and

$$|\mathbb{E}\psi(X_1(t)) - \mathbb{E}\psi(X_2(t))| \leq c\,|\psi|_\infty\, e^{-\beta t}. \tag{1.2}$$

Now we introduce the notion of coupling. Let (μ_1, μ_2) be two distributions on a same space $(E, \mathcal{E})$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (Z_1, Z_2) be two random variables $(\Omega, \mathcal{F}) \to (E, \mathcal{E})$. We say that (Z_1, Z_2) is a coupling of (μ_1, μ_2) if $\mu_i = \mathcal{D}(Z_i)$ for $i = 1, 2$.

Remark 1.1. *Although the marginal laws of (Z_1, Z_2) are imposed, we have a lot of freedom when choosing the law of the couple (Z_1, Z_2). For instance, let us consider (W_1, W_2) a two-dimensional brownian motion. Let μ be the Wiener measure on $\mathbb{R}$, which means that $\mu = \mathcal{D}(W_1) = \mathcal{D}(W_2)$. Then (W_1, W_2), $(W_1', W_2') = (W_1, W_1)$ and $(W_1'', W_2'') = (W_1, -W_1)$ are three couplings of (μ, μ). These three couplings have very different laws. In the one hand, W_1 and W_2 are independent and $W_1 \neq \pm W_2$ a.s. and in the other hand $W_1' = W_2'$ and $W_1'' = -W_2''$.*

In order to establish (1.2), we remark that it is sufficient to build (X_1, X_2) a coupling of $(\mathcal{D}(X(\cdot, x_1)), \mathcal{D}(X(\cdot, x_2)))$ on $\mathbb{R}^+$ such that for any $t \geq 0$

$$\mathbb{P}(X_1(t) \neq X_2(t)) \leq c e^{-\beta t}. \tag{1.3}$$

By induction, it suffices to construct a coupling on a fixed interval $[0, T]$. Indeed, we first set

$$X_i(0) = x_i\ ,\ i = 1, 2.$$

Then we build a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and a measurable function $(\omega', t, x_1, x_2) \to Z_i(t, x_1, x_2)$ such that for any (x_1, x_2), $(Z_i(\cdot, x_1, x_2))_{i=1,2}$ is a coupling of $(X(\cdot, x_i))_{i=1,2}$ on $[0, T]$.

The induction argument is then as follows. Assuming that we have built (X_1, X_2) on $[0, nT]$, we take (Z_1, Z_2) as above independant of (X_1, X_2) on $[0, nT]$ and set

$$X_i(nT + t) = Z_i(t, X_1(nT), X_2(nT)), \qquad \text{for } t \in (0, T].$$

The Markov property of X implies that (X_1, X_2) is a coupling of $(\mathcal{D}(X(\cdot, x_1)), \mathcal{D}(X(\cdot, x_2)))$ on $[0, (n+1)T]$.

The coupling (Z_1, Z_2) on $[0, T]$ constructed below satisfies the following properties

$$\begin{aligned}
&\mathbb{P}(Z_1(T, x_1, x_2) = Z_2(T, x_1, x_2)) && \geq p_0 > 0, && \text{if } x_1 \neq x_2, && (1.4)\\
&\mathbb{P}(Z_1(., x_1, x_2) = Z_2(., x_1, x_2)) && = 1, && \text{if } x_1 = x_2. && (1.5)
\end{aligned}$$

Invoking (1.5), we obtain that

$$\mathbb{P}(X_1(nT) \neq X_2(nT) | X_1((n-1)T) = X_2((n-1)T)) = 0.$$

Thus it follows

$$\begin{aligned}
\mathbb{P}(X_1(nT) \neq X_2(nT)) \quad \leq \quad & \mathbb{P}(X_1((n-1)T) \neq X_2((n-1)T)) \times \\
& \mathbb{P}(X_1(nT) \neq X_2(nT) | X_1((n-1)T) \neq X_2((n-1)T)).
\end{aligned}$$

We easily get from (1.4) and (1.5)

$$\mathbb{P}(X_1(t) \neq X_2(t), \ \text{ for some } t \geq nT) \leq (1 - p_0)^n,$$

which implies (1.3) and allows us to conclude.

Before building (Z_1, Z_2) such that (1.4) and (1.5) hold, we need to define some notions. Let μ, μ_1 and μ_2 be three probability measures on a space $(E, \mathcal{E})$ such that μ_1 and μ_2 are absolutely continuous with respect to μ. We set

$$\begin{array}{lcl} d\,|\mu_1-\mu_2| & = & \left|\frac{d\mu_1}{d\mu}-\frac{d\mu_2}{d\mu}\right| d\mu, \\ d(\mu_1\wedge\mu_2) & = & (\frac{d\mu_1}{d\mu}\wedge\frac{d\mu_2}{d\mu})d\mu, \\ d(\mu_1-\mu_2)^+ & = & (\frac{d\mu_1}{d\mu}-\frac{d\mu_2}{d\mu})^+ d\mu. \end{array}$$

These definitions do not depend on the choice of μ. Moreover we have

$$\|\mu_1-\mu_2\|_{var} = \frac{1}{2}\,|\mu_1-\mu_2|\,(E) = (\mu_1-\mu_2)^+(E) = \frac{1}{2}\int_E \left|\frac{d\mu_1}{d\mu}-\frac{d\mu_2}{d\mu}\right| d\mu. \tag{1.6}$$

The following Lemma is the key of our proof.

Lemma 1.2. *Let (μ_1, μ_2) be two probability measures on $(E, \mathcal{E})$. Then*

$$\|\mu_1-\mu_2\|_{var} = \min \mathbb{P}(Z_1 \neq Z_2).$$

The minimum is taken over the coupling (Z_1, Z_2) of (μ_1, μ_2). Such a coupling exists and is called a maximal coupling and has the following property:

$$\mathbb{P}(Z_1 = Z_2, Z_1 \in \Gamma) = (\mu_1 \wedge \mu_2)(\Gamma) \ \ \textit{for any } \Gamma \in \mathcal{E}.$$

The proof of Lemma 1.2 is given in the Appendix. We consider W' a Wiener process. If $x_1 = x_2 = x$, we choose the trivial coupling $(Z_i(.,x,x))_{i=1,2}$ on $[0,T]$. In other words, we set $Z_1(.,x,x) = Z_2(.,x,x) = X'(.,x)$ on $[0,T]$ where $X'(.,x)$ is the solution of (1.1) associated with W'. Thus (1.5) is clear.

For $x_1 \neq x_2$, the idea is borrowed from [16]. We consider $(\widetilde{Z}_1(.,x_1,x_2), Z_2(.,x_1,x_2))$ the maximal coupling of $\big(\mathcal{D}(X(\cdot,x_1)+\frac{T-\cdot}{T}(x_2-x_1)), \mathcal{D}(X(.,x_2))\big)$ on $[0,T]$ and we set $Z_1(t,x_1,x_2) = \widetilde{Z}_1(t,x_1,x_2) - \frac{T-t}{T}(x_2-x_1)$. Then it is easy to see that $(Z_i(.,x_1,x_2))_{i=1,2}$ is a coupling of $(\mathcal{D}(X(.,x_i))_{i=1,2}$ on $[0,T]$ and we have

$$\mathbb{P}\left(Z_1(T,x_1,x_2) = Z_2(T,x_1,x_2)\right) \geq \mathbb{P}\left(\widetilde{Z}_1(.,x_1,x_2) = Z_2(.,x_1,x_2)\right). \tag{1.7}$$

We need the following result which is lemma D.1 of [18]

Lemma 1.3. *Let μ_1 and μ_2 be two probability measures on a space $(E, \mathcal{E})$. Let A be an event of E. Assume that $\mu_1^A = \mu_1(A \cap .)$ is equivalent to $\mu_2^A = \mu_2(A \cap .)$. Then for any $p > 1$ and $C > 1$*

$$\int_A \left(\frac{d\mu_1^A}{d\mu_2^A}\right)^{p+1} d\mu_2 \leq C < \infty \quad \textit{implies} \quad (\mu_1 \wedge \mu_2)(A) \geq \left(1-\frac{1}{p}\right)\left(\frac{\mu_1(A)^p}{pC}\right)^{\frac{1}{p-1}}.$$

Using (1.7) and Lemma 1.2 and 1.3 with $E = C([0,T];\Pi)$, we obtain that

$$\mathbb{P}\left(Z_1(T,x_1,x_2) = Z_2(T,x_1,x_2)\right) \geq \left(1-\frac{1}{p}\right)\left(p\int_E \left(\frac{d\widetilde{\mu}_1}{d\mu_2}\right)^{p+1} d\mu_2\right)^{-\frac{1}{p-1}}, \tag{1.8}$$

where $(\widetilde{\mu}_1, \mu_2) = (\mathcal{D}(X(\cdot,x_1)+\frac{T-\cdot}{T}(x_2-x_1)), \mathcal{D}(X(.,x_2)))$ on $[0,T]$.

We use a Girsanov formula to estimate $\int_E \left(\frac{d\widetilde{\mu}_1}{d\mu_2}\right)^{p+1} d\mu_2$. Setting $\widetilde{X}(t) = X(t,x_1) + \frac{T-t}{T}(x_2-x_1)$, we obtain that $\widetilde{\mu}_1$ is the distribution of $\widetilde{X}$ under the probability $\mathbb{P}$ and that $\widetilde{X}$ is the unique solution of

$$\frac{d\widetilde{X}}{dt}-\frac{1}{T}(x_2-x_1)+f(\widetilde{X}(t)+\frac{T-t}{T}(x_2-x_1))=\frac{dW}{dt},\quad \widetilde{X}(0)=x_2.$$

We set $W'(t)=W(t)+\int_0^t d(s)dt$, where

$$d(t)=\frac{1}{T}(x_2-x_1)+f(\widetilde{X}(t))-f(\widetilde{X}(t)+\frac{T-t}{T}(x_2-x_1)). \tag{1.9}$$

Then $\widetilde{X}$ is a solution of

$$\frac{d\widetilde{X}}{dt}+f(\widetilde{X})=\frac{dW'}{dt},\quad \widetilde{X}(0)=x_2, \tag{1.10}$$

We are working on the torus and f is continuous, therefore d is uniformly bounded:

$$|d(t)|\leq\frac{1}{T}+2\left|f\right|_\infty.$$

Hence, the Novikov condition is satisfied and the Girsanov formula can be applied. Then we set

$$d\mathbb{P}'=\exp\left(\int_0^t d(s)dW(s)-\frac{1}{2}\int_0^t\left|d(s)\right|^2dt\right)d\mathbb{P}$$

We deduce from the Girsanov formula that $\mathbb{P}'$ is a probability measure under which W' is a brownian motion and $\widetilde{X}$ is a solution of (1.10), then the law of $\widetilde{X}$ under $\mathbb{P}'$ is μ_2. Moreover

$$\int_E\left(\frac{d\widetilde{\mu}_1}{d\mu_2}\right)^{p+1}d\mu_2\leq\exp\left(c_p\left(\frac{1}{T}+\left|f\right|_\infty^2T\right)\right), \tag{1.11}$$

which allows us to conclude this example. Indeed, by applying (1.7), (1.8) and (1.11) we get (1.4).

1.2. **A representative two-dimensionnal example.**

The example we consider now is a two dimensional system which mimics the decomposition of a stochastic partial differential equation according to low and high modes of the solution. This example allows the introduction of the main ideas in a simplified context, the system has the form

$$\left\{\begin{array}{lcl}dX+2Xdt+f(X,Y)dt&=&\sigma_l(X)d\beta,\\ dY+2Ydt+g(X,Y)dt&=&\sigma_h(X)d\eta,\\ X(0)=x_0,\quad Y(0)=y_0.&&\end{array}\right. \tag{1.12}$$

We set $u=(X,Y)$ and $W=(\beta,\eta)$. We use the following assumptions

$$\left\{\begin{array}{ll}\text{i)} & f,\ g,\ \sigma_l \text{ and } \sigma_h \text{ are bounded and Lipschitz},\\ \text{ii)} & \text{There exists } K_0>0 \text{ such that},\\ & f(x,y)x+g(x,y)y\geq-(\left|x\right|^2+\left|y\right|^2+K_0),\quad (x,y)\in\mathbb{R}^2.\end{array}\right. \tag{1.13}$$

Condition i) ensures existence and uniqueness of a solution to (1.12) once the initial data $u_0=(x_0,y_0)$ is given. It is also classical that weak existence and uniqueness holds. We denote by $X(\cdot,u_0)$, $Y(\cdot,u_0)$, $u(\cdot,u_0)$ the solution where $u_0=(x_0,y_0)$

and $u = (X, Y)$. Moreover, it is easy to see that, by ii), there exists an invariant measure ν.

Contrary to section 1.1, we want to allow degenerate noises. More precisely, we want to treat the case when the noise on the second equation may vanish. This possible degeneracy is compensated by a dissipativity assumption. We use the following assumptions.

$$(1.14)\qquad \begin{cases} \text{i)} & \text{There exists } \sigma_0 > 0 \text{ such that, } \sigma_l(x) \geq \sigma_0,\ x \in \mathbb{R}. \\ \text{ii)} & |g(x,y_1) - g(x,y_2)| \leq |y_1 - y_2|\,, \quad (x,y_1,y_2) \in \mathbb{R}^2. \end{cases}$$

By the dissipativity method (see [4] section 11.5), ii) implies exponential convergence to equilibrium for the second equation if X is fixed. Whilst the coupling argument explained in section 1.1 can be used to treat the first equation when Y is fixed. Note however that we need a more sophisticated coupling here. Indeed, the simple coupling explained above seems to be usefull only for additive noise.

Here, we explain how these two arguments may be coupled to treat system (1.12). The essential tool which allows to treat system (1.12) is the so-called Foias-Prodi estimate which reflects the dissipativity property of the second equation. It is a simple consequence of (1.14)ii)

Proposition 1.4. *Let $(u_i, W_i)_{i=1,2}$ be two weak solutions of* (1.12) *such that*

$$X_1(s) = X_2(s), \quad \eta_1(s) = \eta_2(s), \quad s \in [0,t],$$

then

$$|u_1(t) - u_2(t)| \leq |u_1(0) - u_2(0)|\, e^{-t}$$

Since the noise on the second equation might be degenerate, there is no hope to use Girsanov formula on the full system. We can use it to modify the drift of the first equation only and it is not possible to derive a strong estimate as (1.3).

Recall that in section 1.1, we have built the coupling (X_1, X_2) of $(\mathcal{D}(X(\cdot, x_0^1)), \mathcal{D}(X(\cdot, x_0^2)))$ by induction on $[0, kT]$ by using a coupling $(Z_i(\cdot, x_0^1, x_0^2))_{i=1,2}$ of $(\mathcal{D}(X(\cdot, x_0^i)))_{i=1,2}$ on $[0,T]$ which satisfies (1.5). Then if (X_1, X_2) were coupled at time kT, (X_1, X_2) would be coupled on $[kT, \infty)$ with probability one. Thus to conclude, it was sufficient to establish (1.4).

In this section, since we couple (X_1, X_2), but not (Y_1, Y_2), then there is no hope that a couple (X_1, X_2) coupled at time kT remains coupled at time $(k+1)T$ with probability one.

However, coupling the X's and using Foias-Prodi estimates, we obtain a coupling (u_1, u_2) of $(\mathcal{D}(u(\cdot, u_0^1)), \mathcal{D}(u(\cdot, u_0^2)))$ on $\mathbb{R}^+$ such that

$$(1.15)\qquad \mathbb{P}\left(|u_1(t) - u_2(t)| > ce^{-\beta t}\right) \leq ce^{-\beta t}(1 + \left|u_0^1\right|^2 + \left|u_0^2\right|^2).$$

This estimate does not imply the decay of the total variation of $\mathcal{P}_t^* \delta_{u_0^1} - \mathcal{P}_t^* \delta_{u_0^2}$, but the decay of this quantity in the Wasserstein distance $|\cdot|_{Lip_b}^*$ which is the dual norm of the lipschitz and bounded functions. Indeed, for ψ lipschitz and bounded, we clearly have

$$\begin{aligned} \left|\mathbb{E}\psi(u(t,u_0^1)) - \mathbb{E}\psi(u(t,u_0^2))\right| &= |\mathbb{E}\psi(u_1(t)) - \mathbb{E}\psi(u_2(t))|\,, \\ &\leq 2\,|\psi|_\infty\, \mathbb{P}\left(|u_1(t) - u_2(t)| > ce^{-\beta t}\right) + |\psi|_{Lip}\, ce^{-\beta t}, \end{aligned}$$

and then by (1.15)

$$\left|\mathbb{E}\psi(u(t,u_0^1)) - \mathbb{E}\psi(u(t,u_0^1))\right| \leq c\,|\psi|_{Lip_b}\, e^{-\beta t}(1 + \left|u_0^1\right|^2 + \left|u_0^2\right|^2). \tag{1.16}$$

The idea of the proof is the following. We couple $(\mathcal{D}(X(\cdot,u_0^i),\eta))_{i=1,2}$. Then using the Foias-Prodi estimate, we control $Y_1 - Y_2$ which is equivalent to control $u_1 - u_2$. By controlling $u_1 - u_2$, we control the probability to remain coupled.

Remark 1.5. *In the general case f, g are not globally lipschitz and bounded and a cut-off has to be used. This further difficulty will be treated in the context of the CGL equation below.*

It is convenient to introduce the following functions:

$$l_0(k) = \min\left\{l \in \{0,...,k\} | P_{l,k}\right\},$$

where $\min\phi = \infty$ and

$$(P_{l,k}) \begin{cases} X_1(t) = X_2(t), \quad \eta_1(t) = \eta_2(t), \quad \forall\, t \in [lT,kT], \\ |u_i(lT)| \leq d^*, \quad i = 1,2. \end{cases}$$

The first requirement in $(P_{l,k})$ states that the two solutions of the first equation are coupled on $[lT,kT]$. Notice that Proposition 1.4 gives

$$l_0(k) = l \quad \text{implies} \quad |u_1(t) - u_2(t)| \leq 2d^* e^{-(t-lT)}, \quad \text{for any } t \in [lT,kT]. \tag{1.17}$$

From now on we say that (X_1,X_2) are coupled at kT if $l_0(k) \leq k$, in other words if $l_0(k) \neq \infty$.

We set

$$d_0 = 4(d^*)^2.$$

We prove the two following properties.

For any $d_0 > 0$

$$\begin{cases} \exists\, p_0(d_0) > 0,\ (p_i)_{i\geq 1},\ T_0(d_0) > 0 \ \text{ such that for any } l \leq k, \\ \mathbb{P}\left(l_0(k+1) = l \mid l_0(k) = l\right) \geq p_{k-l},\ \text{for any } T \geq T_0(d_0), \\ 1 - p_i \leq e^{-iT},\ i \geq 1, \end{cases} \tag{1.18}$$

and, for any (R_0,d_0) sufficiently large,

$$\begin{cases} \exists\, T^*(R_0) > 0 \text{ and } p_{-1} > 0 \text{ such that for any } T \geq T^*(R_0) \\ \mathbb{P}\left(l_0(k+1) = k+1 \mid l_0(k) = \infty,\ \mathcal{H}_k \leq R_0\right) \geq p_{-1}, \end{cases} \tag{1.19}$$

where

$$\mathcal{H}_k = |u_1(kT)|^2 + |u_2(kT)|^2.$$

(1.18) states that the probability that two solutions decouples at kT is very small, (1.19) states that, inside a ball, the probability that two solutions get coupled at $(k+1)T$ is uniformly bounded below.

In the particular case where $\sigma_l(x)$ does not depend on x and where $K_0 = 0$, one can apply a similar proof as in section 1.1 to establish a result closely related to (1.18), (1.19). This technic has been developped in [16]. But it does not seem to work in the general case.

Consequently, we use some tools developped in [18] to establish (1.18), (1.19). Note that in (1.19), we use only starting points in a ball of radius R_0. This is due to the fact that to prove (1.19), we need to estimate some terms which cannot be controlled on $\mathbb{R}^2$ but only inside a ball. This further difficulty is due to the fact that contrary to the simple example of section 1.1, we work on an unbounded phase space and is overcomed thanks to another ingredient which is the so-called Lyapunov structure. It allows the control of the probability to enter the ball of radius R_0. In our example, it is an easy consequence of (1.13)ii). More precisely, we use the property that for any solution $u(\cdot, u_0)$

$$\left\{ \begin{array}{lcl} \mathbb{E}\,|u(t,u_0)|^2 & \leq & e^{-2t}\,|u_0|^2 + \frac{K_1}{2}, \\ \mathbb{E}\left(|u(\tau',u_0)|^4\, 1_{\tau'<\infty}\right) & \leq & K'\left(|u_0|^4 + 1 + \mathbb{E}\left(\tau' 1_{\tau'<\infty}\right)\right), \end{array} \right. \tag{1.20}$$

for any stopping times τ'.

The following Proposition is a consequence of Theorem 1.8 given in a more general setting below.

Proposition 1.6. *If there exists a coupling of* $\mathcal{D}(u(\cdot,u_0^i),W)$ *such that* (1.18), (1.19) *are satisfied, then* (1.15) *is true. Thus there exists a unique invariant measure* ν *of* $(\mathcal{P}_t)_t$*. Moreover there exist* C *and* α *such that*

$$\|\mathcal{P}_t\mu - \nu\|^*_{Lip_b(\mathbb{R}^2)} \leq Ce^{-\alpha t}\left(1 + \int_{\mathbb{R}^2} |u|\, d\mu(u)\right).$$

To obtain (1.18) and (1.19), we introduce three more ingredients. First in order to build a coupling $((u_1,W_1),(u_2,W_2))$ such that $((X_1,\eta_1),(X_2,\eta_2))$ is a maximal coupling, we use the following results contained in [18], although not explicitly stated. Its proof is postponed to the appendix.

Proposition 1.7. *Let* E *and* F *be two polish spaces,* $f_0 : E \to F$ *be a measurable map and* (μ_1,μ_2) *be two probability measures on* E*. We set*

$$\nu_i = f_0^*\mu_i, \quad i = 1,2.$$

Then there exist a coupling (V_1,V_2) *of* (μ_1,μ_2) *such that* $(f_0(V_1), f_0(V_2))$ *is a maximal coupling of* (ν_1,ν_2)*.*

We also remark that given (X,η) on $[0,T]$, there exists a unique solution $Y(\cdot,u_0)$ of

$$dY + 2Y dt + g(X,Y)dt = \sigma_h(X)d\eta, \quad Y(0,u_0) = y_0.$$

We set

$$Y(\cdot,u_0) = \Phi(X,\eta,u_0)(\cdot).$$

It is easy to see that Y is adapted to the filtration associated to η and X.

Proposition 1.4 implies that for any given (X,η)

$$\left|\Phi(X,\eta,u_0^1)(t) - \Phi(X,\eta,u_0^2)(t)\right| \leq e^{-t}\left|u_0^1 - u_0^2\right|. \tag{1.21}$$

Then we rewrite the equation for X as follows

$$\left\{ \begin{array}{lcl} dX + 2X dt + f(X,\Phi(X,\eta,u_0))dt & = & \sigma_l(X)d\beta, \\ X(0) = x_0. & & \end{array} \right. \tag{1.22}$$

The Girsanov formula can then be used on (1.22) as in section 1.1.

We finally remark that by induction, it suffices to construct a probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ and two measurable couples of functions $(\omega_0, u_0^1, u_0^2) \to (V_i(\cdot, u_0^1, u_0^2))_{i=1,2}$ and $(V_i'(\cdot, u_0^1, u_0^2))_{i=1,2}$ and such that, for any (u_0^1, u_0^2), $(V_i(\cdot, u_0^1, u_0^2))_{i=1,2}$ and $(V_i'(\cdot, u_0^1, u_0^2))_{i=1,2}$ are two couplings of $(\mathcal{D}(u(\cdot, u_0^i), W))_{i=1,2}$ on $[0, T]$. Indeed, we first set

$$u_i(0) = u_0^i, \quad W_i(0) = 0, \quad i = 1, 2.$$

Assuming that we have built $(u_i, W_i)_{i=1,2}$ on $[0, kT]$, then we take $(V_i)_i$ and $(V_i')_i$ as above independant of $(u_i, W_i)_{i=1,2}$ on $[0, kT]$ and set

$$(u_i(kT+t), W_i(kT+t)) = \begin{cases} V_i(t, u_1(kT), u_2(kT)) & \text{if } l_0(k) \leq k, \\ V_i'(t, u_1(kT), u_2(kT)) & \text{if } l_0(k) = \infty, \end{cases} \tag{1.23}$$

for any $t \in [0, T]$.

Proof of (1.18)**.**

To build $(V_i(\cdot, u_0^1, u_0^2))_{i=1,2}$, we apply Proposition 1.7 to $E = C((0,T); \mathbb{R}^2)^2$, $F = C((0,T); \mathbb{R})^2$,

$$f_0\,(u, W) = (X, \eta), \quad \text{where } u = \begin{pmatrix} X \\ Y \end{pmatrix}, \ W = \begin{pmatrix} \beta \\ \eta \end{pmatrix},$$

and to

$$\mu_i = \mathcal{D}(u(\cdot, u_0^i), W), \qquad \text{on } [0, T].$$

Remark that if we set $\nu_i = f_0^* \mu_i$, we obtain

$$\nu_i = \mathcal{D}(X(\cdot, u_0^i), \eta), \qquad \text{on } [0, T].$$

We write

$$(Z_i, \xi_i) = f_0(V_i), \quad i = 1, 2.$$

Then $(V_i(\cdot, u_0^1, u_0^2))_{i=1,2}$ is a coupling of (μ_1, μ_2) such that $((Z_i, \xi_i)(\cdot, u_0^1, u_0^2))_{i=1,2}$ is a maximal coupling of (ν_1, ν_2).

We first use a Girsanov formula to estimate I_p, where

$$I_p = \int_F \left(\frac{d\nu_2}{d\nu_1} \right)^{p+1} d\nu_2.$$

Then, using Lemma 1.2, we establish (1.18).

We consider a couple $(u_i, W_i)_{i=1,2}$ consisting of two solutions of (1.12) on $[0, kT]$. From now on, we are only concerned with a trajectory of $(u_i, W_i)_{i=1,2}$ such that $l_0(k) = l \leq k$. We set

$$x = X_1(kT) = X_2(kT), \quad y_i = Y_i(kT), \quad i = 1, 2.$$

Let (β, ξ) be a two-dimensionnal brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by Z the unique solution of

$$\begin{cases} dZ + 2Zdt + f(Z, \Phi(Z(\cdot), \xi(\cdot), (x, y_1)))dt \;=\; \sigma_l(Z)d\beta, \\ Z(0) = x. \end{cases} \tag{1.24}$$

Taking into account (1.24), we obtain that ν_1 is the distribution of (Z, ξ) under the probability $\mathbb{P}$.

We set $\widetilde{\beta}(t) = \beta(t) + \int_0^t d(s)dt$ where

$$d(t) = \frac{1}{\sigma_l(Z(t))}\left(f(Z(t), \Phi(Z, \xi, (x, y_2))(t)) - f(Z(t), \Phi(Z, \xi, (x, y_1))(t))\right). \tag{1.25}$$

Then Z is a solution of

$$\begin{cases} dZ + 2Zdt + f(Z, \Phi(Z(\cdot), \xi(\cdot), (x, y_2)))dt & = & \sigma_l(Z)d\widetilde{\beta}, \\ Z(0) = x. & & \end{cases} \tag{1.26}$$

Since f is bounded and σ_l is bounded below, then d is uniformly bounded. Hence, the Novikov condition is satisfied and the Girsanov formula can be applied. Then we set

$$d\widetilde{\mathbb{P}} = \exp\left(\int_0^T d(s)dW(s) - \frac{1}{2}\int_0^T |d(s)|^2\, dt\right) d\mathbb{P}$$

We deduce from the Girsanov formula that $\widetilde{\mathbb{P}}$ is a probability under which $(\widetilde{\beta}, \xi)$ is a brownian motion and since Z is a solution of (1.26), then the law of (Z, ξ) under $\widetilde{\mathbb{P}}$ is ν_2. Moreover

$$I_p \leq \mathbb{E}\exp\left(c_p\int_0^T |d(s)|^2\, dt\right). \tag{1.27}$$

Since f is Lipschitz, then we infer from (1.25) and (1.14)i) that

$$|d(t)| \leq \sigma_0^{-1}\, |f|_{Lip}\, |\Phi(Z(\cdot), \xi(\cdot), (x, y_1))(t) - \Phi(Z(\cdot), \xi(\cdot), (x, y_2))(t)|\,.$$

Now we use the Foias-Prodi estimate. Applying (1.17) and (1.21), it follows from $l_0(k) = l$ that

$$|d(t)|^2 \leq d_0\sigma_0^{-2}\, |f|_{Lip}^2 \exp\left(-2(k-l)T\right).$$

Then it follows that

$$I_p \leq \exp\left(c_p\sigma_0^{-2} d_0\, |f|_{Lip}^2\, e^{-2(k-l)T}\right). \tag{1.28}$$

Note that

$$\|\nu_1 - \nu_2\|_{var} = \int_F \left|\frac{d\nu_2}{d\nu_1} - 1\right| d\nu_2 \leq \sqrt{\int \left(\frac{d\nu_2}{d\nu_1}\right)^2 d\nu_2 - 1}.$$

We infer from (1.28) that, for $T \geq T_0(d_0) = (\sigma_0^{-2}c_p d_0\, |f|_{Lip}^2)^{-1}$,

$$\|\nu_1 - \nu_2\|_{var} \leq e^{-(k-l)T}.$$

Applying Lemma 1.2 to the maximal coupling $(Z_1, Z_2)_{i=1,2}$ of (ν_1, ν_2) gives

$$\mathbb{P}\left((Z_1, \xi_1) \neq (Z_2, \xi_2)\right) \leq \|\nu_1 - \nu_2\|_{var} \leq e^{-(k-l)T}. \tag{1.29}$$

Using (1.23) and (1.29), we obtain that on $l_0(k) = l$

$$\mathbb{P}\left((X_1, \eta_1) \neq (X_2, \eta_2) \text{ on } [kT, (k+1)T] \mid \mathcal{F}_{kT}\right) \leq e^{-(k-l)T}.$$

Noticing that

$$\{l_0(k+1) = l\} = \{l_0(k) = l\} \cap \{(X_1, \eta_1) = (X_2, \eta_2) \text{ on } [kT, (k+1)T]\}.$$

and integrating over $l_0(k) = l$ gives for $T \geq T_0(d)$ and for $k > l$

$$\mathbb{P}\left(l_0(k+1) \neq l \,|\, l_0(k) = l\right) \leq e^{-(k-l)T}. \tag{1.30}$$

Now, it remains to consider the case $k=l$, we apply Lemmas 1.2 and 1.3 to $(Z_i,\xi_i)_{i=1,2}$ which gives

$$\mathbb{P}\left((Z_1,\xi_1)=(Z_2,\xi_2)\right)=\left(\nu_1\wedge\nu_2\right)(F)\geq\left(1-\frac{1}{p}\right)(pI_p)^{-\frac{1}{p-1}}.$$

Applying (1.27) and fixing $p>1$, we obtain

$$\mathbb{P}\left((Z_1,\xi_1)=(Z_2,\xi_2)\right)\geq p_0(d_0)=\left(1-\frac{1}{p}\right)p^{-\frac{1}{p-1}}\exp\left(-c_pd_0\,|f|^2_{Lip}\right). \tag{1.31}$$

To conclude, we notice that (1.30) and (1.31) imply (1.18).

Proof of (1.19)**.**

Assume that we have $d_0>0$, $\widetilde{p}>0$, $T_1>0$, $R_1>4K_1$ and a coupling $(\widetilde{V}_i(\cdot,u_0^1,u_0^2))_{i=1,2}$ of (μ_1,μ_2), where

$$\mu_i=\mathcal{D}(u(\cdot,u_0^i),W),\quad\text{on }[0,T_1],\quad i=1,2,$$

and such that for any (u_0^1,u_0^2) which satisfies $\left|u_0^1\right|^2+\left|u_0^2\right|^2\leq R_1$

$$\mathbb{P}\left(Z_1(T_1,u_0^1,u_0^2)=Z_2(T_1,u_0^1,u_0^2),\ \sum_{i=1}^2\left|u_i(T_1,u_0^1,u_0^2)\right|^2\leq d_0\right)\geq\widetilde{p}, \tag{1.32}$$

where

$$\widetilde{V}_i(\cdot,u_0^1,u_0^2)=\left(u_i(\cdot,u_0^1,u_0^2),W_i(\cdot,u_0^1,u_0^2)\right),\ u_i(\cdot,u_0^1,u_0^2)=\begin{pmatrix}Z_i\\G_i\end{pmatrix},\ i=1,2.$$

By applying the Lyapunov structure (1.20), we obtain that for any $\theta\geq T_2(R_0,R_1)$

$$\mathbb{P}\left(|u(\theta,u_0)|^2\geq\frac{R_1}{2}\right)\leq\frac{1}{4},\quad\text{for any }u_0\text{ such that }|u_0|^2\leq\frac{R_0}{2}. \tag{1.33}$$

In order to build (V_1',V_2') such that (1.19) happens, we set $T^*(R_0)=T_1+T_2(R_0)$ and for any $T\geq T^*(R_0)$, we set $\theta=T-T_1$ and we remark that $\theta\geq T_2(R_0)$. Then we construct the trivial coupling $(V"_1,V"_2)$ on $[0,\theta]$. Finally, we consider $(\widetilde{V}_1,\widetilde{V}_2)$ as above independant of $(V"_1,V"_2)$ and we set

$$V_i'(t,u_0^1,u_0^2)=\begin{cases}V"_i(t,u_0^1,u_0^2) & \text{if } t\leq\theta,\\ \widetilde{V}_i(t-\theta,V"_1(\theta,u_0^1,u_0^2),V"_2(\theta,u_0^1,u_0^2)) & \text{if } t\geq\theta.\end{cases}$$

Combining (1.32) and (1.33), we obtain (1.19) with $p_{-1}=\frac{1}{2}\widetilde{p}$.

To build $(\widetilde{V}_i(\cdot,u_0^1,u_0^2))_{i=1,2}$, we apply Proposition 1.7 to $E=C((0,T_1);\mathbb{R}^2)^2$, $F=\mathbb{R}$,

$$f_0\,(u,W)=X(T_1),\quad\text{where }u=\begin{pmatrix}X\\Y\end{pmatrix},\ W=\begin{pmatrix}\beta\\\eta\end{pmatrix},$$

and to (μ_1,μ_2). Remark that if we set $\nu_i=f_0^*\mu_i$, we obtain

$$\nu_i=\mathcal{D}(X(T_1,u_0^i)).$$

Then $(\widetilde{V}_i(\cdot,u_0^1,u_0^2))_{i=1,2}$ is a coupling of (μ_1,μ_2) such that $(Z_i(T_1,u_0^1,u_0^2))_{i=1,2}$ is a maximal coupling of (ν_1,ν_2).

Now we notice that if we have $(\hat{\nu}_1, \hat{\nu}_2)$ two equivalent measures such that ν_i is equivalent to $\hat{\nu}_i$ for $i = 1, 2$, then by applying two Schwartz inequality, we obtain that

$$I_p \leq \left(J^1_{2p+2}\right)^{\frac{1}{2}} \left(J^2_{4p}\right)^{\frac{1}{4}} \left(\hat{I}_{4p+2}\right)^{\frac{1}{4}}, \tag{1.34}$$

where $A = [-d_1, d_1]$ and

$$\begin{array}{rclrcl} I_p &=& \int_A \left(\frac{d\nu_1}{d\nu_2}\right)^{p+1} d\nu_2, & J^1_p &=& \int_A \left(\frac{d\nu_1}{d\hat{\nu}_1}\right)^{p} d\hat{\nu}_1, \\ \hat{I}_p &=& \int_A \left(\frac{d\hat{\nu}_1}{d\hat{\nu}_2}\right)^{p} d\hat{\nu}_2, & J^2_p &=& \int_A \left(\frac{d\hat{\nu}_2}{d\nu_2}\right)^{p} d\hat{\nu}_2 \end{array}$$

Recall that Z_i the unique solution of

$$\left\{ \begin{array}{rcl} dZ_i + 2Z_i dt + f(Z_i, \Phi(Z_i(\cdot), \xi_i(\cdot), u^i_0))dt &=& \sigma_l(Z_i) d\beta_i, \\ Z_i(0) = x^i_0. && \end{array} \right. \tag{1.35}$$

We set $\widetilde{\beta}_i(t) = \beta_i(t) + \int_0^t d_i(s)dt$ where

$$d_i(t) = -\frac{1}{\sigma_l(Z_i(t))} f(Z_i(t), \Phi(Z_i(\cdot), \xi_i(\cdot), u^i_0)(t)). \tag{1.36}$$

Then Z_i is a solution of

$$\left\{ \begin{array}{rcl} dZ_i + 2Z_i dt &=& \sigma_l(Z_i) d\widetilde{\beta}_i, \\ Z_i(0) = x^i_0. && \end{array} \right. \tag{1.37}$$

Since f is bounded and σ_l is bounded below, then d_i is uniformly bounded. Hence, the Novikov condition is satisfied and the Girsanov formula can be applied. Then we set

$$d\widetilde{\mathbb{P}}_i = \exp\left(\int_0^T d_i(s) dW(s) - \frac{1}{2}\int_0^T |d_i(s)|^2 \, dt\right) d\mathbb{P}$$

We deduce from the Girsanov formula that $\widetilde{\mathbb{P}}_i$ is a probability under which $(\widetilde{\beta}_i, \xi_i)$ is a brownian motion. We denote by $\hat{\nu}_i$ the law of $Z_i(T_1)$ under $\widetilde{\mathbb{P}}_i$. Moreover

$$J^i_p \leq \exp\left(c_p \int_0^T |d_i(s)|^2 \, dt\right) \leq \exp\left(c_p \sigma_0^{-2} |f|^2_\infty\right). \tag{1.38}$$

It is classical that since σ_l is bounded below, then $\hat{\nu}_i$ has a density $q(x^i_0, z)$ with respect to lebesgue measure dz, that q is continuous with respect to the couple (x^i_0, z), where x^i_0 is the initial value and where z is the target value and that $q > 0$. Then, we can bound q and q^{-1} uniformly on $\left|x^i_0\right| \leq R_1$ and $z \in A = [-d_1, d_1]$, which allows us to bound $\hat{I}_p$ and then I_p. Actually:

$$I_p \leq C'(p, d_1, T_1, R_1) < \infty. \tag{1.39}$$

Now we apply Lemmas 1.3 and 1.2:

$$\mathbb{P}\left(Z_1(T_1) = Z_2(T_1),\ |Z_1(T_1)| \leq d_1\right) \geq \left(1 - \frac{1}{p}\right) p^{-\frac{1}{p-1}} I_p^{-\frac{1}{p-1}} \nu_1([-d_1, d_1])^{\frac{p}{p-1}}. \tag{1.40}$$

If we fix $d_1 > 4K_1$, then we obtain from the Lyapunov structure (1.20) that there exists $T_1 = T_1(R_1, d_1)$ such that

$$\nu_1([-d_1, d_1]) \geq \frac{1}{2}. \tag{1.41}$$

Combining (1.39), (1.40) and (1.41) gives

$$\mathbb{P}\left(Z_1(T_1) = Z_2(T_1),\ |Z_1(T_1)| \leq d_1\right) \geq C(p, d_1, T_1, R_1) > 0. \tag{1.42}$$

Note that

$$\begin{aligned} \mathbb{P}(\quad Z_1(T_1) = \ & Z_2(T_1),\ |u_i(T_1)| \leq d_1 + d_2,\ i = 1, 2) \geq \\ & \mathbb{P}\left(Z_1(T_1) = Z_2(T_1),\ |Z_1(T_1)| \leq d_1\right) - \textstyle\sum_{i=1}^{2} \mathbb{P}\left(|u_i(T_1)| \geq d_2\right). \end{aligned} \tag{1.43}$$

Using the Lyapunov structure (1.20), we obtain that

$$\mathbb{P}\left(|u_i(T_1)| \geq d_2\right) \leq \frac{R_1 + K_1}{d_2^2}. \tag{1.44}$$

Combining (1.42), (1.43) and (1.44), we can choose d_2 sufficiently high such that, by setting $d^* = d_1 + d_2$, $d_0 = (2d^*)^2$ and $\hat{p} = \frac{1}{2} C(2, d_1, T_1, R_1)$, (1.32) holds.

1.3. **Abstract Result.**

We now state and prove an abstract result which allows to reduce the proof of exponential convergence to equilibrium to the verification of some conditions, as was done in the previous section.

This result is closely related to the abstract result of [18]. Our proof has some similarity with the one in the reference but, in fact, is closer to arguments used in [23]. Our abstract result could be used in articles [12], [14], [15], [16] and [18] to conclude.

In fact, in [18] a family (r_k, s_k) of subprobability are used, whereas in [12], [16] a family of subsets $Q(l, k)$ are introduced. Here, we use a random integer valued process $l_0(k)$. The three points of view are equivalent, the correspondance is given by

$$s_{k+1} = \mathbb{P}\left(\{l_0(k+1) = 1\} \cap \cdot\right), \quad r_{k+1} = \mathbb{P}\left(\{l_0(k+1) = 1\}^c \cap \cdot\right),$$

and

$$Q(l, k) = \{l_0(k) = l\}.$$

The result has already been applied in section 1.2, the function used below is

$$\mathcal{H}(u_0) = |u_0|^2,$$

in this example. In fact, in most of the application and in particular for the CGL equation in the first case treated below, $\mathcal{H}$ wil be the square of the norm. We are concerned with $v(\cdot, (u_0, W_0)) = (u(\cdot, u_0), W(\cdot, W_0))$, a couple of strongly Markovian process defined on polish spaces (E, d_E) and (F, d_F). We denote by $(\mathcal{P}_t)_{t \in I}$ the markovian transition semigroup of u, where $I = \mathbb{R}^+$ or $T\mathbb{N} = \{kT,\ k \in \mathbb{N}\}$.

We consider for any initial conditions (v_0^1, v_0^2) a coupling (v_1, v_2) of $(\mathcal{D}(v(\cdot, v_0^1)), \mathcal{D}(v(\cdot, v_0^2)))$ and a random integer valued process $l_0 : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$

which has the following properties

$$(1.45)\qquad \begin{cases} l_0(k+1)=l \text{ implies } l_0(k)=l, \text{ for any } l\le k,\\ l_0(k)\in\{0,1,2,...,k\}\cup\{\infty\},\\ l_0(k) \text{ depends only of } v_1|_{[0,kT]} \text{ and } v_2|_{[0,kT]},\\ l_0(k)=k \text{ implies } \mathcal{H}_k\le d_0,\end{cases}$$

where

$$\mathcal{H}_k=\mathcal{H}(u_1(kT))+\mathcal{H}(u_2(kT)),\quad \mathcal{H}:\,E\to\mathbb{R}^+.$$

We write $v_i=(u_i,W_i)$. From now on we say that (v_1,v_2) are coupled at kT if $l_0(k)\le k$, in other words if $l_0(k)\neq\infty$.

Now we see four conditions on the coupling. The first condition states that when (v_1,v_2) have been coupled for a long time then the probability that (u_1,u_2) are close is high.

$$(1.46)\qquad \begin{cases} \text{There exist } c_0 \text{ and } \alpha_0>0 \text{ such that}\\ \mathbb{P}\left(d_E(u_1(t),u_2(t))>c_0e^{-\alpha_0(t-lT)} \text{ and } l_0(k)=l\right)\le c_0e^{-\alpha_0(t-lT)},\end{cases}$$

for any $t\in[lT,kT]\cap I$.

The following property states that the probability that two solutions decouples at kT is very small

$$(1.47)\qquad \begin{cases} \text{There exist } (p_k)_{k\in\mathbb{N}},\ c_1>0,\ \alpha_1>0 \text{ such that,}\\ \mathbb{P}\left(l_0(k+1)=l\mid l_0(k)=l\right)\ge p_{k-l}, \text{ for any } l\le k,\\ 1-p_k\le c_1e^{-\alpha_1kT},\ p_k>0 \text{ for any } k\in\mathbb{N}.\end{cases}$$

Next condition states that, inside a ball, the probability that two solutions get coupled at $(k+1)T$ is uniformly bounded below.

$$(1.48)\qquad \begin{cases} \text{There exist } p_{-1}>0,\ R_0>0 \text{ such that}\\ \mathbb{P}\left(l_0(k+1)=k+1\mid l_0(k)=\infty,\ \mathcal{H}_k\le R_0\right)\ge p_{-1}.\end{cases}$$

The last ingredient is the so-called Lyapunov structure. It allows the control of the probability to enter the ball of radius R_0. It states that there exists $\gamma>1$, such that for any solution v_0

$$(1.49)\qquad \begin{cases} \mathbb{E}\mathcal{H}(v(t,v_0)) \le e^{-\alpha_3t}\mathcal{H}(v_0)+\frac{K_1}{2},\\ \mathbb{E}\left(\mathcal{H}(v(\tau',v_0))^\gamma 1_{\tau'<\infty}\right) \le K'\left(\mathcal{H}(v_0)+1+\mathbb{E}\left(\tau' 1_{\tau'<\infty}\right)\right)^\gamma,\\ \text{for any stopping times } \tau' \text{ taking value in } \{kT,\ k\in\mathbb{N}\}\cup\{\infty\}.\end{cases}$$

The process $V=(v_1,v_2)$ is said to be l_0–Markovian if the laws of $V(kT+\cdot)$ and of $l_0(k+\cdot)-k$ on $\{l_0(k)\in\{k,\infty\}\}$ conditionned by $\mathcal{F}_{kT}$ only depend on $V(kT)$ and are equal to the laws of $V(\cdot,V(kT))$ and l_0, respectively.

Notice that in the example of the previous section or in the CGL case below, the process $(u_i,W_i)_{i=1,2}$ is l_0–Markovian but not Markovian. However, in both cases, if we choose $d_0 = R_0$, we can modify the coupling such that the couple is Markovian at discrete times $T\mathbb{N} = \{kT,\ k \in \mathbb{N}\}$. But it does not seem to be possible to modify the coupling to become Markovian at any times.

Theorem 1.8. *Assume that* (1.45), (1.46), (1.47), (1.48) *and* (1.49) *hold whith* $R_0 > 4K_1$ *and* $R_0 \geq d_0$ *and that* $V = (v_1, v_2)$ *is* l_0*–Markovian. Then there exist* $\alpha_4 > 0$ *and* $c_4 > 0$ *such that*

$$\mathbb{P}\left(d_E(u_1(t), u_2(t)) > c_3 e^{-\alpha_4 t}\right) \leq c_3 e^{-\alpha_4 t}\left(1 + \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2)\right). \tag{1.50}$$

Moreover there exists a unique stationnary probability mesure ν *of* $(\mathcal{P}_t)_{t\in I}$ *on* E. *It satisfies,*

$$\int_E \mathcal{H}(u) d\nu(u) \leq \frac{K_1}{2}, \tag{1.51}$$

and there exists $c_4 > 0$ *such that for any* $\mu \in \mathcal{P}(E)$

$$|\mathcal{P}_t^* \mu - \nu|^*_{Lip_b(E)} \leq c_4 e^{-\alpha_4 t}\left(1 + \int_E \mathcal{H}(u) d\mu(u)\right). \tag{1.52}$$

Proposition 1.6 is an easy consequence of Theorem 1.8. Actually (1.45) is clear and (1.46) and (1.49) are consequence of (1.17) and (1.20) if $R_0 \geq d_0$. Finally, since, for any (R_0, d_0, T) sufficiently high, there exists a coupling such that (1.18) and (1.19) hold, we can choose (R_0, d_0, T) such that all our assumptions are true.

Remark 1.9. *Inequality* (1.52) *means that for any* $f \in Lip_b(E)$ *and any* $u_0 \in E$

$$\left|\mathbb{E} f(u(t, u_0)) - \int_E f(u) d\nu(u)\right| \leq c_4 \left|f\right|_{lip_b(E)} e^{-\alpha_4 t}(1 + \mathcal{H}(u_0)).$$

1.4. **Proof of Theorem 1.8.**

Reformulation of the problem

We rewrite our problem in the form on a exponential estimate.

As in the example, it is sufficient to establish (1.50). Then (1.51) is a simple consequence of (1.49) and (1.52) follows from (1.16). Assume that $t > 8T$. We denote by k the unique integer such that $t \in (2(k-1)T, 2kT]$. Notice that

$$\begin{aligned}
&\mathbb{P}(d_E(u_1(t), u_2(t)) > c_0 e^{-\alpha_0 (t-(k-1)T)}) \\
&\qquad \leq \mathbb{P}\left(l_0(2k) \geq k\right) + \mathbb{P}\left(d_E(u_1(t), u_2(t)) > c_0 e^{-\alpha_0(t-(k-1)T)} \text{ and } l_0(2k) < k\right).
\end{aligned}$$

Thus applying (1.46), using $2(t-(k-1)T) > t$, it follows

$$\mathbb{P}\left(d_E(u_1(t), u_2(t)) > c_0 \exp\left(-\tfrac{\alpha_0}{2} t\right)\right) \leq \mathbb{P}\left(l_0(2k) \geq k\right) + c_0 \exp\left(-\tfrac{\alpha_0}{2} t\right). \tag{1.53}$$

In order to estimate $\mathbb{P}\left(l_0(2k) \geq k\right)$, we introduce the following notation

$$l_0(\infty) = \limsup l_0.$$

Taking into account (1.45), we obtain that for $l < \infty$

$$\{l_0(\infty) = l\} = \{l_0(k) = l, \text{ for any } k \geq l\}.$$

We deduce

$$\mathbb{P}\left(l_0(2k) \geq k\right) \leq \mathbb{P}\left(l_0(\infty) \geq k\right). \tag{1.54}$$

Taking into account (1.53), (1.54) and using a Chebyshev inequality, it is sufficient to obtain that there exist $c_5>0$ and $\delta>0$ such that

$$\mathbb{E}\left(\exp\left(\delta l_0(\infty)\right)\right)\leq c_5\left(1+\mathcal{H}(u_0^1)+\mathcal{H}(u_0^2)\right). \tag{1.55}$$

Then (1.50) follows with

$$\alpha_4=\min\left\{\frac{\alpha_0}{2},\frac{\delta}{2T}\right\}.$$

Definition of a sequence of stopping times

Using the Lyapunov structure (1.49), we prove at the end this subsection that there exist $\delta_0>0$ and $c_6>0$ such that

$$\mathbb{E}\left(\exp\left(\delta_0\tau\right)\right)\leq c_6\left(1+\mathcal{H}(u_0^1)+\mathcal{H}(u_0^2)\right), \tag{1.56}$$

where

$$\tau=\min\left\{t\in T\mathbb{N}\,|\,\mathcal{H}(u_1(t))+\mathcal{H}(u_2(t))\leq R_0\right\}.$$

We set

$$\hat\sigma=\min\left\{k\in\mathbb{N}^*\,|\,l_0(k)>1\right\},\quad \sigma=\hat\sigma T.$$

Clearly $\hat\sigma=1$ if the two solutions do not get coupled at time 0 or T. Otherwise, they get coupled at 0 or T and remain coupled until σ.

Let us assume for the moment that if $\mathcal{H}_0\leq R_0$, then

$$\begin{cases}\mathbb{E}\left(\exp\left(\delta_1\sigma\right)1_{\sigma<\infty}\right)\leq c_7,\\ \mathbb{P}\left(\sigma=\infty\right)\geq p_\infty>0.\end{cases} \tag{1.57}$$

The proof is given after the proof of (1.56) at the end of this subsection.

Now we build a sequence of stopping times

$$\begin{array}{rcl}\tau_0&=&\tau,\\ \hat\sigma_{k+1}&=&\min\left\{l\in\mathbb{N}^*\,|\,lT>\tau_k\text{ and }l_0(l)T>\tau_k+T\right\},\quad \sigma_{k+1}\;=\;\hat\sigma_{k+1}\times T\\ \tau_{k+1}&=&\sigma_{k+1}+\tau o\theta_{\sigma_{k+1}},\end{array}$$

where $(\theta_t)_t$ is the shift operator. The idea is the following. We wait the time τ_k to enter the ball of radius R_0. Then, if we do not start coupling at time τ_k, we try to couple at time τ_k+T. If we fail to start coupling at time τ_k or τ_k+T we set $\sigma_k=\tau_k+T$ else we set σ_k the time the coupling fails ($\sigma_k=\infty$ if the coupling never fails). Then if $\sigma_k<\infty$, we retry to enter the ball of radius R_0. The fact that $R_0\geq d_0$ implies that $l_0(\tau_k)\in\{\tau_k,\infty\}$.

The idea of the l_0–Markovian property is the following. Since $l_0(\tau_k)\in\{\tau_k,\infty\}$ and $l_0(\sigma_k)\in\{\sigma_k,\infty\}$, when these stopping times are finite and since these stopping times are taking value in $T\mathbb{N}\cup\{\infty\}$, then the l_0–Markovian property implies the strong Markovian property when conditionning with respect to $\mathcal{F}_{\tau_k}$ or $\mathcal{F}_{\sigma_k}$. Moreover, we infer from the l_0–Markovian property of V that

$$\sigma_{k+1}=\tau_k+\sigma o\theta_{\tau_k},$$

which implies

$$\tau_{k+1}=\tau_k+\rho o\theta_{\tau_k},\quad\text{where }\rho=\sigma+\tau o\theta_\sigma.$$

Exponential estimate on ρ

Before concluding, we establish that there exist K such that for any V_0 such that $\mathcal{H}_0 \le R_0$ and for any $\delta_2 \le \frac{1}{\gamma'}(\delta_0 \wedge \delta_1)$

$$\mathbb{E}_{V_0}\left(e^{\delta_2 \rho} 1_{\rho<\infty}\right) \le K. \tag{1.58}$$

Notice that for any V_0 such that $\mathcal{H}_0 \le R_0$,

$$\mathbb{E}_{V_0}\left(e^{\delta_2 \rho} 1_{\rho<\infty}\right) = \mathbb{E}_{V_0}\left(e^{\delta_2 \sigma} 1_{\sigma<\infty} \mathbb{E}\left(e^{\delta_2 \tau o \theta_\sigma} 1_{\tau o \theta_\sigma<\infty} | \mathcal{F}_\sigma\right)\right).$$

Applying the l_0–Markovian property and (1.56), we obtain

$$\mathbb{E}\left(e^{\delta_2 \tau o \theta_\sigma} 1_{\tau o \theta_\sigma<\infty} | \mathcal{F}_\sigma\right) \le c_6 \left(1 + \mathcal{H}(u_1(\sigma)) + \mathcal{H}(u_2(\sigma))\right) 1_{\sigma<\infty},$$

which implies

$$\mathbb{E}_{V_0}\left(e^{\delta_2 \rho} 1_{\rho<\infty}\right) \le c_6 \mathbb{E}_{V_0}\left(e^{\delta_2 \sigma} 1_{\sigma<\infty}\left(1 + \mathcal{H}(u_1(\sigma)) + \mathcal{H}(u_2(\sigma))\right)\right).$$

An Hölder inequality gives

$$\mathbb{E}_{V_0}\left(e^{\delta_2 \rho} 1_{\rho<\infty}\right) \le c_6 \left(\mathbb{E}_{V_0} e^{\gamma' \delta_2 \sigma} 1_{\sigma<\infty}\right)^{\frac{1}{\gamma'}} \left(\mathbb{E}_{V_0}\left(1 + \mathcal{H}(u_1(\sigma)) + \mathcal{H}(u_2(\sigma))\right)^{\gamma} 1_{\sigma<\infty}\right)^{\frac{1}{\gamma}}.$$

Applying the Lyapunov structure (1.49) and (1.57), we obtain (1.58).

Conclusion

We remark that

$$\mathbb{E}\left(e^{\delta_2 \tau_{k+1}} 1_{\tau_{k+1}<\infty}\right) = \mathbb{E}\left(e^{\delta_2 \tau_k} 1_{\tau_k<\infty} \mathbb{E}\left(e^{\delta_2 \rho o \theta_{\tau_k}} 1_{\rho o \theta_{\tau_k}<\infty} | \mathcal{F}_{\tau_k}\right)\right).$$

Applying again the l_0-Markov property of V

$$\mathbb{E}\left(e^{\delta_2 \tau_{k+1}} 1_{\tau_{k+1}<\infty}\right) = \mathbb{E}\left(e^{\delta_2 \tau_k} 1_{\tau_k<\infty} \mathbb{E}_{V(\tau_k)}\left(e^{\delta_2 \rho} 1_{\rho<\infty}\right)\right). \tag{1.59}$$

Iterating (1.59) by using (1.58) and (1.56), we obtain

$$\mathbb{E} e^{\delta_2 \tau_n} 1_{\tau_n<\infty} \le c_6 K^n \left(1 + \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2)\right). \tag{1.60}$$

Using the second inequality of (1.57) and that $\tau < \infty$, we obtain from the l_0–Markov property that

$$\mathbb{P}\left(k_0 > n\right) \le \left(1 - p_\infty\right)^n, \tag{1.61}$$

where

$$k_0 = \inf\{k \in \mathbb{N} \,|\, \sigma_{k+1} = \infty\}.$$

Then we obtain that $k_0 < \infty$ almost surely and that

$$l_0(\infty) \in \{\tau_{k_0}, \tau_{k_0} + 1\}.$$

Therefore $l_0(\infty) < \infty$ almost surely and

$$\mathbb{E} \exp\left(\frac{\delta_2}{p} l_0(\infty)\right) \le \sum_{n=1}^{\infty} \mathbb{E} e^{\frac{\delta_2}{p}(\tau_n+1)} 1_{k_0=n},$$

which implies, by applying a Hölder inequality,

$$\mathbb{E} \exp\left(\frac{\delta_2}{p} l_0(\infty)\right) \le e^{\frac{\delta_2}{p}} \sum_{n=1}^{\infty} \left(\mathbb{E} e^{\delta_2 \tau_n} 1_{\tau_n \le \infty}\right)^{\frac{1}{p}} \left(\mathbb{P}\left(k_0 = n\right)\right)^{\frac{1}{p'}}.$$

Applying (1.60) and (1.61), we obtain

$$\mathbb{E} \exp\left(\frac{\delta_2}{p} l_0(\infty)\right) \le c_6 e^{\frac{\delta_2}{p}} \left(\sum_{n=1}^{\infty} \left(K^{\frac{1}{p}} (1 - p_\infty)^{\frac{1}{p'}}\right)^n\right) \left(1 + \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2)\right)^{\frac{1}{p}}.$$

Choosing p such that $K^{\frac{1}{p}}(1 - p_\infty)^{\frac{1}{p'}} < 1$ and setting $\delta = \frac{\delta_2}{p}$, we obtain (1.55)

Proof of (1.56)

Let N be an integer such that

$$e^{-\alpha_3 NT} \leq \frac{1}{8}.$$

We fix $i \in \{1,2\}$ and set

$$B_k = \{\mathcal{H}(u_i(jNT)) \geq 2K_1, \text{ for any } j \leq k\}, \quad C_k = \{\mathcal{H}(u_i(kNT)) \geq 2K_1\}.$$

Combining the Markov property of u_i and the Lyapunov structure (1.49), we obtain

$$\mathbb{E}\left(\mathcal{H}(u_i((k+1)NT)) | \mathcal{F}_{kNT}\right) \leq \frac{1}{4}\mathcal{H}(u_i(kNT)) + \frac{K_1}{2}. \tag{1.62}$$

Hence, applying a Chebyshev inequality, it follows that

$$\mathbb{P}\left(C_{k+1} | \mathcal{F}_{kNT}\right) \leq \frac{1}{8K_1}\mathcal{H}(u_i(kNT)) + \frac{1}{4}. \tag{1.63}$$

Integrating (1.62), (1.63) over B_k, we obtain that

$$\begin{pmatrix} \mathbb{E}\left(\mathcal{H}(u_i((k+1)NT))1_{B_{k+1}}\right) \\ \mathbb{P}(B_{k+1}) \end{pmatrix} \leq A \begin{pmatrix} \mathbb{E}\left(\mathcal{H}(u_i(kNT))1_{B_k}\right) \\ \mathbb{P}(B_k) \end{pmatrix}, \tag{1.64}$$

where

$$A = \begin{pmatrix} \frac{1}{4} & \frac{K_1}{2} \\ \frac{1}{8K_1} & \frac{1}{4} \end{pmatrix}.$$

Since the eigenvalues of A are 0 and $\frac{1}{2}$, we obtain that

$$\mathbb{P}(B_k) \leq \frac{2}{K_1}\left(\frac{1}{2}\right)^k \left(1 + \mathcal{H}(u_0^i)\right).$$

It follows from $R_0 \geq 4K_1$ that

$$\mathbb{P}(\tau > kT) \leq c \exp\left(-\frac{k}{N}\ln 2\right)\left(1 + \mathcal{H}(u_0^i)\right).$$

Hence, taking $\delta_0 < \frac{\alpha_3}{3}$, we have established (1.56) .

