Generalized Block Theory

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À mon père.

Une civilisation sans la science, c'est aussi absurde qu'un poisson sans bicyclette.

Pierre Desproges, *Je baisse*.

It is not how fast you go that matters, it is the object of your journey. It is not how you send a message, it is what the value of the message may be.

Sir Arthur Conan Doyle, *The land of mist*.

Je sers la science et c'est ma joie.

Basile (disciplus simplex).

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Introduction

La théorie des blocs a été introduite au milieu du 20ème siècle par R. Brauer. Si G est un groupe fini et si $\mathrm{Irr}(G)$ est l'ensemble de ses caractères complexes irréductibles, alors à tout nombre premier p est associée une partition de $\mathrm{Irr}(G)$ en *p-blocs*. Les blocs correspondent aux termes d'une décomposition en somme directe d'idéaux bilatères de l'algèbre de groupe sur un anneau de valuation discrète complet. La plupart des propriétés des p-blocs viennent des propriétés arithmétiques p-locales des caractères et sont obtenues grâce à une passerelle entre caractéristique 0 et caractéristique p. Mais les p-blocs peuvent aussi être obtenus élémentairement à partir du découpage de G en *p-sections*. Tout élément g de G a une unique écriture $g = g_p g'_p = g'_p g_p$ où g_p est un *p-élément* et g'_p est *p-régulier* (*i.e.* l'ordre de g_p est une puissance de p et celui de g'_p est premier à p). Deux éléments appartiennent à la même section si les p-éléments associés sont conjugués. On note $\mathcal{C}$ l'ensemble des éléments p-réguliers de G, et on définit un produit intérieur tronqué par

$$< \chi, \psi >_{\mathcal{C}} = \frac{1}{|G|} \sum_{g \in \mathcal{C}} \chi(g)\psi(g^{-1}) \quad \text{pour tous } \chi, \psi \in \mathrm{Irr}(G).$$

Deux caractères χ et ψ sont dits *directement $\mathcal{C}$-liés* si $< \chi, \psi >_{\mathcal{C}} \neq 0$ (sinon, ils sont dits *orthogonaux sur $\mathcal{C}$*). Celle relation est réflexive (car $1 \in \mathcal{C}$) et symétrique. En l'étendant par transitivité, on obtient une relation d'équivalence sur $\mathrm{Irr}(G)$ dont les classes sont les p-blocs. Une conséquence du Deuxième Théorème de Brauer est que, si deux caractères irréductibles sont orthogonaux sur $\mathcal{C}$, alors ils sont orthogonaux sur **chaque** p-section ($\mathcal{C}$ est la p-section de 1).

L'idée développée par B. Külshammer, J. B. Olsson et G. R. Robinson dans [18] est de faire la même construction pour d'autres unions $\mathcal{C}$ de classes de conjugaison. On obtient ainsi une partition de $\mathrm{Irr}(G)$ en *$\mathcal{C}$-blocks*. Le but de cette thèse est l'étude des propriétés de ces *blocs généralisés*. Dans [18], les auteurs ont défini des ℓ-sections et des ℓ-blocs pour les groupes symétriques, où $\ell \geq 2$ est un entier quelconque. Ils prennent pour $\mathcal{C}$ l'ensemble des éléments, dits *ℓ-réguliers*, dont aucun des cycles n'est de longueur divisible par ℓ. Les ℓ-blocs obtenus satisfont un analogue de la Conjecture de Nakayama : deux caractères irréductibles χ^λ et χ^μ du groupe symétrique S_n (λ et μ sont des partitions de n) appartiennent au même ℓ-bloc si et seulement si λ et μ ont le même ℓ-cœur. De plus, ils satisfont un analogue du Deuxième Théorème de Brauer.

La première partie de la thèse expose brièvement la construction et les propriétés des blocs dans le cas classique, puis introduit la généralisation ci-dessus et décrit le cas des groupes symétriques.

La deuxième partie traite d'*isométries parfaites généralisées*. Soit G un groupe fini, U un p-sous-groupe de Sylow de G, et B le normalisateur de U dans G. On note B_0 (respectivement b_0) le p-bloc de G (respectivement de B) qui contient le caractère trivial. Une conjecture de M. Broué prévoit que, si U est abélien, alors il existe une *isométrie parfaite* entre B_0 et b_0 (cf [2]). Une telle isométrie parfaite implique une relation étroite entre les anneaux de caractères associés à B_0 et b_0. Si on prend pour G un groupe de Suzuki $Sz(q)$, de Ree (de type G_2) $Re(q)$ ou spécial unitaire $SU(3, q^2)$, chacun en caractéristique p, alors les p-sous-groupes de Sylow de G ne sont pas abéliens. Il est connu (cf Cliff [8]) que, dans le cas des groupes de Suzuki, il n'y a pas d'isométrie parfaite entre B_0 et b_0. Néanmoins, nous démontrons que, dans chacun de ces cas, si l'on prend pour $\mathcal{C}$ (resp. $\mathcal{D}$) l'ensemble des éléments de G (resp. B) dont l'ordre n'est pas divisible par p^2, alors il existe une bijection I entre B_0 et b_0 et des signes $\{\varepsilon(\chi), \chi \in \mathrm{Irr}(G)\}$ tels que, pour tous $\chi, \psi \in B_0$, on a $< \chi, \psi >_{\mathcal{C}} = < \varepsilon(\chi)I(\chi), \varepsilon(\psi)I(\psi) >_{\mathcal{D}}$. On parle d'*isométrie parfaite généralisée* entre B_0 et b_0. Ce résultat est moins fort que celui annoncé par la conjecture de Broué, mais il met cependant en évidence le lien fort entre les anneaux de caractères de B_0 et b_0. En particulier, leurs *groupes de Cartan* sont les mêmes.

La troisième partie de la thèse traite notamment de ce groupe de Cartan. On suppose que $\mathcal{C}$ est une union *fermée* de classes de conjugaison de G (*i.e.* si $x \in \mathcal{C}$ et si y engendre le même sous-groupe de G que x, alors $y \in \mathcal{C}$). Pour $\chi \in \mathrm{Irr}(G)$, on définit une fonction centrale $\chi^{\mathcal{C}}$ de G par $\chi^{\mathcal{C}}(g) = \chi(g)$ si $g \in \mathcal{C}$ et $\chi^{\mathcal{C}}(g) = 0$ si $g \in G \setminus \mathcal{C}$. On définit le sous-$\mathbf{Z}$-module $\mathcal{R}(\mathcal{C})$ de $\mathbf{C}\mathrm{Irr}(G)$ engendré par $\{\chi^{\mathcal{C}}; \chi \in \mathrm{Irr}(G)\}$, et le sous-$\mathbf{Z}$-module $\mathcal{P}(\mathcal{C})$ de $\mathcal{R}(\mathcal{C})$ des éléments de $\mathcal{R}(\mathcal{C})$ qui sont des caractères généralisés de G (*i.e.* $\mathcal{P}(\mathcal{C}) = \mathcal{R}(\mathcal{C}) \cap \mathbf{Z}\mathrm{Irr}(G)$). Alors $\mathcal{R}(\mathcal{C})$ et $\mathcal{P}(\mathcal{C})$ ont le même rang sur $\mathbf{Z}$. En particulier, $\mathcal{R}(\mathcal{C})/\mathcal{P}(\mathcal{C})$ est un groupe abélien fini, le *$\mathcal{C}$-groupe de Cartan* de G, et, pour tout $\chi \in \mathrm{Irr}(G)$, il existe un entier $d > 0$ tel que $d\chi^{\mathcal{C}} \in \mathbf{Z}\mathrm{Irr}(G)$. Le plus petit tel d est appelé *ordre* de χ.

Le groupe de Cartan et les ordres des caractères sont bien connus dans le cas où $\mathcal{C}$ est l'ensemble des éléments p-réguliers de G. Dans cette partie, nous donnons une description du groupe de Cartan et des ordres des caractères dans le cas où G est abélien et $\mathcal{C}$ est l'ensemble des éléments p^k-réguliers de G (*i.e.* dont l'ordre n'est pas divisible par p^k), où p est premier et $k > 0$. Nous établissons également une formule pour l'ordre des caractères des groupes symétriques, quand $\mathcal{C}$ est l'ensemble des éléments ℓ-réguliers. Il s'avère que cette formule est un ℓ-analogue de la Formule du Crochet pour calculer le degré d'un caractère irréductible. En particulier, elle montre que, si ℓ est une puissance d'un nombre premier p, alors le rapport de l'ordre d'un caractère et de la p-partie de son degré est un invariant de chaque ℓ-bloc. Nous démontrons que cette dernière propriété est vraie en dehors des groupes symétriques sous certaines hypothèses (fortes) sur les blocs considérés. C'est aussi le cas si G est dans l'une des familles étudiées dans la deuxième partie.

La quatrième partie est consacrée au groupe linéaire fini $GL(n, q)$. Nous y donnons des définitions de sections et de blocs, basées sur la séparation en deux de l'ensemble des polynômes irréductibles qui divisent le polynôme minimal d'un élément. Cette séparation est faite en comparant le degré des polynômes à un entier quelconque d. Bien que nous définissions des blocs sur l'ensemble des caractères irréductibles, nous n'obtenons des propriétés que pour les caractères *unipotents*. Ceux-ci sont paramétrés par les partitions de n, et nous montrons que, si deux caractères unipotents appartiennent au même *bloc unipotent*, alors les partitions de n associées ont le même d-coeur. Certaines unions de ces blocs, appelés *blocs combinatoires* satisfont donc un analogue de la Conjecture de Nakayama. Nous montrons qu'ils satisfont également un analogue du Deuxième Théorème de Brauer.

Introduction

The representation theory of finite groups appeared and was first developped by G. Frobenius at the very end of the 19th century. We will assume that the reader is familiar with its ideas and techniques, in particular the notion of groups character and the structures of character rings and character tables. One of the major improvement in this theory was the introduction by Brauer, in the 1950's, of modular representation theory. The study of the representations of a finite group over a field of prime characteristic lead Brauer to develop the *block theory* of this group. This is the starting point of this thesis. If G is a finite group and $\mathrm{Irr}(G)$ is the set of irreducible complex characters of G, then to each prime p is associated a partition of $\mathrm{Irr}(G)$ into *p-blocks*. The blocks correspond to the summands in the decomposition into a direct sum of two-sided ideals of the group algebra of G over a complete discrete p-valuation ring. Most of the properties of p-blocks come from the p-local arithmetic properties of characters, and can be obtained through a bridge between characteristic 0 and characteristic p. This link is obtained via the introduction of *p-modular systems*. However, the p-blocks can also be obtained in an elementary way, based on the distinction between p-regular and p-singular elements of G. If we let $\mathcal{C}$ be the set of p-regular elements of G, we can consider the restriction to $\mathcal{C}$ of the ordinary scalar product on characters of G. For $\chi,\, \psi \in \mathrm{Irr}(G)$, we let

$$< \chi,\, \psi >_{\mathcal{C}} = \frac{1}{|G|} \sum_{g \in \mathcal{C}} \chi(g)\psi(g^{-1}).$$

Then χ and ψ are said to be *directly $\mathcal{C}$-linked* if $< \chi,\, \psi >_{\mathcal{C}} \neq 0$, and *orthogonal across* $\mathcal{C}$ otherwise. Then direct $\mathcal{C}$-linking is a reflexive (since $1 \in \mathcal{C}$) and symmetric binary relation on $\mathrm{Irr}(G)$. Extending it by transitivity, we obtain an equivalence relation (called *$\mathcal{C}$-linking*) on $\mathrm{Irr}(G)$ whose equivalence classes are the p-blocks.

The idea introduced by B. Külshammer, J. B. Olsson and G. R. Robinson in [18] is to do the same construction for other unions $\mathcal{C}$ of conjugacy classes of G. We therefore get a partition of $\mathrm{Irr}(G)$ into *generalized blocks*, or $\mathcal{C}$-

blocks. It is the purpose of this thesis to study some properties of these generalized blocks, and to define interesting blocks in some classes of groups. In [18], the authors have defined ℓ-blocks for the symmetric groups, where $\ell \geq 2$ is any integer. To obtain this, they take $\mathcal{C}$ to be the set of *ℓ-regular* elements, i.e. none of whose cycle has length divisible by ℓ. The ℓ-blocks obtained in this way satisfy an analogue of the Nakayama Conjecture: two irreducible characters χ^λ and χ^μ of the symmetric group S_n (where λ and μ are partitions of n) belong to the same ℓ-block if and only if λ and μ have the same ℓ-core.

In the first part of this thesis, we present shortly the construction and properties of the blocks in the classical case. In particular, we state Brauer's First and Second Main Theorems. A consequence of the latter is that irreducible characters in distinct p-blocks, which are thus orthogonal across the p-section of 1, are in fact orthogonal across each p-section. The end of the part is devoted to the presentation of the ideas and results in [18]. We introduce the generalization we mentionned above, in particular the notion of $(\mathcal{X}, \mathcal{Y})$-section, which generalizes p-sections. This allows us to define a *Second Main Theorem property*, which is an analogue of Brauer's Second Main Theorem in this setting (depending on the set $\mathcal{C}$ of conjugacy classes we start with, this property may be satisfied or not by the $\mathcal{C}$-blocks). We then describe the case of symmetric groups. In particular, the ℓ-blocks of symmetric groups satisfy the Second Main Theorem property.

The second part deals with *generalized perfect isometries*. Let G be a finite group, p a prime, U a Sylow p-subgroup of G, and B the normalizer of U in G. Let B_0 and b_0 be the principal p-blocks (i.e. containing the trivial character) of G and B respectively. Then a conjecture of M. Broué states that, if U is Abelian, then there should be a *perfect isometry* between B_0 and b_0 (cf [2]). Such a perfect isometry induces a close relationship between the character rings associated with B_0 and b_0. Now, if G is a Suzuki group $Sz(q)$, a Ree group (of type G_2) $Re(q)$ or a special unitary group $SU(3, q^2)$, each in characteristic p, then the Sylow p-subgroups of G are not Abelian. Furthermore, it is known (cf Cliff [8]) that, in the case of Suzuki groups, there is no perfect isometry between B_0 and b_0. However, we show that, in each of these cases, if we take $\mathcal{C}$ and $\mathcal{D}$ to be the sets of elements of G and B respectively whose order is not divisible by p^2, then there exists a bijection I between B_0 and b_0 and signs $\{\varepsilon(\chi),\, \chi \in \mathrm{Irr}(G)\}$ such that, for any $\chi,\, \psi \in B_0$, we have $< \chi,\, \psi >_{\mathcal{C}} = < \varepsilon(\chi)I(\chi),\, \varepsilon(\psi)I(\psi) >_{\mathcal{D}}$. We refer to this as a *generalized perfect isometry* between B_0 and b_0. This result is weaker than the one announced by Boué's Conjecture, but still enlighten the strong link between the character rings of B_0 and b_0. In particular, their

Cartan groups are isomorphic.

The Cartan group is presented in the third part of the thesis. We suppose that $\mathcal{C}$ is a *closed* union of conjugacy classes of the finite group G (i.e. if $x \in \mathcal{C}$ and if y generates the same subgroup of G as x, then $y \in \mathcal{C}$). For $\chi \in \text{Irr}(G)$, we define a class function $\chi^{\mathcal{C}}$ of G by $\chi^{\mathcal{C}}(g) = \chi(g)$ if $g \in \mathcal{C}$ and $\chi^{\mathcal{C}}(g) = 0$ if $g \in G \setminus \mathcal{C}$. We define the $\mathbf{Z}$-submodule $\mathcal{R}(\mathcal{C})$ of $\mathbf{C}\text{Irr}(G)$ spanned by $\{\chi^{\mathcal{C}};\, \chi \in \text{Irr}(G)\}$, and the $\mathbf{Z}$-submodule $\mathcal{P}(\mathcal{C})$ of $\mathcal{R}(\mathcal{C})$ consisting of those elements of $\mathcal{R}(\mathcal{C})$ which are generalized characters of G (*i.e.* $\mathcal{P}(\mathcal{C}) = \mathcal{R}(\mathcal{C}) \cap \mathbf{Z}\text{Irr}(G)$). Then $\mathcal{R}(\mathcal{C})$ and $\mathcal{P}(\mathcal{C})$ have the same $\mathbf{Z}$-rank. In particular, $\mathcal{R}(\mathcal{C})/\mathcal{P}(\mathcal{C})$ is a finite Abelian group, called the $\mathcal{C}$-*Cartan group* of G, and, for each $\chi \in \text{Irr}(G)$, there exists an integer $d > 0$ such that $d\chi^{\mathcal{C}} \in \mathbf{Z}\text{Irr}(G)$. The smallest such d is called the *order* of χ.

The Cartan group and the orders of characters are well-known when $\mathcal{C}$ is the set of p-regular elements of G. In this part, we give a description of the Cartan group and of the orders of characters in the case where G is Abelian and $\mathcal{C}$ is the set of elements whose order is not divisible by p^k, where p is a prime and $k > 0$. We also obtain a formula for the orders of characters of the symmetric group, when $\mathcal{C}$ is the set of ℓ-regular elements. It turns out that this formula can be seen as an ℓ-analogue of the Hook-Length Formula for computing the degree of an irreducible character. In particular, it implies that, if ℓ is a power of a prime p, then the quotient of the order of a character by the p-part of its degree is an invariant of each ℓ-block. We show that this last property is true outside the symmetric groups under some (strong) hypothesis on the blocks we consider. It is also the case in the three families of groups we study in the second part.

The fourth part is devoted to the finite general linear group $GL(n, q)$. We give definitions of sections and blocks, based on the splitting into two of the set of irreducible polynomials dividing the minimal polynomial of an element. This splitting is made by comparing the degrees of these irreducible polynomials with any given integer d. Although we define blocks for the whole set of irreducible characters of $GL(n, q)$, we only obtain properties for the *unipotent* characters. These are labelled by the partitions of n, and we show that, if two unipotent characters belong to the same *unipotent generalized block*, then the partitions labelling them have the same d-core. Thus some unions of these blocks, called *combinatorial blocks*, satisfy an analogue of the Nakayama Conjecture, and we show that they also satisfy the Second Main Theorem property.

Notations

Throughout this thesis, we will use the following notations:

$\mathbf{N}$	the natural integers (including 0).
$\mathbf{Z}$	the ring of integers.
$\mathbf{Q}$, $\mathbf{C}$	the fields of rationals and complex numbers.
$\mathbf{F}_q$	the field with q elements (q a prime power).
$(\mathbf{F}_q, +)$	the additive group of $\mathbf{F}_q$.
$\mathbf{F}_q^\times$	the multiplicative group of $\mathbf{F}_q$ (the non-zero elements of $\mathbf{F}_q$).
$\mathbf{F}_q[X]$	the ring of polynomials in one indeterminate with coefficients in $\mathbf{F}_q$.
$\delta(f)$	the degree of the polynomial $f \in \mathbf{F}_q[X]$.
n_p	the p-part of n (n a positive integer, p a prime).
$n_{p'}$	the p'-part of n.
$\gcd(m, n)$	the greatest common divisor of the integers m and n.
$\mathrm{lcm}(m, n)$	the least common multiple of the (positive) integers m and n.
δ_{xy}	the Kronecker delta function on x and y.
$A \coprod B$	the disjoint union of the sets A and B.
$\|A\|$	the order of the finite set A.
$rk_{\mathbf{Z}}(M)$	the rank over $\mathbf{Z}$ of a rational matrix M or of a $\mathbf{Z}$-module M.
$V \oplus W$	the direct sum of the vector spaces V and W.
$GL(V)$	the field of automorphisms of the vector space V over the ground field.
Id or 1	the identity map or the identity matrix.
I_n	the n by n identity matrix.
$g\|_W$	the restriction to the subspace W of V of an element g in $GL(V)$.
$Min(g)$	the minimal polynomial of $g \in GL(V)$ over the ground field.
$Char(g)$	the characteristic polynomial of $g \in GL(V)$ over the ground field.
C_d	the cyclic group of order d.
$G \times H$	the direct product of two (finite) groups G and H (written $G \oplus H$ if G and H are additive groups).
$G \rtimes H$	a semi-direct product of G by H.
$Z(G)$	the center of G.
G'	the derived subgroup of G.
$H \leq G$	H is a subgroup of G.

$H \lhd G$	H is a normal subgroup of G.
$[G : H]$	the index of the subgroup H in G.
$N_G(H)$	the normalizer of H in G.
$C_G(g)$	the centralizer in G of an element g of G.
$C_G(H)$	the centralizer in G of a subgroup H of G.
$Cl(G)$	the set of conjugacy classes of G.
$k(\mathcal{C})$	the number of conjugacy classes in a union $\mathcal{C}$ of conjugacy classes of G.
$Cl_G(g)$	the conjugacy class of g in G.
$o(g)$	the order of g in G.
$< g >$	the subgroup of G generated by g.
g_p	the p-part of g (p a prime).
$g_{p'}$	the p-regular part of g.
G_p	the set of p-elements of G.
$G_{p'}$	the set of p-regular elements of G.
$Syl_p(G)$	the set of Sylow p-subgroups of G.
$O_p(G)$	the largest normal p-subgroup of G.
$\mathrm{Irr}(G)$	the set of irreducible complex characters of G.
1_G	the trivial character of G.
$\mathbf{Z}\mathrm{Irr}(G)$	the ring of generalized characters of G.
$< \, , \, >_G$	the scalar product for complex class functions of G.
$\mathrm{Res}^G_H(\chi)$ or $\chi\vert_H$	the restriction to the subgroup H of the character χ of G.
$\mathrm{Ind}^G_H(\psi)$	the character of G induced from the character ψ of the subgroup H.
$\chi \otimes \chi'$	the tensor product of the characters (of the same group) χ and χ'.

Part 1

Generalized Block Theory

In this part, we introduce the notions and objects we will use in this thesis. In a first place, we give a quick overview of the "ordinary" block theory, introduced by R. Brauer. Then we give the definitions and first properties of generalized block theory, as presented in [18], and finally describe the generalized blocks for the symmetric groups defined in this latter article.

1.1 Block Theory

In this section, we let G be a finite group. For all the results in this section, and unless specified otherwise, we refer to Navarro [20]. Most of the notions and results in block theory come from arithmetic constructions and properties. It is this approach we use to describe Brauer characters and, later, blocks, even though, when generalizing, we will use another characterization of blocks.

1.1.1 Modular systems

If k is any field, the group algebra kG is semi-simple if and only if k is of characteristic 0 or p prime to $|G|$. For any field k, we write $\mathrm{Irr}(kG)$ the set of characters of irreducible kG-modules. If kG is semi-simple and if, for every simple kG-module V, V is absolutely simple (i.e. $End_{kG}(V) = k.Id$), then the structure of kG is enlighted by Wedderburn's theorem on semi-simple algebras: kG is a direct sum of matrix algebras. For example, this is the case when k is algebraically closed.

When we move to characteristic p dividing $|G|$, we lose the semi-simplicity of the group algebra, but we want to keep the "absolute simplicity property". This will be the motivation for the introduction of large enough p-modular systems. We first observe :

Proposition 1.1. *There exists a field $K \subset \mathbf{C}$ such that $[K : \mathbf{Q}] < \infty$ and such that any finite-dimensional representation of $\mathbf{C}G$ can be realised over K.*

A theorem of Brauer makes this more precise by stating that, if m is the exponent of G, then $K = \mathbf{Q}(e^{\frac{2i\pi}{m}})$ is sufficient.

Similarly, for any prime p, there exists a finite field $k_0 \subset \overline{\mathbf{F}}_p$ such that any finite-dimensional irreducible representation of $\overline{\mathbf{F}}_p G$ can be realised over k_0.

Now, let K be a field, $K \subset \mathbf{C}$, $[K : \mathbf{Q}] < \infty$, and let $\mathbf{A}_K = \mathbf{A} \cap K$ be the ring of algebraic integers in K (where $\mathbf{A}$ is the subring of $\mathbf{C}$ of algebraic integers). Then $K = Frac(\mathbf{A}_K)$ and we have

Theorem 1.2. *Let $\{0\} \neq \mathcal{P} \lhd \mathbf{A}_K$ be a prime ideal, and*

$$\mathcal{O} = \{\frac{a}{b} \in K = Frac(\mathbf{A}_K) \,|\, b \notin \mathcal{P}\}$$

be the localisation of $\mathbf{A}_K$ in $\mathcal{P}$. Then $\mathcal{O}$ is a subring of K, a local ring (i.e. with a unique maximal ideal $\mathcal{M}$). Moreover, $\mathcal{O}$ is principal, and $k = \mathcal{O}/\mathcal{M}$ is a finite field of characteristic $p > 0$, where $\mathbf{Z} \cap \mathcal{P} = p\mathbf{Z}$. If $\mathcal{M} = (\pi)$ with $\pi \in \mathcal{O}$, then the ideals of $\mathcal{O}$ are exactly the $(\pi^i)_{i\geq 1}$, and $\bigcap_{i\geq 1}(\pi^i) = \{0\}$.

Furthermore, we have $K = Frac(\mathcal{O})$ and every finite-dimensional KG-module can be realised over $\mathcal{O}$. For this theorem, presented like this, we refer to the graduate course on modular representation theory given in Lyon by M. Geck in 2001-2002, and, for another presentation, we refer to [1].

Definition 1.3. *If p is a prime, K is a subfield of $\mathbf{C}$, $[K : \mathbf{Q}] < \infty$, and k and $\mathcal{O}$ are as in the above theorem, then $(K, \mathcal{O}, k)$ is called a p-**modular system** for G.*

If $p|\,|G|$, and without further information about k, not all simple $\overline{\mathbf{F}}_p G$-modules need to be absolutely simple. However, adding to K a suitable $|G|$-th root of unity, we can suppose that $k_0 \subset k$, so that any finite dimensional irreducible $\overline{\mathbf{F}}_p G$-module can be realised over k.

Definition 1.4. *A p-modular system is said to be* **large enough** *for G if any $V \in Irr(\mathbf{C}G)$ can be realised over K and any $M \in Irr(\overline{\mathbf{F}}_p G)$ can be realised over k, and if K and k contain all $|G|$-th roots of unity.*

We have just shown that such a system exists, so from now on we let p be a prime and $(K, \mathcal{O}, k)$ be a p-modular system large enough for G. In particular, any simple KG- (resp. kG-) module is absolutely simple (and thus KG and kG are split algebras).

The ring $\mathcal{O}$ is often taken by authors to be complete (for some p-adic norm). This allows the lifting of idempotents from kG to $\mathcal{O}G$. However, if the p-modular system is big enough for G, then the group algebra KG is split semisimple, and kG is a split algebra. Then, by a result of Heller, the Krull-Schmidt-Azuyama Theorem holds for $\mathcal{O}G$-lattices (cf Curtis-Reiner [9], Theorem 30.18), and, furthermore, idempotents can be lifted from kG to $\mathcal{O}G$ (cf [9], Ex. 6.16).

In the sequel, we will note $z \longmapsto z^*$ the canonical surjection $\mathcal{O} \longrightarrow k$, also called *reduction* modulo p.

1.1.2 Brauer characters

Now we would like to turn to characters of kG-modules. However, because of the characteristic p, we have to use something else than just traces. We will describe the Brauer characters of G, which take values in $\mathcal{O}$, thus allowing us to connect them with the elements of $\mathrm{Irr}(KG)$.

We need first a result about roots of unity :

Proposition 1.5. *Let $\mathcal{U}_{p'} = \{x \in K^\times \,|\, x^n = 1$ for some $n \geq 1,$ with $n|\,|G|$ and $p \nmid n\}$. Then $\mathcal{U}_{p'} \leq K^\times$, $\mathcal{U}_{p'} \leq \mathcal{O}^\times$ and, if $|G| = p^a m$ $(a \geq 0,\, p \nmid m)$, then the restriction modulo p*

$$\{x \in K^\times \,|\, x^m = 1\} \subset \mathcal{U}_{p'} \longrightarrow \{x^* \in k^\times \,|\, (x^*)^m = 1\} \subset k^\times$$

is a bijection.

Let $G_{p'}$ be the set of p-regular elements of G. If M is a kG-module, associated to the representation ρ of dimension d, and $g \in G$ is p-regular of order $o(g)$, then the eigenvalues $(\xi_i)_{1 \leq i \leq d}$ of $\rho(g)$ are $o(g)$-th roots of unity, so are in $k^\times$. Therefore, there are unique $w_i \in \mathcal{U}_{p'}$ such that $w_i^* = \xi_i$, $1 \leq i \leq d$. We define the *Brauer character* φ_M of M by

$$\begin{array}{rccc} \varphi_M : & G_{p'} & \longrightarrow & \mathbf{A}_K \subset \mathcal{O} \\ & g & \longmapsto & \sum_{i=1}^d w_i \end{array}.$$

Then φ_M is a class function of $G_{p'}$, and, writing χ the character afforded by ρ, we have

$$\forall g \in G_{p'},\ \varphi_M(g)^* = \chi(g).$$

Moreover, if $M \cong_{kG} M'$, then $\varphi_M = \varphi_{M'}$.

We say that φ_M is *irreducible* if ρ is irreducible, and let $\mathrm{IBr}_p(G)$ be the set of irreducible Brauer characters of G.

The next theorem illustrates the importance of the Brauer characters. We write $CF(G)$ and $CF(G_{p'})$ the sets of complex class functions of G and $G_{p'}$ respectively. For $\chi \in CF(G)$, we denote by $\widehat{\chi}$ the restriction of χ to $G_{p'}$ (then $\hat{\chi} \in CF(G_{p'})$). We have:

Theorem 1.6. *(i) The set $IBr_p(G)$ is a $\mathbf{C}$-basis for $CF(G_{p'})$.*
(ii) For all $\chi \in Irr(KG)$, there exist unique non-negative integers $(d_{\chi\varphi})_{\varphi \in IBr_p(G)}$ such that

$$\widehat{\chi} = \sum_{\varphi \in IBr_p(G)} d_{\chi\varphi}\varphi.$$

Then (i) implies in particular that $|\mathrm{IBr}_p(G)|$ is the number of p-regular classes of G.

The (integral) matrix $D = ((d_{\chi\varphi}))_{\chi \in \mathrm{Irr}(KG), \varphi \in \mathrm{IBr}_p(G)}$ is called the *decomposition matrix* of G. It has (maximum) rank $|\mathrm{IBr}_p(G)|$ (in particular, each column of D has at least one non-zero entry). We deduce that, if $p \nmid |G|$, then $\mathrm{IBr}_p(G) = \{\widehat{\chi},\, \chi \in \mathrm{Irr(KG)}\}$, and the elements of $\mathrm{Irr}(kG)$ are obtained by taking the reduction modulo p of the elements of $\mathrm{Irr}(KG)$.

We let $C = D^t D$, and call C the *Cartan matrix* of G. Then $C = ((c_{\varphi,\psi}))_{\varphi,\psi \in \mathrm{IBr}_p(G)}$ is a positive definite symmetric matrix with non-negative integer coefficients.

For each $\varphi \in \mathrm{IBr}_p(G)$, we define a class function Φ_φ of G via

$$\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(KG)} d_{\chi\varphi}\chi$$

(Φ_φ corresponds to the column of φ in the decomposition matrix). We call Φ_φ the *principal indecomposable character* associated to φ. We then have the following:

Theorem 1.7. *(i) For each $\varphi \in IBr_p(G)$, Φ_φ vanishes outside $G_{p'}$.*
(ii) For all $\varphi,\, \psi \in IBr_p(G)$, we have

$$< \Phi_\varphi, \psi >_{G_{p'}} := \frac{1}{|G|} \sum_{g \in G_{p'}} \Phi_\varphi(g^{-1})\psi(g) = \delta_{\varphi\psi}.$$

(iii) The set $\{\Phi_\varphi,\, \varphi \in IBr_p(G)\}$ is a $\mathbf{Z}$-basis for the $\mathbf{Z}$-module $\mathbf{Z}Irr_{p'}(G)$ of generalized characters of G vanishing outside $G_{p'}$.

Note that, for $\varphi \in \mathrm{IBr}_p(G)$,

$$\widehat{\Phi_\varphi} = \sum_{\chi \in \mathrm{Irr}(KG)} d_{\chi\varphi}\widehat{\chi} = \sum_{\psi \in \mathrm{IBr}_p(G)} c_{\varphi\psi}\psi,$$

so that (i) and (ii) implie that, for φ, $\psi \in \mathrm{IBr}_p(G)$,

$$< \Phi_\varphi, \Phi_\psi >_G = < \Phi_\varphi, \Phi_\psi >_{G_{p'}} = c_{\varphi\psi}.$$

Together with (iii), this shows that the Cartan matrix corresponds to the generalized definition of Cartan matrix we will give later.

1.1.3 Blocks

We now turn to the definitions of p-blocks of $\mathrm{Irr}(KG)$ and $\mathrm{IBr}_p(G)$. To each $\chi \in \mathrm{Irr}(KG)$ are associated a primitive idempotent e_χ of $Z(KG)$ and an irreducible (linear) representation ω_χ of $Z(KG)$ given by:

$$e_\chi = \frac{\chi(1)}{|G|}\sum_{g \in G}\chi(g^{-1})g$$

and

$$\omega_\chi(\widetilde{\mathcal{C}}) = \frac{|\mathcal{C}|\chi(g_\mathcal{C})}{\chi(1)},$$

where $\mathcal{C}$ is any conjugacy class of G, with representative $g_\mathcal{C}$, and $\widetilde{\mathcal{C}}$ is the sum of the elements in $\mathcal{C}$.

Then, for all χ, $\chi' \in \mathrm{Irr}(KG)$, we have $\omega_\chi(e_{\chi'}) = \delta_{\chi\chi'}$. Furthermore, for each $\chi \in \mathrm{Irr}(KG)$, the restriction ω_χ^* is an irreducible representation of $Z(kG)$.

The p-blocks we define correspond to the decomposition of $\mathcal{O}G$ into a direct sum of ideals $\mathcal{O}Ge$, where e is a primitive idempotent of $Z(\mathcal{O}G)$. We define the *Brauer graph* of G: the vertices are the $\chi \in \mathrm{Irr}(KG)$, and two characters $\chi \neq \chi' \in \mathrm{Irr}(KG)$ are linked if there exists $\varphi \in \mathrm{IBr}_p(G)$ such that $d_{\chi\varphi} \neq 0 \neq d_{\chi'\varphi}$. We then have the following:

Theorem/Definition 1.8. *(i) Two characters $\chi \neq \chi' \in Irr(KG)$ are linked in the Brauer graph if and only if* $\displaystyle\frac{1}{|G|}\sum_{g \in G_{p'}}\chi(g)\chi'(g^{-1}) \neq 0$.
*(ii) An element $e \in Z(\mathcal{O}G)$ is a primitive idempotent of $Z(\mathcal{O}G)$ if and only if $e = \sum_{\chi \in B} e_\chi$ for some connected component B of the Brauer graph. The set $B \subset Irr(KG)$ is then called a p-**block** of $Irr(KG)$ (or just of G).*
(iii) Two characters χ, $\chi' \in Irr(KG)$ belong to the same p-block of G if and only if $\omega_\chi^* = \omega_{\chi'}^*$.

Note that the p-blocks do not depend on the p-modular system we choose (cf [20]). We let $Bl(G)$ be the set of p-blocks of G. We have

Proposition 1.9. *For $B \in Bl(G)$, we let $\omega_B = \omega_\chi^*$ for any $\chi \in B$, and $e_B = \sum_{\chi \in B} e_\chi$. Then*
(i) $\{\omega_B,\, B \in Bl(G)\}$ is the complete set of (distinct) irreducible representations of $Z(kG)$.
(ii) $\{e_B^*,\, B \in Bl(G)\}$ is the complete set of primitive idempotents of $Z(kG)$.
(iii) For all B, $B' \in Bl(G)$, we have $\omega_B(e_B^*) = \delta_{BB'}$.

For $B \in Bl(G)$, the set $\{\varphi \in \mathrm{IBr}_p(G) \,|\, \exists \chi \in B,\, d_{\chi\varphi} \neq 0\}$ is called a *p-block* of $\mathrm{IBr}_p(G)$, and written $\mathrm{IBr}_p(B)$. Up to reordering lines and columns, we see that the p-blocks of $\mathrm{Irr}(KG)$ and $\mathrm{IBr}_p(G)$ correspond to a diagonal block decomposition of the decomposition matrix, and thus of the Cartan matrix. For $B \in Bl(G)$, we define in a natural way the decomposition matrix D_B and Cartan matrix C_B of B.

1.1.4 Defect

We now give the definitions of *defect* of a character, a block, or a conjugacy class. With these, we obtain informations on generalized characters and the invariant factors of the Cartan matrices. The results we mention give answers in the prime case to the questions we will study in the third part of this thesis.

Definition 1.10. *For $\chi \in Irr(KG)$, we define the* **defect** *$d(\chi)$ of χ to be the exact power of p dividing the integer $\frac{|G|}{\chi(1)}$, i.e.*

$$p^{d(\chi)} = \left| \frac{|G|}{\chi(1)} \right|_p .$$

We then define the defect $d(B)$ of a block B of $Irr(KG)$ to be

$$d(B) = \max_{\chi \in B} d(\chi).$$

For $\mathcal{C}$ a conjugacy class of G, we define the defect $d(\mathcal{C})$ of $\mathcal{C}$ by

$$p^{d(\mathcal{C})} = |C_G(g)|_p \ \text{ for } g \in \mathcal{C}.$$

For B a block of G, we let n_B and r_B be the numbers of irreducible (complex) characters in B and irreducible Brauer characters in B respectively.

The case of blocks of defect 0 is described by the following

Theorem 1.11. *Let B be a p-block of G and $\chi \in B$. Then the following are equivalent:*
(i) $d(B) = 0$.
(ii) $n_B = r_B$.
(iii) $d(\chi) = 0$.
(iv) χ *is the only irreducible character in* B.
(v) χ *vanishes on* p*-singular elements.*
(vi) $\chi(1)_p = |G|_p$.

To prove the next theorem, one needs the following lemma, which is interesting in itself for our work:

Lemma 1.12. *Let $\chi \in Irr(KG)$. We define a class function $\widetilde{\chi}$ of G by letting, for $g \in G$,*

$$\widetilde{\chi}(g) = \begin{cases} \chi(g) \text{ if } g \text{ is } p\text{-regular} \\ 0 \text{ if } g \text{ is } p\text{-singular} \end{cases} .$$

Then $p^{d(\chi)}\widetilde{\chi}$ is a generalized character, while $p^{d(\chi)-1}\widetilde{\chi}$ is not.

We then have the following result on Cartan matrices:

Theorem 1.13. *Let B be a p-block of G. Then*
(i) The elementary divisors of the Cartan matrix C_B divide $p^{d(B)}$, and at least one of them is equal to $p^{d(B)}$. In particular, $n_B > r_B$ unless $d(B) = 0$.
(ii) C_B has exactly one elementary divisor equal to $p^{d(B)}$.

Finally, we mention the following:

Theorem 1.14. *The elementary divisors of the Cartan matrix C of G are*

$$\{p^{d(\mathcal{C})} \,|\, \mathcal{C} \text{ } p\text{-regular class of } G\}.$$

1.1.5 Brauer's First and Second Main Theorems

Now, we want to state Brauer's First and Second Main Theorems, which relate blocks of a group G and of certain subgroups of G. These theorems have many applications in representation theory. In order to state them, we first introduce the notion of defect group and the Brauer correspondence. For this presentation, we refer to Goldschmidt [14] and Navarro [20].

Defect groups

The idea of defect groups is to associate to each p-block of G a certain conjugacy class of p-groups of G.

For subgroups H, L, M of G, we will write $H \subset_L M$ (resp. $H =_L M$) if $H^l \subset M$ (resp. $H^l = M$) for some $l \in L$.

For each conjugacy class $\mathcal{C}$ of G with $g_{\mathcal{C}} \in \mathcal{C}$, we define a *defect group* $\delta(\mathcal{C})$ of $\mathcal{C}$ to be a Sylow p-subgroup of $C_G(g_{\mathcal{C}})$ ($\delta(\mathcal{C})$ is therefore determined up to g-conjugacy).

For B a p-block of G and $\mathcal{C}$ a conjugacy class of G, we let

$$a_B(\widetilde{\mathcal{C}}) = \frac{1}{|G|} \sum_{\chi \in B} \chi(1)\chi(g_{\mathcal{C}}^{-1}),$$

where $g_{\mathcal{C}} \in \mathcal{C}$ and $\widetilde{C} = \sum_{g \in \mathcal{C}} g$. Then $e_B = \sum_{\mathcal{C}} a_B(\widetilde{\mathcal{C}})\widetilde{\mathcal{C}}$, and $a_B(\widetilde{\mathcal{C}}) = 0$ if $\mathcal{C}$ is a p-singular class.

Proposition/Definition 1.15. *If B is any p-block of G, then there exists a p-regular class $\mathcal{C}$ such that $\omega_B(\widetilde{\mathcal{C}}) \neq 0 \neq a_B(\widetilde{\mathcal{C}})^*$. Such a class is called a* **defect class** *for B, and $\delta(\mathcal{C})$ is called a* **defect group** *for B.*

Proposition 1.16. *Let $\delta(B)$ be a defect group for the block B and $\mathcal{C}$ a conjugacy class of G. Then*
(i) If $\omega_B(\widetilde{\mathcal{C}}) \neq 0$, then $\delta(B) \subset_G \delta(\mathcal{C})$.
(ii) If $a_B(\widetilde{\mathcal{C}})^* \neq 0$, then $\delta(\mathcal{C}) \subset_G \delta(B)$.
(iii) $\delta(B)$ is determined up to G-conjugation.
(iv) $|\delta(B)| = p^{d(B)}$.

Finally,

Proposition 1.17. *If P is a normal p-subgroup of G and B is a block of G, then $P \subset_G \delta(B)$.*

The Brauer correspondence

Let H be a subgroup of G. If λ is any linear functional on $Z(kH)$ and $\mathcal{C}$ is a conjugacy class of G, we define $\lambda^G(\widetilde{\mathcal{C}}) = \lambda((\mathcal{C} \cap H)^{\sim})$ and extend λ^G linearly to a linear functional on $Z(kG)$.

