Core and Dependent Balancedness: théorie et applications

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Cœur et balancement dépendant : théorie et applcations

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RÉSUMÉ

Cette thèse est composée de quatre articles. L'objet principal de la thèse est d'énoncer des conditions suffisantes de non vacuité pour le cœur dans les jeux sans paiements latéraux. Rappelons que, dans un système social où des comportements coopératifs entrent en jeu, le cœur est l'ensemble qui vérifie des propriétés naturelles de stabilité et de rationalité. Pour assurer la non vacuité, les principaux résultats de la littérature supposent qu'une condition de balancement sur le jeu est satisfaite. La première partie de la thèse concerne les résultats théoriques de non vacuité, ces résultats reposent sur une nouvelle condition de balancement, dite de balancement dépendant. Le deuxième partie de la thèse propose des applications de ces différents résultats pour le problème des tarifications stables dans les marchés contestables.

Le premier article s'intéresse à un résultat de non-vacuité du cœur dans un jeu coopératif sans paiements latéraux. On utilise une condition de balancement dépendant des paiements à partir de règles de taux de transfert qui généralise les versions précédentes de balancement avec poids constant. Des extensions du concept de cœur sont proposées incluant des cœurs satisfaisant un équilibre sur les règles de taux de transfert et des cœurs-équilibre dans le cadre de jeux coopératifs paramétrés. Les preuves d'existence empruntent des outils mathématiques à la théorie de l'équilibre général avec non-convexités. Des applications variées à des résultats de théorie des jeux et d'économie théorique sont données.

Le deuxième article propose quelques applications directes d'un résultat d'existence de l'article premier où il est montré, en particulier, l'existence d'allocations du cœur dans les jeux NTU qui satisfont un équilibre de taux de transfert sous une condition de balancement dépendant. Il s'avère que la notion de balancement dépendant procure en fait un outil manipulable pour sélectionner le cœur. Pour illustrer ce fait, nous montrons que cette notion permet d'unifier une partie de la littérature et d'obtenir des résultats d'existence dans des modèles de cœur avec partenariat, cœur socialement stable, prekernel moyen intersecté avec le cœur et de cœur interne faible.

Le troisième concerne l'existence de tarifications sans subventions croisées et soutenable dans un marché contestable multiproduit où les firmes ont la possibilité de discriminer les marchés locaux, composés d'une partie de la ligne commerciale et d'une partie d'agents. Les résultats sont obtenus sous une hypothèse de fonction de coût à partage équitable, et sous des conditions de bord des fonctions de demandes. Le problème de tarification est modélisé par des cœurs-équilibres de jeux de coût paramétrés par les prix.

Enfin, dans le quatrième article, nous revenons sur le problème de l'existence de tarification soutenables sur un marché contestable monoproduit avec discrimination par les prix. La définition de soutenabilité est généralisée dans un premier temps au cas de coalitions bloquantes pouvant décider d'un mécanisme de financement volontaire ; puis dans un second temps au cas de règle de valuation donnée a priori. Les résultats d'existence sont prouvés sous des hypothèses de fonctions de coût à partage équitable généralisées. Le probléme est modélisé en terme de sélections de cœur dans les jeux NTU.

ABSTRACT

The dissertation is composed by four articles. The central topic of the dissertation deals with sufficient conditions for non-emptiness of the core in games without side payments. Let us recall that, in a social system where cooperative behaviors are at stake, the core satisfies rationality and stability. To guarantee the non-emptiness, the main results of the literature always assume that a balancedness condition is satisfied in the game.The first part of the dissertation deals with theoretical results for non-emptiness, these results rely on a new condition of balncedness, called dependent balancedness. The second part deals with applications to the case of stable pricing in contestable markets.

In the first paper, we prove the non-emptiness of the core in NTU games, using a payoff-dependent balancedness condition, based on transfer rate mappings. Going beyond the non-emptiness of the standard core, we prove the existence of core allocations with transfer rate rule equilibrium and equilibrium-core allocations in a parameterized cooperative game. The proofs borrow mathematical tools and geometric constructions from general equilibrium theory with non convexities. Applications to various extant results taken from game theory and economic theory are given, like the partnered core, the social coalitional equilibrium and the core for economies with non-ordered preferences.

Different kinds of asymmetries between players can occur in core allocations, in that case the stability of the concept is questioned. One remedy consists in selecting robust core allocations. We review, in the second article, results that all select core allocations in NTU games with different concepts of robustness. Within a unified approach, we deduce the existence of allocations in: the partnered core, the social stable core, the core intersected with average prekernel, the weak inner core. We use a recent contribution of Bonnisseau and Iehlé (2003) that states the existence of core allocations with a transfer rate rule equilibrium under a dependent balancedness assumption. It shall turn out to be manipulable tools for selecting the core.

In the third article, we prove the existence of subsidy free and sustainable pricing schedule in multiproduct contestable markets. We allow firms to discriminate the local markets that are composed by a set of the products line and a set of agents. Results are obtained under an assumption of fair sharing cost and under boundary condition of demand functions. The pricing problem is modelled in terms of equilibrium-core allocations of parameterized cost games.

Lastly, one defines in the fourth article a general notion of sustainable price schedules, where,

firstly, the participants of a blocking coalition can agree for voluntary financing device, or secondly follow a valuation rule given by a coordinating center. We consider a single output market where price discrimination is allowed. Existence results are provided in both cases under assumptions of generalized fair sharing cost function. The strategy of the proof is based on recent developments of cooperative game theory about core selections in NTU games.

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1. INTRODUCTION GÉNÉRALE

Introduction

Dans un système social où des comportements coopératifs entrent en jeu, le cœur est l'ensemble qui vérifie des propriétés naturelles de stabilité et de rationalité. Une allocation appartient au cœur si elle est réalisable par la coalition formée de tous les joueurs du système et si aucune sous-coalition de joueurs ne peut réaliser d'allocation meilleure pour tous ses membres. Autrement dit, une fois qu'un accord dans le cœur est trouvé, aucun individu ou groupe d'individus ne gagne à sortir de la grande coalition formée de tous les joueurs.

> L'objet de cette thèse est de proposer des résultats théoriques et des applications pour la non vacuité du cœur.

Historiquement, l'idée de cœur apparaît dans l'ouvrage de Edgeworth (21), qui énonce le concept à partir de la notion de courbe de contrats qui correspond aux situations non mutuellement améliorables dans une économie. La notion trouve ainsi une première justification dans les modèles économiques avant d'être analysée plus précisément avec le développement de la théorie des jeux au début des années cinquante. Le concept est ainsi repris dans l'ouvrage fondateur de la théorie des jeux de von Neumann et Morgenstern (76).

La théorie des jeux coopératifs est supposée donner dès l'origine un cadre d'analyse mathématique précis à l'idée de système social avec comportements collectifs. Le cœur s'impose alors naturellement comme l'un des concepts majeurs car il prédit la stabilité du système et la formation éventuelle de la coalition formée de tous les joueurs.

Un jeu coopératif a une définition simple : pour chaque coalition, un ensemble de paiements décrit les situations atteignables par les joueurs de la coalition, de plus il existe un critère de dominance sur les paiements de chaque coalition.

On peut décrire la théorie des jeux coopératifs à travers deux grandes familles de jeux, d'une part, les jeux à paiements latéraux (TU), de l'autre, les jeux sans paiements latéraux (NTU).

Dans les jeux TU, l'activité collective est décrite par une fonction caractéristique qui associe à chaque coalition une valeur (utilité, richesse...). Une allocation est alors réalisable par une coalition si elle partage au plus cette richesse entre les joueurs de la coalition.

Dans les jeux NTU, l'activité collective est plus complexe, elle est décrite par une fonction caractéristique qui associe à chaque coalition un ensemble de vecteurs de paiements. Les allocations réalisables sont alors définies par ces vecteurs représentant les paiements individuels des joueurs de la coalition. Les joueurs coopèrent mais ne transforment pas leur activité en une richesse commune.

A l'inverse du cas TU, étant donnée une allocation réalisable par une coalition, un nouveau partage qui donne le même niveau de richesse globale n'est pas forcément réalisable dans un jeu NTU. Notons également que les jeux TU forment une classe particulière de jeux NTU.

Dans le cas TU, la première formalisation du cœur est dûe à Gilles (26) ; Aumann (2) en propose une formalisation pour le cas NTU. L'enjeu de l'analyse est alors de décrire les conditions sur le jeu pour lesquelles le système social associé admet une solution stable de type cœur. Ainsi fut introduite la condition dite de jeu balancé. Cette condition emblématique établit que toute allocation appartenant à une famille balancée de coalitions doit également être réalisable par la grande coalition ; la notion de famille balancée de coalitions étant, quant à elle, une généralisation de la notion de partition.

Shapley (63) et Bondareva (12) ont ainsi montré que le cœur d'un jeu TU est non vide si et seulement si le jeu est balancé. Dans le cas NTU, l'approche mathématique est plus complexe. La difficulté principale repose sur la structure de la fonction caractéristique et des ensembles de paiements associés. En effet, à la différence des jeux TU où l'ensemble de paiements peut être vu comme un hyperplan supporté par le vecteur unitaire, les ensembles de paiements d'un jeu NTU vérifient seulement une propriété de fermeture et de compréhensivité par le bas. Néanmoins, Scarf (60) transpose la condition de balancement au cas NTU et prouve que la condition est suffisante pour la non vacuité du cœur dans un jeu NTU. Le résultat est généralisé plus tard par Billera (10) et Keiding et Thorlund-Petersen (42). Ces résultats reposent cependant tous sur la même notion originale de balancement telle qu'elle est définie dans Scarf (60).

> Le premier résultat de cette thèse consiste à montrer la non vacuité du cœur dans les jeux NTU sous une hypothèse dite de balancement dépendant qui généralise les notions précédentes (Chapitre 1).

Pour donner une première interprétation de la notion de balancement dépendant, revenons au cas de jeux TU. En effet, le principe de balancement au sens de Scarf reprend une propriété des jeux TU selon laquelle les transferts d'utilité peuvent se faire au taux constant de un pour un. Ainsi, on peut associer un vecteur de taux de transfert à chaque coalition de joueurs, ce vecteur est défini comme le vecteur dont les coordonnées valent 1 pour les membres de la coalition et 0 pour les autres joueurs. Etant donnés ces vecteurs de taux de transfert, une famille de coalitions est balancée si le taux de transfert de la grande coalition est positivement généré par les taux de transfert des coalitions de la famille. Ainsi un jeu TU est balancé si tout paiement réalisable pour une famille balancée est réalisable pour la grande coalition formée de tous les joueurs.

Pour les jeux NTU, le taux de transfert peut être défini mais il ne correspond pas toujours à un transfert réalisable entre les joueurs de la coalition. La notion de balancement dépendant étend la notion de balancement, en remplaçant les taux de transfert par une règle de taux de transfert qui associe des vecteurs de taux de transfert à tous paiements efficaces de la coalition. Ainsi, une famille de coalition n'est plus balancée intrinsèquement mais relativement à un paiement. La définition d'un jeu balancé dépendant découle alors naturellement de cette nouvelle notion de balancement.

Par ailleurs, il s'avère que cette condition de balancement dépendant est nécessaire pour la non vacuité du cœur comme cela est montré indépendamment par Predtetchinski et Herings (53) dans une récente contribution. Ainsi, à l'image de la caractérisation donnée par Bondareva et Shapley dans les jeux TU, le balancement dépendant fournit une condition nécessaire et suffisante pour la non vacuité du cœur dans les jeux NTU.

Une première généralisation porte sur la définition même d'un jeu NTU. Il s'agit d'étendre le modèle au cas paramétré. Cette idée initiée dans Ichiishi (30), consiste à définir un ensemble de paramètres et d'associer à chaque paramètre un jeu NTU. Les ensembles de paiements deviennent alors des correspondances définies sur l'ensemble des paramètres. Dans ce cadre, la définition du cœur est enrichie d'une condition d'équilibre supplémentaire ; ce nouveau concept est appelé cœur-équilibre. Le coeur-équilibre sont les paires formées d'une allocation et d'un paramètre, l'allocation appartient au cœur du jeu associé au paramètre, tandis que le paramètre vérifie une condition d'équilibre dépendant elle-même de l'allocation. La condition de balancement dépendant est également étendue au cadre paramétré si bien que les résultats de non vacuité pour ce type de modèle généralisent les théorèmes de non vacuité pour le modèle simple de jeux NTU. Cette formalisation permet, entre autres, d'analyser des situations de type : équilibre social, économies avec rendements croissants, sans préférences ordonnées, ou avec information asymétrique, voir Border (18), Debreu (19), Ichiishi et Quinzii (34), Ichiishi et Radner (35), Kajii (40).

> Dans le cadre paramétré, un résultat de non vacuité pour un cœur-équilibre est énoncé.
> Il constitue le résultat abstrait le plus général de cette thèse (Chapitre 1).

Aussi naturelle soit-elle, la solution de type cœur est parfois insatisfaisante. Il peut s'avérer trop "gros" et certains éléments du cœur s'avèrent peu pertinents. Une stratégie consiste alors à renforcer le concept initial en sélectionnant des allocations du cœur vérifiant certaines propriétés supplémentaires. On retrouve cette idée dans plusieurs travaux de l'économie théorique ou de la théorie des jeux. Par exemple, l'un des premiers résultats dans ce sens concerne le cœur avec partenariat de Reny et Wooders (58). Il s'agit de définir à l'intérieur du cœur des propriétés de partenariat entre les différents joueurs. La motivation est claire : si le cœur est stable, il n'est pas forcément équitable et cette inéquité peut in fine provoquer l'instabilité. Ainsi, si certains joueurs contribuent plus que d'autres dans la réalisation de l'allocation du cœur, ces gros contributeurs sont en droit de réclamer un dû particulier. D'autres travaux proposent de raffiner le cœur suivant des motivations différentes, voir par exemple Herings et al. (28), Orshan et al. (49), Qin (54; 55).

Il semble intéressant de noter que lorsque le cœur est vide, un problème symétrique se pose. Que peut-on dire de la stabilité du système social ? Quelles solutions faut-il alors suggérer ? Ces questions ne sont pas nouvelles, et plusieurs réponses ont été apportées, souvent conjointement avec le développement des analyses sur le cœur.

Une première famille de solutions alternatives concernent les ensembles de marchandage (bargaining sets dans la littérature). Ces ensembles contiennent le cœur. La justification de ces concepts se fondent sur l'idée qu'un certain nombre d'allocations réalisables par une coalition ne vont pas se former en pratique car elles sont elles-mêmes dominées par d'autres coalitions. Les contributions principales (dans le cas NTU qui nous intéresse) sont Vohra (75), Mas-Colell (47), Zhou (77).

Une deuxième alternative, lorsque le cœur est vide, est de supposer que la solution de type cœur reste la seule solution pertinente. On considère alors comme stables les situations qui devraient appartenir au cœur si, toutes choses égales par ailleurs, on renforçait le pouvoir de la grande coalition de joueurs. Ces solutions sont appelées dans la littérature, quasi-cœur ou cœur étendu. On retrouve ces idées dans les contributions de Gomez (27), Keiding et Pankratova (41), Shapley et Shubik (65), Shimomura (73).

Ces remèdes pour les situations à cœur vide ne sont pas traitées dans la thèse, mais l'articulation autour des sélections et des extensions du cœur (ensemble de marchandages, quasi-cœur, cœur étendu) enrichissent les analyses sur la stabilité des systèmes sociaux avec comportements collectifs.

> Sous l'hypothèse de balancement dépendant, on montre qu'il existe des allocations du cœur avec équilibre de taux de transfert. Ces allocations spécifiques permettent de sélectionner le cœur. Ce résultat unifie une partie des résultats de la littérature. (Chapitre 2)

Notons que ce théorème de sélection a été utilisé récemment par Allouch (1) et Predtetchinski (52). Les auteurs montrent ainsi la non vacuité du cœur flou d'une économie d'échange qui s'exprime en effet comme une sélection du cœur dans le jeu initial déduit de l'économie.

Les domaines d'application du concept de cœur sont extrêmement nombreux, notamment en économie théorique, voir Aumann et Hart (3) et Shubik (74). La théorie des marchés contestables fournit un terrain d'application naturel pour le cœur. Il s'agit d'étudier, sur un marché multiproduit, la structure industrielle qui peut se dégager en fonction de la concurrence entre les firmes. Les politiques tarifaires des firmes constituent alors un facteur d'explication important. En effet, la firme en place qui fixe les prix de ses marchés peut décider de se positionner sur une tarification qui joue le rôle de barrière à l'entrée.

Ce type de tarification peut expliquer en partie la structure industrielle d'un marché où pourtant la libre entrée est possible. Les premières analyses remontent à Baumol et al. (5) (voir aussi Sharkey (72)). Il s'avère que la théorie des jeux coopératifs est l'outil central dans ce type d'analyse, les tarifications stables sont généralement modélisables en terme de cœur, cœur d'équilibre ou sélections du cœur. On peut ainsi déduire certaines conditions suffisantes d'existence de tarifications sans subventions croisées et soutenables, dans la lignée des résultats de Ten Raa (56; 57), Mirman et al. (48), Bendali et al. (6).

Une tarification qui réalise l'équilibre budgétaire est dite sans subventions croisées si les dépenses de tout sous-marché sont inférieures au coût effectivement induit par ce sous-marché. La notion plus forte de soutenabilité impose que l'entrée bénéficiaire sur tout sous-marché est impossible à d'autres niveaux de tarifications préférées. Dans la littérature, les analyses se limitent aux tarifications pour des firmes qui ne différencient pas leur clientèle ou leurs marchés locaux. Dans ce cadre très général, des conditions suffisantes d'existence d'un prix stable n'ont pu être mises en évidence pour des demandes élastiques et individualisées (voir Baumol et al. (5)).

Ces modèles de marchés contestables sont généralisés dans deux directions. Premièrement, on élargit les possibilités d'entrées pour les firmes grâce à une articulation à deux niveaux, elles choisissent une part de marché parmi les coalitions d'agents et une partie de la ligne de produits, tandis qu'habituellement les analyses ont tendance à traiter isolément biens et agents. La formulation très générale englobe les différents modèles de marchés contestables. Pour prouver l'existence de tarifications stables, on modélise le problème de la firme comme un jeu paramétré où le cœur-équilibre est exactement la tarification sans subventions croisées. On déduit ensuite la tarification soutenable sous une hypothèse de régularité.

> On exhibe des conditions d'existence de tarifications sans subventions croisées et soutenables sur un modèle de marché contestable général avec firmes multiproduits, discrimination par les prix et possibilité d'entrée double autour de la ligne de produits et des coalitions d'agents. (Chapitre 3).

Deuxièmement, à la différence du modèle usuel de contestabilité, les entrants ont la possibilité de financer la production du bien par un système plus élaboré de taxes et subventions. Deux classes de financement sont ainsi proposées, premièrement on suppose qu'un accord est trouvé sur la base du volontariat entre les membres d'une coalition de marchés locaux, deuxièmement, le projet de financement est organisée par un centre de coordination en charge de définir des règles d'évaluation pour chaque coalition. Pour ce type de soutenabilité généralisée, l'approche est basée sur les jeux NTU. La tarification soutenable correspond en fait à une sélection du cœur d'un jeu NTU associé.

> Dans un marché contestable mono-produit avec discrimination par les prix, on montre l'existence de tarifications soutenables où les agents ont la possibilité d'utiliser des projets de financement. (Chapitre 4).

Les preuves des chapitres 3 et 4 utilisent donc pleinement les résultats établis dans les chapitres 1 et 2 sur l'existence d'allocations du cœur-équilibre, et le théorème de sélection du cœur.

La suite de cette introduction générale est consacrée à la présentation formelle des résultats obtenus dans cette thèse. La Partie A regroupe les Chapitres 1 et 2 concernant les résultats de non vacuité du cœur dans les jeux NTU, l'existence de sélections du cœur dans les jeux NTU, la non vacuité du cœur d'équilibre dans les jeux paramétrés. La Partie B regroupe les Chapitres 3 et 4 concernant les applications des résultats théoriques à des tarifications stables dans les marchés contestables, premièrement dans un marché avec firme multiproduit, et deuxièmement pour des firmes monoproduits avec possibilité de projets de financement alternatifs.

Partie A.

Non vacuité du cœur et balancement dépendant

Jeux coopératifs : principales définitions

On définit formellement la notion de jeux sans paiements latéraux (NTU), puis la notion de cœur qui est la solution centrale de stabilité, étudiée par la suite.

Jeux sans paiements latéraux. Les notations suivantes seront utilisées pour définir un jeu NTU (pour non transferable utility). N est un ensemble fini de joueurs, $\mathcal{N}$ est l'ensemble des parties non vides de N. Pour tout $S \in \mathcal{N}$, L_S est le sous espace de dimension $|S|$ défini par $L_S = \{x \in \mathbb{R}^N \mid x_i = 0,\ \forall i \notin S\}$. L_{S+} (L_{S++}) est l'orthant positif (strictement positif) de L_S ; pour tout $x \in \mathbb{R}^N$, x^S est la projection de x sur L_S ; $\mathbf{1}$ est le vecteur de $\mathbb{R}^N$ dont les coordonnées sont 1 ; $\mathbf{1}^\perp$ est l'hyperplan $\{s \in \mathbb{R}^N \mid \sum_{i\in N} s_i = 0\}$; proj est la projection orthogonale sur $\mathbf{1}^\perp$; $\Sigma_S = \mathrm{co}\{\mathbf{1}^{\{i\}} \mid i \in S\}$; $m^S = \frac{\mathbf{1}^S}{|S|}$; $\Sigma = \Sigma_N$ et $\Sigma_{++} = \Sigma \cap \mathbb{R}^N_{++}$.

Un jeu $(V_S, S \in \mathcal{N})$ est une collection de sous-ensembles de $\mathbb{R}^N$ indéxés par $\mathcal{N}$. $x \in \mathbb{R}^N$ est un paiement ; $V_S \subset \mathbb{R}^N$ est l'ensemble des paiements réalisables de S ; $\mathcal{S}(x) = \{S \in \mathcal{N} \mid x \in \partial V_S\}$ est l'ensemble des coalitions, pour lesquelles $x \in \mathbb{R}^N$ est un paiement efficace ; $W := \cup_{S\in\mathcal{N}} V_S$ est l'union des ensembles de paiements.

Les hypothèses canoniques sur les ensembles de paiements. Dans tous les jeux considérés on supposera, sauf mention contraire, que les deux hypothèses suivantes sont satisfaites : les ensembles de paiements sont des cylindres compréhensifs par le bas et fermés, les paiements individuellement rationnels sont bornés supérieurement. Formellement :

(H1) (i) $V_{\{i\}}$, $i \in N$, et V_N sont non vides.

(ii) Pour tout $S \in \mathcal{N}$, V_S est fermé, $V_S - \mathbb{R}^N_+ = V_S$, $V_S \neq \mathbb{R}^N$, et, pour tout $(x, x') \in (\mathbb{R}^N)^2$, si $x \in V_S$ et $x^S = x'^S$, alors $x' \in V_S$.

(H2) Il existe $m \in \mathbb{R}$ tel que, pour tout $S \in \mathcal{N}$, pour tout $x \in V_S$, si $x \notin \mathrm{int}\, V_{\{i\}}$ pour tout $i \in S$, alors $x_j \leq m$ pour tout $j \in S$.

Des projections sur les bords du jeu seront utilisées de manière récurrente. En effet, sous l'hypothèse H1, il existe des fonctions continues p_N de $\mathbb{R}^N$ dans ∂V_N, p_W de $\mathbb{R}^N$ dans ∂W, λ_N et λ_W de $\mathbf{1}^\perp$ dans $\mathbb{R}$ telles que, pour tout $x \in \mathbb{R}^N$, $p_N(x) = \text{proj}(x) - \lambda_N(\text{proj}(x))\mathbf{1}$ et $p_W(x) = \text{proj}(x) - \lambda_W(\text{proj}(x))\mathbf{1}$. Ce type d'outil est développé dans (14; 15; 17).

Le cœur. Le principal concept étudié dans les sections suivantes est le cœur. Cet ensemble correspond à tous les paiements du jeu réalisables par la grande coalition et tels qu'il n'existe pas de coalition qui réalise un paiement strictement supérieur pour tous ses membres. Le cœur peut se réécrire formellement :

Définition 1.0.1 *Soit $(V_S, S \in \mathcal{N})$ un jeu. Un paiement x est dans le cœur du jeu si $x \in V_N \setminus \mathit{int}\, W$.*

La définition fait seulement appel à deux ensembles, cette formulation s'avère cruciale dans la suite, orientant une partie des constructions géométriques.

Il semble également intéressant de noter que cet ensemble peut être défini de manière équivalente grâce aux outils introduits précédemment. Le paiement $x \in \mathbb{R}^N$ est dans le cœur du jeu si l'une des conditions suivantes est vérifiée :

1. $x \in \partial W \cap V_N$.
2. $x \in \partial W$ et $N \in \mathcal{S}(x)$.
3. $x \in V_N$ et $p_W(x) = x$.
4. $x \in \partial W$ et $p_{V_N}(x) = x$.

Le cas des jeux avec paiements latéraux. Dans ces jeux, aussi appelés jeux TU (pour transferable utility), on suppose que les agents peuvent transférer leur utilité à l'intérieur d'une coalition au taux unitaire. Dans un jeu TU, on suppose qu'il existe, pour chaque $S \in \mathcal{N}$, un paiement $v_S \in \mathbb{R}$. Le jeu TU, noté $(v_S, S \in \mathcal{N})$ peut alors s'écrire comme un jeu NTU, il correspond au jeu $(V_S, S \in \mathcal{N})$ défini par les ensembles de paiements :

$$V_S = \{x \in \mathbb{R}^N \mid \sum_{i \in S} x_i \leq v_S\}$$

Notons que les hypothèses H1 et H2 données plus haut sont toutes deux satisfaites. Pour cette classe de jeux particulière, le cœur du jeu peut se réécrire comme l'ensemble des vecteurs $q \in \mathbb{R}^N$ tels que :

$$\begin{cases} q \cdot \mathbf{1} = v_N \\ q \cdot \mathbf{1}^S \geq v_S, \ \forall S \in \mathcal{N} \end{cases}$$

1.1 Balancement dépendant et cœurs

Chapitre 1, référence (16) dans la bibliographie.

1.1.1 Jeux NTU

La première partie de cette section concerne les jeux coopératifs sans paiements latéraux (jeux NTU définis en introduction de la partie A) pour lesquels une condition de non-vacuité du cœur est énoncée. Plus précisément le résultat spécifie l'existence d'allocations du cœur satisfaisant une condition d'équilibre sur les règles de taux de transfert que nous exploitons plus explicitement dans la section 1.2.

Premier résultat de non vacuité. Le résultat repose sur les notions de règles de taux de transfert et de jeu balancé dépendant.

Définition 1.1.1 *Soit* $(V_S, S \in \mathcal{N})$ *un jeu satisfaisant l'hypothèse H1.*

(i) Une règle de taux de transfert est une famille de correspondances $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ *telles que pour tout* $S \in \mathcal{N}$*,* φ_S *est semi-continue supérieurement à valeurs convexes non vides et compactes de* ∂V_S *dans* Σ_S*, et,* ψ *est semi-continue supérieurement à valeurs convexes non vides et compactes de* ∂V_N *dans* Σ*.*

(ii) Le jeu $(V_S, S \in \mathcal{N})$ *est balancé dépendant s'il existe une règle de taux de transfert* $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ *telle que, pour tout* $x \in \partial W$*,*

$$\textit{Si } \mathrm{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset, \textit{ alors } x \in V_N.$$

Dans la littérature, la règle de transfert usuelle est constante. A chaque élément x sur ∂V_S, elle associe le vecteur m^S. L'innovation vient ici du fait que la règle de transfert, définie comme une correspondance, peut dépendre du paiement x. Il est à noter également que la condition de balancement porte seulement sur le bord de W.

Le premier résultat de la section est le suivant :

Théorème 1.1.1 *Soit* $(V_S, S \in \mathcal{N})$ *un jeu satisfaisant les hypothèses H1 et H2. Si le jeu est balancé dépendant par rapport à la règle de taux de transfert* $((\varphi_S)_{S\in\mathcal{N}}, \psi)$*, il existe une allocation du cœur* x *vérifiant :*

$$\mathrm{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(x) \neq \emptyset.$$

Idée de la preuve. Un résultat abstrait de Bonnisseau et Cornet (15, Théorème 1 p.67) permet d'obtenir le Théorème 1.1.1 comme corollaire d'un résultat d'existence d'équilibre économique dans une économie à secteurs de productions non-convexes. La preuve du théorème 1.1.1 repose sur la construction d'une économie fictive avec deux ensembles de production non convexes, construits à partir des ensembles de paiements du jeu initial, et de règles de tarification construites à partir des règles de taux de transfert. Un aspect intéressant de la preuve consiste également relier deux conditions occupant chacune une place emblématique dans la théorie des jeux coopératifs et dans la théorie de l'équilibre économique, respectivement, la condition de balancement et l'hypothèse de survie. Dans Shapley et Vohra (68), on retrouve ce type d'analogies dans les arguments de points fixes entre la théorie du cœur et celle de l'équilibre général avec non convexités.

La classe des jeux balancés dépendants inclut tous les jeux balancés de la littérature, (10; 42; 60), et les jeux convexes à la Billera (10). L'introduction d'une correspondance supplémentaire ψ permet par ailleurs d'obtenir des raffinements intéressants de la notion usuelle de cœur. En effet, le théorème 1.1.1 exhibe une allocation du cœur x telle que la famille de coalitions, pour laquelle x est réalisable, est balancée par rapport au vecteur de taux de transfert donné par la règle de taux de transfert. Cette condition est pertinente lorsque ψ diffère de φ_N, puisque sinon elle est trivialement satisfaite. Cette condition trouve plusieurs terrains d'applications, l'exemple du cœur avec partenariat de Reny et Wooders (58) est donné.

Première famille de corollaires. Non vacuité du cœur. Il existe principalement trois versions de balancement dans la litérature des jeux NTU, Scarf (60), Billera (10), Keiding et Thorlund-Petersen (42). Dans (60), une famille de coalition $\mathcal{B} \subset \mathcal{N}$ est balancée si pour tout $S \in \mathcal{B}$, il existe $\lambda_S \in \mathbb{R}_+$ tel que $\sum_{S\in\mathcal{B}} \lambda_S \mathbf{1}^S = \mathbf{1}$. Le jeu est balancé si pour toute famille balancée de coalitions $\mathcal{B} \subset \mathcal{N}$, $\cap_{S\in\mathcal{B}} V_S \subset V_N$.

Dans Billera (10), un vecteur de taux de transfert $b_S \in L_{S+} \setminus \{0\}$ est associé à chaque coalition S. Une famille de coalitions $\mathcal{B} \subset \mathcal{N}$ est b-balancée si pour tout $S \in \mathcal{B}$, il existe $\lambda_S \in \mathbb{R}_+$ telle que $\sum_{S\in\mathcal{B}} \lambda_S b_S = b_N$. Le jeu est b-balancé si pour toute famille b-balancée de coalitions $\mathcal{B} \subset \mathcal{N}$, $\cap_{S\in\mathcal{B}} V_S \subset V_N$.

Enfin, on est amené à la définition de balancement la plus raffinée, les jeux $(\partial - b)$-balancés. Le jeu est dit $(\partial - b)$-balancé si pour toute famille b-balancée de coalitions $\mathcal{B} \subset \mathcal{N}$, $\partial W \bigcap (\cap_{S\in\mathcal{B}} V_S) \subset V_N$. On obtient le corollaire suivant.

Corollaire 1.1.1 *Pour tout $S \in \mathcal{N}$, soit $b_S \in L_{S+} \setminus \{0\}$. Le cœur du jeu est non vide si le jeu est $(\partial - b)$-balancé et satifait les hypothèses H1 et H2.*

Le résultat de Billera (10) est immédiatement déduit du corollaire précédent, le résultat de Keiding et Thorlund-Petersen (42) est également obtenu en corollaire direct. Leur notion de jeu $(b- <)$-balancé étant lui aussi un cas particulier de jeu $(\partial - b)$-balancé. Dans Scarf (60), l'auteur prouve la non vacuité du cœur sous les hypothèses H1 et,

> *(WH2)* Il existe $m \in \mathbb{R}$ tel que, pour tout $x \in V_N$, si $x \notin \text{int } V_{\{i\}}$ pour tout $i \in N$, alors $x \leq m\mathbf{1}$.

Sous les hypohèses H1 et WH2, l'hypothèse H2 est vérifiée si le jeu est balancé, ainsi le résultat de Scarf est également déduit du Corollaire 1.1.1.

Pour les jeux convexes, Billera (10) donne une condition nécessaire et suffisante de non vacuité. Il utilise la notion de fonction support. Pour tout $S \in \mathcal{N}$, σ_S est la fonction support de V_S, c'est à dire, la fonction de $\mathbb{R}^N$ dans $\mathbb{R} \cup \{+\infty\}$ définie par $\sigma_S(p) = \sup\{p \cdot v \mid v \in V_S\}$.

Corollaire 1.1.2 (Billera (1970)) *Le cœur d'un jeu $(V_S, S \in \mathcal{N})$ est non vide si les hypothèses H1 et H2 sont satisfaites, si V_N est convexe et si pour tout $S \in \mathcal{N} \setminus \{N\}$, il existe $b_S \in \mathbb{R}^N \setminus \{0\}$ tel que $\sigma_S(b_S)$ est fini et pour tout $b \in cone\{b_S \mid S \in \mathcal{N} \setminus N\}$, $\sigma_N(b) \geq \max\{\sum_{S \in \mathcal{N}\setminus\{N\}} \lambda_S \sigma_S(b_S) \mid \forall S \in \mathcal{N} \setminus \{N\}, \lambda_S \geq 0,\ \sum_{S \in \mathcal{N}\setminus\{N\}} \lambda_S b_S = b\}$.*

On obtient également le résultat en corollaire du Théorème 1.1.1, il suffit de considérer la règle de taux de transfert : $\varphi_S = b_S$ pour tout $S \in \mathcal{N} \setminus N$ et $\varphi_N = \psi = N_{V_N}$ où N_{V_N} désigne le cone normal à l'ensemble V_N.

Remarque 1.1.1 *La condition est nécessaire si tous les paiements sont convexes. Notons également qu'un jeu TU peut être vu comme un jeu convexe à la Billera. Les jeux hyperplans (supportés par des vecteurs positifs quelconque) sont également des jeux convexes.*

Les résultats précédents ne sont pas surprenants au vu de la récente contribution de Predtetchinski et Herings (53). Ces auteurs prouvent un résultat similaire de non vacuité du cœur. En fait, ils utilisent l'hypothèse plus faible de bornitude WH2 tandis que la condition de balancement dépendant est plus forte. La contribution importante de leur article repose sur le fait qu'ils montrent que la condition de balancement dépendant est également nécessaire pour la non vacuité du cœur. A l'image de Bondareva (12) et Shapley (63) pour le cas TU, ils obtiennent donc une complète caractérisation du cœur dans les jeux NTU.

Néanmoins, ils n'obtiennent pas dans leur résultat la deuxième partie de la conclusion du Théorème 1.1.1, c'est à dire que la famille est balancée par rapport aux vecteurs de taux de transfert donnés par la règle de taux de transfert. Nous en donnons maintenant une première application. Considérons en effet le cas où la correspondance ψ diffère de φ_N. Dans ce cas, l'énoncé du Théorème 1.1.1 nous donne une allocation particulière du cœur satisfaisant une condition d'équilibre sur les taux de transfert.

Deuxième famille de corollaires. Cœur avec condition d'équilibre de taux de transfert. Le résultat qui suit est dû à Reny et Wooders (58). Il se déduit du Théorème 1.1.1 en construisant un règle de taux de transfert adéquate (d'autres possibilités seront évoquées en section 1.2).

Corollaire 1.1.3 (Reny-Wooders (1996)) *Soit $(V_S, S \in \mathcal{N})$ un jeu balancé satisfaisant les hypothèses H1 et H2. Supposons que pour toute paire de joueurs i et j, il existe une fonction continue $c_{ij} : \partial W \to \mathbb{R}_+$ telle que c_{ij} vaut zéro sur $V(S) \cap \partial W$ quand $i \notin S$ et $j \in S$. Alors il existe une allocation du cœur telle que, pour tout $i \in N$, $\eta_i(x) := \sum_{j \in N}(c_{ij}(x) - c_{ji}(x)) = 0$.*

On peut interpréter les fonctions c_{ij} comme des fonctions de crédits, $\eta_i(x)$ est alors la mesure du crédit net du joueur i par rapport à la grande coalition. La preuve du corollaire est très intuitive, la règle de transfert ψ va prendre en compte les contributions individuelles de chaque joueur face à la grande coalition. Il s'agit alors de montrer l'existence d'un indice constant $\psi(x)$ entre les agents.