Proof of (1.57)

Now we establish (1.57). There are two cases. The first case is $l_0(0) = 0$. Then, applying (1.47), we obtain that

$$\mathbb{P}(\sigma = \infty) \geq \Pi_{k=0}^{\infty}\mathbb{P}\left(l_0(k+1) = 0 | l_0(k) = 0\right) \geq \Pi_{k=0}^{\infty} p_k.$$

The second case is $l_0(0) = \infty$. Then

$$\mathbb{P}(\sigma = \infty) \geq \mathbb{P}(l_0(1) = 1)\,\Pi_{k=1}^{\infty}\mathbb{P}\left(l_0(k+1) = 1 | l_0(k) = 1\right).$$

Since $\mathcal{H}_0 \leq R_0$, then applying (1.47) and (1.48)

$$\mathbb{P}(\sigma = \infty) \geq \Pi_{k=-1}^{\infty} p_k.$$

Since $p_k > 0$ and $1 - p_k$ exponentially decreases, then the product converges and in the two cases

$$\mathbb{P}(\sigma = \infty) \geq p_\infty = \Pi_{k=-1}^{\infty} p_k > 0. \tag{1.65}$$

Notice that (1.47) implies

$$\mathbb{P}(\sigma = n) \leq \mathbb{P}\left(l_0(n+1) \neq n \,|\, l_0(n) = 0\right) + \mathbb{P}\left(l_0(n+1) \neq n \,|\, l_0(n) = 1\right) \leq 2c_1 e^{-\alpha_1(n-1)T},$$

which gives the first inequality of (1.56) and allows to conclude

2. Properties of the CGL equation

We are concerned with the stochastic Complex Ginzburg–Landau (CGL) equations with Dirichlet boundary conditions:

$$\begin{cases} \frac{du}{dt} - (\varepsilon + \mathrm{i})\Delta u + (\eta + \lambda \mathrm{i})\left|u\right|^{2\sigma} u &= b(u)\frac{dW}{dt} + f, \\ u(t,x) &= 0, \quad \text{for } x \in \delta D, \\ u(0,x) &= u_0(x), \end{cases} \tag{2.1}$$

where $\varepsilon > 0$, $\eta > 0$, $\lambda \in \{-1, 1\}$ and where D is an open bounded set of $\mathbb{R}^d$ with sufficiently regular boundary or $D = [0,1]^d$. Also f is the deterministic part of the forcing term. For simplicity in the redaction, we consider the case $f = 0$. The generalisation to a square integrable f is easy. We say that it is the defocusing or the focusing equation when λ is equal to 1 or -1, respectively.

We set

$$A = -\Delta, \quad D(A) = H_0^1(D) \cap H^2(D).$$

Now we can write problem (2.1) in the form

$$\frac{du}{dt} + (\varepsilon + \mathrm{i})Au + (\eta + \lambda \mathrm{i})\left|u\right|^{2\sigma} u = b(u)\frac{dW}{dt}, \tag{2.2}$$

$$u(0) = u_0, \tag{2.3}$$

where W is a cylindrical Wiener process of $L^2(D)$.

The aim of this section is to prove some properties which will be used in Section 3 to build a coupling such that the assumptions of Theorem 1.8 are true.

2.1. **Notations and main result.**

We consider $(e_n, \mu_n)_{n\in\mathbb{N}^*}$ the couples of eigenvalues and eigenvectors of A ($Ae_n = \mu_n$) such that $(e_n)_n$ is an Hilbertian basis of $L^2(D)$ and such that $(\mu_n)_n$ is an increasing sequence. We denote by P_N and Q_N the orthogonal projection in $L^2(D)$ on the space $Sp(e_k)_{1\leq n}$ and on its complementary, respectively.

The first condition is a condition on the smoothness of the noise and a condition ensuring existence and uniqueness of solutions.

We will sometimes consider the $L^2(D)$ sub-critical condition:

H1 *We assume that $0 < \sigma < \frac{2}{d} \wedge \frac{3}{2}$. Moreover $u_0 \in L^2(D)$ and b is bounded Lipschitz*

$$b : L^2(D) \to \mathcal{L}_2(L^2(D), H^2(D)).$$

We also consider the $H^1(D)$ sub-critical condition when the equation is defocusing.

H1' *If $d \leq 2$ we assume that $\sigma > 0$. If $d > 2$, we assume that $0 < \sigma < \frac{2}{d-2}$. Moreover $\lambda = 1$, $u_0 \in H^1(D)$ and b is bounded Lipschitz*

$$b : L^2(D) \to \mathcal{L}_2(L^2(D), H^2(D)).$$

We set, for $s \leq 2$,

$$B_s = \sup_u \left|b(u)\right|^2_{\mathcal{L}_2(L^2(D), H^s(D))}.$$

The second assumption means that b only depends on its low modes.

H2 *There exists N_1 such that*

$$b(u) = b(P_{N_1} u).$$

The third condition is a structure condition on b. It is a slight generalisation of the usual assumption that $b(u)$ is diagonal in the basis $(e_n)_n$.

H3 *There exists $N \geq N_1$, such that for any u,*

$$P_N b(u) Q_N = 0, \quad Q_N b(u) P_N = 0.$$

Moreover $P_N b(u) P_N$ is invertible on $P_N H$ and

$$\sup_u \left| (P_N b(u) P_N)^{-1} \right| < \infty.$$

In this section, we define by $|\cdot|$, $|\cdot|_p$, $\|\cdot\|$ and $\|\cdot\|_s$ the norm of $L^2(D)$, $L^p(D)$, $H^1(D)$ and $H^s(D)$.

The Lyapunov structures are defined by

$$\begin{array}{rcl} \mathcal{H}^{L^2} & = & |\cdot|^2, \\ \mathcal{H}^{H^1} & = & \frac{1}{2} \|\cdot\|^2 + \frac{1}{2\sigma+2} |\cdot|_{2\sigma+2}^{2\sigma+2}. \end{array}$$

The energies are defined by

$$E_u^{L^2}(t,T) = |u(t)|^2 + \varepsilon \int_T^t \|u(s)\|^2 \, ds,$$

and

$$E_u^{H^1}(t,T) = \left\{ \begin{array}{l} \mathcal{H}^{H^1}(u(t)) + \frac{\varepsilon}{2} \int_T^t \|u(s)\|_2^2 \, ds + \frac{\eta}{2} \int_T^t |u(s)|_{4\sigma+2}^{4\sigma+2} \, ds \\ +(\eta+\varepsilon) \int_T^t \int_D |u(s,x)|^{2\sigma} |\nabla u(s,x)|^2 \, dx ds, \end{array} \right.$$

When $T = 0$, we simply write $E_u(t) = E_u(t,0)$.

The first case is the L^2–subcritical focusing or defocusing CGL equation with initial condition in $L^2(D)$:

Case 1:

- **H1**, **H2** and **H3** hold,
- $\lambda \in \{-1, 1\}, \quad H = L^2(D)$,
- $\mathcal{H} = \mathcal{H}^{L^2} = |\cdot|_{L^2(D)}^2, \quad E_u = E_u^{L^2}$.

The second case is the H^1–subcritical defocusing CGL equation with initial condition in $H^1(D)$.

Case 2:

- **H1'**, **H2** and **H3** hold,
- $\lambda = 1, \quad H = H^1(D)$,
- $\mathcal{H} = \mathcal{H}^{H^1} = \frac{1}{2} \|\cdot\|_{H^1(D)}^2 + \frac{1}{2\sigma+2} |\cdot|_{L^{2\sigma+2}(D)}^{2\sigma+2}, \quad E_u = E_u^{H^1}$.

When it is not precised, the results stated are true in both cases. It is well known that we have existence and uniqueness of the solutions in both cases and that the solutions are strongly Markov process. We denote by $(\mathcal{P}_t)_{t \in \mathbb{R}^+}$ the Markov transition semi-group associated to the solutions of (2.2).

The aim of this article is to establish the following result

Theorem 2.1 (MAIN THEOREM)**.** *There exists $N_0(B_2,\eta,\varepsilon,\sigma,D)$ such that if $N\geq N_0$, then in cases 1 and 2, there exists a unique stationnary probability measure ν of $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ on $L^2(D)$. Moreover, ν satisfies*

$$\int_H \|u\|^2_{H^2(D)}\,d\nu(u)<\infty, \tag{2.4}$$

and for any $s\in[0,2)$, there exists $C_s>0$ and α_s such that for any $\mu\in\mathcal{P}(H)$

$$|\mathcal{P}^*_t\mu-\nu|^*_{Lip_b(H^s(D))}\leq C_s e^{-\alpha_s t}\left(1+\int_H |u|^2_{L^2(D)}\,d\mu(u)\right). \tag{2.5}$$

Furthermore, if (u,W) is a weak solution of (2.2), (2.3)*, with u_0 taking value in $L^2(D)$ then for any $f\in Lip_b(H^s(D))$*

$$\left|\mathbb{E}f(u(t))-\int_H f(u)d\nu(u)\right|\leq C_s\,|f|_{Lip_b(H^s(D))}\,e^{-\alpha_s t}\left(1+\mathbb{E}\,|u_0|^2_{L^2(D)}\right). \tag{2.6}$$

Remark 2.2. *In case 1,* (2.5) *is equivalent to* (2.6)*. But in case 2, the Markovian transition semi-group make sense only if u_0 is taking value in $H=H^1(D)$ because strong existence and weak uniqueness may cause problem when $u_0\in L^2(D)$. Hence* (2.5) *make sense only if $\mu\in\mathcal{P}(H^1(D))$ which means that $u_0\in H^1(D)$.*

Remark 2.3. *Assume that $B_s<\infty$ for s sufficiently high. Let k be a positive integer such that*

$$k\leq 2\sigma+2,\ \text{ if } \sigma\notin\mathbb{N},\ \text{ and } k\in\mathbb{N} \text{ if } \sigma\in\mathbb{N}.$$

Applying Remark 2.15 below and adapting the proof of Theorem 2.1, we obtain that (2.4) *can be replaced by*

$$\int_H \|u\|^2_{H^k(D)}\,d\nu(u)<\infty, \tag{2.7}$$

and (2.5) *is true for any s real number such that*

$$s<[2\sigma+2],\ \text{ if } \sigma\notin\mathbb{N},\ \text{ and } s\in\mathbb{R} \text{ if } \sigma\in\mathbb{N},$$

where $[\cdot]$ denote the integer part.

The condition on k and s comes from the lack of derivability of the non-linear part of the CGL equation. Assume that we replace $|u|^{2\sigma}u$ by $g(|u|^2)u$ where

- *g is infinitely continuously differientiable,*
- *$g(x)=x^\sigma$ for $x\geq x_0$,*
- *g is increasing and $g(0)=0$.*

Hence Theorem 2.1, (2.7) *and* (2.5) *are true for any k and s.*

2.2. **Properties of the solutions.**

In this subsection, we state some properties proved in the next subsections. These are used in Section 3 to apply Theorem 1.8 in order to establish Theorem 2.1.

First, we recall the following result.

Proposition 2.4. *In the two previous cases, there exists a mesurable map*

$$\Phi: C((0,T);P_N H)\times C((0,T);Q_N H^{\frac{d+1}{2}}(D))\times H\to C((0,T);Q_N H),$$

such that for any (u,W) solution of (2.2) *and* (2.3)

$$Q_N u=\Phi(P_N u,Q_N W,u_0)\qquad \text{on } [0,T].$$

Moreover Φ *is a non-anticipative functions of* $(P_N u, Q_N W)$.

Proposition 2.4 can be proved by applying a fix point argument and by taking into account that the limit of a sequence of measurable maps is measurable.

We have the so-called Foias-Prodi estimates.

Proposition 2.5 (Foias-Prodi estimate)**.** *Let* u_1 *and* u_2 *be two solutions of the CGL system* (2.2) *associated with Wiener process* W_1 *and* W_2 *respectively. If*

$$P_N u_1(t) = P_N u_2(t), \quad Q_N W_1(t) = Q_N W_2(t), \text{ for } T_0 \le t \le T, \tag{2.8}$$

where N *is a non-negative integer, then*

$$|r(t)|_H \le |r(T_0)|_H \exp\left(-\frac{\varepsilon \mu_{N+1}}{2}(t-T_0) + c_1 \sum_{i=1}^{2} E_{u_i}(t,T_0)\right), \tag{2.9}$$

where $r = u_1 - u_2$ *and* $T_0 \le t \le T$ *and where* $c_1 > 0$ *only depends on* ε, η, σ, D.

We deduce immediately a very usefull Corollary.

Corollary 2.6. *For any* B, *there exists* $N_0'(B,\eta,\varepsilon,D,\sigma)$ *such that under the assumptions of Proposition 2.5, under the assumption* $N \ge N_0'$ *and under the assumption*

$$E_{u_i}(t,T_0) \le \rho + B(t-T_0), \quad i = 1,2$$

we obtain that

$$|r(t)|_H \le |r(T_0)|_H \exp\left(-2(t-T_0) + c_1\rho\right).$$

where c_1 *is the constant of Proposition 2.5.*

Then, by proving analogous result to the previous Corollary, we obtain the Drift estimate which, in Section 3, will ensures the Novikov condition and will allow to apply the Girsanov Formula.

Lemma 2.7 (Drift estimate)**.** *For any* B, *there exists* $N_0"(B,\eta,\varepsilon,D,\sigma)$ *such that for any* u_1, u_2 *solutions of the CGL system* (2.2) *associated with* W_1 *and* W_2 *and for any* $N > N_0"$

$$\int_{T_0}^{\tau} \left| P_N(|u_1(s)|^{2\sigma} u_1(s) - |u_2(s)|^{2\sigma} u_2(s)) \right|^2 ds \le K_N |r(0)|^2 e^{c\rho - 3T_0}, \tag{2.10}$$

where $T > T_0 \ge 0$ *and* $\rho, C, \alpha > 0$, *where* K_N, c *only depend on* B, C, α, ε, η, σ, D, N *and where we have denoted by* τ *the value*

$$\tau = T_0 \vee \inf\left(t \in [0,T] \,\middle|\, \begin{array}{l} E_{u_1}(t) \ge \rho + Bt \text{ or } E_{u_2}(t) \ge \rho + C(1+t^\alpha) \text{ or} \\ P_N u_1(t) \ne P_N u_2(t) \text{ or } Q_N W_1(t) \ne Q_N W_2(t) \end{array}\right).$$

Now we set

$$N_0 = N_0' \vee N_0".$$

In order to apply the previous Lemmas and Corollary, we establish the two following results.

Proposition 2.8 (Exponential estimate for the growth of solution)**.** *Assume that* u *is a solution of* (2.2), (2.3) *associated with a Wiener process* W. *Then, for any* $0 \le T_0 < T \le \infty$

$$\mathbb{P}\left(\sup_{t\in[T_0,T[} (E_u(t) - Bt) \ge \mathcal{H}(u_0) + \rho\right) \le e^{-\gamma_0\rho - 3T_0},$$

where B *only depends on* B_2, σ, η, ε.

Proposition 2.9. *Assume that u is a solution of* (2.2), (2.3) *associated with a Wiener process W. For any u_0^2, we define $\widetilde{u}$ by*

$$\widetilde{u} = P_N u + \phi\left(P_N u, Q_N W, u_0^2\right).$$

Then, there exists $\alpha \geq 1$ such that for any N, there exists C_N,

$$\mathbb{P}\left(\sup_{t\in[0,T[}\left(E_{\widetilde{u}}(t) - C_N t^\alpha\right) \geq C_N\left(1 + \mathcal{H}(u_0) + \mathcal{H}(u_0^2)^\alpha + \rho\right)\right) \leq 2e^{-\gamma_0\rho},$$

for any $0 \leq T \leq \infty$ and any u_0^2.

Let u_1 and u_2 be two solutions of (2.2) that correspond to deterministic intial value u_0^1 and u_0^2, respectively.

Lemma 2.10 (The Lyapunov structure)**.** *There exists $\alpha > 0$ and $C_k > 0$ such that for any k*

$$\mathbb{E}\mathcal{H}(u_i(t))^k \leq \mathcal{H}(u_0^i)^k e^{-\alpha k t} + \frac{C_k}{2},$$

and for any stopping time τ

$$\mathbb{E}\mathcal{H}(u_i(\tau))^k 1_{\tau<\infty} \leq \mathcal{H}(u_0^i)^k + C_k\left(1 + \mathbb{E}\left(\tau 1_{\tau<\infty}\right)\right).$$

Using Lemma 2.10 and Chebyshev's inequality, we obtain

Lemma 2.11. *If $R_0 \geq (\mathcal{H}(u_0^1) + \mathcal{H}(u_0^2)) \vee C_1$, then*

$$\mathbb{P}\left(\mathcal{H}(u_1(t)) + \mathcal{H}(u_2(t)) \geq 4C_1\right) \leq \frac{1}{2},$$

providing $t \geq \theta_1(R_0) = \frac{1}{\alpha}\ln\frac{R_0}{C_1}$.

Then, in the second case, we control $\mathcal{H}(u(t))$ by $|u_0|^2$.

Proposition 2.12. *It is assumed that u is a solution of* (2.2), (2.3) *associated with a Wiener process W. Then, for any $T > 0$*

$$\mathbb{E}\mathcal{H}(u(T)) \leq A + BT + \frac{C}{T}\left|u_0\right|^2,$$

where A, B and C only depends on B_2, σ, η, ε.

Now, we claim that in the two cases, we can control the norm of solutions in Sobolev spaces by the norm in L^2.

Proposition 2.13. *Let k be a positive integer less than 2. There exist $\gamma_k > 1$ only depending on k, σ and d and $C_k > 0$ and $c_k > 0$ only depending on k, $(B_s)_s$, σ, d, ε and η such that for any $T > 0$ and $t > 0$*

$$\mathbb{E}\left(\left\|u(T+t)\right\|_k^2 + \int_T^{T+t}\left\|u(s)\right\|_{k+1}^2 ds\right)^{\frac{2}{\gamma_k}} \leq c_k\frac{1}{T}\left|u_0\right|^2 + C_k(1+T+t).$$

Hence, applying a Chebyshev inequality, we obtain

Corollary 2.14. *Let k be a positive integer less than 2 and $\delta > 0$. There exist $\gamma > 0$ only depending on k, σ and d and $C_\delta > 0$ only depending on δ, k, $(B_s)_s$, σ, d, ε and η such that for any $t > 0$*

$$\mathbb{P}\left(\left\|u(t)\right\|_k \geq e^{\delta t}\right) \leq C_\delta e^{-\frac{\delta}{\gamma}t}\left(\left|u_0\right|^2 + 1\right)$$

Remark 2.15. *Assume that $B_s < \infty$ for s sufficiently high. The proof of Proposition 2.13 can be adapted to k a positive integer such that*

$$k \leq 2\sigma + 2, \text{ if } \sigma \notin \mathbb{N}, \text{ and } k \in \mathbb{N} \text{ if } \sigma \in \mathbb{N},$$

and then Corollary 2.14 is true for such a k.

The condition on k comes from the fact that $|\cdot|^\sigma$ is not C^∞ on 0. As in Remark 2.3, if we replace $|\cdot|^\sigma$ by a nice function which coincides with $|\cdot|^\sigma$ on $[x_0, \infty)$, we can establish those results for any k.

2.3. **Foias-Prodi and Drift estimates.**

The proofs in the first case are closely related to the proofs in the second case, but are simpler. That is the reason why we only give the proof in the second case.

Proof of Proposition 2.5 in the second case.

We denote $u_1 - u_2$ by r.

Step 1. This step is devoted to the proof of

$$I = ((\eta + \mathrm{i})(|u_2|^{2\sigma} u_2 - |u_1|^{2\sigma} u_1), Ar) \leq \frac{\varepsilon}{2} \|r\|_2^2 + c\|r\|^2 \sum_i |u_i|_{4\sigma+2}^{4\sigma+2}. \tag{2.11}$$

We recall the following estimate

$$\left| |x|^{2\sigma} x - |y|^{2\sigma} y \right| \leq c\,|x - y|\,(|x|^{2\sigma} + |y|^{2\sigma}). \tag{2.12}$$

Applying Hölder inequality and then (2.12) gives

$$I \leq \|r\|_2 \left| |u_2|^{2\sigma} u_2 - |u_1|^{2\sigma} u_1 \right|_2 \leq \|r\|_2 \left| \left(\sum_{i=1}^2 |u_i|^{2\sigma} \right) r \right|_2.$$

Let $s \in (1, 2)$ such that $\frac{4\sigma}{2-s} = 4\sigma + 2$. Applying once more Hölder inequality and then the Sobolev embedding $H^s(D) \subset L^{4\sigma+2}(D)$ gives

$$I \leq \|r\|_2 \, |r|_{4\sigma+2} \sum_{i=1}^2 |u_i|_{4\sigma+2}^{2\sigma} \leq \|r\|_2 \, \|r\|_s \sum_{i=1}^2 |u_i|_{4\sigma+2}^{2\sigma},$$

which yields by the interpolatory inequality $\|.\|_s \leq \|.\|_2^{s-1} \|.\|^{2-s}$ and then an arithmetic-geometric inequality

$$I \leq \|r\|_2^s \|r\|^{2-s} \sum_i |u_i|_{4\sigma+2}^{2\sigma} \leq \frac{\varepsilon}{2} \|r\|_2^2 + c\|r\|^2 \sum_i |u_i|_{4\sigma+2}^{4\sigma+2}.$$

Step 2. We now establish (2.9).

Taking into account (2.8), we see that r satisfies the equation

$$\frac{dr}{dt} + (\varepsilon + \mathrm{i})Ar = (\eta + \mathrm{i}) Q_N (|u_2|^{2\sigma} u_2 - |u_1|^{2\sigma} u_1). \tag{2.13}$$

Taking the scalar product of (2.13) by $-2Ar$, we obtain:

$$\frac{d\|r\|^2}{dt} + 2\varepsilon \|r\|_2^2 = 2((\eta + \mathrm{i})(|u_2|^{2\sigma} u_2 - |u_1|^{2\sigma} u_1), Ar). \tag{2.14}$$

Taking into account (2.11), (2.14) gives :

$$\frac{d\|r\|^2}{dt} + \varepsilon \|r\|_2^2 \leq c\|r\|^2 \sum_i |u_i|_{4\sigma+2}^{4\sigma+2}. \tag{2.15}$$

Since $r \in Q_N H$, then $\mu_{N+1} \|r\|^2 \leq \|r\|_2^2$ and it follows from (2.15) that

$$\frac{d\|r\|}{dt} + \varepsilon\mu_{N+1}\|r\|^2 \leq c\|r\|^2 \sum_i |u_i|_{4\sigma+2}^{4\sigma+2}. \tag{2.16}$$

Applying Gromwall Lemma to (2.16), we obtain (2.9).

□

Proof of Lemma 2.7 in the second case.
We first state the following Lemma which strengthen Proposition 2.5.

Lemma 2.16. *Let u_1 and u_2 be two solutions of the CGL system* (2.2) *associated with W_1 and W_2 respectively. If*

$$P_N u_1(s) = P_N u_2(s), \quad Q_N W_1(s) = Q_N W_2(s), \text{ for any } s \in (T_0, t), \tag{2.17}$$

where N is a non-negative integer, then

$$|r(t)|_{L^2} \leq |r(0)|_{L^2} \exp\left(-\frac{\varepsilon\mu_{N+1}}{2}t + c_1 E_{u_1}(t)\right), \tag{2.18}$$

where $r = u_1 - u_2$ and where $c_1 > 0$ only depends on ε, η, σ, D. Moreover, for any B, there exists $N_0"(B, \eta, \varepsilon, D, \sigma)$ such that $N \geq N_0"$ and

$$E_{u_1}(t) \leq \rho + Bt \tag{2.19}$$

imply

$$|r(t)|_{L^2} \leq |r(0)|_{L^2} \exp\left(-2t + c_1\rho\right), \tag{2.20}$$

where c_1 is the constant of Proposition 2.5.

For the first case, this result is Proposition 1.1.6 of [22]. For the second case the proof is the same.
Sketch of the proof of Lemma 2.16.
The proof of Lemma 2.16 is similar to the proof of Proposition 2.5. Indeed it is sufficient to prove

$$I' = -((\eta + \lambda i)(|u_2|^{2\sigma} u_2 - |u_1|^{2\sigma} u_1), r) \leq c \left| |u_1|^{2\sigma} |r|^2 \right|_1. \tag{2.21}$$

to establish Lemma 2.16. We prove (2.21) as follows. Remarking that

$$|u_2|^{2\sigma} u_2 - |u_1|^{2\sigma} u_1 = |u_2|^{2\sigma} r + u_1(|u_2|^{2\sigma} - |u_1|^{2\sigma}),$$

and

$$\left| u_1(|u_2|^{2\sigma} - |u_1|^{2\sigma}) \right| \leq c' |u_1| (|u_2|^{2\sigma-1} + |u_1|^{2\sigma-1}) |r|,$$

we obtain

$$I' \leq -\eta \left| |u_2|^{2\sigma} |r|^2 \right|_1 + c' \left| |u_1|^{2\sigma} |r|^2 \right|_1 + c' \left| u_1 |u_2|^{2\sigma-1} r^2 \right|_1.$$

Applying arithmetico-geometric inequality to the last term of the previous equality, we obtain for $\sigma \geq \frac{1}{2}$

$$c' \left| u_1 |u_2|^{2\sigma-1} r^2 \right|_1 \leq \eta \left| |u_2|^{2\sigma} |r|^2 \right|_1 + c" \left| |u_1|^{2\sigma} |r|^2 \right|_1.$$

We infer (2.21) for $\sigma \geq \frac{1}{2}$ from the two previous inequalities.
To obtain (2.21) when $\sigma < \frac{1}{2}$, one remark that D is the union of $\{x| \, |u_1(x)| \geq |u_2(x)|\}$ and $\{x| \, |u_1(x)| < |u_2(x)|\}$. Treating the first set is trivial. The treatement done before works for the second set.

□

Let us set

$$I=\int_{T_0}^{\tau}\left|P_N(|u_1(s)|^{2\sigma}u_1(s)-|u_2(s)|^{2\sigma}u_2(s))\right|^2 ds.$$

Applying Lemma 2.16 with the same N_0", we obtain

$$|r(t)|\leq |r(0)|\exp\left(-2t+c_1\rho\right),\ \text{ for } \tau\geq t\geq 0. \tag{2.22}$$

Noticing that, since we work in a finite dimensional space, all the norm are equivalent. Hence there exists K_N such that

$$I\leq K_N\int_{T_0}^{\tau}\left||u_1(s)|^{2\sigma}u_1(s)-|u_2(s)|^{2\sigma}u_2(s)\right|_1^2 ds. \tag{2.23}$$

It follows from (2.12) and Hölder inequality that

$$\begin{aligned}\left||u_1(s)|^{2\sigma}u_1(s)-|u_2(s)|^{2\sigma}u_2(s)\right|_1^2 &\leq c\left|\left(\sum_{i=1}^2|u_i(s)|^{2\sigma}\right)|r(s)|\right|_1^2,\\ &\leq c\left(\sum_{i=1}^2|u_i(s)|_{4\sigma}^{4\sigma}\right)|r(s)|^2,\end{aligned}$$

which yields, by applying an arithmetico-geometric inequality,

$$\left||u_1(s)|^{2\sigma}u_1(s)-|u_2(s)|^{2\sigma}u_2(s)\right|_1^2\leq c\left(1+\sum_{i=1}^2|u_i(s)|_{4\sigma+2}^{4\sigma+2}\right)|r(s)|^2. \tag{2.24}$$

Combining (2.22), (2.23) and (2.24) and then an integration by parts, we obtain

$$\begin{aligned} I &\leq K_N|r(0)|^2\int_{T_0}^{\tau}\exp\left(-4t+c_1\rho\right)\left(1+\sum_{i=1}^2|u_i(s)|_{4\sigma+2}^{4\sigma+2}\right)ds,\\ &\leq K_N|r(0)|^2\int_{T_0}^{\tau}\exp\left(-4t+c_1\rho\right)\left(1+\sum_{i=1}^2\int_{T_0}^{t}|u_i(s)|_{4\sigma+2}^{4\sigma+2}ds\right)dt,\\ &\leq K_N|r(0)|^2\int_{T_0}^{\tau}\exp\left(-4t+c_1\rho\right)\left(1+2\rho+Bt+C(1+t^{\alpha})\right)dt,\\ &\leq K_N|r(0)|^2\int_{T_0}^{\tau}\exp\left(-3t+2c_1\rho\right)dt,\end{aligned}$$

which allows us to conclude.

□

2.4. **An exponential estimate for the growth of solution.**

As in the previous subsection, we only give the proof of Propositions 2.8 in the second case.

We set

$$E_u'(t)=\begin{cases}\frac12\|u(t)\|^2+\frac{1}{2\sigma+2}|u(t)|_{2\sigma+2}^{2\sigma+2}+\varepsilon\int_0^t\|u(s)\|_2^2ds+\eta\int_0^t|u(s)|_{4\sigma+2}^{4\sigma+2}ds\\ +\\ (\eta+\varepsilon)\int_0^t\int_D(1+\chi(u\nabla\bar u))\,|u(s,x)|^{2\sigma}|\nabla u(s,x)|^2dxds,\end{cases}$$

where $\chi(z)=2\sigma\Re e\left(\frac{\Re e z}{z}\right)$. Applying Ito's Formula to $\mathcal{H}(u)=\frac12\|u\|^2+\frac{1}{2\sigma+2}|u|_{2\sigma+2}^{2\sigma+2}$, we obtain

$$E_u'(t)=\mathcal{H}(u_0)+M_1(t)+M_2(t)+I_1(t)+I_2(t), \tag{2.25}$$

where we have denoted

$$M_1(t) = \int_0^t(-\Delta u(s), b(u(s))dW(s)), \qquad M_2(t) = \int_0^t(|u(s)|^{2\sigma} u(s), b(u(s)))dW(s)),$$
$$I_1(t) = \frac{1}{2}\int_0^t |b(u(s))|^2_{\mathcal{L}^2(L^2(D),H^1(D))}\, ds, \quad I_2(t) = \frac{1}{2}\int_0^t \sum_{i=1}^2 |g_i(u(s))|^2_{\mathcal{L}^2(L^2(D))}\, ds,$$

where

$$g_i(u)(k) = f_i(u)(b(u)h) \quad f_1(u)(k) = |u|^\sigma \times k, \quad f_2(u)(k) = \sqrt{2\sigma}\, |u|^{\sigma-1}\, \Re e(\bar{u} \times k).$$

Hölder estimate and Sobolev Embedding give

$$\sum_{i=1}^2 |f_i(u)|^2_{\mathcal{L}(H^1(D,L^2(D)))} \leq c\, |u|^{4\sigma}_{4\sigma+2}\,,$$

which yields

$$\sum_{i=1}^2 |g_i(u)|^2_{\mathcal{L}^2(L^2(D))} \leq c\, |u|^{4\sigma}_{4\sigma+2}\, B_1,$$

and thus by an arithmetico-geometric inequality

$$I_2(t) \leq cB_1 t + \frac{\eta}{4}\int_0^t |u|^{4\sigma+2}_{4\sigma+2}\, ds \tag{2.26}$$

Notice that

$$< M_1 > (t) = \int_0^t |b(u(s)^* Au(s)|^2\, ds,$$

which gives

$$< M_1 > (t) \leq B_0 \int_0^t \|u(s)\|_2^2\, ds. \tag{2.27}$$

Moreover

$$< M_2 > (t) = \int_0^t \left|b(u(s))^*\, |u(s)|^{2\sigma}\, u(s)\right|^2 ds.$$

Since

$$\left|b(u(s))^*\, |u(s)|^{2\sigma}\, u(s)\right|^2 \leq B_0 \left||u(s)|^{2\sigma}\, u(s)\right|^2 \leq cB_0\, |u|^{4\sigma+2}_{4\sigma+2}\,,$$

we obtain

$$< M_2 > (t) \leq cB_0 \int_0^t |u|^{4\sigma+2}_{4\sigma+2}\, ds. \tag{2.28}$$

Noticing that $< M_1 + M_2 > \leq 2(< M_1 > + < M_2 >)$, $I_1(t) \leq B_1 t$ and $\chi(z) \geq 0$ for any $z \in \mathbb{C}$, it follows from (2.25), (2.26), (2.27) and (2.28) that

$$E_u(t) - \mathcal{H}(u_0) - Bt \leq M(t) - \frac{\gamma_0}{2} < M > (t), \tag{2.29}$$

where $M = M_1 + M_2$, $B' = c(B_0 + B_1)$ and $\gamma_0 = \frac{\eta \vee \varepsilon}{8B_0(1+c)}$. Thus

$$\mathbb{P}(\sup_{t\in\mathbb{R}^+} (E_u(t) - Bt) \geq \mathcal{H}(u_0) + \rho') \leq e^{-\gamma_0 \rho'}\mathbb{E}e^{\gamma_0 M(t) - \frac{\gamma_0^2}{2} < M > (t)} \leq e^{-\gamma_0\rho'},$$

which allows to conclude by setting $\rho' = \rho + 3\frac{T_0}{\gamma_0}$ and $B' = B + \frac{3}{\gamma_0}$.

We do not give the proof of Proposition 2.9 because it is easilly deduced from the proof of Proposition 2.8. Actually, Ito Formulas associated to a solution u are also true if we replace u by $\widetilde{u}$ and $b(P_N u)dW$ by $b(P_N\widetilde{u})dW + P_N(|u|^{2\sigma}\, u - |\widetilde{u}|^{2\sigma}\, \widetilde{u})dt$. Hence to establish Proposition 2.9, it is sufficient to bound the additionnal term by using the equivalence of the norms in finite-dimensionnal spaces and by applying Proposition 2.9 to bound terms containing u.

2.5. **The Lyapunov structure.**
Now, we prove Lemma 2.10 in the second case. Using the computation of the energy previously done, we obtain that there exixts C_1 such that

$$d\mathcal{H}(u_i(t)) + \frac{\varepsilon}{2}\left\|u_i(t)\right\|_2^2 dt + \frac{\eta}{4\sigma+2}\left|u_i(t)\right|_{4\sigma+2}^{4\sigma+2} dt \leq dM + C_1 dt$$

Applying Ito Formula to $\mathcal{H}(u_i)^k$ and controlling $d < M >$ as above by $\left\|u_i(t)\right\|_2^2 dt$ and $\left|u_i(t)\right|_{4\sigma+2}^{4\sigma+2} dt$, we obtain that there exists α_0 such that

$$\begin{aligned} d\mathcal{H}(u_i(t))^k + \alpha_0 k\mathcal{H}(u_i)^{k-1} & \left(\left\|u_i(t)\right\|_2^2 + \left|u_i(t)\right|_{4\sigma+2}^{4\sigma+2}\right) dt \\ & \leq k\mathcal{H}(u_i(t))^{k-1} dM + C_k dt. \end{aligned} \tag{2.30}$$

Taking into account that $\mu_1 \left\|.\right\|^2 \leq \left\|.\right\|_2^2$ and that there exist $\beta > 0$ such that $\beta\left|.\right|_{2\sigma+2}^{2\sigma+2} \leq \left\|.\right\|_2^2 + \left|.\right|_{4\sigma+2}^{4\sigma+2}$, we obtain that there exists $\alpha > 0$ such that

$$d\mathcal{H}(u_i(t))^k + \alpha k\mathcal{H}(u_i)^k dt \leq k\mathcal{H}(u_i(t))^{k-1} dM + C_k dt, \tag{2.31}$$

which yields, by integrating and taking the expectation, the second inequality of Lemma 2.10.
Now, applying (2.31), we obtain that

$$\mathcal{H}(u_i(t))^k \leq \mathcal{H}(u_0^i)^k e^{-\alpha kt} + k\int_0^t e^{-\alpha k(t-s)}\mathcal{H}(u_i(s))^{k-1} dM(s) + C_k. \tag{2.32}$$

which yields, by taking the expectation, the first inequality of Lemma 2.10.

2.6. **Control of $\mathcal{P}_T\mathcal{H}$ by $|.|^2$ in the second case.**
Now, we prove Proposition 2.12. Taking the expectation on (2.29), we obtain that for any $T > t > 0$

$$\mathbb{E}\mathcal{H}(u(T)) \leq \mathbb{E}\mathcal{H}(u(t)) + B(T-t).$$

Integrating over $[0, T]$ gives

$$\mathbb{E}\mathcal{H}(u(T)) \leq \frac{1}{T}\mathbb{E}\int_0^T \mathcal{H}(u(t))dt + BT. \tag{2.33}$$

Applying Ito Formula to $|u|^2$ and taking the expectation, we obtain

$$\mathbb{E}\left|u(t)\right|^2 + 2\varepsilon\int_0^t \mathbb{E}\left\|u(s)\right\|^2 ds + 2\eta\int_0^t \mathbb{E}\left|u(s)\right|_{2\sigma+2}^{2\sigma+2} ds = \left|u_0\right|^2 + \int_0^t \mathbb{E}\left|b(u(s))\right|_{\mathcal{L}_2(L^2(D))}^2 ds.$$

Applying **H1'**, we obtain

$$\mathbb{E}\int_0^T \mathcal{H}(u(t))dt \leq C\left|u_0\right|^2 + AT,$$

and by 2.33

$$\mathbb{E}\mathcal{H}(u(T)) \leq A + BT + \frac{C}{T}\left|u_0\right|^2.$$

2.7. H^1 **and** H^2 **estimates.**

We first establish that

$$\mathbb{E}\,\|u(T)\|^2 + \varepsilon \int_0^T \mathbb{E}\,\|u(s)\|_2^2\, ds \leq \|u_0\|^2 + c_1\, |u_0|^{\alpha_1} + B_1' T, \tag{2.34}$$

and that

$$\mathbb{E}\,\|u(T)\|^2 \leq c\left(1 + \frac{1}{T}\,|u_0|^2 + |u_0|^{2k} + T\right). \tag{2.35}$$

In the second part of the proof, we establish that there exists $\gamma_0 > 0$ such that

$$\mathbb{E}\,\|u(t)\|_2^2 + \varepsilon \int_0^t \mathbb{E}\,\|u(s)\|_3^2\, ds \leq \|u_0\|_2^2 + c\,\|u_0\|^{\gamma_0} + C(t+1). \tag{2.36}$$

We deduce from Hölder inequality that

$$\mathbb{E}\left(\|u(t)\|_2^2 + \varepsilon \int_0^t \|u(s)\|_3^2\, ds\right)^{\frac{2}{\gamma_0}} \leq c\,\|u_0\|_2^2 + C(t+1). \tag{2.37}$$

and

$$\mathbb{E}\,\|u(T)\|_2^2 \leq c\left(1 + \frac{1}{T}\,\|u_0\|^2 + T\right). \tag{2.38}$$

Hence, combining (2.35), (2.37) and (2.38), we obtain

$$\mathbb{E}\left(\|u(T+t)\|_2^2 + \varepsilon \int_T^{T+t} \|u(s)\|_3^2\, ds\right)^{\frac{2}{\gamma_0}} \leq c\frac{1}{T}\,|u_0|^2 + |u_0|^{2k} + C(T+t+1). \tag{2.39}$$

Applying Hölder inequality allows to conclude.

Proof of (2.34) **and** (2.35)

Note that (2.34) and (2.35) have already been demonstrated in the second case. Then it remains to establish (2.34) in the first case, when $\lambda = -1$.

Remark that Ito's Formula applied to $|u|^{2k}$ gives

$$\mathbb{E}\left(|u(t)|^{2k} + \eta k \int_0^t |u(s)|^{2(k-1)}\, |u|_{2\sigma+2}^{2\sigma+2}\, ds\right) \leq |u_0|^{2k} + B_k" t. \tag{2.40}$$

Taking the scalar product between (2.2) and $2(-\Delta)u$ gives

$$d\,\|u\|^2 + 2\varepsilon\,\|u\|_2^2\, dt \leq 2((-\Delta u), b(u)dW) + 2(\Delta u, (\eta+\lambda \mathrm{i})\,|u|^{2\sigma}\, u)dt + B_1 dt. \tag{2.41}$$

We deduce from Schwartz inequality that

$$2(\Delta u, (\eta+\lambda \mathrm{i})\,|u|^{2\sigma}\, u) \leq c\,\|u\|_2\, |u|_{4\sigma+2}^{2\sigma+1}.$$

The Gagliardo-Niremberg inequality gives

$$2(\Delta u, (\eta+\lambda \mathrm{i})\,|u|^{2\sigma}\, u) \leq c\,\|u\|_2^{1+\frac{\sigma d}{2}}\, |u|^{2\sigma+1-\frac{\sigma d}{2}}.$$

Finally, since $\sigma d < 2$, then we can deduce from a arithmetico-geometric inequality that

$$2(\Delta u, (\eta+\lambda \mathrm{i})\,|u|^{2\sigma}\, u) \leq \varepsilon\,\|u\|_2^2 + c\,|u|^{2\,\frac{4\sigma+2-\sigma d}{2-\sigma d}}.$$

We infer from (2.41) that

$$d\,\|u\|^2 + \varepsilon\,\|u\|_2^2\, dt \leq 2((-\Delta u), b(u)dW) + c\,|u|^{2\,\frac{4\sigma+2-\sigma d}{2-\sigma d}}\, dt + B_1 dt,$$

and then

$$\mathbb{E}\left\|u(t)\right\|^2+\varepsilon\int_0^t \mathbb{E}\left\|u(s)\right\|_2^2 ds\leq \left\|u_0\right\|^2+c\int_0^t \mathbb{E}\left|u(s)\right|^{2\frac{4\sigma+2-\sigma d}{2-\sigma d}} ds+B_1t.$$

Applying (2.40), we obtain for a well-chosen k'

$$\mathbb{E}\left\|u(t)\right\|^2+\varepsilon\int_0^t \mathbb{E}\left\|u(s)\right\|_2^2 ds\leq c\left(\left\|u_0\right\|^2+\left|u_0\right|^{2k'}+T\right). \tag{2.42}$$

Using the same argument as in the last subsection gives (2.35).

Proof of (2.36)**,** (2.37) **and** (2.38)

Taking the scalar product between (2.2) and $2(-\Delta)^2u$ gives

$$d\left\|u\right\|_2^2+2\varepsilon\left\|u\right\|_3^2 dt\leq 2((-\Delta u)^2,b(u)dW)-2((-\Delta)^2u,(\eta+\lambda\mathrm{i})\left|u\right|^{2\sigma}u)dt+B_2dt. \tag{2.43}$$

We deduce from an integration by part and Schwartz inequality that

$$-2((-\Delta)^2u,(\eta+\lambda\mathrm{i})\left|u\right|^{2\sigma}u)\leq c\left\|u\right\|_3\left|\nabla\left(u\left|u\right|^{2\sigma}\right)\right|. \tag{2.44}$$

Hölder inequality gives

$$\left|\nabla\left(u\left|u\right|^{2\sigma}\right)\right|\leq\left|\nabla u\right|_p\left|u\right|_{2\sigma q}^{2\sigma},$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. We choose s, p and q such that

$$\frac{1}{p}=\frac{1}{2}-\frac{s}{d},\quad \frac{1}{2\sigma q}=0\vee\left(\frac{1}{2}-\frac{1}{d}\right).$$

Since $\sigma\leq\frac{2}{d-2}$, then $s\in[0,2)$. Hence the Sobolev embeddings $H^s(D)\to L^p(D)$ and $H^1(D)\to L^{2\sigma q}(D)$ imply

$$\left|\nabla u\left|u\right|^{2\sigma}\right|\leq\left\|u\right\|_{1+s}\left\|u\right\|^{2\sigma},$$

Then, we deduce from (2.44), an interpolatory inequality that

$$-2((-\Delta)^2u,(\eta+\lambda\mathrm{i})\left|u\right|^{2\sigma}u)\leq c\left\|u\right\|_3^{1+\frac{s}{2}}\left\|u\right\|^{2\sigma+1-\frac{s}{2}}.$$

An arithmetico-geometric inequality gives

$$-2((-\Delta)^2u,(\eta+\lambda\mathrm{i})\left|u\right|^{2\sigma}u)\leq\varepsilon\left\|u\right\|_3^2+c\left\|u\right\|^{\beta}, \tag{2.45}$$

with $\beta>0$. We infer from (2.43) and (2.45) that

$$d\left\|u\right\|_2^2+\varepsilon\left\|u\right\|_3^2 dt\leq 2((-\Delta u)^2,b(u)dW)+c\left\|u\right\|^{\beta}dt+B_2dt. \tag{2.46}$$

Hence, we deduce (2.36) from (2.46). Then, applying Hölder inequality, we obtain (2.37). Using the same argument as in the last subsection gives (2.38).

3. The coupling of CGL

Recall that, as in the last section, we consider the two cases devellopped in subsection 2.1 and use the properties stated in subsection 2.2. In this section, we make an other assumption

H4 $\qquad N\geq N_0,$

where N_0 has been defined after Corollary 2.6 and Lemma 2.7.

In this section, we apply Theorem 1.8. Then we obtain there exists a unique invariant probability measure on H and that there exists $c>0$ and $\alpha>0$

$$\mathbb{P}\left(\left|u_1(t)-u_2(t)\right|_H>ce^{-\alpha t}\right)\leq ce^{-\alpha t}\left(1+\mathcal{H}(u_0^1)+\mathcal{H}(u_0^2)\right). \tag{3.1}$$

Recalling Corollary 2.14, we obtain for any $\delta > 0$,

$$\mathbb{P}\left(\|u_i(t)\|_{H^2(D)} \geq e^{\delta t}\right) \leq C_\delta e^{-\frac{\delta}{\gamma}t}\left(\left|u_0^i\right|^2_{L^2(D)} + 1\right) \tag{3.2}$$

Combining (3.1), (3.2) and using an interpolatory inequality between $L^2(D)$ and $H^2(D)$, we obtain that for any $s \in [0,2)$, there exists $\alpha_s > 0$ and $C_s > 0$ such that

$$\mathbb{P}\left(\|u_1(t) - u_2(t)\|_{H^s(D)} > ce^{-\alpha_s t}\right) \leq C_s e^{-\alpha_s t}\left(1 + \sum_{i=1}^2 \left(\left|u_0^i\right|^2_{L^2(D)} + \mathcal{H}(u_0^i)\right)\right),$$

which implies

$$\|\mathcal{P}_t^*\mu - \nu\|^*_{Lip_b(H^s(D))} \leq C_s e^{-\alpha_s t}\left(1 + \int_H \left(|u|^2_{L^2(D)} + \mathcal{H}(u)\right) d\mu(u)\right).$$

Now it remains to conclude the second case, we consider (u, W) a weak solution and we apply Proposition 2.12

$$\mathbb{E}\left(\mathcal{H}(u(T)) + |u(T)|^2_{L^2(D)}\right) \leq \frac{1}{T}\mathbb{E}\,|u_0|^2_{L^2(D)} + C(1+T).$$

which implies for all cases

$$\left|\mathbb{E}f(u(t)) - \int_H f(u)d\nu(u)\right| \leq c_s\,|f|_{Lip_b(H^s(D))}\, e^{-\alpha_s t}\left(1 + \mathbb{E}\,|u_0|^2_{L^2(D)}\right),$$

for any $s < 2$, for any $f \in Lip_b(H^s(D))$.

It follows from this discussion that it suffices to prove that Theorem 1.8 can be applied and that (3.1) holds. Then Theorem 2.1 is proved.

3.1. **Preliminaries.**

We set $|\cdot| = |\cdot|_H$ and

$$X = P_N u, \quad Y = Q_N u, \quad \beta = P_N W, \quad \eta = Q_N W, \quad \sigma_l = P_N b P_N, \quad \sigma_h = Q_N b Q_N,$$

and

$$\begin{aligned} f(X,Y) &= (\eta + \lambda \mathrm{i})P_N\left(|X+Y|^{2\sigma}(X+Y)\right), \\ g(X,Y) &= (\eta + \lambda \mathrm{i})Q_N\left(|X+Y|^{2\sigma}(X+Y)\right). \end{aligned}$$

Now, taking into account **H2** and **H3**, the system has the form

$$\begin{cases} dX + (\varepsilon + \mathrm{i})AX dt + f(X,Y)dt = \sigma_l(X)d\beta, \\ dY + (\varepsilon + \mathrm{i})AY dt + g(X,Y)dt = \sigma_h(X)d\eta, \\ X(0) = x_0, \quad Y(0) = y_0. \end{cases} \tag{3.3}$$

Recall that **H3** states that

(3.4) There exists $\sigma_0 > 0$ such that, $\left|(\sigma_l(x))^{-1}\right| \leq \dfrac{1}{\sigma_0}$, for any $x \in P_N H$.

Now we can define l_0

$$l_0(k) = \min\left\{l \in \{0,...,k\} | P_{l,k}\right\},$$

where $\min \phi = \infty$ and

$$(P_{l,k})\begin{cases} X_1(t) = X_2(t), \quad \eta_1(t) = \eta_2(t), \quad \forall\, t \in [lT, kT], \\ \mathcal{H}_l \le d_0, \quad i = 1, 2, \\ E_{u_i}(t + lT, lT) \le \aleph 1_{t<T} + Bt + 1_{i=2} 1_{t\le T} C_N (1 + t^\alpha), \quad \forall\, t \in [0, (k-l)T], \end{cases}$$

where B, α, C_N are defined in Propositions 2.8 and 2.9, where $\aleph$ will be chosen later and where

$$\mathcal{H}_k = \mathcal{H}(u_1(kT)) + \mathcal{H}(u_2(kT)).$$

Notice that (1.45) is obvious. Corollary 2.6 and **H4** gives

$$l_0(k) = l \quad \text{implies} \quad |u_1(t) - u_2(t)| \le C(d_0) e^{-(t-lT)}, \quad \text{for any } t \in [lT, kT], \tag{3.5}$$

and we have establish (1.46). Lemma 2.10 implies the Lyapunov structure (1.49).

From now on we say that (X_1, X_2) are coupled at kT if $l_0(k) \le k$, in other words if $l_0(k) \ne \infty$. Now it remains to build a coupling such that (3.6) and (3.7) holds, where

$$\begin{cases} \forall\, d_0,\ \exists\, p_0(d_0) > 0,\ (p_i)_{i\in\mathbb{N}^*},\ T_0(d_0) > 0 \ \text{ such that for any } l \le k, \\ \mathbb{P}\left(l_0(k+1) = l \mid l_0(k) = l\right) \ge p_{k-l},\ \text{ for any } T \ge T_0(d_0), \\ 1 - p_i \le e^{-iT},\ i \in \mathbb{N}^*, \end{cases} \tag{3.6}$$

and, for any (R_0, d_0) sufficiently large,

$$\begin{cases} \exists\, T^*(R_0) > 0 \text{ and } p_{-1} > 0 \text{ such that for any } T \ge T^*(R_0) \\ \mathbb{P}\left(l_0(k+1) = k+1 \mid l_0(k) = \infty,\ \mathcal{H}_k \le R_0\right) \ge p_{-1}, \end{cases} \tag{3.7}$$

These properties imply (1.47) and (1.48) and Theorem 1.8 can be applied.

As in the example of section 1.2, we remark that by induction, it suffices to construct a probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ and two measurable couples of functions $(\omega_0, u_0^1, u_0^2) \to (V_i(\cdot, u_0^1, u_0^2))_{i=1,2}$ and $(V_i'(\cdot, u_0^1, u_0^2))_{i=1,2}$ and such that, for any (u_0^1, u_0^2), $(V_i(\cdot, u_0^1, u_0^2))_{i=1,2}$ and $(V_i'(\cdot, u_0^1, u_0^2))_{i=1,2}$ are two couplings of $(\mathcal{D}(u(\cdot, u_0^i), W))_{i=1,2}$ on $[0, T]$. Indeed, we first set

$$u_i(0) = u_0^i, \quad W_i(0) = 0, \quad i = 1, 2.$$

Assuming that we have built $(u_i, W_i)_{i=1,2}$ on $[0, kT]$, then we take $(V_i)_i$ and $(V_i')_i$ as above independant of $(u_i, W_i)_{i=1,2}$ on $[0, kT]$ and set

$$(u_i(kT + t), W_i(kT + t)) = \begin{cases} V_i(t, u_1(kT), u_2(kT)) & \text{if } l_0(k) \le k, \\ V_i'(t, u_1(kT), u_2(kT)) & \text{if } l_0(k) = \infty, \end{cases} \tag{3.8}$$

for any $t \in [0, T]$.

3.2. **Proof of** (3.6)**.**

The essential difference between this proof and the proof of (1.18) in the example in section 1.2 is that a cut-off is used to control the energy.

To build $(V_i(\cdot,u_0^1,u_0^2))_{i=1,2}$, we apply Proposition 1.7 to

$$E = C((0,T);H)\times C((0,T);H^{-\frac{d}{2}-1}(D)),$$
$$F = C((0,T);P_N H)\times C((0,T);Q_N H^{-\frac{d}{2}-1}(D)),$$
$$f_0\,(u,W) = (X,\eta),$$
$$\mu_i = \mathcal{D}(u(\cdot,u_0^i),W), \qquad \text{on } [0,T].$$

Remark that if we set $\nu_i = f_0^*\mu_i$, we obtain

$$\nu_i = \mathcal{D}(X(\cdot,u_0^i),\eta), \qquad \text{on } [0,T].$$

We set

$$(Z_i,\xi_i) = f_0(V_i), \quad i=1,2.$$

Then $(V_i(\cdot,u_0^1,u_0^2))_{i=1,2}$ is a coupling of (μ_1,μ_2) such that $((Z_i,\xi_i)(\cdot,u_0^1,u_0^2))_{i=1,2}$ is a maximal coupling of (ν_1,ν_2).

We first use a Girsanov formula to estimate I_p, where

$$\begin{array}{lcl} I_p & = & \int_{A_{k,l}} \left(\frac{d\nu_2}{d\nu_1}\right)^{p+1} d\nu_2, \\ A_{k,l} & = & \{(Z,\xi)\,|\,\tau_{k,l}=T\}, \\ \tau_{k,l} & = & \inf\{t\in[0,T]\mid E_{\hat{u}_i}(t+kT,lT) > \aleph 1_{k=l} + B(t+(k-l)T) \\ & & \qquad\qquad +1_{i=2}1_{k=l}C_N(1+t^\alpha),\ i\in\{1,2\}\}, \end{array}$$

where

$$\hat{u}_i = u_i \ \text{ on } [0,kT], \quad \hat{u}_i(kT+\cdot) = Z+\Phi(Z,\xi,u_0^i) \ \text{ on } [0,T].$$

Then, using Lemma 1.2, we establish (3.6).

We consider a couple of $(u_i,W_i)_{i=1,2}$, two solutions of (3.3) on $[0,kT]$ and a trajectory of $(u_i,W_i)_{i=1,2}$ such that $l_0(k)=l$. We set

$$x = X_1(kT) = X_2(kT), \quad y_i = Y_i(kT), \quad i=1,2.$$

Let $W=(\beta,\xi)$ a cylindrical Wiener process defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. We denote by Z the unique solution of the truncated equation

$$\text{(3.9)} \qquad \begin{cases} dZ + (\varepsilon+\mathrm{i})AZdt + 1_{t\le\tau_{k,l}} f(Z,\Phi(Z,\xi,(x,y_1)))dt = \sigma_l(Z)d\beta, \\ Z(0)=x. \end{cases}$$

We denote by λ_1 the distribution of (Z,ξ) under the probability $\mathbb{P}$.

We set $\widetilde{\beta}(t) = \beta(t) + \int_0^t d(s)dt$ where

$$d(t) = 1_{t\le\tau_{k,l}}\left(\sigma_l(Z(t))\right)^{-1}\left(f(Z(t),\Phi(Z,\xi,(x,y_2))(t)) - f(Z(t),\Phi(Z,\xi,(x,y_1))(t))\right).$$

Then Z is a solution of

$$\text{(3.10)} \qquad \begin{cases} dZ + (\varepsilon+\mathrm{i})AZdt + 1_{t\le\tau_{k,l}} f(Z,\Phi(Z,\xi,(x,y_2)))dt = \sigma_l(Z)d\widetilde{\beta}, \\ Z(0)=x. \end{cases}$$

The drift estimate in Lemma 2.7 ensures that

$$\int_0^T |d(t)|^2\, dt \le c d_0 \sigma_0^{-2} \exp\left(-3(k-l)T + c\aleph 1_{k=l}\right). \tag{3.11}$$

Hence the Novikov condition is satisfied and the Girsanov formula can be applied. Then we set

$$d\widetilde{\mathbb{P}} = \exp\left(\int_0^T d(s)dW(s) - \frac{1}{2}\int_0^T |d(s)|^2\, dt\right) d\mathbb{P}$$

We deduce from the Girsanov formula that $\widetilde{\mathbb{P}}$ is a probability under which $(\widetilde{\beta}, \xi)$ is a cylindrical Wiener process and we denote by λ_2 the law of (Z, ξ) under $\widetilde{\mathbb{P}}$. Moreover, remarking that

$$\lambda_i(A_{k,l} \cap \cdot) = \nu_i(A_{k,l} \cap \cdot), \quad i = 1, 2, \tag{3.12}$$

we obtain

$$I_p \le I'_p \le \mathbb{E} \exp\left(c_p \int_0^T |d(s)|^2\, dt\right), \tag{3.13}$$

where

$$I'_p = \int_F \left(\frac{d\lambda_2}{d\lambda_1}\right)^{p+1} d\lambda_2,$$

Then it follows from (3.11) that

$$I_p \le I'_p \le \exp\left(c_p \sigma_0^{-2} d_0 e^{-3(k-l)T + c\aleph 1_{k=l}}\right). \tag{3.14}$$

Notice that

$$\|\lambda_1 - \lambda_2\|_{var} = \int_F \left|\frac{d\lambda_2}{d\lambda_1} - 1\right| d\lambda_2 \le \sqrt{\int \left(\frac{d\lambda_2}{d\lambda_1}\right)^2 d\lambda_2 - 1}.$$

We infer from (3.14) that, for $T \ge T_3(d_0) = 2\ln\left(c_p \sigma_0^{-2} d_0\right)$,

$$\|\lambda_1 - \lambda_2\|_{var} \le \frac{1}{2} e^{-2(k-l)T}.$$

Using (3.12), we obtain for $k > l$

$$\|\nu_1 - \nu_2\|_{var} \le \|\lambda_1 - \lambda_2\|_{var} + \sum_{i=1}^{2} \nu_i(A^i_{k,l}) \le \frac{1}{2} e^{-2(k-l)T} + \sum_{i=1}^{2} \nu_i(A^i_{k,l}).$$

where

$$A^i_{k,l} = \left\{(Z,\xi) \;\Big|\; E_{Z+\phi(Z,\xi,u_0^i)}(t, lT) \le B(t + (k-l)T) \text{ for any } t \in [0,T]\right\}.$$

Applying Lemma 1.2 to the maximal coupling $(Z_1, Z_2)_{i=1,2}$ of (ν_1, ν_2) gives for $k > l$

$$\mathbb{P}\left((Z_1, \xi_1) \ne (Z_2, \xi_2)\right) \le \|\nu_1 - \nu_2\|_{var} \le \frac{1}{2} e^{-2(k-l)T} + \sum_{i=1}^{2} \nu_i(A^i_{k,l}). \tag{3.15}$$

Using (3.8) and (3.15), we obtain that on $l_0(k) = l$

$$\mathbb{P}\left((X_1, \eta_1) \ne (X_2, \eta_2) \text{ on } [kT, (k+1)T] \mid \mathcal{F}_{kT}\right) \le \frac{1}{2} e^{-2(k-l)T} + 2\mathbb{P}(B_{l,k} | \mathcal{F}_{kT}),$$

where

$$B_{l,k} = \left\{E_{u_i}(t, lT) \le B(t - lT), \text{ for any } t \in [kT, (k+1)T],\ i = 1, 2\right\}.$$

Noticing that for $k > l$

$$\{l_0(k+1) = l\} = \{l_0(k) = l\} \cap \{(X_1, \eta_1) = (X_2, \eta_2) \text{ on } [kT, (k+1)T]\} \cap B_{l,k}.$$

and integrating over $l_0(k) = l$ gives for $T \geq T_1(d_0)$ and for $k > l$

$$\mathbb{P}\left(l_0(k+1) \neq l \,|\, l_0(k) = l\right) \leq \frac{1}{2} e^{-2(k-l)T} + 3\mathbb{P}(B_{l,k} \,|\, l_0(k) = l),$$

and then

$$\mathbb{P}\left(l_0(k+1) \neq l,\ l_0(k) = l \,|\, l_0(l) = l\right) \leq \frac{1}{2} e^{-2(k-l)T} + 3\mathbb{P}(B_{l,k} \,|\, l_0(l) = l).$$

The exponential estimate for growth of the solution (Proposition 2.8) gives that for T sufficiently high

$$\mathbb{P}\left(l_0(k+1) \neq l,\ l_0(k) = l \,|\, l_0(l) = l\right) \leq \exp(-2(k-l)T). \tag{3.16}$$

Now, it remains to consider the case $k = l$, we apply Lemmas 1.2 and 1.3 to $(Z_i, \xi_i)_{i=1,2}$ which gives

$$\mathbb{P}\left((Z_1, \xi_1) = (Z_2, \xi_2),\ A_{l,l}^2\right) \geq (\nu_1 \wedge \nu_2)\left(A_{l,l}\right) \geq \left(1 - \frac{1}{p}\right)(pI_p)^{-\frac{1}{p-1}} \nu_1(A_{l,l})^{\frac{p}{p-1}}.$$

Choosing $\aleph$ sufficiently high and applying the exponential for growth of the solution (Propositions 2.8 and 2.9), we obtain

$$\nu_1(A_{l,l}) \geq \frac{1}{2},$$

and then applying (3.13) and fixing $p > 1$,

$$\mathbb{P}\left((Z_1, \xi_1) = (Z_2, \xi_2),\ A_{l,l}\right) \geq p_0(d_0) > 0.$$

That gives

$$\mathbb{P}\left(l_0(l+1) = l \,|\, l_0(l) = l\right) \geq p_0(d_0) > 0. \tag{3.17}$$

Since

$$\mathbb{P}\left(l_0(k) \neq l | l_0(l) = l\right) \leq \sum_{n=l}^{k-1} \mathbb{P}\left(l_0(n+1) \neq l,\ l_0(n) = l \,|\, l_0(l) = l\right),$$

then, by applying (3.16) and (3.17), we obtain

$$\mathbb{P}\left(l_0(k) \neq l | l_0(l) = l\right) \leq 1 - p_0 + \sum_{n=1}^{\infty} \exp(-2nT) \leq 1 - p_0 + \frac{\exp(-2T)}{1 - \exp(-2T)},$$

which implies that for $T \geq T_0(d_0)$

$$\mathbb{P}\left(l_0(k) = l | l_0(l) = l\right) \geq \frac{p_0}{2}, \tag{3.18}$$

Combining (3.16), (3.17) and (3.18), we establish (3.6) for T sufficiently high.