If $\lambda = \omega_b$ for some block b of H, then ω_b^G may or may not be an algebra homomorphism. If it is, then $\omega_b^G = \omega_B$ for some $B \in Bl(G)$; we say that b^G is defined and $b^G = B$. The correspondence $b \longmapsto b^G$ is called the *Brauer correspondence*.

Defect groups behave well under the Brauer correspondence :

Proposition 1.18. *Suppose H is a subgroup of G with a block b for which b^G is defined. Then there exists a p-subgroup P of G such that*

$$\delta(b) \subset_H P =_G \delta(b^G).$$

We will obtain a sufficient condition for the existence of b^G. Suppose H, subgroup of G, is such that $C_G(P) \lhd H$ for some p-subgroup P of G. For any conjugacy class $\mathcal{C}$ of G, define $\mu(\widetilde{\mathcal{C}}) = (\mathcal{C} \cap C_G(P))^{\sim}$. Since $C_G(P) \lhd H$, $\mathcal{C} \cap C_G(P)$ is a union of conjugacy classes of H, and μ extends linearly to a map $Z(kG) \longrightarrow Z(kH)$.

Proposition/Definition 1.19. *The map $\mu\colon Z(kG) \longrightarrow Z(kH)$ is an algebra homomorphism, called the* **Brauer homomorphism***.*

Theorem 1.20. *Suppose P is a p-subgroup of G and $PC_G(P) \subset H \subset N_G(P)$, then b^G is defined for all blocks b of H, and $\omega_{b^G} = \omega_b \circ \mu$. Moreover, if B is a block of G, then $B = b^G$ for some block b of H if and only if $P \subset_G \delta(B)$, in which case*

$$\mu(e_B^*) = \sum_{b^G = B} e_b^*.$$

Brauer's Theorems

We can now state Brauer's First and Second Main Theorems.

Theorem 1.21. *(Brauer's First Main Theorem) Suppose P is a p-subgroup of G. Then the Brauer correspondence is a bijection between the set of blocks of $N_G(P)$ with defect group P and the set of blocks of G with defect group P.*

As a consequence, we mention the following theorem concerning *principal blocks*. The principal block is the one containing the trivial character. It is therefore of maximal defect, so any of its defect groups is a Sylow p-subgroup of G. The following theorem can also be seen as an immediate consequence of Brauer's Third Main Theorem, as stated in [20].

Theorem 1.22. *Suppose H is a subgroup of G, b is a block of H, and $C_G(\delta(b)) \subset H$. Then b^G (which is defined) is the principal block of G if and only if b is the principal block of H.*

In particular, if B_0 is the principal block of G and $\delta(B_0) =_G P \in Syl_p(G)$, then B_0 is the image under the Brauer correspondence of the principal block b_0 of $N_G(P)$. The idea of Broué's Conjecture (cf Part 2) is that, if P is Abelian, there should be a deeper correspondence, at the level of characters, namely a perfect isometry, between B_0 and b_0.

Before stating Brauer's Second Main Theorem, we need the following:

Lemma 1.23. *Let x be a p-element of G, and let $H = C_G(x)$. For $\chi \in Irr(KG)$ and $\varphi \in IBr_p(H)$, there exist unique $d^x_{\chi\varphi} \in \mathbf{C}$ such that*

$$\chi(xy) = \sum_{\varphi \in IBr_p(H)} d^x_{\chi\varphi}\varphi(y)$$

for all p-regular $y \in H$.

We have

$$d^x_{\chi\varphi} = \sum_{\psi \in \mathrm{Irr}(KH)} \frac{< \mathrm{Res}^G_H(\chi), \psi >_H \psi(x)}{\psi(1)} d_{\psi\varphi}$$

(where the $d_{\psi\varphi}$'s are the "ordinary" decomposition numbers). The $d^x_{\chi\varphi}$'s are called the *generalized decomposition numbers*.

Theorem 1.24. *(Brauer's Second Main Theorem) Let x be a p-element of G and b be a p-block of $C_G(x)$. Then, if $\chi \in Irr(G)$ is not in b^G, then $d^x_{\chi\varphi} = 0$ for each $\varphi \in IBr_p(b)$.*

(Note that, with the above hypotheses, b^G is always defined.)

Corollary 1.25. *Let x be a p-element of G, and y a p-regular element of $C_G(x)$. Suppose B is a block of G and $\chi \in B$. Then*

$$\chi(xy) = \sum_{b \in Bl(C_G(x)),\, b^G = B} \ \sum_{\mu \in IBr_p(b)} d^x_{\chi\mu}\mu(y).$$

Using the definitions of the $d^x_{\chi\mu}$'s, then the definition of the $d_{\psi\mu}$'s and the fact that x is central in $C_G(x) = H$, this can be reformulated as:

Corollary 1.26. *Suppose B is a block of G and $\chi \in B$. Then, for any p-element $x \in G$ and p-regular element $y \in H = C_G(x)$, we have*

$$\chi(xy) = \sum_{b \in Bl(H),\, b^G = B} \ \sum_{\psi \in b} < Res^G_H(\chi), \psi >_H \psi(xy).$$

If θ is a class function of G and B is a block of G, we let

$$\theta^B = \sum_{\chi \in B} < \theta, \chi >_G \chi.$$

Then $\theta = \sum_{B \in Bl(G)} \theta^B$. Another useful consequence of Brauer's Second Main Theorem is the following:

Theorem 1.27. *Let θ be a class function of G and let $x \in G$ be a p-element. Suppose $\theta(xy) = 0$ for all p-regular $y \in C_G(x)$. Then, for each block B of G and p-regular $y \in C_G(x)$, we have $\theta^B(xy) = 0$.*

Let B be a block of G, $\chi \in B$, and let $x \in G$ be a p-element and $H = C_G(x)$. For any block b of H, we write

$$\chi^{(b)} = \sum_{\psi \in b} < \mathrm{Res}_H^G(\chi), \psi >_H \psi$$

($\chi^{(b)}$ is a generalized character of H). Then $\mathrm{Res}_H^G(\chi) = \sum_{b \in Bl(H)} \chi^{(b)}$, and, by Corollary 1.26

$$\sum_{b \in Bl(H),\, b^G \neq B} \chi^{(b)}(xy) = 0$$

for all p-regular $y \in H$.

Applying Theorem 1.27 in H (in which x is central), we get that, if $b \in Bl(H)$ and $b^G \neq B$, then $\chi^{(b)}(xy) = 0$ for all p-regular $y \in H$. Hence, if there exists a p-regular $y \in H$ such that $\chi^{(b)}(xy) \neq 0$ for some $b \in Bl(H)$, then $b^G = B$ (see the Second Main Theorem property later).

Theorem 1.27 has an important corollary:

Corollary 1.28. *(Block Orthogonality) Let $g, h \in G$ be such that g_p and h_p are not conjugate in G. Then, for each p-block B of G, $\sum_{\chi \in B} \chi(g^{-1})\chi(h) = 0$.*

This has a very important consequence. Before stating it, we define the p-sections of G. For $g \in G$, the *p-section* of g in G is the set of elements of G whose p-part is conjugate to g_p. Then G is a disjoint union of p-sections. The p-section of 1 is just the set of p-regular elements of G.

Take $g \in G$, and write π the p-section of g in G. The last corollary can then be written:

$$\forall B \in Bl(G),\, \forall h \in G \setminus \pi,\, \sum_{\chi \in B} \chi(g^{-1})\chi(h) = 0.$$

We say that each block of G *separates π from its complement*.

If π is a p-section of G, we say that two class functions η and θ of G are *orthogonal across π* if

$$\frac{1}{|G|} \sum_{g \in \pi} \theta(g)\eta(g^{-1}) = 0.$$

By construction, we have seen that, if two irreducible characters of G belong to distinct p-blocks, then they are orthogonal across p-regular elements. We have in fact much better:

Theorem 1.29. *If two irreducible characters of G belong to distinct p-blocks, then they are orthogonal across* **each** *p-section of G.*

(This can be proved by using Corollary and adapting the argument given in [18] after Corollary 1.2)

1.2 Generalized Block Theory

In this section, we introduce the notion of *generalized blocks*, as introduced in [18]. The basis for "ordinary" block theory was the separation of a group between p-regular and p-singular elements. Now the idea is to use another set of conjugacy classes to split the group. This leads to a new definition of block, which we present in a first paragraph. We then give a definition of *sections* to partition the group, and state what an analogue of Brauer's Second Main Theorem could look like with this definition (depending on the set of conjugacy classes we start with, the generalized blocks we obtain may or may not satisfy this property). For all the definitions and results in sections 1.2.1, 1.2.2 and 1.2.3, we refer to Külshammer, Olsson and Robinson [18]. We then give in section 1.2.4 an overview of the first natural generalization, the case of π-blocks.

1.2.1 Generalized blocks

Let G be a finite group, and $\mathcal{C}$ be a union of conjugacy classes of G. We say that $\mathcal{C}$ is *closed* if the following is true: for any $x \in \mathcal{C}$, whenever y is an element of G which generates the same subgroup as x, then $y \in \mathcal{C}$. The notion of closed set of conjugacy classes was introduced by M. Suzuki in [24]. Now let us fix a closed set $\mathcal{C}$ of conjugacy classes of G, and assume that $1 \in \mathcal{C}$. We let $\mathcal{C}' = G \setminus \mathcal{C}$. Let $\mathrm{Irr}(G)$ be the set of complex irreducible characters of G. For any complex class function φ of G, we define $\varphi^{\mathcal{C}}$ to be the class function of G which agrees with φ on $\mathcal{C}$ and vanishes outside $\mathcal{C}$.

Two characters χ, $\psi \in \mathrm{Irr}(G)$ are said to be *directly $\mathcal{C}$-linked* if

$$< \chi, \psi >_{\mathcal{C}} := \frac{1}{|G|} \sum_{x \in \mathcal{C}} \chi(x)\overline{\psi(x)} \neq 0.$$

If $< \chi, \psi >_{\mathcal{C}} = 0$, then χ and ψ are said to be *orthogonal across* $\mathcal{C}$. We call $< \chi, \psi >_{\mathcal{C}}$ the *$\mathcal{C}$-contribution* associated to χ and ψ. Note that $< \chi, \psi >_{\mathcal{C}} = < \chi^{\mathcal{C}}, \psi >_G = < \chi, \psi^{\mathcal{C}} >_G = < \chi^{\mathcal{C}}, \psi^{\mathcal{C}} >_G$. Direct $\mathcal{C}$-linking is a symmetric and reflexive (since $1 \in \mathcal{C}$) binary relation on $\mathrm{Irr}(G)$. Extending it by transitivity to an equivalence relation (called $\mathcal{C}$-linking), we obtain a partition of $\mathrm{Irr}(G)$

into *$\mathcal{C}$-blocks*. We refer to $\mathcal{C}$-blocks of $\mathrm{Irr}(G)$ as $\mathcal{C}$-blocks of G. Two characters χ, $\psi \in \mathrm{Irr}(G)$ thus belong to the same $\mathcal{C}$-block of G if there exists a sequence $\chi_0 = \chi, \chi_1, \ldots, \chi_r = \psi \in \mathrm{Irr}(G)$ such that, for each $0 \leq i \leq r-1$, the characters χ_i and χ_{i+1} are directly $\mathcal{C}$-linked. We say that χ and ψ are *$\mathcal{C}$-linked*.

Note that if we take $\mathcal{C}$ to be the set of p-regular elements of G (p a prime), then the $\mathcal{C}$-blocks of G are just its p-blocks.

We have the following:

Proposition 1.30. *If $g \in \mathcal{C}$ and $h \in \mathcal{C}'$, then, for each $\mathcal{C}$-block B of G,*

$$\sum_{\chi \in B} \chi(g^{-1})\chi(h) = 0.$$

Hence $\mathcal{C}$-blocks separate $\mathcal{C}$ from $\mathcal{C}'$.

The matrix $\Gamma(\mathcal{C}, G) = ((< \chi, \psi >_{\mathcal{C}}))_{\chi,\psi \in \mathrm{Irr}(G)}$ is called the *$\mathcal{C}$-contribution matrix* of G. It has (if we list the elements of $\mathrm{Irr}(G)$ block by block) a diagonal block decomposition corresponding to the $\mathcal{C}$-blocks of G. If B is a $\mathcal{C}$-block of G, we let $\Gamma(\mathcal{C}, B) = ((< \chi, \psi >_{\mathcal{C}}))_{\chi,\psi \in B}$ be the *$\mathcal{C}$-contribution matrix* of B.

Using Galois Theory, and since $\mathcal{C}$ is closed, we see that the $\mathcal{C}$-contributions are **rationals**.

1.2.2 Sections

In the same way we can partition G into p-sections, we would like a more general definition of sections, with the same properties, and which we can adapt to our situation. This includes relating these sections to some $\mathcal{C}$-blocks as defined above, and defining blocks for the centralizers of certain elements (corresponding to the p-elements in the p-case).

Let $\mathcal{X}$ be a union of conjugacy classes of G containing the identity. Suppose that, for each $x \in \mathcal{X}$, there is a union $\mathcal{Y}(x)$ of conjugacy classes of $C_G(x)$ such that:
(i) $1 \in \mathcal{Y}(x)$,
(ii) Two elements of $x\mathcal{Y}(x)$ are G-conjugate if and only if they are $C_G(x)$-conjugate,
(iii) $C_G(xy) \leq C_G(x)$ for each $y \in \mathcal{Y}(x)$.
Suppose also that $\mathcal{Y}(x^g) = \mathcal{Y}(x)^g$ for all $x \in \mathcal{X}$ and $g \in G$, and that G is the disjoint union of the conjugacy classes $(xy)^G$, as x runs through a set of representatives for the G-conjugacy classes in $\mathcal{X}$ and y runs through a set of representatives for the $C_G(x)$-conjugacy classes in $\mathcal{Y}(x)$.

For example, if we take $\mathcal{X}$ to be the set of p-elements of G, then, for each $x \in \mathcal{X}$, $\mathcal{Y}(x)$ can be taken to be the set of p-regular elements of $C_G(x)$.

For any $x \in \mathcal{X}$, we call the union of the G-conjugacy classes meeting $x\mathcal{Y}(x)$ the $\mathcal{Y}$-*section* of x. The hypotheses we made ensure that, for each $x \in \mathcal{X}$, induction of complex class functions gives an isometry from the space of class functions of $C_G(x)$ vanishing outside $x\mathcal{Y}(x)$ onto the space of class functions of G vaniching outside the $\mathcal{Y}$-section of x.

We define an $(\mathcal{X}, \mathcal{Y})$-*block* of G as a $\mathcal{Y}(1)$-block of G in the sense we defined above. For $x \in \mathcal{X}$, we define the $(\mathcal{X}, \mathcal{Y})$-*blocks* of $C_G(x)$ to be the smallest (non-empty) subsets of $\mathrm{Irr}(C_G(x))$ such that irreducible characters in distinct subsets are orthogonal across $x\mathcal{Y}(x)$.

Note that, because of Proposition 1.30, $(\mathcal{X}, \mathcal{Y})$-blocks of G separate $\mathcal{Y}(1)$ from its complement. For $x \in \mathcal{X}$, we can equally define $(\mathcal{X}, \mathcal{Y})$-blocks of $C_G(x)$ to be non-empty subsets of $\mathrm{Irr}(C_G(x))$ which are minimal subject to separating $\mathcal{Y}(x)$ from its complement in $C_G(x)$ (or, equivalently, to separating $x\mathcal{Y}(x)$ from its complement (since x is central in $C_G(x)$)).

1.2.3 Second Main Theorem property

Suppose we have sets $\mathcal{X} \subset G$ and $\mathcal{Y}(x)$ for $x \in \mathcal{X}$ as above. Suppose $\chi \in \mathrm{Irr}(G)$ and β is a union of $(\mathcal{X}, \mathcal{Y})$-blocks of $C_G(x)$ for some $x \in \mathcal{X}$. We define a generalized character $\chi^{(\beta)}$ of $C_G(x)$ via

$$\chi^{(\beta)} = \sum_{\mu \in \beta} < \mathrm{Res}^G_{C_G(x)}(\chi), \mu > \mu.$$

Definition 1.31. *Let $x \in \mathcal{X}$ and b be an $(\mathcal{X}, \mathcal{Y})$-block of $C_G(x)$. We say that an $(\mathcal{X}, \mathcal{Y})$-block B of G* **dominates** *b if there exist $\chi \in B$ and $y \in \mathcal{Y}(x)$ such that $\chi^{(b)}(xy) \neq 0$.*

We see that, for $x \in \mathcal{X}$, if $\chi \in B$ for some $(\mathcal{X}, \mathcal{Y})$-block B of G, then, for each $y \in \mathcal{Y}(x)$, we have $\chi(xy) = \sum_b \chi^{(b)}(xy)$, where b runs through the set of $(\mathcal{X}, \mathcal{Y})$-blocks of $C_G(x)$ dominated by B.

Note that, for $x \in \mathcal{X}$, each $(\mathcal{X}, \mathcal{Y})$-block of $C_G(x)$ is dominated by at least one $(\mathcal{X}, \mathcal{Y})$-block of G.

Definition 1.32. *We say that the $(\mathcal{X}, \mathcal{Y})$-blocks of G satisfy the Second Main Theorem property if, for each $x \in \mathcal{X}$ and each $(\mathcal{X}, \mathcal{Y})$-block b of $C_G(x)$, b is dominated by a unique $(\mathcal{X}, \mathcal{Y})$-block of G.*

As we have remarked in the previous section, if, for some prime p, $\mathcal{X}$ is the set of p-elements of G and if, for each $x \in \mathcal{X}$, $\mathcal{Y}(x)$ is the set of p-regular elements of $C_G(x)$, then the $(\mathcal{X}, \mathcal{Y})$-blocks of G (i.e. the p-blocks of G) have the Second Main Theorem property.

Using the fact that, for $x \in \mathcal{X}$, irreducible characters of $C_G(x)$ in distinct $(\mathcal{X}, \mathcal{Y})$-blocks are orthogonal across $x\mathcal{Y}(x)$, one proves easily the following:

Proposition 1.33. *The $(\mathcal{X}, \mathcal{Y})$-blocks of G satisfy the Second Main Theorem property if and only if, for each $(\mathcal{X}, \mathcal{Y})$-block B of G, there is, for each $x \in \mathcal{X}$, a (possibly empty) union of $(\mathcal{X}, \mathcal{Y})$-blocks $\beta(x, B)$ of $C_G(x)$ such that, for each irreducible character $\chi \in B$ and each character $\mu \in \beta(x, B)$, we may find a complex number $c_{\chi,\mu}$ such that, for each $y \in \mathcal{Y}(x)$, we have*

$$\chi(xy) = \sum_{\mu \in \beta(x,B)} c_{\chi,\mu}\mu(xy),$$

and, furthermore, $\beta(x, B)$ and $\beta(x, B')$ are disjoint whenever B and B' are distinct $(\mathcal{X}, \mathcal{Y})$-blocks of G.

The following theorem enlighten the link between the Second Main Theorem property and Brauer's Second Main Theorem:

Theorem 1.34. *Suppose that the $(\mathcal{X}, \mathcal{Y})$-blocks of G satisfy the Second Main Theorem property. Then:*
(i) Irreducible characters of G which are in distinct $(\mathcal{X}, \mathcal{Y})$-blocks are orthogonal across **each** *$\mathcal{Y}$-section of G.*
(ii) If $x \in \mathcal{X}$ and $\sum_{\chi \in Irr(G)} a_\chi \chi$ is a class function which vanishes identically on the $\mathcal{Y}$-section of x in G, then, for each $(\mathcal{X}, \mathcal{Y})$-block B of G, $\sum_{\chi \in B} a_\chi \chi$ also vanishes identically on the $\mathcal{Y}$-section of x in G.
(iii) $(\mathcal{X}, \mathcal{Y})$-blocks of G separate $\mathcal{Y}$-sections of G.

1.2.4 π-blocks

One of the main problems we meet when moving from p-blocks to $\mathcal{C}$-blocks is that we generally lose all the arithmetic on which the modular representation theory is based. In particular, the discrete valuation ring $\mathcal{O}$ has no analogue in general, and $\mathcal{C}$-blocks are not related to idempotents of a group algebra.

One case in which arithmetic arguments can still be used is that of π-*blocks*, where π is a set of primes. These were initially introduced and studied by Brauer (even though he didn't publish his results), and then considered by many authors. From now on, we let G be a finite group and π be a set of primes. Different, but equivalent, definitions of π-blocks can be given. The first might be the more natural.

Definition 1.35. *(Staszewski [23], definition 1.1) A non-empty subset B of Irr(G) is called a π-**block** of G if, for every $p \in \pi$, B is a union of p-blocks, and if B is minimal for this property.*

Note that, if $\pi = \{p\}$, then a π-block is just a p-block.

Now the π-blocks can also be seen in terms of idempotents. For this definition, we refer to Staszewski [23] and Robinson [21]. Let $\omega = e^{2\pi i/|G|}$. For any $p \in \pi$, we let
$\mathcal{O}_p = \{\frac{\alpha}{\beta} \,|\, \alpha, \beta \in \mathbf{Z}[\omega]$ and, for any ideal ρ of $\mathbf{Z}[\omega]$ containing p, $\beta \notin \rho\}$ be the p-adic integer ring of $\mathbf{Q}[\omega]$ (with relative residue field $\mathbf{F}_p$). We set $\mathcal{O}_\pi = \bigcap_{p\in\pi} \mathcal{O}_p$. Note that $\mathcal{O}_\pi$ is no longer a local ring, but that it has only finitely many prime ideals. Then we have

Definition 1.36. *A non-empty subset B of Irr(G) is a π-**block** of G if*

$$\frac{1}{|G|} \sum_{\chi\in B} \sum_{g\in G} \chi(1)\chi(g^{-1})g$$

is a primitive idempotent of $Z(\mathcal{O}_\pi G)$.

It can be shown (cf [21], Lemma 2) that π-blocks are, with the terminology we introduced before, $\mathcal{C}$-blocks, where $\mathcal{C}$ is the set of π-regular elements of G (π-blocks are subsets of Irr(G) which are minimal subject to being orthogonal across π-regular elements). We also see that the π-sections of G satisfy the properties for $(\mathcal{X}, \mathcal{Y})$-sections, $\mathcal{X}$ being the set of π-elements and $\mathcal{Y}(x)$ being the set of π-regular elements of $C_G(x)$ for $x \in \mathcal{X}$. In this setting, an $(\mathcal{X}, \mathcal{Y})$-block of G is a π-block of G, and, for $x \in \mathcal{X}$, an $(\mathcal{X}, \mathcal{Y})$-block of $C_G(x)$ is a π-block of $C_G(x)$.

The π-blocks of a finite group have properties very similar to those of p-blocks. In particular, it is possible to define analogues of the principal indecomposable characters, the Brauer characters, the decomposition numbers and Cartan numbers, the corresponding *decomposition matrix* and *Cartan matrix* having the same kind of block structure as in the p-case (cf [23]). Staszewski also studies the *defect* of a π-block, and determines the elementary divisors of the Cartan matrices of G and of any given π-block of G.

The generalization of the notion of defect group appears to be more tricky. However, this can be done if G has a nilpotent Hall π-subgroup, and defect groups are unique up to conjugacy if G has furthermore a normal π-complement (cf [23]).

It is shown in Robinson [21] that π-blocks satisfy an analogue of Brauer's Second Main Theorem. In particular, π-blocks separate π-sections, and irreducible characters in distinct π-blocks are orthogonal across each π-section (cf [21], Corollary 8). Robinson and Staszewski also show in [22] that, if G has a cyclic Hall π-subgroup, then the π-blocks of G satisfy an analogue of Brauer's Third Main Theorem (cf [22], Theorem 2.1).

Finally, Broué and Michel have studied the π-blocks of finite reductive groups in [3]. Using the Deligne-Lusztig Theory, the irreducible (complex) characters of such a group G^F are partitioned into *geometric conjugacy classes* $\mathcal{E}(G^F,(s))$, labelled by the conjugacy classes of semisimple elements s in the dual group G^{*F^*}. Broué and Michel show in particular that, if the defining characteristic of G^F does not belong to π, then the union $\mathcal{E}_\pi(G^F,(s))$ of such geometric conjugacy classes for the parameter describing a π'-section, i.e. $\mathcal{E}_\pi(G^F,(s)) = \bigcup_{x\in C_{G^*}(s)^{F^*}} \mathcal{E}(G^F,(sx))$ for some semisimple π'-element s of G^{*F^*}, is a union of π-blocks (cf [3], Theorem 2.2).

1.3 Generalized Blocks for Symmetric Groups

In this section, we give an overview of some of the results obtained in [18]. In this article, the authors define ℓ-blocks for the symmetric group S_n, where $\ell \geq 2$ is an arbitrary integer. These blocks can be related to two different definitions of $(\mathcal{X},\mathcal{Y})$-sections, which give distinct $(\mathcal{X},\mathcal{Y})$-blocks for the centralizers, but the same $(\mathcal{X},\mathcal{Y})$-blocks for S_n. The authors show that the ℓ-blocks of S_n thus defined satisfy an ℓ-analogue of the Nakayama Conjecture, and that they satisfy the Second Main Theorem property.

In all this section, we consider the symmetric group S_n for some $n \geq 1$, and let $\ell \geq 2$ be an integer. We let π be the set of primes dividing ℓ.

1.3.1 Sections, blocks

Recall each element of a symmetric group can be written uniquely as a product of disjoint cycles.

Definition 1.37. *An element of a symmetric group is said to be:*
• *an ℓ-**cycle element** if all its non-trivial cycles have length divisible by ℓ;*
• *ℓ-**regular** if it has no cycle of length divisible by ℓ (and ℓ-**singular** otherwise);*
• *an ℓ-**element** if it is an ℓ-cycle element such that each non-trivial cycle has length dividing a power of ℓ;*
• *π-**regular** if its order is not divisible by any prime in π.*

If two elements $x, y \in S_n$ are disjoint (i.e. each one fixing the points moved by the other), we write $x * y$ for the product xy. We have:

Proposition/Definition 1.38. *Each element $z \in S_n$ has unique factorizations:*

$$z = x * y = rs = sr$$

where x is an ℓ-cycle element, y is ℓ-regular, r is the π-part of x (thus r is an ℓ-element) and s is ℓ-regular. Any element commuting with z commutes with each of x, y, r and s (in particular, these elements all commute with each other). We call x the **ℓ-cycle part** *of z, and r the* **ℓ-part** *of z.*

Two elements of S_n are said to belong to the same *ℓ-cycle section* (respectively *ℓ-section*) of S_n if their ℓ-cycle parts (respectively ℓ-parts) are conjugate in S_n. Then each ℓ-section of S_n is a union of ℓ-cycle sections. The ℓ-section of 1 is the set of ℓ-regular elements of S_n, and is also an ℓ-cycle section.

We now define $(\mathcal{X}, \mathcal{Y})$-blocks for S_n. We let $\mathcal{X}$ be the set of ℓ-elements of S_n. For each $r \in \mathcal{X}$, the centralizer $C = C_{S_n}(r)$ of r can be written $C = C_0 \times C_1$, where C_1 is the pointwise stabilizer of the points moved by r (and thus a symmetric group), and C_0 is the pointwise stabilizer of the points fixed by r. Then $r \in C_0$. We let $\mathcal{Y}(r)$ be the set of elements $s_0 * s_1 \in C_0 \times C_1$ such that s_0 is a π'-element of C_0 and s_1 is an ℓ-regular element of C_1.

We let $\mathcal{X}'$ be the set of ℓ-cycle elements of S_n. For each $x \in \mathcal{X}'$, we let $\mathcal{Y}'(x)$ be the set of ℓ-regular elements of S_n which are disjoint from x.

Then the $\mathcal{Y}$-sections of ℓ-elements are the ℓ-sections of S_n, and the $\mathcal{Y}'$-sections of ℓ-cycle elements are the ℓ-cycle sections of S_n.

Note that $\mathcal{Y}(1)$ and $\mathcal{Y}'(1)$ are both equal to the set of ℓ-regular elements of S_n, so that the $(\mathcal{X}, \mathcal{Y})$-blocks and the $(\mathcal{X}', \mathcal{Y}')$-blocks of S_n coincide, and could be defined by linking across ℓ-regular elements. We call them *ℓ-blocks* of S_n. An $(\mathcal{X}, \mathcal{Y})$-block of the centralizer C of an ℓ-element is called an *ℓ-block* of C, and an $(\mathcal{X}', \mathcal{Y}')$-block of the centralizer C of an ℓ-cycle element is called an *ℓ-cycle block* of C.

1.3.2 Nakayama Conjecture

The irreducible (complex) characters and conjugacy classes of S_n are labelled canonically by the partitions of n. For λ a partition of n (which we write $\lambda \vdash n$), we let χ_λ be the irreducible character of S_n labelled by λ.

Given $\lambda \vdash n$ and any integer d, we obtain the *d-core* γ_λ of λ by removing from (the Young diagram of) λ all the d-hooks. We have the following (cf James and Kerber [17])

Theorem 1.39. *(Nakayama Conjecture) Let p be a prime. Then two irreducible characters χ_λ and χ_μ of S_n belong to the same p-block if and only if λ and μ have the same p-core.*

It is proved in [18] that the ℓ-blocks of S_n satisfy an ℓ-anologue of this theorem. We will give a sketch of the proof of this. The basis to start is the following (cf [17])

Theorem 1.40. *(Murnaghan-Nakayama Formula) Let $n = l + m$, and let $\rho * \sigma \in S_n$, where ρ is an m-cycle and σ is a permutation on the remaining l symbols. Let λ be a partition of n. Then*

$$\chi_\lambda(\rho * \sigma) = \sum_\mu (-1)^{L_{\lambda\mu}} \chi_\mu(\sigma),$$

where μ runs over all partitions μ of l which can be obtained from λ by deleting an m-hook, and $L_{\lambda\mu}$ is the leg length of the deleted hook.

Now take an ℓ-cycle element x of S_n. Suppose that the cycle type of x is $(\ell h_1, \ldots, \ell h_t)$ (where we omit cycles of length 1). We call $\rho = (h_1, \ldots, h_t)$ the *ℓ-type* of x (and of $x*y$ if y is ℓ-regular). Then, if y is any element disjoint from x and λ is a partition of n, repeated use of the Murnaghan-Nakayama Formula gives

$$\chi_\lambda(x * y) = \sum_{\mu \vdash n - v\ell} m^\rho_{\lambda\mu} \chi_\mu(y), \qquad (\dagger)$$

where $v = h_1 + \cdots + h_t$, and the coefficients $m^\rho_{\lambda\mu}$ are integers. The coefficient $m^\rho_{\lambda\mu}$ corresponds in some way to the set of paths in the lattice of partitions obtained by removing first an ℓh_1-hook, then an ℓh_2-hook, and so on, to obtain μ from λ. Now one can show that the removal of an ℓh-hook can be obtained by removing a sequence of h ℓ-hooks, and this implies that, if $m^\rho_{\lambda\mu} \neq 0$, then λ and μ have the same ℓ-core.

Now, each ℓ-cycle section is characterized by the (common) ℓ-type of its elements, and ($\dagger$) allows us to compute the contribution of two irreducible characters of S_n across any ℓ-cycle section. Then an induction argument shows that, if χ_λ and $\chi_{\lambda'}$ are (directly) linked across any ℓ-cycle section, then λ and λ' have the same ℓ-core. This proves one implication of the Nakayama Conjecture:

Theorem 1.41. *If $\chi_\lambda, \chi_{\lambda'} \in Irr(S_n)$ belong to the same ℓ-block of S_n, then λ and λ' have the same ℓ-core.*

This implies that, for a given ℓ-core γ, the set $B_\gamma = \{\chi_\lambda \in \mathrm{Irr}(S_n) \mid \gamma_\lambda = \gamma\}$ is a union of ℓ-blocks. We want to prove that B_γ is in fact a single ℓ-block. For this, we refer to Maróti [19]. For $\chi_\lambda \in B_\gamma$, we let w be the number of ℓ-hooks which must be removed from λ to get γ_λ. Then w is independant on $\chi_\lambda \in B_\gamma$, and is called the *$\ell$-weight* of B_γ (and of any λ such that $\chi_\lambda \in B_\gamma$).

Note that, if $w = 0$, then B_γ consists of a single character χ_λ (and $\lambda = \gamma$), and thus is trivially a single ℓ-block of S_n. We therefore fix an ℓ-core γ and suppose that $w \geq 1$. The main ingredient of the proof is that there exists a *generalized perfect isometry* (this will be defined in Part 2) between B_γ and the set of irreducible characters of the wreath product $\mathbf{Z}_\ell \wr S_w$. We will give some details about the wreath product and the generalized perfect isometry in Part 3. At this point, we only want to give some ideas of how the proof goes.

The first point is that there exists a canonically defined bijection Q between B_γ and $\mathrm{Irr}(\mathbf{Z}_\ell \wr S_w)$ (we write Q for "quotient", as will be seen in Part 3). The second ingredient is that there is an analogue of the Murnaghan-Nakayama Formula in $\mathbf{Z}_\ell \wr S_w$. As in the case of symmetric groups, this allows us to relate the contribution of two characters across some set of elements of $\mathbf{Z}_\ell \wr S_w$ to the contributions of characters of a smaller wreath product across a smaller set of elements, and thus to build an induction argument. Comparing the results in S_n and $\mathbf{Z}_\ell \wr S_w$, Maróti then shows that the contribution of two characters χ_λ and χ_μ in B_γ across ℓ-regular elements is equal, up to a sign, to the contribution of $Q(\chi_\lambda)$ and $Q(\chi_\mu)$ across the so-called *regular elements* of $\mathbf{Z}_\ell \wr S_w$. This gives us a very powerful bridge between $\mathbf{Z}_\ell \wr S_w$ and S_n. To conclude the proof, one proves that, in $\mathbf{Z}_\ell \wr S_w$, every irreducible character is directly linked across regular elements to the trivial character. Using our bridge, this implies that any two characters in B_γ are both directly linked across ℓ-regular elements to a third character (whose image under Q is the trivial character of $\mathbf{Z}_\ell \wr S_w$), and thus belong to the same ℓ-block of S_n. Finally, we have:

Theorem 1.42. *(Generalized Nakayama Conjecture) Two irreducible characters χ_λ and $\chi_{\lambda'}$ of S_n belong to the same ℓ-block if and only if λ and λ' have the same ℓ-core.*

1.3.3 Second Main Theorem property

It is shown in [18] that the ℓ-blocks of S_n satisfy the Second Main Theorem property. It is only at this point, when considering centralizers, that the difference between $(\mathcal{X}, \mathcal{Y})$-blocks and $(\mathcal{X}', \mathcal{Y}')$-blocks becomes apparent. We start with $(\mathcal{X}', \mathcal{Y}')$-blocks, which are somewhat easier to manipulate. This is why only the case of $(\mathcal{X}, \mathcal{Y})$-blocks is presented in [18].

Take $x \in \mathcal{X}'$. With the notations we used above, if x has ℓ-type ρ, where ρ is a partition of v, then we can see x as an element of $S_{\ell v}$. Then the centralizer of x is $C_{S_n}(x) = C_0 \times C_1$, where $C_0 = C_{S_{\ell v}}(x)$ and $C_1 = S_{n-\ell v}$, and $\mathcal{Y}'(x)$ is the set of ℓ-regular elements of C_1. If we take $\mu_0 \otimes \mu_1$ and $\nu_0 \otimes \nu_1$

in $\mathrm{Irr}(C_0 \times C_1)$, it is then easy to see, since x is central in C_0, that $\mu_0 \otimes \mu_1$ and $\nu_0 \otimes \nu_1$ are directly linked across $x\mathcal{Y}'(x)$ if and only if μ_1 and ν_1 are directly linked across ℓ-regular elements of $C_1 = S_{n-\ell v}$. Thus the $(\mathcal{X}', \mathcal{Y}')$-blocks of $C_{S_n}(x)$ are the $\mathrm{Irr}(C_0) \otimes b$'s, where b runs through the ℓ-blocks of C_1.

Now, for any $(\mathcal{X}', \mathcal{Y}')$-block B of S_n, labelled by the ℓ-core γ, we set $\beta(x, B) = \mathrm{Irr}(C_0) \otimes b$, where b is the ℓ-block of C_1 labelled by γ, and, for any $\chi \in B$ and $\mu_0 \otimes \mu_1 \in \mathrm{Irr}(C_0) \otimes b$, we set

$$c_{\chi,\mu_0\otimes\mu_1} = \begin{cases} m^{\rho}_{\chi\mu_1} \text{ if } \mu_0 = 1_{C_0} \\ 0 \text{ otherwise} \end{cases},$$

where the $m^{\rho}_{\chi\mu_1}$ are the coefficients appearing in formula (†). It is then clear, by formula (†), that, for each $x \in \mathcal{X}'$, the $\beta(x, B)$'s and $c_{\chi,\mu_0\otimes\mu_1}$'s thus defined satisfy the hypotheses of Proposition 1.33. Hence the $(\mathcal{X}', \mathcal{Y}')$-blocks of S_n satisfy the Second Main Theorem property.

The case of $(\mathcal{X}, \mathcal{Y})$-blocks is a bit more tricky. However, the main ingredients are the same. It can be shown that, for $r \in \mathcal{X}$, writing $C_{S_n}(r) = C_0 \times C_1$ as indicated when we defined $(\mathcal{X}, \mathcal{Y})$-sections, each ℓ-block of $C_{S_n}(r)$ has the form $\mathrm{Irr}(C_0) \times b$ for some ℓ-block b of C_1. The difference with the previous case is that, given $s_0 * s_1 \in \mathcal{Y}(r)$, we can consider formula (†) applied to $x = rs_0$ and $y = s_1$, but the coefficients appearing on the right side may depend on s_0 (since the ℓ-types of r and rs_0 may differ). However, even in this case, formula (†) can be used to show that, if an $(\mathcal{X}, \mathcal{Y})$-block B of S_n dominates the $(\mathcal{X}, \mathcal{Y})$-block $\mathrm{Irr}(C_0) \otimes b$ of $C_{S_n}(r)$, then B and b must be labelled by the same ℓ-core (cf [18]). Finally, we have

Theorem 1.43. *The $(\mathcal{X}, \mathcal{Y})$-blocks and the $(\mathcal{X}', \mathcal{Y}')$-blocks of S_n satisfy the Second Main Theorem property.*

Part 2

Generalized Perfect Isometries

In [18], the authors give a definition of *generalized perfect isometry* between (unions of) generalized blocks of two groups, and use it in the case of symmetric groups to prove that the ℓ-blocks of S_n satisfy an analogue of the Nakayama Conjecture (cf Part 1). In this part, we investigate some families of groups of Lie rank one (namely $SU(3, q^2)$, $Sz(q)$ and $Re(q)$), exhibiting generalized perfect isometries where there are none in the sense given originally by M. Broué, and thus proving a (weaker) analogue of one of Broué's Conjectures in these cases.

2.1 p^k-blocks

We start by observing that some generalized blocks (namely p^k-blocks) can always be defined in a finite group. We take G any finite group, p a prime, and $k \geq 1$ an integer. An element of G is said to be p^k*-regular* if its order is not divisible by p^k, and p^k*-singular* otherwise. We let $\mathcal{C}_k$ be the set of p^k-regular elements of G. Then $\mathcal{C}_k$ is a closed set of conjugacy classes of G, and, using the definitions of the previous part, we can define the $\mathcal{C}_k$-blocks of G. We call them the p^k*-blocks* of G.

Take $g =\in G$. We define the p^k*-section* of g to be its p-section if g is p^k-singular, and the set $\mathcal{C}_k$ of p^k-regular elements of G if g is p^k-regular. Then each p^k-section of G is a union a p-section. This is clear for p^k-singular sections, for they are already p-sections, and $\mathcal{C}_k$ is the union of the sections of elements of G whose p-part has order 1 or p.

Now the p-blocks of G satisfy the Second Main Theorem property (by Brauer's Second Main Theorem), so that irreducible characters in distinct p-blocks of G are orthogonal across each p-section, and thus across each p^k-section. In particular, if two irreducible characters of G are directly $\mathcal{C}_k$-linked, then they belong to the same p-block of G. This proves that any p-block of G is a union of p^k-blocks.

In the families of groups we will study in this part, we will be interested in p^2-blocks, where p is the defining characteristic.

2.2 (Generalized) Perfect Isometries

We find in [18] a definition of *perfect isometry* which generalizes the context from p-blocks to $\mathcal{C}$ -blocks, but which, when specialized again to p-blocks is a bit weaker than the definition of M. Broué (cf [2]).

Broué's definition goes as follows: we let G and H be two finite groups, $(K, \mathcal{O}, k)$ be a p-modular system large enough for both G and H, e and f be central idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively, and we set $A = \mathcal{O}Ge$ and $B = \mathcal{O}Hf$. To any generalized character μ of $G \times H$ (written $\mu \in \mathbf{Z}\mathrm{Irr}(K(G \times H))$), we can associate a linear map $I_\mu \colon \mathbf{Z}\mathrm{Irr}(KH) \longrightarrow \mathbf{Z}\mathrm{Irr}(KG)$ by letting, for $\zeta \in \mathrm{Irr}(KH)$ and $g \in G$,

$$I_\mu(\zeta)(g) = \frac{1}{|H|} \sum_{h \in H} \mu(g, h^{-1})\zeta(h).$$

Definition 2.1. *(cf [2]) A generalized character μ of $G \times H$, is* **perfect** *if the following two conditions are satisfied:*
(per. 1) For all $g \in G$ and $h \in H$, $|C_G(g)|_p$ and $|C_H(h)|_p$ divide $\mu(g, h)$.
(per. 2) If $\mu(g, h) \neq 0$, then g and h are either both p-regular, or both p-singular.

If furthermore the map I_μ defined by μ induces a bijective isometry between $\mathbf{Z}Irr(KB)$ and $\mathbf{Z}Irr(KA)$, then I_μ is said to be a **perfect isometry** *between B and A, and B and A are said to be* **perfectly isometric***.*

We then have the following (cf [2]):

Theorem 2.2. *Suppose that, with the notations above, B and A are perfectly isometric via I_μ. Then:*
(i) I_μ defines a bijection between primitive idempotents of the centers $Z(KHf)$ and $Z(KGe)$, which in turn induces an algebra isomorphism between $Z(\mathcal{O}Hf)$ and $Z(\mathcal{O}Ge)$.
(ii) I_μ induces a bijection between the blocks of H and G associated to f and e respectively, which preserves the defect and number of ordinary irreducible characters, the number of irreducible Brauer characters, the height of ordinary irreducible characters, and the elementary divisors of the Cartan matrix.