Littérature associée. Les allocations du cœur avec équilibre de taux de transfert trouvent un terrain d'application dans les schémas de division équitable. En effet, comme cela est noté dans Reny et Wooders (58), la notion de collections de coalitions en partenariat est proche des concepts de kernel et prékernel. La notion de partenariat a également été exploitée pour des théorèmes de couvertures de type KKMS, voir Reny et Wooders (59), et Ichiishi et Idzik (33). Dernièrement, Page et Wooders (50) étendent la notion à un modèle d'équilibre compétitif et aux cœurs d'économies. Récemment, Herings et al. (28) définissent des allocations socialement stables, pour lesquelles les indices de pouvoir sont également répartis parmi les joueurs. Ce type d'allocations peuvent également être vues comme des allocations du cœur avec équilibre de taux de transfert (voir section 1.2).

1.1.2 Jeux NTU paramétrés

La deuxième partie de la section concerne les jeux coopératifs NTU paramétrés. Après avoir défini le concept d'allocation cœur-équilibre, un résultat analogue au Théorème 1.1.1 est énoncé dans un modèle canonique de jeux paramétrés. Puis, des résultats de non vacuité dans des économies à rendements croissants ou sans préférences ordonnées, (30; 18) sont obtenus en corollaire.

Ce modèle est proche de celui de Ichiishi (30), où le concept de d'équilibre social coalitionnel est défini. L'ensemble des paramètres est Θ et, un jeu NTU est associé à chaque $\theta \in \Theta$, c'est à dire que les ensembles de paiements V_S sont maintenant définis comme des correspondances de Θ dans $\mathbb{R}^N$. On introduit un ensemble de paiements additionnel V de Θ dans $\mathbb{R}^N$, qui peut éventuellement différer de V_N (voir les justifications dans Border (18) ; Boehm (11), Ichiishi (30)). Enfin, la condition d'équilibre sur les paramètres est représentée par une correspondance G de $\Theta \times \mathbb{R}^N$ dans $\mathbb{R}^N$. Nous notons $W(\theta)$ l'union des ensembles de paiements $\cup_{S\in\mathcal{N}} V_S(\theta)$.

Deuxième résultat de non vacuité.

Définition 1.1.2 *Une allocation du cœur-équilibre est un vecteur $(\theta^*, x^*) \in \Theta \times \mathbb{R}^N$ tel que :*

$$x^* \in \partial V(\theta^*) \setminus int\ W(\theta^*) \ \ et\ \theta^* \in G(\theta^*, x^*).$$

Les hypothèses H1 et H2 sont étendues à ce nouveau cadre paramétré.

(PH0) Θ est un sous-espace non vide, convexe, et compact d'un espace Euclidien. G est une correspondance semi-continue supérieurement à valeurs non vides et convexes.

(PH1) (i) Les correspondances $V_{\{i\}}$, $i \in N$, et V sont à valeurs non vides. V_S, $S \in \mathcal{N}$, et V sont des correspondances semi-continues inférieurement de graphe fermé.

(ii) Pour tout $\theta \in \Theta$, $V_S(\theta)$, $S \in \mathcal{N}$, et $V(\theta)$ satisfont l'hypothèse H1(ii).

(PH2) Pour tout $\theta \in \Theta$, il existe $m(\theta) \in \mathbb{R}$ tel que, pour tout $S \in \mathcal{N}$, pour tout $x \in V_S(\theta)$, si $x \notin \text{int } V_{\{i\}}(\theta)$, pour tout $i \in S$, alors $x_j \leq m(\theta)$ pour tout $j \in S$. Pour tout $\theta \in \Theta$, pour tout $x \in V(\theta)$, si $x_i \notin \text{int } V_{\{i\}}(\theta)$, pour tout $i \in N$, alors $x \leq m(\theta)\mathbf{1}$.

Comme précédemment, on peut définir p_V et λ_V des fonctions continues définies respectivement dans $\Theta \times \mathbb{R}^N$ et $\Theta \times \mathbf{1}^\perp$ telles que : $p_V(\theta, x) = \text{proj}(x) - \lambda_V(\theta, \text{proj}(x))\mathbf{1} \in \partial V_N(\theta)$. On définit de manière similaire p_W et λ_W asociés à W.

La règle de transfert $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ est étendue à un jeu paramétré, i.e. les correspondances qui définissent la règle de taux de transfert sont définies sur le graphe des correspondances de paiements.

Définition 1.1.3 *Soit $(V, (V_S)_{S\in\mathcal{N}}, \Theta)$ un jeu paramétré satisfaisant l'hypothèse PH1. Il est balancé dépendant s'il existe une règle de taux de transfert $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ telle que, pour tout $(\theta, x) \in$ Gr ∂W,*

$$\textit{Si } \text{co}\{\varphi_S(\theta, x) \mid S \in \mathcal{S}_\theta(x)\} \cap \psi(\theta, p_V(\theta, x)) \neq \emptyset, \textit{ alors } x \in V(\theta).$$

Le deuxième résultat de la section est le suivant :

Théorème 1.1.2 *Soit $(V, (V_S)_{S\in\mathcal{N}}, \Theta)$ un jeu paramétré satisfaisant les hypothèses PH0,PH1,PH2. S'il est balancé dépendant par rapport à la règle de transfert $((\varphi_S)_{S\in\mathcal{N}}, \psi)$, il existe une allocation du cœur-équilibre (θ^*, x^*) telle que :*

$$\text{co}\left\{\varphi_S(\theta^*, p_W(\theta^*, x^*)) \mid S \in \mathcal{S}_{\theta^*}(p_W(\theta^*, x^*))\right\} \bigcap \psi(\theta^*, x^*) \neq \emptyset.$$

Bien entendu, ce cadre paramétré comprend le cas où les ensembles de paiements sont constants par rapport à l'environnement. Les résultats de cette section couvrent ainsi tous les résultats de la sous-section 1.2. La preuve du Théorème 1.1.2 suit la même construction géométrique que le Théorème 1.1.1 avec un argument de point fixe, sans faire appel à un résultat de la théorie de l'équilibre général. L'argument de point fixe est néanmoins très proche de celui utilisé dans Bonnisseau (13) qui sert à montrer à l'existence d'un équilibre général dans des économies avec externalités et non-convexités. Nous explicitons la preuve en annexe de la section 1.

Liens avec les cœurs d'économies. On otient, en corollaire du Théorème 2, les résultats de non vacuité de Ichiishi (1981) et Border (1984).

L'équilibre social coalitionnel de Ichiishi (30) est le résultat de référence dans ce cadre de jeux coopératifs paramétrés. De plus, sa formulation générale de l'équilibre, où les agents peuvent réaliser un partition englobe à la fois le cœur et l'équilibre social de Debreu (19) comme cas particuliers. L'auteur développe également le concept autour d'une théorie générale de la firme dans (31).

Une structure coalitionnelle est une partition de N. Soit $\mathcal{P}$ un ensemble non vide de structures coalitionnelles, un élément de $\mathcal{P}$ est dénoté P. A chaque joueur est associé un ensemble de paramètres Θ_i ($\Theta_S = \prod_{i \in S} \Theta_i$, $\Theta = \Theta_N$). Pour tout $S \in \mathcal{N}$, soit F^S une fonction de Θ dans Θ_S. La relation de préférence de chaque joueur i d'une coalition S est représentée par une fonction d'utilité ; v_S^i : Gr $F^S \to \mathbb{R}$.

Un équilibre social coalitionnel est une paire constituée d'un paramètre $\theta^* \in \Theta$ d'une structure coalitionnelle $P^* \in \mathcal{P}$, tels que : (i) Pour tout $D \in P^*$, $\theta^{*D} \in F^D(\theta^*)$. (ii) Il n'est pas vrai qu'il existe $S \in \mathcal{N}$ et $\theta' \in F^S(\theta^*)$ tels que $v_S^i(\theta^*, \theta') > v_{D(i)}^i(\theta^*, \theta^{*D(i)})$ pour tout $i \in S$, où $D(i) \in P^*$ et $i \in D(i)$.

Corollaire 1.1.4 (Ichiishi(1981)) *Un équilibre social coalitionnel existe si : (1) Pour tout $i \in N$, Θ_i est sous-ensemble non vide, convexe et compact d'un espace Euclidien. (2) Pour tout $S \in \mathcal{N}$, F^S est une correspondance semi continue inférieurement et supérieurement à valeurs non vides. (3) Pour tout $S \in \mathcal{N}$, v_S^i est continue sur Gr F^S. (4) Pour tout $\theta \in \Theta$ et tout $v \in \mathbb{R}$, s'il existe une famille balancée $\mathcal{B}$ telle que pour tout $S \in \mathcal{B}$ il existe $\theta(S) \in F^S(\theta)$ pour qui $v_i \leq u_S^i(\theta, \theta(S))$ pour tout $i \in S$, alors il existe $P \in \mathcal{P}$ et $\theta'^D \in F^D(\theta)$ pour tout $D \in P$ tel que $v_i \leq u_D^i(\theta, \theta'^D)$ pour tout $i \in D$. (5) Pour tout $\theta \in \Theta$, et pour tout $v \in \mathbb{R}^N$, l'ensemble*

$$\bigcup_{P \in \mathcal{P}} \left\{ \theta' \in \Theta \mid \forall D \in P, \theta'^D \in F^D(\theta) \text{ et } v \leq (v_{D(i)}^i(\theta, \theta'^{D(i)}))_{i \in N} \right\}$$

est convexe.

Ichiishi et Quinzii (34) utilisent une variante du corollaire précédent pour établir la non vacuité du cœur d'économies avec rendements croissants.

Un second corollaire est obtenu : il s'agit d'un résultat de Border (18) sur la non vacuité du cœur dans une économie sans préférences ordonnées. Nous définissons pour ce faire un jeu paramétré tel que le cœur de l'économie correspond exactement à la définition des allocations du cœur-équilibre de la Définition 1.1.2.

Soit Ξ_i, $i \in N$, les ensembles de paiements des joueurs, $\Xi_S = \prod_{i \in S} \Xi_i$ et $\Xi = \Xi_N$. Pour tout $S \in \mathcal{N}$, soit F^S la fonction de réalisabilité de Ξ dans Ξ_S et soit $\Theta \subset \Xi$ l'ensemble des allocations réalisables globalement dans l'économie. La relation de préférence de chaque joueur est représentée par une correspondance P_i de Ξ_i dans Ξ_i.

Un élément $\xi \in \Xi$ est dans le cœur si : (i) $\xi \in \Theta$. (ii) Il n'existe pas $S \in \mathcal{N}$ et $\xi' \in F^S(\xi)$ satisfaisant $\xi'_i \in P_i(\xi_i)$ pout tout $i \in S$.

Corollaire 1.1.5 (Border (1984)) *Le cœur est non vide si : (1) Pour tout i,* Ξ_i *est un sous-ensemble convexe d'un espace Euclidien. (2) Pour tout* $S \in \mathcal{N}$*,* $F^S : \Xi \rightarrow \Xi_S$ *est une correspondance semi continue supérieurement et inférieurement à valeurs compactes et* F^i*,* $i \in N$*, est à valeurs non vides. (3)* Θ *est compact et convexe. (4) Pour tout i,* P_i *a un graphe ouvert dans* $\Xi_i \times \Xi_i$*, pour tout* $\xi_i \in \Xi_i$*,* $P_i(\xi_i)$ *est convexe et* $\xi_i \notin P_i(\xi_i)$*. (5) Le jeu est balancé : pour tout* $\xi' \in \Xi$*, pour toute famille balancée* β *avec les poids* $(\lambda_B)_{B\in\beta}$*, s'il existe* $(\xi^B)_{B\in\beta}$ *tels que* $\xi^B \in F^B(\xi')$*,* $B \in \beta$*, alors* $\xi \in \Theta$ *où* $\xi_i = \sum_{B\in\beta, i\in B} \lambda_B \xi_i^B$*.*

Littérature associée. Pour le cas des économies avec rendements croissants, des résultats de non vacuité d'économies se basent toujours sur des hypothèses de convexités implicites ; par exemple dans Ichiishi et Quinzii (34), les auteurs se ramènent à une notion d'ensembles distributifs introduite par Scarf (62), qui généralise la notion de convexité. Le résultat de non vacuité que nous énonçons élargit la classe de jeux dans lesquels les cœurs sont non vides. Le résultat négatif de Scarf (62, Theorem 5 p.426) délimite cependant la portée des nouveaux développements (voir aussi des détails dans Mas-Colell (46)). Des approches directes du cœur dans des économies ont été développées par Florenzano (24). Dans ce cadre, sans structure de jeu coopératif, il faut redéfinir la notion de balancement dépendant directement sur les fondamentaux de l'économie, en l'occurence les ensembles de production (voir aussi (43)).

Une deuxième piste concerne la question de représentation des marchés. Elle consiste en la détermination du marché que peut générer un jeu coopératif. Le cas TU a été résolu par Shapley et Shubik (66). Dans le cas jeux NTU, la question est toujours ouverte (le résultat de Mas-Colell (44) est basée sur une notion de balancement avec slacks). La flexibilité du concept de balancement dépendant pourrait permettre d'obtenir des résultats supplémentaires.

En sus, d'autres champs de l'économie théorique sont concernés par les jeux paramétrés : le cœur incitatif en information asymétrique, voir Ichiishi et Idzik (32), Ichishi et Radner (35), ces deux travaux utilisant le résultat originel de Ichiishi (30) ; le α-cœur, défini par Scarf (61) et généralisé par Kajii (40), ce dernier résultat s'inspirant du résultat de Border (18). Notons, néanmoins, que dans ces champs de recherche, des contre-exemples robustes existent, voir respectivement Forges et al. (25) et Holly (29)).

Annexe : preuve du Théorème 1.1.2, points fixes et non convexités

Les preuves de non vacuité de résultats la section 1.1, Théoèmes 1.1.1 et 1.1.2 reposent sur des arguments de points fixes. La difficulté principale vient du fait que les ensembles de paiements des jeux NTU sont non convexes. Pour appliquer un argument de point de fixe de type Kakutani, il faut alors se ramener à un ensemble convexe. Pour ce faire, on utilise le fait que les ensemble sont fermés et compréhensifs par le bas, on peut ainsi utiliser un homéomorphisme entre les bords des ensembles de paiements et l'espace orthogonal au vecteur unité. Les arguments de point fixes seront alors développés sur cet espace.

Nous exposons un aperçu de la preuve du Théorème 1.1.2 qui est le résultat abstrait le plus général de la thèse. On se place dans le cadre de jeux paramétrés et sous les hypothèses associées, définis en sous-section 1.1.2.

Nous introduisons premièrement des bornes uniformes par rapport à l'ensemble des paramètres. D'après la semi-continuité inférieure et l'hypothèse de graphe fermé des correspondances $V_{\{i\}}$, il est immédiat que les fonctions v_i, $i \in N$, sont continues. Dénotons $v = \min\{v_i(\theta) \mid \theta \in \Theta,\ i \in N\}\mathbf{1}$. La borne $m(\theta)$ donnée dans l'hypothèse PH2 peut également être choisie continûment puisque les correspondances V_S, $S \in \mathcal{N}$, sont semi-continues inférieurement avec graphe fermé. Soit $m = \max\{m(\theta) \mid \theta \in \Theta\}$, cette borne est bien définie puisque Θ est compact.

Nous définissons la correspondance Y_2 de Θ dans $\mathbb{R}^N$ par :

$$Y_2(\theta) = -\text{int }(W(\theta)^c).$$

Notons que Y_2 est semi-continue inférieurement avec graphe fermé, et, pour tout $\theta \in \Theta$, $Y_2(\theta) - \mathbb{R}^N_+ = Y_2(\theta)$ et $Y_2(\theta) \neq \mathbb{R}^N$. Soit $\tilde{\varphi}_2$ la correspondance de Gr ∂Y_2 dans Σ définie par :

$$\tilde{\varphi}_2(\theta, y_2) = \text{co}\left\{\varphi_S(\theta, -y_2) \mid S \in \mathcal{S}_\theta(-y_2)\right\}.$$

Lemme 1.1.1 *Soit $(\theta, y_2) \in Gr\ \partial Y_2$, si $y_{2i} < -m$ pour un $i \in N$, alors $p_i = 0$ pour tout $p \in \tilde{\varphi}_2(\theta, y_2)$.*

Puisque $Y_2(\theta)$ est bornée uniformément supérieurement par v, il existe ρ tel que $\text{proj}(\theta, y_2) \in \bar{B}_{\mathbf{1}^\perp}(0, \rho)$ pour tout $(\theta, y_2) \in$ Gr ∂Y_2 tel que $y_2 \in (\{-me\} + \mathbb{R}^N_+)$. Définissons la correspondance Y_1 de Θ dans $\mathbb{R}^N$ by :

$$Y_1(\theta) = \left\{p_V(s, \theta) \mid s \in \bar{B}_{\mathbf{1}^\perp}(0, \rho)\right\} - \mathbb{R}^N_+.$$

Puisque la fonction p_V est continue, notons que Y_1 est semi-continue inférieurement avec graphe fermé, et, pour tout $\theta \in \Theta$, $Y_1(\theta) - \mathbb{R}^N_+ = Y_1(\theta)$ et $Y_1(\theta) \neq \mathbb{R}^N$. Alors, la compacité de Θ implique l'existence de deux nombres réels α_1 et β_1 tels que pour tout $y_1 \in \{z_1 \in \partial Y_1(\theta) \mid \|\text{proj}(z_1)\| \leq \rho,\ \theta \in \Theta\}$, $\alpha_1 \mathbf{1} \leq y_1 \leq \beta_1 \mathbf{1}$. Notons également que, pour tout $\theta \in \Theta$, pour tout $y_1 \in \partial Y_1(\theta)$, si $\|\text{proj}(y_1)\| \leq \rho$, alors $y_1 \in \partial V(\theta)$. Choisissons $\underline{y}' \in$ int $Y_1(\theta)$ pour tout $\theta \in \Theta$. De tels éléments existent puisque chaque élément inférieur strictement à $\alpha_1 \mathbf{1}$ satisfait cette condition.

Lemme 1.1.2 *Il existe une fonction continue c de $Gr\ \partial Y_1$ dans Σ_{++} telle que $c(\theta, y_1) \cdot (y_1 - \underline{y}') \geq 0$ pour tout $(\theta, y_1) \in Gr\ \partial Y_1$.*

Soit $\tilde{\varphi}_1$ la correspondance de Gr ∂Y_1 dans Σ définie par :

$$\tilde{\varphi}_1(\theta, y_1) = \begin{cases} \psi(\theta, y_1) & \text{si } \|\text{proj}(y_1)\| < \rho \\ \text{co}\{\psi(\theta, y_1), c(\theta, y_1)\} & \text{si } \|\text{proj}(y_1)\| = \rho \\ c(\theta, y_1) & \text{si } \|\text{proj}(y_1)\| > \rho \end{cases}$$

Lemme 1.1.3 *Il existe $\alpha \in \mathbb{R}$ tel que, pour tout $(\theta, y_1, y_2) \in Gr\ (\partial Y_1 \times \partial Y_2)$, $(p_1, p_2) \in \tilde{\varphi}(\theta, y_1) \times \tilde{\varphi}_2(\theta, y_2)$, on a $p_1 \cdot y_1 + p_2 \cdot y_2 \geq \alpha$.*

Puisque les valeurs de Y_1 et Y_2 sont respectivement uniformément bornées supérieurement par $\beta_1 \mathbf{1}$ et $-v$, il existe un ensemble compact et convexe $\bar{B} \in (\mathbf{1}^\perp)^2$ tel que : $B(0,\rho) \times B(0,\rho) \subset \bar{B}$ et pour tout $(\theta, y_1, y_2) \in \text{Gr}\ (\partial Y_1 \times \partial Y_2)$ tel que $y_1 + y_2 - \alpha \mathbf{1} \in \mathbb{R}^N_{++} \cup \{0\}$, $(\text{proj}(y_1), \text{proj}(y_2)) \in \text{int}\ \bar{B}$.

Enfin, en utilisant (13, Lemme 3.1 p.217), on peut introduire les fonctions continues λ_1 et λ_2 de $\Theta \times \mathbf{1}^\perp$ dans $\mathbb{R}$ associées à Y_1 et Y_2. Fixons $\eta > 0$ arbitrairement, et définissons Σ_η comme l'ensemble $\left\{p \in \mathbb{R}^N \mid \sum_{i \in N} p_i = 1;\ \ p_i \geq -\eta,\ i \in N\right\}$.

Soit F la correspondance de $\Theta \times B \times \Sigma_\eta \times \Sigma^2$ dans lui-même. $F = \prod_{j=1}^4 F_j$.

$F_1(\theta, (s_1, s_2), p, (p_1, p_2)) = G(\theta, y_1)$

$F_2(\theta, (s_1, s_2), p, (p_1, p_2)) = \{\sigma \in B \mid \sum_{i=1}^2 (p - p_i) \cdot \sigma_i \geq \sum_{i=1}^2 (p - p_i) \cdot \sigma'_i,\ \forall \sigma' \in B\}$

$F_3(\theta, (s_1, s_2), p, (p_1, p_2)) = \{q \in \Sigma_\eta \mid (q - q') \cdot (y_1 + y_2) \leq 0,\ \forall q' \in \Sigma_\eta\}$

$F_4(\theta, (s_1, s_2), p, (p_1, p_2)) = (\tilde{\varphi}_1(\theta, y_1), \tilde{\varphi}_2(\theta, y_2))$

où pour $i = 1; 2$, $y_i = s_i - \lambda_i(\theta, s_i)\mathbf{1}$.

Lemme 1.1.4 *La correspondance F satisfait les conditions du théorème de Kakutani.*

D'après le lemme précédent et après calculs, on peut exhiber des équations un couple (θ^*, x^*) qui satisfait les conclusions du théorème 1.1.2.

1.2 Règles de taux de transfert et sélections du cœur

Chapitre 2, référence (39) dans la bibliographie.

Le cadre. On se place dans la classe de jeux NTU définie dans la section 1.1.

Motivations. Il apparaît que le concept du cœur peut être remis en question dans certaines situations où des dépendances asymétriques entre les joueurs engendrent une instabilité. Ici, on propose d'explorer plus systématiquement quelques conséquences directes du Théorème 1.1.1 qui répondent à cette critique.

Quatre corollaires. Le Théorème 1.1.1 établit l'existence d'allocations du cœur satisfaisant additionnellement une condition d'équilibre de taux de transfert. Il fournit en fait un outil flexible pour contracter/diminuer le cœur en un ensemble d'allocation *plus stables*. Ce fait est illustrer à partir d'exemples empruntés à une littérature récente, où des résultats d'existence d'allocations de cœur satisfont des propriétés additionnelles de stabilité.

La condition :

$$\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(x) \neq \emptyset.$$

est ainsi réinterprétée en termes d'équilibre de crédits, de pouvoir ou de partage. On obtient ainsi les résultats suivants :

1. Le cœur en partenariat de Reny et Wooders (58). Il s'agit d'allocations appartenant au cœur telles que les coalitions qui réalisent cette allocation réalisent une condition de partenariat. Nous rappelons le résultat de Reny et Wooders déjà obtenu dans la section 1.1.
2. Le cœur socialement stable de Herings et al (28). Ces auteurs introduisent un indice de pouvoir pour chaque joueur dans les coalitions. Ils prouvent alors la non vacuité de l'ensemble des allocations du cœur équipotentes : le cœur socialement stable.
3. On considère également la notion de prékernel moyen, qui est l'extension NTU de la notion usuel de prékernel dans les jeux TU. Nous prouvons la non vacuité du cœur intersecté avec le prékernel moyen.
4. Enfin, on propose un résultat d'existence pour une solution stable dans l'esprit du cœur interne defini dans Qin (54; 55). Le concept repose sur l'efficacité de la solution dans des jeux fictifs à λ-transferts (voir le travail originel de Shapley (64)) (voir aussi Shapley et Shubik (67)).

Nous présentons succintement les 3 applications supplémentaires énoncées précédemment.

Cœur socialement stable. Pour chaque coalition $S \in \mathcal{N}$, il existe un nombre fini k_S d'organisations internes. On note $I^S = (I_1^S ... I_{k_S}^S)$ ces organisations. Soit $\mathcal{I}$ l'union sur S de toutes les organisations internes. Pour chaque $S \in \mathcal{N}$, chaque $I \in I^S$, on définit l'ensemble de paiements $v_I \in \mathbb{R}^N$. La fonction p, $\mathcal{I} \to \mathbb{R}_+^N \setminus \{0\}$ désigne la fonction de pouvoir. Pour chaque $S \in \mathcal{N}$, chaque $I \in I^S$, $p(I) \in L_{S+} \setminus \{0\}$. Un jeu socialement structuré est décrit par $(N, \mathcal{I}, v, p)$. Dans (28), les auteurs reformulent les hypothèses H1 et H2 en fonction de ce jeu généralisé. Cela revient exactement à considérer que le jeu $(V_S, S \in \mathcal{N})$, où $V_S = \cup_{I \in I^S} v_I$, satisfait les hypothèses H1 et H2. Définissons le cone de pouvoir pour le paiement x : $PC(x) = \{y \in \mathbb{R}^N \mid y = \sum_{I \in \mathcal{I}(x)} \lambda_I p(I), \lambda_I \geq 0, \text{ pour tout } I\}$, où $\mathcal{I}(x) = \{I \in \mathcal{I} \mid x \in \partial v_I\}$.

Définition 1.2.1 *Soit un jeu socialement structuré, $(N, \mathcal{I}, v, p)$, un paiement $x \in \mathbb{R}^N$ est socialement stable si :*

$$\mathbf{1} \in PC(x).$$

Un allocation $x \in \mathbb{R}^N$ est dans le cœur du jeu si $x \in v_I$ pour un $I \in I^N$ et $x \notin int\, v_{\overline{I}}$ pour tout $\overline{I} \in \mathcal{I}$. Le cœur socialement stable est l'ensemble des allocations du cœur qui sont socialement stables.

(SSG) Si un paiement est socialement stable alors $x \in v_I$ pour un $I \in I^N$.

On déduit le résultat suivant dû à Herings et al. (28).

Corollaire 1.2.1 *Soit $(N, \mathcal{I}, v, p)$ un jeu socialement structuré et supposons que $(V_S, S \in \mathcal{N})$, où $V_S = \cup_{I \in I^S} v_I$, satisfait les Hypothèses H1 et H2. Sous SSG, le cœur socialement stable est non vide*

Prékernel moyen intersecté avec le cœur. Soit $(V_S, S \in \mathcal{N})$ un jeu NTU. On introduit deux hypothèses sur les paiements du jeu

- *(NL)* Pour chaque $S \in \mathcal{N}$, ∂V_S est non-leveled : si $x, y \in \partial V_S$, $x \geq y$ et $y \in I_S$, alors $x_i = y_i$.
- *(SM)* Pour tout $x \in \partial I_N$, il existe un vecteur unique $p(x)$ tel que $\sum_{i \in N} p_i(x) = 1$. De plus, pour tout $x \in \partial I_N$, $p(x) > 0$ et p est une fonction continue.

On peut maintenant définir les fonctions suivantes :

Pour chaque $x \in \mathbb{R}^N$, pour chaque $S \in \mathcal{N}$, soit $k \in S$, l'excès individuel de k par rapport à S au point x :

$$e_k(S, x) = \begin{cases} \max\{y_k - x_k \mid (y_k, x_{-k}) \in V_S\} & \text{si } \{y_k \mid (y_k, x_{-k}) \in V_S\} \neq \emptyset \\ -\infty & \text{sinon} \end{cases}$$

Pour tout $k, \ell \in N$, $k \neq \ell$, le surplus of k par rapport à ℓ au point x est : $s_{k\ell}(x) = \max\{e_k(S, x) \mid S \in \mathcal{N}, k \in S, l \notin S\}$.
Pour tout $k \in N$ et $x \in \partial I_N$, soit $f_k(x) = \sum_{\ell \neq k} (p_k(x) s_{k\ell}(x) - p_\ell(x) s_{\ell k}(x))$ la perte totale du joueur k au point x. Soit $f(x) = (f_1(x), ..., f_n(x))$.

Définition 1.2.2 *Le prékernel moyen du jeu est l'ensemble :*

$$\{x \in \partial V_N \mid f(x) = 0\}.$$

Corollaire 1.2.2 *Soit $(V_S, S \in \mathcal{N})$ satisfaisant les hypothèses H1, H2, NL et SM. S'il est ∂-balancé alors il existe un allocation du cœur qui appartient au prékernel moyen.*

Cœur interne faible. On définit un ensemble de transfert induit par une règle de taux de transfert : pour chaque $x \in \partial W$, $TS(x) = \text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\}$. Ainsi, $(\lambda, x) \in \text{Gr}\, TS$ dit que λ est un taux de transfert admissible au point x. λ va définir également le paiement d'un jeu hyperplan $v_\lambda(N) = \max\left\{\sum_{i \in N} \lambda_i \cdot y_i \mid y \in V_N\right\}$.

Définition 1.2.3 *La paire $(\lambda, x) \in Gr\, TS$ est dite intérieurement stable si :*

$$(\lambda, x) \in Gr\, TS \text{ et } \lambda \cdot x \geq v_\lambda(N).$$

Une allocation x est dans le cœur interne faible du jeu $(V_S, S \in \mathcal{N})$ si x appartient au cœur du jeu et s'il existe au moins un taux $\lambda \in \Sigma$ tel que (λ, x) est stable intérieurement.

Corollaire 1.2.3 *Soit* $(V_S, S \in \mathcal{N})$ *un jeu satisfaisant les hypothèses H1 et H2. Supposons également que* V_N *est un ensemble convexe. Si le jeu est balancé dépendant par rapport à la règle de transfert* $((\varphi_S)_{S\in\mathcal{N}}, N_{V_N} \cap \Sigma)$*, alors le cœur interne faible du jeu est non vide.*

Littérature associée. Les problèmes de partage des gains dans les modèles coopératifs ont déjà été étudiés par Bennett (7; 8) et Bennett et Zame (9). Des récents travaux peuvent également être associées à cette littérature (Gomez (27), Keiding et Pankratova (41)). Au lieu de sélectionner le cœur, ces auteurs proposent de définir un concept élargi de cœur, en étendant l'ensemble de paiements réalisables. Cet ensemble est systématiquement non vide et permet donc d'apporter une réponse quant à la stabilité du système social lorsque le cœur est vide. On dispose ainsi d'une articulation assez riche autour du cœur grâce à des mécanismes de sélection ou d'extension.

Partie B.

Tarifications stables dans les marchés contestables

Marchés contestables

Présentation. La théorie des marchés contestables a été développée dans les années soixante-dix par Bailey, Baumol, Panzar et Willig (4; 51). Il faut également associer au développement de la théorie les travaux suivants, Faulhaber (22), Faulhaber et Levinson (23), Sharkey (69; 70; 71), en particulier pour les questions de tarification stable. On peut préciser l'idée de marché contestable, nous suivons la définition originelle de Baumol et al. (5). Un marché contestable est un marché qui est accessible à des entrants potentiels et qui a les propriétés suivantes : les entrants potentiels peuvent servir sans restriction les mêmes marchés et utiliser les mêmes techniques de production que ceux disponibles pour la firme en place. Nous supposons en sus que les firmes peuvent maintenir séparer leurs marchés, autrement dit, discriminer les tarifications. Par ailleurs, on élargit les possibilités d'entrées pour les firmes grâce à une articulation à deux niveaux, elles choisissent une part de marché parmi un groupe d'agents et un panier de biens.

Le principe de contestabilité repose sur l'idée que l'entrée libre agit comme une force d'auto régulation si bien qu'une seule firme sur le marché peut éventuellement subsister sur le marché, i.e. garantir son équilibre budgétaire et écarter toute entrée viable de firme rivale. Une telle éventualité correspond à un positionnement tarifaire spécifique d'une firme. Nous montrons ici sous quelles conditions une telle tarification peut exister sur un marché contestable. On se concentre principalement sur les tarifications sans subventions croisées (subsidy free) et soutenables (sustainable).

Le modèle de base du marché contestable. On peut définir formellement le marché contestable que l'on considère à partir des quatre hypothèses suivantes.

- *(C1)* Le marché contestable est défini par un ensemble fini indexé de biens L et un ensemble fini et indexé de demandes individualisées N. Pour chaque agent $a \in N$, la fonction de demande D_a est une fonction continue, définie sur $\mathbb{R}_+^{LN}$, à valeurs dans $\mathbb{R}_+^L$.
- *(C2)* Toutes les firmes, qui ont accès au marché contestable, sont dotées de la même technologie de production, matérialisée par une fonction de coût C, fonction continue définie sur $\mathbb{R}_+^L$ et à valeurs dans $\mathbb{R}_+$.
- *(C3)* L'espace des prix de chaque firme est $\mathbb{R}_+^{LN}$, c'est à dire que les firmes sont en mesure de fixer un prix pour tous les marchés séparés.
- *(C4)* Les sous-marchés x accessibles par les firmes entrantes sont les sous parties non vides de l'ensemble produit des biens et agents : $x \in \{(A, B) \mid A \in \mathcal{N},\ B \in \mathcal{L}\}$ où $\mathcal{N}$ et $\mathcal{L}$ dénotent respectivement les ensembles de sous parties non vides de N et L.

$D_a(p) \in \mathbb{R}_+^L$ (resp. $D_a(p)_b$) est le vecteur des demandes des clients de type a (la demande pour le type a en bien b) lorsque la firme pose un prix $p \in \mathbb{R}_+^{LN}$), on notera également $D_a(p)^B$ le vecteur de demande pour le panier composé des biens B, pour $B \subset L$: $D_a(p)_b^B = D_a(p)_b$ si $b \in B$ et zéro sinon. Dans la suite $\mathcal{NL}$ désigne l'ensemble des sous-marchés accessibles.

1.3 *Tarifications stables et cœur-équilibre*

Chapitre 3, référence (37) dans la bibliographie.

Le cadre. On se place dans le cadre de marchés contestables défini précédemment.

Les tarifications stables. Des résultats de tarifications stables sont proposés pour les concepts de stabilité suivants. Premièrement, on considérera la tarification sans subventions croisées :

Définition 1.3.1 *Un prix $p \in \mathbb{R}_+^{LN}$ est sans subventions croisées si :*

1. $\sum_{a \in N} p_a \cdot d_a = C(\sum_{a \in N} d_a)$
2. $d_a = D_a(p)$ *pour tout* $a \in N$.
3. $\sum_{a \in A} p_a \cdot d_a^B \leq C(\sum_{a \in A} d_a^B)$ *pour tout* $AB \in \mathcal{NL}$.

Puis, la notion plus forte de soutenabilité :

Définition 1.3.2 *Un vecteur prix $p \in \mathbb{R}_+^{LN}$ est soutenable si :*

1. $\sum_{a \in N} p_a \cdot d_a = C(\sum_{a \in N} d_a)$.
2. $d_a = D_a(p)$ *pour tout* $a \in N$.
3. *Il n'existe pas* $(AB, p') \in \mathcal{NL} \times \mathbb{R}_+^{LN}$ *tels que :*

$$\begin{cases} \sum_{a \in A} p'_a \cdot {d'}_a^B = C(\sum_{a \in A} {d'}_a^B) \\ d'_a = D_a(p') \ \text{ pour tout } a \in N. \\ p'_a <_B p_a \ \text{ pour tout } a \in A. \end{cases}$$

Résultats. Ces résultats exhibent des conditions suffisantes d'existence de tarification en ramenant le problème à une condition sur les cœurs des jeux de coût. Ils nécessitent une propriété particulière de la fonction de coût qui doit vérifier qu'un partage équitable est possible pour toute structure de demande fixe inélastique par rapport aux prix :

Définition 1.3.3 *La fonction de coût C vérifie la propriété de partage équitable si pour tout $d_a \in \mathbb{R}_+^L$, $a \in N$, il existe $q \in \mathbb{R}_+^{LN}$ tel que :*

$$\begin{cases} \sum_{ab \in NL} q_{ab} = C(\sum_{a \in N} d_a) \\ \sum_{ab \in AB} q_{ab} \leq C(\sum_{a \in A} d_a^B) \ \text{ pour tout } AB \in \mathcal{NL}. \end{cases}$$

On suposera par ailleurs que les conditions de bord suivantes sont vérifiées pour les fonctions de demandes.

(D) Les fonctions de demande D_a sont bornées supérieurement, et il existe un seuil de consommation $\epsilon > 0$, i.e. $D_a \geq \epsilon \mathbf{1}^L$ pour tout $a \in N$.

On obtient le premier résultat de la section.