3.3. **Proof of** (3.7)**.**

As in the example of Section 1.2, The Lyapunov structure gives that it is sufficient to find $d_0>0$, $\widetilde{p}>0$, $R_1>4K_1$ and a coupling $(V_i(\cdot,u_0^1,u_0^2))_{i=1,2}$ of (μ_1,μ_2), where

$$\mu_i=\mathcal{D}(u(\cdot,u_0^i),W),\quad\text{on }[0,1],\qquad i=1,2,$$

and such that

$$\mathbb{P}\left(Z_1(1,u_0^1,u_0^2)=Z_2(1,u_0^1,u_0^2),\ \sum_{i=1}^2\mathcal{H}(u_i(1,u_0^1,u_0^2))\le d_0\right)\ge\widetilde{p}, \tag{3.19}$$

where

$$V_i(\cdot,u_0^1,u_0^2)=\left(u_i(\cdot,u_0^1,u_0^2),W_i(\cdot,u_0^1,u_0^2)\right),\ u_i(\cdot,u_0^1,u_0^2)=\begin{pmatrix}Z_i\\ G_i\end{pmatrix},\ i=1,2.$$

Now we fix $R_1>4K_1$ and consider a cimetery value Δ (some people prefer calling it a heaven value). To build $(V_i(\cdot,u_0^1,u_0^2))_{i=1,2}$, we apply Proposition 1.7 to

$$\begin{aligned}
&E=C((0,1);H)\times C((0,1);H^{-\frac{d}{2}-1}(D)),\\
&F=\left(P_NH\times C((0,1);Q_NH^{-\frac{d}{2}-1}(D))\right)\cup\{\Delta\},\\
&f_0\,(u,W)=X(1)1_A(X,\eta)+\Delta 1_{A^c}(X,\eta),
\end{aligned}$$

and to μ_i where

$$\begin{aligned}
A&=\{(X,\eta)\,|\,\tau=1\},\\
\tau&=\inf\left\{t\in[0,1]\mid E_{X+\Phi(X,\eta,u_0^i)}(t)>\aleph+Bt+1_{i=2}C_N(1+t^\alpha),\ i\in\{1,2\}\right\}.
\end{aligned}$$

We set $\nu_i=f_0^*\mu_i$. Then $(V_i(\cdot,u_0^1,u_0^2))_{i=1,2}$ is a coupling of (μ_1,μ_2) such that $(Z_i(1,u_0^1,u_0^2))_{i=1,2}$ is a maximal coupling of (ν_1,ν_2).

Now, we define

$$f_1\,(u,W)=(X,\eta)\qquad\text{and}\qquad f_2\,(X,\eta)=X(1)1_A(X,\eta)+\Delta 1_{A^c}(X,\eta),$$

and we set $\theta_i=f_1^*\mu_i$ for $i=1,2$. Now we consider $(\hat\theta_1,\hat\theta_2)$ such that $\theta_i(A\cap\cdot)$ is equivalent to $\hat\theta_i(A\cap\cdot)$ for $i=1,2$ and such that $(\hat\nu_1,\hat\nu_2)=(f_2^*\hat\theta_1,f_2^*\hat\theta_2)$ are two equivalent measures. Then by applying two Schwartz inequalities, we obtain that

$$I_p\le\left(J_{2p+2}^1\right)^{\frac12}\left(J_{4p}^2\right)^{\frac14}\left(\hat I_{4p+2}\right)^{\frac14}, \tag{3.20}$$

where

$$\begin{aligned}
I_p&=\int_{B'}\left(\frac{d\nu_1}{d\nu_2}\right)^{p+1}d\nu_2, & J_p^1&=\int_A\left(\frac{d\theta_1}{d\hat\theta_1}\right)^p d\hat\theta_1,\\
\hat I_p&=\int_{B'}\left(\frac{d\hat\nu_1}{d\hat\nu_2}\right)^p d\hat\nu_2, & J_p^2&=\int_A\left(\frac{d\hat\theta_2}{d\theta_2}\right)^p d\hat\theta_2,
\end{aligned}$$

Let us consider $\bar Z_i$ the unique solution of

$$\left\{\begin{aligned}
&d\bar Z_i+(\varepsilon+\mathrm{i})A\bar Z_idt+1_{t\le\tau}f(\bar Z_i,\Phi(\bar Z_i(\cdot),\xi(\cdot),u_0^i))dt=\sigma_l(\bar Z_i)d\beta_i,\\
&\bar Z_i(0)=x_0^i.
\end{aligned}\right. \tag{3.21}$$

Taking into account (3.9), we denote by λ_i the distribution of $(\bar Z_i,\xi_i)$ under the probability $\mathbb{P}$ and we obtain

$$\theta_i(A\cap\cdot)=\lambda_i(A\cap\cdot). \tag{3.22}$$

We set $\widetilde{\beta}_i(t) = \beta_i(t) + \int_0^t d_i(s)dt$ where

$$d_i(t) = -1_{t\leq\tau}(\sigma_l(\bar{Z}_i(t)))^{-1} f(\bar{Z}_i(t), \Phi(\bar{Z}_i(\cdot), \xi(\cdot), u_0^i)(t)). \tag{3.23}$$

Then $\bar{Z}_i$ is a solution of

$$\begin{cases} d\bar{Z}_i + (\varepsilon + \mathrm{i})A\bar{Z}_i dt &= \sigma_l(\bar{Z}_i)d\widetilde{\beta}_i, \\ \bar{Z}_i(0) = x_0^i. \end{cases} \tag{3.24}$$

Since the energy is bounded and σ_l is bounded below, then d is uniformly bounded. Hence, the Novikov condition is satisfied and the Girsanov formula can be applied. Then we set

$$d\widetilde{\mathbb{P}}_i = \exp\left(\int_0^T d_i(s)dW(s) - \frac{1}{2}\int_0^T |d(s)|^2\, dt\right) d\mathbb{P}$$

We deduce from the Girsanov formula that $\widetilde{\mathbb{P}}$ is a probability under which $(\widetilde{\beta}, \xi)$ is a cylindrical Wiener process. We denote by $\hat{\theta}_i$ the law of $(\bar{Z}_i, \xi_i)$ under $\widetilde{\mathbb{P}}_i$. Moreover using (3.22), we obtain

$$J_p^1 \vee J_p^2 \leq \mathbb{E}\exp\left(c_p \int_0^T |d(s)|^2\, dt\right) \leq C(p, \aleph, R_1). \tag{3.25}$$

We set $\hat{\nu}_i = f_2^* \hat{\theta}_i$ for $i = 1, 2$. It is classical that $\hat{\nu}_i$ has a density $q(x_0^i, z)$ with respect to lebesgue measure dz, that q is continuous for the couple (x_0^i, z), where x_0^i is the initial value and where z is the target value and that $q > 0$. Then, we can bound q and q^{-1} uniformly on $\mathcal{H}(x_0^i) \leq R_1$ and on $z \in B' = \{\mathcal{H}(z) \leq C\}$ provided $C = C(\aleph)$. It allows us to bound $\hat{I}_p$ and then I_p. Actually, $d_1 \geq d_1(\aleph)$ implies

$$A \subset B, \tag{3.26}$$

where

$$B = \left\{(Z, \xi) \,\middle|\, \mathcal{H}(Z(1) + \phi(Z, \xi, u_0^i)(1)) \leq d_1,\ i = 1, 2\right\}.$$

Hence it follows that for $d_1 \geq d_1(\aleph)$

$$I_p \leq C'(p, \aleph, R_1) < \infty. \tag{3.27}$$

Now we apply Lemma 1.3 and 1.2:

$$\mathbb{P}\left(Z_1(1) = Z_2(1),\ (A \cap B')^2\right) \geq \left(1 - \frac{1}{p}\right)(pI_p)^{-\frac{1}{p-1}} \nu_1(B')^{\frac{p}{p-1}}. \tag{3.28}$$

We deduce from Propositions 2.8 and 2.9 and from $C(\aleph) \to \infty$ when $\aleph \to \infty$ that $\aleph$ sufficiently high gives

$$\nu_1(B') \geq \frac{1}{2}. \tag{3.29}$$

Combining (3.26), (3.27), (3.28) and (3.29) gives for $d_1 \geq d_1(\aleph)$

$$\mathbb{P}\left(Z_1(1) = Z_2(1),\ B^2\right) \geq \widetilde{p} = \widetilde{p}(p, \aleph, R_1) > 0. \tag{3.30}$$

Taking into account the definition of ϕ and choosing $d_0 = 2d_1$, it follows that (3.19) holds.

Appendix A. Proof of Lemma 1.2

Let $(Y_i)_i$ be a coupling of $(\mu_i)_i$. Let Γ be a measurable set. There exists $(\Gamma_i)_i$ such that

$$\Gamma=\bigcup_i \Gamma_i,\quad \bigcap_i \Gamma_i=\emptyset\quad (\mu_2-\mu_1)^+(\Gamma_2)=0,\quad (\mu_1-\mu_2)^+(\Gamma_1)=0.$$

It follows from $a\wedge b=a-(a-b)^+$ and $(\mu_1-\mu_2)^+(\Gamma_1)=0$ that

$$(\mu_1\wedge\mu_2)(\Gamma_1)=\mu_1(\Gamma_1)-(\mu_1-\mu_2)^+(\Gamma_1)=\mu_1(\Gamma_1)=\mathbb{P}(Y_1\in\Gamma_1).$$

Symetricly, we obtain $(\mu_1\wedge\mu_2)(\Gamma_2)=\mathbb{P}(Y_2\in\Gamma_2)$.

Thus, it follows from $\Gamma=\bigcup_i\Gamma_i$ and $\bigcap_i\Gamma_i=\emptyset$ that

$$(\mu_1\wedge\mu_2)(\Gamma)=\mathbb{P}(Y_1\in\Gamma_1)+\mathbb{P}(Y_2\in\Gamma_2)\geq\sum_{i=1}^{2}\mathbb{P}(Y_1=Y_2,\ Y_1\in\Gamma_i).$$

Since $\Gamma=\bigcup_i\Gamma_i$ and $\bigcap_i\Gamma_i=\emptyset$, then

$$(\mu_1\wedge\mu_2)(\Gamma)\geq\mathbb{P}(Y_1=Y_2,Y_1\in\Gamma). \tag{A.1}$$

Then it follows from $\|\mu_1-\mu_2\|_{var}=1-(\mu_1\wedge\mu_2)(E)$ that

$$\|\mu_1-\mu_2\|_{var}\leq\mathbb{P}(Y_1\neq Y_2).$$

We have equality only if (A.1) appears for $\Gamma=E$, which is true only if (A.1) appears for any Γ. For any measure μ on $(E,\mathcal{E})$, we denote by μ the measure on $(E,\mathcal{E})\otimes(E,\mathcal{E})$ define by

$$\mu(\mathbf{A})=\mu(\{a\in E|(a,a)\in\mathbf{A}\}).$$

If $\mu_1=\mu_2$, we set $\mathbb{P}=\mu_{\mathbf{1}}$. Else we set

$$\mathbb{P}=\mu_{\mathbf{1}}\wedge\mu_{\mathbf{2}}+\frac{1}{\|\mu_2-\mu_1\|_{var}}(\mu_1-\mu_2)^+\otimes(\mu_2-\mu_1)^+. \tag{A.2}$$

Noticing that $a=a\wedge b+(a-b)^+$ and using $\|\mu_1-\mu_2\|_{var}=(\mu_1-\mu_2)^+(E)$, we obtain that $\mathbb{P}(.\times E)=\mu_1\wedge\mu_2+(\mu_1-\mu_2)^+=\mu_1$ and $\mathbb{P}(E\times .)=\mu_2$. Thus if we denote by $(Y_i)_i$ the projectors, we obtain that $(Y_i)_i$ is a coupling of $(\mu_i)_i$. Moreover,

$$\mathbb{P}(Y_1=Y_2,Y_1\in A)=(\mu_1\wedge\mu_2)(A).$$

So it is the desired maximal coupling

□

Remark A.1. *Moreover, in all this article, we admit that the maximal coupling* $(Y_i(u_0^i))_i$ *could be chosen such that* $(Y_i(u_0^1,u_0^2))_i$ *depend measurably on the initials conditions* $(u_0^i)_i$*. The idea is the following. Since we only work in nice spaces, we can consider that we are working on the real line. It can be seen that the laws we use depend measurably on* $(u_0^i)_i$ *and then the law define by* (A.2) *will do it too. Then its repartition function* $F_{(u_0^1,u_0^2)}$ *is measurable too and finally the pseudo-inverse of the repartition function* $F^{-1}_{(u_0^1,u_0^2)}$ *is measurable with respect to* (u_0^1,u_0^2)*. We consider* $([0,1],\mathcal{B}_{[0,1]},\lambda)$*, where* λ *is the Lebesgue measure and we set* $Y_i(u_0^1,u_0^2,\omega)=F^{-1}_{(u_0^1,u_0^2)}(\omega)$*. Then* $(Y_i)_i$ *is measurable with respect to* (u_0^1,u_0^2,ω) *and for every* (u_0^1,u_0^2)*, it is a coupling of* $(\mu_i(u_0^i))_i$*. For a proof see* [14].

Appendix B. Proof of Proposition 1.7

We set
$$\Omega = E^2, \quad \mathcal{F} = \mathcal{B}(E^2),$$
and V_i the i^{th} projector on Ω:
$$V_i(v_1, v_2) = v_i, \quad i = 1, 2.$$
Let (U_1, U_2) be a coupling of (μ_1, μ_2).

In order to establish Proposition 1.7, we build a probability measure Q on $(\Omega, \mathcal{F})$ such that

$$\text{(B.1)} \qquad \begin{cases} \alpha) \quad Q(\cdot \times E) = \mu_1, \quad Q(E \times \cdot) = \mu_2, \\ \beta) \quad Q(f_0(V_1) = f_0(V_2)) \geq (\nu_1 \wedge \nu_2)(E). \end{cases}$$

Then (V_1, V_2) seen as a couple of random variables defined on $(\Omega, \mathcal{F}, Q)$ is a coupling of (μ_1, μ_2) such that $(f(V_1), f(V_2))$ is a maximal coupling of (ν_1, ν_2).

Recall that

$$\text{(B.2)} \qquad \nu_i = \nu_1 \wedge \nu_2 + \left((-1)^i(\nu_1 - \nu_2)\right)^+, \quad i = 1, 2,$$

and that since E, F are polish spaces, then there exists a version of $\mathbb{P}(U_i \in A \,|\, f_0(U_i) = x)$ which is measurable for any $A \in \mathcal{B}(E)$ and which is probability measure for any $x \in F$. Moreover

$$\text{(B.3)} \qquad \mu_i(A) = \int_F \mathbb{P}(U_i \in A \,|\, f_0(U_i) = x)\nu_i(dx), \quad i = 1, 2,$$

Combining (B.2) and (B.3), we obtain

$$\text{(B.4)} \qquad \mu_i = \mu_i^s + \mu_i^r, \quad i = 1, 2,$$

where

$$\begin{array}{rcll} \mu_i^s(A) & = & \int_F \mathbb{P}(U_i \in A \,|\, f_0(U_i) = x)\,(\nu_1 \wedge \nu_2)\,(dx), & i = 1, 2, \\ \mu_i^r(A) & = & \int_F \mathbb{P}(U_i \in A \,|\, f_0(U_i) = x)\left((-1)^i(\nu_1 - \nu_2)\right)^+(dx), & i = 1, 2. \end{array}$$

Remark that

$$\text{(B.5)} \qquad \begin{cases} \mu_i^s, \mu_i^r \geq 0, \quad i = 1, 2, \\ \mu_i^s(E) = (\nu_1 \wedge \nu_2)\,(E), \\ \mu_i^r(E) = \|\nu_1 - \nu_2\|_{var}\,. \end{cases}$$

Taking into account (B.4) and (B.5), we can write problem (B.1) in the form

$$\text{(B.6)} \qquad \begin{cases} \text{Find } r, s \text{ two positive measures on } (\Omega, \mathcal{F}) \text{ such that} \\ i) \quad s(\cdot \times E) = \mu_1^s, \quad s(E \times \cdot) = \mu_2^s, \\ ii) \quad r(\cdot \times E) = \mu_1^r, \quad r(E \times \cdot) = \mu_2^r, \\ iii) \quad s(f_0(V_1) \neq f_0(V_2)) = 0. \end{cases}$$

Once (B.6) is true, we can set
$$Q = r + s.$$

Then (B.1)α) is an obvious consequence of (B.4). Furthermore, since $r \geq 0$, then (B.6)iii), (B.6)i) and (B.5) gives

$$Q(f_0(V_1) = f_0(V_2)) \geq s(f_0(V_1) = f_0(V_2)) = s(\Omega) = \mu_i^s(E) = (\nu_1 \wedge \nu_2)(E).$$

Now we build r by setting

$$r = \frac{1}{\|\nu_1 - \nu_2\|_{var}} \mu_1^r \times \mu_2^r.$$

Notice that $r \geq 0$ and (B.6)ii) are obvious consequence of (B.5).

Now we build s by setting

$$s(A \times B) = \int_F \mathbb{P}(U_1 \in A \,|\, f_0(U_1) = x) \times \mathbb{P}(U_2 \in B \,|\, f_0(U_2) = x)\,(\nu_1 \wedge \nu_2)\,(dx).$$

Notice that (B.6)i) and (B.6)iii) are obvious.

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Annexe B

Preuve de l'ergodicité d'une équation de Schrödinger non linéaire amortie

Résumé: Nous étudions une équation Schrödinger non linéaire (NLS) stochastique et faiblement amortie. Elle est munie d'un bruit blanc pour la variable temporelle et régulier pour la variable spatiale. En utilisant une méthode de couplage, nous établirons la convergence du semi-groupe de Markov vers une unique mesure de probabilité invariante. Ce type de méthode a été originellement développée pour prouver le caractère exponentiellement mélangeant d'équations fortement dissipatives telles que les équations de Navier-Stokes en dimension 2. On considère ici une équation faiblement dissipative, l'équation de Schrödinger amortie dans le cas cubique unidimensionnel. Nous établirons le caractère mélangeant des solutions et nous montrerons que la vitesse de convergence est plus rapide que n'importe quelle puissance négative du temps.

Mots clés : Équation de Schrödinger non linéaire, semi-groupe de transition de Markov, mesures invariantes, ergodicité, méthode de couplage, Formule de Girsanov, inégalité de type Foias–Prodi en espérance.

ERGODICITY FOR THE WEAKLY DAMPED STOCHASTIC NON-LINEAR SCHRÖDINGER EQUATIONS

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Abstract: We study a damped stochastic non-linear Schrödinger (NLS) equation driven by an additive noise. It is white in time and smooth in space. Using a coupling method, we establish convergence of the Markov transition semi-group toward a unique invariant probability measure. This kind of method was originally developed to prove exponential mixing for strongly dissipative equations such as the Navier-Stokes equations. We consider here a weakly dissipative equation, the damped nonlinear Schrödinger equation in the one-dimensional cubic case. We prove that the mixing property holds and that the rate of convergence to equilibrium is at least polynomial of any power.

Key words: Non-linear Schrödinger equations, Markov transition semi-group, invariant measure, ergodicity, coupling method, Girsanov's formula, expectational Foias–Prodi estimate.

Introduction

The non-linear Schrödinger (NLS) equation models the propagation of non-linear dispersive waves in various areas of physics such as hydrodynamics [24], [25], optics, plasma physics, chemical reaction [16]...

When studying the propagation in random media, a noise can be introduced. For instance in [9], [10], the cubic nonlinear Schrödinger equation with additive white noise and damping is introduced. There, the propagation of waves over very long distance is studied. Damping effect cannot be neglected in this case and has to be counterbalanced by amplifiers. The white noise is a model for the description of the randomness in these amplifiers. Such model is valid if the distance between amplifiers is small compared to propagation distance.

Our aim in this work is to study ergodicity for this type of equation. We consider the one-dimensional case with cubic focusing nonlinearity. It has the form

$$(0.1)\quad \left\{\begin{array}{rcll} du+\alpha u\,dt-\mathrm{i}\Delta u\,dt-\mathrm{i}\,|u|^2\,u\,dt &=& bdW, & \\ u(t,x) &=& 0, & \text{for } x\in\{0,1\},\ t>0,\\ u(0,x) &=& u_0(x), & \text{for } x\in[0,1],\end{array}\right.$$

where $\alpha>0$. The unknown u is a complex valued process depending on $x\in[0,1]$ and $t\geq 0$. Dirichlet boundary conditions are considered but we could also use Neumann or periodic boundary condition.

Existence and uniqueness of solutions for (0.1) is not very difficult to prove using straightforward generalization of deterministic arguments. Note that the damping term is necessary to have an invariant measure. Indeed, if $\alpha = 0$ and $b \neq 0$ then the $L^2(0,1)$ norm grows linearly in time.

The Complex Ginzburg-Landau (CGL) is also a form of dissipative NLS equation. The exponential mixing of the stochastic CGL equation has been establish in [14] in a particular case and in [26] in the general case. The method was inspired by the so-called coupling method. This method has been introduced in [3], [14], [19], [20], [21], [23] and [27]. In all these articles, a strongly dissipative stochastic partial differential equations driven by a noise which may be degenerate is considered. Due to the possible degeneracy of the noise Doob Theorem cannot to be applied (see [5] for the theory of ergodicity when Doob Theorem can be applied). Indeed, it requires the strong Feller property, which can be proved only when the noise lives in a space of spatially irregular functions, which is clearly not true for a degenerate noise. The main idea is to compensate the degeneracy of the noise by dissipativity arguments, the so-called Foias-Prodi estimates. Roughly speaking, the process can be decomposed into the sum of a strongly dissipative process and another one driven by a non-degenerate noise. The strongly dissipative part is treated as in [4] section 11.5, while the non-degenerate part is treated thanks to probabilistic arguments. The difficulty is of course in the fact that the two parts of the process do not evolve independently so that the two methods have to be used simultaneously. The probabilistic part can be treated either by a generalization of Doob Theorem (see [8], [15], [18]) or by coupling argument (see [19], [20], [21], [23]). Each method has its advantages. The first one allows treating very degenerate noises while the coupling method proves also exponential convergence to equilibrium.

In the case of the NLS equation, it seems hopeless to use Doob Theorem. Indeed, due to the lack of smoothing effect of the deterministic part of equations, only spatially smooth noises can be treated (see [6], [7]). Note that this equation is not strongly dissipative, indeed the eigenvalues of the linear part do not grow to infinity. However, it is known that Foias-Prodi type estimates hold for the deterministic damped NLS equation (see [13]) and we will see that these can be generalized to the stochastic case and it is reasonable to think that the above ideas can be generalized.

In this article, we show that the method based on coupling argument is applicable. However it requires substantial adaptations. For instance, contrary to the strongly dissipative case treated in the above-mentioned articles, we are only able to prove a weaker form of the Foias-Prodi estimates. Indeed, here, we prove that it holds in average and not path-wise. This causes many technical difficulties when trying to use the coupling method. Moreover, another important ingredient in the argument is an exponential estimate on the growth of the solution, which we are unable to prove in our case. This is due to the fact that the Lyapunov structure is more complicated here. It is not a quadratic functional. We only prove polynomial estimate on the growth of the solutions and it results that we can only prove that convergence to equilibrium holds with polynomial speed at any order. Thus, we develop a general result, which gives sufficient conditions for polynomial mixing.

Note also that a crucial step in [21] is the fact that the probability that a solution enters a ball of small radius is controlled precisely. This fact is still true for the damped NLS equation considered here. However, its proof is more difficult than in the case of the Navier-Stokes equations (see Proposition 2.6 and section 4 hereafter).

The remaining of the article is divided into four parts. First, we give the notations, and state our main result. Its proof is given in section 2. Section 3, 4 and 5 are devoted to the proofs of intermediate results.

1. Notation and Main result

We set

$$A=-\Delta,\quad D(A)=H_0^1(0,1)\cap H^2(0,1)$$

and write problem (0.1) in the form

$$du+\alpha u\,dt+\mathrm{i}Au\,dt-\mathrm{i}\,|u|^2\,u\,dt = bdW, \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

where W is a cylindrical Wiener process on $L^2(0,1)$ and b is a linear operator on $L^2(0,1)$.

We denote by $(\mu_n)_n$ the increasing sequence of eigenvalues of A and by $(e_n)_{n\in\mathbb{N}}$ the associated eigenvectors. Also, P_N and Q_N are the eigenprojectors onto the space $Sp(e_k)_{1\le k\le n}$ and onto its complementary space. Recall that for $s\ge 0$, $D(A^{s/2})$ is a closed subspace of $H^s(0,1)$ and that $\|\cdot\|_s=|A^{s/2}\cdot|_{L^2(0,1)}$ is equivalent to the usual $H^s(0,1)$ norm on this space. Moreover

$$D(A^{s/2})=\{u=\sum_{k\in\mathbb{N}}u_ke_k\in L^2(0,1)\mid \sum_{n\in\mathbb{N}}\mu_k^su_k^2<\infty\}\text{ and }\|u\|_s=\sum_{n\in\mathbb{N}}\mu_k^su_k^2.$$

We denote by $|\cdot|$, $|\cdot|_p$, $\|\cdot\|$ the norms of $L^2(0,1)$, $L^p(0,1)$, $H_0^1(0,1)$.

The operator b is supposed to commute with A, therefore it is diagonal in the basis $(e_n)_{n\in\mathbb{N}}$ and we have

$$be_n=b_ne_n.$$

We assume that b is Hilbert-Schmidt from $L^2(0,1)$ with values in $D(A^{3/2})$. For any $s\in[0,3]$, we set

$$B_s=|b|^2_{\mathcal{L}_2(L^2(0,1),D(A^{s/2}))}=\sum_{n=0}^{\infty}\mu_n^sb_n^2.$$

To study ergodic properties, we assume that there exists N_* such that

$$b_n>0,\ \text{ for } n\le N_*. \tag{1.3}$$

The Hamiltonian plays an important role in the study of the nonlinear Schrödinger equation. It is a conserved quantity in the absence of noise and damping. It is given by

$$\mathcal{H}_*(v)=\frac12\|v\|^2-\frac14|v|_4^4,\ v\in H_0^1(0,1).$$

In our study, it is the basic tool to derive a priori estimates. Recall that the Gagliardo-Nirenberg inequality gives a constant $c_0>0$ such that

$$|v|_4^4\le\frac14\|v\|^2+\frac{c_0}{2}|v|^6,\ v\in H_0^1(0,1).$$

It follows that, setting

$$\mathcal{H}=\frac12\|\cdot\|^2-\frac14|\cdot|_4^4+c_0|\cdot|^6,$$

we have

$$\mathcal{H}(v) \geq \frac{1}{4}\left\|v\right\|^2 + \frac{1}{4}\left|v\right|_4^4 + \frac{c_0}{2}\left|v\right|^6, \ v \in H_0^1(0,1). \tag{1.4}$$

In our computations, we will also use the following quantities which involve the k^{th} power of the energy:

$$E_{u,k}(t,s) = \mathcal{H}(u(t))^k + \alpha k \int_s^t \mathcal{H}(u(\sigma))^k d\sigma, \ t \geq s,$$

when there is no ambiguity we set $E_{u,k}(t) = E_{u,k}(t,s)$.

In the following, α, B_s for $s \in [0,3]$ are fixed. All the constants appearing below may depend on them as well as on A, b.

Well-posedness of equations (1.1), (1.2) is easily proved. Indeed, let $S(t) = e^{-\mathrm{i}At-\alpha t}$, $t \in \mathbb{R}$, be the group generated by the linear equation. We look for a mild solution, that is a process u with paths in $C(\mathbb{R}^+; H_0^1(0,1))$ satisfying

$$u(t) = S(t)u_0 + \mathrm{i}\int_0^t S(t-s)\left|u(s)\right|^2 u(s)ds + \int_0^t S(t-s)bdW(s).$$

Since $(S(t))_{t\geq 0}$ is a contraction semi-group on $H_0^1(0,1)$ and the linear term is locally Lipschitz, local in time existence and uniqueness is straightforward. Note that $\int_0^t S(t-s)bdW(s)$ lives in $D(A^{\frac{3}{2}})$. An a priori estimate is obtained thanks to Ito formula applied to $\mathcal{H}$ and thanks to (1.4). This use of Ito formula is not rigorous since Au is not sufficiently smooth. However, an approximation argument can be used to prove rigorously this point. For instance, the initial data can be approximated by a sequence in $D(A)$ and it is classical that if the initial data is in $D(A)$ then the solution is continuous with values in $D(A)$.

Note that in the following and especially in section 4 and 5, several computations are not rigorous due to the lack of regularity of the solutions. The same approximation argument should be applied.

By classical arguments, the solutions are strong Markov processes. We denote by $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ the Markov transition semi-group associated to the solutions of (1.1).

Also, given a Banach space E, the space $Lip_b(E)$ consists of all the bounded and Lipschitz real valued functions on E. Its norm is given by

$$\|\varphi\|_L = \|\varphi\|_\infty + L_\varphi, \ \varphi \in Lip_b(E),$$

where $\|\cdot\|_\infty$ is the sup norm and L_φ is the Lipschitz constant of φ. The space of probability measures on E is denoted by $\mathcal{P}(E)$. It can be endowed with the metric defined by the total variation

$$\left\|\mu\right\|_{var} = \sup\left\{\left|\mu(\Gamma)\right| \mid \Gamma \in \mathcal{B}(E)\right\},$$

where we denote by $\mathcal{B}(E)$ the set of the Borelian subsets of E. It is well known that $\|.\|_{var}$ is the dual norm of $|.|_\infty$. We can also use a Wasserstein type metric

$$\|\mu-\lambda\|_W = \sup_{\varphi\in Lip_b(E),\ \|\varphi\|_L\leq 1}\left|\int_E \varphi(u)d(\mu-\lambda)(u)\right|$$

which is the dual norm of $\|\cdot\|_L$. We also use the notation $\mathcal{D}(Z)$ for the distribution of a random variable Z.

The aim of this article is to establish the following result

Theorem 1.1. *There exists N_0 such that, if (1.3) holds with $N_* \geq N_0$, then there exists a unique stationary probability measure ν of $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ on $H_0^1(0,1)$. Moreover, for any $p \in \mathbb{N}\backslash\{0\}$, ν satisfies*

$$\int_{H_0^1(0,1)} \|u\|^{2p}\, d\nu(u) < \infty, \tag{1.5}$$

and there exists $C_p > 0$ such that for any $\mu \in \mathcal{P}(H_0^1(0,1))$

$$\|\mathcal{P}_t^*\mu - \nu\|_W \leq C_p\,(1+t)^{-p}\left(1 + \int_{H_0^1(0,1)} \|u\|^2\, d\mu(u)\right). \tag{1.6}$$

Remark 1.2. *Note that the existence of a stationary measure is a byproduct of the proof of the mixing property. It could be proved directly by the standard argument involving the Krylov-Boboliubov theorem. However, this would require more a priori estimates on the solutions of the stochastic nonlinear Schrödinger equation.*

Remark 1.3. *In many articles and books, a family $(W_p(\cdot,\cdot))_{p\in[1,\infty)}$ of Wasserstein type metrics is used. Actually, given a polish space (E, d_E), these metrics are defined by*

$$W_p(\mu,\lambda) = \inf\left(\int_{E^2} d_E(x,y)^p\, \mathbb{P}(dx,dy)\right)^{\frac{1}{p}} \qquad \textit{for any} \quad p \in [1,\infty),$$

where the infimum is taken over all probability measures $\mathbb{P}$ on E^2 whose marginal laws are λ and μ.

Let $(E, \|\cdot\|_E)$ be a separable Banach space. We set $d_E(x,y) = \|x-y\|_E \wedge 1$. Then (E, d_E) is a polish space. Moreover, W_1 is equivalent to $\|\cdot\|_W$. We have chosen not to use the notation W_1 because it might lead to some confusion with the usual notation W for a Wiener process.

The proof of our result is based on coupling arguments. These arguments have initially been used in the context of stochastic partial differential equations in [19], [20], [21], [23] . The main difficulty here is that the nonlinear Schrödinger equation is not strongly dissipative and several modifications are needed.

The strategy is the following. If the noise is non-degenerate, we observe that starting from different initial data u_0^1, u_0^2, Girsanov transform can be used to show that there exists a coupling (u_1, u_2) of the law of the solutions $u(\cdot, u_0^1)$, $u(\cdot, u_0^2)$ such that, for some time T, $u_1(T) = u_2(T)$ with positive probability. Iterating this argument, exponential convergence to equilibrium follows (see section 1.1 in [26]). Here, as well as in the references above, the noise is assumed to be non-degenerate in the low modes only e_k, $1 \leq k \leq N_*$ and this argument gives a coupling such that $P_{N_*}u_1(T) = P_{N_*}u_2(T)$ with positive probability. Another ingredient is used. It is based on the observation that if two solutions are such that their low modes have been equal for a long time then they are very close (see section 1.1 in [26]). In the case of a parabolic equation, this is known as Foias-Prodi estimate. This can be generalized to dispersive equations such as the Schrödinger equation considered here. In [13] this has been used to prove a property of asymptotic smoothing in the deterministic case.

The main difference with the result in the parabolic case is that we are not able to prove a path-wise Foais-Prodi estimate, we only prove that this property holds in average. We need to introduce a substantial change in the construction of the coupling. (See Remark 2.12). Moreover, here we only get polynomial convergence to equilibrium. This comes from the fact that the Lyapunov functional adapted to the nonlinear Schrödinger equation is more complicated, it is not a quadratic functional. We are not able to get exponential estimates on the growth of the solutions.

2. Proof of Theorem 1.1

We define G by

$$D(G) = D(A), \quad Gv = \alpha v + \mathrm{i}Av,$$

and set

$$\begin{aligned}
&X = P_{N_*}u,\ Y = Q_{N_*}u,\ \beta = P_{N_*}W,\ \eta = Q_{N_*}W,\\
&\sigma_l = P_{N_*}bP_{N_*},\ \sigma_h = Q_{N_*}bQ_{N_*},\\
&f(X,Y) = -\mathrm{i}P_{N_*}\left(|X+Y|^2(X+Y)\right),\\
&g(X,Y) = -\mathrm{i}Q_{N_*}\left(|X+Y|^2(X+Y)\right).
\end{aligned}$$

Then the nonlinear Schrödinger equation has the form

$$\left\{\begin{aligned}
&dX + GXdt + f(X,Y)dt &=& \ \sigma_l d\beta,\\
&dY + GYdt + g(X,Y)dt &=& \ \sigma_h d\eta,\\
&X(0) = x_0, \quad Y(0) = y_0.
\end{aligned}\right. \tag{2.1}$$

Clearly (1.3) states that σ_l is invertible. We set

$$\sigma_0 = \|\sigma_l^{-1}\|^{-1}_{\mathcal{L}(P_{N_*}L^2(0,1))} > 0. \tag{2.2}$$

Given two initial data $u_0^i = (x_0^i, y_0^i)$, $i = 1,2$, we will construct a coupling $(u_1,u_2) = ((X_1,Y_1),(X_2,Y_2))$ of the laws of the two solutions $u(\cdot,u_0^i) = (X(\cdot,u_0^i), Y(\cdot,u_0^i))$, $i = 1,2$, of (2.1). Recall that (u_1,u_2) is a coupling of the laws of $u(\cdot,u_0^i)$, $i=1,2$, if the distribution of u_i is the distribution of $u(\cdot,u_0^i)$.

In fact we are going to construct a coupling $(V_1,V_2) = ((u_1,W_1),(u_2,W_2))$ of $\left\{\mathcal{D}\left((u(\cdot,u_0^i),W)\right)\right\}_{i=1,2}$ such that $X_i = P_{N_*}u_i$, $\eta_i = Q_{N_*}W_i$ satisfy good properties. More precisely, we want that $X_1 = X_2$ and $\eta_1 = \eta_2$ for as many trajectories as possible. Clearly, we obtain a coupling of $\mathcal{D}(u(\cdot,u_0^1))$ and $\mathcal{D}(u(\cdot,u_0^2))$. Since the noise may degenerate in the equation for Y, we are not able to require that $u_1 = u_2$. The difference between $Y_1 = Q_{N_*}u_1$ and $Y_2 = Q_{N_*}u_2$ will be controlled thanks to a Foias-Prodi estimate. Note that W is a cylindrical process in $L^2(0,1)$ and does not live in $L^2(0,1)$. This is not a problem. Indeed, it is well-known that $W \in C(\mathbb{R}^+; H^{-1}(0,1))$ a.s. and we consider its distribution in this space.

We define an integer valued random process l_0 which is particularly convenient when deriving properties of the coupling:

$$l_0(k) = \min\left\{l \in \{0,...,k\} \mid P_{l,k} \ \text{ holds }\right\},$$

where $\min\emptyset=\infty$ and

$$(P_{l,k})\begin{cases} X_1(t)=X_2(t), \quad \eta_1(t)=\eta_2(t), \quad \forall\, t\in[lT,kT],\\ \mathcal{H}_l\le d_0,\\ E_{u_i,4}(t,lT)\le \kappa+1+d_0^4+d_0^8+B(t-lT), \quad \forall\, t\in[lT,kT],\ i=1,2,\end{cases}$$

where we have set

$$\mathcal{H}_k=\mathcal{H}(u_1(kT))+\mathcal{H}(u_2(kT)).$$

We say that (X_1,X_2) are coupled at kT if $l_0(k)\le k$, in other words if $l_0(k)\neq\infty$.

The coupling constructed below will be such that, for any $q\in\mathbb{N}\backslash\{0\}$, the following two properties hold

$$(2.3)\qquad \begin{cases}\exists\, d_0,\ \kappa,\ B,\ T_q>0 \text{ such that for any } l\le k\ ,\ T\ge T_q,\\ \mathbb{P}\left(l_0(k+1)\neq l \mid l_0(k)=l\right)\le \frac12\left(1+(k-l)T\right)^{-q}.\end{cases}$$

This says that the probability that the trajectories decouple is small. Moreover, the longer they have been coupled, the smaller this probability is.

The second property is that, for any $R_0,d_0>0$,

$$(2.4)\qquad \begin{cases}\exists\, T^*(R_0,d_0)>0 \text{ and } p_{-1}(d_0)>0 \text{ such that for any } T\ge T^*(R_0,d_0)\\ \mathbb{P}\left(l_0(k+1)=k+1 \mid l_0(k)=\infty,\ \mathcal{H}_k\le R_0\right)\ge p_{-1}(d_0).\end{cases}$$

In other words, inside a ball, the probability that two trajectories get coupled is bounded below.

The construction can be done by induction. At each step, we construct a probability space $(\Omega_0,\mathcal{F}_0,\mathbb{P}_0)$ and a measurable couple of functions $(\omega_0,u_0^1,u_0^2)\to(V_i(\cdot,u_0^1,u_0^2))_{i=1,2}$ such that, for any (u_0^1,u_0^2), $(V_i(\cdot,u_0^1,u_0^2))_{i=1,2}$ is a coupling of $\left\{\mathcal{D}\left((u(\cdot,u_0^i),W)\right)\right\}_{i=1,2}$ on $[0,T]$. Recall that the processes $(V_i)_{i=1,2}$ live in the space $C(0,T;H_0^1(0,1))\times C(0,T;H^{-1}(0,1))$. We first set

$$u_i(0)=u_0^i, \quad W_i(0)=0, \quad i=1,2.$$

Assuming that we have built $(u_i,W_i)_{i=1,2}$ on $[0,kT]$, then we take $(V_i)_{i=1,2}$ as above independent of $(u_i,W_i)_{i=1,2}$ on $[0,kT]$ and set

$$(u_i(kT+t),W_i(kT+t))=V_i(t,u_1(kT),u_2(kT))$$

for any $t\in[0,T]$.

The construction of $(V_i)_{i=1,2}$ depends on whether $l_0(k)\le k$ or $l_0(k)=\infty$. The two cases are treated separately in sections 2.5. We first state and prove the Foias-Prodi estimates and give some a priori estimates. We then recall some results on coupling and give a general result implying polynomial mixing. Sections 3, 4 and 5 are devoted to the proof of some results used in the course of the proof.

2.1. **The Foias-Prodi estimates.** We define for any $(u_1,u_2,r)\in H_0^1(0,1)$

$$J_*(u_1,u_2,r)=\frac12\|r\|^2-\frac14\int_{[0,1]}\left(\left(|u_1|^2+|u_2|^2\right)|r|^2+\left(\Re((u_1+u_2)\bar r)\right)^2\right)dx,$$

where $\Re(z)$ is the real part of the complex number z, and

$$J(u_1,u_2,r) = J_*(u_1,u_2,r) + c_1\left(\sum_{i=1}^2 \mathcal{H}(u_i)\right)|r|^2.$$

We infer from the Sobolev Embedding $H^1(0,1)\subset L^\infty(0,1)$ that there exists $c>0$ such that

$$\int_{[0,1]}\left(\left(|u_1|^2+|u_2|^2\right)|r|^2+\left(\Re((u_1+u_2)\bar r)\right)^2\right)dx \le c(\|u_1\|^2+\|u_2\|^2)|r|^2.$$

Therefore, by (1.4), there exists $c_1>0$ such that

$$J(u_1,u_2,r)\ge \frac14\|r\|^2.$$

We set

$$l(u_1,u_2) = 1+\sum_{i=1}^2\mathcal{H}(u_i)^4.$$

For $N\ge 1$, given u_1,u_2, two solutions of (1.1), we define $J_{FP}^N = J_{FP}^N(u_1,u_2)$ by

$$J_{FP}^N(t) = J(u_1(t),u_2(t),r(t))\exp\left(2\alpha t-\frac{\Lambda}{\mu_{N+1}^{\frac18}}\int_0^t l(u_1,u_2)ds\right),$$

where $r=u_1-u_2$. The following result will be proved in section 5. It is the Foias-Prodi estimates adapted to the nonlinear Schrödinger equation. It states that two solutions having the same low modes are close. The main difference with similar results in the parabolic case is that we are not able to derive a path-wise estimate. Moreover, we introduce a slight generalization to allow the perturbation of the Wiener process by a drift in the low modes. This generalization is essential in our argument below.

Proposition 2.1. *For any $\kappa_0>0$, there exists $\Lambda>0$ depending only on κ_0 such that for any $N\in\mathbb{N}\backslash\{0\}$, we have the following property:*

Let W_1, W_2 be two cylindrical Wiener processes and h be an adapted process with continuous paths in $P_N L^2(0,1)$. Let u_1 be a solution in $C(0,T;H_0^1(0,1))$ of

$$\begin{cases} du_1+\alpha u_1\,dt+iAu_1\,dt-i|u_1|^2u_1\,dt &= bdW_1+hdt,\\ u_1(0) &= u_0^1,\end{cases}$$

and u_2 be the solution of (1.1), (1.2) for $u_0=u_0^2$ and $W=W_2$. Let τ be a stopping time and assume that

$$P_Nu_1=P_Nu_2,\quad Q_NW_1=Q_NW_2 \text{ on } [0,\tau], \tag{2.5}$$

and

$$\|h(t)\|^2\le\kappa_0 l(u_1(t),u_2(t))^{3/4} \text{ on } [0,\tau], \tag{2.6}$$

then we have

$$\mathbb{E}\left(J_{FP}^N(u_1,u_2)(t\wedge\tau)\right)\le J(u_0^1,u_0^2,r_0),\ t>0, \tag{2.7}$$

where $r_0=u_0^1-u_0^2$.

We deduce a very useful Corollary.

Corollary 2.2. *For any B, d_0, $\kappa_0 > 0$, there exists $N_1(B,\kappa_0)$ and $C^*(d_0)$ such that, with the notations of Proposition 2.1, if* (2.5) *and* (2.6) *hold with $N \geq N_1$, and for some $\rho > 0$,*

$$E_{u_i,4}(t) \leq \rho + 1 + d_0^4 + d_0^8 + Bt \text{ on } [0,\tau], \text{ for } i = 1,2, \tag{2.8}$$

then for any u_0^1, u_0^2 such that $d_0 \geq \sum_{i=1}^2 \mathcal{H}(u_0^i)$ and for any $a \in \mathbb{R}$,

$$\mathbb{P}\left(\|r(T)\| > C^*\left(d_0\right)\exp\left(a - \frac{\alpha}{4}T + \rho\right) \text{ and } T \leq \tau\right) \leq \exp\left(-a - \frac{\alpha}{4}T\right).$$

Moreover, there exists $c > 0$ such that

$$C^*(d_0) \leq c d_0 e^{c d_0^8}.$$

Then, integrating (2.7) in Proposition 2.1 and applying the inequality

$$1 + x \leq C_\delta e^{\delta x} \quad \text{for any } x \geq 0,$$

we obtain the following result which, in Section 3, ensures that the Novikov condition holds and allows the use of the Girsanov Formula.

Lemma 2.3. *For any B, d_0, $\kappa_0 > 0$ and any $a \in \mathbb{R}$, there exists $N_2(B,\kappa_0,a)$ and $C^{**}(d_0,B)$ such that, with the notations of Proposition 2.1, if* (2.5) *and* (2.6) *hold with $N \geq N_2$ and* (2.8) *holds for some $\rho > 0$, we obtain that for any T*

$$\mathbb{P}\left(\int_T^\tau l(u_1(s),u_2(s))\,\|r(s)\|^2\,ds > C^{**}(d_0,B)\exp\left(a - \tfrac{\alpha}{2}T + \rho\right) \text{ and } T \leq \tau\right)$$
$$\leq \exp\left(-a - \tfrac{\alpha}{2}T\right).$$

provided $d_0 \geq \sum_{i=1}^2 \mathcal{H}(u_0^i)$ holds. Moreover, there exists $c > 0$ such that

$$C^{**}(d_0,B) \leq C(B) d_0 e^{c d_0^8}.$$

We set

$$N_0 = \max(N_1,N_2). \tag{2.9}$$

2.2. **A priori estimates.** We first give an estimate proven in section 4 on the growth of the solutions of the stochastic nonlinear Schrödinger equation.

Proposition 2.4. *Assume that u is a solution of* (1.1), (1.2) *associated with a Wiener process W. Then, for any $(k,p) \in (\mathbb{N}\backslash\{0\})^2$, there exists C'_k and $K_{k,p}$ depending only on k and p such that for any $\rho > 0$ and $0 \leq T < \infty$*

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,T]}\left(E_{u,k}(t) - C'_k t\right) \geq \mathcal{H}(u_0)^k + \rho\left(\mathcal{H}(u_0)^{2k} + T\right)\right) &\leq K_{k,p}\rho^{-p},\\
\mathbb{P}\left(\sup_{t\in[T,\infty)}\left(E_{u,k}(t) - C'_k t\right) \geq \mathcal{H}(u_0)^k + \mathcal{H}(u_0)^{2k} + 1 + \rho\right) &\leq K_{k,p}\left(\rho + T\right)^{-p}.
\end{aligned}$$

The following result uses the Hamiltonian as a Lyapunov functional and is also proven in section 4.

Lemma 2.5. *There exists $C_k > 0$ such that for any $k \in \mathbb{N}\backslash\{0\}$, for any $t \in \mathbb{R}^+$ and for any stopping time τ*

$$\begin{cases}
\mathbb{E}\left(\mathcal{H}(u(t,u_0))^k\right) & \leq \mathcal{H}(u_0)^k e^{-\alpha k t} + \frac{C_k}{2},\\
\mathbb{E}\left(\mathcal{H}(u(\tau,u_0))^k 1_{\tau<\infty}\right) & \leq \mathcal{H}(u_0)^k + C_k \mathbb{E}(\tau 1_{\tau<\infty}).
\end{cases}$$

The following result states that we control the probability of entering a small ball.

Proposition 2.6. *For any* $R_0, R_1 > 0$*, there exists* $T_{-1}(R_0, R_1) \geq 0$ *and* $\pi_{-1}(R_1) > 0$ *such that*

$$\mathbb{P}\left(\mathcal{H}(u(t,u_0^1)) + \mathcal{H}(u(t,u_0^2)) \leq R_1\right) \geq \pi_{-1}(R_1),$$

provided $\mathcal{H}(u_0^1) + \mathcal{H}(u_0^2) \leq R_0$ *and* $t \geq T_{-1}(R_0, R_1)$.

2.3. **Basic properties of couplings.**

Let (μ_1, μ_2) be two distributions on a same space $(E, \mathcal{E})$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (Z_1, Z_2) be two random variables $(\Omega, \mathcal{F}) \to (E, \mathcal{E})$. Recall that (Z_1, Z_2) is a coupling of (μ_1, μ_2) if $\mu_i = \mathcal{D}(Z_i)$ for $i = 1, 2$ and that we have denoted by $\mathcal{D}(Z_i)$ the law of the random variable Z_i.

Let μ, μ_1 and μ_2 be three probability measures on a space $(E, \mathcal{E})$ such that μ_1 and μ_2 are absolutely continuous with respect to μ. We set

$$d(\mu_1 \wedge \mu_2) = (\frac{d\mu_1}{d\mu} \wedge \frac{d\mu_2}{d\mu})d\mu.$$

This definition does not depend on the choice of μ and we have

$$\|\mu_1 - \mu_2\|_{var} = \frac{1}{2}\int_E \left|\frac{d\mu_1}{d\mu} - \frac{d\mu_2}{d\mu}\right| d\mu.$$

Remark that if μ_1 is absolutely continuous with respect to μ_2, we have

$$\|\mu_1 - \mu_2\|_{var} \leq \frac{1}{2}\sqrt{\int \left(\frac{d\mu_1}{d\mu_2}\right)^2 d\mu_2 - 1}. \tag{2.10}$$

Next result is a fundamental result in the coupling methods, the proof is given for instance in the Appendix of [26].

Lemma 2.7. *Let* (μ_1, μ_2) *be two probability measures on* $(E, \mathcal{E})$. *Then*

$$\|\mu_1 - \mu_2\|_{var} = \min \mathbb{P}(Z_1 \neq Z_2).$$

The minimum is taken over all couplings (Z_1, Z_2) *of* (μ_1, μ_2). *There exists a coupling which reaches the minimum value. It is called a maximal coupling and has the following property:*

$$\mathbb{P}(Z_1 = Z_2, Z_1 \in \Gamma) = (\mu_1 \wedge \mu_2)(\Gamma) \quad \textit{for any } \Gamma \in \mathcal{E}.$$

Next result is a refinement of Lemma 2.7 used in [23] (see also Proposition 1.7 in [26]).

Proposition 2.8. *Let* E *and* F *be two Polish spaces,* $f_0 : E \to F$ *be a measurable map and* (μ_1, μ_2) *be two probability measures on* E. *We set*

$$\nu_i = f_0^* \mu_i, \quad i = 1, 2.$$

Then there exist a coupling (V_1, V_2) *of* (μ_1, μ_2) *such that* $(f_0(V_1), f_0(V_2))$ *is a maximal coupling of* (ν_1, ν_2).

2.4. **Sufficient conditions for polynomial mixing.**

We now state and prove a general result which allows to reduce the proof of polynomial convergence to equilibrium to the verification of some conditions. This result is a polynomial version of Theorem 1.8 of subsection 1.3 in [26] which gives sufficient conditions for exponential mixing.

We are concerned with $v(\cdot,(u_0,W_0)) = (u(\cdot,u_0),W(\cdot,W_0))$ a couple of strongly Markov processes defined on Polish spaces (E,d_E) and (F,d_F). We denote by $(\mathcal{P}_t)_{t\in I}$ the Markov transition semigroup associated to u, where $I=\mathbb{R}^+$ or $T\mathbb{N}=\{kT,\ k\in\mathbb{N}\}$. We are also given a real valued function $\mathcal{H}$ defined on E.

We consider for any couple of initial conditions (v_0^1,v_0^2) a coupling (v_1,v_2) of $\left\{\mathcal{D}(v(\cdot,v_0^1)),\mathcal{D}(v(\cdot,v_0^2))\right\}$. We write $v_i=(u_i,W_i)$. Let $l_0:\mathbb{N}\to\mathbb{N}\cup\{\infty\}$ be a random integer valued process which has the following properties

$$\begin{cases} l_0(k+1)=l \text{ implies } l_0(k)=l, \text{ for any } l\le k,\\ l_0(k)\in\{0,1,2,...,k\}\cup\{\infty\},\\ l_0(k) \text{ depends only of } v_1|_{[0,kT]} \text{ and } v_2|_{[0,kT]},\\ l_0(k)=k \ \text{ implies } \mathcal{H}_k\le d_0, \end{cases} \tag{2.11}$$

where

$$\mathcal{H}_k=\mathcal{H}(u_1(kT))+\mathcal{H}(u_2(kT)),\quad \mathcal{H}:\,E\to\mathbb{R}^+,$$

and $d_0>0$.

We now give four conditions on the coupling. The first condition states that when (v_1,v_2) have been coupled for a long time then the probability that (u_1,u_2) are close is high. It will be a consequence of the Foias-Prodi estimate.

$$\begin{cases} \text{There exist } c_0 \text{ and } q>0 \text{ such that for any } t\in[lT,kT]\cap I\\ \mathbb{P}\left(d_E(u_1(t),u_2(t))>c_0\,(t-lT)^{-q} \text{ and } l_0(k)\le l\right)\le c_0\,(t-lT)^{-q}, \end{cases} \tag{2.12}$$

The next two properties are exactly (2.3) and (2.4).

$$\begin{cases} \exists\,(p_k)_{k\in\mathbb{N}},\ c_1>0,\ q_0>1+q \ \text{ such that,}\\ \mathbb{P}\left(l_0(k+1)=l\mid l_0(k)=l\right)\ge p_{k-l}, \text{ for any } l\le k,\\ 1-p_k\le c_1\left((k+1)T\right)^{-q_0},\ p_k>0 \ \text{ for any } k\in\mathbb{N}. \end{cases} \tag{2.13}$$

$$\begin{cases} \text{There exist } p_{-1}>0,\ R_0>0 \text{ such that}\\ \mathbb{P}\left(l_0(k+1)=k+1\mid l_0(k)=\infty,\ \mathcal{H}_k\le R_0\right)\ge p_{-1}. \end{cases} \tag{2.14}$$

The last ingredient is the so-called Lyapunov structure and follows from Lemma 2.5. It allows the control of the probability to enter the ball of radius R_0. It states that there exist K_1 and K' constants such that for any initial data v_0 and any stopping time τ' taking values in $\{kT,\ k\in\mathbb{N}\}\cup\{\infty\}$

$$(2.15)\qquad \begin{cases} \mathbb{E}\mathcal{H}(v(t,v_0)) & \leq \ e^{-\alpha t}\mathcal{H}(v_0)+\frac{K_1}{2}, \quad t\geq 0,\\ \mathbb{E}\left(\mathcal{H}(v(\tau',v_0))1_{\tau'<\infty}\right) & \leq \ K'\left(\mathcal{H}(v_0)+\mathbb{E}(\tau' 1_{\tau'<\infty})\right). \end{cases}$$

The process $V=(v_1,v_2)$ is said to be l_0–Markov if the laws of $V(kT+\cdot)$ and of $l_0(k+\cdot)-k$ on $\{l_0(k)\in\{k,\infty\}\}$ conditioned by $\mathcal{F}_{kT}$ only depend on $V(kT)$ and are equal to the laws of $V(\cdot,V(kT))$ and l_0, respectively.

In this article, we construct a coupling $(u_i,W_i)_{i=1,2}$ of two solutions which is l_0–Markov but not Markov. We could modify the construction so that it is Markov at discrete times $T\mathbb{N}=\{kT,\ k\in\mathbb{N}\}$. However, it does not seem to be possible to modify the coupling to be Markov at any times. As shown below, the following result implies Theorem 1.1. Its proof is given in section 3.

Theorem 2.9. *Assume that for any (u_0^1,W_0^1), (u_0^2,W_0^2) there exists a coupling $V=(v_1,v_2)$ of the laws of $(u(\cdot,u_0^1),W(\cdot,W_0^1))$ and $(u(\cdot,u_0^2),W(\cdot,W_0^2))$ which is l_0–Markov and satisfies (2.11), (2.12), (2.13), (2.14) and (2.15) with $R_0>4K_1$ and $R_0\geq d_0$. Then there exists $c_4>0$ such that, for any $\varphi\in Lip_b(E)$ and any $u_0^1,\ u_0^2\in E$,*

$$(2.16)\qquad \left|\mathbb{E}\varphi(u(t,u_0^1))-\mathbb{E}\varphi(u(t,u_0^2))\right|\leq c_4\,(1+t)^{-q}\,\|\varphi\|_L(1+\mathcal{H}(u_0^1)+\mathcal{H}(u_0^2)).$$

Corollary 2.10. *Under the same assumptions as Theorem 2.9, there exists a unique stationary probability measure ν of $(\mathcal{P}_t)_{t\in I}$ on E. It satisfies,*

$$(2.17)\qquad \int_E \mathcal{H}(u)d\nu(u)\leq\frac{K_1}{2}.$$

Moreover for any $\mu\in\mathcal{P}(E)$

$$(2.18)\qquad \|\mathcal{P}_t^*\mu-\nu\|_W\leq 2c_4\,(1+t)^{-q}\left(1+\int_E\mathcal{H}(u)d\mu(u)\right).$$

To prove Theorem 2.9, we first note that it is sufficient to prove that, for any initial data u_0^1 and u_0^2, the coupling satisfies

$$(2.19)\qquad \mathbb{P}\left(d_E(u_1(t),u_2(t))>c_3\,(1+t)^{-q}\right)\leq c_3\,(1+t)^{-q}\left(1+\mathcal{H}(u_0^1)+\mathcal{H}(u_0^2)\right)$$

where, as above, $v_i=(u_i,W_i)$. Indeed we have, since (u_1,u_2) is a coupling of $\left\{\mathcal{D}(u(\cdot,u_0^1)),\mathcal{D}(u(\cdot,u_0^2))\right\}$,

$$\begin{aligned}
&|\mathbb{E}\varphi(u(t,u_0^1))-\mathbb{E}\varphi(u(t,u_0^2))|=|\mathbb{E}\varphi(u_1(t))-\mathbb{E}(\varphi(u_2(t)))|\\
&\leq L_\varphi c_3(1+t)^{-q}+2\|\varphi\|_\infty\mathbb{P}\left(d_E(u_1(t),u_2(t))>c_3\,(1+t)^{-q}\right)\\
&\leq L_\varphi c_3(1+t)^{-q}+2\|\varphi\|_\infty c_3\,(1+t)^{-q}\left(1+\mathcal{H}(u_0^1)+\mathcal{H}(u_0^2)\right)
\end{aligned}$$

so that (2.16) follows. The existence and uniqueness of a stationary measure is then straightforward. Moreover, (2.18) is an easy consequence of (2.16) and (2.17) follows from (2.15).

2.5. **Construction of the coupling.** We first state the following result.

Proposition 2.11. *There exists a measurable map*

$$\Phi : C((0,T);P_{N_*}H_0^1(0,1))\times C((0,T);Q_{N_*}H^{-1}(0,1))\times H_0^1(0,1) \to C((0,T);Q_{N_*}H_0^1(0,1)),$$

such that for any (u,W) *solution of* (1.1) *and* (1.2)

$$Y=\Phi(X,\eta,u_0) \quad \text{on } [0,T], \text{ where } X=P_{N_*}u,\ Y=Q_{N_*}u,\ \eta=Q_{N_*}W.$$

Moreover Φ *is a non-anticipative functions of* (X,η).

To prove this result, we note that the equation

$$y(t)=S(t)y_0-\int_0^t S(t-s)g(x(s),y(s))ds+\int_0^t S(t-s)dz(s),$$

can be solved by a fixed point argument in $C(0,T;H_0^1(0,1))$ for any deterministic functions $x\in C(0,T;P_{N_*}H_0^1(0,1))$ and $z\in C(0,T;D(A^{\frac32}))$. The last term is defined thanks to an integration by part. Clearly $y=\Psi(x,z,y_0)$ for a measurable function Ψ. Thus $Y=\Psi(X,\sigma_h\eta,Q_{N_*}u_0)$. We set $\Phi(x,\widetilde z,u_0)=\Psi(x,\sigma_h\widetilde z,Q_{N_*}u_0)$ for $\widetilde z$ such that $\sigma_h\widetilde z\in C(0,T;D(A^{\frac32}))$ and 0 otherwise. It is clear that Φ is not anticipative.

As already explained, the coupling (u_1,u_2) is constructed by induction and we start by constructing a coupling for two solutions $u(\cdot,u_0^i),\ i=1,2$ on an interval $[0,T]$. In fact, we construct three different couplings. At time kT, we choose between these depending on whether $l_0(k)=\infty$ and $\mathcal H(u_1(kT))+\mathcal H(u_2(kT))\le R_0$ (case a) or $l_0(k)\le k$ (case b). In this latter case, $P_{N_*}u_1(kT)=P_{N_*}u_2(kT)$. In the third case, $l_0(k)=\infty$ and $\mathcal H(u_0^1)+\mathcal H(u_0^2)>R_0$, we choose the trivial coupling.

Case a: $l_0(k)=\infty$ and $\mathcal H(u_0^1)+\mathcal H(u_0^2)\le R_0$. We construct a coupling such that (2.4) holds.