In particular, under the same hypotheses, there exist a bijection $I\colon \mathrm{Irr}(B) \longrightarrow \mathrm{Irr}(A)$, and signs $\{\varepsilon(\zeta),\, \zeta \in \mathrm{Irr}(B)\}$ such that, for all $\zeta \in \mathrm{Irr}(B)$, $I_\mu(\zeta) = \varepsilon(\zeta)I(\zeta)$.

We now turn to the definition introduced in [18]. If G and H are finite groups, $\mathcal{C}$ and $\mathcal{D}$ are closed unions of conjugacy classes of G and H respectively, and if b (resp. b') is a union of $\mathcal{C}$-blocks of G (resp. $\mathcal{D}$-blocks of H), then we say that there is a *generalized perfect isometry* between b and b' (with respect to $\mathcal{C}$ and $\mathcal{D}$) if there exists a bijection with signs between b and b', which furthermore preserves contributions; i.e. there exists a bijection $I\colon b \longmapsto b'$ such that, for each $\chi \in b$, there is a sign $\varepsilon(\chi)$, and such that

$$\forall \chi,\, \psi \in b,\ < I(\chi),\, I(\psi) >_{\mathcal{D}} = < \varepsilon(\chi)\chi,\, \varepsilon(\psi)\psi >_{\mathcal{C}} .$$

Note that this is equivalent to $< I(\chi),\, I(\psi) >_{\mathcal{D}'} = < \varepsilon(\chi)\chi,\, \varepsilon(\psi)\psi >_{\mathcal{C}'}$, where $\mathcal{C}' = G \setminus \mathcal{C}$ and $\mathcal{D}' = H \setminus \mathcal{D}$.

We define $\mathcal{R}(\mathcal{C}, b)$ to be the $\mathbf{Z}$-submodule of the space of complex class functions of G generated by $\{\chi^{\mathcal{C}} \mid \chi \in b\}$, and $\mathcal{P}(\mathcal{C}, b)$ to be the $\mathbf{Z}$-submodule of $\mathcal{R}(\mathcal{C}, b)$ consisting of generalized characters. The fact that $\mathcal{C}$ is closed implies (using Galois theory) that the modules $\mathcal{R}(\mathcal{C}, b)$ and $\mathcal{P}(\mathcal{C}, b)$ have the same $\mathbf{Z}$-rank, and that this rank is the number of conjugacy classes in $\mathcal{C}$(cf [24]). Given a $\mathbf{Z}$-basis $\{\varphi_1,\, \ldots,\, \varphi_s\}$ for $\mathcal{P}(\mathcal{C}, b)$, we let $C(b)$ be the $s \times s$ matrix with (i, j)-entry $< \varphi_i,\, \varphi_j >_{\mathcal{C}}$, and call $C(b)$ the *Cartan matrix* of b. A different choice of $\mathbf{Z}$-basis for $\mathcal{P}(\mathcal{C}, b)$ leads to a Cartan matrix with the same elementary divisors. We define similarly $\mathcal{R}(\mathcal{D}, b')$, $\mathcal{P}(\mathcal{D}, b')$ and $C(b')$. We then have the following:

Theorem 2.3. *(Proposition 1.4 in [18]) With the above notations, if there is a generalized perfect isometry between b and b' with respect to $\mathcal{C}$ and $\mathcal{D}$, then the Abelian groups $\mathcal{R}(\mathcal{C}, b)$ and $\mathcal{R}(\mathcal{D}, b')$ are isomorphic via an isomorphism which restricts to an isomorphism between $\mathcal{P}(\mathcal{C}, b)$ and $\mathcal{P}(\mathcal{D}, b')$. With suitable choice of $\mathbf{Z}$-bases, the Cartan matrices $C(b)$ and $C(b')$ are equal.*

In particular, we see that, if $\mathcal{C}$ (resp. $\mathcal{D}$) is the set of p-regular elements of G (resp. H), then such a generalized perfect isometry induces a bijection between the p-blocks in b and b', which preserves the numbers of ordinary irreducible characters and irreducible Brauer characters, and the elementary divisors of the Cartan matrix. However, with this more general definition, one doesn't get the algebra isomorphism of Theorem 2.2 (i).

One of Broué's conjectures (which is just the shadow, at the level of characters, of much deeper equivalences conjectured by Broué) states that, if G is a finite group with Abelian Sylow p-subgroup U, and if $B = N_G(U)$ is the normalizer of U in G, then the principal p-blocks of B and G are perfectly isometric (in Broué's sense).

This conjecture doesn't apply to the three families of groups of Lie rank one we will consider when p is the defining characteristic, for then the Sylow p-subgroups are not Abelian. Furthermore, it is shown in Cliff [8] that, if G is a Suzuki group, then there isn't any perfect isometry between the principal 2-blocks of B and G. This was the starting point of this work. In this part, we show that, in the case of Suzuki groups and in two other (and quite similar) families, there is a generalized perfect isometry between the principal p-blocks of B and G, with respect to p^2-regular (or p^2-singular) elements.

Section 2.3 deals with special unitary groups $SU(3,q^2)$, section 2.4 with Suzuki groups $Sz(q)$, and section 2.5 with Ree groups (of type G_2) $Re(q)$.

In two of the families of groups we are studying, we will use two theorems concerning p-blocks. The first, due to Green, states that any defect group of a p-block of a (finite) group G is the intersection of two Sylow p-subgroups of G (see e.g. [20], Corollary (4.21)). An immediate consequence of this is that, if the Sylow p-subgroups of G have trivial intersection, then any p-block of G has either maximal defect or defect 0. The second result, proved by Fong in the case of p-solvable groups (see e.g. Isaacs [16], Problem (4.9)), but which still holds in general, states that, if a finite group G is such that $C_G(O_p(G)) \leq O_p(G)$, where $O_p(G)$ is the largest normal p-subgroup of G, then G has exactly one p-block.

2.3 Special Unitary Groups

In this section, we denote by $G = SU(3,q^2)$ the 3-dimensional special unitary group on $\mathbf{F}_{q^2}$, where $q = p^n$ for some prime p and $n \geq 1$. We write $d = \gcd(3, q+1)$. Then the center $Z(G)$ of G is cyclic of order d. Note that, if $p = 3$, then $d = 1$. Thus, whatever the value of p, the integers p and d are coprime, and their least common multiple is pd. The order of G is $|G| = q^3(q+1)^2(q-1)(q^2-q+1)$. We consider U a Sylow p-subgroup of G and its normalizer $B = N_G(U)$, which is a semi-direct product of U with a cyclic group H of order q^2-1. We have $|U| = q^3$ and $|B| = q^3(q^2-1)$. For a complete description of G and B, their conjugacy classes and ordinary and modular characters, we refer to Geck [13], whose notations are now in force.

2.3.1 Conjugacy classes

The tables of conjugacy classes of G and B are taken from Geck [13]. Only the order of their elements have been added (these can be easily computed using the canonical forms given in [13]).

The group G has $q^2+q+1+d^2$ conjugacy classes, parametrized as follows:

name	parameters	length	order
$C_1^{(k)}$	$0 \le k \le d-1$	1	$1\,(k=0)$ or d
$C_2^{(k)}$	$0 \le k \le d-1$	$(q^3+1)(q-1)$	$p\,(k=0)$ or pd
$C_3^{(k,l)}$	$0 \le k, l \le d-1$	$\frac{1}{d}q(q^2-1)(q^3+1)$	$2p\,(k=0)$ or $2dp$
$C_4^{(k)}$	$0 \le k \le q$ $\frac{q+1}{d} \nmid k$	$q^2(q^2-q+1)$	$\frac{q+1}{\gcd(k,\, q+1)}$
$C_5^{(k)}$	$0 \le k \le q$ $\frac{q+1}{d} \nmid k$	$q^2(q-1)(q^3+1)$	$\frac{2(q+1)}{\gcd(k,\, q+1)}$
$C_6^{(k,l,m)}$	$0 \le k < l < m \le q$ $k+l+m \equiv 0 (\mathrm{mod}\, q+1)$	$q^3(q-1)(q^2-q+1)$	dividing $q+1$
$C_7^{(k)}$	$0 \le k \le q^2-2$ $k \not\equiv 0 (\mathrm{mod}\, q-1)$; if $k_1 \equiv -qk (\mathrm{mod}\, q^2-1)$ then $C_7^{(k)} = C_7^{(k_1)}$	$q^3(q^3+1)$	$\frac{(q+1)(q-1)}{\gcd((q+1)(q-1),\, k)}$
$C_8^{(k)}$	$0 \le k \le q^2-q$ $\frac{q^2-q+1}{d} \nmid k$; if $k_1 \equiv -kq$ or $k_2 \equiv kq^2$ $(\mathrm{mod}\, q^2-q+1)$, then $C_8^{(k)} = C_8^{(k_1)} = C_8^{(k_2)}$	$q^3(q+1)^2(q-1)$	$\frac{q^2-q+1}{\gcd(q^2-q+1,\, k)}$

B has q^2+q+d conjugacy classes, parametrized as follows:

name	fusion into G	parameters	length
$B_1^{(k)}$	$C_1^{(k)}$	$0 \le k \le d-1$	1
$B_2^{(k)}$	$C_2^{(k)}$	$0 \le k \le d-1$	$q-1$
$B_3^{(k,l)}$	$C_3^{(k,l)}$	$0 \le k, l \le d-1$	$\frac{1}{d}q(q^2-1)$
$B_4^{(k)}$	$C_4^{(k)}$	$0 \le k \le q\,,\ \frac{q+1}{d} \not| k$	q^2
$B_5^{(k)}$	$C_5^{(k)}$	$0 \le k \le q\,,\ \frac{q+1}{d} \not| k$	$q^2(q-1)$
$B_6^{(k)}$	$C_7^{(k)}$	$0 \le k \le q^2-2\,,\ q-1 \not| k$	q^3

We are only interested in elements of order divisible by p^2. We see from the tables above that, unless $p=2$, there is no such element in G.

Remark: this can be seen without actually computing the orders of the elements of G. Take any p-element g in G, of order p^k say. Then its eigenvalues, which lie in some finite extension of $\mathbf{F}_{q^2}$, $\mathbf{F}_{p^l}$ say, must be p^k-th roots of 1.

But the only p^k-th root of 1 in $\mathbf{F}_{p^l}$ (in which each x is a root of $x^{p^l} - x$) is 1 itself. Thus the characteristic polynomial of g is $(X-1)^3$, and thus $(g - Id)^3 = 0$. Then, if $p \geq 3$, then $g^p - Id = (g - Id)^p = 0$, so that $g^p = Id$. Hence, if $p \geq 3$, then any non-trivial p-element of G has order p.

As a consequence, if $p \neq 2$, then every element of G is p^2-regular, so that the scalar product on p^2-regular elements is just the ordinary scalar product, and there exists a generalized perfect isometry between the principal p-blocks B_0 and b_0 of G and B with respect to p^2-regular elements if and only if there exists a bijection between B_0 and b_0 (any bijection will give a generalized perfect isometry).

If $p = 2$, we will only consider the values of irreducible characters on elements of order divisible by 4, that is, we will consider

$$\mathcal{C} = \{g \in G \,|\, 4|o(g)\} = \bigcup_{0 \leq k,\, l \leq d-1} C_3^{(k,l)}$$

and

$$\mathcal{D} = \{h \in B \,|\, 4|o(h)\} = \bigcup_{0 \leq k,\, l \leq d-1} B_3^{(k,l)}.$$

Both $\mathcal{C}$ and $\mathcal{D}$ are closed unions of conjugacy classes (of G and B respectively).

2.3.2 Irreducible characters, principal blocks

The irreducible characters of G can be listed as follows (The index used in the notation for a character indicates its degree; characters of the same degree are then parametrized by parameters u and v. Note that for characters of degree q^3+1 and $(q+1)^2(q-1)$, different choices of parameters can yield the same character (cf [13]); we therefore indicate the number of characters in these families.) :

$$\chi_1,\, \chi_{q^2-q},\, \chi_{q^3},\, \{\chi^{(u)}_{q^2-q+1},\, \chi^{(u)}_{q(q^2-q+1)} \,|\, 1 \leq u \leq q\},$$

$$\{\chi^{(u,v)}_{(q-1)(q^2-q+1)} \,|\, 1 \leq u \leq (q+1)/3,\, u < v < 2(q+1)/3\},$$

$$\{\chi^{(u)}_{q^3+1} \,|\, 1 \leq u \leq q^2-1,\, q-1 \nmid u\} \text{ (} (q+1)(q-2)/2 \text{ characters)},$$

$$\{\chi^{(u)}_{(q+1)^2(q-1)} \,|\, 0 \leq u \leq q^2-q,\, q^2-q+1 \nmid u\} \text{ (} (q^2-q+1-d)/3 \text{ characters)},$$

and, if $d = 3$, we have in addition

$$\{\chi^{(u)}_{(q-1)(q^2-q+1)/3} \,|\, 0 \leq u \leq 2\} \text{ and } \{\chi^{(u,v)}_{(q+1)^2(q-1)/3} \,|\, 0 \leq u \leq 2,\, v = 1, 2\}.$$

The irreducible characters of B can be listed as follows:

$$\{\theta_1^{(u)} \,|\, 0 \le u \le q^2-2\},\ \{\theta_{q(q-1)}^{(u)} \,|\, 0 \le u \le q\},\ \{\theta_{q^2-1}\} \text{ (only if } d=1),$$

$$\text{and } \{\theta_{(q^2-1)/3}^{(u,v)} \,|\, 0 \le u,\, v \le 2\} \text{ (only if } d=3).$$

We now compute the principal p-blocks B_0 and b_0 of G and B respectively.

We compute them directly, using the fact that two (complex) irreducible characters belong to the same p-block if and only if the associated central characters are the same when reduced to a field of characteristic p (cf [20], definition (3.1) and Problem (3.3)):

Theorem 2.4. *If N is a finite group, $(K, \mathcal{O}, k)$ a p-modular system large enough for H, $\mathcal{O}$ being the localization of A_K (algebraic integers of K) in $\mathcal{P}$, then $\chi \in Irr(KN)$ belongs to the principal p-block of N if and only if*

$$|C|\frac{\chi(C)}{\chi(1)} \equiv |C| \,(mod\, \mathcal{P})$$

for each p-regular conjugacy class C of N.

Most of the work is done using the following observation: with the notations of the previous theorem, and given C a conjugacy class of N, if $|C|$ is divisible by p, then $|C| \in \mathcal{P}$ (since $\mathcal{P} \cap \mathbf{Z} = p\mathbf{Z}$), so that we only need to check whether $|C|\frac{\chi(C)}{\chi(1)} \in \mathcal{P}$. We distinguish two special cases:

a) If $\chi(C) = 0$, then the property is clearly verified.

b) If $\frac{|C|}{\chi(1)} \in \mathcal{P}$, then, since $\chi(C) \in A_K$ and $\mathcal{P} \lhd A_K$, $|C|\frac{\chi(C)}{\chi(1)} \in \mathcal{P}$.

It turns out that, for any p-regular conjugacy class of length divisible by p and any irreducible character of G or B, one of these two cases apply:

We first consider the group G. The p-regular conjugacy classes of length divisible by p are those of type C_4, C_6, C_7 and C_8. We examine only characters distinct from χ_1 and χ_{q^3} (the first one belonging to B_0 and the second one being of defect 0 and thus not belonging to B_0).

C_8 has length $q^3(q+1)^2(q-1)$, so that case b) applies to characters of degree q^2-q, $(q+1)^2(q-1)$ and $(q+1)^2(q-1)/3$; case a) applies to any other character.

C_7 has length $q^3(q^3+1) = q^3(q+1)(q^2-q+1)$, so that case b) applies to characters of degree q^2-q+1, $q(q^2-q+1)$ and q^3+1; case a) applies to any other character.

C_6 has length $q^3(q-1)(q^2-q+1)$, so that case b) applies to characters of degree q^2-q, q^2-q+1, $q(q^2-q+1)$, $(q-1)(q^2-q+1)$ and $(q-1)(q^2-q+1)/3$; case a) applies to any other character.

C_4 has length $q^2(q^2-q+1)$, so that case b) applies to characters of degree q^2-q+1 and $q(q^2-q+1)$; case a) applies to characters of degree $(q+1)^2(q-1)$ and $(q+1)^2(q-1)/3$; we check the other cases by hand :

$$|C_4|\frac{\chi_{q^2-q}(C_4)}{\chi_{q^2-q}(1)} = \frac{q^2(q^2-q+1)(1-q)}{q^2-q} = -q(q^2-q+1) \in \mathcal{P}$$

$$|C_4|\frac{\chi_{(q-1)(q^2-q+1)}(C_4)}{\chi_{(q-1)(q^2-q+1)}(1)} = \frac{q^2(q^2-q+1)(q-1)\alpha}{(q-1)(q^2-q+1)} = q^2\alpha \in \mathcal{P}\ (\alpha \in A_K)$$

$$|C_4|\frac{\chi_{q^3+1}(C_4)}{\chi_{q^3+1}(1)} = \frac{q^2(q^2-q+1)(q+1)\alpha'}{(q^3+1)} = q^2\alpha' \in \mathcal{P}\ (\alpha' \in A_K)$$

$$|C_4|\frac{\chi_{(q-1)(q^2-q+1)/3}(C_4)}{\chi_{(q-1)(q^2-q+1)/3}(1)} = \frac{q^2(q^2-q+1)(q-1)}{(q-1)(q^2-q+1)/3} = 3q^2 \in \mathcal{P}.$$

Now, in B, the p-regular conjugacy classes of length divisible by p are those of type B_4 and B_6. We have

$$\frac{|B_4|}{\theta_1(1)} = q^2 \in \mathcal{P} \text{ and } \frac{|B_6|}{\theta_1(1)} = q^3 \in \mathcal{P}$$

so that case b) applie to these two cases. Furthermore,

$$|B_4|\frac{\theta_{q(q-1)}(B_4)}{\theta_{q(q-1)}(1)} = \frac{q^2(1-q)\alpha''}{q(q-1)} = -q\alpha'' \in \mathcal{P}\ (\alpha'' \in A_K)$$

and, in every other cases, case a) applies.

In order to compute the principal blocks B_0 and b_0, we thus only have to study the values of irreducible characters on p-regular conjugacy classes of G and B of length not divisible by p. These are the $C_1^{(k)}$'s and the $B_1^{(k)}$'s respectively, and they have length 1 (since lie in the center). For C one of these classes, and ψ an irreducible character of the corresponding group, we thus have to check whether $\frac{\psi(C)}{\psi(1)}$ is congruent to 1 modulo $\mathcal{P}$. But, since C lies in the center, which has order d, $\frac{\psi(C)}{\psi(1)}$ is a d-th root of 1. And it can be shown that, if ε is a p'-root of 1, then $\mathbf{Z}[\varepsilon] \cap \mathcal{P} = p\mathbf{Z}[\varepsilon]$. In particular, we get that $\frac{\psi(C)}{\psi(1)} - 1 \in \mathcal{P}$ if and only if $\frac{\psi(C)}{\psi(1)} = 1$.

Principal block of B

We see that, for fixed degree, the value of $\theta \in \mathrm{Irr}(B)$ on $B_1^{(k)}$ only depends on the parameter u in θ (and on k); writing $\beta = e^{2i\pi/d}$, we get, in all the cases,

$$\theta_u \in b_0 \Longleftrightarrow (\beta^{uk} = 1 \text{ for all } 0 \leq k \leq d-1).$$

This is always true if $d = 1$ (for then $\beta = 1$), so that $b_0 = \mathrm{Irr}(B)$ and $|b_0| = q^2 + q + 1$.
If $d = 3$, then $\beta = e^{2i\pi/3}$, and thus

$$\begin{aligned} \theta_u \in b_0 &\iff \beta^0 = \beta^u = \beta^{2u} = 1 \\ &\iff \beta^u = 1 \\ &\iff u \equiv 0 \ (\mathrm{mod}\ 3) \end{aligned}$$

Hence

$$\begin{aligned} b_0 &= \{\theta_1^{(u)} \,|\, 0 \le u \le q^2-2,\, 3|u\} \cup \{\theta_{q(q-1)}^{(u)} \,|\, 0 \le u \le q,\, 3|u\} \\ &\quad \cup \{\theta_{(q^2-1)/3}^{(u,v)} \,|\, 0 \le u,\, v \le 2,\, 3|u\} \\ &= \{\theta_1^{(u)} \,|\, 0 \le u \le q^2-2,\, 3|u\} \cup \{\theta_{q(q-1)}^{(u)} \,|\, 0 \le u \le q,\, 3|u\} \\ &\quad \cup \{\theta_{(q^2-1)/3}^{(0,v)} \,|\, 0 \le v \le 2\} \end{aligned}$$

Now $d = \gcd(3,\, q+1) = 3$, so that $q \equiv -1 (\mathrm{mod}\, 3)$, $q^2 \equiv 1 (\mathrm{mod}\, 3)$ and $q^2 - 2 \equiv -1 (\mathrm{mod}\, 3)$, and thus $|b_0| = \frac{q^2-1}{3} + \frac{q+1}{3} + 3 = \frac{q(q+1)}{3} + 3$.

Principal block of G

Similarly, we must check, for $\chi \in \mathrm{Irr}(G) \setminus \{\chi_{q^3}\}$ and $0 \le k \le d-1$, whether $\frac{\chi(C_1^{(k)})}{\chi(1)} = 1$. We let again $\beta = e^{2i\pi/d}$.
If $d = 1$, then $\beta = 1$, and this is true for all $\chi \in \mathrm{Irr}(G) \setminus \{\chi_{q^3}\}$, so that $B_0 = \mathrm{Irr}(G) \setminus \{\chi_{q^3}\}$ and $|B_0| = q^2 + q + 2 - 1 = q^2 + q + 1$.
If $d = 3$, then $\beta = e^{2i\pi/3}$. The result is true for χ_1, χ_{q^2-q} and $\chi \in \{\chi^{(u)}_{(q-1)(q^2-q+1)/3} \,|\, 0 \le u \le 2\}$, whose values on the $C_1^{(k)}$'s do not depend on d.

For $\chi_u \in \mathrm{Irr}(G) \setminus \Big(\{\chi_1,\, \chi_{q^3},\, \chi_{q^2-q}\} \cup \{\chi^{(u,v)}_{(q-1)(q^2-q+1)}\} \cup \{\chi^{(u,v)}_{(q+1)^2(q-1)/3}\}$ $\cup \{\chi^{(u)}_{(q-1)(q^2-q+1)/3}\}\Big)$, we have

$$\begin{aligned} \chi_u \in B_0 &\iff \beta^{uk} = 1 \ \text{ for all } k = 0,\, 1,\, 2 \\ &\iff 3|u. \end{aligned}$$

Finally

$$\begin{aligned} \chi^{(u,v)}_{(q-1)(q^2-q+1)} \in B_0 &\iff \beta^{(u+v)k} = 1 \ \text{ for all } k = 0,\, 1,\, 2 \\ &\iff 3|u+v \end{aligned}$$

and

$$\begin{aligned} \chi^{(u,v)}_{(q+1)^2(q-1)/3} \in B_0 &\iff \beta^{vk} = 1 \ \text{ for all } k = 0,\, 1,\, 2 \\ &\iff 3|v, \end{aligned}$$

which is excluded.

Hence

$$B_0 = \{\chi_1,\, \chi_{q^2-q}\} \cup \{\chi^{(u)}_{q^2-q+1},\, \chi^{(u)}_{q(q^2-q+1)} \,|\, 1 \le u \le q,\, 3|u\} \cup \{\chi^{(u,v)}_{(q-1)(q^2-q+1)} \,|\, 1 \le u \le (q{+}1)/3,\, u < v < 2(q{+}1)/3,\, 3|u{+}v\} \cup \{\chi^{(u)}_{q^3+1} \,|\, 1 \le u \le q^2{-}1\,(\ldots),\, 3|u\} \cup \{\chi^{(u)}_{(q+1)^2(q-1)} \,|\, 0 \le u \le q^2-q\,(\ldots),\, 3|u\} \cup \{\chi^{(u)}_{(q-1)(q^2-q+1)/3} \,|\, 0 \le u \le 2\}$$

and it can be verified that, in this case, $|B_0| = 2 + 2(\frac{q+1}{3} - 1) + \frac{(q+1)(q-2)}{18} + \frac{(q+1)(q-2)}{6} + \frac{q^2-q-2}{9} + 3 = q\frac{q+1}{3} + 3$.

We thus notice that $|b_0| = |B_0|$, independantly of d. In particular, if $p \neq 2$, then there is a generalized perfect isometry between B_0 and b_0 with respect to p^2-regular elements. In the rest of this section, we will therefore only consider the case $p = 2$.

2.3.3 A generalized perfect isometry

We start by observing that the number of conjugacy classes of elements of order divisible by 4 is the same in G and B. Moreover, if $x \in B$ is such an element, then $|C_G(x)| = |C_B(x)| = dq^2$. As a consequence, if we can find a bijection with signs between B_0 and b_0 which furthermore preserves the values of characters on 4-singular elements, then it will also preserve contributions, and thus will be a generalized perfect isometry between B_0 and b_0 with respect to 4-singular elements (or 4-regular elements).

If $d = 1$, we have the following fragment of the character table of B:

	$\theta^{(u)}_{q(q-1)}$, $0 \le u \le q$	$\theta^{(u)}_1$, $0 \le u \le q^2-2$	θ_{q^2-1}
$B^{(0,0)}_3$	0	1	-1

and the corresponding fragment for G:

	χ_{q^2-q}	$\chi^{(u)}_{q(q^2-q+1)}$ $1 \le u \le q$	$\chi^{(u)}_{q^2-q+1}$ $1 \le u \le q$	$\chi^{(u,v)}_{(q-1)(q^2-q+1)}$ $1 \le u \le (q+1)/3$ $u < v < 2(q+1)/3$
$C^{(0,0)}_3$	0	0	1	-1

	$\chi^{(u)}_{q^3+1}$ $1 \le u \le q^2-1$ $q-1 \nmid u$	$\chi^{(u)}_{(q+1)^2(q-1)}$ $0 \le u \le q^2-q$ $q^2-q+1 \nmid u$	χ_1
$C^{(0,0)}_3$	1	-1	1

In view of the remark above, we can see that there is a generalized perfect isometry between b_0 and B_0. Indeed, listing the irreducible characters of b_0 and B_0 in the same order as in the above tables, and relabelling them so that $\mathrm{Irr}(b_0) = \{\Theta_1, \ldots, \Theta_{q^2+q+1}\}$ and $\mathrm{Irr}(B_0) = \{X_1, \ldots, X_{q^2+q+1}\}$, then a generalized perfect isometry between b_0 and B_0 is given by

$$I\colon\ \Theta_i \longmapsto \varepsilon_i X_i\ ,\ i = 1, \ldots q^2 + q + 1,$$

where $\varepsilon_i = -1$ if X_i has degree $(q-1)(q^2-q+1)$, $(q+1)^2(q-1)$ or 1, and $\varepsilon_i = 1$ otherwise.

If $d = 3$, we have the following fragment of the character table of B:

	$\theta^{(u)}_{q(q-1)}$ $0 \le u \le q,\ 3\|u$	$\theta^{(u)}_1$ $0 \le u \le q^2-2,\ 3\|u$	$\theta^{(0,v)}_{(q^2-1)/3}$ $0 \le v \le 2$
$B_3^{(k,l)}$ $0 \le k,l \le 2$	0	1	$q\delta_{vl} - \frac{q+1}{3}$

and the corresponding fragment for G:

	χ_{q^2-q}	$\chi^{(u)}_{q(q^2-q+1)}$ $1 \le u \le q$ $3\|u$	χ_1	$\chi^{(u)}_{q^2-q+1}$ $1 \le u \le q$ $3\|u$	$\chi^{(u,v)}_{(q-1)(q^2-q+1)}$ $1 \le u \le (q+1)/3$ $u < v < 2(q+1)/3$ $3\|(u+v)$
$C_3^{(k,l)}$ $0 \le k,l \le 2$	0	0	1	1	-1

	$\chi^{(u)}_{q^3+1}$ $1 \le u \le q^2-1$ $q-1 \nmid u$ $3\|u$	$\chi^{(u)}_{(q+1)^2(q-1)}$ $0 \le u \le q^2-q$ $q^2-q+1 \nmid u$ $3\|u$	$\chi^{(u)}_{(q-1)(q^2-q+1)/3}$ $0 \le u \le 2$
$C_3^{(k,l)}$ $0 \le k,l \le 2$	1	-1	$q\delta_{lu} - \frac{q+1}{3}$

Then, here again, we can see that b_0 and B_0 are perfectly isometric (with respect to 4-regular elements): listing the irreducible characters of b_0 and B_0 in the same order as in the above tables, and relabelling them so that $\mathrm{Irr}(b_0) = \{\Theta_1, \ldots, \Theta_{q(q+1)/3+3}\}$ and $\mathrm{Irr}(B_0) = \{X_1, \ldots, X_{q(q+1)/3+3}\}$, then we have the following generalized perfect isometry between b_0 and B_0:

$$I\colon\ \Theta_i \longmapsto \varepsilon_i X_i\ ,\ i = 1, \ldots q(q+1)/3 + 3,$$

where $\varepsilon_i = -1$ if X_i has degree $(q-1)(q^2-q+1)$ or $(q+1)^2(q-1)$, and $\varepsilon_i = 1$ otherwise.

2.4 Suzuki Groups

In this section, we let $G = Sz(q)$ be a Suzuki group, where $q = 2^{2n+1}$ for some $n \geq 1$, U a Sylow 2-subgroup of G, and $B = N_G(U)$.

2.4.1 Conjugacy classes

The conjugacy classes of G can be found in Burkhardt [4]. We have $|G| = q^2(q-1)(q^2+1)$, and, writing $r^2 = 2q$ ($r \in \mathbf{N}$), we have representatives for the conjugacy classes of G : t (order 2), f, f^{-1} (order 4) for the elements of even order, and

$$1,\ \{x^a,\ 1 \leq a \leq \frac{q-2}{2}\},\ \{y^b,\ 1 \leq b \leq \frac{q+r}{4}\},\ \{z^c,\ 1 \leq c \leq \frac{q-r}{4}\}$$

for the elements of odd order (dividing $q-1$, $q+r+1$ and $q-r+1$ respectively).

We have $|C_G(f)| = |C_G(f^{-1})| = 2q$.

For the following facts about the structure of B and references for proofs, we refer to Cliff [8].

We have $|U| = q^2$, and $B = N_G(U)$ is a semi-direct product of U by a cyclic group A of order $q-1$. U is a TI (trivial intersection) set in G with respect to B, and B is a Frobenius group of order $q^2(q-1)$ (U is the Frobenius kernel and A is the Frobenius complement). Letting C be the center of U, we have $C \cong U/C \cong (\mathbf{F}_q, +)$, and A acts regularly on the sets of non-identity elements of both C and U/C. We have the following non-trivial conjugacy classes in B :

- a unique class of involutions, consisting of the non-identity elements of C. For τ in this class, $|C_B(\tau)| = q^2$.

- two classes of elements of order 4, $Cl_B(\rho)$ and $Cl_B(\rho^{-1})$, where $\rho \in U \setminus C$. We have $|C_B(\rho)| = |C_B(\rho^{-1})| = 2q$.

- $q-2$ classes of elements of odd order : $\{Cl_B(\pi),\ \pi \in A \setminus \{1\}\}$. For such a π, $C_B(\pi) = A$ has order $q-1$.

We can now compute the conjugacy classes of U. For any involution τ in U, we have $\tau \in C = Z(U)$ and $Cl_U(\tau) = \{\tau\}$, which gives us $q-1$ classes of involutions: $\{\{\tau_j\},\ 2 \leq j \leq q\}$. Using the fact that the centralizer in B of an element of U of order 4 is a 2-group, and thus is equal to its centralizer in U, we see that U must have $2(q-1)$ classes of elements of order 4, each of length $q/2$: $\{Cl_U(f_2), Cl_U({f_2}^{-1}), \ldots, Cl_U(f_q), Cl_U({f_q}^{-1})\}$, where (without loss of generality) ${f_j}^2 = \tau_j$ for all $j = 2, \ldots, q$. This makes a total of $3q-2$ conjugacy classes in U. Furthermore, $C = Z(U)$ is also the commutator subgroup of U, so that U has $[U : U'] = [U : C] = q$ linear irreducible (complex) characters.

2.4.2 Irreducible characters of B

First step and Clifford's Theorem

From the semi-direct product structure, we have a group homomorphism $\pi\colon B \longrightarrow A$, which gives us $q-1$ distinct linear (irreducible) characters of B with U in their kernel : $\pi \circ \lambda_1 = 1_B$, $\pi \circ \lambda_2 = \varphi_2, \ldots, \pi \circ \lambda_{q-1} = \varphi_{q-1}$, where Irr($A$)= $\{\lambda_1 = 1_A, \lambda_2, \ldots, \lambda_{q-1}\}$. Therefore 3 elements μ_1, μ_2, μ_3 of Irr(B) are missing, and these don't have U in their kernel (for otherwise they could be lifted to B from irreducible characters of A). Whence, since B is a Frobenius group with Frobenius kernel U, μ_1, μ_2 and μ_3 are induced by non-trivial irreducible characters of U (cf Curtis-Reiner [9], Proposition (14.4)). We write $\mu_i = \mathrm{Ind}_U^B(\nu_i)$ for some $\nu_i \in \mathrm{Irr}(U) \setminus \{1_U\}$, $i = 1, 2, 3$. Writing $|B| = q^2(q-1) = (q-1)1 + \mu_1(1)^2 + \mu_2(1)^2 + \mu_3(1)^2$, we see that $\nu_1(1)^2+\nu_2(1)^2+\nu_3(1)^2 = q+1$ **odd**. The $\nu_i(1)$'s being powers of 2, this implies that, without loss of generality, $\nu_1(1) = 1$, $\mu_1(1) = q-1$ and $\nu_2(1)^2+\nu_3(1)^2 = q = 2^{2n+1}$, so that $\nu_2(1) = \nu_3(1) = 2^n$ and $\mu_2(1) = \mu_3(1) = (q-1)2^n$.

Finally, B has the following irreducible characters:

$\varphi_1 = 1_B, \ldots, \varphi_{q-1}$ degree 1	μ_1 degree $q-1$	μ_2, μ_3 degree $(q-1)2^n$
$\varphi_i = \pi \circ \lambda_i$, $i = 1, \ldots q-1$, $\pi\colon B = U \rtimes A \twoheadrightarrow A$, $\mathrm{Irr}(A) = \{\lambda_1, \ldots, \lambda_{q-1}\}$.	$\mu_1 = \mathrm{Ind}_U^B(\nu_1)$, $\nu_1 \in \mathrm{Irr}(U) \setminus \{1_U\}$, $\nu_1(1) = 1$.	$\mu_i = \mathrm{Ind}_U^B(\nu_i)$, $\nu_i \in \mathrm{Irr}(U) \setminus \{1_U\}$, $\nu_i(1) = 2^n$, $i = 2, 3$.

Now, U acts (trivially) by conjugation on C, and thus on Irr(C). By Clifford's Theorem, we see that, if $\chi \in \mathrm{Irr}(U)$ and if $< \mathrm{Res}_C^U(\chi), \mu >_C \neq 0$ for some $\mu \in \mathrm{Irr}(C)$, then $\mathrm{Res}_C^U(\chi) = e\mu$ for some $e \in \mathbf{N}$. Furthermore, since every element of Irr(C) is linear, $e = \chi(1)$, a power of 2. In particular, $\mathrm{Res}_C^U(\nu_1) = \alpha_1$, $\mathrm{Res}_C^U(\nu_2) = 2^n\alpha_2$ and $\mathrm{Res}_C^U(\nu_3) = 2^n\alpha_3$ for some α_1, α_2, α_3 linear (irreducible) characters of C.

Furthermore, using Frobenius Reciprocity, we see that we can define an **injective** map

$$\begin{array}{ccl} \mathrm{Irr}(C) & \longrightarrow & \mathrm{Irr}(U) \\ \alpha & \longmapsto & \nu \text{ s.t. } < \mathrm{Ind}_C^U(\alpha), \nu >_U \neq 0 \end{array}$$

Actions of A and B

We have $C \lhd B$, and B acts by conjugation on the conjugacy classes of C: for $c \in C$, $u \in U$ and $a \in A$, we have $(u, a)^{-1}(c, 1)(u, a) = (c^{a^{-1}}, 1)$

where $c^{a^{-1}}$ denotes the image of c under a^{-1} in the action of A on U (which action preserves C which is characteristic in U). Hence the action of B in fact reduces to an action of A (by this we mean that $\{B\text{-orbits of } C\} = \{A\text{-orbits of } C\}$).

Now B also acts by conjugation on $\mathrm{Irr}(C) = \{\beta_1 = 1_C, \beta_2, \ldots, \beta_q\}$ via

$$\beta_i^h(c) = \beta_i(c^{h^{-1}}) \text{ for} c \in C,\ h \in B, \text{and } i = 1, \ldots, q,$$

where $c^{h^{-1}}$ denotes the image of c under h^{-1} in the B-action we just described. And, here again, $\{B\text{-orbits of } \mathrm{Irr}(C)\} = \{A\text{-orbits of } \mathrm{Irr}(C)\}$. We can now apply Corollary (11.10) in Curtis-Reiner [9] which states that, if B acts by conjugation on the conjugacy classes of $C \lhd B$ and on $\mathrm{Irr}(C)$, then the number of B-orbits of these two actions coincide.
But, in the first action, we only have two orbits : $\{1\}$ and $C \setminus \{1\}$ (since A acts regularly on $C \setminus \{1\}$). Since $\{1_C\}$ is clearly an orbit in the action on $\mathrm{Irr}(C)$, we must have

$$\{\beta_2^a,\ a \in A\} = \{\beta_2, \ldots \beta_q\} (= \{\beta_j^a,\ a \in A\} \text{ for all } j = 2, \ldots q).$$

And, since $|A| = q - 1$, for all $i = 2, \ldots q$, if $\beta_i^a = \beta_i^b$, then $a = b$.

Now B acts by conjugation on $\mathrm{Irr}(U)$ and, the (underlying) action of U being trivial, this also reduces to an action of A. Furthermore, for any $\chi, \psi \in \mathrm{Irr}(U)$, for all $a \in A$, since $< \chi^a, \psi^a >_U = < \chi, \psi >_U$, we have

$$\chi^a = \psi^a \iff \chi = \psi \qquad (\triangle)$$

Now take any $i \in \{2, \ldots q\}$. Then $\mathrm{Ind}_C^U(\beta_i)$ has, say, k distinct irreducible summands in $\mathrm{Irr}(U)$. If $\chi \in \mathrm{Irr}(U)$ is any of these, then, for all $a \in A$, χ^a is an irreducible summand of $\mathrm{Ind}_C^U(\beta_i^a)$. This also implies that, for $a, b \in A$, $\chi^a = \chi^b \Rightarrow \beta_i^a = \beta_i^b \Rightarrow a = b$. Hence the orbit of χ under the action of A has size $q - 1$ (and all the elements in this orbit have the same degree).

Now, because of $(\triangle)$, we see that, for all $a \in A$, $\mathrm{Ind}_C^U(\beta_i^a)$ must have at least k distinct irreducible summands in $\mathrm{Irr}(U)$. And, since i was arbitrary, and since $\{\beta_i^a,\ a \in A\} = \{\beta_2, \ldots \beta_q\}$, we get that, for any $i \in \{2, \ldots q\}$, $\mathrm{Ind}_C^U(\beta_i)$ has exactly k irreducible summands in $\mathrm{Irr}(U)$. Notice that all the elements of $\mathrm{Irr}(U)$ which arose are distinct because of the injectivity of the map we constructed in the last paragraph. Since all irreducible characters of U appear as irreducible summands of characters induced from C to U, this gives us

$$|\mathrm{Irr}(U)| = |\{\text{irr. summ. of } \mathrm{Ind}_C^U(1_C)\}| + (q - 1)k.$$

Irreducible characters of U

By Frobenius Reciprocity, 1_U is an irreducible summand of $\mathrm{Ind}_C^U(1_C)$. Furthermore, by Frobenius Reciprocity and Clifford's Theorem, we can write $\mathrm{Ind}_C^U(1_C) = \sum \chi(1)\chi$. In particular, any linear irreducible summand of $\mathrm{Ind}_C^U(1_C)$ appears with multiplicity 1.

Thus $\sum(\chi(1))^2 = q$ **even** yields that there must be (at least) one other (than 1_U) linear irreducible summand of $\mathrm{Ind}_C^U(1_C)$, α say.

Now U has exactly q linear irreducible characters. Supposing one of these is an irreducible summand of some $\mathrm{Ind}_C^U(\beta_i)$, $\beta_i \neq 1_C$, then we would find one **distinct** linear irreducible character in each of $\{\mathrm{Ind}_C^U(\beta_j) \,|\, j \neq i,\, \beta_j \neq 1_C\}$, each distinct from 1_U and α, thus giving $2 + q - 1 = q + 1$ distinct linear irreducible characters of U. This is a contradiction. Hence all linear irreducible characters of U appear as irreducible summands of $\mathrm{Ind}_C^U(1_C)$, and nowhere else. For the degrees to match, we must have $\mathrm{Ind}_C^U(1_C) = \sum_{i=1}^q \xi_i$, where the ξ_i's are the (only) q linear irreducible characters of U.

This implies that $k = 2$, i.e. for any $1_C \neq \beta_i \in \mathrm{Irr}(C)$, $\mathrm{Ind}_C^U(\beta_i)$ has exactly 2 irreducible summands in $\mathrm{Irr}(U)$. Writing $\mathrm{Ind}_C^U(\beta_2) = \chi(1)\chi + \psi(1)\psi$, we get $\chi(1) = \psi(1) = 2^n$.