Théorème 1.3.1 *Sous (D), tout marché contestable avec technologie de partage équitable admet une tarification sans subventions croisées.*

Nous réalisons un lien avec la soutenabilité en introduisant une hypothèse de marché régulier. Soit Π_{AB} la fonction de profit associée à la coalition $AB \in \mathcal{NL}$:

$$\Pi_{AB} : \ p \longrightarrow \sum_{a \in A} p_a \cdot D_a(p)^B \ - \ C(\sum_{a \in A} D_a(p)^B)$$

Soit $B \in \mathbb{R}_+$, la borne uniforme supérieure sur $C(D_a(p)_b$ pour tout $p \in \mathbb{R}_+^{LN}$ et $a, b \in NL$. Le marché est régulier si :

(R) Soit $\mathcal{K}$ le pavé multidimensionnel $\prod_{ab \in NL}[0, \frac{B}{\epsilon}]$, pour tout sous-marché AB : $\mathcal{K} \cap \{\Pi_{AB} > 0\} = \mathcal{K} \cap (\{\Pi_{AB} = 0\} + \mathbb{R}_{++}^{LN})$.

L'hypothèse précédente stipule seulement que le profit reste positif ou nul au delà des niveaux de profit zéro.

Théorème 1.3.2 *Sous (D) et (R), tout marché contestable avec technologie à partage équitable admet une tarification soutenable.*

Remarque 1.3.1 *On énonce également une série de résultats parallèles pour le cas de marché où s'engagent à fournir soit toute la ligne commerciale, soit l'intégralité des agents. Ces deux cas de figure correspondent respectivement au cas où $\mathcal{L} = \{L\}$ ou $\mathcal{N} = \{N\}$;*

Approche par les cœurs-équilibre de jeux de coût paramétrés. On utilise un corollaire du Théorème 1.1.2. On peut en effet établir un résultat de non vacuité pour le cœur-équilibre dans un jeu TU (voir section 1.2).

Corollaire 1.3.1 *Soit $((v_S)_{S \in \mathcal{X}}, \Theta)$ un jeu TU paramétré satisfaisant les Hypothèses (PH0) et (PH1) et tel que pour tout $\theta \in \Theta$, le jeu TU $(v_S(\theta), S \in \mathcal{X})$ est balancé, alors il existe un allocation cœur équilibre.*

Une première approche mathématique consiste à paramétrer par les prix des cost games afin de montrer l'existence d'une tarification spécifique. En effet, en considérant le cœur de ces jeux associés à chaque prix comme l'image d'une correspondance, on peut obtenir le résultat par un argument de point fixe (voir (36) par exemple). Ici on utilise la version TU du Théorème 1.1.2, i.e. le Corollaire 1.3.1. Plus précisément on montre que les tarifications sans subventions croisées correspondent exactement aux allocations de cœur- équilibre d'un jeux de coût paramétré par les prix.

Littérature associée. Faulhaber (22) propose déjà des résultats d'existence pour des tarifications subsidy free en reprenant le formalisme des jeux coopératifs, notamment en introduisant la notion de cost games. La tarification subsidy free est habituellement vue comme un point fixe d'une famille de cœurs paramétrés. Ce type d'outil mathématique a déjà été utilisé dans la littérature des marchés contestables, Sharkey (69; 70) et Ten Raa (56; 57). L'originalité tient ici à expliciter complètement les tarifications comme des cœurs-équilibres. Par ailleurs, le modèle de cœur généralisé permet également de traiter des modèles de type welfare games et benefit games à la Sharkey (72).

1.4 *Soutenabilité versus projet de financement*

Chapitre 4, référence (38) dans la bibliographie.

Le cadre. On considère un marché contestable tel qu'il est défini en introduction de la partie B. On se limite à des firmes monoproduits : $|L| = 1$.

Motivations. Etant donnée la tarification de la firme en place, les firmes concurrentes peuvent entrer sur le marché. A la différence du modèle usuel de contestabilité, les entrants ont la possibilité de financer la production du bien par un système plus élaboré de taxes et subventions. Ainsi, au lieu de payer exactement le montant de leurs demandes individuelles, un système d'évaluation définit les contributions des agents dans la production du bien. On retrouve ainsi l'idée de système d'évaluation tel qu'il est défini dans Mas-Colell (45). Dans un premier cas, appelé Λ-soutenabilité, les agents peuvent trouver un accord pour tout système de partage du coût, sur une base volontaire. Dans un second temps, la v-soutenabilité suppose que ce partage est organisé par un centre de coordination qui fixe une règle de répartition a priori que les agents s'engagent à suivre. On adopte les terminologies de projet de financement volontaire dans le premier cas et de règle d'évaluation pour le deuxième cas. On supposera dans la suite que l'hypothèse suivante est satisfaite.

(DD) Les fonctions de demandes D_a sont décroissantes strictement et bornées en zéro et indépendantes, i.e. $D(p) = D(p_a, p{-}a'$ pour tout $p, p' \in \mathbb{R}_+^N$ (dans lasuite on utilisera $D(p_a)$ pour désigner la demande induite par les prix). .

Λ-sustainability. Par commodité, nous posons $C : \mathcal{N} \times \mathbb{R}_+^N \to \mathbb{R}_+$, la fonction de coût. Soit $A \in \mathcal{N}$ et $p \in \mathbb{R}_+^N : C(A,p) = c(\sum_{a \in A} D_a(p_a))$. Un projet de financement volontaire est une paire (p, v) formée d'un prix $p \in \mathbb{R}_+^N$ et d'un vecteur de partage $v \in \Lambda$, où $\Lambda := \left\{x \in \mathbb{R}_+^N \mid \sum_{a \in \text{supp}(x)} x_a = |\text{supp}(x)|\right\}$. Dans la suite, les agents s'accordent sur la base du volontariat pour un projet de financement déterminé par le vecteur de partage v : une unité du bien est payée par l'agent $a \in \text{supp}(v)$ au niveau $v_a p_a$.

Soit Π, $\mathbb{R}_+^N \times \Lambda \to \mathbb{R}$, la fonction de profit généralisé, pour tout $p \in \mathbb{R}_+^N$ et $v \in \Lambda$:

$$\Pi(p, v) = \sum_{a \in \text{supp}(v)} v_a p_a D_a(p_a) - C(\text{supp}(v), p)$$

Définition 1.4.1 (Λ-soutenabilité) *Un prix p est dit Λ-soutenable si :*

1. $\Pi(p, \mathbf{1}) = 0$.
2. *Il n'existe pas de projet de financement (p', v) tel que $\Pi(p', v) \geq 0$ pour tous les agents $a \in$ supp(v), $D_a(p'_a) > D_a(p_a)$.*

Les hypothèses sur le marché contestable sont les suivantes.

Définition 1.4.2 (Marché à Λ-partage équitable) *Le marché est dit à Λ-partage équitable si, pour tout $p \in \mathbb{R}_+^N$, il existe $q \in \mathbb{R}_+^N$ satisfaisant :*

$$\begin{cases} q \cdot \mathbf{1} = C(N,p) \\ q \cdot v \leq C(supp(v),p), \ \ \forall v \in \Lambda \end{cases} \tag{1.1}$$

(A1) Pour tout $v \in \Lambda$, l'ensemble $\{\Pi(.,v) \geq 0\}$ est compréhensif pr le haut.

(A2) Pour tout $a \in N$, $\{\Pi(.,\mathbf{1}^{\{a\}}) \geq 0\} \neq \emptyset$.

$(A1)$ est une hypothèse de régularité, par exemple, elle est vérifiée si la fonction profit est croissante par rapport aux prix. Notons cependant que $(A1)$ est plus faible qu'une propriété de croissance. $(A2)$ établit que chaque marché local peut dégager un profit positif. On ne peut pas se dispenser de ces deux hypothèses qui sont nécessaires pour définir le jeu NTU associé que nous utilisons pour montrer l'existence de tarifications soutenables.

On peut alors établir un résultat d'existence pour notre définition de soutenabilité

Théorème 1.4.1 *Sous $(A1),(A2)$, tout marché à Λ-partage équitable admet un prix Λ-soutenable.*

v-sustainability. Une règle d'évaluation v est désormais donnée a priori. Les agents s'engagent à suivre cette règle pour payer le bien produit. Considérons pour chaque $A \in \mathcal{N}$, l'ensemble $\Lambda_A = \{x \in \Lambda \mid \text{supp}(x) = A\}$. Une règle d'évaluation est une famille de fonctions continues $v_A : \mathbb{R}_+^N \to \Lambda_A$, pour chaque $A \in \mathcal{N}$.

Nous utilisons la propriété suivante sur le marché.

Définition 1.4.3 (Marché à v-partage équitable) *Le marché est dit à v-partage équitable si, pour tout $p \in \mathbb{R}_+^N$, il existe $q \in \mathbb{R}_+^N$ satifaisant :*

$$\begin{cases} q \cdot \mathbf{1} = C(N,p) \\ q \cdot v_A(p) \leq C(A,p), \ \ \forall A \in \mathcal{N} \end{cases} \tag{1.2}$$

Notons que, lorsque la règle d'évaluation est réduite aux fonctions constantes $(\mathbf{1}^A)_{A \in \mathcal{N}}$, (1.2) est exactement la propriété dite de partage équitable que l'on trouve dans la section 1.3 pour le concept de soutenabilité usuel.

Définition 1.4.4 (v-soutenabilité) *Un prix p est v-soutenable si :*

1. $\Pi(p,\mathbf{1}) = 0$.
2. *Il n'existe pas $(p',A) \in \mathbb{R}_+^N \times \mathcal{N}$ tel que $\Pi(p',v_A(p')) \geq 0$ et $D_a(p') > D_a(p)$ pour tout $a \in A$.*

Pour ce modèle, l'hypothèse $(A1)$ se réécrit en :

(A1') Pour tout $A \in \mathcal{N}$, the set $\{\Pi(., v_A(.)) \geq 0\}$ est compréhensif par le haut.

Le résultat d'existence peut alors être énoncé pour ce modèle.

Théorème 1.4.2 *Sous $(A1'), (A2)$, tout marché à v-partage équitable admet un prix v-soutenable.*

Les définitions précédentes étendent la définition originale de soutenabilité au cas de fonction de profit généralisé avec projet de financement. Dans la définition usuelle, la condition 2 est remplacée par :

2'. Il n'existe pas $(p', A) \in \mathbb{R}_+^N \times \mathcal{N}$ tel que : $\Pi(p', \mathbf{1}^A) \geq 0$ et $D_a(p')_a > D_a(p_a)$ pour tout $a \in A$.

La proposition suivante décrit les relations entre les deux concepts de soutenabilité les hypothèses de marché à partage équitable.

Proposition 1.4.1 *1. p est un prix Λ-soutenable si et seulement si p is v-soutenable pour toute règle d'évaluation.*

2. $q \in \mathbb{R}_+^N$ satisfait (1.1) si et seulement si q satisfait (1.2) pour toute règle d'évaluation.

3. La propriété de marché à Λ-partage équitable est équivalente à : pour tout $p \in \mathbb{R}_+^N$, $\frac{C(A,p)}{|A|} \geq \frac{C(N,p)}{|N|}$ pour tout $A \in \mathcal{N}$..

La dernière équivalence de la proposition précédente montre que la condition de partage équitable associée au concept de Λ-soutenabilité correspond exactement à une condition de rendements croissants par rapport aux coalitions (voir aussi Demange (20)). Cette hypothèse de rendements croissants est également utilisée en section 1.3 afin de montrer que la fonction de coût vérifie une propriété de partage équitable. Ici, l'étude de la Λ-soutenabilité permet d'illustrer le cas limite pour lequel cette condition sur les coûts permet d'obtenir la stabilité du monopole en place.

Approche par les sélections du cœur. Les deux concepts de soutenabilité énoncés ici sont modélisés à l'aide de jeux NTU. On peut montrer que les tarifications soutenables correspondent alors à des sélections particulières du cœur. On peut alors utiliser le résultat d'existence de sélection du cœur dans les jeux NTU sous l'hypothèse de balancement dépendant (Théorème 1.1.1). La condition de balancement est vérifiée dans les deux cas sous les hypothèses de marché à partage équitable.

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2. BALANCEMENT DÉPENDANT ET CœURS

Résumé

Nous proposons un résultat de non-vacuité du cœur dans un jeu coopératif sans paiements latéraux. On utilise une condition de balancement dépendant des paiements à partir de règles de taux de transfert qui généralise les versions précédentes de balancement avec poids constant. Des extensions du concept de cœur sont proposées incluant des cœurs satisfaisant une condition supplémentaire et des cœurs avec équilibre dans le cadre de jeux coopératifs paramétrés. Les preuves d'existence empruntent des outils mathématiques à la théorie de l'équilibre général avec non-convexités. Des applications variées à des résultats de théorie des jeux et d'économie théorique sont données.

Payoff-dependent balancedness and cores [0]

Jean-Marc Bonnisseau and Vincent Iehlé

Abstract. We prove the non-emptiness of the core in NTU games, using a payoff-dependent balancedness condition, based on transfer rate mappings. Going beyond the non-emptiness of the standard core, we prove the existence of core allocations with transfer rate rule equilibrium and equilibrium-core allocations in a parameterized cooperative game. The proofs borrow mathematical tools and geometric constructions from general equilibrium theory with non convexities. Applications to various extant results taken from game theory and economic theory are given, like the partnered core, the social coalitional equilibrium and the core for economies with non-ordered preferences. *Journal of Economic Literature* Classification Numbers : C60, C62, C71, D50, D51.

Key words: cooperative games, balancedness, non-emptiness, core concepts, parameterized games.

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2.1 Introduction

The core of an n-person cooperative game is the set of feasible outcomes that cannot be improved by any coalition of players. The stability of a model where social interactions are at stake is guaranteed when the core is non-empty, since an allocation in the core is coalition-proof. Bondareva-Shapley's result states that the core of a TU game is non-empty if and only if the game is balanced and Scarf's theorem states that the core of a NTU balanced game is non-empty.

Several improvements have been made since Scarf's result, but the balancedness conditions in the literature always rely on the same principle. Let us first consider a TU game. In such a game, within a coalition, transfers of utility can be made from one player to another at a constant one-to-one rate. Then, a transfer rate vector is naturally associated to each coalition. The vector is defined as the vector whose coordinates are 1 for the members of the coalition and 0 for the other players. Given these transfer rate vectors, a family of coalitions is balanced if the transfer rate vector of the grand coalition is positively generated by the transfer rate vectors of the coalitions in the family. Then the TU game is balanced if any feasible payoff for a balanced family is feasible for the grand coalition.

For NTU games, the transfer rate vector can be defined but it does not always correspond to a feasible transfer among the members of the coalition. Nevertheless, the balancedness condition remains sufficient for the non-emptiness of the core. Billera (1) generalizes Scarf's result by considering transfer rate vectors with any positive coordinates. In this paper, we extend the notion of balancedness. For each coalition, we replace the transfer rate vector by a transfer rate rule, which associates transfer rate vectors to each efficient payoff of the coalition. Consequently, a family of coalitions is no more balanced intrinsically but relatively to a payoff. We deduce from this definition of balancedness the notion of a payoff-dependent balanced game. Our first result, Theorem 2.2.1, states that any payoff-dependent balanced game has a non-empty core.

The games, which are balanced in the extant literature (including the standard balancedness (Scarf (27)); b-balancedness (Billera (1)); balancedness for convex games (Billera (1)); $(b, <)$-balancedness (Keiding and Thorlund Petersen(20))), are all payoff-dependent balanced. Hence, Theorem 2.2.1 generalizes the previous non-emptiness results.

Independently of our work, Predtetchinski and Herings (24) define the class of Π-balanced games, which is almost identical to the class of payoff-dependent balanced games. They prove that the condition is not only sufficient for non-emptiness of the core in NTU games but also necessary. Thus, they characterize core non-emptiness in NTU games, in the likeness of Bondareva-Shapley's result for TU games.

In this paper, using the flexibility of the payoff-dependent balancedness, Theorem 2.2.1 goes beyond the non-emptiness of the core. Indeed, Theorem 2.2.1 shows the existence of a core allocation x satisfying an additional equilibrium condition. This equilibrium condition involves the transfer rate vectors of the coalitions for which x is feasible. Then, a number of results in the literature involving core allocations with additional requirements can be deduced from Theorem 2.2.1.

For example, we deduce Reny and Wooders (25) key lemma, which exhibits a core allocation satisfying an equilibrium condition for credit/debit mappings. To do this, we consider a transfer rate rule that takes into account the individual contributions of the agents within the different coalitions. Consequently, the non-emptiness of the partnered core of Reny and Wooders (25), and, the existence of the average prekernel intersected with the core (Orshan and al. (21)) can be deduced from Theorem 2.2.1.

Lastly, we consider parameterized cooperative games. This is very much in the spirit of Ichiishi (13) even if he does not explicitly use this abstract framework. The stake is the following: the payoffs sets, taken as set-valued mappings, depend on parameters, which stand for an abstract environment; furthermore, an equilibrium condition on the parameters is represented by a set-valued mapping depending on the parameters and the payoffs. We define an equilibrium-core allocation, which is a payoff-environment pair, the allocation belongs to the core of the game associated to the environment, and, the environment is a fixed point of the equilibrium set-valued mapping. We prove the existence of equilibrium-core allocations, in Theorem 2.3.1, under our payoff-dependent balancedness condition. The existence of a Social Coalitional Equilibrium as stated in the benchmark work of Ichiishi (13) is a consequence of our result.[1] Let us recall that Social Equilibrium of Debreu (8) is a particular Social Coalitional Equilibrium. In economies without ordered preferences, or in economies with increasing returns, Border (7) or Ichiishi and Quinzii (17) did already use the parametric framework as intermediate steps to show the non-emptiness of the core. We show how these results can be deduced from Theorem 2.3.1.

The geometric intuition behind our proof is borrowed from the existence of a general pricing rule equilibrium in an economy with a non-convex production sectors, see Bonnisseau and Cornet (4; 5) and (3). We show in the proof of Theorem 2.2.1 that a core allocation may actually be considered as an equilibrium of a two production set economy. Moreover, the price equilibrium condition given by the pricing rules may be restated as a transfer rate rule equilibrium condition satisfied by the core allocation. Finally, the analogy between core and equilibrium allocations puts a light on the close relationship between the two key assumptions into both theories, namely balancedness, at stake in cooperative games, and the survival assumption in general equilibrium. Note that Shapley and Vohra (32) did already quote similarities between the fixed point mappings they use to show the non-emptiness of the core, and, the fixed point mappings at stake in general economic equilibrium.[2] However these authors did not investigate further in this direction.

Since our intuition comes from the general economic equilibrium, the main results are naturally

[1] Note that we limit ourself to a space of environment in a finite dimensional Euclidean space, whereas Ichiishi considers a locally convex Hausdorff topological vector space.

[2] See Vohra (33) to convince oneself of this fact for Shapley–Vohra mappings.

obtained through Kakutani's fixed-point theorem. Note that one usually associates the question of non-emptiness of cores with KKMS covering theorems or Fan's coincidence theorems, but binding the concept of core with Kakutani's theorem makes sense due to its intimate link with Walrasian economies.[3]

The paper is organized as follows: in Section 2.2, Theorem 2.2.1 states the existence of core allocations with transfer rate rule equilibrium in payoff-dependent balanced games. Then, we show how one can deduce number of results involving balancedness. Section 2.3 is devoted to the model with parameters and its related topics. Under our condition of payoff-dependent balancedness, we state a result for the existence of equilibrium-core allocation in a parameterized game, Theorem 2.3.1, covering Theorem 2.2.1. Quoted examples of applications will follow. In the body of the paper, the proofs consist mainly of geometric constructions, which are of constant use, the proofs of Theorems and technical lemmas are given in Appendix. Except for some notations and basic assumptions given below, Sections 2.2 and 2.3 can be taken independently. We discuss the related literature and possible directions for future works in Section 2.4.

2.2 *Core solutions in NTU Games*

Notations.[4]

N is the finite set of players;

$\mathcal{N}$ is the set of the non-empty subsets of N, i.e. the coalitions of players;

For each $S \in \mathcal{N}$, L_S is the $|S|$-dimensional subspace of $\mathbb{R}^N$ defined by

$$L_S = \{x \in \mathbb{R}^N \mid x_i = 0, \ \forall i \notin S\}$$

L_{S+} (L_{S++}) is the non negative orthant (positive orthant) of L_S;

For each $x \in \mathbb{R}^N$, x^S is the projection of x into L_S;

$\mathbf{1}$ is the vector of $\mathbb{R}^N$ whose coordinates are equal to 1;

$\mathbf{1}^\perp$ is the hyperplane $\{s \in \mathbb{R}^N \mid \sum_{i \in N} s_i = 0\}$;

proj is the orthogonal projection mapping on $\mathbf{1}^\perp$;

$\Sigma_S = \text{co}\{\mathbf{1}^{\{i\}} \mid i \in S\}$;

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[3] See the discussion about these links in Ichiishi (14, p.118-125).

[4] For any set $Y \subset \mathbb{R}^N$, co(Y), ∂Y, int Y and cone$Y$, will denote respectively its convex hull, boundary, interior and the conic hull, which is the smallest convex cone containing Y. For any set-valued mapping Γ, Gr Γ will denote its graph.

$m^S = \frac{\mathbf{1}^S}{|S|}$;

$\Sigma = \Sigma_N$ and $\Sigma_{++} = \Sigma \cap \mathbb{R}^N_{++}$.

Game description. A game $(V_S, S \in \mathcal{N})$ is a collection of subsets of $\mathbb{R}^N$ indexed by $\mathcal{N}$.

$x \in \mathbb{R}^N$ is called a payoff;

$V_S \subset \mathbb{R}^N$ is the set of feasible payoffs of the coalition S;

$\mathcal{S}(x) = \{S \in \mathcal{N} \mid x \in \partial V_S\}$ is the set of coalitions, for which $x \in \mathbb{R}^N$ is an efficient payoff;

$W := \cup_{S \in \mathcal{N}} V_S$ is the union of the payoffs sets.

Definition 2.2.1 Let $(V_S, S \in \mathcal{N})$ be a game. A payoff x is in the core of the game if $x \in V_N \setminus \operatorname{int} W$.

It is worth noting that, in this formulation, the core only involves two sets: V_N and W. A feasible payoff for the grand coalition, which is obviously in W, belongs to the core if this payoff lies on the boundary of the set W. This formulation is crucial in the remainder of the paper, leading most of our geometric constructions. Equivalently, note also that x belongs to the core if and only if $x \in \partial W$ and $N \in \mathcal{S}(x)$.

We now posit two basic assumptions on the game. Roughly speaking, Assumption H1 states that the payoffs sets of any coalition, ranked as cylinders of $\mathbb{R}^N$ for convenience, satisfy the free disposal, and Assumption H2 states the boundedness of the individually rational payoffs of any coalition.[5]

(H1) (i) $V_{\{i\}}$, $i \in N$, and V_N are non-empty.

(ii) For each $S \in \mathcal{N}$, V_S is closed, $V_S - \mathbb{R}^N_+ = V_S$, $V_S \neq \mathbb{R}^N$, and, for all $(x, x') \in (\mathbb{R}^N)^2$, if $x \in V_S$ and $x^S = x'^S$, then $x' \in V_S$.

(H2) There exists $m \in \mathbb{R}$ such that, for each $S \in \mathcal{N}$, for each $x \in V_S$, if $x \notin \operatorname{int} V_{\{i\}}$ for all $i \in S$, then $x_j \leq m$ for all $j \in S$.

2.2.1 The main theorem

Before stating the main result of this section, we note that, under Assumption H1, V_N and W satisfy the assumptions of Bonnisseau and Cornet (5, Lemma 5.1. p.139). Therefore, there exist

[5] Assumption H1 implies that there exists, for each $i \in N$, $v_i \in \mathbb{R}$ such that $V_{\{i\}} = \{x \in \mathbb{R}^N \mid x_i \leq v_i\}$. The non-emptiness of V_S is not necessary since it suffices to put $V_S := \{x \in \mathbb{R}^N \mid x_i \leq v_i, \forall i \in S\}$ for the empty payoffs sets, then Assumptions H1 and H2 are satisfied, all payoffs sets are non-empty, and, the core is unchanged. We do not normalize the game as usually done. For instance, in Shapley and Vohra (32), the game is normalized without loss of generality by imposing $v_i > 0$ for each $i \in N$.

continuous mappings p_N from $\mathbb{R}^N$ to ∂V_N, p_W from $\mathbb{R}^N$ to ∂W, λ_N and λ_W from $\mathbf{1}^\perp$ to $\mathbb{R}$ such that, for all $x \in \mathbb{R}^N$, $p_N(x) = \text{proj}(x) - \lambda_N(\text{proj}(x))\mathbf{1}$ and $p_W(x) = \text{proj}(x) - \lambda_W(\text{proj}(x))\mathbf{1}$.

We are now in position to define the notions of transfer rate rule and of payoff-dependent balanced game.

Definition 2.2.2 Let $(V_S, S \in \mathcal{N})$ be a game satisfying Assumption H1.

(i) A transfer rate rule is a collection of set-valued mappings $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ such that for all $S \in \mathcal{N}$, φ_S is upper semi-continuous with non-empty compact and convex values from ∂V_S to Σ_S, and, ψ is upper semi-continuous with non-empty compact and convex values from ∂V_N to Σ.

(ii) The game $(V_S, S \in \mathcal{N})$ is payoff-dependent balanced if there exists a transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ such that, for each $x \in \partial W$,

$$\text{if } \text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset, \text{ then } x \in V_N.$$

The usual transfer rate rule is constant and single-valued. To each element x on ∂V_S, it associates the vector m^S. Our innovation comes from the fact that we allow the transfer rates to depend on the payoff x.

To give some intuition on the balancedness condition, let us first consider the case where $\psi = \varphi_N$. For any given efficient payoff $x \in \partial W$, for all coalition $S \in \mathcal{S}(x)$, $\varphi_S(x)$ defines a set of admissible transfer rates between agents within the coalition. Let $b_S \in \varphi_S(x)$ for each $S \in \mathcal{S}(x)$ and $b_N \in \varphi_N(p_N(x))$. Let us consider the hyperplane game $(\widetilde{V}_S, S \in \mathcal{N})$ where $\widetilde{V}_S = \{\xi \in \mathbb{R}^N \mid b_S \cdot \xi \leq b_S \cdot x\}$ and $\widetilde{V}_N = \{\xi \in \mathbb{R}^N \mid b_N \cdot \xi \leq b_S \cdot p_N(x)\}$. Thus, the game is payoff-dependent balanced at x if $x \in \widetilde{V}_N$ whenever the family $\mathcal{S}(x)$ is balanced in the standard sense, that is, b_N belongs to the convex hull of the family $(b_S)_{S\in\mathcal{S}(x)}$.

The class of payoff-dependent balanced games includes all traditional balanced games ((27; 1; 20)) and convex games à la Billera (1) (see Sub-section 2.2.3). The introduction of an additional set-valued mapping ψ allows us to get some refinements to the standard core solution (see Sub-section 2.2.6). Indeed, we prove the existence of a core allocation x, such that the family of coalitions, for which x is feasible, is balanced with respect to the transfer rate vector given by the transfer rate rule. This condition is relevant whenever ψ differs from φ_N, since, otherwise, it is obviously satisfied for $N \in \mathcal{S}(x)$.

We also point out that the balancedness condition holds only on ∂W, the (*weakly*) efficient frontier of the game.

The following theorem is the first result of the paper. Its proof is referred to Appendix.

Theorem 2.2.1 *Let $(V_S, S \in \mathcal{N})$ be a game satisfying Assumptions H1 and H2. If it is payoff-dependent balanced with respect to the transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$, there exists a core allocation* *x such that:*

$$\mathrm{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(x) \neq \emptyset.$$

Remark 2.2.1 Independently, Predtetchinski and Herings (24) prove a similar result on the non-emptiness of the core. Actually, their boundedness assumption is weaker but the payoff-dependent balancedness is slightly stronger. A very interesting contribution of their paper is to show that the payoff-dependent balancedness is a necessary condition. Nevertheless, they do not get the second part of the conclusion of Theorem 2.2.1, that is the fact that the family $\mathcal{S}(x)$ is balanced with respect to the transfer rate vectors given by the transfer rate rule.

2.2.2 About the proof

We briefly comment the main outline of the proof to emphasize the geometric intuition that leads our reasonings. We will use an abstract result of Bonnisseau and Cornet (5, Theorem 1 p.67), to obtain Theorem 2.2.1 as a corollary of an existence result of equilibrium in an economy with non-convex production sets. Actually, we only need a weak version with only two production sets.

Before recalling the general equilibrium result, we first posit some notations. Let C be a closed, convex cone included in $\mathbb{R}^N_{++} \cup \{0\}$ such that $\mathbf{1}$ is an element of its interior. Let $\Delta = \{x \in -C^\circ \mid x \cdot \mathbf{1} = 1\}$, where C° is the negative polar cone of C. Note that $\Sigma \subset \mathrm{int}\,\Delta$.

Theorem 2.2.2 (Bonnisseau–Cornet (1991)) *Let Y_1 and Y_2 two subsets of $\mathbb{R}^N$. For each $j = 1; 2$, let $\tilde{\varphi}_j$ be a set-valued mapping from ∂Y_j to Δ. We assume the following assertions:*

- *(P)* For $j = 1; 2$, Y_j is closed, non-empty and $Y_j - C = Y_j$; $\tilde{\varphi}_j$ is upper semi-continuous with non-empty convex values; there exists $\alpha_j \in \mathbb{R}$ such that for all $y_j \in \partial Y_j$, for all $p \in \tilde{\varphi}_j(y_j)$, $p \cdot y_j \geq \alpha_j$.
- *(B)* For each $t \geq 0$, $A_t = \{(y_1, y_2) \in Y_1 \times Y_2 \mid y_1 + y_2 + t\mathbf{1} \in C\}$ is bounded.
- *(S)* For each $t > 0$, for each $(p, y_1, y_2) \in \Delta \times \partial Y_1 \times \partial Y_2$, if $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$ and $y_1 + y_2 + t\mathbf{1} \in C$, then $p \cdot (y_1 + y_2 + t\mathbf{1}) > 0$.

Then there exists $(y_1, y_2, p) \in \partial Y_1 \times \partial Y_2 \times \Delta$ such that $y_1 + y_2 \in C$ and $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$.

In the original statement of Assumption S, one has $t \geq 0$. In the case of pure production economies, one needs only to consider a positive number t. Indeed, when Assumption S does not hold for $t = 0$, then the conclusion of Theorem 2.2.2 obviously holds true. To do the link precisely with Bonnisseau and Cornet (5, Theorem 1 p.67), the reader must consider $C = X = \mathbb{R}^N_+$. It is an easy matter to check that the proof works with a general convex cone C.[6] The use of the cone C is

[6] See Bonnisseau and Jamin (6).

necessary to show that Assumption S holds true, which is not true for the cone $\mathbb{R}^N_+$. See Appendix, proof of Lemma 2.5.4.

The proof of Theorem 2.2.1 relies on the construction of a fictitious economy with two (*non convex*) production sets built from the payoff sets of the coalitions and pricing rules derived from the transfer rate rule. Then, one gets the existence of a core allocation from an equilibrium, which exists thanks to Theorem 2.2.2. The last conclusion of Theorem 2.2.1, $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(x) \neq \emptyset$, comes from the equilibrium condition $p \in \tilde{\varphi}(y_1) \cap \tilde{\varphi}(y_2)$.

In the body of the proof of Lemma 2.5.4 (Claim 3), we show that the fictitious economy satisfies Assumption S thanks to the payoff-dependent balancedness of the game. This is the unique line of arguments where we need it. Surprisingly (or not?), the argument binds intimately the most questionable assumptions of general equilibrium and cooperative games theories, respectively the survival and the balancedness.

2.2.3 About the balancedness condition

In this section, we focus on the particular case of Theorem 2.2.1, where $\psi = \varphi_N$. We show how it generalizes the existing results on the non-emptiness of the core in the literature. The proofs of the corollaries consist only in defining the "right" transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$. We also provide an example of a game that is not balanced for the previous definitions. Nevertheless, its core is non-empty and we exhibit a transfer rate rule such that the game is payoff-dependent balanced with respect to these rules.

There are mainly three versions of balancedness in NTU games literature, Scarf (27), Billera (1) and Keiding and Thorlund-Petersen (20). In (27), a family of coalition $\mathcal{B} \subset \mathcal{N}$ is balanced if for each $S \in \mathcal{B}$, there exists $\lambda_S \in R_+$ such that $\sum_{S\in\mathcal{B}} \lambda_S \mathbf{1}^S = \mathbf{1}$. The game is balanced if for any balanced family of coalition $\mathcal{B} \subset \mathcal{N}$, $\cap_{S\in\mathcal{B}} V_S \subset V_N$.

In Billera (1), a transfer rate vector $b_S \in L_{S+} \setminus \{0\}$ is associated to each coalition S. A family of coalition $\mathcal{B} \subset \mathcal{N}$ is b-balanced if for each $S \in \mathcal{B}$, there exists $\lambda_S \in \mathbb{R}_+$ such that $\sum_{S\in\mathcal{B}} \lambda_S b_S = b_N$. The game is b-balanced if for any b-balanced family of coalition $\mathcal{B} \subset \mathcal{N}$, $\cap_{S\in\mathcal{B}} V_S \subset V_N$.

Lastly, one is led to the most refined definition of $(\partial - b)$-balanced games. The game is said to be $(\partial - b)$-balanced if for any b-balanced family of coalitions $\mathcal{B} \subset \mathcal{N}$, $\partial W \bigcap (\cap_{S\in\mathcal{B}} V_S) \subset V_N$.

Corollary 2.2.1 *For each $S \in \mathcal{N}$, let $b_S \in L_{S+} \setminus \{0\}$. The core of the game is non-empty if it is $(\partial - b)$-balanced and satisfies Assumptions H1 and H2.*

Proof of Corollary 2.2.1. We show that the game is payoff-dependent balanced. For each $S \in \mathcal{N}$, let φ_S be the constant mapping which associates $\frac{1}{\sum_{i\in S} b_{Si}} b_S$ to each $x \in \partial V_S$ and let $\psi = \varphi_N$. Let $x \in \partial W$ such that $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \varphi_N(x) \neq \emptyset$. This means that there exist $\lambda_S \in \mathbb{R}_+$, $S \in \mathcal{S}(x)$, which satisfy $\sum_{S\in\mathcal{S}(x)} \lambda_S \frac{1}{\sum_{i\in S} b_{Si}} b_S = \frac{1}{\sum_{i\in N} b_{Ni}} b^N$. Consequently, $\sum_{S\in\mathcal{S}(x)} \lambda_S \frac{\sum_{i\in N} b_{Ni}}{\sum_{i\in S} b_{Si}} b_S =$

b_N, which implies that the family $\mathcal{S}(x)$ is b-balanced. Then, the $(\partial - b)$-balancedness implies that $x \in V_N$. □

The result of Billera (1) is immediately deduced from Corollary 2.2.1. In Scarf (27), the author proves that the core of a balanced game is non-empty under Assumptions H1 and,

> *(WH2)* There exists $m \in \mathbb{R}$ such that, for each $x \in V_N$, if $x \notin \text{int } V_{\{i\}}$ for all $i \in N$, then $x \leq m\mathbf{1}$.

We remark that, under Assumptions H1 and WH2, Assumption H2 also holds true if the game is balanced, so, Scarf's result is a consequence of Corollary 2.2.1. Indeed, let $S \in \mathcal{N}$, we remark that the family $\{S, (\{i\})_{i \notin S}\}$ is a balanced family. Now let $x \in V_S$ such that $x_i \geq v_i$, $i \in S$. Let x' defined by $x'_i = x_i$, $i \in S$ and $x'_i = v_i$, $i \notin S$. From Assumption H1, $x' \in V_S \bigcap (\cap_{i \notin S} V_{\{i\}})$. From the balancedness of the game, $x' \in V_N$ and clearly, $x'_i \geq v_i$, $i \in N$. Consequently, from Assumption WH2, $x' \leq m\mathbf{1}$, which implies $x_i \leq m$, $i \in S$. Thus, Scarf's result is obtained as a corollary of Theorem 2.2.1.

We now come to the result of Keiding and Thorlund-Petersen (20). We let the reader check that $(\partial - b)$-balancedness is weaker than the $(b, <)$-balancedness introduced by the authors, so that their existence result (Theorem 2.1 p.277) is also a consequence of Corollary 2.2.1 obtained as a corollary. The authors advance the idea that some dominated payoffs should not be taken into account when examining the non-emptiness of the core, then a procedure of elimination is proposed. This intuition will be made useless with the $(\partial - b)$balancedness condition since the irrelevant payoffs are straightaway disregarded by considering the efficient payoffs on ∂W.

However, Keiding and Thorlund-Petersen (20) characterize the class of game with a non-empty core as the class of weakly $(b, <)$-balanced games. One can clarify their result by replacing the notion of $(b, <)$-balancedness by $(\partial - b)$-balancedness.