In this case, we consider $u_0^1,\ u_0^2$ such that $\mathcal H(u_0^1)+\mathcal H(u_0^2)\le R_0$. The construction of the coupling is done in two steps. We set

$$\mu_i=\mathcal D\left((u(\cdot,u_0^i),W)\right), \quad \text{on } [0,T_1], \quad i=1,2.$$

Step 1:

We first prove that, for any $d_0>0$, there exist $T_1(d_0)>0$, $R_1=R_1(d_0)>0$ and a coupling $(\widetilde V_i(\cdot,u_0^1,u_0^2))_{i=1,2}$ of (μ_1,μ_2) such that for any (u_0^1,u_0^2) satisfying $\sum_{i=1}^2\mathcal H(u_0^i)\le R_1$ we have

$$\mathbb P\left(\widetilde X_1(T_1,u_0^1,u_0^2)=\widetilde X_2(T_1,u_0^1,u_0^2),\ \sum_{i=1}^2\mathcal H(\widetilde u_i(T_1,u_0^1,u_0^2))\le d_0\right)\ge\frac12, \tag{2.20}$$

where

$$\widetilde V_i(\cdot,u_0^1,u_0^2)=\left(\widetilde u_i(\cdot,u_0^1,u_0^2),\widetilde W_i(\cdot,u_0^1,u_0^2)\right),\ \widetilde X_i(\cdot,u_0^1,u_0^2)=P_{N_*}\widetilde u_i(\cdot,u_0^1,u_0^2),\ i=1,2.$$

To construct $\widetilde{V}_i$ such that (2.20) holds, we take $R_1, T_1 > 0$ and we set

$$\begin{aligned}
&E = C((0,T); H_0^1(0,1)) \times C((0,T); H^{-1}(0,1)),\\
&F = C((0,T); P_{N_*}H_0^1(0,1)) \times C((0,T); Q_{N_*}H^{-1}(0,1)),\\
&f_0\,(u,W) = (P_{N_*}u, Q_{N_*}W) = (X,\eta),\\
&\hat{\mu}_1 = \mathcal{D}\left((u(\cdot,u_0^1) + \tfrac{T_1-\cdot}{T_1}P_{N_*}(u_0^2-u_0^1), W)\right) \text{ on } [0,T_1],\\
&\nu_i = f_0^*\mu_i, \quad \hat{\nu}_1 = f_0^*\hat{\mu}_1.
\end{aligned}$$

We apply Proposition 2.8 to $(E, F, f_0, (\hat{\mu}_1, \mu_2))$ and obtain $(\hat{V}_1(\cdot,u_0^1,u_0^2), \widetilde{V}_2(\cdot,u_0^1,u_0^2))$ a coupling of $(\hat{\mu}_1,\mu_2)$. Moreover, setting

$$(\widetilde{X}_2, \widetilde{\eta}_2) = f_0(\widetilde{V}_2(\cdot,u_0^1,u_0^2)),\ (\hat{X}_1,\eta_1) = f_0(\hat{V}_1(\cdot,u_0^1,u_0^2)),$$

$((\widetilde{X}_2,\widetilde{\eta}_2),(\hat{X}_1,\eta_1))$ is a maximal coupling of $(\hat{\nu}_1,\nu_2)$.

Finally, we set

$$\widetilde{V}_1 = \left(\hat{u}_1 - \frac{T_1-\cdot}{T_1}P_{N_*}(u_0^2-u_0^1), W_1\right) \text{ on } [0,T_1], \text{ where } \hat{V}_1 = (\hat{u}_1, W_1)\,.$$

We also write

$$\beta_1 = P_{N_*}W_1,\ \widetilde{V}_1 = (\widetilde{u}_1, W_1)\,,\ \ \widetilde{V}_2 = (\widetilde{u}_2, W_2)\,.$$

To prove (2.20) we first remark that since $\hat{u}_1(T_1) = \widetilde{u}_1(T_1)$ and $\hat{X}_1 = P_{N_*}\hat{u}_1$, $\widetilde{X}_i = P_{N_*}\widetilde{u}_i$, then

$$\begin{aligned}
&\mathbb{P}\left(\widetilde{X}_1(T_1) = \widetilde{X}_2(T_1) \text{ and } \textstyle\sum_{i=1}^2 \mathcal{H}(\widetilde{u}_i(T_1))^6 \le \kappa'(\rho,T_1,R_1)\right)\\
&\ge \mathbb{P}\left(\hat{X}_1 = \widetilde{X}_2 \text{ on } [0,T_1] \text{ and } \textstyle\sum_{i=1}^2 E_{\widetilde{u}_i,6}(t) \le \kappa'(\rho,t,R_1) \text{ on } [0,T_1]\right),
\end{aligned} \tag{2.21}$$

where

$$\kappa'(\rho,t,R_1) = 2\left(R_1^6 + C_6't + \rho(R_1^{12}+t)\right),\ t>0,$$

ρ to be chosen below.

Let us consider $\bar{X}_1$ the unique solution of

$$\begin{cases} d\bar{X}_1 + G\bar{X}_1dt - \delta(t) + 1_{t\le\tau}f(\bar{X}_1 - \hat{\delta}, \Phi(\bar{X}_1-\hat{\delta},\eta_1,u_0^1))dt = \sigma_l d\beta_1,\\ \bar{X}_1(0) = x_0^2, \end{cases} \tag{2.22}$$

where $\delta(t) = \left(\frac{T_1-t}{T_1} - \frac{1}{T_1}G\right)P_{N_*}(u_0^2-u_0^1)$, $\hat{\delta}(t) = \frac{T_1-t}{T_1}P_{N_*}(u_0^2-u_0^1)$, and $\tau = \tau_1\wedge\tau_2$ where

$$\begin{cases} \tau_1 = \inf\left\{t\in[0,T_1] \mid E_{\bar{X}_1-\hat{\delta}+\Phi(\bar{X}_1-\hat{\delta},\eta_1,u_0^1),6}(t) > \kappa'(\rho,t,R_1)\right\},\\ \tau_2 = \inf\left\{t\in[0,T_1] \mid E_{\bar{X}_1+\Phi(\bar{X}_1,\eta_1,u_0^2),6}(t) > \kappa'(\rho,t,R_1)\right\}. \end{cases}$$

Clearly, $\bar{X}_1 = \hat{X}_1 = P_{N_*}\widetilde{u}_1 + \hat{\delta}$ on $[0,\tau]$. We denote by λ_1 the distribution of $(\bar{X}_1,\eta_1)$ under the probability $\mathbb{P}$. We set $\widetilde{\beta}_1(t) = \beta_1(t) + \int_0^t d(s)dt$ where

$$d(t) = \delta(t) + 1_{t\le\tau}\sigma_l^{-1}\begin{pmatrix} f(\bar{X}_1(t)-\hat{\delta}(t), \Phi(\bar{X}_1-\hat{\delta},\eta_1,u_0^1)(t))\\ -f(\bar{X}_1(t),\Phi(\bar{X}_1,\eta_1,u_0^2)(t))\end{pmatrix}. \tag{2.23}$$

Then $\bar{X}_1$ is a solution of

$$\left\{\begin{array}{lcl} d\bar{X}_1 + G\bar{X}_1 dt + 1_{t\leq\tau} f(\bar{X}_1, \Phi(\bar{X}_1, \eta_1, u_0^2))dt & = & \sigma_l d\widetilde{\beta}_1, \\ \bar{X}_1(0) = x_0^2. & & \end{array}\right. \tag{2.24}$$

It is not difficult to see that since σ_l is bounded below and by the definition of τ, the Novikov condition is satisfied:

$$\mathbb{E}\left(\exp\left(\int_0^T |d(t)|^2 dt\right)\right) < \infty$$

and the Girsanov formula can be applied. Then we set

$$d\widetilde{\mathbb{P}} = \exp\left(\int_0^T d(s)dW(s) - \frac{1}{2}\int_0^T |d(s)|^2\, dt\right) d\mathbb{P}$$

and deduce that $\widetilde{\mathbb{P}}$ is a probability under which $(\widetilde{\beta}_1, \eta_1)$ is a cylindrical Wiener process. We denote by λ_2 the law of $(\bar{X}_1, \eta_1)$ under $\widetilde{\mathbb{P}}$.

We prove below that

(2.25)

$$\begin{array}{l} \mathbb{P}\left(\hat{X}_1(t) \neq \widetilde{X}_2(t) \text{ or } \sum_{i=1}^2 E_{\widetilde{u}_i,6}(t) > \kappa'(\rho, t, R_1) \text{ for some } t < T_1\right) \\ \leq \|\lambda_1 - \lambda_2\|_{var} + \mathbb{P}\left(E_{\widetilde{u}_1,6}(\tau) \geq \frac{1}{2}\kappa'(\rho,\tau,R_1)\right) + \mathbb{P}\left(E_{\widetilde{u}_2,6}(\tau) \geq \frac{1}{2}\kappa'(\rho,\tau,R_1)\right) \end{array}$$

We choose

$$\rho = 8K_{6,1}$$

in the definition of $\kappa'(\rho, t, R_1)$ and deduce from Proposition 2.4 that

$$\mathbb{P}\left(E_{\widetilde{u}_1,6}(\tau) \geq \frac{1}{2}\kappa'(\rho,\tau,R_1)\right) + \mathbb{P}\left(E_{\widetilde{u}_2,6}(\tau) \geq \frac{1}{2}\kappa'(\rho,\tau,R_1)\right) \leq \frac{1}{4}. \tag{2.26}$$

Moreover using (2.10), we obtain

$$\|\lambda_1 - \lambda_2\|_{var} \leq \frac{1}{2}\sqrt{\mathbb{E}\exp\left(c\int_0^T |d(s)|^2\, dt\right) - 1},$$

and then, for $T_1,\ R_1$ sufficiently small,

$$\|\lambda_1 - \lambda_2\|_{var} \leq 2(R_1\,(T_1+1)\,(1+R_1^2) + \kappa'(\rho, T_1, R_1)).$$

We choose

$$T_1 = R_1,$$

and deduce

$$\|\lambda_1 - \lambda_2\|_{var} \leq cR_1\left(1 + R_1^{11}\right). \tag{2.27}$$

Taking into account (2.21), (2.25), (2.26) and (2.27), we can choose $R_1^0 > 0$ sufficiently small such that for any $R_1 \leq R_1^0$

$$\mathbb{P}\left(\widetilde{X}_1(T_1) = \widetilde{X}_2(T_1) \text{ and } \sum_{i=1}^2 \mathcal{H}(\widetilde{u}_i(T_1))^6 \leq \kappa'(\rho, R_1, R_1)\right) \geq \frac{1}{2}. \tag{2.28}$$

Remark that there exists $R_1(d_0) \in (0, R_1^0)$ such that $R_1 \leq R_1(d_0)$ implies

$$\left\{\sum_{i=1}^{2} \mathcal{H}(\widetilde{u}_i(T_1))^6 \leq \kappa'(\rho, R_1, R_1)\right\} \subset \left\{\sum_{i=1}^{2} \mathcal{H}(\widetilde{u}_i(T_1)) \leq d_0\right\},$$

so that (2.20) follows.

It remains to prove (2.25). We write

$$\begin{aligned}
&\mathbb{P}\left(\hat{X}_1(t) \neq \widetilde{X}_2(t) \text{ or } \textstyle\sum_{i=1}^2 E_{\widetilde{u}_i,6}(t) > \kappa'(\rho, t, R_1) \text{ for some } t < T_1\right) \\
&= \mathbb{P}\left(\hat{X}_1|_{[0,\tau]} \neq \widetilde{X}_2|_{[0,\tau]} \text{ or } \textstyle\sum_{i=1}^2 E_{\widetilde{u}_i,6}(\tau) = \kappa'(\rho, \tau, R_1)\right) \\
&\leq \mathbb{P}\left(\hat{X}_1|_{[0,\tau]} \neq \widetilde{X}_2|_{[0,\tau]}\right) + \mathbb{P}\left(E_{\widetilde{u}_1,6}(\tau) \geq \tfrac{1}{2}\kappa'(\rho, \tau, R_1)\right) \\
&+\mathbb{P}\left(E_{\widetilde{u}_2,6}(\tau) \geq \tfrac{1}{2}\kappa'(\rho, \tau, R_1)\right).
\end{aligned}$$

Let $\bar{X}_2$ is the solution of equation (2.24) where β_1 is replaced by $\beta_2 = P_{N_*} W_2$ then, with the probability $\mathbb{P}$, $\bar{X}_2$ has the same law as $\bar{X}_1$ under the probability $\widetilde{\mathbb{P}}$ and

$$\mathbb{P}\left(P_{N_*}\hat{u}_1|_{[0,\tau]} \neq P_{N_*}\widetilde{u}_2|_{[0,\tau]}\right) \leq \mathbb{P}(\bar{X}_1 \neq \bar{X}_2).$$

Thus, (2.25) would follow if $((\bar{X}_1, \eta_1), (\bar{X}_2, \eta_2))$ was a maximal coupling of (λ_1, λ_2) (here, we have set $\eta_2 = Q_{N_*} W_2$). However, we only know that $((\hat{X}_1, \eta_1), (\widetilde{X}_2, \tilde{\eta}_2))$ is a maximal coupling of $(\hat{\nu}_1, \nu_2)$. It is not difficult to remedy this problem. Indeed, the above result holds for any coupling of $(\hat{\nu}_1, \nu_2)$. Thus, instead of $((\hat{X}_1, \eta_1), (\widetilde{X}_2, \tilde{\eta}_2))$, we choose another coupling such that the processes constructed as $((\bar{X}_1, \eta_1), (\bar{X}_2, \eta_2))$ above is a maximal coupling of (λ_1, λ_2). Then, the right hand side is equal to the right hand side of (2.25) while, by Lemma 2.7, the left hand side is larger than the left hand side of (2.25).

Step 2: Construction of the coupling under the assumptions of case a.

Thanks to Proposition 2.6, we know that there exists $T_{-1}(R_0, R_1) > 0$ and $\pi_{-1}(R_1) > 0$ such that

$$\mathbb{P}\left(\sum_{i=1}^{2} \mathcal{H}(u(\theta, u_0^i)) \leq R_1\right) \geq \pi_{-1}(R_1), \tag{2.29}$$

provided $\sum_{i=1}^2 \mathcal{H}(u_0^i) \leq R_0$ and $\theta \geq T_{-1}(R_0, R_1)$.

We set $T^*(R_0, d_0) = T_{-1}(R_0, R_1(d_0)) + T_1(d_0)$ and assume that $T \geq T^*(R_0, d_0)$. We also write $\theta = T - T_1$. Then on $[0, \theta]$, we take the trivial coupling which we denote by (V_1', V_2'). Finally, we consider $(\widetilde{V}_1, \widetilde{V}_2)$ as above independent of (V_1', V_2') and we set

$$V_i^a(t, u_0^1, u_0^2) = \begin{cases} V_i'(t, u_0^1, u_0^2) & \text{if } t \leq \theta, \\ \widetilde{V}_i(t - \theta, V_1'(\theta, u_0^1, u_0^2), V_2'(\theta, u_0^1, u_0^2)) & \text{if } t \geq \theta. \end{cases}$$

Combining (2.20) and (2.29) and setting

$$p_{-1}(d_0) = \frac{1}{2}\pi_{-1}(R_1(d_0)),$$

we obtain, for any $u_0^1,\ u_0^2$ such that $\mathcal{H}(u_0^1)+\mathcal{H}(u_0^2)\le R_0$,

$$\mathbb{P}\left(X_1^a(T,u_0^1,u_0^2)=X_2^a(T,u_0^1,u_0^2),\ \sum_{i=1}^2\mathcal{H}(u_i^a(T,u_0^1,u_0^2))\le d_0\right)\ge p_{-1}(d_0), \tag{2.30}$$

where now

$V_i^a(\cdot,u_0^1,u_0^2)=\left(u_i^a(\cdot,u_0^1,u_0^2),W_i^a(\cdot,u_0^1,u_0^2)\right),\ X_i^a(\cdot,u_0^1,u_0^2)=P_{N_*}u_i^a(\cdot,u_0^1,u_0^2),\ i=1,2.$

Clearly, (2.30) implies (2.4).

Case b: $l_0(k)\le k$. We now construct a coupling so that (2.3) holds. Since (2.3) depends on the whole history of the coupling and not only on the latest step, (2.3) is proved afterwards when the coupling is constructed on $[0,\infty)$.

In this case, we have $P_{N_*}u_0^1=P_{N_*}u_0^2$. We write $x=P_{N_*}u_0^1=P_{N_*}u_0^2$, $y_1=Q_{N_*}u_0^1$ and $y_2=Q_{N_*}u_0^2$.

We apply Proposition 2.8 to

$$E=C((0,T);H_0^1(0,1))\times C((0,T);H^{-1}(0,1)),$$

$$F=C((0,T);P_{N_*}H_0^1(0,1))\times C((0,T);Q_{N_*}H^{-1}(0,1)),$$

$$f_0\left(u,W\right)=(P_{N_*}u,Q_{N_*}W)=(X,\eta),$$

$$\mu_i=\mathcal{D}\left((u(\cdot,u_0^i),W)\right),\qquad\text{on }[0,T].$$

We set $\nu_i=f_0^*\mu_i=\mathcal{D}\left((X(\cdot,u_0^i),\eta)\right)$ on $[0,T]$. We obtain $(V_i^b(\cdot,u_0^1,u_0^2))_{i=1,2}=(u_i^b(\cdot,u_0^1,u_0^2),W_i^b(\cdot,u_0^1,u_0^2))_{i=1,2}$, a coupling of (μ_1,μ_2) such that if we set

$$(X_i^b,\eta_i^b)=f_0(V_i^b),\quad i=1,2.$$

Then $(X_i^b,\eta_i^b)(\cdot,u_0^1,u_0^2))_{i=1,2}$ is a maximal coupling of (ν_1,ν_2). We define $Y_i^b=Q_{N_*}u_i^b$, $\beta_i^b=P_{N_*}W_i^b$

Let τ be a stopping time associated to the process (X,η).

Let $\widetilde{X}_1^b$ be the unique solution of the truncated equation

$$\begin{cases} d\widetilde{X}_1^b+G\widetilde{X}_1^bdt+1_{t\le\tau}f(\widetilde{X}_1^b,\Phi(\widetilde{X}_1^b,\eta_1^b,(x,y_1)))dt=\sigma_l d\beta_1^b,\\ \widetilde{X}_1^b(0)=x.\end{cases} \tag{2.31}$$

Clearly $\widetilde{X}_1^b=X_1^b$ on $[0,\tau]$. We denote by λ_1 the distribution of $(\widetilde{X}_1^b,\eta)$ under the probability $\mathbb{P}$.

Let $\widetilde{\beta_1^b}(t)=\beta_1^b(t)+\int_0^t d(s)dt$ where

$$d(t)=1_{t\le\tau}\left(\sigma_l\right)^{-1}\left(f(\widetilde{X}_1^b(t),\Phi(\widetilde{X}_1^b,\eta_1^b,(x,y_2))(t))-f(\widetilde{X}_1^b(t),\Phi(\widetilde{X}_1^b,\eta_1^b,(x,y_1))(t))\right).$$

We take below a stopping time τ such that

$$\int_0^T|d(t)|^2dt\le M, \tag{2.32}$$

for a constant M. Thus Novikov condition holds and Girsanov formula applies. Setting

$$d\widetilde{\mathbb{P}}=\exp\left(\int_0^T d(s)dW(s)-\frac12\int_0^T\left|d(s)\right|^2dt\right)d\mathbb{P},$$

we know that $\widetilde{\mathbb{P}}$ is a probability under which $(\widetilde{\beta},\eta)$ is a cylindrical Wiener process.

Furthermore, with such a stopping time τ, $\widetilde{X}_1^b$ is the solution of

$$\begin{cases} d\widetilde{X}_1^b + G\widetilde{X}_1^b dt + 1_{t\leq\tau} f(\widetilde{X}_1^b, \widetilde{\Phi}(\widetilde{X}_1^b, \eta_1^b, (x,y_2)))dt = \sigma_l d\widetilde{\beta}, \\ \widetilde{X}_1^b(0) = x. \end{cases} \tag{2.33}$$

We denote by λ_2 the law of $(\widetilde{X}_1^b,\eta)$ under $\widetilde{\mathbb{P}}$. As in the case a, it is not difficult to see that

$$\|\lambda_1-\lambda_2\|_{var} \leq \frac{1}{2}\sqrt{\mathbb{E}\exp\left(c\int_0^T |d(s)|^2\, dt\right) - 1}. \tag{2.34}$$

This will be helpful to estimate $\|\nu_1-\nu_2\|_{var}$.

Definition of the coupling on $[0,\infty)$**.**

We first set

$$u_i(0) = u_0^i, \quad W_i(0) = 0, \quad i = 1,2.$$

Assuming that we have built $(u_i,W_i)_{i=1,2}$ on $[0,kT]$, then we take $(V_i^a)_i$ and $(V_i^b)_i$ as above independent of $(u_i,W_i)_{i=1,2}$ on $[0,kT]$ and set for any $t\in[0,T]$

$$(u_i(kT+t), W_i(kT+t)) =$$

$$\begin{cases} V_i^a(t,u_1(kT),u_2(kT)) & \text{if } l_0(k)=\infty \text{ and } \mathcal{H}(u_0^1)+\mathcal{H}(u_0^2)\leq R_0, \\ V_i^b(t,u_1(kT),u_2(kT)) & \text{if } l_0(k)\leq k, \\ V_i^0(t,u_1(kT),u_2(kT)) & \text{if } l_0(k)=\infty \text{ and } \mathcal{H}(u_0^1)+\mathcal{H}(u_0^2)> R_0, \end{cases} \tag{2.35}$$

where $V_i^0(t,u_1(kT),u_2(kT))$ is the trivial coupling. In other words, we take a cylindrical Wiener process W independent of $(u_i,W_i)_{i=1,2}$ on $(0,kT)$ and set

$$V_i^0(t,u_1(kT),u_2(kT)) = \left((u(t-kT,u_0^1),W),(u(t-kT,u_0^2),W)\right).$$

Remark that, when $l_0(k)=\infty$ and $\mathcal{H}(u_0^1)+\mathcal{H}(u_0^2)>R_0$, the choice of the coupling is not very important.

Clearly, $(u_i,W_i)_{i=1,2}$ is a coupling of $(u(\cdot,u_0^i))_{i=1,2}$ which is l_0–Markov. In the following, we write

$$X_i = P_{N_*}u_i,\ Y_i = Q_{N_*}u_i,\ \beta_i = P_{N_*}W_i,\ \eta_i = Q_{N_*}W_i,\ i=1,2.$$

It remains to prove that (2.3) holds.

Proof of (2.3)

We are in the situation where the coupling on $[kT,(k+1)T]$ has been constructed in case b. We use the notation used in the construction of the coupling.

Let us define for $i=1,2$

$$\hat{\tau}_{k,l}^i = \inf\left\{t\in[0,T]\ \middle|\ E_{\hat{u}_i,4}(kT+t,lT) > \kappa+1+d_0^4+d_0^8+C_4'(t+(k-l)T)\right\},$$

where C_4' is given in Proposition 2.1, and

$$\hat{\tau}_{k,l}^3 = \inf\left\{t\leq T\ \middle|\ \int_{kT}^{kT+t\wedge\hat{\tau}_{k,l}^1\wedge\hat{\tau}_{k,l}^2} l(\hat{u}_1(s),\hat{u}_2(s))\,\|\hat{r}(s)\|^2\, ds > C_*(d_0)e^{a-\frac{\alpha}{2}(k-l)T}\right\},$$

where $a,\ d_0,\ \kappa$ are chosen below, $C_*(d_0) = C^{**}(C_4', d_0)$ is given in Lemma 2.3 and

$$\hat{u}_i = u_i \text{ on } [0, kT], \quad \hat{u}_i(kT + \cdot) = \widetilde{X}_1^b + \Phi(\widetilde{X}_1^b, \eta_1^b, u_i(kT)) \text{ on } [kT, (k+1)T],$$
$$\hat{r} = \hat{u}_1 - \hat{u}_2.$$

We also take $B = C_4'$ in the definition of $l_0(k)$.

Note that, with the notation of case b, $\hat{u}_1$ (resp. $\hat{u}_2$) is a solution of a truncated NLS equation under the the probability $\mathbb{P}$ (resp. $\widetilde{\mathbb{P}}$). It follows that when $(\widetilde{X}_1^b, \eta_1^b)$ has law λ_1 (resp. λ_2) then $\hat{u}_1$ (resp. $\hat{u}_2$) is a solution in law of a truncated NLS equation. But if $(\widetilde{X}_1^b, \eta_1^b)$ has law λ_1, $\hat{u}_2$ is a solution of a truncated NLS equation with a drift term.

We wish to use the construction described in case b with the stopping time $\tau = \tau_{k,l}$ given by

$$\tau_{k,l} = \hat{\tau}_{k,l}^1 \wedge \hat{\tau}_{k,l}^2 \wedge \hat{\tau}_{k,l}^3.$$

Then

$$|d(t)| \le 1_{t \le \tau_{k,l}} \sigma_0 |f(\widetilde{X}_1^b(t), \Phi(\widetilde{X}_1^b, \eta_1^b, (x, y_2))(t)) - f(\widetilde{X}_1^b(t), \Phi(\widetilde{X}_1^b, \eta_1^b, (x, y_1))(t))|$$

and it is not difficult to see that

$$|d(t)|^2 \le c 1_{t \le \tau_{k,l}} l(\hat{u}_1(t), \hat{u}_2(t)) \|\hat{r}(t)\|^2.$$

So that, by the definition of $\tau_{k,l}$, we get

$$\int_0^T |d(t)|^2\, dt \le C^*(d_0) \sigma_0^{-2} \exp\left(a - \frac{\alpha}{2}(k - l)T\right). \tag{2.36}$$

Hence the Novikov condition is satisfied and (2.10) holds.

Moreover, using the same argument as in the proof of (2.25), we obtain

$$\mathbb{P}\left((X_1^b, \eta_1^b) \ne (X_2^b, \eta_2^b) \text{ or } \tau < T\right) \le \|\lambda_1 - \lambda_2\|_{var} + \nu_1(\hat{A}_1^c) + \nu_1(\hat{A}_3^c) + \nu_2(\hat{A}_2^c). \tag{2.37}$$

where

$$\hat{A}_i = \{(X, \eta) \mid \hat{\tau}_i = T\},\ i = 1, 2, 3.$$

It can be seen that for $i = 1, 2$

$$\nu_i(\hat{A}_i^c) = \mathbb{P}\left(\sup_{t \in [0,T]} (E_{u_i,4}(kT + t, lT) - C_4'(t + (k - l)T)) > \kappa + 1 + d_0^4 + d_0^8 \,|\, \mathcal{F}_{kT}\right). \tag{2.38}$$

The third term $\nu_1(\hat{A}_3^c)$ cannot be written in terms of u_1 and u_2. Indeed, when $(\widetilde{X}_1^b, \eta_1^b)$ has law ν_1, $\hat{u}_2$ is a solution of an equation with a drift term.

Remark 2.12. *We remark here that Proposition 2.1 is not the Foias-Prodi estimate which is usually used in the coupling method. Here, we have also a drift term h. This modification is introduced precisely to treat the term $\nu_1(\hat{A}_3^c)$. We take $h(\cdot) = bd(kT + \cdot) = \sigma_l d(kT + \cdot)$. This additional term is due to the fact that we introduce a term depending on r in the truncation. In the preceding papers using this kind of coupling method, this was not necessary and the Foias-Prodi estimate was used to get* (2.36)*. However, this requires a path-wise Foias-Prodi estimate and we do not know if it holds in our situation.*

By (2.38), we have

$$\nu_1(\hat{A}_1^c) + \nu_1(\hat{A}_3^c) + \nu_2(\hat{A}_2^c) \leq 3\mathbb{P}(B_{l,k} \,\big|\, \mathcal{F}_{kT})$$

with

$$B_{l,k} = \left\{ \begin{array}{l} \sup_{t\in[0,T]} \left(E_{u_i,4}(kT+t,lT) - C_4'(t+(k-l)T)\right) > \kappa + 1 + d_0^4 + d_0^8,\ i \in \{1,2\} \\ \text{or } \int_{kT}^{kT+\tau_{k,l}} \left(\sum_{i=1}^2 \mathcal{H}(\hat{u}_i(s))^2\right) \|\hat{r}(s)\|^2\, ds \geq C_*(d_0) e^{2\kappa - \frac{\alpha}{4}(k-l)T} \end{array} \right\}.$$

Let us write

$$\begin{aligned} &\mathbb{P}(B_{l,k} \,\big|\, l_0(l) = l) \\ &\leq \sum_{i=1,2} \mathbb{P}\left(\sup_{t\in[0,T]} \left(E_{u_i,4}(kT+t,lT) - C_4'(t+(k-l)T)\right) > \kappa + 1 + d_0^4 + d_0^8 \,\big|\, l_0(l) = l \right) \\ &+\mathbb{P}\left(\int_{kT}^{kT+\tau_{k,l}} \left(\sum_{i=1}^2 \mathcal{H}(\hat{u}_i(s))^2\right) \|\hat{r}(s)\|^2\, ds \geq C_*(d_0) e^{2\kappa - \frac{\alpha}{4}(k-l)T} \,\big|\, l_0(l) = l \right). \end{aligned}$$

Using Proposition 2.4 with $\kappa = \rho$ and solutions starting at lT and replacing T by kT we get, since $l_0(l) = l$ implies $\mathcal{H}(u_i(lT) \leq d_0$,

$$\begin{aligned} &\mathbb{P}\left(\sup_{t\in[0,T]} \left(E_{u_i,4}(kT+t,lT) - C_4'(t+(k-l)T)\right) > \kappa + 1 + d_0^4 + d_0^8 \,\big|\, \mathcal{F}_{lT} \right) \\ &\leq \mathbb{P}\left(\sup_{t\in[0,T]} \left(E_{u_i,4}(kT+t,lT) - C_4'(t+(k-l)T)\right) > \kappa + 1 + \mathcal{H}(u_i(lT)^4 + \mathcal{H}(u_i(lT)^8 \,\big|\, \mathcal{F}_{lT} \right) \\ &\leq K_{4,q+1} \left(\kappa + (k-l)T\right)\right)^{-q-1}. \end{aligned}$$

It follows

$$\begin{aligned} &\mathbb{P}\left(\sup_{t\in[0,T]} \left(E_{u_i,4}(kT+t,lT) - C_4'(t+(k-l)T)\right) > \kappa + 1 + d_0^4 + d_0^8 \,\big|\, l_0(l) = l \right) \\ &\leq K_{4,q+1} \left(\kappa + (k-l)T\right)\right)^{-q-1}. \end{aligned}$$

Similarly, by Lemma 2.3, with $h(t) = \sigma_l d(kT+t) 1_{t\leq\tau}$ which clearly satisfies (2.6) and $\rho = a = \kappa$, we have

$$\begin{aligned} &\mathbb{P}\left(\int_{kT}^{kT+\tau_{k,l}} \left(\sum_{i=1}^2 \mathcal{H}(\hat{u}_i(s))^2\right) \|\hat{r}(s)\|^2\, ds \geq C_*(d_0) e^{2\kappa - \frac{\alpha}{4}(k-l)T} \,\big|\, l_0(l) = l \right) \\ &\leq e^{-\kappa - \frac{\alpha}{2}(k-l)T} \\ &\leq c \left(\kappa + (k-l)T\right)\right)^{-q-1}. \end{aligned}$$

Gathering these estimates, we obtain

$$\mathbb{P}(B_{l,k} \,\big|\, l_0(l) = l) \leq c \left(\kappa + (k-l)T\right)\right)^{-q-1}.$$

By (2.37), (2.10), and (2.36), we obtain for $k \geq l$ and on $l_0(k) = l$

$$\begin{aligned} &\mathbb{P}\left((X_1,\eta_1) \neq (X_2,\eta_2) \text{ on } [kT,(k+1)T] \text{ or } B_{k,l} \,\left|\, \mathcal{F}_{kT} \right.\right) \\ &\leq \|\lambda_1 - \lambda_2\|_{var} + 3\mathbb{P}(B_{l,k} \big| \mathcal{F}_{kT}) \\ &\leq \sqrt{\mathbb{E}\exp\left(c \int_0^T |d(s)|^2\, dt\right) - 1} + 3\mathbb{P}(B_{l,k} \big| \mathcal{F}_{kT}) \\ &\leq C^*(d_0) \sigma_0^{-1} e^{\kappa - \frac{\alpha}{4}(k-l)T} + 3\mathbb{P}(B_{l,k} \big| \mathcal{F}_{kT}). \end{aligned}$$

We have

$$\{l_0(k)=l\}\cap\{(X_1,\eta_1)=(X_2,\eta_2)\text{ on }[kT,(k+1)T]\}\cap B_{l,k}^c\subset\{l_0(k+1)=l\}.$$

Therefore, integrating over $l_0(k)=l$ gives for $T\geq T_1(d_0)$ and for $k>l$

$$\mathbb{P}\left(l_0(k+1)\neq l,\ l_0(k)=l\,|\,l_0(l)=l\right)\leq C^*(d_0)\sigma_0^{-1}e^{\kappa-\frac{\alpha}{4}(k-l)T}+3\mathbb{P}(B_{l,k}\,|\,l_0(l)=l).$$

Which implies that there exists $\kappa>0$ sufficiently large and $d_0>0$ sufficiently small such that for any $T>0$

$$\mathbb{P}\left(l_0(k+1)\neq l,\ l_0(k)=l\,|\,l_0(l)=l\right)\leq\frac{1}{4}\left(1+(k-l)T\right)\right)^{-q}. \tag{2.39}$$

Remark that

$$\mathbb{P}\left(l_0(k)\neq l|l_0(l)=l\right)\leq\sum_{n=l}^{k-1}\mathbb{P}\left(l_0(n+1)\neq l,\ l_0(n)=l\,|\,l_0(l)=l\right),$$

so that, applying (2.39), we obtain

$$\mathbb{P}\left(l_0(k)\neq l|l_0(l)=l\right)\leq\frac{1}{4}+\frac{1}{T^{q+1/2}}\sum_{n=1}^{\infty}\frac{1}{k^q}\leq\frac{1}{4}+C_q\frac{1}{T^q},$$

which implies that there exists $T_q>0$ such that for $T\geq T_q$

$$\mathbb{P}\left(l_0(k)=l|l_0(l)=l\right)\geq\frac{1}{2}, \tag{2.40}$$

Combining (2.39) and (2.40), we establish (2.3).

2.6. **Conclusion.** We have just shown that the coupling constructed in section 2.5 satisfies (2.3) and (2.4) which are precisely (2.13) and (2.14). The constants used in (2.3) have been chosen in the preceeding subsection. The random variables $l_0(k)$ clearly satisfy (2.11) and, as already mentioned, (2.15) is implied by Lemma 2.5. Finally, (2.12) is a consequence of Proposition 2.1 with $h=0$ and Tchebychev inequality.

We deduce that Theorem 2.9 can be applied. Moreover (1.5) is a consequence of Lemma 2.5. This ends the proof of Theorem 1.1.

3. Proof of Theorem 2.9

3.1. **Reformulation of the problem.** We already noticed that it is sufficient to establish (2.19).

Let us denote by k the unique integer such that $t\in(2(k-1)T,2kT]$. Then

$$\begin{aligned}\mathbb{P}\left(d_E(u_1(t),u_2(t))>c_0\left(1+t-(k-1)T\right)^{-q}\right)&\leq\mathbb{P}\left(l_0(2k)\geq k\right)+\\ &\mathbb{P}\left(d_E(u_1(t),u_2(t))>c_0\left(1+t-(k-1)T\right)^{-q}\text{ and }l_0(2k)<k\right).\end{aligned}$$

Thus applying (2.12), using $2(t-(k-1)T)>t$, it follows

$$\mathbb{P}\left(d_E(u_1(t),u_2(t))>2^qc_0\left(1+t\right)^{-q}\right)\leq\mathbb{P}\left(l_0(2k)\geq k\right)+2^qc_0\left(1+t\right)^{-q}. \tag{3.1}$$

In order to estimate $\mathbb{P}\left(l_0(2k)\geq k\right)$, we introduce the following notation

$$l_0(\infty)=\limsup l_0.$$

Taking into account (2.11), we obtain that for $l<\infty$

$$\{l_0(\infty)=l\}=\{l_0(k)=l,\ \text{for any } k\geq l\}.$$

We deduce

$$\mathbb{P}\left(l_0(2k)\geq k\right)\leq \mathbb{P}\left(l_0(\infty)\geq k\right). \tag{3.2}$$

Taking into account (3.1), (3.2) and using a Chebyshev inequality, it is sufficient to obtain that there exist $c_5>0$ such that

$$\mathbb{E}\left(l_0(\infty)^q\right)\leq c_5\left(1+\mathcal{H}(u_0^1)+\mathcal{H}(u_0^2)\right). \tag{3.3}$$

3.2. **Definition of a sequence of stopping times.** Let

$$\tau=\min\left\{t\in T\mathbb{N}\,|\,\mathcal{H}(u_1(t))+\mathcal{H}(u_2(t))\leq R_0\right\}.$$

Then, the Lyapunov structure (2.15) implies that there exist $\delta_0>0$ and $c_6>0$ such that

$$\mathbb{E}\left(\exp\left(\delta_0\tau\right)\right)\leq c_6\left(1+\mathcal{H}(u_0^1)+\mathcal{H}(u_0^2)\right). \tag{3.4}$$

For a proof, see the proof of (1.56) at the end of the subsection 1.4 of [26].

We set

$$\hat{\sigma}=\min\left\{k\in\mathbb{N}\backslash\{0\}\,|\,l_0(k)>1\right\},\quad \sigma=\hat{\sigma}T.$$

Clearly $\hat{\sigma}=1$ if the two solutions do not get coupled at time 0 or T. Otherwise, they get coupled at 0 or T and remain coupled until σ.

From now, we fix $q_1\in(q,q_0-1)$. Let us assume for the moment that there exists p_∞ such that if $\mathcal{H}_0\leq R_0$, then

$$\begin{cases}\mathbb{E}\left(\sigma^{q_1}1_{\sigma<\infty}\right)\leq K,\\[2mm] \mathbb{P}\left(\sigma=\infty\right)\geq p_\infty>0.\end{cases} \tag{3.5}$$

The proof is given at the end of this section.

Now we build a sequence of stopping times

$$\begin{aligned}\tau_0&=\tau,\\ \hat{\sigma}_{k+1}&=\min\left\{l\in\mathbb{N}\backslash\{0\}\,|\,lT>\tau_k\text{ and } l_0(l)T>\tau_k+T\right\},\quad \sigma_{k+1}=\hat{\sigma}_{k+1}T\\ \tau_{k+1}&=\sigma_{k+1}+\tau o\theta_{\sigma_{k+1}},\end{aligned}$$

where $(\theta_t)_t$ is the shift operator. The idea is the following. We wait the time τ_k to enter the ball of radius R_0. Then, if we do not start coupling at time τ_k, we try to couple at time τ_k+T. If we fail to start coupling at time τ_k or τ_k+T we set $\sigma_k=\tau_k+T$ else we set σ_k the time the coupling fails ($\sigma_k=\infty$ if the coupling never fails). Then if $\sigma_k<\infty$, we retry to couple after entering in the ball of radius R_0. The fact that $R_0\geq d_0$ implies that $l_0(\tau_k)\in\{\tau_k,\infty\}$.

Note that we clearly have $l_0(\tau_k)\in\{\tau_k,\infty\}$ and $l_0(\sigma_k)\in\{\sigma_k,\infty\}$, and the l_0–Markov property implies the strong Markov property when conditioning with respect to $\mathcal{F}_{\tau_k}$ or $\mathcal{F}_{\sigma_k}$.

We infer from the l_0–Markov property of V that

$$\sigma_{k+1}=\tau_k+\sigma o\theta_{\tau_k},$$

which implies

$$\tau_{k+1}=\tau_k+\rho o\theta_{\tau_k},\quad\text{where } \rho=\sigma+\tau o\theta_\sigma.$$

3.3. **Polynomial estimate on ρ.** We first establish that there exist K_0 such that for any V_0 such that $\mathcal{H}_0 \leq R_0$

$$\mathbb{E}_{V_0}\left(\rho^{q_1} 1_{\rho<\infty}\right) \leq K_0. \tag{3.6}$$

Notice that for any V_0 such that $\mathcal{H}_0 \leq R_0$,

$$\mathbb{E}_{V_0}\left(\rho^{q_1} 1_{\rho<\infty}\right) \leq c\left(\mathbb{E}_{V_0}\left(\sigma^{q_1} 1_{\sigma<\infty}\right) + \mathbb{E}\left((\tau o\theta_\sigma)^{q_1} 1_{\tau o\theta_\sigma<\infty} 1_{\sigma<\infty}\right)\right). \tag{3.7}$$

Applying the l_0–Markovian property and (3.4), we obtain

$$\mathbb{E}\left((\tau o\theta_\sigma)^{q_1} 1_{\tau o\theta_\sigma<\infty} 1_{\sigma<\infty} | \mathcal{F}_\sigma\right) \leq c_6\left(1 + \mathcal{H}(u_1(\sigma)) + \mathcal{H}(u_2(\sigma))\right) 1_{\sigma<\infty},$$

which implies by applying the Lyapunov structure (2.15)

$$\mathbb{E}\left((\tau o\theta_\sigma)^{q_1} 1_{\tau o\theta_\sigma<\infty}\right) \leq c_6(1 + 2K'(R_0 + \mathbb{E}(\sigma 1_{\sigma<\infty})). \tag{3.8}$$

Applying (3.5) and (3.8) to (3.7), we obtain (3.6).

3.4. **Conclusion.** Applying a convexity inequality, we obtain

$$\mathbb{E}\left(\tau_k^{q_1} 1_{\tau_k<\infty}\right) \leq (k+1)^{(q_1-1)^+}\left(\mathbb{E}\tau^{q_1} + \sum_{n=0}^{k-1} \mathbb{E}(\rho o\theta_{\tau_n})^{q_1} 1_{\rho o\theta_{\tau_n}<\infty}\right).$$

Combining the l_0–Markov property, (3.4) and (3.6) gives

$$\mathbb{E}\left(\tau_k^{q_1} 1_{\tau_k<\infty}\right) \leq C(k+1)^{1\vee q_1}\left(1 + \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2)\right). \tag{3.9}$$

Now, we are able to estimate $\mathbb{E}\left(l_0(\infty)^q\right)$

$$\mathbb{E}\left(l_0(\infty)^q\right) \leq c\left(1 + \sum_{n=0}^{\infty} \mathbb{E}\left(\tau_n^q 1_{\tau_n<\infty} 1_{k_0=n}\right)\right),$$

where

$$k_0 = \inf\{k \in \mathbb{N} \,|\, \sigma_{k+1} = \infty\}.$$

Then, applying an Holder inequality, we obtain

$$\mathbb{E}\left(l_0(\infty)\right)^q \leq c\left(1 + \sum_{n=0}^{\infty} \mathbb{E}\left(\tau_n^{pq} 1_{\tau_n<\infty}\right)^{\frac{1}{p}} \left(\mathbb{P}\left(k_0 = n\right)\right)^{\frac{1}{p'}}\right).$$

Using the second inequality of (3.5) and $\tau < \infty$, we obtain from the l_0–Markov property that

$$\mathbb{P}\left(k_0 > n\right) \leq \left(1 - p_\infty\right)^n. \tag{3.10}$$

It follows that $k_0 < \infty$ almost surely and that

$$l_0(\infty) \in \{\tau_{k_0}, \tau_{k_0} + 1\}.$$

Therefore $l_0(\infty) < \infty$ almost surely and applying (3.9), we obtain that if $pq = q_1$

$$\mathbb{E}\left(l_0(\infty)\right)^q \leq C\left(\sum_{n=0}^{\infty} (n+1)^{\frac{1}{p}\vee q}(1-p_\infty)^{\frac{n}{p'}}\right)\left(1 + \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2)\right).$$

Thus (3.3) is established and we can conclude.

3.5. **Proof of** (3.5)**.** Now we establish (3.5). There are two cases. The first case is $l_0(0)=0$. Then, applying (2.13), we obtain that

$$\mathbb{P}\left(\sigma=\infty\right)\geq\Pi_{k=0}^{\infty}\mathbb{P}\left(l_0(k+1)=0|l_0(k)=0\right)\geq\Pi_{k=0}^{\infty}p_k.$$

The second case is $l_0(0)=\infty$. Then

$$\mathbb{P}\left(\sigma=\infty\right)\geq\mathbb{P}\left(l_0(1)=1\right)\Pi_{k=1}^{\infty}\mathbb{P}\left(l_0(k+1)=1|l_0(k)=1\right).$$

Since $\mathcal{H}_0\leq R_0$, then applying (2.13) and (2.14)

$$\mathbb{P}\left(\sigma=\infty\right)\geq\Pi_{k=-1}^{\infty}p_k.$$

Since $p_k>0$ and $1-p_k$ decreases to 0 faster than k^{-q_0} with $q_0>1$, then the product converges and in the two cases

$$\mathbb{P}\left(\sigma=\infty\right)\geq p_\infty=\Pi_{k=-1}^{\infty}p_k>0. \tag{3.11}$$

Notice that (2.13) implies

$$\begin{array}{rcl}\mathbb{P}\left(\sigma=n\right) & \leq & \mathbb{P}\left(l_0(n+1)\neq n\,|\,l_0(n)=0\right)+\mathbb{P}\left(l_0(n+1)\neq n\,|\,l_0(n)=1\right),\\ & \leq & 2c_1\left(1+(n-1)T\right)^{-q_0},\end{array}$$

which gives the first inequality of (3.5) and allows to conclude because $q_1<q_0-1$.

4. Proof of the a priori estimates

As already mentioned, the various computations made in this section are not rigorous. Indeed, the solutions are not smooth enough to apply Ito formula. A suitable approximation could be used to justify the result rigorously.

Ito Formula for $|u|^6$

Applying Ito Formula to $|u|^6$, we obtain

$$d\left|u\right|^6+6\alpha\left|u\right|^6dt=6\left|u\right|^4(u,bdW)+12\left|u\right|^2\left|b^*u\right|^2dt+3B_0\left|u\right|^4dt.$$

Since b^* is a bounded operator on $L^2(0,1)$,

$$12\left|u\right|^2\left|b^*u\right|^2\leq 12B_0\left|u\right|^4.$$

We deduce

$$12\left|u\right|^2\left|b^*u\right|^2+3B_0\left|u\right|^4\leq\alpha\left|u\right|^6+C,$$

and

$$d\left|u\right|^6+5\alpha\left|u\right|^6dt\leq 6\left|u\right|^4(u,bdW)+Cdt. \tag{4.1}$$

Ito Formula for $\mathcal{H}$

Applying Ito Formula to $\mathcal{H}_*$, we obtain

$$d\mathcal{H}_*(u)+\alpha\left(\|u\|^2-|u|_4^4\right)dt=dM_*+B_1dt+I_*dt, \tag{4.2}$$

where

$$\begin{array}{rcl}dM_* & = & \left(Au-\left|u\right|^2u,bdW\right),\\ I_* & = & -\sum_{n=1}^{\infty}b_n^2\int_{[0,1]}\left(2\Re(u(t,x)\overline{e_n}(x))^2+\left|e_n(x)\right|^2\left|u(t,x)\right|^2\right)dx.\end{array}$$

Note that, since b^*A is a bounded operator from $L^2(0,1)$ to $H_0^1(0,1)$, M_* is well defined. Recalling that $|e_n|_\infty=1$, we obtain

$$I_*\leq 3B_0\left|u\right|^2\leq\alpha c_0\left|u\right|^6+C.$$

Recalling that $|\cdot|_4^4 \leq \frac{1}{4}\|\cdot\|^2 + c_0 |\cdot|^6$, we infer from (4.1), (4.2) and the last inequality that

$$d\mathcal{H}(u) + \frac{3}{2}\alpha\mathcal{H}(u)dt \leq dM_1 + C_1 dt, \tag{4.3}$$

where

$$dM_1 = dM_* + 6c_0 |u|^4 (u, bdW).$$

Ito Formula for $\mathcal{H}^k$

Applying Ito Formula to $\mathcal{H}^k$ for $k \in \mathbb{N}\backslash\{0\}$, we obtain similarly as above

$$d\mathcal{H}(u)^k + \frac{3}{2}\alpha k\mathcal{H}(u)^k dt \leq dM_k + k\mathcal{H}(u)^{k-1}C_1 dt + \frac{k(k-1)}{2}\mathcal{H}(u)^{k-2} d\langle M_1\rangle, \tag{4.4}$$

where

$$dM_k = k\mathcal{H}(u)^{k-1}dM_1.$$

Note that, since b^*A is a bounded operator from $L^2(0,1)$ to $H_0^1(0,1)$ and b^* is bounded from $L^1(0,1)$ to $L^2(0,1)$,

$$\left|b^*\left(Au - |u|^2 u\right)\right|^2 \leq 4B_1\|u\|^2 + cB_1 |u|_3^6,$$

it follows from a Gagliardo-Nirenberg inequality

$$\left|b^*\left(Au - |u|^2 u\right)\right|^2 \leq cB_1\left(\|u\|^2 + |u|^{10}\right).$$

Now, we write

$$d\langle M_1\rangle \leq 2\left|b^*\left(Au - |u|^2 u\right)\right|^2 + 72c_0^2 |u|^8 |b^* u|^2,$$

and deduce that

$$d\langle M_1\rangle \leq cB_1\left(\|u\|^2 + |u|^{10}\right),$$

and

$$d\langle M_1\rangle \leq cB_1\left(1 + \mathcal{H}(u)^{\frac{5}{3}}\right). \tag{4.5}$$

Gathering (4.4) and (4.5) and using once more an arithmetico-geometric inequality, we obtain

$$d\mathcal{H}(u)^k + \alpha k\mathcal{H}(u)^k dt \leq dM_k + C_k'' dt. \tag{4.6}$$

Proof of Lemma 2.5

Multiplying (4.6) by $e^{\alpha kt}$ yields

$$d\left(e^{\alpha kt}\mathcal{H}(u)^k\right) \leq e^{\alpha kt}dM_k + e^{\alpha kt}C_k'' dt.$$

By integration we obtain

$$e^{\alpha kt}\mathcal{H}(u(t))^k \leq \mathcal{H}(u_0)^k + \int_0^t e^{\alpha ks}dM_k(s) + \frac{C_k''}{\alpha k}e^{\alpha kt}$$

and

$$\mathcal{H}(u(t))^k \leq \mathcal{H}(u_0)^k e^{-\alpha kt} + \int_0^t e^{-\alpha k(t-s)}dM_k(s) + \frac{C_k''}{\alpha k},$$

which yields, by taking the expectation, the first inequality of Lemma 2.5.
Let $M > 0$ and $\tau \leq M$ be a bounded stopping time. Then, integrating (4.6) between 0 and $\tau$and taking the expectation yields

$$\mathbb{E}\left(\mathcal{H}(u(\tau))^k\right) \leq \mathcal{H}(u_0)^k + C"_k\mathbb{E}(\tau),$$

which is the second inequality of Lemma 2.5 for bounded stopping times.
Assume now that τ is a general stopping time. We consider the second inequality of Lemma 2.5 for the stopping time $\tau \wedge M$ and we take the limit when $M \to \infty$. The second inequality of Lemma 2.5 for τ follows from Fatou Lemma.

Proof of Proposition 2.4

We first note that

$$d\langle M_k\rangle = k^2\mathcal{H}(u)^{2(k-1)}d\langle M_1\rangle,$$

so that, taking into account (4.5),

$$d\langle M_k\rangle \leq c_k\left(1+\mathcal{H}(u)^{2k}\right)ds \tag{4.7}$$

Taking the expectation of (4.6), we obtain for any $k \geq 1$

$$\mathbb{E}\int_0^t \mathcal{H}(u(s))^k dt \leq C_k\left(\mathcal{H}(u_0)^k + t\right). \tag{4.8}$$

Hence, for any $p \geq 1$,

$$\mathbb{E}\langle M_k\rangle^p(t) \leq C_{k,p}\left(\mathcal{H}(u_0)^{2kp} + t^p\right). \tag{4.9}$$

Applying the maximal martingale inequality and taking into account (4.6), we infer from (4.9) the first inequality of Proposition 2.4.

Applying the maximal martingale inequality on $[n, n+1]$, $n \geq 0$, we have

$$\mathbb{P}\left(\sup_{[n,n+1]} M_k > a + \mathcal{H}(u_0)^{2k} + n + 1\right) \leq c_p \frac{\mathbb{E}\langle M_k\rangle^{p+1}(n+1)}{\left(a+\mathcal{H}(u_0)^{2k}+n+1\right)^{2p+2}}.$$

It follows from (4.9) that

$$\mathbb{P}\left(\sup_{[n,n+1]} M_k > a + \mathcal{H}(u_0)^{2k} + n + 1\right) \leq \frac{c_p C'_{k,p+1}}{\left(a+\mathcal{H}(u_0)^{2k}+n+1\right)^{p+1}}. \tag{4.10}$$

Now, summing (4.10) over $n \geq T$, for T integer, we obtain that for any $(p,k) \in (\mathbb{N}\backslash\{0\})^2$ there exists $K_{k,p}$ such that

$$\mathbb{P}\left(\sup_{t\in[T,\infty)} M_k(t) > 1 + a + \mathcal{H}(u_0)^{2k} + t\right) \leq K_{k,p}\left(a+T\right)^{-p},\ T > 0. \tag{4.11}$$

Taking into account (4.6), this implies the second inequality of Proposition 2.4.

Proof of Proposition 2.6

Combining Lemma 2.5 applied to $\tau = t$ and Chebyshev's inequality, we obtain

Lemma 4.1. *Let $(u_i, W_i)_{i=1,2}$ be a couple of solutions of* (1.1), (1.2) *such that W_1 and W_2 are two cylindrical Wiener process on $L^2([0,1])$. If $R_0 \geq \left(\sum_{i=1}^2 \mathcal{H}(u_0^i)\right) \vee C_1$, then*

$$\mathbb{P}\left(\mathcal{H}(u_1(t)) + \mathcal{H}(u_2(t)) \geq 4C_1\right) \leq \frac{1}{2},$$

provided $t \geq \theta_1(R_0) = \frac{1}{\alpha}\ln\frac{R_0}{C_1}$.

It follows from Lemma 4.1 that it is sufficient to establish Proposition 2.6 for $R_0 = 4C_1$ and $t = T_{-1}(R_0, R_1)$ (instead of $t \geq T_{-1}(R_0, R_1)$). From now on, we only consider the case $R_0 = 4C_1$.

Let $T,\delta>0$. Applying Chebyshev inequality, we obtain $N_{-2}=N_{-2}(T,\delta)\in\mathbb{N}$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T]}\left\|bQ_{N_{-2}}W(t)\right\|_3>\frac{\delta}{2}\right)\leq\frac{2}{\delta}\sum_{n>N_2}\mu_n^3b_n^2\leq\frac{1}{2}.$$

Moreover $P_{N_{-2}}W$ is a finite dimensional brownian motion and it is classical that

$$\pi_{-3}(T,\delta,N_{-2})=\mathbb{P}\left(\sup_{t\in[0,T]}\left|P_{N_{-2}}W(t)\right|\leq\frac{\delta}{2}\left\|b\right\|^{-1}_{\mathcal{L}_2(L_2(D),H^3(D))}\right)>0.$$

Writing

$$\mathbb{P}\left(\sup_{t\in[0,T]}\left\|bW(t)\right\|_3\leq\delta\right)\geq\mathbb{P}\left(\sup_{t\in[0,T]}\left\|bQ_{N_{-2}}W(t)\right\|_3\leq\frac{\delta}{2}\right)\pi_{-3}(T,\delta,N_{-2}),$$

it follows

$$\pi_{-2}(T,\delta)=\mathbb{P}\left(\sup_{t\in[0,T]}\left\|bW(t)\right\|_3\leq\delta\right)>0. \tag{4.12}$$

It thus suffices to prove that there exists $T_{-1}(R_1),\delta_{-1}(R_1)>0$ such that

$$\left\{\sup_{t\in[0,T_{-1}]}\left\|bW(t)\right\|_3\leq\delta_{-1}\right\}\subset\left\{\mathcal{H}(u(T_{-1},u_0))\leq\frac{1}{2}R_1\right\}, \tag{4.13}$$

provided $\mathcal{H}(u_0)\leq R_0$.

Proof of (4.13)

Let us set

$$v=u(\cdot,u_0)-bW,$$

then

$$\frac{dv}{dt}+\alpha v+\mathrm{i}Av-\mathrm{i}\left|bW+v\right|^2(bW+v)=-\left(\alpha+\mathrm{i}A\right)bW. \tag{4.14}$$

Taking the scalar product between (4.14) and $2v$, we obtain

$$\frac{d\left|v\right|^2}{dt}+2\alpha\left|v\right|^2=2\left(v,\mathrm{i}\left|bW+v\right|^2(bW+v)-\left(\alpha+\mathrm{i}A\right)bW\right).$$

Since

$$\left(v,\mathrm{i}\left|bW+v\right|^2v\right)=0,$$

applying Hölder inequalities and Sobolev Embedding $H^1(D)\subset L^\infty(0,1)$, we deduce

$$\frac{d\left|v\right|^2}{dt}+2\alpha\left|v\right|^2\leq c\left\|bW\right\|_3\left(1+\left\|bW\right\|_3^2\right)\left(1+\left\|v\right\|^3\right).$$

Applying Ito Formula to $\left|v\right|^6$, we deduce

$$\frac{d\left|v\right|^6}{dt}+6\alpha\left|v\right|^6\leq c\left\|bW\right\|_3\left(1+\left\|bW\right\|_3^2\right)\left(1+\left\|v\right\|^9\right). \tag{4.15}$$

Taking the scalar product between (4.14) and $Av-\left|v\right|^2v$, we obtain

$$\frac{d\mathcal{H}_*(v)}{dt}+\alpha\left\|v\right\|^2=-\left(Av-\left|v\right|^2v,\left(\alpha+\mathrm{i}A\right)bW\right)+\alpha\left(\left|v+bW\right|^2(v+bW),v\right).$$

Since

$$I_1=\alpha\left(\left(\left|v+bW\right|^2(v+bW),v\right)-\left|v\right|_4^4\right)=\alpha\left(\left|v+bW\right|^2(v+bW)-\left|v\right|^2v,v\right),$$

we obtain

$$\frac{d\mathcal{H}_*(v)}{dt} + \alpha\left(\|v\|^2 - |v|_4^4\right) = I_1 + I_2, \tag{4.16}$$

where

$$I_2 = -\left(Av - |v|^2 v, (\alpha + \mathrm{i}A)\, bW\right).$$

Recalling that for any $z, h \in \mathbb{C}^2$

$$\left||z+h|^2 (z+h) - |z|^2 z\right| \leq |h| \left(|z|^2 + |h|^2\right),$$

and applying Hölder inequality and the Sobolev Embedding $H^1(D) \subset L^\infty(0,1)$, we obtain

$$I = I_1 + I_2 \leq c \|bW\|_3 \left(1 + \|v\|^3\right)\left(1 + \|bW\|_3^2\right).$$

It follows from (4.15), (4.16) and the last inequality that

$$\frac{d\mathcal{H}(v)}{dt} + 2\alpha\mathcal{H}(v) \leq c\|bW\|_3 \left(1 + \|bW\|_3^2\right)\left(1 + \mathcal{H}(v)^5\right). \tag{4.17}$$

Let $T, \delta, M > 0$ and assume that

$$\sup_{t\in[0,T]} \|bW(t)\|_3 \leq \delta.$$

We set

$$\tau = \inf\left\{t \in [0,T] \mid \mathcal{H}(v) \leq 3R_0\right\}.$$

Integrating (4.17), we obtain

$$\mathcal{H}(v(t)) \leq e^{-2\alpha t} R_0 + \frac{c}{2\alpha}\delta\left(1+\delta^2\right)\left(1 + R_0^5\right), \tag{4.18}$$

provided $t \leq \tau$.

Now we choose $\delta \leq \delta_{-2}(R_1') > 0$ such that

$$\frac{c}{2\alpha}\delta\left(1+\delta^2\right)\left(1+R_0^5\right) \leq R_1' \wedge R_0.$$

It follows from (4.18) that

$$\tau = T,$$

and that

$$\mathcal{H}(v(T)) \leq 2R_1',$$

provided

$$T \geq \frac{1}{2\alpha}\ln\left(\frac{R_1'}{R_0}\right).$$

In order to conclude, we remark that

$$\mathcal{H}(u(T)) \leq c\left(\mathcal{H}(bW(T)) + \mathcal{H}(v(T))\right) \leq c(\delta^2(1+\delta^4) + R_1').$$

Then, choosing δ and R_1' sufficiently small, we obtain (4.13).

5. Proof of the Foias-Prodi estimates

The aim of this section is to establish Proposition 2.1.