So far, we got that $\mathrm{Irr}(U) = \{\xi_1, \ldots, \xi_q, \chi_2, \ldots, \chi_q, \psi_2, \ldots, \psi_q\}$. Now the ξ_i's are **real** (indeed, for any $j \in \{2, \ldots, q\}$, $\xi_i(\tau_j) = 1_C(\tau_j) = 1$, hence $\xi_i(f_j) \in \{\pm 1\}$). If all $\chi_2, \ldots, \chi_q$ were real, then, since the β_i's are real, $\chi_i + \psi_i = \mathrm{Ind}_C^U(\beta_i)$ is real for all $i = 2, \ldots, q$, and thus all $\chi_2, \ldots, \chi_q, \psi_2, \ldots, \psi_q$ would be real. But this is not true since (e.g.) f_2 and f_2^{-1} are not conjugate in U. Thus, without loss of generality, χ_2 is not real and, since $\{\chi_2, \ldots, \chi_q\} = \{\chi_2^a,\, a \in A\}$, **none** of $\chi_2, \ldots, \chi_q$ is real.

Now, for all $i = 2, \ldots, q$, $\mathrm{Res}_C^U(\overline{\chi_i}) = 2^n\overline{\beta_i} = 2^n\beta_i = \mathrm{Res}_C^U(\chi_i)$, so that $\overline{\chi_i} \in \{\chi_i, \psi_i\}$. From the above remark, we deduce that $\overline{\chi_i} = \psi_i$ for all $i = 2, \ldots, q$. Finally, the irreducible characters of U are as follows:

$1_U = \xi_1, \ldots, \xi_q$ degree 1	$\chi_2, \ldots, \chi_q, \overline{\chi_2}, \ldots, \overline{\chi_q}$ degree 2^n
$\xi_i = \overline{\xi_i}$, $i = 1, \ldots, q$, $\mathrm{Ind}_C^U(1_C) = \sum_{i=1}^q \xi_i$.	$\mathrm{Ind}_C^U(\beta_i) = 2^n(\chi_i + \overline{\chi_i})$, $i = 1, \ldots, q$, $\mathrm{Irr}(C) \setminus \{1_C\} = \{\beta_2, \ldots, \beta_q\}$.

Values on 4-elements

Characters of U:

Take any $k \in \{2, \ldots, q\}$ and consider χ_k. We have $\mathrm{Res}_C^U(\chi_k) = 2^n\beta_k$, $\beta_k \in \mathrm{Irr}(C)$. We define $I_1 = \{j \in \{2, \ldots, q\} \,|\, \beta_k(\tau_j) = 1\}$ and

$I_{-1} = \{j \in \{2, \ldots, q\} \mid \beta_k(\tau_j) = -1\}$. Then $|I_1| = \frac{q}{2} - 1$ and $|I_{-1}| = \frac{q}{2}$. Take $\rho_k \colon U \longmapsto GL_{2^n}(\mathbf{C})$ the representation associated to χ_k. Using the fact $\chi_k(\tau_j) = \pm 2^n$, depending on if $j \in I_1$ or $j \in I_{-1}$, we see that $Sp(\rho_k(\tau_j)) = \{1\}$ (the set of eigenvalues of $\rho_k(\tau_j)$) if $j \in I_1$, and $Sp(\rho_k(\tau_j)) = \{-1\}$ if $j \in I_{-1}$. Thus

$$Sp(\rho_k(f_j)) \subset \left\{ \begin{array}{l} \{-1,\, 1\} \text{ if } j \in I_1 \\ \{-i,\, i\} \text{ if } j \in I_{-1} \end{array} \right.$$

and

$$\chi_k(f_j) = \left\{ \begin{array}{l} n_{kj} \in \mathbf{Z} \text{ if } j \in I_1 \\ n_{kj} i \in \mathbf{Z} i \text{ if } j \in I_{-1} \end{array} \right. .$$

Now $< \chi_k,\, 1_U >_U = 0 = \sum_{u \in U} \chi_k(u) = \sum_{u \in C} \chi_k(u) + \sum_{u \in U \setminus C} \chi_k(u)$. But

$$\sum_{u \in C} \chi_k(u) = |C| < \mathrm{Res}^U_C(\chi_k),\, 1_C >_C = |C| < 2^n \beta_k,\, 1_C >_C = 0$$

and

$$\begin{aligned} \sum_{u \in U \setminus C} \chi_k(u) &= \sum_{j \in I_{-1}} |Cl_U(f_j)|(\chi_k(f_j) + \chi_k(f_j^{-1})) + \sum_{j \in I_1} |Cl_U(f_j)|(\chi_k(f_j) + \chi_k(f_j^{-1})) \\ &= \sum_{j \in I_{-1}} \frac{q}{2}(n_{kj} i - n_{kj} i) + \sum_{j \in I_1} q \chi_k(f_j) \end{aligned}$$

whence we finally get that

$$\sum_{j \in I_1} \chi_k(f_j) = 0 \text{ for any } 2 \leq k \leq q.$$

(Notice that, even though it is not clearly indicated, the set I_1 in the above formula depends on k.)

Now, similarly, for any $2 \leq k \leq q$, ξ_k is orthogonal to 1_U :

$$0 = \sum_{u \in U} \xi_k(u) = \sum_{u \in C} \xi_k(u) + \sum_{u \in U \setminus C} \xi_k(u) = |C| + \sum_{j=2}^{q} 2 \frac{q}{2} \xi_k(f_j)$$

whence we get that

$$\sum_{j=2}^{q} \xi_k(f_j) = -1 \text{ for any } 2 \leq k \leq q.$$

Induction from U to B:

We write $B = Ut_1 \cup \cdots \cup Ut_{q-1}$, where the t_i's are representatives for the cosets of U in B. For any $\chi \in \mathrm{Irr}(U)$ and $2 \leq k \leq q$, we have (since $U \lhd B = N_G(U)$) $\mathrm{Ind}_U^B(\chi)(f_k) = \sum_{i=1}^{q-1} \chi(t_i f_k t_i^{-1})$. Comparing the conjugacy classes of elements of order 4 in U and B, we see that, up to relabelling the f_j's, we can suppose that $f_2, \ldots, f_q$ are conjugate in B to ρ, and $f_2^{-1}, \ldots, f_q^{-1}$ are conjugate in B to ρ^{-1}. In particular, f_k is conjugate in B to each of $f_2, \ldots, f_q$. Since conjugation of f_k by elements of a given coset of U in B gives a unique conjugacy class in U, we see that, as i runs through $\{1, \ldots, q-1\}$, $f_k^{Ut_i}$ must run through $\{Cl_U(f_j), 2 \leq j \leq q\}$. This implies that

$$\{\chi(t_i f_k t_i^{-1}), 1 \leq i \leq q-1\} = \{\chi(f_j), 2 \leq j \leq q\}$$

and

$$\mathrm{Ind}_U^B(\chi)(f_k) = \sum_{j=2}^{q} \chi(f_j) \text{ for all } \chi \in \mathrm{Irr}(U),\ 2 \leq k \leq q.$$

Last step

Now we can finish our description of the irreducible characters of B. Remember only three were missing, μ_1, μ_2 and μ_3, respectively induced from U to B by (non-trivial) irreducible characters ν_1, ν_2 and ν_3 of U of degree 1, 2^n and 2^n. Thus, w.l.o.g., $\mu_1 = \mathrm{Ind}_U^B(\xi_2)$ and $\mu_2 = \mathrm{Ind}_U^B(\chi_2)$. Thus, writing ρ and ρ^{-1} representatives for the two classes of elements of order 4 in B, we have

$$\mu_1(\rho) = \sum_{j=2}^{q} \xi_2(f_j) = -1$$

$$\mu_2(\rho) = \sum_{j=2}^{q} \chi_2(f_j) = \sum_{j \in I_1} \chi_2(f_j) + \sum_{j \in I_{-1}} \chi_2(f_j)$$

which, since $\sum_{j \in I_1} \chi_2(f_j) = 0$, gives $\mu_2(\rho) = ai \in \mathbf{Z}i$. Similarly, $\mu_3(\rho) = bi \in \mathbf{Z}i$.

Writing the second orthogonality relation in H for the column of ρ, we get $\sum_{\mu \in \mathrm{Irr}(H)} |\mu(\rho)|^2 = q + a^2 + b^2 = |C_H(\rho)| = 2q$. Thus $a^2 + b^2 = q = 2^{2n+1}$; this forces both a and b to be non-zero. But then $\mu_2(\rho) \in \mathbf{Z}i \setminus \{0\}$, so that $\overline{\mu_2} \neq \mu_2$, and therefore $\overline{\mu_2} = \mu_3$. Furthermore, $|a| = |b| = 2^n$, and, since $bi = \mu_3(\rho) = \overline{\mu_2(\rho)} = \overline{ai} = -ai$, we have $b = -a$ and, up to exchanging μ_2 and μ_3, we have $a = 2^n$ and $b = -2^n$.

Remark: A description of the irreducible characters of B, as well as their values on elements of order 4 can actually be found in Suzuki [25].

2.4.3 A generalized perfect isometry

We want to show that there exists a generalized perfect isometry with respect to elements of order (divisible by) 4 between the principal 2-blocks B_0 and b_0 of G and B respectively.

First note that, if $x \in B$ has order 4, then $x \in U$ and $C_B(x) \leq U$, so that $C_B(U) \leq U$. Since U is the largest normal 2-subgroup of B, this implies that B has only one 2-block, i.e. $b_0 = \mathrm{Irr}(B)$. Furthermore, the fact that U is a TI set with respect to B implies that the 2-blocks of G have either full defect or defect 0. Since B has only one 2-block, Brauer's First Main Theorem implies that G has exactly one block of full defect, B_0, and any other block has defect 0. Using the notations of Burkhardt [4], we see that G has exactly one character of defect 0, written Π, and thus $B_0 = \mathrm{Irr}(G) \setminus \{\Pi\}$.

We take from [4] the following fragment of the character table of G (corresponding to characters in B_0):

	1	f	f^{-1}
1_G	1	1	1
$\Omega_s,\ 1 \leq s \leq \frac{q-2}{2}$	q^2+1	1	1
$\Theta_l,\ 1 \leq l \leq \frac{q+r}{4}$	$(q-r+1)(q-1)$	-1	-1
$\Lambda_u,\ 1 \leq u \leq \frac{q-r}{4}$	$(q+r+1)(q-1)$	-1	-1
Γ_1	$(q-1)2^n$	$2^n i$	$-2^n i$
Γ_2	$(q-1)2^n$	$-2^n i$	$2^n i$

(Note that $|B_0| = q+2 = |\mathrm{Irr}(B)| = |b_0|$.) And, with what we have done before, we can compute the same fragment of the character table of B :

	1	ρ	ρ^{-1}
1_B	1	1	1
$\varphi_k,\ 2 \leq k \leq q-1$	1	1	1
μ_1	$(q-1)$	-1	-1
μ_2	$(q-1)2^n$	$2^n i$	$-2^n i$
$\mu_3 = \overline{\mu_2}$	$(q-1)2^n$	$-2^n i$	$2^n i$

Finally, note that, if $x \in B$ has order 4, then $C_G(x) = C_B(x)$, so that, as in the previous section, if a bijection with signs between B_0 and b_0 preserves the values of characters on 4-elements, it will also preserve contributions. We then see from the two tables above that such a bijection exists; one can consider for example

$$I: \begin{cases} 1_G & \longmapsto & 1_B \\ \Omega_1 & \longmapsto & \varphi_2 \\ \vdots & \vdots & \vdots \\ \Omega_{(q-2)/2} & \longmapsto & \varphi_{q/2} \\ \Theta_1 & \longmapsto & -\varphi_{q/2+1} \\ \vdots & \vdots & \vdots \\ \Theta_{(q+r)/4} & \longmapsto & -\varphi_{(3q+r)/4} \\ \Lambda_1 & \longmapsto & -\varphi_{(3q+r)/4+1} \\ \vdots & \vdots & \vdots \\ \Lambda_{(q-r)/4-1} & \longmapsto & -\varphi_{q-1} \\ \Lambda_{(q-r)/4} & \longmapsto & \mu_1 \\ \Gamma_1 & \longmapsto & \mu_2 \\ \Gamma_2 & \longmapsto & \mu_3 \end{cases}$$

2.5 Ree Groups

In this section, we consider $G = \mathrm{Re}(q) = {}^2G_2(q)$ a Ree group, where $q = 3^{2m+1}$, for some $m \in \mathbf{N}$. We consider U a Sylow 3-subgroup of G and $B = N_G(U)$.

2.5.1 The groups G, B and U

The group G is a twisted group of Lie type G_2, simple if $m \geq 1$, and a description based on a root system of type G_2 can be found in Carter [7]. In particular, we find in [7] that $|G| = q^3(q-1)(q^3+1)$, so that $|U| = q^3$. Moreover, U can be written $U = \{x(t,u,v) \mid t,u,v \in \mathbf{F}_q\}$, with multiplication given by

$$x(t_1,u_1,v_1)x(t_2,u_2,v_2) = x(t_1+t_2, u_1+u_2-t_1t_2^{3\theta}, v_1+v_2-t_2u_1+t_1t_2^{3\theta+1}-t_1^2t_2^{3\theta})$$

where θ is the automorphism of $\mathbf{F}_q$ given by $\lambda^\theta = \lambda^{3^m}$ for all $\lambda \in \mathbf{F}_q$ (and thus $3\theta^2 = 1$). We will sometimes write (abusively) $\theta = 3^m$.

There exists a subgroup H of B such that $B = UH$ and $|H| = q-1$. We can write $H = \{h(w)\,,\, w \in \mathbf{F}_q^\times\}$ and conjugation of U by H is given by: for any $t,u,v,w \in \mathbf{F}_q$, $w \neq 0$,

$$h(w)x(t,u,v)h(w)^{-1} = x(w^{2-3\theta}t, w^{3\theta-1}u, wv).$$

We find some other structure results about Ree groups in Ward [26]. There can be found the following:

Theorem 2.5. *U is disjoint from its conjugates. Its center $Z(U)$ is elementary Abelian of order q. U is of class 3 and contains a normal Abelian subgroup U' of order q^2 containing $Z(U)$ which is both the derived group and the Frattini subgroup of U.*
The members of $U \setminus U'$ have order 9, their cubes forming $Z(U) \setminus \{1\}$.
We have $B = N_G(U) = UH$ where H is cyclic of order $q-1$.

The character table of G is also given in this article. G has $q+8$ conjugacy classes, with representatives 1, $R^a \neq 1$ for some R of order $(q-1)/2$ which is prime to 3, $S^a \neq 1$ for some S of order $(q+1)/4$ which is prime to 3, V and W of order dividing respectively $q+1-3^{m+1}$ and $q+1+3^{m+1}$ (the orders of their centralizers, cf [26], p 85) which are prime to 3, X, T and T^{-1} of order 3 ($X \in Z(U)$ and $T, T^{-1} \in U \setminus Z(U)$, cf [26], pp 78-80), Y, YT and YT^{-1} of order 9 (cf [26], p 82), J of order 2, JT and JT^{-1} which belong to groups whose order has 3-valuation 1 (cf [26], p 83) so which order is not divisible by 9, and $JR^A \neq J$ and $JS^a \neq J$ of respective orders $\mathrm{o}(J).\mathrm{o}(R^a)$ and $\mathrm{o}(J).\mathrm{o}(S^a)$ which are not divisible by 3.

Note that, following Carter, we write $q = 3^{2m+1}$, while Ward writes $q = 3^{2k+1}$ and $m = 3^k$. The m in Ward's paper therefore corresponds to 3^m in our notations.

The number and lengths of the conjugacy classes of U, as well as the number and degrees of the irreducible characters of U and B can be found in Eaton [11]. However, in order to obtain the values we need, we must compute some details Eaton doesn't give about the irreducible characters of B. We therefore compute again everything explicitly.

2.5.2 Conjugacy classes of U

Conjugation in U, centralizers

We see from the formula for multiplication in U that the identity element of U is $x(0,0,0)$, and that, since the center $Z(U)$ of U has order q, it can be identified with $\{x(0,0,v)\,,\, v \in \mathbf{F}_q\}$. We also obtain informations about the order of elements : $x(t,u,v)^2 = x(-t, -u-t^{3\theta+1}, -v-tu)$ and $x(t,u,v)^3 = x(0,0,-t^{3\theta+2})$, so that $x(t,u,v)^9 = 1$, and $x(t,u,v)$ has order 3 if and only if $t = 0$. Moreover, as $\gcd(3^{m+1}-1, 3\theta+2) = 1$, we have $\{-t^{3\theta+2}\,,\, t \in \mathbf{F}_q^\times\} = \mathbf{F}_q^\times$, so that the cubes of the elements of order 9 of U are $Z(U) \setminus \{1\} = \{x(0,0,v)\,,\, v \in \mathbf{F}_q^\times\}$.

Furthermore, the inverse of an element $x(t,u,v)$ of U is

$$x(t,u,v)^{-1} = x(-t, -u-t^{3\theta+1}, -v-tu+t^{3\theta+2}).$$

Hence conjugation in U is given by

$$x(t_1,u_1,v_1)x(t_2,u_2,v_2)x(t_1,u_1,v_1)^{-1} =$$

$$x(t_2, u_2 - t_2^{3\theta}t_1 + t_2t_1^{3\theta}, v_2 - t_2u_1 + t_1u_2 + t_2^2t_1^{3\theta} - 2t_2^{3\theta}t_1^2 + t_2^{3\theta+1}t_1).$$

In particular, conjugation in U preserves the first "coordinate".

Now take any $t_1, t_2, u_1, u_2, v_1, v_2 \in \mathbf{F}_q$. Then $x(t_1,u_1,v_1)x(t_2,u_2,v_2) = x(t_2,u_2,v_2)x(t_1,u_1,v_1)$ if and only if

$$\begin{cases} t_1 + t_2 = t_2 + t_1 \\ t_1t_2^{3\theta} = t_2t_1^{3\theta} \\ t_2u_1 - t_1t_2^{3\theta+1} + t_1^2t_2^{3\theta} = t_1u_2 - t_2t_1^{3\theta+1} + t_2^2t_1^{3\theta} \ (\dagger) \end{cases}$$

Writing $\omega = t_1t_2^{3\theta} = t_2t_1^{3\theta}$, we get that $(\dagger) \iff t_1(u_2 - 2\omega) = t_2(u_1 - 2\omega)$ $(\dagger')$.

If t_1, u_1 and v_1 are now fixed, we can solve the above system of equations in t_2, u_2, $v_2 \in \mathbf{F}_q$, and we eventually obtain:
$C_U(x(0, u_1 \neq 0, v_1)) = \{x(0,u_2,v_2) \,|\, u_2, v_2 \in \mathbf{F}_q\}$ has order q^2, and thus $Cl_U(x(0, u_1 \neq 0, v_1))$ has length q.
$C_U(x(t \neq 0, u, v_1)) = \{x(0,0,v_2)\,,\, x(t,u,v_2)\,,\, x(-t, -t^{3\theta+1} - u, v_2) \,|\, v_2 \in \mathbf{F}_q\}$ has order $3q$, and $Cl_U(x(t \neq 0, u, v_1))$ has length $q^2/3$.

Using the formula for conjugation in U and the lengths we found, we see that, given $t, u, v \in \mathbf{F}_q$, $Cl_U(x(0,u,v)) = \{x(0, u, v + t_1u)\,,\, t_1 \in \mathbf{F}_q\}$, so that, if $u \neq 0$, then $Cl_U(x(0,u,v)) = \{x(0,u,v_2)\,,\, v_2 \in \mathbf{F}_q\}$, giving us $q-1$ such classes, uniquely determined by u.
Moreover, $x(t,u_2,v_2)x(t,u,v)x(t,u_2,v_2)^{-1} = x(t,u,v+t(u-u_2))$, so that, if $t \neq 0$, then $Cl_U(x(t,u,v)) \supset \{x(t,u,v_2)\,,\, v_2 \in \mathbf{F}_q\}$. Hence, by conjugation, the first coordinate is fixed and the last can be changed to any one without modifying the second one. We obtain that $x(t,u_2,v_2) \in Cl_U(x(t,u,v))$ if and only if there exists $t_2 \in \mathbf{F}_q$ such that $u_2 = u - t_2t^{3\theta} + tt_2^{3\theta}$. We can show that there are $q/3$ such u_2's. Indeed, take any $t_2, t_3 \in \mathbf{F}_q$, and the associated u_2 and u_3. Then,

$$\begin{aligned} u_2 = u_3 &\iff tt_2^{3\theta} - t_2t^{3\theta} = tt_3^{3\theta} - t_3t^{3\theta} \\ &\iff t(t_2^{3\theta} - t_3^{3\theta}) = (t_2 - t_3)t^{3\theta} \\ &\iff t(t_2 - t_3)^{3\theta} = (t_2 - t_3)t^{3\theta} \end{aligned}$$

(this last equivalence being true since the caracteristic is 3 and $(-1)^{3\theta} = -1$). And this has 3 solutions, $t_2 - t_3 \in \{0, \pm t\}$. Hence we obtain $|\mathbf{F}_q|/3 = q/3$ distinct values for u_2. Thus, we get

$$Cl_U(x(t \neq 0, u, v)) = \{x(t, u - t_2t^{3\theta} + tt_2^{3\theta}, v_2) \,|\, t_2, v_2 \in \mathbf{F}_q\},$$

giving us $3(q-1)$ such classes, determined by t and 3 distinct values u_1, u_2 and $u_3 \in \mathbf{F}_q$.

This gives us the whole of U :

- q classes of length 1, type $x(0,0,v)$, parametrized by $v \in \mathbf{F}_q$
- $q-1$ classes of length q, type $x(0,u \neq 0,v)$, parametrized by $u \in \mathbf{F}_q^\times$
- $3(q-1)$ classes of length $q^2/3$, type $x(t \neq 0,u,v)$, parametrized by $t \in \mathbf{F}_q^\times$ and 3 values of $u \in \mathbf{F}_q$.

Thus U has $5q-4$ conjugacy classes.

H-conjugacy classes of U

First, recall that, for any $t,u,v,w \in \mathbf{F}_q$, $w \neq 0$,

$$h(w)x(t,u,v)h(w)^{-1} = x(w^{2-3\theta}t, w^{3\theta-1}u, wv),$$

so that conjugation by elements of H preserves the three types of conjugacy classes of U we just described.

Now, H acts by conjugation on the conjugacy classes of U. We find the number and sizes of the orbits under this action.

- Take $v \in \mathbf{F}_q$. If $v \neq 0$, then $\{wv\,,\, w \in \mathbf{F}_q^\times\} = \mathbf{F}_q^\times$, so that we get **two** H-orbits : $\{x(0,0,0)\}$ and $Z(U) \setminus \{1\} = \{x(0,0,v)\,,\, v \in \mathbf{F}_q^\times\}$.
- Take $u \in \mathbf{F}_q^\times$ and $w \in \mathbf{F}_q^\times$. Then

  $$h(w)Cl_U(x(0,u,v))h(w)^{-1} = Cl_U(x(0,u,v)) \Longleftrightarrow w^{3\theta-1}u = u$$

  and this has 2 solutions $w \in \mathbf{F}_q^\times$ $(w = \pm 1)$. This gives us **two** H-orbits, of size $(q-1)/2$.
- Take $t \in \mathbf{F}_q^\times$ and $w \in \mathbf{F}_q^\times$. Then

  $$h(w)Cl_U(x(t,u,v))h(w)^{-1} = Cl_U(x(t,u,v)) \Longrightarrow w^{2-3\theta}t = t$$

  Now $w^{2-3\theta}t = t \Leftrightarrow w^{3\theta-2} = 1 \Leftrightarrow w^{3^{m+1}-2} = 1$. However, we can see that $\gcd(3^{2m+1}-1, 3^{m+1}-2) = 1$, and thus $w^{3^{m+1}-2} = 1$ has a unique solution $w \in \mathbf{F}_q$ $(w = 1)$. This gives us **three** H-orbits, of size $q-1$.

Finally, there are 7 H-orbits on the conjugacy classes of U.

2.5.3 Irreducible characters of U

Actions of B

If $L = U$ or $Z(U)$, then $L \lhd B$, and B acts by conjugation on the conjugacy classes of L (via $x^b = b^{-1}xb$) and on $\mathrm{Irr}(L)$ (via $\chi^b(x) = \chi(bxb^{-1})$). Corollary (11.10) in Curtis-Reiner [9] implies that the number of orbits in these two actions are the same. Furthermore, since U is normal in $B = UH$ and U acts trivially on the conjugacy classes of $Z(U)$ and of U, the B-orbits in these actions are in fact the H-orbits. The study of H-orbits we made before shows that there are 2 H-orbits in $Z(U)$, and thus in $\mathrm{Irr}(Z(U))$, and 7 H-orbits in $Cl(U)$, and thus in $\mathrm{Irr}(U)$.

Induction from $Z(U)$ to U

We write $C = Z(U)$. Since $\{1_C\}$ is clearly one H-orbit of $\mathrm{Irr}(C)$, we see that H must act transitively on $\mathrm{Irr}(C) \setminus \{1_C\}$, and Clifford's Theorem gives us an **injective** map

$$\begin{array}{ccl} \mathrm{Irr}(C) & \longrightarrow & \mathrm{Irr}(U) \\ \beta_i & \longmapsto & \chi \text{ s.t. } < \mathrm{Ind}_C^U(\beta_i), \chi >_U \neq 0 \end{array}$$

where $\mathrm{Irr}(C) = \{\beta_1 = 1_C, \beta_2, \ldots, \beta_q\}$. Furthermore, since $|H| = |\mathrm{Irr}(C) \setminus \{1_C\}|$, we have, as in Suzuki groups,

$$|\mathrm{Irr}(U)| = |\{\text{Irred. summ. of } \mathrm{Ind}_C^U(1_C)\}| + (q-1)k$$

where k is the (common) number of irreducible summands of any $\mathrm{Ind}_C^U(\beta_i)$, $i \geq 2$.

We can write $\mathrm{Ind}_C^U(1_C) = \sum_{\chi \in I_1 \subset \mathrm{Irr}(U)} \chi(1)\chi$. Now 1_U appears with multiplicity 1 in $\mathrm{Ind}_C^U(1_C)$, and so does any linear irreducible summand of $\mathrm{Ind}_C^U(1_C)$. Since $\mathrm{Ind}_C^U(1_C)(1) = [U : C] = q^2$ is a power of 3, and since any non-linear irreducible summand of $\mathrm{Ind}_C^U(1_C)$ has degree a non-trivial power of 3, we see that there must be (at least) 2 other linear irreducible summands in $\mathrm{Ind}_C^U(1_C)$, α and β say. Supposing some $\mathrm{Ind}_C^U(\beta_i)$ ($\beta_i \neq 1_C$) has a linear irreducible summand, we would get an orbit of $q-1$ distinct linear irreducible characters of U, all distinct from 1_U, α and β (since, by Clifford, $1_U|_C = \alpha|_C = \beta|_C = 1_C$), giving us $q+2$ linear irreducible characters of U. This is a contradiction, since $[U : U'] = q$. Hence all linear irreducible characters of U appear as irreducible summands of $\mathrm{Ind}_C^U(1_C)$ and nowhere else. Writing $\mathrm{Ind}_C^U(1_C) = \sum_{i=1}^q \lambda_i + \sum_{i \in I} \chi_i(1)\chi_i$, where the λ_i's are linear, we have $|\mathrm{Irr}(U)| = 5q - 4 = q + |I| + (q-1)k$. Thus $4q - 4 = |I| + (q-1)k$, and $k \leq 4$.

$k = 4$ implies $I = \emptyset$ and $\mathrm{Ind}_C^U(1_C) = \sum_{i=1}^q \lambda_i$, which is impossible since the left has degree q^2 and the right has degree q.

Suppose $k = 2$. Then we can write (using Clifford) $\mathrm{Ind}_C^U(\beta_2) = \chi(1)\chi + \psi(1)\psi$ and (degrees) $3^{2a} = q^2 = \chi(1)^2 + \psi(1)^2 = 3^{2b} + 3^{2c}$. Then $b = c$ implies $2|3^{2a}$, which is impossible, and $b < c$ implies $3^{2a} = 3^{2b}(1 + 3^{2(c-b)})$, which is even, so this is impossible too.

Suppose $k = 3$. Then, $q^2 = \chi(1)^2 + \psi(1)^2 + \eta(1)^2$, which can similarly be shown to be impossible.

We deduce that $k = 1$. Hence there exist $\psi_2, \ldots, \psi_q \in \mathrm{Irr}(U)$ such that

$$\mathrm{Ind}_C^U(\beta_i) = \psi_i(1)\psi_i \text{ for } i = 2, \ldots, q$$

and thus $\psi_i(1) = q$ for $i = 2, \ldots, q$.

Furthermore, $|I| = 3q - 3$, $\mathrm{Ind}_C^U(1_C) = \sum_{i=1}^q \lambda_i + \sum_{i=1}^{3q-3} \chi_i(1)\chi_i$, and, by Clifford's Theorem, the χ_i's have $C = Z(U)$ in their kernel, so are in fact irreducible characters of the quotient $U/Z(U)$.

The quotient $V = U/Z(U)$

From the central series $\{1\} \lhd Z(U) \lhd U' \lhd U$, we see that $V = U/Z(U)$ has order q^2, and has a normal Abelian subgroup $(U'/Z(U))$ of order q such that the corresponding quotient is Abelian of order q.

We have seen that $Z(U) = \{x(0, 0, v)\,,\ v \in \mathbf{F}_q\}$, and, from the formula for multiplication in U, we see that $U/Z(U)$ can be parametrized by $\{y(t, u)\,|\, t, u \in \mathbf{F}_q\}$, where multiplication is given by

$$y(t_1, u_1)y(t_2, u_2) = y(t_1 + t_2, u_1 + u_2 - t_1 t_2^{3\theta}).$$

We have $1_{U/Z(U)} = y(0, 0)$, $y(t, u)^{-1} = y(-t, -u - t^{3\theta+1})$, and

$$y(t_1, u_1)y(t_2, u_2)y(t_1, u_1)^{-1} = y(t_2, u_2 - t_1 t_2^{3\theta} + t_2 t_1^{3\theta}).$$

From the results in U, we see that the conjugacy classes of V are as follows :

- $\{y(0, 0)\}$
- $q - 1$ classes of length 1 : $\{y(0, u)\}\,,\ u \in \mathbf{F}_q^\times$
- $3(q - 1)$ classes of length $q/3$, of type $y(t, u)$ for $t \in \mathbf{F}_q^\times$ and 3 values of $u \in \mathbf{F}_q$.

The subgroup $W = \{y(0, u)\,,\ u \in \mathbf{F}_q\}$ is both the center and the derived group of V. Thus V has q linear irreducible characters and $3q - 3$ non-linear irreducible characters, the ones we are looking for, $\chi_1, \ldots, \chi_{3q-3}$. As

V has **odd** order q^2, each non-trivial irreducible character of V is **non-real**. Furthermore, by Clifford's Theorem, every irreducible character of V appears as irreducible summand of some character induced from W, with multiplicity its degree.

Irreducible characters of $W = Z(V)$:

We have $V = U/Z(U) \lhd B/Z(U) = (U/Z(U))H$. Thus $B/Z(U)$ acts by conjugation on $Z(V) = W$, and this action reduces to an action of H. From the study of H-classes of U, we see that there are 3 orbits under this action, giving us 3 orbits under the derived action of $B/Z(U)$ on $\mathrm{Irr}(W) = \{\gamma_1 = 1_W, \gamma_2, \ldots, \gamma_q\}$, where the γ_i's are linear. Since $\{1_W\}$ is one H-orbit, $\{\gamma_2, \ldots, \gamma_q\}$ must be a union of 2 H-orbits. Now the length of an orbit divides the order of H, that is $q-1$, and this forces each of these 2 orbits to have length $(q-1)/2$.
Now $q-1 \equiv (-1)^{2m+1} - 1 \equiv -2 \pmod 4$, so that $(q-1)/2$ is **odd**. Moreover, W has **odd** order q. Thus the only real irreducible character of W is 1_W, and $\gamma_i \neq \overline{\gamma_i}$ for all $i = 2, \ldots, q$. Supposing that $\overline{\gamma_2}$ belongs to the same H-orbit as γ_2, $\overline{\gamma_2} = \gamma_2^h$ say, and taking any γ_i in this orbit, it is easy to see (using the definitions of the actions of H) that then $\gamma_i^h = \overline{\gamma_i}$, so that $\overline{\gamma_i}$ belongs to the same H-orbit. The orbit of γ_2 would then have **even** length, which is a contradiction. This shows that, up to relabelling the γ_i's, the H-orbits of $\mathrm{Irr}(W)$ are $\{1_W\}$, $\{\gamma_2, \ldots \gamma_{(q+1)/2}\}$ and $\{\overline{\gamma_2}, \ldots \overline{\gamma_{(q+1)/2}}\}$.

Induction from W to V:

If γ_i and γ_j belong to the same H-orbit of $\mathrm{Irr}(W)$, then $\mathrm{Ind}_W^V(\gamma_i)$ and $\mathrm{Ind}_W^V(\gamma_j)$ have the same number of irreducible components, with the same degrees (that is, we can build a bijection preserving the degree between the sets of irreducible components of $\mathrm{Ind}_W^V(\gamma_i)$ and $\mathrm{Ind}_W^V(\gamma_j)$). Furthermore, complex conjugation gives the same kind of bijection between irreducible components of $\mathrm{Ind}_W^V(\gamma_i)$ and $\mathrm{Ind}_W^V(\overline{\gamma_i})$, whence all the $\mathrm{Ind}_W^V(\gamma_i)$'s, $2 \leq i \leq q$, have the same number of irreducible components, with the same degrees. Moreover, the subsets of $\mathrm{Irr}(V)$ obtained in this way are **disjoint**.

Now 1_V appears with multiplicity 1 in $\mathrm{Ind}_W^V(1_W)$, which has degree q. Hence at least two other linear characters of V must appear (each with multiplicity 1). Supposing a linear character appears in some $\mathrm{Ind}_W^V(\gamma_i)$, $i \geq 2$, we would get one in each $\mathrm{Ind}_W^V(\gamma_j)$, $j \geq 2$, giving us (at least) $q+2$ distinct linear characters of V, which is impossible. Hence each linear character of V appears in $\mathrm{Ind}_W^V(1_W)$ and nowhere else. Comparing degrees, we see that $\mathrm{Ind}_W^V(1_W) = \sum_{i=1}^q \mu_i$, where the μ_i's are (all) the irreducible linear characters of V. Writing k' the number of irreducible components of any $\mathrm{Ind}_W^V(\gamma_i)$, $i \geq 2$, we have $|\mathrm{Irr}(V)| = 4q - 3 = q + k'(q-1)$, so that $k' = 3$.

Writing $\mathrm{Ind}_W^V(\gamma_2) = \chi_1(1)\chi_1 + \chi_2(1)\chi_2 + \chi_3(1)\chi_3$, we obtain $q = 3^{2m+1} = \chi_1(1)^2 + \chi_2(1)^2 + \chi_3(1)^2$, which implies $\chi_1(1) = \chi_2(1) = \chi_3(1) = 3^m$.

Finally, V has q linear irreducible characters, and $3q-3$ irreducible characters of degree 3^m.

Irreducible characters of U

We summarize the results we obtained so far about the character table of U. By convention, in character tables, we will take the first line to correspond to the values on 1, and the first column to correspond to the trivial character.

	$1_U = \lambda_1, \ldots, \lambda_q$ $\mathrm{Irr}(U/U')$ degree 1	$\psi_2, \ldots, \psi_q$ degree q
$x(0,0,v),\ v \in \mathbf{F}_q$ length 1 order 3	1	$q\psi_i = \mathrm{Ind}_{Z(U)}^U(\beta_i)$ $\beta_i \in \mathrm{Irr}(Z(U)) \setminus \{1_{Z(U)}\}$ $Z(U) \cong (\mathbf{F}_q, +)$
$x(0,u,v),\ u \in \mathbf{F}_q^\times$ length q order 3	1	0
$x(t,u_i,v),\ t \in \mathbf{F}_q^\times$, $i = 1,2,3$ length $q^2/3$ order 9	three times each non-first line of the character table of U/U' Same line $\Leftrightarrow$ same t $U/U' = \{x(t),\ t \in \mathbf{F}_q\}$ $U/U' \cong (\mathbf{F}_q, +)$	0

	$\chi_1, \ldots, \chi_{3q-3}$ irreducible characters of $U/Z(U)$ degree 3^m
$x(0,0,v)$, $v \in \mathbf{F}_q$ length 1 order 3	3^m
$x(0,u,v)$, $u \in \mathbf{F}_q^\times$ length q order 3	$3^m. \left(\begin{array}{c} \text{three times each non-first column of the} \\ \text{character table of } W = Z(U/Z(U)) \\ \text{Same column } \Leftrightarrow \chi_i, \chi_j \text{ belong} \\ \text{to the same } \mathrm{Ind}_W^V(\gamma_k) \\ W = \{x(0,u),\ u \in \mathbf{F}_q\} \cong (\mathbf{F}_q, +) \end{array} \right)$
$x(t,u_i,v)$, $t \in \mathbf{F}_q^\times$, $i = 1,2,3$ length $q^2/3$ order 9	???????

H-orbits of $\mathbf{Irr}(U)$

We have seen that there are two orbits in the actions of H by conjugation on $\mathrm{Irr}(Z(U))$ and $Z(U)$. The orbits of $\mathrm{Irr}(Z(U))$ must be $\{1_{Z(U)}\}$ and $\{\beta_2, \ldots, \beta_q\}$. Now H acts by conjugation on $\mathrm{Irr}(U)$ and on $Cl(U)$, and the underlying action on $Cl_U(Z(U)) = Z(U)$ is the one before. Via Clifford's Theorem, we have that, for $h \in H$, $\psi \in \mathrm{Irr}(U)$ and $2 \leq i \leq q$,

$$< \psi, \mathrm{Ind}_{Z(U)}^U(\beta_i) >_U \neq 0 \Longleftrightarrow < \psi^h, \mathrm{Ind}_{Z(U)}^U(\beta_i^h) >_U \neq 0$$

(indeed, both are equal to $< \psi|_{Z(U)}, \beta_i >_{Z(U)} = < \psi^h|_{Z(U)}, \beta_i^h >_{Z(U)} = \psi(1) = \psi^h(1)$). As $\mathrm{Ind}_{Z(U)}^U(\beta_i) = q\psi_i$ for each $i = 2, \ldots, q$, we see that the ψ_i's form **one** H-orbit of $\mathrm{Irr}(U)$. $\{1_U\}$ is another **one**. However, we know that $\mathrm{Irr}(U)$ has 7 H-orbits. Thus $\{\lambda_2, \ldots, \lambda_q, \chi_1, \ldots, \chi_{3q-3}\}$ is a union of 5 H-orbits. Furthermore, because of the preservation of the degree under the action of H, $\{\lambda_i,\ i \geq 2\}$ and $\{\chi_i,\ 1 \leq i \leq 3q-3\}$ are unions of H-orbits.

Now the χ_i's are characters of $V = U/Z(U)$. They are all non-real, and can be written as $\{\chi_1, \ldots, \chi_{(3q-3)/2}, \overline{\chi_1}, \ldots, \overline{\chi_{(3q-3)/2}}\}$, the separation between the χ_i's and the $\overline{\chi_i}$'s corresponding to the one we introduced between the γ_i's and the $\overline{\gamma_i}$'s respectively. For $\gamma_i \in \mathrm{Irr}(Z(V))$, $i \geq 2$, and $\chi \in \mathrm{Irr}(V)$, we have

$$< \chi, \mathrm{Ind}_{Z(V)}^V(\gamma_i) >_V = < \overline{\chi}, \overline{\mathrm{Ind}_{Z(V)}^V(\gamma_i)} >_V .$$

Since $\{\gamma_2, \ldots, \gamma_{(q+1)/2}\}$ and $\{\overline{\gamma_2}, \ldots, \overline{\gamma_{(q+1)/2}}\}$ are **two** H-orbits, this shows that χ and $\overline{\chi}$ have distinct orbits. Furthermore, for $\chi, \psi \in \mathrm{Irr}(V)$ and $h \in H$, we have

$$<\chi^h, \psi^h>_V = <\chi, \psi>_V = <\overline{\chi}, \overline{\psi}>_V = <\overline{\chi}^h, \overline{\psi}^h>_V$$

and $\overline{\chi}^h = \overline{\chi^h}$. Hence the orbits of χ and $\overline{\chi}$ have the same length and consist of 2 by 2 conjugate characters. Moreover, since $<\chi^h|_{Z(V)}, \gamma_i^h>_{Z(V)} = <\chi|_{Z(V)}, \gamma_i>_{Z(V)}$, we see that $\{\chi_1, \ldots, \chi_{(3q-3)/2}\}$ and $\{\overline{\chi_1}, \ldots, \overline{\chi_{(3q-3)/2}}\}$ are unions of H-orbits, and that there is a bijection preserving the length between the two corresponding sets of orbits (namely, complex conjugation). Moreover, the orbits have size at least $(q-1)/2$, which is the size of γ_i^H.

This shows that $\{\chi_i, 1 \leq i \leq 3q-3\}$ is a union of an **even** number of H-orbits, which is strictly less than 5, and must therefore be 2 or 4. If it were 2, these orbits would have length $(3q-3)/2 > q-1 = |H|$, which is impossible. Thus, there are 4 orbits, of 2 lengths l and l', and we can write $3q-3 = 2l+2l'$, with $l, l' \geq (q-1)/2$. Also $l, l' \mid |H| = q-1$, so that $l, l' \in \{(q-1)/2, q-1\}$, and this implies that $l = (q-1)/2$ say, and $l' = q-1$.

Finally, the λ_i's must form **one** H-orbit, and, relabelling the $\chi_i' s$, we have the following 7 H-orbits of $\mathrm{Irr}(U)$:

$$\{1_U\}\ ,\ \{\lambda_2, \ldots, \lambda_q\}\ ,\ \{\psi_2, \ldots, \psi_q\}$$

$$\{\chi_{i,j},\ i = 1, 2 \text{ and } 1 \leq j \leq (q-1)/2\}\ ,\ \{\chi_{3,j},\ 1 \leq j \leq (q-1)/2\}$$

$$\{\overline{\chi_{i,j}},\ i = 1, 2 \text{ and } 1 \leq j \leq (q-1)/2\}\ ,\ \{\overline{\chi_{3,j}},\ 1 \leq j \leq (q-1)/2\}$$

where $\mathrm{Ind}_{Z(V)}^V(\gamma_{j+1}) = 3^m(\chi_{1,j} + \chi_{2,j} + \chi_{3,j})$ for $1 \leq j \leq (q-1)/2$.