For all coalition S, let $V_{S*} = \{x \in V_S \mid x \notin \text{int } V_{\{i\}}, \forall i \in S\}$, that is, the set of individually rational feasible payoffs. A game is weakly $(\partial - b)$-balanced if there exists a sequence $\{V^\tau\}_{\tau=1}^{\infty}$ of $(\partial - b)$-balanced games such that: $V_N = V_N^\tau$ for all τ, and, for all $S \in \mathcal{N}$ the sequence $\{V_{S*}^\tau\}_{\tau=1}^{\infty}$ converges to the set V_{S*} for the Hausdorff topology on the non-empty compact sets of $\mathbb{R}^N$. Using the arguments of Keiding and Thorlund-Petersen(20, Proof of Theorem 5.1. p.286), one shows the following result:

Corollary 2.2.2 *Let $(V_S, S \in \mathcal{N})$ be a game satisfying Assumptions H1 and H2. It has a non-empty core if and only if there exists a weakly $(\partial - b)$-balanced game (V', N) such that $V_N = V'_N$ and $V_S \subset V'_S$, $S \in \mathcal{N}$.*

2.2.4 *An example*

We end the discussion on balancedness by considering an example. The following 3-player game with a non-empty core is not $(\partial - b)$-balanced. Let $N = \{1, 2, 3\}$, and define:

$V_{\{i\}} = \{x \in \mathbb{R}^3 \mid x_i \leq 1\}$ for all $i = 1, 2, 3$;

$V_{\{ij\}i \neq j} = (\{x \in \mathbb{R}^3 \mid x_i \leq 1\} \cup \{x \in \mathbb{R}^3 \mid x_j \leq 1\}) \cap \{x \in \mathbb{R}^3 \mid x_i \leq 2; x_j \leq 2\}$;

$V_{\{123\}} = \{x \in \mathbb{R}^3 \mid \sum_{i=1}^{3} x_I \leq 3\}$.

The game satisfies Assumptions H1 and H2 (consider $m = 2$), and, the core is non-empty and reduced to the element $(1, 1, 1)$,

Proposition 2.2.1 *The game is not $(\partial - b)$-balanced.*

Proof of Proposition 2.2.1. Consider the two points $(1, 2, 1)$ and $(1, 1, 2)$. They lie outside V_N, and, they belong to ∂W. One checks that

$$\mathcal{S}((1, 2, 1)) = \{\{1\}, \{3\}, \{12\}, \{13\}, \{23\}\},$$

$$\mathcal{S}((1, 1, 2)) = \{\{1\}, \{2\}, \{12\}, \{13\}, \{23\}\}.$$

Now, remark that, one of the families: $C_1 = \{\{1\}, \{3\}, \{23\}\} \subset \mathcal{S}((1, 2, 1))$ and $C_2 = \{\{1\}, \{2\}, \{23\}\} \subset \mathcal{S}((1, 1, 2))$ must be b-balanced. Indeed, let $(b_S, S \in \mathcal{N})$ a family of transfer rate vectors. Since $b_{\{2,3\}} \in L_{\{2,3\}} \setminus \{0\}$, one easily checks that the simplex Σ is the union of the convex hulls of $\{b'_{\{1\}} = (1, 0, 0), b'_{\{3\}} = (0, 0, 1), b'_{\{2,3\}}\}$ and of $\{b'_{\{1\}} = (1, 0, 0), b'_{\{2\}} = (0, 1, 0), b'_{\{2,3\}}\}$, where b'_S is equal to $(1/\sum_{i=1}^{3} b^i_S) b_S$. Consequently, $b'_{\{1,2,3\}}$ belongs to at least one of these convex hulls, and, the game cannot be $(\partial - b)$-balanced since $(1, 2, 1)$ and $(1, 1, 2)$ do not belong to V_N. □

The game is not $(\partial - b)$-balanced, hence, neither b-balanced nor balanced. But, the game is payoff-dependent balanced.

Proposition 2.2.2 *For all $S \in \mathcal{N}$, let φ_S be defined on ∂V_S as follows:*

$$\varphi_S(x) = \{t_S \in \Sigma_S \mid t_S \cdot x = 1\}$$

Then the game is payoff-dependent balanced with respect to $(\varphi_S)_{S \in \mathcal{N}}$ and $\psi = \varphi_N$.

Proof of Proposition 2.2.2. The set-valued mappings φ_S, $S \in \mathcal{N}$, have convex values, which are non-empty since $(1, 1, 1) \in \partial V_S$, for each $S \in \mathcal{N}$, and from the free disposal assumption. Furthermore, it is routine to check that these set-valued mappings are upper semi-continuous.[7]

[7] We remark that the mappings $(\varphi_S)_{S \in \mathcal{N}}$ can be seen as the exact analogues of average cost pricing rules up to a translation (we consider the point $(1, 1, 1)$ instead of $(0, 0, 0)$). An example of such rules is given in Bonnisseau and Cornet (5, Corollary 3.3 p.130).

Suppose that $x \in \partial W$, such that $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \varphi_N(p_N(x)) \neq \emptyset$. Then, there exist, for all $S \in \mathcal{S}(x)$, $\lambda_S \in \mathbb{R}_+$, $b_S \in \varphi_S(x)$, and, $b_N \in \varphi_N(p_N(x))$, such that $\sum_{S\in\mathcal{S}(x)} \lambda_S = 1$ and $\sum_{S\in\mathcal{S}(x)} \lambda_S b_S = b_N$. Suppose now, that $N \notin \mathcal{S}(x)$, it implies that $x = p_N(x) + \alpha\mathbf{1}$ with $\alpha > 0$. Therefore, $b_N \cdot x > b_N \cdot p_N(x)$. But, $b_N \cdot x = (\sum_{S\in\mathcal{S}(x)} \lambda_S b_S) \cdot x = \sum_{S\in\mathcal{S}(x)} \lambda_S b_S \cdot x = \sum_{S\in\mathcal{S}(x)} \lambda_S = 1$ and $b_N \cdot p_N(x) = 1$, a contradiction. □

So far, it is worth noticing that the transfer rate rules $(\varphi_S)_{S\in\mathcal{N}}$ have been taken as constant in the proof of Corollary 2.2.1. In this example, the transfer rate rules must depend on the payoffs in order to get the balancedness. For example, $\varphi_{\{2,3\}}(1,2,1) = \{(0,0,1)\}$, $\varphi_{\{1,2,3\}}(p_N(1,2,1)) = \{(t_1,t_2,t_3) \in \Sigma \mid t_2 = 1/3\}$, $\varphi_{\{2,3\}}(1,1,2) = \{(0,1,0)\}$ and $\varphi_{\{1,2,3\}}(p_N(1,1,2)) = \{(t_1,t_2,t_3) \in \Sigma \mid t_3 = 1/3\}$.

2.2.5 Convex games

We now consider a case involving convexity in payoffs sets, and, for which Billera (1) gives a necessary and sufficient condition for non-emptiness of the core. He uses the notion of the support function. For all $S \in \mathcal{N}$, σ_S denotes the support function of V_S, that is, the mapping from $\mathbb{R}^N$ to $\mathbb{R} \cup \{+\infty\}$ defined by $\sigma_S(p) = \sup\{p \cdot v \mid v \in V_S\}$.

Corollary 2.2.3 (Billera (1970)) *The core of a game $(V_S, S \in \mathcal{N})$ is non-empty if Assumptions H1 and H2 are satisfied, if V_N is convex and if for all $S \in \mathcal{N} \setminus \{N\}$, there exists $b_S \in \mathbb{R}^N \setminus \{0\}$ such that $\sigma_S(b_S)$ is finite and for all $b \in cone\{b_S \mid S \in \mathcal{N} \setminus N\}$, $\sigma_N(b) \geq \max\{\sum_{S\in\mathcal{N}\setminus\{N\}} \lambda_S \sigma_S(b_S) \mid \forall S \in \mathcal{N} \setminus \{N\}, \lambda_S \geq 0,\ \sum_{S\in\mathcal{N}\setminus\{N\}} \lambda_S b_S = b\}$.*

Remark 2.2.2 The condition is necessary if all payoffs sets are convex. Note also that a TU game enters in the class of convex games à la Billera.[8] In this case, $\sigma_S(b_S)$ is finite if and only if b_S is positively proportional to $\mathbf{1}^S$, which leads back to the standard balancedness. Hyperplane games are particular cases of convex games as well.

Proof of Corollary 2.2.3. It suffices to prove that the game is payoff-dependent balanced. For each $S \in \mathcal{N} \setminus \{N\}$, we let φ_S be the constant mapping which associates $\frac{1}{\sum_{i\in N} b_{Si}} b_S$ to each $x \in \partial V_S$ and we let $\varphi_N(x) = N_{V_N}(x) \cap \Sigma$, where $N_{V_N}(x)$ is the normal cone of convex analysis to V_N at x. From the convexity of V_N and Assumption H1, the set-valued mapping φ_N has convex values and it is upper semi-continuous. From Assumption H1, $b_S \in L_{S+}$, $S \in \mathcal{N}$, since $\sigma_S(b_S)$ is finite and V_S is a cylinder. Let $x \in \partial W$ such that $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \varphi_N(p_N(x)) \neq \emptyset$. Suppose that $N \notin \mathcal{S}(x)$. From the definition of φ_S, this implies that there exists $b \in N_{V_N}(p_N(x)) \cap \text{cone}\{b_S \mid S \in \mathcal{S}(x)\}$. Note that $b \cdot p_N(x) = \sigma_N(b)$. Remark also that, if x does not belong to V_N, then $x = p_N(x) + \alpha\mathbf{1}$ with $\alpha > 0$. Consequently, $b \cdot x > b \cdot p_N(x) = \sigma_N(b)$. On the other hand, for all $S \in \mathcal{S}(x)$, $b_S \cdot x \leq \sigma_S(b_S)$.

[8] In TU case, there exists a payment $v_S \in \mathbb{R}$ for each coalition, in other terms $V_S = \{x \in \mathbb{R}^N \mid \sum_{i\in S} x_i \leq v_S\}$.

Since $b \in \text{cone}\{b_S \mid S \in \mathcal{S}(x)\}$, there exists $\lambda_S \geq 0$, $S \in \mathcal{S}(x)$, such that $b = \sum_{S \in \mathcal{S}(x)} \lambda_S b_S$. From our assumption, one has $\sigma_N(b) \geq \sum_{S \in \mathcal{S}(x)} \lambda_S \sigma_S(b_S) \geq (\sum_{S \in \mathcal{S}(x)} \lambda_S b_S) \cdot x = b \cdot x$. Therefore, it leads to a contradiction, which proves that $x \in V_N$. □

2.2.6 Core allocation with transfer rate rule equilibrium

We now consider the case where the mapping ψ can differ from the mapping φ_N. In this case, the statement of Theorem 2.2.1 allows us to pick up a particular element of the core with an equilibrium condition of the transfer rate rule.

The following result is due to Reny and Wooders (25). We deduce it from Theorem 2.2.1 by constructing a well-chosen transfer rate rule. In particular, ψ will depend on the cooperative commitments of each player in all the coalitions. We then apply this corollary to demonstrate the existence of solutions concepts closely related to fair division schemes, namely the partnered core and the core intersected with the average prekernel.

Corollary 2.2.4 (Reny-Wooders (1996)) *Let $(V_S, S \in \mathcal{N})$ be a ∂-balanced game satisfying Assumptions H1 and H2.*[9] *Suppose that for each pair of players i and j, there is a continuous mapping $c_{ij}\colon \partial W \to \mathbb{R}_+$ such that c_{ij} is zero on $V(S) \cap \partial W$ whenever $i \notin S$ and $j \in S$. Then there exists a core allocation x such that, for each $i \in N$, $\eta_i(x) := \sum_{j \in N}(c_{ij}(x) - c_{ji}(x)) = 0$.*

It makes sense to interpret the mappings c_{ij} as credit/debit mappings. Then, one can see $\eta_i(x)$ as the measure of the grand coalition's net indebtness to i or as i's net credit against the grand coalition. If $c_{ij} = 0$ for each $i, j \in N$ then one gets Scarf's result. We provide a direct and intuitive proof for Corollary 2.2.4. The set-valued mapping ψ will take into account individual contributions in the payoff of the grand coalition, then ψ^i stands for the cooperation index of the agent i. One is led to show the existence of a constant index among the agents.

Proof of Corollary 2.2.4. Firstly, notice that, for each $x \in \partial W$, $\eta(x) \in \mathbf{1}^\perp$. Then, put for each $x \in \partial W$: $\eta_*(x) = \max_{i \in N}\{|\eta_i(x)|\}$ and let $\tilde{\eta}_i$, $i \in N$, be mappings from ∂W to $\mathbb{R}_+$: $\tilde{\eta}_i(x) := \frac{1}{\eta_*(x)+1} \frac{\eta_i(x)}{|N|}$. The idea of the proof is to include in a suitable way the net credit and normalized mapping $\tilde{\eta}$ into ψ. Define for each $x \in \partial V_S$, $S \in \mathcal{N}$, and each $x' \in \partial V_N$:

$$\varphi_S(x) = m^S \text{ and } \psi(x') = m^N - \tilde{\eta}(p_W(x'))$$

It appears clearly that, for each $S \in \mathcal{N}$, φ_S is valued into Σ_S, and, ψ is valued into Σ. Furthermore, the mappings are all upper semi-continuous with non-empty compact and convex values. It stems from the continuity of the mappings c_{ij} and the normalization.

Lemma 2.2.1 *For each $x \in \partial W$, if $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$, then $\psi(p_N(x)) = m^N$.*

[9] The original result is stated for balanced games, it is slightly improved by considering ∂-balanced games. A game is ∂-balanced if for any balanced family of coalitions $\mathcal{B}$, $\partial W \cap (\cap_{S \in \mathcal{B}} V_S) \subset V_N$.

We provide in Appendix the detailed proof of Lemma 2.2.1. Let $x \in \partial W$, then from Lemma 2.2.1 the condition $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$ says that the family $\mathcal{S}(x)$ is balanced and since the game is ∂-balanced one deduces that the game is payoff-dependent balanced. Now, applying Theorem 2.2.1, there exists x in the core of the game and such that $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$. Noticing that $x = p_N(x) = p_W(x)$ and using once again Lemma 2.2.1, this implies that $\tilde{\eta}(x) = 0$, that is $\eta(x) = 0$, so Corollary 2.2.4 is proved. $\square$

We briefly recall two applications of Corollary 2.2.4. We refer the reader respectively to the works of Reny and Wooders (25) and Orshan and al. (21) for a complete presentation.

The partnerships properties have been firstly described and, later, carried out in other fields by Reny and Wooders, see (23; 26). A payoff $x \in \partial W$ is said to be partnered if the family $\mathcal{S}(x)$ satisfies, for all $i, j \in N$, $\mathcal{S}_i(x) \subset \mathcal{S}_j(x) \Rightarrow \mathcal{S}_j(x) \subset \mathcal{S}_i(x)$, where $\mathcal{S}_i(x) = \{S \in \mathcal{S}(x) \mid i \in S\}$. Then, the partnered core is the core intersected with the set of partnered outcomes. Reny and Wooders (25) apply Corollary 2.2.4 to suitable mappings c_{ij} to prove the existence of an element in the partnered core.[10] The mappings are defined as follows: $c_{ij}(x) = \min\{\text{dist }(x_S, V(S)) \mid S \in \mathcal{N} \text{ and } i \notin S \ni j\}$ for each $x \in \partial W$ where dist is the Euclidean distance.

As a second direct application, one can also prove the existence of an element lying in the core intersected with the average prekernel (also called bilateral consistent prekernel) as defined in Orshan and al. (22), see also Serrano and Shimomura (30). To define the average prekernel, we need to introduce two additional assumptions on the game, namely non-levelness and smoothness. The average prekernel is the consistent extension of the usual prekernel at stake in TU games. It also generalizes the Nash bargaining solution. The result can be deduced from Corollary 2.2.4 by considering suitable credit mappings. Indeed, the average prekernel may be rewritten as the set of elements $x \in \partial V_N$ such that $\sum_{j \in N}(c_{ij}(x) - c_{ji}(x)) = 0$ for some credit mappings satisfying the requirements of Corollary 2.2.4. In this context, $c_{ij}(x)$ can be seen as the weighted surplus of agent i with respect to agent j at the point x. Orshan and al. (21) have shown the non-emptiness of the core intersected with the average prekernel in ∂-separating games, here the result is improved by considering the larger class of ∂-balanced games.[11]

[10] The core of the game is not tight if there exists a payoff x in the core, which does not belong to V_S for all $S \neq N$. In that case, every element satisfying this property has the partnership property. Indeed since $x \notin V_S$ for each $S \in \mathcal{N} \setminus N$, then $\mathcal{S}_i(x) = \mathcal{S}_j(x) = \{N\}$ for each $i, j \in N$. The statement of Theorem 2.2.1 is in the same spirit since in the case of non tight core one gets: $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} = \varphi_N(x)$ and thus $\varphi_N(x) \cap \psi(x) \neq \emptyset$ that is, for the transfer rate rules in the proof of Corollary 2.2.4, $m^N = \varphi_N(x) = \psi(x) = m^N - \tilde{\eta}(x)$, then $\eta(x) = 0$.

[11] Note that a ∂-separating game is ∂-balanced.

2.3 Parameterized cooperative game

2.3.1 Existence of Equilibrium-core allocations

This section is intended to unify the literature, which uses explicitly or implicitly parameterized games. To take into account the environment and the possible interactions between players payoffs, we introduce here a canonical version of a parameterized game and the associated solution concept, called equilibrium-core allocation.

The parameter set is Θ and, a game is associated to each $\theta \in \Theta$, that is, one has a set-valued mapping V_S from Θ to $\mathbb{R}^N$ for each coalition S. To encompass several results in the literature, we add a set-valued mapping V from Θ to $\mathbb{R}^N$, which possibly differs from V_N. It represents the feasible set of the whole economy in Border (7); in Boehm (2), it differs from V_N du to costs of forming a coalition; in Ichiishi (13), V represents the set of all feasible allocations in coalition structure. Finally, the equilibrium condition on the parameters is represented by a set-valued mapping G from $\Theta \times \mathbb{R}^N$ to Θ. We denote by $W(\theta)$ the union of the payoffs sets $\cup_{S \in \mathcal{N}} V_S(\theta)$.

Definition 2.3.1 An equilibrium-core allocation is a vector $(\theta^*, x^*) \in \Theta \times \mathbb{R}^N$ such that:

$$x^* \in \partial V(\theta^*) \setminus \operatorname{int} W(\theta^*) \text{ and } \theta^* \in G(\theta^*, x^*).$$

The model is closely linked up to the model of Ichiishi (13), who defines a Social Coalitional Equilibrium. We show that Ichiishi's Equilibrium is a corollary of our main result. Another example of application is provided by Border (7) for the core of an economy without ordered preferences. Note that both results originally use the same mathematical tool in their proofs. These authors have applied Fan's coincidence theorem to exhibit the non-emptiness of the core at stake.

Furthermore the general framework allows us to investigate some other topics of economic theory. For instance, Ichiishi and Idzik (15) have shown the non-emptiness of the incentive compatible core in incomplete information framework, using Ichiishi's Social Coalitional Equilibrium existence.

Remark 2.3.1 Obviously, the parametric framework encompasses the case of constant payoffs sets with respect to the environment, then the results of Section 2.2 are all covered by Theorem 2.3.1 presented below.

The assumptions on the game are the following. They stand for the former Assumptions H1 and H2 of Section 2.2. Actually we just add continuous dependencies with respect to the environment.

(PH0) Θ is a non-empty, convex, compact subset of an Euclidean space. G is an upper semi-continuous set-valued mapping with non-empty and convex values.

(PH1) (i) The set-valued mappings $V_{\{i\}}$, $i \in N$, and V are non-empty valued. V_S, $S \in \mathcal{N}$, and V are lower semi-continuous set-valued mappings with closed graph.
(ii) For each $\theta \in \Theta$, $V_S(\theta)$, $S \in \mathcal{N}$, and $V(\theta)$ satisfy Assumption H1(ii).

(PH2) For all $\theta \in \Theta$, there exists $m(\theta) \in \mathbb{R}$ such that, for each $S \in \mathcal{N}$, for each $x \in V_S(\theta)$, if $x \notin \text{int } V_{\{i\}}(\theta)$, for all $i \in S$, then $x_j \leq m(\theta)$ for each $j \in S$. For all $\theta \in \Theta$, for each $x \in V(\theta)$, if $x_i \notin \text{int } V_{\{i\}}(\theta)$, for all $i \in N$, then $x \leq m(\theta)\mathbf{1}$.

Assumption PH1 implies that there exist functions v_i, $i \in N$, from Θ to $\mathbb{R}$, such that, for each $\theta \in \Theta$, $V_{\{i\}}(\theta) = \{z \in R^N \mid z_i \leq v_i(\theta)\}$. Bonnisseau (3, Lemma 3.1. p.217) states that if a set-valued mapping M from Θ to $\mathbb{R}^N$ is lower semi-continuous with non-empty values, has a closed graph and satisfies Assumption H1(ii) for all $\theta \in \Theta$, then there exists a continuous mapping λ from $\Theta \times \mathbf{1}^\perp$ to $\mathbb{R}$ such that, for all $(\theta, s) \in \Theta \times \mathbf{1}^\perp$, $s - \lambda(\theta, s)\mathbf{1} \in \partial M(\theta)$. Let p_V and λ_V be the continuous mappings defined respectively on $\Theta \times \mathbb{R}^N$ and $\Theta \times \mathbf{1}^\perp$ such that: $p_V(\theta, x) = \text{proj}(x) - \lambda_V(\theta, \text{proj}(x))\mathbf{1} \in \partial V_N(\theta)$. We define similarly the mappings p_W and λ_W associated to W.

We can extend the notion of a transfer rate rule $((\varphi_S)_{S \in \mathcal{N}}, \psi)$ to a parameterized game. The unique difference is that the set-valued mappings are now defined on the graphs Gr ∂V_S or Gr ∂V.

Definition 2.3.2 Let $(V, (V_S)_{S \in \mathcal{N}}, \Theta)$ be a parameterized game satisfying Assumption PH1. It is payoff-dependent balanced if there exists a transfer rate rule $((\varphi_S)_{S \in \mathcal{N}}, \psi)$ such that, for each $(\theta, x) \in \text{Gr } \partial W$,

$$\text{if co}\{\varphi_S(\theta, x) \mid S \in \mathcal{S}_\theta(x)\} \cap \psi(\theta, p_V(\theta, x)) \neq \emptyset, \text{ then } x \in V(\theta).$$

The second result of the paper is the following.

Theorem 2.3.1 *Let $(V, (V_S)_{S \in \mathcal{N}}, \Theta)$ be a parameterized game satisfying Assumptions PH0, PH1 and PH2. If it is payoff-dependent balanced with respect to the transfer rate rule $((\varphi_S)_{S \in \mathcal{N}}, \psi)$, there exists an equilibrium-core allocation (θ^*, x^*) such that:*

$$\text{co}\left\{\varphi_S(\theta^*, p_W(\theta^*, x^*)) \mid S \in \mathcal{S}_{\theta^*}(p_W(\theta^*, x^*))\right\} \bigcap \psi(\theta^*, x^*) \neq \emptyset.$$

The proof, referred to Appendix, follows the geometric construction given in the proof of Theorem 2.2.1. But, note that, contrary to the proof of Theorem 2.2.1, the proof is self contained in the sense that we do not appeal an existence result for a general economic equilibrium.

Let us consider the parameterized game $(V_N, (V_S)_{S \in \mathcal{N}}, \Theta)$ where for all $\theta \in \Theta$, $(V_N(\theta), (V_S(\theta))_{S \in \mathcal{N}})$ is the game defined in Sub-section 2.2.4. If Θ and G satisfy Assumption PH0, then there exists an equilibrium-core allocation, $(\theta^*, (1, 1, 1))$ where θ^* is a fixed-point of $G(., (1, 1, 1))$. Nevertheless, $(V_N(\theta), (V_S(\theta))_{S \in \mathcal{N}})$ is not $(\partial - b)$- balanced for each $\theta \in \Theta$. Once again, we need to consider a non-constant transfer rate rule to prove the existence of equilibrium-core allocations.

2.3.2 Two applications of Theorem 2.3.1

In both applications below, we do not make full use of Theorem 2.3.1 since for any given parameter, the game, $(V(\theta), (V_S(\theta))_{S \in \mathcal{N}})$, will be balanced with a constant transfer rate rule. So we mostly focus on the role of the parameterization in these models. Both applications will clarify the usefulness of the mapping G, which is explicitly given in each case. Elementary proofs are provided thanks to Theorem 2.3.1.

Ichiishi's Social Coalitional Equilibrium

The Social Coalitional Equilibrium of Ichiishi (13) is the benchmark work in the framework of parameterized games. Furthermore, the general formulation of the Equilibrium, where the agents can realize a coalition structure, encompasses Social Equilibrium of Debreu (8) and the usual core as special cases. Number of applications of this seminal result are reviewed in Ichiishi (14).

A coalition structure is a partition of N. Let $\mathcal{P}$ be a non-empty collection of coalition structure, a member of $\mathcal{P}$ is denoted P. Each player has a parameter set Θ_i ($\Theta_S = \prod_{i\in S}\Theta_i$, $\Theta = \Theta_N$). For each $S \in \mathcal{N}$, let F^S be a mapping from Θ into Θ_S. Preference relation of each player i in a coalition S is represented by a utility function; $v^i_S : \text{Gr } F^S \to \mathbb{R}$.

A Social Coalitional Equilibrium of a society is a pair of a parameter $\theta^* \in \Theta$ and admissible coalition structure $P^* \in \mathcal{P}$, such that: (i) For each $D \in P^*$, $\theta^{*D} \in F^D(\theta^*)$. (ii) It is not true that there exists $S \in \mathcal{N}$ and $\theta' \in F^S(\theta^*)$ such that $v^i_S(\theta^*, \theta') > v^i_{D(i)}(\theta^*, \theta^{*D(i)})$ for every $i \in S$, where $D(i) \in P^*$ and $i \in D(i)$.

Corollary 2.3.1 (Ichiishi(1981)) *A Social Coalitional Equilibrium exists if: (1) For every $i \in N$, Θ_i is a non-empty, convex compact subset of an Euclidean space. (2) For every $S \in \mathcal{N}$, F^S is a lower and upper semi-continuous set-valued mapping with non-empty values. (3) For every $S \in \mathcal{N}$, v^i_S is continuous on Gr F^S. (4) For every $\theta \in \Theta$ and every $v \in \mathbb{R}$, if there exists a balanced family $\mathcal{B}$ such that for each $S \in \mathcal{B}$ there exists $\theta(S) \in F^S(\theta)$ for which $v_i \le u^i_S(\theta, \theta(S))$ for each $i \in S$, then there exist $P \in \mathcal{P}$ and $\theta'^D \in F^D(\theta)$ for every $D \in P$ such that $v_i \le u^i_D(\theta, \theta'^D)$ for all $i \in D$. (5) For every $\theta \in \Theta$, and for every $v \in \mathbb{R}^N$, the set*

$$\bigcup_{P\in\mathcal{P}} \left\{\theta' \in \Theta \mid \forall D \in P, \theta'^D \in F^D(\theta) \text{ and } v \le (v^i_{D(i)}(\theta, \theta'^{D(i)}))_{i\in N}\right\}$$

is convex.

In the original paper of Ichiishi, the strategy sets are taken in Hausdorff topological vector spaces, we limit here the framework within Euclidean spaces.

Proof of Corollary 2.3.1. For each $S \in \mathcal{N}$, let us define:

$$V_S(\theta) = \left\{u \in \mathbb{R}^N \mid \exists \theta' \in F^S(\theta),\ u_i \le v^i_S(\theta, \theta'),\ i \in S\right\}$$

$$\widetilde{V}_S(\theta) = \{u \in V_S(\theta) \mid \forall i \in N \setminus S, u_i = 0\}$$

$$V(\theta) = \bigcup_{P\in\mathcal{P}} \sum_{D\in P} \widetilde{V}_D(\theta)$$

And let $G(\theta, x)$ be equal to

$$\bigcup_{P \in \mathcal{P}} \left\{ \theta' \in \Theta \mid \forall D \in P, \theta'^D \in F^D(\theta) \text{ and } p_V(\theta, x) \leq (v^i_{D(i)}(\theta, \theta'^{D(i)}))_{i \in N} \right\}.$$

Consider the parameterized game defined above, we show that the game meets the requirements of Theorem 2.3.1. We begin with the condition of balancedness. For each $S \in \mathcal{N}$, let φ_S be the constant mapping equals to m^S and $\psi = \varphi_N$. Let (θ, x) be in Gr ∂W such that $\text{co}\{\varphi_S(\theta, x) \mid S \in \mathcal{S}_\theta(x)\} \cap \psi(\theta, p_V(\theta, x)) \neq \emptyset$. As seen before, one easily checks that the family $\mathcal{S}_\theta(x)$ is balanced. For each $S \in \mathcal{S}_\theta(x)$, there exists $\theta(S) \in F^S(\theta)$ for which $x_i \leq u^i_S(\theta, \theta(S))$, $i \in S$. Then, from (4), there exist $P \in \mathcal{P}$ and $\theta'^D \in F^D(\theta)$ for every $D \in P$ such that $x_i \leq u^i_D(\theta, \theta'^D)$, $i \in D \in P$. It means that $x \in \sum_{D \in P} \widetilde{V}_D(\theta) \in V(\theta)$.

Ichiishi proved the continuity of the set-valued mappings V_S and V (Ichiishi (13, Proof of Lemma, step 1, p.372)), so PH1 clearly holds true. G is non-empty from the definitions of p_V and V, and, convex valued from (5). It suffices to prove that G has a closed graph to imply that it is upper semi-continuous. It is straightforward, from the finiteness of $\mathcal{P}$ and (2), (3), so PH0 holds. (1) and the continuity of v^i_S guarantees that PH2 is satisfied.

We apply Theorem 2.3.1, then there exists (θ^*, x^*) such that $\theta^* \in G(\theta^*, x^*)$ and $x^* \in \partial V(\theta^*) \setminus \text{int } W(\theta^*)$. Hence, there exists $P \in \mathcal{P}$ such that for all $D \in P$, $\theta^{*D} \in F^D(\theta^*)$ and $p_V(\theta^*, x^*) \leq (v^i_{D(i)}(\theta^*, \theta^{*D(i)}))_{i \in N}$, furthermore $x^* = p_V(\theta^*, x^*)$. Necessarily $(v^i_{D(i)}(\theta^*, \theta^{*D(i)}))_{i \in N} \in \partial V(\theta^*) \setminus \text{int } \cup_{S \in \mathcal{N}} V_S(\theta^*)$, satisfying the requirement of Social Coalitional Equilibrium. □

Ichiishi and Quinzii (17) use a variant of Corollary 2.3.1 to prove the non-emptiness of the core for economies with increasing returns. The authors split the parameter set into an abstract parameter set and action sets for each individual. Moreover, they do not need the agents realize a coalition structure, they only use the benchmark partition N. Therefore, $V = V_N$, and, the feasibility condition in the definition of Social Equilibrium must hold only on the coarsest coalition N. Using our own materials, one can prove directly the result as stated in (17, Lemma A p.406).

One can combine results from Sub-section 2.2.6 with the solution concept of Social Equilibrium to show, for instance, the existence of a Social Equilibrium with partnerships. Indeed, by considering the equilibrium condition on the transfer rate rule $((\varphi_S)_{S \in \mathcal{N}}, \psi)$ and under assumptions given in Ichiishi and Quinzii (17, Lemma A p.406), one is led to a Social Equilibrium such that the social outcome is a partnered outcome. Though the concept of partnered structure is much weaker than any partition structure, it avoids the use of the ad hoc Assumption (4) in Corollary 2.3.1. Then, the emerging structure is obtained as the outcome of an endogenous process. To the best of our knowledge, such solutions have not been explored any further in this parameterized framework.

Remark 2.3.2 In a standard cooperative game, one can also define solution concepts dealing with coalition structure. Using Theorem 2.3.1 when the parameter set is reduced to a point, one provides a sufficient condition for non-emptiness of the core with a coalition structure, in NTU games, if V

is the super-additive cover of the game $(V_S, S \in \mathcal{N})$ defined by $\cup_{P\in\mathcal{P}} \sum_{S\in P}\{x \in V_S \mid x_i = 0,\ i \in N \setminus S\}$.

Core allocations for non-ordered preferences

Border proves the non-emptiness of the core of an economy where the preferences are non ordered. One exhibits a parameterized game such that the core of the economy is exactly the equilibrium-core allocation stated in Definition 2.3.1. Hence, one can recover Border (7)'s result from our abstract result. This shall turn out to be true for the generalization given by Kajii (19), which is also briefly described.

Let Ξ_i, $i \in N$, be the payoffs set of agent i, $\Xi_S = \prod_{i\in S}\Xi_i$ and $\Xi = \Xi_N$. For each $S \in \mathcal{N}$, let F^S be the feasibility mapping from Ξ into Ξ_S and let $\Theta \subset \Xi$ be the set of all jointly feasible allocations. Preference relation of each player is represented by a set-valued mapping P_i from Ξ_i into Ξ_i.

An element $\xi \in \Xi$ is said to be in the core if : (i) $\xi \in \Theta$. (ii) There is no $S \in \mathcal{N}$ and $\xi' \in F^S(\xi)$ satisfying $\xi'_i \in P_i(\xi_i)$ for all $i \in S$.

Corollary 2.3.2 (Border (1984)) *The core is non-empty if: (1) For each i, Ξ_i is a non-empty convex subset of an Euclidean space. (2) For each $S \in \mathcal{N}$, $F^S : \Xi \to \Xi_S$ is a lower and upper semi-continuous set-valued mapping with compact values and F^i, $i \in N$, is non-empty valued. (3) Θ is compact and convex. (4) For each i, P_i has an open graph in $\Xi_i \times \Xi_i$, for all $\xi_i \in \Xi_i$, $P_i(\xi_i)$ is convex and $\xi_i \notin P_i(\xi_i)$. (5) The game is balanced: for all $\xi' \in \Xi$, for any balanced family β with the balancing weights $(\lambda_B)_{B\in\beta}$, if there exist $(\xi^B)_{B\in\beta}$ such that $\xi^B \in F^B(\xi')$, $B \in \beta$, then $\xi \in \Theta$ where $\xi_i = \sum_{B\in\beta, i\in B}\lambda_B \xi_i^B$.*

Proof of Corollary 2.3.2. We define pseudo-utility functions $v_i : \Xi_i \times \Xi_i \to \mathbb{R}$, $i \in N$, as follows: $v_i(\xi'_i, \xi_i) = \text{dist}[(\xi_i, \xi'_i), (\text{Gr } P_i)^c]$. The convexity of $P_i(\xi_i)$ implies that v_i is quasi-concave in its first argument (see Border's Appendix).

The parameterized game is defined on the compact set Θ. For each $\xi \in \Theta$, for each $S \in \mathcal{N}$, let $V_S(\xi) = \{u \in \mathbb{R}^N \mid \exists \xi' \in F^S(\xi),\ u_i \le v_i(\xi'_i, \xi_i),\ i \in S\}$, and, $V(\xi) = \{u \in \mathbb{R}^N \mid \exists \xi' \in \Theta,\ u_i \le v_i(\xi'_i, \xi_i),\ i \in N\}$. Let $W(\xi) = \cup_{S\in\mathcal{N}} V_S(\xi)$. Remark that $\xi \in \Theta$ is in the core if and only if $0 \in V(\xi) \setminus \text{int } W(\xi)$. Let:

$$G(\xi, x) = \left\{\xi' \in \Theta \mid p_V(\xi, x) \le (v_i(\xi'_i, \xi_i))_{i\in N}\right\}.$$

Consider the game above. We provide in Appendix the detailed and rather technical proof that the assumptions of Theorem 2.3.1 are all fulfilled. Then, there exists a bundle (θ^*, x^*) such that $x^* \in \partial V(\theta^*) \setminus \text{int } W(\theta^*)$ and $\theta^* \in G(\theta^*, x^*)$. Hence, $x^* = p_V(\theta^*, x^*) \le (v_i(\theta_i^*, \theta_i^*))_i = 0$. From the free-disposal assumption, since $\theta^* \in \Theta$, $0 \in V(\theta^*)$ and $x^* \in \partial V(\theta^*)$, $x^* \le 0$ implies that

$0 \in \partial V(\theta^*)$. Furthermore, $x^* \notin \text{int } W(\theta^*)$ and $x^* \leq 0$ implies that $0 \notin \text{int } W(\theta^*)$. That is to say that $0 \in \partial V(\theta^*) \setminus \text{int } W(\theta^*)$ and $\theta^* \in \Theta$, so the core is non-empty. □

This model has been carried out more generally. In (19), Kajii proposes a generalization of both Border's result and of Scarf's α-core non-emptiness result ((28)). The same construction as in the previous proof can be applied to show Kajii's result. The difference comes from the fact that preferences are interdependent, that is, the mappings P^i are defined from Ξ into Ξ. Consequently the pseudo utility mappings are defined on $\Xi \times \Xi$ but still verify quasi-concavity in their first variables (see (19, p.196)).