L^2 estimates

Taking into account (2.5), we deduce that the difference of the two solutions $r = u_1 - u_2$ satisfies the equation

$$\frac{dr}{dt} + \alpha r + \mathrm{i}Ar = \mathrm{i}Q_N\left(|u_1|^2 u_1 - |u_2|^2 u_2\right). \tag{5.1}$$

Applying Ito Formula to $|r|^2$, we obtain

$$\frac{d\,|r|^2}{dt} + 2\alpha\,|r|^2 = 2\left(\mathrm{i}r, |u_2|^2 u_2 - |u_1|^2 u_1\right).$$

Since

$$\left||u_2|^2 u_2 - |u_1|^2 u_1\right| \le c\left(\sum_{i=1}^{2} |u_i|^2\right)|r|,$$

it follows

$$\frac{d\,|r|^2}{dt} + 2\alpha\,|r|^2 \le c\int_{[0,1]}\left(\sum_{i=1}^{2} |u_i|^2\right)|r|^2\,dx.$$

Using the Sobolev Embedding $H^1(0,1) \subset L^\infty(0,1)$, we obtain

$$\frac{d\,|r|^2}{dt} + 2\alpha\,|r|^2 \le c\left(\sum_{i=1}^{2}\mathcal{H}(u_i)\right)|r|^2. \tag{5.2}$$

We deduce as in the proof of (4.3)

$$d\mathcal{H}(u_i) + \frac{3}{2}\alpha\mathcal{H}(u_i)dt \le dM_1^i + C_1 dt + 1_{i=1}(G,h)dt,$$

where

$$\left\{\begin{array}{rcl} dM_1^i &=& \left(Au_i - |u_i|^2 u_i, bdW_i\right) + 6c_0\,|u_i|^4\,(u_i, bdW_i), \\ G &=& Au_1 - |u_1|^2 u_1 + 6c_0\,|u_1|_{L^2}^4\,u_1. \end{array}\right.$$

It follows from Sobolev Embeddings and Hölder inequalities that

$$\|G\|_{-1} \le c\,(1 + \mathcal{H}(u_1))^{\frac{5}{6}}.$$

Hence we deduce from (2.6) that

$$(G,h) \le c\,(1 + \mathcal{H}(u_1) + \mathcal{H}(u_2))^4.$$

Taking into account (5.2), it follows

$$dZ_1 + 2\alpha Z_1 dt \le c\left(1 + \sum_{i=1}^{2}\mathcal{H}(u_i)^4\right)|r|^2\,dt + |r|^2\,dM_\#, \tag{5.3}$$

where

$$Z_1 = \left(\sum_{i=1}^{2}\mathcal{H}(u_i)\right)|r|^2$$

and

$$dM_\# = dM_1^1 + dM_1^2.$$

Ito Formula for J

Now we rewrite (5.1) in the form

$$\frac{dr}{dt}+\alpha r+\mathrm{i}Ar=-\mathrm{i}\frac{1}{2}Q_N\left((|u_1|^2+|u_2|^2)r+\Re\left((u_1+u_2)\bar r\right)(u_1+u_2)\right). \tag{5.4}$$

Applying Ito Formula to $J_*(u_1,u_2,r)$, we obtain

$$\begin{aligned} dJ_*+2\alpha J_*dt &= g(u_1,u_2,r)dt+g(u_2,u_1,r)dt+\psi(u_1,u_2,r)(bdW_1)\\ &+\psi(u_1,u_2,r)(h(t))dt+\psi(u_2,u_1,r)(bdW_2)+I_1(r)dt+dI_2(r,dt), \end{aligned} \tag{5.5}$$

where

$$\begin{array}{lcl} g(u_1,u_2,r) & = & \left\{\begin{array}{cc} & 2\int_{[0,1]}\left(\Re\left(\bar u_1(\alpha u_1+\mathrm{i}Au_1-\mathrm{i}\,|u_1|^2\,u_1)\right)|r|^2\right)dx\\ + & 2\int_{[0,1]}\Re\left(\bar r(u_1+u_2)\right)\Re\left(\bar r(\alpha u_1-\mathrm{i}Au_1+\mathrm{i}\,|u_1|^2\,u_1)\right)dx\end{array}\right. ,\\ \psi(u_1,u_2,r)(h) & = & 2\int_{[0,1]}\left(\Re\left(\bar u_1 h\right)|r|^2\right)dx+2\int_{[0,1]}\Re\left(\bar r(u_1+u_2)\right)\Re\left(\bar r h\right)dx,\\ I_1(r) & = & -\sum_{n=1}^{\infty}b_n^2\int_{[0,1]}\left(|e_n|^2\,|r|^2+\Re(e_n\bar r)^2\right)dx,\\ dI_2(r,t) & = & -\sum_{p,q=1}^{\infty}b_pb_q\left(\left(\int_{[0,1]}\Re\left(e_p\bar r\right)\Re\left(e_q\bar r\right)dx\right)\ d\left\langle (W_1,e_p),(W_2,e_q)\right\rangle\right). \end{array}$$

Applying an integration by part to Au_1, Hölder inequality and the Sobolev Embedding $H^{\frac{3}{4}}(0,1)\subset L^\infty(0,1)$, we obtain

$$g(u_1,u_2,r)\le\left(1+\sum_{i=1}^{2}\|u_i\|^6\right)\|r\|\,\|r\|_{\frac{3}{4}}. \tag{5.6}$$

We deduce from Hölder inequality that

$$\psi(u_1,u_2,r)(h(t))\le\left(\sum_{i=1}^{2}|u_i|_\infty\right)|h(t)|\,|r|_4^2.$$

Taking into account (2.6) and applying the Sobolev Embeddings $H^1(0,1)\subset L^\infty(0,1)$ and $H^{\frac{1}{2}}(0,1)\subset L^4(0,1)$, we obtain

$$\psi(u_1,u_2,r)(h(t))\le c\kappa_0\left(1+\sum_{i=1}^{2}\mathcal{H}(u_i)^{\frac{7}{2}}\right)\|r\|_{\frac{1}{2}}^2. \tag{5.7}$$

Recalling that $|e_n|_\infty=1$, we obtain

$$I_1(r)\le 3B_0\,|r|^2. \tag{5.8}$$

Note that we have no information on the law of the couple (W_1,W_2). Hence, we cannot compute $d\left\langle (W_1,e_p),(W_2,e_q)\right\rangle$. However we know that

$$d\left|\left\langle (W_1,e_p),(W_2,e_q)\right\rangle\right|\le dt.$$

Hence

$$d\,|I_2(r,t)|=\left(\int_{[0,1]}\Re\left(\sum_{n=1}^{\infty}(b_ne_n)\,\bar r\right)^2dx\right)dt.$$

Applying the following Schwartz inequality

$$\left(\sum_{n=1}^{\infty}b_n\right)^2\le\left(\sum_{n=1}^{\infty}\mu_nb_n^2\right)\left(\sum_{n=1}^{\infty}\frac{1}{\mu_n}\right)\le cB_1,$$

we deduce from $|e_n|_\infty = 1$ that

$$d\,|I_2(r,t)| \leq cB_1\,|r|^2\,dt. \tag{5.9}$$

Combining (5.5), (5.6), (5.7), (5.8), and (5.9) , we obtain

$$dJ_* + 2\alpha J_* dt \leq c\left(1 + \sum_{i=1}^{2} \mathcal{H}(u_i)^4\right) \|r\|\,\|r\|_{\frac{3}{4}}\,dt + dM_{\#\#}, \tag{5.10}$$

where

$$dM_{\#\#} = \left(\psi(u_1,u_2,r)(bdW_1) + \psi(u_2,u_1,r)(bdW_2)\right).$$

Summing (5.3) and (5.10), we obtain

$$dJ + 2\alpha J dt \leq c\left(1 + \sum_{i=1}^{2} \mathcal{H}(u_i)^4\right) \|r\|\,\|r\|_{\frac{3}{4}}\,dt + dM, \tag{5.11}$$

where

$$dM = dM_{\#\#} + c_1\,|r|^2\,dM_{\#}.$$

Conclusion

Since $\|r\|_{\frac{3}{4}} \leq \mu_{N+1}^{-\frac{1}{8}}\|r\|$ then there exists $\Lambda > 0$ such that

$$dJ + \left(2\alpha - \frac{\Lambda}{\mu_{N+1}^{\frac{1}{8}}} l(u_1,u_2)\right) J dt \leq dM. \tag{5.12}$$

Multiplying (5.12) by $\exp\left(2\alpha s - \Lambda\mu_{N+1}^{-\frac{1}{8}}\int_0^s l(u_1(s'),u_2(s'))ds'\right)$, we obtain that

$$J_{FP}^N(t\wedge\tau) \leq \int_0^{t\wedge\tau} \exp\left(\frac{3}{2}\alpha s - \Lambda\mu_{N+1}^{-\frac{1}{8}}\int_0^s l(u_1(s'),u_2(s'))ds'\right) dM(s). \tag{5.13}$$

Fatou Lemma allows to conclude.

Acknowledgments: We are very grateful to the referee for his many suggestions, which have led to many improvements in this article . We also would like to thank A. Shirikyan for many fruitful discussions.

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Annexe C

Preuve du caractère exponentiellement mélangeant des équations aux dérivées partielles stochastiques dotées d'un bruit non additif

Résumé: Nous établissons ici un critère général qui assure le caractère exponentiellement mélangeant des équations aux dérivées partielles paraboliques stochastiques dotées d'un bruit non additif blanc en temps et régulier par rapport à la variable d'espace.

On appliquera ce critère à deux exemple représentatif : Les équations de Navier-Stokes bidimensionnelles (NS) et celles de Ginzburg-Landau Complexe (CGL) muni d'un bruit localement Lipschitz. Du fait de la possible dégénérescence du bruit, le théorème de Doob ne peut être appliqué. Ainsi une méthode de couplage est utilisée dans l'esprit de [18], [41] et [45].
Les résultats précédents requieraient des hypothèses sur la covariance du bruit qui peuvent sembler restrictives et artificielles. Par exemple, pour NS et CGL, l'opérateur de covariance était supposé commuter avec le Laplacien et ne pas dépendre de la base des hauts modes des solutions.

La méthode développée ici s'affranchit de ces hypothèses et requiert seulement que l'image de l'opérateur recouvre les bas modes.

Mots clés : Équations de Navier-Stokes bidimensionnelles, Équations de Ginzburg-Landau Complexe, Semi-groupe de transition de Markov, mesures invariantes, ergodicité, méthode de couplage, Transformation de Girsanov, Inégalité de type Foias–Prodi en espérance.

Exponential Mixing for Stochastic PDEs: The Non-Additive Case.

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Abstract: We establish a general criterion which ensures exponential mixing of parabolic Stochastic Partial Differential Equations (SPDE) driven by a non additive noise which is white in time and smooth in space. We apply this criterion on two representative examples: 2D Navier-Stokes (NS) equations and Complex Ginzburg-Landau (CGL) equation with a locally Lipschitz noise. Due to the possible degeneracy of the noise, Doob theorem cannot be applied. Hence a coupling method is used in the spirit of [9], [22] and [24].
Previous results require assumptions on the covariance of the noise which might seem restrictive and artificial. For instance, for NS and CGL, the covariance operator is supposed to be diagonal in the eigenbasis of the Laplacian and not depending on the high modes of the solutions. The method developped in the present paper gets rid of such assumptions and only requires that the range of the covariance operator contains the low modes.

Key words: Two-dimensional Navier-Stokes equations, Complex Ginzburg-Landau equations, Markov transition semi-group, invariant measure, ergodicity, coupling method, Girsanov Formula, expectational Foias–Prodi estimate.

Introduction

We investigate ergodic properties of parabolic Stochastic Partial Differential Equations (SPDE) driven by a noise which is white in time and smooth in space. Such systems are difficult to handle with the standard theory because the phase spaces are infinite dimensional. Moreover the noise is allowed to be degenerate and the conditions required to apply Doob theorem are not always verified (see [7] for the theory of ergodicity when Doob Theorem can be applied).

The idea of compensating the degeneracy of the noise on some subspaces by dissipativity arguments has been introduced in [18], and then in [3], [9] . In the same spirit, we consider systems which have only a finite number of unstable directions. In other words, the unstable manifold is finite-dimensional. Dissipative SPDEs such as the stochastic 2D Navier–Stokes (NS) and Complex Ginzburg–Landau (CGL) equations have this structure. The main requirement on the noise is that it is non degenerate in the unstable directions. Later, coupling methods have been introduced to prove exponential convergence to equilibrium (see [14], [20], [21], [22], [24] and [29]).

These articles mainly deal with additive noises. Only in [24], the noise is allowed to have some dependence on the solution but it has to be of a very special form - see below for more details. Moreover, the noise is assumed to be diagonal in the eigenbasis of the linear part of the equation.

In this article, we wish to get rid of these assumptions. This requires substantial adaptations in the method, for instance an auxiliary process is introduced. We develop a general ergodic criterion which ensures exponential mixing of the solution provided the image of the covariance operator of the noise contains the unstable modes.

Roughly speaking, our method allows to treat SPDEs perturbed by a noise of the type $\phi(u)dW$ where u is the unknown of the equation and W is the driving noise. Denoting by P_N the projection onto the unstable modes, our main assumption is that the range of $\phi(u)$ contains the unstable modes $P_N H$. We think that this is a very natural condition. Note that with these notations, the above cited articles treat noises of the type ϕdW, ϕ being constant and diagonal and with the main assumption that the range of ϕ contains the unstable modes $P_N H$. In [24] (see also [28]), the noise has the form $\phi(P_N u)dW$ with $(I - P_N)\phi(P_N u) = 0$.

Our method is very general. Given a SPDE, it is sufficient to build an auxilliary process with good properties to apply our method and establish exponential convergence of the solutions to equilibrium. The technic to build this process depends on the type of SPDE. In fact, we distinguish three types of SPDEs. Examples of the two first types are given by NS and CGL. The third type of SPDE is more complicated to treat. It includes weakly damped but not strongly dissipative SPDEs. An example is the weakly damped Non-Linear Schrödinger (NLS) equation (see [8] for the case of an additive noise). We will study this equation in a forthcoming article.

The NS equations describe the time evolution of an incompressible fluid. It has been widely studied. Most of the articles cited above have been motivated by the application to this equation.

Originally introduced to describe a phase transition in superconductivity [12], the CGL equation also models the propagation of dispersive non-linear waves in various areas of physics such as hydrodynamics [26], [27], optics, plasma physics, chemical reaction [16]... The CGL equation arises in the same areas of physics as the non-linear Schrödinger (NLS) equation. In fact, the CGL equation is obtained by adding two viscous terms to the NLS equation. The inviscid limits of the deterministic and stochastic CGL equation to the NLS equation are established in [2] and [23], respectively.

Ergodicity of the stochastic CGL equation is established in [1] when the noise is invertible and in [14] for the one-dimensional cubic case when the noise is diagonal, does not depend on the solution and is smooth in space. In [28], we have established exponential mixing of CGL driven by a noise which verifies the additional assumptions mentioned above under the L^2 or the H^1–subcritical conditions.

We hope that the method developped here can be combined with other recent ideas. For instance, in [15], [25], the case of NS perturbed by a four dimensional noise is treated. Hopefully, a four dimensional noise depending on the unknown could be studied. Another topic of interest is to try to prove exponential mixing in the three dimensional case for the transition semigroup constructed in [5]. This latter problem will be treated in a forthcoming paper.

The remaining of article is divided into four sections. First, we consider NS. We build its auxiliary process and we state (Theorem 1.3) that its solutions converge exponentially fast to equilibrium. This result is a trivial consequence of the criterion (Theorem 2.1) established in section 2, which ensures exponential mixing of a stochastic process provided there exists an auxiliary process which verifies some properties. In section 3, we establish exponential mixing of the solutions of CGL (Theorem 3.2). Section 4 is devoted to the proof of properties given in Section 1.2 and 3.2.

1. The Two-dimensional Navier-Stokes equations

In this section, we investigate properties of the two-dimensional Navier-Stokes (NS) equations with Dirichlet boundary conditions. These equations describe the time evolution of an incompressible fluid and are given by

$$(1.1)\qquad \left\{\begin{array}{rcll} du+\nu(-\Delta)u\,dt+(u,\nabla)u\,dt+\nabla p\,dt &=& \phi(u)dW+f dt, & \\ (\mathrm{div}\; u)\,(t,x) &=& 0, & \text{for } x\in D,\ \ t>s,\\ u(t,x) &=& 0, & \text{for } x\in \delta D, t>s,\\ u(s,x) &=& u_s(x), & \text{for } x\in D,\end{array}\right.$$

where $u(t,x)\in\mathbb{R}^2$ denotes the velocity field at time t and position x, p denotes the pressure, $\phi(u)dW$ is a random external force field acting on the fluid, $\nu>0$ is the viscosity of the fluid and D is an open bounded domain of $\mathbb{R}^2$ with regular boundary or $D=(0,1)^2$. Also f is the deterministic part of the forcing term. For simplicity in the redaction, we consider the case $f=0$. The generalization to a square integrable f is easy.
In Section 1.1, we rewrite problem (1.1) and we state an ergodic result about NS (Theorem 1.3). In order to establish this result, we introduce an auxiliary process and we state some properties in Section 1.2. These properties allow to apply the general criterion (Theorem 2.1) stated in Section 2.1.

1.1. **Notations and Main result.**

We denote by $|\cdot|$ the norm of $L^2(D;\mathbb{R}^2)$ and by $\|\cdot\|$ the norm of $H^1_0(D;\mathbb{R}^2)$. Let H and V be the closure of the space of smooth functions on D with compact support and free divergence for the norm $|\cdot|$ and $\|\cdot\|$, respectively.
Let Π be the orthogonal projection in $L^2(D;\mathbb{R}^2)$ onto the space H. Setting

$$A=\Pi\left(-\Delta\right),\quad D(A)=V\cap H^2(D;\mathbb{R}^2)\ \text{ and }\ B(u)=\Pi\left((u,\nabla)u\right),$$

we can write problem (1.1) in the form

$$(1.2)\qquad \left\{\begin{array}{rcl} du+\nu Audt+B(u)dt &=& \phi(u)dW,\\ u(s) &=& u_s,\end{array}\right.$$

where W is a cylindrical Wiener process on a Hilbert space U.
In order to have existence and uniqueness of the solution of (1.2), we make the following assumption

H0 *The function $\phi:\ H\to\mathcal{L}_2(U;H)$ is bounded Lipschitz.*

Here $\mathcal{L}_2(K_1;K_2)$ denotes the space of Hilbert-Schmidt operators from the Hilbert space K_1 to K_2.

We set

$$B_0 = 1 + \sup_{u \in H} \|\phi(u)\|^2_{\mathcal{L}_2(U;H)}, \quad L = L_\phi^2,$$

where L_ϕ is the Lipschitz constant of ϕ.
Under **H0**, we have existence and uniqueness of the solution of (1.2) in H when $u_s \in H$. Moreover there exists a measurable map u such that $u(\cdot, s, W, u_s)$ is the unique solution of (1.2). This result ensures the strong Markov property of the solutions of (1.2). We denote by $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ the Markov transition semi-group associated to the solutions of (1.2).
In our computations, we use the energy:

$$E_u(T, t_0) = |u(T)|^2 + \nu \int_{t_0}^T \|u(t)\|^2 \, dt, \qquad \text{for any } T \geq t_0.$$

It is well-known that $(A, \mathcal{D}(A))$ is a self-adjoint operator with discrete spectrum. See [4], [30]. We consider $(e_n)_n$ an eigenbasis of H associated to the increasing sequence $(\mu_n)_n$ of eigenvalues of $(A, \mathcal{D}(A))$. We denote by P_N and Q_N the orthogonal projection in H onto the space $Sp(e_k)_{1\leq N}$ and onto its complementary, respectively.

Now, we make the assumption which is used to prove the exponential mixing of $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$.

H1 *There exist $N \in \mathbb{N}^*$ and a bounded measurable map $g : H \to \mathcal{L}(H; U)$ such that for any $u \in H$*

$$\phi(u)g(u) = P_N.$$

We assume in our main result that **H1** holds with N sufficiently high.

Remark 1.1 (Sufficient conditions to satisfy **H1**).
A sufficient condition to satisfy **H1** *is for instance that U is the orthogonal sum $U_1 \oplus U_2$ and there exists two measurable maps $(\phi_i : H \to \mathcal{L}_2(U_i; H))_{i=1,2}$ such that ϕ_2 verifies* **H1** *and*

$$\phi(u)W = \phi_1(u)W_1 + \phi_2(u)W_2,$$

for any $u \in H$ and any $W = (W_1, W_2) \in U$. Moreover, if ϕ_2 is a constant map, we can omit the orthogonality condition on U_1 and U_2.
A very interesting consequence is the case of the sum of a multiplicative noise and an additive noise which covers the low modes. Namely, for any measurable map $f : \mathbb{R}^2 \to \mathbb{R}$ and $(\phi_1, \phi_2) \in \mathcal{L}_2(U_1; H^2(D)) \times \mathcal{L}_2(U_2; H)$, one define ϕ by

$$\phi(u)W = f(u)(\phi_1 W_1) + \phi_2 W_2,$$

for any $u \in H$ and any $W = (W_1, W_2) \in U$. The operator ϕ verifies **H1** *provided*

$$P_N H \subset Im \ \phi_2.$$

Another sufficient condition is that $U = H$ and there exists an invertible operator ψ on the low modes and a constant ε such that for any $u \in H$

$$Q_N \phi(u) P_N = 0 \quad \text{and} \quad |P_N \phi(u) P_N - \psi|_{\mathcal{L}(P_N H)} < \varepsilon < \left|\psi^{-1}\right|^{-1}_{\mathcal{L}(P_N H)}.$$

Thus, our result holds when the covariance operator is a small perturbation of a constant in the low modes.
We think that these examples might be physically relevant.

Remark 1.2. *The existence of a map $\widetilde{g}(u)$ such that $\phi(u)\widetilde{g}(u) = P_N$, is equivalent to the following property*

$$P_N H \subset Im(\phi(u)).$$

Hence **H1** *can be seen as a non degeneracy condition on the low modes in the spirit of* [3], [8], [9], [14], [18], [20], [21], [22], [24], [28] *and* [29]. *The lack of surjectivity of $\phi(u)$ on the high modes is counterbalanced by the dissipativity of* (1.2).
Moreover, if there exists such a map $\widetilde{g}$, then there exists a measurable map g such that $\phi(u)g(u) = P_N$ and $|g(u)|_{\mathcal{L}(H;U)} \leq |\widetilde{g}(u)|_{\mathcal{L}(H;U)}$. This mapping g is constructed by similar ideas as in the construction of the pseudo inverse. Hence the assumption of measurability of g in **H1** *is superfluous.*

Given a polish space E, the space $Lip_b(E)$ consists of all the bounded and Lipschitz real valued functions on E. Its norm is given by

$$\|\varphi\|_L = |\varphi|_\infty + L_\varphi, \ \varphi \in Lip_b(E),$$

where $|\cdot|_\infty$ is the sup norm and L_φ is the Lipschitz constant of φ. The space of probability measures on E is denoted by $\mathcal{P}(E)$. It can be endowed with the norm defined by the total variation

$$\|\mu\|_{var} = \sup\left\{|\mu(\Gamma)| \ | \ \Gamma \in \mathcal{B}(E)\right\},$$

where we denote by $\mathcal{B}(E)$ the set of the Borelian subsets of E. It is well known that $\|.\|_{var}$ is the dual norm of $|.|_\infty$. We can also use the Wasserstein norm

$$\|\mu\|_* = \sup_{\varphi \in Lip_b(E),\ \|\varphi\|_L \leq 1} \left|\int_E \varphi(u) d\mu(u)\right|,$$

which is the dual norm of $\|\cdot\|_L$.
One of the main results of this article is the following.

Theorem 1.3. *Assume that* **H0** *holds. There exists $N_0(B_0, \nu, D, L)$ such that, if* **H1** *holds with $N \geq N_0$, then there exists a unique stationary probability measure μ of $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ on H. Moreover, for any $q \in \mathbb{N}^*$, μ satisfies*

$$\int_H |u|^{2q} \, d\mu(u) < \infty, \tag{1.3}$$

and there exist $C, \gamma > 0$, such that for any $\lambda \in \mathcal{P}(H)$

$$\|\mathcal{P}_t^* \lambda - \mu\|_* \leq C e^{-\gamma t} \left(1 + \int_H |u|^2 \, d\lambda(u)\right). \tag{1.4}$$

Theorem 1.3 is a consequence of Theorem 2.1 below and of auxiliary results given in section 1.2.

Remark 1.4. *Theorem 1.3 could be improved and the assumptions could be weakened. We chose to restrict to this statement for clarity and readability. For instance, it is possible to replace* **H1** *by*
H1' *There exist $n \in \mathbb{N}^*$, a measurable map $g : H \to \mathcal{L}(H;U)$ and two constants σ, C such that for any $u \in H$*

$$\phi(u)g(u) = P_N, \quad |g(u)|_{\mathcal{L}(H;U)} \leq C \exp\left(\sigma |u|^2\right).$$

In this case N_0 depends on σ. Moreover, it is easy to strengthen (1.3) *into*

$$\int_H \exp\left(\sigma_1(B_0, \nu, D) |u|^2\right) d\mu(u) < \infty.$$

Remark that if we assume **H0** *and* **H1'** *for* N *sufficiently high and*

$$B_1 = \sup_{u\in H} \|\phi(u)\|^2_{\mathcal{L}_2(U;V)} < \infty,$$

then we can strengthen (1.3) *by*

$$\int_H \exp\left(\sigma_2(B_1,\nu,D)\|u\|^2\right) d\mu(u) < \infty,$$

and in (1.4) *we can replace* $\|\cdot\|_*$*, the Wassertein norm in* H*, by the Wassertstein norm in* $H^s(D;\mathbb{R}^2)$ *for any* $s<1$*. Moreover, if* $\phi : H \to \mathcal{L}_2(U;V)$ *is bounded Lipschitz, then in* (1.4) *we can replace* $\|\cdot\|_*$ *the Wassertein norm in* H *by the Wassertstein norm in* V*.*
In **H0***, the boundedness of* ϕ *could be replaced by* $|\phi(u)| \le C(1+|u|^\gamma)$ *with* $\gamma<1$*. In this case, the rate of convergence becomes greater than any power of time instead of being exponential in time. Moreover for any* p *there exists* c_p *such that if there exists* C *such that* $|\phi(u)| \le C + c_p|u|$*, then the rate of convergence is greater than* $(1+t)^{-p}$ *instead of being exponential.*

The proof of our result is based on coupling arguments. These arguments have initially been used in the context of dissipative SPDEs in [22], [24]. For a better understanding of this kind of method, see section 1 of [24] and the two first subsections of [28]. There the coupling method is explained on two examples which are simpler but contain all the difficulties.
The method developed in [22] and [24] requires the three following assumptions for a N sufficiently high.
The first assumption is a structure condition on ϕ. It is a slight generalization of the usual assumption that $\phi(u)$ is diagonal in the basis $(e_n)_n$.
Ha *The Hilbert space* U *is* H*. Moreover, for any* $u\in H$*, we have*

$$P_N\phi(u)Q_N = 0,\ Q_N\phi(u)P_N = 0.$$

The second assumption means that ϕ only depends on its low modes.
Hb *For any* $u \in H$

$$\phi(u) = \phi(P_N u).$$

The third assumption is **H1**. Under **Ha**, it could be written in the form.
Hc *The linear map* $P_N\phi(u)P_N$ *is invertible on* P_NH*. Moreover*

$$\sup_{u\in H}\left|(P_N\phi(u)P_N)^{-1}\right|_{\mathcal{L}(P_NH;P_NH)} < \infty.$$

In these papers, the proof is divided in two steps.
Step 1: Starting from initial data (u_s^1,u_s^2) in a ball of radius R_0, Girsanov transform can be used to show that there exists a couple of Wiener processes (W_1,W_2) such that

$$P_Nu(T+s,s,W_1,u_s^1) = P_Nu(T+s,s,W_2,u_s^2)$$

with positive probability (see section 1.1 and 1.2 in [28]).
Step 2: Another ingredient, the so-called Foias-Prodi estimate, is used. It is based on the observation that if two solutions are such that

$$(P_Nu(\cdot,s,W_1,u_s^1),Q_NW_1) = (P_Nu(\cdot,s,W_2,u_s^2),Q_NW_2),$$

for a long time then they become very close. Girsanov transform can again be used to show that, if we start from initial conditions which have the same low modes, then there exists a couple of Wiener processes (W_1,W_2) such that the low modes

of the solutions remain equal for all times with a positive probability (see section 1.2 in [28]).
Conclusion: Since the time of entering a ball of radius R_0 admits an exponential moment, we are able to combine and to iterate the two steps and then to conclude.

In the above mentioned articles, **Ha** and **Hb** are essential when using the Girsanov transform. They are also necessary to prove that the Foais-Prodi estimate holds pathwise. This latter point is not important. We have shown in [8] how to use a Foias-Prodi estimate in expectation and it is not difficult to see that such an estimate holds in our case.
On the contrary, it seems very difficult to use Girsanov Transform without **Ha** and **Hb**. We think that these are artificial whereas **H1** is very natural.
The idea developed in this article is to separate the use of the Foias-Prodi estimate and the Girsanov Transform by introducing an auxiliary process $\widetilde{u}(\cdot, s, W, u_s^1, u_s^2)$. Our method requires that the auxiliary process verifies two properties.
First fundamental property: The first property is a variation of the Foias-Prodi estimate. It states that $u(t, s, W, u_s^1)$ and $\widetilde{u}(t, s, W, u_s^1, u_s^2)$ become close exponentially fast in probability.
Second fundamental property: The second property states that there exists $h(\cdot, s, W, u_s^1, u_s^2) \in L^2\left((s, \infty); U\right)$ such that

$$\widetilde{u}(t, s, W, u_s^1, u_s^2) = u(t, s, W + \int_s^{\cdot} h(r, s, W, u_s^1, u_s^2) dr\,, u_s^2). \tag{1.5}$$

Hence, taking into account **H1**, a Girsanov Transform can be used to build a couple of Wiener processes (W_1, W_2) such that

$$\widetilde{u}(\cdot, s, W_1, u_s^1, u_s^2) = u(\cdot, s, W_2, u_s^2),$$

with a positive probability.
Conclusion: Iterating and combining the two properties, we can conclude by remarking that it allows to control the probability of $u(t, 0, W_1, u_0^1)$ and $u(t, 0, W_2, u_0^2)$ being very close.

1.2. **Construction of the auxiliary process.**
Now, we build the auxiliary process. We set

$$F(u) = \nu A(u) + B(u).$$

Taking into account **H1**, we remark that (1.5) is a consequence of

$$\left\{ \begin{array}{rcl} d\widetilde{u} + F(\widetilde{u})dt + P_N \delta(t, s, W, u_s^1, u_s^2)dt &=& \phi(\widetilde{u})dW, \\ \widetilde{u}(s, s, W, u_s^1, u_s^2) &=& u_s^2, \end{array} \right. \tag{1.6}$$

with

$$h(t, s, W, u_s^1, u_s^2) = -g(\widetilde{u}(t, s, W, u_s^1, u_s^2))\delta(t, s, W, u_s^1, u_s^2). \tag{1.7}$$

Since we want that $\widetilde{u}(t, s, W, u_s^1, u_s^2)$ and $u(t, s, W, u_s^1)$ become very close with a large probability, it is natural to build $\widetilde{u}$ such that (1.6) and (1.7) hold with

$$\delta(t, s, W, u_s^1, u_s^2) = K P_N (\widetilde{u}(t, s, W, u_s^1, u_s^2) - u(t, s, W, u_s^1)). \tag{1.8}$$

Hence we consider the following equation

$$\left\{ \begin{array}{rcl} d\widetilde{u} + F(\widetilde{u})dt + K P_N (\widetilde{u} - u(t, s, W, u_s^1))dt &=& \phi(\widetilde{u})dW, \\ \widetilde{u}(s) &=& u_s^2, \end{array} \right. \tag{1.9}$$

where $K>0$ will be chosen later and N is the integer used in **H1**.
It is not difficult to deduce from **H0** that there exists a unique H–valued solution to (1.9) when $(u_s^1,u_s^2)\in H^2$. Moreover, there exists a measurable map $\widetilde{u}$ such that $\widetilde{u}(\cdot,s,W,u_s^1,u_s^2)$ is the unique solution of (1.9). It follows that $(u(\cdot,s,W,u_s^1),\widetilde{u}(\cdot,s,W,u_s^1,u_s^2))$ is a strong Markov process.
Taking into account **H1** and the uniqueness of the solution of (1.2) under **H0**, we deduce (1.5) with (1.7) and (1.8).
Now, it remains to prove that $\widetilde{u}(t,s,W,u_s^1,u_s^2)$ and $u(t,s,W,u_s^1)$ become close exponentially fast. In [22], [24] and [28], a pathwise Foias-Prodi is used. Here, $u-\widetilde{u}$ does not seem to tend to 0, pathwise. That is the reason why we adapt an idea we have already used in [8] to prove polynomial mixing for the weakly damped Non-linear Schrödinger (NLS) equations. Since it seems that there is no pathwise Foias-Prodi estimate for NLS, a Foias-Prodi estimate in expectation was used. Using analogous technics, we have the following result.

Proposition 1.5. *Assume that* **H0** *holds. There exists $\Lambda>0$ depending only on ν and D such that for any finite stopping time $\tau\geq s$*

$$\mathbb{E}\left(|r(\tau)|^2\exp\left(\left(K\vee(\nu\mu_{N+1})-L\right)(\tau-s)-\Lambda E_{u(\cdot,s,W,u_s^1)}(\tau,s)\right)\right)\leq|r(s)|^2,$$

where

$$r=\widetilde{u}(\cdot,s,W,u_s^1,u_s^2)-u(\cdot,s,W,u_s^1).$$

The proof is postponed to Section 4.2.

Remark 1.6. *Proposition 1.5 displays a nice and well known property of Navier-Stokes equations. Indeed, we see that the difference between u and $\widetilde{u}$ can be estimated using only the energy of u and not the energy of $\widetilde{u}$ which is much more difficult to estimate. No control on the probability of the linear growth of the energy of the auxiliary process is required. This property holds also for the one-dimensional Burger equation and for the Complex Ginzburg-Landau equation with a globally Lipschitz noise in the subcritical case. However it does not hold in general and in this case the construction of the auxiliary process is more involved. For instance, in the case of the Non-Linear Schrödinger equations, it is not possible to prove a result similar to Proposition 1.5. It is important to show that our method can also be used for these equations. The Non-Linear Schrödinger equations will be treated in a forthcoming article and we consider the Complex Ginzburg-Landau with a locally Lipschitz noise in section 3.*

We immediately deduce the two following very useful corollaries. The first corollary states that, under an energy condition, $\widetilde{u}(t,s,W,u_s^1,u_s^2)$ and $u(t,s,W,u_s^1)$ become close exponentially fast.

Corollary 1.7. *Assume that* **H0** *holds. There exists $N_1(B_0,\nu,D,L)$ and $K_1(B_0,\nu,D,L)$ such that if $N\geq N_1$ and $K\geq K_1$, then for any $(a,s)\in\mathbb{R}^2$ and $t\geq s$*

$$\mathbb{P}\left(|r(t)|\geq|r(s)|\exp\left(a-(t-s)+\Lambda\left(\rho+\left|u_s^1\right|^2\right)\right)\text{ and }A\right)\leq e^{-a-(t-s)},$$

where

$$\left\{\begin{array}{rcl} A&=&\left\{W\;\middle|\;E_{u(\cdot,s,W,u_s^1)}(t,s)\leq\rho+\left|u_s^1\right|^2+B_0(t-s)\right\},\\ r&=&\widetilde{u}(\cdot,s,W,u_s^1,u_s^2)-u(\cdot,s,W,u_s^1).\end{array}\right.$$

Corollary 1.7 is deduced from Proposition 1.5 by applying a Chebyshev inequality. Integrating the formula of Proposition 1.5, applying a Chebyshev inequality and setting

$$C_\rho^*(d_0) = c_0 d_0 \exp\left(\Lambda(d_0+\rho)\right),$$

we obtain the following result

Corollary 1.8. *Assume that* **H0** *and* **H1** *hold. There exists* $N_2(B_0,\nu,D,L)$ *and* $K_2(B_0,\nu,D,L)$ *such that if* $N \geq N_2$ *and* $K \geq K_2$ *hold, then for any* $(s,a)\in\mathbb{R}^2$, $t_0 \geq s$

$$\mathbb{P}\left(\int_{t_0}^{\tau} \left|h(t,s,W,u_s^1,u_s^2)\right|_U^2 dt \geq C_\rho^*\left(\left|u_s^1\right|^2+\left|u_s^2\right|^2\right)e^{a-(t_0-s)}\right) \leq e^{-a-(t_0-s)},$$

provided $\tau \geq t_0$ *is a stopping time such that*

$$E_{u(\cdot,s,W,u_s^1)}(t,s) \leq \rho + \left|u_s^1\right|^2 + B_0(t-s) \ \text{ for any } t\in(t_0,\tau),$$

and where h *is defined in* (1.7) *and* (1.8).

Taking into account (1.5), Corollary 1.8 ensures that the Novikov condition holds. Girsanov Formula can be applied to transform $\widetilde{u}(\cdot,s,W,u_s^1,u_s^2)$ into $u(\cdot,s,\hat{W},u_s^2)$ on $[s,\tau]$ by the change of Wiener process (1.5).
We take

$$N_0 = \max(N_1,N_2),\ K = \max(K_1(B_0,\nu,D,L),K_2(B_0,\nu,D,L)). \tag{1.10}$$

Assumption **H0** implies the two following easy results. They are used when applying Corollary 1.7 and 1.8.

Lemma 1.9 (Exponential estimate for the growth of the solution)**.** *Assume that* **H0** *holds. There exists* $\gamma_0>0$ *only depending on* ν, B_0 *and* D *such that*

$$\mathbb{P}\left(\sup_{t\geq t_0}\left(E_{u(\cdot,s,W,u_s)}(t,s)-B_0(t-s)\right) \geq \rho + |u_s|^2\right) \leq e^{-\gamma_0(\rho+t_0-s)},$$

for any $u_s\in H$, $\rho\geq 0$ *and* $t_0\geq s$.

Lemma 1.10 (The Lyapunov structure)**.** *Assume* **H0**. *For any* $q\in\mathbb{N}$, *there exists* $C_q>0$ *only depending on* q,ν,B_0 *and* D *such that*

$$\begin{cases} \mathbb{E}\left(|u(t,s,W,u_s)|^{2q}\right) & \leq & e^{-\nu\mu_1 q(t-s)}\,|u_s|^{2q}+C_q, \\ \mathbb{E}\left(|u(\tau,s,W,u_s)|^{2q}\right) & \leq & |u_s|^{2q}+C_q\mathbb{E}\left(\tau-s\right), \end{cases}$$

for any $t\geq s$ *and any finite stopping time* $\tau\geq s$.

The proofs are given in Section 4.1.

2. A general criterion

This section is devoted to the proof of a general criterion -Theorem 2.1- which ensures exponential mixing of a Markov process $u(\cdot,s,W,u_s^1)$, provided there exists an auxiliary process $\widetilde{u}(\cdot,s,W,u_s^1,u_s^2)$ which verifies some properties. In particular, Theorem 1.3 is a Corollary of Theorem 2.1 and of the properties stated in Section 1.2.

2.1. **Statement of the criterion.**
Let $(U, |\cdot|_U)$ be a Hilbert space, W be a cylindrical Wiener processes on U and (H, d_H) be a polish space. We consider three measurable mappings u, $\widetilde{u}$ and h such that for any $(s, u_s^1, u_s^2) \in \mathbb{R} \times H^2$, $\left(u(\cdot, s, W, u_s^1), \widetilde{u}(\cdot, s, W, u_s^1, u_s^2), h(\cdot, s, W, u_s^1, u_s^2)\right)$ is in $(C([s,\infty); H))^2 \times L^2([s,\infty); U)$ with probability one. Moreover $u(T, s, W, u_s^1)$, $\widetilde{u}(T, s, W, u_s^1, u_s^2)$ and $h(T, s, W, u_s^1, u_s^2)$ are only depending on $(W(t) - W(s))_{t\in[s,T]}$ and are supposed to be homogenous in time. This means that

$$u(T + \cdot, T + s, W(T + \cdot), u_s^1) = u(\cdot, s, W, u_s^1),$$

and similar relations for $\widetilde{u}$ and h.
We first impose a Markov condition on $(u, \widetilde{u})$
A0 *For any* $(T, t, s, u_s^1, u_s^2) \in \mathbb{R}^3 \times H^2$ *such that* $T \geq t \geq s$,

$$\begin{cases} \left(u(s, s, W, u_s^1), \widetilde{u}(s, s, W, u_s^1, u_s^2)\right) = \left(u_s^1, u_s^2\right), \\ u(T, s, W, u_s^1) = u(T, t, W, u(t, s, W, u_s^1)), \\ \widetilde{u}(T, s, W, u_s^1, u_s^2) = \widetilde{u}(T, t, W, u(t, s, W, u_s^1), \widetilde{u}(t, s, W, u_s^1, u_s^2)). \end{cases}$$

We denote by $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ the Markov transition semi-group associated to the Markov family $(u(\cdot, 0, W, u_0))_{u_0\in H}$.
The next assumptions involve a positive functional $\mathcal{H}$, which plays the role of a Lyapunov functional. We assume that there exists $\gamma > 0$ and a constant $C^*(\cdot)$ depending on an argument $d_0 \geq 0$ such that the following hold.
A1 *There exist* $q > 1$ *and* $C_1, C_q > 0$, *such that for any* $(s, u_s) \in \mathbb{R} \times H$, *any* $t \geq s$ *and any finite stopping time* $\tau \geq s$,

$$\begin{cases} \mathbb{E}\left(\mathcal{H}(u(t, s, W, u_s))^k\right) & \leq & e^{-\nu\mu_1 q(t-s)}\mathcal{H}(u_s)^k + C_k, \\ \mathbb{E}\left(\mathcal{H}(u(\tau, s, W, u_s))^k\right) & \leq & \mathcal{H}(u_s)^k + C_k\mathbb{E}(\tau - s), \end{cases}$$

for $k = 1, q$.
A2 *For any* $(u_s^1, u_s^2) \in H^2$, *for any couple* (W_1, W_2) *of cylindrical Wiener processes of* U *and for any* $(s, t) \in \mathbb{R}^2$ *such that* $t \geq s$, *we have*

$$\mathbb{P}\left(d_H(u_1(t), u_2(t)) \geq C^*(d_0)e^{-\gamma(t-s)} \text{ and } \widetilde{u} = u_2 \text{ on } [s,t]\right) \leq Ce^{-\gamma(t-s)},$$

where

$$\begin{cases} u_i(t) & = & u(t, s, W_i, u_s^i) \quad \text{for} \quad i = 1, 2, \\ \widetilde{u}(t) & = & \widetilde{u}(t, s, W_1, u_s^1, u_s^2), \\ d_0 & \geq & \mathcal{H}(u_s^1) + \mathcal{H}(u_s^2). \end{cases}$$

A3 *For any* $(t, s, u_s^1, u_s^2) \in \mathbb{R}^2 \times H^2$ *such that* $t \geq s$

$$\widetilde{u}(t, s, W, u_s^1, u_s^2) = u(t, s, W + \int_s^\cdot h(r, s, W, u_s^1, u_s^2)dr\,, u_s^2).$$

A4 *For any couple* (W_1, W_2) *of cylindrical Wiener processes of* U *and for any* $(t_0, s, u_s^1, u_s^2) \in \mathbb{R}^2 \times H^2$ *such that* $t_0 \geq s$

$$\mathbb{P}\left(\int_{t_0}^\tau |h(t)|_U^2\, dt \geq C^*(d_0)\, e^{-\gamma(t_0-s)} \text{ and } \widetilde{u} = u_2 \text{ on } [s,\tau]\right) \leq Ce^{-\gamma(t_0-s)},$$

where $(\widetilde{u}, u_2)$ *are defined in* **A2**, *where* $\tau \geq t_0$ *is any stopping time and where*

$$\begin{cases} h(t) &= h(t, s, W_1, u_s^1, u_s^2), \\ d_0 &\geq \mathcal{H}(u_s^1) + \mathcal{H}(u_s^2). \end{cases}$$

A5 *For any* $(s, u_s^1, u_s^2) \in \mathbb{R} \times H^2$, *we have*

$$\mathbb{P}\left(\int_s^\infty |h(t)|_U^2 \, dt \leq C^*(d_0) \right) \geq \frac{1}{2},$$

where

$$\begin{cases} h(t) &= h(t, s, W, u_s^1, u_s^2), \\ d_0 &\geq \mathcal{H}(u_s^1) + \mathcal{H}(u_s^2). \end{cases}$$

We now state our criterion.

Theorem 2.1. *Under the above assumptions, there exists a unique stationary probability measure* μ *of* $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ *on* H. *Moreover,* μ *satisfies*

$$\int_H \mathcal{H}(u)^q d\mu(u) \leq C_q, \tag{2.1}$$

and there exist $C, \gamma' > 0$ *such that for any* $\lambda \in \mathcal{P}(H)$

$$\|\mathcal{P}_t^* \lambda - \mu\|_* \leq C e^{-\gamma' t} \left(1 + \int_H \mathcal{H}(u) d\lambda(u) \right). \tag{2.2}$$

Theorem 2.1 is proved in sections 2.2, 2.4 and 2.5 hereafter.
Assumption **A1** is standard and ensures that (2.1) holds and that the time of return of the process in any ball of radius greater than $2C_1$ admits an exponential moment. For the Navier-Stokes equations, **A1** holds with $\mathcal{H} = |\cdot|^2$ as shown in Lemma 1.10. For the Complex Ginzburg-Landau equation treated below, we use the same functional. More complicated choices may be necessary as in the case of the weakly damped Non Linear Schrödinger equation treated in [8].
A2, **A3**, **A4** and **A5** translate the two fundamental properties required on the auxiliary process $\widetilde{u}$; **A2** states that u and $\widetilde{u}$ get close exponentially fast.
For the Navier-Stokes equations, we bound the probability of the left-hand side by $\mathbb{P}\left(d_H(u_1(t), \widetilde{u}(t)) \geq C^*(d_0)e^{-\gamma(t-s)}\right)$. This is then bounded thanks to Corollary 1.7 and Lemma 1.9. For the Complex Ginzburg-Landau equation we also majorize $\mathbb{P}\left(d_H(u_1(t), \widetilde{u}(t)) \geq C^*(d_0)e^{-\gamma(t-s)}\right)$ but it is more complicated since the analogous of Corollary 1.7 involves the energy of $\widetilde{u}$.
For these two applications, a stronger version of **A2** therefore holds. However, **A2** is the natural assumption needed in the proof of Theorem 2.1 and might be easier to check in some application.
A3, **A4** and **A5** allow to change the Wiener process thanks to Girsanov Formula used in the construction of the coupling. **A4** and **A5** imply that the Novikov condition holds for a truncation of $\widetilde{u}$.
For the Navier-Stokes equations, **A3** is (1.5) which has been proved at the beginning of section 1.2. We choose ρ in Lemma 1.9 such that $e^{-\gamma_0 \rho} = \frac{1}{4}$ and then use Corollary 1.8 with $t_0 = s$ and $e^{-a} = \frac{1}{4}$. We obtain **A5** for $C^*(d_0) \geq C_\rho^*(d_0)e^a$. Similarly **A4** follows from Corollary 1.8 and Lemma 1.9, we can take $\rho = a = 0$. Theorem 1.3 is thus proved.

Remark 2.2. *It we replace the term $e^{-\gamma t}$ by a negative power of t in our assumptions, we obtain a Theorem analogous to Theorem 2.1, but where convergence is polynomial instead of being exponential.*
We have seen in [8] *that for the stochastic Non–Linear Schrödinger (NLS) equation the control of the energy is polynomial. Hence, we think that the polynomial version of our criterion is the good framework when studying weakly damped SPDE.*
Moreover we have seen in Remark 1.4 that there exist some variations of Theorem 1.3 whose convergence is polynomial. To establish such results, we need the polynomial version of Theorem 2.1.

Remark 2.3. *This result is of the same type as the results obtained in* [9], [22], [24] *and* [28] *where SPDEs with additive noise only depending on the low modes are studied. Since the decrease of $\mathcal{P}_t^*\lambda - \mu$ is measured in Wasserstein norm, we know that $\mathcal{P}_t\phi$ converges to its average with respect to μ for any Lipschitz function ϕ. In fact, in the above mentioned articles, it is also true if ϕ is only Lipschitz with respect to the high modes. We do not know if this holds in our situation.*

2.2. **Proof of Theorem 2.1.**
Let us denote by (u_0^1, u_0^2) two initial conditions in H. For a given couple of cylindrical processes (W_1, W_2), we set

$$u_i(t) = u(t, 0, W_i, u_0^i),\ \widetilde{u}(t) = \widetilde{u}(t, (k-1)T, W_1, u_1((k-1)T), u_2((k-1)T)),$$

where k is the unique integer such that $t \in [(k-1)T, kT)$.
The main idea is to build the couple of cylindrical processes (W_1, W_2) recursively on $[kT, (k+1)T]$ so that, at each step, we have $\widetilde{u} = u_2$ for more and more trajectories. The distance between $\widetilde{u}$ and u_1 will be controlled thanks to **A2**.
We define an integer valued random process l_0 which is particularly convenient when deriving properties of the coupling:

$$l_0(k) = \min \{l \in \{0, ..., k\} | P_{l,k}\},$$

where $\min \phi = \infty$ and

$$(P_{l,k}) \begin{cases} \widetilde{u} = u_2 \ \text{ on } [lT, kT], \\ \mathcal{H}(u_1(lT)) + \mathcal{H}(u_2(lT)) \le d_0, \end{cases}$$

where d_0 will be chosen later.
Fixing k, l, T, t such that $t \in [lT, kT]$ and noticing that $l_0(k) = l$ implies

$$u_2(t) = \widetilde{u}(t) = \widetilde{u}(t, lT, W_1, u_1(lT), u_2(lT)), \quad u_1(t) = u(t, lT, W_1, u_1(lT)),$$

it follows from **A2** that

$$\mathbb{P}\left(d_H(u_1(t), u_2(t)) > C^*(d_0)e^{-\gamma(t-lT)} \text{ and } l_0(k) = l\right) \le Ce^{-\gamma(t-lT)}. \tag{2.3}$$

We fix $d_0 = 16C_1$ and take into account **A3**, **A4** and **A5** to build in the next subsections a couple (W_1, W_2) such that

$$\begin{cases} \exists\, T_0 > 0,\, p_0 > 0 \ \text{ such that for any } l \le k\ ,\ T \ge T_0, \\ \mathbb{P}\left(l_0(k+1) \ne l \mid l_0(k) = l\right) \le (1-p_0)e^{-\frac{\gamma}{4}(k-l)T}. \end{cases} \tag{2.4}$$

It is now easy to check that the assumptions of Theorem 1.8 in [28] are satisfied. Note that by **A1** with the notation of [28], we can take $p_{-1} = \frac{1}{4}$ since $(P_{l,l})$ is

reduced to $\mathcal{H}(u_1(lT)) + \mathcal{H}(u_2(lT)) \leq d_0$. Thus Theorem 2.1 is proved provided we construct (W_1, W_2) such that (2.4) is established.
For the sake of completeness, let us briefly recall how we can prove Theorem 2.1.
The Lyapunov structure **A1** implies that the time $\tau \in \mathbb{N}$ of entering the ball $\mathcal{H}(u_1(\tau T) + \mathcal{H}(u_2(\tau T)) \leq d_0$ admits an exponential moment. We deduce from $u_2(kT) = \widetilde{u}(kT)$ that τ is the time of having $l_0(\tau) = \tau$ for the first time.
Now, assume that $l_0(l) = l$. We consider σ the time of failure of the coupling ($\sigma = \min\{k \geq l \,|\, l_0(k) \neq l\}$). We infer from (2.4) that there exists $T > 0$ and $p_\infty > 0$ such that $\mathbb{P}(\sigma = \infty) \geq p_\infty$. Moreover (2.4) implies that $\sigma 1_{\sigma \neq \infty}$ admits an exponential moment. It means that, when $l_0(l) = l$, then the probability of having $\widetilde{u}(t) = u_2(t)$ for all times $t \geq lT$ is bounded below and that, when this property fails, the time of failure admits an exponential moment.
Combining τ and σ and iterating this combination, we build a finite random time $l_0(\infty)$ such that $l_0(k) = l_0(\infty)$ for any $k \geq l_0(\infty)$ and such that $l_0(\infty)$ admits an exponential moment. Hence it follows from (2.3) that

$$\mathbb{E}\left(d_H(u_1(t), u_2(t)) \wedge 1\right) \leq C e^{-\gamma' t} \left(1 + \left|u_0^1\right|^2 + \left|u_0^2\right|^2\right). \tag{2.5}$$

Remarking that

$$\left|\mathcal{P}_t f(u_0^2) - \mathcal{P}_t f(u_0^1)\right| = \left|\mathbb{E}\left(f(u_2(t)) - f(u_1(t))\right)\right| \leq \|f\|_L \, \mathbb{E}\left(d_H(u_1(t), u_2(t)) \wedge 1\right),$$

we deduce Theorem 2.1 from (2.5).

2.3. **Basic properties of couplings.**

In order to build a couple of cylindrical processes (W_1, W_2) such that (2.4) holds, we recall some basic results about the coupling. Such results are developed and motivated in subsections 1.1 and 1.2 of [28] through two simple and representative examples.
Let (Λ_1, Λ_2) be two distributions on a same space $(E, \mathcal{E})$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (Z_1, Z_2) be two random variables $(\Omega, \mathcal{F}) \to (E, \mathcal{E})$. We say that (Z_1, Z_2) is a coupling of (Λ_1, Λ_2) if $\Lambda_i = \mathcal{D}(Z_i)$ for $i = 1, 2$. We have denoted by $\mathcal{D}(Z_i)$ the law of the random variable Z_i.
Let Λ, Λ_1 and Λ_2 be three probability measures on a space $(E, \mathcal{E})$ such that Λ_1 and Λ_2 are absolutely continuous with respect to Λ. We set

$$d(\Lambda_1 \wedge \Lambda_2) = \left(\frac{d\Lambda_1}{d\Lambda} \wedge \frac{d\Lambda_2}{d\Lambda}\right) d\Lambda.$$

This definition does not depend on the choice of Λ and we have

$$\|\Lambda_1 - \Lambda_2\|_{var} = \frac{1}{2} \int_E \left|\frac{d\Lambda_1}{d\Lambda} - \frac{d\Lambda_2}{d\Lambda}\right| d\Lambda.$$

Next result is a fundamental result in the coupling methods, the proof is given for instance in the Appendix of [28].

Lemma 2.4. *Let (Λ_1, Λ_2) be two probability measures on $(E, \mathcal{E})$. Then*

$$\|\Lambda_1 - \Lambda_2\|_{var} = \min \mathbb{P}(Z_1 \neq Z_2).$$

The minimum is taken over all couplings (Z_1, Z_2) of (Λ_1, Λ_2). There exists a coupling which reaches the minimum value. It is called a maximal coupling and has the following property:

$$\mathbb{P}(Z_1 = Z_2, Z_1 \in \Gamma) = (\Lambda_1 \wedge \Lambda_2)(\Gamma) \quad \textit{for any } \Gamma \in \mathcal{E}.$$

It is interesting to remark that if Λ_1 is absolutely continuous with respect to Λ_2, we have

$$\|\Lambda_1-\Lambda_2\|_{var}\le\frac{1}{2}\sqrt{\int\left(\frac{d\Lambda_1}{d\Lambda_2}\right)^2 d\Lambda_2-1}. \tag{2.6}$$

In order to estimate the bound given in Lemma 2.4, we use either (2.6) or the following result which is lemma D.1 of [24] and which is very useful in order to bound below the probability that a maximal coupling get coupled.

Lemma 2.5. *Let Λ_1 and Λ_2 be two probability measures on a space $(E,\mathcal{E})$. Let A be an event of E. Assume that $\Lambda_1^A=\Lambda_1(A\cap .)$ is equivalent to $\Lambda_2^A=\Lambda_2(A\cap .)$. Then for any $p>1$*

$$I_p=\int_A\left(\frac{d\Lambda_1^A}{d\Lambda_2^A}\right)^{p+1}d\Lambda_2<\infty \quad \textit{implies} \quad (\Lambda_1\wedge\Lambda_2)(A)\ge\left(1-\frac{1}{p}\right)\left(\frac{\Lambda_1(A)^p}{pI_p}\right)^{\frac{1}{p-1}}.$$

Next result is a refinement of Lemma 2.4 used in [24] (see also Proposition 1.7 in [28]).

Proposition 2.6. *Let E and F be two polish spaces, $f_0:E\to F$ be a measurable map and (Λ_1,Λ_2) be two probability measures on E. We set*

$$\lambda_i=f_0^*\Lambda_i,\quad i=1,2.$$

Then there exist a coupling (V_1,V_2) of (Λ_1,Λ_2) such that $(f_0(V_1),f_0(V_2))$ is a maximal coupling of (λ_1,λ_2).

2.4. **Construction of the coupling.**

Now, using **A3**, **A4** and **A5**, we build (W_1,W_2) such that (2.4) holds. The construction can be done by induction. At each step, we construct a probability space $(\Omega_k,\mathcal{F}_k,\mathbb{P}_k)$ and a measurable couple of functions $(\omega_k,u^1_{kT},u^2_{kT},l)\to(V_i^k(\cdot,u^1_{kT},u^2_{kT},l))_{i=1,2}$ such that, for any (u^1_{kT},u^2_{kT},l), $(V_i^k(\cdot,u^1_{kT},u^2_{kT},l))_{i=1,2}$ is a coupling of $(\mathcal{D}(u(\cdot,kT,W,u^i_{kT}),W))_{i=1,2}$ on $[kT,(k+1)T]$.
Indeed, we first set

$$u_i(0)=u_0^i,\quad W_i(0)=0,\quad i=1,2.$$

Assuming that we have built $(u_i,W_i)_{i=1,2}$ on $[0,kT]$, we take $(V_i^k)_{i=1,2}$ as above independent of $(u_i,W_i)_{i=1,2}$ on $[0,kT]$ and we set

$$(u_i,W_i)=V_i^k(\cdot,u_1(kT),u_2(kT),l_0(k)\wedge k)\ \text{ on }\ [kT,(k+1)T].$$

Recall **A3**

$$\widetilde{u}(t,kT,W,u^1_{kT},u^2_{kT})=u(t,kT,W+H(\cdot,W),u^2_{kT}), \tag{2.7}$$

where

$$H(\cdot,W)=\int_{kT}^{\cdot}h(r,kT,W,u^1_{kT},u^2_{kT})dr.$$

The natural idea we have is to apply Proposition 2.6 to build a coupling $((\widetilde{u},W_1),(u_2,W_2))$ of $\left(\mathcal{D}\left(\widetilde{u}(\cdot,kT,W,u^1_{kT},u^2_{kT}),W\right),\mathcal{D}\left(u(\cdot,kT,W,u^2_{kT}),W\right)\right)$ on $[kT,(k+1)T]$ such that $(\widetilde{u},u_2)$ is a maximal coupling. The problem is that we do not known if the Novikov condition is verified. Hence it is difficult to estimate the distance between $\mathcal{D}\left(u(\cdot,kT,W,u^2_{kT})\right)$ and $\mathcal{D}\left(\widetilde{u}(\cdot,kT,W,u^1_{kT},u^2_{kT})\right)$. That is the reason why we introduce a truncation.

We set

$$\begin{cases} J(t,W) &= \int_{kT}^{t} \left|h(s,kT,W,u^1_{kT},u^2_{kT})\right|_U^2 ds, \\ \tau(W) &= \inf\left\{t\in[kT,(k+1)T] \;\middle|\; J(t,W) > C^*(d_0)e^{-\gamma(k-l)T} \right\}, \end{cases}$$

and we introduce the following laws on $[kT,(k+1)T]$

$$\Lambda_1 = \mathcal{D}\left(u(\cdot,kT,W+H(\tau(W)\wedge\cdot,W),u^2_{kT}),W\right),\ \Lambda_2 = \mathcal{D}\left(u(\cdot,kT,W,u^2_{kT}),W\right).$$

We denote by f_0 the first projection $f_0(u,W)=u$ and we set $\lambda_i = f_0^*\Lambda_i$. Now we can apply Proposition 2.6 and we obtain a coupling

$$((\bar u,W_1),(u_2,W_2)) \quad \text{of} \quad (\Lambda_1,\Lambda_2),$$

such that $(\bar u,u_2)$ is a maximal coupling of (λ_1,λ_2). Hence we obtain that for any $t\in[kT,(k+1)T]$, we have

$$u_2(t) = u(t,kT,W_2,u^2_{kT}),$$

and

$$\bar u(t) = u(t,kT,W_1+H(\tau(W_1)\wedge\cdot,W_1),u^2_{kT}). \tag{2.8}$$

We define

$$u_1 = u(\cdot,kT,W_1,u^1_{kT}) \text{ and } \widetilde u = \widetilde u(\cdot,kT,W_1,u^1_{kT},u^2_{kT}) \text{ on } [kT,(k+1)T]$$

and we deduce from (2.7) that

$$\widetilde u(t) = u(t,kT,W_1+H(\cdot,W_1),u^2_{kT}). \tag{2.9}$$

We compare (2.8) with (2.9) and we obtain that

$$\bar u = \widetilde u \text{ on } [kT,\tau(W_1)]. \tag{2.10}$$

2.5. **Proof of** (2.4)**.**
Recall that considering a trajectory $((u_1,W_1),(u_2,W_2))$ on $[0,kT]$ such that $l_0(k)=l$, we build the processes on $[kT,(k+1)T]$. Remind that for the probability $\mathbb{P}_k$, the trajectory is fixed on $[0,kT]$ and that the probability $\mathbb{P}$ is obtained by integrating $\mathbb{P}_k$ over the trajectories on $[0,kT]$.
In order to apply Lemmas 2.4 and 2.5 and inequality (2.6), we will establish that λ_1 and λ_2 are absolutely continuous and estimate

$$I_p = \int\left(\frac{d\lambda_1}{d\lambda_2}\right)^{p+1} d\lambda_2,\ p\ge 1.$$

The Novikov condition is obviously satisfied and we apply the following Girsanov Transform

$$d\hat{\mathbb{P}}_k = \exp\left(M(\tau(W_2),W_2) - \frac{1}{2}J(\tau(W_2),W_2)\right)d\mathbb{P}_k,$$

where

$$M(t,W_2) = \int_{kT}^{t} h(s,kT,W_2,u^1_{kT},u^2_{kT})dW_2(s).$$

Then

$$\hat W_2 = W_2 + H(\tau(W_2)\wedge\cdot,W_2),$$

is a cylindrical Wiener process on U for the probability $\hat{\mathbb{P}}_k$. The law of u_2 under $\hat{\mathbb{P}}_k$ is λ_1. Hence λ_1 and λ_2 are absolutely continuous and

$$I_p \le \mathbb{E}\left(\exp\left(c_pJ(\tau(W_2),W_2)\right)\right) \le \exp\left(C_p(d_0)e^{-\gamma(k-l)T}\right). \tag{2.11}$$

We first establish (2.4) when $k = l$. Applying Lemmas 2.4 and 2.5 to the maximal coupling $(\bar{u}, u_2)$ of (λ_1, λ_2), we obtain for any $p > 1$

$$\begin{aligned}\mathbb{P}_k\left(\bar{u} = u_2 \text{ and } J((k+1)T, W_1) \leq C^*(d_0)\right)\\ \geq \left(1 - \tfrac{1}{p}\right)(pI_p)^{-\frac{1}{p-1}}\left(\mathbb{P}_k\left(J((k+1)T, W_1) \leq C^*(d_0)\right)\right)^{\frac{p}{p-1}}.\end{aligned} \tag{2.12}$$

Since $J(\cdot, W_1)$ is increasing, we deduce from (2.10) that

$$J((k+1)T, W_1) \leq C^*(d_0) \quad \text{implies} \quad \widetilde{u} = \bar{u} \text{ on } [kT, (k+1)T].$$

Hence, taking into account (2.11), (2.12) and **A5**, we obtain there exists $p_0 > 0$ not depending on k, T such that

$$\mathbb{P}_k\left(\widetilde{u} = u_2\right) \geq \mathbb{P}_k\left(\bar{u} = u_2 \text{ and } J((k+1)T, W_1) \leq C^*(d_0)\right) \geq p_0.$$

Integrating the previous inequality over $l_0(l) = l$, it follows

$$\mathbb{P}\left(l_0(l+1) = l \mid l_0(l) = l\right) \geq p_0. \tag{2.13}$$

We now treat the case $k > l$. It follows from the definition of τ that

$$J(\tau(W_1), W_1) < C^*(d_0)e^{-\gamma(k-l)T} \text{ implies } \tau(W_1) = (k+1)T.$$

Then we infer from (2.10) that

$$J(\tau(W_1), W_1) < C^*(d_0)e^{-\gamma(k-l)T} \text{ implies } \widetilde{u} = \bar{u} \text{ on } [kT, (k+1)T].$$

Hence

$$\mathbb{P}_k\left(\widetilde{u} \neq u_2\right) \leq \mathbb{P}_k\left(J(\tau(W_1), W_1) \geq C^*(d_0)e^{-\gamma(k-l)T} \text{ and } \bar{u} = u_2\right) + \mathbb{P}_k\left(\bar{u} \neq u_2\right).$$

We deduce from (2.10) that

$$\mathbb{P}_k\left(\widetilde{u} \neq u_2\right) \leq \mathbb{P}_k\left(\bar{u} \neq u_2\right) + \mathbb{P}_k\left(B\right), \tag{2.14}$$

where

$$B = \left\{(W_1, W_2) \,\middle|\, J(\tau(W_1), W_1) \geq C^*(d_0)e^{-\gamma(k-l)T} \text{ and } \widetilde{u} = u_2 \text{ on } [kT, \tau(W_1)]\right\}.$$

Applying successively Lemma 2.4 to the maximal coupling $(\bar{u}, u_2)$ of (λ_1, λ_2) and inequality (2.6), we obtain

$$\mathbb{P}_k\left(\bar{u} \neq u_2\right) \leq \|\lambda_1 - \lambda_2\|_{var} \leq \frac{1}{2}\sqrt{I_1 - 1}.$$

It follows from (2.11) that

$$\mathbb{P}_k\left(\bar{u} \neq u_2\right) \leq C_1(d_0)e^{-\frac{\gamma}{2}(k-l)T}. \tag{2.15}$$

Combining (2.15) and (2.14), we obtain

$$\mathbb{P}_k\left(\widetilde{u} \neq u_2\right) \leq C_1(d_0)e^{-\frac{\gamma}{2}(k-l)T} + \mathbb{P}_k\left(B\right).$$

Integrating over $l_0(k) = l$, we deduce from **A4** that

$$\mathbb{P}\left(l_0(k+1) \neq l \text{ and } l_0(k) = l \mid l_0(l) = l\right) \leq C'(d_0)e^{-\frac{\gamma}{2}(k-l)T}. \tag{2.16}$$

We choose T_1 such that

$$1 - p_0 + \sum_{q=1}^{\infty} C'(d_0)e^{-\frac{\gamma}{2}qT_1} \leq 1 - \frac{p_0}{2},$$

and take into account (2.13) and (2.16), we obtain for $T \geq T_1$

$$\mathbb{P}\left(l_0(k) \neq l \mid l_0(l) = l\right) \leq \sum_{i=l}^{k-1} \mathbb{P}\left(l_0(i+1) \neq l \text{ and } l_0(i) = l \mid l_0(l) = l\right) \leq 1 - \frac{p_0}{2}.$$

Hence, we deduce from (2.16) that

$$\mathbb{P}\left(l_0(k+1) \neq l \mid l_0(k) = l\right) \leq 2\frac{C'(d_0)}{p_0} e^{-\frac{\gamma}{2}(k-l)T}. \tag{2.17}$$

Choosing $T_0 \geq T_1$ sufficiently high allows to conclude the case $k > l$.