2.5.4 Irreducible characters of $B = N_G(U)$

Clifford's Theory

The following description of Clifford's Theory can be found in Isaacs [16]. Recall that B acts on $\mathrm{Irr}(U)$ like H does. Take any $\eta \in \mathrm{Irr}(B)$. If, for some $\chi \in \mathrm{Irr}(U)$, $<\eta, \mathrm{Ind}_U^B(\chi)>_B = <\eta|_U, \chi>_U \neq 0$, then $\eta|_U = e\sum_{i=1}^t \chi^{h_i}$, where $e = <\eta|_U, \chi>_U$, $t = [B : I_B(\chi)]$ with $I_B(\chi)$ the inertial subgroup of χ, and $\{\chi^{h_i};\ i = 1 \ldots t\}$ is the H-orbit of χ.

Now take any $\theta \in \mathrm{Irr}(U)$, and write $\mathrm{Ind}_U^B(\theta) = \sum_{i\in I} e_i\mu_i$, with $\mu_i \in \mathrm{Irr}(B)$ for $i \in I$ and $\mu_i \neq \mu_j$ if $i \neq j \in I$ (and thus, as above, $\mu_i|_U = e_i \sum_{h_j} \theta^{h_j}$). Writing $T = I_B(\theta)$, we then have $\mathrm{Ind}_U^T(\theta) = \sum_{i\in I} e_i\nu_i$, with $0 \neq e_i \in \mathbf{N}$ and $\nu_i \in \mathrm{Irr}(T)$; moreover, for $i \in I$, $\mu_i = \mathrm{Ind}_T^B(\nu_i)$ and thus $\mu_i(1) = [B : T]\nu_i(1)$.

To study the e_i's, it suffices to restrict our attention to $T = I_B(\theta)$, $U \lhd T$ and θ is T-invariant. We have $\mathrm{Ind}_U^T(\theta) = \sum_{i\in I} e_i\nu_i$, and $\nu_i|_U = e_i\theta$ (since the

T-orbit of θ is just $\{\theta\}$). Hence $\nu_i(1) = e_i\theta(1)$, and

$$[T : U]\theta(1) = \mathrm{Ind}_U^T(\theta)(1) = \sum_{i\in I} e_i\nu_i(1) = \sum_{i\in I} e_i^2\theta(1).$$

Thus $\sum_{i\in I} e_i^2 = [T : U]$.

Irreducible characters of B

We now apply the previous results to the irreducible characters of U, and obtain all the irreducible characters of B.

If $\theta \in \{\lambda_2, \ldots, \lambda_q, \psi_2, \ldots, \psi_q, \chi_{1,j}, \chi_{2,j}, \overline{\chi_{1,j}}, \overline{\chi_{2,j}}\}$, then the H-orbit of θ has length $q-1 = |H|$, so that $I_H(\theta) = \{1\}$ and $I_B(\theta) = U$. Hence $\mathrm{Ind}_U^B(\theta)$ is irreducible, of degree $(q-1)\theta(1)$.

If $\theta \in \{\chi_{3,j}, \overline{\chi_{3,j}} \,|\, 1 \le j \le (q-1)/2\}$, then the H-orbit of θ has length $(q-1)/2$, so that $|I_H(\theta)| = 2$, and thus $I_H(\theta) =< J >$ where J is **the** involution of H, and $T = I_B(\theta) = U < J >$. Hence $\mathrm{Ind}_U^T(\theta) = \sum_{i\in I} e_i\nu_i$ and $\sum_{i\in I} e_i^2 = [T : U] = 2$, so that $|I| = 2$ and $e_1 = e_2 = 1$. Thus $\mathrm{Ind}_U^T(\theta)$ is the sum of **two** irreducible characters ν_1 and ν_2 of T, each with multiplicity 1, and of same degree as θ. Then $\mathrm{Ind}_U^B(\theta) = \mu_1 + \mu_2$, with $\mu_1, \mu_2 \in \mathrm{Irr}(B)$ distinct, and $\mu_1(1) = \mu_2(1) = [B : T]\nu_{1,2}(1) = \theta(1)(q-1)/2$.

Furthermore, two characters of the same H-orbit of $\mathrm{Irr}(U)$ give the same induced character of B (and thus the same subset or $\mathrm{Irr}(B)$), and two distinct orbits give two **disjoint** subsets of $\mathrm{Irr}(B)$. We obtain

$\mathrm{Irr}(U)$		$\mathrm{Irr}(B)$
$\{\lambda_2, \ldots, \lambda_q\}$	$\longrightarrow$	$\lambda,$ degree $q-1$
$\{\psi_2, \ldots, \psi_q\}$	$\longrightarrow$	$\psi,$ degree $(q-1)q$
$\{\chi_{i,j},\, i = 1, 2$ and $1 \le j \le \frac{q-1}{2}\}$	$\longrightarrow$	$\chi,$ degree $(q-1)3^m$
$\{\overline{\chi_{i,j}},\, i = 1, 2$ and $1 \le j \le \frac{q-1}{2}\}$	$\longrightarrow$	$\overline{\chi},$ degree $(q-1)3^m$
$\{\chi_{3,j},\, 1 \le j \le \frac{q-1}{2}\}$	$\longrightarrow$	$\mu_1, \mu_2,$ degree $\frac{q-1}{2}3^m$
$\{\overline{\chi_{3,j}},\, 1 \le j \le \frac{q-1}{2}\}$	$\longrightarrow$	$\overline{\mu_1}, \overline{\mu_2},$ degree $\frac{q-1}{2}3^m$

Finally, B has (at least) $q-1$ linear irreducible characters $\alpha_1 = 1_B, \ldots \alpha_{q-1}$ given by $\alpha_i = \pi \circ \tilde{\alpha}_i$, where $\pi \colon B = UH \longrightarrow H$ is the natural homomorphism and $\mathrm{Irr}(H) = \{\tilde{\alpha}_i,\, 1 \le i \le q-1\}$. Each of the α_i's appears with multiplicity 1 in $\mathrm{Ind}_U^B(1_U)$, and, since $\mathrm{Ind}_U^B(1_U)(1) = q-1$, we have $\mathrm{Ind}_U^B(1_U) = \sum_{i=1}^{q-1} \alpha_i$, and the α_i's are the only linear irreducible characters of B.

Values on elements of order 9

From the results on H-classes of U, we see that there are three classes of elements of order 9 (of type $x(t \neq 0, u_i, v)$) in B. We take representatives x_1, x_2 and x_3, and we want the values of $\mathrm{Irr}(B)$ on the x_i's.

The α_i's are lifted from H, so have U in their kernel, so that $\alpha_i(x_j) = 1$ for all $i = 1, \ldots, q-1$ and $j = 1, 2, 3$.

We have $\psi|_U = \sum_{i=2}^{q} \psi_i$, and the ψ_i's are 0 on $U \setminus Z(U)$. Thus $\psi(x_j) = 0$ for all $j = 1, 2, 3$.

We have $\lambda|_U = \sum_{i=2}^{q} \lambda_i$. For each $j = 1, 2, 3$, using the fragment of the character table of U we found, the second orthogonality relation applied (in U/U') to x_j and 1 gives $\sum_{i=1}^{q} \lambda_i(x_j)\lambda_i(1) = 0$, and $\lambda_i(1) = 1$ for all i, so that $\lambda(x_j) = -\lambda_1(x_j) = -1$ (note that this doesn't depend on the U-conjugacy class in which we took x_j).

Now we have $(\chi + \mu_1)|_U = \sum_{k=1}^{(q-1)/2}(\chi_{1,k} + \chi_{2,k} + \chi_{3,k})$, a character of $U/Z(U)$, and

$$3^m(\chi + \mu_1)|_{U/Z(U)} = \sum_{k=1}^{\frac{q-1}{2}} 3^m(\chi_{1,k} + \chi_{2,k} + \chi_{3,k}) = \sum_{k=1}^{\frac{q-1}{2}} \mathrm{Ind}_W^{U/Z(U)}(\gamma_{k+1}),$$

where $\mathrm{Irr}(W) = \{\gamma_1 = 1_W, \gamma_2, \ldots, \gamma_q\}$, and, since $W = Z(U/Z(U))$, $3^m(\chi + \mu_1)|_U \equiv 0$ on $(U/Z(U)) \setminus W$. Hence $(\chi + \mu_1)(x_j) = 0$ for $j = 1, 2, 3$. Since $\mu_1|_U = \mu_2|_U$, we have, for some $a, b, c \in \mathbf{C}$,

	χ	μ_1, μ_2	$\overline{\chi}$	$\overline{\mu_1}, \overline{\mu_2}$	$\alpha_1, \ldots, \alpha_{q-1}$	λ	ψ
x_1	a	$-a$	$\bar{a}$	$-\bar{a}$	1	-1	0
x_2	b	$-b$	$\bar{b}$	$-\bar{b}$	1	-1	0
x_3	c	$-c$	$\bar{c}$	$-\bar{c}$	1	-1	0

The second orthogonality relation applied to each line gives, since $|C_B(x_i)| = 3q$, $|a| = |b| = |c| = 3^m$.

Now we have, for any $t, u, v \in \mathbf{F}_q$, $h(-1)x(t,u,v)h(-1)^{-1} = x(-t, u, -v)$. If we take $u = t = 1$, then $x(t,u,v)^{-1} = x(-1, 1, -v) = x(-t, u, -v)$. Hence $x(1,1,v)$ and $x(1,1,v)^{-1}$ are conjugate in B, to x_1 say. Thus a is **real**, and $a = \pm 3^m$.

Now the second orthogonality relation applied to the first and second line and to the first and third line gives that $\mathrm{Re}(b) = \mathrm{Re}(c) = \mp 3^m/2$. Together with $|b| = |c| = 3^m$, this gives us that $b, c \notin \mathbf{R}$, so that $c = \bar{b}$ (and x_2^{-1} is conjugate to x_3), and, writing $\omega = e^{2i\pi/3}$, we have $b3^{-m} \in \{\omega, \bar{\omega}, -\omega, -\bar{\omega}\}$. Hence, we see that, up to exchanging χ and $\overline{\chi}$ (and thus μ_i and $\overline{\mu_i}$), or x_2 and x_3, there exists $\varepsilon \in \{\pm 1\}$ such that $a = \varepsilon 3^m$, $b = \varepsilon 3^m \bar{\omega}$ and $c = \varepsilon 3^m \omega$.

2.5.5 A generalized perfect isometry

Since the Sylow 3-subgroup U of G is disjoint from its conjugates, the 3-blocks of G have either full defect or defect 0. Furthermore, we have seen that the elements of $Z(U) \setminus \{1\}$ form one H-orbit. Since $H = q-1 = |Z(U) \setminus \{1\}|$, this shows that each element of $Z(U) \setminus \{1\}$ is centralized by no non-identity element of H. Thus we get that, for $z \in Z(U) \setminus \{1\}$, $C_B(z) = U$, so that $C_B(U) \leq U$, and, as $U = O_3(B)$, B has only **one** 3-block, the principal one b_0. This implies that G has only one block B_0 of maximal defect. Since G has only one character ξ_3 of defect 0, we get $b_0 = \mathrm{Irr}(B)$ and $B_0 = \mathrm{Irr}(G) \setminus \{\xi_3\}$. In particular, we see that $|B_0| = |b_0| = q+7$.

In G, the only elements of order divisible by 9 have order precisely 9. Hence the same must be true in B. We have representatives for the 9-singular classes Y, YT and YT^{-1} in G, and x_1, x_2 and x_3 in B. We have already seen that $|C_B(x_i)| = 3q$ for $i = 1, 2, 3$. Using the character table of G in [26], we find that $|C_G(Y)| = |C_G(YT)| = |C_G(YT^{-1})| = 3q$. Here again, it is thus sufficient to find a bijection with signs between B_0 and b_0 wich preserves the values of characters on 9-singular elements.

We now consider the following fragments of character table for B_0 (cf [26]) and b_0:

	ξ_4	$\xi_1, \xi_2, \eta_r, \eta'_r$	$\eta_t, \eta'_t, \eta_i^-, \eta_i^+$	ξ_5, ξ_6	ξ_7, ξ_8	ξ_9	ξ_{10}
Y	0	1	-1	3^m	3^m	-3^m	-3^m
YT	0	1	-1	$3^m\bar{\omega}$	$3^m\omega$	$-3^m\bar{\omega}$	$-3^m\omega$
YT^{-1}	0	1	-1	$3^m\omega$	$3^m\bar{\omega}$	$-3^m\omega$	$-3^m\bar{\omega}$

(where the numbers of exceptionnal characters are
$|\{\eta_r, \eta'_r\}| = (q-3)/2$ and $|\{\eta_t, \eta'_t, \eta_i^-, \eta_i^+\}| = (q-1)/2$)

	ψ	$\alpha_1, \ldots, \alpha_{q-1}$	λ	μ_1, μ_2	$\overline{\mu_1}, \overline{\mu_2}$	χ	$\overline{\chi}$
x_1	0	1	-1	$-\varepsilon 3^m$	$-\varepsilon 3^m$	$\varepsilon 3^m$	$\varepsilon 3^m$
x_2	0	1	-1	$-\varepsilon 3^m\bar{\omega}$	$-\varepsilon 3^m\omega$	$\varepsilon 3^m\bar{\omega}$	$\varepsilon 3^m\omega$
x_3	0	1	-1	$-\varepsilon 3^m\omega$	$-\varepsilon 3^m\bar{\omega}$	$\varepsilon 3^m\omega$	$\varepsilon 3^m\bar{\omega}$

It is then easy to see that there is a generalized perfect isometry between B_0 and b_0: listing the irreducible characters of B_0 and b_0 in the same order as in the above tables, and relabelling them so that $\mathrm{Irr}(B_0) = \{\Xi_1, \ldots, \Xi_{q+7}\}$ and $\mathrm{Irr}(b_0) = \{X_1, \ldots, X_{q+7}\}$, then the following is a generalized perfect isometry between B_0 and b_0:

$$I\colon\ \Xi_i \longmapsto \varepsilon_i X_i\ ,\ i = 1, \ldots, q+7,$$

where $\varepsilon_i = -1$ if $X_i \in \{\alpha_{(q+3)/2}, \ldots, \alpha_{q-1}\}$, $\varepsilon_i = -\varepsilon$ if $X_i \in \{\mu_1, \mu_2, \overline{\mu_1}, \overline{\mu_2}, \chi, \overline{\chi}\}$, and $\varepsilon_i = 1$ otherwise.

We summarize the results we obtained in the following

Theorem 2.6. *Suppose $G = SU(3, q^2)$, $Sz(q)$ or $Re(q)$, and p is the defining characteristic of G. Let U be a Sylow p-subgroup of G, and $B = N_G(U)$. Let B_0 and b_0 be the principal p-blocks of G and B respectively. Then there is a generalized perfect isometry with respect to p^2-regular elements between B_0 and b_0.*

Part 3

Cartan Group, Generalized Characters

3.1 Cartan Group, Factors

In this section, we introduce the notions of Cartan group and Cartan matrix, as presented in [18].

Take G a finite group. Take $\mathcal{C}$ a *closed* set of conjugacy classes of G, and $\mathcal{C}' = G \setminus \mathcal{C}$. Let $\Gamma(\mathcal{C}, G) = ((< \chi, \psi >_{\mathcal{C}}))_{\chi,\psi \in \mathrm{Irr}(G)}$ be the $\mathcal{C}$-contribution matrix of G. Write $ch(G) = \mathbf{Z}\mathrm{Irr}(G)$ the set of generalized characters of G. We define two $\mathbf{Z}$-submodules of the space of complex class-functions of G: let

$$\mathcal{R}(\mathcal{C}) = \{\alpha^{\mathcal{C}},\, \alpha \in ch(G)\} =< \chi^{\mathcal{C}},\, \chi \in \mathrm{Irr}(G) >_{\mathbf{Z}}$$

and

$$\mathcal{P}(\mathcal{C}) = \{\beta \in ch(G) |\, \beta \equiv 0 \text{ outside } \mathcal{C}\}$$

(then $\mathcal{P}(\mathcal{C})$ is the $\mathbf{Z}$-submodule of $\mathcal{R}(\mathcal{C})$ consisting of generalized characters).

The fact that $\mathcal{C}$ is closed implies (via Galois Theory) that the modules $\mathcal{R}(\mathcal{C})$ and $\mathcal{P}(\mathcal{C})$ have the same $\mathbf{Z}$-rank, and that this rank is the number of conjugacy classes in $\mathcal{C}$ (cf [24]): we have

$$s := k(\mathcal{C}) = rk_{\mathbf{Z}}(\Gamma(\mathcal{C}, G)) = rk_{\mathbf{Z}}(\mathcal{R}(\mathcal{C})) = rk_{\mathbf{Z}}(\mathcal{P}(\mathcal{C})).$$

The quotient $\mathcal{R}(\mathcal{C})/\mathcal{P}(\mathcal{C})$ is thus an Abelian group, which we call the *Cartan group* of G (with respect to $\mathcal{C}$), and denote by $Cart(\mathcal{C}, G)$, or just $Cart(\mathcal{C})$. We have $Cart(\mathcal{C}) \cong C_{d_1} \times \cdots \times C_{d_s}$ (a product of s cyclic groups).

Now $\mathcal{R}(\mathcal{C})$, $\mathcal{P}(\mathcal{C})$ and $Cart(\mathcal{C})$ have decompositions into direct sums corresponding to the $\mathcal{C}$-blocks of G. For B a union of $\mathcal{C}$-blocks of G, we denote by $\mathcal{R}(\mathcal{C}, B)$, $\mathcal{P}(\mathcal{C}, B)$ and $Cart(\mathcal{C}, B)$ the corresponding direct summands of $\mathcal{R}(\mathcal{C})$, $\mathcal{P}(\mathcal{C})$ and $Cart(\mathcal{C})$ respectively.

For a given $\mathbf{Z}$-basis $\{\varphi_i,\ 1 \leq i \leq r\}$ of $\mathcal{P}(\mathcal{C}, B)$, we define the *Cartan matrix* of B to be the matrix $C(B) = ((< \varphi_i, \varphi_j >_{\mathcal{C}}))_{1 \leq i,j \leq r}$. A different choice of $\mathbf{Z}$-basis leads to a Cartan matrix which is equivalent over $\mathbf{Z}$ to $C(B)$ (more precisely, if $C'(B)$ is the new Cartan matrix, then there exists a unimodular integral matrix A such that $C'(B) = A^t C(B) A$). In particular, both Cartan matrices have the same invariant factors. The invariant factors are precisely the orders of the cyclic factors of $Cart(\mathcal{C}, B)$.

Now the invariant factors of $C(B)$ are also linked to the invariant factors of the contribution matrix. There exists a $\mathbf{Z}$-basis $\{\psi_i,\ 1 \leq i \leq r\}$ for $\mathcal{R}(\mathcal{C}, B)$ such that $\{d_i \psi_i,\ 1 \leq i \leq r\}$ is a $\mathbf{Z}$-basis for $\mathcal{P}(\mathcal{C}, B)$. Then d_r is the smallest positive integer such that $d_r \Gamma(\mathcal{C}, B)$ has integer entries, and the non-zero invariant factors of $d_s \Gamma(\mathcal{C}, G)$ are $1 = \frac{d_s}{d_r}, \frac{d_r}{d_{r-1}}, \ldots, \frac{d_r}{d_1}$ (cf [18], Lemma 1.3).

All these definitions and properties also apply to $\mathcal{C}'$. From the definitions for $\mathcal{P}$ and $\mathcal{R}$ and the definition of blocks, we see that $\mathcal{P}(\mathcal{C}, B) \oplus \mathcal{P}(\mathcal{C}', B) \subset \mathcal{R}(G, B)$. Furthermore, using the fact that $\mathcal{C}$ is closed, Galois Theory implies that $|G| \mathcal{R}(G, B) \subset \mathcal{P}(\mathcal{C}, B) \oplus \mathcal{P}(\mathcal{C}', B)$. Hence we get that $rk_{\mathbf{Z}}(\mathcal{P}(\mathcal{C}, B)) + rk_{\mathbf{Z}}(\mathcal{P}(\mathcal{C}', B) = rk_{\mathbf{Z}}(\mathcal{R}(\mathcal{G}, B)) = |B|$. Thus, if $\Gamma(\mathcal{C}, B)$ has rank r, then $\Gamma(\mathcal{C}, B)$ has rank $|B| - r$.

Finally, note that, if $\mathcal{C}$ is the set of p-regular elements of G for some prime p, then the invariant factors of the Cartan matrix are known to be the orders of the p-defect groups of p-regular classes (cf e.g. Isaacs [16]).

3.2 Generalized Characters

3.2.1 Order in the Cartan group

We have seen that the exponent d_s of the Cartan group $Cart(\mathcal{C})$ is the smallest positive integer such that $d_s \Gamma(\mathcal{C}, G)$ has integral entries. This implies that, for any $\chi \in \mathrm{Irr}(G)$, the class function $d_s \chi^{\mathcal{C}}$ is a generalized character. Indeed, if $\chi \in \mathrm{Irr}(G)$, then $\chi^{\mathcal{C}}$ is a class function of G, and we can write

$$\begin{aligned} \chi^{\mathcal{C}} &= \sum_{\psi \in \mathrm{Irr}(G)} < \chi^{\mathcal{C}}, \psi >_G \psi \\ &= \sum_{\psi \in \mathrm{Irr}(G)} < \chi^{\mathcal{C}}, \psi^{\mathcal{C}} >_G \psi \\ &= \sum_{\psi \in \mathrm{Irr}(G)} < \chi, \psi >_{\mathcal{C}} \psi. \end{aligned}$$

And, since $d_s < \chi, \psi >_{\mathcal{C}} \in \mathbf{Z}$ for all $\psi \in \mathrm{Irr}(G)$, we see that $d_s\chi^{\mathcal{C}}$ is a $\mathbf{Z}$-linear combination of irreducible characters of G, i.e. a generalized character.

Given any $\chi \in \mathrm{Irr}(G)$, we would like to find the **smallest** positive integer d such that $d\chi^{\mathcal{C}}$ is a generalized character. The integer d will be called the *order* of χ in $Cart(\mathcal{C})$.

From what we wrote above, we deduce that, for $\chi \in \mathrm{Irr}(G)$ and $d \in \mathbf{N}$,

$$\begin{aligned} d\chi^{\mathcal{C}} \in ch(G) &\iff d < \chi^{\mathcal{C}}, \psi^{\mathcal{C}} > \in \mathbf{Z},\ \forall \psi \in \mathrm{Irr}(G) \\ &\iff \begin{cases} d < \chi^{\mathcal{C}}, \psi^{\mathcal{C}} > \in \mathbf{Z},\ \forall \psi \in \mathrm{Irr}(G) \text{ such} \\ \quad \text{that } \psi \text{ is directly } \mathcal{C}\text{-linked to } \chi \end{cases} \end{aligned}$$

Thus the order of χ in $Cart(\mathcal{C})$ is the smallest positive integer d such that d times the column of $\Gamma(\mathcal{C}, G)$ corresponding to χ has integral entries. Then, the decomposition of $d\chi^{\mathcal{C}}$ as a linear combination of irreducible characters of G can be read from the contribution matrix. Only characters which are directly linked to χ will appear with non-zero coefficients. Finally, notice that, for $\chi \in \mathrm{Irr}(G)$, since $\chi = \chi^{\mathcal{C}} + \chi^{\mathcal{C}'}$, then χ has the same order in $Cart(\mathcal{C})$ and $Cart(\mathcal{C}')$.

3.2.2 First observations

Order of the trivial character

First note that, for each $\chi \in \mathrm{Irr}(G)$, we can write $\chi^{\mathcal{C}} = \chi \otimes 1_G^{\mathcal{C}}$, so that, for any $d \in \mathbf{N}$, we have $d\chi^{\mathcal{C}} = \chi \otimes (d1_G^{\mathcal{C}})$. Hence, if $d1_G^{\mathcal{C}} \in ch(G)$, then $d\chi^{\mathcal{C}} \in ch(G)$ for all $\chi \in \mathrm{Irr}(G)$. This implies that the trivial character 1_G has maximal order in $Cart(\mathcal{C})$, and thus that the order of 1_G in $Cart(\mathcal{C})$ is the exponent d_s, and that, for each $\chi \in \mathrm{Irr}(G)$, the order of χ in $Cart(\mathcal{C})$ divides d_s.

Characters of order 1

We next study $\mathcal{C}$-blocks of G consisting of a single character. Suppose $\{\chi\}$ is such a $\mathcal{C}$-block of G. Then $< \chi, \psi >_{\mathcal{C}} = 0$ for all $\psi \in \mathrm{Irr}(G)$, $\psi \neq \chi$. Hence we have $\chi^{\mathcal{C}} = t\chi$ for some $t \in \mathbf{C}$. If we suppose furthermore that $1 \in \mathcal{C}$, then $\chi(1) = \chi^{\mathcal{C}}(1) = t\chi(1)$ leads to $t = 1$ and $\chi^{\mathcal{C}} = \chi$. Hence χ vanishes outside $\mathcal{C}$, and χ has order 1 in $Cart(\mathcal{C})$.

Conversely, if χ vanishes outside $\mathcal{C}$, then $\chi^{\mathcal{C}} = \chi$, so that χ has order 1 in $Cart(\mathcal{C})$, and $< \chi, \psi >_{\mathcal{C}} = < \chi^{\mathcal{C}}, \psi >_G = < \chi, \psi >_G = 0$ for all $\psi \in \mathrm{Irr}(G) \setminus \{\chi\}$, so that $\{\chi\}$ is a $\mathcal{C}$-block of G.

Finally, note that, if $\{\chi\}$ is a $\mathcal{C}$-block of G, since $< \chi, \psi >_G = < \chi, \psi >_{\mathcal{C}} + < \chi, \psi >_{\mathcal{C}'}$ for all $\psi \in \mathrm{Irr}(G)$, we obtain that $< \chi, \psi >_{\mathcal{C}'} = 0$ for all

$\psi \in \mathrm{Irr}(G) \setminus \{\chi\}$, so that $\{\chi\}$ is also a $\mathcal{C}'$-block of G. Hence, if $1 \notin \mathcal{C}$, then $1 \in \mathcal{C}'$ and $\chi^{\mathcal{C}'} = \chi$.

Hence, for $\chi \in \mathrm{Irr}(G)$, $\{\chi\}$ is a $\mathcal{C}$-block of G **if and only if** χ vanishes outside $\mathcal{C}$ (if $1 \in \mathcal{C}$) or χ vanishes outside $\mathcal{C}'$ (if $1 \in \mathcal{C}'$).

Now suppose that $\chi \in \mathrm{Irr}(G)$ has order 1 in $Cart(\mathcal{C})$. Then $\chi^{\mathcal{C}} \in ch(G)$ and, in particular, $< \chi^{\mathcal{C}}, \chi >_G \in \mathbf{Z}$. However, $< \chi^{\mathcal{C}}, \chi >_G = \frac{1}{|G|} \sum_{g \in \mathcal{C}} |\chi(g)|^2$, so that $0 \leq < \chi^{\mathcal{C}}, \chi >_G \leq 1$. This yields that $< \chi^{\mathcal{C}}, \chi >_G \in \{0, 1\}$. Supposing $1 \in \mathcal{C}$, we have $< \chi^{\mathcal{C}}, \chi >_G \neq 0$, so that $< \chi^{\mathcal{C}}, \chi >_G = 1$. But then $< \chi^{\mathcal{C}'}, \chi >_G = < \chi, \chi >_G - < \chi^{\mathcal{C}}, \chi >_G = 0$ and, since $< \chi^{\mathcal{C}'}, \chi >_G = \frac{1}{|G|} \sum_{g \in \mathcal{C}'} |\chi(g)|^2$, we obtain that χ vanishes identically on $\mathcal{C}'$. Supposing $1 \in \mathcal{C}'$, we would obtain that χ vanishes identically on $\mathcal{C}$.

Finally, we obtain

Proposition 3.1. *Suppose* $1 \in \mathcal{C}$. *Then, for* $\chi \in Irr(G)$, *the following are equivalent:*

- *(i)* $\{\chi\}$ *is a* $\mathcal{C}$*-block of* G.
- *(ii)* χ *vanishes outside* $\mathcal{C}$.
- *(iii)* χ *has order 1 in* $Cart(\mathcal{C})$.

Blockwise considerations

We have seen that at least one character (namely the trivial one) must have maximal order d_s, which is the biggest invariant factor of the cartan matrix. However, it is not clear if, for example, for each invariant factor d_i of the Cartan matrix, there is an irreducible character whose order is d_i (and, in fact, it is false in general). However, by looking at things block by block, we can obtain a bit more information. Namely, if $\mathcal{C}$ is the set of p^k-regular elements, where p is a prime and k is a positive integer, then, in each $\mathcal{C}$-block of G, there is (at least) one character whose order in $Cart(\mathcal{C})$ is an invariant factor of the Cartan matrix.

Let $B_1, \ldots, B_l$ be the $\mathcal{C}$-blocks of G. Then $\Gamma(\mathcal{C}, G)$ has a block-structure corresponding to the blocks of G:

$$\Gamma(\mathcal{C}, G) = \begin{pmatrix} \Gamma_1(\mathcal{C}) & & (0) \\ & \ddots & \\ (0) & & \Gamma_l(\mathcal{C}) \end{pmatrix}$$

where we write $\Gamma_i(\mathcal{C})$ for the $\mathcal{C}$-contribution matrix of B_i.

We have a similar decomposition for the Cartan group:

$$Cart(\mathcal{C}, G) = \bigoplus_{i=1}^{l} Cart(\mathcal{C}, B_i).$$

We let $M(\mathcal{C}) = d_s\Gamma(\mathcal{C}, G) \in M_n(\mathbf{Z})$. The Smith Normal Form of $M(\mathcal{C})$ is

$$D = \begin{pmatrix} 1 & & & & & & (0) \\ & \frac{d_s}{d_{s-1}} & & & & & \\ & & \ddots & & & & \\ & & & \frac{d_s}{d_1} & & & \\ & & & & 0 & & \\ & (0) & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

i.e. there exists $U, V \in M_n(\mathbf{Z})$ unimodular integral matrices such that $M(\mathcal{C}) = d_s\Gamma(\mathcal{C}, G) = UDV$. And, in fact, because of the block-structure of $M(\mathcal{C})$, there exist $U_1, \ldots, U_l$ and $V_1, \ldots, V_l$ unimodular integral matrices such that

$$M(\mathcal{C}) = \begin{pmatrix} U_1 & & (0) \\ & \ddots & \\ (0) & & U_l \end{pmatrix} \begin{pmatrix} D_1 & & (0) \\ & \ddots & \\ (0) & & D_l \end{pmatrix} \begin{pmatrix} V_1 & & (0) \\ & \ddots & \\ (0) & & V_l \end{pmatrix}$$

where D_i is the Smith Normal Form of $\Gamma_i(\mathcal{C})$. (Note that, in general, $V \neq \begin{pmatrix} V_1 & & (0) \\ & \ddots & \\ (0) & & V_l \end{pmatrix}$ (and similarly for U), since the columns and rows may need to be reordered to obtain D from $\begin{pmatrix} D_1 & & (0) \\ & \ddots & \\ (0) & & D_l \end{pmatrix}$.)

Hence, for $i \in \{1, \ldots, l\}$, we have

$$M_i(\mathcal{C}) := d_s\Gamma_i(\mathcal{C}) = U_iD_iV_i \qquad (\dagger)$$

and

$$D_i = \begin{pmatrix} \frac{d_s}{d_{i,1}} & & & & & \\ & \ddots & & & & \\ & & \frac{d_s}{d_{i,k_i}} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

where

$$\begin{cases} k_i = rk_{\mathbf{Z}}(\Gamma_i(\mathcal{C})) \text{ (and thus } s = \sum_{i=1}^{l} k_i), \\ \{d_{i,1}, \ldots, d_{i,k_i}\} \subset \{d_1, \ldots, d_s\} \text{ and } d_{i,k_i} | d_{i,k_{i-1}} | \cdots | d_{i,1} \text{ for all } 1 \leq i \leq l, \\ \text{and } \{d_1, \ldots, d_s\} = \bigcup_{i=1}^{l} \{d_{i,1}, \ldots, d_{i,k_i}\}. \end{cases}$$

Multiplying both sides of (†) by $\frac{1}{d_s} Id$, we obtain

$$\Gamma_i(\mathcal{C}) = U_i \begin{pmatrix} \frac{1}{d_{i,1}} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{d_{i,k_i}} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} V_i, \text{ for all } 1 \leq i \leq l.$$

And, as U_i and V_i are invertible in some $M_{n_i}(\mathbf{Z})$, we have

$$\begin{pmatrix} \frac{1}{d_{i,1}} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{d_{i,k_i}} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} = U_i^{-1} \Gamma_i(\mathcal{C}) V_i.$$

Hence, for $d \in \mathbf{N}$, $d\Gamma_i(\mathcal{C})$ is integral if and only if $\frac{d}{d_{i,1}}, \ldots, \frac{d}{d_{i,k_i}} \in \mathbf{Z}$, and thus if and only if $d_{i,1} | d$. Hence $d_{i,1}$ is the smallest positive integer such that $d_{i,1}\Gamma_i(\mathcal{C})$ is integral.

In particular, if $\mathcal{C}$ is the set of p^k-regular elements of G, where p is a prime and k is a positive integer, then, since the order of any irreducible character of G in $Cart(\mathcal{C})$ is a power of p (cf 3.2.3), we see that at least one character in B_i has order $d_{i,1}$ in $Cart(\mathcal{C})$ (the others having order dividing $d_{i,1}$).

3.2.3 The prime case

Let G be a finite group, and p be a prime number. We have the following classical result:

Theorem 3.2. *If $\mathcal{C}$ is the set of p-regular elements of G and $\chi \in Irr(G)$, then the order of χ in the Cartan group $Cart(\mathcal{C})$ is $p^{d(\chi)}$, where $d(\chi)$ is the p-defect of χ (i.e. $p^{d(\chi)} = \frac{|G|_p}{\chi(1)_p}$).*

A proof of this can be found for example in [20]. The first step is to show that $|G|_p\chi^{\mathcal{C}}$ is a generalized character, and is just an application of Brauer's Characterization of Characters (cf [20], Lemma (2.15)). Thus the order of χ in $Cart(\mathcal{C})$ is a power of p. It follows from this and the properties of the discrete valuation ring used in the construction of the p-blocks of G that $p^{d(\chi)}\chi^{\mathcal{C}}$ is a generalized character, while $p^{d(\chi)-1}\chi^{\mathcal{C}}$ is not (cf [20], Lemma (3.23)).

We now turn to the situation where $\mathcal{C}$ is the set of p^k-regular elements of G, for some $k \geq 1$. Then $\mathcal{C}$ is closed, and is a union of p-sections. This implies that $|G|_p 1_{\mathcal{C}}$ is a generalized character of G (where $1_{\mathcal{C}}$ is the characteristic function of $\mathcal{C}$). Now, if $\chi \in \mathrm{Irr}(G)$, then $|G|_p\chi^{\mathcal{C}} = \chi(|G|_p 1_{\mathcal{C}})$ is also a generalized character.

On the other hand, for any $\psi \in \mathrm{Irr}(G)$, we have $< \chi^{\mathcal{C}}, \psi >_G \in \mathbf{Q}$, so that $\frac{|G|}{\chi(1)} < \chi^{\mathcal{C}}, \psi >_G \in \mathbf{Q}$. But

$$\frac{|G|}{\chi(1)} < \chi^{\mathcal{C}}, \psi >_G = \sum_{i=1}^{s} [G : C_G(y_i)] \frac{\chi(y_i)}{\chi(1)} \overline{\psi(y_i)}$$

(where $y_1, \ldots, y_s$ are representatives for the conjugacy classes in $\mathcal{C}$), so that $\frac{|G|}{\chi(1)} < \chi^{\mathcal{C}}, \psi >_G$ is an algebraic integer.

Hence $\frac{|G|}{\chi(1)} < \chi^{\mathcal{C}}, \psi > \in \mathbf{Z}$ for all $\psi \in \mathrm{Irr}(G)$, and thus $\frac{|G|}{\chi(1)}\chi^{\mathcal{C}}$ is a generalized character.

Hence $|G|_p\chi^{\mathcal{C}}$ and $\frac{|G|}{\chi(1)}\chi^{\mathcal{C}}$ are generalized characters, and, since $\frac{|G|_p}{\chi(1)_p} = gcd(|G|_p, \frac{|G|}{\chi(1)})$, we see that $\frac{|G|_p}{\chi(1)_p}\chi^{\mathcal{C}} = p^{d(\chi)}\chi^{\mathcal{C}}$ is a generalized character. Thus the order of χ in $Cart(\mathcal{C})$ divides $p^{d(\chi)}$.

3.3 First Examples

In this section, we obtain information about the p^k-Cartan group and the orders of irreducible characters in some easy examples. The first is the case of Abelian groups; then we compute these informations for the three families of groups we studied in the previous part; finally, we study the particular case when the contribution matrix has $\mathbf{Z}$-rank one.

3.3.1 Abelian groups

Let G be an Abelian finite group, and p be a prime number such that $p^k||G|$ for some integer $k \geq 1$. We take $\mathcal{C}$ to be the set of p^k-regular elements of G.

Then $\mathcal{C}$ is in fact a (normal) subgroup of G. Indeed, write $G = G_pG_{p'}$, and take $g, h \in \mathcal{C}$; we can (uniquely) write $g = g_pg_{p'} = g_{p'}g_p$ and $h = h_ph_{p'} = h_{p'}h_p$, with, as g and h are p^k-regular, $o(g_p) < p^k$ and $o(h_p) < p^k$.
Then $gh = (g_ph_p)(g_{p'}h_{p'}) = (g_{p'}h_{p'})(g_ph_p)$, with $g_ph_p \in G_p$, $g_{p'}h_{p'} \in G_{p'}$, and

$$(o(gh))_p = o(g_ph_p) \leq Max(o(g_p), o(h_p)) \leq p^k$$

(the first inequality being true since both $o(g_p)$ and $o(h_p)$ are powers of p, so that $\text{lcm}(o(g_p), o(h_p)) = Max(o(g_p), o(h_p)))$.
Of course, $g^{-1} \in \mathcal{C}$ and $1 \in \mathcal{C}$.

Hence $\mathcal{C}$ is a normal subgroup of G, and, furthermore, $G = C_G(\mathcal{C})$. Thus, by Clifford's Theory, we see that, for all $\lambda \in$Irr$(\mathcal{C})$, $\text{Ind}_{\mathcal{C}}^G(\lambda)$ has $[G : \mathcal{C}]$ irreducible summands, each appearing with multiplicity 1 (since G is Abelian and all its irreducible characters have degree 1). And, if $\chi \in \text{Irr}(G)$ is such an irreducible summand, then $\text{Res}_{\mathcal{C}}^G(\chi) = \lambda$.

Furthermore, each irreducible character of G appears in this way. We can label each irreducible character of G with an irreducible character of $\mathcal{C}$ and an integer i, $1 \leq i \leq [G : \mathcal{C}]$. We write

$$\text{Irr}(G) = \{\chi_\lambda^{(i)} |\, \lambda \in \text{Irr}(\mathcal{C}),\ 1 \leq i \leq [G : \mathcal{C}]\},$$

where, for each $\lambda \in \text{Irr}(\mathcal{C})$, $\text{Ind}_{\mathcal{C}}^G(\lambda) = \sum_{i=1}^{[G:\mathcal{C}]} \chi_\lambda^{(i)}$.

Now, for all $\chi_\lambda^{(i)}, \chi_\mu^{(j)} \in \text{Irr}(G)$,

$$\begin{aligned} < \chi_\lambda^{(i)}, \chi_\mu^{(j)} >_{\mathcal{C}} &= \frac{1}{|G|} \sum_{g \in \mathcal{C}} \chi_\lambda^{(i)}(g)\chi_\mu^{(j)}(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in \mathcal{C}} \text{Res}_{\mathcal{C}}^G(\chi_\lambda^{(i)})(g)\text{Res}_{\mathcal{C}}^G(\chi_\mu^{(j)})(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in \mathcal{C}} \lambda(g)\mu(g^{-1}). \end{aligned}$$

Hence

$$< \chi_\lambda^{(i)}, \chi_\mu^{(j)} >_{\mathcal{C}} = \frac{1}{[G : \mathcal{C}]} < \lambda, \mu >_{\mathcal{C}} . \qquad (\dagger)$$

We see that $\chi_\lambda^{(i)}$ and $\chi_\mu^{(j)}$ belong to the same $\mathcal{C}$-block of G if and only if $\lambda = \mu$. Hence G has $|\mathcal{C}|$ $\mathcal{C}$-blocks, which we write $\{B_\lambda,\ \lambda \in \text{Irr}(\mathcal{C})\}$, each containing $[G : \mathcal{C}]$ irreducible characters:

$$B_\lambda = \{\chi_\lambda^{(i)},\ 1 \leq i \leq [G : \mathcal{C}]\},\ \forall \lambda \in \text{Irr}(\mathcal{C}).$$

It is therefore clear that all irreducible characters of G in a given $\mathcal{C}$-block have the same order in the Cartan group $Cart(\mathcal{C}, G)$ (since they have the same restriction to $\mathcal{C}$). It is also clear from (†) that this order is $[G : \mathcal{C}]$.

Note that, from (†), we deduce that all entries in the contribution matrix of any given $\mathcal{C}$-block of G will be the same, namely $\frac{1}{[G:\mathcal{C}]}$ (in particular, the contribution matrix of each $\mathcal{C}$-block has rank 1). This in turn implies that the elementary divisors of the $\mathcal{C}$-Cartan matrix (there are $|\mathcal{C}|$ of them) will all be equal to $[G : \mathcal{C}]$, and that the Cartan group $Cart(\mathcal{C}, G)$ is (isomorphic to) the direct product of $|\mathcal{C}|$ copies of the cyclic group of order $[G : \mathcal{C}]$.