In Kajii's setting, a coalition S blocks a feasible allocation $\xi \in \Theta$ if there exists $\xi' \in F^S(\xi)$ such that for all ξ'' with $\xi''_i = \xi'_i$ all $i \in S$, one has $\xi'' \in P_i(\xi)$. Then, the payoffs sets are naturally defined as: $V_S(\xi) = \{u \in \mathbb{R}^N \mid \exists \xi' \in F^S(\xi) \text{ such that for all } \xi''_i = \xi'_i, i \in S,\ u_i \leq v_i(\xi'', \xi),\ i \in S\}$, for all $S \in \mathcal{N}$ and $V(\xi) = \{u \in \mathbb{R}^N \mid \exists \xi' \in \Theta\ \ u_i \leq v_i(\xi', \xi),\ i \in N\}$. These set-valued mappings satisfy the expected properties of continuities as in Border's setting. We obtain the result of Kajii (19, Corollary p.201) (he additionally assumes that $F^N(\xi) = \Theta$ and $\Xi_i = \Theta_i$) if we posit:

$$G(\xi, x) = \bigcap_{i \in N} \left\{ \xi' \in \Theta \mid p_V^i(\xi, x) \leq v_i(\xi', \xi) \right\}.$$

The mapping G is an upper semi-continuous set-valued mapping with convex values as a finite intersection of upper semi-continuous set-valued mappings with convex values. G has non-empty values since, for all $(\theta, x) \in \Theta \times \mathbb{R}^N$, $p_V(\theta, x) \in V(\theta)$ by definition. Therefore the parameterized game meets the requirements of Theorem 2.3.1.

2.4 Further developments

We discuss the literature that could be submitted to a similar treatment. In non-convex economies, the most achieved results for non-emptiness of cores are deeply relying on elasticities conditions of the demand functions (Ichiishi and Quinzii (17)). These works consist mostly in recovering some convexity properties, as far as possible, to draw nearer to the notion of distributive sets introduced by Scarf (29), and then, they comprehend the non-convexity thanks to convexifying assumptions.

New developments could follow from Theorem 2.3.1 for non-convex economies. Indeed the payoff-dependent balancedness enlarges the geometric possibilities to get a non-empty core. The negative result (Scarf (29, Theorem 5 p.426)) delimits, however, the range of new results. Direct approaches for the core of an economy have been carried out by Florenzano (9). In this framework where no cooperative game structure is defined, one should restate payoff-dependent balancedness by defining transfer rates directly on the fundamentals of the economy.

An second concern is the problem of markets representation. It consists of the determination of the market that can generate a given game. The TU case has been solved by Shapley and Shubik

(31). In the case of NTU games, the question is still open for the general case. The flexibility of payoff-dependent balancedness could help to get further results on this problem.

In addition, one can cite related topics partially evoked in this paper. For the games in parametric form: the incentive cores in asymmetric information, see Ichiishi and Idzik (15) and Radner (18), both using the seminal result of Ichiishi (13); the α-core, as seen before with Scarf (28) and Kajii (19). We have shown that such recent works can receive a positive treatment by computations of equilibrium-core allocations. Note however that, in both research fields, there exist robust counterexamples of empty cores (see respectively Forges et al. (10) and Holly (12)).

Core allocations with transfer rate rule equilibrium are, in particular, linked up to the fair division schemes. Indeed, as quoted by Reny and Wooders (25), the notion of partnered collections of sets is closely related to the concept of kernel (prekernel) at stake in NTU games. We have clearly illustrate this point in Sub-section 2.2.6 showing the connection between the partnered core and core intersected with average prekernel. Besides, the notion of partnership gave also rise to a literature on the side of covering theorems as developed by Reny and Wooders (26), see also Ichiishi and Idzik (16). Lastly, Page and Wooders (23) extended the notion of partnership to competitive equilibrium and cores in economies. Very recently, Herings et al. (11) define a notion of Social Stable core allocations, for which power indexes are equally shared out among the coalitions. Social Stable core allocations can also be seen as core allocations with transfer rate rule equilibrium.

2.5 Appendix

2.5.1 Proof of Theorem 2.2.1

Let $Y_2 = -(\text{int } W)^c$ where $(\text{int } W)^c$ denotes the complementary of the interior of the set W. Note that Y_2 is bounded above by $-v$. Let $\tilde{\varphi}_2$ be the set-valued mapping from ∂Y_2 to Σ defined by

$$\tilde{\varphi}_2(y_2) = \text{co}\,\{\varphi_S(-y_2) \mid S \in \mathcal{S}(-y_2)\}\,.$$

Lemma 2.5.1 *For all $(y_2, p) \in Gr\ \tilde{\varphi}_2$ and such that $y_{2i} < -m$ for some $i \in N$, then $p_i = 0$.*

Proof of Lemma 2.5.1. Let $y_2 \in \partial Y_2$ such that $y_2 \notin \{-m\mathbf{1}\} + \mathbb{R}^N_+$. Let $i \in N$ such that $y_{2i} < -m$. Then, for all $S \in \mathcal{S}(-y_2)$ we show that $i \notin S$. Indeed, recalling that $-y_2 \geq v$, Assumption H2 states that if $-y_2 \in V_S$, then $-y_{2j} \leq m$ for all $j \in S$. Thus, $i \notin S$. Consequently, for all $S \in \mathcal{S}(-y_2)$, for all $p \in \varphi_S(-y_2)$, $p_i = 0$ since φ_S takes its values in Σ_S. Hence, for all $p \in \tilde{\varphi}_2(y_2)$, $p_i = 0$. □

Let m be the upper bound given in Assumption H2. Since Y_2 is bounded above by $-v$ and from Assumption H2, the set $Y_2 \cap (\{-m\mathbf{1}\} + \mathbb{R}^N_+)$ is compact, therefore there exists $\rho > 0$ such that $\text{proj}(y) \in B_{\mathbf{1}^\perp}(0, \rho)$ for all $y \in Y_2 \cap (\{-m\mathbf{1}\} + \mathbb{R}^N_+)$.

Define $Y_1 = \{p_N(s) \mid s \in \bar{B}_{\mathbf{1}^\perp}(0, \rho)\} - \mathbb{R}^N_+$, we remark that, for all $y_1 \in \partial Y_1$, if $\text{proj}(y_1) \in \bar{B}_{\mathbf{1}^\perp}(0, \rho)$ then $y_1 \in \partial V_N$. Let us choose $\underline{y} \in \text{int } Y_1$.

Lemma 2.5.2 *There exists a continuous mapping c from ∂Y_1 to Σ_{++} such that $c(y_1) \cdot (y_1 - \underline{y}) \geq 0$ for all $y_1 \in \partial Y_1$.*

Proof of Lemma 2.5.2. Since Y_1 satisfies the free disposal condition and $\underline{y} \in \text{int } Y_1$, for all $y_1 \in \partial Y_1$, there exists a vector $p \in \Sigma_{++}$ such that $p \cdot (y_1 - \underline{y}) > 0$. Define the set valued mapping Γ from ∂Y_1 to Σ_{++} as $\Gamma(y_1) = \{p \in \Sigma_{++} \mid p \cdot (y_1 - \underline{y}) > 0\}$, this set-valued mapping is non-empty valued from the argument above. It is an easy matter to check that it has open graph and convex values. One gets the existence of a continuous selection of Γ applying a weak version of Michael's selection theorem. $\square$

Let $\tilde{\varphi}_1$ be the set-valued mapping from ∂Y_1 to Σ defined by:

$$\tilde{\varphi}_1(y_1) = \begin{cases} \psi(y_1) & \text{if } \|\text{proj}(y_1)\| < \rho \\ \text{co}\{\psi(y_1), c(y_1)\} & \text{if } \|\text{proj}(y_1)\| = \rho \\ c(y_1) & \text{if } \|\text{proj}(y_1)\| > \rho \end{cases}$$

Lemma 2.5.3 *For all $(y_1, y_2) \in \partial Y_1 \times \partial Y_2$ and $p \in \Sigma$ such that proj$(y_1) = -$proj(y_2) and $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$, one has $p \in \psi(y_1)$ and $y_1 \in \partial V_N$.*

Proof of Lemma 2.5.3. We first prove that $\|\text{proj}(y_1)\| \leq \rho$. Indeed, if it is not true, then $\varphi_1(y_1) = c(y_1) \in \Sigma_{++}$ and, from Lemma 2.5.1 and the choice of ρ, since $\|\text{proj}(y_2)\| = \|\text{proj}(y_1)\| > \rho$, one has $\varphi_2(y_2) \notin \Sigma_{++}$. But, this contradicts $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$. Now, the above remark implies that $y_1 \in \partial V_N$. If $p \notin \psi(y_1)$, $\|\text{proj}(y_1)\| = \rho$ and $p \in \Sigma_{++}$. The same argument leads again to a contradiction. $\square$

Given these materials, one can state the following technical lemma. The proof consists of the verifications of Theorem 2.2.2 assumptions with respect to the construction above.

Lemma 2.5.4 *Y_1,Y_2 and $\tilde{\varphi}_1$, $\tilde{\varphi}_2$ satisfy the requirements of Theorem 2.2.2 for any closed convex cone C included in $\mathbb{R}^N_{++} \cup \{0\}$ such that $\mathbf{1} \in$ int C.*

Proof of Lemma 2.5.4. We check with the three following claims that the assumptions of Theorem 2.2.2 hold true for the sets Y_1 and Y_2 and the mappings $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$.

Claim 2.5.1 *Y_1 and $\tilde{\varphi}_1$ satisfies Assumptions P, PR and BL.*

Proof of Claim 2.5.1. Y_1 clearly satisfies Assumption P, Assumption PR is also clearly satisfied from the definition of the set-valued mapping $\tilde{\varphi}_1$ and the continuity of the function c.

Assumption BL also holds true. Indeed, there exists $\alpha \in \mathbb{R}$ such that for all $s \in \bar{B}(0, \rho)$ and for all $p \in \Sigma$, $p \cdot (s - \lambda_N(s)\mathbf{1}) \geq \alpha$ and $p \cdot \underline{y} \geq \alpha$. Thus one gets that $p \cdot y_1 \geq \alpha$ for all $(y_1, p) \in \text{Gr } \tilde{\varphi}_1$, using Lemma 2.5.2. $\square$

Claim 2.5.2 *Y_2 and $\tilde{\varphi}_2$ satisfies Assumptions P, PR and BL.*

Proof of Claim 2.5.2. One easily checks that Y_2 is closed and that it satisfies the free-disposal assumption so P is satisfied. $\tilde{\varphi}_2$ has obviously convex compact values from the assumption of payoff-dependent balancedness. $\tilde{\varphi}_2$ has non-empty values since $y_2 \in \partial Y_2$ implies that $-y_2 \in \partial V_S$ for at least one $S \in \mathcal{N}$. Since Σ is compact, it suffices to show that the set-valued mapping $\tilde{\psi}_2$ defined by $\tilde{\psi}_2(y_2) = \cup_{S \in \mathcal{S}(-y_2)} \varphi_S(-y_2)$ has a closed graph in order to prove that $\tilde{\varphi}_2$ is upper semi-continuous. Let (y_2^ν, p^ν) a sequence of $\partial Y_2 \times \Sigma$ which converges to (y_2, p) and such that $p^\nu \in \tilde{\psi}_2(y_2^\nu)$ for all ν. From the definition of $\mathcal{S}$, for ν large enough, $\mathcal{S}(-y_2^\nu) \subset \mathcal{S}(-y_2)$. Consequently, for all ν large enough, there exists $S^\nu \in \mathcal{S}(-y_2)$ such that $p^\nu \in \varphi_{S^\nu}(-y^\nu)$. Since $\mathcal{S}(-y_2)$ is a finite set, there exists a subsequence such that S^ν is constant equal to S. Since φ_S is upper semi-continuous, this implies that $p \in \varphi_S(-y_2) \subset \tilde{\psi}_2(y_2)$. It ends the proof.

Assumption BL also holds true. Indeed, for all $(y_2, p) \in$ Gr $\tilde{\varphi}_2$, from Lemma 2.5.1, one has $p \cdot y_2 \geq \sum_{i \in N, y_{2i} \geq -m} p_i y_{2i} \geq -m \sum_{i \in N, y_{2i} \geq -m} p_i = -m \sum_{i \in N} p_i = -m$. □

Claim 2.5.3 *Assumptions B and S are satisfied.*

Proof of Claim 2.5.3. Assumption B is satisfied since Y_2 is bounded above by $-v$ and Y_1 is also bounded above since $\{p_N(s) \mid s \in \bar{B}_{\mathbf{1}^\perp}(0, \rho)\}$ is a compact set.

Assumption S holds true. If it is not the case, there exists $t > 0$, $(y_1, y_2) \in \partial Y_1 \times \partial Y_2$ and $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$ such that $y_1 + y_2 + t\mathbf{1} \in C$ and $p \cdot (y_1 + y_2 + t\mathbf{1}) = 0$. Since $p \in \mathbb{R}^N_+ \setminus \{0\}$ and $C \subset \mathbb{R}^N_{++} \cup \{0\}$, one deduces that $y_1 + y_2 + t\mathbf{1} = 0$. Let $s_1 = \text{proj}(y_1)$ and $s_2 = \text{proj}(y_2)$. Clearly, $s_1 = -s_2$. Then one can apply Lemma 2.5.3 which states that $y_1 \in \partial V_N$ and $p \in \psi(y_1)$.

Let $x = -y_2$, thus $x \in \partial W$. $s_1 = -s_2$ implies that $p_N(x) = y_1$ consequently $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$. Thus, since the game is payoff-dependent balanced, one has $x \in V_N$. But $y_1 = x - t\mathbf{1}$ contradicts the fact that $y_1 \in \partial V_N$ from the free disposal property of V_N. □

From the previous claims, the conclusion of Lemma 2.5.4 is satisfied. □

Theorem 2.2.2 implies that there exists a vector $(y_1, y_2, p) \in \partial Y_1 \times \partial Y_2 \times \Sigma$ such that: $y_1 + y_2 \in C \subset \{0\} \cup \mathbb{R}^N_{++}$, and, $p \in \tilde{\varphi}_1(y_1) \cap \tilde{\varphi}_2(y_2)$.

We first show that $y_1 + y_2 = 0$. If it is not true, $-y_2 \ll y_1$. But $y_1 \in Y_1 \subset V_N$, and, thus, $-y_2 \in \text{int } V_N \subset \text{int } W$, which contradicts that $-y_2 \in (\text{int } W)^c$. Since $\text{proj}(y_1) = -\text{proj}(y_2)$, applying Lemma 2.5.3, we get $p \in \text{co}\{\varphi_S(y_1) \mid S \in \mathcal{S}(y_1)\} \cap \psi(y_1) \neq \emptyset$ and $y_1 \in \partial V_N$. Since $y_1 = -y_2 \in \partial W$, $y_1 \notin \text{int } V_S$ for all $S \in \mathcal{N}$. As was to be proved, y_1 satisfies the conclusion of Theorem 2.2.1. □

2.5.2 Proof of Lemma 2.2.1.

Suppose for some $x \in \partial W$ that $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$. Then there exists some non-negative λ_S for each $S \in \mathcal{S}(x)$ such that:

$\sum_{S \in \mathcal{S}(x)} \lambda_S = 1$, $x = p_W(p_N(x))$ and $m^N - \sum_{S \in \mathcal{S}(x)} \lambda_S m_S = \tilde{\eta}(p_W(p_N(x)))$ $(*)$.

To end the proof, we show that : $\eta^* := \tilde{\eta}(x) = 0$. Putting $M := \{m \in N \mid \eta^*_m = \max_{i\in N} \eta^*_i\}$, we only need to show that $\sum_{i\in M} \eta^*_i \leq 0$ since $\eta^* = \tilde{\eta}(x) \in \mathbf{1}_\perp$. If $i \in M$ and $j \in N \setminus M$, that is $\eta^*_j < \eta^*_i$, then, from $(*)$, there exists $R \in \mathcal{S}(x)$ such that $j \in R$ and $i \notin R$. Therefore, $c_{ij}(x) = 0$ from the definition of the mapping c_{ij}. Using the following argument extracted from (16), we get, denoting by $t(x) = 1/(|N|(\eta_*(x)+1))$:[12]

$$\begin{aligned}\sum_{i\in M} \eta^*_i &= \sum_{i\in M} \tilde{\eta}_i(x) = t(x) \sum_{i\in M} \sum_{j\in N} (c_{ij}(x) - c_{ji}(x)) \\ &= t(x)(\sum_{i\in M} \sum_{j\in M} (c_{ij}(x) - c_{ji}(x)) + \sum_{i\in M} \sum_{j\in N\setminus M} (c_{ij}(x) - c_{ji}(x))) \\ &= 0 + t(x) \sum_{i\in M} \sum_{j\in N\setminus M} (-c_{ji}(x)) \leq 0.\end{aligned}$$

Consequently $\eta^* = 0$. □

2.5.3 Proof of Theorem 2.3.1

We first introduce some uniform bounds with respect to the parameter set. From the lower semi-continuity and closed graph assumptions of the set valued $V_{\{i\}}$, it is immediate to see that the mappings v_i, $i \in N$, are continuous. Let us denote $v = \min\{v_i(\theta) \mid \theta \in \Theta,\ i \in N\}\mathbf{1}$. The bound $m(\theta)$ given in Assumption PH2 can also be chosen continuously since the set-valued mapping V_S, $S \in \mathcal{N}$, are lower semi-continuous with closed graph. Let $m = \max\{m(\theta) \mid \theta \in \Theta\}$. These elements exist since Θ is compact.

We define the set-valued mapping Y_2 from Θ into $\mathbb{R}^N$ by:

$$Y_2(\theta) = -\text{int } (W(\theta)^c).$$

Note that Y_2 is lower semi-continuous with a closed graph, and, for all $\theta \in \Theta$, $Y_2(\theta) - \mathbb{R}^N_+ = Y_2(\theta)$ and $Y_2(\theta) \neq \mathbb{R}^N$. Let $\tilde{\varphi}_2$ be the set-valued mapping from Gr ∂Y_2 into Σ defined by:

$$\tilde{\varphi}_2(\theta, y_2) = \text{co}\left\{\varphi_S(\theta, -y_2) \mid S \in \mathcal{S}_\theta(-y_2)\right\}.$$

Lemma 2.5.5 *Let $(\theta, y_2) \in Gr\ \partial Y_2$, if $y_{2i} < -m$ for some $i \in N$, then $p_i = 0$ for all $p \in \tilde{\varphi}_2(\theta, y_2)$.*

Proof of Lemma 2.5.5. We apply Lemma 2.5.1 to the set-valued mappings $\tilde{\varphi}_2(\theta, .)$ and the set $Y_2(\theta)$. □

Since $Y_2(\theta)$ is uniformly bounded above by $-v$, there exists ρ such that $\text{proj}(\theta, y_2) \in \bar{B}_{\mathbf{1}^\perp}(0, \rho)$ for all $(\theta, y_2) \in \text{Gr } \partial Y_2$ such that $y_2 \in (\{-me\} + \mathbb{R}^N_+)$. Let us define the set-valued mapping Y_1 from Θ into $\mathbb{R}^N$ by:

$$Y_1(\theta) = \left\{p_V(s, \theta) \mid s \in \bar{B}_{\mathbf{1}^\perp}(0, \rho)\right\} - \mathbb{R}^N_+.$$

Since p_V is continuous, note that Y_1 is lower semi-continuous with a closed graph, and, for all $\theta \in \Theta$, $Y_1(\theta) - \mathbb{R}^N_+ = Y_1(\theta)$ and $Y_1(\theta) \neq \mathbb{R}^N$. Then, the compacity of Θ implies the existence of two real numbers α_1 and β_1 such that for all $y_1 \in \{z_1 \in \partial Y_1(\theta) \mid \ \|\text{proj}(z_1)\| \leq \rho,\ \theta \in \Theta\}$,

[12] They provide a new proof of an extension of the KKMS theorem proposed by (26), which was also exhibiting the partnership property.

$\alpha_1\mathbf{1} \le y_1 \le \beta_1\mathbf{1}$. Note also that, for all $\theta \in \Theta$, for all $y_1 \in \partial Y_1(\theta)$, if $\|\text{proj}(y_1)\| \le \rho$, then $y_1 \in \partial V(\theta)$. Let us choose $\underline{y}' \in \text{int } Y_1(\theta)$ for all $\theta \in \Theta$. Such element exists since every element strictly inferior to $\alpha_1\mathbf{1}$ satisfies this condition.

Lemma 2.5.6 *There exists a continuous mapping c from Gr ∂Y_1 to Σ_{++} such that $c(\theta, y_1) \cdot (y_1 - \underline{y}') \ge 0$ for all $(\theta, y_1) \in$ Gr ∂Y_1.*

Proof of Lemma 2.5.6. Define a mapping Γ' on Gr ∂Y_1 such that $\Gamma'(\theta, y_1) = \{p \in \Sigma_{++} \mid p \cdot (y_1 - \underline{y}') > 0\}$ and use the arguments given in the proof of Lemma 2.5.2. □

Let $\tilde{\varphi}_1$ be the set-valued mapping from Gr ∂Y_1 into Σ defined by:

$$\tilde{\varphi}_1(\theta, y_1) = \begin{cases} \psi(\theta, y_1) & \text{if } \|\text{proj}(y_1)\| < \rho \\ \text{co}\{\psi(\theta, y_1), c(\theta, y_1)\} & \text{if } \|\text{proj}(y_1)\| = \rho \\ c(\theta, y_1) & \text{if } \|\text{proj}(y_1)\| > \rho \end{cases}$$

Lemma 2.5.7 *There exists $\alpha \in \mathbb{R}$ such that, for all $(\theta, y_1, y_2) \in$ Gr $(\partial Y_1 \times \partial Y_2)$, $(p_1, p_2) \in \tilde{\varphi}(\theta, y_1) \times \tilde{\varphi}_2(\theta, y_2)$, one has $p_1 \cdot y_1 + p_2 \cdot y_2 \ge \alpha$.*

Proof of Lemma 2.5.7. For all $(\theta, y_2) \in$ Gr ∂Y_2, for all $p \in \tilde{\varphi}_2(\theta, y_2)$, from Lemma 2.5.5, one has $p \cdot y_2 \ge \sum_{i \in N, y_{2i} \ge -m} p_i y_{2i} \ge -m \sum_{i \in N, y_{2i} \ge -m} p_i = -m \sum_{i \in N} p_i = -m$ since $p \in \Sigma$. Secondly, for all $(\theta, y_1) \in$ Gr ∂Y_1, $p \in \tilde{\varphi}_1(\theta, y_1)$, if $\|\text{proj}(y_1)\| \le \rho$ then $p \cdot y_1 \ge p \cdot \alpha_1\mathbf{1} = \alpha_1$; if $\|\text{proj}(y_1)\| > \rho$, from Lemma 2.5.6, $p \cdot y_1 = c(\theta, y_1) \cdot y_1 \ge c(\theta, y_1) \cdot \underline{y}' \ge \min\{q \cdot \underline{y}' \mid q \in \Sigma\}$. Hence $p \cdot y_1$ is bounded below, which proves the result. □

Since the values of Y_1 and Y_2 are respectively uniformly bounded above by $\beta_1\mathbf{1}$ and $-v$, there exists a convex and compact set $\bar{B} \in (\mathbf{1}^\perp)^2$ such that: $B(0, \rho) \times B(0, \rho) \subset \bar{B}$ and for all $(\theta, y_1, y_2) \in$ Gr $(\partial Y_1 \times \partial Y_2)$ such that $y_1 + y_2 - \alpha\mathbf{1} \in \mathbb{R}^N_{++} \cup \{0\}$, $(\text{proj}(y_1), \text{proj}(y_2)) \in \text{int } \bar{B}$.

Finally, using again Bonnisseau (3, Lemma 3.1 p.217), one introduces the continuous mappings λ_1 and λ_2 from $\Theta \times \mathbf{1}^\perp$ to $\mathbb{R}$ associated to Y_1 and Y_2. We fix $\eta > 0$ arbitrary, let Σ_η be the set $\left\{p \in \mathbb{R}^N \mid \sum_{i \in N} p_i = 1;\ p_i \ge -\eta,\ i \in N\right\}$.

Let F be the set-valued mapping from $\Theta \times B \times \Sigma_\eta \times \Sigma^2$ into itself. $F = \prod_{j=1}^4 F_j$.

$F_1(\theta, (s_1, s_2), p, (p_1, p_2)) = G(\theta, y_1)$

$F_2(\theta, (s_1, s_2), p, (p_1, p_2)) = \{\sigma \in B \mid \sum_{i=1}^2 (p - p_i) \cdot \sigma_i \ge \sum_{i=1}^2 (p - p_i) \cdot \sigma'_i,\ \forall \sigma' \in B\}$

$F_3(\theta, (s_1, s_2), p, (p_1, p_2)) = \{q \in \Sigma_\eta \mid (q - q') \cdot (y_1 + y_2) \le 0,\ \forall q' \in \Sigma_\eta\}$

$F_4(\theta, (s_1, s_2), p, (p_1, p_2)) = (\tilde{\varphi}_1(\theta, y_1), \tilde{\varphi}_2(\theta, y_2))$

where for $i = 1; 2$, $y_i = s_i - \lambda_i(\theta, s_i)\mathbf{1}$.

Lemma 2.5.8 *The mapping F satisfies Kakutani's fixed point theorem conditions.*

Proof of Lemma 2.5.8. F is a set valued mapping from a non-empty, convex, compact set into itself. Actually, it suffices to verify for F_4 that the assumptions of Kakutani's fixed point theorem are satisfied since the others components meet obviously the expected conditions.

By construction, $\tilde{\varphi}_1$ is a non-empty, convex valued and upper semi-continuous. $\tilde{\varphi}_2$ is obviously convex valued and it has non-empty values since $(\theta, y_2) \in \mathrm{Gr}\ \partial Y_2$ implies that $-y_2 \in \partial V_S(\theta)$ for at least one $S \in \mathcal{N}$. Since Σ is compact, it suffices to show that the set-valued mapping $\tilde{\psi}_2$ defined by $\tilde{\psi}_2(\theta, y_2) = \cup_{S \in \mathcal{S}_\theta(-y_2)} \varphi_S(\theta, -y_2)$ has a closed graph in order to prove that $\tilde{\varphi}_2$ is upper semi-continuous. Let $(\theta^\nu, y^\nu, p^\nu)$ a sequence of $\mathrm{Gr}\ \partial Y_2 \times \Sigma$ which converges to (θ, y, p) and such that $p^\nu \in \tilde{\psi}_2(\theta^\nu, y^\nu)$ for all ν. From the definition of $\mathcal{S}$, for ν large enough, $\mathcal{S}_{\theta^\nu}(-y_2^\nu) \subset \mathcal{S}_\theta(-y_2)$. Indeed, it is not true, since $\mathcal{N}$ is a finite set, there exists $S \in \mathcal{N}$ and a subsequence (θ^ν, y^ν) such that for ν large enough $-y_2^\nu \in \partial V_S(\theta^\nu)$, $-y_2 \notin \partial V_S(\theta)$. Since V_S is a lower semi-continuous set-valued mapping with a closed graph and $V_S(\theta) - \mathbb{R}^N_+ = V_S(\theta)$, the set-valued mapping $\theta \to \partial V_S(\theta)$ has a closed graph. Since $-(y_2^\nu)$ converges to $-y_2$, one gets a contradiction. Consequently, for all ν large enough, there exists $S^\nu \in \mathcal{S}_\theta(-y_2)$ such that $p^\nu \in \varphi_{S^\nu}(\theta^\nu, -y_2^\nu)$. Since $\mathcal{S}_\theta(-y_2)$ is a finite set, there exists a subsequence such that S^ν is constant equal to S. Since φ_S is upper semi-continuous, this implies that $p \in \varphi_S(\theta, -y_2)$, which is included in $\tilde{\psi}_2(\theta, y_2)$ since $S \in \mathcal{S}_\theta(-y_2)$. $\square$

From the previous lemma, there exists $(\theta^*, (s_1^*, s_2^*), p^*, (p_1^*, p_2^*))$ such that, if, for $i = 1; 2$, $y_i^* = s_i^* - \lambda_i(\theta^*, s_i^*)\mathbf{1}$:

$$\theta^* \in G(\theta^*, y_1^*) \tag{2.1}$$

$$(s_1^*, s_2^*) = (\mathrm{proj}(y_1^*), \mathrm{proj}(y_2^*)) \text{ and } (y_1^*, y_2^*) \in \partial Y_1(\theta^*) \times \partial Y_2(\theta^*) \tag{2.2}$$

$$\sum_{i=1}^{2} (p^* - p_i^*) \cdot s_i^* \geq \sum_{i=1}^{2} (p^* - p_i^*) \cdot \sigma_i' \text{ for each } \sigma' \in B \tag{2.3}$$

$$(p^* - q') \cdot (y_1^* + y_2^*) \leq 0 \text{ for each } q' \in \Sigma_\eta \tag{2.4}$$

$$(p_1^*, p_2^*) \in (\tilde{\varphi}_1(\theta^*, y_1^*), \tilde{\varphi}_2(\theta^*, y_2^*)) \tag{2.5}$$

We now exhibit from the above equations an element satisfying the conclusion of Theorem2.3.1.[13] Let $\gamma^* = -p^* \cdot (y_1^* + y_2^*)$, remark that $p^* \cdot (y_1^* + y_2^* + \gamma^* e) = 0$ and $\gamma^* \leq -\alpha$. Indeed, $p^* \cdot \sum_{i=1}^2 y_i^* = p^* \cdot (\sum_{i=1}^2 s_i^* - \lambda_i(\theta^*, s_i^*)e)$. From (3) with $\sigma' = 0$, one gets: $p^* \cdot \sum_{i=1}^2 y_i^* \geq \sum_{i=1}^2 p_i^* \cdot s_i^* - \lambda_i(\theta^*, s_i^*) = \sum_{i=1}^2 p_i^* \cdot y_i^* \geq \alpha$ from Lemma 2.5.7.

From (4), for each $q' \in S_\eta$, $q' \cdot (y_1^* + y_2^* + \gamma^* \mathbf{1}) = q' \cdot (y_1^* + y_2^*) + \gamma^* \geq p^* \cdot (y_1^* + y_2^*) + \gamma^* = 0$. Therefore, $y_1^* + y_2^* + \gamma^* \mathbf{1} \in \{0\} \cup \mathbb{R}^N_{++}$ and it follows that $(s_j^*) \in \mathrm{int}\ \bar{B}$ by construction of the set $\bar{B}$. Then $p^* = p_1^* = p_2^* \in \Sigma$, from (3), since the maximum of a linear function is interior only if it is a null mapping. $p^* \in \Sigma$ implies $y_1^* + y_2^* + \gamma^* \mathbf{1} = 0$. From (2) that means $s_1^* = \mathrm{proj}(y_1^*) = -\mathrm{proj}(y_2^*) = -s_2^*$.

It remains to show that $y_1^* \in \partial V(\theta^*)$ and $p^* \in \psi(\theta^*, y_1^*)$. The argument is exactly the same as the one in the proof of Lemma 2.5.3.

Let $\xi^* = -y_2^*$ and $x^* = y_1^*$. It implies that $\xi^* \in \partial W(\theta^*)$ and therefore, from $x^* - \xi^* + \gamma^* \mathbf{1} = 0$, it follows that $p_W(\theta^*, x^*) = \xi^*$, or, equivalently, $p_V(\theta^*, \xi^*) = x^*$. From (5), $p^* \in \psi(\theta^*, p_V(\theta^*, \xi^*)) \cap \mathrm{co}\{\varphi_S(\theta^*, \xi^*) \mid S \in \mathcal{S}_{\theta^*}(\xi^*)\}$. So we deduce from the condition of payoff-dependent balancedness

[13] Bonnisseau (3) used a similar argument to show the existence of a general equilibrium with externalities.

that $\xi^* \in V(\theta^*)$. Since $\xi^* \in \partial W(\theta^*) \cap V(\theta^*)$, $x^* = p_V(\theta^*, \xi^*) \in \partial V(\theta^*) \setminus \text{int } W(\theta^*)$. Using (1), one can say that (θ^*, x^*) is an equilibrium-core allocation, and, $p^* \in \psi(\theta^*, x^*) \cap \text{co}\{\varphi_S(\theta^*, p_W(\theta^*, x^*)) \mid S \in \mathcal{S}_{\theta^*}(p_W(\theta^*, x^*))\}$ as was to be proved. □

2.5.4 *Proof of Corollary 2.3.2.*

Assumption PH0: G is convex valued from the quasi-concavity of v_i with respect to the first variable, non-empty valued since from the definition of V_S, for all $\theta \in \Theta$, for all $x \in \partial V_N(\theta)$, there exists $\theta' \in \Theta$ such that $x \leq (v_i(\theta'_i, \theta_i))_i$. G is clearly an upper semi-continuous set-valued mapping from the continuity of the mappings v_i and p_N.

Assumption PH1: Since F^i is non-empty valued for all $i \in N$ and from the balancedness Assumption, taking the balanced family $(\{i\}, i \in N)$ one can prove the non-emptiness of Θ. Now, the lower-semi continuity and closed graph assumption of the set-valued mappings V_S, $S \in \mathcal{N}$, are proved.

(*l.s.c.*) For all $\theta^\nu \in \Theta$ a sequence converging to $\theta \in \Theta$, we show that, for all $x \in V_S(\theta)$, there exists a sequence (x^ν) converging to x with $x^\nu \in V_S(\theta^\nu)$ for ν large enough. Since $x \in V_S(\theta)$, there exists $\theta' \in F^S(\theta)$ such that $x_i \leq v_i(\theta'_i, \theta_i)$, $i \in S$. Since F^S is lower semi-continuous, there exists a sequence (θ'^ν) converging to θ' with $\theta'^\nu \in F^S(\theta^\nu)$ for ν large enough. Then, from the continuity of the mapping v_i, one has $v_i(\theta'^\nu_i, \theta^\nu_i)$ tends to $v_i(\theta'_i, \theta_i)$, $i \in S$. Let T be a subset of S such that for each $i \in T$ one has $x_i = v_i(\theta'_i, \theta_i)$. Now, it suffices to take $x^\nu_i = v_i(\theta'^\nu_i, \theta^\nu)$, $i \in T$, and $x^\nu_i = x_i$, $i \in S \setminus T$. This ends the proof.

(*closed graph*) Let (θ^ν) be a sequence converging to θ, and show that if $x^\nu \in V_S(\theta^\nu)$ converges to $x \in \mathbb{R}^N$, then $x \in V_S(\theta)$. For all $\nu \geq 0$, there exists $\theta'^\nu \in F^S(\theta^\nu)$ such that $x^\nu_i \leq v_i(\theta'^\nu_i, \theta^\nu)$. Since F^S is upper-semi continuous with compact values, $F^S(\Theta)$ is compact. Then, the sequence (θ'^ν) remains in a compact. So taking a subsequence if we need to, one can say that (θ'^ν) tends to an element $\theta \in F^S(\theta)$. Taking the limit and from the continuity of the mappings v_i, one gets $\theta' \in F^S(\theta)$ such that $x_i \leq v_i(\theta'_i, \theta)$ for all $i \in S$, that is to say that $x \in V_S(\theta)$, as was to be proved.

Remark now, that from a well known argument relying on the quasi-concavity of the functions $v_i(., \xi_i)$, the balancedness condition given in (5) is equivalent the balancedness of the game $(V(\theta), V_S(\theta)_{S \in \mathcal{N}})$ for each $\theta \in \Theta$, that is, for each balanced family $\mathcal{B}$, $\cap_{S \in \mathcal{B}} V_S(\theta) \subset V(\theta)$. This fact is used in the two paragraphs below.

We check that Assumption PH2 holds true. Let $S \in \mathcal{N}$. The family $\{S, (\{i\})_{i \notin S}\}$ is a balanced family. Let $(\theta, x) \in \text{Gr } V_S$, since $F^i(\theta)$ is non-empty, there exist $\xi^i \in F^i(\theta)$, $i \notin S$. Let x' be defined by $x'_i = x_i$, $i \in S$ and $x'_i = v_i(\xi^i, \theta_i)$, $i \notin S$. Clearly, $x' \in V_S(\theta) \cap (\cap_{i \notin S} V_{\{i\}}(\theta))$. From the balancedness of the game, $x' \in V(\theta)$. Consequently, from the compactness of Θ and the continuity of v_i, $i \in N$, there exists $m(\theta)$ such that $x' \leq m(\theta)\mathbf{1}$, hence $x_i \leq m(\theta)$, $i \in S$.

Finally, the parameterized game is payoff-dependent balanced since the game is balanced (see the argument in the proof of Corollary 2.2.1). □

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3. RÈGLES DE TAUX DE TRANSFERT ET SÉLECTIONS DU CœUR

Résumé

On propose quelques applications directes d'un résultat d'existence de Bonnisseau et Iehlé (2003). Ces auteurs ont montré l'existence d'allocations du cœur dans les jeux NTU qui satisfont un équilibre de taux de transfert sous une condition de balancement dépendant. Il s'avère que la notion de balancement dépendant procure en fait un outil manipulable pour sélectionner le cœur. Pour illustrer ce fait, nous montrons que cette notion permet d'obtenir des résultats d'existence dans des modèles de cœur avec partenariat, cœur socialement stable, prekernel moyen intersecté avec le cœur et de cœur interne faible.