3. The Complex Ginzburg–Landau equation with a locally Lipschitz noise

The aim of this section is to apply our method to the stochastic CGL equation with Dirichlet boundary conditions and with a locally Lipschitz noise.
Let us recall that it has the form

$$\left\{ \begin{array}{rcl} du + (\varepsilon + \mathrm{i})(-\Delta)u\, dt + (\eta + \lambda \mathrm{i}) \left|u\right|^{2\sigma} u\, dt & = & \phi(u)dW + f dt, \\ u(t,x) & = & 0, \quad \text{for } x \in \delta D, t > s, \\ u(s,x) & = & u_s(x), \text{ for } x \in D, \end{array} \right. \tag{3.1}$$

where D is an open bounded domain of $\mathbb{R}^d$ with regular boundary or $D = (0,1)^d$, where $\varepsilon > 0$, $\eta > 0$, $\lambda \in \{-1, 1\}$ and where we impose the L^2–subcritical condition $\sigma d < 2$. For simplicity, we also assume $\sigma < \frac{3}{2}$. For simplicity in the redaction, we consider the case $f = 0$, where f is the deterministic part of the forcing term $\phi(u)dW + f dt$. The generalization to a square integrable f is easy.
Ergodicity for the stochastic CGL equation is established in [1] when the noise is invertible and in [14] for the one-dimensional cubic case when the noise is diagonal, does not depend on the solution and is smooth in space. Then, in [28], we have established exponential mixing of CGL driven by a noise which verifies **Ha**, **Hb** and **Hc** under the L^2 or the H^1–subcritical conditions.
As explained in Remark 1.6 and in section 2.1, technics of section 1 can easily be applied to the stochastic CGL equation with a globally Lipschitz noise. It gives exponential mixing in L^2 under the L^2–subcritical condition $\sigma d < 2$. Moreover, one can obtain the exponential mixing in H^1 under the H^1–subcritical condition $\sigma < \frac{2}{d-2}$ when $\lambda = 1$. Using a polynomial version of our criterion, one can obtain polynomial mixing in H^1 under the L^2–subcritical condition $\sigma d < 2$ when $\lambda = -1$.

As explained in Remark 1.6, it seems that such technics can not always be applied when there is no analogous property to Proposition 1.5. For instance, the case of the stochastic non-linear Schrödinger equation requires more sophisticated tools and will be treated in a forthcoming paper. We study the CGL equation with a locally Lipschitz noise because it gives a simple example of SPDE for which the difference of two solutions cannot be estimated with the help of only one energy, an essential ingredient in Proposition 1.5.

Remark 3.1. *The condition $\sigma < \frac{3}{2}$ is artificial. We have introduced it because Corollary 3.9 below is not not true for $\sigma \geq \frac{3}{2}$. But, the proof could be adapted when working in H^1. Actually, when $\sigma < \frac{2}{d-2}$ and $\lambda = 1$, we have exponential mixing in H^1. Moreover, when $\sigma d < 2$ and $\lambda = -1$, we have polynomial mixing in H^1. The reason why we do not establish those results is that the computations are longer.*

3.1. **Notations and Main result.**
We set

$$H = L^2(D;\mathbb{C}),\quad A = -\Delta,\quad D(A) = H_0^1(D;\mathbb{C})\cap H^2(D;\mathbb{C}),$$

and we denote by $|\cdot|$, $|\cdot|_p$, $\|\cdot\|$ and $\|\cdot\|_s$ the norm of $\mathbb{C}$, $L^p(D;\mathbb{C})$, $H_0^1(D;\mathbb{C})$ and $H^s(D;\mathbb{C})$. The norm of H will be denoted by $|\cdot|$ when no confusion is possible or $|\cdot|_H$ otherwise.
Now we can write problem (3.1) in the form

$$\left\{\begin{array}{rcl} du + (\varepsilon+\mathrm{i})Au\,dt + (\eta+\lambda\mathrm{i})\,|u|^{2\sigma}\,u\,dt &=& \phi(u)dW,\\ u(s) &=& u_s,\end{array}\right. \tag{3.2}$$

where W is a cylindrical Wiener process on a Hilbert U.
In order to have existence and uniqueness of the solution of (3.2), we make the following assumption
H0' *The function $\phi : H \to \mathcal{L}_2(U;H)$ is bounded and local Lypschitz. More precisely, we assume there exists $L>0$ such that for any $(u_1,u_2)\in H^2$*

$$\|\phi(u_2)-\phi(u_1)\|^2_{\mathcal{L}_2(U;H)} \le L\,|u_2-u_1|^2\left(1+|u_1|^{2\sigma}+|u_2|^{2\sigma}\right).$$

We set

$$B_1 = 1 + \sup_{u\in H}\|\phi(u)\|^2_{\mathcal{L}_2(U;H)}.$$

Under **H0'**, we have existence and uniqueness of the solution of (3.2) in H when $u_s\in H$. Moreover there exists a measurable map u such that $u(\cdot,s,W,u_s)$ is the unique solution of (3.2). This result ensures the strong Markov property of the solutions of (3.2). We denote by $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ the Markov transition semi-group associated to the solutions of (3.2).
In our computations, we use the following energy

$$E_u(T,t_0) = |u(T)|^2 + \varepsilon\int_{t_0}^T \|u(t)\|^2\,dt + \eta\int_{t_0}^T |u(t)|^{2\sigma+2}_{2\sigma+2}\,dt,\qquad \text{for any } T\ge t_0.$$

It is well-known that $(A,\mathcal{D}(A))$ is a self-adjoint operator with discrete spectrum. We consider $(e_n)_n$ an eigenbasis of H associated to the increasing sequence $(\mu_n)_n$ of eigenvalues of $(A,\mathcal{D}(A))$. We denote by P_N and Q_N the orthogonal projection in H onto the space $Sp(e_k)_{1\le N}$ and onto its complementary, respectively.
Now, we state the assumption which gives the exponential mixing of $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ provided it holds for N sufficiently high.
H1 *There exists a bounded measurable map $g : H \to \mathcal{L}(H;U)$ such that for any $u\in H$*

$$\phi(u)g(u) = P_N.$$

The aim of this section is to establish the following result.

Theorem 3.2. *Assume* **H0'**. *There exists $N_0(B_1,\varepsilon,\eta,\sigma,D,L)$ such that, if* **H1** *holds with $N\ge N_0$, then there exists a unique stationary probability measure μ of $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ on H. Moreover, for any $q\in\mathbb{N}^*$, μ satisfies*

$$\int_H |u|^{2q}\,d\mu(u) < \infty, \tag{3.3}$$

and there exist $C,\gamma>0$ such that for any $\lambda\in\mathcal{P}(H)$

$$\|\mathcal{P}_t^*\lambda - \mu\|_* \le Ce^{-\gamma t}\left(1+\int_H |u|^2\,d\lambda(u)\right). \tag{3.4}$$

Remark 3.3. *There exists a lot of variations of Theorem 3.2. Hence* (3.3) *could be strengthened into*

$$\int_H \exp\left(\alpha_1(B_1,\varepsilon,\eta,D)\,|u|^2\right)d\mu(u)<\infty.$$

Moreover, one can work in $H_0^1(D;\mathbb{C})$ in the defocusing case under the H^1–subcritical condition $\sigma<\frac{2}{d-2}$ and in the focusing case under the subcritical condition $\sigma<\frac{2}{d}$. In the focusing case, the rate of convergence is greater than any power of time instead of being exponential in time.
In **H1**, *the boundedness of g could be replaced by the existence of C such that*

$$|g(u)|_{\mathcal{L}(H;U)}\le C\exp\left(\frac{\varepsilon\mu_1}{4}\,|u|^2\right).$$

Contrary to Navier–Stokes, the coefficient in the exponential cannot be as high as we want because it seems that for the locally Lipschitz CGL, there is no property analogous to Proposition 1.5.
In **H0'**, *the boundedness of ϕ could be replaced by $|\phi(u)|\le C(1+|u|^\gamma)$. If $\gamma\le\sigma$, then the rate of convergence remains exponential. If $\gamma<\sigma+1$, then the rate of convergence becomes greater than any power of time instead of being exponential in time. Moreover for any p there exists c_p such that if there exists C such that $|\phi(u)|\le C+c_p\,|u|^{\sigma+1}$, then the rate of convergence is greater than $(1+t)^{-p}$.*

3.2. **Construction of the auxiliary process.**
Now, we build the auxiliary process $\widetilde{u}$ such that assumptions **A0**,...,**A5** are true. This will allow to deduce Theorem 3.2 from Theorem 2.1.
We first recall the following classical result which ensures **A1**

Lemma 3.4 (The Lyapunov structure)**.** *Assume* **H0'**. *For any $q\in\mathbb{N}$, there exists $C_q>0$ only depending on q,ε,η,B_1 and D such that*

$$\begin{cases}\mathbb{E}\left(\mathcal{H}(u(t,s,W,u_s))^q\right) & \le & e^{-\nu\mu_1 q(t-s)}\mathcal{H}(u_s)^q+C_q,\\ \mathbb{E}\left(\mathcal{H}(u(\tau,s,W,u_s))^q\right) & \le & \mathcal{H}(u_s)^q+C_q\mathbb{E}\left(\tau-s\right),\end{cases}$$

for any $t\ge s$ and any finite stopping time $\tau\ge s$.

This result is a consequence of **H0'**. Its proof is analogous to the one of Lemma 1.10.
Let K be a positive number. We set

$$F(u)=(\varepsilon+\mathrm{i})Au+(\eta+\lambda\mathrm{i})\,|u|^{2\sigma}u,$$

and

$$\begin{aligned}\delta(\widetilde{u},u_1)=P_N\Big((\eta+\lambda\mathrm{i})\left(|u_1|^{2\sigma}u_1-|\widetilde{u}|^{2\sigma}\widetilde{u}\right)+K\,|P_N\widetilde{u}|^{2\sigma}P_N\left(\widetilde{u}-u_1\right)\Big)\\ +L\left(1+|\widetilde{u}|_H^{2\sigma}+|u_1|_H^{2\sigma}\right)P_N\left(\widetilde{u}-u_1\right).\end{aligned}\tag{3.5}$$

We now consider the following equation

$$\begin{cases}d\widetilde{u}+F(\widetilde{u})dt+\delta(\widetilde{u},u(t,s,W,u_s^1))dt & = & \phi(\widetilde{u})dW,\\ \widetilde{u}(s) & = & u_s^2.\end{cases}\tag{3.6}$$

It is not difficult to deduce from **H0'** that there exists a unique H–valued solution to (3.6) when $(u_s^1, u_s^2) \in H^2$. Moreover, there exists a measurable map $\widetilde{u}$ such that $\widetilde{u}(\cdot, s, W, u_s^1, u_s^2)$ is the unique solution of (3.6). Actually, we deduce that $(u(\cdot, s, W, u_s^1), \widetilde{u}(\cdot, s, W, u_s^1, u_s^2))$ is a strong Markov process. Hence **A0** is true.
Taking into account **H1** and the uniqueness of the solution of (3.2) under **H0'**, we deduce **A3** by setting

$$h(t, s, W, u_s^1, u_s^2) = -g(\widetilde{u}(t, s, W, u_s^1, u_s^2))\delta(\widetilde{u}(t, s, W, u_s^1, u_s^2), u(t, s, W, u_s^1)). \tag{3.7}$$

We now state a result which replaces Proposition 1.5.

Proposition 3.5. *Assume* **H0'**. *There exists $(\Lambda_q)_q, \alpha > 0$ only depending on L, ε, η and D such that for any finite stopping time $\tau \geq s$ and any $q \in \mathbb{N}$*

$$\mathbb{E}\left(|r(\tau)|^{2q} \exp\left(\varepsilon q \mu_1(\tau - s) - \frac{\Lambda_q}{\mu_{N+1}^{\alpha}}\left(\tau - s + E_{\widetilde{u}}(\tau, s) + E_{u_1}(\tau, s)\right)\right)\right) \leq |r(s)|^{2q},$$

where

$$r = \widetilde{u} - u_1, \quad \widetilde{u} = \widetilde{u}(\cdot, s, W, u_s^1, u_s^2), \quad u_1 = u(\cdot, s, W, u_s^1).$$

The proof is postponed to Section 4.3. Remark that Proposition 3.5 is weaker than Proposition 1.5 because it involves energies of both u_1 and $\widetilde{u}$. Then, in order to apply it, we have to establish energy control of both u_1 and $\widetilde{u}$. That is the reason why the building of the auxiliary process is more complicated than for NS or for CGL with a global Lipschitz noise. For instance, if we set

$$\delta(\widetilde{u}, u_1) = K P_N(\widetilde{u} - u_1),$$

then we would have an energy Lemma on $\widetilde{u}$ and a Proposition analogous to Proposition 3.5. The problem is that the constants of both results would depend on the value of K. Then it does not seem to be possible to establish that there exists K and N such that we can combine both results.
Since Proposition 3.5 is not depending on K, it is possible to combine it with the two following energy Lemmas.

Lemma 3.6 (Exponential estimate for the growth of the solution)**.** *Assume* **H0**. *There exists $\gamma_0 > 0$ only depending on ε, η, B_1 and D such that*

$$\mathbb{P}\left(\sup_{t \geq T+s}\left(E_{u(\cdot, s, W, u_s)}(t, s) - B_1(t - s)\right) \geq \rho + |u_s|^2\right) \leq e^{-\gamma_0(\rho + T)},$$

for any $u_s \in H$ and $(\rho, T, s) \in (\mathbb{R}^+) \times \mathbb{R}$.

The proof of Lemma 3.6 is analogous to the proof of Lemma 1.9.

Lemma 3.7 (Exponential estimate for the growth of the auxiliary process)**.** *Assume* **H0**. *There exists K_L, $c_{K,L}$ not depending on N such that if $K \geq K_L$*

$$\mathbb{P}\left(\sup_{t \geq T+s}\left(E_{\widetilde{u}}(t, s) - c_{K,L}(E_{u_1}(t, s) + B_1(t - s))\right) \geq \rho + \left|u_s^2\right|^2\right) \leq e^{-\gamma_0(\rho + T)},$$

where

$$\widetilde{u} = \widetilde{u}(\cdot, s, W, u_s^1, u_s^2), \quad u_1 = u(\cdot, s, W, u_s^1),$$

for any $(u_s, u_s^2) \in H^2$ and $(\rho, T, s) \in (\mathbb{R}^+)^2 \times \mathbb{R}$.

The proof is postponed to section 4.5.
We immediately deduce the two following very useful corollaries of Proposition 3.5. The first Corollary combined with Lemmas 3.6 and 3.7 implies **A2**. We set

$$C_\rho^*(d_0) = d_0 \exp\left(\frac{\varepsilon\mu_1}{4}(d_0+\rho)\right).$$

Corollary 3.8. *Assume* **H0'**. *There exists* $N_1(B_1,\varepsilon,\eta,D,L)$ *such that if* $N \geq N_1$, *then for any* $(a,s)\in\mathbb{R}^2$ *and* $t \geq s$

$$\mathbb{P}\left(|r(t)| \geq C_\rho^*(d_0)\, e^{a-\frac{\varepsilon\mu_1}{4}(t-s)} \text{ and } A_1 \text{ and } \hat{A}\right) \leq e^{-a-\frac{\varepsilon\mu_1}{4}(t-s)},$$

where

$$\begin{cases} d_0 & \geq & \left|u_s^1\right|^2 + \left|u_s^2\right|^2, \\ A_1 & = & \left\{W \,\middle|\, E_{u_1}(t,s) \leq \rho + \left|u_s^1\right|^2 + B_1(t-s)\right\}, \\ \hat{A} & = & \left\{W \,\middle|\, E_{\widetilde{u}}(t,s) \leq \rho + \left|u_s^2\right|^2 + B_1(t-s)\right\}. \end{cases}$$

Corollary 3.8 is deduced from Proposition 3.5 by applying a Chebyshev inequality. In order to prove **A4** and **A5**, we give the following Corollary of Proposition 3.5 whose proof is given in Section 4.4.

Corollary 3.9. *Assume* **H0'** *and* **H1**. *There exists* $N_2(B,B_1,\varepsilon,\eta,D,L)$ *and* K'_N *such that if* $N\geq N_2$ *holds, then for any* $(s,a)\in\mathbb{R}^2$, $T\geq s$ *and any* $B,c>0$

$$\mathbb{P}\left(\int_T^\tau \left|h(t,s,W,u_s^1,u_s^2)\right|_U^2 dt \geq K'_N C_\rho^*(cd_0)\, e^{a-\frac{\varepsilon\mu_1}{4}(T-s)}\right) \leq e^{-a-\frac{\varepsilon\mu_1}{4}(T-s)},$$

provided $\tau \geq T$ *is a stopping time such that for any* $t\in(T,\tau)$

$$\begin{cases} d_0 & \geq & \left|u_s^1\right|^2 + \left|u_s^2\right|^2, \\ E_{u_1}(t,s) & \leq & \rho + \left|u_s^1\right|^2 + B_1(t-s), \\ E_{\widetilde{u}}(t,s) & \leq & \rho + c\left(\left|u_s^1\right|^2 + \left|u_s^2\right|^2\right) + B(t-s). \end{cases}$$

Taking

$$\begin{cases} B_0 = 2c_{K_L,L}B_1, \quad K = K_L, \\ N_0(B_1,\varepsilon,\eta,D,L) = \max(N_1(B_1,\varepsilon,\eta,D,L), N_2(B_0,B_1,\varepsilon,\eta,D,L)), \end{cases} \tag{3.8}$$

we have **A0**,..., **A5**. Hence, applying Theorem 2.1, we obtain Theorem 3.2.

4. Proof of technical results.

This section is devoted to the proof of results concerning NS and CGL.

4.1. **Proof of Lemmas 1.9 and 1.10.**

We set $u = u(\cdot,s,W,u_s)$. We apply Ito Formula to $|u|^2$

$$d|u|^2 + 2\nu\|u\|^2 dt + 2(u,B(u))\,dt = 2(u,\phi(u)dW) + \|\phi(u)\|^2_{\mathcal{L}_2(U;H)}dt.$$

Recall that

$$(u,B(u)) = 0.$$

Hence, taking into account **H0**, we deduce

$$d|u|^2 + 2\nu \|u\|^2 dt = 2(u, \phi(u)dW) + (B_0 - 1)dt. \tag{4.1}$$

Applying Ito formula to $|u|^{2q}$, it follows from (4.1) that

$$\begin{aligned} d|u|^{2q} + 2\nu q |u|^{2(q-1)} \|u\|^2 dt \leq 2q |u|^{2(q-1)} (u, \phi(u)dW) \\ +2q(q-1)|u|^{2(q-2)} |\phi(u)^* u|^2 dt + q(B_0 - 1)|u|^{2(q-1)} dt. \end{aligned}$$

Hence, we deduce from **H0** and a arithmetico-geometric inequality that there exists C_q such that

$$d|u|^{2q} + \nu q |u|^{2(q-1)} \|u\|^2 dt \leq 2q |u|^{2(q-1)} (u, \phi(u)dW) + C_q dt.$$

Integrating and taking the expectation, we establish Lemma 1.10.
We set for any γ_0

$$M(t) = 2\int_s^t (u(r), \phi(u(r))dW(r)), \quad \mathcal{M}(t) = M(t) - \frac{\gamma_0}{2} \langle M \rangle (t).$$

Remarking that

$$d\langle M \rangle = 4 |\phi(u)^* u|^2 dt \leq 4cB_0 \|u\|^2 dt,$$

and setting $\gamma_0 = \frac{\nu}{2cB_0}$, we deduce from (4.1) that

$$E_u(t,s) \leq \mathcal{M}(t) + |u_s|^2 + (B_0 - 1)(t - s). \tag{4.2}$$

Remarking that $\left(e^{\gamma_0 \mathcal{M}(t)}\right)_{t \geq s}$ is a positive supermartingal whose value is 1 at time s, we deduce from maximal supermartingal inequality that

$$\mathbb{P}\left(\sup_{t\geq s} \mathcal{M}(t) \geq \rho'\right) \leq \mathbb{P}\left(\sup_{t\geq s} e^{\gamma_0 \mathcal{M}(t)} \geq e^{\gamma_0 \rho'}\right) \leq e^{-\gamma_0 \rho'}. \tag{4.3}$$

Hence, applying (4.3) to (4.2) with $\rho' = \rho + T_0$, We deduce Lemma 1.9.

4.2. **Proof of Proposition 1.5.**
For any function f, we denote by $\delta f(u)$ the value $f(\widetilde{u}) - f(u_1)$. Taking the difference between (1.2) and (1.9), we obtain

$$dr + \nu A r dt + K P_N r dt + \delta B(u) dt = \delta\phi(u) dW.$$

Hence, applying Ito Formula to $|r|^2$, we have

$$d|r|^2 + 2\left(\nu \|r\|^2 + K|P_N r|^2\right) dt = 2(r, \delta\phi(u)dW) + I(t)dt, \tag{4.4}$$

where

$$I(t) = I_1(t) + I_2(t), \quad I_1(t) = -2(r, \delta B(u)), \quad I_2(t) = \|\delta\phi(u)\|^2_{\mathcal{L}_2(U;H)}.$$

Remarking that

$$\delta B(u) = \pi\left((\widetilde{u}, \nabla)r + (r, \nabla)u_1\right), \quad (r, (\widetilde{u}, \nabla)r) = 0,$$

we deduce from a Schwartz inequality that

$$I_1(t) \leq c\left|r^2\right| \|u_1\| = c|r|_4^2 \|u_1\|.$$

It follows from Sobolev Embedding $H^{\frac{1}{2}}\left(D, \mathbb{R}^2\right) \subset L^4\left(D, \mathbb{R}^2\right)$ and interpolatory inequality that

$$I_1(t) \leq c\|r\|^2_{\frac{1}{2}} \|u_1\| \leq c\|r\| |r| \|u_1\|.$$

We infer from an arithmetico-geometric inequality that

$$I_1(t) \leq \nu \|r\|^2 + c\,|r|^2 \|u_1\|^2 . \tag{4.5}$$

Applying **H0**, we obtain

$$I_2(t) \leq L\,|r|^2 . \tag{4.6}$$

Remarking that

$$(K \wedge (\nu\mu_{N+1}))\,|r|^2 \leq \nu \|Q_N r\|^2 + K\,|P_N r|^2 \leq \nu \|r\|^2 + K\,|P_N r|^2 ,$$

we deduce from (4.4), (4.5) and (4.6) that there exists Λ such that

$$d\,|r|^2 + \left((K \wedge (\nu\mu_{N+1}) - L) - \Lambda \|u_1\|^2\right) |r|^2\, dt \leq 2\,(r, \delta\phi(u)dW) .$$

Integrating this formula and taking the expectation, we establish Proposition 1.5.

4.3. **Proof of Proposition 3.5.**

For any function f, we denote by $\delta f(u)$ the value $f(\widetilde{u}) - f(u_1)$. Taking the difference between (1.2) and (1.9), we obtain

$$\begin{aligned} dr + (\varepsilon + \mathrm{i})Ardt + L\left(1 + |\widetilde{u}|_H^{2\sigma} + |u_1|_H^{2\sigma}\right) P_N r dt + K P_N \left(|P_N\widetilde{u}|^{2\sigma} P_N r\right) dt \\ = \delta\phi(u)dW - (\eta + \lambda\mathrm{i}) Q_N \delta\left(|u|^{2\sigma} u\right) dt. \end{aligned}$$

Hence, applying Ito Formula to $|r|^2$, we have

$$\begin{aligned} d\,|r|^2 + 2\varepsilon \|r\|^2 dt + 2L\left(1 + |\widetilde{u}|^{2\sigma} + |u_1|^{2\sigma}\right) |P_N r|^2 dt \leq 2\,(r, \delta\phi(u)dW) \\ -2\left(Q_N r, (\eta + \lambda\mathrm{i})\delta\left(|u|^{2\sigma} u\right)\right) dt + \|\delta\phi(u)\|^2_{\mathcal{L}_2(U;H)}\, dt. \end{aligned} \tag{4.7}$$

Remarking that for any $(x, y) \in \mathbb{C}^2$

$$\left| |x|^{2\sigma} x - |y|^{2\sigma} y \right| \leq C_\sigma \left(|x|^{2\sigma} + |y|^{2\sigma}\right) |x - y|^2 , \tag{4.8}$$

it follows from Hölder inequality

$$-\left(Q_N r, (\eta + \lambda\mathrm{i})\delta\left(|u|^{2\sigma} u\right)\right) \leq c\,|Q_N r|_{2\sigma+2}\, |r|_{2\sigma+2} \left(|\widetilde{u}|^{2\sigma}_{2\sigma+2} + |u_1|^{2\sigma}_{2\sigma+2}\right).$$

Setting

$$s_0 = \frac{\sigma d}{2\sigma + 2}, \quad s_+ = \frac{1}{\sigma + 1},$$

we deduce from Sobolev Embedding $H^{s_0}(D;\mathbb{C}) \subset L^{2\sigma+2}(D;\mathbb{C})$ that

$$-\left(Q_N r, (\eta + \lambda\mathrm{i})\delta\left(|u|^{2\sigma} u\right)\right) \leq c\,\|Q_N r\|_{s_0} \|r\|_{s_0} \left(|\widetilde{u}|^{2\sigma}_{2\sigma+2} + |u_1|^{2\sigma}_{2\sigma+2}\right).$$

Remarking that $s_0 < s_+ < 1$ and that $\|Q_N r\|_{s_+} \leq \mu_{N+1}^{\frac{s_+ - s_0}{2}} \|Q_N r\|_{s_0}$, we deduce from interpolatory inequality

$$-\left(Q_N r, (\eta + \lambda\mathrm{i})\delta\left(|u|^{2\sigma} u\right)\right) \leq c\mu_{N+1}^{-\frac{s_+ - s_0}{2}} \|r\|^{2s_+} |r|^{2(1-s_+)} \left(|\widetilde{u}|^{2\sigma}_{2\sigma+2} + |u_1|^{2\sigma}_{2\sigma+2}\right).$$

Hence, it follows from arithmetico-geometric inequality that there exists $\alpha \in (0,1)$ only depending on σ and d such that

$$-\left(Q_N r, (\eta + \lambda\mathrm{i})\delta\left(|u|^{2\sigma} u\right)\right) \leq \frac{\varepsilon}{2} \|r\|^2 + \frac{c}{\mu_{N+1}^{\alpha}} |r|^2 \left(|\widetilde{u}|^{2\sigma+2}_{2\sigma+2} + |u_1|^{2\sigma+2}_{2\sigma+2}\right). \tag{4.9}$$

Recall **H0'**,

$$\|\delta\phi(u)\|^2_{\mathcal{L}_2(U;H)} \leq L\left(1+|\widetilde{u}|^{2\sigma}+|u_1|^{2\sigma}\right)|r|^2 .$$

Remarking that

$$|r|^2 \leq |P_N r|^2 + \frac{1}{\mu_{N+1}^{s_+}}\|Q_N r\|^2_{s_+},$$

and making interpolatory inequality analogous to those done to obtain (4.9), we obtain

$$\begin{aligned}\|\delta\phi(u)\|^2_{\mathcal{L}_2(U;H)} \leq \tfrac{\varepsilon}{4}\|r\|^2 + \tfrac{c}{\mu_{N+1}^{\alpha}}|r|^2\left(|\widetilde{u}|^{2\sigma+2}_{2\sigma+2}+|u_1|^{2\sigma+2}_{2\sigma+2}\right)\\ +L\left(1+|\widetilde{u}|^{2\sigma}+|u_1|^{2\sigma}\right)|P_N r|^2 .\end{aligned} \tag{4.10}$$

Hence, setting

$$V(t) = 1 + |\widetilde{u}(t)|^{2\sigma+2}_{2\sigma+2} + |u_1(t)|^{2\sigma+2}_{2\sigma+2},$$

we deduce from (4.7), (4.9) and (4.10) that there exists $\Lambda_1 > 0$ such that

$$\begin{aligned}d|r|^2 + \left(\tfrac{5}{4}\varepsilon\|r\|^2 - \tfrac{\Lambda_1}{\mu_{N+1}^{\alpha}}V|r|^2 + L\left(1+|\widetilde{u}|^{2\sigma}+|u_1|^{2\sigma}\right)|P_N r|^2\right)dt\\ \leq 2\left(r,\delta\phi(u)dW\right)dt.\end{aligned}$$

We infer from Ito Formula of $|r|^{2q}$ that

$$\begin{aligned}d|r|^{2q} + \tfrac{5}{4}\varepsilon q|r|^{2(q-1)}\|r\|^2 dt + qL\left(1+|\widetilde{u}|^{2\sigma}+|u_1|^{2\sigma}\right)|r|^{2(q-1)}|P_N r|^2 dt\\ -q\tfrac{\Lambda_1}{\mu_{N+1}^{\alpha}}V dt \leq dM + 2q(q-1)|r|^{2(q-2)}|\delta\phi(u)^* r|^2 dt,\end{aligned} \tag{4.11}$$

where

$$dM = 2q|r|^{2(q-1)}\left(r,\delta\phi(u)dW\right).$$

Applying **H0'** and making analogous calculus than those done to obtain (4.10), we obtain

$$\begin{aligned}2q(q-1)|\delta\phi(u)^* r|^2_{\mathcal{L}_2(U;H)} \leq \tfrac{\varepsilon}{4}|r|^2\|r\|^2 + \tfrac{c}{\mu_{N+1}^{\alpha}}|r|^4\left(|\widetilde{u}|^{2\sigma+2}_{2\sigma+2}+|u_1|^{2\sigma+2}_{2\sigma+2}\right)\\ +qL\left(1+|\widetilde{u}|^{2\sigma}+|u_1|^{2\sigma}\right)|r|^2|P_N r|^2 .\end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12), we obtain that there exists Λ_q such that

$$d|r|^{2q} + \left(\varepsilon q\mu_1 - \tfrac{\Lambda_q}{\mu_{N+1}^{\alpha}}V\right)|r|^{2q}dt \leq 2q|r|^{2(q-1)}\left(r,\delta\phi(u)dW\right). \tag{4.13}$$

Integrating this formula and taking the expectation, we establish Proposition 3.5.

4.4. **Proof of Corollary 3.9.**

Taking into account **H1** and (3.7) and applying a chebyshev inequality, we remark that it is sufficient to estimate

$$\mathbb{E}\int_T^{\tau}|\delta(\widetilde{u}(t),u_1(t))|^2 dt.$$

Recalling (3.5), we deduce that it is sufficient to estimate the three following values

$$\begin{array}{rcl} I_1 & = & \mathbb{E}\int_T^\tau \left|P_N\left(|u_1|^{2\sigma}u_1-|\widetilde{u}|^{2\sigma}\widetilde{u}\right)\right|^2 dt,\\ I_2 & = & \mathbb{E}\int_T^\tau \left|P_N\left(|P_N\widetilde{u}|^{2\sigma}P_Nr\right)\right|^2 dt,\\ I_3 & = & \mathbb{E}\int_T^\tau \left(1+|\widetilde{u}|^{4\sigma}+|u_1|^{4\sigma}\right)|P_Nr|^2 dt. \end{array}$$

Recalling the definition of τ, we remark that

$$I_3 \le 4\mathbb{E}\int_T^\tau (\rho+(B+B_1)(t-s))^{2\sigma}|P_Nr|^2 dt.$$

Integrating on time the result of Proposition 3.5, it follows that for N sufficiently high

$$I_3 \le C_\rho^*(d_0)e^{-\frac{\varepsilon\mu_1}{2}(T-s)}. \tag{4.14}$$

Applying successively Hölder inequality and the equivalence of the norm in finite-dimensional spaces, it follows

$$\left|P_N\left(|P_N\widetilde{u}|^{2\sigma}P_Nr\right)\right|^2 \le |P_N\widetilde{u}|_{4\sigma+2}^{4\sigma}|P_Nr|_{4\sigma+2}^2 \le K_N'|\widetilde{u}|^{4\sigma}|r|^2.$$

We infer from the last inequality that

$$I_2 \le K_N'\mathbb{E}\int_T^\tau |\widetilde{u}|^{4\sigma}|r|^2 dt.$$

Integrating on time the result of Proposition 3.5 and taking into account the definition of τ, we obtain for N sufficiently high

$$I_2 \le K_N'C_\rho^*(d_0)e^{-\frac{\varepsilon\mu_1}{2}(T-s)}. \tag{4.15}$$

The equivalence of the norm in finite-dimensional spaces gives

$$\left|P_N\left(|u_1|^{2\sigma}u_1-|\widetilde{u}|^{2\sigma}\widetilde{u}\right)\right|^2 \le K_N'\left||u_1|^{2\sigma}u_1-|\widetilde{u}|^{2\sigma}\widetilde{u}\right|_1^2.$$

Hence, we deduce from (4.8) that

$$\left|P_N\left(|u_1|^{2\sigma}u_1-|\widetilde{u}|^{2\sigma}\widetilde{u}\right)\right|^2 \le K_N'\left|\left(|u_1|^{2\sigma}+|\widetilde{u}|^{2\sigma}\right)r\right|_1^2$$

A Schwartz inequality gives

$$\left|P_N\left(|u_1|^{2\sigma}u_1-|\widetilde{u}|^{2\sigma}\widetilde{u}\right)\right|^2 \le K_N'\left(|u_1|_{4\sigma}^{4\sigma}+|\widetilde{u}|_{4\sigma}^{4\sigma}\right)|r|^2.$$

It follows that

$$I_1 \le c\int_T^\tau \mathbb{E}\left(\left(|u_1|_{4\sigma}^{4\sigma}+|\widetilde{u}|_{4\sigma}^{4\sigma}\right)|r|^2\right)dt. \tag{4.16}$$

Applying Hölder inequality, we obtain for any conjugate pair $(q,q')\in(1,\infty)^2$

$$I_1 \le c\int_T^\tau \left(\mathbb{E}\left(|u_1|_{4\sigma}^{4\sigma}+|\widetilde{u}|_{4\sigma}^{4\sigma}\right)^{q'}\right)^{\frac{1}{q'}}\left(\mathbb{E}|r|^{2q}\right)^{\frac{1}{q}}dt.$$

Hence taking into account the definition of τ and Proposition 3.5, we obtain that for any q

$$I_1 \le c(d_0,\rho)\int_T^\tau \left(\mathbb{E}\left(|u_1|_{4\sigma}^{4\sigma}+|\widetilde{u}|_{4\sigma}^{4\sigma}\right)^{q'}\right)^{\frac{1}{q'}}e^{-\frac{3\varepsilon\mu_1}{4}(t-s)}dt.$$

Applying Sobolev Embedding and Hölder inequality, we remark that there exits $c, \beta \geq 0$ and $q' > 1$ such that

$$|\cdot|_{4\sigma}^{4\sigma q'} \leq c\left(1 + |\cdot|_{2\sigma+2}^{2\sigma+2} + \|\cdot\|^2 + |\cdot|^\beta\right),$$

which yields by applying arithmetico-geometric inequality

$$I_1 \leq c(d_0, \rho) \int_T^\tau \left(1 + |u|_{2\sigma+2}^{2\sigma+2} + \|u\|^2 + |u|^\beta\right) e^{-\frac{3\varepsilon\mu_1}{4}(t-s)} dt.$$

Hence, we infer from integrating by part that

$$I_1 \leq c(d_0, \rho) \int_T^\tau e^{-\frac{3\varepsilon\mu_1}{4}(t-s)} \left(\int_s^t \left(1 + |u|_{2\sigma+2}^{2\sigma+2} + \|u\|^2 + |u|^\beta\right) dr\right) dt.$$

Applying the definition of τ, we obtain

$$I_1 \leq C"(d_0, \rho) e^{-\frac{\varepsilon\mu_1}{2}(T-s)}. \tag{4.17}$$

Combining (4.14), (4.15) and (4.17), we deduce Corollary 3.9 from a Chebyshev inequality.

4.5. **Proof of Lemma 3.7.**

For any function f, we denote by $\delta f(u)$ the value $f(\widetilde{u}) - f(u_1)$. Moreover, we set $r = \widetilde{u} - u_1$.

Taking the Ito Formula of $|\widetilde{u}|^2$, we obtain

$$\begin{aligned} &d\,|\widetilde{u}|^2 + 2\varepsilon\,\|\widetilde{u}\|^2\,dt + 2\eta\,|\widetilde{u}|_{2\sigma+2}^{2\sigma+2}\,dt + 2K\,|P_N\widetilde{u}|_{2\sigma+2}^{2\sigma+2}\,dt = 2\,(\widetilde{u}, \phi(\widetilde{u})dW) \\ &\qquad + \|\phi(u)\|_{\mathcal{L}_2(U;H)}^2\,dt + I(t)dt + 2\left(P_N\widetilde{u}, (\eta + \lambda \mathrm{i})\delta\left(|u|^{2\sigma} u\right)\right) dt, \end{aligned} \tag{4.18}$$

where $I = I_1 + I_2$ and

$$\begin{aligned} I_1(t) &= -2L\left(1 + |\widetilde{u}|^{2\sigma} + |u_1|^{2\sigma}\right)(P_N\widetilde{u}, P_N r), \\ I_2(t) &= 2K\left(|P_N\widetilde{u}|^{2\sigma} P_N\widetilde{u}, P_N u_1\right). \end{aligned}$$

Applying arithmetico-geometric inequalities, we obtain

$$I_2(t) \leq K\,|P_N\widetilde{u}|_{2\sigma+2}^{2\sigma+2} + cK\,|u_1|_{2\sigma+2}^{2\sigma+2}. \tag{4.19}$$

Remarking that

$$(P_N\widetilde{u}, P_N r) = |P_N\widetilde{u}|^2 - (P_N\widetilde{u}, P_N u_1),$$

we deduce from Schwartz inequality

$$I_1(t) \leq 2L\left(1 + |\widetilde{u}|^{2\sigma} + |u_1|^{2\sigma}\right)|P_N\widetilde{u}|\,|P_N u_1|,$$

which implies by applying an arithmetico-geometric inequality

$$I_1(t) \leq 1 + \frac{K_L}{2}\,|P_N\widetilde{u}|_{2\sigma+2}^{2\sigma+2} + \frac{\eta}{2}\left(|u_1|_{2\sigma+2}^{2\sigma+2} + |\widetilde{u}|_{2\sigma+2}^{2\sigma+2}\right). \tag{4.20}$$

Applying arithmetico-geometric inequality, it follows

$$2\left(P_N\widetilde{u}, (\eta + \lambda \mathrm{i})\delta\left(|u|^{2\sigma} u\right)\right) \leq \frac{K_L}{2}\,|P_N\widetilde{u}|_{2\sigma+2}^{2\sigma+2} + \frac{\eta}{2}\left(|u_1|_{2\sigma+2}^{2\sigma+2} + |\widetilde{u}|_{2\sigma+2}^{2\sigma+2}\right). \tag{4.21}$$

Combining (4.18), (4.19), (4.20), (4.21) and **H0'**, we obtain that if $K \geq K_0$

$$E_{\widetilde{u}}(t, s) \leq \mathcal{M}(t) + c_1(1 + K_L + K)E_{u_1}(t, s) + (K_L B_1 - 1)t, \tag{4.22}$$

where $\mathcal{M}$ has been defined as in Section 4.1

$$M(t) = 2\int_s^t (u(r), \phi(u(r))dW(r)), \quad \mathcal{M}(t) = M(t) - \frac{\gamma_0}{2}\langle M\rangle(t).$$

As in Section 4.1, we have

$$\mathbb{P}\left(\sup_{t\geq s}\mathcal{M}(t) \geq \rho'\right) \leq e^{-\gamma_0\rho'}. \tag{4.23}$$

Combining (4.22) and (4.23), we obtain Lemma 3.7.

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Annexe D

Preuve de la régularité spatiale des solutions stationnaires des équations de Navier-Stokes tridimensionnelles

Résumé: On considère des solutions stationnaires des équations de Navier-Stokes tridimensionnelles (NS3D) soumises à un champs de force extérieure muni de composantes déterministe et aléatoires. La composante aléatoire de la force est supposée est un bruit blanc par rapport à la variable temporelle et régulier par rapport à la variable spatiale. On focalisera notre attention sur la régularité spatiale des solutions stationnaires.

Les techniques classiques pour étudier la régularité d'équations aux dérivées partielles stochastiques semblent inopérantes car l'existence global des solutions fortes reste un problème ouvert. Nous utiliserons l'opérateur de Kolmogorov et les approximations de Galerkin. Nous supposerons d'abord que le bruit vit dans l'espace de Sobolev H^p. Dans ce cas, pour tout temps t fixé, la loi d'une solution stationnaire au temps t est à support dans H^{p+1}.

Puis, en utilisant une technique totalement différente, nous établirons que si le bruit à une régularité Gevrey, alors, pour tout temps t fixé, la loi d'une solution stationnaire au temps t est à support dans un espace de Gevrey. Nous en déduirons certaines informations sur l'échelle de dissipation de Kolmogorov (K41).

Mots clés : Équations de Navier-Stokes tridimensionnelles, mesures invariantes, espaces de Gevrey, opérateur de Kolmogorov, K41.

Spatial smoothness of the stationary solutions of the 3D Navier–Stokes equations

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Abstract: We consider stationary solutions of the three dimensional Navier–Stokes equations (NS3D) with periodic boundary conditions and driven by an external force which might have a deterministic and a random part. The random part of the force is white in time and very smooth in space. We investigate smoothness properties in space of the stationary solutions.

Classical technics for studying smoothness of stochastic PDEs do not seem to apply since global existence of strong solutions is not known. We use the Kolmogorov operator and Galerkin approximations. We first assume that the noise has spatial regularity of order p in the L^2 based Sobolev spaces, in other words that its paths are in H^p. Then we prove that at each fixed time the law of the stationary solutions is supported by H^{p+1}.

Then, using a totally different technic, we prove that if the noise has Gevrey regularity then at each fixed time, the law of a stationary solution is supported by a Gevrey space. Some informations on the Kolmogorov dissipation scale are deduced.
Key words: Stochastic three-dimensional Navier-Stokes equations, invariant measure, Gevrey spaces, Kolmogorov operator, Kolmogorov dissipation scale.

Introduction

We are concerned with the stochastic Navier–Stokes equations in dimension 3 (NS3D) with periodic boundary conditions and zero mean value. These equations describe the time evolution of an incompressible fluid and are given by

$$(0.1)\qquad \begin{cases} dX+\nu(-\Delta)X\,dt+(X,\nabla)X\,dt+\nabla p\,dt=\phi(X)dW+g(X)dt, \\ (\mathrm{div}\ X)\,(t,\xi) \;=\; 0, \qquad \text{for } \xi\in D,\ \ t>0, \\ \int_D X(t,\xi)d\xi \;=\; 0, \qquad \text{for } t>0, \\ X(0,\xi) \;=\; x_0(\xi), \qquad \text{for } \xi\in D, \end{cases}$$

where $D=(0,2\pi)^3$. We have denoted by $X(t,\xi)$ the velocity and by $p(t,\xi)$ the pressure at time t and at the point $\xi\in D$, also ν denotes the viscosity. The external force acting on the fluid is the sum of a random force of white noise type $\phi(X)dW$ and a determinist one $g(X)dt$.

As is well known, in the deterministic case, global existence of weak (in the PDE sense) solutions and uniqueness of strong solutions hold for the Navier-Stokes equations. In space dimension two, weak solutions are strong and global existence and uniqueness follows. Such a result is an open problem in dimension three. (See for instance [4], [7], [16], [20], [21], [22], [23], [28] and [32]).

In the stochastic case, the situation is similar. However due to the lack of uniqueness, we have to work with global weak (in the PDE sense) solutions of the martingale problem. (See for instance [1], [2], [3], [5], [6], [13], [14], [24], [30] and [31]). Roughly speaking, this means that in (0.1), we take X, p and W for unknown.

As is usual in the context of the incompressible Navier-Stokes equation, we get rid of the pressure thanks to the Leray projector. Let us denote by (X, W) a weak (in the PDE sense) stationary solution of the martingale problem (0.1) and by μ the the law of $X(t)$, which is an invariant measure if we can prove that (0.1) defines a Markov evolution. In this article, we establish that μ admits a moment in spaces of smooth functions provided the external force is sufficiently smooth. We think that this is an interesting question to study. First, it can be seen that if we were able to prove that μ has a moment of sufficiently high order in a well chosen Sobolev norm (order 4 in H^1 or 2 in H^2 for instance) then this would imply global existence of strong solutions for μ almost every initial data.

Moreover, this result is an important ingredient if one tries to follow the method of [8] to construct a Markov transition semi-group in $H^p(D)$ under suitable conditions on ϕ and f. Since even uniqueness in law is not known for NS3D, such result might be important.

We first prove that if the external force is in $H^{p-1}(D)$ and the noise term has paths in $H^p(D)$ then μ admits a moment in the Sobolev space $H^{p+1}(D)$

Note that analogous results are well-known for the two dimensional Navier–Stokes equations (NS2D). Actually a stronger result is true for NS2D. Namely, for any square integrable x_0, the unique solution of NS2D is continuous from $(0, \infty)$ into $H^p(D)$ and is square integrable from (t_0, t_1) into $H^{p+1}(D)$. It follows that μ admits moments of any orders in $H^p(D)$ and a moment of order 2 in $H^{p+1}(D)$. This stronger result is linked to the global existence of strong solutions for NS2D.

This kind of idea cannot be used for NS3D and we use a generalization of an idea used in [8] for the case $p = 1$. The method is based on the use of the Kolmogorov operator applied to suitable Lyapunov functional. These functionals have already been used in the deterministic case in [29], chapter 4.

Using a totally different method, we establish also that the invariant measure μ admits a moment in a Gevrey class of functions provided the external force is also in such a class. Gevrey regularity has been studied in the deterministic case in [15] and [17]. Our method is based on tools developed in [15]. In [27], these tools have been used to obtain an exponential moment for the invariant measure in Gevrey norms in the two dimensional case. The arguments used in [27] do not generalize to the three dimensional case since there strong existence and uniqueness is used. The three dimensional case NS3D requires substantial adaptations. We develop a framework which gives a control on a Gevrey norm by using a control of the $H^1(D)$–norm of X only at fixed time.

In this way, we are able to generalize to NS3D the results of [27]. However, we do not have exponential moments. We deduce that the Kolmogorov dissipation scale is larger than $\nu^{-(4+\delta)}$. This is certainly not optimal since it is expected that the scale is of order $\nu^{-\frac{3}{4}}$. Note that our result is rigorous and does not use any heuristic argument.

1. Notations

For $m \in \mathbb{N}$, we denote by $\mathbb{H}^m_{\text{per}}(D)$ the space of functions which are restrictions of periodic functions in $H^m_{loc}(D;\mathbb{R}^3)$ and whose average on D is zero. We set

$$H = \left\{ X \in \mathbb{H}^0_{\text{per}}(D) \mid \text{ div } X = 0 \text{ on } D \right\},$$

and

$$V = H \cap \mathbb{H}^1_{\text{per}}(D).$$

Let π be the orthogonal projection in $L^2(D;\mathbb{R}^3)$ onto the space H. We set

$$A = \pi\left(-\Delta\right), \quad D(A) = V \cap \mathbb{H}^2_{\text{per}}(D) \text{ and } B(u) = \pi\left((u,\nabla)u\right).$$

It is convenient to endow $\mathbb{H}^m_{\text{per}}(D)$ with the inner product $((\cdot,\cdot))_m = (A^m\cdot,\cdot)_{L^2(D;\mathbb{R}^3)}$. The corresponding norm is denoted by $\|\cdot\|_m$. It is classical that this defines a norm which is equivalent to the usual one. For $m = 0$ we write $|\cdot| = \|\cdot\|_0$ and for $m = 1$ we write $\|\cdot\| = \|\cdot\|_1$. Note that, since we work with functions whose average is zero on $(0,2\pi)^3$, we have the following Poincaré type inequality

$$\|x\|_{m_1} \le \|x\|_{m_2}, \ m_1 \le m_2, \ x \in \mathbb{H}^{m_2}_{\text{per}}(D;\mathbb{R}^3).$$

We also use the spaces $L^p(D;\mathbb{R}^3)$ endowed with their usual norm denoted by $|\cdot|_p$. Moreover, given two Hilbert spaces K_1 and K_2, $\mathcal{L}_2(K_1;K_2)$ is the space of Hilbert-Schmidt operators from K_1 to K_2 .

The noise is described by a cylindrical Wiener process W defined on a Hilbert space U and a mapping ϕ defined on H with values in $\mathcal{L}_2(U;H)$. We also consider a deterministic forcing term described by a mapping g from H into H. More precise assumptions on ϕ and g are made below.
Now, we can write problem (0.1) in the form

$$\left\{\begin{array}{rcl} dX + \nu AX dt + B(X)dt &=& \phi(X)dW + g(X)dt, \\ X(0) &=& x_0, \end{array}\right. \tag{1.1}$$

where W is a cylindrical Wiener process on a Hilbert space U.
In all the paper, we consider (X,W) a H–valued stationary solution of the martingale problem (1.1). Existence of such a solution has been proved in [13]. We denote by μ the law of $X(t)$. We do not consider any stationary solutions but only those which are limit in distribution of stationary solutions of Galerkin approximations of (1.1). More precisely, for any $N \in \mathbb{N}$, we denote by P_N the eigenprojector of A associated to the first N eigenvalues and consider the following approximation of (1.1)

$$\left\{\begin{array}{rcl} dX_N + \nu AX_N dt + P_N B(X_N)dt &=& P_N\phi(X_N)dW + P_N g(X_N)dt, \\ X_N(0) &=& P_N x_0. \end{array}\right. \tag{1.2}$$

It is easily shown that (1.2) has a stationary solution X_N. Proceeding as in [14], we can see that their laws are tight in a well chosen functional space and, for a subsequence, (X_N, W) converges in law to (X, W) a stationary solution of (1.1). We only consider stationary solutions constructed in that way.

All the computations made in this paper are derived formally and are not totally rigorous. A rigorous proof could easily be obtained by making all the computations on the Galerkin approximations and then letting N go to infinity in the final estimate. This type of argument is classical and, in order to lighten the redaction, we chose to write all the computations directly on (X, W).
Some of our results described properties of μ in Gevrey type spaces. According to the setting given in [15], we set for any $(\alpha, \beta) \in \mathbb{R}_*^+ \times (0,1]$

$$\begin{cases} (\cdot,\cdot)_{G(\alpha,\beta)} = \left(A^{\frac{1}{2}} e^{\alpha A^{\frac{\beta}{2}}} \cdot, A^{\frac{1}{2}} e^{\alpha A^{\frac{\beta}{2}}} \cdot\right), & \|\cdot\|_{G(\alpha,\beta)} = \left|A^{\frac{1}{2}} e^{\alpha A^{\frac{\beta}{2}}} \cdot\right|, \\ G(\alpha,\beta) = \left\{ X \in H \;\middle|\; \|X\|_{G(\alpha,\beta)} < \infty \right\}. \end{cases}$$

Clearly, $\left(G(\alpha,\beta), (\cdot,\cdot)_{G(\alpha,\beta)}\right)$ is a Hilbert space.

We are not interested in large viscosities and in all the article it is assumed that $\nu \le 1$. We will use various constants which may depend on some parameter such as $p, \nu, \dots$ When this dependance is important, we make it explicit.

2. $\mathbb{H}^p_{\mathrm{PER}}(D)$–REGULARITY

2.1. **Statement of the result.** Let $p \in \mathbb{N}$. We now make the following smoothness assumptions on the forcing terms.

Hypothesis 2.1. *The mapping ϕ (resp. g) takes values in $\mathcal{L}_2\left(U; H \cap \mathbb{H}^p_{per}(D)\right)$ (resp. $H \cap \mathbb{H}^{p-1}_{per}(D)$) and $\phi : H \to \mathcal{L}_2\left(U; H \cap \mathbb{H}^p_{per}(D)\right)$ and $g : H \to H \cap \mathbb{H}^{p-1}_{per}(D)$ are bounded.*

We set, when Hypothesis 2.1 holds,

$$B_p = \sup_H \left(\|\phi\|^2_{\mathcal{L}_2(U;\mathbb{H}^p_{\mathrm{per}}(D))} + \|g\|^2_{p-1} \right).$$

It is also convenient to define

$$\bar{B}_p = \sup_H \|\phi\|^2_{\mathcal{L}_2(U;\mathbb{H}^p_{\mathrm{per}}(D))} + \left(\frac{2}{\nu} \sup_H \|g\|^2_{p-1} \right) \wedge \left(\sup_H \|g\|^2_{p+1} \right).$$

The aim of this section is to establish the following result.

Theorem 2.2. *Let μ be the invariant law of a stationary solution of the three dimensional Navier-Stokes equation and assume that Hypothesis 2.1 holds for some $p \ge 1$. There exists $c_{p,\nu}$ depending on p, ν and B_p such that for any $\nu \le 1$*

$$\int_H \|x\|_{p+1}^{\frac{2}{2p+1}} \, d\mu(x) \le c_{p,\nu}.$$

This result is proved in sections 2.2 to 2.5. We now wish to make few comments.

Note that it would be very important to obtain an estimate on $\int_H \|x\|_{p+1}^{\delta_p}\ d\mu(x)$ with $p\delta_p > 3$. Indeed, by Agmon inequality , we have

$$\int_H |x|_\infty^2\, d\mu(x) \le c \int_H |x|^{2-3/p} \|x\|_p^{\frac{3}{p}}\ d\mu(x)$$

and this would give an estimate on the right hand side. Since uniqueness is easily shown to hold for solutions in $L^2(0,T;L^\infty(D;\mathbb{R}^3))$, a classical argument could be used to deduce that for μ almost every initial data there exists a unique global weak solution. Combining with the result in [12], this would partially solve Leray's conjecture.

Consider the case $g = 0$, $U = H$ and $\phi = A^{-s-\frac{3}{2}}$. Then Hypothesis 2.1 holds for any $p < s$ and the unique invariant measure of the three dimensional linear stochastic Stokes equation in H is in $\mathbb{H}^{r+1}_{\text{per}}(D)$ with probability zero if $r > s$. Therefore it seems that $\|\cdot\|_{p+1}$ is the strongest norm we can control under Hypothesis 2.1.

Remark that in the two dimensional case a much stronger result holds. Indeed, standard arguments imply that under Hypothesis 2.1 we have for any invariant measure μ and any $q \in \mathbb{N}^*$

$$\int_H \|x\|_p^{2q}\ d\mu(x) < \infty, \quad \int_H \|x\|_{p+1}^2\ d\mu(x) < \infty.$$

In the proof, we use ideas developped in [29]. Similar but more refined techniques have been used in [17] to derive interesting properties on the decay of the Fourier spectrum of smooth solutions of the deterministic Navier-Stokes equations. Using such techniques does not seem to yield great improvement of our result. Indeed, trying to do so, we have been able to precise the estimate of Theorem 2.2 as follows

$$\nu \int_H \|x\|_{p+1}^{\frac{c_*}{p}}\ d\mu(x) \le \bar{B}_p + c2^p(1+\bar{B}_0),$$

where c and c_* are positive constants and c_* is close to 1.02. We have not been able to derive very interesting results from this improved estimate and therefore have preferred to give the simpler one which follows from easier arguments.

2.2. **Proof of Theorem 2.2.**

The proof is based on the fact that for any suitable f

$$\int_H Lf(x)d\mu(x) = 0, \tag{2.1}$$

where L is the Kolmogorov operator associated to the stochastic Navier-Stokes equations (1.1)

$$Lf(x) = \frac{1}{2} tr\left(\phi(x)\phi^*(x)D^2f(x)\right) - (\nu Ax + B(x) - g(x), Df(x)).$$

As already mentionned, this is not fully rigorous. Indeed, (2.1) is valid only for the Galerkin approximations and for smooth and bounded functions. The sequence of Galerkin approximations described in section 1 should be used. Moreover, for a better understanding, we only consider the case $g = 0$. The generalization is easy.

Step 1: $p = 0$

Applying (2.1) to $f = |\cdot|^2$, we obtain

$$\int_H (\nu Ax + B(x), x)\ d\mu(x) = \frac{1}{2}\int_H tr\left(\phi(x)\phi^*(x)\right)\ d\mu(x),$$

which yields, since $(B(x), x) = 0$,

$$\nu \int_H \|x\|^2 \, d\mu(x) \quad \le \quad \frac{1}{2}\bar{B}_0. \tag{2.2}$$

Step 2: Estimate of a quotient of norms.
We set

$$\varepsilon_p = \frac{1}{2p-1}$$

and

$$R'_p = \nu \int_H \frac{\|x\|_{p+1}^2}{\left(1+\|x\|_p^2\right)^{1+\varepsilon_p}} d\mu(x),$$

$$R_p = \nu \int_H \frac{1+\|x\|_{p+1}^2}{\left(1+\|x\|_p^2\right)^{1+\varepsilon_p}} d\mu(x),$$

In order to establish Theorem 2.2 for $p \ge 1$, we prove now that there exists c_p such that

$$R_p \le \bar{B}_p + c_p \bar{B}_0 + 1 \tag{2.3}$$

Let us set

$$f = \frac{1}{\left(1+\|\cdot\|_p^2\right)^{\varepsilon_p}}.$$

We have

$$Df(x) = -2\varepsilon_p \frac{A^p x}{\left(1+\|x\|_p^2\right)^{1+\varepsilon_p}},$$

and

$$D^2 f(x) = -2\varepsilon_p \frac{A^p}{\left(1+\|x\|_p^2\right)^{1+\varepsilon_p}} + 4\varepsilon_p(1+\varepsilon_p)\frac{A^p x \otimes A^p x}{\left(1+\|x\|_p^2\right)^{2+\varepsilon_p}},$$

which yields

$$\begin{aligned}(Lf)(x) \quad &= \frac{2\varepsilon_p}{\left(1+\|x\|_p^2\right)^{1+\varepsilon_p}}\left(\nu\|x\|_{p+1}^2 + (A^p x, B(x)) - \frac{1}{2}\|\phi(x)\|^2_{\mathcal{L}_2(U;\mathbb{H}^p_{\mathrm{per}}(D))}\right) \\ &\quad + \frac{2\varepsilon_p(1+\varepsilon_p)}{\left(1+\|x\|_p^2\right)^{2+\varepsilon_p}} \sum_n |(A^p x, \phi(x).e'_n)|^2,\end{aligned}$$

where $(e'_n)_n$ is an orthonormal basis of U. Hence we have

$$2\nu \frac{\|x\|_{p+1}^2}{\left(1+\|x\|_p^2\right)^{1+\varepsilon_p}} \le \frac{1}{\varepsilon_p} Lf(x) + \bar{B}_p - \frac{2}{\left(1+\|x\|_p^2\right)^{1+\varepsilon_p}}\left(A^p x, B(x)\right).$$

Integrating with respect to μ and applying (2.1), it follows

$$2R'_p \le \bar{B}_p - 2\int_H \frac{(A^p x, B(x))}{\left(1+\|x\|_p^2\right)^{1+\varepsilon_p}} d\mu(x). \tag{2.4}$$

It is proved in [29] (see equation (4.8) chapter 4) that there exists a constant $c_{p,\nu}$ such that

$$-\left(A^p x, B(x)\right) \le c_{p,\nu} \|x\|^2 \|x\|_p^{4p/2p-1} + \frac{\nu}{2} \|x\|_{p+1}^2 . \tag{2.5}$$

Then taking into account (2.2) and (2.4), (2.3) follows.