Finally, note that, writing $|G| = p^a m$, with $gcd(p, m) = 1$, we have

$$|\mathcal{C}| = mp^{(k-1)\begin{pmatrix} a \\ a-k+1 \end{pmatrix}} = mp^{(k-1)\begin{pmatrix} a \\ k-1 \end{pmatrix}}$$

and

$$[G : \mathcal{C}] = p^{a-(k-1)\begin{pmatrix} a \\ k-1 \end{pmatrix}}.$$

3.3.2 Special unitary groups, Suzuki groups, Ree groups

Using the results of the previous part, it is easy to compute the $\mathcal{C}$-contributions in the case where G is $SU(3, q^2)$, $Sz(q)$ or $Re(q)$, p is the defining characteristic, and $\mathcal{C}$ is the set of p^2-singular or p^2-regular elements of G. The orders of the irreducible characters of the principal p-block B_0 of G in $Cart(\mathcal{C}, B_0)$ can be read directly from the contribution matrix, and the structure of the Cartan group can be obtained by reducing a multiple of the contribution matrix to Smith normal form (cf 3.1.1). The reduction to Smith normal form is performed by elementary row and column operation.

We will use the notations of the previous part: in each case, we write U a Sylow p-subgroup of G and $B = N_G(U)$; we let B_0 and b_0 be the principal p-blocks of G and B respectively; we let $\mathcal{C}$ and $\mathcal{D}$ be the sets of p^2-singular elements of G and B respectively, and we let $\mathcal{C}' = G \setminus \mathcal{C}$ and $\mathcal{D}' = B \setminus \mathcal{D}$.

In the three cases, we prefer to compute the Smith normal form of the matrix associated to the $\mathcal{D}$-contributions of $\mathrm{Irr}(b_0)$, as this is easier to manipulate than for B_0. Because of the perfect isometry we exhibited, this doesn't change anything to the results we're looking for.

The case of $SU(3, q^2)$

In this section, we let $G = SU(3, q^2)$ and $d = gcd(3, q + 1)$. With the values taken by the irreducible characters of b_0 on the elements of order divisible by

4, we compute the contribution matrix $\Gamma(\mathcal{D}, b_0)$. For any χ, $\psi \in b_0$, we have

$$< \chi, \psi >_{\mathcal{D}} = \frac{1}{|B|} \sum_{y \in \mathcal{D}} \chi(y) \overline{\psi(y)}.$$

If $d = 1$, then $\mathcal{D}$ is only **one** conjugacy class (namely $B_3^{(0,0)}$), so that

$$\begin{array}{rcl} < \chi, \psi >_{\mathcal{D}} & = & \frac{|\mathcal{D}|}{|B|} \chi(\mathcal{D}) \overline{\psi(\mathcal{D})} \\ & = & \frac{q(q^2-1)}{q^3(q^2-1)} \chi(\mathcal{D}) \psi(\mathcal{D}) \\ & = & \frac{1}{q^2} \chi(B_3^{(0,0)}) \psi(B_3^{(0,0)}). \end{array}$$

If $d = 3$, then we have, for any χ, $\psi \in b_0$,

$$\begin{array}{rcl} < \chi, \psi >_{\mathcal{D}} & = & \displaystyle\frac{1}{|B|} \sum_{y \in \mathcal{D}} \chi(y) \overline{\psi(y)} \\ & = & \displaystyle\frac{1}{|B|} \sum_{0 \leq k,l \leq 2} |B_3^{(k,l)}| \chi(B_3^{(k,l)}) \psi(B_3^{(k,l)}) \\ & = & \displaystyle\frac{1}{3q^2} \sum_{0 \leq k,l \leq 2} \chi(B_3^{(k,l)}) \psi(B_3^{(k,l)}). \end{array}$$

In particular, for any $0 \leq v \neq v' \leq 2$,

$$< \theta_{(q^2-1)/3}^{(0,v)}, \theta_1^{(u)} >_{\mathcal{D}} = \frac{1}{3q^2} \left(3(q - \frac{q+1}{3}) + 6(-\frac{q+1}{3}) \right) = \frac{1}{3q^2}(-3)$$

$$< \theta_{(q^2-1)/3}^{(0,v)}, \theta_{(q^2-1)/3}^{(0,v)} >_{\mathcal{D}} = \frac{1}{3q^2} \left(3(q - \frac{q+1}{3})^2 + 6(-\frac{q+1}{3})^2 \right) = \frac{1}{3q^2}(2q^2+1)$$

$$< \theta_{(q^2-1)/3}^{(0,v)}, \theta_{(q^2-1)/3}^{(0,v')} >_{\mathcal{D}} = \frac{1}{3q^2} \left(6(-\frac{q+1}{3})(q - \frac{q+1}{3}) + 3(-\frac{q+1}{3})^2 \right) = \frac{1}{3q^2}(1-q^2)$$

We obtain, labelling the lines and columns in the order we used to list the irreducible characters of b_0 in the previous part:

if $d = 1$,

$$\Gamma(\mathcal{D}, b_0) = \frac{1}{q^2}\left(\begin{array}{ccc|ccc|c} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 & -1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 & -1 \\ \hline 0 & \cdots & 0 & -1 & \cdots & -1 & 1 \end{array}\right)$$

$$\underbrace{\qquad}_{q+1} \quad \underbrace{\qquad}_{q^2-1} \quad \underbrace{\quad}_{1}$$

and if $d = 3$,

$$\Gamma(\mathcal{D}, b_0) = \frac{1}{3q^2}\left(\begin{array}{ccc|ccc|c|c|c} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline 0 & \cdots & 0 & 9 & \cdots & 9 & -3 & -3 & -3 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 9 & \cdots & 9 & -3 & -3 & -3 \\ \hline 0 & \cdots & 0 & -3 & \cdots & -3 & 2q^2+1 & 1-q^2 & 1-q^2 \\ \hline 0 & \cdots & 0 & -3 & \cdots & -3 & 1-q^2 & 2q^2+1 & 1-q^2 \\ \hline 0 & \cdots & 0 & -3 & \cdots & -3 & 1-q^2 & 1-q^2 & 2q^2+1 \end{array}\right)$$

$$\underbrace{\qquad}_{\frac{q+1}{3}} \quad \underbrace{\qquad}_{\frac{q^2-1}{3}} \quad \underbrace{\qquad\qquad}_{3}$$

In the array below, we show, for each $\mathcal{C}$-block of B_0, the 2-defect of the characters belonging to the block and their order in the Cartan group (these turn out to be the same for each character in a given $\mathcal{C}$-block). Recall that, for each character $\chi \in b_0$, the order of χ in $Cart(\mathcal{C}, B_0)$ (which is also its order in $Cart(\mathcal{C}', B_0)$) is the smallest positive integer δ such that δ times the column of $\Gamma(\mathcal{C}, B_0)$ is integral. Because of the generalized perfect isometry between B_0 and b_0, the corresponding character of b_0 has the same order in $Cart(\mathcal{D}, b_0)$. It also turns out that this doesn't depend on the value of d.

$\mathcal{C}$-block	$p^{d_2(\chi)}$	order in $Cart(\mathcal{C}, B_0)$
$\{\chi_{q^2-q}\}$; $\{\chi^{(u)}_{q(q^2-q+1)}\}$, $1 \leq u \leq q$	q^2	1
{All the rest of B_0}	q^3	q^2

We want now to compute the invariant factors $\{d_1, \ldots, d_s\}$ of the Cartan matrix $C_0(\mathcal{D})$ of b_0. The biggest one, d_s, is the smallest positive integer such that $d_s\Gamma(\mathcal{D}, b_0) = M(\mathcal{D})$ is an integral matrix.

It is easy to see that, whatever the value of d is, $d_s = q^2$ (just notice that, if $d = 3$, then $q \equiv -1 \pmod 3$, so that $2q^2 + 1$ and $1 - q^2$ are divisible by 3). The non-zero invariant factors of $M(\mathcal{D})$ are $\{1, \frac{d_s}{d_{s-1}}, \ldots \frac{d_s}{d_1}\}$.

If $d = 1$, the last column (resp. line) of $M(\mathcal{D})$ is the opposite of any other non-zero column (resp. line) of $M(\mathcal{D})$. Hence there exists an integral matrix V such that

$$M(\mathcal{D}) = V^t \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} V.$$

Thus $C_0(\mathcal{D})$ has exactly **one** non-zero invariant factor, $d_s = q^2$. We have $Cart(\mathcal{D}, b_0) \cong Cart(\mathcal{C}, B_0) \cong C_{q^2}$.

Suppose now that $d = 3$. We will denote by C_i and R_i the i-th column and row of any matrix.
Performing $C_i \longleftarrow C_i - C_{(q+q^2)/3}$ and $R_i \longleftarrow R_i - R_{(q+q^2)/3}$ for $\frac{q+1}{3} + 1 \leq i \leq \frac{q+q^2}{3} - 1$, $M(\mathcal{D})$ becomes

$$\left(\begin{array}{c|cccc} (0) & & (0) & & \\ \hline & 3 & -1 & -1 & -1 \\ (0) & -1 & \frac{2q^2+1}{3} & \frac{1-q^2}{3} & \frac{1-q^2}{3} \\ & -1 & \frac{1-q^2}{3} & \frac{2q^2+1}{3} & \frac{1-q^2}{3} \\ & -1 & \frac{1-q^2}{3} & \frac{1-q^2}{3} & \frac{2q^2+1}{3} \end{array}\right)$$

Then, $R_{(q+q^2)/3} \longleftarrow R_{(q+q^2)/3} + \sum(\text{non-zero } R_i\text{'s})$ and $C_{(q+q^2)/3} \longleftarrow C_{(q+q^2)/3} + \sum(\text{non-zero } C_i\text{'s})$ give

$$\left(\begin{array}{c|ccc} (0) & & (0) & \\ \hline & \frac{2q^2+1}{3} & \frac{1-q^2}{3} & \frac{1-q^2}{3} \\ (0) & \frac{1-q^2}{3} & \frac{2q^2+1}{3} & \frac{1-q^2}{3} \\ & \frac{1-q^2}{3} & \frac{1-q^2}{3} & \frac{2q^2+1}{3} \end{array}\right)$$

We then work on the only lower right block. $C_1 \longleftarrow C_1 + 2C_2$ gives

$$\left(\begin{array}{c|c|c} 1 & \frac{1-q^2}{3} & \frac{1-q^2}{3} \\ \hline q^2+1 & \frac{2q^2+1}{3} & \frac{1-q^2}{3} \\ \hline 1-q^2 & \frac{1-q^2}{3} & \frac{2q^2+1}{3} \end{array}\right) \xrightarrow{C_{2,3} \leftarrow C_{2,3} - \frac{1-q^2}{3}C_1} \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline q^2+1 & \frac{q^2(q^2+2)}{3} & \frac{q^2(q^2-1)}{3} \\ \hline 1-q^2 & \frac{q^2(1-q^2)}{3} & \frac{q^2(4-q^2)}{3} \end{array}\right)$$

Performing $R_2 \longleftarrow R_2 - (q^2+1)R_1$ and $R_3 \longleftarrow R_3 - (1-q^2)R_1$ produces

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \frac{q^2(q^2+2)}{3} & \frac{q^2(q^2-1)}{3} \\ \hline 0 & \frac{q^2(1-q^2)}{3} & \frac{q^2(4-q^2)}{3} \end{array}\right) \xrightarrow{C_2 \leftarrow C_2 - C_3} \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & q^2 & \frac{q^2(q^2-1)}{3} \\ \hline 0 & -q^2 & \frac{q^2(4-q^2)}{3} \end{array}\right)$$

Finally, $R_3 \longleftarrow R_3 + R_2$ and $C_3 \longleftarrow C_3 - \frac{q^2-1}{3}C_2$ give

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & q^2 & 0 \\ \hline 0 & 0 & q^2 \end{array}\right)$$

Whence we find that $s = rk_{\mathbf{Z}}(\Gamma(\mathcal{D}, b_0) = 3$, and the non-zero invariant factors of the Cartan matrix $C_0(\mathcal{D})$ are $d_3 = q^2$ and $d_2 = d_1 = 1$. Here again, we have $Cart(\mathcal{D}, b_0) \cong Cart(\mathcal{C}, B_0) \cong C_{q^2}$.

We now turn to elements of order not divisible by 4. We have seen from the definition that b_0 and B_0 are also perfectly isometric with respect to $\mathcal{C}'$ and $\mathcal{D}'$. Hence the Cartan groups $Cart(\mathcal{C}', B_0)$ and $Cart(\mathcal{D}', b_0)$ are isomorphic. We have $\Gamma(\mathcal{D}', b_0) = Id - \Gamma(\mathcal{D}, b_0)$, and we shall find the $|\mathrm{Irr}(b_0)| - d = \frac{q^2+q}{d}$ non-zero invariant factors of $q^2\Gamma(\mathcal{D}', b_0) = N(\mathcal{D}')$.

If $d = 1$, then

$$N(\mathcal{D}') = \left(\begin{array}{ccc|ccc|c} q^2 & & (0) & & & & 0 \\ & \ddots & & & (0) & & \vdots \\ (0) & & q^2 & & & & 0 \\ \hline & & & q^2-1 & & (-1) & 1 \\ & (0) & & & \ddots & & \vdots \\ & & & (-1) & & q^2-1 & 1 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 & q^2-1 \end{array}\right)$$

We reduce the lower-right (q^2 by q^2) block to Smith normal form. Reversing the order of the columns, we get

$$\begin{pmatrix}
1 & & & & q^2-1 \\
\vdots & (-1) & & q^2-1 & \\
\vdots & & \nearrow & & \\
1 & q^2-1 & & (-1) & \\
q^2-1 & 1 & \cdots & \cdots & 1
\end{pmatrix}$$

Then, $C_i \longleftarrow C_i + C_1$, $1 < i \leq q^2$ gives

$$\begin{pmatrix}
1 & & & & q^2 \\
\vdots & (0) & & q^2 & \\
\vdots & & \nearrow & & \\
1 & q^2 & & (0) & \\
q^2-1 & q^2 & \cdots & \cdots & q^2
\end{pmatrix}$$

$R_{q^2} \longleftarrow R_{q^2} - \sum_{i=1}^{q^2-1} R_i$ gives

$$\begin{pmatrix}
1 & & & & q^2 \\
\vdots & (0) & & q^2 & \\
\vdots & & \nearrow & & \\
1 & q^2 & & (0) & \\
0 & 0 & \cdots & \cdots & 0
\end{pmatrix}$$

$R_i \longleftarrow R_i - R_1$, $1 < i < q^2$, gives

$$\begin{pmatrix}
1 & 0 & \cdots & 0 & q^2 \\
0 & (0) & & q^2 & -q^2 \\
\vdots & & \nearrow & & \\
\vdots & q & & (0) & -q^2 \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}$$

$C_{q^2} \longleftarrow C_{q^2} - q^2 C_1$ gives

$$\begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & & & q^2 & -q^2 \\
\vdots & & \nearrow & & \vdots \\
\vdots & q^2 & & & -q^2 \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}$$

$C_{q^2} \longleftarrow C_{q^2} + \sum_{i=2}^{q^2-1} C_i$ gives

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & & & q^2 & 0 \\ \vdots & & \nearrow & & \vdots \\ \vdots & q^2 & & & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

which, reversing the order of the rows, apart from the first, gives finally

$$\begin{pmatrix} 1 & & & & \\ & q^2 & & & \\ & & \ddots & & \\ & & & q^2 & \\ & & & & 0 \end{pmatrix}$$

We obtain that $N(\mathcal{D}')$ is equivalent (over $\mathbf{Z}$) to

$$\left(\begin{array}{ccc|cccc|c} q^2 & & (0) & & & & & 0 \\ & \ddots & & & (0) & & & \vdots \\ (0) & & q^2 & & & & & 0 \\ \hline & & & 1 & & (0) & & 0 \\ & (0) & & & q^2 & & & \vdots \\ & & & & & \ddots & & \vdots \\ & & & (0) & & & q^2 & 0 \\ \hline 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 \end{array}\right)$$

so that the non-zero invariant factors of the Cartan matrix $C_0(\mathcal{D}')$ of b_0 are $d'_1 = \ldots = d'_{q^2+q-1} = 1$ and $d'_{q^2+q} = q^2$, and $Cart(\mathcal{D}', b_0) \cong Cart(\mathcal{C}', B_0) \cong C_{q^2}$.

If $d=3$, then

$$q^2\Gamma(\mathcal{D}',b_0)=\left(\begin{array}{ccc|ccc|c}
q^2 & & & & & & \\
& \ddots & & & (0) & & (0)\\
& & q^2 & & & & \\ \hline
& & & q^2-3 & & (-3) & \\
& (0) & & & \ddots & & (1)\\
& & & (-3) & & q^2-3 & \\ \hline
& (0) & & & (1) & & \left(\frac{q^2-1}{3}\right)
\end{array}\right)$$

$$\underset{\frac{q+1}{3}}{\longleftrightarrow}\qquad\underset{\frac{q^2-1}{3}}{\longleftrightarrow}\qquad\underset{3}{\longleftrightarrow}$$

By working on the last 3 rows and columns (which are the same), this can be reduced to

$$\left(\begin{array}{ccc|ccc|c|cc}
q^2 & & & & & & & & \\
& \ddots & & & & (0) & & & \\
& & q^2 & & & & & & \\ \hline
& & & q^2-3 & & (-3) & 1 & & \\
& & & & \ddots & & \vdots & (0) & \\
& & & (-3) & & q^2-3 & 1 & & \\ \cline{4-9}
& (0) & & 1 & \cdots & 1 & \frac{q^2-1}{3} & (0) & \\ \cline{4-9}
& & & & (0) & & (0) & 0 & 0\\
& & & & & & & 0 & 0
\end{array}\right)$$

Subtracting to the last non-zero column the sum of the preceeding $\frac{q^2-1}{3}$ columns, and then doing the same thing to the last non-zero row, we obtain(leaving aside the upper left (diagonal) block)

$$\left(\begin{array}{ccc|c}
q^2-3 & & (-3) & \\
& \ddots & & (0)\\
(-3) & & q^2-3 & \\ \hline
& (0) & & (0)
\end{array}\right)$$

Now we work separately on the non-zero block. $C_1 \longleftarrow C_1+\sum_{i>1} C_i$ gives

$$\begin{pmatrix}
1 & -3 & \cdots & -3\\
1 & q^2-3 & & (-3)\\
\vdots & & \ddots & \\
1 & (-3) & & q^2-3
\end{pmatrix}
\xrightarrow{C_i\leftarrow C_i+3C_1,\ i>1}
\begin{pmatrix}
1 & & & \\
1 & q^2 & & (0)\\
\vdots & & \ddots & \\
1 & (0) & & q^2
\end{pmatrix}$$

And, finally, $R_i \longleftarrow R_i - R_1$ for $i > 1$ gives

$$\begin{pmatrix} 1 & & & \\ & q^2 & & \\ & & \ddots & \\ & & & q^2 \end{pmatrix}$$

Putting all the blocks back together, we see that the non-zero invariant factors of the Cartan matrix $C_0(\mathcal{D}')$ are $d'_1 = \ldots = d'_{\frac{q+q^2}{3}-1} = 1$ and $d'_{\frac{q+q^2}{3}} = q^2$, and, here again, $Cart(\mathcal{D}', b_0) \cong Cart(\mathcal{C}', B_0) \cong C_{q^2}$.

The case of $Sz(q)$

In this section, we let $G = Sz(q)$, with $q = 2^{2n+1}$. With the values taken by the irreducible characters of b_0 on the elements of order divisible by 4, we compute the contribution matrix $\Gamma(\mathcal{D}, b_0)$. We have, for any $\chi,\ \psi \in \mathrm{Irr}(B)$,

$$\begin{aligned} <\chi,\ \psi>_{\mathcal{D}} &= \frac{1}{|B|}\sum_{y\in\mathcal{D}}\chi(y)\overline{\psi(y)} \\ &= \frac{1}{|B|}\left(\frac{q(q-1)}{2}(\chi(\rho)\overline{\psi(\rho)} + \chi(\rho^{-1})\overline{\psi(\rho^{-1})})\right) \\ &= \frac{1}{2q}(\chi(\rho)\psi(\rho^{-1}) + \chi(\rho^{-1})\psi(\rho)) \end{aligned}$$

We obtain, labelling the lines and columns in the order we used to list the irreducible characters of b_0 in the previous part:

$$\Gamma(\mathcal{D},\ b_0) = \frac{1}{q}\left(\begin{array}{c|c|cc} 1 & -1\cdots\cdots-1 & 0 & 0 \\ \hline -1 & & & \\ \vdots & (1)_{q-1,q-1} & (0) & \\ -1 & & & \\ \hline 0 & 0\cdots\cdots0 & \frac{q}{2} & \frac{-q}{2} \\ 0 & 0\cdots\cdots0 & \frac{-q}{2} & \frac{q}{2} \end{array}\right).$$

In the array below, we show, for each $\mathcal{C}$-block of B_0, the 2-defect of the characters belonging to the block and their order in the Cartan group. Here again, these turn out to be the same for each character in a given $\mathcal{C}$-block.

$\mathcal{C}$-block	$p^{d_2(\chi)}$	order in $Cart(\mathcal{C}, B_0)$
$\{1_G,\ \Omega_s,\ \Theta_l,\ \Lambda_u\}$	q^2	q
$\{\Gamma_1,\ \Gamma_2\}$	$q2^{n+1}$	2

The only irreducible character of G which doesn't belong to the principal 2-block is Π. It has 2-defect 0, so is a 2-block and a $\mathcal{C}$-block by itself, and has order 1 in the Cartan group $Cart(\mathcal{C}, G)$.

We want now to compute the invariant factors $\{d_1, \ldots, d_s\}$ of the Cartan matrix $C_0(\mathcal{D})$ of b_0. The biggest, d_s, is the smallest positive integer such that $d_s\Gamma(\mathcal{D}, b_0) = M(\mathcal{D})$ is an integral matrix. It is easy to see that, $d_s = q$. The non-zero invariant factors of $M(\mathcal{D})$ are $\{1, \frac{d_s}{d_{s-1}}, \ldots \frac{d_s}{d_1}\}$.

We will denote by C_i and R_i the i-th column and row of any matrix. We have

$$q\Gamma(\mathcal{D}, b_0) = \left(\begin{array}{c|c|cc} 1 & -1 \cdots\cdots -1 & 0 & 0 \\ \hline -1 & & & \\ \vdots & (1)_{q-1,q-1} & (0) & \\ -1 & & & \\ \hline 0 & 0 \cdots\cdots 0 & \frac{q}{2} & \frac{-q}{2} \\ 0 & 0 \cdots\cdots 0 & \frac{-q}{2} & \frac{q}{2} \end{array}\right)$$

Performing $C_i \longleftarrow C_i + C_1$ for $i > 1$, we obtain

$$\left(\begin{array}{c|c|cc} 1 & 0 \cdots\cdots 0 & 0 & 0 \\ \hline -1 & & & \\ \vdots & (0)_{q-1,q-1} & (0) & \\ -1 & & & \\ \hline 0 & 0 \cdots\cdots 0 & \frac{q}{2} & \frac{-q}{2} \\ 0 & 0 \cdots\cdots 0 & \frac{-q}{2} & \frac{q}{2} \end{array}\right)$$

Performing $R_i \longleftarrow R_i + R_1$ for $i > 1$, we get

$$\left(\begin{array}{c|c|cc} 1 & 0 \cdots\cdots 0 & 0 & 0 \\ \hline 0 & & & \\ \vdots & (0)_{q-1,q-1} & (0) & \\ 0 & & & \\ \hline 0 & 0 \cdots\cdots 0 & \frac{q}{2} & \frac{-q}{2} \\ 0 & 0 \cdots\cdots 0 & \frac{-q}{2} & \frac{q}{2} \end{array}\right)$$

Similarly, $C_{q+2} \longleftarrow C_{q+2} + C_{q+1}$ and $R_{q+2} \longleftarrow R_{q+2} + R_{q+1}$ lead to

$$\left(\begin{array}{c|c|cc} 1 & 0 \cdots\cdots 0 & 0 & 0 \\ \hline 0 & & & \\ \vdots & (0)_{q-1,q-1} & (0) & \\ 0 & & & \\ \hline 0 & 0 \cdots\cdots 0 & \frac{q}{2} & 0 \\ 0 & 0 \cdots\cdots 0 & 0 & 0 \end{array}\right).$$

Finally, there exists an integral matrix A such that

$$q\Gamma(\mathcal{D}, b_0) = A^t \begin{pmatrix} 1 & & & & \\ & \frac{q}{2} & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} A$$

whence $s = 2$, and the non-zero invariant factors of $C_0(\mathcal{D})$ are $d_2 = q$ and $d_1 = 2$. We have $Cart(\mathcal{C}, B_0) \cong Cart(\mathcal{D}, b_0) \cong C_q \times C_2$.

We now turn to the sets $\mathcal{C}'$ and $\mathcal{D}'$ of elements of order not divisible by 4 in G and B respectively. We want to compute the (common) invariant factors $\{d'_1, \ldots, d'_{s'}\}$ of $C_0(\mathcal{C}')$ and $C_0(\mathcal{D}')$. From the equality

$$rk_{\mathbf{Z}}(\mathcal{R}(b_0, \mathcal{D})) + rk_{\mathbf{Z}}(\mathcal{R}(b_0, \mathcal{D}')) = |b_0| = |B_0| = q + 2$$

we see that $s' = q$.

We have

$$q\Gamma(\mathcal{D}', b_0) = \left(\begin{array}{c|ccc|cc} q-1 & 1 & \cdots\cdots & 1 & 0 & 0 \\ \hline 1 & q-1 & & (-1) & & \\ \vdots & & \ddots & & \multicolumn{2}{c}{(0)} \\ 1 & (-1) & & q-1 & & \\ \hline 0 & 0 & \cdots\cdots & 0 & \frac{q}{2} & \frac{q}{2} \\ 0 & 0 & \cdots\cdots & 0 & \frac{q}{2} & \frac{q}{2} \end{array}\right)$$

so that it is clear that the biggest invariant factor we will find is $d'_q = q$. We work separately on the two blocks of $q\Gamma(\mathcal{D}, b_0)$. The upper-left block has the same structure as the matrix $N(\mathcal{D}')$ we obtained for $SU(3, q^2)$ in the case $d = 1$. Reducing it to Smith normal form, we obtain

$$\begin{pmatrix} 1 & & & & \\ & q & & & \\ & & \ddots & & \\ & & & q & \\ & & & & 0 \end{pmatrix}$$

As for the second block, it is easy to see that, performing $C_2 \longleftarrow C_2 - C_1$ and $R_2 \longleftarrow R_2 - R_1$, we obtain

$$\begin{pmatrix} \frac{q}{2} & \frac{q}{2} \\ \frac{q}{2} & \frac{q}{2} \end{pmatrix} \longmapsto \begin{pmatrix} \frac{q}{2} & 0 \\ 0 & 0 \end{pmatrix}$$

Whence we get that $q\Gamma(\mathcal{D}', b_0)$ is equivalent (over $\mathbf{Z}$) to

$$\begin{pmatrix} 1 & & & & & & \\ & \frac{q}{2} & & & & & \\ & & q & & & & \\ & & & \ddots & & & \\ & & & & q & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$$

This gives us that the invariant factors of the Cartan matrices $C_0(\mathcal{D}')$ and $C_0(\mathcal{C}')$ are $d'_1 = \cdots = d'_{q-2} = 1$, $d'_{q-1} = 2$, $d'_q = q$, and $Cart(\mathcal{C}', B_0) \cong Cart(\mathcal{D}', b_0) \cong C_q \times C_2$.

The case of $Re(q)$

In this section, we let $G = Re(q)$, with $q = 3^{2m+1}$. Then $\mathcal{D}$ is the set of elements of order divisible by 9 in B. We first compute the contribution matrix $\Gamma(\mathcal{D}, b_0)$. For any $\gamma, \delta \in b_0$, the $\mathcal{D}$-contribution of γ and δ is given by

$$< \gamma, \delta >_{\mathcal{D}} = \frac{1}{|B|} \sum_{y \in \mathcal{D}} \gamma(y)\overline{\delta(y)} = \frac{1}{3q} \sum_{i=1}^{3} \gamma(x_i)\overline{\delta(x_i)}.$$

In particular, $< \psi, \delta >_{\mathcal{D}} = 0$ for all $\delta \in b_0$. Furthermore, if $\gamma \in \{\mu_{1,2},\ \overline{\mu_{1,2}},\ \chi,\ \overline{\chi}\}$ and $\delta \in \{\psi,\ \alpha_i,\ \lambda\}$, then $< \gamma, \delta >_{\mathcal{D}} = 0$ (for we can factorize by $1+\omega+\bar{\omega} = 0$). We then compute

$$< \alpha_i, \alpha_j >_{\mathcal{D}} = \frac{1}{3q}3\,,\ < \lambda, \lambda >_{\mathcal{D}} = \frac{1}{3q}3\,,\ < \alpha_i, \lambda >_{\mathcal{D}} = \frac{1}{3q}(-3)\,,$$

$$< \mu_i, \mu_j >_{\mathcal{D}} = \frac{1}{3q}3^{2m}(1+1+1) = \frac{1}{3q}q\,,\ < \mu_i, \overline{\mu_j} >_{\mathcal{D}} = \frac{1}{3q}3^{2m}(1+\bar{\omega}^2+\omega^2) = 0\,,$$

$$< \chi, \chi >_{\mathcal{D}} = \frac{1}{3q}3^{2m}3 = \frac{1}{3q}q\,,\ < \chi, \overline{\chi} >_{\mathcal{D}} = 0\,,$$

$$< \mu_i, \chi >_{\mathcal{D}} = \frac{1}{3q}(-3^{2m})(1+1+1) = \frac{1}{3q}(-q)\,,\ < \mu_i, \overline{\chi} >_{\mathcal{D}} = \frac{1}{3q}(-3^{2m})(1+\bar{\omega}^2+\omega^2) = 0.$$

The other contributions are found using these and complex conjugation. Finally, we obtain, labelling the lines and columns in the order used in the character table we gave in the previous part,

$$\Gamma(\mathcal{D},b_0)=\frac{1}{3q}\left(\begin{array}{c|c|c|cc|cc|c|c}
0 & 0\cdots 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \hline
0 & 3\cdots 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0\\
\vdots & \vdots\quad\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 3\cdots 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0\\ \hline
0 & -3\cdots -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0\\ \hline
0 & 0\cdots 0 & 0 & q & q & 0 & 0 & -q & 0\\
0 & 0\cdots 0 & 0 & q & q & 0 & 0 & -q & 0\\ \hline
0 & 0\cdots 0 & 0 & 0 & 0 & q & q & 0 & -q\\
0 & 0\cdots 0 & 0 & 0 & 0 & q & q & 0 & -q\\ \hline
0 & 0\cdots 0 & 0 & -q & -q & 0 & 0 & q & 0\\ \hline
0 & 0\cdots 0 & 0 & 0 & 0 & -q & -q & 0 & q
\end{array}\right)$$

We can now read the $\mathcal{C}$-blocks of B_0, and, for each given block, the 3-defect of its characters and their order in the Cartan group $Cart(\mathcal{C},B_0)$ (once again, defect and order are in fact the same for any character in the block):

$\mathcal{C}$-block	$p^{d_3(\chi)}$	order in $Cart(\mathcal{C},B_0)$
$\{\xi_4\}$	q^2	1
$\{\xi_5,\,\xi_6,\,\xi_9\}\,;\,\{\xi_7,\,\xi_8,\,\xi_{10}\}$	$q^2 3^{k+1}$	3
$\{\xi_1,\,\xi_2,\,\eta_r,\,\eta_r',\,\eta_t,\,\eta_t',\,\eta_i^-,\,\eta_i^+\}$	q^3	q

We now turn to the invariant factors of the Cartan matrix. We see that the smallest (positive) integer d_s such that $M(\mathcal{D})=d_s\Gamma(\mathcal{D},b_0)$ is an integral matrix is $d_s=q$, and that $s=rk(\Gamma(\mathcal{D},b_0))=3$ (it suffices to consider the columns corresponding to χ, $\overline{\chi}$ and λ, the others being opposite to these columns). It is then easy to reduce $M(\mathcal{D})=q\Gamma(\mathcal{D},b_0)$ to Smith normal form; $M(\mathcal{D})$ is equivalent over $\mathbf{Z}$ to

$$\left(\begin{array}{ccc|c}
1 & & (0) & \\
 & \frac{q}{3} & & (0)\\
(0) & & \frac{q}{3} & \\ \hline
 & (0) & & (0)
\end{array}\right)$$

so that $\{\frac{d_s}{d_s}=1,\,\frac{d_s}{d_{s-1}},\,\frac{d_s}{d_{s-2}}\}=\{1,\,\frac{q}{3},\,\frac{q}{3}\}$, the invariant factors of the Cartan matrix $C_0(\mathcal{D})$ are $d_1=d_2=3$ and $d_3=q$, and $Cart(\mathcal{C},B_0)\cong Cart(\mathcal{D},b_0)\cong C_q\times C_3\times C_3$.

Let us now turn to elements whose order is not divisible by 9. We have $\Gamma(\mathcal{D}',b_0)=Id-\Gamma(\mathcal{D},b_0)$, whence we see that $d'_{s'}=q$ (where, this time,

$s' = rk(\Gamma(\mathcal{D}', b_0)) = q + 4)$, and, after having re-ordered lines and columns, we have, writing $N(\mathcal{D}') = d'_{s'}\Gamma(\mathcal{D}', b_0)$,

$$N(\mathcal{D}') = \begin{pmatrix} \boxed{q} & & & \\ & \begin{array}{|c|ccc|} \hline q-1 & 1 & \cdots & 1 \\ \hline 1 & q-1 & & (-1) \\ \vdots & & \ddots & \\ 1 & (-1) & & q-1 \\ \hline \end{array} & & \\ & & \begin{array}{|ccc|} \hline \frac{2q}{3} & \frac{q}{3} & \frac{q}{3} \\ \frac{q}{3} & \frac{2q}{3} & -\frac{q}{3} \\ \frac{q}{3} & -\frac{q}{3} & \frac{2q}{3} \\ \hline \end{array} & \\ & & & \begin{array}{|ccc|} \hline \frac{2q}{3} & \frac{q}{3} & \frac{q}{3} \\ \frac{q}{3} & \frac{2q}{3} & -\frac{q}{3} \\ \frac{q}{3} & -\frac{q}{3} & \frac{2q}{3} \\ \hline \end{array} \end{pmatrix}$$

The second block on the diagonal is the same as the one we considered in the case of Suzuki groups, and has Smith normal form

$$\begin{array}{|ccccc|} \hline 1 & & & & \\ & q & & (0) & \\ & & \ddots & & \\ & (0) & & q & \\ & & & & 0 \\ \hline \end{array}$$

It just remains to study the lower right block. We write C_i (resp. R_i) for the i-th column (resp. row) of any matrix.

$$\begin{array}{|ccc|} \hline \frac{2q}{3} & \frac{q}{3} & \frac{q}{3} \\ \frac{q}{3} & \frac{2q}{3} & -\frac{q}{3} \\ \frac{q}{3} & -\frac{q}{3} & \frac{2q}{3} \\ \hline \end{array} \xrightarrow{C_1 \leftarrow C_1 - (C_2 + C_3)} \begin{array}{|ccc|} \hline 0 & \frac{q}{3} & \frac{q}{3} \\ 0 & \frac{2q}{3} & -\frac{q}{3} \\ 0 & -\frac{q}{3} & \frac{2q}{3} \\ \hline \end{array} \xrightarrow{R_1 \leftarrow R_1 - (R_2 + R_3)} \begin{array}{|ccc|} \hline 0 & 0 & 0 \\ 0 & \frac{2q}{3} & -\frac{q}{3} \\ 0 & -\frac{q}{3} & \frac{2q}{3} \\ \hline \end{array}$$

Then $C_2 \longleftarrow C_2 - C_3$ gives

$$\begin{array}{|ccc|} \hline 0 & 0 & 0 \\ 0 & q & -\frac{q}{3} \\ 0 & -q & \frac{2q}{3} \\ \hline \end{array} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{array}{|ccc|} \hline 0 & 0 & 0 \\ 0 & q & -\frac{q}{3} \\ 0 & 0 & \frac{q}{3} \\ \hline \end{array} \xrightarrow{R_2 \leftarrow R_2 + R_3} \begin{array}{|ccc|} \hline 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & \frac{q}{3} \\ \hline \end{array}$$

Finally, we obtain that $N(\mathcal{D}')$ has Smith normal form

$$\begin{pmatrix} 1 &&&&&&&& \\ & \frac{q}{3} &&&&&&& \\ && \frac{q}{3} &&&&&& \\ &&& q &&&&& \\ &&&& \ddots &&&& \\ &&&&& q &&& \\ &&&&&& 0 && \\ &&&&&&& 0 & \\ &&&&&&&& 0 \end{pmatrix}$$

so that the invariant factors of the Cartan matrix $C_0(\mathcal{D}')$ are $d'_1 = \ldots = d'_{q+1} = 1$, $d'_{q+2} = d'_{q+3} = 3$ and $d'_{q+4} = q$, and $Cart(\mathcal{C}', B_0) \cong Cart(\mathcal{D}', b_0) \cong C_q \times C_3 \times C_3$.

3.3.3 Defect and order in the Cartan group

We would like to find some relation between the p-defect $d(\chi)$ of a character χ and its order $o(\chi)$ in the Cartan group $Cart(\mathcal{C})$, when $\mathcal{C}$ is the set of p^k-regular elements of the finite group G. We have mentioned that, for any $\chi \in \mathrm{Irr}(G)$, the order $o(\chi)$ is a power of p which divides $p^{d(\chi)}$.

We have seen in the previous section that, when G is $Sz(q)$, $Re(q)$ or $SU(3, q^2)$ (with q a power of the prime p) and $\mathcal{C}$ is the set of p^2-regular of G, then all the characters in a given $\mathcal{C}$-block of the principal p-block have the same order in the Cartan group. This is also true when G is Abelian and $\mathcal{C}$ is the set of p^k-regular elementsof G.

We see that in these particular cases (where all characters in a given $\mathcal{C}$-block have the same p-defect) the quotient $p^{d(\chi)}/o(\chi)$ is independant on the choice of χ in a given $\mathcal{C}$-block. We show that this is also true under strong hypotheses:

Theorem 3.3. *Let G be a finite group, p be a prime number (dividing $|G|$), and let $\mathcal{C}$ be the set of p^k-regular elements of G, for some $k \geq 1$. Suppose that B is a $\mathcal{C}$-block of G such that the contribution matrix $\Gamma(\mathcal{C}, B)$ has rank 1 over $\mathbf{Q}$. Then*

$$\forall \chi,\, \xi \in B, \frac{o(\xi)}{o(\chi)} = \frac{p^{d(\xi)}}{p^{d(\chi)}}.$$

Proof. Take χ, $\xi \in B$. Since the rank of $\Gamma(\mathcal{C}, B)$ is 1, we see that there exists $a \in \mathbf{Q} \setminus \{0\}$ such that $\xi^{\mathcal{C}} = a\chi^{\mathcal{C}}$ ($a \neq 0$ since $1 \in \mathcal{C}$). We can write $a = \frac{b}{c}$, with

b, $c \in \mathbf{Z}$ and $\gcd(b, c) = 1$. Furthermore, we can suppose that $\gcd(b, p) = 1$ (if this is not true, it suffices to exchange the roles of ξ and χ).

Let d be the order of χ in $Cart(\mathcal{C}, B)$. Then

$$\forall \varphi \in B,\ d < \chi^{\mathcal{C}}, \varphi^{\mathcal{C}} > \in \mathbf{Z},$$

and

$$\exists \psi \in B \text{ such that } d < \chi^{\mathcal{C}}, \psi^{\mathcal{C}} > \not\equiv 0 \pmod p.$$

Now, for all $\varphi \in B$,

$$\begin{aligned} d < \chi^{\mathcal{C}}, \varphi^{\mathcal{C}} > &= \frac{da}{a} < \chi^{\mathcal{C}}, \varphi^{\mathcal{C}} > \\ &= \frac{d}{a} < \xi^{\mathcal{C}}, \varphi^{\mathcal{C}} > \\ &= \frac{dc}{b} < \xi^{\mathcal{C}}, \varphi^{\mathcal{C}} > \in \mathbf{Z} \end{aligned}$$

and $\frac{dc}{b} < \xi^{\mathcal{C}}, \psi^{\mathcal{C}} > \not\equiv 0 \pmod p$.

Then, for all $\varphi \in B$, $dc < \xi^{\mathcal{C}}, \varphi^{\mathcal{C}} > \in \mathbf{Z}$, and, since $\gcd(b, p) = 1$, $dc < \xi^{\mathcal{C}}, \psi^{\mathcal{C}} > \not\equiv 0 \pmod p$. This implies that the order of ξ in $Cart(\mathcal{C}, B)$ is $|dc|_p = d|c|_p$, so that $|c|_p = \frac{d|c|_p}{d} = \frac{o(\xi)}{o(\chi)}$.

On the other hand, $1 \in \mathcal{C}$, so that $\xi(1) = a\chi(1) = \frac{b}{c}\chi(1)$, and $|\xi(1)|_p = \frac{|\chi(1)|_p}{|c|_p}$ (since $\gcd(b, p) = 1$). Hence $|c|_p = \frac{|\chi(1)|_p}{|\xi(1)|_p}$. Therefore

$$|c|_p = \frac{o(\xi)}{o(\chi)} = \frac{|\chi(1)|_p}{|\xi(1)|_p} = \frac{p^{d(\xi)}}{p^{d(\chi)}},$$

which is the result we wanted.

□

This result seems to be true in many more cases than the very restrictive situation of the above theorem (cf for example the case of Symmetric groups which we study in the next section). However, it is not true in general, as can be seen for example in the case where $G = M_{24}$ or J_2 and $p^k = 4$ (this can be shown using GAP on a computer).

3.4 Symmetric Groups

In this section, we study the case of ℓ-blocks of the Symmetric groups. In [18], the authors give the exponent of the Cartan group of an ℓ-block of a given weight, and they conjecture its structure. They also give the determinant of the Cartan matrix of such a block.