Transfer rate rules and core selections in NTU games [0]

Vincent Iehlé

Abstract. Different kinds of asymmetries between players can occur in core allocations, in that case the stability of the concept is questioned. One remedy consists in selecting robust core allocations. We review, in this note, results that all select core allocations in NTU games with different concepts of robustness. Within a unified approach, we deduce the existence of allocations in: the partnered core, the social stable core, the core intersected with average prekernel, the weak inner core. We use a recent contribution of Bonnisseau and Iehlé (2003) that states the existence of core allocations with a transfer rate rule equilibrium under a dependent balancedness assumption. It shall turn out to be manipulable tools for selecting the core.*Journal of Economic Literature* Classification Numbers: C60, C71.

Keywords: cooperative games, balancedness, non-emptiness, core allocation with transfer rate rule equilibrium.

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3.1 Introduction

The core of a cooperative game is the set of efficient payoffs for the grand coalition that cannot be improved by any coalition of players. One critic arising from the core concept in cooperative NTU games is that some core allocations may exhibit asymmetric dependencies (Reny and Wooders (13)) or inefficiencies if players agree for a transfer of utility (Qin (11; 12)).[1] For example, at a core allocation, some players could contribute more than others. Then, the stability of the concept is questioned since the best contributors are likely to receive rewards for their participation. A remedy consists in selecting the core, that is, defining a criterion to find a utility vector that each member of the grand coalition finds acceptable. Hence, by selecting the core, we mean prescribing specific core allocations satisfying division schemes or stable matchings that are robust with respect to asymmetries or inefficiencies. We review, in this note, non-emptiness results with different concerns and frameworks that all selects specific core allocations in NTU games.

We propose a unified treatment based on a recent notion of balancedness with a transfer rate rule, generalizing the extant notions, and called dependent balancedness. The idea behind the notion is to consider a transfer rate rule depending on the payoffs to define a notion of balancedness whereas the usual transfer rate of the literature is supposed to be constant. It turns out that the class of dependent balancedness games is exactly the class of games with non empty cores.[2]

Going beyond the non-emptiness of the core, it is also proved in Bonnisseau and Iehlé (3) that dependent balancedness is a sufficient condition to get the existence of core allocations with a transfer rate rule equilibrium. All the following selections of the core will coincide with a core allocation with a transfer rate rule equilibrium. For instance, the authors deduce from their existence result the non-emptiness of the partnered core of Reny and Wooders (13). This specific core selection is the set of core allocations such that, for any pair players i, j, if the player i cannot achieve her core payoff without player j then player j cannot either achieve her core payoff without player j.

As further applications, three other results will illustrate the role of the transfer rate rule equilibrium, giving hints on its manipulation.

First, we turn to the non-emptiness of a social stable core. To define such a concept, Herings et al. (5) introduce a power index for each players in the coalitions. And then, they prove the non-emptiness of the set of equipotent allocations in the core: the social stable core.

The second core selection is the average prekernel intersected with the core. The prekernel is the NTU extension of the usual notion of prekernel at stake in TU games. Though no interpretation is attached to the prekernel, it can be seen as a fair sharing allocation with respect to a surplus measure of the players. We improve the existence result originally given in Orshan et al. (8), by considering the class of ∂-balanced games.

[1] See also Bennett (2) and Bennett and Zame (1) for further developments on conflicts over gains from cooperation.
[2] See details in (3; 10).

Lastly, we propose an existence result for a core allocation in the spirit of the original notion of inner core of Qin (11; 12). x is in the inner core if it is feasible for the grand coalition, and there exists a transfer rate λ such that x is in the core of the λ-transfer game. The inner core is included in the core, hence it can be seen as a selection of the core. We exhibit a result for the non-emptiness of a weak inner core.

To deduce these results as corollaries of the abstract result of Bonnisseau and Iehlé (3), we construct explicitly a transfer rate rule for which the games are dependent balanced.

3.1.1 Game description and the general result

We will use the following notations:[3] $N = \{1, ..., n\}$ is the finite set of players; $\mathcal{N}$ is the set of the non-empty subsets of N, i.e. the coalitions of players; for each $S \in \mathcal{N}$, L_S is the $|S|$-dimensional subspace of $\mathbb{R}^N$ defined by $L_S = \{x \in \mathbb{R}^N \mid x_i = 0,\ \forall i \notin S\}$; L_{S+} (L_{S++}) is the non negative orthant (positive orthant) of L_S; for each $x \in \mathbb{R}^N$, x^S is the projection of x into L_S; $\mathbf{1}$ is the vector of $\mathbb{R}^N$ whose coordinates are equal to 1; $\mathbf{1}^\perp$ is the hyperplane $\{s \in \mathbb{R}^N \mid \sum_{i \in N} s_i = 0\}$; proj is the orthogonal projection mapping on $\mathbf{1}^\perp$; $\Sigma_S = \text{co}\{\mathbf{1}^{\{i\}} \mid i \in S\}$; $m^S = \frac{\mathbf{1}^S}{|S|}$; $\Sigma = \Sigma_N$ and $\Sigma_{++} = \Sigma \cap \mathbb{R}^N_{++}$.

A game $(V_S, S \in \mathcal{N})$ is a collection of subsets of $\mathbb{R}^N$ indexed by $\mathcal{N}$. $x \in \mathbb{R}^N$ is called a payoff; $V_S \subset \mathbb{R}^N$ is the set of feasible payoffs of the coalition S; $\mathcal{S}(x) = \{S \in \mathcal{N} \mid x \in \partial V_S\}$ is the set of coalitions, for which $x \in \mathbb{R}^N$ is an efficient payoff; $W := \cup_{S \in \mathcal{N}} V_S$ is the union of the payoffs sets.

We will assume in the remainder of the paper that the two following assumptions are satisfied.

- *(H1)* (i) $V_{\{i\}}$, $i \in N$, and V_N are non-empty. (ii) For each $S \in \mathcal{N}$, V_S is closed, $V_S - \mathbb{R}^N_+ = V_S$, $V_S \neq \mathbb{R}^N$, and, for all $(x, x') \in (\mathbb{R}^N)^2$, if $x \in V_S$ and $x^S = x'^S$, then $x' \in V_S$.
- *(H2)* There exists $m \in \mathbb{R}$ such that, for each $S \in \mathcal{N}$, for each $x \in V_S$, if $x \notin \text{int } V_{\{i\}}$ for all $i \in S$, then $x_j \leq m$ for all $j \in S$.

Note that under Assumption H1, there exist continuous mappings p_N from $\mathbb{R}^N$ to ∂V_N, p_W from $\mathbb{R}^N$ to ∂W, λ_N and λ_W from $\mathbf{1}^\perp$ to $\mathbb{R}$ such that, for all $x \in \mathbb{R}^N$, $p_N(x) = \text{proj}(x) - \lambda_N(\text{proj}(x))\mathbf{1}$ and $p_W(x) = \text{proj}(x) - \lambda_W(\text{proj}(x))\mathbf{1}$.[4] Let us recall now the definitions of core and dependent balancedness, and, the main result obtained by Bonnisseau and Iehlé (3).

Definition 3.1.1 Let $(V_S, S \in \mathcal{N})$ be a game. A payoff x is in the core of the game if $x \in V_N \setminus \text{int } W$.

[3] For any set $Y \subset \mathbb{R}^N$, co(Y), ∂Y, int Y will denote respectively its convex hull, boundary, interior. For any set-valued mapping Γ, Gr Γ will denote its graph.

[4] See (3) for more details.

Definition 3.1.2 Let $(V_S, S \in \mathcal{N})$ be a game satisfying Assumption H1: (i) A transfer rate rule is a collection of set-valued mappings $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ such that for all $S \in \mathcal{N}$, φ_S is upper semi-continuous with non-empty compact and convex values from ∂V_S to Σ_S, and, ψ is upper semi-continuous with non-empty compact and convex values from ∂V_N to Σ. (ii) The game $(V_S, S \in \mathcal{N})$ is dependent balanced if there exists a transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ such that, for each $x \in \partial W$, if $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(p_N(x)) \neq \emptyset$, then $x \in V_N$.

Theorem 3.1.1 *Let $(V_S, S \in \mathcal{N})$ be a game satisfying Assumptions H1 and H2. If it is dependent balanced with respect to the transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$, there exists a core allocation x such that:* $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(x) \neq \emptyset$.

3.2 Applications: four selections of the core

Theorem 3.1.1 downsizes the core into specific core allocations with transfer rate rule equilibrium. In the following applications, the stake is to define indexes for contribution, power or transfer and to prescribe an allocation in the core of the game satisfying a division scheme with respect to these indexes. Thanks to Theorem 3.1.1, we unify different models where such a prescription is proposed. The following results are all deduced as corollaries, the proofs are given in Appendix.

3.2.1 Partnered core

To get a first application of this result, consider the following corollary of Theorem 3.1.1, due to Reny and Wooders (13) and already proved in Bonnisseau and Iehlé (3). We recall before the notion of ∂-balancedness.

> *∂-balancedness* The game is ∂-balanced if for all $x \in \partial W$ and any balanced family of coalitions $\mathcal{B} \subset \mathcal{N}$ such that $x \in \cap_{S\in\mathcal{B}} V_S$ then $x \in V_N$.

Corollary 3.2.1 *Let $(V_S, S \in \mathcal{N})$ be a ∂-balanced game satisfying Assumptions H1 and H2.*[5] *Suppose that for each pair of players i and j, there is a continuous mapping $c_{ij}\colon \partial W \to \mathbb{R}_+$ such that c_{ij} is zero on $V(S) \cap \partial W$ whenever $i \notin S$ and $j \in S$. Then there exists a core allocation x such that, for each $i \in N$:*

$$\eta_i(x) := \sum_{j\in N} (c_{ij}(x) - c_{ji}(x)) = 0.$$

The mappings c_{ij} can be interpreted as credit/debit mappings. Then, one can see $\eta_i(x)$ as the measure of the grand coalition's net indebtness to i or as i's net credit against the grand coalition. The previous result states the existence of a core allocation where the net credits of the players

[5] In Reny and Wooders (13), the result is stated for balanced games, it is slightly improved by considering ∂-balanced games.

are all equal to 0. The result of Reny and Wooders(13) has been originally applied to a stable matching problem. They state indeed that any balanced game has a non-empty partnered core, which is the set of core allocations such that, for any pair players i, j, if the player i cannot achieve her core payoff without player j then player j cannot either achieve her core payoff without player j. Formally, a payoff $x \in \partial W$ is said to be partnered if the family $\mathcal{S}(x)$ satisfies, for all $i, j \in N$, $\mathcal{S}_i(x) \subset \mathcal{S}_j(x) \Rightarrow \mathcal{S}_j(x) \subset \mathcal{S}_i(x)$, where $\mathcal{S}_i(x) = \{S \in \mathcal{S}(x) \mid i \in S\}$, and the partnered core is the set of all partnered core allocations.

3.2.2 Social stable core

In Herings et al. (5), the authors propose a generalization of NTU games. Firstly, they assume the possibility of internal organizations, that is, inside a given coalition, the members can choose among several possibilities of organization, it give rise to a multiplicity of possible payoffs sets for the given coalition.[6] Secondly, a power mapping that describes the power of agents inside each organization is introduced. Under a balancedness condition, it is shown that there exists an allocation lying in the core of the generalized NTU game and such that the agents are equally powerful.

For each coalition $S \in \mathcal{N}$, there is a finite number k_S of possible internal organizations. Denote $I^S = (I_1^S ... I_{k_S}^S)$ these organizations. Let $\mathcal{I}$ be the union over S of all internal organizations. For each $S \in \mathcal{N}$, each $I \in I^S$, define a payoff set $v_I \in \mathbb{R}^N$. Now, define the power of an agent within an internal organization by a power vector function p from $\mathcal{I}$ to $\mathbb{R}_+^N \setminus \{0\}$. For each $S \in \mathcal{N}$, each $I \in I^S$, $p(I) \in L_{S+} \setminus \{0\}$. A socially structured game is described by $(N, \mathcal{I}, v, p)$. In Herings et al. (5), the authors restate Assumptions H1 and H2 for this generalized game. We omit their statements, it is an easy matter to check that it amounts to consider that the game $(V_S, S \in \mathcal{N})$, where for each $S \in \mathcal{N}$, $V_S = \cup_{I \in I^S} v_I$, satisfies Assumption H1 and H2. Define the power cone of a payoff x as: $PC(x) = \{y \in \mathbb{R}^N \mid y = \sum_{I \in \mathcal{I}(x)} \lambda_I p(I), \lambda_I \geq 0, \text{ for all } I\}$, where $\mathcal{I}(x) = \{I \in \mathcal{I} \mid x \in \partial v_I\}$.

Definition 3.2.1 For a socially structured game, $(N, \mathcal{I}, v, p)$, a payoff vector $x \in \mathbb{R}^N$ is socially stable if:

$$\mathbf{1} \in PC(x).$$

A core allocation is a payoff vector $x \in \mathbb{R}^N$ such that $x \in v_I$ for some $I \in I^N$ and $x \notin \text{int } v_{\overline{I}}$ for all $\overline{I} \in \mathcal{I}$. A socially stable core is the set of socially stable core allocations.

(SSG) If a payoff vector x is socially stable then $x \in v_I$ for some $I \in I^N$.[7]

We deduce the following result given in Herings et al. (5).

[6] The reader can imagine that possibilities of a coalition are described by special pairwise links between its members that give rise to different networks, (e.g. see networks formation in Jackson (6)).

[7] This balancedness notion is sandwiched between the notion of b-balancedness of Billera and that of dependent balancedness.

Corollary 3.2.2 *Let $(N, \mathcal{I}, v, p)$ be a socially structured game and suppose that $(V_S, S \in \mathcal{N})$, where for each $S \in \mathcal{N}$, $V_S = \cup_{I \in I^S} v_I$, satisfies Assumption H1 and H2. Under SSG, the socially stable core is non-empty.*

To prove the result, we will consider the transfer rate rule $((\varphi_S)_{S \in \mathcal{N}}, \psi)$ where: $\psi = m^N$, and, for all $S \in \mathcal{N}$ and all $x \in \partial V_S$, $\varphi_S(x) = \text{co}\{\frac{p(I)}{\sum_{i \in S} p_i(I)} \mid I \in \mathcal{I}(x) \cap I^S\}$.

It is an easy matter to extend the result to parameterized games. It suffices to apply Theorem 3.1 of Bonnisseau and Iehlé (3). Furthermore, the parameterization could allow us to define a sharper model of internal organization, continuously depending on the parameter set.

3.2.3 Average prekernel

As another application, one can also prove the existence of an element lying in the core intersected with the average prekernel (also called bilateral consistent prekernel) as defined in Orshan and Zarzuelo (9), see also Serrano and Shimomura (14). The average prekernel is the consistent extension of the usual prekernel at stake in TU games. Furthermore, the most interesting feature is that the following concept for multi-player games coincides with the Nash solution and intersects the core in a general class of games. We show how we can deduce this existence result under an assumption of balancedness.

Define additionally, for each coalition, the set of individually rational payoffs, $I_S = V_S \cap (\cap_{i \in S} (\text{int } V_{\{i\}})^c)$. Before introducing the average prekernel, we need two additional assumptions on the game, namely non-levelness (NL) and smoothness (SM).

- *(NL)* For each $S \in \mathcal{N}$, ∂V_S is non-leveled, that is: if $x, y \in \partial V_S$, $x \geq y$ and $y \in I_S$, then $x_i = y_i$.
- *(SM)* At each point $x \in \partial I_N$, there exists a unique vector $p(x)$ such that $\sum_{i \in N} p_i(x) = 1$. Moreover, for all $x \in \partial I_N$, $p(x) > 0$ and p is a continuous map.

Let us now define the individual excess functions, bilateral surplus functions and total loss functions as follows:

For each $x \in \mathbb{R}^N$, for each $S \in \mathcal{N}$, for $k \in S$, the individual excess of k with respect to S at x is :

$$e_k(S, x) = \begin{cases} \max\{y_k - x_k \mid (y_k, x_{-k}) \in V_S\} & \text{if } \{y_k \mid (y_k, x_{-k}) \in V_S\} \neq \emptyset \\ -\infty & \text{otherwise} \end{cases}$$

For every $k, \ell \in N$, $k \neq \ell$, define the surplus of k with respect to ℓ at x to be $s_{k\ell}(x) = \max\{e_k(S, x) \mid S \in \mathcal{N}, k \in S, l \notin S\}$.

For every $k \in N$ and $x \in \partial I_N$ denote $f_k(x) = \sum_{\ell \neq k} (p_k(x) s_{k\ell}(x) - p_\ell(x) s_{\ell k}(x))$ the total loss of player k at x. Let $f(x)$ be the vector $(f_1(x), ..., f_n(x))$.

Definition 3.2.2 The average prekernel of $(V_S, S \in \mathcal{N})$ is the set:

$$\{x \in \partial V_N \mid f(x) = 0\}.$$

In Orshan et al. (8), the authors have shown the non-emptiness of the core intersected with the average prekernel in ∂-separating games, here the result is improved by considering the larger class of ∂-balanced games.[8]

Corollary 3.2.3 *Let $(V_S, S \in \mathcal{N})$ be a game satisfying Assumptions H1, H2, NL and SM. If it is a ∂-balanced game then there exists a core allocation that belongs to the average prekernel.*

We will deduce the result from Corollary 3.2.1, but it amounts to consider the transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ where: $\psi(x) = m^N - \tilde{f}(x)$, ($\tilde{f}(x) = f(x)$ up to a normalization), and for all $S \in \mathcal{N}$, $\varphi_S = m^S$.

3.2.4 Inner core

The inner core is an alternative way of downsizing the core. The main results have been obtained by Qin (11; 12).

Let $(V_S, S \in \mathcal{N})$ be a game compactly generated. And let $\lambda \in \mathbb{R}^N_+$. Define a real valued set function v_λ on the set of all non-empty subsets of N by $v_\lambda(S) = \max\left\{\sum_{i\in S} \lambda_i \cdot y_i \mid y \in V_S\right\}$. Define the fictitious λ-transfer game $(V_S^\lambda, S \in \mathcal{N})$ associated with v_λ: for each $S \in \mathcal{N}$, the fictitious λ-transfer payoffs sets are: $V_S^\lambda = \left\{y \in \mathbb{R}^N \mid \sum_{i\in S} \lambda_i \cdot y_i \leq v_\lambda(S)\right\}$.

An allocation x is in the inner core of the game $(V_S, S \in \mathcal{N})$ if $x \in V_N$ and there exists at least one $\lambda \in \Sigma$ such that x is in the core of $(V_S^\lambda, S \in \mathcal{N})$. Note that the inner core is included in the core of $(V_S, S \in \mathcal{N})$. The requirements imposed in the definition of the inner core are very strong. However, Qin (12) proposes a class of balanced games for which the inner core is non-empty.[9] In the following, we relax the definition and deduce from Theorem 3.1.1 the non-emptiness of a weak inner core.

The interpretation of this weak concept of inner core differs from the initial inner core. We consider a group of players who agree for transfer rate rules within each coalition. Then a global transfer rate rule is prescribed, this rule must belong to the set of admissible transfer rates, defined below. At this prescribed rate λ, the players can transfer utility among themselves. The core allocation x is in the weak inner core if x is an efficient point in the fictitious λ-transfer payoff set of the grand coalition.

Formally, the transfer set induced by the transfer rate mappings is defined as follows: for each $x \in \partial W$, $TS(x) = \text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\}$. Then, $(\lambda, x) \in \text{Gr}\, TS$ means that λ is an admissible

[8] A ∂-separating game is ∂-balanced. See (8).

[9] The non-emptiness of the inner core is proved in games that cover the class of compactly generated and balanced-with-slacks games.

transfer rate at the point x. λ defines a fictitious transfer game and one is led to the following definition.

Definition 3.2.3 A pair $(\lambda \in \Sigma, x \in \partial W) \in \text{Gr } TS$ is said to be internally stable if:

$$(\lambda, x) \in \text{Gr } TS \text{ and } \lambda \cdot x \geq v_{\lambda}(N).$$

An allocation x is in the weak inner core of the game $(V_S, S \in \mathcal{N})$ if x belongs to the core of the game and there exists at least one $\lambda \in \Sigma$ such that (λ, x) is internally stable.

Suppose the players can transfer utility at a prescribed rate λ. The pair $(\lambda, x) \in \text{Gr } TS$ is not internally stable if each player can get a strictly better payoff in the fictitious λ-transfer payoff set V_N^{λ}. Denote N_{V_N} the normal cone of convex analysis of the set V_N.

Corollary 3.2.4 *Let $(V_S, S \in \mathcal{N})$ be a game satisfying Assumptions H1 and H2. Suppose also that V_N is a convex set. If it is dependent balanced with respect to the transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, N_{V_N} \cap \Sigma)$, then the weak inner core is non-empty.*

We omit the proof of the last result which is a direct application of Theorem 3.1.1. To apply Theorem 3.1.1, it suffices to notice that the transfer rate mapping $\psi = N_{V_N} \cap \Sigma$ satisfies the conditions of Definition 3.1.2(i).

3.3 A concluding remark

As a related topic, we want to mention that a recent stream of research defines the notion of extended core, see Gomez(4) and Keiding and Pankratova (7). The problematic is the following: in the case of an empty core, which feasible allocations should be considered as potential candidates for guaranteeing the stability? Briefly, the construction of the extended core consists in blowing up the feasible payoff set of the grand coalition to get at least an allocation in the core of the extended game. Then, two tools based on mechanisms, that either downsize/select the core or blow up the payoffs are now available. They provide a rich articulation around the core concept.

3.4 Appendix

Proof of Corollary 3.2.2. Let define the induced coalitional game $(V_S, S \in \mathcal{N})$ where for each $S \in \mathcal{N}$, $V_S = \cup_{I \in I^S} v_I$. Let us normalize the power mappings by setting $\overline{p}(I) = \frac{p(I)}{\sum_{i\in S} p_i(I)}$. Now, define the transfer rates rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$. For each $x \in \partial V_N$, $\psi(x) = m^N$ and for all $S \in \mathcal{N}$ and all $x \in \partial V_S$, $\varphi_S(x) = \text{co}\{\overline{p}(I) \mid I \in \mathcal{I}(x) \cap I^S\}$. It is now routine to check that the assumptions of Theorem 3.1.1 are all fulfilled (Assumption SSG is actually a special case of dependent balancedness).

Then there exists a core allocation (for the game $(V_S, S \in \mathcal{N})$) x such that $\text{co}\{\varphi_S(x) \mid S \in \mathcal{S}(x)\} \cap \psi(x) \neq \emptyset$. It can be rewritten as: there exist, for all $S \in \mathcal{S}(x)$, $\rho_S \in \mathbb{R}_+$, $b_S \in \varphi_S(x)$, such that $\sum_{S \in \mathcal{S}(x)} \rho_S = 1$ and $\sum_{S \in \mathcal{S}(x)} \rho_S b_S = m^N$. Furthermore, $b_S \in \varphi_S(x)$ is equivalent to: for each $I \in I^S$, there exist $\nu_I^S \in \mathbb{R}_+$ such that $\sum_{I \in \mathcal{I}(x) \cap I^S} \nu_I^S = 1$ and $b_S = \sum_{I \in \mathcal{I}(x) \cap I^S} \nu_I^S \overline{p}(I)$. If we consider for each $S \in \mathcal{N}$ and each $I \in I^S$, $\lambda_I = \frac{|N| \rho_S \nu_I^S}{\sum_{i \in S} p_i(I)}$, then one gets the result. Indeed, firstly we remark that:
$\sum_{S \in \mathcal{S}(x)} \rho_S \sum_{I \in \mathcal{I}(x) \cap I^S} \nu_I^S \overline{p}(I) = \sum_{I \in \mathcal{I}(x)} \sum_{\{S | I \in I^S\}} \rho_S \nu_I^S \overline{p}(I) = \sum_{I \in \mathcal{I}(x)} \frac{\lambda_I p(I)}{|N|} = m^N$, that is $\sum_{I \in \mathcal{I}(x)} \lambda_I p(I) = \mathbf{1}$. Secondly it is an easy matter to check that x is in the core of the game $(V_S, S \in \mathcal{N})$ if and only if x is in the core of the socially structured game $(N, \mathcal{I}, v, p)$ in the sense of Definition 2, as was to be proved. □

Proof of Corollary 3.2.3. First, we remark that, without any loss of generality, one can extend Assumption SM on the whole boundary of the set V_N since the core solution lies on the set of individually rational payoffs.

Lemma 3.4.1 *If the game satisfies the non levelness assumption (NL), the mappings $s_{k\ell}$ are non positive and continuous on ∂W.*

Proof of Lemma 3.4.1. First, remark that for each $x \in \partial W$, one has $x \in I_S$, so non-levelness applies. The non positivity is straightforward from Assumption H1. Furthermore, the mappings $s_{k\ell}$, $k, \ell \in N$, are well defined on ∂W (consider the coalition $T := \{k\}$). We now show the continuity of the mappings $s_{k\ell}$ which derives from Assumption NL. Let $k, \ell \in N$, $x \in \partial W$ and denote S^* the set of coalitions (satisfying $k \in S^*$ and $\ell \notin S^*$) maximizing $e_k(., x)$, i.e. $S^* := argmax\, \{e_k(S, x) \mid S \in \mathcal{N}, k \in S, \ell \notin S\}$; let x^ν be a sequence in ∂W converging toward x and denote, for each ν, S^ν the set of coalitions (satisfying $k \in S^\nu$ and $\ell \notin S^\nu$) maximizing $e_k(., x^\nu)$, i.e. $S^\nu := argmax\, \{e_k(S, x^\nu) \mid S \in \mathcal{N}, k \in S, \ell \notin S\}$. We first remark that, since S^* is a finite set there exists a real m such that for all $\nu \geq m$, $S^\nu \subseteq S^*$. Consider now some $T \in \mathcal{N}$ such that $T \in \mathcal{S}^\nu$ for each ν big enough (taking a subsequence if necessary), then from the definition of the mappings e_k, there exist two real numbers y_k and y_k^ν respectively solutions of $e_k(S, x)$ and $e_k(S^\nu, x^\nu)$ and satisfying $(y_k, x_{-k}) \in \partial V_T$ and $(y_k^\nu, x_{-k}^\nu) \in \partial V_T$. Now suppose we do not have the convergence, that is, there exists $\epsilon > 0$ such that $|y_k - y_k^\nu| > \epsilon$ for all ν sufficiently high. Then, taking the limit components by components, this contradicts assumption NL. Indeed, it implies that $(\lim_{\nu \to \infty} y_k^\nu, x_{-k})$ and (y_k, x_{-k}) belong to ∂V_T, but $|y_k - \lim_{\nu \to \infty} y_k^\nu| > \epsilon$, which is impossible. □

The mappings $s_{k\ell}$ are non positive on the boundary of the game ∂W and continuous from Lemma 3.4.1. Let $x \in \partial W \cap V_S$ for some $S \in \mathcal{N}$ with $j \in S$ and $i \notin S$, since: $argmax\{y_j - x_j \mid (y_j, x^{S \setminus \{j\}}) \in V_S\} = \{x_j\}$. We deduce that $s_{ji}(x) = 0$. Let $c_{ij}(x) := -p_j(p_W(x)) s_{ji}(x)$. Obviously, from the assumption SM which guarantees the positivity and continuity of the mapping p, we deduce that, for each pair of players i and j, the mapping c_{ij}: $\partial W \to \mathbb{R}_+$ is continuous and satisfies: c_{ij} is zero on $V(S) \cap \partial W$ whenever $i \notin S$ and $j \in S$.

We deduce from Corollary 3.2.1 that there exists a core allocation x such that for each $i \in N$, $\sum_{j \in N}(c_{ij}(x) - c_{ji}(x)) = 0$.

Equivalently, remarking that $p_W(x) = x$ on ∂W, $\sum_{j \in N} p_i(x)s_{ij}(x) - p_j(x)s_{ji}(x) = f_i(x) = 0$. Hence, x is a core allocation that belongs to the average prekernel, as was to be proved. □

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4. TARIFICATION STABLE ET CœUR-ÉQUILIBRE

Résumé

L'existence de tarifications sans subventions croisées et soutenables est prouvée dans un marché contestable multiproduit où les firmes ont la possibilité de discriminer les marchés locaux, composés d'une partie de la ligne commerciale et d'une partie d'agents. Les résultats sont obtenus sous une hypothèse de fonction de coût à partage équitable, et sous des conditions de bord des fonctions de demandes. Le problème de tarification est modélisé par des cœurs-équilibres de jeux de coût paramétrés.

Stable pricing in monopoly and equilibrium-core of parameterized cost games[0]

Vincent Iehlé

Abstract. We prove the existence of subsidy free and sustainable pricing schedule in multiproduct contestable markets. We allow firms to discriminate the local markets that are composed by a set of the products line and a set of agents. Results are obtained under an assumption of fair sharing cost and under boundary condition of demand functions. The pricing problem is modelled in terms of equilibrium-core allocations of parameterized cost games. *Journal of Economic Literature* Classification Numbers: C71, L11, L12.

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4.1 Introduction

The paper lies in the continuation of the theory of contestable markets, paradigm developed in the seventies by Bailey, Baumol, Panzar and Willig (1; 9). One of the main concern of the theory deals with stable pricing. In essence, the different definitions of stability all satisfy coalition proof properties, in other terms, the market is to be protected against entries from rivals. Such stability concepts shall therefore explain the emergence of industrial structure, and those price strategies may be expected in the process of market regulation. Indeed, it is likely that the incumbent firm is willing to protect its market against rivals through a stable pricing. Note however that, as quoted by Faulhaber (5, p. 967), stable pricing does not consist of *" welfare maximizing; nor are we entitled to assume that such prices are socially superior on grounds of social justice"*. Hence, even if normative questions are still open, the issue remains whether there exists such a pricing that explains a given industrial structure.

The literature on this topic can be divided into four different classes (following the terminology of contestable markets):

- Existence of subsidy free pricing, (2; 3; 5).
- Links between sustainable and subsidy free pricing, (2; 3; 8).
- Existence of anonymous equitable pricing, (6; 10; 12).
- Links between sustainable and anonymous equitable pricing, (8; 11; 12).

We address the two first problems and a new point gives an additional lighting on the extant literature. It deals with the assumption of price discrimination: firms maintain separated the local markets through individual prices.

Let us describe briefly the markets we analyze. A market is said to be a natural monopoly market if any production bundle can be produced at the lowest cost by one firm. Therefore, the incumbent firm has an advantage on any combination of firms committed in the production of the same bundle of goods. That kind of markets is thus characterized by the existence of sub-additive costs:

$$C(y) + C(y') \geq C(y + y')$$

When the incumbent and entrant firms share the same productive techniques given by a cost function, and the entry is free, the market is said to be contestable. It is well known that the advantage of the incumbent firm given by the existence of sub-additive costs is not sufficient to guarantee stable pricing. Our objective is to provide sufficient conditions for the existence of stable pricing in a contestable market.

We put our attention on the notions of subsidy free pricing and sustainable pricing. A pricing is said to be subsidy free if the incumbent achieves the budget balance condition for that pricing and the expenditures of any coalition is lower than the cost of that coalition .

The stronger definition of sustainability states that the incumbent firm can set a price schedule such that the budget is balanced on the market, and, no entrant with a non negative profit can attract a group of agents, or in a more general sense a sub-market of goods and agents. Attracting in the sense that the entrant price schedule is preferred by the chosen sub-market. We mention that in a companion paper, Iehlé (7) studies a more general definition of sustainability.

The existence of sustainable and subsidy free pricing is proved in discriminatory markets for elastic demand functions under assumptions of regular markets and fair sharing cost function. Roughly speaking, the regularity assumption states that above the zero profit price level, the profit remains non negative on a given set. Besides, all along the paper, the fair sharing cost property will play a central role. The property ensures that for any outputs levels, there exists a sharing of the cost such that no coalition will be charged more than the cost of serving that coalition alone. It shall turn out to be a necessary condition for the existence of stable pricing. In other terms, the property states a stable condition for any constant demands. Hence to go a step further towards elastic demands in the analysis of contestable markets, one needs to assume that stability holds for inelastic demands. It is shown that a sufficient condition to realize the fair sharing cost property is given by an assumption of increasing returns to coalitions on the cost function.

In our contestable market modelling, entrants have a large choice of strategies. Entrants can decide to enter on a subset of the products line and/or a subset of the agents. Henceforth, the entrants commit to supplying this sub-market. Thus, our formulation of contestable markets is sufficiently general to take into account, as special cases, contestable markets where there is: no discrimination and a global demand (one agent) as in Baumol et al. (2) and Faulhaber and Levinson (6); no discrimination and a full product line supply as in Faulhaber and Levinson (6); discrimination and single output firms as in Bendali et al. (3). All along the paper, we state two different classes of results. First, results for the case of double articulation entries (i.e. sub-market composed by a subset of agents and goods); second, results within a single articulation entries where firms commit to providing, either, the full products line, or all the agents. These two classes of results are close but not comparable.

The paper follows the usual stream of cooperative game modelling of stable pricing in monopoly. Faulhaber (5) has first proposed existence results for subsidy free pricing within this formalism of TU game, by considering cost games. Our treatment is going further and consists of the construction of a family of parameterized cost games, where the games are parameterized by prices. The subsidy free pricing is restated as an equilibrium-core of the parameterized games. Then, the results are deduced from an existence result of Bonnisseau and Iehlé (4) that states the existence of an equilibrium-core in NTU games. We restate the result for the TU case. Contrasting with a usual strategy in the literature, our proof does not appeal Kakutani's fixed point theorem. The advantage stems from the fact that our strategy to show the existence of an equilibrium-core of parameterized cost games may be extended to a more general case of cost games where the transfers of wealth are limited.

Henceforth, our strategy to deduce sustainability is usual in the literature, see Mirman et al. (8) for instance. We first state an existence result for a subsidy free pricing, then a regular market condition leads to the stronger notion of sustainability.

All the results rely on boundary conditions on demand functions and/or cost function. In the literature, such conditions are always assumed and slightly differ from ours. Up to these technical boundary conditions, our results generalize the extant ones of the literature since our model is able to encompass different contestable markets of Baumol et al. (2), Bendali et al. (3), Faulhaber and Levinson (6).

The paper can be divided into two parts, it deals first with the notions stable pricing in contestable markets, second with the cooperative game modelling. In Section 4.2, the model of contestable market with discrimination and the stability concepts are defined. In Section 4.3, one states the existence of stable pricing. First, we state existence results, Theorems 4.3.1, 4.3.2 for a subsidy free pricing under an assumption of fair sharing cost function and boundary conditions. Secondly, we prove that any subsidy free pricing is sustainable under an assumption of regular market. We deduce from this fact the existence of a sustainable pricing, Theorems 4.3.3, 4.3.4. These results are all obtained as corollaries of technical results given in Section 4.4, where we define parameterized cost games and equilibrium-cores. Omitted proofs in the body of the paper are referred to Appendix.

4.2 The model

We consider a multiproduct contestable market with price discrimination.[1]

(C1) There are $\ell < \infty$ goods. Goods are denoted by script $b \in L := \{1...\ell\}$. The commodities space is $\mathbb{R}^L_+$. There are $n < \infty$ agents (or local markets). Agents are denoted by script $a \in N := \{1...n\}$.

(C2) Price space is $\mathbb{R}^{LN}_+$, that is, firms can set a price $p_a \in \mathbb{R}^L_+$ for each agent $a \in N$. Each agent $a \in N$ is endowed with a continuous demand function $D_a : \mathbb{R}^{LN}_+ \to$

[1] We will use the following notations: for any set-valued mapping Γ, denote by $\mathrm{Gr}\,\Gamma$ its graph. $X \setminus A = \{x \in X \mid x \notin A\}$. Vector partial order: for all $x, y \in E$, Euclidean space of dimension L, $x \geq y$ if for all $k = 1...L$, $x_k \geq y_k$; $x > y$ if for all $k = 1...L$, $x_k \geq y_k$ and there exists j such that $x_j > y_j$; $x \gg y$ if for all $k = 1...L$, $x_k > y_k$. Let X be a finite set, we denote by $\mathbb{R}^X$ the Euclidean space whose components are indexed by the elements of X, $\mathbb{R}^{XY} := (\mathbb{R}^X)^Y$. For any $S \subset X$, E^X_S is the $\mid S \mid$-dimensional subspace of $\mathbb{R}^X$ defined as $\{x \in R^X \mid x_i = 0, i \notin S\}$. We denote by x^S the projection of an element of $\mathbb{R}^X$ on E^X_S. We denote by $<_S$ the partial order on E_S, i.e. $p' <_S p$ iff $p'_i \leq p_i$ for all $i \in S$, with at least one strict inequality.