Step 3: Estimate of $\int_H \|x\|_p^{2/2p-1} d\mu(x)$.

We estimate $A_p = \nu \int_H \left(1 + \|x\|_p^2\right)^{1/2p-1} d\mu(x)$ by induction. The case $p = 1$ has been treated in step 1.

Using Hölder inequality yields

$$\begin{aligned}
&\nu \int_H \left(1 + \|x\|_{p+1}^2\right)^{1/2p+1} d\mu(x) \\
&= \nu \int_H \left(\frac{\left(1 + \|x\|_{p+1}^2\right)}{\left(1 + \|x\|_{p+1}^2\right)^{1+\varepsilon_p}} \right)^{1/2p+1} \left(1 + \|x\|_{p+1}^2\right)^{(1+\varepsilon_p)/2p+1} d\mu(x) \\
&\le R_p^{1/2p+1} \left(\nu \int_H \left(1 + \|x\|_{p+1}^2\right)^{(1+\varepsilon_p)/2p} d\mu(x) \right)^{2p/2p+1} .
\end{aligned}$$

Since $(1+\varepsilon_p)/2p = 1/2p - 1$, we deduce

$$A_{p+1} \le R_p^{1/2p+1} A_p^{2p/2p+1}.$$

The result follows.

3. Gevrey regularity

3.1. **Statement of the result.** We prove in this section that if the external force is bounded in a Gevrey class of functions, then μ have support in another Gevrey class of function.
The main assumption in this section is the following

Hypothesis 3.1. *There exists $(\alpha, \beta) \in \mathbb{R}_+^* \times (0,1]$ such that the mappings $g : H \to H \cap G(\alpha,\beta)$ and $\phi : H \to \mathcal{L}_2\left(U; H \cap G(\alpha,\beta)\right)$ are bounded.*

We set

$$B_0' = \sup_{x \in H} \|\phi(x)\|_{\mathcal{L}_2(U;G(\alpha,\beta))}^2 + \sup_{x\in H} \|g(x)\|_{G(\alpha,\beta)}^2 .$$

The aim of this section is to establish the following results proved in the following subsections.

Theorem 3.2. *Assume that Hypothesis 3.1 holds. There exist a family $(C_\gamma)_\gamma$ only depending on (α, β, B_0') and a family $(\alpha_\nu)_{\nu\in(0,1)}$ of measurable mappings $H \to (0,\alpha)$ such that for any $\nu \in (0,1)$*

$$\int_H \|x\|_{G(\nu\alpha_\nu(x),\beta)}^{2\gamma} d\mu(x) \quad\le\quad C_\gamma \nu^{-\frac{3}{4}}, \tag{3.1}$$

$$\int_H \left(\alpha_\nu(x)\right)^{-\frac{\gamma}{4}} d\mu(x) \quad\le\quad C_\gamma \nu^{-\frac{3}{4}}, \tag{3.2}$$

for any $\gamma \in (0,1)$.

This result gives some informations on the Kolmogorov dissipation scale. Indeed, it follows from (3.1), (3.2) that

$$|\hat{x}(k)| \leq \|x\|_{G(\nu\alpha_\nu(x),\beta)} |k| e^{-\nu\alpha_\nu(x)|k|^\beta},$$

where $(\hat{x}(k))_{k\in\mathbb{Z}^3}$ are the Fourier coefficients of x.
Hence, if Hypothesis 3.1 holds with $\beta = 1$, then $|\hat{x}(k)|$ decreases faster than any powers of $|k|$ for $|k| \gg (\nu\alpha_\nu(x))^{-1}$. By (3.2), for any $\delta > 0$

$$\frac{1}{\alpha_\nu(x)} \leq c_{\delta,\nu}(x)\nu^{-3(1+\delta)} \quad \text{with} \quad \int c_{\delta,\nu}(x)^{\frac{1}{4(1+\delta)}} \mu(dx) \leq \Theta_\delta < \infty,$$

and Θ_δ not depending on ν. It follows that $|\hat{x}(k)|$ decreases faster than any powers of $|k|$ for $|k| \gg \nu^{-(4+3\delta)}$. This indicates that the Kolmogorov dissipation scale is larger than $\nu^{-(4+3\delta)}$. Note that by physical arguments it is expected that the Kolmogorov dissipation scale is of order of $\nu^{-\frac{3}{4}}$.
We can also control a moment of a fixed Gevrey norm.

Corollary 3.3. *Under the same assumptions, there exists a family $(C_{\gamma,\alpha',\beta',\nu})_{\gamma,\alpha',\beta',\nu}$ only depending on (α,β,B_0',ν) such that*

$$\int \left(\ln^+ \|x\|^2_{G(\alpha',\beta')}\right)^\gamma d\mu(x) \leq C_{\gamma,\alpha',\beta',\nu}, \tag{3.3}$$

where $\ln^+ r = \max\{0, \ln r\}$ and provided $\alpha' > 0$ and $\beta', \gamma > 0$ verify

$$\beta' < \beta \quad \text{and} \quad 4\gamma < \frac{\beta}{\beta'} - 1.$$

3.2. **Estimate of blow-up time in Gevrey spaces.**
Before proving Theorem 3.2, we establish the following result which implies that the time of blow-up of the solution in Gevrey spaces admits a negative moment.

Lemma 3.4. *Assume that Hypothesis 3.1 holds. For any stationary solution of the Navier-Stokes equations and any $\nu \in (0,1)$, there exist K only depending on (α,β,B_0') and a stopping time $\tau > 0$ such that*

$$\mathbb{E}\left(\sup_{t\in(0,\tau)} \|X(t)\|^2_{G(\nu t,\beta)}\right) \leq 4(\bar{B}_0 + 1), \tag{3.4}$$

$$\mathbb{P}(\tau \leq t) \leq K t^{\frac{1}{4}} \nu^{-\frac{3}{4}}. \tag{3.5}$$

This result is a refinement of the result developed by Foias and Temam in [15] and is strongly based on the tools developed in this latter article. Let us set

$$\tau = \inf\left\{t \geq 0 \,\middle|\, \|X(t)\|^2_{G(\nu t,\beta)} > 4\left(\|X(0)\|^2 + 1\right)\right\}. \tag{3.6}$$

Clearly

$$\mathbb{E}\left(\sup_{t\in(0,\tau)} \|X(t)\|^2_{G(\nu t,\beta)}\right) \leq 4\mathbb{E}\left(\|X(0)\|^2 + 1\right)$$

and (3.4) follows from (2.2). It remains to prove (3.5).
We apply Ito Formula to $\|X(t)\|^2_{G(\nu t,\beta)}$ for $t \in (0,\alpha)$
(3.7)

$$d\|X(t)\|^2_{G(\nu t,\beta)} + 2\nu \left\|A^{\frac{1}{2}}X(t)\right\|^2_{G(\nu t,\beta)} dt = \nu \left\|A^{\frac{\beta}{4}}X(t)\right\|^2_{G(\nu t,\beta)} dt + dM(t) + I(t)dt,$$

where

$$\begin{cases} I(t) &= 2I_g(t)+2I_B(t)+I_\phi(t), \quad I_B(t) = -\left(X(t),B(X(t))\right)_{G(\nu t,\beta)}, \\ I_\phi(t) &= \|\phi(X(t))\|^2_{\mathcal{L}_2(U;G(\nu t,\beta))}, \quad I_g(t) = \left(g(X(t)),X(t)\right)_{G(\nu t,\beta)}, \\ M(t) &= 2\int_0^t \left(X(s),\phi(X(s))dW(s)\right)_{G(\nu t,\beta)}. \end{cases}$$

The following inequality is proved in [15] for $\beta \le 1$

$$2I_B(t) \le \nu \left\| A^{\frac{1}{2}} X(t) \right\|^2_{G(\nu t,\beta)} + \frac{c}{\nu^3} \|X(t)\|^6_{G(\nu t,\beta)}. \tag{3.8}$$

By Hypothesis 3.1 we have

$$I_\phi(t) + 2I_g(t) \le \|X(t)\|^6_{G(\nu t,\beta)} + B_0' + 1. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we obtain since $\beta, \nu \le 1$

$$d\,\|X(t)\|^2_{G(\nu t,\beta)} \le dM(t) + \frac{c}{\nu^3}\|X(t)\|^6_{G(\nu t,\beta)}\,dt + (B_0'+1)dt. \tag{3.10}$$

Applying Ito formula to $\left(1+\|X(t)\|^2_{G(\nu t,\beta)}\right)^{-2}$, we then deduce from (3.10) and from Hypothesis 3.1 that for any $t \in (0,\alpha)$ and any $\nu \le 1$

$$-d\left(1+\|X(t)\|^2_{G(\nu t,\beta)}\right)^{-2} \le d\mathcal{M}(t) + C_0 \nu^{-3} dt, \tag{3.11}$$

where $C_0 = c(B_0'+1)$ and

$$\mathcal{M}(t) = 4\int_0^t \left(1+\|X(s)\|^2_{G(\nu t,\beta)}\right)^{-3} \left(X(s)\,,\,\phi(X(s))dW(s)\right)_{G(\nu t,\beta)}.$$

Setting

$$\begin{cases} \tau_0 &= \inf\left\{ t \in (0,\alpha) \,\middle|\, \mathcal{M}(t) > \frac{1}{4\left(1+\|X(0)\|^2\right)^2} \right\}, \\ \tau_1 &= \tau_0 \wedge \left(\frac{\nu^3}{4C_0\left(1+\|X(0)\|^2\right)^2} \right), \end{cases}$$

we obtain by integration of (3.11) on $[0,t]$ for $t \in (0,\tau_1)$

$$\|X(t)\|^2_{G(\nu t,\beta)} \le 4\left(1+\|X(0)\|^2\right).$$

We deduce that $\tau \ge \tau_1$ and

$$\mathbb{P}\left(\tau \le t\right) \le \mathbb{P}\left(\tau_0 \le t\right) + \mathbb{P}\left(\left(1+\|X(0)\|^2\right)^2 \ge \frac{\nu^3}{4C_0 t} \right). \tag{3.12}$$

Since μ is the law of $X(0)$, we have

$$\mathbb{P}\left(\left(1+\|X(0)\|^2\right)^2 \ge \frac{\nu^3}{4C_0 t} \right) = \mu\left(x \,\middle|\, \sqrt{1+\|x\|^2} \ge \frac{\nu^{\frac{3}{4}}}{(4C_0 t)^{\frac{1}{4}}} \right).$$

Applying Chebyshev and Schwartz inequalities, we deduce from (2.2)

$$\begin{aligned}
&\mathbb{P}\left(\left(1+\|X(0)\|^2\right)^2 \geq \frac{\nu^3}{4C_0 t}\right)\\
&\leq 2\nu^{-\frac{3}{4}}\left(C_0 t\right)^{\frac{1}{4}}\sqrt{\int_H (1+\|x\|^2)d\mu(x)} \qquad (3.13)\\
&\leq 2\nu^{-\frac{3}{4}}\sqrt{1+\bar{B}_0}\left(C_0 t\right)^{\frac{1}{4}}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\mathbb{P}\left(\tau_0\leq t\right) &= \mathbb{P}\left(4\left(\left(1+\|X(0)\|^2\right)^2 \sup_{s\in[0,t]}\mathcal{M}(s)\right)>1\right)\\
&\leq 2\mathbb{E}\left(\left(1+\|X(0)\|^2\right)\sup_{s\in[0,t]}\left|\mathcal{M}(s)\right|^{\frac{1}{2}}\right).
\end{aligned}$$

Conditionning by $\mathcal{F}_0$ the σ-algebra generate by $X(0)$, it follows

$$\mathbb{P}\left(\tau_0\leq t\right)\leq 2\mathbb{E}\left(\left(1+\|X(0)\|^2\right)\mathbb{E}\left(\sup_{s\in[0,t]}\left|\mathcal{M}(s)\right|^{\frac{1}{2}}\,\middle|\,\mathcal{F}_0\right)\right).$$

By Burkholder-Davis-Gundi inequality (see Theorem 3.28 page 166 in [19]) we obtain

$$\mathbb{E}\left(\sup_{s\in[0,t]}\left|\mathcal{M}(s)\right|^{\frac{1}{2}}\,\middle|\,\mathcal{F}_0\right)\leq c\mathbb{E}\left(\left\langle\mathcal{M}\right\rangle^{\frac{1}{4}}(t)\,\middle|\,\mathcal{F}_0\right),$$

and

$$\mathbb{P}\left(\tau_0\leq t\right)\leq 2\mathbb{E}\left(\left(1+\|X(0)\|^2_{G(\nu t,\beta)}\right)\mathbb{E}\left(\left\langle\mathcal{M}\right\rangle^{\frac{1}{4}}(t)\,\middle|\,\mathcal{F}_0\right)\right). \qquad (3.14)$$

We have

$$\left\langle\mathcal{M}\right\rangle(t)=4\int_0^t\left(1+\|X(s)\|^2_{G(\nu t,\beta)}\right)^{-6}\left|\left(A^{\frac{1}{2}}e^{\nu tA^{\frac{\beta}{2}}}\phi(X(s))\right)^*\left(A^{\frac{1}{2}}e^{\nu tA^{\frac{\beta}{2}}}X(s)\right)\right|_U^2 ds.$$

Therefore

$$\left\langle\mathcal{M}\right\rangle(t)\leq 4\int_0^t\left(1+\|X(s)\|^2_{G(\nu t,\beta)}\right)^{-6}\|\phi(X(s))\|^2_{\mathcal{L}(U;G(\nu t,\beta))}\|X(s)\|^2_{G(\nu t,\beta)}\,ds\leq 4B_0't.$$

Hence we infer from (3.14) that

$$\mathbb{P}\left(\tau_0\leq t\right)\leq c\left(B_0't\right)^{\frac{1}{4}}\nu\int_H\left(1+\|x\|^2\right)d\mu(x),$$

which yields by (2.2)

$$\mathbb{P}\left(\tau_0\leq t\right)\leq c(1+\bar{B}_0)(B_0')^{1/4}t^{\frac{1}{4}}. \qquad (3.15)$$

Combining (3.12), (3.13) and (3.15), we deduce (3.5) from $\bar{B}_0\leq B_0'$.

3.3. **Proof of Theorem 3.2.**

Setting

$$\alpha_\nu(x)=\inf\left\{s\geq 0\,\middle|\,\|x\|^2_{G(\nu s,\beta)}>\frac{4}{s^{\frac{1}{4}}}\left(\bar{B}_0+1\right)\right\},$$

it follows that for any $\gamma\in(0,1)$

$$\int\|x\|^{2\gamma}_{G(\nu\alpha_\nu(x),\beta)}\,d\mu(x)\leq 4^\gamma\left(\bar{B}_0+1\right)^\gamma\int\left(\alpha_\nu(x)\right)^{-\frac{\gamma}{4}}d\mu(x). \qquad (3.16)$$

Hence (3.1) is consequence of (3.2) and of $\bar{B}_0 \leq B_0'$. Then in order to establish Theorem 3.2, it is sufficient to prove (3.2).
Clearly

$$\mathbb{P}\left(\|X(t)\|^2_{G(\nu t,\beta)} > \frac{4}{t^{\frac{1}{4}}}\left(\bar{B}_0+1\right)\right) \leq \mathbb{P}\left(\sup_{s\in[0,\tau]}\|X(s)\|^2_{G(\nu s,\beta)} > \frac{4}{t^{\frac{1}{4}}}\left(\bar{B}_0+1\right)\right)+\mathbb{P}\left(\tau<t\right),$$

where τ is defined in section 3.2. Applying Chebyshev inequality, we infer from Lemma 3.4 that for any $t>0$

$$\mathbb{P}\left(\|X(t)\|^2_{G(\nu t,\beta)} > \frac{4}{t^{\frac{1}{4}}}\left(\bar{B}_0+1\right)\right) \leq (K+1)t^{\frac{1}{4}}\nu^{-\frac{3}{4}}. \tag{3.17}$$

Since μ is the law of $X(t)$, we have

$$\mu\left(x \mid \alpha_\nu(x) \leq t\right) = \mathbb{P}\left(\alpha_\nu(X(t)) \leq t\right),$$

which yields

$$\mu\left(x \mid \alpha_\nu(x) \leq t\right) = \mathbb{P}\left(\|X(t)\|^2_{G(\nu t,\beta)} > \frac{4}{t^{\frac{1}{4}}}\left(\bar{B}_0+1\right)\right).$$

Hence we deduce from (3.17) that for any $t>0$

$$\mu\left(x \mid \alpha_\nu(x) \leq t\right) \leq (K+1)t^{\frac{1}{4}}\nu^{-\frac{3}{4}}. \tag{3.18}$$

It is well-known that (3.18) for any $t>0$ implies (3.2), which yields Theorem 3.2.

3.4. **Proof of Corollary 3.3.**

To deduce Corollary 3.3 from Theorem 3.2, it is sufficient to prove that for any $(\alpha',\alpha,\beta',\beta)\in(0,\infty)^2\times(0,1]^2$ such that $\beta'<\beta$, we have

$$\|x\|_{G(\alpha',\beta')} \leq \exp\left(c(\beta,\beta')\left(\alpha'\right)^{\frac{\beta}{\beta-\beta'}}\left(\alpha\right)^{-\frac{\beta'}{\beta-\beta'}}\right)\|x\|_{G(\alpha,\beta)}. \tag{3.19}$$

Indeed, (3.19) implies that for any $\gamma\in\mathbb{R}^+_*$

$$\left(\ln^+\|x\|^2_{G(\alpha',\beta')}\right)^\gamma \leq c_\gamma\left(c(\beta,\beta')+\left(\alpha'\right)^{\frac{\gamma\beta}{\beta-\beta'}}\left(\nu\alpha_\nu(x)\right)^{-\frac{\gamma\beta'}{\beta-\beta'}}+\left(\ln^+\|x\|^2_{G(\nu\alpha_\nu(x),\beta)}\right)^\gamma\right),$$

which yields Corollary 3.3 provided Theorem 3.2 is true.
We now establish (3.19). It follows from arithmetico-geometric inequality that for any $k\in\mathbb{Z}^3$

$$\alpha'\left|k\right|^{\beta'} \leq c(\beta,\beta')\left(\alpha'\right)^{\frac{\beta}{\beta-\beta'}}\left(\alpha\right)^{-\frac{\beta'}{\beta-\beta'}}+\alpha\left|k\right|^{\beta}. \tag{3.20}$$

Recalling that

$$\|x\|^2_{G(\alpha',\beta')} = \sum_{k\in\mathbb{Z}^3}\left|k\right|^2\exp\left(2\alpha'\left|k\right|^{\beta'}\right)\left|\hat{x}(k)\right|^2,$$

we infer (3.19) from (3.20).

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Annexe E

Preuve du caractère exponentiellement mélangeant des équations de Navier-Stokes tridimensionnelles

Résumé: Nous étudierons les équations de Navier-Stokes tridimensionnelles (NS3D) muni d'un bruit blanc en temps. Nous établirons que si le bruit est à la fois suffisamment régulier et non dégénéré par rapport à la variable d'espace, alors les solutions faibles convergent exponentiellement vite vers l'équilibre.

Nous utiliserons une méthode de couplage. La preuve est un peu plus difficile que dans le cas bidimensionnel car l'unicité des solutions reste un problème ouvert pour NS3D. Cependant, certaines simplifications apparaissent car nous travaillons avec des bruits non dégénérés.

Mots clés : Équations de Navier-Stokes tridimensionnelles, semi-groupe de transition de Markov, mesures invariantes, ergodicité, méthode de couplage, mélange exponentiel, approximations de Galerkin.

Exponential mixing for the $3D$ stochastic Navier–Stokes equations

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Abstract: We study the Navier-Stokes equations in dimension 3 (NS3D) driven by a noise which is white in time. We establish that if the noise is at same time sufficiently smooth and non degenerate in space, then the weak solutions converge exponentially fast to equilibrium.

We use a coupling method. The arguments used in dimension two do not apply since, as is well known, uniqueness is an open problem for NS3D. New ideas are introduced. Note however that many simplifications appears since we work with non degenerate noises.

Key words: Stochastic three-dimensional Navier-Stokes equations, Markov transition semi-group, invariant measure, ergodicity, coupling method, exponential mixing, galerkin approximation.

Introduction

We are concerned with the stochastic Navier–Stokes equations on a three dimensional bounded domain (NS3D) with Dirichlet boundary conditions. These equations describe the time evolution of an incompressible fluid subjected to a determinist and a random exterior force and are given by

$$
(0.1)\qquad \begin{cases} dX + \nu(-\Delta)X\,dt + (X,\nabla)X\,dt + \nabla p\,dt = \phi(X)dW + f\,dt, \\ (\mathrm{div}\ X)\,(t,\xi) = 0, \qquad \text{for } \xi \in D, \quad t>0, \\ X(t,\xi) = 0, \qquad \text{for } \xi \in \partial D, \quad t>0, \\ X(0,\xi) = x_0(\xi), \qquad \text{for } \xi \in D. \end{cases}
$$

Here D is an open bounded domain of $\mathbb{R}^3$ with smooth boundary ∂D or $D=(0,1)^3$. We have denoted by X the velocity, by p the pressure and by ν the viscosity. The external force field acting on the fluid is the sum of a random force field of white noise type $\phi(X)dW$ and a determinist one $f\,dt$.

In the deterministic case ($\phi=0$), there exists a global weak solution (in the PDE sense) of (0.1), but uniqueness of such solution is not known. On another hand, there exists a unique local strong solution when x_0 is a smooth map, but global existence is an open problem (See for instance [5], [8], [18], [23], [24], [31], [32], [41] and [46]).

In the stochastic case, there exists a global weak solution of the martingale problem, but pathwise uniqueness and uniqueness in law remain open problems. (See for instance [1], [2], [4], [6], [7], [14], [16], [35], [44] and [45])

The main result of the present article is to establish that, if ϕ is at the same time sufficiently smooth and non degenerate, then the solutions converge exponentially fast to equilibrium. More precisely, given a solution, there exists a stationary solution (which might depends on the given solution), such that the total variation distance between the laws of the given solution and of the stationary solution tends to zero exponentially fast.

Due to the lack of uniqueness, it is not straightforward to define a Markov evolution associated to (0.1). Some recent progress have been obtained in this direction. In [9], under suitable conditions on ϕ and f very similar to ours, a Markov transition semi-group associated to (0.1) has been constructed. Moreover it is the limit of Galerkin approximations. Uniqueness in law is not known but we think that this result is a step in this direction. Our result combined with this result implies that the transition semi-group constructed in [9] is exponentially mixing.

Note also that recently, a Markov selection argument has allowed the construction of a Markov evolution in [17]. Our result does not directly apply since we only consider solutions which are limit of Galerkin approximations. However, suitable modifications of our proof might imply that under suitable assumptions on the noise, the Markov semi-group constructed in [17] is also exponentially mixing.

Our proof relies on coupling arguments. These have been introduced recently in the context of stochastic partial differential equations by several authors (see [19], [25], [28], [29], [30], [33], [37], [38] and [40]). The aim was to prove exponential mixing for degenerate noise. It was previously observed that the degeneracy of the noise on some subspace could be compensated by dissipativity arguments [3], [13], [26]. More recently, highly degenerate noise noises have been considered in [20], [34].

In all these articles, global well posedness of the stochastic equation is strongly used in many places of the proof. As already mentioned, this is not the case for the three dimensional Navier-Stokes equations considered here. Thus substantial changes in the proof have to be introduced. However, we require that the noise is sufficiently non degenerate and many difficulties of the above mentioned articles disappear.

The main idea is that coupling of solutions can be achieved for initial data which are small in a sufficiently smooth norm. A coupling satisfying good properties is constructed thanks to the Bismuth-Elworthy-Li formula. Another important ingredient in our proof is that any weak solution enters a small ball in the smooth norm and that the time of entering in this ball admits an exponential moment. We overcome the lack of uniqueness of solutions by working with Galerkin approximations. We prove exponential mixing for these with constants which are controlled uniformly. Taking the limit, we obtain our result for solutions which are limit of Galerkin approximations.

1. Preliminaries and main result

1.1. **Weak solutions.**

Here $\mathcal{L}(K_1;K_2)$ (resp $\mathcal{L}_2(K_1;K_2)$) denotes the space of bounded (resp Hilbert-Schmidt) linear operators from the Hilbert space K_1 to K_2.

We denote by $|\cdot|$ and $(\cdot,\cdot)$ the norm and the inner product of $L^2(D;\mathbb{R}^3)$ and by $|\cdot|_p$ the norm of $L^p(D;\mathbb{R}^3)$. Recall now the definition of the Sobolev spaces $H^p(D;\mathbb{R}^3)$ for $p\in\mathbb{N}$

$$\left\{\begin{array}{l} H^p(D;\mathbb{R}^3)=\left\{X\in L^2(D;\mathbb{R}^3)\;\middle|\;\partial_\alpha X\in L^2(D;\mathbb{R}^3)\;\text{ for }\;|\alpha|\le p\right\},\\ |X|_{H^p}^2=\sum_{|\alpha|\le p}|\partial_\alpha X|^2.\end{array}\right.$$

It is well known that $(H^p(D;\mathbb{R}^3),|\cdot|_{H^p})$ is a Hilbert space. The Sobolev space $H_0^1(D;\mathbb{R}^3)$ is the closure of the space of smooth functions on D with compact support by $|\cdot|_{H^1}$. Setting $\|X\|=|\nabla X|$, we obtain that $\|\cdot\|$ and $|\cdot|_{H^1}$ are two equivalent norms on $H_0^1(D;\mathbb{R}^3)$ and that $(H_0^1(D;\mathbb{R}^3),\|\cdot\|)$ is a Hilbert space.

Let H and V be the closure of the space of smooth functions on D with compact support and free divergence for the norm $|\cdot|$ and $\|\cdot\|$, respectively.
Let π be the orthogonal projection in $L^2(D;\mathbb{R}^3)$ onto the space H. We set

$$A=\pi\,(-\Delta)\,,\;D(A)=V\cap H^2(D;\mathbb{R}^3),\;B(u,v)=\pi\left((u,\nabla)v\right)\;\text{ and }\;B(u)=B(u,u).$$

Let us recall the following useful identities

$$\left\{\begin{array}{rcll}(B(u,v),v)&=&0,&u,\,v\in V,\\ (B(u,v),w)&=&-(B(u,w),v),&u,\,v,\,w\in V.\end{array}\right.$$

As is classical, we get rid of the pressure and rewrite problem (0.1) in the form

$$\left\{\begin{array}{rcl} dX+\nu AX\,dt+B(X)dt&=&\phi(X)dW+f\,dt,\\ X(0)&=&x_0,\end{array}\right.\tag{1.1}$$

where W is a cylindrical Wiener process on H and with a slight abuse of notations, we have denoted by the same symbols the projections of ϕ and f.

It is well-known that $(A,\mathcal{D}(A))$ is a self-adjoint operator with discrete spectrum. See [8], [41]. We consider $(e_n)_n$ an eigenbasis of H associated to the increasing sequence $(\mu_n)_n$ of eigenvalues of $(A,\mathcal{D}(A))$. It will be convenient to use the fractionnal power $(A^s,\mathcal{D}(A^s))$ of the operator $(A,\mathcal{D}(A))$ for $s\in\mathbb{R}$

$$\left\{\begin{array}{rcl}\mathcal{D}(A^s)&=&\left\{X=\sum_{n=1}^\infty x_ne_n\;\middle|\;\sum_{n=1}^\infty \mu_n^{2s}|x_n|^2<\infty\right\},\\ A^sX&=&\sum_{n=1}^\infty\mu_n^s x_ne_n\;\text{ where }\;X=\sum_{n=1}^\infty x_ne_n.\end{array}\right.$$

We set for any $s\in\mathbb{R}$

$$\|X\|_s=\left|A^{\frac{s}{2}}X\right|,\quad \mathbb{H}_s=\mathcal{D}(A^{\frac{s}{2}}).$$

It is obvious that $(\mathbb{H}_s,\|\cdot\|_s)$ is a Hilbert space, that $(\mathbb{H}_0,\|\cdot\|_0)=(H,|\cdot|)$ and that $(\mathbb{H}_1,\|\cdot\|_1)=(V,\|\cdot\|)$. Moreover, recall that, thanks to the regularity theory of the Stokes operator, $\mathbb{H}_s$ is a closed subspace of $H^s(D,\mathbb{R}^3)$ and $\|\cdot\|_s$ is equivalent to the usual norm of $H^s(D;\mathbb{R}^3)$ when D is an open bounded domain of $\mathbb{R}^3$ with smooth boundary ∂D. When $D=(0,1)^3$, it remains true for $s\le 2$.

Let us define

$$\begin{cases} \mathcal{X} &=& L^\infty_{\text{loc}}(\mathbb{R}^+;H)\cap L^2_{\text{loc}}(\mathbb{R}^+;V)\cap C(\mathbb{R}^+;\mathbb{H}_s),\\ \mathcal{W} &=& C(\mathbb{R}^+;\mathbb{H}_{-2}),\\ \Omega_* &=& \mathcal{X}\times\mathcal{W},\end{cases}$$

where s is any fixed negative number. Remark that the definition of $\mathcal{X}$ is not depending on $s<0$. Let X_* (resp W_*) be the projector $\Omega_*\to\mathcal{X}$ (resp $\Omega_*\to\mathcal{W}$). The space Ω_* is endowed with its borelian σ-algebra $\mathcal{F}^*$ and with $(\mathcal{F}^*_t)_{t\ge 0}$ the filtration generated by (X_*,W_*).

Recall that W is said to be a $(\mathcal{F}_t)_t$–cylindrical Wiener process on H if W is $(\mathcal{F}_t)_t$–adapted, if $W(t+\cdot)-W(t)$ is independant of $\mathcal{F}_t$ for any $t\ge 0$ and if W is a cylindrical Wiener process on H. Let E be a Polish space. We denote by $P(E)$ the set of probability measure on E endowed with the borelian σ–algebra.

Definition 1.1 (Weak solutions)**.** *A probability measure $\mathbb{P}_\lambda$ on $(\Omega_*,\mathcal{F}^*)$ is said to be a weak solution of* (1.1) *with initial law $\lambda\in P(H)$ if the three following properties hold.*

i) *The law of $X_*(0)$ under $\mathbb{P}_\lambda$ is λ.*
ii) *The process W_* is a $(\mathcal{F}^*_t)_t$–cylindrical Wiener process on H under $\mathbb{P}_\lambda$.*
iii) *We have $\mathbb{P}_\lambda$-almost surely*

$$(1.2)\qquad \begin{aligned}&(X_*(t),\psi)+\nu\int_0^t(X_*(s),A\psi)ds-\int_0^t(X_*(s),(X_*(s),\nabla)\psi)ds\\ &\qquad\qquad=(X_*(0),\psi)+t\,(f,\psi)+\int_0^t(\psi,\phi(X_*(s))dW_*(s)),\end{aligned}$$

for any $t\in\mathbb{R}^+$ and any ψ smooth map on D with compact support and free divergence.

When the initial value λ is not specified, x_0 is the initial value of the weak solution $\mathbb{P}_{x_0}$ (i.e. λ is equal to δ_{x_0} the Dirac mass at point x_0).

These solutions are weak in both probability and PDE sense. On the one hand, these are solutions in law. Existence of solutions in law does not imply that, given a Wiener process W and an initial condition x_0, there exist a solution X associated to W and x_0. On the other hand, these solutions live in H and it is not known if they live in $\mathbb{H}_1$. This latter fact causes many problems when trying to apply Ito Formula on $F(X_*(t))$ when F is a smooth smap. Actually, we do not know if we are allowed to apply it.

That is the reason why we do not consider any weak solution but only those which are limit in distribution of solutions of Galerkin approximations of (1.1). More precisely, for any $N\in\mathbb{N}$, we denote by P_N the eigenprojector of A associated to the first N eigenvalues. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and W be a cylindrical Wiener process on H for $\mathbb{P}$. We consider the following approximation of (1.1)

$$(1.3)\qquad \begin{cases} dX_N+\nu AX_Ndt+P_NB(X_N)dt &=& P_N\phi(X_N)dW+P_Nfdt,\\ X_N(0) &=& P_Nx_0.\end{cases}$$

In order to have existence of a weak solution, we use the following assumption.

Hypothesis 1.2. *The mapping ϕ is bounded Lipschitz $H\to\mathcal{L}_2(H;\mathbb{H}_1)$ and $f\in H$.*

It is easily shown that, given $x_0 \in H$, (1.3) has a unique solution $X_N = X_N(\cdot, x_0)$. Proceeding as in [14], we can see that the laws $(\mathbb{P}^N_{x_0})_N$ of $(X_N(\cdot, x_0), W)$ are tight in a well chosen functional space. Then, for a subsequence $(N_k)_k$, (X_{N_k}, W) converges in law to $\mathbb{P}_{x_0}$ a weak solution of (1.1). Hence we have existence of the weak solutions of (1.1), but uniqueness remains an open problem.

Remark 1.3. *We only consider weak solutions constructed in that way, because it allows to make some computations and to obtain many estimates. For instance, when trying to estimate the L^2-norm of $X_*(t)$ under a weak solution $\mathbb{P}_{x_0}$, we would like to apply the Ito Formula on $|X_*|^2$. This would give*

$$d\,|X_*|^2 + 2\nu \,\|X_*\|^2\, dt = 2\,(X_*, \phi(X_*)dW_*) + 2(f, X_*)dt + |\phi(X_*(t))|^2_{\mathcal{L}_2(H;H)}\, dt.$$

Integrating and taking the expectation, we would deduce that, if $f = 0$ and ϕ constant,

$$\mathbb{E}_{x_0}\left(|X_*(t)|^2 + 2\nu \int_0^t \|X_*(s)\|^2\, dt\right) = |x_0|^2 + t\,|\phi|^2_{\mathcal{L}_2(H;H)}\,.$$

Unfortunately, those computations are not allowed. However, analogous computations become true if we replace $\mathbb{P}_{x_0}$ by $\mathbb{P}^N_{x_0}$, which yields

$$\mathbb{E}\left(|X_N(t)|^2 + 2\nu \int_0^t \|X_N(s)\|^2\, dt\right) = |P_N x_0|^2 + t\,|P_N\phi|^2_{\mathcal{L}_2(H;H)}\,.$$

Then, we take the limit and we infer from Fatou Lemma and from the semi-continuity of $|\cdot|$, $\|\cdot\|$ in $\mathbb{H}_s$ that

$$\mathbb{E}_{x_0}\left(|X_*(t)|^2 + 2\nu \int_0^t \|X_*(s)\|^2\, dt\right) \le |x_0|^2 + t\,|\phi|^2_{\mathcal{L}_2(H;H)}\,,$$

provided $f = 0$ and ϕ constant and provided $\mathbb{P}_{x_0}$ is limit in distribution of solutions of (1.3).

Let $\mathbb{P}'$ and Y be a probability measure and a random variable on $(\Omega_*, \mathcal{F}^*)$, respectively. The distribution $\mathcal{D}_{\mathbb{P}'}(Y)$ denotes the law of Y under $\mathbb{P}'$.

A weak solution $\mathbb{P}_\mu$ with initial law μ is said to be stationary if, for any $t \ge 0$, μ is equal to $\mathcal{D}_{\mathbb{P}_\mu}(X_*(t))$.

We define

$$\left(\mathcal{P}^N_t \psi\right)(x_0) = \mathbb{E}\left(\psi(X_N(t, x_0))\right) = \mathbb{E}^N_{x_0}\left(\psi(X_*(t)\right),$$

where $\mathbb{E}^N_{x_0}$ is the expectation associated to $\mathbb{P}^N_{x_0}$.

It is easily shown that $X_N(\cdot, x_0)$ verifies the strong Markov property, which obviously implies that $(\mathcal{P}^N_t)_{t\in\mathbb{R}^+}$ is a Markov transition semi-group on $P_N H$.

Ito Formula on $|X_N(\cdot, x_0)|^2$ gives

$$d\,|X_N|^2 + 2\,\|X_N\|^2\, dt = 2\,(X_N, \phi(X_N)dW) + 2(X_N, f)dt + |P_N\phi(X_N)|^2\, dt,$$

which yields, by applying arithmetico-geometric inequality and Hypothesis 1.4,

$$d\,|X_N|^2 + \|X_N\|^2\, dt \le 2\,(X_N, \phi(X_N)dW) + cB_0 dt. \tag{1.4}$$

Integrating and taking the expectation, we obtain

$$\mathbb{E}\left(|X_N(t)|^2\right) \le e^{\mu_1 t}\,|x_0|^2 + \frac{c}{\mu_1}B_0. \tag{1.5}$$

Hence, applying the Krylov-Bogoliubov Criterion (see [10]), we obtain that $(\mathcal{P}^N_t)_t$ admits an invariant measure μ_N and that every invariant measure has a moment of order two in H. Let X_0^N be a random variable whose law is μ_N and which is independent of W, then $X_N = X_N(\cdot, X_0^N)$ is a stationary solution of (1.3). Integrating (1.4), we obtain

$$\mathbb{E}\,|X_N(t)|^2 + \mathbb{E}\int_0^t \|X_N(s)\|^2\, ds \le \mathbb{E}\,|X_N(0)|^2 + cB_0 t.$$

Since the law of $X_N(s)$ is μ_N for any $s \ge 0$ and since μ_N admits a moment of order 2, it follows

$$\int_{P_N H} \|x\|^2\, \mu_N(dx) \le cB_0. \tag{1.6}$$

Moreover the laws $(\mathbb{P}^N_{\mu_N})_N$ of $(X_N(\cdot, X_0^N), W)$ are tight in a well chosen functional space. Then, for a subsequence $(N'_k)_k$, $\mathbb{P}^{N_k}_{\mu_{N_k}}$ converges in law to $\mathbb{P}_\mu$ a weak stationary solution of (1.1) with initial law μ (See [14] for details). We deduce from (1.6) that

$$\int_H \|x\|^2\, \mu(dx) \le cB_0,$$

which yields (See [16])

$$\mathbb{P}_\mu\left(X_*(t) \in \mathbb{H}_1\right) = 1 \quad \text{for any} \quad t \ge 0. \tag{1.7}$$

We do not know if $X_*(t) \in \mathbb{H}_1$ for all t holds $\mathbb{P}_\mu$–almost surely. This would probably imply strong uniqueness ν–almost surely. Remark that it is not known if μ is an invariant measure because, due to the lack of uniqueness, it is not known if (1.1) defines a Markov evolution.

1.2. **Exponential convergence to equilibrium.**

In the present article, the covariance operator ϕ of the noise is assumed to be at the same time sufficiently smooth and non degenerate with bounded derivatives. More precisely, we use the following assumption.

Hypothesis 1.4. *There exist $\varepsilon > 0$ such that $f \in \mathbb{H}_\varepsilon$ and a family $(\phi_n)_n$ of continuous maps $H \to \mathbb{R}$ with continuous derivatives such that*

$$\begin{cases} \phi(x)dW = \sum_{n=1}^\infty \phi_n(x) e_n dW_n \quad \text{where} \quad W = \sum_{n=0}^\infty W_n e_n, \\ \kappa_0 = \sum_{n=1}^\infty \sup_{x\in H} |\phi_n(x)|^2\, \mu_n^{1+\varepsilon} < \infty. \end{cases}$$

Moreover there exists κ_1 such that for any x, $\eta \in \mathbb{H}_2$

$$\sum_{n=1}^\infty |\phi_n'(x)\cdot\eta|^2\, \mu_n^2 < \kappa_1 \|\eta\|_2^2.$$

For any $x \in H$ and $N \in \mathbb{N}$, we have $\phi_n(x) > 0$ and

$$\kappa_2 = \sup_{x\in H} \left|\phi^{-1}(x)\right|^2_{\mathcal{L}(\mathbb{H}_3;H)} < \infty,$$

where

$$\phi(x)^{-1}\cdot h = \sum_{n=1}^\infty \phi_n(x)^{-1} h_n e_n \quad \text{for} \quad h = \sum_{n=0}^\infty h_n e_n.$$

For instance, $\phi = A^{-\frac{s}{2}}$ fulfills Hypothesis 1.4 provided $s \in \left(\frac{5}{2}, 3\right]$.
We set

$$B_0 = \kappa_0 + \kappa_1 + \kappa_2 + \|f\|_\varepsilon^2.$$

Remark 1.5 (Additive noise). *If the noise is additive, Hypothesis 1.4 simplifies. Indeed in this case, we do not need to assume that ϕ and A commute. This require a different but simpler proof of Lemma 3.2 below.*

Remark 1.6 (Large viscosity). *Another situation where we can rid of the assumption that the noise is diagonal is when the viscosity ν is sufficiently large. The proof is simpler in that case.*

Remark 1.7. *It is easily shown that Hypothesis 1.4 implies Hypothesis 1.2. Therefore, solution of* (1.3) *are well-defined and, for a subsequence, they converges to weak solution of* (1.1).

The aim of the present article is to establish that, under Hypothesis 1.4, the law of $X_*(t)$ under a weak solution $\mathbb{P}_{x_0}$ converges exponentially fast to equilibrium provided $\mathbb{P}_{x_0}$ is limit in distribution of solutions of (1.3).

Before stating our main result, let us recall some definitions. Let E be Polish space. The set of all bounded measurable (resp uniformly continuous) maps from E to $\mathbb{R}$ is denoted by $B_b(E;\mathbb{R})$ (resp $UC_b(E;\mathbb{R})$). The total variation $\|\mu\|_{var}$ of a finite real measure λ on E is given by

$$\|\lambda\|_{var} = \sup\left\{|\lambda(\Gamma)| \mid \Gamma \in \mathcal{B}(E)\right\},$$

where we denote by $\mathcal{B}(E)$ the set of the Borelian subsets of E.

The main result of the present article is the following. Its proof is given in section 4 after several preliminary results.

Theorem 1.8. *Assume that Hypothesis 1.4 holds. There exists δ^0, C and $\gamma > 0$ only depending on ϕ, D, ε and ν such that, for any weak solution $\mathbb{P}_\lambda$ with initial law $\lambda \in \mathcal{P}(H)$ which is limit of solutions of* (1.3)*, there exists a weak stationary solution $\mathbb{P}_\mu$ with initial law μ such that*

$$\|\mathcal{D}_{\mathbb{P}_\lambda}(X_*(t)) - \mu\|_{var} \le Ce^{-\gamma t}\left(1 + \int_H |x|^2\,\lambda(dx)\right), \tag{1.8}$$

provided $\|f\|_\varepsilon^2 \le \delta^0$ and where $\|\cdot\|_{var}$ is the total variation norm associated to the space $\mathbb{H}_s$ for $s<0$.

Moreover, for a given $\mathbb{P}_\lambda$, μ is unique and $\mathbb{P}_\mu$ is limit of solutions of (1.3).

It is well known that $\|\cdot\|_{var}$ is the dual norm of $|\cdot|_\infty$ which means that for any finite measure λ' on $\mathbb{H}_s$ for $s<0$

$$\|\lambda'\|_{var} = \sup_{|g|_\infty \le 1}\left|\int_{\mathbb{H}_s} g(x)\,\lambda'(dx)\right|,$$

where the supremum is taken over $g \in UC_b(\mathbb{H}_s)$ which verifies $|g|_\infty \le 1$. Hence (1.8) is equivalent to

$$\left|\mathbb{E}_\lambda\left(g(X_*(t))\right) - \int_H g(x)\,\mu(dx)\right| \le C\,|g|_\infty\left(1 + \int_H |x|^2\,\lambda(dx)\right), \tag{1.9}$$

for any $g \in UC_b(\mathbb{H}_s)$.

Remark 1.9 (Topology associated to the total variation norm). *Remark that if λ' is a finite measure of $\mathbb{H}_{s_0}$, then the value of the total variation norm of λ' associated to the space $\mathbb{H}_s$ is not depending of the value of $s \le s_0$.*

Hence, since $\mathcal{D}_{\mathbb{P}_\lambda}(X_*(t))$ is a probability measure on H then (1.8) *(resp* (1.9)*) remains true when $\|\cdot\|_{var}$ is the total variation norm associated to the space H (resp for any $g \in B_b(H;\mathbb{R})$).*

Moreover, we see below that if λ is a probability measure on $\mathbb{H}_2$, then $\mathcal{D}_{\mathbb{P}_\lambda}(X_*(t))$ is still a probability measure on $\mathbb{H}_2$. It follows that (1.8) *(resp* (1.9)*) remains true when $\|\cdot\|_{var}$ is associated to $\mathbb{H}_2$ (resp for any $g \in B_b(\mathbb{H}_2;\mathbb{R})$).*

We deduce the following result.

Corollary 1.10 (Regularization of the solutions)**.** *Assume that Hypothesis 1.4 holds. There exist $\delta^0 = \delta^0(B_0, D, \varepsilon, \nu)$, $C = C(\phi, D, \varepsilon, \nu) > 0$ and $\gamma = \gamma(\phi, D, \varepsilon, \nu) > 0$ such that if $\|f\|_\varepsilon^2 \le \delta^0$ then, for any weak solution $\mathbb{P}_\lambda$ with initial law $\lambda \in \mathcal{P}(H)$,*

$$\mathbb{P}_\lambda\left(X_*(t) \notin \mathbb{H}_2\right) \le Ce^{-\gamma t}\left(1 + \int_H |x|^2 \, \lambda(dx)\right), \tag{1.10}$$

provided $\mathbb{P}_\lambda$ is limit of solutions of (1.3).

The proof of this result is postponed to section 1.4. This is a remarkable result because X_* living in $\mathbb{H}_1$ when starting from $\mathbb{H}_1$ remains an open problem.

Remark 1.11. *It is well-known that Hypothesis 1.2 implies that*

$$\mathbb{P}_\lambda\left(X_*(t) \in \mathbb{H}_2\right) = 1 \quad \textit{almost surely in time for the Lebesgue measure},$$

provided $\lambda \in P(H)$.

Inequality (1.10) *of Corollary 1.10 is true for any $t \in \mathbb{R}^+$. Moreover, we see below that if $\lambda \in P(\mathbb{H}_2)$, then*

$$\mathbb{P}_\lambda\left(X_*(t) \in \mathbb{H}_2\right) = 1 \quad \textit{for any } t \in \mathbb{R}^+,$$

provided f and ϕ verifies suitable conditions.

Our method is not influenced by the size of the viscosity ν. Then, for simplicity in the redaction, we now assume that $\nu = 1$.

1.3. **Markov evolution.**

Here, we take into account the remarkable result of [9] and we rewrite Theorem 1.8. This section is not necessary in the understanding of the proof of Theorem 1.8.

Let $(N_k')_k$ be an increasing sequence of integer. In [9], it is established that it is possible to extract a subsequence $(N_k)_k$ of $(N_k')_k$ such that $(\mathcal{P}_t^{N_k})_{t\ge 0}$ converges in some sense to a family $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ provided the following assumption holds.

Hypothesis 1.12. *There exist $\varepsilon, \delta > 0$ such that the mapping ϕ is constant and lives in $\mathcal{L}_2\left(H;\mathbb{H}_{1+\varepsilon}\right)$. Moreover ker $\phi = \{0\}$ and there exits $\phi^{-1} \in \mathcal{L}\left(\mathbb{H}_{3-\delta}; H\right)$ such that*

$$\phi \cdot \phi^{-1} \cdot h = h \quad \textit{ for any } h \in \mathbb{H}_{3-\delta}.$$

The method to extract $(N_k)_k$ is based on the investigation of the properties of the Kolmogorov equation associated to (1.1) perturbed by a very irregular potential. Moreover, for any $x_0 \in \mathbb{H}_2$, a subsequence of $(N_k)_k$ such that $\mathbb{P}_{x_0}^N$ converges in distribution to a weak solution $\mathbb{P}_{x_0}$ of (1.1) is extracted. We have

$$\mathbb{E}_{x_0}\left(\psi(X_*(t))\right) = \left(\mathcal{P}_t\psi\right)(x_0), \tag{1.11}$$

provided $t \ge 0$ and $\psi \in UC_b(\mathbb{H}_s;\mathbb{R})$ where s is any fixed negative number. In this way, we have constructed a family of weak solutions $(\mathbb{P}_x)_{x\in\mathbb{H}_2}$.

Moreover it is proved that $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ is a Markov transition semi-group on $\mathbb{H}_2$.

Namely, it is shown that $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ is a family of operators $B_b(\mathbb{H}_2;\mathbb{R}) \to B_b(\mathbb{H}_2;\mathbb{R})$ which verifies

- $\mathcal{P}_0 = Id_{B_b(\mathbb{H}_2;\mathbb{R})}$,
- $\mathcal{P}_{t+s} = \mathcal{P}_t\mathcal{P}_s$ for any $(t,s) \in (\mathbb{R}^+)^2$,
- $\mathcal{P}_t^*\delta_{x_0}$ is a probability measure on $\mathbb{H}_2$, for any $(t,x_0) \in \mathbb{R}^+ \times \mathbb{H}_2$.

Furthermore, $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ is strong Feller and strongly mixing.

An important consequence is the following. Assume that Hypothesis 1.12 holds and let $x_0 \in \mathbb{H}_2$ and $\mathbb{P}'_{x_0}$ be a weak solution of (1.1) which is limit in distribution of solutions of (1.3). Then, extracting a subsequence, we can build $(\mathcal{P}_t)_{t\geq 0}$ and $(\mathbb{P}_x)_{x\in\mathbb{H}_2}$ as above such that

$$\mathbb{P}'_{x_0} = \mathbb{P}_{x_0}.$$

Hence, although it is not known if X_* has the weak Markov property under $\mathbb{P}'_{x_0}$, some Markov properties can be used.

Another important remark is that, for any $x_0 \in \mathbb{H}_2$ and any weak solution $\mathbb{P}_{x_0}$ limit of solutions of (1.3), we have

$$\mathbb{P}_{x_0}\left(X_*(t) \in \mathbb{H}_2\right) = 1 \quad \text{for any } t \geq 0.$$

Note that this result was known only for a stationary solution (see [16]).

We believe that the existence of such transition semi-group still holds when ϕ is non constant with bounded derivative. The proof is an extension of method in [9] and will be treated in a future work. That is the reason why, under Hypothesis 1.4, it is natural to expect that the following assumption holds

Hypothesis 1.13. *There exist a Markov transition semi-group $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ on $\mathbb{H}_2$ and a family $(\mathbb{P}_{x_0})_{x_0\in\mathbb{H}_2}$ of weak solutions of (1.1) that are limit in distribution of solutions of (1.3) and such that, for any $(t,x_0) \in \mathbb{R}^+ \times \mathbb{H}_2$, $\mathcal{P}_t^*\delta_{x_0}$ is the law of $X_*(t)$ under $\mathbb{P}_{x_0}$.*

Hence, we immediately deduce the following corollary from Theorem 1.8.

Corollary 1.14. *Assume that Hypothesis 1.4 and 1.13 hold. Then there exit a unique invariant measure μ for $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ and $C,\gamma > 0$ such that for any $\lambda \in \mathcal{P}(\mathbb{H}_2)$*

$$\|\mathcal{P}_t^*\lambda - \mu\|_{var} \leq Ce^{-\gamma t}\left(1 + \int_{\mathbb{H}_2} |x|^2\,\lambda(dx)\right), \tag{1.12}$$

provided $\|f\|_\varepsilon^2 \leq \delta^0$ and where $\|\cdot\|_{var}$ is the total variation norm associated to the space $\mathbb{H}_2$.

A particular case is the following result.

Corollary 1.15. *Assume that Hypothesis 1.12 holds. Then there exit a unique invariant measure μ for $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ and $C,\gamma > 0$ such that for any $\lambda \in \mathcal{P}(\mathbb{H}_2)$*

$$\|\mathcal{P}_t^*\lambda - \mu\|_{var} \leq Ce^{-\gamma t}\left(1 + \int_{\mathbb{H}_2} |x|^2\,\lambda(dx)\right), \tag{1.13}$$

where $\|\cdot\|_{var}$ is the total variation norm associated to the space $\mathbb{H}_2$.

Remark 1.16 (Uniqueness of the invariant measure μ)**.** *Assume that Hypothesis 1.12 holds. Let $\mathbb{P}_{x_0}$ and $\mathbb{P}'_{x_0}$ be two weak solutions of (1.1) which are limit in distribution of solutions of (1.3). Then we build $(\mathcal{P}_t)_t$ and $(\mathcal{P}'_t)_t$ as above associated to $\mathbb{P}_{x_0}$ and $\mathbb{P}'_{x_0}$, respectively. It follows that there exists a couple (μ,μ')*

such that (1.13) *and* (1.9) *hold for* $((\mathcal{P}_t)_t, \mathbb{P}_{x_0}, \mu)$ *and* $((\mathcal{P}'_t)_t, \mathbb{P}'_{x_0}, \mu')$*. Moreover we have uniqueness of the invariant measures* μ *and* μ' *associated to* $(\mathcal{P}_t)_t$ *and* $(\mathcal{P}'_t)_t$ *respectively. However we do not know if* μ *and* μ' *are equal.*

1.4. **Coupling methods.**

The proof of Theorem 1.8 is based on coupling arguments. We now recall some basic results about the coupling and we deduce Corollary 1.10 from Theorem 1.8. Moreover, in order to explain the coupling method in the case of non degenerate noise, we briefly give the proof of exponential mixing for equation (1.3).

Let (λ_1, λ_2) be two distributions on a same polish space $(E, \mathcal{E})$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (Z_1, Z_2) be two random variables $(\Omega, \mathcal{F}) \to (E, \mathcal{E})$. We say that (Z_1, Z_2) is a coupling of (λ_1, λ_2) if $\lambda_i = \mathcal{D}(Z_i)$ for $i = 1, 2$. We have denoted by $\mathcal{D}(Z_i)$ the law of the random variable Z_i.

Next Lemma is a fundamental result in the coupling methods, the proof is given for instance in the Appendix of [37].

Lemma 1.17. *Let* (λ_1, λ_2) *be two probability measures on* $(E, \mathcal{E})$*. Then*

$$\|\lambda_1 - \lambda_2\|_{var} = \min \mathbb{P}(Z_1 \neq Z_2).$$

The minimum is taken over all couplings (Z_1, Z_2) *of* (λ_1, λ_2)*. There exists a coupling which reaches the minimum value. It is called a maximal coupling.*

Corollary 1.10 is an immediate consequence of Theorem 1.8 and Lemma 1.17. Indeed, let (Z_1, Z_2) be a maximal coupling of $(\mathcal{D}_{\mathbb{P}_\lambda}(X_*(t)), \mu)$. Combining Theorem 1.8 and Lemma 1.17, we obtain

$$\mathbb{P}\left(Z_1 \neq Z_2\right) \leq Ce^{-\gamma t}\left(1 + \int_H |x|^2 \ \lambda(dx)\right).$$

Recall that, as explained in section 1.3, $Z_2 \in \mathbb{H}_2$ almost surely. Hence

$$\mathbb{P}\left(Z_1 \notin \mathbb{H}_2\right) \leq Ce^{-\gamma t}\left(1 + \int_{\mathbb{H}_2} |x|^2 \ \lambda(dx)\right).$$

Since $\mathcal{D}(Z_1) = \mathcal{D}_{\mathbb{P}_\lambda}(X_*(t))$, Corollary 1.10 follows.

Let us now treat the case of the solutions of (1.3). Assume that Hypothesis 1.4 holds. Let $N \in \mathbb{N}$ and $(x_0^1, x_0^2) \in \mathbb{R}^2$. Combining the method explained in [37] section 1.1 and a simple truncation argument, it can be shown that there exists a decreasing map $p_N(\cdot) > 0$ such that

$$\left\| \left(\mathcal{P}_1^N\right)^* \delta_{x_0^2} - \left(\mathcal{P}_1^N\right)^* \delta_{x_0^1} \right\|_{var} \leq 1 - p_N\left(\left|x_0^1\right| + \left|x_0^2\right|\right). \tag{1.14}$$

Applying Lemma 1.17, we build a maximal coupling $(Z_1, Z_2) = (Z_1(x_0^1, x_0^2), Z_2(x_0^1, x_0^2))$ of $(\left(\mathcal{P}_1^N\right)^* \delta_{x_0^1}, \left(\mathcal{P}_1^N\right)^* \delta_{x_0^2})$. It follows

$$\mathbb{P}\left(Z_1 = Z_2\right) \geq p_N\left(\left|x_0^1\right| + \left|x_0^2\right|\right) > 0. \tag{1.15}$$

Let $(W, \widetilde{W})$ be a a couple of independent cylindrical Wiener processes and $\delta > 0$. We denote by $X_N(\cdot, x_0)$ and $\widetilde{X}_N(\cdot, x_0)$ the solutions of (1.3) associated to W and $\widetilde{W}$, respectively. Now we build a couple of random variables $(V_1, V_2) =$

$(V_1(x_0^1,x_0^2),V_2(x_0^1,x_0^2))$ on P_NH as follows

$$(1.16)\quad (V_1,V_2)=\begin{cases}(X_N(\cdot,x_0),X_N(\cdot,x_0)) & \text{if } x_0^1=x_0^2=x_0,\\ (Z_1(x_0^1,x_0^2),Z_2(x_0^1,x_0^2)) & \text{if } (x_0^1,x_0^2)\in B_H(0,\delta)\backslash\{x_0^1=x_0^2\},\\ (X_N(\cdot,x_0^1),\widetilde{X}_N(\cdot,x_0^2)) & \text{else,}\end{cases}$$

where $B_H(0,\delta)$ is the ball of $H\times H$ with radius δ.

Then $(V_1(x_0^1,x_0^2),V_2(x_0^1,x_0^2))$ is a coupling of $(\left(\mathcal{P}_1^N\right)^*\delta_{x_0^1},\left(\mathcal{P}_1^N\right)^*\delta_{x_0^2})$. It can be shown that it depends measurably on (x_0^1,x_0^2). We now build a coupling (X^1,X^2) of $(\mathcal{D}(X_N(\cdot,x_0^1)),\mathcal{D}(X_N(\cdot,x_0^2)))$ by induction on $\mathbb{N}$. Indeed, we first set $X^i(0)=x_0^i$ for $i=1,2$. Then, assuming that we have built (X^1,X^2) on $\{0,1,\dots,k\}$, we take (V_1,V_2) as above independent of (X^1,X^2) and set

$$X^i(k+1)=V_i(X^1(k),X^2(k))\quad \text{for } i=1,2.$$

Taking into account (1.5), it is easily shown that the time of return of (X^1,X^2) in $B(0,4(c/\mu_1)B_0)$ admits an exponential moment. We choose $\delta=4(c/\mu_1)B_0$. It follows from (1.15), (1.16) that, $(X^1(n),X^2(n))\in B(0,\delta)$ implies that the probability of having (X^1,X^2) coupled (i.e. equal) at time $n+1$ is bounded below by $p_N(2\delta)>0$. Finally, remark that if (X^1,X^2) are coupled at time $n+1$, then they remain coupled for any time after. Combining these three properties and using the fact that $(X^1(n),X^2(n))_{n\in\mathbb{N}}$ is a discrete strong Markov process, it is easily shown that

$$(1.17)\qquad \mathbb{P}\left(X^1(n)\neq X^2(n)\right)\le C_Ne^{-\gamma_N n}\left(1+\left|x_0^1\right|^2+\left|x_0^2\right|^2\right),$$

with $\gamma_N>0$.

Recall that (X^1,X^2) is a coupling of $(\mathcal{D}(X_N(\cdot,x_0^1)),\mathcal{D}(X_N(\cdot,x_0^2)))$ on $\mathbb{N}$. It follows that $(X^1(n),X^2(n))$ is a coupling of $((\mathcal{P}_n^N)^*\delta_{x_0^1},(\mathcal{P}_n^N)^*\delta_{x_0^2})$. Combining Lemma 1.17 and (1.17), we obtain, for $n\in\mathbb{N}$,

$$\left\|\left(\mathcal{P}_n^N\right)^*\delta_{x_0^2}-\left(\mathcal{P}_n^N\right)^*\delta_{x_0^1}\right\|_{var}\le C_Ne^{-\gamma_N n}\left(1+\left|x_0^1\right|^2+\left|x_0^2\right|^2\right).$$

Setting $n=\lfloor t\rfloor$ and integrating (x_0^2,x_0^1) over $((\mathcal{P}_{t-n}^N)^*\lambda)\otimes\mu_N$ where μ_N is an invariant measure, it follows that, for any $\lambda\in P(P_NH)$,

$$(1.18)\qquad \left\|\left(\mathcal{P}_t^N\right)^*\lambda-\mu_N\right\|_{var}\le C_Ne^{-\gamma_N t}\left(1+\int_{P_NH}|x|^2\,\lambda(dx)\right).$$

This result is useless when considering equation (1.1) since the constants C_N, γ_N strongly depend on N. If one tries to apply directly the above arguments to the infinite dimensional equation (1.1), one faces several difficulties. First it is not known whether $\mathbb{P}_{x_0}$ is Markov. We only know that, as explained in section 1.3, a Markov transition semi-group can be constructed. This is a major difficulty since this property is implicitely used in numerous places above. Another strong problem is that Girsanov transform is used in order to obtain (1.14). Contrary to the two dimensional case, no Foias-Prodi estimate is available for the three dimensional Navier-Stokes equations and the Girsanov transform should be done in the infinite dimensional equation. This seems impossible. We will show that we are able to prove an analogous result to (1.14) by a completely different argument. However, this will hold only for small initial data in $\mathbb{H}_2$. Another problem will occur since it is not known whether solutions starting in $\mathbb{H}_2$ remain in $\mathbb{H}_2$.