We will compute the order of any character in the Cartan group. In all the sequel of this section, we take two integers $2 \leq \ell \leq n$, and we let $\mathcal{C}_\ell$ be the set of ℓ-regular elements of the symmetric group S_n.

3.4.1 Some facts about ℓ-blocks of the symmetric group

Take B an ℓ-block of S_n of weight $w \neq 0$. The characters of B are labelled by the partitions of n which have a given ℓ-core, γ say. We write $B = \{\chi_\lambda,\, \gamma_\lambda = \gamma\}$. One of the key ingredients in [18] is that there is a generalized perfect isometry between B and the set of irreducible characters of the wreath product $\mathbf{Z}_\ell \wr S_w$. One way to see $\mathbf{Z}_\ell \wr S_w$ is to represent it as the set of monomial w by w matrices whose non-zero entries are ℓ-th roots of unity.

We will see that the elements of $\mathrm{Irr}(\mathbf{Z}_\ell \wr S_w)$ are labelled by *ℓ-quotients*: for $\chi_\lambda \in B$, the quotient β_λ is a sequence $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(\ell-1)})$ such that, for each $0 \leq i \leq \ell - 1$, $\lambda^{(i)}$ is a partition of some k_i, $0 \leq k_i \leq w$, and $\sum_{i=0}^{\ell-1} k_i = w$ (the quotient β_λ "stores" the information about λ and how to remove w ℓ-hooks from λ to get γ_λ). Each partition is uniquely determined by its ℓ-core and ℓ-quotient.

The subgroup (isomorphic to) S_w of $\mathbf{Z}_\ell \wr S_w$ acts on the base group $\mathbf{Z}_\ell^w \lhd \mathbf{Z}_\ell \wr S_w$ by conjugation: for $(a_1, \ldots, a_w) \in \mathbf{Z}_\ell^w$ and $\sigma \in S_w$, we have

$$\sigma^{-1}(a_1, \ldots, a_w)\sigma = (a_{\sigma(1)}, \ldots, a_{\sigma(w)}).$$

We write $\mathrm{Irr}(\mathbf{Z}_\ell) = \{\alpha_0, \ldots, \alpha_{\ell-1}\}$. Then S_w also acts by conjugation on $\mathrm{Irr}(\mathbf{Z}_\ell^w)$ via

$$(\alpha_{i_1} \otimes \cdots \otimes \alpha_{i_w})^{\sigma^{-1}} = \alpha_{i_{\sigma(1)}} \otimes \cdots \otimes \alpha_{i_{\sigma(w)}},$$

for all $\sigma \in S_w$ and $\alpha_{i_1} \otimes \cdots \otimes \alpha_{i_w} \in \mathrm{Irr}(\mathbf{Z}_\ell^w)$.

The number of conjugacy classes (and of irreducible complex characters) of $\mathbf{Z}_\ell \wr S_w$ is the number of ℓ-quotients of partitions of n (cf James and Kerber [17], Lemma 4.2.9). In [17], the authors give a description of the conjugacy classes of $\mathbf{Z}_\ell \wr S_w$ (cf [17], Theorem 4.2.8). If $(a_1, \ldots, a_w)\sigma \in \mathbf{Z}_\ell \wr S_w$, then $(a_1, \ldots, a_w)$ can be "endowed" with a cycle structure, given by that of σ (as it is easy to see in the monomial representation of $\mathbf{Z}_\ell \wr S_w$). Then, for each cycle in $(a_1, \ldots, a_w)$, the product of the ℓ-th roots of unity corresponding to the a_i's in the cycle is another ℓ-th root of unity. Such a product is called a *cycle product* in $(a_1, \ldots, a_w)$. Then the classification of conjugacy classes of $\mathbf{Z}_\ell \wr S_w$ is as follows: two elements $(a_1, \ldots, a_w)\sigma$ and $(b_1, \ldots, b_w)\tau$ of $\mathbf{Z}_\ell \wr S_w$ are conjugate if and only if σ and τ have the same cycle structure, and, for each cycle length k and each ℓ-th root of unity ζ, the numbers of cycle products equal to ζ and associated to cycles of length k in $(a_1, \ldots, a_w)$ and $(b_1, \ldots, b_w)$ respectively are the same.

Now take any $\chi \in \mathrm{Irr}(\mathbf{Z}_\ell \wr S_w)$. Then, by Clifford's Theorem (Isaacs [16], Theorem (6.2)), we have $\chi|_{\mathbf{Z}_\ell^w} = e \sum_{i=1}^t \theta^{\sigma_i}$ for some integer e and some $\theta \in \mathrm{Irr}(\mathbf{Z}_\ell^w)$, where $\{\theta^{\sigma_i},\, 1 \leq i \leq t\}$ is the conjugacy class of θ.

Because of the way S_w acts by conjugation on $\mathrm{Irr}(\mathbf{Z}_\ell^w)$, we see that exactly one conjugate of θ has the form $\alpha_0 \otimes \cdots \otimes \alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_1 \otimes \cdots \otimes \alpha_{\ell-1} \otimes \cdots \otimes \alpha_{\ell-1}$, where the multiplicity k_i of α_i is the same as in θ for each $0 \leq i \leq \ell-1$. We may assume that $\theta = \alpha_0 \otimes \cdots \otimes \alpha_0 \otimes \cdots \otimes \alpha_{\ell-1} \otimes \cdots \otimes \alpha_{\ell-1}$.

Again from the action of S_w on $\mathrm{Irr}(\mathbf{Z}_\ell^w)$, we see that the inertial subgroup $I_{\mathbf{Z}_\ell \wr S_w}(\theta)$ of θ is $(\mathbf{Z}_\ell \wr S_{k_0}) \times \cdots \times (\mathbf{Z}_\ell \wr S_{k_{l-1}})$.

Suppose first that $I_{\mathbf{Z}_\ell \wr S_w}(\theta) = \mathbf{Z}_\ell \wr S_w$. This means that all the k_i's are 0, expect one. Thus we have $\theta = \alpha_i \otimes \cdots \otimes \alpha_i$ for some $0 \leq i \leq \ell-1$. Using the description we made of the conjugacy classes of $\mathbf{Z}_\ell \wr S_w$, it is easy to show that $(\alpha_i \otimes \cdots \otimes \alpha_i)1_{S_w} \in \mathrm{Irr}(\mathbf{Z}_\ell \wr S_w)$ (where, for $\psi \in \mathrm{Irr}(S_w)$ and $(z, \sigma) \in \mathbf{Z}_\ell \wr S_w$, we let $((\alpha_i \otimes \cdots \otimes \alpha_i)\psi)((z,\sigma)) = (\alpha_i \otimes \cdots \otimes \alpha_i)(z)\psi(\sigma))$. One first shows, using the description above, that $(\alpha_i \otimes \cdots \otimes \alpha_i)1_{S_w}$ is a class function of $\mathbf{Z}_\ell \wr S_w$; then it is clear that $< (\alpha_i \otimes \cdots \otimes \alpha_i)1_{S_w}, (\alpha_i \otimes \cdots \otimes \alpha_i)1_{S_w} >_{\mathbf{Z}_\ell \wr S_w} = 1$, so that $(\alpha_i \otimes \cdots \otimes \alpha_i)1_{S_w} \in \mathrm{Irr}(\mathbf{Z}_\ell \wr S_w)$. Furthermore, $(\alpha_i \otimes \cdots \otimes \alpha_i)1_{S_w}|_{\mathbf{Z}_\ell^w} = (\alpha_i \otimes \cdots \otimes \alpha_i)$. Thus, by a result by Gallagher ([16], Corollary (6.17)), the characters $(\alpha_i \otimes \cdots \otimes \alpha_i)\psi$ for $\psi \in \mathrm{Irr}(S_w)$ are irreducible, distinct for distinct ψ, and are all the irreducible characters of $\mathbf{Z}_\ell \wr S_w$ which lie over $(\alpha_i \otimes \cdots \otimes \alpha_i)$. In particular, χ has the form $(\alpha_i \otimes \cdots \otimes \alpha_i)\chi_\lambda$ for some $\lambda \vdash w$, and i and χ_λ are uniquely determined by χ. We label χ by the quotient $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(\ell-1)})$ where $\lambda^{(i)} = \lambda$ and, for each $0 \leq i \neq j \leq \ell - 1$, $\lambda^{(j)} = \emptyset$.

Now suppose $I_{\mathbf{Z}_\ell \wr S_w}(\theta) < \mathbf{Z}_\ell \wr S_w$. Then, by Clifford's Theory ([16], Theorem 6.11), χ is induced by a unique irreducible character of $I_{\mathbf{Z}_\ell \wr S_w}(\theta)$ which lies over θ. By the same argument as above, we see that such a character must have the form $(\alpha_0 \otimes \cdots \otimes \alpha_0)\chi_{\lambda^{(0)}} \otimes \cdots \otimes (\alpha_{\ell-1} \otimes \cdots \otimes \alpha_{\ell-1})\chi_{\lambda^{(\ell-1)}}$ where, for each $0 \leq i \leq \ell - 1$, $\lambda^{(i)} \vdash k_i$. And, here again, the sequence $(\chi_{\lambda^{(0)}}, \ldots, \chi_{\lambda^{(\ell-1)}})$ is uniquely determined by χ. We therefore label χ by the quotient $(\lambda^{(0)}, \ldots, \lambda^{(\ell-1)})$.

We see that, in this way, as χ runs through $\mathrm{Irr}(\mathbf{Z}_\ell \wr S_w)$, all the ℓ-quotients appear (once). Thus, to each quotient β_λ can be associated a uniquely determined irreducible character of $\mathbf{Z}_\ell \wr S_w$, which we denote by χ_{β_λ}.

From the description of χ_{β_λ} we gave, we compute easily its degree (using the fact that the α_i's have degree 1). We have, writing θ_λ for θ and η_λ for $\bigotimes_{i=0}^{\ell-1}(\alpha_i \otimes \cdots \otimes \alpha_i)\chi_{\lambda^{(i)}}$,

$$\begin{aligned}
\chi_{\beta_\lambda}(1) &= (\mathrm{Ind}_{I_{\mathbf{Z}_\ell \wr S_w}(\theta_\lambda)}^{\mathbf{Z}_\ell \wr S_w}(\eta_\lambda))(1) \\
&= \frac{|\mathbf{Z}_\ell \wr S_w|}{|I_{\mathbf{Z}_\ell \wr S_w}(\theta_\lambda)|}\eta_\lambda(1) \\
&= \frac{\ell^w w!}{\prod_{i=0}^{\ell-1} \ell^{k_i} k_i!}\chi_{\lambda^{(0)}}(1)\dots\chi_{\lambda^{(\ell-1)}}(1) \\
&= \frac{w!}{\prod_i k_i!}\prod_{i=0}^{\ell-1}\frac{k_i!}{\prod \text{hook lengths in } \lambda^{(i)}}
\end{aligned}$$

(this last equality being true because of the Hook Length Formula (cf [17])). Hence

$$\chi_{\beta_\lambda}(1) = \frac{w!}{\prod \text{hook lengths in } \beta_\lambda}$$

(where we call *hook* in β_λ a hook in any of the partitions appearing in β_λ), and

$$\frac{|\mathbf{Z}_\ell \wr S_w|}{\chi_{\beta_\lambda}(1)} = \ell^w \prod \text{hook lengths in } \beta_\lambda. \qquad (\dagger\dagger)$$

The key point for the reductions in [18] and in the end of this section is the following: the map $\chi_\lambda \longmapsto \chi_{\beta_\lambda}$ is a generalized perfect isometry between B and $\mathrm{Irr}(\mathbf{Z}_\ell \wr S_w)$ with respect to ℓ-regular elements of S_n and regular elements of $\mathbf{Z}_\ell \wr S_w$ (where an element of $\mathbf{Z}_\ell \wr S_w$ is *regular* if 1 is not an eigenvalue of the corresponding monomial matrix). This is proved in [18]. For χ_λ, $\chi_\mu \in B$, there exist signs σ_λ and σ_μ such that $< \chi_\lambda, \chi_\mu >_{\mathcal{C}_\ell} = < \sigma_\lambda \chi_{\beta_\lambda}, \sigma_\mu \chi_{\beta_\mu} >_{reg}$ (where $<,>_{\mathcal{C}_\ell}$ and $<,>_{reg}$ denote the scalar products across ℓ-regular elements of S_n and across regular elements of $\mathbf{Z}_\ell \wr S_w$ respectively). In particular, χ_λ and χ_{β_λ} have the same order in the corresponding Cartan groups ("note" that the set of regular elements in $\mathbf{Z}_\ell \wr S_w$ is a closed set of conjugacy classes).

Note that an ℓ-block B of S_n of weight 0 consists of a single character. We have $B = \{\chi_\lambda\}$ where λ is its own ℓ-core. In this case, we have $\mathbf{Z}_\ell \wr S_w = \{1\}$, and we define *reg*, the set of *regular* elements of $\mathbf{Z}_\ell \wr S_0$ to be $\{1\}$. We have $\mathrm{Irr}(\mathbf{Z}_\ell \wr S_w) = \{1_{\mathbf{Z}_\ell \wr S_w}\}$. Now χ_λ vanishes outside $\mathcal{C}_\ell$ (by Proposition 3.1), so that $\chi_\lambda^{\mathcal{C}_\ell} = \chi_\lambda$, and we again get a generalized perfect isometry between B and $\mathrm{Irr}(\mathbf{Z}_\ell \wr S_0)$ (with respect to $\mathcal{C}_\ell$ and *reg*). The corresponding Cartan groups $Cart(\ell, B)$ and $Cart(\ell, 0)$ are both (isomorphic to) the trivial group C_1.

3.4.2 Invariant factors of the Cartan matrix

The following results can be found in [18]. Let B be an ℓ-block of S_n of weight w. Then there is a generalized perfect isometry between B $\mathrm{Irr}(\mathbf{Z}_\ell \wr S_w)$

(with respect to ℓ-regular classes in S_n and regular classes in $\mathbf{Z}_\ell \wr S_w$). This implies that the corresponding Cartan groups, $Cart(\mathcal{C}_\ell, B) = Cart(\ell, B)$ and $Cart(\text{reg}, \mathbf{Z}_\ell \wr S_w) = Cart(\ell, \mathbf{Z}_\ell \wr S_w) = Cart(\ell, w)$, are isomorphic, and thus that the Cartan matrices $C(\ell, B)$ and $C(\ell, w)$ have the same invariant factors. This allows the authors to work in $\mathbf{Z}_\ell \wr S_w$, and then transfer the results back to B.

In the course of their work, they show that every irreducible character of $\mathbf{Z}_\ell \wr S_w$ is directly linked across regular elements to the trivial character (cf [18], Theorem 5.12). More precisely, we have

$$\forall \chi \in \text{Irr}(\mathbf{Z}_\ell \wr S_w), \quad \mathbf{Z} \ni \frac{\ell^w w! < \chi, 1_{\mathbf{Z}_\ell \wr S_w} >_{reg}}{\chi(1)} \equiv (-1)^w \pmod{\ell}.$$

Writing π the set of primes dividing ℓ, every positive integer m factors uniquely as $m = m_\pi m_{\pi'}$, where every prime factor of m_π belongs to π and no prime factor of $m_{\pi'}$ is contained in π. We call m_π the π-part of m. It can be shown (cf Donkin [10]) that $Cart(\ell, w)$ is a π-group. Together with the above result, this gives the following

Theorem 3.4. *([18] Theorem 6.2 and Corollary 6.3) The exponent of $Cart(\ell, w)$ (i.e. the order of $1_{\mathbf{Z}_\ell \wr S_w}$ in $Cart(\ell, w)$) is $\ell^w w!_\pi$. This in turns implies that the exponent of $Cart(\mathcal{C}_\ell, S_n)$ is $\ell^{\lfloor \frac{n}{\ell} \rfloor} \lfloor \frac{n}{\ell} \rfloor !_\pi$.*

We next turn to the invariant factors of the $\mathcal{C}_\ell$-Cartan matrix C_n of S_n. A partition λ of n is called an *ℓ-class regular* partition of n if no part of λ is divisible by ℓ. We then write $\lambda \vdash_\ell n$.
If m is a positive integer, we define $\ell_m = \frac{\ell}{\gcd(\ell, m)}$, and let π_m be the set of primes dividing ℓ_m. If a is also a positive integer, we let

$$r_\ell(m, a) = \ell_m^{\lfloor \frac{a}{\ell} \rfloor} \lfloor \frac{a}{\ell} \rfloor !_{\pi_m}.$$

Finally, if $\lambda \vdash_\ell n$ is written exponentially $\lambda = (1^{a_1(\lambda)}, 2^{a_2(\lambda)}, \ldots)$, we let

$$r_\ell(\lambda) = \prod_{m \geq 1} r_\ell(m, a_m(\lambda)).$$

We then have the following

Conjecture 3.5. *([18] Conjecture 6.4) The $\mathcal{C}_\ell$-Cartan group $Cart(\ell, S_n)$ is a direct product of cyclic groups of order $r_\ell(\lambda)$, where λ runs through the set of ℓ-class regular partitions of n. In particular, the determinant of an ℓ-Cartan matrix C_n of S_n is*

$$det(C_n) = \prod_{\lambda \vdash_\ell n} r_\ell(\lambda).$$

This conjecture is supported by a number of examples computed by the authors. Furthermore, they show that the consequence they mention is true:

Theorem 3.6. *([18] Theorem 6.13) We have*

$$det(C_n) = \prod_{\lambda \vdash_\ell n} r_\ell(\lambda).$$

3.4.3 Orders in the Cartan group

We now compute, for an ℓ-block B of S_n, the orders of the characters of B in the Cartan group $Cart(\ell, B)$. We will need the following three lemmas:

Lemma 3.7. *If $m \geq 1$ and λ, partition of n, has m-weight w, then there are w hooks in λ whose length is divisible by m.*

Lemma 3.8. *For any $0 \neq m \in \mathbf{N}$ and partition λ, there is a bijection between the set of m-hooks in β_λ (the ℓ-quotient of λ) and the set of $m\ell$-hooks in λ (where a m-hook (resp. $m\ell$-hook) is a hook whose length is exactly m (resp. $m\ell$)).*

These two results can be seen by using the abacus, as presented in [17].

Lemma 3.9. *If $n \in \mathbf{N}$, $2 \leq \ell \leq n$, and if $m|\ell$ for some $m \geq 2$, then any m-block of S_n is a union of ℓ-blocks.*

Proof. Two proofs can be given:
1. Write $\ell = mr$. Suppose $\chi_\lambda, \chi_\mu \in \mathrm{Irr}(S_n)$ belong to the same ℓ-block. Then λ and μ have the same ℓ-core, and this can be obtained by removing from λ and μ a certain number of ℓ-hooks (this number being the weight of the ℓ-block). However, the removal of an ℓ-hook, i.e. an mr-hook, can be obtained by removing r m-hooks (cf for example the proof of Lemma 3.2 in [18]). Hence the common ℓ-core γ of λ and μ can be obtained by removing m-hooks, and, removing m-hooks from γ, one must eventually obtain an m-core, which will thus be the same for λ and μ. Hence χ_λ and χ_μ belong to the same m-block, giving the result.
2. It is "easy" to see from the definitions that any ℓ-cycle section is a union of m-cycle sections (for, if the m-cycle parts of $\alpha, \beta \in S_n$ are conjugate, then certainly their ℓ-cycle parts are conjugate too. Hence, if α and β belong to the same m-cycle section, then they belong to the same ℓ-cycle section).

Now suppose $\chi, \psi \in \mathrm{Irr}(S_n)$ belong to distinct m-blocks. Then, the m-blocks of S_n having the Second Main Theorem Property, χ and ψ are orthogonal across **each** m-cycle section of S_n. By the above remark, we see that χ and ψ are thus also orthogonal across each ℓ-cycle section, in particular $\mathcal{C}_\ell$, and thus belong to distinct ℓ-blocks. A priori, we just get that χ and ψ are not **directly** linked across ℓ-regular elements. But then, if $\chi_0 = \chi, \chi_1, \ldots, \chi_r$ is a sequence of characters such that χ_i is directly $\mathcal{C}_\ell$-linked to χ_{i+1} for $0 \leq i \leq r-1$, then, by the above, all the χ_i's must lie in the same m-block. In particular, the ℓ-block containing χ is contained in the m-block containing χ. Hence the result.

□

Some notations: We take $2 \leq \ell \leq n \in \mathbf{N}$. For λ a partition of n, we will write HL_λ (or just HL when the context is clear) for a hook length in λ. For any integer m dividing ℓ, we will write $(m) - HL$ for a hook length divisible by m, and $\{m\} - HL$ for a hook length whose π-part is precisely m (where π is the set of primes dividing ℓ). With a slight abuse of notation, we define similarly HL_{β_λ} to be a hook length in any of the partitions appearing in the quotient β_λ. Finally, a hook whose length is divisible by m will be called an (m)-hook.

Definition 3.10. *For any (finite) group G and $\chi \in Irr(G)$, and for $\ell \geq 2$ an integer dividing $|G|$, writing π the set of primes dividing ℓ, we define the* π**-defect** *$d_\pi(\chi)$ of χ, via*

$$d_\pi(\chi) = (d_{p_1}(\chi), \ldots, d_{p_s}(\chi))$$

where $\pi = \{p_1, \ldots, p_s\}$ and, for $1 \leq i \leq s$, $d_{p_i}(\chi)$ is the p_i-defect of χ.
We write (abusively)

$$\pi^{d_\pi(\chi)} := \prod_{i=1}^{s} p_i^{d_{p_i}(\chi)} = \frac{|G|_\pi}{\chi(1)_\pi}$$

(where $|G|_\pi$ and $\chi(1)_\pi$ are the π-parts of $|G|$ and $\chi(1)$ respectively).

We first establish the following:

Proposition 3.11. *If $2 \leq \ell \leq n$, if B is an ℓ-block of S_n of weight $w \neq 0$, if $\chi_\lambda \in B$ and if χ_λ corresponds to χ_{β_λ} under one of the perfect isometries we described between B and $Irr(\mathbf{Z}_\ell \wr S_w)$, then*

$$\pi^{d_\pi(\chi_{\beta_\lambda})} = (\prod (\ell) - HL_\lambda)_\pi.$$

Proof. First note that (††) implies that

$$\pi^{d_\pi(\chi_{\beta_\lambda})} = (\ell^w \prod HL_{\beta_\lambda})_\pi.$$

Now there are precisely w hooks in β_λ, so that

$$\pi^{d_\pi(\chi_{\beta_\lambda})} = (\prod \ell.HL_{\beta_\lambda})_\pi.$$

On the other hand, by applying Lemma 3.7 for each hook length appearing in β_λ, we see that there is a bijection between the set of hooks in β_λ and the set of hooks of length divisible by ℓ in λ (which is, in fact, the statement of Lemma 3.6), and that, if two hooks correspond to each other under this bijection, then $HL_\lambda = \ell.HL_{\beta_\lambda}$. Hence

$$\prod \ell.HL_{\beta_\lambda} = \prod(\ell) - HL_\lambda.$$

Taking π-parts, we finally obtain

$$\pi^{d_\pi(\chi_{\beta_\lambda})} = (\prod(\ell) - HL_\lambda)_\pi.$$

□

We next prove a result in the special case where ℓ is a prime power. It is this proposition which will in the end give us the result we announced in the previous section (i.e. that the quotient of the defect of a character by its order in the Cartan group is an invariant of the block).

Proposition 3.12. *Let p be a prime and $k \geq 1$ be such that $2 \leq p^k \leq n$. If B is a p^k-block of S_n and if $\chi_\lambda, \chi_\mu \in B$, then*

$$\frac{(\prod(p^k) - HL_\lambda)_p}{(\prod(p^k) - HL_\mu)_p} = \frac{p^{d_p(\chi_\lambda)}}{p^{d_p(\chi_\mu)}}.$$

Proof. By Lemma 3.8, for all $1 \leq i \leq k$, χ_λ and χ_μ belong to the same p^i-block of S_n. Thus, by Lemma 3.6, λ and μ have the same number of (p^i)-hooks for each $1 \leq i \leq k$.

Now, if $1 \leq i \leq k-1$, λ and μ have the same number of (p^i)-hooks and the same number of (p^{i+1})-hooks. Since the set of (p^i)-hooks is the **disjoint** union of the set of (p^{i+1})-hooks and the set of $\{p^i\}$-hooks, we see that λ and μ also have the same number of $\{p^i\}$-hooks (note that, by definition, for such a hook, the p-part of the hook length is exactly p^i).

By the Hook Length Formula, we have that

$$p^{d_p(\chi_\lambda)} = (\prod HL_\lambda)_p = (\prod(p) - HL_\lambda)_p.$$

Thus

$$\begin{aligned} p^{d_p(\chi_\lambda)} &= \left(\prod_{i=1}^{k-1} \prod(\{p^i\} - HL_\lambda)_p\right) (\prod(p^k) - HL_\lambda)_p \\ &= \left(\prod_{i=1}^{k-1} p^{i|\{\{p^i\}\text{-hooks in } \lambda\}|}\right) (\prod(p^k) - HL_\lambda)_p \end{aligned}$$

And, since, for all $1 \leq i \leq k-1$, $|\{\{p^i\}\text{-hooks in } \lambda\}| = |\{\{p^i\}\text{-hooks in } \mu\}|$, it is easy to see that

$$\frac{p^{d_p(\chi_\lambda)}}{p^{d_p(\chi_\mu)}} = \frac{(\prod(p^k) - HL_\lambda)_p}{(\prod(p^k) - HL_\mu)_p}.$$

□

We now turn to the actual computation of the order of an irreducible character of the wreath product in the Cartan group. It turns out that it suffices to know the product across regular classes with the trivial character, and this has been computed in [18].

Proposition 3.13. *Take* $2 \leq \ell \in \mathbf{N}$ *and* $0 \neq w \in \mathbf{N}$. *If* $\chi \in Irr(\mathbf{Z}_\ell \wr S_w)$, *then the order* $o(\chi)$ *of* χ *in the Cartan group* $Cart(\ell, \mathbf{Z}_\ell \wr S_w)$ *is* $\dfrac{\ell^w (w!)_\pi}{\chi(1)_\pi} = \pi^{d_\pi(\chi)}$.

Proof. Take $\chi \in \mathrm{Irr}(\mathbf{Z}_\ell \wr S_w)$. Recall that

$$\mathbf{Z} \ni \frac{\ell^w w!}{\chi(1)} < \chi, 1_{\mathbf{Z}_\ell \wr S_w} >_{reg} \equiv (-1)^w \pmod{\ell} \quad (\bullet)$$

Now $o(\chi)$ is a π-number, so that $< \chi, 1_{\mathbf{Z}_\ell \wr S_w} >_{reg}$ is a rational whose (reduced) denominator is a π-number. This implies that

$$\frac{\ell^w (w!)_\pi}{\chi(1)_\pi} < \chi, 1_{\mathbf{Z}_\ell \wr S_w} >_{reg} \in \mathbf{Z}.$$

Furthermore, from $(\bullet)$, we also deduce that, **for each** $p \in \pi$,

$$\frac{\ell^w w!}{\chi(1)} < \chi, 1_{\mathbf{Z}_\ell \wr S_w} >_{reg} \not\equiv 0 \pmod{p}.$$

Thus, for any $p \in \pi$,

$$\frac{\ell^w (w!)_\pi}{\chi(1)_\pi} < \chi, 1_{\mathbf{Z}_\ell \wr S_w} >_{reg} \not\equiv 0 \pmod{p}.$$

Hence $\dfrac{\ell^w (w!)_\pi}{\chi(1)_\pi}$ is the **smallest** positive integer d such that $d < \chi, 1_{\mathbf{Z}_\ell \wr S_w} >_{reg} \in \mathbf{Z}$. This implies that

$$\frac{\ell^w (w!)_\pi}{\chi(1)_\pi} | o(\chi)$$

(indeed, by definition, $o(\chi) < \chi, 1_{\mathbf{Z}_\ell \wr S_w} >_{reg} \in \mathbf{Z}$, and $o(\chi)$ is a π-number).

Now, conversely, if $\psi \in \mathrm{Irr}(\mathbf{Z}_\ell \wr S_w)$, then $< \chi, \psi >_{reg} \in \mathbf{Q}$, so also (since $\chi(1)$ divides $|\mathbf{Z}_\ell \wr S_w| = \ell^w w!$)

$$\frac{\ell^w w!}{\chi(1)} < \chi, \psi >_{reg} \in \mathbf{Q}.$$

However,

$$\frac{\ell^w w!}{\chi(1)} < \chi, \psi >_{reg} = \frac{\ell^w w!}{\ell^w w!} \sum_{g \in reg/\sim} \frac{K_g \chi(g)}{\chi(1)} \psi(g^{-1})$$

(where the sum is taken over representatives for the regular classes, and, for g such a representative, K_g is the size of the conjugacy class of g). And, for each g in the sum, $\frac{K_g \chi(g)}{\chi(1)}$ and $\psi(g^{-1})$ are both algebraic integers. Hence $\frac{\ell^w w!}{\chi(1)} < \chi, \psi >_{reg}$ is also an algebraic integer, and thus an integer. Thus

$$\forall \psi \in \mathrm{Irr}(\mathbf{Z}_\ell \wr S_w), \quad \frac{\ell^w w!}{\chi(1)} < \chi, \psi >_{reg} \in \mathbf{Z}$$

and this implies that $o(\chi)$ divides $\frac{\ell^w w!}{\chi(1)}$, and, $o(\chi)$ being a π-number,

$$o(\chi) | \frac{\ell^w (w!)_\pi}{\chi(1)_\pi}.$$

Hence we finally get

$$o(\chi) = \frac{\ell^w (w!)_\pi}{\chi(1)_\pi} = \pi^{d_\pi(\chi)}.$$

□

Putting together the results of these three propositions, we obtain the following

Theorem 3.14. *Let $2 \leq \ell \leq n$, and let B be an ℓ-block of S_n of weight w. Then*

- *(i) If $w = 0$, then $B = \{\chi_\lambda\}$ for some partition λ of n, and χ_λ has order 1 in the Cartan group $Cart(\ell, S_n)$.*
- *(ii) If $w \neq 0$, and if $\chi_\lambda \in B$, then the order of χ_λ in $Cart(\ell, S_n)$ is $(\prod(\ell) - HL_\lambda)_\pi$, where π is the set of primes dividing ℓ (i.e. $o(\chi_\lambda)$ is the π-part of the product of the hook lengths divisible by ℓ in λ).*
- *(iii) If $\ell = p^k$ for some prime p and $k \geq 1$, and if $\chi, \psi \in B$, then $\frac{o(\chi)}{o(\psi)} = \frac{p^{d_p(\chi)}}{p^{d_p(\psi)}}$.*

Part 4

Finite General Linear Group

In all this part, we let n be a positive integer and q be a power of a prime p. We will work in the finite general linear group G, which can be seen as the group $GL(V)$ of automorphisms of an n-dimensional vector space V over the finite field $\mathbf{F}_q$, or as its natural matrix representation $GL(n, q)$, the group of invertible n by n matrices with entries in $\mathbf{F}_q$. The irreducible (complex) characters of G have been described by Green in [15], using deep combinatorial arguments. Then, using in particular the Deligne-Lusztig theory, Fong and Srinivasan have classified the blocks of G (cf [12]). They show in particular that unipotent characters behave nicely with respect to blocks. The unipotent characters of $GL(n, q)$ are parametrized by the partitions of n. It turns out that, if r is an odd prime not dividing q, then two unipotent characters belong to the same r-block of G if and only if the partitions labelling them have the same e-core, where e is the multiplicative order of q modulo r. This result is shown using analogues of the Murnaghan-Nakayama rule for irreducible characters of G. Our aim is to use these analogues to obtain properties of some generalized blocks we define in G. So far, we did this only for unipotent irreducible characters of G. However, we have good hope that this could be generalized to arbitrary irreducible characters. We also believe that the same thing should work in the unitary groups. We construct *unipotent blocks* for G which satisfy one sense of an analogue of the Nakayama Conjecture. Unions of these unipotent blocks then satisfy this analogue, and they have the Second Main Theorem property.

For a more general survey of the representation theory of finite reductive groups, we refer to Cabanes and Enguehard [5].

4.1 Conjugacy classes

4.1.1 Rational canonical form

We first introduce the theory of elementary divisors and the rational canonical form in G, which give a parametrization of the conjugacy classes. We present this in the matrix representation $GL(n,q)$ of G, where things are seen more easily. For the results we give in this section, we refer to Green [15].

The conjugacy classes of G are parametrized by the sequences $(\dots f_i^{\nu_i} \dots)$, where $\mathcal{F} = \{f_i,\, i \in \mathcal{I}\}$ is the set of irreducible, monic polynomials distinct from X and of degree at most n in $\mathbf{F}_q[X]$ (so that $\mathcal{I}$ is finite), and the ν_i's are partitions of non-negative integers k_i such that $\sum_i k_i = n$. We will write $\nu \vdash k$ to say that ν is a partition of k, and we will write $|\nu| = k$.

In all the sequel, it will be assumed that, when we write $(f_1^{\nu_1}, \dots, f_r^{\nu_r})$ for a conjugacy class, then polynomials which don't contribute have been omitted (i.e. $|\nu_1| \neq 0, \dots, |\nu_r| \neq 0$).

Take a conjugacy class $c = (\dots f_i^{\nu_i} \dots) = (\dots f_i^{\nu(f_i)} \dots)$ of G, and $g \in c$. Then, the characteristic polynomial of g over $\mathbf{F}_q$ is

$$Char(g) = \prod_{i \in \mathcal{I}} f_i^{k_i}.$$

Writing, for each i, $\nu(f_i) = \nu_i = (\lambda_{i,1} \geq \lambda_{i,2} \geq \dots \geq \lambda_{i,s})$ (where s can be chosen to be big enough for all $f_i's$ appearing in $Char(g)$ (and even for all $g \in G$), by taking for example $s = n$), the minimal polynomial of g over $\mathbf{F}_q$ is

$$Min(g) = \prod_{i \in \mathcal{I}} f_i^{\lambda_{i,1}}.$$

For $k = 1, \dots, s$, the polynomial $ED_k(g) = \prod_i f_i^{\sum_{j=1}^k \lambda_{i,j}}$ is the k-th elementary divisor of g over $\mathbf{F}_q$.

For any irreducible monic polynomial f over $\mathbf{F}_q$, we let $U(f)$ be the companion matrix of f, i.e. if $f(X) = X^d - a_{d-1}X^{d-1} - \dots - a_0$, then

$$U(f) = U_1(f) = \begin{pmatrix} 0 & 1 & & & \\ \vdots & \ddots & \ddots & (0) & \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & \cdots & a_{d-1} \end{pmatrix}.$$

For any positive integer λ, we write

$$U_\lambda(f) = \begin{pmatrix} U(f) & I_d & & & \\ & U(f) & I_d & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_d \\ & & & & U(f) \end{pmatrix},$$

where I_d is the d by d identity matrix, and there are λ blocks $U(f)$ on the diagonal. Note that $U_\lambda(f)$ is equivalent over $\mathbf{F}_q$ to the companion matrix of f^λ. Finally, for $\nu = (\lambda_1 \geq \ldots \geq \lambda_s)$ any partition (of an integer k say), we write

$$U_\nu(f) = \begin{pmatrix} U_{\lambda_1}(f) & & \\ & \ddots & \\ & & U_{\lambda_s}(f) \end{pmatrix}.$$

If $g \in c = (f_1^{\nu_1}, \ldots, f_r^{\nu_r})$, then the matrix of g is equivalent over $\mathbf{F}_q$ to

$$U_{(f_1^{\nu_1}, \ldots, f_r^{\nu_r})} = \begin{pmatrix} U_{\nu_1}(f_1) & & \\ & \ddots & \\ & & U_{\nu_r}(f_r) \end{pmatrix}.$$

This is the rational canonical form of g over $\mathbf{F}_q$.

Note that the theory of elementary divisors is independant on the representation of G we take, so that we don't change anything by, for example, taking $G = GL(V)$ and taking any basis for V.

4.1.2 The Jordan decomposition

An element $g \in G = GL(n,q)$ is *semisimple* if and only if it is diagonalizable over an algebraic extension of $\mathbf{F}_q$ (i.e. some $\mathbf{F}_{q^k}$, $k \geq 1$), i.e. $Min(g)$ splits over $\mathbf{F}_{q^k}$ and has only simple roots. Let c be the conjugacy class of g. If $c = (\ldots f^{\nu(f)} \ldots)$, then g is semisimple if and only if for any f appearing in c (i.e. $|\nu(f)| \neq 0$), $\nu(f) = (1, \ldots, 1)$. We write $\Delta((q_i))$ for the diagonal matrix with diagonal blocks the q_i's. Suppose $g = \Delta((U_{\nu_i}(f_i)))$. Then $g = g_S + g_N = \Delta((U_{\tilde{\nu_i}}(f_i))) + \Delta((\tilde{I}_{\nu_i}(f_i)))$, where $\tilde{\nu_i}(f_i) = (1, \ldots, 1) \vdash |\nu_i(f_i)| = k_i$ and

$$\tilde{I}_{\nu_i}(f_i) = \begin{pmatrix} (0) & I_{d_i} & & & \\ & (0) & I_{d_i} & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_{d_i} \\ & & & & (0) \end{pmatrix},$$

with $|\nu_i| = k_i$ diagonal d_i by d_i blocks, where f_i has degree d_i.

Thus $g_S = \Delta((U_{k_i}(f_i)))$ is semisimple and commutes with g_N, which is nilpotent. Hence g_S is the semisimple part of g (cf e.g. the introduction of Carter [6]). The unipotent part of g is $g_U = I_n + g_S^{-1} g_N$.

Now, if $\tilde{g} = hgh^{-1}$ for some $h \in G$, then $\tilde{g} = (hg_S h^{-1})(hg_U h^{-1})$, and we see that the semisimple and unipotent parts of $\tilde{g}$ are $hg_S h^{-1}$ and $hg_U h^{-1}$ respectively.

4.1.3 Primary decomposition

Now, we consider G as the group of automorphisms $GL(V)$ of an n-dimensional vector space over $\mathbf{F}_q$. We first mention a fact about centralizers in $GL(V)$. Take $g \in GL(V)$. If, in some matrix representation corresponding to the decomposition $V = V_1 \oplus V_2$, we have

$$g = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}$$

with, for $i \in \{1, 2\}$, $g_i \in G_i = GL(V_i)$, and $\gcd(Min(g_1), Min(g_2)) = 1$, then

$$C_G(g) = \left\{ \begin{pmatrix} a & \\ & b \end{pmatrix},\ a \in C_{G_1}(g_1),\ b \in C_{G_2}(g_2) \right\},$$

and thus $C_G(g) \cong C_{G_1}(g_1) \times C_{G_2}(g_2)$ via $h \longmapsto (h|_{V_1}, h|_{V_2})$ (which is independant on the matrix representation).

We also give the following lemma

Lemma 4.1. *Let $g \in G = GL(V)$. Suppose V_1 and V_2 are g-stable subspaces of V such that $V = V_1 \oplus V_2$. If $g|_{V_1} \in (\dots f^{\nu_1(f)} \dots)_{f \in \mathcal{F}}$ and $g|_{V_2} \in (\dots f^{\nu_2(f)} \dots)_{f \in \mathcal{F}}$, then $g \in (\dots f^{\nu_1(f) \cup \nu_2(f)} \dots)_{f \in \mathcal{F}}$, where, by $\cup$, we denote the concatenation of partitions (i.e. the components of $\nu_1 \cup \nu_2$ are those of ν_1 together with those of ν_2. If $\nu_1 \vdash n_1$ and $\nu_2 \vdash n_2$, then $\nu_1 \cup \nu_2 \vdash n_1 + n_2$.)*

Proof. Taking any bases for V_1 and V_2 (which then add to give a basis of V), we can write

$$g = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}$$

(where g_1 (resp. g_2) is the matrix of $g|_{V_1}$ (resp. $g|_{V_2}$)). The idea is that we can obtain the rational canonical form of g by reducing to this form g_1 and g_2. There exist $h_1 \in GL(V_1)$ and $h_2 \in GL(V_2)$ such that

$$g = \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}^{-1} \begin{pmatrix} U_{(\dots f^{\nu_1(f)} \dots)} & \\ & U_{(\dots f^{\nu_2(f)} \dots)} \end{pmatrix} \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}.$$

Thus, for some permutation matrix P,

$$g = \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}^{-1} P^{-1} U_{(\ldots f^{\nu_1(f)\cup\nu_2(f)}\ldots)} P \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix},$$

so that $g \in (\ldots f^{\nu_1(f)\cup\nu_2(f)}\ldots)_{f\in\mathcal{F}}$. □

We now turn to the primary decomposition of elements of G. Take $g \in GL(V)$ and suppose $g \in (f_1^{\nu_1}, \ldots, f_r^{\nu_r})$. Then there exists a unique decomposition $V = V_1 \oplus \cdots \oplus V_r$, where the V_i's are g-stable subspaces of V and, for $1 \leq i \leq r$, $g|_{V_i} \in (f_i^{\nu_i}) \subset GL(V_i)$. For each $1 \leq i \leq r$, V_i is given by

$$V_i = \{v \in V \mid f_i^k(g)v = 0 \text{ for some } k > 0\}.$$

We have $C_G(g) \cong C_{GL(V_1)}(g|_{V_1}) \times \cdots \times C_{GL(V_r)}(g|_{V_r})$. Then there exists a unique writing $g = g_1 \ldots g_r$, where, for each $1 \leq i \leq r$, $g_i \in GL(V)$, V_i is g_i-stable and $g_i|_{V_j} = 1$ for all $j \neq i$. Indeed, we must have $g_i|_{V_i} = g|_{V_i}$ for each $1 \leq i \leq r$, and g_i is uniquely determined by this and its properties listed before. Furthermore, the g_i's are pairwise commuting elements. We say that $g_1 \ldots g_r$ is the *primary decomposition* of g. The g_i's are the *primary components* of g.