$\mathbb{R}^L_+$.[2]

(C3) Incumbent and entrant firms share the same productive techniques given by a continuous cost function $C: \mathbb{R}^L_+ \to \mathbb{R}_+$.

(C4) Accessible sub-markets for entrants are the elements of the set $\{(A,B) \mid A \in \mathcal{N},\ B \in \mathcal{L}\}$, where $\mathcal{N}$ and $\mathcal{L}$ are respectively the non empty subsets of N and L.

In the remainder, the contestable markets will satisfy the above set of conditions (C1–4).

Let us now describe the scenario that takes place in the contestable market. There exists an incumbent firm that maintains separated the different sub-markets of agents $a \in N$, and provides the entire products line L to each of them with a technology described by the cost function C (for each $d \in \mathbb{R}^L_+$, $C(d)$ is the induced cost to produce the commodity bundle d). The firm is supposed to be full supplier, that is, it provides fully the demand induced by the price (set by itself). Therefore, $d_a = D_a(p)$ where d_a is the quantity of outputs produced for the agent a, and, $D_a(p) \in \mathbb{R}^L_+$ is the demand of agent a for a given price $p \in \mathbb{R}^{LN}_+$.[3]

On the side of entrants behavior, the entrants can choose a sub-market among a subset of agents and goods. As supposed before for the incumbent, the entrants are also full supplier on the chosen sub-market. Let AB be an element in $\mathcal{NL} := \{(S,T) \mid S \in \mathcal{N},\ T \in \mathcal{L}\}$, then the entrant, choosing AB, commits to providing fully the demands of agents $a \in A$ for the products line B: $D_a(p)^B \in E^L_{B+}$. Henceforth, the contestability takes place with respect to the prices.

4.2.1 Stable pricing

We are now in position to present formally two criteria of stability for a natural monopoly market. First, the notion of subsidy free pricing is defined, it can be seen as an internal stability concept where cross subsidies between sub-markets are made impossible. Second, the stronger notion of sustainability is introduced, it can be seen as an external criterion where no profitable entry, at any preferred pricing schedule, can be achieved.[4]

Definition 4.2.1 (Subsidy free) [5]

A pricing $p = (p_a)_{a\in N} \in \mathbb{R}^{LN}_+$ is subsidy free if:

---

[2] Note that demands for goods are inter-dependent between agents since they are defined on the whole price space.

[3] See Mirman et al (8) or Baumol et al. (2) for partial supplier analysis, i.e. markets where the firms can provide $d_a \le D_a(p)$.

[4] The analysis is mostly focused on the two notions of subsidy free and sustainable pricing, we mention however that others close notions exist, as anonymous equitability and supportability, see (10).

[5] Faulhaber and Levinson (6) have studied a similar model without price discrimination and full products line supply, i.e. a price $p \in \mathbb{R}^L_+$ satisfying:

$$\begin{cases} p \cdot \sum_{a\in N} D_a(p) = C(\sum_{a\in N} D_a(p)) \\ p \cdot \sum_{a\in A} D_a(p) \le C(\sum_{a\in A} D_a(p)) \text{ for each } A \in \mathcal{N}. \end{cases}$$

1. $\sum_{a\in N} p_a \cdot d_a = C(\sum_{a\in N} d_a)$.

2. $d_a = D_a(p)$ for each $a \in N$.

3. $\sum_{a\in A} p_a \cdot d_a^B \leq C(\sum_{a\in A} d_a^B)$ for each $AB \in \mathcal{N}\mathcal{L}$.

We will refer hereafter to subsidy free on $\mathcal{N}$ (resp. $\mathcal{L}$) if $\mathcal{L}$ (resp. $\mathcal{N}$) is fixed and equal to $\{L\}$ (resp. $\{N\}$) in the definition above.

From the point of view of the firm, the no cross subsidies and budget balance conditions mean that there is no jurisdiction (a sub-market) achieving a positive profit and subsidizing another jurisdiction facing losses. In that way, subsidy free pricing is an internal criterion of stability, no jurisdiction is entitled to claim a reward for its participation.

Besides, subsidy free notion is in the heart of controversy for antitrust policies, the subsidy free pricing can stand for a first criterion to be prescribed to a natural monopoly.[6] Note also that the concept is appealing for the practice, it is an easy matter for the regulators to check whether or not the pricing is subsidy free. It consists of the verification of a finite number of inequalities. It is not the case for the second criterion of stability we introduce now.

When entrants are free to enter the market, one can additionally suppose that they will set their own pricing, unlikely to the previous definition of subsidy free pricing, where the pricing is fixed. This is the essence of sustainability, to enter a sub-market, the entrants set a pricing that is likely to be preferred by the agents of the chosen sub-market. Roughly, the pricing p' is preferred to p by agent a if $p'_a < p_a$.

Definition 4.2.2 (Sustainability) [7] A price $p = (p_a)_{a\in N} \in \mathbb{R}_+^{LN}$ is sustainable if:

1. $\sum_{a\in N} p_a \cdot d_a = C(\sum_{a\in N} d_a)$.

2. $d_a = D_a(p)$ for all $a \in N$.

3. There is no $(AB, p') \in \mathcal{N}\mathcal{L} \times \mathbb{R}_+^{LN}$ such that:

$$\begin{cases} \sum_{a\in A} p'_a \cdot {d'}_a^B = C(\sum_{a\in A} {d'}_a^B) \\ d'_a = D_a(p') \text{ for all } a \in A. \\ p'_a <_B p_a \text{ for all } a \in A. \end{cases}$$

We will refer hereafter to sustainability on $\mathcal{N}$ (resp. $\mathcal{L}$) if $\mathcal{L}$ (resp. $\mathcal{N}$) is fixed and equal to $\{L\}$ (resp. $\{N\}$) in the definition above.

[6] The aircraft industry provides a good example of pricing with cross subsidies.

[7] Definitions slightly differ with respect to authors, see Baumol et al. (9), Mirman et al. (8) or Ten Raa (11).

The first condition guarantees the budget balance, the second is the full supplier requirement. The third formalizes the stability concept: no firm can enter a sub-market AB, by setting a preferred pricing for agents in A and achieving the budget balance condition. The definition of sustainability is much stronger than one of subsidy free (it is proved in Lemma 4.3.1 that a sustainable pricing is a subsidy free pricing). Here, the market is fully protected against entry from rivals, and the incumbent is in invulnerable situation.

4.3 Existence of stable pricing

Using results of Appendix for equilibrium-cores in parameterized cost games, we state two existence results for subsidy free pricing and sustainability pricing. Firstly, we exhibit a subsidy free pricing, then we deduce the sustainable pricing under the restriction of regular markets. The subsidy free pricing can be restated as allocations of an equilibrium-core of parameterized cost games. To study the non-emptiness of the core of these games, we will suppose that the cost function satisfies a condition of fair sharing for any output structure:[8]

Definition 4.3.1 The cost function C satisfies the property of fair sharing if for all $d_a \in \mathbb{R}^L_+$, $a \in N$, there exists $q \in \mathbb{R}^{LN}_+$ such that:

$$\begin{cases} \sum_{ab \in NL} q_{ab} = C(\sum_{a \in N} d_a) \\ \sum_{ab \in AB} q_{ab} \leq C(\sum_{a \in A} d_a^B) \text{ for all } AB \in \mathcal{N}\mathcal{L}. \end{cases} \tag{4.1}$$

We will refer hereafter to fair sharing on $\mathcal{N}$ (resp. $\mathcal{L}$) if $\mathcal{L}$ (resp. $\mathcal{N}$) is fixed and equal to $\{L\}$ (resp. $\{N\}$) in the definition above.

The above property states the existence of a fair sharing $(q_a)_{a \in N}$ among the agents for any structure of outputs $(d_a)_{a \in N}$. In other terms, no coalition can achieve a positive profit if its members decide to secede. We will obtain the subsidy free pricing from this property on the costs since the subsidy free property can be seen as the extension of the fair sharing cost property to the case of inelastic demand functions. Indeed, the two notions are linked up together, for any subsidy free pricing p, the vector $(q_{ab})_{ab \in NL}$ given by the expenditures of the agents at price p: $(p_{ab} D_a(p)_b)_{ab \in NL}$ satisfies the fair sharing property for the output level given by $(D_a(p))_{a \in N}$. Conversely, a fair sharing implementation $(\sum_{b \in L} q_{ab})_{a \in N}$ associated with a positive outputs level $(d_a)_{a \in N}$ can be seen as the expenditures of the agents with inelastic demand functions $(d_a)_{a \in N}$ at the subsidy free pricing: $p_a^b = \frac{q_a^b}{d_a^b}$.

[8] In the literature, this evaluation of fairness, is also called the stand alone test, as in Faulhaber (5), Faulhaber and Levinson (6).

Note also that the condition of fair sharing states equivalently that for all outputs $(d_a)_{a\in N}$: for all balanced family $\mathcal{B}$ and balancing weights $(\gamma_{AB})_{AB\in\mathcal{B}}$,

$$\sum_{AB\in\mathcal{B}} \gamma_{AB} C(\sum_{a\in A} d_a^B) \geq C(\sum_{a\in N} d_a).$$

Sharkey (13) and Baumol et al. (2) put a great attention on fair sharing cost functions, they exhibit technological properties such that the condition fair sharing always exists. For instance, they provide a sufficient condition on the cost function to be fair sharing.

(WC) $C(x+z) - C(x) \geq C(x+y+z) - C(x+y)$ for all $x, y, z \in \mathbb{R}^L_+$.

(WC) is called weak cost complementarities condition.

Proposition 4.3.1 (Baumol et al.) *Under (WC), the cost function is fair sharing.*

For the proof, see (2, Proposition 8C10, p.203).

One can provide another sufficient condition that involves a notion of increasing returns to coalitions. Given an output structure and a coalition, the grand coalition is able to purchase the output at the lowest cost per capita. coalition.

(IRC) For each $(d_a)_{a\in N} \in \mathbb{R}^{LN}_+$, $\frac{C(\sum_{a\in A} d_a^B)}{|A|} \geq \frac{C(\sum_{a\in N} d_a)}{|N|}$ for all $AB \in \mathcal{NL}$.

Proposition 4.3.2 *Under (IRC), the cost function is fair sharing.*[9]

Proof of Proposition 4.3.2. In order to show the implication it suffices to show equivalently that the following inequality holds true: for all $(d_a)_{a\in N}$ and all balanced family of coalition $\mathcal{B}$, with balancing weights $(\gamma_{AB})_{AB\in\mathcal{B}}$, $\sum_{A\in\mathcal{B}} \gamma_A C(\sum_{a\in A} d_a^B) \geq C(\sum_{a\in N} d_a)$. Indeed from (IRC),

$$\sum_{A\in\mathcal{B}} \gamma_A C(\sum_{a\in A} d_a^B) \geq \frac{C(\sum_{a\in N} d_a)}{|N|} \sum_{A\in\mathcal{B}} \gamma_A |A|$$

Furthermore by computations of the balancing weights, we deduce:

$$\sum_{A\in\mathcal{B}} \gamma_A |A| = \sum_{a\in N} \sum_{A\in\mathcal{B}, A\ni a} \gamma_A = \sum_{a\in N} 1 = |N|$$

□

In the remainder, the fair sharing property will play a central role, since the results are all stated under this condition. It shall turn out that the condition is also a necessary condition for the existence of stable pricing. The following proposition illustrates the point for the case of subsidy free pricing. Since any sustainable pricing is also a subsidy free pricing (see Lemma 4.3.1), the condition of fair sharing cost is consequently a necessary condition for sustainability.

[9] One can state the analogue of Propositions 4.3.1 and 4.3.2 for the cases of full products line supply or all agents supply.

Proposition 4.3.3 *If the cost function is not fair sharing, there exist demand functions such that there is no subsidy free pricing.*

Proof of Proposition 4.3.3. The proof is straightforward. Suppose that there exists $(d_a)_{a \in N} \in \mathbb{R}_+^{LN}$ such that (4.1) is not satisfied for any $q \in \mathbb{R}_+^{LN}$. It suffices to consider the constant demand functions $D_a = d_a$ for all $a \in N$. These demand functions cannot be supported by a subsidy free pricing, otherwise set $q_{ab} = p_{ab} D_a(p)_b$ that leads to a contradiction. □

We posit the following assumption on the demand functions.

(D) Demand functions D_a are bounded above, and there exists a minimal threshold $\epsilon > 0$ in the consumption, i.e. $D_a \geq \epsilon \mathbf{1}$ for each $a \in N$.[10]

One can state the first main results of the paper:

Theorem 4.3.1 *Under (D), any contestable market with fair sharing cost admits a subsidy free pricing.*

Theorem 4.3.2 *Under (D),*

- *Any contestable market admits a subsidy free pricing on $\mathcal{N}$ if firms commit to providing entirely the products line, $\mathcal{L} = \{L\}$, and the cost function is fair sharing on $\mathcal{N}$.*
- *Any contestable market admits a subsidy free pricing on $\mathcal{L}$ if firms commit to providing all the agents, $\mathcal{N} = \{N\}$, and the cost function is fair sharing on $\mathcal{L}$.*

The statements above are very close. Firstly, it is worth pointing out that Theorem 4.3.2 is not a corollary of Theorem 4.3.1. Obviously, a subsidy free pricing in the sense of Theorem 4.3.1 is also subsidy free for the specific cases of Proposition 4.3.2, but remark that the fair sharing assumption is weaker in these cases. Thus, the two results are not comparable. We exhibit the detailed proof for Theorem 4.3.1, which is a direct corollary of Proposition 4.4.1, whereas the proof of Theorem 4.3.2 is omitted (it follows the same lines of arguments, see also Remark 1 in Section 4.5).

We state our results under a condition of fair sharing cost function. Additionally, we need boundary conditions (D) on demand functions and/or cost function. In the literature, such conditions are always assumed and slightly differ from ours. Up to these technical boundary conditions, our results generalize the extant ones of the literature since our model is able to encompass different contestable markets of Baumol et al. (2), Bendali et al. (3), Faulhaber and Levinson (6).

Let us turn to the notion of sustainability. We make the link with the sustainability by introducing the assumption of regular market, this way of doing is usual in the literature, see Bendali et al. (3), Mirman et al. (8) or Ten Raa (11). The condition of regularity implies that the subsidy

[10] Ten Raa (10) uses the condition of threshold in the consumption. See Remark 5 in Section 4.5.

free pricing is not dominated by another pricing to get eventually the notion of sustainability. The regularity condition holds on the profit function:

For any sub-market $AB \in \mathcal{NL}$, the profit function is given by:

$$\Pi_{AB} : p \longrightarrow \sum_{a\in A} p_a \cdot D_a(p)^B \; - \; C(\sum_{a\in A} D_a(p)^B)$$

From (D) and the continuity of the function C, we know that there exists an upper bound $B \in \mathbb{R}_+$ such that $C(D_a(p)_b) \leq B$ for any $p \in \mathbb{R}_+^{LN}$ and $ab \in NL$. The market is said to be regular if it satisfies the following condition:

(R) Let $\mathcal{K}$ be the multidimensional cube $\prod_{ab\in NL}[0, \frac{B}{\epsilon}]$, for any sub-market AB it holds that: $\mathcal{K} \cap \{\Pi_{AB} > 0\} = \mathcal{K} \cap \{\{\Pi_{AB} = 0\} + \mathbb{R}_{++}^{LN}\}$.

We will say that (R) is satisfied on $\mathcal{N}$ (resp. $\mathcal{L}$) if $\mathcal{L}$ (resp. $\mathcal{N}$) is fixed and equal to $\{L\}$ (resp. $\{N\}$) in the definition above.

The condition can be seen as a notion of positive comprehensiveness from above. Above zero profit level, profits remain positive on a compact set $\mathcal{K}$.

In Bendali et al. (3), the authors suppose that the expenditures functions are strictly increasing with respect to prices. In Mirman et al. (8), a weaker assumption is at stake, the profits are strictly increasing with respect to prices. Regularity is the weakest version, note also that it is consistent with a bell curve profit mapping.

Theorem 4.3.3 *Under (D) and (R), any contestable market with fair sharing cost admits a sustainable pricing.*

Proof of Theorem 4.3.3. From Theorem 4.3.1, there exists a subsidy free pricing p^*. Firstly, note that $p^* \in \mathcal{K}$. Indeed, consider the definition of subsidy free pricing. As a special case one has: for all $ab \in NL$,

$$p_{ab} D_a(p)_b \leq C(D_a(p)_b)$$

Using the boundedness and threshold conditions of Theorem 4.3.1, one gets the result. The following lemma will end up the proof.

Lemma 4.3.1 *Under Assumptions of Theorem 4.3.3, the following two propositions are equivalent:*

1. *p is subsidy free.*
2. *p is sustainable.*

Proof of Lemma 4.3.1. Suppose that the regular contestable market admits a subsidy free pricing p. Since $\sum_{a\in A} p_a \cdot D_a(p)^B \leq C(\sum_{a\in A} D_a(p)^B)$ and $p \in \mathcal{K}$, it follows from regularity that for all

p' such that $p_a <_B p'_a$ for all $a \in A$, it holds that $\Pi_{AB} < 0$. Hence, the pricing p is sustainable. Conversely, suppose that the regular contestable market admits a subsidy free pricing p, then for all $p' \in \mathbb{R}_+^{LN}$ such that $p_a >_B p'_a$ for all $a \in A$, it holds that $\Pi_{AB}(p') < 0$. From the continuity of the mappings C and $(D_a)_{a \in N}$, by taking the limit, it is clear that for $p' = p$, one gets $\Pi_{AB}(p') \leq 0$. Hence, the pricing p is subsidy free. □

We deduce that the subsidy free pricing p^* is sustainable. □

Symmetrically, one states results for the case of fixed product line or fixed number of agents in the provision of the goods. The proof is omitted (the result can be deduced straightforwardly from Theorem 4.3.2).

Theorem 4.3.4 *Under (D),*

- *Any contestable market admits a sustainable pricing on $\mathcal{N}$ if $\mathcal{L} = \{L\}$, the cost is fair sharing on $\mathcal{N}$ and (R) is satisfied on $\mathcal{N}$, or*
- *Any contestable market admits a sustainable pricing on $\mathcal{L}$ if $\mathcal{N} = \{N\}$, the cost function is fair sharing on $\mathcal{L}$ and (R) is satisfied on $\mathcal{L}$.*

4.4 *Equilibrium-core of parameterized cost games*

In this Section, the notion of parameterized TU games is defined. We state a non-emptiness result for the associated concept of core, called equilibrium-core. Then, this abstract result is appealed to show the non-emptiness of a generalized core of cost games, where additionally to the core stability requirement, the core allocation must satisfy a consistency property with respect to a parameter.[11] We state Proposition 4.4.1 from which we deduce Theorem 4.3.1, related to subsidy free pricing.

Let us first define a TU cooperative game. Consider a finite set of players X, $\mathcal{X}$ the set of non empty subsets of X. The characteristic function is $v : \mathcal{X} \to \mathbb{R}_+$. Thus, for each $S \in \mathcal{X}$, v_S is the payoff of the coalition S. Let $(v_S, S \in \mathcal{X})$ denote a TU game.

An imputation in S is a vector $q = (q_x)_{x \in S}$ such that $\sum_{x \in S} q_x \leq v_S$. Consider an imputation q in N, it is dominated by q' in S if q' is an imputation in S and $q'_x > q_x$ for each $x \in S$. The core of a game is the set of all imputations that are undominated in any coalition. A family of coalitions $B \subset \mathcal{X}$ is balanced if there exists $\lambda_S \in \mathbb{R}_+$ for each $S \in \mathcal{B}$ such that:

$$\sum_{S \in \mathcal{B}} \lambda_S \mathbf{1}^S = \mathbf{1}^X$$

A TU game is said to be balanced if for any balanced family of coalitions $\mathcal{B}$ with balancing coefficients $(\lambda_S)_{S \in \mathcal{B}}$, it holds that:

$$\sum_{S \in \mathcal{B}} \lambda_S v_S \geq v_N$$

[11] In the literature, the results are usually obtained directly from an application of a fixed point theorem, we adopt a different strategy, which is closely related, see comments in Section 4.5.

We need to consider now a more general model of TU games called parameterized TU games.[12] The parameter set is Θ and, a game is associated to each $\theta \in \Theta$, that is, one has a mapping v_S from Θ to $\mathbb{R}$ for each coalition S. Let $((v_S)_{S\in\mathcal{X}}, \Theta)$ denote a parameterized TU game.

Finally, the equilibrium condition on the parameters is represented by a set-valued mapping G from $\Theta \times \mathbb{R}^X$ to Θ.

(PH0) Θ is a non-empty, convex, compact subset of an Euclidean space. G is an upper semi-continuous set-valued mapping with non-empty and convex values.

(PH1) v_S, $S \in \mathcal{X}$, are continuous mappings on Θ.

Definition 4.4.1 Let $((v_S)_{S\in\mathcal{X}}, \Theta)$ be a parameterized game. An equilibrium-core allocation is a pair $(q \in \mathbb{R}^X, \theta \in \Theta)$ such that q belongs to the core of the game $(v_S(\theta), S \in \mathcal{X})$ and $\theta \in G(\theta, q)$.

We can state now a weak version of the abstract result of Bonnisseau and Iehlé (4). The original version holds in NTU games. We let the reader check that Theorem 4.4.1 coincides to the TU case. Moreover, we point out that the result can be obtained directly as an application of Kakutani's fixed point theorem for this specific case of TU games.

Theorem 4.4.1 *Let $((v_S)_{S\in\mathcal{X}}, \Theta)$ be a parameterized TU game satisfying Assumptions (PH0) and (PH1) and such that, for each $\theta \in \Theta$, the TU game $(v_S(\theta), S \in \mathcal{X})$ is balanced, then there exists an equilibrium-core allocation.*

It is well known that the core of a TU game can be restated as a set of vectors $(q_x)_{x\in X}$ satisfying a finite number of inequalities, as follows:

$$\begin{cases} \sum_{x\in X} q_x = v_X \\ \sum_{x\in S} q_x \geq v_S \text{ for all } S \in \mathcal{X}. \end{cases}$$

We introduce now a generalized notion of the core for cost games. In cost games the worth of a coalition is supposed to model the cost associated to the coalition, so that we use converse inequalities. Consider the following families of mappings: $\psi_{AB} : \mathbb{R}_+^{LN} \to \mathbb{R}_+$ for all $AB \in \mathcal{NL}$, $\phi_a : \mathbb{R}_+^{LN} \to \mathbb{R}_+^L$ for all $a \in N$.

Definition 4.4.2 The generalized core $\mathcal{M}$ is the set of vectors $p \in \mathbb{R}_+^{LN}$ such that:

$$\begin{cases} \sum_{a\in N} p_a \cdot \phi_a(p) = \psi_{LN}(p) \\ \sum_{a\in A} p_a \cdot \phi_a(p)^B \leq \psi_{AB}(p) \text{ for all } AB \in \mathcal{NL}. \end{cases}$$

For each $p \in \mathbb{R}_+^{LN}$, let $\mathcal{M}^p$ denote the core of the TU game $(-\psi_S(p), S \in \mathcal{NL})$. The following result is deduced from Theorem 4.4.1, we prove that the generalized core is exactly an equilibrium-core of parameterized cost games. The definition below of generalized core is no more than a general formulation of the subsidy free problem.

[12] The reader is referred to (4) for further details.

Proposition 4.4.1 *Under Assumptions:*

1. *There exists* $\epsilon > 0$*, such that, for all* $a \in N$*,* $\phi_a \geq \epsilon \mathbf{1}^L$*. For all* $ab \in NL$*,* $\psi_{\{ab\}}$ *is bounded above by* $B \in \mathbb{R}_+$*.*
2. ψ_{AB}*,* $AB \in \mathcal{NL}$*, and* ϕ_a*,* $a \in N$*, are continuous.*
3. *For* $p \in \mathbb{R}_+^{LN}$*,* $\mathcal{M}^p$ *is non empty.*

$$\mathcal{M} \neq \emptyset.$$

Proof of Proposition 4.4.1.[13] We want to prove that $\mathcal{M}$ is non empty, it amounts to show that there exists a vector p^* satisfying: $(p_a^* \cdot \phi_a(p^*))_{a\in N} \in \mathcal{M}^{p^*}$. We use Theorem 4.4.1 to deduce the existence of such a vector from the existence of an equilibrium-core allocation.

Let Θ be the compact and convex subset of $\mathbb{R}^{NL}$, defined by $\Theta = \prod_{ab\in NL} \Theta_{ab}$ where: for each $ab \in NL$, $\Theta_{ab} = [0, \frac{B}{\epsilon}]$. Define the parameterized game as follows: given $S \in \mathcal{NL}$ and $\theta \in \Theta$, $v_S(\theta) = -\psi_S(\theta)$.

Consider now the following set-valued mappings. Firstly, let V be the comprehensive hull of the individually rational payoffs for the grand coalition.

$$V(\theta) := \left\{ x \in \mathbb{R}^{NL} \mid \sum_{i\in NL} x_i = v_N(\theta),\ x_i \geq v_{\{i\}}(\theta) \text{ for all } i \in NL \right\} - \mathbb{R}_+^{NL}$$

And, let $\bar{V}_N(\theta)$ be the non dominated payoff of $V(\theta)$:

$$\bar{V}(\theta) = \{x \in V(\theta) \mid \nexists y > x,\ y \in V(\theta)\}$$

Note that, for each $\theta \in \Theta$, $V(\theta)$ is a comprehensive from below and closed set of $\mathbb{R}^{NL}$. From the continuity of the mappings v_N and $v_{\{i\}}, i \in NL$, one can define a continuous mapping p_V : $\Theta \times \mathbb{R}^{NL} \to \partial V$, such that $p_V(\theta, q) = q$ for all $(q, \theta) \in \text{Gr } \partial V$.

The following mapping selects, for each pair (θ, q), in the non dominated set of individual rational payoff that are above the pseudo projection of q in the game parameterized by θ.

$$I(\theta, q) = \left\{ t \in \bar{V}(\theta) \mid t \geq p_V(\theta, q) \right\}$$

Lastly, one is led to the definition of the mapping $G: \Theta \times \mathbb{R}^{NL} \to \Theta$. Given $(\theta, q) \in \Theta \times \mathbb{R}_+^{NL}$:

$$G(\theta, q) = \{u \in \Theta \mid -(u_i \phi_i(\theta))_{i\in NL} \in I(\theta, q)\}$$

Now, suppose that Assumptions of Theorem 4.4.1 hold true for the parameterized game $((v_S)_{S\in\mathcal{NL}}, \Theta)$ and the set-valued mapping G, then there exists (θ^*, q^*) such that q^* belongs to the core of the game

[13] Note that the proof also holds true for a more general model, where the induced game is not a transferable utility game. Indeed, we do not use the fact the core is polyhedral in the proof. In NTU games, the core of the game has not such a structure.

$(v_S(\theta^*), S \in \mathcal{N}\mathcal{L})$ and $\theta^* \in G(\theta^*, q^*)$. Thus $I(\theta^*, q^*) = \{q^*\}$, that is, we have $-(\theta_i^* \phi_i(\theta^*))_{i \in NL} = q^*$ and $-q^* \in \mathcal{M}^{\theta^*}$. Hence $(\theta_i^* \phi_i(\theta^*))_{i \in NL} \in \mathcal{M}^{\theta^*}$ and we are done.

Lemma 4.4.1 *The parameterized game $((v_S)_{S \in \mathcal{N}}, \Theta)$ and the set-valued mapping G meet Assumptions of Theorem 4.4.1.*

This ends up the proof of Proposition 4.4.1. □

4.5 Concluding remarks

1. In Section 4.4, analog results are obtained if we consider now slightly different statements of the generalized core where $\mathcal{L}$ is set to $\{L\}$ or $\mathcal{N}$ is set to $\{N\}$. We omit the statements of these propositions. To prove the results, we shall need to consider a similar game where the strategy space is restricted to $\mathbb{R}^L$ for the case $\mathcal{N} = \{N\}$ and to $\mathbb{R}^N$ for the case $\mathcal{L} = \{L\}$. For instance, for the latter case: the generalized core $\mathcal{M}_N$ is the set of vectors $p \in \mathbb{R}_+^{LN}$ such that:

$$\begin{cases} \sum_{a \in N} p_a \cdot \phi_a(p) = \psi_N(p) \\ \sum_{a \in A} p_a \cdot \phi_a(p) \leq \psi_A(p) \text{ for all } A \in \mathcal{N}. \end{cases}$$

With respect to the families of mappings $\psi_A : \mathbb{R}_+^{LN} \rightarrow \mathbb{R}_+$ for all $A \in \mathcal{N}$, $\phi_a : \mathbb{R}_+^{LN} \rightarrow \mathbb{R}_+^L$ for all $a \in N$. Then, define the parameterized game as follows: given $S \in \mathcal{N}$ and $\theta \in \Theta$, $v_S(\theta) = -\psi_S(\theta)$. And,

$$V(\theta) := \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v_N(\theta),\ x_i \geq v_{\{i\}}(\theta) \text{ for all } i \in N \right\} - \mathbb{R}_+^N$$

Henceforth, the lines of arguments are similar to Proof of Proposition 4.4.1.

2. If $n = 1$, Definition 4.2.2 can be restated as the set of prices $p \in \mathbb{R}_+^L$ such that:

1. $p \cdot D(p) = C(D(p))$.

2. There is no $(B \in \mathcal{L}, p' \in \mathbb{R}_+^L)$ such that:

$$\begin{cases} p' \cdot D(p')^B = C(D(p')^B) \\ p' <_B p. \end{cases}$$

Hence, it leads back to the definition of Baumol et al. (2) for a contestable market with global demands, these authors have not exhibited general conditions for sustainability existence. Theorem 4.3.3 provides an existence result for that family of contestable market.

3. Sustainability conditions have been largely studied by Mirman et al. (8), Panzar and Willig (9) and Ten Raa (11). The most achieved results are given in Mirman (8) and Ten Raa (11). In Ten Raa, the sustainability is proved under assumptions of complementarities costs and independent goods, also it is assumed that any output levels can be supplied at a profit. The strategy differs from ours, since the sustainability is deduced from an anonymous equity pricing, the proof does not involve a regularity condition.

4. The above results can be obtained as a direct application of Kakutani's fixed point theorem. The set-valued is the following:

$$\Gamma^*(p) = \left\{u \in \mathbb{R}_+^{LN} \mid (u_{ab} D_a(p)_b)_{ab \in NL} \in \Gamma(p)\right\}$$

Where $\Gamma(p)$ is the core of the cost game associated to the pricing p. This idea is also at stake in Bendali et al. (3). The authors deduce a subsidy free pricing in a contestable market with a single output technology. Their boundary assumptions slightly differ from ours. They assume that demand functions are positive and that the cost of any output bundle is bounded above by the norm of the output bundle.

5. Two other fixed point approaches are given in Faulhaber and Levinson (6) and Ten Raa (10). In Faulhaber and Levinson (6), it is proved that an anonymous equitable pricing exists under a cost complementarity property, independent demands and compactness assumption. A closely related approach is Ten Raa (10), the author deduces the existence of an anonymous equity pricing from a condition of supportability. This way of doing is analog to our strategy to deduce subsidy free pricing from fair sharing cost. Furthermore, the author also assumes that there exists consumption threshold $\epsilon > 0$ such that $D(p) \geq \epsilon$. Formally, let us consider the set-valued mapping P:

$$P(d) = \left\{s \in \mathbb{R}^N \mid \ s \cdot d = C(d);\ s \cdot d' \leq C(d'),\ \forall d' \leq d\right\}$$

If P has non-empty values, then the cost function is said to be supportable. On suitable compact sets $\mathcal{E}$ and $\mathcal{F}$, a mapping is defined on $\mathcal{E} \times \mathcal{F}$ into itself: $F(p,d) = P(d) \times D(p)$. The mapping admits a fixed point which is exactly the anonymous equitable pricing.

6. As quoted in Sharkey (13), the analysis of cost games to comprehend the stability on natural monopoly market makes abstraction of the preferences of agents. To remedy this drawback, the author defines general cost games where preferences are taken into account in the characteristic function of the games. One is led to the definitions of so called benefit and welfare games. For instance, the characteristic function of a benefit game is given by :

$$v_T = \max_{S \in X} \left\{ \sum_{i \in T} u_i(S) - C(S) \right\}$$

The general formulations of Section 4.4 also allow to consider that family of games as special cases.

7. In a companion paper, Iehlé (7) uses a different approach based on NTU game theoretical modelling to show the existence of generalized sustainable pricing in a single output contestable markets. It amounts to define an associated NTU game in the price space, the payoffs sets of any coalition being the prices for which the coalition profit is non negative. The sustainable pricing coincides exactly with the core of the NTU game. Then, the existence result follows from Scarf's theorem, indeed the game shall turn out to be balanced under the fair sharing market assumption. It is worth pointing out that in our paper the fair sharing market is also central to get the existence, it guarantees the non-emptiness of the cores of the TU games, parameterized by the prices.

4.6 Appendix

Proof of Theorem 4.3.1. The theorem is a direct corollary of Proposition 4.4.1. Consider the mappings $(\psi_{AB})_{AB\in\mathcal{NL}}$ and $(\phi_a)_{a\in N}$ defined as follows:

$$\psi_{AB}(p) = C(\sum_{a\in A} D_a(p)^B) \text{ and } \phi_a(p) = D_a(p)$$

We show that Assumptions of Proposition 4.4.1 are all fulfilled. Assumption 1 follows from the continuity of the mapping C and Assumption (D). Continuity Assumption follows from the continuity of the mappings C and $(D_a)_{a\in N}$. Finally, the fair sharing property of the cost function implies that $\mathcal{M}^p$ is non empty for all $p \in \mathbb{R}_+^{LN}$. □

Proof of Lemma 4.4.1. Assumption (PH1) is satisfied from the continuity of the mappings ψ_A for all $A \in \mathcal{N}$. The game $(v_S(\theta), S \in \mathcal{X})$ is balanced for each $\theta \in \Theta$, from Assumption 3 of Proposition 4.4.1. We need to check now that Assumption $(PH0)$ is fulfilled. It suffices to show that G is an upper semi-continuous set-valued mapping with non-empty and convex values. Firstly, it is straightforward that the equation in u: $-(u_i\phi_i(\theta))_{i\in NL} \in I(\theta, q)$ has a solution in $\mathbb{R}_+^{NL}$ since I has also non-empty values. The latter stems from the fact that the partition of the singletons forms a balanced family, therefore we deduce that $\sum_{i\in NL} v_{\{i\}}(\theta) \leq v_N(\theta)$ for all $\theta \in \Theta$ from the balancedness condition. Thus, the set of individual rational allocations is non-empty.

Let us check that for all $u \in G(\theta, q)$, $u \in \Theta$. Indeed, for all $i \in NL$, $-u_i\phi_i(\theta) = t_i$, for some $t \in I(\theta, q)$, i.e. $u_i\phi_i(\theta) = -t_i \leq -v_{\{i\}}(\theta) = \psi_{\{i\}}(\theta) \leq B$ since $t \in I(\theta, q)$. Furthermore, $u_i\phi_i(\theta) \geq \epsilon u_i$. Therefore, $u \in \Theta$ and G has non-empty values into Θ.

G has convex values: let $u^1, u^2 \in G(\theta, q)$ and consider $v = \lambda u^1 + (1-\lambda)u^2$ for $\lambda \in [0, 1]$. Since $(u_i^1\phi_i(\theta))_{i\in NL} \in I(\theta, q)$, and, $(u_i^2\phi_i(\theta))_{i\in NL} \in I(\theta, q)$, $(v_i\phi_i(\theta))_{i\in NL} \in \text{co}I(\theta, q)$. It is straightforward to verify that I has convex values, hence we have proved that $v \in G(\theta, q)$.

For upper-semi continuity of G, we need to prove a closed graph assumption, since G is valued into a compact set. It follows directly from the fact that I is upper-semi continuous since p_V is a continuous mapping. Consider the following sequences $(\theta^\nu, q^\nu) \in \Theta \times \mathbb{R}_+^{NL}$ and $u^\nu \in \Theta$ converging respectively to (θ, q) and u such that $u^\nu \in G(\theta^\nu, q^\nu)$ for all ν. Note also, from the continuity of the mappings $(\phi_a)_{a\in N}$, that the vector $(u_i^\nu\phi_i(\theta^\nu))_{i\in NL}$ converges to $(u_i \cdot \phi_i(\theta))_{i\in NL}$. For all ν,

$(u_i^\nu \phi_i(\theta^\nu))_{i\in NL} \in I(\theta^\nu, q^\nu)$, since I has a closed graph, we deduce that $(u_i \cdot \phi_i(\theta))_{i\in NL} \in I(\theta, q)$, equivalently $u \in G(\theta, q)$. □

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5. SOUTENABILITÉ VERSUS PROJET DE FINANCEMENT

Résumé

On prouve l'existence de tarification soutenable sur un marché contestable monoproduit avec discrimination par les prix. La définition de soutenabilité est généralisée dans un premier temps au cas de coalitions bloquantes pouvant décider d'un mécanisme de financement volontaire ; puis dans un second temps au cas de règle de valuation donnée a priori. Les résultats d'existence sont prouvés sous des hypothèses de fonctions de coût à partage équitable généralisées. Nous utilisons des développements récents de la théorie des jeux sur des sélections de cœur dans les jeux NTU.