We will remedy the lack of Markov property by working only on Galerkin approximations and prove that (1.18) holds with constants uniform in N. As already mentioned, we prove that (1.14) is true for x_0^1, x_0^2 in a small ball of $\mathbb{H}_2$ and uniformly in N. Then, following the above argument, it remains to prove that the time of return in this small ball admits an exponential moment. Note that the smallness assumption on f is used at this step. In the following sections, we prove

Proposition 1.18. *Assume that Hypothesis 1.4 holds. Then there exist* $\delta^0 = \delta^0(B_0, D, \varepsilon, \nu)$, $C = C(\phi, D, \varepsilon, \nu) > 0$ *and* $\gamma = \gamma(\phi, D, \varepsilon, \nu) > 0$ *such that if* $\|f\|_\varepsilon^2 \leq \delta^0$ *holds, then, for any* $N \in \mathbb{N}$, *there exists a unique invariant measure* μ_N *for* $\left(\mathcal{P}_t^N\right)_{t\in\mathbb{R}^+}$. *Moreover, for any* $\lambda \in P(P_N H)$

$$\left\| \left(\mathcal{P}_t^N\right)^* \lambda - \mu_N \right\|_{var} \leq Ce^{-\gamma t}\left(1 + \int_{P_N H} |x|^2 \, \lambda(dx)\right). \tag{1.19}$$

We now explain why this result implies Theorem 1.8.

Let $\lambda \in P(H)$ and X_λ be a random variable on H whose law is λ and which is independant of W. Since $\|\cdot\|_{var}$ is the dual norm of $|\cdot|_\infty$, then (1.19) implies that

$$\left| \mathbb{E}\left(g(X_N(t, X_\lambda))\right) - \int_{P_N H} g(x)\, \mu_N(dx) \right| \leq C\, |g|_\infty \left(1 + \int_H |x|^2 \, \lambda(dx)\right), \tag{1.20}$$

for any $g \in UC_b(\mathbb{H}_s)$ for $s < 0$.

Assume that, for a subsequence $(N_k')_k$, $X_N(t, X_\lambda)$ converges in distribution in $\mathbb{H}_s$ to the law $X_*(t)$ under the weak solution $\mathbb{P}_\lambda$ of (1.1). Recall that the family $(\mathbb{P}_{\mu_N}^N)_N$ is tight. Hence, for a subsequence $(N_k)_k$ of $(N_k')_k$, $\mathbb{P}_{\mu_{N_k}}$ converges to $\mathbb{P}_\mu$ a weak stationary solution of (1.1) with initial law μ. Taking the limit, (1.9) follows from (1.20), which yields Theorem 1.8.

2. Coupling of solutions starting from small initial data

The aim of this section is to establish the following result. A result analogous to (1.15) but uniform in N.

Proposition 2.1. *Assume that Hypothesis 1.4 holds. Then there exist* $(T, \delta) \in (0,1)^2$ *such that, for any* $N \in \mathbb{N}$, *there exists a coupling* $(Z_1(x_0^1, x_0^2), Z_2(x_0^1, x_0^2))$ *of* $((\mathcal{P}_T^N)^*\delta_{x_0^1}, (\mathcal{P}_T^N)^*\delta_{x_0^2})$ *which measurably depends on* $(x_0^1, x_0^2) \in \mathbb{H}_2$ *and which verifies*

$$\mathbb{P}\left(Z_1(x_0^1, x_0^2) = Z_2(x_0^1, x_0^2)\right) \geq \frac{3}{4} \tag{2.1}$$

provided

$$\left\|x_0^1\right\|_2^2 \vee \left\|x_0^2\right\|_2^2 \leq \delta. \tag{2.2}$$

Assume that Hypothesis 1.4 holds. Let $T \in (0,1)$. Applying Lemma 1.17, we build $(Z_1(x_0^1, x_0^2), Z_2(x_0^1, x_0^2))$ as the maximal coupling of $(\mathcal{P}_T^*\delta_{x_0^1}, \mathcal{P}_T^*\delta_{x_0^2})$. Measurable dependance follows from a slight extension of Lemma 1.17 (see [37], remark A.1).

In order to establish Proposition 2.1, it is sufficient to prove that there exists $c(B_0, D)$ not depending on $T \in (0,1)$ and on $N \in \mathbb{N}$ such that

$$\left\| \left(\mathcal{P}_T^N\right)^* \delta_{x_0^2} - \left(\mathcal{P}_T^N\right)^* \delta_{x_0^1} \right\|_{var} \leq c(B_0, D)\sqrt{T}, \tag{2.3}$$

provided

$$\left\|x_0^1\right\|_2^2 \vee \left\|x_0^2\right\|_2^2 \le B_0 T^3. \tag{2.4}$$

Then it suffices to choose $T \le 1/(4c(B_0, D))^2$ and $\delta = B_0 T^3$.

Since $\|\cdot\|_{var}$ is the dual norm of $|\cdot|_\infty$, (2.3) is equivalent to

$$\left|\mathbb{E}\left(g(X_N(T, x_0^2)) - g(X_N(T, x_0^1))\right)\right| \le 8\,|g|_\infty\, c(B_0, D)\sqrt{T}. \tag{2.5}$$

for any $g \in UC_b(P_N H)$.

It follows from the density of $C_b^1(P_N H) \subset UC_b(P_N H)$ that, in order to establish Proposition 2.1, it is sufficient to prove that (2.5) holds for any $N \in \mathbb{N}$, $T \in (0,1)$ and $g \in C_b^1(P_N H)$ provided (2.4) holds.

The proof of (2.5) under this condition is splitted into the next three subsections.

2.1. **A priori estimate.**

For any process X, we define the $\mathbb{H}_1$–energy of X at time t by

$$E_X^{\mathbb{H}_1}(t) = \|X(t)\|^2 + \int_0^t \|X(s)\|_2^2\, ds.$$

Now we establish the following result which will be useful in the proof of 2.5.

Lemma 2.2. *Assume that Hypothesis 1.4 holds. There exist $K_0 = K_0(D)$ and $c = c(D)$ such that for any $T \le 1$ and any $N \in \mathbb{N}$, we have*

$$\mathbb{P}\left(\sup_{(0,T)} E^{\mathbb{H}_1}_{X_N(\cdot,x_0)} > K_0\right) \le c\left(1 + \frac{B_0}{K_0}\right)\sqrt{T},$$

provided $\|x_0\|^2 \le B_0 T$.

Let $X_N = X_N(\cdot, x_0)$. Ito Formula on $\|X_N\|^2$ gives

$$d\,\|X_N\|^2 + 2\,\|X_N\|_2^2\, dt = dM_{\mathbb{H}_1} + I_{\mathbb{H}_1} dt + \|P_N\phi(X_N)\|^2_{\mathcal{L}_2(H;\mathbb{H}_1)}\, dt + I_f dt, \tag{2.6}$$

where

$$\begin{cases} I_{\mathbb{H}_1} = -2\left(AX_N, B(X_N)\right),\ I_f = 2\left(AX_N, f\right), \\ M_{\mathbb{H}_1}(t) = 2\int_0^t \left(AX_N(s), \phi(X_N(s))dW(s)\right). \end{cases}$$

Combining a Hölder inequality, a Agmon inequality and a arithmetico-geometric inequality gives

$$I_{\mathbb{H}_1} \le 2\,\|X_N\|_2\,|X_N|_\infty\,\|X_N\| \le c\,\|X_N\|_2^{\frac{3}{2}}\,\|X_N\|^{\frac{3}{2}} \le \frac{1}{4}\,\|X_N\|_2^2 + c\,\|X_N\|^6. \tag{2.7}$$

Similarly, using Poincaré inequality and Hypothesis 1.4,

$$I_f \le \frac{1}{4}\,\|X_N\|_2^2 + c\,|f|^2 \le \frac{1}{4}\,\|X_N\|_2^2 + cB_0. \tag{2.8}$$

We deduce from (2.6), (2.7), (2.8), Hypothesis 1.4 and Poincaré inequality that

$$d\,\|X_N\|^2 + \|X_N\|_2^2\, dt \le dM_{\mathbb{H}_1} + cB_0 dt + c\,\|X_N\|^2\left(\|X_N\|^4 - 4K_0^2\right) dt, \tag{2.9}$$

where

$$K_0 = \sqrt{\frac{\mu_1}{8c}}. \tag{2.10}$$

Setting

$$\sigma_{\mathbb{H}_1} = \inf\left\{ t \in (0,T) \;\middle|\; \|X_N(t)\|^2 > 2K_0 \right\},$$

we infer from $\|x_0\|^2 \le B_0 T$ that for any $t \in (0, \sigma_{\mathbb{H}_1})$

$$E^{\mathbb{H}_1}_{X_N}(t) \le cB_0T + M_{\mathbb{H}_1}(t). \tag{2.11}$$

We deduce from Hypothesis 1.4 and from Poincaré inequality that $\phi(x)^*A$ is bounded in $\mathcal{L}(\mathbb{H}_1;\mathbb{H}_1)$ by cB_0. It follows that for any $t \in (0,\sigma_{\mathbb{H}_1})$

$$\langle M_{\mathbb{H}_1}\rangle(t) = 4\int_0^t \|P_N\phi(X_N(s))^*AX_N(s)\|^2\,dt \le cB_0\int_0^t \|X_N(s)\|^2\,ds \le 2cK_0B_0T.$$

Hence a Burkholder-Davis-Gundy inequality gives

$$\mathbb{E}\left(\sup_{(0,\sigma_{\mathbb{H}_1})} M_{\mathbb{H}_1}\right) \le c\mathbb{E}\sqrt{\langle M_{\mathbb{H}_1}\rangle(\sigma_{\mathbb{H}_1})} \le c\sqrt{K_0B_0T} \le c(K_0+B_0)\sqrt{T}.$$

It follows from (2.11) and $T \le 1$ that

$$\mathbb{E}\left(\sup_{(0,\sigma_{\mathbb{H}_1})} E^{\mathbb{H}_1}_{X_N}\right) \le c(B_0+K_0)\sqrt{T},$$

which yields, by a Chebyshev inequality,

$$\mathbb{P}\left(\sup_{(0,\sigma_{\mathbb{H}_1})} E^{\mathbb{H}_1}_{X_N} > K_0\right) \le c\left(1+\frac{B_0}{K_0}\right)\sqrt{T}.$$

Now, since $\sup_{(0,\sigma_{\mathbb{H}_1})} E^{\mathbb{H}_1}_{X_N} \le K_0$ implies $\sigma_{\mathbb{H}_1} = T$, we deduce Lemma 2.2.

2.2. **Estimate of the derivative of X_N.**

Let $N \in \mathbb{N}$ and $(x_0,h) \in (\mathbb{H}_2)^2$. We are concerned with the following equation

$$\left\{\begin{array}{rcl} d\eta_N + A\eta_N\,dt + P_N\widetilde{B}(X_N,\eta_N)\,dt &=& P_N(\phi'(X_N)\cdot\eta_N)\,dW, \\ \eta_N(s,s,x_0)\cdot h &=& P_Nh, \end{array}\right. \tag{2.12}$$

where $\widetilde{B}(X_N,\eta_N) = B(X_N,\eta_N) + B(\eta_N,X_N)$, $X_N = X_N(\cdot,x_0)$ and $\eta_N(t) = \eta_N(t,s,x_0)\cdot h$ for $t \ge s$.

Existence and uniqueness of the solutions of (2.12) are easily shown. Moreover if $g \in C^1_b(P_NH)$, then, for any $t \ge 0$, we have

$$\left(\nabla\left(\mathcal{P}^N_t g\right)(x_0), h\right) = \mathbb{E}\left(\nabla g(X_N(t,x_0)), \eta_N(t,0,x_0)\cdot h\right). \tag{2.13}$$

For any process X, we set

$$\sigma(X) = \inf\left\{ t \in (0,T) \;\middle|\; \int_0^t \|X(s)\|_2^2\,ds \ge K_0+1 \right\}, \tag{2.14}$$

where K_0 is defined in Lemma 2.2. We establish the following result.

Lemma 2.3. *Assume that Hypothesis 1.4 holds. Then there exists $c = c(B_0,D)$ such that for any $N \in \mathbb{N}$, $T \le 1$ and $(x_0,h) \in (\mathbb{H}_2)^2$*

$$\mathbb{E}\int_0^{\sigma(X_N(\cdot,x_0))} \|\eta_N(t,0,x_0)\cdot h\|_3^2\,dt \le c\,\|h\|_2^2.$$

For a better readability, we set $\eta_N(t) = \eta_N(t,0,x_0)\cdot h$ and $\sigma = \sigma(X_N(\cdot,x_0))$. Ito Formula on $\|\eta_N(t)\|_2^2$ gives

$$d\,\|\eta_N\|_2^2 + 2\,\|\eta_N\|_3^2\,dt = dM_{\eta_N} + I_{\eta_N}\,dt + \|P_N\,(\phi'(X_N)\cdot\eta_N)\|^2_{\mathcal{L}_2(U;\mathbb{H}_2)}\,dt, \tag{2.15}$$

where

$$\begin{cases} M_{\eta_N}(t) &=& 2\int_0^t \left(A^2\eta_N, (P_N\phi'(X_N)\cdot\eta_N)\,dW\right)\,ds,\\ I_{\eta_N} &=& -2\left(A^{\frac{3}{2}}\eta_N, A^{\frac{1}{2}}\widetilde{B}(X_N,\eta_N)\right). \end{cases}$$

It follows from Hölder inequalities, Sobolev Embedding and a arithmetico-geometric inequality

$$I_{\eta_N} \le c\,\|\eta_N\|_3\,\|\eta_N\|_2\,\|X_N\|_2 \le \|\eta_N\|_3^2 + c\,\|\eta_N\|_2^2\,\|X_N\|_2^2.$$

Hence, we deduce from (2.15) and Hypothesis 1.4

$$d\,\|\eta_N\|_2^2 + \|\eta_N\|_3^2\,dt \le dM_{\eta_N} + c\,\|\eta_N\|_2^2\,\|X_N\|_2^2 + B_0\,\|\eta_N\|_2^2\,dt.$$

Integrating and taking the expectation, we obtain

$$\mathbb{E}\left(\mathcal{E}(\sigma,0)\,\|\eta_N(\sigma)\|_2^2 + \int_0^\sigma \mathcal{E}(\sigma,t)\,\|\eta_N(t)\|_3^2\,dt\right) \le \|h\|_2^2, \tag{2.16}$$

where

$$\mathcal{E}(t,s) = e^{-B_0 t - c\int_s^t \|X_N(r)\|_2^2\,dr}.$$

Applying the definition of σ, we deduce

$$\mathbb{E}\int_0^\sigma \|\eta_N(t)\|_3^2\,dt \le \|h\|_2^2 \exp\left(c(K_0+1) + B_0 T\right), \tag{2.17}$$

which yields Lemma 2.3.

2.3. **Proof of** (2.5)**.**
Let $\psi \in C^\infty(\mathbb{R};[0,1])$ such that

$$\psi = 0 \ \text{ on } \ (K_0+1,\infty) \quad \text{ and } \quad \psi = 1 \ \text{ on } \ (-\infty, K_0).$$

For any process X, we set

$$\psi_X = \psi\left(\int_0^T \|X(s)\|_2^2\,ds\right).$$

Remark that

$$\left|\mathbb{E}\left(g(X_N(T,x_0^2)) - g(X_N(T,x_0^1))\right)\right| \le I_0 + |g|_\infty\,(I_1+I_2), \tag{2.18}$$

where

$$\begin{cases} I_0 &=& \left|\mathbb{E}\left(g(X_N(T,x_0^2))\psi_{X_N(\cdot,x_0^2)} - g(X_N(T,x_0^1))\psi_{X_N(\cdot,x_0^1)}\right)\right|,\\ I_i &=& \mathbb{P}\left(\int_0^T \left\|X_N(s,x_0^i)\right\|_2^2 ds > K_0\right). \end{cases}$$

For any $\theta \in [1,2]$, we set

$$\begin{cases} x_0^\theta = (2-\theta)x_0^1 + (\theta-1)x_0^2, & X_\theta = X_N(\cdot,x_0^\theta),\\ \eta_\theta(t) = \eta_N(t,0,x_0^\theta), & \sigma_\theta = \sigma(X_\theta). \end{cases}$$

Recall that σ was defined in (2.14). For a better readability, the dependance on N has been omitted. Setting

$$h = x_0^2 - x_0^1,$$

we have

$$I_0 \leq \int_1^2 |J_\theta|\, d\theta \quad J_\theta = \left(\nabla \mathbb{E}\left(g(X_\theta(T))\psi_{X_\theta}\right), h\right). \tag{2.19}$$

To bound J_θ, we apply a truncated Bismuth-Elworthy formula (See appendix A)

$$J_\theta = \frac{1}{T} J'_{\theta,1} + 2 J'_{\theta,2}, \tag{2.20}$$

where

$$\begin{cases} J'_{\theta,1} &= \mathbb{E}\left(g(X_\theta(T))\psi_{X_\theta} \int_0^{\sigma_\theta} (\phi^{-1}(X_\theta(t)) \cdot \eta_\theta(t)\cdot h, dW(t))\right), \\ J'_{\theta,2} &= \mathbb{E}\left(g(X_\theta(T))\psi'_{X_\theta} \int_0^{\sigma_\theta} \left(1-\frac{t}{T}\right)(AX_\theta(t), A(\eta_\theta(t)\cdot h))\, dt\right), \\ \psi'_X &= \psi'\left(\int_0^T \|X_\theta(s)\|_2^2\, ds\right). \end{cases}$$

It follows from Hölder inequality that

$$\left|J'_{\theta,2}\right| \leq |g|_\infty |\psi'|_\infty \sqrt{\mathbb{E}\int_0^{\sigma_\theta} \|X_\theta(t)\|_2^2\, dt}\sqrt{\mathbb{E}\int_0^{\sigma_\theta} \|\eta_\theta(t)\cdot h\|_2^2\, dt}.$$

and from Hypothesis 1.4 that

$$\left|J'_{\theta,1}\right| \leq |g|_\infty B_0 \sqrt{\mathbb{E}\int_0^{\sigma_\theta} \|\eta_\theta(t)\cdot h\|_3^2\, dt}.$$

Hence for any $T \leq 1$

$$|J_\theta| \leq c(B_0, D)\, |g|_\infty \frac{1}{T}\sqrt{\mathbb{E}\int_0^{\sigma_\theta} \|\eta_\theta(t)\cdot h\|_3^2\, dt}. \tag{2.21}$$

Combining (2.21) and Lemma 2.3, we obtain

$$|J_\theta| \leq c(B_0, D)\, |g|_\infty \frac{\|h\|_2}{T},$$

which yields, by (2.4) and (2.19),

$$I_0 \leq c(B_0, D)\, |g|_\infty \sqrt{T}.$$

Since $B_0 T^3 \leq B_0 T$, we can apply Lemma 2.2 to control $I_1 + I_2$ in (2.18) if (2.4) holds. Hence (2.5) follows provided (2.4) holds, which yields Proposition 2.1.

3. Time of return in a small ball of $\mathbb{H}_2$

Assume that Hypothesis 1.4 holds. Let $N \in \mathbb{N}$ and T, δ, Z_1, Z_2 be as in Proposition 2.1. Let $(W, \widetilde{W})$ be a couple of independant cylindrical Wiener processes on H. We denote by $X_N(\cdot, x_0)$ and $\widetilde{X}_N(\cdot, x_0)$ the solutions of (1.3) associated to W and $\widetilde{W}$, respectively. We build a couple of random variables

$(V_1,V_2)=(V_1(x_0^1,x_0^2),V_2(x_0^1,x_0^2))$ on P_NH as follows

$$(3.1)\quad (V_1,V_2)=\begin{cases}(X_N(\cdot,x_0),X_N(\cdot,x_0)) & \text{if } x_0^1=x_0^2=x_0,\\ (Z_1(x_0^1,x_0^2),Z_2(x_0^1,x_0^2)) & \text{if } (x_0^1,x_0^2)\in B_{\mathbb{H}_2}(0,\delta)\backslash\{x_0^1=x_0^2\},\\ (X_N(\cdot,x_0^1),\widetilde{X}_N(\cdot,x_0^2)) & \text{else,}\end{cases}$$

We then build (X^1,X^2) by induction on $T\mathbb{N}$. Indeed, we first set $X^i(0)=x_0^i$ for $i=1,2$. Then, assuming that we have built (X^1,X^2) on $\{0,T,2T,\dots,nT\}$, we take (V_1,V_2) as above independent of (X^1,X^2) and we set

$$X^i((n+1)T)=V_i(X^1(nT),X^2(nT))\quad \text{ for } i=1,2.$$

It follows that (X^1,X^2) is a discrete strong Markov process and a coupling of $(\mathcal{D}(X_N(\cdot,x_0^1)),\mathcal{D}(X_N(\cdot,x_0^2)))$ on $T\mathbb{N}$. Moreover, if (X^1,X^2) are coupled at time nT, then they remain coupled for any time after.

We set

$$(3.2)\qquad \tau=\inf\left\{t\in T\mathbb{N}\backslash\{0\}\ \Big|\ \left\|X^1(t)\right\|_2^2\vee\left\|X^2(t)\right\|_2^2\le\delta\right\}.$$

The aim of this section is to establish the following result.

Proposition 3.1. *Assume that Hypothesis 1.4 holds. There exist $\delta^3=\delta^3(B_0,D,\varepsilon,\delta)$, $\alpha=\alpha(\phi,D,\varepsilon,\delta)>0$ and $K"=K"(\phi,D,\varepsilon,\delta)$ such that for any $(x_0^1,x_0^2)\in H\times H$ and any $N\in\mathbb{N}$*

$$\mathbb{E}\left(e^{\alpha\tau}\right)\le K"\left(1+\left|x_0^1\right|^2+\left|x_0^2\right|^2\right),$$

provided $\|f\|_\varepsilon^2\le\delta^3$.

The result is based on the fact that, in the absence of noise and forcing term, all solutions go to zero exponentially fast in H. A similar idea is used in the works of the two dimensional Navier-Stokes equations by Kuksin and Shirikyan. The proof is based on the following four Lemmas. The first one allows to control the probability that the contribution of the noise is small. Its proof strongly uses the assumption that the noise is diagonal in the eigenbasis of A. As already mentioned, in the additive case, the proof is easy and does not need this assumption.

Lemma 3.2. *Assume that Hypothesis 1.4 holds. For any $t,M>0$, there exists $p_0(t,M)=p_0(t,M,\varepsilon,(|\phi_n|_\infty)_n,D)>0$ such that for any adapted process X*

$$\mathbb{P}\left(\sup_{(0,t)}\|Z\|_2^2\le M\right)\ge p_0(t,M),$$

where

$$Z(t)=\int_0^t\phi(X(s))dW(s).$$

It is proved in section 3.1.

Then, using this estimate and the smallness assumption on the forcing term, we estimate the moment of the first return time in a small ball in H.
Let $\delta_3>0$. We set

$$\tau_{L^2}=\tau\wedge\inf\left\{t\in T\mathbb{N}^*\ \Big|\ \left|X^1(t)\right|^2\vee\left|X^2(t)\right|\ge\delta_3\right\}.$$

Lemma 3.3. *Assume that Hypothesis 1.4 holds. Then, for any $\delta_3>0$, there exist $C_3(\delta_3)$, $C_3'(\delta_3)$ and $\gamma_3(\delta_3)$ such that for any $(x_0^1,x_0^2)\in(\mathbb{H}_2)^2$*

$$\mathbb{E}\left(e^{\gamma_3\tau_{L^2}}\right)\le C_3\left(1+\left|x_0^1\right|^2+\left|x_0^2\right|^2\right),$$

provided

$$\|f\|_\varepsilon\le C_3'.$$

The proof is postponed to section 3.2.

Then, we need to get a finer estimate in order to control the time necessary to enter a ball in stronger topologies. To prove the two next lemmas, we use an argument similar to one used in the determinist theory (see [43], chapter 7).

Lemma 3.4. *Assume that Hypothesis 1.4 holds. Then, for any δ_4, there exist $p_4(\delta_4)>0$, $C_4'(\delta_4)>0$ and $R_4(\delta_4)>0$ such that for any x_0 verifying $|x_0|^2\le R_4$, we have for any $T\le 1$*

$$\mathbb{P}\left(\|X_N(T,x_0)\|^2\le\delta_4\right)\ge p_4,$$

provided

$$\|f\|_\varepsilon\le C_4'.$$

The proof is postponed to section 3.3.

Lemma 3.5. *Assume that Hypothesis 1.4 holds. Then, for any δ_5, there exist $p_5(\delta_5)>0$, $C_5'(\delta_5)>0$ and $R_5(\delta_5)>0$ such that for any x_0 verifying $\|x_0\|^2\le R_5$ and for any $T\le 1$*

$$\mathbb{P}\left(\|X_N(T,x_0)\|_2^2\le\delta_5\right)\ge p_5.$$

provided

$$\|f\|_\varepsilon\le C_5'.$$

The proof is postponed to section 3.4.

Proof of Proposition 3.1: We set

$$\delta_5=\delta,\quad \delta_4=R_5(\delta_5),\quad \delta_3=R_4(\delta_4),\quad p_4=p_4(\delta_4),\quad p_5=p_5(\delta_5),\quad p_1=(p_4p_5)^2,$$

and

$$\delta^3=C_3'(\delta_3)\wedge C_4'(\delta_4)\wedge C_5'(\delta_5).$$

By the definition of τ_{L^2}, we have

$$\left|X^1(\tau_{L^2})\right|^2\vee\left|X^2(\tau_{L^2})\right|^2\le R_4(\delta_4).$$

We distinguish three cases.

The first case is $\left\|X^1(\tau_{L^2})\right\|_2^2\vee\left\|X^2(\tau_{L^2})\right\|_2^2\le\delta$, which obviously yields

$$\mathbb{P}\left(\min_{k=0,\dots,2}\max_{i=1,2}\left\|X^i(\tau_{L^2}+kT)\right\|_2^2\le\delta\ \middle|\ \left(X^2(\tau_{L^2}),X^2(\tau_{L^2})\right)\right)\ge p_1. \tag{3.3}$$

We now treat the case $x_0=X^1(\tau_{L^2})=X^2(\tau_{L_2})$ with $\|x_0\|_2^2>\delta$. Combining Lemma 3.4 and Lemma 3.5, we deduce from the weak Markov property of X_N that

$$\mathbb{P}\left(\|X_N(2T,x_0)\|_2^2\le\delta\right)\ge p_5p_4,$$

provided $|x_0|^2 \leq R_4$. Recall that, in that case, $X^1(\tau_{L^2}+2T) = X^2(\tau_{L^2}+2T)$. Hence, since the law of $X^1(\tau_{L^2}+2T)$ conditioned by $(X^1(\tau_{L^2}), X^2(\tau_{L_2}))$ is $\mathcal{D}(X_N(2T,x_0))$, it follows

$$\mathbb{P}\left(\max_{i=1,2}\left\|X^i(\tau_{L^2}+2T)\right\|_2^2 \leq \delta \mid \left(X^1(\tau_{L^2}), X^2(\tau_{L_2})\right)\right) \geq p_4 p_5 \geq p_1,$$

and then (3.3)

The last case is $X^1(\tau_{L^2}) \neq X^2(\tau_{L_2})$ and $\left\|X^1(\tau_{L^2})\right\|_2^2 \vee \left\|X^2(\tau_{L^2})\right\|_2^2 > \delta$. In that case, $(X^1(\tau_{L^2}+T), X^2(\tau_{L^2}+T))$ conditioned by $(X^1(\tau_{L^2}), X^2(\tau_{L_2}))$ are independent. Hence, since the law of $X^i(\tau_{L^2}+T)$ conditioned by $(X^1(\tau_{L^2}), X^2(\tau_{L_2})) = (x_0^1, x_0^2)$ is $\mathcal{D}(X_N(T,x_0^i))$, it follows from Lemma 3.4 that

$$\mathbb{P}\left(\max_{i=1,2}\left\|X^i(\tau_{L^2}+T)\right\|_1^2 \leq \delta_4 \mid \left(X^1(\tau_{L^2}), X^2(\tau_{L_2})\right)\right) \geq p_4^2.$$

Then, we distinguish the three cases $(\left\|X^i(\tau_{L^2}+T)\right\|_1^2)_{i=1,2}$ in the small ball of $\mathbb{H}_2$, equal or different and we deduce from Lemma 3.5 by the same method

$$\mathbb{P}\left(\min_{k=1,2}\max_{i=1,2}\left\|X^i(\tau_{L^2}+kT)\right\|_2^2 \leq \delta \mid \left(X^1(\tau_{L^2}+T), X^2(\tau_{L_2}+T)\right)\right) \geq p_5^2,$$

provided

$$\max_{i=1,2}\left\|X^i(\tau_{L^2}+T)\right\|_1^2 \leq \delta_4.$$

Combining the two previous inequalities, we deduce (3.3) for the latter case. We have thus proved that (3.3) is true almost surely.

Integrating (3.3), we obtain

$$\mathbb{P}\left(\min_{k=0,\dots,2}\max_{i=1,2}\left\|X^i(\tau_{L^2}+kT)\right\|_2^2 \leq \delta\right) \geq p_1. \tag{3.4}$$

Combining Lemma 3.3 and (3.4), we conclude.

3.1. **Probability of having a small noise.**

We now establish Lemma 3.2.
We deduce from Hölder inequality and from $\sum_n \mu_n^{-2} < \infty$ that Hypothesis 1.4 implies the following result. For any $\varepsilon_0 \in (0,\varepsilon)$, there exists $\alpha \in (0,1)$, a family $(\bar{\phi}_n)_n$ of measurable maps $H \to \mathbb{R}$ and a family $(b_i)_i$ of positive numbers such that

$$\begin{cases} \phi(x)\cdot e_n = b_n \bar{\phi}_n(x) e_n, \\ \sup_{x\in H}\left|\bar{\phi}_n(x)\right| \leq 1, \quad B^* = \sum_n \mu_n^{1+\varepsilon_0}\,(b_n)^{2(1-\alpha)} < \infty. \end{cases} \tag{3.5}$$

For simplicity we restrict our attention to the case $t=1$. The generalization is easy.
Remark that

$$Z(t) = \sum_n b_n Z_n(t) e_n,$$

where

$$Z_n(t) = \int_0^t e^{-\mu_n(t-s)}\bar{\phi}_n(X(s))dW_n(s), \quad \text{where} \quad W = \sum_n W_n e_n.$$

It follows from $\|Z\|_2^2=\sum_n b_n^2\mu_n\left|\sqrt{\mu_n}Z_n\right|^2$ and from (3.5) that

$$\mathbb{P}\left(\sup_{(0,1)}\|Z\|_2^2\le B^*M\right)\ge\mathbb{P}\left(\sup_{(0,1)}|\sqrt{\mu_n}Z_n|^2\le M\,\mu_n^{\varepsilon_0}\,(b_n)^{-2\alpha},\ \forall\, n\right). \tag{3.6}$$

Setting

$$W_n'(t)=\sqrt{\mu_n}W_n\left(\frac{t}{\mu_n}\right),$$

we obtain $(W_n')_n$ a family of independent brownian motions. Moreover we have

$$\sqrt{\mu_n}Z_n(t)=Z_n'(\mu_n t),$$

where

$$Z_n'(t)=\int_0^t e^{-(t-s)}\psi_n(s)dW_n'(s),\quad \psi_n(s)=\bar\phi_n\left(X\left(\frac{s}{\mu_n}\right)\right).$$

Hence, it follows from (3.6) that

$$\mathbb{P}\left(\sup_{(0,1)}\|Z\|_2^2\le B^*M\right)\ge\mathbb{P}\left(\sup_{(0,\mu_n)}|Z_n'|^2\le M\,\mu_n^{\varepsilon_0}\,(b_n)^{-2\alpha},\ \forall\, n\right). \tag{3.7}$$

Let $W_{n,i}'=W_n'(i+\cdot)-W_n'(i)$ on $(0,1)$. We set

$$M_{n,i}(t)=\begin{cases}0 & \text{if}\quad t\le 0,\\ \int_0^{1\wedge t}e^s\,\psi_n(i+s)dW_{n,i}'(s) & \text{if}\quad t\ge 0.\end{cases}$$

Remark that

$$Z_n'(t)=\sum_{i=1}^{\infty}e^{-(t-i)}M_{n,i}(t-i),$$

which yields for any $q\in\mathbb{N}$

$$\sup_{(0,q)}\|Z_n'\|_2\le\left(\frac{e}{e-1}\right)\max_{i=0,\dots,q-1}\sup_{(0,1)}|M_{n,i}|. \tag{3.8}$$

Remark that $(W_{n,k}')_{n,k}$ is a family of independant brownian motions on $(0,1)$. It follows that $(M_{n,k})_{n,k}$ are martingales verifying $\langle M_{n,k},M_{n',k'}\rangle=0$ if $(n,k)\neq(n',k')$.
Hence, combining a Theorem by Dambis, Dubins and Schwartz (Theorem 4.6 page 174 of [22]) and a Theorem by Knight (Theorem 4.13 page 179 of [22]), we obtain a family $(B_{n,k})_{n,k}$ of independent brownian motions verifying

$$M_{n,k}(t)=B_{n,k}(\langle M_{n,k}\rangle(t)). \tag{3.9}$$

Remark 3.6. *In the two previous Theorem, they assume that $<M>_\infty=\infty$ almost surely. However, as explained in Problem 4.7 of [22], it is easy to adapt the proof for M such that $<M>_\infty<\infty$ with a positive probability.*

Remarking that for any $t\in(0,1)$

$$\langle M_{n,k}\rangle(t)=\int_0^t|\psi_n(k+s)|^2\,ds\le 1,$$

we deduce from (3.8) and (3.9) that for any $q\in\mathbb{N}^*$

$$\sup_{(0,q)}\|Z_n'\|_2\le\left(\frac{e}{e-1}\right)\max_{i=0,\dots,q-1}\sup_{(0,1)}|B_{n,i}|.$$

Hence it follows from (3.7) that

$$\mathbb{P}\left(\sup_{(0,1)} \|Z\|_2^2 \leq M\right) \geq \mathbb{P}\left(\sup_{(0,1)} |B_{n,i}|^2 \leq cM\,\mu_n^{\varepsilon_0}\,(b_n)^{-2\alpha}\,,\ \forall n,\ \forall i \leq \mu_n + 1\right),$$

where $c = \left(\frac{e-1}{e}\right)^2 \frac{1}{B^*}$.
We deduce from the independence of $(B_{n,k})_{n,k}$ that

$$\mathbb{P}\left(\sup_{(0,1)} \|Z\|_2^2 \leq M\right) \geq \prod_{n\in\mathbb{N}^*}\left(P\left(cM\,\mu_n^{\varepsilon_0}\,(b_n)^{-2\alpha}\right)^{\mu_n+1}\right), \tag{3.10}$$

where

$$P(d_0) = \mathbb{P}\left(\sup_{(0,1)} |B_{1,1}|^2 \leq d_0\right).$$

Recall there exists a family $(c_p)_p$ such that

$$\mathbb{E}\left(\sup_{(0,1)} |B_{1,1}|^{2p}\right) \leq c_p.$$

It follows from Chebyshev inequality and from $1 - x \geq e^{-ex}$ for any $x \leq e^{-1}$ that for any $d_0 \leq d_p = \left(e^{-1}c_p\right)^{\frac{1}{p}}$

$$P(d_0) \geq 1 - c_p d_0^{-p} \geq e^{-ec_p d_0^{-p}}.$$

Applying (3.10), we obtain for any $p > 0$

$$\mathbb{P}\left(\sup_{(0,1)} \|Z\|_2^2 \leq M\right) \geq C_p(M)\exp\left(-\frac{c'_p}{M^p}\sum_{n>\mathbb{N}(p,M)}\left(\frac{\mu_n+1}{\mu_n^{\varepsilon_0 p}}\right)b_n^{2\alpha p}\right), \tag{3.11}$$

where

$$\begin{cases} N(p,M) &= \sup\left\{n\in\mathbb{N}\backslash\{0\}\ \middle|\ M\,\mu_n^{\varepsilon_0}\,(b_n)^{-2\alpha} \leq d_p\right\}, \\ C_p(M) &= \Pi_{n\leq\mathbb{N}(p,M)}\left(P\left(cM\,\mu_n^{\varepsilon_0}\,(b_n)^{-2\alpha}\right)^{\mu_n+1}\right). \end{cases}$$

Choosing p sufficiently high, we deduce from **H0** that

$$\sum_n \left(\frac{\mu_n+1}{\mu_n^{\varepsilon_0 p}}\right) b_n^{2\alpha p} \leq C'_p < \infty,$$

which yields, by (3.11), that for any $M > 0$ and for p sufficiently high

$$\mathbb{P}\left(\sup_{(0,1)} \|Z\|_2^2 \leq M\right) \geq C_p(M)\exp\left(-\frac{c_p"}{M^p}\right), \tag{3.12}$$

Remark that for any p, ε_0 we have $N(p,M) < \infty$. Moreover, it is well-known that for any $d_0 > 0$, $P(d_0) > 0$, which yields $C_p(M) > 0$ and then Lemma 3.2.

3.2. **Proof of Lemma 3.3.**
For simplicity in the redaction, we restrict our attention to the case $f = 0$. The generalisation is easy.

Recall (1.5)

$$\mathbb{E}\left|X_N(t)\right|^2 \leq e^{-\mu_1 t}\left|x_0\right|^2 + \frac{c}{\mu_1}B_0.$$

Since (X^1, X^2) is a coupling of $(\mathcal{D}(X_N(\cdot, x_0^1)), \mathcal{D}(X_N(\cdot, x_0^1)))$ on $T\mathbb{N}$, we obtain

$$\mathbb{E}\left(\left|X^1(nT)\right|^2 + \left|X^2(nT)\right|^2\right) \leq e^{-\mu_1 nT}\left(\left|x_0^1\right|^2 + \left|x_0^2\right|^2\right) + 2\frac{c}{\mu_1}B_0. \tag{3.13}$$

Since (X^1, X^2) is a strong Markov process, it can be deduced that there exist C_6 and γ_6 such that for any $x_0 \in H$

$$\mathbb{E}\left(e^{\gamma_6 \tau'_{L^2}}\right) \leq C_6\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right), \tag{3.14}$$

where

$$\tau'_{L^2} = \inf\left\{t \in T\mathbb{N}\backslash\{0\} \;\Big|\; \left|X^1(t)\right|^2 + \left|X^2(t)\right|^2 \geq 4cB_0\right\}.$$

Taking into account (3.14), a standard argument gives that, in order to establish Lemma 3.3, it is sufficient to prove that there exist (p_7, T_7) such that

$$\mathbb{P}\left(\left|X_N(t, x_0)\right|^2 \leq \delta_3\right) \geq p_7(\delta_3, t) > 0, \tag{3.15}$$

provided $N \in \mathbb{N}$, $t \geq T_7(\delta_3)$ and $\left|x_0\right|^2 \leq 4cB_0$.

We set

$$Z(t) = \int_0^t \phi(X_N(s))dW(s), \quad Y_N = X_N - P_N Z, \quad M = \sup_{(0,t)}\left\|Z\right\|_2^2.$$

Assume that there exist $M_7(\delta_3) > 0$ and $T_7(\delta_3)$ such that

$$M \leq M_7(\delta_3) \quad \text{implies} \quad \left|Y_N(t)\right|^2 \leq \frac{\delta_3}{4}, \tag{3.16}$$

provided $t \geq T_7(\delta_3)$ and $\left|x_0\right|^2 \leq 4cB_0$. Then (3.15) results from Lemma 3.2 with

$$M = \min\left\{M_7(\delta_3), \frac{\delta_3}{4}\right\}.$$

We now prove (3.16). Remark that

$$\frac{d}{dt}Y_N + AY_N + P_N B(Y_N + P_N Z) = 0. \tag{3.17}$$

Taking the scalar product of (3.17) with Y_N, it follows that

$$\frac{d}{dt}\left|Y_N\right|^2 + 2\left\|Y_N\right\|^2 = -2(Y_N, B(Y_N + P_N Z)). \tag{3.18}$$

Recalling that $(B(y,x),x) = 0$, we obtain

$$-2(Y_N, B(Y_N + P_N Z)) = -2(Y_N, (Y_N, \nabla)P_N Z) - 2(Y_N, B(P_N Z)).$$

We deduce from Hölder inequalities and Sobolev embedding that

$$-(z, (x, \nabla)y) \leq c\left\|z\right\|\left\|x\right\|\left\|y\right\|.$$

Hence it follows from (3.18) that

$$\frac{d}{dt}\left|Y_N\right|^2 + 2\left\|Y_N\right\|^2 \leq c\left\|Z\right\|^2\left\|Y_N\right\| + c\left\|Z\right\|\left\|Y_N\right\|^2,$$

which yields, by an arithmetico-geometric inequality,

$$\frac{d}{dt}|Y_N|^2 + 2\|Y_N\|^2 \le cM^{\frac12}\|Y_N\|^2 + cM^{\frac32}.$$

It follows that $M \le \frac{1}{c^2}$ implies

$$\frac{d}{dt}|Y_N|^2 + \|Y_N\|^2 \le cM^{\frac32} \quad \text{on } (0,t). \tag{3.19}$$

Integrating, we deduce from $|x_0|^2 \le 4cB_0$ that

$$|Y_N(t)|^2 \le 4ce^{-\mu_1 t}B_0 + c\left(\frac{M^{\frac32}}{\mu_1}\right).$$

Choosing t sufficiently large and M sufficiently small we obtain (3.16) which yields (3.15) and then Lemma 3.3.

Remark 3.7. *In order to avoid a lengthy proof, we have not splitted the arguments in several cases as in the proof of Proposition 3.1. The reader can complete the details.*

3.3. **Proof of Lemma 3.4.**

We use the decomposition $X_N = Y_N + P_N Z$ defined in section 3.2 and set

$$M = \sup_{(0,T)} \|Z\|_2^2.$$

Integrating (3.19), we obtain for M satisfying the same assumption $M \le \frac{1}{c^2}$

$$\frac1T\int_0^T \|Y_N(t)\|^2 dt \le \frac1T|x_0|^2 + cM^{\frac32},$$

which yields, by a Chebyshev inequality,

$$\lambda\left(t\in(0,T)\;\middle|\;\|Y_N(t)\|^2 \le \frac2T|x_0|^2 + 2cM^{\frac32}\right) \ge \frac T2, \tag{3.20}$$

where λ denotes the Lebesgue measure on $(0,T)$.

Setting

$$\tau_{\mathbb{H}_1} = \inf\left\{t\in(0,T)\;\middle|\;\|Y_N(t)\|^2 \le \frac2T|x_0|^2 + 2cM^{\frac32}\right\},$$

we deduce from (3.20) and the continuity of Y_N that

$$\|Y_N(\tau_{\mathbb{H}_1})\|^2 \le \frac2T|x_0|^2 + 2cM^{\frac32}. \tag{3.21}$$

Taking the scalar product of $2AY$ and (3.17), we obtain

$$\frac{d}{dt}\|Y_N\|^2 + 2\|Y_N\|_2^2 = -2(AY_N, B(Y_N + P_N Z)). \tag{3.22}$$

It follows from Hölder inequalities, Sobolev Embeddings and Agmon inequality that

$$-2(Ay,\widetilde{B}(x,z)) \le c\|y\|_2\|z\|^{\frac12}\|z\|_2^{\frac12}\|x\|,$$

where $B(x,y) = (x,\nabla)y + (y,\nabla)x$. Hence, we obtain by applying arithmetico-geometric inequalities

$$\begin{cases} -2(AY_N, B(Y_N)) & \leq \quad c\left\|Y_N\right\|_2^{\frac{3}{2}}\left\|Y_N\right\|^{\frac{3}{2}} & \leq \quad \frac{1}{4}\left\|Y_N\right\|_2^2 + c\left\|Y_N\right\|^6, \\ -2(AY_N, B(P_N Z)) & \leq \quad c\left\|Y_N\right\|_2\left\|Z\right\|^{\frac{3}{2}}\left\|Z\right\|_2^{\frac{1}{2}} & \leq \quad \frac{1}{4}\left\|Y_N\right\|_2^2 + c\left\|Z\right\|_2^4, \\ -2(AY_N, \widetilde{B}(Y_N, P_N Z)) & \leq \quad c\left\|Y_N\right\|_2^{\frac{3}{2}}\left\|Y_N\right\|^{\frac{1}{2}}\left\|Z\right\| & \leq \quad c\left\|Z\right\|\left\|Y_N\right\|_2^2 \end{cases}$$

Remarking that $B(Y_N + P_N Z) = B(Y_N) + \widetilde{B}(Y_N, P_N Z) + B(P_N Z)$, it follows from (3.22) that $M \leq \frac{1}{4c}$ implies

$$\frac{d}{dt}\left\|Y_N\right\|^2 + \left\|Y_N\right\|_2^2 \leq c\left\|Y_N\right\|^2\left(\left\|Y_N\right\|^4 - 4K_0^2\right) + cM^2, \tag{3.23}$$

where K_0 is defined in (2.10). Let us set

$$\sigma_{\mathbb{H}_1} = \inf\left\{t \in (\tau_{\mathbb{H}_1}, T)\ \middle|\ \left\|Y_N(t)\right\|^2 > 2K_0\right\},$$

and remark that on $(\tau_{\mathbb{H}_1}, \sigma_{\mathbb{H}_1})$, we have

$$\frac{d}{dt}\left\|Y_N\right\|^2 + \left\|Y_N\right\|_2^2 \leq cM^2. \tag{3.24}$$

Integrating, we obtain that

$$\left\|Y_N(\sigma_{\mathbb{H}_1})\right\|^2 + \int_{\tau_{\mathbb{H}_1}}^{\sigma_{\mathbb{H}_1}}\left\|Y_N(t)\right\|_2^2 dt \leq \left\|Y_N(\tau_{\mathbb{H}_1})\right\|^2 + cM^2. \tag{3.25}$$

Combining (3.21) and (3.25), we obtain that, for M and $|x_0|^2$ sufficiently small,

$$\left\|Y_N(\sigma_{\mathbb{H}_1})\right\|^2 \leq \frac{\delta_4}{4} \wedge K_0,$$

which yields $\sigma_{\mathbb{H}_1} = T$. It follows that

$$\left\|X_N(T)\right\|^2 \leq \delta_4, \tag{3.26}$$

provided M and $|x_0|^2$ sufficiently small. It remains to use Lemma 3.2 to get Lemma 3.4.

3.4. **Proof of Lemma 3.5.**

It follows from (3.24) that

$$\int_0^T \left\|Y_N(t)\right\|_2^2 dt \leq \left\|x_0\right\|^2 + cM^2,$$

provided $M \leq \frac{1}{4c}$ and $\left\|x_0\right\|^2 + cM^2 \leq K_0$.
Applying the same argument as in the previous subsection, it is easy to deduce that there exists a stopping times $\tau_{\mathbb{H}_2} \in (0,T)$ such that

$$\left\|Y_N(\tau_{\mathbb{H}_2})\right\|_2^2 \leq \frac{2}{T}\left(\left\|x_0\right\|^2 + cM^2\right), \tag{3.27}$$

provided M and $\left\|x_0\right\|$ are sufficiently small.

Taking the scalar product of (3.17) and $2A^2 Y_N$, we obtain

$$\frac{d}{dt}\left\|Y_N\right\|_2^2 + 2\left\|Y_N\right\|_3^2 = -2\left(A^{\frac{3}{2}}Y_N, A^{\frac{1}{2}}B(Y_N + P_N Z)\right). \tag{3.28}$$

Applying Hölder inequality, Sobolev Embeddings $\mathbb{H}_2 \subset L^\infty$ and $\mathbb{H}_1 \subset L^4$ and arithmetico-geometric inequality, we obtain

$$-2\left(A^{\frac{3}{2}}y, A^{\frac{1}{2}}B(x,y)\right) \leq c\,\|y\|_3\,\|x\|_2\,\|y\|_2 \leq \frac{1}{4}\,\|y\|_3^2 + c\left(\|x\|_2^4 + \|y\|_2^4\right).$$

Hence we deduce from (3.28) and from $B(Y_N + P_N Z) = B(Y_N) + B(Y_N, P_N Z) + B(P_N Z)$

$$\frac{d}{dt}\,\|Y_N\|_2^2 + \|Y_N\|_3^2 \leq c\,\|Y_N\|_2^2\,(\|Y_N\|_2^2 - 2K_1) + c\,\|Z\|_2^4\,, \tag{3.29}$$

where K_1 is defined as K_0 in (2.10) but with a different c. We set

$$\sigma_{\mathbb{H}_2} = \inf\left\{t \in (\tau_{\mathbb{H}_2}, T)\ \Big|\ \|Y_N(t)\|_2^2 > 2K_1\right\},$$

Integrating (3.29), we obtain

$$\|Y_N(\sigma_{\mathbb{H}_2})\|_2^2 + \int_{\tau_{\mathbb{H}_2}}^{\sigma_{\mathbb{H}_2}} \|Y_N(t)\|_3^2\,dt \leq \|Y_N(\tau_{\mathbb{H}_2})\|_2^2 + cM^2.$$

Taking into account (3.27) and choosing $\|x_0\|^2$ and M^2 sufficiently small, we obtain

$$\|Y_N(\sigma_{\mathbb{H}_2})\|_2^2 \leq \frac{\delta}{4} \wedge K_1.$$

It follows that $\sigma_{\mathbb{H}_2} = T$ and that

$$\|X_N(T)\|^2 \leq \delta, \tag{3.30}$$

provided M and $\|x_0\|$ sufficiently small, which yields (2.2).

4. Proof of Theorem 1.8

As already explained, Theorem 1.8 follows from Proposition 1.18. We now prove Proposition 1.18. Let $(x_0^1, x_0^2) \in (\mathbb{H}_2)^2$. Let us recall that the process (X^1, X^2) is defined at the beginning of section 3.

Let $\delta > 0$, $T \in (0,1)$ be as in Proposition 2.1 and τ defined in (3.2), setting

$$\tau_1 = \tau, \quad \tau_{k+1} = \inf\left\{t > \tau_k\ \Big|\ \left\|X^1(t)\right\|_2^2 \vee \left\|X^2(t)\right\|_2^2 \leq \delta\right\}.$$

it can be deduced from the strong Markov property of (X^1, X^2) and from Proposition 3.1 that

$$\mathbb{E}\left(e^{\alpha\tau_{k+1}}\right) \leq K"\mathbb{E}\left(e^{\alpha\tau_k}\left(1 + \left|X^1(\tau_k)\right|^2 + \left|X^2(\tau_k)\right|^2\right)\right),$$

which yields, by the Poincaré inequality,

$$\begin{cases} \mathbb{E}\left(e^{\alpha\tau_{k+1}}\right) & \leq & cK"(1+2\delta)\mathbb{E}\left(e^{\alpha\tau_k}\right), \\ \mathbb{E}\left(e^{\alpha\tau_1}\right) & \leq & K"\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right). \end{cases}$$

It follows that there exists $K > 0$ such that

$$\mathbb{E}\left(e^{\alpha\tau_k}\right) \leq K^k\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right).$$

Hence, applying Jensen inequality, we obtain that, for any $\theta \in (0,1)$

$$\mathbb{E}\left(e^{\theta\alpha\tau_k}\right) \leq K^{\theta k}\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right). \tag{4.1}$$

We deduce from Proposition 2.1 and from (3.1) that

$$\mathbb{P}\left(X^1(T) \neq X^2(T)\right) \leq \frac{1}{4},$$

provided (x_0^1, x_0^2) in the ball of $(\mathbb{H}^2)^2$ with radius δ.

Setting

$$k_0 = \inf\left\{k \in \mathbb{N} \;\middle|\; X^1(\tau_k + T) = X^2(\tau_k + T)\right\},$$

it follows that $k_0 < \infty$ almost surely and that

$$\mathbb{P}\left(k_0 > n\right) \leq \left(\frac{1}{4}\right)^n. \tag{4.2}$$

Let $\theta \in (0,1)$. We deduce from Schwartz inequality that

$$\mathbb{E}\left(e^{\frac{\theta}{2}\alpha\tau_{k_0}}\right) = \sum_{n=1}^{\infty} \mathbb{E}\left(e^{\frac{\theta}{2}\alpha\tau_n} 1_{k_0=n}\right) \leq \sum_{n=0}^{\infty} \sqrt{\mathbb{P}\left(k_0 \geq n\right) \mathbb{E}\left(e^{\theta\alpha\tau_n}\right)}.$$

Combining (4.1) and (4.2), we deduce

$$\mathbb{E}\left(e^{\frac{\theta}{2}\alpha\tau_{k_0}}\right) \leq \left(\sum_{n=0}^{\infty}\left(\frac{K^\theta}{2}\right)^n\right)\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right).$$

Hence, choosing $\theta \in (0,1)$ sufficiently small, we obtain that there exists $\gamma > 0$ non depending on $N \in \mathbb{N}$ such that

$$\mathbb{E}\left(e^{\gamma\tau_{k_0}}\right) \leq 4\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right). \tag{4.3}$$

Recall that if (X^1, X^2) are coupled at time $t \in T\mathbb{N}$, then they remain coupled for any time after. Hence $X^1(t) = X^2(t)$ for $t > \tau_{k_0}$. It follows

$$\mathbb{P}\left(X^1(nT) \neq X^2(nT)\right) \leq 4e^{-\gamma nT}\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right).$$

Since $(X^1(nT), X^2(nT))$ is a coupling of $((\mathcal{P}_{nT}^N)^*\delta_{x_0^1}, (\mathcal{P}_{nT}^N)^*\delta_{x_0^2})$, we deduce from Lemma 1.17

$$\left\|\left(\mathcal{P}_{nT}^N\right)^*\delta_{x_0^1} - \left(\mathcal{P}_{nT}^N\right)^*\delta_{x_0^2}\right\|_{var} \leq 4e^{-\gamma nT}\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right), \tag{4.4}$$

for any $n \in \mathbb{N}$ and any $(x_0^1, x_0^2) \in (\mathbb{H}_2)^2$.

Recall that the existence of an invariant measure $\mu_N \in P(P_N H)$ is justified in section 1.3. Let $\lambda \in P(H)$ and $t \in \mathbb{R}^+$. We set $n = \lfloor \frac{t}{T} \rfloor$ and $C = 4e^{\gamma T}$. Integrating (x_0^1, x_0^2) over $((\mathcal{P}_{t-nT}^N)^*\lambda) \otimes \mu$ in (4.4), we obtain

$$\left\|\left(\mathcal{P}_t^N\right)^*\lambda - \mu_N\right\|_{var} \leq Ce^{-\gamma t}\left(1 + \int_H \left|x_0\right|^2 \lambda(dx)\right),$$

which establishes (1.19).

Appendix A. Proof of (2.20)

For simplicity in the redaction, we omit θ and N in our notations.

Remark that

$$J = \left(\nabla\mathbb{E}\left(g(X(T))\psi_X\right), h\right) = J_1 + 2J_2, \tag{A.1}$$

where

$$\left\{\begin{array}{lcl} J_1 & = & \mathbb{E}\left((\nabla g(X(T)), \eta(T,0)\cdot h)\,\psi_X\right), \\ J_2 & = & \mathbb{E}\left(g(X(T))\psi_X' \int_0^T (AX(t), A(\eta(t,0)\cdot h))\,ds\right). \end{array}\right.$$

According to [36], let us denote by $D_s F$ the Malliavin derivative of F at time s. We have the following formula of the Malliavin derivative of the solution of a stochastic differential equation

$$D_s X(t) = 1_{t\geq s}\eta(t,s)\cdot\phi(X(s)),$$

which yields

$$\int_0^t D_s X(t)\cdot m(s)\,ds = G(t)\cdot m, \tag{A.2}$$

where

$$G(t)\cdot m = \int_0^t \eta(t,s)\cdot\phi(X(s))\cdot m(s)\,ds.$$

The uniqueness of the solutions gives

$$\eta(t,0)\cdot h = \eta(t,s)\cdot(\eta(s,0)\cdot h) \text{ for any } 0\leq s\leq t,$$

which yields

$$\eta(T,0)\cdot h = \frac{1}{T}\int_0^T \eta(t,s)\cdot(\eta(s,0)\cdot h)\,ds.$$

Setting

$$w(s) = \phi^{-1}(X(s))\cdot\eta(s,0)\cdot h,$$

we infer from (A.2)

$$\eta(T,0)\cdot h = \frac{1}{T}G(T)\cdot w = \frac{1}{T}\int_0^T D_s X(T)\cdot w\,ds, \tag{A.3}$$

which yields

$$(\nabla g(X(T)), \eta(T,0)\cdot h) = \frac{1}{T}\int_0^T (\nabla g(X(T)), D_s X(T)\cdot w)\,ds.$$

Remark that

$$(D_s g(X(T)), w) = (\nabla g(X(T)), D_s X(T)\cdot w).$$

It follows

$$(\nabla g(X(T)), \eta(T,0)\cdot h) = \frac{1}{T}\int_0^T (D_s g(X(T)), w)\,ds,$$

which yields

$$J_1 = \frac{1}{T}\mathbb{E}\int_0^T \psi_X\,(D_s g(X(T)), w)\,ds. \tag{A.4}$$

Recall that the Skohorod integral is the dual operator of the Malliavin derivative (See [36]). It follows

$$J_1 = \frac{1}{T}\mathbb{E}\left(g(X(T))\int_0^T \psi_X(w(t), dW(t))\right). \tag{A.5}$$

Recall the formula of integration of a product

$$\int_0^T \psi_X(w(t), dW(t)) = \psi_X\int_0^T (w(t), dW(t)) - \int_0^T (D_s\psi_X, w(s))\,ds. \tag{A.6}$$

Remark that

$$D_s\psi_X = 2\psi'_X \int_0^T AD_sX(t)\cdot (AX(t))\,dt,$$

which yields, by $(AX(t), AD_sX(t)\cdot w(s)) = (w(s), AD_sX(t)\cdot(AX(t)))$,

$$\int_0^T (D_s\psi_X, w(s))\,ds = 2\psi'_X \int_0^T\int_0^T (AX(t), AD_sX(t)\cdot w(s))\ dtds.$$

We deduce from (A.3) that

$$\int_0^T (D_s\psi_X, w(s))\,ds = 2\psi'_X \int_0^T t\,(AX(t), A\eta(t,0)\cdot h)\ dt. \tag{A.7}$$

Remark that

$$\psi_X = \psi'_X = 0 \quad \text{if} \quad \sigma < T.$$

Hence combining (A.6) and (A.7), we obtain

$$\int_0^T \psi_X(w(t), dW(t)) = \psi_X \int_0^\sigma (w(t), dW(t)) - 2\psi'_X \int_0^\sigma t\,(AX(t), A\eta(t,0)\cdot h)\ dt.$$

Thus, (2.20) follows from (A.1) and (A.5).

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Méthodes de couplage pour des équations stochastiques de type Navier-Stokes et Schrödinger

Résumé: Nous nous intéresserons d'abord aux équations stochastiques de Navier-Stokes bidimensionnelles (NS), de Ginzburg-Landau Complexes (CGL) et de Schrödinger non-linéaires (NLS) munies d'un bruit blanc en temps et régulier pour la variable spatiale. En nous appuyant sur des méthodes de couplages, nous établirons le caractère exponentiellement (resp polynomialement) mélangeant de NS et CGL (resp NLS) lorseque le bruit recouvre un nombre suffisant de bas modes. Deux des innovations majeures de ces résultats sont le fait que l'on s'autorise à traiter des équations non-dissipatives telles que NLS et que l'on considère des bruits non additifs.

Dans un deuxième temps, nous considérerons les équations de Navier-Stokes stochastiques tridimensionnelles (NS3D). Nous établirons la régularité H^p et Gevrey des solutions stationnaires de NS3D et nous en déduirons des informations sur l'échelle de dissipation de Kolmogorov (K41). Puis, nous établirons le caractère exponentiellement mélangeant des solutions de NS3D lorsque le bruit est à la fois suffisament régulier et non-dégénéré.

Mots clés: Équations de Navier-Stokes 2D et 3D, équations de Schrödinger non-linéaires, équations de Ginzburg-Landau Complexes, semi-groupe de transition de Markov, mesures invariantes, ergodicité, méthodes de couplage, mélange exponentiel, espaces de Gevrey, échelle de dissipation de Kolmogorov (K41).

Coupling methods for stochastic equations such as Navier-Stokes and Schrödinger

Abstract: In a first part, we are concerned with stochastic two-dimensional Navier-Stokes (NS), non-linear Schrödinger (NLS) and Complex Ginzburg-Landau (CGL) equations driven by a noise which is white in time and smooth in space. Using coupling arguments, we establish exponential (resp polynomial) mixing of NS and CGL (resp NLS) provided the noise is non degenerate on the low modes. Although, this kind of method was originally developed for strongly dissipative equations with additive noise, we are able to treat non dissipative equation (NLS) and general non additive noise.

In a second part, we are concerned with stochastic three-dimensional Navier-Stokes equations (NS3D). We first investigate smoothness properties in space of the stationary solutions. Some informations on the Kolmogorov dissipation scale (K41) are deduced. Then we establish exponential mixing of the solutions of NS3D provided the noise is at the same time sufficiently smooth and non degenerate.

Keywords: Stochastic 2D and 3D Navier-Stokes equations, non-linear Schrödinger equation, Complex Ginzburg-Landau equation, Markov transition semigroup, invariant measure, ergodicity, coupling method, exponential mixing, Gevrey spaces, Kolmogorov dissipation scale (K41).

2000 Mathematics Subject Slassification: 35Q60, 37H99, 37L99, 60H10, 60H15, 35Q55, 35Q60, 35Q30, 47B06.