More generally, an element of G is said to be *primary* if its characteristic polynomial is divisible by at most one irreducible polynomial distinct from $X - 1$. We have the following (cf Fong-Srinivasan [12])

Proposition 4.2. *Suppose h is a semisimple primary element of some general linear group $GL(m, q)$, and that $h \in (f^\nu)$ for some $f \neq X - 1$. Writing d the degree of f, we thus have $m = kd$ where $\nu = (1, \ldots, 1) \vdash k$. Then*

$$C_{GL(kd,q)}(h) \cong GL(k, q^d).$$

From the primary decomposition of $g \in G$, we deduce the following: if $\mathcal{F}$ is the disjoint union of $\mathcal{F}_1$ and $\mathcal{F}_2$, then there is a unique decomposition $V = V_1 \oplus V_2$ where V_1 and V_2 are g-stable, $g|_{V_1} \in (\ldots f^{\nu(f)}\ldots)_{f\in\mathcal{F}_1}$ and $g|_{V_2} \in (\ldots f^{\nu(f)}\ldots)_{f\in\mathcal{F}_2}$. Then g has a unique decomposition $g = g_1g_2 = g_2g_1$ where $g_1, g_2 \in GL(V)$, $g_1|_{V_2} = 1$ and $g_2|_{V_1} = 1$. Indeed, under these hypotheses, necessarily, for $i \in \{1, 2\}$, V_i is g_i-stable and $g_i|_{V_i} = g|_{V_i}$. We then have $C_G(g) \cong C_{GL(V_1)}(g|_{V_1}) \times C_{GL(V_2)}(g|_{V_2})$.

4.2 $(\mathcal{X}, \mathcal{Y})$-sections

We will give four definitions of sections in G. The first two will give us information on blocks defined only on the unipotent characters of G. The other two give us bigger sections, and thus smaller blocks, but they should allow us to work on the whole of $Irr(G)$.

The idea of the following definitions is, in the rational canonical form we gave above, to isolate blocks corresponding to irreducible polynomials whose degree is equal to or divisible by a given d (and which, in the last two definitions, are semisimple). We let, writing $\delta(f)$ the degree of any polynomial f,

$$\mathcal{F}_d = \{f \in \mathcal{F} \,|\, f \neq X-1 \text{ and } \delta(f) = d\}\ ,\ \mathcal{F}_d^0 = \mathcal{F} \setminus \mathcal{F}_d$$

$$\mathcal{F}_{(d)} = \{f \in \mathcal{F} \,|\, f \neq X-1 \text{ and } d|\delta(f)\} \text{ and } \mathcal{F}_{(d)}^0 = \mathcal{F} \setminus \mathcal{F}_{(d)}.$$

We define four unions of conjugacy classes of G:

$$\mathcal{X}_d = \{(\dots f^{\nu(f)} \dots) \,|\, f \in \mathcal{F}_d \cup \{X-1\} \text{ and } \nu(X-1) = (0) \text{ or } (1,\dots,1)\},$$

$$\mathcal{X}_{(d)} = \{(\dots f^{\nu(f)} \dots) \,|\, f \in \mathcal{F}_{(d)} \cup \{X-1\} \text{ and } \nu(X-1) = (0) \text{ or } (1,\dots,1)\},$$

$$\mathcal{X}_d^{(s)} = \{(\dots f^{\nu(f)} \dots) \,|\, f \in \mathcal{F}_d \cup \{X-1\} \text{ and } \nu(f) = (0) \text{ or } (1,\dots,1)\},$$

$$\mathcal{X}_{(d)}^{(s)} = \{(\dots f^{\nu(f)} \dots) \,|\, f \in \mathcal{F}_{(d)} \cup \{X-1\} \text{ and } \nu(f) = (0) \text{ or } (1,\dots,1)\}$$

(Then $\mathcal{X}_d^{(s)}$ (resp. $\mathcal{X}_{(d)}^{(s)}$) is the set of semisimple elements of $\mathcal{X}_d$ (resp. $\mathcal{X}_{(d)}$).) We have $1 \in \mathcal{X}_d \cap \mathcal{X}_{(d)} \cap \mathcal{X}_d^{(s)} \cap \mathcal{X}_{(d)}^{(s)}$.

For $x \in \mathcal{X}_d$ (resp. $x \in \mathcal{X}_{(d)}$), we let $\mathcal{F}_x = \mathcal{F}_d$ and $\mathcal{F}_x^0 = \mathcal{F}_d^0$ (resp. $\mathcal{F}_x = \mathcal{F}_{(d)}$ and $\mathcal{F}_x^0 = \mathcal{F}_{(d)}^0$). For $x \in \mathcal{X}_d^{(s)}$ (resp. $x \in \mathcal{X}_{(d)}^{(s)}$), we let $\mathcal{F}_x = \{f \in \mathcal{F}_d \,|\, \nu_x(f) \neq (0)\}$ (resp. $\mathcal{F}_x = \{f \in \mathcal{F}_{(d)} \,|\, \nu_x(f) \neq (0)\}$) and $\mathcal{F}_x^0 = \mathcal{F} \setminus \mathcal{F}_x$.

We let $\mathcal{X} = \mathcal{X}_d$, $\mathcal{X}_{(d)}$, $\mathcal{X}_d^{(s)}$ or $\mathcal{X}_{(d)}^{(s)}$. Then, for each $x \in \mathcal{X}$, the set $\mathcal{F}$ is the disjoint union of $\mathcal{F}_x$ and $\mathcal{F}_x^0$. For each $x \in \mathcal{X}$, there exists a unique decomposition $V = V_x \oplus V_x^0$ such that V_x is x-stable, $x|_{V_x^0} = 1$ and $x|_{V_x} \in (\dots f^{\nu(f)} \dots)_{f \in \mathcal{F}_x}$. We then have $C_G(x) \cong C_{GL(V_x)}(x|_{V_x}) \times GL(V_x^0)$. We let

$$\mathcal{Y}(x) = \{y \in GL(V) \,|\, V_x^0 \text{ is } y\text{-stable, } y|_{V_x} = 1 \text{ and } y|_{V_x^0} \in (\dots f^{\nu(f)} \dots)_{f \in \mathcal{F}_x^0}\}.$$

From the definitions, we see that $\mathcal{Y}(x) \subset C_G(x)$ for any $x \in \mathcal{X}$. We also see, using the remarks we made on the primary decomposition, that, for any $g \in G$, there exists **unique** $x \in \mathcal{X}$, $y \in \mathcal{Y}(x)$ such that $g = xy$. Indeed, if $g \in (f_1^{\nu_1}, \dots, f_r^{\nu_r})$ and if we write, as above, $V = V_1 \oplus \dots \oplus V_r$ and $g = g_1 \dots g_r$, then we necessarily have $V_x = \oplus_{i \in I} V_i$, where $I \subset \{1, \dots, r\}$ is the set of indices i such that $f_i^{\nu_i}$ has the property defining $\mathcal{X}$, $V_x^0 = \oplus_{i \notin I} V_i$, $x|_{V_x} = g|_{V_x}$ and $y|_{V_x^0} = g|_{V_x^0}$ (and $x \in \mathcal{X}$ and $y \in \mathcal{Y}(x)$ are uniquely determined by these conditions). We have $x = \prod_{i \in I} g_i$ and $y = \prod_{i \notin I} g_i$.

We show that these definitions allow us to define $(\mathcal{X}, \mathcal{Y})$-sections.

Proposition 4.3. *Take any integer $d > 0$, and let $\mathcal{X} = \mathcal{X}_d$, $\mathcal{X}_{(d)}$, $\mathcal{X}_d^{(s)}$ or $\mathcal{X}_{(d)}^{(s)}$. Then, for any $x \in \mathcal{X}$,*
(i) $\mathcal{Y}(x)$ is a union of conjugacy classes of $C_G(x)$.
(ii) For all $y \in \mathcal{Y}(x)$, $C_G(xy) \leq C_G(x)$.
(iii) For all $g \in G$, $\mathcal{Y}(x^g) = \mathcal{Y}(x)^g$.
(iv) Two elements of $x\mathcal{Y}(x)$ are G-conjugate if and only if they are $C_G(x)$-conjugate.
(v) $G = \coprod_{x \in \mathcal{X}/G} \{(xy)^G, \; y \in \mathcal{Y}(x)/C_G(x)\}$.

Proof. (i) We have $C_G(x) = H_x \times H_x^0$, and we see $C_G(x)$ as a subgroup of $GL(V_x) \times GL(V_x^0)$. For $y \in \mathcal{Y}(x)$, $y = (y|_{V_x}, y|_{V_x^0}) = (1, y_x^0)$. Then, for all $h = (h_x, h_x^0) \in C_G(x)$, $h^{-1}yh = (1, (h_x^0)^{-1} y_x^0 h_x^0) \in \mathcal{Y}(x)$.
(ii) For all $y \in \mathcal{Y}(x)$, we have $C_G(xy) \cong C_{GL(V_x)}(x|_{V_x}) \times C_{GL(V_x^0)}(y|_{V_x^0}) \leq C_{GL(V_x)}(x|_{V_x}) \times GL(V_x^0) \cong C_G(x)$, and $C_G(xy) \leq C_G(x)$ since the isomorphism on the left is the restriction to $C_{GL(V_x)}(x|_{V_x}) \times C_{GL(V_x^0)}(y|_{V_x^0})$ of the isomorphism on the right.
(iii) Take $g \in G$ and $y \in \mathcal{Y}(x)$. Then $x^g \in \mathcal{X}$ and $\mathcal{F}_{x^g} = \mathcal{F}_x$ (and thus $\mathcal{F}_{x^g}^0 = \mathcal{F}_x^0$). We have $V = g^{-1}V = g^{-1}V_x \oplus g^{-1}V_x^0$. Furthermore, $x^g|_{g^{-1}V_x^0} = 1$, $g^{-1}V_x$ is x^g-stable, and $x^g|_{g^{-1}V_x} \in (...f^{\nu(f)}...)_{f \in \mathcal{F}_x}$. Thus $g^{-1}V_x = V_{x^g}$ and $g^{-1}V_x^0 = V_{x^g}^0$. Now, since $y \in \mathcal{Y}(x)$, we see that $y^g|_{g^{-1}V_x} = 1$, $g^{-1}V_x^0$ is y^g-stable, and $y^g|_{g^{-1}V_x^0} \in (...f^{\nu(f)}...)_{f \in \mathcal{F}_x^0}$. Hence $y^g \in \mathcal{Y}(x^g)$, and $\mathcal{Y}(x)^g \subset \mathcal{Y}(x^g)$ for all $g \in G$. Then, for any $g \in G$, $\mathcal{Y}(x^g)^{g^{-1}} \subset \mathcal{Y}(x)$ so that $\mathcal{Y}(x^g) \subset \mathcal{Y}(x)^g$. Hence the result.
(iv) Suppose that, for some $y, z \in \mathcal{Y}(x)$, there exists $h \in G$ such that $xy = h^{-1}xzh$. Writing $g = xy$, we also have $g = x^h z^h$, and $x^h \in \mathcal{X}$ (since $\mathcal{X}$ is a union of G-conjugacy classes) and $z^h \in \mathcal{Y}(x^h)$ (by (iii)). By the uniqueness of such a writing for g, we have $x^h = x$ and $z^h = y$. Hence $h \in C_G(x)$. In particular, xy and xz are $C_G(x)$-conjugate.
(v) For any $g \in G$, there exists a unique $x \in \mathcal{X}$ such that $g \in x\mathcal{Y}(x)$. Thus

$$\begin{aligned} G &= \coprod_{x \in \mathcal{X}} x\mathcal{Y}(x) \\ &= \coprod_{x \in \mathcal{X}/G} (x\mathcal{Y}(x))^G \text{ (because of (iii))} \\ &= \coprod_{x \in \mathcal{X}/G} \bigcup_{y \in \mathcal{Y}(x)} (xy)^G \\ &= \coprod_{x \in \mathcal{X}/G} \coprod_{y \in \mathcal{Y}(x)/C_G(x)} (xy)^G, \end{aligned}$$

this last equality being true by (iv) (if $(xy)^g = (xz)^{g'}$, then $xy = xz^{g'g^{-1}}$ so that, by (iv), y and z are $C_G(x)$-conjugate).

□

This proposition shows that the $(\mathcal{X}, \mathcal{Y})$-sections we defined correspond to the notion of $(\mathcal{X}, \mathcal{Y})$-sections presented in [18].

For any $x \in \mathcal{X}$, we call the union of the G-conjugacy classes meeting $x\mathcal{Y}(x)$ the $\mathcal{Y}$-*section* of x. We remark that the $\mathcal{Y}$-sections of G are quite different from ordinary prime sections. Take any $1 \neq x \in \mathcal{X}$. Then, by definition, $\mathcal{Y}(x) \subset \mathcal{Y}(1) \cap C_G(x)$. However, if $d \neq 1$, then there exists $\lambda \in \mathbf{F}_q^\times$ such that $\lambda I_n \in \mathcal{Y}(1) \cap C_G(x)$ but $\lambda I_n \notin \mathcal{Y}(x)$, so that $\mathcal{Y}(x) \neq \mathcal{Y}(1) \cap C_G(x)$ (while this equality holds when we define $(\mathcal{X}, \mathcal{Y})$-sections to be the ordinary ℓ-sections for some prime ℓ).

Furthermore, still supposing $d \neq 1$, if x is a (non trivial) ℓ-element of G for some prime ℓ, then, most of the time (that is, when $q - 1$ is not a power of ℓ), there exists an ℓ-regular element $\lambda \in \mathbf{F}_q^\times$ such that $\lambda I_n \notin \mathcal{Y}(x)$, so that $x\lambda I_n \notin (x\mathcal{Y}(x))^G$. But $x\lambda I_n$ belongs to the ℓ-section of x. Hence the $\mathcal{Y}$-section of x is **not** a union of ℓ-sections.

4.3 $(\mathcal{X}, \mathcal{Y})$-blocks

4.3.1 Blocks

Take any integer $d > 0$, and let $\mathcal{X} = \mathcal{X}_d$, $\mathcal{X}_{(d)}$, $\mathcal{X}_d^{(s)}$ or $\mathcal{X}_{(d)}^{(s)}$. We define on Irr(G) the relation $\sim$ of direct $\mathcal{Y}(1)$-linking: for χ, $\psi \in$Irr(G), $\chi \sim \psi$ if and only if

$$< \chi, \psi >_{\mathcal{Y}(1)} = \frac{1}{|G|} \sum_{y \in \mathcal{Y}(1)} \chi(y)\overline{\psi(y)} \neq 0.$$

Extending $\sim$ by transitivity, we obtain an equivalence relation $\approx$ on Irr(G). We define an $(\mathcal{X}, \mathcal{Y})$-block of G to be an equivalence class of the equivalence relation $\approx$. We will also consider the restriction of $\approx$ to the subset Unip(G) of unipotent characters of G. The equivalence classes of Unip(G) will be called *unipotent* $(\mathcal{X}, \mathcal{Y})$-*blocks* of G. It is clear that, for any $(\mathcal{X}, \mathcal{Y})$-block B of G, $B \cap$ Unip(G) is a **union** of unipotent $(\mathcal{X}, \mathcal{Y})$-blocks.

4.3.2 Irreducible characters of G

For the results we give here, we refer to [12].

The unipotent characters of $G = GL(n, q)$ are the irreducible components of the permutation representation of G on the cosets of a Borel subgroup (i.e. the normalizer in G of a Sylow p-subgroup of G, where p is the defining characteristic). They are labelled by the partitions of n. For $\lambda \vdash n$, we will write χ_λ the unipotent (irreducible) character of G labelled by λ.

Each irreducible character of G can be parametrized as $\chi_{s,\mu}$, where s is a semisimple element of G and χ_μ is a unipotent character of $C_G(s)$ (note that, since s is semisimple, $C_G(s)$ is a product of linear groups $GL(k, q^\delta)$ where δ is the degree of an irreducible factor of $Char(s)$ and k is its multiplicity in $Char(s)$; in particular, μ may be seen as a sequence of partitions $(\mu_1, \dots, \mu_t)$ (of $k_1, \dots, k_t$ respectively), and $\chi_\mu = \chi_{\mu_1} \otimes \cdots \otimes \chi_{\mu_t}$). We obtain a complete minimal set of representatives by taking representatives (s, μ) for the G-conjugacy classes of such pairs. The set of characters $\chi_{s,\mu}$, for μ varying, is the *geometric conjugacy class* s_G of s.

More precisely, if $\chi \in$Irr(G), then there exists $s \in G$ semisimple, there exists ψ, unipotent character of $H = C_G(s)$, and there exist signs ε_G and ε_H such that

$$\chi = \varepsilon_G \varepsilon_H R_H^G(\hat{s}\psi),$$

where R_H^G is the additive operator, defined in the Deligne-Lusztig theory, from $X(H)$ to $X(G)$ (character rings of representations of H and G respectively over $\overline{\mathbf{Q}_\ell}$, an algebraic closure of the ℓ-adic field $\mathbf{Q}_\ell$), and $\hat{s}$ is the linear character of the center $Z(H)$ of H given by:

$$\forall \varphi \in s_H,\ \forall h \in H,\ \forall t \in Z(H),\ \varphi(th) = \hat{s}(t)\varphi(h).$$

If the unipotent character ψ is labelled by the tuple of partitions μ, we write $\psi = \chi_\mu$, and $\chi = \chi_{s,\mu} = \pm R_H^G(\hat{s}\chi_\mu)$. We introduce two class functions χ^μ and $\chi^{s,\mu}$ (of H and G respectively) such that $\chi_\mu = \pm\chi^\mu$, and $\chi^{s,\mu} = R_H^G(\hat{s}\chi^\mu) = \mp\chi_{s,\mu}$.

4.3.3 Murnaghan-Nakayama Rule for unipotent characters

Pick $g \in G$, and write $Char(g) = \prod_i f_i^{k_i}$ and the corresponding decomposition $g = \prod_i g_i$. Then pick i_0, and write $g = \rho\sigma$, where

$$\rho = g_{i_0} \text{ and } \sigma = \prod_{i \neq i_0} g_i.$$

Writing d the degree of f_{i_0}, $m = k_{i_0} d$, and $l = n - m$, we have, writing $\backsim$ for equivalence of matrices over $\mathbf{F}_q$,

$$g \backsim \begin{pmatrix} \ddots & & \\ & U_{\nu_i}(f_i) & \\ & & \ddots \end{pmatrix},$$

$$\rho \backsim \begin{pmatrix} I_l & \\ & U_{\nu_{i_0}}(f_{i_0}) \end{pmatrix}$$

(and we may consider that $\rho \in G_0 = GL(m, q)$),

$$\sigma \backsim \begin{pmatrix} () & & & \\ & \ddots & & \\ & & () & \\ & & & I_m \end{pmatrix}$$

(and we may consider that $\sigma \in G_1 = GL(l, q)$).

Then, using the results on the Jordan decomposition given in the first section, we see that the semi-simple part ρ_S of ρ is equivalent to

$$\begin{pmatrix} I_l & \\ & U_{k_{i_0}}(f_{i_0}) \end{pmatrix}.$$

We have $\rho_S \in GL(n, q)$ and $Char(\rho_S) = f_{i_0}^{k_{i_0}}(X-1)^l$, and we may consider that $\rho_S \in G_0 = GL(m, q)$ and $Char(\rho_S) = f_{i_0}^{k_{i_0}}$.

We have $C_G(\rho_S) = H = H_0 \times H_1$, where $H_1 \cong GL(l, q)$ and $H_0 = C_{GL(m,q)}(\rho_S) \cong GL(k_{i_0}, q^d)$.

Theorem 4.4 (Murnaghan-Nakayama Rule)**.** *(cf [12])*

Let $g \in G$, and ρ and σ be as above, and let $\nu \vdash n$. Then

$$\chi^\nu(\rho\sigma) = \sum_{\lambda \in \mathcal{L}_\nu} a^\rho_{\nu\lambda} \chi^\lambda(\sigma),$$

where $\mathcal{L}_\nu$ is the set of partitions λ of l which can be obtained from ν by removing k_{i_0} d-hooks, and $a^\rho_{\nu\lambda} \in \mathbf{Z}[q^d]$.
If $\mathcal{L}_\nu = \emptyset$, then $\chi^\nu(\rho\sigma) = 0$.
We have $a^\rho_{\nu\lambda} \neq 0$ for $\lambda \in \mathcal{L}_\nu$.
(The coefficients of $a^\rho_{\nu\lambda}$ depend on the characters of the symmetric group $S_{k_{i_0}}$ and the Green functions of $GL(k_{i_0}, q^d) \cong H_0$ (applied to the unipotent part ρ_U of ρ), and all the non-zero coefficients of $a^\rho_{\nu\lambda}$ have the same sign).

Remark: it is easy to see that, if $a^\rho_{\nu\lambda} \neq 0$, then ν and λ have the same d-core.

The idea is to use this theorem recursively so as to be able to obtain information about $< \chi^\lambda, \chi^\mu >_{x\mathcal{Y}(x)}$, for λ, $\mu \vdash n$ and $x \in \mathcal{X}_d$ or $x \in \mathcal{X}_{(d)}$. We first use it to obtain a formula for $\chi^\mu(xy)$, where $\mu \vdash n$ and $y \in \mathcal{Y}(x)$. We let $\mathcal{X} = \mathcal{X}_d$ or $\mathcal{X}_(d)$, and take $1 \neq x \in \mathcal{X}$. Suppose $x \in c_x = (\dots f_i^{\lambda_i} \dots)$

where, for each i, $\lambda_i \vdash k_i$, and $\delta(f_i) = d$ (resp. $d|\delta(f_i)$) or $f_i = X-1$. If $f_i \neq X-1$, we let $\delta(f_i) = m_i d$. From the definition of $\mathcal{X}$, we see that, in the primary decomposition of x, we may omit the component corresponding to $X-1$, because it is necessarily the identity. We therefore write $x = x_1 \dots x_r$ where each x_i has exatly one elementary divisor distinct from $X-1$, and with the same multiplicity as in $Char(x)$. We will say that x has *d-type* $(k_1m_1, \dots, k_rm_r)$.

By repeated use of the Murnaghan-Nakayama Rule, we obtain that, for any $y \in \mathcal{Y}(x)$ and for $\mu \vdash n$,

$$\begin{aligned}\chi^\mu(xy) &= \sum_{\mu_1\in\mathcal{L}_\mu} a^{x_1}_{\mu\mu_1}\chi^{\mu_1}(x_2\dots x_r y)\\ &= \sum_{\mu_1\in\mathcal{L}_\mu} a^{x_1}_{\mu\mu_1}\Big(\sum_{\mu_2\in\mathcal{L}_{\mu_1}} a^{x_2}_{\mu_1\mu_2}\chi^{\mu_2}(x_3\dots x_r y)\Big)\\ &= \sum_{\mu_1\in\mathcal{L}_\mu}\cdots\sum_{\mu_r\in\mathcal{L}_{\mu_{r-1}}} a^{x_1}_{\mu\mu_1}\dots a^{x_r}_{\mu_{r-1}\mu_r}\chi^{\mu_r}(y)\end{aligned}$$

which can be written

$$\chi^\mu(xy) = \sum_{\lambda\in\mathcal{L}^\mu_{(k_1m_1,\dots,k_rm_r)d}} \alpha^{(x_1,\dots,x_r)}_{\mu\lambda}\chi^\lambda(y),$$

where the $\alpha^{(x_1,\dots,x_r)}_{\mu\lambda}$'s are integers and $\mathcal{L}^\mu_{(k_1m_1,\dots,k_rm_r)d}$ is the set of partitions of $n-(\sum_i k_im_i)d$ which can be obtained from μ by removing k_1 m_1d-hooks, then k_2 m_2d-hooks, ..., and finally k_r m_rd-hooks. We will call such a sequence of removals a $(k_1m_1,\dots,k_rm_r)d$*-path* from μ to λ.

Note that, in this sum, each λ can appear several times, if there is more than one $(k_1m_1,\dots,k_rm_r)d$-path from μ to λ. Note also that, in the right side of this equality, y has implicitely been seen as an element of $GL(l,q)$, where $l = n - (\sum_i k_im_i)d$.

If $\mathcal{L}^\mu_{(k_1m_1,\dots,k_rm_r)d} = \emptyset$, then $\chi^\mu(xy) = 0$. We have $\alpha^{(x_1,\dots,x_r)}_{\mu\lambda} \in \mathbf{Z}[q^d]$, and, if we separate the possibly multiple occurences of each λ in the sum, then, for $\lambda \in \mathcal{L}^\mu_{(k_1m_1,\dots,k_rm_r)d}$, each of the $\alpha^{(x_1,\dots,x_r)}_{\mu\lambda}$'s is non-zero (these are indexed by the $(k_1m_1,\dots,k_rm_r)d$-paths from μ to λ).

If $\alpha^{(x_1,\dots,x_r)}_{\mu\lambda} \neq 0$, then, since there is a $(k_1m_1,\dots,k_rm_r)d$-path from μ to λ, and since the removal of a hook of length md can be obtained by the removal of a sequence of m hooks of length d, we see that μ and λ have the same d-core.

We call the $\alpha^{(x_1,\dots,x_r)}_{\mu\lambda}$'s the MN-coefficients, and we will from now on write $\alpha^x_{\mu\lambda}$ for $\alpha^{(x_1,\dots,x_r)}_{\mu\lambda}$.

4.3.4 Nakayama Conjecture for unipotent blocks

Now, we are able to prove that the unipotent characters of G satisfy one sense of a generalized Nakayama Conjecture. The proof we give is an adaptation to our case of the proof given by Külshammer, Olsson and Robinson in the case of symmetric groups. We let $\mathcal{X} = \mathcal{X}_d$ or $\mathcal{X}_{(d)}$ and take $x \in \mathcal{X}$ of d-type $\mathbf{km} = (k_1m_1, \ldots, k_rm_r)$. We let $l = n - \mathbf{km}d = l - (\sum_{i=1}^r k_im_i)d$. Writing $G_0 = GL(\mathbf{km}d, q) = \prod_{i=1}^r GL(k_im_id, q)$ and $G_1 = GL(l, q)$, we have $x = (x_0, x_1) = (x_0, 1) \in G_0 \times G_1$. Then $C_G(x) = H_0 \times H_1$, where $H_1 \cong GL(l, q)$ and $H_0 = C_{G_0}(x_0)$ (note that, if x is semi-simple, then $H_0 \cong GL(\mathbf{k}, q^{\mathbf{m}d}) \cong \prod_{i=1}^r GL(k_i, q^{m_id})$).

Now take $y \in \mathcal{Y}(x)$. Then, as an element of $C_G(x) = H_0 \times H_1$, we have $y = (y_0, y_1) = (1, y_1)$.

Writing $\mathcal{Y}_t(u)$ for $\mathcal{Y}(u)$ when $u \in GL(t, q)$, we have that y, element of $C_G(x)$, belongs to $\mathcal{Y}_n(x)$ if and only if $y = (1, y_1) \in H_0 \times H_1$ where y_1 belongs to $\mathcal{Y}_l(1)$. Hence $\mathcal{Y}_n(x)$ is in natural one to one correspondence with $\mathcal{Y}_l(1)$. Now we consider $\mu, \mu' \vdash n$, and $x \in \mathcal{X}$ of d-type $\mathbf{km} = (k_1m_1, \ldots, k_rm_r)$. We have

$$\begin{aligned} < \chi^\mu, \chi^{\mu'} >_{x\mathcal{Y}(x)} &= \frac{1}{|G|} \sum_{y \in \mathcal{Y}(x)} \chi^\mu(xy)\overline{\chi^{\mu'}(xy)} \\ &= \frac{1}{|G|} \sum_{y=(y_0,y_1) \in \mathcal{Y}(x)} \left[\left(\sum_{\lambda \in \mathcal{L}^\mu_{\mathbf{km}}} \alpha^x_{\mu\lambda}\chi^\lambda(y_1) \right) \left(\sum_{\lambda' \in \mathcal{L}^{\mu'}_{\mathbf{km}}} \alpha^x_{\mu'\lambda'}\overline{\chi^{\lambda'}(y_1)} \right) \right] \\ &= \frac{1}{|G|} \sum_{y_1 \in Y_l(1)} \sum_{\lambda \in \mathcal{L}^\mu_{\mathbf{km}},\, \lambda' \in \mathcal{L}^{\mu'}_{\mathbf{km}}} \alpha^x_{\mu\lambda}\alpha^x_{\mu'\lambda'}\chi^\lambda(y_1)\overline{\chi^{\lambda'}(y_1)} \end{aligned}$$

(by the above remark on $\mathcal{Y}(x)$).

We write $A^x_{\mu\mu'} =< \chi^\mu, \chi^{\mu'} >_{x\mathcal{Y}(x)}$. Then

$$\begin{aligned} A^x_{\mu\mu'} &= \frac{1}{|G|} \sum_{\lambda \in \mathcal{L}^\mu_{\mathbf{km}},\, \lambda' \in \mathcal{L}^{\mu'}_{\mathbf{km}}} \alpha^x_{\mu\lambda}\alpha^x_{\mu'\lambda'} \sum_{y_1 \in \mathcal{Y}_l(1)} \chi^\lambda(y_1)\overline{\chi^{\lambda'}(y_1)} \\ &= \frac{|H_1|}{|G|} \sum_{\lambda \in \mathcal{L}^\mu_{\mathbf{km}},\, \lambda' \in \mathcal{L}^{\mu'}_{\mathbf{km}}} \alpha^x_{\mu\lambda}\alpha^x_{\mu'\lambda'} < \chi^\lambda \chi^{\lambda'} >_{\mathcal{Y}_l(1)} \end{aligned}$$

$$\text{i.e. } A^x_{\mu\mu'} = \frac{|H_1|}{|G|} \sum_{\lambda \in \mathcal{L}^\mu_{\mathbf{km}},\, \lambda' \in \mathcal{L}^{\mu'}_{\mathbf{km}}} \alpha^x_{\mu\lambda}\alpha^x_{\mu'\lambda'} A^1_{\lambda\lambda'}.$$

We use induction on n to prove that, if $A^x_{\mu\mu'} \neq 0$, then μ and μ' have the same d-core. We may assume that $\mu \neq \mu'$.

If $n < d$, then each partition is its own d-core. Furthermore, in this case, $\mathcal{X} = \{1\}$ and $\mathcal{Y}(1) = G$. Thus, for all $x \in \mathcal{X}$, we have $A^x_{\mu\mu'} = A^1_{\mu\mu'}$, and $A^1_{\mu\mu'} =< \chi^\mu, \chi^{\mu'} >_G = 0$ (since $\mu \neq \mu'$). Hence the result is true in this case. Thus, we suppose $n \geq d$. First suppose $x \neq 1$ and $A^x_{\mu\mu'} \neq 0$. Then

$$A^x_{\mu\mu'} = \frac{|H_1|}{|G|} \sum_{\lambda \in \mathcal{L}^\mu_{\mathbf{km}},\, \lambda' \in \mathcal{L}^{\mu'}_{\mathbf{km}}} \alpha^x_{\mu\lambda}\alpha^x_{\mu'\lambda'}A^1_{\lambda\lambda'} \neq 0.$$

Thus there exist $\lambda \in \mathcal{L}^\mu_{\mathbf{km}}$ and $\lambda' \in \mathcal{L}^{\mu'}_{\mathbf{km}}$ such that $\alpha^x_{\mu\lambda}\alpha^x_{\mu'\lambda'}A^1_{\lambda\lambda'} \neq 0$. Then $\alpha^x_{\mu\lambda} \neq 0$ implies that μ and λ have the same d-core, and $\alpha^x_{\mu'\lambda'} \neq 0$ implies that μ' and λ' have the same d-core. And, by the induction hypothesis (applied to $n - \mathbf{km}d < n$), $A^1_{\lambda\lambda'} \neq 0$ implies that λ and λ' have the same d-core.

Now, if $x = 1$, we see, by the existence and uniqueness of the decomposition we introduced, that

$$0 =< \chi^\mu,\, \chi^{\mu'} >_G = \sum_{x \in \mathcal{X}} < \chi^\mu,\, \chi^{\mu'} >_{x\mathcal{Y}(x)} = \sum_{x \in \mathcal{X}} A^x_{\mu\mu'}.$$

Hence, if $A^1_{\mu\mu'} \neq 0$, then there exists an $x' \in \mathcal{X} \setminus \{1\}$ such that $A^{x'}_{\mu\mu'} \neq 0$. This in turn implies, by the previous case, that μ and μ' have the same d-core. Skipping back from class functions to irreducible characters, we see that we have proved the following

Theorem 4.5. *If two unipotent (irreducible) characters χ_μ and $\chi_{\mu'}$ of $G = GL(n,q)$ are directly linked across some $x\mathcal{Y}(x)$, where $x \in \mathcal{X} = \mathcal{X}_d$ or $\mathcal{X}_{(d)}$, then μ and μ' have the same d-core (and this is true in particular for $x = 1$).*

Extending by transitivity the relation of direct $\mathcal{Y}(1)$-linking, we obtain

Theorem 4.6. *Let $\mathcal{X} = \mathcal{X}_d$ or $\mathcal{X}_{(d)}$. If two unipotent characters χ_μ and $\chi_{\mu'}$ of $G = GL(n,q)$ are in the same unipotent $(\mathcal{X},\mathcal{Y})$-block of G, then μ and μ' have the same d-core.*

Each unipotent $(\mathcal{X},\mathcal{Y})$-block of G is therefore associated to a d-core. For each given d-core γ, we can consider the union of the (possibly many) unipotent $(\mathcal{X},\mathcal{Y})$-blocks associated to γ. The (a priori) bigger blocks obtained in this way are parametrized by the set of d-cores of partitions of n, and they satisfy the following equivalent of the Nakayama Conjecture. In accordance with the terminology used in [18], we call them *combinatorial unipotent $(\mathcal{X},\mathcal{Y})$-blocks.*

Theorem/Definition 4.7. *Let $\mathcal{X} = \mathcal{X}_d$ or $\mathcal{X}_{(d)}$. Two unipotent (irreducible) characters χ_μ and $\chi_{\mu'}$ belong to the same* **combinatorial unipotent $(\mathcal{X},\mathcal{Y})$-block** *of $GL(n,q)$ if and only if μ and μ' have the same d-core.*

We can find in [12] an analogue of the Murnaghan-Nakayama rule which applies to **any** irreducible character of G, with the extra hypothesis that the element ρ is semisimple and primary, with an elementary divisor of degree $d = 1$. It is also mentioned that this theorem can be generalized to an arbitrary d. This should allow us to derive the same properties as above for $(\mathcal{X}, \mathcal{Y})$-blocks of the whole of $\mathrm{Irr}(G)$, but taking this time $\mathcal{X} = \mathcal{X}_d^{(s)}$ or $\mathcal{X}_{(d)}^{(s)}$. The $(\mathcal{X}, \mathcal{Y})$-blocks of G obtained in this way should satisfy one sense of a generalized Nakayama Conjecture, and we should be able to define combinatorial $(\mathcal{X}, \mathcal{Y})$-blocks of G satisfying this generalized Nakayama Conjecture. Note that the properties we proved for unipotent characters hold if we take $\mathcal{X} = \mathcal{X}_d^{(s)}$ or $\mathcal{X}_{(d)}^{(s)}$.

In any case, it remains to study the possible difference between $(\mathcal{X}, \mathcal{Y})$-blocks and combinatorial $(\mathcal{X}, \mathcal{Y})$-blocks.

4.4 Second Main Theorem property

We now want to show that the combinatorial unipotent $(\mathcal{X}, \mathcal{Y})$-blocks of G satisfy the Second Main Theorem property. We therefore need to define, for $x \in \mathcal{X}$, the $(\mathcal{X}, \mathcal{Y})$-blocks of $C_G(x)$, and then to look at their behaviour with respect to domination.

4.4.1 $(\mathcal{X}, \mathcal{Y})$-blocks of centralizers

We take any positive integer d, and $\mathcal{X} = \mathcal{X}_d$, $\mathcal{X}_{(d)}$, $\mathcal{X}_d^{(s)}$ or $\mathcal{X}_{(d)}^{(s)}$. Even though we don't know what they look like, this defines $(\mathcal{X}, \mathcal{Y})$-blocks of $\mathrm{Irr}(G)$ (which are just $\mathcal{Y}(1)$-blocks, built using $\mathcal{Y}(1)$-linking), and we can define $(\mathcal{X}, \mathcal{Y})$-blocks of $C_G(x)$ for each $x \in \mathcal{X}$.

For $x \in \mathcal{X} \setminus \{1\}$, the $(\mathcal{X}, \mathcal{Y})$-blocks of $C_G(x)$ are the equivalence classes of the transitive closure of direct $x\mathcal{Y}(x)$-linking. Equivalently, irreducible characters in distinct $(\mathcal{X}, \mathcal{Y})$-blocks of $C_G(x)$ are orthogonal across $x\mathcal{Y}(x)$, and the blocks are minimal for this property. If x has d-type $\mathbf{km}$, then, writing $l = n - \mathbf{km}d$, we have $C_G(x) = H_0 \times H_1$ where $H_1 \cong GL(l, q)$ and $H_0 \leq G_0 \cong GL(\mathbf{km}d, q)$. Then $\mathrm{Irr}(C_G(x)) = \mathrm{Irr}(H_0) \otimes \mathrm{Irr}(H_1)$. Note that, as we noted in the previous section, we may consider x as an element of H_0, and then $x\mathcal{Y}(x) = \{(x, y) \in H_0 \times H_1,\, y \in \mathcal{Y}_l(1)\}$.

Take $\chi_0, \psi_0 \in \mathrm{Irr}(H_0)$ and $\chi_1, \psi_1 \in \mathrm{Irr}(H_1)$. We have

$$\begin{aligned} < \chi_0 \otimes \chi_1, \psi_0 \otimes \psi_1 >_{x\mathcal{Y}(x)} &= \frac{1}{|C_G(x)|} \sum_{y \in \mathcal{Y}(x)} (\chi_0 \otimes \chi_1)(xy)\overline{(\psi_0 \otimes \psi_1)(xy)} \\ &= \frac{1}{|C_G(x)|} \sum_{y \in \mathcal{Y}_l(1)} \chi_0(x)\chi_1(y)\overline{\psi_0(x)\psi_1(y)} \\ &= \frac{\chi_0(x)\overline{\psi_0(x)}}{|C_G(x)|} \sum_{y \in \mathcal{Y}_l(1)} \chi_1(y)\overline{\psi_1(y)} \\ &= \frac{\chi_0(x)\overline{\psi_0(x)}}{|H_0|} < \chi_1, \psi_1 >_{\mathcal{Y}_l(1)} . \end{aligned}$$

Since x is central in H_0, we have $\chi_0(x)\overline{\psi_0(x)} \neq 0$, and we see that $\chi_0 \otimes \chi_1$ and $\psi_0 \otimes \psi_1$ are directly $x\mathcal{Y}(x)$-linked if and only if χ_1 and ψ_1 are directly $\mathcal{Y}_l(1)$-linked. Extending by transitivity, we obtain that the $(\mathcal{X}, \mathcal{Y})$-blocks of $C_G(x)$ are the $\mathrm{Irr}(H_0) \otimes b_i$'s, where b_i runs through the $\mathcal{Y}_l(1)$-blocks of $H_1 \cong GL(l, q)$.

In analogy with this, we define the unipotent $(\mathcal{X}, \mathcal{Y})$-blocks and combinatorial unipotent $(\mathcal{X}, \mathcal{Y})$-blocks of $C_G(x)$ to be the $\mathrm{Irr}(H_0) \otimes b_i$'s, where b_i runs through the unipotent $\mathcal{Y}_l(1)$-blocks and combinatorial unipotent $\mathcal{Y}_l(1)$-blocks of $H_1 \cong GL(l, q)$ respectively.

4.4.2 Second Main Theorem property for combinatorial unipotent $(\mathcal{X}, \mathcal{Y})$-blocks

We take any positive integer d, $\mathcal{X} = \mathcal{X}_d$ or $\mathcal{X}_{(d)}$, and $x \in \mathcal{X} \setminus \{1\}$. We write $C_G(x) = H_0 \times H_1$ as above. For any combinatorial unipotent $(\mathcal{X}, \mathcal{Y})$-block B of G, labelled by the d-core γ, we set $\beta(x, B) = \mathrm{Irr}(H_0) \otimes b$, where b is the $\mathcal{Y}_l(1)$-block of H_1 labelled by γ. For any $\chi_\mu \in B$ and $\psi_0 \otimes \psi_\lambda \in \mathrm{Irr}(H_0) \otimes b$, we set

$$c_{\chi_\mu, \psi_0 \otimes \psi_\lambda} = \begin{cases} \alpha^x_{\mu\lambda} \text{ if } \psi_0 = 1_{H_0} \\ 0 \text{ otherwise} \end{cases} ,$$

where the $\alpha^x_{\mu\lambda}$'s are the MN-coefficients, obtained from the Murnaghan-Nakayama rule for unipotent characters. Then the definition of the $\alpha^x_{\mu\lambda}$'s shows that, for each $x \in \mathcal{X}$, the $\beta(x, B)$'s and $c_{\chi_\mu, \psi_0 \otimes \psi_\lambda}$'s satisfy the hypotheses of Proposition 1.33. Indeed, for each combinatorial unipotent $(\mathcal{X}, \mathcal{Y})$-block B, for each $\chi_\mu \in B$ and for each $y \in \mathcal{Y}(x)$, we have

$$\chi(xy) = \sum_{\psi_0 \otimes \psi_\lambda \in \beta(x, B)} c_{\chi_\mu, \psi_0 \otimes \psi_\lambda} \psi_0 \otimes \psi_\lambda(xy),$$

and, furthermore, $\beta(x, B)$ and $\beta(x, B')$ are disjoint whenever B and B' are distinct combinatorial unipotent $(\mathcal{X}, \mathcal{Y})$-blocks of G. This implies that, for each $x \in \mathcal{X}$, each combinatorial unipotent $(\mathcal{X}, \mathcal{Y})$-block of $C_G(x)$ is dominated by a unique combinatorial unipotent $(\mathcal{X}, \mathcal{Y})$-block of G. Hence the combinatorial unipotent $(\mathcal{X}, \mathcal{Y})$-blocks of G satisfy the Second Main Theorem property.

(...), which just shows that the human brain is ill-adapted for thinking and was probably originally designed for cooling the blood.

Terry Pratchett, *The last hero*.

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