Sustainability versus financing device[0]

Vincent Iehlé

Abstract. We prove the existence of general sustainable price schedules, where, firstly, the participants of a blocking coalition can agree for voluntary financing device, or secondly follow a valuation rule given by a coordinating center. We consider a single output market where price discrimination is allowed. Existence results are provided in both cases under assumptions of generalized fair sharing cost function. The strategy of the proof is based on recent developments of cooperative game theory about core selections in NTU games. *Journal of Economic Literature* Classification Numbers: C71, L11, L12.

Keywords: sustainability, contestable markets, core selections in NTU games.

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5.1 Introduction

The theory of contestable markets addresses, among others, the question of the existence of stable price schedule in the case of free entry. In essence, the different definitions of stability all satisfy coalition proof properties. Sustainability is considered as the strongest requirement of stability. We prove, in this note, the existence of a general sustainable price in a contestable market.

The usual definition of sustainability states that the incumbent firm can set a price schedule such that a zero profit is achieved on the market, and, no entrant firm with a non negative profit can attract a coalition of agents. Attracting in the sense that the entrant price schedule is preferred by the agents. In the remainder, we adopt the terminologies of incumbent coalition and blocking coalition to describe respectively the incumbent firm and the entrant firm since firms are totally characterized by the coalitions of agents they supply.

Many authors have addressed the question of existence of sustainable price in different contestable markets, (for a survey, see Baumol et al. (1)). These models differ from each others: the markets can be taken with multiproduct technology, elastic or inelastic demands, price discrimination or global demands. In a companion paper, Iehlé (4) exhibits conditions for the existence of sustainable prices for that definition of sustainability in contestable markets with multiproduct technology and price discrimination. Here, we focus on a rather simple model of contestable market with elastic demands with single output technology and price discrimination. And, the innovation is coming from the definition of sustainability.

In the usual definition of sustainability, blocking coalitions and incumbent coalition behave symmetrically, the output is fully supplied with respect to the demand functions, and the participants pay the expenditures corresponding to these demands. Here, we assume that the blocking coalitions behave differently and may refine the way of financing a project.

We suppose indeed that participants within a coalition can agree for devices to finance the amount of the cost induced by their demands. Thus, the participants of the blocking coalition will not pay the expenditures corresponding to their individual demand but rather share the global cost induced by the coalition with respect to some device, whereas the incumbents are modelled as usual, that is, they face the usual budget balance constraint.

We consider two different classes of financing device to model blocking coalitions' behaviors.

(1) An agreement for any voluntary financing device is found among the participants.

(2) A valuation rule for any coalition is given a priori by a coordinating center in charge of enforcing it.

To give some intuition into the concept, consider the following illustration of an industry organized into jurisdictions. Let participants in an incumbent jurisdiction be involved in the production of a public service. The participants can produce the service within any jurisdiction, composed by a coalition of participants. Suppose now the incumbent jurisdiction sets a price such that the production of the public service satisfies the budget balance condition, each participant pays the amount of his respective demand. Let there be a jurisdiction willing to secede, that is to organize the production of the service within the jurisdiction (the service being consumed and paid by its participants). Our analysis deals with two cases.

In Case (1), the blocking jurisdiction also needs to insure a viable budget in the production of the service but participants can choose among any financing device, that is, share the total cost of their demands, without restriction (one participant can pay for all). Therefore, the jurisdiction has two tools in hands to realize the secession, first setting a price for the service for any participant of the jurisdiction, (the prices induce the demands), second deciding the shares within the jurisdiction for financing the service (induce by the demands).

In Case (2), the financing device is given a priori as a valuation rule by a coordinating center. That is, one fixes participants' shares for each price level for each jurisdiction. The share will define the price to pay for one unit of the service. The coordinating center can, in that way, interfere to give priority to the incumbent's situation or to other regulation policies by choosing a suitable valuation rule. The set of strategies for a blocking jurisdiction is thus restricted, only one tool remains: the price schedule; and no freedom is given for the choice of the agent's participation.

In both cases, a sustainable price schedule satisfies the budget balance condition for the incumbent jurisdiction, and, is such that there is no viable financing device that gives more quantity of the service to each of the participants in the financing. That is, the blocking jurisdiction is willing to increase the consumption of the service throughout an alternative system of financing that reaches the budget balance, but the incumbent strategy makes it impossible.

It is worth pointing out that the usual definition of sustainability coincides to the particular situation of Case (2), where the coordinating center decides for a share, equal for all participants. It leads back to the financing where participants pay their own demands at the actual price as the incumbent coalition does. The definition of sustainability for Case (1) clearly strengthens the usual concept, a coalition setting a price such that all the participants are better off may fail to satisfy the usual budget balance condition whereas a financing device may be found to finance eventually the service. Contrasting, Case (2) provides an alternative definition but not comparable to the usual one, except for equal share rule where they coincide as mentioned above.

In the theory of public goods, the financing of a public good may follow similar rules. Since analogies apply, we use terminologies taken from Mas-Colell (5): for Case (1), voluntary financing device; for Case (2), valuation rules.

We prove the existence of sustainable price schedules in both cases of valuation systems under assumptions of fair sharing market, which are exactly the generalization of the so called stand alone cost property at stake in the literature of contestable markets. The cost function satisfies the stand alone cost property if, there exists a sharing of the total cost for the incumbent coalition such that no coalition is charged more than the cost of serving that coalition alone.

In Iehlé (4), the stand alone cost property shall turn out to be central to show the existence of a sustainable price (in the usual definition). The idea behind is the following: to deal with elastic demands, one cannot dispense to use stand alone cost property, which amounts to a stable property for inelastic demands. It is also proved that a so called assumption of increasing returns to coalitions (IRC) of the cost function is sufficient to guarantee the property of stand aloe cost.

Here, the generalization of the property of stand alone cost takes naturally into account the possibilities of different financing devices. But the idea remains: we need a cost assumption for inelastic demands to get the existence results of sustainable price with financing device in elastic demands environment. It is worth pointing out that, for Case (1), one proves that the associated fair sharing market condition is exactly Assumption (IRC). Therefore, the natural extension of stand alone cost property into the so called fair sharing market actually leads to the limit case where the cost function has to exhibit necessarily increasing returns to coalitions. That is, the highest requirement of sustainability of Case (1) puts a light on the technological condition on the cost that arises to guarantee sustainability, namely increasing returns to coalitions.

We will deduce the existence result from a recent contribution of Bonnisseau and Iehlé (2), who state the existence of a core allocation with transfer rate rule equilibrium under a general balancedness condition, called dependent balancedness. Hence, the paper follows the usual stream of cooperative game theoretical approach of sustainable pricing in monopoly. The difference stems from the fact that, contrasting with the usual TU cost games analysis (see details in Iehlé (4)), we restate our stable concepts in terms of core selections of NTU games.

The paper is organized as follows: in Section 5.2, we introduce the contestable market and define our two notions of sustainability for voluntary financing device and valuation rule. We provide two existence results for these price schedules, Theorems 5.2.1 and 5.2.2. Section 5.3 is devoted to the proof of the existence results. We present the formal framework of NTU games and the weak version of an existence result for core allocations with transfer rate rule equilibrium, Theorem 5.3.1. Then, we define suitable NTU games for which the specific core allocations coincide with the sustainable price schedules, and, we show how we can deduce Theorems 5.2.1 and 5.2.2 from Theorem 5.3.1.

5.2 Model and existence results

We will use the following notations: for any set $Y \subset \mathbb{R}^N$, $\text{co}(Y)$, ∂Y, $\text{int}\, Y$ will denote respectively its convex hull, boundary, interior. For any set-valued mapping Γ, $\text{Gr}\,\Gamma$ will denote its graph. For each $A \in \mathcal{N}$, L_A is the $|A|$-dimensional subspace of $\mathbb{R}^N$ defined by $L_A = \{x \in \mathbb{R}^N \mid x_i = 0, \forall i \notin A\}$; L_{A+} (L_{A++}) is the non negative orthant (positive orthant) of L_A; for each $x \in \mathbb{R}^N$, x^A is the projection of x into L_A; $\mathbf{1}$ is the vector of $\mathbb{R}^N$ whose coordinates are equal to 1; $\Sigma_A = \text{co}\{\mathbf{1}^{\{i\}} \mid i \in A\}$; $\Sigma = \Sigma_N$.

Consider Baumol and al. (1) definition of a contestable market :

> *"We define a perfectly contestable market as one that is accessible to potential entrants and has the following two properties: the potential entrants can without restriction serve the same market demands and use the same productive techniques as those available to the incumbent coalition."*

In this paper, additional specifications on the contestable market are price discrimination and single output technology.

- One good is produced. There are $n < \infty$ participants (or local markets). The participants are denoted by script $a \in N := \{1...n\}$.
- Price space is $\mathbb{R}_+^N$, that is, any coalition of participants can set a price $p_a \in \mathbb{R}_+$ for each agent $a \in N$. Each agent $a \in N$ is endowed with a decreasing and continuous demand function $D_a : \mathbb{R}_+ \to \mathbb{R}_+$, bounded above in 0.
- Incumbent and blocking coalitions share the same productive techniques given by a continuous cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$.

In the remainder, denote by $C : \mathcal{N} \times \mathbb{R}_+^N \to \mathbb{R}_+$, the cost function. Given $A \in \mathcal{N}$ and $p \in \mathbb{R}_+^N$: $C(A, p) = c(\sum_{a \in A} D_a(p_a))$.

Given the incumbent price, rivals enter a subset of the market. For the blocking coalitions, the production of the good within a coalition is financed throughout a system of taxes and subsidies. This financing is very much in the spirit of valuation system of Mas-Colell (5). In a first case, we consider that participants are able to find an agreement on a voluntary basis for any system of subsidies-taxes to purchase the good. In a second case, the system is given a priori through a valuation rule. In both cases, a definition of sustainable price is given, the sustainable price is supposed to be stable against the financing device.

We follow Mas-Colell terminologies of voluntary financing device and valuation rule since the analogies apply.

However, a central difference with the above idea stems from the absence of preferences in the usual sense (preference relation or utility function). Indeed, our criteria to decide wether or not a situation is enviable is the rough criterion of maximal quantity. That is, participants will consider the quantity of good they will consume and therefore a situation attract the participants of a coalition if it leads to a strict increase in the consumption of the good for each participant of the coalition. Nevertheless, the rough maximal quantity criterion is usual in the contestable market analysis.

5.2.1 Λ-sustainability

A financing device is a pair (p, v) formed by a price $p \in \mathbb{R}_+^N$ and a sharing vector $v \in \Lambda$, where $\Lambda := \left\{x \in \mathbb{R}_+^N \mid \sum_{a \in \text{supp}(x)} x_a = |\text{supp}(x)|\right\}$. Henceforth, participants can agree for any voluntary financing given by the sharing vector v: a unit of the good is paid by agent $a \in \text{supp}(v)$ at level $v_a p_a$ (See (5, Definitions 11 and 12)).

> *"... we shall regard them (financing devices) as willingness-to-pay functions under the direct control of agents". (Mas-Colell (5, p.631))*

Let Π, $\mathbb{R}_+^N \times \Lambda \to \mathbb{R}$, is the generalized profit function, given $p \in \mathbb{R}_+^N$ and $v \in \Lambda$: $\Pi(p, v) = \sum_{a \in \text{supp}(v)} v_a p_a D_a(p_a) - C(\text{supp}(v), p)$.

Definition 5.2.1 (Λ-sustainability) A price p is said to be Λ-sustainable if:

1. $\Pi(p, \mathbf{1}) = 0$.
2. There is no financing device (p', v) such that $\Pi(p', v) \geq 0$ and for all participants a in $\text{supp}(v)$, $D_a(p'_a) > D_a(p_a)$.

In the above definition, Condition 1. states that the incumbent reaches the budget balance at price p, where participants pay for the demands at the actual price. Condition 2. allows the blocking coalitions to use refined means to reach budget balance by following a financing device. Additionally, we require the blocking coalitions to increase the individual consumption of each of the participants in financing device.

We posit the following assumptions on the contestable market. Let us recall that a set B of $\mathbb{R}^N$ is said to be comprehensive from above if $B + \mathbb{R}_+^N \subset B$.

- *(A1)* For each $v \in \Lambda$, the set $\{\Pi(., v) \geq 0\}$ is comprehensive from above.
- *(A2)* For each $a \in N$, $\{\Pi(., \mathbf{1}^{\{a\}}) \geq 0\} \neq \emptyset$.

$(A1)$ is a regularity assumption, for instance if the profit function is non decreasing with respect to the prices then $(A1)$ is satisfied. Note however that the comprehensiveness condition is weaker than a non decreasing profit function. $(A2)$ states that for any single agent $a \in N$, there is a price p such that the induced expenditures $pD(p)_a$ is greater than the cost to produce that demand. In other terms, each agent can reach a viable project for at least one price. We cannot dispense ourselves from the both assumptions that are necessary for the definition of the NTU game in our game theoretical approach.

The central condition to get the existence of Λ-sustainability is the following:

- *(IRC)* For each $p \in \mathbb{R}_+^N$, $\frac{C(A,p)}{|A|} \geq \frac{C(N,p)}{|N|}$ for all $A \in \mathcal{N}$.

(IRC) is a form of increasing returns to coalition: at any given price, the larger coalition, for inclusion order, is able to purchase the output at the lower cost per capita. Another version for increasing returns to coalition can be found in Demange and Henriet (3). We discuss further the condition at the end of Section 5.2.2.

We are now in position to state the first result of this paper.

Theorem 5.2.1 *Under* $(A1), (A2)$, *and* (IRC) *any contestable market admits a* Λ*-sustainable price.*

5.2.2 v-sustainability

A valuation rule v is now given a priori. Participants are committed to follow the rule for the purchase of the good. Consider, for each $A \in \mathcal{N}$, the set $\Lambda_A = \{x \in \Lambda \mid \text{supp}(x) = A\}$. A valuation rule is a family of continuous mappings $v_A : \mathbb{R}_+^N \to \Lambda_A$, for each $A \in \mathcal{N}$ (See (5, Definitions 7 and 8)).

> *"As a source of valuation system we should think of a coordinating center in charge of announcing and enforcing them." (Mas-Colell (5, p.628))*

We will need the following property on the market.

Definition 5.2.2 (v-Fair Sharing Market) Let $(v_A)_{A \in \mathcal{N}}$ be a valuation rule. The market is said to be v-fair sharing if, for any $p \in \mathbb{R}_+^N$, there exists $q \in \mathbb{R}_+^N$ satisfying:

$$\begin{cases} q \cdot \mathbf{1} = C(N, p) \\ q \cdot v_A(p) \leq C(A, p), \ \ \forall A \in \mathcal{N} \end{cases} \tag{5.1}$$

Hence, the market is v-fair sharing if for any demands, there exists a sharing of the cost such that no coalition is charged, with respect to the valuation rule, more than the total cost of the coalition. When the valuation is reduced to constant mappings $(\mathbf{1}^A)_{A \in \mathcal{N}}$, (5.1) is exactly the so called stand alone cost property used in the analysis of contestable markets (See (1) and (4)).

Definition 5.2.3 (v-sustainability) Let $(v_A)_{A \in \mathcal{N}}$ be a valuation rule. A price p is v-sustainable:

1. $\Pi(p, \mathbf{1}) = 0$.
2. There is no $(p', A) \in \mathbb{R}_+^N \times \mathcal{N}$ such that $\Pi(p', v_A(p')) \geq 0$ and $D_a(p') > D_a(p)$ for all $a \in A$.

The difference with Definition 5.2.1 stems from the obligation for the blocking coalitions to follow the given valuation rule $(v_A)_{A \in \mathcal{N}}$. Let us state the counterpart of Assumption $(A1)$ for that concept of sustainability.

(A1') For each $A \in \mathcal{N}$, the set $\{\Pi(., v_A(.)) \geq 0\}$ is comprehensive from above.

The second existence result for sustainable price is the following.

Theorem 5.2.2 *Under $(A1')$, $(A2)$, any v-fair sharing contestable market admits a v-sustainable price.*

Previous Definitions 5.2.1 and 5.2.3 extend the usual definition of sustainability to the case of a general profit function with financing device. In the definition of sustainability, Condition 2 is replaced by the weaker:

2′. There is no $(p', A) \in \mathbb{R}^N_+ \times \mathcal{N}$ such that: $\Pi(p', \mathbf{1}^A) \geq 0$ and $D_a(p')_a > D_a(p_a)$ for all $a \in A$.

Therefore, Theorem 5.2.2 provides as corollary, an existence result for sustainability in a contestable market with stand alone cost property:

Corollary 5.2.1 *Let the valuation rule be constant and equal to $(\mathbf{1}^A)_{A\in\mathcal{N}}$. Under $(A1')$, $(A2)$, if the market is v-fair sharing then there exists a sustainable price.*

Let us state the definition of Λ-fair sharing market, which is exactly the counterpart of the v-fair sharing market for the case of Λ-sustainability.

Definition 5.2.4 (Λ-Fair Sharing Market) The market is said to be Λ-fair sharing if, for any $p \in \mathbb{R}^N_+$, there exists $q \in \mathbb{R}^N_+$ satisfying:

$$\begin{cases} q \cdot \mathbf{1} = C(N, p) \\ q \cdot v \leq C(\text{supp}(v), p), \ \ \forall v \in \Lambda \end{cases} \tag{5.2}$$

The next proposition puts a light on the relations between the two concepts of sustainability and the two assumptions of fair sharing market.

Proposition 5.2.1 *The following assertions are satisfied:*

1. *p is a Λ-sustainable price if and only if p is v-sustainable for any valuation rule.*
2. *$q \in \mathbb{R}^N_+$ satisfies (5.2) if and only if q satisfies (5.1) for any valuation rule.*
3. *Assumption (IRC) is equivalent to the condition of Λ-fair market.*

Proof of Proposition 5.2.1. The two first equivalences are trivial, to show the last one, let $p \in \mathbb{R}^N_+$, and consider the constant vector $\bar{q} \in \mathbb{R}^N_+$ where $\bar{q}_a = \frac{C(N,p)}{|N|}$ for all $a \in N$. Now, remark that if the condition of Λ-fair sharing holds then necessarily $q = \bar{q}$. Indeed if we consider the grand coalition N, one gets that $\sum_{a\in N} q_a = C(N, p) \leq q \cdot v$ for all $v \in \Lambda$ such that $\text{supp}(v) = N$, it implies that $q = \bar{q}$. Hence, one deduces that for any $A \in \mathcal{N}$, one gets that $\bar{q} \cdot v \leq C(A, p)$ for any

$v \in \Lambda$ such that $\mathrm{supp}(v) = A$, then $\frac{C(N,p)}{|N|} \sum_{a \in A} v_a \leq C(A,p)$, that is $\frac{C(A,p)}{|A|} \geq \frac{C(N,p)}{|N|}$. Conversely, suppose that (IRC) holds, then by considering the vector $\bar{q}$, one can deduce straightforwardly that the condition of Λ-fair sharing is satisfied. □

The first two assertions deserve no special comments. The last equivalence is insightful. We know from Iehlé (4), that the condition of stand alone cost is central to prove the existence of sustainable price, in the usual sense (the stand alone cost is exactly the v-fair sharing market for the constant valuation rule $(\mathbf{1}^A)_{A \in \mathcal{N}}$). Furthermore, it is proved that a sufficient condition for the property is given by increasing returns to coalitions (IRC).

In Sub-Section 5.2.2, the stand alone cost property is generalized into the notion of v-fair sharing market for any valuation rules to get predictably the existence of the stronger notion of v-sustainability. Going a step further, the analysis of Λ-sustainability is the highest requirement of sustainability one can define. For that definition, the natural extension of v-fair sharing market into the so called Λ-fair sharing market actually leads to the limit case where the cost function has to exhibit increasing returns to coalitions (Assumption (IRC)). Hence, the Λ-sustainability puts a light on the technological condition that arises to guarantee sustainability, and Assumption (IRC) is somehow a minimal condition one should expect in the analysis of Λ-sustainability.

5.3 Proof of Theorems

We use a recent contribution of Bonnisseau and Iehlé (2) who states that there exists a core allocation with a transfer rate rule equilibrium in dependent balanced NTU games. We restate the existence of a sustainable price as the existence of a core allocation with a transfer rate rule equilibrium. The idea behind the notion of dependent balancedness is to consider a transfer rate rule depending on the payoffs to define a notion of balancedness whereas the usual transfer rate of the literature is supposed to be constant. In the following subsection, we recall formally the definition of dependent balancedness and the non-emptiness result.

5.3.1 A core selection in NTU games

A game $(V_A, A \in \mathcal{N})$ is a collection of subsets of $\mathbb{R}^N$ indexed by $\mathcal{N}$. $x \in \mathbb{R}^N$ is called a payoff; $V_A \subset \mathbb{R}^N$ is the set of feasible payoffs of the coalition A; $\mathcal{S}(x) = \{A \in \mathcal{N} \mid x \in \partial V_A\}$ is the set of coalitions, for which $x \in \mathbb{R}^N$ is an efficient payoff; $W := \cup_{A \in \mathcal{N}} V_A$ is the union of the payoffs sets.

(H1) (i) $V_{\{i\}}$, $i \in N$, and V_N are non-empty. (ii) For each $A \in \mathcal{N}$, V_A is closed, $V_A + \mathbb{R}^N_+ = V_A$, $V_A \neq \mathbb{R}^N$, and, for all $(x, x') \in (\mathbb{R}^N)^2$, if $x \in V_A$ and $x^A = x'^A$, then $x' \in V_A$.

(H2) There exists $m \in \mathbb{R}$ such that, for each $A \in \mathcal{N}$, for each $x \in V_A$, if $x \notin \mathrm{int}\, V_{\{i\}}$ for all $i \in A$, then $x_j \geq m$ for all $j \in A$.

Definition 5.3.1 Let $(V_A, A \in \mathcal{N})$ be a game. A payoff x is in the core of the game if $x \in V_N \setminus \text{int}\, W$.

We introduce now the notion of **1**-dependent balancedness, which is a special case of the more general notion of dependent balancedness as defined in (2).

Definition 5.3.2 Let $(V_A, A \in \mathcal{N})$ be a game satisfying Assumption H1: (i) A transfer rate rule is a collection of set-valued mappings $(\varphi_A)_{A \in \mathcal{N}}$ such that for all $A \in \mathcal{N}$, φ_A is upper semi-continuous with non-empty compact and convex values from ∂V_A to Σ_A (ii) The game $(V_A, A \in \mathcal{N})$ is **1**-dependent balanced if there exists a transfer rate rule $(\varphi_A)_{A \in \mathcal{N}}$ such that, for each $x \in \partial W$, if $\frac{\mathbf{1}}{|N|} \in \text{co}\{\varphi_A(x) \mid A \in \mathcal{S}(x)\}$, then $x \in V_N$.

Consider now a weak statement of a result obtained in (2).

Theorem 5.3.1 *Let $(V_A, A \in \mathcal{N})$ be a game satisfying Assumptions H1 and H2. If it is **1**-dependent balanced with respect to the transfer rate rule $(\varphi_A)_{A \in \mathcal{N}}$, there exists a core allocation x such that: $\frac{\mathbf{1}}{|N|} \in \text{co}\{\varphi_A(x) \mid A \in \mathcal{S}(x)\}$.*

5.3.2 End of the Proof of Theorem 5.2.1

Consider, for each $A \in \mathcal{N}$, the set $\Lambda_A = \{x \in \Lambda \mid \text{supp}(x) = A\}$. Note that a price p is Λ-sustainable if: $\Pi(p, \mathbf{1}) = 0$ and there is no $(A \in \mathcal{N}, v_A \in \Lambda_A, p' \in \mathbb{R}^N_+)$ such that: $\Pi(p', v_A) \geq 0$ and $p'_a < p_a$ for all $a \in A$. Define the following NTU game $(V_A, A \in \mathcal{N})$ and transfer rate rule $(\varphi_A)_{A \in \mathcal{N}}$. For each $A \in \mathcal{N}$:

$$V_A = \left\{p \in \mathbb{R}^N_+ \mid \exists v \in \Lambda_A \; s.t. \Pi(p, v) \geq 0\right\}$$

For each $A \in \mathcal{N}$, $\varphi_A : \partial V_A \to \Sigma_A$. Given $p \in \partial V_A$:

$$\varphi_A(p) = \{t \in \Sigma_A \mid \Pi(p, t|A|) \geq 0\}$$

Lemma 5.3.1 *The game $(V_A, A \in \mathcal{N})$ satisfies Assumptions H1,H2.*

Proof of Lemma 5.3.1. From $(A2)$, $V_{\{a\}} \neq \emptyset$. To show that V_N is non-empty, consider $\{\Pi(., \mathbf{1}^{\{a\}}) \geq 0\} \neq \emptyset$ for all $a \in N$, let $p \in \cap_{a \in N}\{\Pi(., \mathbf{1}^{\{a\}}) \geq 0\}$, then $\Pi(p, \mathbf{1}^N) \geq 0$. Indeed, from Assumption (IRC), there exists $q \in \mathbb{R}^N_+$ such that (consider $v = \mathbf{1}^{\{a\}}$ for each $a \in N$): $\sum_{a \in N} p_a D_a(p_a) \geq \sum_{a \in A} C(\{a\}, p) \geq \sum_{a \in N} \frac{C(N,p)}{|N|} = C(N, p)$. For $H1(ii)$, it follows from the fact that demands of participants are independent, and comprehensiveness from above follows from Assumption $(A1)$. Assumption $(H2)$ follows from the fact that the game is bounded below by 0. $\square$

Lemma 5.3.2 *For all $A \in \mathcal{N}$, φ_A is upper semi-continuous with non-empty compact and convex values from ∂V_A to Σ_A.*

Proof of Lemma 5.3.2. The rules φ_A have non empty values by construction. It is obvious that the mappings have compact values. We check that the mappings have convex values. Let $t, t' \in \varphi_A(p)$. Consider $\bar{t} = \nu t + (1-\nu)t'$ for $\nu \in [0; 1]$, then $\bar{t} \in \Sigma_A$, and furthermore $\Pi(p, \bar{t}|A|) \geq 0$ since we have got that $\sum_{a \in A} \bar{t}_a |A| p_a D_a(p_a) = \sum_{a \in A} (\nu t_a + (1-\nu)t'_a)|A| p_a D_a(p_a) = \sum_{a \in A} \nu t_a |A| p_a D_a(p_a) + \sum_{a \in A} (1-\nu) t'_a |A| p_a D_a(p_a) \geq C(A, p)$. Hence $\bar{t} \in \varphi_A(p)$.

It just remains to show the closed graph assumption of φ_A for each $A \in \mathcal{N}$. Let $p^\nu \in \partial V_A$ and $t^\nu \in \Sigma_A$ be two sequences respectively converging to p and t and such that $t^\nu \in \varphi_A(p^\nu)$ for each ν. Then, from the continuity of the mappings $C(A, .)$ and $(D_a)_{a \in A}$, we get that $\Pi(p, t|A|) = 0$, that is $t \in \varphi_A(p)$, hence the closed graph assumption is satisfied. □

Lemma 5.3.3 *Let $p^* \in \partial W$, $\lambda_A \in \mathbb{R}_+$ and $t_A \in \varphi_A(p^*)$ for each $A \in \mathcal{S}(p^*)$, such that:*

$$\sum_{A \in \mathcal{S}(p^*)} \lambda_A = 1 \text{ and } \sum_{A \in \mathcal{S}(p^*)} \lambda_A t_A = \frac{\mathbf{1}}{|N|}.$$

Then $\Pi(p, \mathbf{1}) \geq 0$.

Proof of Lemma 5.3.3.

From the assumptions of the lemma:

$$\frac{1}{|N|} \sum_{a \in N} p^*_a D_a(p^*_a) = \sum_{a \in N} \sum_{A \in \mathcal{S}(p^*) | A \ni a} \lambda_A t^a_A p^*_a D_a(p^*_a)$$

$$= \sum_{A \in \mathcal{S}(p^*)} \sum_{a \in A} \lambda_A t^a_A p^*_a D_a(p^*_a) = \sum_{A \in \mathcal{S}(p^*)} \lambda_A \sum_{a \in A} t^a_A p^*_a D_a(p^*_a)$$

Since $t_A \in \varphi_A(p^*)$,

$$\geq \sum_{A \in \mathcal{S}(p^*)} \frac{\lambda_A}{|A|} C(A, p^*) \tag{5.3}$$

From (IRC), $\sum_{A \in \mathcal{S}(p^*)} \frac{\lambda_A}{|A|} C(A, p^*) \geq \sum_{A \in \mathcal{S}(p^*)} \frac{\lambda_A}{|N|} C(N, p^*) = \frac{C(N, p^*)}{|N|}$. Hence, Lemma 5.3.3 is proved. □

From Lemmas 5.3.2 and 5.3.3, the game $(V_A, A \in \mathcal{N})$ is $\mathbf{1}$-dependent balanced with respect to the transfer rate rule $(\varphi_A)_{A \in \mathcal{N}}$. Hence, from Lemma 5.3.1, an application of Theorem 5.2.1 shows the existence of a core allocation p^* such that $\text{co}\{\varphi_A(p^*) \mid A \in \mathcal{S}(p^*)\} \cap \{\frac{\mathbf{1}}{|N|}\} \neq \emptyset$. Using again Lemma 5.3.3, we deduce first that $\Pi(p^*, \mathbf{1}) \geq 0$. Since the demands functions are bounded above in 0, necessarily, $p^* \neq 0$ because $\Pi(0, \mathbf{1}) < 0$. Then, from the continuity of all the mappings at stake, we can consider without loss of generality that, $\Pi(p^*, \mathbf{1}) = 0$. Furthermore, p^* is a core allocation, thus it is also true from the definition of the game that Condition 2 of Definition 5.2.1 is satisfied. This ends up Proof of Theorem 5.2.1. □

5.3.3 *End of the Proof of Theorem 5.2.2*

The game slightly differs from the previous one. Define the following NTU game $(V_A, A \in \mathcal{N})$ and transfer rate rule $(\varphi_A)_{A\in\mathcal{N}}$. The transfer rate rule is exactly the normalized valuation rule given by the property of v-fair sharing market. For each $A \in \mathcal{N}$:

$$V_A = \left\{p \in \mathbb{R}_+^N \mid \Pi(p, v_A(p)) \geq 0\right\}$$

For each $A \in \mathcal{N}$, $\varphi_A : \partial V_A \to \Sigma_A$. Given $p \in \partial V_A$:

$$\varphi_A(p) = \left\{\frac{v_A(p)}{|A|}\right\}$$

From now on, the proof is following the stream of Proof of Theorem 5.2.1. It is an easy matter to check that Lemmas 5.3.1 and 5.3.2 also hold true for this game and this transfer rate rule. Lemma 5.3.3 is also verified, using (5.3) and by a direct application of Assumption of v-fair sharing market. After (3), from the property of Λ-fair sharing where $v_A = t_A|A|$, for each $A \in \mathcal{S}(p^*)$, there exists $q \in \mathbb{R}_+^N$ such that:
$\sum_{A\in\mathcal{S}(p^*)} \frac{\lambda_A}{|A|} C(p^*, A) = \sum_{A\in\mathcal{S}(p^*)} \frac{\lambda_A}{|A|} q \cdot v_A(p^*)$.

$$\frac{1}{|N|} \sum_{a\in N} p_a^* D_a(p_a^*) \geq \sum_{A\in\mathcal{S}(p^*)} \lambda_A \sum_{a\in A} q_a t_A^a$$

Hence,

$$= \sum_{a\in N} q_a \sum_{A\in\mathcal{S}(p^*)|A\ni a} \lambda_A t_A^a = q \cdot \frac{\mathbf{1}}{|N|} = \frac{C(p^*, N)}{|N|}$$

Hence, Lemma 5.3.3 is verified. We end the proof of Theorem 5.2.2 by considering the arguments given in the last paragraph of Proof of Theorem 5.2.1. □

5.4 *A concluding remark*

One weakness of our NTU game approach in price space is that it enforces us to consider independent demands between participants in order to get suitable payoffs sets. The usual approach by TU games (cost games) can deal with more inter-dependencies and multiproduct technology (see Iehlé (4)). But, in that approach, one needs boundary conditions to apply a fixed point argument.

For example, up to boundary conditions, it proved in Iehlé that a sustainable price (in the usual sense) exists under the property of stand alone cost (v-fair sharing for the constant valuation rule $(\mathbf{1}^A)_{A\in\mathcal{N}}$). The problem is reduced to a fixed point argument within a TU game approach. One can get a similar result within NTU game approach, the game turns out to be balanced (in Scarf's sense).

For v-sustainability, one needs the more refined condition of dependent balancedness, we have proved that the NTU game is dependent balanced under v-fair sharing market condition. Conversely, the fixed point argument in TU games can also lead to the existence of v-sustainability under v-fair sharing market condition.

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Cœur et balancement dépendant: théorie et applications

Chapitre 1. Balancement dépendant et cœurs. *Ecrit en collaboration avec Jean-Marc Bonnisseau.*
Article: *Payoff-dependent balancedness and cores*, en révision dans *Games and Economic Behavior.*
Résumé. Nous proposons un résultat de non-vacuité du cœur dans un jeu coopératif sans paiements latéraux. On utilise une condition de balancement dépendant des paiements à partir de règles de taux de transfert qui généralise les versions précédentes de balancement avec poids constant. Des extensions du concept de cœur sont proposées incluant des cœurs satisfaisant une condition supplémentaire et des cœurs avec équilibre dans le cadre de jeux coopératifs paramétrés. Les preuves d'existence empruntent des outils mathématiques à la théorie de l'équilibre général avec non-convexités. Des applications variées à des résultats de théorie des jeux et d'économie théorique sont données. *Journal of Economic Literature* Classification Numbers : C60, C62, C71, D50, D51.
Mots-clé: condition de balancement, concepts de cœurs, jeux paramétrés, non vacuité.

Chapitre 2. Règles de transfert et sélections du cœur.
Article: *Transfer rate rules and core selections in NTU games*, publié dans *Economics Bulletin.*
Résumé. On propose quelques applications directes d'un résultat d'existence de Bonnisseau et Iehlé (2003). Ces auteurs ont montré l'existence d'allocations du cœur dans les jeux NTU qui satisfont un équilibre de taux de transfert sous une condition de balancement dépendant. Il s'avère que la notion de balancement dépendant procure en fait un outil manipulable pour sélectionner le cœur. Pour illustrer ce fait, nous montrons que cette notion permet d'obtenir des résultats d'existence dans des modèles de cœur avec partenariat, cœur socialement stable, prekernel moyen intersecté avec le cœur et de cœur interne faible. *Journal of Economic Literature* Classification Numbers: C60, C71.
Mots-clés: jeux coopératifs, balancement dépendant, sélections du cœur dans les jeux NTU.

Chapitre 3. Tarification stable et cœur-équilibre.
Article: *Stable pricing and equilibrium-core in parameterized cost games*, soumis à *Annals of Operations Research.*
Résumé. L'existence de tarifications sans subventions croisées et soutenable est prouvée dans un marché contestable multiproduit où les firmes ont la possibilité de discriminer les marchés locaux, composés d'une partie de la ligne commerciale et d'une partie d'agents. Les résultats sont obtenus sous une hypothèse de fonction de coût à partage équitable, et sous des conditions de bord des fonctions de demandes. Le problème de tarification est modélisé par des cœurs-équilibres de jeux de coût paramétrés. *Journal of Economic Literature* Classification Numbers: C71, L11, L12.
Mots-clés: jeux coopératifs, marchés contestables, soutenabilité, subventions croisées, jeux de coût paramétrés.

Chapitre 4. Soutenabilité versus projet de financement.
Article: *Sustainability versus financing device.*
Résumé. Nous prouvons l'existence de tarification soutenables sur un marché contestable monoproduit avec discrimination par les prix. La définition de soutenabilité est généralisée dans un premier temps au cas de coalitions bloquantes pouvant décider d'un mécanisme de financement volontaire; puis dans un second temps au cas de règle de valuation donnée a priori. Les résultats d'existence sont prouvés sous des hypothèses de fonctions de coût à partage équitable généralisées. Nous utilisons des développements récents de la théorie des jeux sur des sélections de cœur dans les jeux NTU. *Journal of Economic Literature* Classification Numbers: C71, L11, L12.
Mots-clés: soutenabilité, marché contestable, sélections du cœur dans les jeux NTU.